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Packing Chromatic Number of Square Lattice

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Declaration

I declare that this dissertation titled ”Packing Chromatic Number of a Square Lattice” has been composed by myself in the Faculty of Science, School of Mathematics at the University of the Witwatersrand, Johannesburg and that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or processional qualification except as specified.

Signature **Date**

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Abstract

The notion of packing coloring comes from the area of frequency planning in wireless networks, and was introduced by Goddard et al [1], under the name broadcast coloring. Broadcast colouring seems to have been renamed packing colouring in the work in [8].If two vertices of a graph are at distance d and the vertices represent radio stations, and if the radio stations are close enough so that they assign frequencies that interfere with each other, a coloring can be used to reassign non-interfering frequencies to the stations. The United States Federal Communications Commission has come up with many rules and regulation with regards to the assignment of broadcast frequencies to radio stations. Specifically, two radio stations which have received the same broadcast frequency must be located sufficiently far apart so that neither broadcast interferes with the reception of the other.

A function π: $V \rightarrow \{1,2,...,k\}$ is *packing coloring of order k* if $\pi(u) = \pi(v)$ means that the distance between u and v is more than $\pi(u)$. The minimum number of colors with which the vertices of a graph G can be packing colored is called the *packing chromatic number*, denoted by $\chi_\rho(G)$. In this dissertation, we will be investigating the upper bound and the lower bound of the packing chromatic number of graphs, in particular, that of the 2-dimensional infinite square lattice.

Suppose n is an integer $n \geq 1$. Define $c = 4n$. We will also investigate the packing chromatic number of the Cartesian product of C_4 and C_c. Since the sub-grid $P_4 \square P_c$ is a sub-graph of the torus $C_4 \square C_c$, we know that $\chi_\rho(P_4 \square P_c) \leq \chi_\rho(C_4 \square C_c)$. Generally speaking, $\chi_\rho(P_r \square P_c) \leq \chi_\rho(C_r \square C_c)$ for r rows and c columns where r and c are in $\mathbb{N}$, where r, $c \geq 3$. The relationship between the packing chromatic numbers of the grid and torus will be remarked on further in the conclusion of this study.

Chapter 1

Introduction

1.1 Introduction

We define a ***graph*** G as a nonempty finite set V of objects called ***vertices*** (vertex in singular form) together with a set E (possible empty) of 2-element subsets of V called ***edges***. Vertices are fundamental units of which graphs are formed. An ***edge*** is an ordered pair of vertices. We say that u and v are ***adjacent*** if uv is an edge in G. If two vertices are adjacent in a graph G, then they are called ***neighbors*** of each other. An ***open neighborhood*** of a vertex v, denoted by $N(v)$, is a set of neighbors of v. A ***closed neighborhood*** of v is a set $N[v] = N(v) \cup \{v\}$. We say that uv and vw are ***adjacent edges*** if uv and vw are different edges in G. The edge uv and a vertex u are ***incident*** with each other.

We define the ***order*** of a graph G as the number of vertices in G, denoted by n, and the ***size*** of a graph G is the number of edges in G, denoted by m. The number of vertices in G that are adjacent to v is said to be the ***degree*** of the vertex v in a graph G. An ***isolated vertex*** is a vertex with degree zero and an ***end vertex*** or a ***leaf*** is a vertex with degree one. A ***pendant edge*** is defined as an edge incident to the end vertex. A ***maximum degree*** of a graph G, denoted by $\Delta(G)$ is the largest degree among the degrees of G while ***minimum degree***, $\delta(G)$ is the smallest degree among the degrees of G.

A ***trivial*** graph is a graph of order one and a graph with two or more vertices is therefore called a ***nontrivial*** graph. An ***empty*** graph G is a graph with size zero and at least one vertex and a graph with one or more edges is called a ***nonempty*** graph. An empty graph does not contain any adjacent vertices. On the other hand, if every two distinct vertices are adjacent we say that the graph G is ***complete*** and the size of a complete graph of order n is $\binom{n}{2} = n(n-1)/2$. If the degree of a vertex v is even in G, then v is an ***even vertex*** in a graph G and if the degree of a vertex v is odd, then v is an ***odd vertex***. If $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is said to be a ***sub-graph*** of a graph G.

A graph G is a ***super-graph*** of H if H is a sub-graph of G. H is a ***spanning sub-graph*** of G if $V(H) = V(G)$. H is a ***proper sub-graph*** of G if H is a sub-graph of a graph G and $V(H)$ is a proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$. Let $G[S]$ denote a sub-graph of G induced by S. If there exist a nonempty subset S of $V(G)$ such that $H = G[S]$, then H is called an ***induced sub-graph***. If there exist a nonempty subset X of $E(G)$ such that $H = G[X]$, then we say that a sub-graph H of G is ***edge-induced***. In a graph G, if the vertices of G have the same degree, then G is ***regular*** and is regular of degree r if this degree is r. We sometimes call these graphs r***-regular***.

A ***cubic*** graph is a 3-regular graph. We define graph G to be a ***bipartite graph*** if $V(G)$ can be partitioned into two sets U and W (called ***partite sets***) so that every edge of G joins a vertex of U and a vertex of W. Suppose $V(G)$ can be partitioned into two sets U and W, then a graph G is a ***complete bipartite*** graph if uv is an edge of G for every $u \in U$ and for every $w \in W$. A ***complete*** k***-partite*** graph for some integer $k \geqslant 2$ is called a ***complete multipartite*** graph. A graph denoted by $\bar{G}$ with vertex set $V(G)$ such that two vertices are adjacent in $\bar{G}$ if and only if these vertices are not adjacent in a graph G, we call the ***complement*** of G. A ***self-complementary*** graph G is a graph isomorphic to $\bar{G}$, for example, graphs P_4 and C_5. If there exists a bijective function $\theta : V(G_1) \to V(G_2)$ such that vertices v and u are adjacent in G_1 if and only if $\theta(u)$ and $\theta(v)$ are adjacent in G_2, then the function θ is called an ***isomorphism*** from G_1 to G_2.

The ***Cartesian product*** $G_1 \square G_2$ of graphs G_1 and G_2 is a graph such that the vertex set of $G_1 \square G_2$ is the Cartesian product $V(G_1) \times V(G_2)$ and any two vertices (u, v) and (w, z) are adjacent in $G_1 \square G_2$ if and only if either $u = w$ and v is adjacent to z in G_2 or $v = z$ and u is adjacent to w in G_1. Let u and v be vertices in a graph G not necessarily different. Then we define a $u - v$ ***walk*** W in G as a sequence of vertices in G, starting from u and finishing at v such that consecutive vertices in W are adjacent in G. The ***length*** of W is the number of edges that are in W. An ***open walk*** is one whose first and end vertices are distinct, otherwise it is a ***closed walk***. If no edges are repeated in a nontrivial closed (open) walk, this walk is called a ***circuit (trail)*** in a graph G.

A finite or infinite sequence of edges in a graph G which connect a sequence of vertices which are all distinct from each other is called a ***path*** of G. A circuit S: $v = v_0, v_1, ..., v_k = v$, $k \geq 2$, for which the vertices $v_i, 0 \leq i \leq k-1$ are different is a ***cycle*** C_k in G. A cycle of length $k \geq 2$ is a k-cycle. We also call a 3-cycle a ***triangle***. If a cycle has an even length, it is then called an ***even cycle***; while a cycle of odd length is called an ***odd cycle***. A graph G is ***connected*** if every two vertices in G are connected. A ***disconnected*** graph is a graph that is not connected. We define a ***distance*** from a vertex u to vertex v in a graph G, denoted by $d_G(u, v)$, as the minimum length of a $u - v$ path in G. A $u - v$ path with length $d_G(u, v)$ is a $u - v$ ***geodesic***. The maximum distance between a vertex v and any other vertex u of G in a connected graph G is called the **eccentricity** of vertex v, denoted by $e(v)$.

The greatest eccentricity among the vertices of G is called the ***diameter*** of G, denoted by $diam(G)$, while the smallest eccentricity among the vertices of G is called the **radius** of G, denoted by $rad(G)$. We define the ***directed*** graph or a ***digraph*** G as a finite nonempty set of vertices together with set of directed edges or arcs or just edges. An ***undirected*** graph without multiple edges is called a ***simple graph***. In graph theory, graph coloring is an assignment of labels called "colors" to elements of a graph subject to certain constraints. The coloring of vertices of a graph G such that no two adjacent vertices share the same color is called a ***vertex coloring***. A ***minimum vertex coloring*** seeks to minimize the number of colors for a given graph and this is the most general type of vertex coloring. The minimum number of colors with which the vertices of a graph G may be colored is called the ***chromatic number***, denoted by $\chi(G)$.

In Boolean logic and computer science, a ***SAT solver*** is a tool that takes as input a conjunctive normal form formula and outputs either a satisfying Boolean assignment to the variables used in the conjunctive normal form formula if the formula is consistent or UNSAT if it is not. The ***Boolean satisfiability problem (SAT)*** is, given a formula, to check whether it is satisfiable.

The idea of packing coloring developed from the area of frequency assignment in wireless networks. In graphs, we seek for a partition of the vertex set of a graph G containing independent sets $X_1, ..., X_k$ (representing frequency usage) according to the following constraints: every color class X_i must be an ***i-packing***, which means that the set of vertices has the property that for any different pair $u, v \in X_i$, it holds that $d(u, v) > i$.

The notion of the packing chromatic number is essential in this dissertation and it was introduced by Goddard [1] under the name "broadcast chromatic number of graph". A partition of a vertex set $V(G)$ into independent sets $X_1, ..., X_k$, for each $i \in \{1, ..., k\}$ and $x, y \in X_i$ such that the minimum distance between x and y in G is greater than i, is called a ***packing coloring***. The minimum number of colors with which the vertices of a graph G can be packing colored is called the ***packing chromatic number***, denoted by $\chi_\rho(G)$.

A type of a lattice in a two-dimensional space, $P_r \square P_c$ where r, $c \geq 3$, is called a ***square lattice***. We will put our focus on simple undirected graphs, specifically the 2-dimensional infinite square lattice. We seek to find the precise packing chromatic number of the square lattice. Bernard, Jiri and Sandi showed in [2] that $\chi_\rho(\mathbb{Z}^2) \geq 10$, which improves the lower bound $\chi_\rho(\mathbb{Z}^2) \geq 9$ [1]. Also it has been proven that $\chi_\rho(\mathbb{Z}^2) \geq 12$ [2] which improves Bernard and et al lower bound of $\chi_\rho(\mathbb{Z}^2) \geq 10$. The latest lower bound improvement was done by Barnaby, Franco, Chen and Jos in [5] where the lower bound is 13. Schwenk [4] improved the upper bound of Goddard of 23 [1] to 22, and later Holub and Soukal [5] improved it to 17. Thereafter, Barnaby and others improved it to 16 [5]. The latest upper bound improvement was done by Barnaby, Franco, Chen and Jos in [5] where the upper bound is 15.

We will first consider the lower bound of the packing chromatic number of grid and then we will later consider the upper bound of the packing chromatic number of a grid.

Chapter 2

Graphs with small packing chromatic numbers

2.1 The Packing chromatic number for paths and cycles

Proposition 2.1.1. [1] *For $n \geq 4$, $\chi_\rho(P_n) = 3$ and $2 \leq n \leq 3$, $\chi_\rho(P_n) = 2$.*

Proof. The packing coloring for P_n is shown by the first n-entries below.

1 2 1 3 | 1 2 1 3 |...

This is the best possible pattern.

□

Proposition 2.1.2. [1] *Suppose n is 3 or a multiple of 4, for $n \geq 3$, then $\chi_\rho(C_n) = 3$; otherwise $\chi_\rho(C_n) = 4$.*

Proof. As the cycle has either P_4 or K_3, $\chi_\rho(C_n) \geq 3$. Let v_0, v_1, v_2, ..., v_{n-1}, v_0 be the vertices of C_n. Assume there exists a packing coloring of order 3 with $n \geq 4$. Hence, no two consecutive vertices has color 1. Suppose that v_2 receives color 2 and v_3 receives color 3. It follows that neither v_0 nor v_1 can get color 2 or 3, and only one between them receives color 1, which is a contradiction. Note that vertices which contain color 1 cannot be consecutive, hence the vertices which contain color 1 alternate, in particular, n is even.

Suppose the even numbered vertices contain color 1. Since no two consecutive odd-numbered vertices have the same color, it follows the colors must alternate between colors 2 and 3. If n is a multiple of 4, $\chi_\rho(C_n) \geq 3$. If n is not a multiple of 4, we have that $\chi_\rho(C_n) \geq 4$.

Let us have a look at optimal packing colorings. For cycles with order n a multiple of 4, the pattern;

$$1, 2, 1, 3, 1, 2, 1, 3, ..., 1, 2, 1, 3$$

is a packing coloring. On the other hand when n is not a multiple of 4, the pattern consists of repeating blocks of "$1, 2, 1, 3$" with an adjustment at the very end:

$$\begin{aligned} n = 4r + 1 &: 1, 2, 1, 3, 1, 2, 1, 3, ..., 1, 2, 1, 3, 4 \\ n = 4r + 2 &: 1, 2, 1, 3, 1, 2, 1, 3, ..., 1, 2, 1, 3, 1, 4 \\ n = 4r + 3 &: 1, 2, 1, 3, 1, 2, 1, 3, ..., 1, 2, 1, 3, 1, 2, 4 \end{aligned}$$

Therefore, if n is a multiple of 4, it follows that $\chi_\rho(C_n) \leq 3$; otherwise $\chi_\rho(C_n) \leq 4$. □

2.2 Graphs with small packing chromatic numbers

A ***vertex cover*** is a set of vertices of a graph that includes at least one endpoint of every edge of the graph. We use $\alpha_0(G)$ for the ***vertex cover number***, which is the size of a minimum vertex cover. We define a ***edge cover*** is a set of edges of a graph such every vertex of a the graph is incident to at least one edge of the set. The ***edge cover number***, indicated by $\alpha_1(G)$, is the minimum of the cardinalities of all edge covers. An ***independent vertex set*** in a graph is defined as a set where no two vertices in the set are adjacent. We indicate the ***vertex independence number*** by $\beta_0(G)$, which is the cardinality of the maximum independent vertex set. $\beta_1(G)$ is used for the ***edge independence number***, sometimes called the ***maximum matching number***, which is the size of the largest independent edge set.

In computational complexity theory, NP-hard refers to the property of a class of problems that are informally harder than those that can be solved by a nondeterministic Turing machine in polynomial time.

Theorem 2.2.1. *(Gallai) For any connected and nontrivial graph $G = (V, E)$ with n vertices,*
(i) $\beta_0(G) + \alpha_0(G) = n$
(ii) $\beta_1(G) + \alpha_1(G) = n$

It is NP-hard to determine if the packing chromatic number of a graph is at most 4.

Proposition 2.2.2. [1] *Suppose a graph G has diameter two, then*

$$\chi_\rho(G) \leq \alpha_0(G) + 1.$$

Proof. We begin by giving color 1 to every vertex in a maximum independent set in G. We then give each vertex not in the independent set a different color. The equality $n - \beta_0(G) = \alpha_0(G)$ follows from Gallai's theorem. Suppose a graph has diameter two, it follows that no two vertices receive the same color i, for any $i \geq 2$. Also, since the vertices which receive the color 1 form an independent set, we have at most $\beta_0(G)$ such vertices.

□

We begin with graphs with packing chromatic number 2.

Proposition 2.2.3. [1] *Given a connected graph G, G is a star if and only if $\chi_\rho(G) = 2$.*

Proof. The packing chromatic number of the star $K_{1,m}$ is greater or equal to 2 because the internal node is adjacent of all leaves and hence two colors are needed. Suppose $\chi_\rho(G) = 2$. This means that P_4 is not contained in G and $diam(G) \leq 2$. It follows from Proposition 2.2.2 that $\alpha_0(G) = 1$; ie G is a star.

□

With regards to the graphs with $\chi_\rho(G) = 3$, we start by characterizing blocks with this property. Suppose G is a graph, then we denote by $S(G)$ the ***sub-division*** graph of G, which is the result of subdividing every edge once. In $S(G)$, the vertices of G are called the ***original vertices***; while the other vertices are called ***sub-division vertices***. A ***vertex-connectivity*** of a connected graph G is defined as the least number of vertices whose removal reduces G to a trivial or disconnected graph, denoted by $K(G)$. A connected graph G is called ***2-connected*** if for every vertex $x \in V(G)$, $G{-}x$ is connected. A ***block*** is a maximal 2-connected sub-graph of a graph.

Proposition 2.2.4. [1] *Assume G is a 2-connected graph. Then $\chi_\rho(G) = 3$ if and only G is either $S(H)$ for some bipartite multi-graph H or the join of K_2 and an independent set.*

Proof. Suppose that $\chi_\rho(G) = 3$. Let π: V(G) $\rightarrow$ {1, 2 ,3} be a packing coloring and assume $V_i = \pi^{-1}(i)$, for $1 \leq i \leq 3$. Therefore V_i is an i-packing for each i. Assume $v \in V_1$. Set V_1 is an independent set, so $N(v) \subseteq V_2 \cup V_3$. But v contains at most one neighbor in V_3. Since v contains at least two neighbors, we have that v has degree 2 and is adjacent to exactly one vertex in each of V_2 and V_3. From Proposition 2.1.2, it follows that the length of any cycle in G is either 3 or a multiple of 4.

Suppose that G has a triangle x, y, z. These vertices get different colors; say $x \in V_2$ and $y \in V_3$. Therefore, for every neighbor m of x apart from y, $m \in V_1$ and since it is too close to y to have another neighbor of color 3, $N(m) = \{x, y\}$. Since y cannot be a cut-vertex, this is the whole G. Suppose that every cycle length is a multiple of 4. From the proof of Proposition 2.1.2, in any packing coloring of order 3 of a cycle, every alternating vertex gets color 1. Therefore $V_2 \cup V_3$ is an independent set. Note that since G is a block, every edge lies in a cycle. In particular, G is a sub-division of some multi-graph H where every subdivision vertex gets color 1. Moreover, since V_2 and V_3 are 2-packings in G, this means they are independent sets in H; that is, (V_2, V_3) is a bi-partition of H.

Conversely, to packing color the sub-division of a bipartite multi-graph, consider V_1 as the sub-division vertices and (V_2, V_3) as the original bi-partition.

□

In order to categorize general graphs with packing chromatic number 3, we define a *T-odd* operation to a vertex v as introducing a vertex w_v and a set X_v of independent vertices, and adding the edge vw_v, and some of the edges between v, w_v and X_v. By extending the above result, one can show the following result.

Proposition 2.2.5. [1] *Let G be a graph. Then $\chi_\rho(G) = 3$ if and only G can be formed by taking some bipartite multi-graph H with bipartition (V_2, V_3), sub-dividing every edge exactly once, adding leaves to some vertices in $V_2 \cup V_3$, and then performing a single T-odd operation to some vertices in V_3.*

2.3 Trees

We now investigate trees with large packing chromatic numbers. We first discuss trees with small packing chromatic number. A tree with diameter 2, that is, a star, has a packing chromatic number 2. Also, a tree with diameter 3 has packing chromatic number 3. P_4 is the only tree on 4 vertices that needs 3 colors. We will mention the result of the packing chromatic number for trees with diameter 4. A vertex is ***large*** if it has degree 4 or more, otherwise it is ***small***. A vertex is ***central*** in a graph G if its eccentricity is less or equal the eccentricity of any other vertex of the graph.

Proposition 2.3.1. [1] *Suppose T is a tree of diameter 4 with central vertex v. We denote n_i to be the number of neighbors of v of degree i, for $i = 1, 2, 3$ and assume L is the number of large neighbors of v, that is the number of vertices with degree greater than 3. If $L = 0$ then*

$$\chi_\rho(T) = \begin{cases} 4 \text{ if } n_3 \geq 2 \text{ and } n_1 + n_2 + n_3 \geq 3 \\ 3 \text{ otherwise} \end{cases} .$$

and if $L > 0$ then

$$\chi_\rho(T) = \begin{cases} L + 3 \text{ if } n_3 \geq 1 \text{ and } n_1 + n_2 + n_3 \geq 2 \\ L + 1 \text{ if } n_1 = n_2 = n_3 = 0 \\ L + 2 \text{ otherwise} \end{cases}$$

We also mention the following results without proof.

Proposition 2.3.2. [1] *The minimum order of a tree with packing chromatic number 2 is 2; for 3 it is 4 and for 4 it is 8. Moreover, P_4*

is the only tree on 4 vertices that needs 3 colors. Now the two trees on 8 vertices that need 4 colors are:

(i) the diameter-4 tree with $n_3 = 2$, $n_1 = 1$ and $L = n_2 = 0$ called A_8
(ii) the diameter-6 tree where the two central vertices have degree-3 and for each centre vertex its three neighbors have degrees $1, 2$ and 3 respectively called B_8. See Figure 2.1 below as these are depicted.

Figure 2.1: The smallest trees with $\chi_\rho(T) = 4$. [1]

Theorem 2.3.3. [1] *For all trees T of order n it is true that $\chi_\rho(T) \leq (n + 7)/4$ except for $n = 4$ or 8 where the bound is $\frac{1}{4}$ more and these bounds are sharp.*

Chapter 3

The lower bound of the chromatic number of grids part 1

3.1 The lower bound of the packing chromatic number of a square lattice is at least 9

We shall analyze the packing coloring of grids, $G_{r,c}$, with r rows and c columns. The following results show the values for $r \leq 5$.

Proposition 3.1.1. [1] *$\chi_\rho(G_{2,c}) = 5$ for $c \geq 6$; $\chi_\rho(G_{3,c}) = 7$ for $c \geq 12$; $\chi_\rho(G_{4,c}) = 8$ for $c \geq 10$ and $\chi_\rho(G_{5,c}) = 9$ for $c \geq 10$.* The table below shows the values for smaller grids.

r ; c	2	3	4	5	6	7	8	9	10	11	12
2	3	4	4	4	5	...					
3		4	5	5	6	6	6	6	6	6	7 ...
4			5	7	7	7	7	7	8	...	
5				7	7	8	8	9	...		

Figure 3.1

Proof. We consider the following coloring pattern. For $c \geq 2$, the first c columns show the values for $\chi_\rho(G_{2,c})$ in Figure 3.1. Consider the coloring pattern below.

$$\left|\begin{array}{cccccc} 2 & 1 & 4: & 1 & 3 & 1 \\ 1 & 3 & 1: & 2 & 1 & 5 \end{array}\right| \begin{array}{c} ... \\ ... \end{array}$$

For $c \geq 3$, the first c columns show that the values for $\chi_\rho(G_{3,c})$ are upper bounds (Figure 3.1). This is demonstrated in the coloring patterns below.

$$\left|\begin{array}{cccccccccccc} 2 & 1 & 3 & 1 & 2 & 1: & 3 & 1 & 2 & 1 & 3 & 1 \\ 1 & 4 & 1 & 5 & 1 & 6: & 1 & 4 & 1 & 5 & 1 & 7 \\ 3 & 1 & 2 & 1 & 3 & 1: & 2 & 1 & 3 & 1 & 2 & 1 \end{array}\right| \begin{array}{c} ... \\ ... \\ ... \end{array}$$

Similarly for $c \geq 4$, we have;

$$\left|\begin{array}{cccccccccc} 1 & 2 & 1 & 3 & 1: & 2 & 1 & 3 & 1 & 6 \\ 3 & 1 & 5 & 1 & 7: & 1 & 4 & 1 & 2 & 1 \\ 1 & 4 & 1 & 2 & 1: & 3 & 1 & 5 & 1 & 8 \\ 2 & 1 & 3 & 1 & 6: & 1 & 2 & 1 & 3 & 1 \end{array}\right| \begin{array}{c} ... \\ ... \\ ... \\ ... \end{array}$$

Now consider a coloring pattern different from the ones above. Let i and j denote the row and column of a vertex with $i + j$ odd. Use color 1 for all vertices with $i + j$ odd, use color 2 for all the vertices with both i and j odd and where $i + j$ is not a multiple of 4. Finally, use color 3 for every other remaining vertex with i and j odd. See pattern below.

$$\left|\begin{array}{cccc|cccc} 2 & 1 & 3 & 1 & 2 & 1 & 3 & 1 \\ 1 & - & 1 & - & 1 & - & 1 & - \\ 3 & 1 & 2 & 1 & 3 & 1 & 2 & 1 \\ 1 & - & 1 & - & 1 & - & 1 & - \\ 2 & 1 & 3 & 1 & 2 & 1 & 3 & 1 \end{array}\right| \begin{array}{c} ... \\ ... \\ ... \\ ... \\ ... \end{array}$$

Note that the gaps from $G_{5,c}$ form a $G_{2,c}$. For $G_{2,c}$, we color as follows.

$$\begin{vmatrix} 4 & 6 & 5 & 8: & 4 & 7 & 5 & 9 \\ 5 & 7 & 4 & 9: & 5 & 6 & 4 & 8 \end{vmatrix} \begin{matrix} ... \\ ... \end{matrix}$$

Note that both $G_{5,8}$ and $G_{5,9}$ are exceptions. All upper bounds are verified. The lower bound can be found by computer, though some can be done by hand.

Since $\chi_\rho(G_{5,c}) = 9$ for $c \geq 10$, the lower bound of the packing chromatic number of a square lattice is 9. Note that $G_{5,c}$ is a sub grid of the infinite square lattice $(P_r \Box P_c)$ and hence $\chi_\rho(G_{5,c}) \leq \chi_\rho(P_r \Box P_c)$ $\square$

Chapter 4

The lower bound of the chromatic number of grids part 2

4.1 The lower bound of the packing chromatic number of a square lattice is at least 10

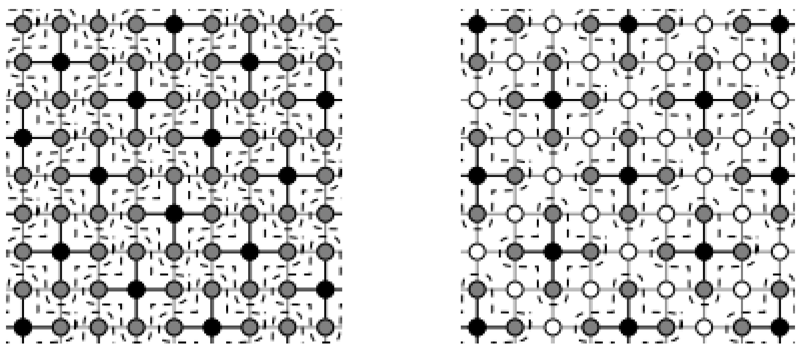
In this section we show that the lower bound of the packing chromatic number of a square lattice is $\chi_\rho(\mathbb{Z}^2) \geq 10$. In proving this, we assign some unit area, denoted by A, to all the vertices of some lattice, L. Therefore we distribute the area to every vertex covered by the packing so that areas at each vertex from the packing are equal, and as large as possible. Proceeding this way, Definition 4.1.2 of density leads to the use of density of a color class as in Lemma 4.1.3 as the reciprocal of the area. Assume X_k is a *k-packing* in L. We denote $N_\ell(x)$ as the set of vertices at distance at most ℓ from x for every x from L and some positive integer ℓ, that is; $N_\ell(x) := \{y : y \in L, dist(x, y) \leq \ell\}$.

Note that for the vertices u and v of X_k, the sets $N_{k/2}(u)$ and $N_{k/2}(v)$ are disjoint because vertices u and v are at distance bigger than k from each other. Let x be a vertex from X_k, where k represents an odd number, and y be a vertex at distance $\lceil k/2 \rceil$ from x. It follows that there exists no vertex from X_k in $N_{\lfloor k/2 \rfloor}(y)$. Therefore for every vertex z from X_k, $N_{\lfloor \frac{k}{2} \rfloor}(z)$ does not contain y. We re-distribute the unit area given to y and to vertices of X_k by sending the reciprocal of its degree to each of its neighboring sets $N_{\lfloor k/2 \rfloor}(z)$ as follows:

Definition 4.1.1. [2] The *k-area* $A(x,k)$ given to a vertex $x \in V(L)$ is defined by

$$A(x,k) = \begin{cases} |N_{k/2}[x]| & \text{for k even} \\ |N_{\lfloor k/2 \rfloor}[x]| + \sum\limits_{\substack{y \in V(G) \\ dist(x,y) = \lceil k/2 \rceil}} \frac{|N_1(y) \cap N_{\lfloor k/2 \rfloor}(x)|}{deg(y)}, & \text{for k odd} \end{cases} \tag{4.1}$$

Figure 4.1 shows the distribution of the 2-area and the 3-area in $\mathbb{Z}^2$. Suppose *k-area* is the same for all vertices of the lattice L. Let $A(k) := A(x,k)$, where x is arbitrary.



$A(2) = 5$ $A(3) = 8$

Figure 4.1: The 2-area and the 3-area in $\mathbb{Z}^2$ [2]

Proposition 4.1.1. [2] *Suppose G is a finite graph with a packing k-coloring and all areas $A(i)$, $1 \leq i \leq k$, are well-defined, then*
$\sum\limits_{i=1}^{k} \frac{1}{A(i)} \geq 1.$

Proof. Suppose G has n vertices. Then any color class X_i can have at most $\frac{n}{A(i)}$ vertices. Hence

$n = |V_G| = |X_1| + |X_2| + ... + |X_k| \leq \frac{n}{A(1)} + \frac{n}{A(2)} + ... + \frac{n}{A(k)}$,

and the results follows.

□

Note, we will use d to indicate density as defined in the definition below. Therefore we use *dist* for the concept distance between two vertices.

Definition 4.1.2. [2] Let G be a graph. Then the density of a set of vertices $X(G)$ is defined as:

$$d(X) := \lim_{\ell \to \infty} \sup \max_{x \in V} \left\{ \frac{|X \cap N_\ell(x)|}{|N_\ell(x)|} \right\}.$$

Lemma 4.1.2. [2] *For every finite packing coloring with k classes X_1, X_2,..., X_k of a graph G, it follows that*

$$\sum_{i=1}^{k} d(X_i) \geq d(X_1 \cup X_2) + \sum_{i=3}^{k} d(X_i) \geq d\left(\bigcup_{i=1}^{k} X_i\right) = 1.$$

Proof. We iteratively apply the following argument. For any vertex x and arbitrary position small ϵ, every sufficiently large ℓ satisfies

$$\frac{|N_\ell(x) \cap (X \cup Y)|}{|N_\ell(x)|} \leq \frac{|N_\ell(x) \cap X|}{|N_\ell(x)|} + \frac{|N_\ell(x) \cap Y|}{|N_\ell(x)|} \leq d(X) + d(Y) + \epsilon$$

□

Let x be a vertex of a graph G. We denote the boundary of $N_\ell(x)$ by

$$\Delta N_\ell(x) = \{y : dist(y, x) = \ell\}.$$

Lemma 4.1.3. [2] Suppose the area of a graph G is well defined. If

$$\lim_{\ell \to \infty} \frac{|\Delta N_\ell(x)|}{|N_\ell(x)|} = 0,$$

then for any k-packing X_k it holds that

$$d(X_k) \leq \frac{1}{A(k)}.$$

Now notice that in a single $\mathbb{Z}^2$ plane for any vertex $x \in \mathbb{Z}^2$, it holds that $|\{y : dist(x,y) = i\}| = 4i$. This observation will help us to prove the lemma below.

Lemma 4.1.4. [2] For any lattice $\mathbb{Z}^2$ and every k it holds that

$$A(k) = \lfloor \frac{k^2}{2} \rfloor + k + 1.$$

Proof. We use the observation above to prove the lemma. In the case of an even $k = 2\ell$ we have,

$$A(k) = |N_\ell[x]| = 1+\sum_{i=1}^{\ell} 4i = 2\ell^2+2\ell+1 = 2(k/2)^2+2(k/2)+1 = \frac{k^2}{2}+k+1.$$

Considering a case of an odd $k = 2\ell + 1$, it follows that four vertices at distance $\ell + 1$ from x that have single neighbor in $N_\ell(x)$ and the remaining 4ℓ vertices at distance $\ell + 1$ have two neighbors in $N_\ell(x)$. We get

$$A(k) = |N_\ell(x)| + 4(\frac{1}{4}) + 4\ell(\frac{2}{4}) = 2\ell^2 + 4\ell + 2 = \lfloor \frac{k^2}{2} \rfloor + k + 1.$$

□

The following lemma shows the best possible coverage of $\mathbb{Z}^2$ by $X_1 \cup X_2$ covers $\frac{5}{8}$ of the lattice which improves the bound $\frac{1}{2} + \frac{1}{6}$ corresponding to the case where X_1 and X_2 are dealt with separately.

Lemma 4.1.5. [2] The density $d(X_1 \cup X_2) \geq \frac{5}{8}$ on $\mathbb{Z}^2$.

Proof. Define a graph A on eight vertices consisting of a cycle $v_1, v_2, ...v_6, v_1$, a chord v_1v_4 and two vertices v_7 and v_8 of degree one adjacent to v_1 and v_4 respectively. This is depicted in Figure 4.2.

We say that the position of A is $[x, y]$ if in such an embedding of A the vertex v_1 is placed at $[x, y]$. The square lattice $\mathbb{Z}^2$ can be divided into copies of A, e.g., the copies of A can be placed at position $[4i+2j, 2j]$ where $i, j \in \mathbb{Z}^2$. This partition is shown in Figure 4.3 and we assume that it is fixed through out the proof.

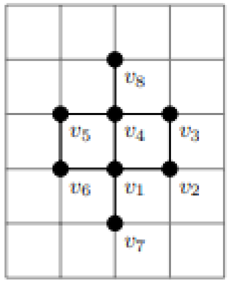


Figure 4.2: The graph A [2]

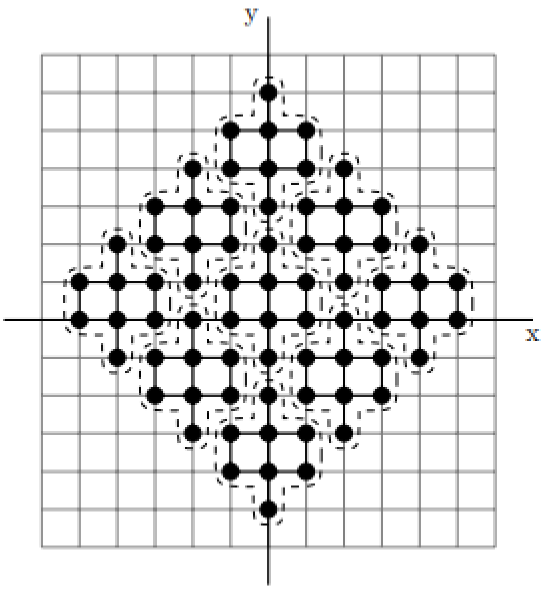


Figure 4.3: A partition of $\mathbb{Z}^2$ into isomorphic copies of A [2]

Suppose the $X_1, X_2 ..., X_k$ is a packing k-coloring of $\mathbb{Z}^2$. Let $X = X_1 \cup X_2$. According to Definition 4.1.3, we can bound the density of X, but we should first display some properties of X and A. For this cause, a copy of A is called a *t-copy* if it contains exactly t vertices of X. The aim here is to prove that on average every copy of A has at most 5 vertices of X.

Let us suppose that the partition has a 6-*copy* $P[x, y]$ and assume (without loss of generality) that $v_3, v_6, v_7, v_8 \in X_1$ and $v_2, v_5 \in X_2$. See Figure 4.4.

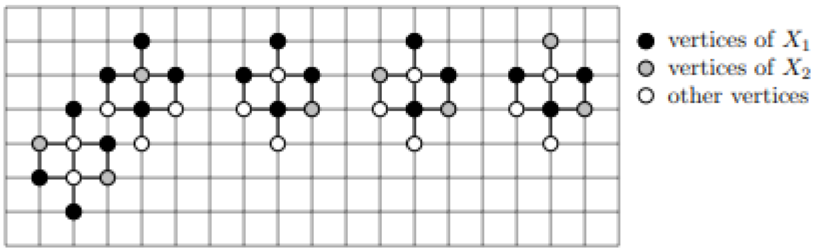


Figure 4.4: A 6-*copy* A$[x; y]$ is the bottom left copy A. The others are possibilities for a 5-*copy* A$[x + 2; y + 2]$ [2]

Suppose it is true that if the partition has another 6-*copy* $A[x+2i, y+2i]$ for some $i \geq 0$, then there is a $j \in [0, i]$ such that $A[x + 2j, y + 2j]$ contains strictly less than 5 vertices of X. See that v_6 and v_7 of $A[x + 2, y + 2]$ are not elements of X. Figure 4.5 shows the four possibilities of extending X such that $A[x+2, y+2]$ has five vertices of X. All these possibilities force v_6 and v_7 from $A[x + 4, y + 4]$ not to belong to X. Therefore it constantly spreads through 1 diagonal up to $A[x+2i, y+2i]$. This is a contradiction and the supposition is proved .

Notice that we went along the up-right diagonal in the prove of the supposition above. This was because of the way we arranged $A[x, y]$. For the other possible arrangement, where $v_3, v_5 \in X_2$, we use the right-down diagonal. It is very important that in either case we continue on the diagonals to the right. We call such a diagonal formed from a 6-*copy* an *A-strip*. This *A-strip* has only 5-*copies*. The *A-strip* can be one-way infinite.

It may happen that two A-*strips* contain different orientations, therefore they cross. Suppose that the partitive has suitable 6-*copies* $A[x-2i, y-2i]$ and $A[x-2j, y+2j]$ for position i, j such that $A[x, y]$ is in the intersection of two corresponding A-*strips*. Also suppose that between $A[x, y]$ and $A[x-2i, y-2i]$ are only 5-*copies* as well as for the other A-*strip*. We refer to the previous paragraph to see that X has no v_5, v_6, v_7 or v_8 of $A[x, y]$. Furthermore, there are at most three vertices of $v_1, ...v_4$ that may belong to X. Therefore $A[x, y]$ has at most three vertices of X, which is shown in the Figure 4.5 below.

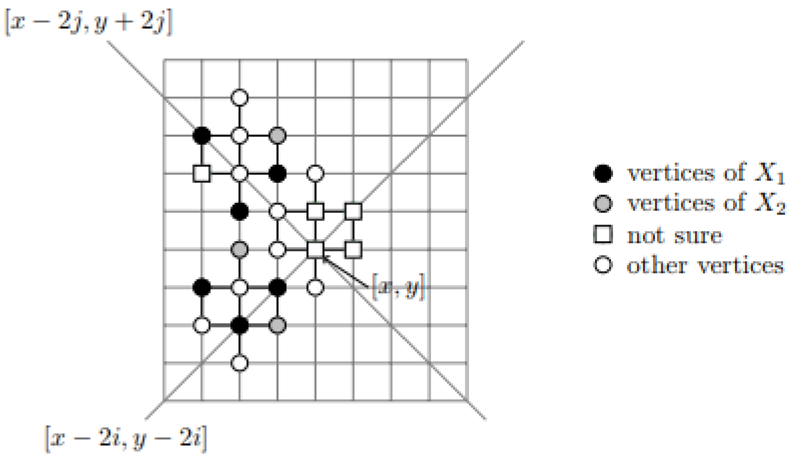


Figure 4.5: Intersection of two A-strips. In every possible intersection some vertices are forced to be in X_1, X_2, or they are not covered all. The square vertices are not forced. [2]

Now we are in the position to prove the limit on the density of X. For every 6-*copy* M, travel across the diagonal while increasing the first coordinate. We will either come across a t-*copy* D where $t < 5$ or the diagonal contains only 5-*copies*. The $t-copy$ D is a pairing copy for M. See that D can be in two pairs but then $t < 4$. Let x be an arbitrary vertex. We can utilize what we know, namely, $\lim_{\ell\to\infty} \frac{|\Delta N_\ell(x)|}{|N_\ell(x)|} = 0$ on $\mathbb{Z}^2$. Let $A_\ell(x)$ denote the set of copies A which are included in $N_\ell(x)$. We can now prove that $|X \cap N_\ell(x)| \leq 5|A_\ell| + n|\Delta N_\ell(x)|$. If both a 6-*copy* and its pair copy are in $A_\ell(x)$, they contribute at most 10 vertices to $X \cap N_\ell(x)$. Surely, if a single copy of A is paired with the two copies, then these three have at most 15 vertices of X.

Note that the number of 6-*copies* which has no pair copy in A_ℓ is linear in $|\Delta N_\ell(x)|$ since travelling across a diagonal of a copy of A without its pair in $A_\ell(x)$ ends on the boundary. $A_\ell(x)$ does not have to cover the whole of $N_\ell(x)$; but it can possibly miss linearly many vertices of the boundary. See Figure 4.6 below. Lastly, the density of X is

$$d(X) \leq \limsup_{\ell \to \infty} \left(\tfrac{5}{8} + \tfrac{n|\Delta N_\ell(x)|}{|N_\ell(x)|} \right) = \tfrac{5}{8}.$$

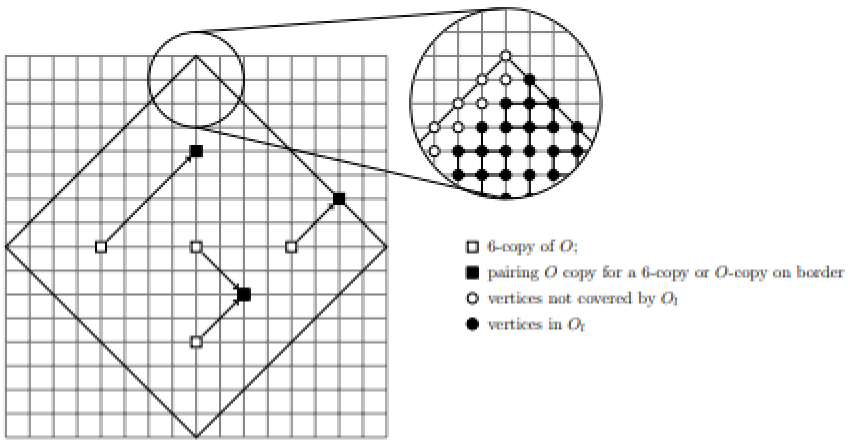


Figure 4.6: Bounding density of X in $N_\ell(x)$ [2]

$\square$

Theorem 4.1.6. [2] *Let G be the infinite square lattice $\mathbb{Z}^2$. Then* $10 \leq \chi_p(\mathbb{Z}^2)$.

Proof. We first calculate an upper bound on the density of the union of packings $X_i, 1 \leq i \leq 9$. Lemma 4.1.5 gives the bound for $X_1 \cup X_2$. By using Lemma 4.1.4, then other packings are bounded separately.

$$d\left(\bigcup_{i=1}^{9} X_i\right) \leq \tfrac{5}{8} + \sum_{i=3}^{9} \left(\tfrac{1}{A(i)}\right) = \tfrac{3830381}{3837600} < 1.$$

Finally, Lemma 3.1.2 implies that $\chi_\rho(\mathbb{Z}^2) \geq 10$.

$\square$

Chapter 5

The lower bound of the chromatic number of grids part 3

5.1 The lower bound of the packing chromatic number of a square lattice is at least 12

The lower bound of 9 of the packing chromatic number of the grid was proven by Goddard [1], then improved by Fiala and et al to 10 [2]. In this chapter we show that the packing chromatic number of a square lattice is at least 12.

Theorem 5.1.1. [3] The packing chromatic number of a square lattice is at least 12.

The algorithm they used to prove Theorem 5.1.1. is a brute force search through all possible configurations on a 15×9 lattice. Brute force search is a systematical technique enumerating all possible candidates for the solution and check if each candidate satisfies the problem statement. It takes too much time to check everything. Therefore the following observation can speed up the process.

Observation : In a finite $n \times m$ grid, we denote the entry at row i and column j by $(i; j)$. Suppose there is a coloring of the square lattice with 11 colors, then it is possible to color a 15×9 lattice where color 9 is at position $[5, 5]$. Instead of color 9 at $[5, 5]$ we can fix any other color at any position. If we place 9 at $[5, 5]$ it just reduces the number of configurations to check. We assume that position $[5,5]$ is pre-colored by 9 in the search through the configurations. The coloring procedure color vertices row by row. Here is the pseudo code for that coloring;

A Pseudocode [3]

```
Function boolean try-color (lattice, [x, y])
    for color : = 1 to 11 do
        if can use color on lattice at [x, y] then lattice [x, y] :=color
            if [x, y] is the last point return true
            else if try-color (lattice, next ([x, y]) then return true)
        end if
    end for
    return false
```

The Pseudocode is a function which contains a For-loop that goes from one to eleven. In fact, it is performing a task of coloring a two dimensional lattice with colors from one to eleven. The third line assign any possible color in a suitable position. If the position is the last possible point in the lattice, we are done. Otherwise we continue like that using those colors. The procedure could not color a 15×9 lattice with 9 at position $[5,5]$ with 11 colors. Therefore we conclude that the packing chromatic number for the square lattice is at least 12 [3].

This procedure uses 11 colors to color a 15×9 lattice, but ignores constraints with torus shapes, meaning that color 11 may be in the middle of the first row as well as the middle of the last row, which may not be allowed in a periodic vertex coloring. The computations performed by J Ekstein, J Fiala, P Holub and B Lidicky show that no such 11-coloring exists where color 9 is placed in position $[5,5]$. This computation in 2009 took 120 days to complete [3].

5.2 The lower bound of the packing chromatic number of a square lattice is at least 13

The Boolean satisfiability problem seeks to know whether there is at least one combination of a binary input variables, say $x_i \in \{false, true\}$ of which a Boolean logic formula returns true. The formula is satisfiable, SAT, if this is the case, otherwise it is unsatisfiability, UNSAT.

A SAT-solver is an algorithm that is used to establish satisfiability by taking the Boolean logic formula as input and returns SAT if it finds a combination of variables that satisfy it or UNSAT if it can show that no such combination exists. It is sometimes possible that it will return without an answer if it cannot determine whether the problem is SAT or UNSAT. When solving SAT problems, we begin by converting Boolean logic formula into a standard form that is more receptive to algorithmic manipulation. Every formula can be re-written as a conjunction of dis-junctions, that is, the logical AND of statements containing OR relations, know as conjunctive normal form. For example:

$$\alpha := (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee \neg x_3).$$

Each and every term in brackets is called a clause and combines variables together and their complements with a series of logical ORs. The clauses themselves are combined with AND relations.

Now, we recall the following. We refer to an entry at row i and column j by (i, j). The lower bound proved in Section 5.1 considers coloring a 15×9 grid with 11 colors but forgetting the toroidal constraints color 11 may appear legally in the middle of the first row as well as in the middle of the last row, which would obviously be prohibited in a periodic coloring. The computation shows that there exists no 11-coloring where color 9 is placed in the position $[5, 5]$. This computation took 120 days. Computations to find the lower bound was replicated with SAT-Solver on a 12×12 grid, with 9 placed in position $[6, 6]$, in just 26952 seconds. This was done in [6] by B Martin, F Raimondi, T Chen and J Martin.

Finally, B Martin, F Raimondi, T Chen and J Martin showed that a 12-coloring on a 14×14 grid with 9 placed at position $[7, 12]$ is impossible. In this case unsatisfiability was found after 15 days. This thoroughly shows the versatility of the SAT-solving method for the lower bound. The SAT instances can be downloaded from the website address at www.bedewell.com/sat, as kindly made available by Barnaby Martin.

Chapter 6

The upper bound of the packing chromatic number of grids

6.1 The upper bound of the packing chromatic number of a square lattice is at most 23

In this chapter we discuss the upper bound of packing chromatic number of the grids. Recall that $G_{r,c}$ denotes a grid with r rows and c columns. Upper bounds are provided in the table [1] below where an asterisk indicates exact value.

$r;c$	6	7	8	9	10	11	12	13	14	15	16
6	8^*	9^*	9^*	9^*	9^*	9^*	10	10	10	11	11
7		9^*	9^*	10	10	11	11	11	11	12	12

The greedy approach mostly produces a bound close to optimal. This approach uses more than 25 colors by the time the grids have 20 rows together with several hundred columns. The following theorem shows that these bounds are not best possible.

Theorem 6.1.1. [1] *For any grid* $G_{r,c}$, $\chi_\rho(G_{r,c}) \leq 23$.

Proof. The packing coloring of infinite grid that uses 23 colors is demonstrated below. This provide a packing coloring of any finite subgrid.

Now, we start coloring $G_{24,24}$ with $1's$, $2's$ and $3's$ just like in the coloring of $G_{5,c}$ in Section 3.1. After that we will notice that the uncolored vertices are in every alternate row and column. See that the gaps form a $G_{12,12}$ as shown in the table [1] below .

4	5	8	4	5	9	4	5	8	4	5	9
10	6	11	7	12	6	10	7	11	6	13	7
5	4	9	5	4	8	5	4	9	5	4	8
14	7	15	6	13	7	16	6	17	7	12	6
4	5	18	4	5	11	4	5	19	4	5	11
20	6	21	7	10	6	14	7	15	6	10	7
5	4	8	5	4	9	5	4	8	5	4	9
13	7	11	6	22	7	12	6	11	7	23	6
4	5	9	4	5	8	4	5	9	4	5	8
12	6	10	7	15	6	13	7	10	6	14	7
5	5	17	5	4	11	5	4	18	5	4	11
16	7	19	6	14	7	20	6	21	7	15	6

Fill now the gaps to form a 24×24 tile, and use this tile periodically to color the plane. This coloring was found by placing the color 4 through 9 in a specific pattern, then they used a computer for the remaining colors [1].

$\square$

6.2 The upper bound of the packing chromatic number of a square lattice is at most 22

The upper bound was improved by Schwenk [4] to 22 in a personal communication with W. Goddard et al., (2002). Thereafter Holub and Soukal improved the upper from 22 to 17.

6.3 The upper bound of the packing chromatic number of a square lattice is at most 17

Goddard et al. proved in [1] that the packing chromatic number of the infinite square lattice is between 9 and 23. Then, Schwenk [4] proved that the packing chromatic number of the infinite square lattice is at most 22. Using a computer, the following results were obtained improving the upper bound of the packing chromatic number for the infinite square lattice [5].

Theorem 6.3.1. [5] *Suppose that G is the infinite square lattice. Then $\chi_\rho(G) \leq 17$.*

Note that the square pattern on 24×24 vertices (see Figure 6.3.1) may be copied periodically for the whole infinite square lattice.

1	7	1	4	1	6	1	3	1	2	1	3	1	8	1	4	1	5	1	3	1	2	1	3
2	1	3	1	2	1	5	1	7	1	4	1	2	1	3	1	2	1	10	1	6	1	11	1
1	5	1	9	1	3	1	2	1	3	1	6	1	5	1	7	1	3	1	2	1	3	1	4
3	1	2	1	15	1	4	1	10	1	2	1	3	1	2	1	17	1	4	1	5	1	2	1
1	6	1	3	1	2	1	3	1	5	1	13	1	4	1	3	1	2	1	3	1	7	1	8
2	1	4	1	5	1	16	1	2	1	3	1	2	1	11	1	5	1	9	1	2	1	3	1
1	3	1	2	1	3	1	6	1	4	1	7	1	3	1	2	1	3	1	12	1	4	1	5
10	1	7	1	11	1	2	1	3	1	2	1	5	1	6	1	4	1	2	1	3	1	2	1
1	2	1	3	1	4	1	5	1	8	1	3	1	2	1	3	1	7	1	5	1	6	1	3
4	1	5	1	2	1	3	1	2	1	9	1	4	1	10	1	2	1	3	1	2	1	13	1
1	3	1	6	1	14	1	7	1	3	1	2	1	3	1	5	1	8	1	4	1	3	1	2
9	1	2	1	3	1	2	1	4	1	5	1	15	1	2	1	3	1	2	1	11	1	5	1
1	8	1	4	1	5	1	3	1	2	1	3	1	7	1	4	1	6	1	3	1	2	1	3
2	1	3	1	2	1	10	1	6	1	11	1	2	1	3	1	2	1	5	1	7	1	4	1
1	5	1	7	1	3	1	2	1	3	1	4	1	5	1	9	1	3	1	2	1	3	1	6
3	1	2	1	17	1	4	1	5	1	2	1	3	1	2	1	14	1	4	1	10	1	2	1
1	4	1	3	1	2	1	3	1	7	1	8	1	6	1	3	1	2	1	3	1	5	1	12
2	1	11	1	5	1	13	1	2	1	3	1	2	1	4	1	5	1	16	1	2	1	3	1
1	3	1	2	1	3	1	9	1	4	1	5	1	3	1	2	1	3	1	6	1	4	1	7
5	1	6	1	4	1	2	1	3	1	2	1	10	1	7	1	11	1	2	1	3	1	2	1
1	2	1	3	1	7	1	5	1	6	1	3	1	2	1	3	1	4	1	5	1	8	1	3
4	1	10	1	2	1	3	1	2	1	12	1	4	1	5	1	2	1	3	1	2	1	9	1
1	3	1	5	1	8	1	4	1	3	1	2	1	3	1	6	1	15	1	7	1	3	1	2
14	1	2	1	3	1	2	1	11	1	5	1	9	1	2	1	3	1	2	1	4	1	5	1

Figure 6.3.1: The Soukal-Holub 17-coloring on 24×24 [5]

Discussion

For the Soukal-Holub 17-coloring, the frequency of each color monotonal decreases as the color number increases.

Method and algorithm.

Simulated Annealing, SA, is a probabilistic approach used for approximating the global optimum of a given function. SA can be applied in solving combinatorial problems, e.g., in the travelling salesman problem it is used to minimize the length of a route that connects all points. SA is one of the is most preferred heuritic methods for solving solids optimization problems.

It was proven in [1] that for a general graph G, $\chi_\rho(G) \leq 4$ is NP-complete. The theorem above is going to be solved using heuristics based on Simulated Annealing (SA). SA is used to determined a suboptimal solution of combinatorial problems, which are NP-complete. This implies that we can have a large space of possible solutions and a globally optimal solution that can not be found in polynomial time. The annealing process has to do with the way in which a metal cools and freezes into a minimum energy crystalline structure and the search for a minimum in a more general system. The algorithm comes from that of Metropolis et al. [13]. This was proposed as a means of finding the equilibrium configuration of a collection of atoms at a given temperature. It was Kirkpatrick et al [10] who suggested that it forms the foundation of the optimization technique for combinatorial problems.

This method has the ability to avoid being stuck at local minima as other methods do. The algorithm makes use of a random search which accepts changes that decrease or increase the objective function f. See [14] for more information on SA. The central idea of SA algorithm is that we iteratively look for $\chi_{min} \in D(f)$ (where $D(f)$ is the definition scope of an objective function f) in the domain of f such that $f(\chi_{min})$ is as near as possible to the minimum of $f(x)$. The process is based on the degression of the temperature t. If t is unchanged, there are many iterations and in each of these iterations we pick an arbitrary χ_ρ from a neighborhood of a current χ_{min}. A new χ_ρ is accepted with a probability $p = e^{\frac{(f(\chi_{min})-f(\chi_\rho))}{t}}$, meaning that a more efficient solution is accepted automatically (with $p \geq 1$), and bad solution is also accommodated with specific probability that decreases with lower temperature t.

Algorithm[5]

In this section we discuss a pseudocode algorithm used. This form offers a packing coding of a 24×24 square and can be used to tile the whole infinite square lattice, to attain a packing coloring thereof. The approach is to color as many vertices as possible of a 24×24 square with a color c. The algorithm lists the best found patterns and memorises each pattern up to symmetric patterns with maximum number of vertices colored with c in this list. We begin with placing color 1 into an empty 24×24 square. Then we add color 2 in this 24×24 grid partially. We continue this way until this 24×24 grid is packing colored completely. We utilise two lists of patterns in this pseudocode of Algorithm 1. List γ has the best found pattern with the most number of vertices colored after coloring with color $(c-1)$. List δ contains currently best found patterns after coloring with c.

We define functions

(i) NonColored (pattern P) - the number of vertices not colored in a pattern P.

(ii) FreeForColor (color c, pattern P) - the number of vertices of P which has color c. Notice that if color c is not used in any of the vertices of P, then

$$\text{FreeForColor (color c, pattern P)} = \text{NonColored (pattern P)},$$

and also, if there exist a vertex in P colored with c, then vertices which are not colored at distance not greater than c cannot be colored with c, therefore,

$$\text{FreeForColor (color c, pattern P)} = \text{NonColored (pattern P)}.$$

(iii) PlantColor (pattern P, color c, temperature t) – place current color c in P under a temperature t and give back a modified pattern L. Notice that no vertex exists in L which can be colored with c. We can use the function PlantColor (pattern P, color c, temperature t) for each P of γ and for the same given color c repeatedly every time with distinct temperature, we recall a new list pattern $\triangle$.

This list $\triangle$ has a pattern in which we mostly put the current color c. If we discover a new pattern in which we utilize the color c the same number of times as in pattern, we can add this pattern to $\triangle$ and form a new list.

This implies that every pattern P from the list $\triangle$ has the same and least found value of the function NonColored (pattern P). The approach for coloring of a pattern P with a color c is based on the following technique, where we put a pattern L as a copy of P . In p iterations, we pick an arbitrary vertex w of L, we get $f(w)$=FreeForColor (color c, pattern L after coloring of w with c) and we accommodate this vertex w and set $v = w$ with a probability $p = e^{\frac{f(v)-f(w)}{t}}$. Notice that in the first iteration we accommodate w automatically. After p iterations we color the vertex v with c and we apply this method again until FreeForColor (color c, pattern L) = 0. When FreeForcolor (color c, pattern L)=0, we return L to Algorithm 1.

Remark: Let us notice that Soukal and Holub [5] used this algorithm also for other shapes of pattern, e.g., rectangular patterns on 16×24, 24×32 vertex and a square pattern on 32×32 vertices. Nevertheless, the square pattern on 24×24 vertices gave the solutions.

Algorithm 1:[5] **This is the main algorithm for determining a pattern of packing coloring of the square lattice.**

Input : size of a pattern n

Simulated annealing algorithm constants

$t_{max} > t_{min} > 0$, $q \in (0, 1)$

Output :Γ - list of the best found patterns

//**initialization step**

Γ := empty list of the best found patterns;

pattern K:= new pattern // a null squared matrix of order n add K to Γ; color c := 1; // c is a current color

// end of the initialization step

// value of function $nonColor(K)$ is the same for all patterns K in Γ

// L is the first pattern in Γ

while $nonColor(L) > 0$ **do**

Δ:= empty list of patterns;

int p:= $nonColor(L)$;

int m:= 0; // m is a maximum number of color vertices with current color c

foreach K in Γ **do**

t:=t_max;

while $t > t_{min}$ **do**

T:=$plantColor(K, c, t)$; // color vertices of K with c. See Algorithm 2

int a := p - $nonColored(T)$; // a is number of vertices colored with c in T

if $a >= m$ **then**

// better possibility for color c than patterns in Δ

if $a > m$ **then**

Δ := empty list list of patterns; // Δ is cleared

$m := a$ // a is a new maximum number of vertices colored with c

end

if $T \notin \Delta$ **then**

add T to Δ;

end

end

$t := t * q$; // decrease temperature t

end

end

$\Gamma := \Delta$; // put all the best found patterns from Δ to Γ

$c := c + 1$; // set a new color c

end

return Γ ; // return a list of the best found patterns

Algorithm 2:[5] Function $plantColor(K, c, t)$.

Input: the original pattern K
the current color c
the simulated annealing parameter t (temperature)

Auxiliary: the simulated annealing constant k (number of iterations)

Output: pattern L - the original pattern K colored with c

$L := K$; // L is a copy of K
// while there is at least one vertex in L which can be colored with c

while $freeForColor(c, L) > 0$ **do**
$v :=$ **null** ; // v is a vertex colorable with c, at the beginning we have no such a vertex
// $f(v)$ is a number of vertices we lose for coloring with c while we color v with c
// at the beginning of $f(v)$ is set as a maximum int value, because it will be minimized *int* $f(v)$:= max int value;

for $i = 1$ to k **do**
$w :=$ randomly chosen vertex from L which can be colored with c;
$//f(w)$ is a number of vertices we lose for colored with c while we color w with c
int $f(w) := freeForColor(c, L)$ - $freeForColor(c, Lwithcolorw)$;
// SA accepts possibility with probability $p = \frac{e}{f(v)-f(w)}t$
// if $f(v)$ ¿ $f(w)$ then $p > 1$ and better possibility is accepted automatically
if $random(0, 1)$ ¡ $\frac{e}{f(v)-f(w)}t$ **then**
$f(v) := f(w)$;
$v := w$;

end

end

color vertex v with c;

end

return L;

6.4 The upper bound of the packing chromatic number of a square lattice is at most 16

In this section, we show that the packing chromatic number of the infinite square lattice is at most 16. Specifically, using a SAT-solver on top of the previous results of the upper bound of the packing chromatic number of a square lattice, they improve this upper bound from 17 to 16 [6].

The previous upper bounds were given a packing coloring based on a finite grid which can be interpreted up-down and left-right to yield a periodic packing coloring of the infinite square lattice. We make significant use of the 17-coloring of the 24×24 block periodically showed by Soukal-Holub in [5], see Figure 6.3.1 in section 6.3.

In Martin et al.[6] 16-coloring, they took the Soukal-Hulob coloring and remove colors 8 and 17, and blow it up from 24×24 to 48×48. Thereafter give the resulting partially colored grid to a SAT-solver to see if a 16-coloring is possible, which turned to be possible. See Figure 6.4.1 below. Take note that the method of SAT-solver does not compile efficiently unless several colors are placed, i.e., some vertices are good for certain colors. For example, using color 1 placed at the maximal density of 1/2 on a 24×24 grid, the computation on the machine that is being used will run indefinitely. Placing colors might be interpreted as a form of pre-coloring, hence objects given to the SAT-solver are partially pre-colored grids.

Figure 6.4.1: 16-coloring on 48×48 [6]

Discussion

We are aware that Simulated Annealing may not have Soukal-Hulob 16-coloring solution and one argument is concluded for this by tables in Figure 6.4.2 below. Soukal-Hulob 16-coloring, this monotonicity is broken in two places. Simulated Annealing allows for some sub optimal searching but algorithm from [5] is greedy color-by-color, in ascending order and so it looks like it will not throw this break of monotonicity. This might be advanced as an argument in favour of SAT-solving for this sort of problem, for as long as the search space is suitably restricted. But we notice that their 15-coloring is monotone. According to Martin et al., the SAT solver program will run more efficiently if the large clauses from the first constraint are broken up into smaller clauses. A large clause can be broken up into $k + 1$ smaller clauses using k new variables known as commander variables. Looking at Figure 6.5.3, we can observe running times for some cases with the basic encoding versus using commander variables.

1	288
2	72
3	72
4	32
5	32
6	16
7	16
8	8
9	8
10	8
11	8
12	3
13	3
14	3
15	3
16	2
17	2
	576

1	1152
2	288
3	288
4	128
5	128
6	64
7	64
8	28
9	32
10	32
11	31
12	16
13	16
14	13
15	14
16	10
	2304

Figure 6.4.2: The frequency table for 16-coloring on 48×48 grid (right) and the Soukal-Holub 17-coloring on 24×24 (left). [6]

6.5 The upper bound of the packing chromatic number of a square lattice is at most 15

In this section, we show that the packing chromatic number of the infinite square lattice is at most 15. Similarly to the above section, using a SAT-solver on top of the previous results of the upper bound of the packing chromatic number of a square lattice, they improve this upper bound from 16 to 15.

In Martin et al.[6] 15-coloring, they took the Soukal-Hulob coloring and remove colors 6 and 17 , thereafter blow it up from 24×24 to 72×72. See Figure 6.5.1 below. The designation, for instance, 72×72 indicates that the grid was planted with colors 1 to 5. They wanted to get a 15-coloring. It is easy to see that the running times were improved greatly by the addition of the commander variable. This improvement is not a surprise given that what is known about the pigeonhole principle which performs much better when using commander encoding.

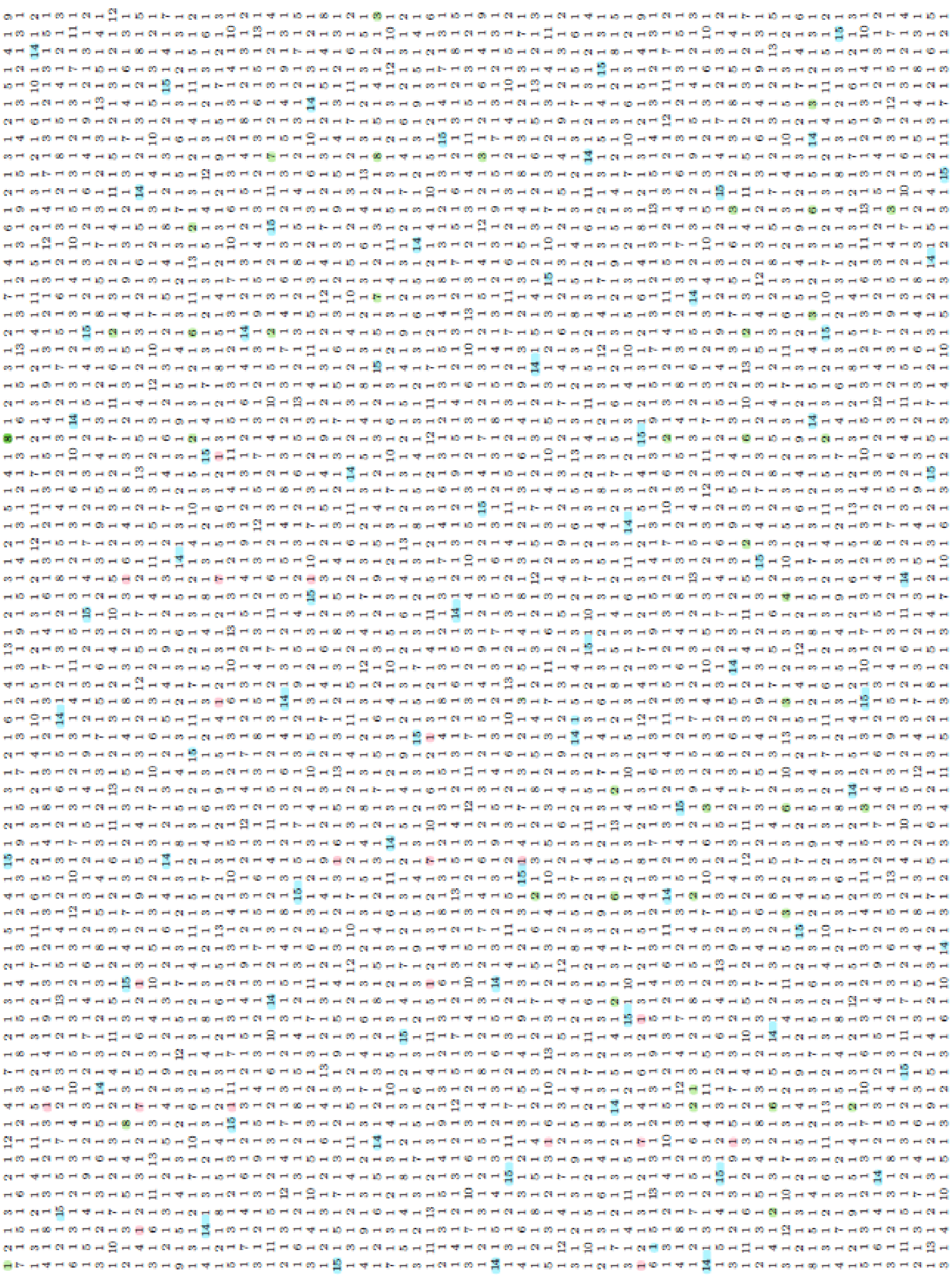


Figure 6.5.1: 15-coloring on 72 × 72 [6]

1	288	1	1152	1	2592
2	72	2	288	2	648
3	72	3	288	3	648
4	32	4	128	4	288
5	32	5	128	5	288
6	16	6	64	6	144
7	16	7	64	7	144
8	8	8	28	8	72
9	8	9	32	9	72
10	8	10	32	10	72
11	8	11	31	11	72
12	3	12	16	12	36
13	3	13	16	13	36
14	3	14	13	14	36
15	3	15	14	15	36
16	2	16	10		
17	2				
	576		2304		5184

Figure 6.5.2: The frequency table for our 15-coloring (right) on 72×72, 16-coloring on 48×48 grid (middle) and the Soukal-Holub 17-coloring on 24×24 (left) [6]

	SAT?	basic G Syrup	cmdr G Syrup
72x72-14(1-4)*	unsat	4913	1290
72x72-15(1-5)	sat	464	420
48x48-16(1-8)	unsat	112	35
48x48-16(1-7)*	sat	18589	10519
24x24-16(1-7)	unsat	82	35

Figure 6.5.3: Running times in seconds on interesting instances [6]

We show the derivation of their results by producing a SAT instance that is found to be unsatisfiable.

Encoding and computation

Let $[n] := 1, \ldots, n$. The basic encoding for investigating the upper bound includes variables $P_{i,j,k}$, which are true asserts that position (i,j), in some m-coloring, periodic on a grid of size $n \times n$, is set to color k. As a result we need huge clauses of the form

$$P_{i,j,1} \text{ V } P_{i,j,2} \text{ V } ... \text{ } P_{i,j,m}$$

for each $(i,j) \in [n]^2$, together with constraints, $\neg P_{i,j,k}$ v $\neg P_{i,j,k}$ whenever the distance between (i,j) and (i',j), $d((i,j),(i',j'))$, is less than k. Thus, the distance should be calculated toroidally ie

d((i,j),(i',j'))= min (i-j' mod n, i'-i mod n) + min(j-j'mod n, j'-jmod n)

To plant color k at position (i,j) we use the singleton clause $P_{i,j,k}$. The commander encoding got identical pair clauses as the fundamental encoding, but divides the big clauses into a system of smaller clauses. Taking into consideration a 16- coloring, each clauses $P_{i,j,1}$ v $P_{i,j,2}$ v ... v $P_{i,j,16}$ is replaced by five clauses, requiring four new command variables $C_{i,j,1}, \ldots C_{i,j,4}$ as

$$\begin{gathered}
\neg C_{i,j,1} \text{ V } P_{i,j,1} \text{ V } P_{i,j,2} \text{ V } P_{i,j,3} \text{ V } P_{i,j,4} \\
\neg C_{i,j,2} \text{ V } P_{i,j,5} \text{ V } P_{i,j,6} \text{ V } P_{i,j,7} \text{ V } P_{i,j,8} \\
\neg C_{i,j,3} \text{ V } P_{i,j,9} \text{ V } P_{i,j,10} \text{ V } P_{i,j,11} \text{ V } P_{i,j,12} \\
\neg C_{i,j,4} \text{ V } P_{i,j,13} \text{ V } P_{i,j,14} \text{ V } P_{i,j,15} \text{ V } P_{i,j,16} \\
\neg C_{i,j,1} \text{ V } C_{i,j,2} \text{ V } C_{i,j,3} \text{ V} C_{i,j,4}
\end{gathered}$$

Surely, this system is equivalent to the single clause, fulfilling assignment in the variables $P_{i,j,k}$ being conserved. On the case of 15-coloring, every one of big clause is replaced by six clauses involving five new commander variable (therefore five of those clauses have width 4, and the fifth, 5) in similar way.

SAT-solver Glucose Syrup used on a quad core Intel(R) Xeon(R) CPU ES-2660 @2020GHz with 40GB RAM, operatively Ubuntu Linux 14.04 [5]. Through the examination they put a maximal limit of 12GB of RAM, which permitted four threads of the SAT solver to run in equidistance, no hyper threading one process per physical CPU and no exchanging. Glucose Syrup produced a wide variance in its results so in the table in Figure 6.5.3 every result is the average over five runs, three runs for the starred rows where times are longer.

Chapter 7

Upper bound of Cartesian products of Cycles

7.1 The upper bound

Suppose n is an integer, $n \geq 3$. Suppose $q = 4n$. We investigate in this chapter the packing chromatic number of the Cartesian products of C_4 and C_q. By using a theoretical approach, it is shown that $9 \leq \chi_\rho(C_4 \square C_q) \leq 11$ [6]. Let us define $D(X;Y) = max\{dist(u,v) : u \in X,\, v \in \text{Y}\}$.

To consider the Cartesian product of cycles with cycles, we make use of a grid-like representation of the graph, then we label the vertices of an $m \times n$ matrix. Observe Figure 7.1 below. Notice that we exclude later on the arches in the sketches of the Cartesian products.

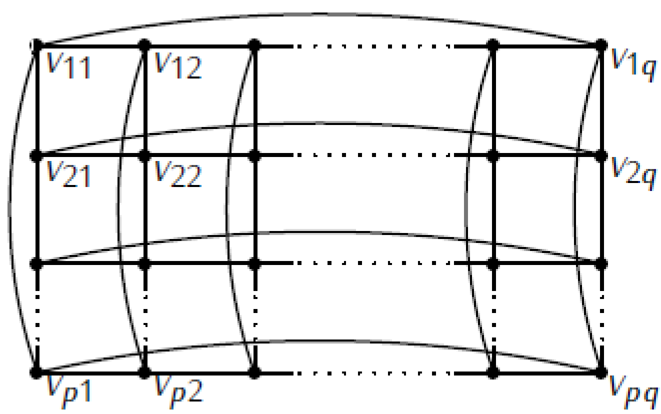


Figure 7.1: $C_p \square C_q$ [7]

We now prove an upper bound for $\chi_\rho(C_4 \square C_q)$. Remember that we defined $q = 4n$, where n is the number of 4×4 blocks.

Theorem 7.1.1. [6] *Suppose G is a graph $C_4 \square C_q$ where $q = 4n$, $n \geq 3$. Then*

$$\chi_\rho(G) \leq \begin{cases} 9 \qquad if \qquad n = 4x, x \leq 1 \\ \\ 11 \qquad if \qquad n = \begin{cases} 4x+1, x \geq 1 \\ 4x+2, x \geq 1 \\ 4x+3, x \geq 0 \end{cases} \end{cases}$$

Proof. To show the results, we depicted packing colorings of $C_4 \square C_q$ where we divide it into blocks of 4×4 vertices.
Step 1: Assign colors 1 to 3 to all vertices possible .
Step 2: Use color 1 on as many vertices possible.
Step 3: Elect two vertices for color 3, and then two vertices with color 2. Refer to Figure 7.2.

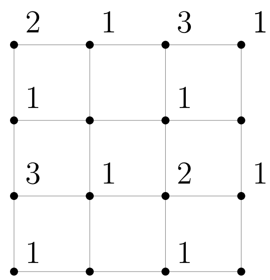


Figure 7.2: Basic coloring for each block.

Figure 2 indicates some basis coloring of a 4×4 block. Four vertices remain uncolored in each block. We arrange n blocks in a row to consider Case 1 and Case 2 in order to packing color of $C_4 \square C_q$.

Case 1: $n = 4x$, $x \geq 1$.
We color the not yet colored vertices in 4×4 blocks in such to form

Blocks 1, 2 and $2a$.

Block 1: Color vertex $v_{2,2}$ with 4 and vertex $v_{4,2}$ with 5. Color vertex $v_{2,4}$ with 6 and vertex $v_{4,4}$ with 7. See Figure 7.3.

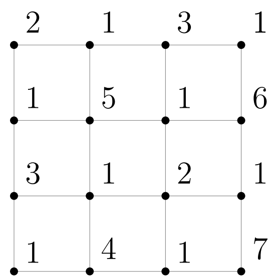


Figure 7.3: Block 1

Block 2: To vertex $v_{2,2}$, assign color 5 and color 4 to vertex $v_{4,2}$. To vertex $v_{2,4}$, assign 8 and to vertex $v_{4,4}$, assign 9. See Figure 7.4 below.

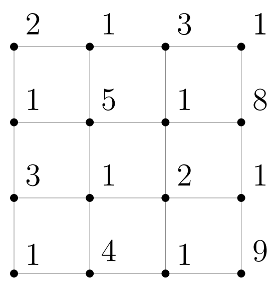


Figure 7.4: Block 2

Block $2a$: Block $2a$ is formed by changing Block 2 slightly. We interchange colors 9 and 8 in column 4 and use the same colors in Block 2 assigned in column 2. See Figure 7.5 below. Now, to complete the proof, use 1 to indicate Block 1 and use 2 to indicate Block 2. Therefore, the sequence 1, 2, 1, $2a$ a total x times to create the pattern 1, 2, 1, $2a$, 1, 2, 1, $2a$,…1, 2, 1, $2a$. It is repeated to packing color $C_4 \Box C_q$. Hence $\chi_p(C_4 \Box C_p) \leq 9$.

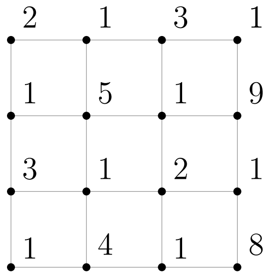


Figure 7.4: Block $2a$

Case 2: $n = 4x + 1$ or $n = 4x + 2$ where $x \geq 1$ or $n = 4x + 3$ where $x \geq 0$.

Case 2.1: We use Block 1, 2 and $2a$ from above and create Block $2b$ by changing Block 2 slightly.

Block $2b$: We keep the basis coloring of Block 2 and color vertex $v_{2,4}$ with 10 and vertex $v_{4,4}$ with 11. To vertex $v_{2,2}$ assign color 5 and color 4 to vertex $v_{4,2}$. See Figure 7.5.

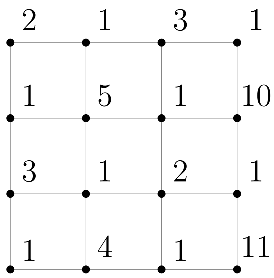


Figure 7.5: Block $2b$.

Arranging the Blocks 1, 2 and 2a in the pattern 1, 2, 1, 2a, 1, 2, 1, 2a. . . 1, 2, 1, 2a, 1, 2b will eventually give the packing coloring of $C_4 \square C_p$. Hence $\chi_p(C_4 \square C_p) \leq 11$.

Case 2.2: $n = 4x + 1$, $x \geq 1$.

In this case, use Blocks 1, 2 and $2a$ and introduce a new block, Block 3, with basis coloring of Block 2.

Block 3: Color vertex $v_{2,2}$ with 7 and vertex $v_{4,2}$ with 6. Assign color 11 to vertex $v_{4,4}$ and color 10 to vertex $v_{2,4}$. See Figure 7.6 below.

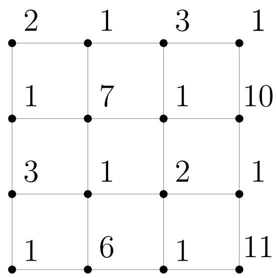


Figure 7.6: Block 3.

Suppose the number of Blocks is $4x + 1$ with $x \geq 1$, then we can arrange Blocks 1, 2, $2a$ and 3, by repeating the sequence 1, 2, 1, $2a$ a total of x times, then end with a single Block 3: 1, 2, 1, $2a$, 1, 2, 1 ,$2a$,... 1, 2, 1, $2a$, 3. Hence $\chi_p(C_4 \square C_p) \leq 11$.

Case 2.3. $n = 4x$, $x \geq 0$.

In this case we use the same analogy as previously. To form a packing coloring of $C_4 \square C_q$, repeat the sequence 1, 2, 1, $2a$ a total of x times, then the sequence 1, 2 3 : 1, 2, 1, $2a$, 1, 2, 1, $2a$,... 1, 2, 1, $2a$, 1, 2, 3. It follows that $\chi_p(C_4 \square C_q) \leq 11$.

$\square$

Chapter 8

Lower Bound of the Cartesian Products of Cycles

8.1 The Lower Bound

Now we consider the lower bound of the packing chromatic number of Cartesian product of cycles given by $G = C_4 \square C_q$, where $q = 4x$ and $x > 2$ is an integer. Let $c(u) = a$ indicates that vertex u is colored with a. The main proof of this section, Theorem 8.1.4, depends on Lemmas 8.1.1 to 8.1.3. The proofs follow later on in the consecutive section.

The colors 2 and 3 are called small colors, while a large color is any color greater than 3. We will represent a color of a vertex by c, e.g., $c(h_1)$ reads or means the color of h_1. We define D as the distance between sets, e.g., $D(A, B)$ means the distance between A and B. We represent N as the neighborhood of a vertex, e.g., $N(v)$ refers to the neighborhood of a vertex v.

Lemma 8.1.1. [7] *There exist two color* 1 *vertices u and v for any packing coloring of the graph G, such that $|N(u) \cap N(v)| = 2$.*

Lemma 8.1.2. [7] *Suppose $|N(u) \cap N(v)| = 2$. Also, if graph G contains two color* 1 *vertices u and v in adjacent columns, then more than eight colors are necessary to packing color G, therefore G will* *need more than eight colors to color the vertices of G.*

Lemma 8.1.3. [7] *Say G contains two color 1 vertices u and v in the same column. Then a packing coloring of G will need more than 8 colors.*

Theorem 8.1.4. [7] *Suppose $G = C_4 \square C_q$, where $q = 4x$ and integer x, where $x \geq 3$. Then $\chi_\rho(G) \geq 9$.*

Proof. By Lemma 8.1.1, G contains two color 1 vertices u and v such that $|N(u) \cup N(v)| = 2$. Suppose u and v lie in adjacent columns; it follows from Lemma 8.1.2 that $\chi_\rho(G) \geq 9$. If u and v lie in the same column, then again $\chi_\rho(G) \geq 9$, by Lemma 8.1.3.

□

Corollary 8.1.4.1. [7] *Suppose that $q = 16x$ where $x \geq 1$ is a positive integer. Then $\chi_\rho(C_4 \square C_q) = 9$.*

Proofs of Lemmas.

We first define a sub-graph H_{10} of G where H_{10}=$C_4 \square P_{10}$. In order to prove the Lemmas mentioned, we indicate the vertices of H_{10} in Figure 8.1.

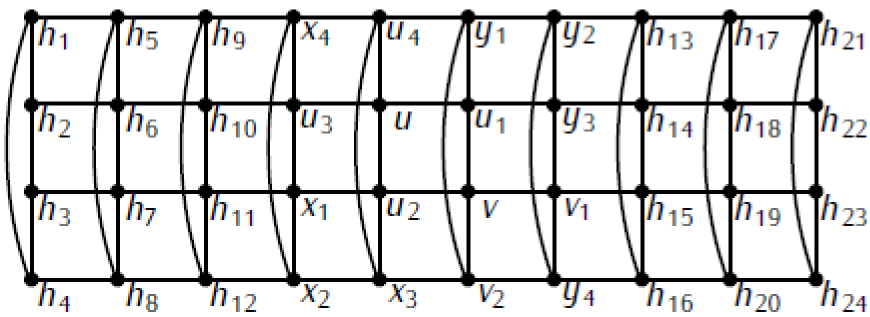


Figure 8.1: The sub-graph H_{10} with vertices u and v [7]

Proof. Assume that there exist no two color 1 vertices u and v for any coloring of the graph G, such that $|N(u) \cap N(v)| = 2$, and that at most 8 colors are needed to color G. Let us consider a sub-graph H of G which is induced by four columns in a row. From our hypothesis, being that H does not contain two color 1 vertices u and v such that $|N(u) \cap N(v)| = 2$, we conclude that there is at most one vertex colored 1 in a column. It means that at most four vertices of H are in color class 1. Also, there are at most one vertex of color 2 in each column. If we have vertices colored with 2 in adjacent columns, there will not be vertices of color 2 in the next and previous columns.

Therefore, H has at most three vertices of color 2. Note that H has at most two vertices colored 3, and at most two vertices colored 4. Also, note that H has at most one vertex of colors 5, 6, 7 and 8 each. This clearly shows that at most 15 vertices can be colored by 1,..., 8 and so a color larger than 8 has to be used. This completes the proof. □

Proof of Lemma 8.1.2

Proof. **Case 1**. $\{c(u_1), c(u_2)\} = \{2, 3\}$, where u_1 and u_2 are as in Figure 8.1.

Let

$$\begin{aligned} F &= N(u) \cup N(v) - \{u_1, u_2\} \\ &= \{u_4, u_1, u_2, u_3, v_2, v_1\} - \{u_1, u_2\} \\ &= \{u_4, u_3, v_1, v_2\} \\ &\quad\text{and} \\ B &= \{y_2, y_3, h_{13}, h_{14}\} \end{aligned}$$

The vertices of $F = N(u) \cup N(v) - \{u_1, u_2\}$ must receive a distinct color greater than 3. We may assume that $c(u_2) = 2$. We now show that a vertex in $A = \{x_1, x_2, h_{11}\}$ should receive a color bigger than 3. Since x_1 cannot receive a small color, x_1 must receive either color 1 or color bigger than 3. If $c(x_1) = 1$, then as $N(x_1)$ has at least two vertices colored with a color bigger than 3, it follows that x_2 or h_{11} must receive a color bigger than 3. Everyone of the vertices in A is at distance at most four from any vertex in F. Therefore, five distinct colors are needed to color the vertices of $A \cup F$.

Note that $D(B, F) \leq 5$ and $B = \{y_2, y_3, y_{14}, h_{13}\}$. As $c(u_1) = 3$, none of the vertices in B can receive color 3. Amongst the vertices of B, color 2 cannot be repeated and color 1 can occur at most twice.

Hence we have at least one vertex in B that receives a large color.

Claim A. We look at the set H of vertices of two adjacent columns. We have two vertices that get large colors in H.

Proof. Consider Figure 8.2.

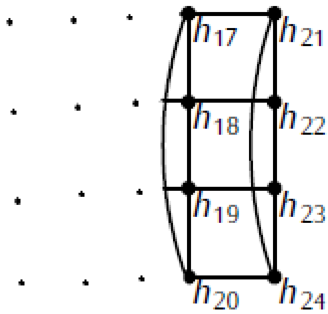


Figure 8.2: The set H

Assume that $H = \{h_{17}, h_{18}, ..., h_{24}\}$. Suppose that there is one vertex that receives a large color in H. We may also suppose that the vertices in $\{h_{17}, h_{18}, h_{21}, h_{22}\}$ do not receive large colors and $c(h_{17}) = c(h_{22}) = 1$ and that h_{18} and h_{21} both receive small colors. Therefore neither h_{23} nor h_{20} can receive colors in the set {1,2,3}. A contradiction.

□

Claim B. Assume one vertex is colored by 4 in $F \cup B \cup \{h_{18}, h_{21}, h_{20}\}$. Then some vertex in B - $\{h_{13}\}$ must get a large color.

Proof. Since there is no vertex in B - $\{h_{13}\}$ that can receive color 3, let every vertex in B - $\{h_{13}\}$ receives a color from the set {1,2}. Therefore, $c(y_2) = c(h_{14}) = 1$ and $c(y_3) = 2$. It then means that $c(y_1) \notin \{1, 2, 3, 4\}$. Since $D(A, F) \leq 4$ and $D(A, \{y_1\}) \leq 5$, we conclude that a color greater than 8 is needed to color some vertices of $F \cup A \cup \{y_1\}$. Hence a vertex in B - $\{h_{13}\}$ has a large color.

□

Let

$$B' = \{h_{17}, ..., h_{24}\},$$
$$B'' = \{h_1, ..., h_8\},$$
$$A'' = \{h_5, ..., h_{12}\},$$

and

$$B''' = \{h_{13}, ..., h_{20}\}$$

Case 1.1: A vertex in A or B receives color 4.

First assume that a vertex in S receives color 4. Note that no vertex in F receives color 4.

Case 1.1.1: $c(h_{11}) \neq 4$.

Notice that B should have a vertex colored by 4, otherwise we need a color greater than 8 to color vertices of $F \cup A \cup B$. Moreover, suppose that the vertices of F are colored $5, 6, 7$ and 8. By Claim A, there exist two vertices of large color in A''. Since $D(A'', A) \leq 4$, none of the vertices in A'' receives color 4. Suppose a vertex in A'' receives a large color that does not occur amongst the vertices of F, as a result we need a color bigger than 8 to color the vertices of $F \cup A'' \cup A$. Therefore there exist vertices in A'' that get large colors and these colors must be 5 and 6. Moreover, $D(A'', \{u_3, u_4\}) \leq 5$ and $D(A'', \{v_3, v_4\}) \leq 7$. We have that the vertices in $\{v_2, v_1\}$ should receive color 6 and 5 respectively. Since $D(B'', \{v_1, v_2\}) \leq 5$, by Claim A, it follows that B''' has a vertex that must receive a color larger than 8.

Case 1.1.2: $c(h_{11}) = 4$.

Notice that if $D(\{h_{11}, F\}) \leq 4$, then none of the vertices in F has color 4. We suppose that the vertices of F are colored with colors 5, 6, 7 and 8. Suppose a vertex in B receives a color greater than 4, then a color bigger than 8 will be needed to color the vertices of $F \cup B$. Therefore, only one vertex in B gets a large color; this color must be 4. Since $D(B'', \{h_{11}, F\}) \leq 4$, we have that none of the vertices of B'' can receive color 4. Suppose at most one large color is repeating amongst the colors of vertices of $B'' \cup F$. Hence a color greater than 8 is needed to color the vertices of $F \cup B'' \cup A$. Since $D(B'', F\}) \leq 8$, color 5, 6 and 7 have to repeat in $F \cup B''$.

Assume that no vertex receives color 4 in $\{h_{13}, h_{14}\}$. We consider the set B''', and note none of the vertices in B''' can receive color 4. Assume at most one large color is repeating amongst the colors of the vertices of B and F; then a color larger than 8 is needed to color the vertices of $F \cup T'' \cup A$. Therefore, since $D(B''', \{v_1, v_2\}) \leq 5$

and $D(B''', F\})\leq 7$, we conclude that colors 5 and 6 are repeating amongst the colors of the vertices of F and B''', and the vertices of $\{u_3, u_4\}$ must receive color 5 and 6. From Claim A, it follows that the set A'' contains a vertex that gets a large color which is neither $4, 5$ or 6. Since $D(A'', \{v_1, v_2\})\leq 7$, it follows that a color greater than 8 is required to color the vertices of $B\cup A''\cup A$.

Suppose that a vertex in $\{h_{13}, h_{14}\}$ receives color 4. It is important to note that none of the vertices in B' can receive a color 4. A color bigger than 8 is needed to color some vertices in of $F\cup B\cup B'$ if B' has at least one vertex that is colored by a large color, that is not anyone of the colors 4, 5, 6 or 7. Therefore, the large colors 5, 6 (note not 7) repeat amongst the vertices of F and B'. Assume that colors 6 and 7 can be repeated amongst the colors of the vertices of F and B', it follows that the vertices of $\{u_4, u_3\}$ receive colors 6 and 7 and hence none of the vertices in B'' can receive either colors 4, 6 or 7 and so a color greater than 8 is needed to color the vertices of $F\cup B''\cup A$. Then there exists a vertex in B' that receives color 5 and there exists a vertex in B'' that receives a color $\gamma\in\{6,7\}$. The color γ and 5 are amongst the vertices of F.

We realize that one vertex in $\{u_4, u_3\}$ must receive color γ. Assume $\alpha\in\{6,7\}$ - $\{\gamma\}$. Note that a vertex in B'' must receive color γ. If none of the vertices of B'' receives color 5 or none of the vertices in B'' can receive color α, then a color greater than 8 is needed to color the vertices of $F\cup B''\cup A$. Therefore, there exist two vertices that receive color α and 5 respectively. Moreover, a vertex in $\{v_1, v_2\}$ receive color α.

Suppose that $c(u_3) = 5$. Note that none of the vertices in B'' can receive colors 4, 5 or γ and so, by Claim A, a color greater 8 is needed to color the vertices of $F\cup B'\cup A$. Suppose now that $c(u_4) = 5$ or $c(v_2) = 5$. Assume that a vertex in $E = \{x_2, h_{12}, x_3, h_9, x_4, x_1\}$ receives a large color, then no vertex in E that receives color 4 and $D(E, \{u_4, v_2\})\leq 5$. There is no vertex in E receives a color 5. Therefore, since $D(E, F)\leq 6$, a color larger than 8 is needed to color the vertices of $F\cup E\cup A$. Also, the vertices of E can get colors only from the set $\{1, 2, 3\}$.

We consider the set E - $\{x_3, x_1\}$. Consequently it follows that either $c(h_{12}) = 1$ or $c(x_2) = 1$. Suppose that $c(x_2) = 1$, x_2 must have two neighbours that receives large colors. Hence a vertex in $E - \{h_9, x_2\}$ receives a large color, which is impossible. Therefore, $c(h_{12}) = 1$ and so h_8 must receive a large color that cannot be 4 or 5. Therefore, $c(h_8) = \alpha$. But then one vertex in $\{v_1, v_2\}$ has a color α and $D(\{h_8\}, \{v_1, v_2\}) \leq 6$, which is impossible. Let $c(v_1) = 5$. No vertex in B' can receive color 4 or 5 and so, by Claim A, a vertex in B' receives a color greater than 8.

Suppose that none of the vertices in A receives color 4 and that some vertex in B gets color 4. Assume that w denote the color 4 vertex in B. Suppose that B has a vertex colored by a large color γ such that $\gamma \neq 4$, then α is repeating amongst vertices of $A \cup B$ and $\gamma \in \{5, 6\}$. The only large color that vertices of F and B can have in common is 4. Assume the vertices B and B contain no large colors in common. It follows that we must use a color larger than 8 to color the vertices of $B \cup \{f\} \cup A$. Therefore, a vertex in F and a vertex in B must both have the same large color and this color should be 4. We let $c(u_3) = c(h_{13}) = 4$ and by Claim B, it follows that some vertex in B - h_{13} and some vertex in A receives a color $\gamma \in \{5, 6\}$.

As $D(A, B - \{h_{13}\}) \leq 6$, we have that $\gamma = 5$. By Claim A, the set T' has a vertex colored by a large color γ not in {4,5}. Suppose γ is not amongst the vertices of F. A color greater than 8 is needed to color the vertices of $F \cup B \cup B'$. Therefore, color γ must appear amongst vertices of $\{u_3, u_4\}$ and as $c(u_3) = 4$ and $D(B', \{u_4\}) \leq 7$, it follows that a color greater that 8 is needed to color the vertices of $F \cup B' \cup B$.

Case 1.2: No vertex in $A \cup B$ gets color 4.

Assume no vertex in F gets color 4. It follows that a color bigger than 8 will be used to color the vertices of $F \cup B$. Therefore, there must be at least one vertex in F colored by 4. The vertices of B and A must have a large color $\gamma \in \{5, 6\}$ in common. It then means that, by Claim B, set B - $\{h_{13}\}$ must have a vertex that gets a large color and as $D(A, B - \{h_{13}\}) \leq 6$, we must have that $\gamma = 5$.

None of the two large distinct colors amongst the vertices of B' (B'' respectively) can be γ and therefore must repeat amongst vertices of F, and be in {4, 6, 7}. We want to show that at least one vertex in B' receives color 4. Assume that none of the vertices in B' receives color 4. If a vertex in B' receives a color that does not appear amongst the colors of the vertices of F, then a color bigger than 8 is needed to color the vertices of $F \cup A \cup B'$. Since $D(B', \{v_1, v_2\}) \leq 6$, it follows that the two vertices in $N(u) - \{u_1, u_2\}$ receive colors 6 and 7. Therefore, a color bigger than 8 is needed to color the vertices of $F \cup A \cup B''$. This means that at least one vertex in B' must get color 4.

We also show that B' has strictly two large colored vertices. Assume that B' has at least three vertices colored by large colors. Then these vertices should be 4, 5 and 7. Since $D(B', \{v_1, v_2\}) \leq 6$, it follows that both vertices in $N(u) - \{u_1, u_2\}$ receive colors 6 and 7. Therefore a color bigger than 8 is needed to color the vertices of $F \cup A \cup B''$. This then means that B' has exactly two large colored vertices.

Claim C. The distance between the two vertices in B' that receive large colors is two.

Proof. Suppose the distance between two vertices of large color, is three. Let $c(h_{18})$ and $c(h_{24})$ be large.

Assume $E' = \{h_{23}, h_{19}, h_{20}, h_{24}\}$ and $E = \{h_{17}, h_{18}, h_{21}, h_{22}\}$. Suppose E' and E contain at most one vertex with color 1. It follows that there exist two vertices of color 3 being at distance at most three from each other, a contradiction. Therefore either $c(h_{22}) = c(h_{17}) = 1$ or $c(h_{20}) = c(h_{23}) = 1$. If $c(h_{20}) = c(h_{23}) = 1$, then $c(h_{17})$, $c(h_{22})$ and $c(h_{19})$ must be small, which is impossible. If $c(h_{22}) = c(h_{17}) = 1$, then $c(h_{20})$, $c(h_{23})$ and $c(h_{21})$ must be small, which is impossible as well.

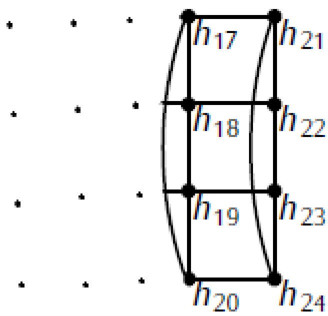


Figure 8.3 : $E' \cup E$

Suppose that two large colors are adjacent and let $c(h_{24})$ and $c(h_{20})$ be large. Let $E' = \{h_{23}, h_{19}, h_{20}, h_{24}\}$ as before. Notice that E' contains two vertices with color 1; let us say h_{22} and h_{17}. It is easy to see that h_{21}, h_{18} and h_{23} receive small colors, which is impossible. Suppose $c(h_{24})$ and $c(h_{22})$ are large. Notice that E' has two vertices of color 1. If $c(h_{21}) = 1$ and $c(h_{18}) = 1$, then h_{19}, h_{22} and h_{17} must receive small colors, but this is impossible. If $c(h_{17}) = c(h_{22}) = 1$, then h_{20}, h_{21} and h_{18} must receive small colors, which is not possible.

□

Claim D. Suppose B' contains two large vertices F and z in the same column with $d(w, z) = 2$. Therefore z and w each have a neighbor colored 1 in the opposite column.

Proof. Consider again the sketch $E' \cup E$ below.

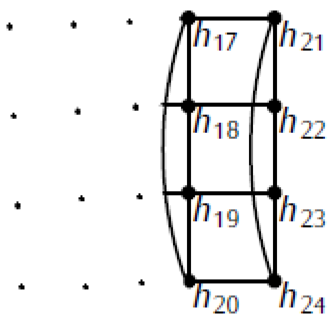


Figure 8.4 : $E' \cup E$

Suppose $c(h_{22})$ and $c(h_{24})$ are large. Assume $E = \{h_{17}, ..., h_{20}\}$. Notice that E has two vertices with 1 since B' has exactly two vertices of large colors. We let $c(h_{17}) = c(h_{19}) = 1$. Then all the vertices remaining get small colors and so two color 3 vertices will be at most distance three from each other; impossible.

□

Case 1.2.1: $c(h_{23}) = 6$.

A vertex in $\{u_3, u_4\}$ should receive color 6. Suppose that h_{20} is colored by 4. No vertex in $\{h_{17}, h_{18}, h_{22}\}$ receives a large color. Therefore either

$$c(h_{21}) = c(h_{18}) = 1$$
$$\text{or}$$
$$c(h_{22}) = c(h_{17}) = 1.$$

Suppose that $c(h_{17}) = c(h_{22}) = 1$. Since a vertex colored 1 is adjacent to two vertices that get large colors, it follows that $c(h_{13})$ is large. Consequently, two large colors and hence a color bigger than 8 is needed to color the vertices of $A \cup F \cup B$. Therefore, $c(h_{21}) = c(h_{18}) = 1$. Assume $w \in N(h_{21}) - \{h_{17}, h_{22}, h_{24}\}$. Since $D(\{w\}, N(u) - \{u_1, u_2\}) \leq 8$, $D(\{w\}, B) \leq 5$, and $D(\{w\}, \{v_1, v_2\}) \leq 6$, it follows that both vertices in $N(u) - \{u_1, u_2\}$ must get colors 6 and 7 respectively, so a color bigger than 8 is required to color the vertices of $F \cup A \cup B''$.

Consider again the sketch $E' \cup E$ below.

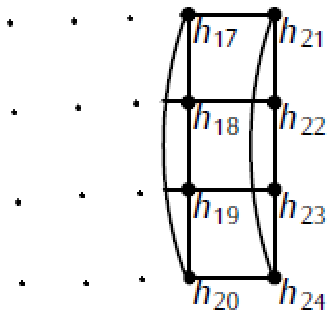


Figure 8.5 : $E' \cup E$

Assume now that $c(h_{18}) = 4$. Since $\{h_{17}, h_{21}, h_{20}, h_{24}\}$ has no vertex of a large color, it follows that either $c(h_{17}) = c(h_{24}) = 1$ or $c(h_{21}) = c(h_{20}) = 1$. Suppose first that $c(h_{17}) = c(h_{24}) = 1$. Thus $c(h_{13})$ is large. Then by Claim B, set B has two vertices of large color. Hence a color bigger than 8 is needed to color vertices of $F \cup B$. Therefore, $c(h_{20}) = c(h_{21}) = 1$ and so $c(h_{19})$ or $c(h_{17})$ is large; not possible. Suppose that $c(h_{21}) = 4$. From Claim D, $c(h_{17}) = c(h_{19}) = 1$ and so $c(h_{13})$ must be large. From Claim B, it follows that T has two vertices with large colors and hence a color bigger than 8 is necessary to color vertices in $F \cup B$.

Case 1.2.2: $c(h_{24}) = \alpha$, where $\alpha \in \{6, 7\}$.

A vertex in $\{u_3, u_4\}$ must receive color α. We first suppose that $\{h_{19}\}$ is colored with 4. No vertex in $\{h_{17}, h_{18}, h_{21}, h_{22}\}$ receives a large color. Therefore, either $c(h_{18}) = c(h_{21}) = 1$ or $c(h_{17}) = c(h_{22}) = 1$. Assume $c(h_{17}) = c(h_{22}) = 1$. Then since any vertex colored 1 is adjacent to two vertices that get large colors, we conclude that $c(h_{18})$ or $c(h_{20})$ is large, which is not possible. Therefore, $c(h_{18}) = c(h_{21}) = 1$. Assume $w \in N(h_{21}) - \{h_{17}, h_{22}, h_{24}\}$. Since $D(\{w\}, N(u) - \{u_1, u_2\}) \leq 8$, $D(\{w\}, B) \leq 5$ and $D(\{w\}, \{v_1, v_2\}) \leq 6$, it means that the vertices in $N(u) - \{u_1, u_2\}$ receive colors 6 and 7 respectively. Hence a color bigger than 8 is required to color vertices of $F \cup A \cup B''$.

Let $c(h_{18}) = 4$. From Claim D, we have $c(h_{20}) = c(h_{18}) = 1$ and so $c(h_{16})$ is large. Note that vertex h_{16} cannot be colored by 4, 5 or α. Since $D(F, \{h_{16}\}) \leq 6$, a color bigger than 8 is needed to color vertices of $F \cup A \cup \{h_{16}\}$. Let $c(h_{17}) = 4$. Notice that either $c(h_{19}) = 1$ or $c(h_{18}) = 1$. Assume $c(h_{19}) = 1$. Then a vertex in $N(h_{19}) - \{h_{15}\}$ receives a large color; impossible. Therefore $c(h_{23}) = c(h_{18}) = 1$, so $c(h_{14})$ must be large and thus $c(h_{14}) = 5$. Suppose $w \in N(h_{23}) - \{h_{19}, h_{24}, h_{22}\}$. The vertex w receives neither color 4 or α. Since $D(\{w\}, \{u_3, u_4\}) \leq 8$ and $D(\{w\}, \{v_1, v_2\}) \leq 6$, we conclude that the vertices in $\{u_3, u_4\}$ receive colors 6, 7 and thus a color bigger than 8 is required for vertices in $F \cup A \cup B''$.

Case 1.2.3: $c(h_{21}) = 6$.

Vertex u_3 receives color 6. Suppose that $c(h_{18}) = 4$. We notice that either $c(h_{15})$ or $c(h_{16})$ is large, and this large color cannot be 4, 5 or 6. Since $D(F, \{h_{16}, h_{15}\}) \leq 6$, a color bigger than 8 is needed to color vertices of $F \cup A \cup \{h_{16}, h_{15}\}$. Consider a case where $c(h_{20}) = 20$. We see that either $c(h_{19}) = 1$ or $c(h_{18}) = 1$. Assume $c(h_{18}) = 1$, then a vertex in $N(h_{18}) - \{h_{14}\}$ receives a large color; impossible. Hence, $c(h_{19}) = 1$ and so $c(h_{15})$ is large. Then a color bigger than 8 is needed to color vertices of $F \cup A \cup \{h_{15}\}$. Suppose that $c(h_{23})$ is large. Therefore, $c(h_{19}) = 1$ and hence $c(h_{15})$ is large. A color bigger than 8 is needed to color vertices of $F \cup A \cup \{h_{15}\}$.

Note the following observation.

Observation 8.2: Suppose u, u_1, v and u_2 indicate the vertices of an induced C_4 cycle C of G, with v and u in different columns. Then we have that $c(w) \notin \{1, 2, 3\}$ for $w \in V(C)$.

Proof. Consider cycle C.

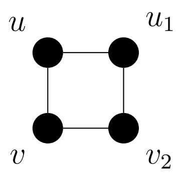


Figure 8.6: Cycle C

This proof uses contradiction. For every $w \in V(C)$, let $c(w) \in \{1, 2, 3\}$. Strictly two vertices of C must receive color 1, say v and u. Therefore, $\{c(u_1), c(u_2)\} = \{2, 3\}$ and so by Case 1, the proof is complete. Thus it is safe to assume that we have $c(w) \notin \{1, 2, 3\}$ for $w \in V(C)$.

$\square$

Case 2: $c(u_1)$, $c(u_2) \in \{4, 5, 6, 7\}$.

Case 2.1: Vertices of $N(u) - \{u_1, u_2\}$ or $N(v) - \{u_1, u_2\}$ receive colors greater than 3.

Without loss of generality, we suppose that vertices of $N(u) - \{u_1, u_2\}$ receive colors bigger than 3. Define

$$A = \{x_1, x_2, h_{11}, h_{12}\}$$
$$B = \{y_2, y_3, h_{13}, h_{14}\}$$
and
$$B' = \{v, v_1, v_2, y_4\}$$

and consider the sketch below.

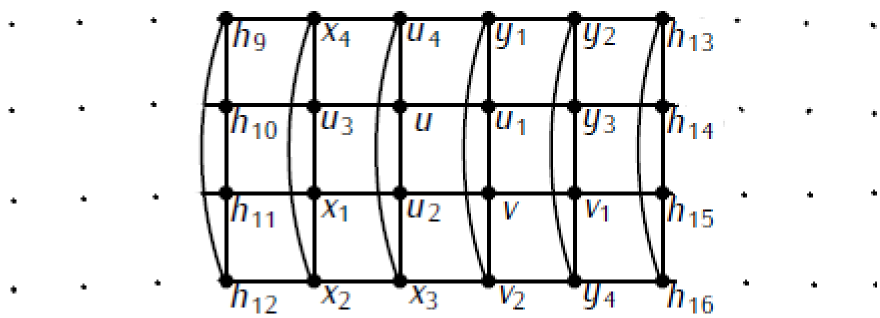


Figure 8.7 : $A \cup B \cup B'$

The sets A, B, B' must each contain a vertex that receives a color bigger that 3, by Observation 8.2. Notice the greatest distance between a vertex $v \in A$ and a vertex $w \in N(u)$ is 5, and it happens if and only if $v = h_{12}$ and $w = u_1$. Also, the greatest distance between a vertex $v \in B$, and a vertex $w \in N(u)$ is 5, and this happens if and only if $v = h_{13}$ and $w = u_2$.

Case 2.1.1: A color bigger than 3 is not repeating in $A \cup N(u) \cup B'$.

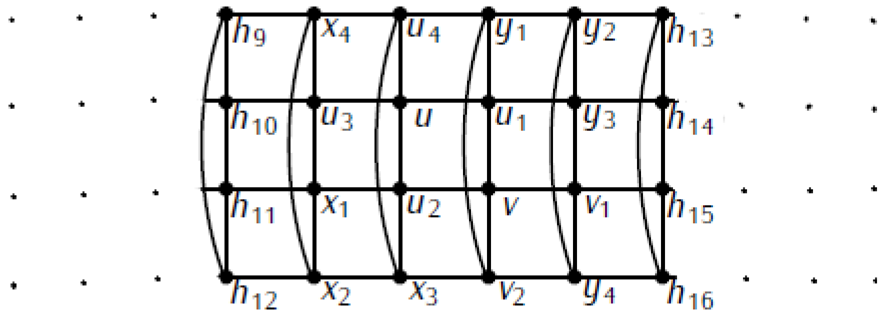


Figure 8.8 : $A \cup N(u) \cup B'$

Since $D(A, B') \leq 5$ and $D(N(u), B') \leq 5$, it follows that a color bigger than 8 is required to color vertices of $N(u) \cup A \cup B'$.

Case 2.1.2: A color bigger than 3 is repeating in $A \cup N(u) \cup B'$.

Then a vertex of $\{v_1, y_4, u_1\}$ must receive color 4. This immediately means that a color bigger than 7 should be used to color the vertices of $N(u) \cup B$, and none of the vertices in B receives color 4.

Case 2.1.2.1: A large color is repeating amongst the vertices of $N(u) \cup B'$.

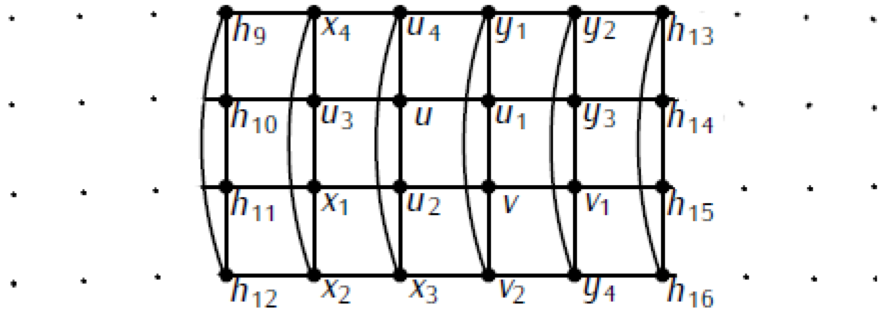


Figure 8.9 : $B \cup A' \cup A \cup B'$

Clearly $c(y_4) = c(u_3) = 4$ and since $D(A, \{u_3\}) \leq 4$ and $D(B, \{y_4\}) \leq 4$, it follows that none of the vertices in $A \cup B$ receives color 4. Assume no large color is repeating amongst vertices of $A \cup B$. Then a color bigger than 8 is needed to color vertices of $N(u) \cup B \cup A$. We let a large color $\gamma \in \{5, 6\}$ be repeating amongst vertices of $A \cup B$. By Observation 8.2, there exists one vertex in $A' = \{h_5, h_6, h_9, h_{10}\}$ that receives a large color and since $D(A', \{u_3, u_4\}) \leq 4$ and $D(A', A \cup \{u_1, u_2\}) \leq 5$, it follows that a color bigger than 8 is required to color vertices of $N(u) \cup A' \cup A$.

Case 2.1.2.2: No large color is repeating in $N(u) \cup B'$.

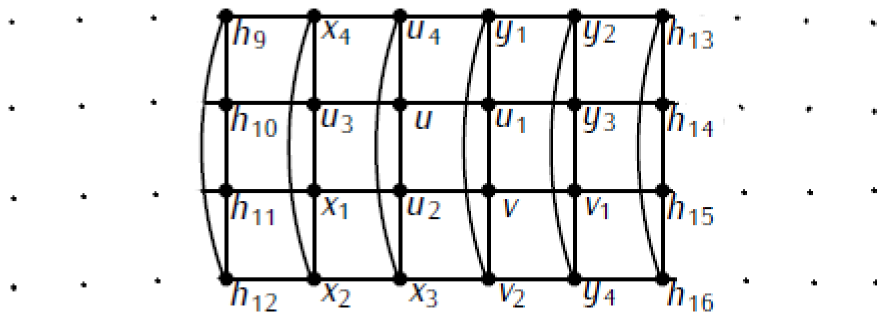


Figure 8.10 : $A \cup B \cup B'$

We then have that a vertex in A and a vertex in $\{u_1, v_1, y_4\}$ receive color 4. Therefore, as $D(B, B' \cup \{u_1\}) \leq 4$ and $D(B, N(u)) \leq 5$, it follows that no vertex in $N(u) \cup B - \{u_1\}$ receives color 4 and so a color bigger than 8 is required for vertices in $N(u) \cup B \cup B'$.

Case 2.2: One vertex in $N(u) - \{u_1, u_2\}$ receives a small color; the other vertex receives a large color.

Case 2.2.1: The vertex u_3 is receiving a large color; the vertex u_4 a small color.

Let

$$A = \{x_4, u_4, x_2, x_3\}$$
$$B = \{y_2, y_3, h_{13}, h_{14}\}$$
$$F = \{u_1, u_2, u_3\}$$
and
$$B' = \{v, v_1, v_2, y_4\}$$

and consider the figure below.

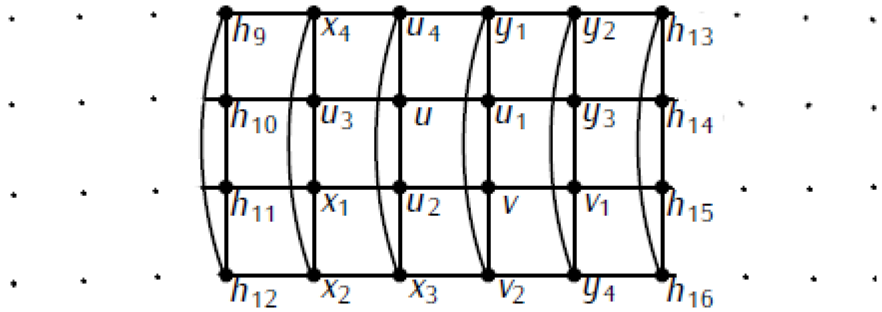


Figure 8.11 : $A \cup B \cup F \cup B'$

Sets A, B, B' contain vertices that receive large colors, by Observation 8.2. Since the distance between vertices in A (B respectively) and vertices in $F \cup B'$ is at most 5, it follows that color 4 may be repeating in $F \cup A$ and $F \cup B$. Since $D(A, B) \leq 6$, we have that colors 4 and 5 are repeating amongst vertices of $A \cup B$.

Case 2.2.1.1: A large color is repeating in $F \cup B'$.

We have that $c(u_3) = 4$ and $c(y_4) = 4$. Notice that none of the vertices in $A \cup B$ receives color 4. Assume first that a large color is not repeating amongst the vertices of $A \cup B$. Let $B'' = \{h_{15}, h_{16}, h_{19}, h_{20}\}$ and $A'' = \{h_6, h_7, h_{10}, h_{11}\}$. Since $D(B, B'' \cup \{y_4\}) \leq 4$, no large color is repeating amongst the vertices of $B \cup B' \cup B''$. Assume no large color is repeating amongst vertices in $A \cup F \cup B''$. It follows that a color greater than 8 is needed to color vertices of $F \cup A \cup B \cup B''$. Therefore, since $D(B'', A \cup F) \leq 7$, it follows that a vertex in B'' and a vertex in $\{x_2, x_4\}$ receive a color $\gamma \in \{5, 6\}$. At least one vertex $w \in \{u_4, y_1, x_3, v_2\}$ receives a large color by Observation 8.2. Hence, a color larger than 8 is required to color vertices in $F \cup \{w\} \cup A \cup B$. A large color is repeating amongst the vertices of $A \cup B$ and we may suppose that this color is 5. Moreover, $c(x_2) = c(h_{14}) = 5$.

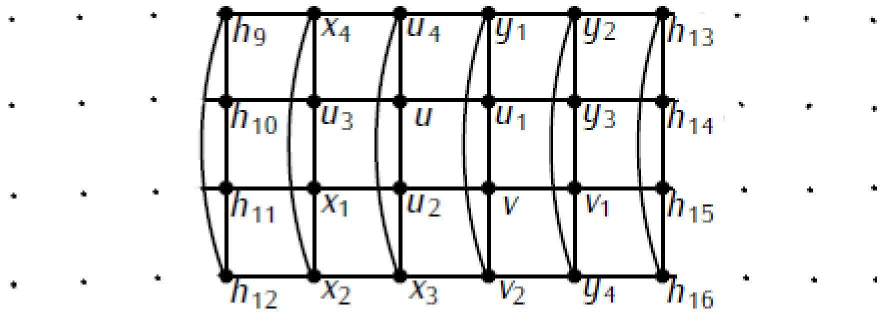


Figure 8.12 : $F \cup A \cup B \cup B''$.

At least one vertex in $\{u_4, y_1, x_3, v_2\}$ must be a large color, by Observation 8.2. Since the distance between a vertex in B'' and any vertex in $\{y_1, x_3, v_2, u_1, u_2\}$ is at most 5 and $D(B'', B \cup B') \leq 4$, it then follows that none of the vertices in B'' receives a color in $\{4{,}5\}$. Note that there is no repetition of any large colors in $B'' \cup \{y_1, x_3, v_1, u_2\}$. A color bigger than 8 is required to color $F \cup B \cup B''$ and $\{y_1, x_3, v_2\}$.

Case 2.2.1.2: No large color is repeating in the vertices set $F \cup B'$.

Suppose first that no large color is repeating in $F \cup B$. Notice that no large color is repeating amongst the vertices of $B \cup B'$. Now suppose that color 4 is repeating in $F \cup B$. Then we have that either $c(u_3) = 4$ or $c(u_2) = 4$ and $c(h_{13}) = 4$.
Let

$$
\begin{aligned}
A'' &= \{h_{11}, h_{12}, x_1, x_2\}, \\
B'' &= \{u_4, y_1, x_3, v_2\}, \\
B''' &= \{v_1, y_3, h_{14}, h_{15}\} \text{ and} \\
A''' &= \{v_1, y_4, h_{15}, h_{16}\}.
\end{aligned}
$$

Assume A has a large color vertex in $\{x_3, x_4\}$, then a color greatest than 8 is required to color vertices in $F \cup A'' \cup B' \cup A$. Therefore, x_2 receives a large color. Suppose that $c(v_2)$ is not large. Then by Observation 8.2, we suppose $c(x_2) = c(h_{14}) = 5$, otherwise a color greater than 8 is needed to color vertices of $A \cup F \cup B' \cup B'''$. Then a color larger than 8 is required for vertices of $F \cup A''' \cup B' \cup A$.

Suppose that none of the colors is repeating in $B \cup F$. Assume no large color is repeating amongst vertices of B' and A. Hence a color greater than 8 is needed to color $A \cup F \cup B \cup B'$. Therefore, since $D(A, B') \leq 5$, it follows that $c(x_4) = c(v_1) = 4$. By Observation 8.2, at least one vertex in $\{u_4, y_1, x_3, v_2\}$ is receiving a large color. Therefore, a color greater than 8 is needed to color $F \cup A \cup B$ and $\{u_4, y_1, x_3, v_2\}$. Suppose some large color γ is repeating amongst vertices of $A \cup B$. This color is in $\{4, 5\}$, since $D(A, B) \leq 6$.

First assume $y = 4$. Suppose that a vertex that receives a color 4 in A is not in $\{x_2, x_4\}$. Assume $A'' = \{h_{11}, x_1, x_2, h_{12}\}$. By Observation 8.2, we have that a vertex in A'' gets a large color other than 4. Since $D(A'', N(u) \cup B') \leq 5$, it follows that a color greater than 8 is required to color vertices in the set $F \cup B \cup B' \cup A''$. Therefore, the vertex receiving a color 4 is in $\{x_4, x_2\}$. By Observation 8.2, one vertex in $\{u_4, y_1, x_3, v_2\}$ must receive a large color. If v_1 or y_4 receives a large color then a color greater than 8 is required to color vertices of $F \cup A \cup B'$ and $\{u_4, y_1, x_3, v_2\}$. Therefore, the only vertex in B' receiving a large color is v_2 .

Assume $A''' = \{h_6, h_7, h_{10}, h_{11}\}$. Some vertex in A''' gets a large color, by Observation 8.2. None of the vertices in A''' receives color 4. Suppose no color is repeating in the vertices of A''' and $F \cup B'$. It follows that a color larger than 8 is needed for vertices in $F \cup A''' \cup \{v_2\} \cup B'''$. Therefore, $c(h_6) = c(v_2) = 5$. Let $B''' = \{v_1, y_4, h_{15}, h_{16}\}$. Once more, a vertex in B''' receives a large color that is neither 4 nor 5. Moreover, since $D(F, T''') \leq 6$, it follows that a color larger than 8 is necessary to color $F \cup B \cup \{v_2\} \cup B'''$.

Let $y = 5$. Therefore, $c(x_2) = c(h_{14}) = 5$. First suppose that a vertex in $\{v_1, y_4\}$ receives a large color. Assume $B''' = \{u_4, y_1, x_3, v_2\}$. One vertex gets a large color in B''' by Observation 8.2 and so a color larger that 8 is needed to color vertices of $F \cup A \cup B' \cup B'''$. Hence, v_2 must get a large color and no vertex in $\{v_1, y_4\}$ can receive such one. Suppose $B''' = \{v_1, y_4, h_{15}, h_{16}\}$ and $A'' = \{h_6, h_7, h_{10}, h_{11}\}$. Both A'' and B''' have a vertex with a large color, by Observation 8.2.

Assume no large color is repeating amongst vertices of A'' and $F \cup B'$. Then a color larger than 8 is required to color vertices of $F \cup A'' \cup B' \cup A$. Therefore, since $D(A'', F \cup \{v_2\}) \leq 6$, it follows that color 4 is repeating amongst vertices of $A'' \cup F \cup \{v_2\}$. Hence either u_1 or v_2 gets 4. We have that none of the vertices in B''' receives any color in $\{4, 5\}$. Since $D(B''', F) \leq 6$, it follows that a color bigger than 8 is needed for vertices in $F \cup A \cup B' \cup B'''$.

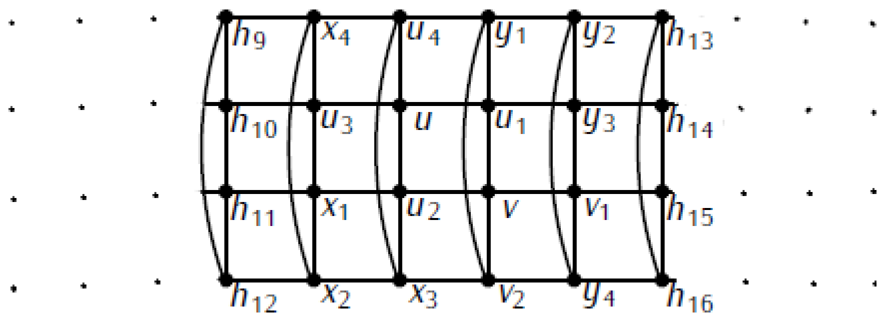


Figure 8.13 : $F \cup B''' \cup A''' \cup B' \cup A \cup B$.

Case 2.2.2: u_4 receives a large color and u_3 a small color.

First assume that v_1 receives a large color and v_2 a small one. Relabel u as v and v as u, rotate the graph H_{10} in such a way that our present Case 2.2.2 changes to Case 2.2.1. We have considered this case. Therefore assume that either v_1 and v_2 both receive small colors with y_4 receiving a large one, or that v_2 receive a large color with v_1 receiving a small one.

Case 2.2.2.1: y_4 requires a large color and u_3 a small color.

Let

$$\begin{gathered} A = \{h_{10}, h_{11}, u_3, x_1\}, \\ A' = \{h_9, x_4, h_{12}, x_2\}, \\ B = \{y_2, y_3, h_{13}, h_{14}\}, \\ B' = \{y_3, h_{14}, h_{15}, v_1\} \\ \text{and} \end{gathered}$$

$$F = \{u_4, u_1, u_2, y_4\}$$

and consider the Figure 8.14 below.

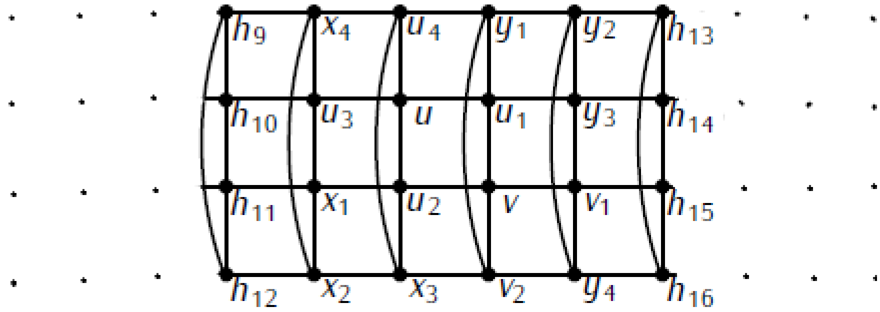


Figure 8.14 : $A \cup A' \cup B \cup B' \cup F$

The sets A, A', B, B' all contain vertices with large colors, by Observation 8.2. Assume that no large color is repeating amongst vertices of $A \cup A'$ and F. Then a color bigger than 8 is needed to color $F \cup A \cup A'$. Suppose that a large color γ is repeating amongst the vertices of $A \cup A'$ and F. Since $D(A \cup A', F) \leq 6$, it follows that $\gamma \in \{4, 5\}$. Say $\gamma = 5$. Hence, $c(h_{10}) = c(y_4) = 5$. Consequently, there is a vertex in A' which receives a large α.

Case 2.2.2.1.1: α is not repeating in $F \cup A'$.

Suppose that α is not repeating in $A' \cup B \cup B'$. Suppose there are two vertices with large colors, or there is one vertex in $B \cup B'$ with a large color that is not appearing in F. Then a color larger than 8 is needed to color $F \cup A' \cup B \cup B'$. Therefore a vertex in $B \cup B'$ gets a large color that occurs in F. This vertex must be in $\{y_3, h_{14}\}$, by Observation 8.2. However, $D(F, \{y_3, h_{14}\}) \leq 4$, and so no large color can repeat in $F \cup \{y_3, h_{14}\}$, which is a contradiction. We then can assume that α is repeating in $A' \cup B \cup B'$.

The only color that may repeat amongst vertices of $F \cup B \cup B'$ is 4. Suppose that color 4 is repeating amongst vertices $F \cup B \cup B'$. Therefore, a vertex in $\{u_4, u_2\}$ and a vertex in $\{h_{13}, h_{15}\}$ both receive color 4 and so $\alpha \geq 6$. Moreover, since $D(A', B \cup B') \leq 7$, it follows that $\alpha \leq 6$. Thus, $\alpha = 6$. Assume $B'' = \{h_{18}, h_{19}, h_{22}, h_{23}\}$. Since $D(B'', B \cup \{h_{13}, h_{15}\}) \leq 5$ and $D(B'', F \cup B \cup B') \leq 7$, it follows that a color larger than 8 is required in $F \cup B \cup B' \cup B''$. Then no large color is repeating in the vertices of $F \cup B \cup B'$.

Assume that $B \cup B'$ contains two vertices of large color, it follows that some color bigger than 8 appears in $F \cup B \cup B'$. Therefore exactly one vertex in $B \cup B'$ receives color α and this vertex must lie in $\{y_3, h_{14}\}$, by Observation 8.2. Since $D(\{y_2\}, \{v_1, v_2\}) \leq 2$, we must have $c(y_2) = 1$. Since the only vertex in $B \cup B'$ of large color lies in $\{y_3, h_{14}\}$. It follows that $c(h_{13})$ is small. Assume $C(y_1)$ is large; then a color bigger than 8 is required for the vertices of $F \cup \{y_1\} \cup A'$. Therefore $c(y_1)$ is small. Since $D(\{y_{13}\}, \{v_1, v_2\}) \leq 3$ and $D(\{y_1\}, \{v_1, v_2\}) \leq 3$, we see that there are two vertices of color 3 at a distance at most three from each other; not possible.

Case 2.2.2.1.2: α is repeating in $F \cup A'$.

It is easy to see that $c(h_{12}) = c(u_1) = 4$ and $\alpha = 4$. Now notice that none of the vertices in $B \cup B'$ receives color 4 or 5. Assume that two vertices receive large colors in $B \cup B'$. Then a color bigger than 8 is used in $F \cup B \cup B'$. Therefore, only one vertex in $B \cup B'$ may receive a large color. This vertex is in $\{y_3, h_{14}\}$ by Observation 8.2. Since $c(y_2) = 1$, $D(\{v_1, v_2\}, \{y_2\}) \leq 2$, we then can deduce again as before that $c(h_{13})$ and $c(h_1)$ are small. Since $D(\{h_{13}\}, \{v_1, v_2\}) \leq 3$ and $D(\{y_1\}, \{v_1, v_2\}) \leq 3$, there are two color 3 vertices at a distance at most three from each other; not possible.

Let $\gamma = 4$. Therefore, either y_4 or u_1 receives color 4. Since $D(B \cup B', \{u_1, y_4\}) \leq 4$ and $D(B \cup B', F) \leq 5$, no vertex in $B \cup B'$ has color 4. Consequently, there is strictly one vertex in $B \cup B'$ that receives a large color. This vertex is found in $\{y_3, h_{14}\}$. Since $c(y_2) = 1$, $D(\{v_1, v_2\}, \{y_2\}) \leq 2$, it follows that $c(h_{13})$ is small and, moreover, $c(y_1)$ is also small otherwise a color bigger than 8 is required to color vertices of $F \cup B \cup \{y_1\}$. By the same argument once more, we find that there are two color 3 vertices in $\{v_1, v_2, h_{13}, y_1\}$ at a distance at most three from each other; not possible.

Case 2.2.2.2: Vertex v_1 gets a small color and vertex v_2 gets a large color.

Let

$$A = \{h_9, ..., h_{12}, x_4, u_3, x_2, x_2\},$$
$$F = \{u_4, u_1, u_2, v_2\},$$
and
$$B = \{h_{13}, ..., h_{16}, y_2, y_3, v_1, y_4\}$$

and consider Figure 8.15 below.

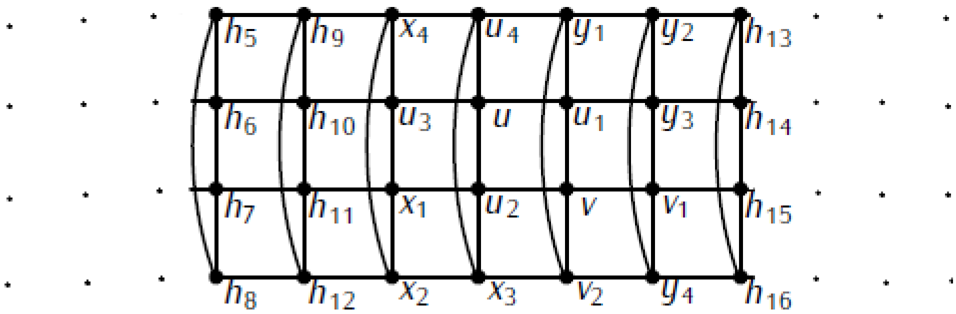


Figure 8.15 : $A \cup F \cup B$

The sets B and A each contain two vertices of large color, by Observation 8.2. Suppose no large color is repeating amongst the vertices of A (B respectively) and F. It follows that a color bigger than 8 is needed to color vertices of $F \cup A$ ($F \cup B$ respectively). Suppose a large color is repeating amongst vertices $A \cup F$. Since $D(A, F) \leq 5$, it follows that color 4 is repeating and so either u_1 or v_1 gets color 4. As $D(F, B) \leq 4$ and $D(B, \{v_2, u_1\}) \leq 4$, it follows that none of the large colors can repeat amongst the vertices of F and B and so a color greater than 8 is necessary for $F \cup B$.

Case 2.3: Both vertices in $N(u) - \{u_1, u_2\}$ can receive small colors.

Suppose that both vertices in $N(v) - \{u_1, u_2\}$ can receive small colors. Let

$$A = \{x_4, u_4, u, u_3\},$$
$$B = \{v, v_1, v_2, y_4\},$$
$$F = \{u_1, u_2\}$$
$$A' = \{u_4, y_1, x_3, v_2\}.$$

and consider the Figure 8.16.

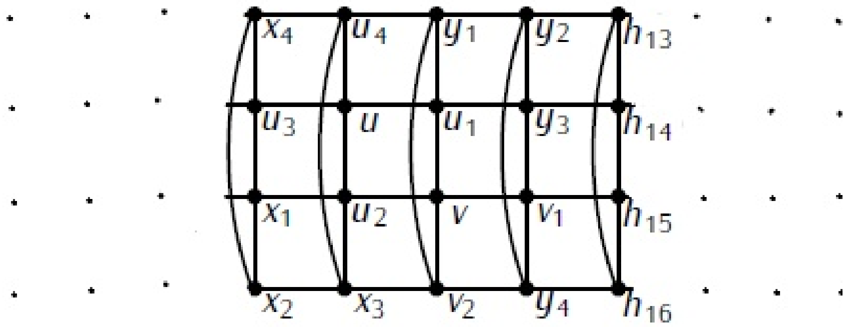


Figure 8.16 : $A \cup B \cup F \cup A'$

The sets A, B, A' contain vertices with large colors that are not repeating, by Observation 8.2. If both x_3 and y_1 receive large colors, we then have that a color bigger than 8 will appear in $F \cup A' \cup A \cup B$. We may suppose that y_1 does not receive a large color by symmetry. Assume that $c(y_2)$ and ($c(y_3)$ respectively) is large, then a color bigger than 8 is required for vertices in $A \cup F \cup B \cup \{y_2\} \cup A'$ ($A \cup F \cup B \cup \{y_3\} \cup A'$ respectively). Since $D(\{y_2\}, \{v_1, v_2\}) \leq 2$ and $c(y_2) = 1$, both $c(y_1)$ and $c(y_3)$ are small. Note that $D(\{y_3\}, \{v_1, v_2\}) \leq 3$ and $D(\{y_1\}, \{v_1, v_2\}) \leq 3$, which is not possible.

Case 3: One of u_1, u_2 is receiving a large color and the other one a small color.

Suppose that $c(u_2)$ is small and $c(u_1)$ is large. Also suppose that a vertex in $N(u) - \{u_1, u_2\}$ and a vertex $N(v) - \{u_1, u_2\}$, both gets large colors. We now make more claims. Assume H is a graph induced by the vertices $\{h_{17}, ..., h_{24}\}$ which is depicted by Figure 8.17 below.

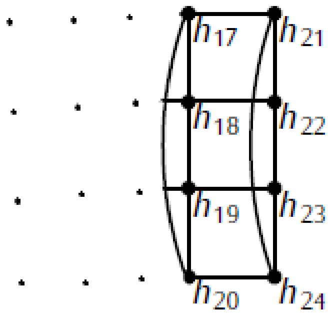


Figure 8.17 : The set H

Claim E: Suppose H has strictly two vertices with large colors. These vertices are at distance two from each other.

Proof. See proof of Claim C.

□

Claim F: Assume H has strictly two vertices in large colors, then these vertices are in the same column.

Proof. Re-consider $H = \{h_{17}, h_{18}, ..., h_{24}\}$ above.
Assume that $c(h_{24})$ and $c(h_{19})$ are large. Then by Observation 8.2, a vertex in $H - \{h_{19}, h_{20}, h_{23}, h_{24}\}$ must receive a large color; not possible.

□

Claim G: Assume H has strictly two vertices with large colors. Then these vertices contain a color 1 neighbor in the opposite column.

Proof. See proof of Claim D

□

Case 3.1: $c(v_2)$ is large.
Let

$$F = \{u_3, u_4, u_1, v_2\},$$
$$A = \{x_1, x_2, x_3, u_2\},$$
$$B' = \{y_2, y_3, v_1, y_4, h_{13}, ...h_{16}\},$$
$$A' = \{h_5, h_6, ...h_{12}\},$$
and
$$B' = \{h_{17}, h_{11}, ..., h_{24}\}$$

and consider the sketch below.

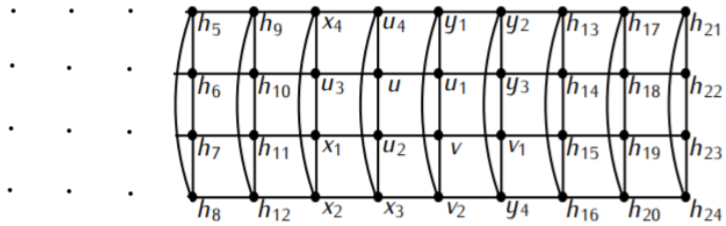


Figure 8.18 : $F \cup A \cup B' \cup A' \cup B'$

The set A contains at least one vertex with a large color. The sets B, B', A' contain two vertices with large colors. Assume no color is repeating in the vertices of $F \cup A$, and B (A' respectively), then we have that a color greater than 8 is necessary for the vertices of $F \cup A \cup B$ ($F \cup A \cup A'$ respectively). Since $D(B, A \cup F) \leq 6$, only colors 4, 5 are repeating amongst vertices of $F \cup A$ and B. Similarly, only colors 4, 5 are repeating amongst vertices of $F \cup A \cup A'$.

Case 3.1.1: The color 5 is repeating on a vertex in $F \cup A$ and a vertex in B.

Suppose that a color 5 is used in $A \cup \{u_3, u_4\}$ and $\{h_{13}, ..., h_{16}\}$. Then no vertex in A' and B' receives color 5. So 4 and 5 are the only large colors which are repeating amongst vertices of B' and $\{u_1, v_2\}$. If B (B' respectively) contains three vertices of large color, it implies that a color bigger than 8 is needed to color the vertices of $\{u_1, v_2\} \cup B \cup B'$. Therefore, B (B' respectively) contains strictly two vertices with large colors. Applying Claim F, both large colors of B must be used in $\{h_{13}, ..., h_{16}\}$. If we let B has a vertex colored with 4, it follows that none of the vertices in $\{u_1, v_2\} \cup B'$ receives 4 or 5, and so a color bigger than 8 is needed to color vertices of $\{u_1, u_2\} \cup B' \cup B$. Suppose that one vertex in B' and one in $\{u_1, v_2\}$ receive color 4.

Case 3.1.1.1: $c(h_{16})$ and $c(h_{15})$ are large.

Applying Claim G, $c(y_2)$ and $c(v_1)$ must have small colors. First assume that color 4 is used in $\{h_{17}, ..., h_{20}\}$. By Claim F, $c(h_{18})$ and $c(h_{20})$ receive large colors. By claim G, $c(h_{22}) = c(h_{24}) = 1$. Since a vertex that gets color 1 contains two large color neighbours, there exist large color vertices w, z such that $w \in N(h_{22})$ and w, $z \notin B'$.

Since $D(\{u_1, v_2\}, \{z, w\}) \leq 7$, it follows that at least one vertex in $\{z, w\}$ receives a color that is not repeating in $\{u_1, v_2\} \cup B \cup B'$ and so a color bigger than 8 is needed. We suppose that B' contains two large colors in $\{h_{21}, ..., h_{24}\}$ and one of these colors must be 4.

Assume that $c(h_{21})$ and $c(h_{23})$ are large. By our claims, $c(h_{18})$ and $c(h_{20})$ are small. But $D(\{h_{18}\}, \{y_2, v_1\}) = 3$ and $D(\{h_{20}\}, \{y_2, v_1\}) = 3$. We suppose $c(h_{22})$ and $c(h_{24})$ are large. Assume that the large colors other than 4 that are used amongst the vertices of $\{h_{22}, h_{24}\}$ be γ. Now we know that γ is not repeating in B and $\{u_1, v_2\}$, and we may suppose that it is used amongst the vertices of $A \cup \{u_3, u_4\}$. Since $D(A \cup \{u_3, u_4\}, \{h_{22}, h_{24}\}) \leq 8$, it follows that $\gamma \in \{6, 7\}$. Remember that none of the vertices in the set A' receives color 5. Therefore, A' must have a large color vertex that is neither 4 or 5. Let us say that this large color is not occurring amongst the vertices of B, it follows that a color greater than 8 is required to color vertices of $F \cup A \cup B \cup A'$. Therefore, since $D(B, A') \leq 8$, the vertices of A' and B share a large color $\alpha \in \{6, 7\}$. Also, suppose that A' has strictly two large color vertices, one of them being color 4.

Assume that $c(h_{22}) = 4$. It follows $c(h_{24}) = \gamma$ and $c(v_2) = 4$. Let $c(h_{16}) = 5$. We then have that $c(u_3) = 5$ and $c(h_{14}) = \alpha$. Hence $c(h_{24}) = c(x_1) = 6$, and so $\alpha = 7$. Consider the set $E = \{x_4, u_4, x_2, x_3\}$ and apply Observation 8.2. It follows that there exists a vertex with large color in this set. Then, a color greater than 8 is necessary to color vertices of the set $E \cup A \cup F \cup B$. Suppose that $c(h_{14}) = 5$ and thus $c(h_{16}) = \alpha$. Therefore, $c(u_3) = \gamma$ and $c(x_2) = 5$. Since $c(h_{16}) = \alpha$, it follows that neither h_{12} nor h_9 can receive color α. We may suppose that the color 4 vertex of A' is used in the column which contains h_9. Now, h_8 cannot receive color 4 or α. Hence, $c(h_7)$ and $c(h_5)$ must be large and by applying Claim G, $c(u_2)$, $c(u_4)$, $c(h_{10})$ and $c(h_{12})$ must be small. But is not possible since $D(\{h_{10}\}, \{u_4, u_2\}) = D(\{h_{12}, \{u_4, u_2\}\}) = 3$.

Now suppose that $c(u_1) = 4$, $c(h_{24}) = 4$ and $c(h_{22}) = \gamma$. Let $c(h_{16}) = 5$. Then $c(u_3) = 5$ and $c(h_{14}) = \alpha$, and either $c(x_1) = \gamma$ or $c(x_2) = \gamma$ or $c(x_3) = \gamma$. Consider the set $E = \{x_4, u_4, x_2, x_3\}$. By Observation 8.2, we then have that there exists a vertex with a large color in this set. Therefore, a color greater than 8 is required to color vertices of the set $E \cup A \cup F \cup B$. It means, $c(x_2) = \gamma$ or $c(x_3) = \gamma$ and we may assume that $c(u_4)$ is small. If $c(h_{12}) = 4$, then

$c(h_{10}) = \alpha$, which is not possible since $c(h_{14}) = \alpha$. Clearly, h_9, h_{10}, h_6, h_{12}, h_{11} cannot receive color 4. Since h_6 cannot be colored by α, we have that $c(h_5)$ and $c(h_7)$ are large colors. Applying our claims, $c(h_{10})$ and $c(h_{12})$ are small colors. However, this is not possible since $D(\{h_{10}\}, \{u_4, u_2\}) = D(\{h_{12}, \{u_4, u_2\}\}) = 3$. We may suppose that $c(h_{14}) = 5$. Hence, $c(x_2) = 5$, $c(h_{16}) = \alpha$.

Assume $c(u_4)$ is large; it follows that the vertices of $E = \{u_3, u, x_1, u_2\}$, by applying Observation 8.2. We conclude that E contains a large color vertex. A color bigger than 8 is necessary to color vertices of $F \cup E \cup B \cup A$. Therefore, $c(u_3)$ is large and the two columns that u and u_3 respectively, contain only two large color vertices. Therefore $c(u_3) = \gamma$, but we have $c(h_{22}) = \gamma$; not possible.

Case 3.1.1.2: $c(h_{13})$ and $c(h_{15})$ are large.

Since B contains strictly two large color vertices, by Claim G, it follows that $c(y_2) = c(v_1) = 1$. However $c(v) = 1$; not possible.

Case 3.1.2: The color 4 is repeating on a vertex in B and on a vertex $F \cup A$.

Suppose B has a vertex of a color 5. It follows that neither u_1 nor v_2 gets a color of $\{4, 5\}$. None of the vertices in T receives color 5. Suppose none of the large colors are repeating in $\{u_1, v_2\} \cup B \cup B'$, then a color greater than 8 is required to color vertices of H_{10} in G. Therefore, a large color is repeating amongst vertices of $B \cup B'$ and this color is 4. Thus, by applying Claim E and Claim F, we have that $c(v_1)$, $c(y_2)$, $c(h_{21})$ and $c(h_{23})$ are all large, or $c(h_{22})$, $c(h_{24})$, $c(y_3)$ and $c(y_4)$ are also large. Applying Claim G we get a contradiction.

Suppose that none of the vertices in B gets color 5. Therefore, there exists a vertex in B which receives a large color bigger than 4 and a vertex that receives 4. Moreover, none of the vertices in $\{u_1, v_2\}$ gets color 4. Assume B contains three large color vertices; it follows that a color bigger than 8 is required to color $F \cup A \cup B$. Therefore, B has strictly two large color vertices. Let set $E = \{x_4, u_3, x_1, x_2, u_4, u, u_2, x_3\}$ has three vertices that receive large colors, it follows that a color larger than 8 must be used in $F \cup E \cup B$. Therefore, E has strictly two large color vertices. We have that $c(x_2)$ and $c(u_3)$ are large, by our claims.

Suppose that a color 4 vertex of B is in the column which has y_2. Assume $B'' = \{h_{13}, ..., h_{20}\}$. The set B'' contains two large color vertices. Since $D(B'', A \cup F) \leq 7$, it follows that only colors 5 and 6 are repeating amongst vertices of $F \cup A$ and B''. Suppose a vertex in B'' gets a color that does not appear in the vertices of $F \cup A$, it follows that a color bigger than 8 is required to color vertices of $F \cup A \cup B \cup B''$. Therefore, colors 5 and 6 appear in the vertices of B'' and $\{u_3, u_4\} \cup A$. As $c(u_3)$ and $c(x_2)$ are large colors and either x_2 or u_3 receives color 4. It follows that at most one large color is repeating amongst vertices of B'' and $F \cup A$ and so color greater than 8 is needed to color vertices of $F \cup A \cup B \cup B''$.

Also suppose that the color 4 vertex is in the column that has h_{13}. None of the vertices of B' receives color 4 and if the vertices of $B' \cup \{u_1, v_2\}$ have no large color in common, then a color bigger than 8 is needed to color the vertices of $B \cup B' \cup \{u_1, v_2\}$. Therefore, the color 5 is repeating on a vertex in B' and on a vertex in $\{u_1, v_2\}$.

Case 3.1.2.1: $c(h_{15})$ and $c(h_{13})$ are large.

Since B contains only two vertices that get large colors, by Claim G, it follows that $c(v_1) = c(y_2) = 1$. Note $c(v) = 1$ is not possible.

Case 3.1.2.2: $c(h_{16})$ and $c(h_{14})$ are large.

As the color 5 is repeating amongst the vertices of B' and $\{u_1, v_2\}$, it then means that the color 5 vertex of B' is in the column that has h_{21}. We may suppose that B' contains strictly two large color vertices, one amongst them is $\gamma \in \{6, 7\}$. The color γ is in the vertices of $\{u_3, x_2\}$, otherwise a color bigger than 8 is required to color vertices of $F \cup A \cup B \cup B'$. Furthermore, one vertex in $\{h_{14}, h_{16}\}$ gets a large color $\alpha \in \{6, 7\}$.

Assume that $c(h_{23})$ and $c(h_{21})$ are large. By applying Claim G, we have $c(v_1)$, $c(y_2)$, $c(h_{18})$ and $c(h_{20})$ are small colors, but then this is not possible as $D(\{h_{18}\}, \{y_2, v_1\}) = D(\{h_{20}\}, \{y_2, v_1\}) = 3$. Suppose that $c(h_{22})$ and $c(h_{24})$ are large. Since either $c(u_3)$ or $c(x_2)$ is 4, none of the of vertices in A' gets a color 4. Assume no large color is repeating in the vertices of F and A'. Hence a color greater than 8 is required to color vertices of $F \cup A \cup A'$. Therefore, a vertex of A'

receives color 5.

Assume that $c(h_{24}) = \alpha$ and $c(h_{22}) = 5$. Thus $c(v_2) = 5$. Suppose that $c(h_{14}) = 4$. Then $c(h_6) = 5$, $c(u_3) = \gamma$ and $c(x_2) = 4$. Moreover, suppose that A' and B contain a vertex with the same large color. Therefore $c(h_{16}) = c(h_8) = \alpha$, which is not possible. Suppose $c(h_{22}) = \gamma$ and $c(h_{24}) = 5$. Therefore, $c(x_2) = \gamma$, $c(u_1) = 5$ and $c(u_3) = c(h_{16}) = 4$. We have again that $c(u_8) = 5$ and $c(u_5) = \alpha$. But this is not possible since $c(h_{14}) = \alpha$.

Case 3.2: $c(v_2)$ is small.

Suppose $c(v_1)$ is large and $c(v_2)$ is small. Therefore $c(u_4)$ must be large.

Let

$$A = \{x_1, x_2, x_3, u_2\}$$
$$B = \{y_1, y_2, y_4, v_2\}$$
$$\text{and}$$
$$F = \{u_4, u_3, u_1, v_1\}$$

and consider the figure below.

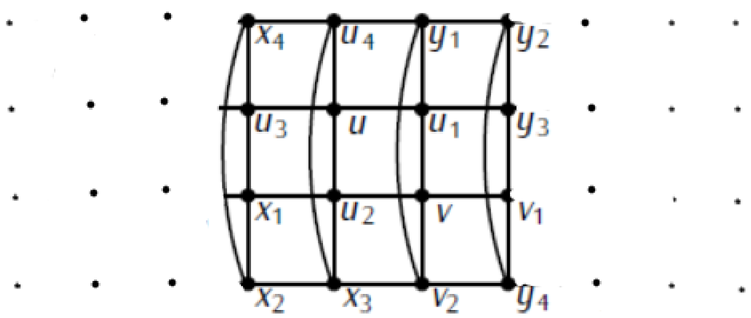


Figure 8.19 : $A \cup B \cup F$

The sets A and B each contain at least one large colored vertex, by applying Observation 8.2. Assume $c(x_3) = 1$, we then have that the vertices x_3 and v are two color 1 vertices that contain two small color neighbors. We are done by applying Case 1. Therefore, $c(x_3)$ is large. We state that either $c(x_1)$ or $c(u_3)$ is large. Suppose $c(u_3)$ is small.

Assume $c(x_1) = 1$, we then have that u and x_1 contain two small color neighbors and we are done by applying Case 1. Therefore, $c(x_1)$ is large; the statement is checked. There exist six large color vertices in $F \cup A \cup B$. Suppose a large color is repeating and this color is 4. Therefore, either $c(y_4) = 4$ or $c(y_1) = 4$.

We may suppose that $c(y_1)$, $c(y_3) \in \{1, 2, 3\}$. Suppose $c(y_1) = 1$ ($c(y_3) = 1$ respectively), it means that y_1 and u (y_3 and v respectively) contain two large color common neighbors. We are done by Case 2. As a result, $c(y_1)$ and $c(y_3)$ are small. Note that $c(v_2) = 3$, $c(u_2) = c(y_1) = 2$ and $c(y_3) = 3$; not possible.

□

Proof of Lemma 8.1.3

Proof. We prove Lemma 8.1.3 by considering a sub-graph H of G induced in a row by four columns and assume that H contains two vertices in a column that both are colored by 1. Suppose a column of H contains two vertices of color 1, then the next and previous columns contain no vertices colored by 1, by Lemma 8.1.2. Therefore, H contains at most four vertices colored by 1. We use the same argument in the proof of Lemma 8.1.1 to conclude that more than 8 colors are required to color vertices of H.

□

Chapter 9

Conclusion

Our aim was to investigate the upper bound and the lower bound of the packing chromatic number of a 2-dimensional square lattice. Goddard et al.[1] introduced the notion of packing coloring under the name "Broadcast coloring". In the paper, he showed that the upper bound of the packing chromatic number is less than or equal to 23 and that the lower bound is greater than or equal to 9. We studied his grid approach and other approaches in papers to understand the building up towards a solution to the problem. Schwenk [4] improved the upper bound to 22 and later, Holub and Soukal [5] improved the upper bound to 17. Afterwards, Barnaby, Franco, Chen and Jos in [6], improved the upper bound to 15 and the lower bound to 13.

We also studied the packing chromatic number of Cartesian products of $C_4 \square C_q$ where $q = 4x$, where $x \geq 3$, $x \in \mathbb{N}$, using theoretical approach [7].

Sarila and Jonck have tried for years to improve on the bounds (lower and upper) of the packing chromatic number of the infinite grids, without success. However, all these attempts were not in vain. The experience we have gained led us to a calculated conjecture.

Conjecture 9.1: The packing chromatic number of the infinite square lattice is not 13 or 14, but it is 15.

Moreover, since $P_r \square P_c \in C_r \square C_c$ where r, $c \geq 3$ and r and c tend to infinity, we have that

$$\chi_\rho(C_r \square C_c) \geq \chi_\rho(P_r \square P_c),$$

Also

$$\chi_\rho(C_r \Box C_c) \leq 15,$$

since the same 72×72 tile that was used in [5] to color the infinite grid periodically, can be used to color the infinite torus periodically.

Conjecture 9.2: The packing chromatic number of the infinite torus is 15.

Considering the methods used in this study, we believe that Simulated Annealing (SA) is the best method to determine lower bounds. Packing colorings can be used to determine upper bounds.

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