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UNIVERSITY OF WITWATERSRAND,
JOHANNESBURG.

MASTERS DISSERTATION

Nonclassical symmetry reductions and conservation laws for reaction-diffusion equations with application to population dynamics

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May 29, 2015

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Abstract

This dissertation analyses the reaction-diffusion equations, in particular the modified Huxley model, arising in population dynamics. The focus is on determining the classical Lie point symmetries, and the construction of the conservation laws and group-invariant solutions for reaction-diffusion equations. The invariance criterion for determination of classical Lie point symmetries results in a system of *linear* determining equations which can be solved analytically. Furthermore, the Lie point symmetries associated with the conservation laws are determined. Reductions by associated Lie point symmetries are carried out. Nonclassical symmetry techniques are also employed. Here the invariance criterion for symmetry determination results in a system of *nonlinear* determining equations which may be solved albeit difficult. Nonclassical symmetries results in exact solutions which may not be constructed by classical Lie point symmetries. The highlight in construction of exact solution using nonclassical symmetries is the introduction of the *modified* Hopf-Cole transformation. In this dissertation, the diffusion term and the coefficient of the source term are given as quadratic functions of space variable in one case, and the coefficient as the generalised power law in the other. These equations admit a number of classical Lie point symmetries. The genuine nonclassical symmetries are admitted when the source term of the reaction-diffusion equation is a cubic.

Declaration

I, Kirsten Ingrid Louw, hereby certify that the work done in this dissertation is wholly my own original work except where due references have been made. It is being submitted for the degree of Masters of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before at any other institution.

Signed:

Date:

Acknowledgment

I would like to thank my supervisor Prof. R. J. Moitsheki for his patience and guidance throughout the course of this degree. The experience led to some new and exciting transformations which allowed construction of exact solutions.

I would like to thank my friends and family for the support they have provided in my journey.

I would also like to express my thanks to the University of Witwatersrand for funding my Masters degree.

Chapter 1

Introduction

In this work, the focus is on partial differential equations arising in population dynamics. These equations are referred to as reaction-diffusion equations. Reaction-diffusion equations describe how the concentration of a substance is distributed in space changes, whereby the diffusion term causes the spread over the surface. The reaction-diffusion equations have appeared in many fields of study. For example, the reaction-diffusion equations arise in heat transfer problems [1], biology [2, 3], and transmission of nerve signals [4]. This dissertation specifically investigates the nonlinear reaction-diffusion equations applied in biology. Fisher (1937) had used a nonlinear reaction-diffusion equation to model the population growth of mutant genes over a period of time. One can take Fisher's equation [2] and with simple modifications, derive the Fitzhugh-Nagumo equation [5]. Moreover, one can make a modification to the Fitzhugh-Nagumo equation [2] in order to obtain Huxley's equation [2].

A significant amount of work has been done in the process of studying the reaction-diffusion equation. In particular, from the symmetry analysis point of view. As a result there are many ways in which to solve for a solution for the reaction-diffusion equation. One of these methods to solve for exact solutions is to find the classical

Lie point symmetries of the equation. Furthermore, invariant exact solutions can be constructed via nonclassical symmetries rather than classical Lie point symmetries. Examples of determining classical Lie point symmetries and nonclassical symmetries are provided in chapter 3.

1.1 Literature Review

In this section, the background knowledge pertaining to the specified topics will be discussed. This literature review provides a firm foundation from which to understand topics covered herein such as ideas and definitions that will be taken as assumed knowledge hereafter. Definitions of biological terminology can be found in Appendix A.

1.1.1 Classical and Nonclassical Symmetries: Historical Background

As a student, when one is first introduced to ordinary differential equations (ODEs), one learns various methods in which to solve for the ODEs. Some of the ODEs one encounters are separable, homogeneous and exact equations, in which the method to solve them are seemingly unrelated in the student's point of view. But these special methods are related. Sophus Lie's [6] ground-breaking work in the nineteenth century proved that these methods were in fact special cases of a general integration procedure based on the invariance of the differential equation under a continuous group of transforms [6]. These continuous groups are now referred to as Lie groups, and the discovery of the Lie groups impacted the mathematically-based sciences. To name a few fields that application of Lie groups include are; algebraic topology, differential geometry, control theory and classical mechanics [6].

Lie's (1881) pioneering work gave rise to the classical symmetries method, also called group analysis [7, 8]. Lie point symmetries are used to find group invariant (exact) solutions for the governing equation in question. The theory on this topic is well documented (see for example [6, 9, 10, 11]).

As time went on, the method of finding symmetries evolved. The notion of nonclassical symmetries was introduced by Bluman and Cole [29] in 1969, which gave rise to a new method of finding symmetries. Calculating nonclassical symmetries provides a method to find possible new symmetries that may not be constructed via the classical route [7, 8]. Due to the nature of nonclassical symmetries, the field of finding nonclassical symmetries and its application has expanded. When solving for the nonclassical symmetries, one has an added condition in the infinitesimal criterion for invariance called the invariant surface condition (ISC). Bluman and Cole [29] analysed the heat equation, comparing and contrasting the the classical and nonclassical symmetry solutions.

The main difficulty in finding the nonclassical symmetries is that the determining equations are nonlinear. This method may yield more solutions than the classical method [8, 12]. Nucci [8, 12] discovered that the nonclassical symmetries method can give rise to some well-known Bäcklund transformations of the governing equations discussed.

When applying the nonclassical symmetries method, Arrigo *et al* [13] not only found new forms of nonclassical symmetries but were also able to linearise the Burgers' system. This was achieved by removing a restriction on the nonclassical symmetry operator that was initiated by Cherniha and Serov [13].

When comparing the solutions obtained from the classical Lie point and nonclassical symmetry analysis, it would have seemed that when comparing solutions the asymptotic analysis was an effective tool [14]. One may write that solutions obtained via nonclassical symmetry techniques may not be obtained via the classical Lie point sym-

metries.

1.1.2 General Ideas on Conservation Laws

There are a few methods to calculating conserved vectors. In fluid mechanics, one uses a conserved quantity in jet flows [15]. The conserved quantity can be derived from conserved vectors. Furthermore, one may determine Lie point symmetries associated with the conservation laws. As such, one can use the double reduction method by Sjöberg [16] once the conserved vectors are and associated Lie point symmetries found. The resulting ordinary differential equation (ODE) will then be integrable [17].

Although one can generate conservation laws with Lie-Bäcklund symmetry operator of the system without having to convert to the canonical Lie-Bäcklund symmetry operator [18], using canonical produces higher order conservation laws. But with point transformations, order is preserved [18].

Conserved vectors can form a basis of conserved vectors for a system. One would like to obtain the basis of conservation laws, the way forward lies within the structure of the symmetry Lie algebra of the equation [18].

Kara and Mahomed [18] have developed a theorem to eliminate nontrivial conserved vectors. However the symmetry operators need to be of Noether nature and also need to be associated with a Lagrangian.

There are several methods in obtaining conserved vectors. Some are listed below [19]:

1. Direct method
2. Noether's approach: methods that make use of Noether's theorem
 - (a) Euler-Lagrange differential equations

- (b) Noether symmetry generator
 - (c) Noether conserved vectors
- 3. Multiplier method
- 4. Variational approach
- 5. Variational approach on space of solutions of the differential equation
- 6. Symmetry and conservation law relation
- 7. Direct construction method for conservation laws
 - (a) Cauchy-Kovalevskaya
 - (b) Conservation law construction formula
- 8. Partial Noether approach
- 9. Conservation theorem

Note that the direct method is the most elementary method of determining conservation laws.

The characteristic method, also known as the multiplier method, can evaluate conserved vectors such that the multipliers of the governing equation have a conserved form. The determining equations are the same as those in the direct method [19]. This method is the simplest and most effective to use [19].

Using the Noether approach is elegant but is constrictive in the sense that differential equations (DEs) require standard Lagrangian and corresponding Noether symmetries [19]. As was the case in [20], as the Noether approach did not work, thus reverting to using the direct method to solve for the conservation laws.

Hence, using the partial Noether approach is more effective as it works in the same manner as that of Noether approach with or without the DEs standard Lagrangian [19]. This is due to the fact that, in real world applications, the standard Lagrangian may not exist.

When one has the conserved vectors, it can form a basis of conserved vectors for a system [15]. For the purpose of this dissertation, only two methods of calculating the conserved vectors are considered, namely the direct method and the multiplier method.

1.1.3 A Note on Reaction-diffusion Equation

Consider the following reaction-diffusion equation:

$$u_t = (D(x, u)u_x)_x + g(x)Q(u), \quad (1.1)$$

where $D(x, u)$ and $Q(u)$ are the diffusivity and source term, respectively. Note that D is dependent on x and u , and Q on u . Furthermore, the coefficient of Q is spatially dependent. In some analysis, the diffusion coefficient is held constant. This leads to $D(x, u) = D_0$, where D_0 is a constant, and $g(x) = 1$. Hence,

$$u_t = (D_0 u_x)_x + Q(u). \quad (1.2)$$

Many solutions to Fisher's equation (1.2) are explicit, given $Q(u) = u(1 - u)$ [2]. In the 1930's, Fisher [21] used a reaction-diffusion equation with a quadratic source term to describe the spread of a recessive advantageous gene. This model is given by

$$u_t = \kappa u_{xx} + mu(1 - u), \quad (1.3)$$

where u is the frequency of the new mutant gene, κ is the coefficient of diffusion, and m is the intensity of selection in favour of the mutant gene. He did not give much

motivation as to support the derivation of the equation [21].

There are also a number of solutions that exist for the Fitzhugh-Nagumo equation, equation (1.2) with $Q(u) = u(1-u)(u-a)$, $a \in \mathbb{R}$ [5]. Newell-Whitehead equation can be given by Fitzhugh-Nagumo equation with $a = -1$. Similarly, Huxley's equation is given by Fitzhugh-Nagumo equation with $a = 0$ [5].

Exact solutions obtained from these equations are important and reveal interesting information that would otherwise not be as evident or revealed by numerical methods [5]. The best way to analyse these equations with the aim of constructing exact solutions is by employing Lie symmetry techniques. In particular, classical Lie point symmetry and using nonclassical symmetry methods. These methods will be discussed in detail in chapter 2.

Arrigo *et al* [5] studied various forms of Q which produced differing profiles. In fact these forms of Q are given as cubic functions of the dependent variable as listed below.

1. $u_t = u_{xx} + a(u - r_1)(u - r_2)(u - r_3).$
2. $u_t = u_{xx} + a(u - r_1)(u - r_2)^2.$
3. $u_t = u_{xx} + a(u - r_1)^3.$
4. $u_t = u_{xx} + a(u - r_1)(u^2 - 2pu + p^2 + q^2).$

Many authors thereafter have also sort to develop the reaction-diffusion equation with a source term. As others have claimed to have confirmed Fisher's results with no real derivation or convincing mathematical proof. Later, a cubic source term was suggested.

For simplicity purposes, one would start off modelling changes in frequency of the alleles at one particular locus within a population. As modelling for a population of sexually reproducing species generates large possibilities for genotypes, this in turn becomes

difficult to solve [22]. In the modelling of gene frequencies in [21], the assumptions made are that all genotypes have the same death rate, the population randomly mates, the population exists in a one dimensional habitat (for instance a shoreline), an individual's genotype does not influence the movement of another and genotypes are independent of migration rates. Bronwyn Bradshaw-Hajek *et al* [21] then went on to model a diploid population, creating three possible genotypes. This model then collapsed into Fitzhugh-Nagumo equation. Allowing the diffusion coefficient to be equal to one, coupled with constant population density, reduced the equation further to a cubic reaction-diffusion equation known as Huxley's equation. With further analysis, [21] proves that Fisher's equation is suitable for asexual reproducing species.

Bradshaw-Hajek *et al* [21] then extend this model to having a single locus having three different alleles (A_1, A_2, A_3), therefore six different genotypes. Bradshaw-Hajek *et al* consider two cases that may arise:

- Case1: A_2 and A_3 are co-dominant with equal reproductive success rates. It is made apparent that Huxley's equation is appropriate for this special case.
- Case 2: A_2 and A_3 are partially dominant. Fitzhugh-Nagumo is appropriate for this special case.

Although, having a cubic source term is robust, therefore having a weak reproductive success rate of the genotype and common death rate, the cubic has some level of validity [21]. Bradshaw-Hajek *et al* [23] went on to study the following equation:

$$p_t = \kappa p_{xx} + k(x)p^2(1 - p). \quad (1.4)$$

When $k(x)$ is constant, equation (1.4) describes a population with two alleles at the locus under consideration and how the recessive advantageous allele spread through the population [23]. Whereas if $k(x)$ is not constant, the above situation still holds but

the birth rates of the various genotypes in the population becomes spatially dependent [23]. To effectively compare alternate reaction-diffusion models, one will take $\kappa = 1$ [22].

In [23], Bradshaw-Hajek *et al* found three forms for a non-constant $k(x)$ using the nonclassical symmetries method. Nucci *et al* [24] furthered the investigation into the non-constant forms of $k(x)$, wherein they were able to derive nonclassical symmetries for three families of the governing equation found in [24].

1.2 Aim and Objective

In this dissertation, the following governing equation is considered

$$u_t = (k(x)u_x)_x + g(x)Q(u),$$

where $k(x)$ is known as the diffusivity term, $Q(u)$ as the cubic source term and $g(x)$ as the x dependent coefficient of the source term. This equation describes the propagation of a mutant gene, $u(t, x)$, where t is time and x is the spatial variable.

The main focus is to analyse the reaction-diffusion equations arising in population dynamics, specifically concentrating on the generalised Huxley equation. Here the diffusivity is given as an arbitrary function of space variable. The exercise is to determine all forms of diffusivity for which the Huxley equation admits the nonclassical symmetries. Furthermore, a Lie point symmetry analysis will be carried out to construct other group invariant solutions.

Conservation laws and associated Lie point symmetries will also be determined and possibly double reduction of the general Huxley equation will be investigated. In particular, we intend on using the multiplier and the direct method to construct con-

servation laws. Note that the diffusion equation does not have a Lagrangian, as such other methods such as the variational approach would fail.

1.3 Outline

Chapter 2 explores the symmetry methods that one can use to reduce the equation in question or to construct exact (group-invariant) solutions. Firstly, the chapter outlines how to find classical Lie point symmetries. Secondly, the description of how to find the nonclassical symmetries of partial differential equations (PDEs) is given. These are the methods employed to find the invariant solutions of a PDE. An illustrative example for calculating nonclassical symmetries is taken from [1] and the detailed steps can be found in Appendix B. Lastly, the chapter will describe how to construct conserved vectors using conservation laws by the Direct and Multiplier method.

Chapter 3 describes how the model for the gene propagation is formulated. This formulation of the reaction-diffusion equation for gene propagation can be derived from difference equations and continuous differential equations. This will reveal how the governing equation for this paper was derived.

Chapter 4 seeks to find the exact solutions of the reaction-diffusion equation by calculating the classical Lie point symmetries. One may note that the admitted symmetry may reduce the $(1+1)$ PDE to a second order ODE which may or may not be solvable.

Chapter 5 then seeks to find the nonclassical symmetries of the reaction-diffusion equation. The nonclassical symmetry reduction will be performed to construct an exact solution.

Chapter 6 provides calculated conserved vectors and the associated Lie point symmetries for the reaction-diffusion equation. Two methods will be employed to find the

conservation laws, namely the direct method and the multiplier method. Consequently, the associated Lie point symmetries of the governing equation will be determined. Furthermore, one may use the double reduction method to construct exact solutions.

Chapter 2

Symmetry Techniques for Differential Equations

A symmetry of a system of differential equations is a transformation that maps any solution to another solution of the system [10]. These transformations form groups that depend on continuous parameters and consist of point or contact symmetries [10]. The most elementary of Lie groups include translations, rotations and scalings.

Lie groups and their infinitesimal generators can be prolonged, produce an overdetermined system of linear homogeneous PDEs. One can usually determine the infinitesimal generators in explicit form.

2.1 Classical Lie Point Symmetries

A brief discussion on calculation of Lie point symmetries is given.

2.1.1 Infinitesimal Transformation

Suppose a PDE with 1 dependent and n independent variables, u and x respectively. Hence $x = (x^1, x^2, \dots, x^n)$, $u = u(x)$. Thus extending the transformations from (x, u) -space to $(x, u, u_1, u_2, \dots, u_k)$ -space, where $u_{(k)}$ denotes all k th-order partial derivatives of u w.r.t. x .

The purpose of finding the infinitesimal transformations or equivalently one-parameter Lie group of transformations,

$$\begin{aligned}\bar{x}^i &= x^i + \epsilon \xi^i(x, u) + O(\epsilon^2), \quad i = 1, 2, \dots, n, \\ \bar{u} &= u + \epsilon \eta(x, u) + O(\epsilon^2),\end{aligned}\tag{2.1}$$

is to apply them to the PDEs, for which one can construct invariant solutions.

The infinitesimal generator is given by:

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta(x, u) \frac{\partial}{\partial u}.\tag{2.2}$$

For which the k^{th} extensions of equation (2.1) given by:

$$\begin{aligned}\bar{x}^i &= x^i + \epsilon \xi^i(x, u) + O(\epsilon^2), \\ \bar{u} &= u + \epsilon \eta(x, u) + O(\epsilon^2), \\ \bar{u}_i &= u_i + \epsilon \zeta_i(x, u, u_{(1)}) + O(\epsilon^2), \\ &\vdots \\ \bar{u}_{i_1, i_2, \dots, i_k} &= u_{i_1, i_2, \dots, i_k} + \epsilon \zeta_{i_1, i_2, \dots, i_k}(x, u, u_{(1)}, u_{(2)}, \dots, u_{(k)}) + O(\epsilon^2).\end{aligned}$$

where $i = 1, 2, \dots, n$, $i_l = 1, 2, \dots, n$ for $l = 1, 2, \dots, k$ with $k \geq 1$.

2.1.2 Symmetry Generators

The corresponding k^{th} extended infinitesimal generator is given by:

$$X^{[k]} = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} + \zeta_i \frac{\partial}{\partial u_i} + \dots + \zeta_{i_1, i_2, \dots, i_k} \frac{\partial}{\partial u_{i_1, i_2, \dots, i_k}}, \quad k \geq 1.$$

where ζ can be calculated by:

$$\begin{aligned} \zeta_i &= D_i \eta - u_j (D_i \xi^j), \quad i = 1, 2, \dots, n, \\ \zeta_{i_1, i_2, \dots, i_k} &= D_{i_k} (\zeta_{i_1, i_2, \dots, i_{k-1}}) - u_{i_1, i_2, \dots, i_{k-1}, j} D_{i_k} (\eta^j), \\ i_l &= 1, 2, \dots, n \quad \text{for } l = 1, 2, \dots, k \quad \text{with } k \geq 2. \end{aligned} \quad (2.3)$$

The total derivative w.r.t. x^i will then be given by:

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ii_1} \frac{\partial}{\partial u_{i_1}} + \dots + u_{ii_1 i_2 \dots i_k} \frac{\partial}{\partial u_{i_1 i_2 \dots i_k}} + \dots, \quad i = 1, 2, \dots, n. \quad (2.4)$$

Given a k^{th} order PDE,

$$F(t, x, u, u_{(1)}, u_{(2)}, \dots, u_{(k)}) = 0, \quad (2.5)$$

A Lie point symmetry is admitted if and only if

$$X^{[k]} F|_{F=0} = 0, \quad (2.6)$$

where $X^{[k]}$ is the k^{th} prolongation of X .

For each Lie point symmetry, X_i , a corresponding one-parameter group, G_{ϵ_i} can be determined via integration of the Lie equations. F is invariant under the transformation

G_{ϵ_i} .

Then to find group invariant solutions for a PDE, one could use the admitted Lie point symmetries. First, take X_i and use its corresponding characteristic equations. Hence using required methods to recover an invariant solution with arbitrary constants.

It is also plausible for one to use the direct method, solving for equation (2.6). The difference here lies that one will likely solve for some function dependent on the dependent variable of the PDE, i.e. if the given PDE is $F = 0$, then within it lies an equation $g(u)$ in which to solve for.

2.1.3 Calculation of Lie Point Symmetries: an Example

For example, given the well-known heat equation,

$$u_t = u_{xx}. \quad (2.7)$$

One seeks transformations of the form,

$$\begin{aligned} t_* &= t + \varepsilon \xi^1(t, x, u) + O(\varepsilon^2), \\ x_* &= x + \varepsilon \xi^2(t, x, u) + O(\varepsilon^2), \\ u_* &= u + \varepsilon \eta(t, x, u) + O(\varepsilon^2), \end{aligned} \quad (2.8)$$

generated by the base vector

$$X = \xi^1(t, x, u) \frac{\partial}{\partial t} + \xi^2(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u}. \quad (2.9)$$

Since equation (2.7) is second order, then X is prolonged appropriately. Acting on the

heat equation with the second prolongation

$$X^{[2]}(u_t - u_{xx})|_{u_{xx}=u_t} = 0, \quad (2.10)$$

$$\Rightarrow \zeta_t - \zeta_{xx} = 0, \text{ whenever } u_t = u_{xx}. \quad (2.11)$$

ζ_t and ζ_{xx} are derived iteratively as given in (2.3). Equation (2.11) yields an overdetermined system of *linear* differential equations which can be solved algorithmically. The final results are given by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= u \frac{\partial}{\partial u}, \\ X_4 &= 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}, \\ X_5 &= 2t \frac{\partial}{\partial x} - xu \frac{\partial}{\partial u}, \\ X_6 &= 4t^2 \frac{\partial}{\partial t} + 4tx \frac{\partial}{\partial x} - u(x^2 + 2t) \frac{\partial}{\partial u}, \\ X_\infty &= H(t, x) \frac{\partial}{\partial x}, \text{ where } H_t = H_{xx}. \end{aligned} \quad (2.12)$$

2.2 Nonclassical Symmetries

The difference in finding nonclassical symmetries from classical symmetries is an added condition on the infinite criterion for invariance, that is, one seeks invariance on the solution of the equation and the invariant surface condition (ISC). Therefore equation (2.6) has an added condition, which can be written as

$$X^{[k]}F|_{F=0, ISC} = 0, \quad (2.13)$$

where ISC is given by,

$$\xi^i(x, u(t, x)) \frac{\partial u}{\partial x^i} = \eta(x, u(t, x)). \quad (2.14)$$

Unlike in the case of classical Lie point symmetries, equation (2.13) leads to a system of nonlinear determining equations. A number of scholars have used nonclassical symmetry techniques to obtain exact solutions of the equation in question (some applications can be found in [1, 22, 23, 24, 29]). The example below is taken from [1] to illustrate how to find nonclassical symmetries. Here provides detailed calculations of nonclassical symmetries for an equation describing heat transfer in functionally-graded material.

2.2.1 Determination of Nonclassical Symmetries: an Example

For example consider a $(1 + 1)$ D quasi-linear PDE [1]

$$u_t = (k(x)u_x)_x - M^2u^{n+1}, \quad 0 \leq x \leq 1. \quad (2.15)$$

where M is a constant. This form of PDE arises in heat transfer problem through fins, population dynamics and so on. Here, an attempt to classify all functions $k(x)$ for which the equation (2.15) admits nonclassical symmetries is exercised. The reader is referred to Appendix B for more details. The infinitesimal criterion for invariance is given by (see example [1])

$$X^{[2]}eqn(2.15)|_{eqn(2.15),ISC} = 0. \quad (2.16)$$

where ISC is given by

$$\xi^1 u_t + \xi^2 u_x = \eta, \quad (2.17)$$

$$u_t = \frac{\eta - \xi^2 u_x}{\xi^1}. \quad (2.18)$$

Equation (2.17) implies that,

$$u_{tx} = D_x \left[\frac{\eta - \xi^2 u_x}{\xi^1} \right]. \quad (2.19)$$

Without loss of generality, it is assumed $\xi^1 = 1$. Alternatively, it is possible to assume $\xi^1 = 0$, as such $\xi^2 u_x = \eta$. Note that this assumption only holds for parabolic PDEs.

Implementing the former assumption for ξ , the following equation holds true,

$$u_{tx} = D_x [\eta - \xi^2 u_x], \quad (2.20)$$

where D_x is the total derivative as given in equation (2.4).

Equation (2.15) leads to a system of *nonlinear* differential equations which may be solved algorithmically. Here MAPLE was used to facilitate the calculations (see Appendix B). It turns out that $k(x) = (\alpha + \beta x)^2$ for equation (2.15) to admit the genuine nonclassical symmetry,

$$X = \frac{\partial}{\partial t} + \left[(\alpha + \beta x) \left(\frac{3M}{\sqrt{2}} u - \beta \right) \right] \frac{\partial}{\partial x} - \frac{3M}{2} u^3 \frac{\partial}{\partial u}. \quad (2.21)$$

The symmetry generator X in equation (2.21) may lead to exact solutions which cannot be constructed using Lie point symmetries. The symmetry generator X in equation (2.21) was somehow overlooked in [1] but has been derived in [30]

2.2.2 Construction of Exact Solutions Using Nonclassical Symmetries

Eliminating u_t in equations (B.57) and (B.59) gives,

$$(\alpha + \beta x)^2 u_{xx} + (\alpha + \beta x)(3\kappa u + \beta)u_x + \kappa^2 u^3 = 0, \quad (2.22)$$

where

$$\kappa = \frac{M}{\sqrt{2}}.$$

Let

$$\begin{aligned} y &= \frac{1}{\beta} \log(\alpha + \beta x), \\ dy &= \frac{dx}{\alpha + \beta x}. \end{aligned}$$

So,

$$\begin{aligned} \frac{d}{dx} &= \frac{1}{\alpha + \beta x} \cdot \frac{d}{dy}, \\ \frac{d^2}{dx^2} &= \frac{d}{dx} \left(\frac{d}{dx} \right), \\ &= \frac{d}{dx} \left(\frac{1}{\alpha + \beta x} \cdot \frac{d}{dy} \right), \\ &= \frac{-\beta}{(\alpha + \beta x)^2} \cdot \frac{d}{dy} + \frac{1}{\alpha + \beta x} \cdot \frac{d}{dx} \cdot \frac{d}{dy}, \\ &= \frac{-\beta}{(\alpha + \beta x)^2} \cdot \frac{d}{dy} + \frac{1}{(\alpha + \beta x)^2} \cdot \frac{d^2}{dy^2}. \end{aligned}$$

Transforming equation (2.22) with $y = \frac{1}{\beta} \log(\alpha + \beta x)$, it reduces to

$$u_{yy} + 3\kappa u u_y + \kappa^2 u^3 = 0. \quad (2.23)$$

The Hopf-Cole transformation $u = \frac{1}{\kappa} \frac{v_y}{v}$ transforms the equation (2.23) into

$$v_{yyy} = 0, \quad (2.24)$$

which has the solution

$$v(t, y) = \frac{1}{2} c_1(t) y^2 + c_2(t) y + c_3(t). \quad (2.25)$$

Working backwards through the substitutions, the final result is given by as,

$$u(t, x) = \frac{\beta c_1(t) \ln(\alpha + \beta x) + \beta^2 c_2(t)}{\kappa(\frac{1}{2} c_1(t) (\ln(\alpha + \beta x))^2 + \beta c_2(t) \ln(\alpha + \beta x) + \beta^2 c_3(t))}. \quad (2.26)$$

To solve for the the arbitrary functions $c_i(t)$, equation (2.26) is substituted into the ISC. If all the arbitrary functions are kept as such, then solving gives $c_1(t) = 0$, $c_2(t) = 0$ and $c_3(t)$ remains an arbitrary function. This results in the trivial solution, i.e. $u(t, x) = 0$.

Thus take $c_1(t)$ to be unity without loss of generality. Then a non-trivial solution for $u(t, x)$ is obtained, where $\kappa = \frac{M}{\sqrt{2}}$,

$$\begin{aligned} c_2(t) &= \beta t + d_1, \\ c_3(t) &= \frac{1}{2} \beta^2 t^2 + \beta d_1 t + 3t + \frac{1}{2} d_2, \end{aligned}$$

such that

$$u(t, x) = \frac{\beta \ln(\alpha + \beta x) + \beta^2(\beta t + d_1)}{\kappa(\frac{1}{2}(\ln(\alpha + \beta x))^2 + \beta(\beta t + d_1) \ln(\alpha + \beta x) + \beta^2(\frac{1}{2}\beta^2 t^2 + \beta d_1 t + 3t + \frac{1}{2}d_2))}. \quad (2.27)$$

Now consider the governing equation,

$$u_t = (k(x) u_x)_x - M^2(u - 1)^3. \quad (2.28)$$

Instead of repeating the process to find exact solution, one can make the transformation,

$$\bar{u} = (u - 1). \quad (2.29)$$

Therefore

$$\bar{u}_t = (k(x) \bar{u}_x)_x - M^2 \bar{u}^3. \quad (2.30)$$

Using the results from the original equation and reversing the transformation, equation (2.29), to obtain

$$u(t, x) = \frac{\beta \ln(\alpha + \beta x) + \beta^2(\beta t + d_1)}{\kappa(\frac{1}{2}(\ln(\alpha + \beta x))^2 + \beta(\beta t + d_1) \ln(\alpha + \beta x) + \beta^2(\frac{1}{2}\beta^2 t^2 + \beta d_1 t + 3t + \frac{1}{2}d_2))} + 1. \quad (2.31)$$

Notice that the change of variables

$$\begin{aligned} z &= \frac{1}{\beta} \log(\alpha + \beta x) + \beta t, \\ t_* &= t, \\ u(t, x) &= v(t_*, z), \end{aligned}$$

such that,

$$\begin{aligned} u_x &= \frac{1}{\alpha + \beta x} v_z, \\ u_{xx} &= -\frac{\beta}{(\alpha + \beta x)^2} v_z + \frac{1}{(\alpha + \beta x)^2} v_{zz}, \\ u_t &= \beta v_z + v_{t_*}, \end{aligned}$$

transforms equation (2.15) to

$$v_{t_*} = v_{zz} - M^2 v^3, \quad (2.32)$$

a well studied equation.

2.3 Conservation Laws

A number of scholars have been focused on the determination of conservation laws of reaction-diffusion equations (see for example [25, 26]). Conservation laws may be constructed. Furthermore, the Kara and Mahomed theorem [18, 27] will be used to determine the associated Lie point symmetries. The aspiration is to use the Sjöberg double reduction method [16] to construct group-invariant solutions.

Corollary [18] *If $\hat{X}$ is the canonical operator of X , i.e. $\hat{X} = X - \xi^i D_i$, and $T_*^i = X(T^i) + T^i D_j(\xi^j) - T^j D_j(\xi^i)$, $i = 1, \dots, n$, then the following diagram commutes*

$$\begin{array}{ccc} X & \rightarrow & \hat{X} \\ \downarrow & & \downarrow \\ T_*^i & \rightarrow & \hat{T}_*^i \end{array}$$

where

$$\hat{T}_*^i = T^i + T^k D_k(\xi^i) - D_k(\xi^k T^i).$$

The Corollary can help generate conservation laws of a system. But if conserved components associated with symmetry X_j is a multiple of X_i , then no new conservation laws are produced [18]. Even with the canonical approach is taken, this doesn't help via the Corollary.

Consider equation (2.5), and given

$$D_i(T^i) = 0, \tag{2.33}$$

where T^i are the differential functions of a finite order. Equation (2.33) is considered a conservation law for equation (2.5) if the following holds,

$$D_i T^i|_{F=0} = 0.$$

The conserved vector is given by $T = (T^1, T^2, \dots, T^n)$. If the following holds,

$$X(T^i) + T^i D_l(\xi^l) - T^l D_l(\xi^i) = 0, \quad i = 1, \dots, n,$$

then equation (2.2) is said to be associated with the conserved vector $T^i = (T^1, \dots, T^n)$.

2.3.1 Direct Method

This method entails that one should use equation (2.33) subjected to equation (2.5) being satisfied as the determining equation for the conserved vectors. Then one is to

separate the powers and derivatives of u such that the components of $T^1, \dots, T^n$ can be found.

For example [25], a nonlinear PDE,

$$u_t = (u^m u_x)_x - M^2 u^{n+1}, \quad (2.34)$$

has conserved vectors

$$T^1 = u, \quad T^2 = -u^m u_x; \quad (2.35)$$

$$T^1 = xu, \quad T^2 = xu^m u_x + \frac{u^{m+1}}{m+1}. \quad (2.36)$$

2.3.2 Multiplier Method

This method requires the variational derivative

$$D_i T^i = \Lambda F,$$

where Λ^α are the characteristics. The characteristics are the multipliers which make the equation exact.

For example [25], a nonlinear PDE,

$$u_t = ((1 + \beta u)u_x)_x - M^2 u^{n+1}, \quad (2.37)$$

has conserved vectors

$$T^1 = u, \quad T^2 = -(1 + \beta u)u_x + M^2 x; \quad (2.38)$$

$$T^1 = xu, \quad T^2 = x \left(\frac{M^2 x}{x} - u_x(1 + \beta u) \right) + u + \frac{\beta u^2}{2}. \quad (2.39)$$

2.4 Concluding remarks

The methodologies highlighted in this chapter will help to determine the Lie point symmetries, the nonclassical symmetries, conserved vectors and with the associated Lie point symmetries of the problem in question. In all, these are methods that determine symmetries in order to find exact (group invariant) solutions.

This chapter has illustrated the determination of both classical Lie point symmetries and nonclassical symmetries with some examples. The detailed calculation of nonclassical symmetries is given in Appendix A.

Chapter 3

Modelling the Population Dynamics

The intention in this dissertation is to study the reaction-diffusion equation given by

$$u_t = (D(x)u_x)_x + g(x)Q(u),$$

where D and Q , called the diffusivity term and source term respectively, are specified. In particular D is given as a quadratic function of x and Q as cubic in u .

This form of equation is found in many fields in science and engineering. However within this dissertation, the focus is on gene propagation. Modelling gene frequencies has significance as it can help in many fields. For example, being able to genetically modify crops and livestock such that they become more resilient to pests and are able to produce a larger yield [21].

Now consider the following reaction-diffusion equation that models the propagation of the mutant gene

$$p_t = p_{xx} + Q(p), \tag{3.1}$$

Bradshaw-Hajek [22] makes several assumptions under which equation (3.1) is derived

- Randomly mating populations - hence no discrimination regardless of genotype.
- One-dimensional habitat.
- The population moves randomly.
- Individual genotypes have no influence of the individual's mobility. This allows for a uniform diffusion coefficient.
- Genotypes have no differences in their death rates, fitness and survival rates - therefore only different phenotypes in variations in reproductive success rates.

The Huxley equation was given by equation (3.1) with $Q(p) = gp^2(1 - p)$ and the Fitzhugh-Nagumo was given by equation (3.1) with $Q(p) = p(1 - p)(g_1 - g_2p)$ [22]. In the numerical analysis of equation (3.1) for two alleles, Bradshaw-Hajek [22] found that the mutant genes takeover significantly distorts the Huxley model compared to the Fisher model.

When the model was expanded to accommodate three alleles, the Huxley model found three cases for $g(x)$ with each form giving two new exact solutions when using the nonclassical symmetries method. The Fitzhugh-Nagumo was too complex to solve and Bradshaw-Hajek [22] left it to be solved later in the future. Bradshaw-Hajek [22] found that the delayed spread of an advantageous allele can be contributed to having a cubic source term.

3.1 Deriving the Reaction-Diffusion Model

There are two approaches in which Bradshaw-Hajek [22] shows how to derive a reaction-diffusion equation that describes the change in frequency of the allele in question.

The first approach implements difference equations, whereas the second approach uses continuous differential equations. When deriving the equations, the above mentioned assumptions are taken into consideration.

The derivations are taken from [22] to explain how the reaction-diffusion equation appears in biology.

3.1.1 Development via Difference Equations

Consider a population with two alleles, A and a , the dominant and recessive allele respectively, at the locus in question. This leads to having three genotypes AA , Aa and aa . The main focus is put onto how the frequency of the recessive allele a changes over time. In the classical Mendel binary scheme, the physical characteristic with the dominant allele are going to be equally apparent in individuals possessing two copies of the gene (AA). This also holds true for individuals possessing one copy of the gene (Aa). The idea is formed that the survival rate of those two genotypes is r_0 . The recessive characteristic can solely be expressed by those individuals possessing two copies of the recessive gene, aa . The survival rate, r_2 , is going to be assumed for these individuals.

Following the notation of Bradshaw-Hajek [22], N_k^* is denoted as the total population at the k^{th} generation. So the entire population for the three genotypes are given as $N_k^*(AA)$, $N_k^*(Aa)$ and $N_k^*(aa)$, wherever the corresponding symbols without the asterisks refer to the population at sexual maturity of the k^{th} generation. $G_k(AA)$, $G_k(Aa)$ and $G_k(aa)$ are the genotype *frequencies* within the k^{th} generation, so that, for instance, $G_k(AA) = N_k(AA)/N_k$. The gene frequencies in the gene pool at the k^{th} generation are denoted by $P_k(A)$ and $P_k(a)$.

By definition,

$$N_{k+1}^*(AA) = G_{k+1}^*(AA)N_{k+1}. \quad (3.2)$$

By the assumption of random mating, the expected value of $G_{k+1}^*(AA)$ is $[P_k(A)]^2$. For a large population, equation (3.2) reduces to,

$$N_{k+1}^*(AA) = [P_k(A)]^2 N_{k+1}. \quad (3.3)$$

Similarly,

$$N_{k+1}^*(Aa) = 2P_k(A)P_k(a)N_{k+1}, \quad (3.4)$$

$$N_{k+1}^*(aa) = [P_k(a)]^2 N_{k+1}. \quad (3.5)$$

By definition,

$$P_{k+1}(a) = \frac{\text{number of } a \text{ alleles}}{\text{total number of alleles in gene pool}}, \quad (3.6)$$

$$= \frac{N_{k+1}(Aa) + 2N_{k+1}(aa)}{2N_{k+1}(AA) + 2N_{k+1}(Aa) + 2N_{k+1}(aa)}. \quad (3.7)$$

Substituting,

$$N_{k+1}^*(AA) = r_0 N_{k+1}^*(AA),$$

$$N_{k+1}^*(Aa) = r_0 N_{k+1}^*(Aa),$$

$$N_{k+1}^*(aa) = r_2 N_{k+1}^*(aa),$$

and

$$P_k(A) = 1 - P_k(a),$$

to obtain,

$$P_{k+1} = \frac{\left(\frac{r_2}{r_0} - 1\right) P_k^2(a) + P_k(a)}{1 + \left(\frac{r_2}{r_0} - 1\right) P_k^2(a)}, \quad (3.8)$$

where r_2/r_0 is the relative fitness of the two phenotypes.

Fisher considered the case where the recessive allele (a) was an advantageous mutant. This means that for the advantageous allele a , the individuals the phenotype associated with this allele (i.e. individuals with genotype aa) will have a greater survival rate, thus $r_2 > r_0$. These beneficial mutations most commonly only result in a small advantage, so that $\left(\frac{r_2}{r_0} - 1\right)$ is usually small, i.e. $0 < \left(\frac{r_2}{r_0} - 1\right) \ll 1$. This means that equation (3.8) implies,

$$P_{k+1}(a) - P_k(a) = \left(\frac{r_2}{r_0} - 1\right) P_k^2(a)(1 - P_k(a)) + O\left(\left[\frac{r_2}{r_0} - 1\right]^2\right). \quad (3.9)$$

Observing equation (3.9), one notices that the source term is cubic as opposed to Fisher's quadratic source term.

If one is to generalise the case above, where there was no dominance of the A allele so that each genotype had differing survival rates, r_0 , r_1 and r_2 (where the subscripts denote the number of a alleles present), we obtain

$$P_{k+1}(a) = \frac{\frac{r_1}{r_0} P_k(a) + \left(\frac{r_2}{r_0} - \frac{r_1}{r_0}\right) P_k^2(a)}{1 + 2\left(\frac{r_1}{r_0} - 1\right) P_k(a) + \left(1 - 2\frac{r_1}{r_0} + \frac{r_2}{r_0}\right) P_k^2(a)}. \quad (3.10)$$

Once more, beneficial mutations usually result in only small advantages, $0 < \frac{r_1}{r_0} - 1 = O\left(\frac{r_2}{r_0} - 1\right) \ll 1$, which implies

$$P_{k+1}(a) - P_k(a) = P_k(a)(1 - P_k(a)) \left[\frac{r_1}{r_0} - 1 + \left(1 - 2\frac{r_1}{r_0} + \frac{r_2}{r_0} \right) P_k(a) \right] + O \left(\left[\frac{r_2}{r_0} - 1 \right]^2 \right). \quad (3.11)$$

If Δt the time between successive generations, then equations (3.9) and (3.11) can be expressed as

$$\frac{P_{k+1}(a) - P_k(a)}{\Delta t} = \left(\frac{r_2}{r_0} - 1 \right) P_k^2(a)(1 - P_k(a)), \text{ and} \quad (3.12)$$

$$\frac{P_{k+1}(a) - P_k(a)}{\Delta t} = P_k(a)(1 - P_k(a)) \left[\frac{r_1}{r_0} - 1 + \left(1 - 2\frac{r_1}{r_0} + \frac{r_2}{r_0} \right) P_k(a) \right]. \quad (3.13)$$

Since the differences in selective advantages are small, the discrete model can be approximated by an interpolating continuous model with

$$\begin{aligned} P_{k+1}(a) &= P([k+1]\Delta t, a), \\ &= P_k(a) + \Delta t \frac{\partial}{\partial t} P(t, a) + O(\Delta t^2), \end{aligned}$$

at $t = k\Delta t$. In the continuum model, a time scale over which P may significantly change is adopted. Then the time between generations, Δt , is small compared to 1. The finite difference quotient in equations (3.12) and (3.13) may then be viewed as the Euler approximation of the time derivative.

Assuming Fickian diffusion with equal mobility for each genotype, rewriting $P(t, a)$ as $p(t)$ and allowing spatial variation, the following equations are obtained

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2} + \left(\frac{r_2}{r_0} - 1\right) p^2(1 - p), \quad (3.14)$$

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2} + p(1 - p) \left(\frac{r_1}{r_0} - 1 + p \left(1 - 2\frac{r_1}{r_2} + \frac{r_2}{r_0} \right) \right). \quad (3.15)$$

For simplicity, mobility is assumed to not enlarge an individual's habitat so much that one's available pool of mating partners has a genetic composition greatly different from that of one's local neighbourhood. Otherwise, the local source terms would be integrals [22].

Equation (3.14) describes how the frequency of a recessive gene changes, while equation (3.15) describes how the frequency of a particular allele changes, regardless of whether or not the allele is dominant or recessive.

3.1.2 Development via Continuous Genotype Differential Equations

For a diploid population having two available alleles at the locus in question (A_1 and A_2), there are three possible genotypes; A_1A_1 , A_1A_2 and A_2A_2 , where A_1 is the allele is under observation. Then the following three equations describe the change in the genotype frequencies $\rho_{11}(t, x)$, $\rho_{12}(t, x)$ and $\rho_{22}(t, x)$,

$$\begin{aligned} \frac{\partial \rho_{11}}{\partial t} &= \frac{\partial}{\partial x} \left(k(x) \frac{\partial \rho_{11}}{\partial x} \right) - \mu \rho_{11} + \gamma_{11} g(x) p^2 \rho, \\ \frac{\partial \rho_{12}}{\partial t} &= \frac{\partial}{\partial x} \left(k(x) \frac{\partial \rho_{12}}{\partial x} \right) - \mu \rho_{12} + \gamma_{12} g(x) p(1 - p) \rho, \\ \frac{\partial \rho_{22}}{\partial t} &= \frac{\partial}{\partial x} \left(k(x) \frac{\partial \rho_{22}}{\partial x} \right) - \mu \rho_{22} + \gamma_{22} g(x) (1 - p)^2 \rho, \end{aligned} \quad (3.16)$$

where γ_{ij} is the reproductive success rate of genotype $A_i A_j$, μ is the common death rate, $\rho(t, x)$ is the total population density, $\rho(t, x) = \rho_{11}(t, x) + \rho_{12}(t, x) + \rho_{22}(t, x)$, and $p(t, x)$ is the frequency of allele A_1 . The frequency of allele A_2 is given by $(1 - p(t, x))$.

The frequency of allele A_1 is

$$p = \frac{2\rho_{11} + \rho_{12}}{2\rho}. \quad (3.17)$$

Differentiating equation (3.17) with respect to t , then the three genotype equations (3.16) collapse into a single equation that describes the change in frequency of the new mutant gene

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x} \left(k(x) \frac{\partial p}{\partial x} \right) + \frac{2k(x)}{\rho} \frac{\partial \rho}{\partial x} \frac{\partial p}{\partial x} + g(x)p(1-p)(h_1 - h_2 p), \quad (3.18)$$

where $h_1 = \gamma_{12} - \gamma_{22}$ and $h_2 = -\gamma_{11} + 2\gamma_{12} - \gamma_{22}$. Equation (3.18) is a reaction-diffusion-convection equation with cubic nonlinearities. If $\frac{\partial \rho}{\partial x} = 0$, the total population density is constant in space, equation (3.18) reduces to the Fitzhugh-Nagumo equation. Comparing equation (3.15) and (3.18), one can see that both approaches produce the same class of equations if $k(x) = g(x) = 1$. When deriving the models with the continuous method, there is an extra convective term due to the assumption that total population density is not uniform spatially.

If one is to consider the conditions that Fisher had examined, the allele in question become completely recessive. This implies that the genotypes AA and Aa have the same phenotype, therefore they have the same reproductive success rate. Let the alleles represented by A and a be set as the alleles A_2 and A_1 respectively, therefore A_1 is the allele under consideration. This implies that $\gamma_{12} = \gamma_{22}$ and equation (3.15) reduces to,

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x} \left(k(x) \frac{\partial p}{\partial x} \right) + \frac{2k(x)}{\rho} \frac{\partial \rho}{\partial x} \frac{\partial p}{\partial x} + h g(x) p^2 (1 - p), \quad (3.19)$$

where $h = \gamma_{11} - \gamma_{22}$. One can see that equation (3.19) is also a reaction-diffusion-convection equation. If $\frac{\partial \rho}{\partial x} = 0$, then equation (3.19) reduces to Huxley's equation, and this equation models the propagation of impulses along nerve axons. Comparing equation (3.14) and (3.19), one can see that again both approaches produce the same class of equations if $k(x) = g(x) = 1$.

3.2 Concluding remarks

In this chapter, prescription of models describing propagation of mutant genes is provided. The development of these models is carried out via difference equations and continuous differential equations. The focus is placed on the latter. These are given in terms of reaction-diffusion. Here the diffusion term and the coefficient of the source term are both assumed to be given as quadratic functions of space variable. Furthermore, the source term is cubic in the dependent variable.

Chapter 4

Classical Lie Point Symmetry Reductions and Exact Solutions

This chapter will look at the resulting Lie point symmetries.

Consider the following equation,

$$u_t = (k(x)u_x)_x + g(x)Q(u), \quad (4.1)$$

where $Q(u) = (u - r_1)(u - r_2)(u - r_3)$.

When investigating the determining equations while keeping the diffusivity and source coefficients, $k(x)$ and $g(x)$ respectively, as arbitrary functions, is computationally expensive. As is the case where $k(x) = g(x)$. Hence, these functions are set to

$$\begin{aligned} k(x) &= \beta^2 x^2, \\ g(x) &= -a^2 x^2. \end{aligned}$$

Note that the source term becomes a sink representing the death rate. Therefore the governing equation becomes,

$$u_t = (\beta^2 x^2 u_x)_x - a^2 x^2 (u - r_1)(u - r_2)(u - r_3). \quad (4.2)$$

4.1 Classical Lie Point Symmetries of Equation (4.2)

To start off, the simplest case is considered first, where $r_1 = r_2 = r_3 = 0$. To find the determining equations, the second prolongation of the infinitesimal generator is evaluated given equation (4.2),

$$X^{[2]}(eqn.(4.2))|_{eqn.(4.2)} = 0. \quad (4.3)$$

The resulting determining equations are produced where u_{xx} is eliminated, and this leads to an overdetermined system of equations. The determining equations are derived from separating the result of equation (4.3) with respect to the derivatives of u , such as u_t , u_x , u_{xt} .

1:

$$\begin{aligned} 0 = & x \frac{\partial}{\partial t} \eta(t, x, u) + 3 a^2 u^2 x^3 \eta(t, x, u) - a^2 u^3 x^3 \frac{\partial}{\partial u} \eta(t, x, u) \\ & + 2 a^2 u^3 x^3 \frac{\partial}{\partial x} \xi^2(t, x, u) - \beta^2 x^3 \frac{\partial^2}{\partial x^2} \eta(t, x, u) - 2 \beta^2 x^2 \frac{\partial}{\partial x} \eta(t, x, u), \end{aligned} \quad (4.4)$$

u_{xt} :

$$0 = \frac{\partial}{\partial x} \xi^1(t, x, u), \quad (4.5)$$

$u_x u_{xt}:$

$$0 = \frac{\partial}{\partial u} \xi^1(t, x, u), \quad (4.6)$$

$u_t:$

$$\begin{aligned} 0 = & -2 \xi^2(t, x, u) + a^2 u^3 x^3 \frac{\partial}{\partial u} \xi^1(t, x, u) + \beta^2 x^3 \frac{\partial^2}{\partial x^2} \xi^1(t, x, u) \\ & + 2 \beta^2 x^2 \frac{\partial}{\partial x} \xi^1(t, x, u) + 2 x \frac{\partial}{\partial x} \xi^2(t, x, u) - x \frac{\partial}{\partial t} \xi^1(t, x, u), \end{aligned} \quad (4.7)$$

$u_t^2:$

$$0 = 0, \quad (4.8)$$

$u_x u_t:$

$$0 = \beta^2 x^2 \frac{\partial^2}{\partial u \partial x} \xi^1(t, x, u) + \frac{\partial}{\partial u} \xi^2(t, x, u), \quad (4.9)$$

$u_x^2 u_t:$

$$0 = \frac{\partial^2}{\partial u^2} \xi^1(t, x, u), \quad (4.10)$$

$u_x:$

$$\begin{aligned} 0 = & 3 a^2 u^3 x^3 \frac{\partial}{\partial u} \xi^2(t, x, u) - 2 \beta^2 x^3 \frac{\partial^2}{\partial u \partial x} \eta(t, x, u) \\ & + \beta^2 x^3 \frac{\partial^2}{\partial x^2} \xi^2(t, x, u) - 2 \beta^2 x^2 \frac{\partial}{\partial x} \xi^2(t, x, u) \\ & + 2 \beta^2 x \xi^2(t, x, u) - \frac{\partial}{\partial t} \xi^2(t, x, u), \end{aligned} \quad (4.11)$$

$u_x^2:$

$$\begin{aligned} 0 = & 2 x \frac{\partial^2}{\partial u \partial x} \xi^2(t, x, u) - x \frac{\partial^2}{\partial u^2} \eta(t, x, u) \\ & - 4 \frac{\partial}{\partial u} \xi^2(t, x, u), \end{aligned} \quad (4.12)$$

$u_x^3:$

$$0 = \frac{\partial^2}{\partial u^2} \xi^2(t, x, u). \quad (4.13)$$

From equations (4.5), (4.6) and (4.10), one can deduce that

$$\xi^1(t, x, u) = f_1(t).$$

Similarly, from equations (4.9) and (4.13)

$$\xi^2(t, x, u) = f_2(t, x),$$

and from equation (4.12),

$$\eta(t, x, u) = f_3(t, x)u + f_4(t, x),$$

where f_i are arbitrary functions. Upon further analysis of the determining equations, the following is observed,

$$\xi^1 = \frac{2c_1}{\beta^2}t + c_3, \quad (4.14)$$

$$\xi^2 = \frac{1}{\beta^2} (c_1 \ln(x) + (c_1 t + c_2)\beta^2) x, \quad (4.15)$$

$$\eta = -\frac{1}{\beta^2} (c_1 + c_1 \ln(x) + (c_1 t + c_2)\beta^2) u. \quad (4.16)$$

Therefore the following symmetries were found,

$$X_1 = \frac{\partial}{\partial t}, \quad (4.17)$$

$$X_2 = x \frac{\partial}{\partial x} - u \frac{\partial}{\partial u}, \quad (4.18)$$

$$X_3 = 2t \frac{\partial}{\partial t} + (\ln(x) + \beta^2 t)x \frac{\partial}{\partial x} - (1 + \ln(x) + \beta^2 t)u \frac{\partial}{\partial u}. \quad (4.19)$$

The Lie Bracket for the admitted symmetry algebra is depicted below,

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	$2X_1 + X_2$
X_2	0	0	$\frac{1}{\beta^2}X_2$
X_3	$-2X_1 - X_2$	$-\frac{1}{\beta^2}X_2$	0

Table 4.1: Lie bracket of symmetries (4.17)-(4.19)

4.2 Symmetry Reductions of Equation (4.2)

Using the admitted Lie point symmetries, one is able to reduce a PDE into an ODE. Three case are considered namely, using the combination of X_1 and X_2 , and using them individually.

4.2.1 Symmetry Reductions Using X_1 and X_2

Symmetry X_1 and X_2 is utilised in the analysis for convenience. Usually it is possible to use any linear combination of admitted symmetries to reduce the equation by one variable. More often, the set of one dimensional optional systems is constructed. This helps in terms reduction that are not connected by any point transformation. To find the exact solution for u , consider the linear combination of symmetries X_1 and X_2 ,

$$\frac{dt}{c_1} = \frac{dx}{c_2x} = \frac{du}{-c_2u}.$$

Firstly, evaluate

$$\frac{dt}{c_1} = \frac{dx}{c_2x},$$

where,

$$I_1 = x \exp\left(-\frac{c_2}{c_1}t\right).$$

Secondly,

$$\frac{dt}{c_1} = \frac{du}{-c_2 u},$$

where,

$$I_2 = \exp\left(\frac{c_2}{c_1}t\right)u.$$

Let $I_1 = \gamma$ and $I_2 = F(\gamma)$, where F is an arbitrary function. Therefore u can be written as,

$$u(t, x) = \exp\left(-\frac{c_2}{c_1}t\right)F(\gamma). \quad (4.20)$$

To determine what form $F(\gamma)$ takes on, equation (4.20) is substituted into the governing equation, equation (4.2) and where $\beta = 1$ for simplicity, to obtain

$$0 = \gamma^2 F''(\gamma) + \left(2 + \frac{c_2}{c_1}\right) \gamma F'(\gamma) + \frac{c_2}{c_1} F(\gamma) - a^2 \gamma^2 F(\gamma)^3. \quad (4.21)$$

Equation (4.21) admits the following symmetry,

$$X = \gamma \frac{\partial}{\partial \gamma} - F \frac{\partial}{\partial F}. \quad (4.22)$$

Implement the method of differential invariants to reduce the equation (4.21) to a first order ODE using equation (4.22). The first prolongation is given by,

$$X^{[1]} = \gamma \frac{\partial}{\partial \gamma} - F \frac{\partial}{\partial F} - 2F_\gamma \frac{\partial}{\partial F_\gamma}. \quad (4.23)$$

The characteristic equations are given by,

$$\frac{d\gamma}{\gamma} = -\frac{dF}{F} = -\frac{dF_\gamma}{2F_\gamma}. \quad (4.24)$$

The invariants are therefore, $\phi = \gamma F$ and $G(\phi) = \gamma^2 F_\gamma$.

Therefore,

$$\begin{aligned} \frac{DG}{D\phi} &= \frac{\frac{\partial G}{\partial \gamma} d\gamma + \frac{\partial G}{\partial F} dF + \frac{\partial G}{\partial F_\gamma} dF_\gamma}{\frac{\partial \phi}{\partial \gamma} d\gamma + \frac{\partial \phi}{\partial F} dF}, \\ &= \frac{2xF_\gamma + x^2 F_{\gamma\gamma}}{F + xF_\gamma}, \\ &= \frac{a^2 \gamma^2 F(\gamma)^3 - \frac{c_2}{c_1} F(\gamma) - \frac{c_2}{c_1} \gamma F'(\gamma)}{F + xF_\gamma}, \\ &= \frac{a^2 \phi^3 - \frac{c_2}{c_1} (G + \phi)}{G + \phi}. \end{aligned} \quad (4.25)$$

A further simplification can be made to equation (4.25), whereby $\bar{G} = G + \phi$. Therefore equation (4.25) can be written as,

$$\bar{G} \frac{D\bar{G}}{D\phi} = a^2 \phi^3 + \left(1 - \frac{c_2}{c_1}\right) \bar{G}. \quad (4.26)$$

Suppose $c_2 = c_1$, then equation (4.26) becomes variable separable and the solution for $\bar{G}$ can be found. Substituting back for G gives,

$$G(\phi) = \frac{1}{2} \sqrt{2a^2 \phi^4 + 4d} - \phi, \quad (4.27)$$

where d is an arbitrary constant.

Substituting back for the given values for G and ϕ into equation (4.27)

$$\gamma^2 F_\gamma = \frac{1}{2} \sqrt{2a^2(\gamma F)^4 + 4d} - \gamma F, \quad (4.28)$$

$$\left(\frac{\partial}{\partial \gamma}(\gamma F) \right)^2 = \frac{1}{2} a^2 \gamma^2 F^4 + d \gamma^{-2}. \quad (4.29)$$

One of the solutions for equation (4.29) is

$$F(\gamma) = \frac{(-2a^2 d)^{1/4}}{a\gamma}. \quad (4.30)$$

From equation (4.30), one can see that d has to be negative. If d_1 is non-negative, F becomes a complex function. Finally, $u(t, x)$ can be expressed as

$$\begin{aligned} u(t, x) &= \exp\left(-\frac{c_2}{c_1}t\right) \frac{(-2a^2 d)^{1/4}}{ax \exp\left(-\frac{c_2}{c_1}t\right)}, \\ &= \frac{(-2d)^{1/4}}{\sqrt{a} x}, \end{aligned} \quad (4.31)$$

where $d < 0$ and $a \neq 0$.

Surprisingly, u does not depend on time. Graphing equation (4.31) one is able to visualise how the mutant genes behave over space.

In Figure 4.1, it is apparent that the population deteriorates faster as it moves along the x -axis. A similar nature can be found when keeping a constant and varying d as seen in Figure 4.2 below. This is tied into the nature of hyperbolic functions.

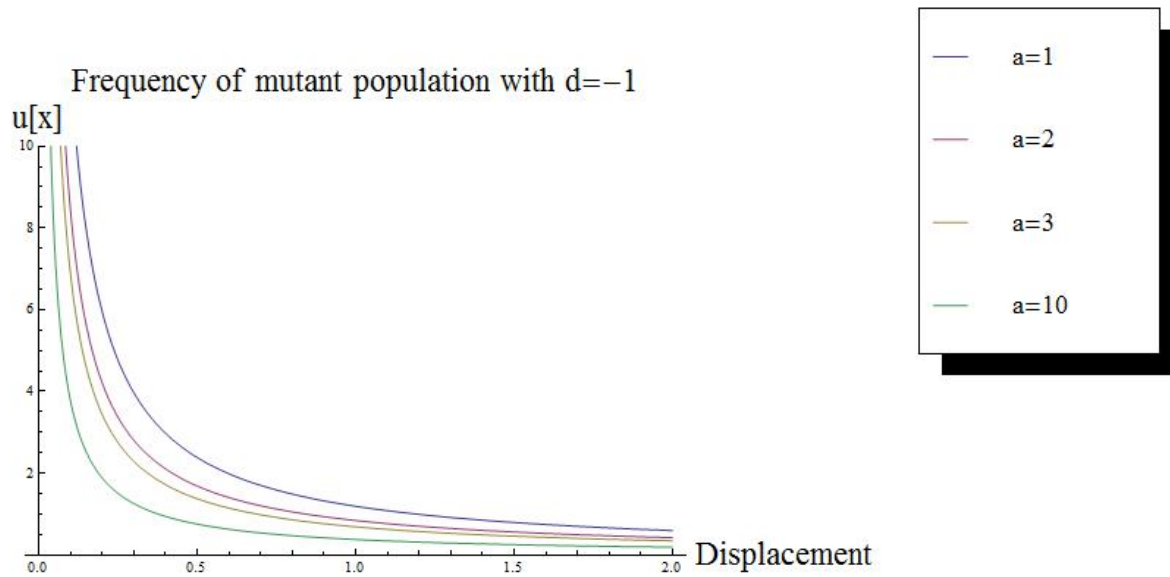


Figure 4.1: Impact on the population frequency by adjusting the coefficient a .

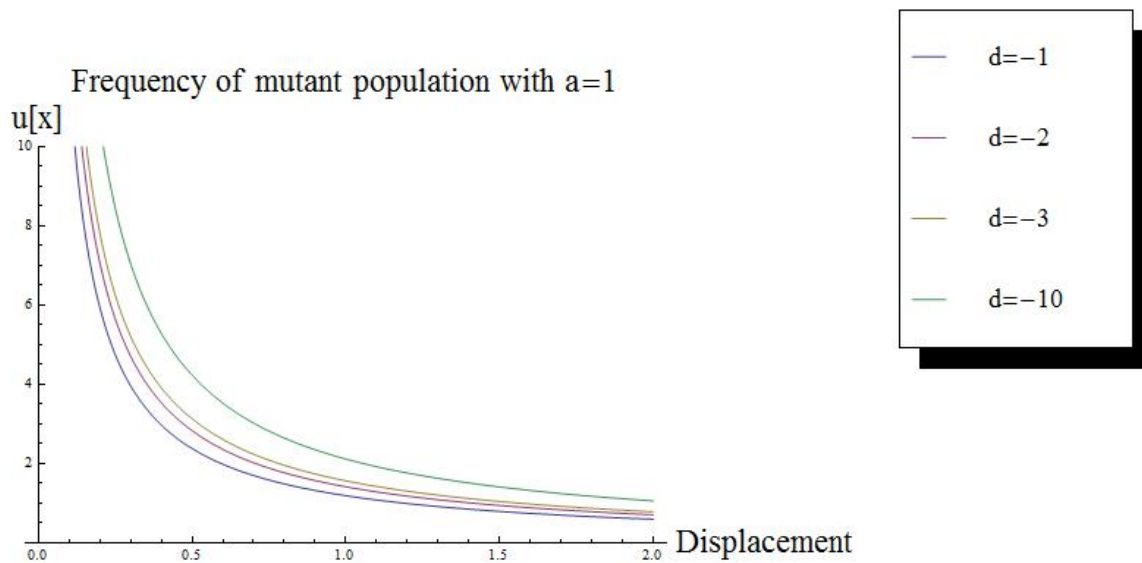


Figure 4.2: Impact on the population frequency by adjusting the coefficient d .

4.2.2 Symmetry Reductions Using X_2

Next, consider the scaling symmetry given by,

$$X_2 = x \frac{\partial}{\partial x} - u \frac{\partial}{\partial u}. \quad (4.32)$$

The corresponding characteristic equations are given by

$$\frac{dx}{x} = -\frac{du}{u}. \quad (4.33)$$

Therefore $u(t, x)$ can be defined as $u(t, x) = \frac{1}{x}f(t)$. Substituting $u(t, x)$ into equation (4.2), gives

$$f'(t) = -a^2 f(t)^3. \quad (4.34)$$

Therefore,

$$f(t) = \left(\frac{1}{2a^2 t + d} \right)^{\frac{1}{2}}, \quad (4.35)$$

and so,

$$u(t, x) = \frac{1}{x} \left(\frac{1}{2a^2 t + d} \right)^{\frac{1}{2}}. \quad (4.36)$$

If one considers the governing equation (4.2), where $r_1 = r_2 = r_3 = \omega$, where ω is some arbitrary constant, given by

$$u_t = (\beta^2 x^2 u_x)_x - a^2 x^2 (u - \omega)^3. \quad (4.37)$$

The transformation, $\bar{u} = u - \omega$ can be made to equation (4.37) to give

$$\bar{u}_t = (\beta^2 x^2 \bar{u}_x)_x - a^2 x^2 \bar{u}^3, \quad (4.38)$$

which is the governing equation that was solved. Therefore the exact solution for equation (4.37) given the scaling symmetry, equation (4.32), is

$$u(t, x) = \frac{1}{x} \left(\frac{1}{2a^2 t + d} \right)^{\frac{1}{2}} + \omega. \quad (4.39)$$

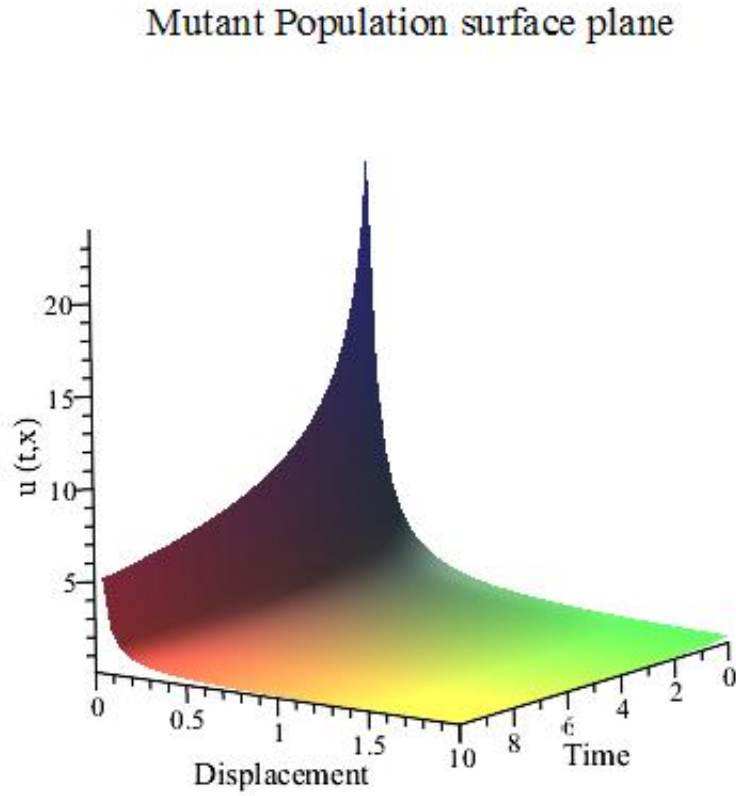


Figure 4.3: Population frequency over time and space.

Examining the surface in Figure 4.3, one notices that in the region $x = 0$ to $x = 0.2$ that the gradient is very steep. This is due to the nature of the exact equation as it is a hyperbolic function, thus making the line $x = 0$ an asymptote.

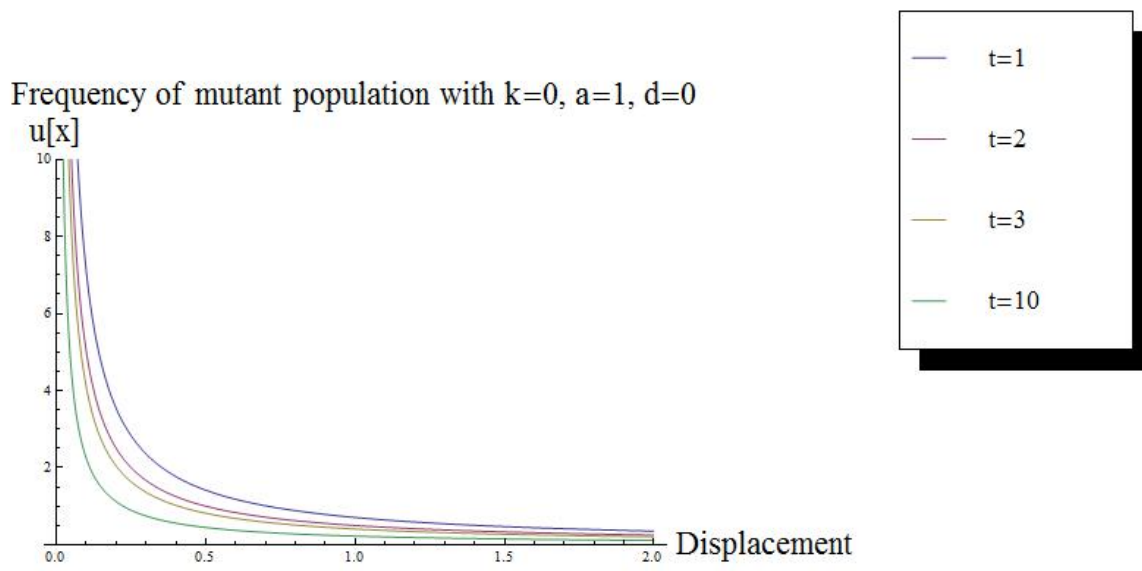


Figure 4.4: Population dynamics as time progresses over space.

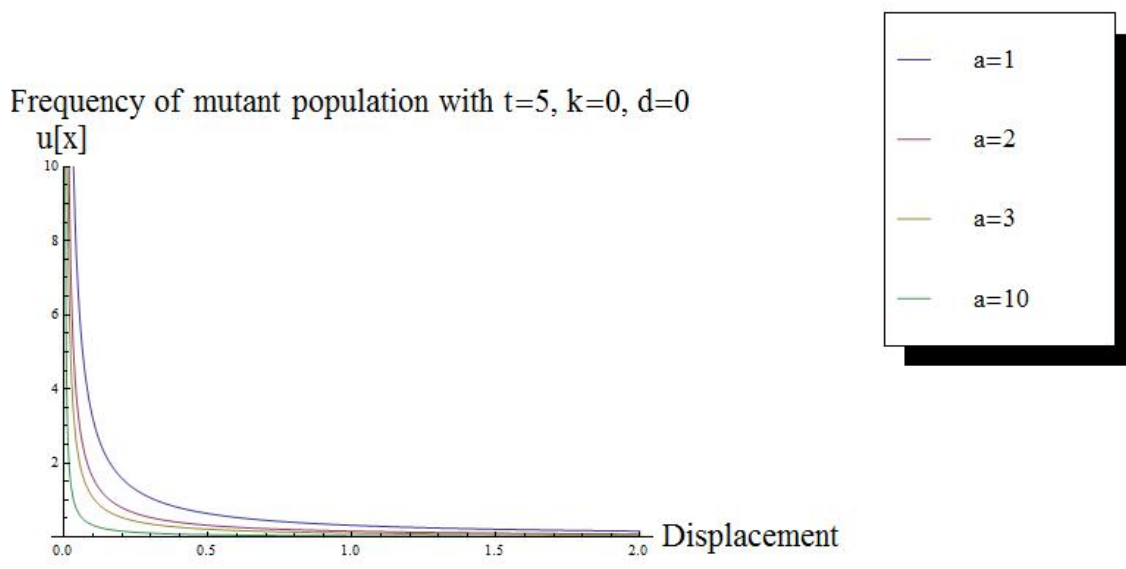


Figure 4.5: Impact on the population frequency by adjusting the coefficient a .

Figure 4.4 shows that as time progresses, the population frequency diminishes. Figure 4.5 shows the same tendencies as Figure 4.1.

4.2.3 Symmetry Reductions Using X_1

Consider the scaling symmetry,

$$X_1 = \frac{\partial}{\partial t}. \quad (4.40)$$

This leads to the steady state of the system, such that equation 4.2 becomes

$$u''(x) + \frac{2}{x}u'(x) - a^2u(x)^3 = 0, \quad (4.41)$$

where $\beta = 1$.

Equation (4.41) admits the following symmetry,

$$X = \gamma \frac{\partial}{\partial x} - u \frac{\partial}{\partial u}. \quad (4.42)$$

Using the method of differential invariants, the first prolongation of the symmetry is given by,

$$X^{[1]} = \gamma \frac{\partial}{\partial x} - u \frac{\partial}{\partial u} - 2u_x \frac{\partial}{\partial u_x}. \quad (4.43)$$

The characteristic equations are given by,

$$\frac{dx}{x} = -\frac{du}{u} = -\frac{du_x}{2u_x}. \quad (4.44)$$

The invariants are therefore, $\phi = x u$ and $G(\phi) = x^2 u_x$.

Therefore,

$$\frac{DG}{D\phi} = \frac{a^2\phi}{G + \phi}, \quad (4.45)$$

and hence

$$\frac{G G_\phi}{a^2 - G_\phi} = \phi. \quad (4.46)$$

Solving for $G(\phi)$ in equation (4.46), the solution can be given as

$$-\frac{1}{2} \ln (G(\phi)^2 + \phi G(\phi) - a^2 \phi^2) + \frac{1}{\sqrt{4a^2 + 1}} \tanh^{-1} \left(\frac{2G(\phi) + \phi}{\sqrt{4a^2 + 1} \phi} \right) = d, \quad (4.47)$$

where d is an arbitrary constant. In terms of the original variables, the first integral is obtained,

$$-\frac{1}{2} \ln (x^4 u_x^2 + x^3 u u_x - a^2 x^2 u^2) + \frac{1}{\sqrt{4a^2 + 1}} \tanh^{-1} \left(\frac{2x u_x + u}{\sqrt{4a^2 + 1} u} \right) = d. \quad (4.48)$$

4.3 Some Discussion and Concluding Remarks

Within this chapter, the three Lie point symmetries were found for the governing equation (4.2). To construct the exact solutions for the governing equation, symmetries X_1 and X_2 were used. The resulting exact solutions found are hyperbolic in nature. Notice that the method of differential invariants was employed to analyse this ODE arising in the reduction of the PDE.

The interval for which the $u(t, x)$ should lie in is $(0, 1)$. The function $u(t, x)$ describes the frequency of the mutated genes on the surface over time. Therefore one should find

that as $x \rightarrow -\infty$, $u(t, x) \rightarrow 1$ and $x \rightarrow \infty$, $u(t, x) \rightarrow 0$. The distortion of the exact solution is attributed to the hyperbolic nature where there is an asymptote at $x = 0$. Asymptotic analysis may be explored.

When varying the constants present within the exact solutions, then one notices that the population frequency deteriorates at a higher rate. It was interesting to find that the exact solution given by equation (4.31) is not time dependent. Unfortunately, determining the exact solution at steady state was not explicitly found. If equation (4.48) could be solved for $u(x)$, it would be interesting to compare the results with equation (4.31).

Chapter 5

Nonclassical Symmetry Reductions and Exact Solutions

In this chapter, the focus is placed on the analysis of the reaction-diffusion equation using nonclassical symmetries. Nonclassical symmetries result in exact solutions which may not be constructed otherwise. The governing equation is given by

$$u_t = (k(x)u_x)_x + g(x)Q(u).$$

where the source term $Q(u) = (u - r_1)(u - r_2)(u - r_3)$, and the diffusivity term $k(x) = \beta^2 x^2$.

When computing for the nonclassical symmetries whilst keeping $g(x)$ arbitrary, proved to be difficult. However, the problem is simplified by assuming that $g(x) = -a^2 x^2$. Hence the governing equation is modified to,

$$u_t = (\beta^2 x^2 u_x)_x - a^2 x^2 (u - r_1)(u - r_2)(u - r_3). \quad (5.1)$$

Solving for the case where r_1 , r_2 and r_3 are distinct, proves to be difficult. There are terms of the form $r_1 r_2 - r_3^2$ in the resulting determining equations, which encourages one to try $r_1 = r_2 = r_3$. When examining the determining equation for r_1 to be an arbitrary constant and $r_2 = r_3 = 0$, or r_1, r_2 is set to the same constant and $r_3 = 0$, suggests that either r_1 , β or a is to be set to zero to find the nonclassical symmetries. Therefore, it is best to proceed with the simplest case first, where $r_1 = r_2 = r_3 = 0$.

The governing equation is then reduced to be,

$$u_t = (\beta^2 x^2 u_x)_x - a^2 x^2 u^3. \quad (5.2)$$

5.1 Nonclassical Symmetries Admitted by Equation (5.2)

To retrieve the determining equations, the second prolongation of the infinitesimal generator is evaluated given equations (2.17), (2.19) and (5.2) where $\xi^1 = 1$,

$$X^{[2]}(eqn.(5.2))|_{eqn.(5.2), ISC} = 0. \quad (5.3)$$

The resulting determining equations that are retrieved by separating the result of equation (5.3) in powers of u_x ,

1:

$$\begin{aligned} 0 = & 2\eta(t, x, u)\xi^2(t, x, u) - 3a^2 u^2 x^3 \eta(t, x, u) + a^2 u^3 x^3 \frac{\partial}{\partial u} \eta(t, x, u) \\ & - 2a^2 u^3 x^3 \frac{\partial}{\partial x} \xi^2(t, x, u) - x \frac{\partial}{\partial t} \eta(t, x, u) - 2x \eta(t, x, u) \frac{\partial}{\partial x} \xi^2(t, x, u) \\ & + \beta^2 x^3 \frac{\partial^2}{\partial x^2} \eta(t, x, u) + 2\beta^2 x^2 \frac{\partial}{\partial x} \eta(t, x, u), \end{aligned} \quad (5.4)$$

u_x :

$$\begin{aligned}
0 = & 3 a^2 u^3 x^3 \frac{\partial}{\partial u} \xi^2(t, x, u) - 2 \beta^2 x^3 \frac{\partial^2}{\partial u \partial x} \eta(t, x, u) \\
& + \beta^2 x^3 \frac{\partial^2}{\partial x^2} \xi^2(t, x, u) - 2 \beta^2 x^2 \frac{\partial}{\partial x} \xi^2(t, x, u) + 2 \beta^2 x \xi^2(t, x, u) \\
& + 2 x \xi^2(t, x, u) \frac{\partial}{\partial u} \xi^2(t, x, u) - 2 x \xi^2(t, x, u) \frac{\partial}{\partial x} \xi^2(t, x, u) \\
& + 2 \left(\xi^2(t, x, u) \right)^2 - x \frac{\partial}{\partial t} \xi^2(t, x, u),
\end{aligned} \tag{5.5}$$

u_x^2 :

$$\begin{aligned}
0 = & 2 \beta^2 x^2 \frac{\partial^2}{\partial u \partial x} \xi^2(t, x, u) - \beta^2 x^2 \frac{\partial^2}{\partial u^2} \eta(t, x, u) \\
& - 4 \beta^2 x \frac{\partial}{\partial u} \eta(t, x, u) - 2 \xi^2(t, x, u) \frac{\partial}{\partial u} \xi^2(t, x, u),
\end{aligned} \tag{5.6}$$

u_x^3 :

$$0 = \frac{\partial^2}{\partial u^2} \xi^2(t, x, u). \tag{5.7}$$

From equation (5.7), ξ^2 can be defined as

$$\xi^2(x, t, u) = f_1(t, x)u + f_2(t, x), \tag{5.8}$$

where f_i are arbitrary functions.

Substituting $\xi^2(x, t, u)$ back into equation (5.5), one can obtain a general form of $\eta(x, t, u)$, given by

$$\begin{aligned}
\eta(x, t, u) = & u^2 \frac{\partial}{\partial x} f_1(x, t) - 2 \frac{1}{x} u^2 f_1(x, t) \\
& - \frac{1}{3} \frac{1}{\beta^2 x^2} u^2 (f_1(x, t))^2 - \frac{1}{\beta^2 x^2} u^2 f_1(x, t) f_2(x, t) \\
& + u f_3(x, t) + f_4(x, t).
\end{aligned} \tag{5.9}$$

Substituting equation (B.20) into equation (5.5), then separating equation (5.5) in

powers of u .

u^3 from equation (5.5) :

$$0 = -\beta^2 x \left(\frac{2}{3} \frac{1}{\beta^2 x} (f_1(x, t))^3 - 3a^2 x^3 f_1(x, t) \right). \quad (5.10)$$

Hence, one can solve for $f_1(x, t)$ to be,

$$f_1(x, t) = 0, \text{ or} \quad (5.11)$$

$$f_1(x, t) = \pm \frac{3}{\sqrt{2}} x^2 \beta \alpha. \quad (5.12)$$

Therefore, u^2 from equation (5.5) simplifies to,

u^2 from equation (5.5) :

$$0 = 9\beta^2 x^4 a^2 (f_2(x, t) - \beta^2 x). \quad (5.13)$$

From equation (5.13), one can solve for $f_2(x, t)$,

$$f(x) = \beta^2 x. \quad (5.14)$$

Simplifying u from equation (5.5) gives,

u from equation (5.5) :

$$0 = 3\sqrt{2}\beta^3 x^4 a (f_3(x, t) + \beta^2). \quad (5.15)$$

From equation (5.15), one can solve for $f_3(x, t)$,

$$f_3(x) = -\beta^2. \quad (5.16)$$

Simplifying the remainder from equation (5.5) gives,

1 from equation (5.5) :

$$0 = 3\sqrt{2}a\beta^3x^4f_4(x, t). \quad (5.17)$$

From equation (5.17), one can see that $f_4(x, t) = 0$. Substituting all the functions of f_i , $i = 1, \dots, 4$, equation (5.4) is solved identically.

Examining the determining equations, where $f_1(x, t)$ is positive, yields the following nonclassical symmetry generator,

$$X = \frac{\partial}{\partial t} + \left(\frac{3}{\sqrt{2}}x^2\beta au + \beta^2x \right) \frac{\partial}{\partial x} + \left(-\frac{3}{\sqrt{2}}x\beta au^2 - \frac{3}{2}x^2a^2u^3 - \beta^2u \right) \frac{\partial}{\partial u}. \quad (5.18)$$

The associated invariant surface condition is,

$$u_t + \left(\frac{3}{\sqrt{2}}x^2\beta au + \beta^2x \right) u_x = -\frac{3}{\sqrt{2}}x\beta au^2 - \frac{3}{2}x^2a^2u^3 - \beta^2u. \quad (5.19)$$

5.2 Nonclassical Symmetries Reductions of Equation (5.2)

Using equation (5.1) and equation (5.19) simultaneously to eliminate u_t gives

$$\beta^2x^2u_{xx} + \left(3\beta^2x + \frac{3}{\sqrt{2}}x^2\beta au \right) u_x + \frac{1}{2}x^2a^2u^3 + \frac{3}{\sqrt{2}}x\beta au^2 + \beta^2u = 0. \quad (5.20)$$

For simplicity, β is set to unity and let $y = \ln(x)$. Therefore equation (5.20) is given by,

$$u_{yy} + \left(2 + \frac{3}{\sqrt{2}} \exp(y) a u\right) u_y + \frac{1}{2} a^2 \exp(2y) u^3 + \frac{3}{\sqrt{2}} \exp(y) a u^2 + u = 0. \quad (5.21)$$

The next step would be to implement the Hopf-Cole transformation, given by

$$u(y) = B \frac{v'(y)}{v},$$

where B is an arbitrary constant.

In this case, the Hopf-Cole transformation does not work. When using the transformation, the resulting equation was nonlinear and the constant B was dependent on y . Therefore a ‘modified’ Hopf-Cole transformation, given by

$$u(y) = f(y) \frac{v'(y)}{v}, \quad (5.22)$$

is used. It turns out that $f(y)$ must satisfy an algebraic equation

$$f(y) \left(\sqrt{2} \exp(y) f(y) a - 2 \right) = 0. \quad (5.23)$$

As such $f(y) = \frac{\sqrt{2}}{a \exp(y)}$. If $f(y) = 0$ is chosen, the trivial solution is obtained. Thus equation (5.21) reduces to a linear third order differential equation

$$v_{yyy} = 0, \quad (5.24)$$

Therefore,

$$v(t, y) = \frac{1}{2} c_1(t) y^2 + c_2(t) y + c_3(t). \quad (5.25)$$

Substituting backwards for $u(t, x)$, produces

$$u(t, x) = \frac{2\sqrt{2}(c_1(t) \ln(x) + c_2(t))}{a x(c_1(t) \ln(x)^2 + 2c_2(t) \ln(x) + 2c_3(t))}. \quad (5.26)$$

Solving for the arbitrary functions, $c_i(t)$, equation (5.26) is substituted back into the ISC given by equation (5.19). Without loss of generality, $c_1(t) = 1$, hence the resulting exact solution is given by,

$$u(t, x) = \frac{2\sqrt{2}(\ln(x) - t + d_1)}{a x(\ln(x)^2 + 2(-t + d_1) \ln(x) + t^2 - 2t d_1 + 6t + 2d_2)}, \quad (5.27)$$

where d_1 and d_2 are arbitrary constants.

If one considers the case where $r_1 = r_2 = r_3 = \omega$, where ω is an arbitrary constant, then the simple transformation given by $\bar{u} = u - \omega$, gives rise to the governing equation

$$\bar{u}_t = (\beta^2 x^2 \bar{u}_x)_x - a^2 x^2 \bar{u}^3. \quad (5.28)$$

Therefore, the exact solution will be given by,

$$u(t, x) = \frac{2\sqrt{2}(\ln(x) - t + d_1)}{a x(\ln(x)^2 + 2(-t + d_1) \ln(x) + t^2 - 2t d_1 + 6t + 2d_2)} + \omega. \quad (5.29)$$

Figure 4.3 and Figure 5.1 share some similarities, whereby the region surrounding the origin has steep gradients and moving along displacement, the gradient decreases. In Figure 5.1, there is a strong difference where along the boundary $x = 0$ from 1 sec to 2 sec, there is an abnormality.

Figure 5.2 below depicts how the frequency of mutant population changes as a changes. One can see that as a increases it constrains the initial growth of the population. This

Mutant Population surface plane

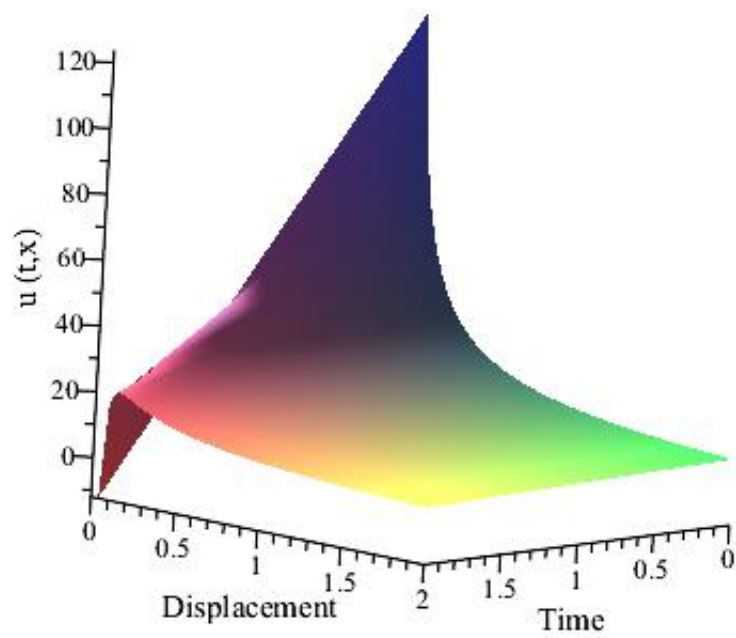


Figure 5.1: Surface plane of exact solution obtained where $d_1 = d_2=5$

is the same effect as the change in a depicted in Figure 4.2.

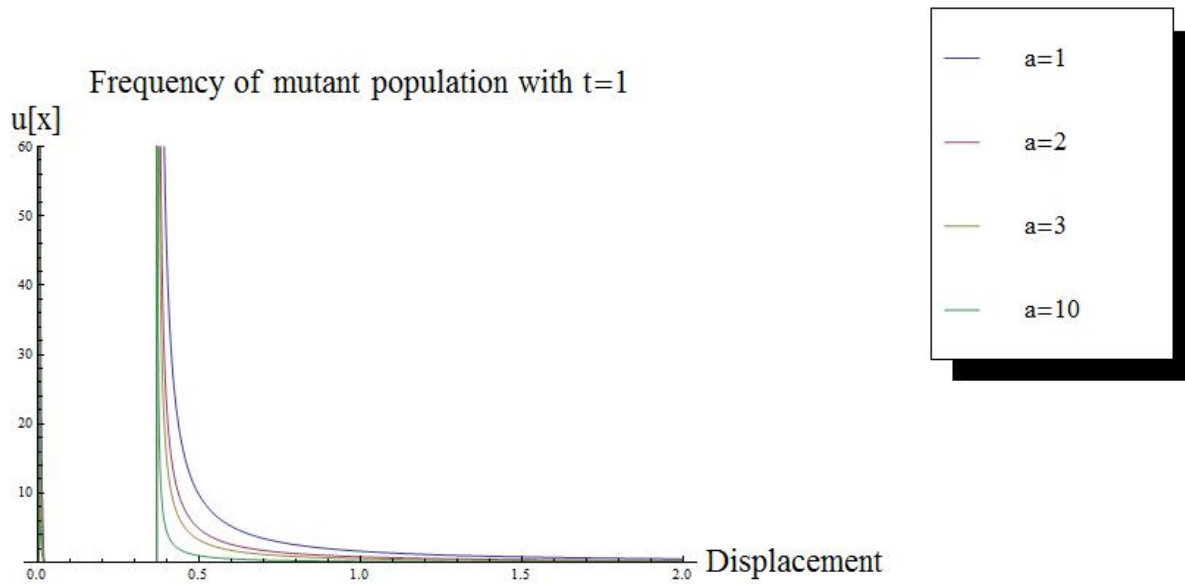


Figure 5.2: Impact on the population frequency by adjusting the coefficient a where $d_1 = d_2=5$.

Figure 5.3 shows the decay of the population as time increases. The negative population values are not physically viable. In Figure 4.4 all values of $u(t, x)$ are positive. Another difference between the figures is that in Figure 5.3 $u(t, x)$ first has a peak before the decay begins. As time progresses, the peaks start to diminish.

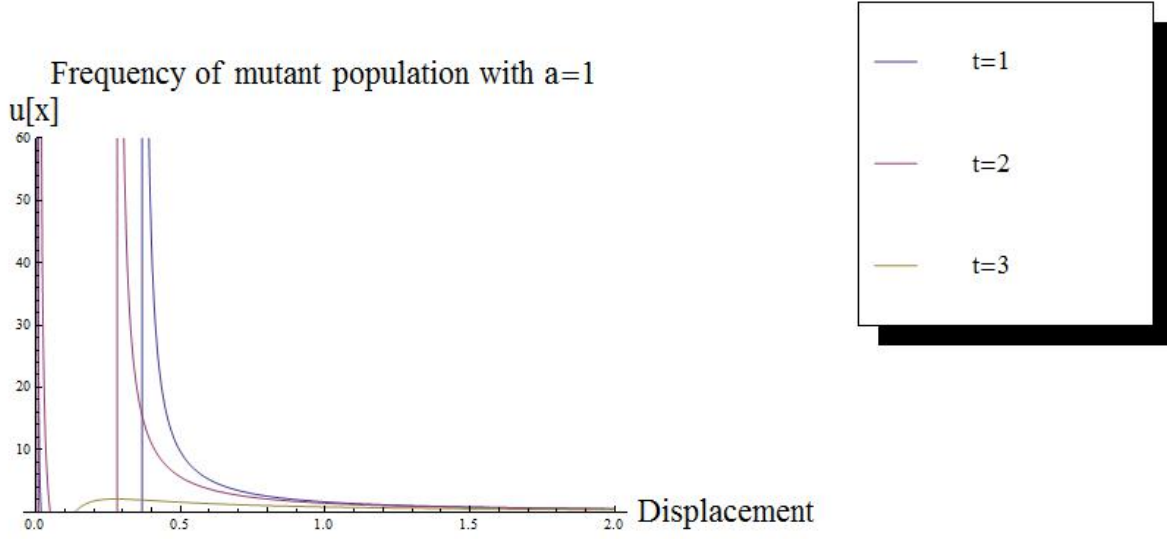


Figure 5.3: Change in population frequency as time changes where $d_1 = d_2 = 5$.

5.3 Nonclassical Symmetry for a More General $g(x)$

Suppose that one considers a more general case, $g(x) = (ax)^{2n}$ where $n \in \mathfrak{R}$. The governing equation then becomes

$$u_t = (x^2 u_x)_x - (ax)^{2n} u^3. \quad (5.30)$$

Following the steps of Section 5.1, the admitted genuine nonclassical symmetry is given by

$$X = \frac{\partial}{\partial t} + \frac{1}{2}x \left(3\sqrt{2}(ax)^n u + 4n - 2 \right) \frac{\partial}{\partial x} - \frac{1}{2}u \left(3(ax)^{2n} u^2 + 3\sqrt{2}(ax)^n n u + n^2 + n \right) \frac{\partial}{\partial u}. \quad (5.31)$$

Similarly, following the steps in Section 5.2 and using the ‘modified’ Hopf-Cole trans-

formation (5.22), it turns out that

$$f(y) = \frac{\sqrt{2}}{(a \exp(y))^n},$$

one obtains an ODE

$$2v'''(y) - (n^2 - n)v'(y) = 0.$$

Hence, in terms of the original variables the exact solution can be given as

$$u(t, x) = \frac{n^{3/2}(n-1)^{3/2} \left(x^{\frac{1}{\sqrt{2}}} \sqrt{n(n-1)} c_2(t) - x^{-\frac{1}{\sqrt{2}}} \sqrt{n(n-1)} c_1(t) \right)}{(a x)^n \left(x^{\frac{1}{\sqrt{2}}} \sqrt{n(n-1)} c_2(t) (n^2 - n) + x^{-\frac{1}{\sqrt{2}}} \sqrt{n(n-1)} c_1(t) (n^2 - n) - 2c_0(t) \right)}, \quad (5.32)$$

where $c_i(t)$ are arbitrary functions of t . Without loss of generality, set $c_2(t) = 1$ and then solving for the rest of the arbitrary functions by substituting equation (5.32) into the ICS pertaining to this case. Then $c_0(t)$ and $c_1(t)$ can be found to be

$$c_0(t) = 0, \quad \text{or} \quad (5.33)$$

$$c_0(t) = d_1 \exp \left(\frac{1}{2} t ((2n-1) \sqrt{2n(n-1)} - 3n(n-1)) \right), \quad (5.34)$$

and

$$c_1(t) = d_2 \exp \left(\sqrt{2n(n-1)} (2n-1) t \right) \quad (5.35)$$

If $n = 0$ or $n = 1$, then the trivial solution is obtained, where $u(t, x) = 0$.

Figures 5.4 -5.7 depict how the surface planes change as n changes, where $c_0(t)$ is trivial and $d_1 = d_2 = 1$. Note that as n increases the plane becomes more boxed shape. Therefore the power of x in the source term can heavily affect the evolution of the population over time.

Frequency of mutant population surface plane with n=3

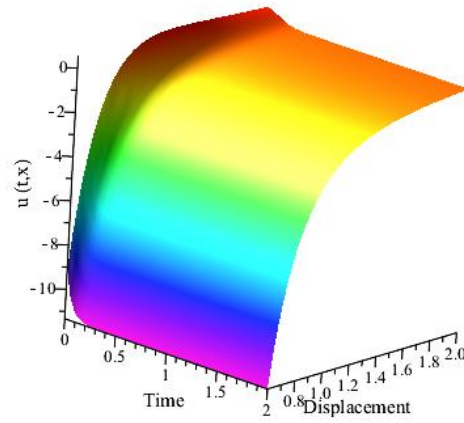


Figure 5.4: Exact solution of with $g(x) = (a x)^3$

Frequency of mutant population surface plane
with n=4

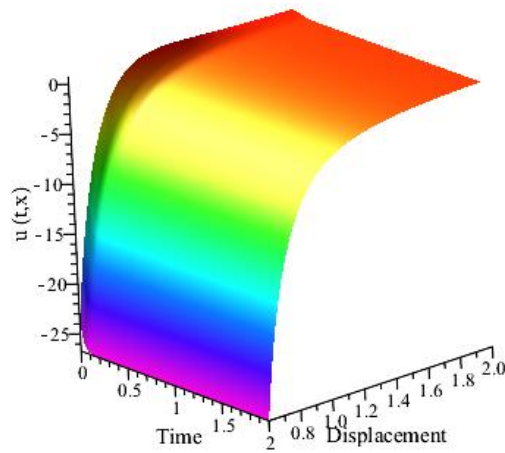


Figure 5.5: Exact solution of with $g(x) = (a x)^4$

Frequency of mutant population surface plane
with $n=5$

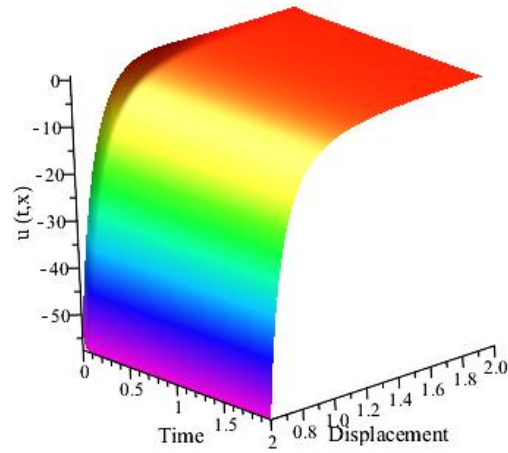


Figure 5.6: Exact solution of with $g(x) = (ax)^5$

Frequency of mutant population surface plane
with $n=10$

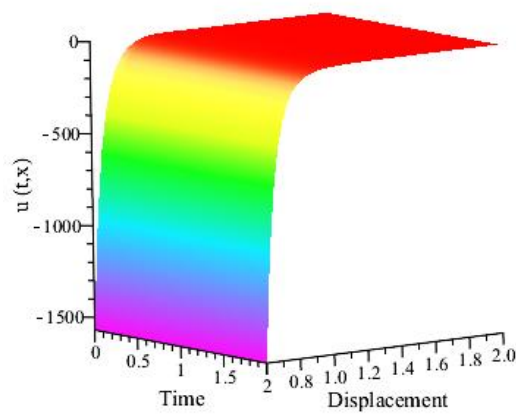


Figure 5.7: Exact solution of with $g(x) = (ax)^{10}$

Zooming in on Figure 5.4 as depicted above in Figure 5.8, one can see that as time increases the plane starts to flatten out. At time $t = 0$, one can see that there is a peak that also appears in all the other figures.

Unfortunately, these cases also have negative values for $u(t, x)$, which does not hold physically. In Figure 5.8, the positive values of $u(t, x)$ lie further away from $x = 0$.

Frequency of mutant population surface plane
with $n=3$

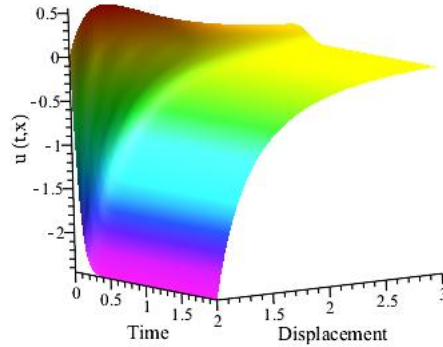


Figure 5.8: Zooming in on Figure 5.4

5.4 Some Discussions and Concluding Remarks

In this chapter the genuine nonclassical symmetry was found for the governing equation. This symmetry was utilised to reduce the governing equation to an ODE. Here the modified Hopf-Cole transformation was introduced. The transformation was needed due to the fact that the coefficient of the source term is dependent on x . This greatly reduced the governing equation to an easily solvable third order ODE. The exact solution could be found.

The graphical representation of the exact solution had its similarities and differences to that of the surface planes obtained by exact solution via classical Lie point symmetries.

The basic structure was evident in Figure 5.1.

Chapter 6

Conservation Laws and Associated Lie Point Symmetries

In this chapter, a focus is given to the construction of conserved vectors and the associated Lie point symmetries. Consider equation (5.2) with $\beta = 1$. Both the direct and the multiplier methods are employed.

6.1 Direct Method

Within this section, the direct method is used to construct the conservation laws. Conservation laws for equation (5.2) have to satisfy

$$D_1 T^1 + D_2 T^2|_{eqn.(5.2)} = 0, \quad (6.1)$$

where D_i is the total derivative.

For convenience and simplicity, we look for conserved vectors of the form,

$$\begin{aligned} T^1 &= T^1(t, x, u, u_x), \\ T^2 &= T^2(t, x, u, u_x). \end{aligned} \quad (6.2)$$

Substitute equation (6.2) into equation (6.1) to obtain,

$$\left(\frac{\partial T^1}{\partial t} + u_t \frac{\partial T^1}{\partial u} + u_{xt} \frac{\partial T^1}{\partial u_x} + \frac{\partial T^2}{\partial x} + u_x \frac{\partial T^2}{\partial u} + u_{xx} \frac{\partial T^2}{\partial u_x} \right) \Big|_{eqn.(5.2)} = 0 \quad (6.3)$$

Evaluate equation (6.3) given equation (5.2) to obtain,

$$0 = \frac{\partial T^1}{\partial t} + [x^2 u_{xx} + 2x u_x - a^2 x^2 u^3] \frac{\partial T^1}{\partial u} + u_{xt} \frac{\partial T^1}{\partial u_x} + \frac{\partial T^2}{\partial x} + u_x \frac{\partial T^2}{\partial u} + u_{xx} \frac{\partial T^2}{\partial u_x} \quad (6.4)$$

Equation (6.4) can be separated with respect to second derivatives of u as T^i is independent of them, giving

$$\underline{u_{xx}:} \quad 0 = x^2 \frac{\partial}{\partial u} T^1(t, x, u, u_x) + \frac{\partial}{\partial u_x} T^2(t, x, u, u_x), \quad (6.5)$$

$$\underline{u_{xt}:} \quad 0 = \frac{\partial}{\partial u_x} T^1(t, x, u, u_x), \quad (6.6)$$

remainder:

$$\begin{aligned} 0 &= -a^2 x^2 u^3 \frac{\partial}{\partial u} T^1(t, x, u, u_x) + 2x u_x \frac{\partial}{\partial u} T^1(t, x, u, u_x) \\ &\quad + u_x \frac{\partial}{\partial u} T^2(t, x, u, u_x) + \frac{\partial}{\partial t} T^1(t, x, u, u_x) + \frac{\partial}{\partial x} T^2(t, x, u, u_x). \end{aligned} \quad (6.7)$$

From equation (6.6), it is implied that $T^1 = T^1(t, x, u)$. Therefore, T^2 can be written

explicitly by integrating equation (6.5) with respect to u_x to obtain

$$T^2(t, x, u, u_x) = -x^2 \left(\frac{\partial}{\partial u} T^1(t, x, u) \right) u_x + f_1(t, x, u), \quad (6.8)$$

where f_1 is an arbitrary function. Substituting T^2 into equation (6.7) and separating it with respect to powers of u_x ,

$$\underline{u_x^2}: \quad 0 = \frac{\partial^2}{\partial u^2} T^1(t, x, u), \quad (6.9)$$

$$\underline{u_x^1}: \quad 0 = -x^2 \frac{\partial^2}{\partial u \partial x} T^1(t, x, u) + \frac{\partial}{\partial u} f_1(t, x, u), \quad (6.10)$$

$$\underline{u_x^0}: \quad 0 = -a^2 x^2 u^3 \frac{\partial}{\partial u} T^1(t, x, u) + \frac{\partial}{\partial t} T^1(t, x, u) + \frac{\partial}{\partial x} f_1(t, x, u). \quad (6.11)$$

From equation (6.9), T^1 can be derived by integrating the equation twice with respect to u , to give

$$T^1(t, x, u) = f_2(t, x)u + f_3(t, x), \quad (6.12)$$

where f_i are arbitrary functions. Then equation (6.10) becomes,

$$0 = -x^2 \frac{\partial}{\partial x} f_2(t, x) + \frac{\partial}{\partial u} f_1(t, x, u). \quad (6.13)$$

Integrating equation (6.13) with respect to u , f_1 can be written as,

$$f_1(t, x, u) = x^2 u \frac{\partial}{\partial x} f_2(t, x) + f_4(t, x). \quad (6.14)$$

Substituting the current solutions of T^1 and f_1 into equation (6.11) and then separating the equation with respect to the powers of u we obtain,

u^3 :

$$0 = a^2 x^2 f_2(t, x), \quad (6.15)$$

u^1 :

$$0 = \frac{\partial}{\partial t} f_2(t, x) + x^2 \frac{\partial^2}{\partial x^2} f_2(t, x) + 2x \frac{\partial}{\partial x} f_2(t, x), \quad (6.16)$$

u^0 :

$$0 = \frac{\partial}{\partial t} f_3(t, x) + \frac{\partial}{\partial x} f_4(t, x). \quad (6.17)$$

From equation (6.15), either $a = 0$ or $f_2(t, x) = 0$. For $f_2(t, x) = 0$, this provides the trivial solution. Therefore taking $a = 0$ and for equation (6.17) to be directly satisfied, we have $f_3(t, x) = 0$ and $f_4(t, x) = 0$. From equation (6.16), $f_2(t, x)$ can be solved to be,

$$f_2(t, x) = c_1 \exp(c_2 t) \left(c_3 x^{-\frac{1}{2} + \frac{1}{2}\sqrt{1-4c_2}} + c_4 x^{-\frac{1}{2} - \frac{1}{2}\sqrt{1-4c_2}} \right). \quad (6.18)$$

where c_i are arbitrary constants.

For simplicity, setting $c_1 = 1$, $c_2 = -\frac{3}{4}$, $c_3 = d_1$ and $c_4 = d_2$, to obtain

$$\begin{aligned} 0 = & D_t \left[x^{-\frac{3}{2}} \exp\left(-\frac{3}{4}t\right) (d_1 x^2 + d_2) u \right] \\ & + D_x \left[-\frac{1}{2} x^{-\frac{1}{2}} \exp\left(-\frac{3}{4}t\right) (2d_1 u_x x^3 - d_1 u x^2 + 2d_2 u_x x + 3d_2 u) \right]. \end{aligned} \quad (6.19)$$

Therefore the conserved vector for equation (5.30) where $a = 0$ is the linear combination of the two conserved vectors

$$T^1 = \exp\left(-\frac{3}{4}t\right) \sqrt{x} u, \quad T^2 = \frac{1}{2} \exp\left(-\frac{3}{4}t\right) x^{\frac{3}{2}} (u - 2x u_x), \quad (6.20)$$

$$T^1 = \exp\left(-\frac{3}{4}t\right) x^{-\frac{3}{2}} u, \quad T^2 = -\frac{1}{2} \exp\left(-\frac{3}{4}t\right) \sqrt{x}(3u + 2x u_x). \quad (6.21)$$

Although the conserved vectors for the nontrivial solution is found, $a = 0$ means that the source term in the governing equation disappears. This is not the ideal situation.

When equation (5.30) is considered, the resulting conserved vectors are given by equations (6.20) and (6.21). The constant a is required to be zero in order to find the non-trivial solution.

6.2 Multiplier Method

To acquire the conserved vectors using the multiplier method, one evaluates

$$D_1 T^1 + D_2 T^2 = \Lambda \left(u_t - (x^2 u_x)_x + a^2 x^2 u^3 \right), \quad (6.22)$$

where Λ is the multiplier that makes the governing equation exact. The determining equation is then,

$$E_u \left[\Lambda \left(u_t - (x^2 u_x)_x + a^2 x^2 u^3 \right) \right] = 0, \quad (6.23)$$

where E_u is the standard Euler operator given by,

$$E_u = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} - D_t \frac{\partial}{\partial u_t} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_x D_t \frac{\partial}{\partial u_{xt}} + D_t^2 \frac{\partial}{\partial u_{tt}} - \dots \quad (6.24)$$

Hence, expanding equation (6.23) gives

$$\begin{aligned}
0 = & \left(\frac{\partial}{\partial u} \Lambda(t, x, u) \right) (u_t - x^2 u_{xx} - 2x u_x + a^2 x^2 u^3) + 3u^2 a^2 x^2 \Lambda(t, x, u) \\
& - 2x \frac{\partial}{\partial x} \Lambda(t, x, u) - \frac{\partial}{\partial t} \Lambda(t, x, u) - u_t \frac{\partial}{\partial u} \Lambda(t, x, u) - x^2 \frac{\partial^2}{\partial x^2} \Lambda(t, x, u) \\
& - x u_x \left[x \frac{\partial^2}{\partial u \partial x} \Lambda(t, x, u) + x \frac{\partial^2}{\partial u \partial x} \Lambda(t, x, u) + 2 \frac{\partial}{\partial u} \Lambda(t, x, u) + x u_x \frac{\partial^2}{\partial u^2} \Lambda(t, x, u) \right].
\end{aligned} \tag{6.25}$$

Since Λ is a function independent of the second derivatives of u , one can separate equation (6.25) with respect to the second derivatives and find the coefficient of u_{xx} , which is given by,

$$0 = 2x^2 \frac{\partial}{\partial u} \Lambda(t, x, u). \tag{6.26}$$

This implies that $\Lambda = \Lambda(t, x)$. Therefore equation (6.25) is simplified to

$$0 = 3u^2 a^2 x^2 \Lambda(t, x) - x^2 \frac{\partial^2}{\partial x^2} \Lambda(t, x) - 2x \frac{\partial}{\partial x} \Lambda(t, x) - \frac{\partial}{\partial t} \Lambda(t, x). \tag{6.27}$$

Equation (6.27) can now be separated with respect to the powers of u , which gives

u^2 :

$$0 = a^2 x^2 \Lambda(t, x), \tag{6.28}$$

u^0 :

$$0 = -2x \frac{\partial}{\partial x} \Lambda(t, x) - \frac{\partial}{\partial t} \Lambda(t, x) - x^2 \frac{\partial^2}{\partial x^2} \Lambda(t, x). \tag{6.29}$$

From equation (6.28), either $a = 0$ or $\Lambda(t, x) = 0$. But $\Lambda(t, x) = 0$, gives the trivial solution. From equation (6.29),

$$\Lambda(t, x) = c_1 \exp(c_2 t) \left(c_3 x^{-\frac{1}{2} + \frac{1}{2}\sqrt{1-4c_2}} + c_4 x^{-\frac{1}{2} - \frac{1}{2}\sqrt{1-4c_2}} \right). \tag{6.30}$$

For simplicity, setting $c_1 = 1$, $c_2 = -\frac{3}{4}$, $c_3 = d_1$ and $c_4 = d_2$, to simplify equation (6.30)

$$\Lambda(t, x) = \exp\left(-\frac{3}{4}t\right) \left(d_1\sqrt{x} + d_2x^{-\frac{3}{2}}\right), \quad (6.31)$$

where d_1 and d_2 are arbitrary constants.

Substituting $\Lambda(t, x)$ back into the equation (6.22), given $a = 0$ and with some manipulation, gives

$$\begin{aligned} & \exp\left(-\frac{3}{4}t\right) \left(d_1\sqrt{x} + d_2x^{-\frac{3}{2}}\right) (u_t - (x^2u_x)_x) \\ &= D_t \left[x^{-\frac{3}{2}} \exp\left(-\frac{3}{4}t\right) (d_1x^2 + d_2) u \right] \\ &+ D_x \left[-\frac{1}{2}x^{-\frac{1}{2}} \exp\left(-\frac{3}{4}t\right) (2d_1u_x x^3 - d_1ux^2 + 2d_2u_x x + 3d_2u) \right], \end{aligned} \quad (6.32)$$

Therefore the conserved vector for equation (5.2) with a multiplier of the form $\Lambda(t, x)$ where $a = 0$ is the linear combination of the two conserved vectors

$$T^1 = \exp\left(-\frac{3}{4}t\right) \sqrt{x} u, \quad T^2 = \frac{1}{2} \exp\left(-\frac{3}{4}t\right) x^{\frac{3}{2}} (u - 2x u_x). \quad (6.33)$$

$$T^1 = \exp\left(-\frac{3}{4}t\right) x^{-\frac{3}{2}} u, \quad T^2 = -\frac{1}{2} \exp\left(-\frac{3}{4}t\right) \sqrt{x} (3u + 2x u_x). \quad (6.34)$$

6.3 Conserved Vectors and Associated Point Symmetries

We obtain the Lie point symmetries, X , associated with the conserved vectors $T = (T^1, T^2)$ calculated in the previous section. They can be determined from,

$$X(T^i) + T^i D_j(\xi^j) - T^j D_j(\xi^i) = 0. \quad (6.35)$$

Equation (6.35) has two components, namely

$$X(T^1) + T^1 D_2(\xi^2) - T^2 D_2(\xi^1) = 0, \quad (6.36)$$

$$X(T^2) + T^2 D_1(\xi^1) - T^1 D_1(\xi^2) = 0. \quad (6.37)$$

Using the conserved vector given by equation (6.20) to substitute into equations (6.36) and (6.37), gives

$$\begin{aligned} 0 = -\exp\left(-\frac{3}{4}t\right) & \left[-4x^3 u_x^2 \frac{\partial}{\partial u} \xi^1(t, x, u) - 4xu u_x \frac{\partial}{\partial u} \xi^2(t, x, u) - 4x^3 u_x \frac{\partial}{\partial x} \xi^1(t, x, u) \right. \\ & + 2x^2 u u_x \frac{\partial}{\partial u} \xi^1(t, x, u) + 3xu \xi^1(t, x, u) - 4xu \frac{\partial}{\partial x} \xi^2(t, x, u) \\ & \left. + 2x^2 u \frac{\partial}{\partial x} \xi^1(t, x, u) - 4x\eta(t, x, u) - 2u\xi^2(t, x, u) \right], \end{aligned} \quad (6.38)$$

and

$$\begin{aligned}
0 = \exp\left(-\frac{3}{4}t\right) & \left[-8x^4 u_t u_x \frac{\partial}{\partial u} \xi^1(t, x, u) + 4x^3 \eta(t, x, u) - 8x^2 u u_t \frac{\partial}{\partial u} \xi^2(t, x, u) \right. \\
& + 6x^4 u_x \xi^1(t, x, u) + 4x^3 u u_t \frac{\partial}{\partial u} \xi^1(t, x, u) - 8x^4 u_x \frac{\partial}{\partial t} \xi^1(t, x, u) - 8x^2 u \frac{\partial}{\partial t} \xi^2(t, x, \\
& \left. u) - 3x^3 u \xi^1(t, x, u) - 20x^3 u_x \xi^2(t, x, u) + 4x^3 u \frac{\partial}{\partial t} \xi^1(t, x, u) + 6x^2 u \xi^2(t, x, u) \right].
\end{aligned} \tag{6.39}$$

Separating equation (6.38) with respect to the derivatives of u ,

$$\underline{u_x^2}: \quad 0 = 4x^3 \exp\left(-\frac{3}{4}t\right) \left[\frac{\partial}{\partial u} \xi^1(t, x, u) \right], \tag{6.40}$$

$$\underline{u_x^1}: \quad 0 = 2x \exp\left(-\frac{3}{4}t\right) \left[2x^2 \frac{\partial}{\partial x} \xi^1(t, x, u) - xu \frac{\partial}{\partial u} \xi^1(t, x, u) \right. \\
\left. + 2u \frac{\partial}{\partial u} \xi^2(t, x, u) \right], \tag{6.41}$$

$$\underline{u_x^0}: \quad 0 = -\exp\left(-\frac{3}{4}t\right) \left[3xu \xi^1(t, x, u) \right. \\
\left. - 4xu \frac{\partial}{\partial x} \xi^2(t, x, u) + 2x^2 u \frac{\partial}{\partial x} \xi^1(t, x, u) - 4x \eta(t, x, u) - 2u \xi^2(t, x, u) \right]. \tag{6.42}$$

Similarly, equation (6.39) is separated with respect to the derivatives of u ,

$$\underline{u_x}: \quad 0 = 2x^3 \exp\left(-\frac{3}{4}t\right) \left[3x \xi^1(t, x, u) - 4x \frac{\partial}{\partial t} \xi^1(t, x, u) - 10 \xi^2(t, x, u) \right], \tag{6.43}$$

u_t :

$$0 = -4 x^2 u \exp \left(-\frac{3}{4} t \right) \left[-x \frac{\partial}{\partial u} \xi^1 (t, x, u) + 2 \frac{\partial}{\partial u} \xi^2 (t, x, u) \right], \quad (6.44)$$

$u_x u_t$:

$$0 = -8 x^4 \exp \left(-\frac{3}{4} t \right) \left[\frac{\partial}{\partial u} \xi^1 (t, x, u) \right], \quad (6.45)$$

1:

$$\begin{aligned} 0 = & -x^2 \exp \left(-\frac{3}{4} t \right) \left[3 x u \xi^1 (t, x, u) \right. \\ & \left. - 4 x u \frac{\partial}{\partial t} \xi^1 (t, x, u) - 6 u \xi^2 (t, x, u) - 4 x \eta (t, x, u) + 8 u \frac{\partial}{\partial t} \xi^2 (t, x, u) \right]. \end{aligned} \quad (6.46)$$

From equations (6.40) and (6.45),

$$\xi^1(t, x, u) = f_1(t, x). \quad (6.47)$$

Equation (6.44) simplifies to,

$$0 = -8 \exp \left(-\frac{3}{4} t \right) x^2 \left(\frac{\partial}{\partial u} \xi^2 (t, x, u) \right) u \quad (6.48)$$

Thus,

$$\xi^2(t, x, u) = f_2(t, x). \quad (6.49)$$

Therefore, equation (6.41) reduces to

$$0 = 4 \exp \left(-\frac{3}{4} t \right) \left(\frac{\partial}{\partial x} f_1 (t, x) \right) x^3, \quad (6.50)$$

implying that $f_1(t, x) = f_1(t)$, which simplifies equation (6.42) to,

$$0 = -3xuf_1(t) + 4xu\frac{\partial}{\partial x}f_2(t, x) + 4x\eta(t, x, u) + 2uf_2(t, x) \quad (6.51)$$

Therefore, η is given as

$$\eta(t, x, u) = \frac{1}{4x}u \left(3xf_1(t) - 4x\frac{\partial}{\partial x}f_2(t, x) - 2f_2(t, x) \right) \quad (6.52)$$

Finally, equations (6.43) and (6.46) reduce to,

$$0 = -2x^3 \left[4x\frac{d}{dt}f_1(t) - 3xf_1(t) + 10f_2(t, x) \right], \quad (6.53)$$

and

$$0 = 4x^2u \left[x\frac{d}{dt}f_1(t) - x\frac{\partial}{\partial x}f_2(t, x) + f_2(t, x) - 2\frac{\partial}{\partial t}f_2(t, x) \right]. \quad (6.54)$$

Simultaneously solving for $f_1(t)$ and $f_2(t, x)$, gives

$$f_1(t) = d_1 + d_2 \exp\left(-\frac{1}{2}t\right), \quad (6.55)$$

and

$$f_2(t, x) = \frac{3}{10}d_1x + \frac{1}{2}d_2x \exp\left(-\frac{1}{2}t\right). \quad (6.56)$$

In summary,

$$\xi^1(t, x, u) = d_1 + d_2 \exp\left(-\frac{1}{2}t\right), \quad (6.57)$$

$$\xi^2(t, x, u) = \frac{1}{10}x \left(5d_2 \exp\left(-\frac{1}{2}t\right) + 3d_1 \right), \quad (6.58)$$

and

$$\eta(t, x, u) = \frac{3}{10}d_1u. \quad (6.59)$$

Hence the Lie point symmetries associated with the conserved vector, equation (6.20),

$$X_1 = \exp\left(-\frac{1}{2}t\right) \frac{\partial}{\partial t} + \frac{1}{2} \exp\left(-\frac{1}{2}t\right) x \frac{\partial}{\partial x}, \quad (6.60)$$

$$X_2 = \frac{\partial}{\partial t} + \frac{3}{10}x \frac{\partial}{\partial x} + \frac{3}{10}u \frac{\partial}{\partial u}. \quad (6.61)$$

Notice that X_1 and X_2 span the two dimensional Lie subalgebra since $[X_1, X_2] = -\frac{1}{2}X_2$.

6.4 Reductions Using the Associated Lie Point Symmetries

The associated Lie point symmetries given by equations (6.60) and (6.61) are for the governing equation,

$$u_t = (x^2u_x)_x. \quad (6.62)$$

6.4.1 Reduction Using X_1

The characteristic equation from X_1 is given by,

$$\frac{dt}{\exp\left(-\frac{1}{2}t\right)} = \frac{dx}{\frac{1}{2} \exp\left(-\frac{1}{2}t\right) x} \equiv \frac{du}{0}.$$

Therefore $u(t, x)$ can be rewritten in the form of

$$u(t, x) = F(\gamma), \quad \text{where } \gamma = x \exp\left(-\frac{1}{2}t\right). \quad (6.63)$$

The reduced PDE is given by,

$$\gamma^2 F''(\gamma) + \frac{5}{2} \gamma F'(\gamma) = 0, \quad (6.64)$$

for which the solution for F is

$$F(\gamma) = d_1 + \frac{d_2}{\gamma^{3/2}}, \quad (6.65)$$

where d_i are arbitrary constants. Therefore the invariant solution can be written as

$$u(t, x) = d_1 + \frac{d_2}{\left(x \exp\left(-\frac{1}{2}t\right)\right)^{3/2}}. \quad (6.66)$$

The graphical representation below is given with $d_i = 1$.

Mutant Population surface plane

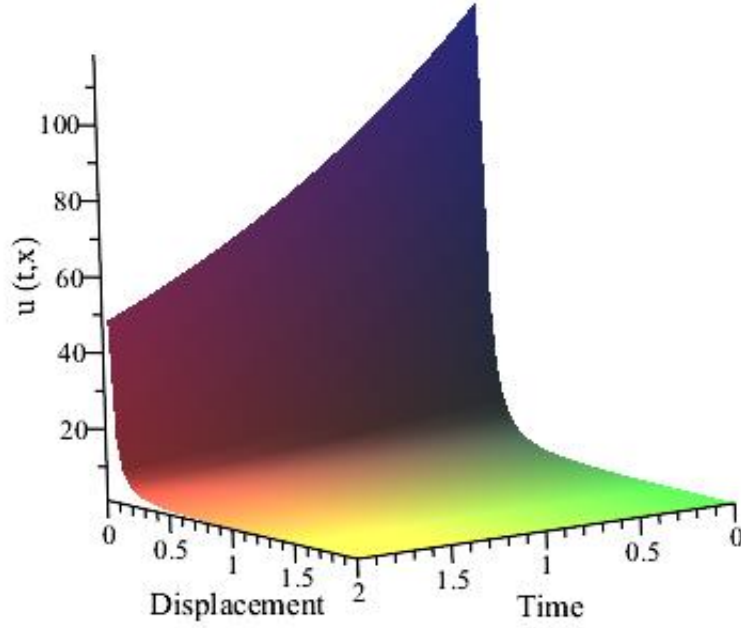


Figure 6.1: Surface plane of $u(t, x)$ reduced by X_1

6.4.2 Reduction Using X_2

The characteristic equation from X_1 is given by,

$$\frac{dt}{1} = \frac{dx}{\frac{3}{10}x} = \frac{du}{\frac{3}{10}u}.$$

Therefore $u(t, x)$ can be rewritten in the form of

$$u(t, x) = \exp\left(\frac{3}{10}t\right) F(\gamma), \quad \text{where } \gamma = x \exp\left(-\frac{3}{10}t\right). \quad (6.67)$$

The reduced equation is given by the Euler equation

$$\gamma^2 F''(\gamma) + \frac{23}{10} \gamma F'(\gamma) - \frac{3}{10} F(\gamma) = 0. \quad (6.68)$$

Using Maple or [28] one obtains the solution for F , namely

$$F(\gamma) = d_1 \gamma^{1/5} + \frac{d_2}{\gamma^{3/2}}. \quad (6.69)$$

where d_i are arbitrary constants. Therefore the invariant solution can be written as

$$u(t, x) = \exp\left(\frac{3}{10}t\right) \left(d_1 \left(x \exp\left(-\frac{3}{10}t\right) \right)^{1/5} + \frac{d_2}{\left(x \exp\left(-\frac{3}{10}t\right) \right)^{3/2}} \right). \quad (6.70)$$

The graphical representation below is given with $d_i = 1$.

6.5 Some Discussion and Concluding Remarks

Not surprisingly both the direct and multiplier methods produced the same conserved vectors. Unfortunately, a was required to be zero in order to construct conserved vectors for the considered reaction-diffusion equation. Thus looking at the governing equation, the source term was lost and the governing equation became the general heat equation with a quadratic diffusivity term.

Notice that, the simplest conserved vector would be $T^1 = u$ and $T^2 = -x^2 u_x$. To prove

Mutant Population surface plane

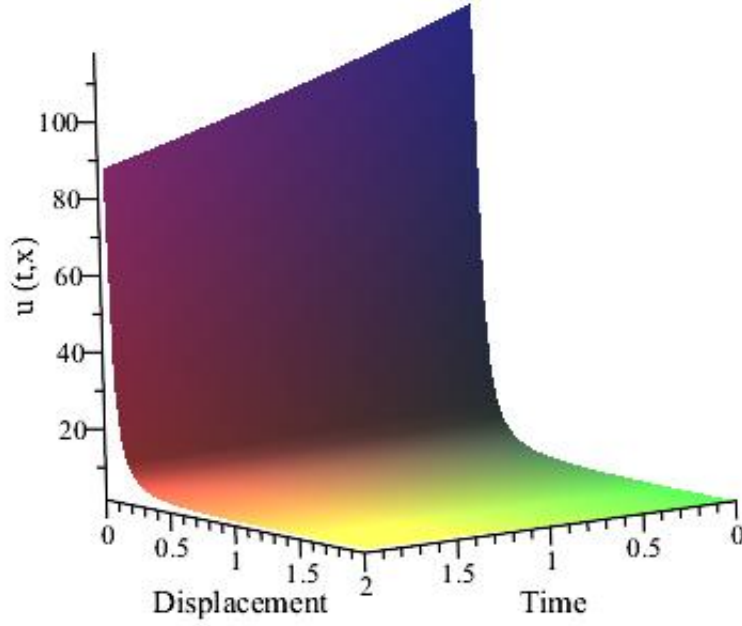


Figure 6.2: Surface plane of $u(t, x)$ reduced by X_2

this, one uses the definition of a conservation law,

$$\begin{aligned}
 D_i T^i &= 0, \\
 D_1 T^1 + D_2 T^2 &= 0, \\
 D_t(u) + D_x(-x^2 u_x) &= 0, \\
 u_t - 2x u_x - x^2 u_{xx} &= 0, \\
 u_t &= (x^2 u_x)_x.
 \end{aligned}$$

Working from the definition, one arrives to the governing equation. Therefore satisfying all necessary conditions.

Notice that the surface planes depicted by equations (6.66) and (6.70) are similar. The main difference being that along the x -axis the gradient is steeper in Figure 6.1 than in Figure 6.2.

Chapter 7

Summary

This dissertation analysed a reaction-diffusion equation that describes the propagation of the recessive advantageous gene over space and time. There are many variations of Fisher's equation that have been analysed. Fisher's equation can be derived using the continuous and the discrete method.

In the dissertation, the governing equation has a cubic source term with the source term and the diffusivity term given as a quadratic function of space. The coefficient of the source term is given by the power law. The aim was to find the classical Lie point symmetries, nonclassical symmetries and conserved vectors and their associated Lie point symmetries. Enabled with the symmetries, finding exact solutions was possible and graphic representations could be used to determine the space propagation properties over space and time.

In Chapter 4 three Lie point symmetries were found. The linear combination of two of the symmetries was used to find exact solutions. In the exact solutions found, it can be seen that the population frequency is inversely proportional to space and increasing the magnitude of the cubic function's coefficient aggravated the decay of the population

frequency. It would be important to determine the steady state solutions.

In Chapter 5 the exact solution was obtained using the genuine nonclassical symmetry. The biggest discovery that was found was being able to modify the Hopf-Cole transformation to better suit the environment. Since the coefficient of the source term was nonlinear, the Hopf-Cole transformation failed to fully simplify.

The coefficient of the source term was generalised, and hence given by the power law the exact solution was possible to construct. Noting the structure of the solution given one can see that it has similar traits to that of the solutions found in Chapter 4. From the solution, one can see that the population frequency is inversely proportional to the squared root of the more general $g(x)$.

In Chapter 6 the conservation laws and their corresponding Lie point symmetries were found. It appeared that the conservation laws exist only when $a = 0$. The corresponding Lie point symmetries were used to construct the exact solutions. The graphical representation of the solutions were similar in nature to that of the other cases. In the first reduction one can see that the population frequency is inversely proportional to the spatial coordinate.

Thorough investigation has been done on the governing equation and as a result an improvement on the Hopf-Cole transformation has been made. Hopefully this will lead to better solutions found in the future.

Appendix A

Definitions and Theorems

Definition [34] Bäcklund transform is typically a system of first order PDEs relating two functions and often depending on an additional parameter. It implies that the two functions separately satisfy the PDEs and each of the two functions is then said to be a Bäcklund transform of the other.

Theorem [35] Noether's first theorem states that any differential symmetry of a physical system has a corresponding conservation law.

Definition [36] Genotype is the genetic makeup of a cell, an organism or an individual usually with reference to a specific characteristic under consideration.

Definition [37] Allele is one of a number of alternative forms of the same gene or the same genetic locus. It is the alternative form of a gene for a character producing different effects.

Definition [38] Locus (in genetics), plural loci, is the specific location of a gene or DNA sequence or position on a chromosome.

Definition [39] Phenotype is the composite of organisms observable characteristics or traits, such as morphology, development, biochemical or physiological properties.

Definition [22] Homozygote possess two identical alleles for a given trait.

Definition [22] Heterozygote possess two different alleles for a given trait.

Definition [22] Dominant allele are allele fully expressed by phenotype.

Definition [22] Recessive allele are allele completely masked in the phenotype.

Appendix B

Step-by-step calculation for the example done in Chapter 3 to find nonclassical symmetries

The total derivative with respect to x and t , respectively, are given below,

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + u_{xxx} \frac{\partial}{\partial u_{xx}} + \dots, \quad (\text{B.1})$$

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{xt} \frac{\partial}{\partial u_x} + \dots \quad (\text{B.2})$$

For the governing equation,

$$u_t = (k(x)u_x)_x - M^2 u^{n+1}, \quad 0 \leq x \leq 1. \quad (\text{B.3})$$

The infinitesimal criterion for invariance is given by,

$$X^{[2]}eqn(B.3)|_{eqn(B.3),ISC} = 0. \quad (\text{B.4})$$

The second prolongation is defined by,

$$\begin{aligned}
X^{[2]} = & \xi^1(t, x, u) \frac{\partial}{\partial t} + \xi^2(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u} + \zeta_x(t, x, u) \frac{\partial}{\partial u_x} \\
& + \zeta_t(t, x, u) \frac{\partial}{\partial u_t} + \zeta_{xx}(t, x, u) \frac{\partial}{\partial u_{xx}} + \zeta_{xt}(t, x, u) \frac{\partial}{\partial u_{xt}} + \zeta_{tt}(t, x, u) \frac{\partial}{\partial u_{tt}} \\
& + \dots
\end{aligned} \tag{B.5}$$

where the partial derivatives of ζ can be calculated by,

$$\begin{aligned}
\zeta_x(t, x, u) &= D_x[\eta(t, x, u)] - u_t D_x[\xi^1(t, x, u)] - u_x D_x[\xi^2(t, x, u)], \\
\zeta_t(t, x, u) &= D_t[\eta(t, x, u)] - u_t D_t[\xi^1(t, x, u)] - u_x D_t[\xi^2(t, x, u)], \\
\zeta_{xx}(t, x, u) &= D_x[\zeta_x(t, x, u)] - u_{xt} D_x[\xi^1(t, x, u)] - u_{xx} D_x[\xi^2(t, x, u)], \\
\zeta_{xt}(t, x, u) &= D_t[\zeta_x(t, x, u)] - u_{xt} D_t[\xi^1(t, x, u)] - u_{xx} D_t[\xi^2(t, x, u)], \\
\zeta_{tt}(t, x, u) &= D_t[\zeta_t(t, x, u)] - u_{tt} D_t[\xi^1(t, x, u)] - u_{xt} D_t[\xi^2(t, x, u)].
\end{aligned}$$

Here the ISC is given by,

$$\xi^1 u_t + \xi^2 u_x = \eta, \tag{B.6}$$

$$u_t = \eta - \xi^2 u_x. \tag{B.7}$$

Without loss of generality, it is assumed $\xi^1 = 1$. Furthermore,

$$u_{tx} = D_x[\eta - \xi^2 u_x], \tag{B.8}$$

Let $n = 2$, then expanding and rearranging the governing equation (B.3) gives,

$$u_t = k(x) u_{xx} + u_x \frac{dk(x)}{dx} - M^2 u^3,$$

$$u_{xx} = (u_t - u_x \frac{dk(x)}{dx} + M^2 u^3)/k(x). \quad (\text{B.9})$$

At every instance where u_{xx} appears in the infinitesimal criterion, equation (B.9) will be substituted in.

The infinitesimal criterion for invariance is given by,

$$\begin{aligned} 0 &= X^{[2]}eqn(B.3)|_{eqn(B.3),ISC}, \\ 0 &= k(x) M^2 u^2 \eta(t, x, u) - k(x) M^2 u^3 \frac{\partial}{\partial u} \eta(t, x, u) + 2 k(x) M^2 u^3 \frac{\partial}{\partial x} \xi^2(t, x, u) \\ &\quad - 2 k(x) u_x^2 k'(x) \frac{\partial}{\partial u} \xi^2(t, x, u) - 2 k(x) u_x^2 \xi^2(t, x, u) \frac{\partial}{\partial u} \xi^2(t, x, u) \\ &\quad - k(x) u_x k'(x) \frac{\partial}{\partial x} \xi^2(t, x, u) + 2 k(x) u_x \eta(t, x, u) \frac{\partial}{\partial u} \xi^2(t, x, u) \\ &\quad - 2 k(x) u_x \xi^2(t, x, u) \frac{\partial}{\partial x} \xi^2(t, x, u) - M^2 k'(x) u^3 \xi^2(t, x, u) \\ &\quad - k(x) u_x k'(x) \xi^2(t, x, u) + k(x) \frac{\partial}{\partial t} \eta(t, x, u) \\ &\quad + (k(x))^2 u_x^3 \frac{\partial^2}{\partial u^2} \xi^2(t, x, u) + 2 (k(x))^2 u_x^2 \frac{\partial^2}{\partial u \partial x} \xi^2(t, x, u) \\ &\quad - (k(x))^2 u_x^2 \frac{\partial^2}{\partial u^2} \eta(t, x, u) - 2 (k(x))^2 u_x \frac{\partial^2}{\partial u \partial x} \eta(t, x, u) \\ &\quad + (k(x))^2 u_x \frac{\partial^2}{\partial x^2} \xi^2(t, x, u) + 2 k(x) \eta(t, x, u) \frac{\partial}{\partial x} \xi^2(t, x, u) \\ &\quad + (k'(x))^2 u_x \xi^2(t, x, u) + k'(x) u_x (\xi^2(t, x, u))^2 \\ &\quad - k'(x) \eta(t, x, u) \xi^2(t, x, u) - k(x) k'(x) \frac{\partial}{\partial x} \eta(t, x, u) \\ &\quad - k(x) u_x \frac{\partial}{\partial t} \xi^2(t, x, u) + 3 k(x) M^2 u^3 u_x \frac{\partial}{\partial u} \xi^2(t, x, u) \\ &\quad - (k(x))^2 \frac{\partial^2}{\partial x^2} \eta(t, x, u). \end{aligned} \quad (\text{B.10})$$

Then, one separates equation (B.10) with respect to the powers of u_x as follows:

u_x^0 :

$$\begin{aligned}
0 = & k(x) M^2 u^2 \eta(t, x, u) - k(x) M^2 u^3 \frac{\partial}{\partial u} \eta(t, x, u) + 2 k(x) M^2 u^3 \frac{\partial}{\partial x} \xi^2(t, x, u) \\
& - M^2 k'(x) u^3 \xi^2(t, x, u) + k(x) \frac{\partial}{\partial t} \eta(t, x, u) - (k(x))^2 \frac{\partial^2}{\partial x^2} \eta(t, x, u) \\
& + 2 k(x) \eta(t, x, u) \frac{\partial}{\partial x} \xi^2(t, x, u) - k'(x) \eta(t, x, u) \xi^2(t, x, u) \\
& - k(x) k'(x) \frac{\partial}{\partial x} \eta(t, x, u),
\end{aligned} \tag{B.11}$$

u_x^1 :

$$\begin{aligned}
0 = & -k(x) k'(x) \frac{\partial}{\partial x} \xi^2(t, x, u) + 2 k(x) \eta(t, x, u) \frac{\partial}{\partial u} \xi^2(t, x, u) \\
& - 2 k(x) \xi^2(t, x, u) \frac{\partial}{\partial x} \xi^2(t, x, u) - k(x) k''(x) \xi^2(t, x, u) \\
& + (k(x))^2 \frac{\partial^2}{\partial x^2} \xi^2(t, x, u) + (k'(x))^2 \xi^2(t, x, u) + k'(x) (\xi^2(t, x, u))^2 \\
& - k(x) \frac{\partial}{\partial t} \xi^2(t, x, u) + 3 k(x) M^2 u^3 \frac{\partial}{\partial u} \xi^2(t, x, u) \\
& - 2 (k(x))^2 \frac{\partial^2}{\partial u \partial x} \eta(t, x, u),
\end{aligned} \tag{B.12}$$

u_x^2 :

$$\begin{aligned}
0 = & -2 k(x) k'(x) \frac{\partial}{\partial u} \xi^2(t, x, u) - 2 k(x) \xi^2(t, x, u) \frac{\partial}{\partial u} \xi^2(t, x, u) \\
& + 2 (k(x))^2 \frac{\partial^2}{\partial u \partial x} \xi^2(t, x, u) - (k(x))^2 \frac{\partial^2}{\partial u^2} \eta(t, x, u),
\end{aligned} \tag{B.13}$$

u_x^3 :

$$0 = (k(x))^2 \frac{\partial^2}{\partial u^2} \xi^2(t, x, u). \tag{B.14}$$

From equation (B.14),

$$\frac{\partial^2}{\partial u^2} \xi^2(t, x, u) = 0, \quad (\text{B.15})$$

$$\Rightarrow \xi^2(t, x, u) = f_1(t, x)u + f_2(t, x). \quad (\text{B.16})$$

If one takes equation (B.16) and substitutes it into the right hand side of equation (B.14), equation (B.14) is identically satisfied.

To simplify equations (B.11), (B.12) and (B.13), one substitutes equation (B.16) into them and they become,

u_x^0 :

$$\begin{aligned} 0 = & 3k(x)M^2u^2\eta(t, x, u) - k(x)M^2u^3\frac{\partial}{\partial u}\eta(t, x, u) \\ & + 2k(x)(M^2u^3 + \eta(t, x, u)) \left[\left(\frac{\partial}{\partial x} f_1(t, x) \right) u + \frac{\partial}{\partial x} f_2(t, x) \right] \\ & - k'(x)(M^2u^3 + \eta(t, x, u))(f_1(t, x)u + f_2(t, x)) \\ & + k(x)\frac{\partial}{\partial t}\eta(t, x, u) - (k(x))^2\frac{\partial^2}{\partial x^2}\eta(t, x, u) \\ & - k(x)k'(x)\frac{\partial}{\partial x}\eta(t, x, u), \end{aligned} \quad (\text{B.17})$$

u_x^1 :

$$\begin{aligned}
0 = & -k(x) k'(x) \left[\left(\frac{\partial}{\partial x} f_1(t, x) \right) u + \frac{\partial}{\partial x} f_2(t, x) \right] + 2k(x) \eta(t, x, u) f_1(t, x) \\
& - 2k(x) (f_1(t, x) u + f_2(t, x)) \left[\left(\frac{\partial}{\partial x} f_1(t, x) \right) u + \frac{\partial}{\partial x} f_2(t, x) \right] \\
& - k(x) k''(x) (f_1(t, x) u + f_2(t, x)) - 2(k(x))^2 \frac{\partial^2}{\partial u \partial x} \eta(t, x, u) \\
& + (k(x))^2 \left[\left(\frac{\partial^2}{\partial x^2} f_1(t, x) \right) u + \frac{\partial^2}{\partial x^2} f_2(t, x) \right] + (k'(x))^2 (f_1(t, x) u + f_2(t, x)) \\
& + k(x) (f_1(t, x) u + f_2(t, x))^2 - k(x) \left[\left(\frac{\partial}{\partial t} f_1(t, x) \right) u + \frac{\partial}{\partial t} f_2(t, x) \right] \\
& + 3k(x) M^2 u^3 f_1(t, x),
\end{aligned} \tag{B.18}$$

u_x^2 :

$$\begin{aligned}
0 = & -2k(x) k'(x) f_1(t, x) - 2k(x) (f_1(t, x) u + f_2(t, x)) f_1(t, x) \\
& + 2(k(x))^2 \frac{\partial}{\partial x} f_1(t, x) - (k(x))^2 \frac{\partial^2}{\partial u^2} \eta(t, x, u).
\end{aligned} \tag{B.19}$$

One can observe that η can be solved using equation (B.19) in terms of f_1 , f_2 and u , to obtain,

$$\begin{aligned}
\eta(t, x, u) = & -\frac{1}{3} \frac{(f_1(t, x))^2 u^3}{k(x)} + \left(\frac{\partial}{\partial x} f_1(t, x) \right) u^2 \\
& - \frac{f_1(t, x) f_2(t, x) u^2}{k(x)} - \frac{f_1(t, x) \left(\frac{d}{dx} k(x) \right) u^2}{k(x)} + f_3(t, x) u \\
& + f_4(t, x).
\end{aligned} \tag{B.20}$$

Summary:

$$\xi^1 = 1, \quad (B.21)$$

$$\xi^2(t, x, u) = f_1(t, x)u + f_2(t, x), \quad (B.22)$$

$$\begin{aligned} \eta(t, x, u) = & -\frac{1}{3} \frac{(f_1(t, x))^2 u^3}{k(x)} + \left(\frac{\partial}{\partial x} f_1(t, x) \right) u^2 \\ & - \frac{f_1(t, x) f_2(t, x) u^2}{k(x)} - \frac{f_1(t, x) \left(\frac{d}{dx} k(x) \right) u^2}{k(x)} + f_3(t, x) u \\ & + f_4(t, x). \end{aligned} \quad (B.23)$$

Now, to simplify equations (B.17) and (B.18), one substitutes equation (B.20) into them, becoming,

u_x^0 :

$$\begin{aligned} 0 = & \frac{1}{3k(x)} [3 f_1(t, x) k'''(x) u^2 (k(x))^2 - (f_1(t, x))^2 u^3 k(x) k''(x) \\ & + 2 (k(x))^2 f_1(t, x) u^3 \frac{\partial^2}{\partial x^2} f_1(t, x) + 3 (k(x))^2 u^2 f_2(t, x) \frac{\partial^2}{\partial x^2} f_1(t, x) \\ & + 3 f_1(t, x) u^2 (k(x))^2 \frac{\partial^2}{\partial x^2} f_2(t, x) + 6 k''(x) u^2 (k(x))^2 \frac{\partial}{\partial x} f_1(t, x) \\ & + 12 u^2 (k(x))^2 \left(\frac{\partial}{\partial x} f_1(t, x) \right) \left(\frac{\partial}{\partial x} f_2(t, x) \right) - 3 k(x) (k'(x))^2 u^2 \frac{\partial}{\partial x} f_1(t, x) \quad (B.24) \\ & + 6 M^2 (k(x))^2 u^3 \frac{\partial}{\partial x} f_2(t, x) + 6 M^2 f_3(t, x) (k(x))^2 u^3 \\ & + 9 M^2 (k(x))^2 u^2 f_4(t, x) - 3 (k(x))^2 k'(x) u \frac{\partial}{\partial x} f_3(t, x) \\ & - 2 k(x) u^4 \left(\frac{\partial}{\partial x} f_1(t, x) \right) (f_1(t, x))^2 - 2 (f_1(t, x))^2 k(x) u^3 \left(\frac{\partial}{\partial x} f_2(t, x) \right) \end{aligned}$$

$$\begin{aligned}
& + 6 (k(x))^2 u^2 f_3(t, x) \frac{\partial}{\partial x} f_1(t, x) + 6 (k(x))^2 u f_3(t, x) \frac{\partial}{\partial x} f_2(t, x) \\
& + 3 f_1(t, x) k'(x)^3 u^2 + 6 f_4(t, x) (k(x))^2 \frac{\partial}{\partial x} f_2(t, x) \\
& + 4 (f_1(t, x))^2 u^3 k'(x)^2 + 8 u^3 (k(x))^2 \left(\frac{\partial}{\partial x} f_1(t, x) \right)^2 \\
& + (f_1(t, x))^3 k'(x) u^4 + 3 (k(x))^2 u^2 \frac{\partial^2}{\partial t \partial x} f_1(t, x) \\
& - 3 k'(x) (k(x))^2 \frac{\partial}{\partial x} f_4(t, x) - 3 u (k(x))^3 \frac{\partial^2}{\partial x^2} f_3(t, x) \\
& + 3 (k(x))^2 u \frac{\partial}{\partial t} f_3(t, x) - 3 u^2 (k(x))^3 \frac{\partial^3}{\partial x^3} f_1(t, x) \\
& - 3 (k(x))^3 \frac{\partial^2}{\partial x^2} f_4(t, x) + 3 (k(x))^2 \frac{\partial}{\partial t} f_4(t, x) \\
& - 6 f_1(t, x) f_2(t, x) k(x) u^2 \frac{\partial}{\partial x} f_2(t, x) - 3 f_3(t, x) f_1(t, x) k'(x) k(x) u^2 \\
& - 3 f_3(t, x) f_2(t, x) k'(x) k(x) u - 3 f_4(t, x) f_1(t, x) k'(x) k(x) u \\
& - 3 M^2 f_1(t, x) f_2(t, x) k(x) u^4 - 11 f_1(t, x) u^3 k'(x) k(x) \frac{\partial}{\partial x} f_1(t, x) \\
& - 6 f_2(t, x) u^2 k'(x) k(x) \frac{\partial}{\partial x} f_1(t, x) - 3 f_1(t, x) f_2(t, x) u^2 k(x) k''(x) \\
& - 9 f_1(t, x) u^2 k'(x) k(x) \frac{\partial}{\partial x} f_2(t, x) - 3 M^2 f_2(t, x) k'(x) k(x) u^3 \\
& - 6 M^2 f_1(t, x) k'(x) k(x) u^4 - 6 f_1(t, x) k'(x) u^2 k''(x) k(x) \\
& - 6 f_1(t, x) f_2(t, x) k(x) u^3 \frac{\partial}{\partial x} f_1(t, x) + 6 f_4(t, x) (k(x))^2 u \frac{\partial}{\partial x} f_1(t, x) \\
& + 4 (f_1(t, x))^2 f_2(t, x) k'(x) u^3 + 3 f_1(t, x) (f_2(t, x))^2 k'(x) u^2 \\
& - 3 f_4(t, x) f_2(t, x) k'(x) k(x) - 2 f_1(t, x) k(x) u^3 \frac{\partial}{\partial t} f_1(t, x) \\
& - 3 f_2(t, x) k(x) u^2 \frac{\partial}{\partial t} f_1(t, x) - 3 k'(x) k(x) u^2 \frac{\partial}{\partial t} f_1(t, x) \\
& - 3 f_1(t, x) k(x) u^2 \frac{\partial}{\partial t} f_2(t, x) + 9 M^2 (k(x))^2 u^4 \frac{\partial}{\partial x} f_1(t, x) \\
& + 6 f_1(t, x) f_2(t, x) u^2 k'(x)^2],
\end{aligned}$$

u_x^1 :

$$\begin{aligned}
0 = & 3k(x)M^2f_1(t,x)u^3 - \frac{2}{3}(f_1(t,x))^3u^3 + 4f_1(t,x)u^2k(x)\frac{\partial}{\partial x}f_1(t,x) \\
& - 2(f_1(t,x))^2f_2(t,x)u^2 - 3(f_1(t,x))^2u^2k'(x) + 2f_1(t,x)uk(x)\frac{\partial}{\partial x}f_2(t,x) \\
& + 2f_3(t,x)f_1(t,x)k(x)u - 3u(k(x))^2\frac{\partial^2}{\partial x^2}f_1(t,x) \\
& + 2f_2(t,x)uk(x)\frac{\partial}{\partial x}f_1(t,x) + 3k'(x)uk(x)\frac{\partial}{\partial x}f_1(t,x) \\
& - 2f_1(t,x)f_2(t,x)uk'(x) + 3f_1(t,x)k''(x)uk(x) - 3f_1(t,x)k'(x)^2u \\
& - 2f_2(t,x)k(x)\frac{\partial}{\partial x}f_2(t,x) - k'(x)k(x)\frac{\partial}{\partial x}f_2(t,x) \\
& + 2f_4(t,x)f_1(t,x)k(x) - 2(k(x))^2\frac{\partial}{\partial x}f_3(t,x) + (k(x))^2\frac{\partial^2}{\partial x^2}f_2(t,x) \\
& - k(x)u\frac{\partial}{\partial t}f_1(t,x) + (f_2(t,x))^2k'(x) - f_2(t,x)k''(x)k(x) \\
& + f_2(t,x)k'(x)^2 - k(x)\frac{\partial}{\partial t}f_2(t,x).
\end{aligned} \tag{B.25}$$

To solve for the undetermined functions, f_1 to f_4 , one can separate equation (B.25) in powers of u .

u^0 from equation(B.25):

$$\begin{aligned}
0 = & -k'(x)k(x)\frac{\partial}{\partial x}f_2(t,x) + 2f_4(t,x)f_1(t,x)k(x) \\
& - 2f_2(t,x)k(x)\frac{\partial}{\partial x}f_2(t,x) - f_2(t,x)k''(x)k(x) - 2(k(x))^2\frac{\partial}{\partial x}f_3(t,x) \\
& + (k(x))^2\frac{\partial^2}{\partial x^2}f_2(t,x) + f_2(t,x)k'(x)^2 + (f_2(t,x))^2k'(x) \\
& - k(x)\frac{\partial}{\partial t}f_2(t,x),
\end{aligned} \tag{B.26}$$

u^1 from equation (B.25):

$$\begin{aligned}
0 = & -k(x)k'(x)\frac{\partial}{\partial x}f_1(t,x) + 2k(x)f_3(t,x)f_1(t,x) \\
& - 2k(x)f_2(t,x)\frac{\partial}{\partial x}f_1(t,x) - 2k(x)f_1(t,x)\frac{\partial}{\partial x}f_2(t,x) - k''(x)f_1(t,x)k(x) \\
& - 2(k(x))^2\left(2\frac{\partial^2}{\partial x^2}f_1(t,x) + 2\frac{f_1(t,x)f_2(t,x)k'(x)}{(k(x))^2} - 2\frac{\left(\frac{\partial}{\partial x}f_1(t,x)\right)f_2(t,x)}{k(x)}\right. \\
& \left.- 2\frac{f_1(t,x)\frac{\partial}{\partial x}f_2(t,x)}{k(x)} + 2\frac{f_1(t,x)k'(x)^2}{(k(x))^2} - 2\frac{\left(\frac{\partial}{\partial x}f_1(t,x)\right)k'(x)}{k(x)}\right. \\
& \left.- 2\frac{f_1(t,x)\frac{d^2}{dx^2}k(x)}{k(x)}\right) + (k(x))^2\frac{\partial^2}{\partial x^2}f_1(t,x) \\
& + f_1(t,x)k'(x)^2 + 2f_1(t,x)f_2(t,x)k'(x) - k(x)\frac{\partial}{\partial t}f_1(t,x),
\end{aligned} \tag{B.27}$$

u^2 from equation (B.25):

$$\begin{aligned}
0 = & 2k(x)\left(\frac{\partial}{\partial x}f_1(t,x) - \frac{f_1(t,x)f_2(t,x)}{k(x)} - \frac{k'(x)f_1(t,x)}{k(x)}\right)f_1(t,x) \\
& - 2k(x)f_1(t,x)\frac{\partial}{\partial x}f_1(t,x) + (f_1(t,x))^2k'(x) \\
& - 2\left(\frac{(f_1(t,x))^2k'(x)}{(k(x))^2} - 2\frac{f_1(t,x)\frac{\partial}{\partial x}f_1(t,x)}{k(x)}\right)(k(x))^2,
\end{aligned} \tag{B.28}$$

u^3 from equation (B.25):

$$0 = -\frac{2}{3}(f_1(t,x))^3 + 3k(x)M^2f_1(t,x), \tag{B.29}$$

From equation (B.29) one can solve for $f_1(t,x)$,

$$0 = \left((f_1(t,x))^2 - \frac{9}{2}k(x)M^2\right)f_1(t,x). \tag{B.30}$$

So the possibly solutions for $f_1(t, x)$,

$$f_1(t, x) = 0, \quad (\text{B.31})$$

$$f_1(t, x) = \pm \sqrt{\frac{9}{2} M^2 k(x)}. \quad (\text{B.32})$$

Set,

$$f_1(t, x) = \sqrt{\frac{9}{2} M^2 k(x)}. \quad (\text{B.33})$$

To simplify equations (B.26), (B.27) and (B.28), equation (B.33) is substituted into them.

u^0 from equation (B.25):

$$\begin{aligned} 0 = & -k'(x) k(x) \frac{\partial}{\partial x} f_2(t, x) + 3 f_4(t, x) \sqrt{2} \sqrt{k(x) M^2} k(x) \\ & - 2 f_2(t, x) k(x) \frac{\partial}{\partial x} f_2(t, x) - f_2(t, x) k''(x) k(x) - 2 (k(x))^2 \frac{\partial}{\partial x} f_3(t, x) \\ & + (k(x))^2 \frac{\partial^2}{\partial x^2} f_2(t, x) + f_2(t, x) k'(x)^2 + (f_2(t, x))^2 k'(x) \\ & - k(x) \frac{\partial}{\partial t} f_2(t, x), \end{aligned} \quad (\text{B.34})$$

u^1 from equation (B.25):

$$\begin{aligned} 0 = & \frac{3\sqrt{2}}{8} M \sqrt{k(x)} \left(8 k(x) \frac{\partial}{\partial x} f_2(t, x) + 8 f_3(t, x) k(x) - 4 f_2(t, x) k'(x) \right. \\ & \left. + 6 k''(x) k(x) - 3 k'(x)^2 \right), \end{aligned} \quad (\text{B.35})$$

u^2 from equation (B.25):

$$0 = -\frac{9}{2} k(x) M^2 (k'(x) + 2 f_2(t, x)). \quad (\text{B.36})$$

From equation (B.36), $f_2(t, x)$ can be solved,

$$f_2(t, x) = -\frac{1}{2}k'(x). \quad (\text{B.37})$$

To simplify equations (B.34) and (B.35), equation (B.37) is substituted into them.

u^0 from equation (B.25):

$$\begin{aligned} 0 = & \frac{1}{2}k''(x)k'(x)k(x) + 3f_4(t, x)\sqrt{2}\sqrt{k(x)M^2}k(x) \\ & - 2\left(\frac{\partial}{\partial x}f_3(t, x)\right)(k(x))^2 - \frac{1}{2}k'''(x)(k(x))^2 - \frac{1}{4}k'(x)^3, \end{aligned} \quad (\text{B.38})$$

u^1 from equation (B.25):

$$0 = \frac{3\sqrt{2}}{8}M\sqrt{k(x)}\left(8f_3(t, x)k(x) + 2k''(x)k(x) - k'(x)^2\right). \quad (\text{B.39})$$

From equation (B.39), $f_3(t, x)$ can be solved,

$$f_3(t, x) = \frac{1}{8k(x)}\left(k'(x)^2 - 2k''(x)k(x)\right). \quad (\text{B.40})$$

Substituting equation (B.40) into the right hand side of equation (B.39), equation (B.39) reduces to zero. To simplify equations (B.38) equation (B.40) is substituted into it.

u^0 from equation (B.25):

$$0 = 3\sqrt{2}\sqrt{k(x)M^2}f_4(t, x)k(x). \quad (\text{B.41})$$

From equation (B.41), $f_4(t, x)$ can be solved,

$$f_4(t, x) = 0. \quad (\text{B.42})$$

Summary:

$$\xi^1 = 1, \quad (\text{B.43})$$

$$\xi^2(t, x, u) = f_1(t, x)u + f_2(t, x), \quad (\text{B.44})$$

$$\begin{aligned} \eta(t, x, u) = & -\frac{1}{3} \frac{(f_1(t, x))^2 u^3}{k(x)} + u^2 \frac{\partial}{\partial x} f_1(t, x) \\ & - \frac{f_1(t, x) f_2(t, x) u^2}{k(x)} - \frac{f_1(t, x) k'(x) u^2}{k(x)} + f_3(t, x) u \\ & + f_4(t, x), \end{aligned} \quad (\text{B.45})$$

$$\text{where,} \quad (\text{B.46})$$

$$f_1(t, x) = \sqrt{\frac{9}{2} M^2 k(x)}, \quad (\text{B.47})$$

$$f_2(t, x) = -\frac{1}{2} k'(x), \quad (\text{B.48})$$

$$f_3(t, x) = \frac{1}{8k(x)} (k'(x)^2 - 2k''(x)k(x)), \quad (\text{B.49})$$

$$f_4(t, x) = 0, \quad (\text{B.50})$$

When taking equations (B.44) - (B.50), and substituting them into the right hand side of equation (B.25), equation (B.25) reduces to zero. Then substituting equations (B.44) - (B.50) to simplify equation (B.24), one obtains,

u_x^0 :

$$0 = -\frac{1}{16k(x)} u \left[4k^{(4)}(x) (k(x))^3 + 2k''(x) (k'(x))^2 k(x) - (k'(x))^4 \right], \quad (\text{B.51})$$

From equation (B.51), one can see that $k(x)$ must satisfy the fourth order nonlinear ODE,

$$0 = 4 \left(\frac{d^4}{dx^4} k(x) \right) (k(x))^3 + 2 \left(\frac{d^2}{dx^2} k(x) \right) \left(\frac{d}{dx} k(x) \right)^2 k(x) - \left(\frac{d}{dx} k(x) \right)^4. \quad (\text{B.52})$$

There are a number of special solutions that satisfy equation (B.52) [1]. Therefore, for the specific case where $k(x)$ [1] is chosen to be,

$$k(x) = (\alpha + \beta x)^2. \quad (\text{B.53})$$

Finally, one can simplify ξ^1 , ξ^2 and η .

Summary:

$$\xi^1 = 1, \quad (\text{B.54})$$

$$\xi^2(t, x, u) = (\alpha + \beta x) \left(\frac{3M}{\sqrt{2}} u - \beta \right), \quad (\text{B.55})$$

$$\eta(t, x, u) = -\frac{3}{2} M^2 u^3 \quad (\text{B.56})$$

Hence, the nonlinear partial differential equation,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left[(\alpha + \beta x) \left(\frac{3M}{\sqrt{2}} u - \beta \right) \frac{\partial u}{\partial x} \right] - M^2 u^3, \quad (\text{B.57})$$

admits the following nonclassical symmetry generator,

$$X = \frac{\partial}{\partial t} + (\alpha + \beta x) \left(\frac{3M}{\sqrt{2}} u - \beta \right) \frac{\partial}{\partial x} - \frac{3}{2} M^2 u^3 \frac{\partial}{\partial u}. \quad (\text{B.58})$$

The associated invariant surface condition is given by

$$\frac{\partial u}{\partial t} + (\alpha + \beta x) \left(\frac{3M}{\sqrt{2}} u - \beta \right) \frac{\partial u}{\partial x} = -\frac{3}{2} M^2 u^3. \quad (\text{B.59})$$

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