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Evaluating determinants of treatment outcomes among tuberculosis patients in the mining district of Butha Buthe Lesotho

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On a family of higher order recurrence relations: symmetries, formula solutions, periodicity and stability analysis

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Abstract In this paper, we present formula solutions of a family of difference equations of higher order. We discuss the periodic nature of the solutions and we investigate the stability character of the equilibrium points. We utilize Lie symmetry analysis as part of our approach together with some number theoretic functions. Our findings generalize certain results in the literature.

Mathematics Subject Classification 39A10 · 39A13 · 39A99

1 Introduction

There has been great attention and focus on difference equations. Just like differential equations, there are techniques that one can use to solve difference equations. One method for solving them is to use Lie symmetry analysis. In this approach, one finds an invariant which can be used to find a simpler form of the equation. Amongst the first people to use Lie symmetry analysis to solve difference equations are Maeda [12] and Hydon [9]. Examples of the use of difference equations in real-life include modeling the proliferation of disease, loan payments, population studies, etc [4,11].

Recurrence equations of a general order have been investigated in the literature and from different approaches by several authors [1–3,5,6,8]. In their work, Almatrafi and others [1] discussed the exact solutions, stability, oscillation and periodic aspects of the the difference equations

$$\eta_{n+1} = \frac{\eta_{n-4k+1}}{\pm 1 \pm \prod_{i=1}^{k} \eta_{n-4i+1}}. \tag{1}$$

These equations are special cases of a more generalized setting

$$\eta_{n+1} = \frac{\eta_{n-4k+1}}{a_n + b_n \prod_{i=1}^{k} \eta_{n-4i+1}}, \tag{2}$$

where a_n and b_n are arbitrary real sequences. One can readily see that with a suitable change of variables, the above equations can be derived from the higher order Berverton–Holt difference [3] equation

$$r_{n+l} = \frac{\mu_n K_n r_n}{K_n + (\mu_n - 1) r_n}, \tag{3}$$

where $\mu_n > 1$ denotes the growth rate, K_n is the carrying capacity and $r_0, r_1, \ldots, r_{l-1}$ are the positive initial values.

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In this paper, we perform a symmetry analysis of the equivalent equation

$$u_{n+4k} = \frac{u_n}{A_n + B_n \prod_{i=1}^{k} u_{n+4(i-1)}}, \tag{4}$$

for some arbitrary real sequences A_n and B_n where $u_0, u_1, \ldots, u_{4k-1}$ are initial values. Using symmetries, we obtain explicit formulas for the solution of (4) and we deduce the solutions of (2) from those of (4). We also study the periodic nature of the solutions and a stability analysis of the difference equation is investigated.

The derivation of symmetries for higher order recurrences involves cumbersome calculations and, to the best of our knowledge, there are no computer packages that generate these symmetries. For more on Lie symmetry analysis of difference equations, the reader can refer to [9] and, among others, the articles [6,7,13].

1.1 Preliminaries

Consider the difference equation:

$$u_{n+4k} = \mathcal{G}(n, u_n, u_{n+4}, u_{n+8} \ldots, u_{n+4k-4}), \tag{5}$$

for some smooth function $\mathcal{G}$ satisfying $\partial\mathcal{G}/\partial u_n \neq 0$. Symmetry groups are connected to the determination of infinitesimal transformations. Let

$$\hat{u}_n = u_n + \varepsilon Q(n, u_n) + O(\varepsilon^2), \tag{6}$$

be the one parameter Lie group of transformations of (5) with the corresponding generator

$$\mathcal{V} = Q(n, u_n)\frac{\partial}{\partial u_n}. \tag{7}$$

Note that the knowledge of the characteristic $Q = Q(n, u_n)$ requires the knowledge of the $(4k-4)$-th prolongation of $\mathcal{V}$:

$$\mathcal{V}^{[4k-4]} = Q\frac{\partial}{\partial u_n} + (S^4 Q)\frac{\partial}{\partial u_{n+4}} + \cdots + (S^{4k-4} Q)\frac{\partial}{\partial u_{n+4k-4}}, \tag{8}$$

where $S^i : n \to n+i$ is the forward shift operator. It is known that (6) is a symmetry group if and only if the condition

$$S^{4k} Q(n, u_n) - \mathcal{V}^{[4k-4]}(\mathcal{G}) = 0\Big|_{u_{n+5k}=\mathcal{G}(n,u_n,u_{n+4},\ldots,u_{n+4k-4})} \tag{9}$$

holds. Suppose that the characteristic is obtained by solving the functional equation (9). One can use the canonical coordinate [10]

$$c_n = \int \frac{\mathrm{d}u_n}{Q(n, u_n)} \tag{10}$$

to derive the invariants which may be used to lower the order of the difference equations. In [9], the author attests that with the choice of canonical coordinate (10), the recurrence equation can, without fail, be represented in the form $c_{n+1} - c_n = d_n$ whose solution takes the shape

$$c_n = \sum_{k=n_0}^{n} d_k + w_1 \tag{11}$$

for some constant w_1. From (11), it is not difficult to find the solution expressed in terms of the original variables. In this paper, our solution is obtained via the use of the canonical coordinate through a different methodology.

The following theorem and definitions [8] are useful for studying local and globally stability aspects of the equilibrium point.

Definition 1.1 The equilibrium point $\bar{u}$ of (5) is said to be locally stable if for any $\epsilon > 0$ such that if $\{u_n\}_{n=0}^{\infty}$ is a solution of (5) with

$$|u_0 - \bar{u}| + |u_1 - \bar{u}| + \cdots + |u_{4k-2} - \bar{u}| + |u_{4k-1} - \bar{u}| < \delta, \tag{12}$$

then

$$|u_n - \bar{u}| < \epsilon \quad \text{for all} \quad n \geq 0. \tag{13}$$

Definition 1.2 The equilibrium point $\bar{u}$ of (5) is said to be a global attractor if for any solution $\{u_n\}_{n=0}^{\infty}$ of (5),

$$\lim_{n \leftarrow \infty} u_n = \bar{u}. \tag{14}$$

Definition 1.3 The equilibrium point $\bar{u}$ of (5) is globally asymptotically stable if $\bar{u}$ is locally stable and is a global attractor of (5).

Let

$$p_i = \frac{\partial f}{\partial u_{n+i}}(\bar{u}, \ldots, \bar{u}), \quad i = 4r, \quad r = 0, 1, \ldots, k-1. \tag{15}$$

It follows that

$$\lambda^{4k} - p_{4k-4}\lambda^{4k-4} - \cdots - p_4\lambda^4 - p_0 = 0 \tag{16}$$

is the corresponding characteristic equation of (5) about the equilibrium point $\bar{u}$.

Theorem 1.4 *Suppose f is a smooth function defined on some open neighborhood of $\bar{u}$. Then the following statements are true:*

(i) *The equilibrium point $\bar{u}$ is locally asymptotically stable if all the roots of (16) have absolute value less than one.*
(ii) *The equilibrium point $\bar{u}$ is unstable if at least one root of (16) has absolute value greater than one.*

Definition 1.5 The equilibrium point $\bar{u}$ of (5) is called non-hyperbolic if there exists a root of (16) with absolute value equal to one.

2 Lie analysis and solutions

To derive the characteristic function Q admitted by (4), that is,

$$u_{n+4k} = \mathcal{G} = \frac{u_n}{A_n + B_n \prod_{i=1}^{k} u_{n+4(i-1)}}, \tag{17}$$

we apply the symmetry constraint equation (9) to (17) to get

$$Q(n+4k, u_{n+4k}) - \sum_{i=0}^{k-1} \mathcal{G}_{,u_{n+4i}} Q(n+4i, u_{n+4i}) = 0, \tag{18}$$

where $f_{,x}$ denotes the partial derivative of f with respect to x. To solve for Q, we first apply the differential operator $\partial/\partial u_{n+4} - (u_{n+8}/u_{n+4})\partial/\partial u_{n+8}$ on (18). This gives

$$\begin{aligned}
&\left(\frac{u_{n+8}}{u_{n+4}}\mathcal{G}_{,u_{n+8}u_{n+4}} - \mathcal{G}_{,u_{n+4}u_{n+4}}\right) Q(n+4, u_{n+4}) + \left(\frac{u_{n+8}}{u_{n+4}}\mathcal{G}_{,u_{n+8}u_{n+8}} - \mathcal{G}_{,u_{n+4}u_{n+8}}\right) Q(n+8, u_{n+8}) \\
&+ \sum_{i\geq 3}\left(\frac{u_{n+8}}{u_{n+4}}\mathcal{G}_{,u_{n+8}u_{n+4i}} - \mathcal{G}_{,u_{n+4}u_{n+4i}}\right) Q(n+4i, u_{n+4i}) - \frac{B_n u_n^2 \prod_{i=3}^{k} u_{n+4(i-1)}}{\left(A_n + B_n \prod_{i=1}^{k} u_{n+4(i-1)}\right)^2}\big(Q'(n+8, u_{n+8}) \\
&- Q'(n+4, u_{n+4})\big) + \left(\frac{u_{n+8}}{u_{n+4}}\mathcal{G}_{,u_{n+8}u_n} - \mathcal{G}_{,u_{n+4}u_n}\right) Q(n, u_n) = 0
\end{aligned} \tag{19}$$

which simplifies to

$$u_{n+8}Q(n+4,u_{n+4})-u_{n+4}Q(n+8,u_{n+8})+u_{n+4}u_{n+8}(Q'(n+8,u_{n+8})-Q'(n+4,u_{n+4}))=0. \quad (20)$$

We then differentiate the above equation with respect to u_{n+4} twice to get

$$\big(u_{n+4}Q'(n+4,u_{n+4})-Q(n+4,u_{n+4})\big)''=0. \quad (21)$$

The general solution of (21) takes the form

$$Q(n,u_n)=\alpha_n u_n+\beta_n(u_n\ln u_n+u_n)+\gamma_n \quad (22)$$

for some functions α_n, β_n and γ_n of n. Next, we substitute Q and its corresponding shifts in (18). Bearing in mind that α_n, β_n and γ_n are independent of u_n and their shifts, we use the method of separation. It turns out that β_n is equal to zero and the system of overdetermined equations resulting from the separation is as follows:

$$\begin{aligned}
1 &: A_n^2\gamma_{n+4k}-A_n\gamma_n=0\\
u_n &: A_n\alpha_{n+4k}-A_n\alpha_n=0\\
u_n^2u_{n+4}\dots u_{n+4k-4} &: B_n(\alpha_{n+4k}+\alpha_{n+4}+\alpha_{n+8}+\cdots+\alpha_{n+4k-8}+\alpha_{n+4k-4})=0\\
u_nu_{n+4}\dots u_{n+4k-4} &: 2A_nB_n\gamma_{n+4k}=0
\end{aligned} \quad (23)$$

which reduces to

$$\gamma_n=0,\quad \alpha_{n+4k}-\alpha_n=0, \quad (24)$$

$$\alpha_n+\alpha_{n+4}+\alpha_{n+8}+\cdots+\alpha_{n+4k-8}+\alpha_{n+4k-4}=0. \quad (25)$$

Equation (25) is a linear difference equation with constant coefficients and has the characteristic equation

$$1+r^4+r^8+\cdots+r^{4k-8}+r^{4k-4}=0. \quad (26)$$

It is well known that if r is a solution of (26), then $\alpha_n=r^n$ is a solution of (25). Multiplying (26) by $1-r^4$ and solving the resulting equation gives

$$r_1(s)=e^{\frac{is\pi}{2k}},\quad r_2(s)=-e^{\frac{is\pi}{2k}},\quad r_3(s)=ie^{\frac{is\pi}{2k}},\quad r_4(s)=-ie^{\frac{is\pi}{2k}}, \quad (27)$$

for $1\le s\le k-1$. Thus, from (22), the finite dimensional Lie algebra is spanned by the vectors fields

$$X_1(s)=e^{\frac{ins\pi}{2k}}u_n\frac{\partial}{\partial u_n},\ X_2(s)=(-1)^ne^{\frac{ins\pi}{2k}}u_n\frac{\partial}{\partial u_n}, \quad (28)$$

$$X_3(s)=i^ne^{\frac{ins\pi}{2k}}u_n\frac{\partial}{\partial u_n},\ X_4(s)=(-i)^ne^{\frac{ins\pi}{2k}}u_n\frac{\partial}{\partial u_n} \quad (29)$$

for $1\le s\le k-1$. To obtain the compatible variable, we use the canonical coordinate

$$C(n)=\int\frac{\mathrm{d}u_n}{\alpha_nu_n}=\frac{1}{\alpha_n}\ln|u_n|, \quad (30)$$

where α_n satisfies (25). Replacing α_n with $C(n)\alpha_n$ in the left hand side of equation in (25) yields the group invariant $I_n=\ln|u_nu_{n+4}\dots u_{n+4k-4}|$ since $(X_r(s))I_n=0$ for $r=1,2,3,4$. Hence, using (17), we have $e^{-I_{n+4}}=A_ne^{-I_n}+B_n$. For the sake of simplicity, we instead use the invariant

$$r_n=\exp(-I_n)=\frac{1}{u_nu_{n+4}\dots u_{n+4k-4}}. \quad (31)$$

On one hand, shifting (31) four times and replacing u_{n+4k} in the resulting equation yields

$$r_{n+4}=A_nr_n+A_n \quad (32)$$

whose iteration gives

$$r_{4n+j} = r_j \left(\prod_{k_1=0}^{n-1} A_{4k_1+j} \right) + \sum_{l=0}^{n-1} \left(B_{4l+j} \prod_{k_2=l+1}^{n-1} A_{4k_2+j} \right) \tag{33}$$

for $j = 0, 1, 2, 3$. On the other hand, using the same relation given in (31), we have

$$u_{n+4k} = \frac{r_n}{r_{n+4}} u_n. \tag{34}$$

We iterate (34) and its solution in closed form takes the form

$$u_{4kn+i} = u_i \left(\prod_{s=0}^{n-1} \frac{r_{4ks+i}}{r_{4ks+4+i}} \right), \quad i = 0, \ldots, 4k-1. \tag{35}$$

From the known fact that any integer r can be written as $r = 4\lfloor r/4 \rfloor + \tau(r),\ 0 \le \tau(r) \le 3$, where $\tau(r)$ is the remainder when r is divided by 4, we can rewrite (35) as follows:

$$u_{4kn+i} = u_i \left(\prod_{s=0}^{n-1} \frac{r_{4(ks+\lfloor \frac{i}{4} \rfloor)+\tau(i)}}{r_{4(ks+\lfloor \frac{i+4}{4} \rfloor)+\tau(i+4)}} \right), \tag{36}$$

where $i = 0, \ldots, 4k-1$ and $0 \le \tau(i) \le 3$. Using (35) in (36), we obtain

$$u_{4kn+i} = u_i \prod_{s=0}^{n-1} \frac{r_{\tau(i)} \left(\prod_{k_1=0}^{ks+\lfloor \frac{i}{4} \rfloor - 1} A_{4k_1+\tau(i)} \right) + \sum_{l=0}^{ks+\lfloor \frac{i}{4} \rfloor - 1} \left(B_{4l+\tau(i)} \prod_{k_2=l+1}^{ks+\lfloor \frac{i}{4} \rfloor - 1} A_{4k_2+\tau(i)} \right)}{r_{\tau(i+4)} \left(\prod_{k_1=0}^{ks+\lfloor \frac{i+4}{4} \rfloor - 1} A_{4k_1+\tau(i+4)} \right) + \sum_{l=0}^{ks+\lfloor \frac{i+4}{4} \rfloor - 1} \left(B_{4l+\tau(i+4)} \prod_{k_2=l+1}^{ks+\lfloor \frac{i+4}{4} \rfloor - 1} A_{4k_2+\tau(i+4)} \right)}, \tag{37}$$

$i = 0, \ldots, 4k-1$. Noting that $\tau(i+4) = \tau(i)$ and $1/r_i = \prod_{j=0}^{k-1} u_{i+4j}$, the above equation simplifies

$$u_{4kn+i} = u_i \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{ks+\lfloor \frac{i}{4} \rfloor - 1} A_{4k_1+\tau(i)} \right) + \left(\prod_{j=0}^{k-1} u_{\tau(i)+4j} \right) \sum_{l=0}^{ks+\lfloor \frac{i}{4} \rfloor - 1} \left(B_{4l+\tau(i)} \prod_{k_2=l+1}^{ks+\lfloor \frac{i}{4} \rfloor - 1} A_{4k_2+\tau(i)} \right)}{\left(\prod_{k_1=0}^{ks+\lfloor \frac{i}{4} \rfloor} A_{4k_1+\tau(i)} \right) + \left(\prod_{j=0}^{k-1} u_{\tau(i)+4j} \right) \sum_{l=0}^{ks+\lfloor \frac{i}{4} \rfloor} \left(B_{4l+\tau(i)} \prod_{k_2=l+1}^{ks+\lfloor \frac{i}{4} \rfloor} A_{4k_2+\tau(i)} \right)}, \tag{38}$$

$i = 0, \ldots, 4k-1$. The solution of (2) is obtained by back shifting (38) $4k-1$ times. Hence, the closed form solution of (2) is given by

$$\eta_{4kn-4k+1+i} = \eta_{i-4k+1} \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{ks+\lfloor \frac{i}{4} \rfloor - 1} a_{4k_1+\tau(i)} \right) + \left(\prod_{j=0}^{k-1} \eta_{\tau(i)-4k+1+4j} \right) \sum_{l=0}^{ks+\lfloor \frac{i}{4} \rfloor - 1} \left(b_{4l+\tau(i)} \prod_{k_2=l+1}^{ks+\lfloor \frac{i}{4} \rfloor - 1} a_{4k_2+\tau(i)} \right)}{\left(\prod_{k_1=0}^{ks+\lfloor \frac{i}{4} \rfloor} a_{4k_1+\tau(i)} \right) + \left(\prod_{j=0}^{k-1} \eta_{\tau(i)-4k+1+4j} \right) \sum_{l=0}^{ks+\lfloor \frac{i}{4} \rfloor} \left(b_{4l+\tau(i)} \prod_{k_2=l+1}^{ks+\lfloor \frac{i}{4} \rfloor} a_{4k_2+\tau(i)} \right)}. \tag{39}$$

Observe that if $\{a_n\}$ and $\{b_n\}$ are constant sequences, i.e. $a_n = a$ for all n and $b_n = b$ for all n, then the solutions of (2) and (4) are given by

$$\eta_{4kn-4k+1+i} = \eta_{i-4k+1} \prod_{s=0}^{n-1} \frac{a^{ks+\lfloor \frac{i}{4} \rfloor} + b \left(\prod_{j=0}^{k-1} \eta_{\tau(i)-4k+1+4j} \right) \sum_{l=0}^{ks+\lfloor \frac{i}{4} \rfloor - 1} a^l}{a^{ks+\lfloor \frac{i}{4} \rfloor + 1} + b \left(\prod_{j=0}^{k-1} \eta_{\tau(i)-4k+1+4j} \right) \sum_{l=0}^{ks+\lfloor \frac{i}{4} \rfloor} a^l} \tag{40}$$

and

$$u_{4kn+i} = u_i \prod_{s=0}^{n-1} \frac{A^{ks+\lfloor \frac{i}{4} \rfloor} + B\left(\prod_{j=0}^{k-1} u_{\tau(i)+4j}\right) \sum_{l=0}^{ks+\lfloor \frac{i}{4} \rfloor - 1} A^l}{A^{ks+\lfloor \frac{i}{4} \rfloor + 1} + B\left(\prod_{j=0}^{k-1} u_{\tau(i)+4j}\right) \sum_{l=0}^{ks+\lfloor \frac{i}{4} \rfloor} A^l}, \tag{41}$$

respectively.

In the following section, we investigate some special cases. One of the aims is to realize some results in [1].

3 Special cases

3.1 The case when $a = 1$ and b is a constant

We investigate the case when $a = 1$ and b constant. In this case, from (40), the solution is given by

$$\eta_{4kn-4k+1+i} = \eta_{i-4k+1} \prod_{s=0}^{n-1} \frac{1 + b\left(\prod_{j=0}^{k-1} \eta_{\tau(i)-4k+1+4j}\right)(ks + \lfloor \frac{i}{4} \rfloor)}{1 + b\left(\prod_{j=0}^{k-1} \eta_{\tau(i)-4k+1+4j}\right)(ks + \lfloor \frac{i}{4} \rfloor + 1)}, \tag{42}$$

$i = 0, \ldots, 4k-1$. Replacing $b = \pm 1$ in (42) yields the results in [1, see Theorems 1 and 6]. In fact, noting that $\tau(4k-1-j) = 3 - \tau(j) = 3 - j + 4\lfloor j/4 \rfloor$, we have:

$$\eta_{4kn-j} = \eta_{4kn-4k+1+(4k-1-j)}, \qquad j = 0, 1, \ldots, 4k-1 \tag{43}$$

$$= \eta_{-j} \prod_{s=0}^{n-1} \frac{1 + b\left(\prod_{r=0}^{k-1} \eta_{\tau(4k-1-j)-4k+1+4r}\right)(ks + \lfloor \frac{4k-1-j}{4} \rfloor)}{1 + b\left(\prod_{r=0}^{k-1} \eta_{\tau(4k-1-j)-4k+1+4r}\right)(ks + \lfloor \frac{4k-1-j}{4} \rfloor + 1)} \tag{44}$$

$$= \eta_{-j} \prod_{s=0}^{n-1} \frac{1 + b\left(\prod_{r=0}^{k-1} \eta_{-j+4\lfloor \frac{j}{4} \rfloor - 4r}\right)(ks + k - 1 - \lfloor \frac{j}{4} \rfloor)}{1 + b\left(\prod_{r=0}^{k-1} \eta_{-j+4\lfloor \frac{j}{4} \rfloor - 4r}\right)(ks + k - \lfloor \frac{j}{4} \rfloor)}. \tag{45}$$

3.2 The case when $a \neq 1$ and b is a constant

Here, from (40), the solution is given by

$$\eta_{4kn-4k+1+i} = \eta_{i-4k+1} \prod_{s=0}^{n-1} \frac{a^{ks+\lfloor \frac{i}{4} \rfloor} + b\left(\prod_{j=0}^{k-1} \eta_{\tau(i)-4k+1+4j}\right)\left(\frac{1-a^{ks+\lfloor \frac{i}{4} \rfloor}}{1-a}\right)}{a^{ks+\lfloor \frac{i}{4} \rfloor + 1} + b\left(\prod_{j=0}^{k-1} \eta_{\tau(i)-4k+1+4j}\right)\left(\frac{1-a^{ks+\lfloor \frac{i}{4} \rfloor + 1}}{1-a}\right)} \tag{46}$$

and, similarly, this can also be written in the form

$$\eta_{4kn-j} = \eta_{-j} \prod_{s=0}^{n-1} \frac{a^{ks+k-1-\lfloor \frac{i}{4} \rfloor} + b\left(\prod_{j=0}^{k-1} \eta_{-j+4\lfloor \frac{j}{4} \rfloor - 4r}\right)\left(\frac{1-a^{ks+k-1+\lfloor \frac{i}{4} \rfloor}}{1-a}\right)}{a^{ks+k-\lfloor \frac{i}{4} \rfloor} + b\left(\prod_{j=0}^{k-1} \eta_{-j+4\lfloor \frac{j}{4} \rfloor - 4r}\right)\left(\frac{1-a^{ks+k-\lfloor \frac{i}{4} \rfloor}}{1-a}\right)}. \tag{47}$$

For $a = -1$, the above equation simplifies to

- For k even,

$$\eta_{4kn-j} = \eta_{-j} \prod_{s=0}^{n-1} \frac{-(-1)^{\lfloor \frac{i}{4} \rfloor} + b\left(\prod_{j=0}^{k-1} \eta_{-j+4\lfloor \frac{j}{4} \rfloor - 4r}\right)\left(\frac{1+(-1)^{\lfloor \frac{i}{4} \rfloor}}{2}\right)}{(-1)^{\lfloor \frac{i}{4} \rfloor} + b\left(\prod_{j=0}^{k-1} \eta_{-j+4\lfloor \frac{j}{4} \rfloor - 4r}\right)\left(\frac{1-(-1)^{\lfloor \frac{i}{4} \rfloor}}{2}\right)} \tag{48}$$

$$= \eta_{-j}\left[-1 + b\left(\prod_{j=0}^{k-1} \eta_{-j+4\lfloor \frac{j}{4} \rfloor - 4r}\right)\right]^{(-1)^{\lfloor \frac{j}{4} \rfloor} n}. \tag{49}$$

This result was obtained in [1] for $b = \pm 1$ (see Theorems 11 and 18).

- For k odd,

$$\eta_{4kn-j} = \eta_{-j} \prod_{s=0}^{n-1} \frac{(-1)^{s-\lfloor \frac{i}{4} \rfloor} + b\left(\prod_{j=0}^{k-1} \eta_{-j+4\lfloor \frac{j}{4} \rfloor - 4r}\right)\left(\frac{1-(-1)^{s-\lfloor \frac{i}{4} \rfloor}}{2}\right)}{-(-1)^{s-\lfloor \frac{i}{4} \rfloor} + b\left(\prod_{j=0}^{k-1} \eta_{-j+4\lfloor \frac{j}{4} \rfloor - 4r}\right)\left(\frac{1+(-1)^{s-\lfloor \frac{i}{4} \rfloor}}{2}\right)} \tag{50}$$

$$= \begin{cases} \eta_{-j}, & \text{if } n \text{ is even} \\ \eta_{-j}\left[-1 + b\left(\prod_{r=0}^{k-1} \eta_{-j+4\lfloor \frac{j}{4} \rfloor - 4r}\right)\right]^{(-1)^{\lfloor \frac{j}{4} \rfloor + 1}}, & \text{if } n \text{ is odd} \end{cases} \tag{51}$$

for $j = 0, 1, \ldots, 4k-1$ and furthermore,

$$\eta_{8kn-j} = \eta_{-j} \tag{52}$$

for all n. More explicitly, the $8k$ periodic solutions are as follows:

$$\eta_i = \eta_{4k(1)-(4k-i)} = \frac{\eta_{-4k+i}}{\left[-1 + b\left(\prod_{r=0}^{k-1} \eta_{-4(r+1)+i}\right)\right]}, i = 1, 2, 3, 4 \tag{53}$$

$$\eta_{4+i} = \eta_{4k(1)-(4k-i-4)} = \eta_{-4k+i+4}\left[-1 + b\left(\prod_{r=0}^{k-1} \eta_{-4(r+1)+i}\right)\right], i = 1, 2, 3, 4 \tag{54}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\eta_{4k-8+i} = \eta_{4k(1)-(8-i)} = \eta_{8-i}\left[-1 + b\left(\prod_{r=0}^{k-1} \eta_{-4(r+1)+i}\right)\right], i = 1, 2, 3, 4 \tag{55}$$

$$\eta_{4k-4+i} = \eta_{4k(1)-(4k-i-4)} = \frac{\eta_{-4k+i+4}}{\left[-1 + b\left(\prod_{r=0}^{k-1} \eta_{-4(r+1)+i}\right)\right]}, \quad i = 1, 2, 3, 4 \tag{56}$$

$$\eta_{4k+1+i} = \eta_{4k(2)-(4k-1-i)} = \eta_{-4k+1+i}, \quad i = 0, 1, \ldots, 4k-1. \tag{57}$$

For this special case, the results were obtained in [1] for $b = \pm 1$ (see Theorems 9, 15 and 16).

4 Periodicity and behavior of the solutions

Theorem 4.1 *Let u_n be a solution of*

$$u_{n+4k} = \frac{u_n}{A + B\prod_{i=1}^{k} u_{n+4(i-1)}}, \tag{58}$$

for some non-zero constants $A \neq 1$ and B. Suppose the initial conditions x_i, $i = 0, \ldots, 4k-1$, are such that $\prod_{j=0}^{k-1} u_{4j+p} = (1-A)/B$, $p = 0, 1, 2, 3$. Then the solution of (58) is periodic with period $4k$.

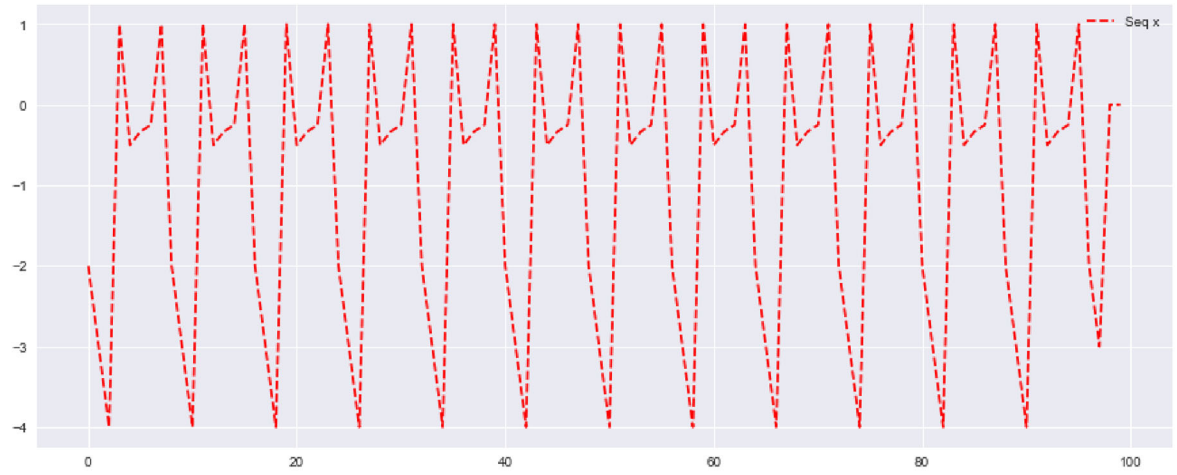


Fig. 1 Graph of $x_{n+8} = \dfrac{x_n}{(2 - x_n x_{n+4})}$, where $x_0 = -2$, $x_1 = -3$, $x_2 = -4$, $x_3 = 1$, $x_4 = -1/2$, $x_5 = -1/3$, $x_6 = -1/4$, $x_7 = 1$ and are such that $x_0x_4 = x_1x_5 = x_2x_6 = x_3x_7 = (1 - A)/B$

Proof Suppose $\prod_{j=0}^{k-1} u_{4j+p} = (1 - A)/B$, $p = 0, 1, 2, 3$. From (41), we get

$$u_{4kn+i} = u_i \prod_{s=0}^{n-1} \frac{A^{ks+\lfloor \frac{i}{4} \rfloor} + B\left(\prod_{j=0}^{k-1} u_{\tau(i)+4j}\right) \sum_{l=0}^{ks+\lfloor \frac{i}{4} \rfloor - 1} A^l}{A^{ks+\lfloor \frac{i}{4} \rfloor + 1} + B\left(\prod_{j=0}^{k-1} u_{\tau(i)+4j}\right) \sum_{l=0}^{ks+\lfloor \frac{i}{4} \rfloor} A^l} \tag{59}$$

$$= u_i \prod_{s=0}^{n-1} \frac{A^{ks+\lfloor \frac{i}{4} \rfloor} + B\left(\frac{1-A}{B}\right)\left(\frac{1-A^{ks+\lfloor \frac{i}{4} \rfloor}}{1-A}\right)}{A^{ks+\lfloor \frac{i}{4} \rfloor + 1} + B\left(\frac{1-A}{B}\right)\left(\frac{1-A^{ks+\lfloor \frac{i}{4} \rfloor + 1}}{1-A}\right)} \tag{60}$$

$$= u_i, \tag{61}$$

for all $i = 0, 1, \ldots, 4k - 1$ since $0 \le \tau(i) \le 3$. □

We plot Fig. 1 to illustrate Theorem 4.1. We note that for $A = -1$ and $B = 1$, we get the result in Theorem 13 in [1] and the result's restriction ($\prod_{r=0}^{k-1} \eta_{-j+4\lfloor \frac{j}{4} \rfloor - 4r} = 2$, $j = 0, 1, \ldots, 4k - 1$ or simply $-j + 4\lfloor \frac{j}{4} \rfloor = 0, 1, 2, 3$) is a special case of the assumption in the above theorem ($\prod_{j=0}^{k-1} u_{4j+p} = (1 - A)/B$, that is, $\prod_{r=0}^{k-1} \eta_{p-4r\lfloor \frac{i}{4} \rfloor - 4r} = (1 - A)/B$, $p = 0, 1, 2, 3$).

Observe that in Theorem 15 in [1], the authors ought to add the restriction $v_j \neq -2$. If this condition is not satisfied, the period will be $4k$ and not $8k$ as they clearly stated in Theorem 20 in [1].

Theorem 4.2 *Let u_n be a solution of*

$$u_{n+4k} = \frac{u_n}{1 + B\prod_{i=1}^{k} u_{n+4(i-1)}}, \tag{62}$$

for some non-zero constant B. The zero equilibrium point is non hyperbolic. Furthermore, if the initial conditions x_i, $i = 0, \ldots, 4k - 1$, and B are positive, then the solution converges to the zero equilibrium point.

Proof The equilibrium point of (62) is $u = 0$. Let

$$f(u_n, u_{n+4}, \ldots, u_{n+4(k-1)}) = u_{n+4k} = \frac{u_n}{1 + B\prod_{i=1}^{k} u_{n+4(i-1)}}. \tag{63}$$

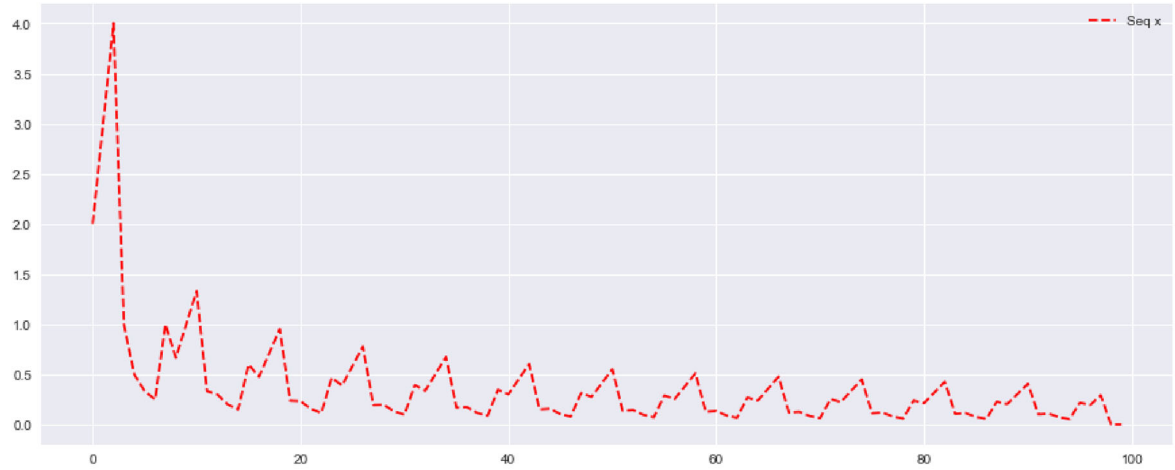


Fig. 2 Graph of $x_{n+8} = x_n/(2 + x_n x_{n+4})$, where $x_0 = 2, x_1 = 3, x_2 = 4, x_3 = 1, x_4 = 1/2, x_5 = 1/3, x_6 = 1/4, x_7 = 1$

So,

$$f_{,u_n} = \frac{1}{\left(1 + B\prod_{i=1}^{k} u_{n+4(i-1)}\right)^2}, \quad f_{,u_{n+4j}} = \frac{-Bu_n^2}{\left(1 + B\prod_{i=1}^{k} u_{n+4(i-1)}\right)^2} \prod_{\substack{i=1 \\ i\neq j}}^{k-1} u_{n+4i}, \quad j = 1, 2, \ldots, k-1. \tag{64}$$

We have $f_{,u_n}(0, 0, \ldots, 0) = 1$ and $f_{,u_{n+4j}}(0, 0, \ldots, 0) = 0$, $j = 1, 2, \ldots, k-1$. Thus, the characteristic equation associated with (58) is $\lambda^{4k} - 1 = 0$ and therefore, $|\lambda_i| = 1$. Therefore, the zero equilibrium point is non-hyperbolic.

Suppose the non-zero initial conditions are all positive. From (41), we get

$$u_{4kn+i} = u_i \prod_{s=0}^{n-1} \frac{1 + B\left(\prod_{j=0}^{k-1} u_{\tau(i)+4j}\right)\left(ks + \left\lfloor \frac{i}{4} \right\rfloor\right)}{1 + B\left(\prod_{j=0}^{k-1} u_{\tau(i)+4j}\right)\left(ks + \left\lfloor \frac{i}{4} \right\rfloor + 1\right)} \tag{65}$$

$$= u_i \prod_{s=0}^{n-1} \left(1 - \frac{B\left(\prod_{j=0}^{k-1} u_{\tau(i)+4j}\right)}{1 + B\left(\prod_{j=0}^{k-1} u_{\tau(i)+4j}\right)\left(ks + \left\lfloor \frac{i}{4} \right\rfloor + 1\right)}\right) \tag{66}$$

$$= u_i \prod_{s=0}^{n-1} \Theta(s). \tag{67}$$

If B is positive, $\Theta(s) < 1$, $s = 0, 1, \ldots, n-1$. Therefore, u_n tends to zero as n tends to infinity. □

We plot Fig. 2 to illustrate Theorem 4.2.

Theorem 4.3 *Assume that B is positive and $u_i \geq 0$, $i = 0, 1, \ldots, k-1$. Then the zero equilibrium point $u = 0$ of (62) is globally asymptotically stable.*

Proof The equilibrium point of (62) satisfies $u(1 + Bu^k) = 0$. Thus, $u = 0$. Let $\epsilon \geq 0$ and suppose the $u_i's$, $i = 0, 1, \ldots, k-1$ are such that

$$|u_i| \leq \frac{\epsilon}{4k(B+1)}, i = 0, 1, \ldots, k-1.$$

We have that

$$|u_0| + |u_1| + \cdots + |u_{k-1}| \leq \frac{\epsilon}{B+1} \tag{68}$$

and (see (67)),

$$u_{4kn+i} \leq u_i, \tag{69}$$

$i = 0, 1, \ldots, k-1$, for all n if $B \geq 0$.

This implies that for $|u_{4kn+i}| \leq |u_i| \leq \frac{\epsilon}{4k(B+1)} \leq \epsilon$, we have found $\delta = \epsilon/(B+1)$ such that $|u_0| + |u_1| + \ldots |u_{k-1}| \leq \delta$. Thus, the zero equilibrium point is locally stable. On the other hand (see Theorem 4.2), x_n tends to zero as n goes to infinity. The zero equilibrium being a global attractor and locally stable, it is globally asymptotically stable. □

Theorem 4.4 *Assume $A \neq 1$. The zero equilibrium point of (58) is asymptotically stable for $|A| > 1$ and unstable for $|A| < 1$. Furthermore, all non zero equilibrium points of (58) are non-hyperbolic.*

Proof The equilibrium points of (58) satisfy $u(A + Bu^k - 1) = 0$. Let

$$f(u_n, u_{n+4}, \ldots, u_{n+4(k-1)}) = u_{n+4k} = \frac{u_n}{A + B\prod_{i=1}^{k} u_{n+4(i-1)}}. \tag{70}$$

We have

$$f_{,u_n} = \frac{A}{\left(A + B\prod_{i=1}^{k} u_{n+4(i-1)}\right)^2}, \quad f_{,u_{n+4j}} = \frac{-Bu_n^2}{\left(A + B\prod_{i=1}^{k} u_{n+4(i-1)}\right)^2} \prod_{\substack{i=1 \\ i\neq j}}^{k-1} u_{n+4i}, \quad j = 1, 2, \ldots, k-1. \tag{71}$$

- For the equilibrium point $u = 0$, we have $f_{,u_n}(0, 0, \ldots, 0) = 1/A$ and $f_{,u_{n+4j}}(0, 0, \ldots, 0) = 0$, $j = 1, 2, \ldots, k-1$. Thus, the characteristic equation associated with (58) is $\lambda^{4k} - \frac{1}{A} = 0$. Therefore, $|\lambda| < 1$ if $|A| > 1$ (that is, locally asymptotically stable) and $|\lambda| > 1$ if $|A| < 1$ (that is, unstable).
- The non-zero equilibrium points u satisfy $A + Bu^k - 1 = 0$. Then $f_{,u_n}(u, u, \ldots, u) = A$ and $f_{,u_{n+4j}}(u, u, \ldots, u) = A - 1$, $j = 1, 2, \ldots, k-1$. Thus, the characteristic equation associated with (58) is

$$\lambda^{4k} - (A-1)\lambda^{4k-4} - \cdots - (A-1)\lambda^4 - A = 0. \tag{72}$$

Multiplying the above equation by $1 - \lambda^4$, we get (after simplification)

$$(1 - \lambda^{4k})(\lambda^4 - A) = 0. \tag{73}$$

It follows that, for $A < 0$, the solutions of (73) are $\lambda_r = -Ae^{i\frac{i(2r+1)\pi}{4}}$, $r = 0, 1, 2, 3$ or $\lambda_r = e^{i\frac{2r\pi}{4k}}$, $p = 1, 2, \ldots, k-1, k+1, \ldots, 2k-1, 2k+1, \ldots, 2k-1, 2k+1, \ldots, 3k-1, 3k+1, \ldots, 4k-1$. For $A > 0$, the solutions of (73) are $\lambda_r = Ae^{i\frac{2r\pi}{4}}$, $r = 0, 1, 2, 3$ or $\lambda_r = e^{i\frac{2r\pi}{4k}}$, $p = 1, 2, \ldots, k-1, k+1, \ldots, 2k-1, 2k+1, \ldots, 2k-1, 2k+1, \ldots, 3k-1, 3k+1, \ldots, 4k-1$. Therefore, for $k > 1$, there exists a root of (72) with modulus equal to one.

□

5 Conclusion

We studied the difference equation $\eta_{n+1} = \eta_{n-4k+1}/(a_n + b_n \prod_{i=1}^{k} \eta_{n-4i+1})$ by performing its symmetry analysis and we used the canonical coordinate to obtain its invariants. These invariants are utilized to derive the solutions in closed form. We demonstrated that all the formula solutions in [1] are special cases of our findings. Some conditions for existence of $4k$ and $8k$ periodic solutions were established. Finally, we investigated the stability of the solution of the difference equation and proved the existence of non-hyperbolic and globally asymptotically stable equilibrium points.

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References

1. Aljoufi, L.S.; Almatrafi, M.B.; Seadawy, A.R.: Dynamical analysis of discrete time equations with a generalized order. Alex. Eng. J. **64**, 937–945 (2023)
2. Banasiak, J.: Mathematical Modelling in One Dimension: An Introduction via Difference and Differential Equation. Cambridge University Press, Cambridge (2013)
3. Bohner, M.; Dannan, F.M.; Streipert, S.: A nonautonomous Beverton–Holt equation of higher order. J. Math. Anal. Appl. **457**, 114–133 (2018)
4. Chen, S.; Jiang, X.: Modeling repayment behavior of consumer loan in portfolio across business cycle: a triplet Markov model approach. Complexity **2020**, 5458941, 11 (2020)
5. Elsayed, E.M.; Ibrahim, T.F.: Periodicity and solutions for some systems of non-linear rational difference equations. Hacettepe J. Math. Stat. **44**(6), 1361–1390 (2015)
6. Folly-Gbetoula, M.; Kgatliso Mkhwanazi, K.; Nyirenda, D.: On a study of a family of higher order recurrence relations. Math. Probl. Eng. **2022**, 6770105, 11 (2022)
7. Folly-Gbetoula, M.; Nyirenda, D.: A generalised two-dimensional system of higher order recursive sequences. J. Differ. Equ. Appl. **26**(2), 244–260 (2020)
8. Grove, E.A.; Ladas, G.: Periodicities in Nonlinear Difference Equations, vol. 4. Chapman & Hall/CRC, Boca Raton (2005)
9. Hydon, P.E.: Difference Equations by Differential Equation Methods. Cambridge University Press, Cambridge (2014)
10. Joshi, N.; Vassiliou, P.: The existence of Lie symmetries for first-order analytic discrete dynamical systems. J. Math. Anal. Appl. **195**, 872–887 (1995)
11. Li, Y.; Li, J.: Stage-structured discrete-time models for interacting wild and sterile mosquitoes with Beverton–Holt survivability. Math. Biosci. Eng. **16**(2), 572–602 (2019)
12. Maeda, S.: The similarity method for difference equations. IMA J. Appl. Math. **38**, 129–134 (1987)
13. Quispel, G.R.W.; Sahadevan, R.: Lie symmetries and the integration of difference equations. Phys. Lett. A **184**, 64–70 (1993)

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