Approximating the Nucleon as a Relativistic Three Particle System

Philippe Alberto Friedrich Ferrer

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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Philippe A. F. Ferrer

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Abstract

This dissertation is divided into two parts: the first part deals with the concepts of angular momentum and spin in classical mechanics and quantum mechanics and relativistic quantum mechanics and their connection with magnetic moments.

In the second part, a model is set up of a relativistic three particle system, based on the previously introduced concepts, which will serve as a template for a nucleon. The spatial component of the Lorentz invariant electromagnetic current is computed, and on the basis of it, the magnetic moment in the non-relativistic limit.

It will be seen that the ratio $-\frac{3}{2}$ for the magnetic moment of the proton to the neutron will be recovered, in accordance with the static quark model, static QCD and very close to experiment.

Contents

0.1 List of Figures

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- γ^{ν} : γ-matrices of the Dirac theory defining the Clifford algebra.
- $\vec{P}_{N(L)}$: 3-momentum of combined 3 particle system in the Lab frame
- $\vec{p}_{i\|(L)}$: 3-momentum of the ith constituent in direction of $\vec{P}_{N(L)}$
- $\vec{p}_{i\perp(N)}$: 3-momentum of the ith constituent in direction perpendicular to $\vec{P}_{N(L)}$
- $a_s^{\dagger}(\vec{p})\,/a_s(\vec{p})$: Fock space creation/annihilation operator of a particle with spin s and momentum $\vec{p}$
- λ_i : Lorentz invariant helicity quantum number.
- m_i : Canonical spin projection quantum number
- $D^{j}_{m_i\lambda_i}$: Rotation operator rotating the quantum number m_i onto λ_i.
- $U\left[L\left(\vec{p}\right)\right]$:Boost operator in direction of $\vec{p}$.
- α, β, γ : Euler angles indicating the direction of the normal to the plane of the three constituents.
- $U\left[R\left(\alpha, \beta, \gamma\right)\right]$: Rotation operator in terms of the Euler angles
- E_i : Total energy of ith constituent.
- $E_{i(N)}$: Total energy of ith constituent in the CMS frame.
- m_{0_i} :restmass of particle i
- μ : Quantum number proportional to parity.
- $T; t_i$: Total isospin quantum number ; isospin quantum number of ith constituent
- $\vec{p}_{N_f}; \vec{p}_{Nin}$: final and initial momentum of the ith constituent respectively
- $\vec{q}_N = \vec{p}_{N_f} - \vec{p}_{Nin}$: Change of the nucleon momentum
- $\vec{\sigma}^{\nu}$:Pauli matrices forming the basis of Su(2)

- $\sigma^{\nu\mu} = \frac{i}{2}\left[\gamma^{\nu}, \gamma^{\mu}\right]$:Tensor used in the Dirac theory for finite boosts and rotations
- We use $c = \hbar = 1$ unless otherwise specified

Chapter 1

Introduction

In recent years, the study of spin has provided valuable insights into the structure of subatomic particles. The quantum variable "spin" cannot be directly observed, but only inferred to from the related quantity "magnetic moment". Experimental results for the value of magnetic momenta exists for a large number of particles. The correct computation of the magnetic moment for a theoretical model can be viewed as a important initial success.

The experiments which study spin hence measure magnetic momenta. Some experiments involved a comparison of the results of polarized to anti-polarized proton-proton scattering cross-sections. These results indicate that for large $P_{\perp}^2$, where the asymptotically free region is probed and the scattering is taken to be a consequence of the hard scattering constituents, the collision parameter, or the "size" of the proton, appears to be larger for collisions between polarized beams [16].

Interpretation of the experimental outcome leads to conclude that only a fraction of proton as well as neutron spin is due to quarks [10][11].

This dissertation aims to pave the way in a direction that may finally produce some clarity to this problem.

In this dissertation, a template for the nucleon will be presented in the dissertation under the assumption that the nucleon is a three body system composed of quarks.

The quark model has its origins in a large number of experimental data on baryon and meson resonances during the 1960's [18]. It was found that the patterns found could be accounted for in terms of quark constituents. Initially, Gell-Mann and Zweig proposed three types of fermion constituents in a baryon,with "flavours" u (up), d (down) and s (strange). The "up" and "down" refers to the relative orientation of the isospin, $I_3 = \frac{1}{2}$ (up) and $I_3 = -\frac{1}{2}$ (down) and $I_3 = 0$ (strange). These constituents were assumed to carry fractional charges, with d and s having charge $\frac{Q}{e} = -\frac{1}{3}$, u has charge $\frac{Q}{e} = \frac{2}{3}$ and each carries spin $\frac{1}{2}$.

By imposing the symmetry of quark flavour [18] as well as spin (i.e. the state does not change under exchange of any pair of quarks), the baryon decuplet with spin-parity $\frac{3}{2}^+$ can be constructed. Each member of the decuplet has the spin of all three constituents aligned. The Ω^- baryon was predicted on this basis 3 years before it was observed.

Since three fermions in the same state with the same spin violate the Pauli exclusion principle, an additional quantum number was necessary to describe the quarks. This quantum number was called "colour", and the quarks can have quantum number red, green and blue. So the added constraint of the baryons being overall "colourless" was imposed.

More important to this disseration is the symmetry operation of simultaneous interchange of flavour and spin of any quark pair although not under each separately [18]. This symmetry gives rise to the baryon octet, of which the proton (uud) and the neutron (udd) are members. Each of the members of the octet has spin-parity $\frac{1}{2}^+$.

Although this group theoretical analysis assumes the quarks to be static, against which there are strong arguments to be presented later, it nevertheless yields satisfactory results and the quark hypothesis is still going strong. We shall compare our results with the results obtained by the static quark model.

A large part of the dissertation deals with the model introduced in [13] and further work on it in [10]. In these papers, the zeroth component of the current $G_E(q^2)$ was calculated and a value was obtained for the quark core radius. It turned out that there was a non-vanishing contribution to $\langle r_E^2 \rangle$ which was purely due to the minimal correlation considered in the model. The numerical values obtained and found to be in the range $0 < \sqrt{\langle r_E^2 \rangle} < 0.16 fm$ in the non relativistic and the extreme relativistic limit respectively. The extreme relativistic value accounted for 19% of the empirical value for the proton radius at rest, which is given as $0.81 \pm 0.04 fm$ or $0.862 \pm 0.012 fm$ in [16]. This contribution to the radius is thought to be due to Thomas precession effect, which arises from the non-commutativity of CMS and perpendicular boosts acting on the system. In this dissertation, it will be shown that the model retrieves the static ratio $-\frac{3}{2}$ of the magnetic moments of the proton and the neutron . To this end, the spatial component of the invariant electromagnetic current will have to be computed.

In order to understand all the components used in constructing the model, it was necessary to study some very important formalisms. The principle work presented in this dissertation was to calculate the ratio of the expectation value of the magnetic moment of the proton and the nucleon.

In Chapters two to five, I give an overview of how spin is described in physics. Chapters six to eight present the new work done.

Chapter two deals with rotational phenomena in general and how they are described mathematically in classical physics and quantum mechanics. We start from classical physics and derive the relationship between angular momentum and magnetic moment. This is a very important result, since as stated earlier, spin as well as angular momentum can only be inferred on a particle level via the observable quantity of magnetic moment. The group structure of rotational phenomena will also be presented, since this will be the stepping stone to the classical description of spin, which is intuitively very appealing. The transition to quantum mechanics is presented in some detail.

Chapter three investigates the origin of spin as an experimentally inferred phenomenon and, moreover, how these results were incorporated in the description of particles. Spin is of special importance to us, since we assume the three partons in our model to be in an initial s-state, a state of zero angular momentum. Hence we would expect, in the static (non-relativistic) limit, all the contribution to the magnetic moment to come from the spin of the constituents.

The fourth Chapter deals with the Dirac equation. The marriage of the two fundamental principles, quantum mechanics and relativity, gives rise to an analytically derived expression for spin. As already mentioned, there are good arguments for assuming the quarks to move with relativistic velocities. The quarks will have to be described quantum-mechanically because of their dimensions and relativistically because of their velocity. Dirac was the first to describe relativistic quantum mechanics successfully for fermions. His description contains some surprising results, which will be presented later.

Following this, Chapter five introduces the helicity formalism. This spin formalism, which has been used initially to describe three particle decays [12], is employed in a new context the use of the helicity formalism in this new context leads to the formulation of a new model of the nucleon described in Chapter six. The helicity formalism has the advantage of making the spin projection quantum number relativistically invariant. This in turn will help us to couple the spins of the three constituents, since we can still use LS-coupling rules. A two particle system will be presented in chapter five on the basis of which we shall build a three body system in the following chapter.

The invariant vector electromagnetic current as well as the ratio of the magnetic moment expectation value of the proton to the neutron are computed in Chapter seven. The current will be derived, first in the Schrödinger picture, which is intuitively appealing and then by analogy in the Dirac picture. From the point particle current, the general baryon current will be derived and finally the Gordon decomposition. The non-relativistic will be taken and the magnetic moment ratio calculated.

Finally, in Chapter eight, conclusions are drawn and future possibilities are discussed.

The Appendices contain explicit details of some of the calculations.

Chapter 2

Angular Momentum

In this second chapter, we shall start by introducing some basics of angular momentum theory. The relationship between angular momentum and magnetic moment will be derived. We identify the symmetry and conserved quantity of relevance, on the basis of which a classical Lie-group can be constructed.

We then show how the classical theory can be generalized to quantum mechanics.

The chapter is concluded by giving a brief overview of the representation of the rotation group relevant to quantum mechanics and a physical interpretation of the mathematical results.

Many of the concepts introduced in this chapter will be used when discussing spin.

2.1 Classical Angular Momentum

2.1.1 Magnetic moment and orbital angular momentum

Angular momentum is classically defined as :

$$\begin{aligned} \vec{L} &= \vec{r} \times m\vec{v} \\ &= m\,|r|\,|v|\cos\theta \end{aligned} \tag{2.1}$$

The AM (angular momentum) vector $\vec{L}$ is the vector perpendicular to the velocity vector $\vec{v}$ of the particle and the position vector $\vec{r}$ as seen from the axis of rotation as origin. In other words, the rotation occurs in the plane defined by the vector $\vec{L}$. The angle θ defines the angle between the vectors $\vec{r}$ and $\vec{v}$.

The classical orbital magnetic dipole moment is defined as [19]

$$\mu_l = IA \tag{2.2}$$

where I is the electric current and A the area of the loop the current flows through. The current is assumed to be perpendicular to the position vector of the moving charge.

Assuming $\cos\theta = 1$ in (2.1), we rewrite the relation as [19]

$$L = mrv \tag{2.3}$$

where the quantities L, r and v are taken as magnitudes of their vectors.

When describing, for instance, an electron orbiting an atom, the current is given by the single charge e circling the atom with a period $T = \frac{2\pi r}{v}$, where r is the radius of the orbital and v the magnitude of velocity of the charge. The current is then

$$I = \frac{e}{T} = \frac{ev}{2\pi r} \tag{2.4}$$

Solving (2.3) for v and substituting into (2.4) and then substituting (2.4) into (2.2) yields the result [19]

$$\mu_l = \frac{eL}{2m}$$

or, in vector form

$$\vec{\mu}_l = -\frac{e}{2m}\vec{L}$$

where the minus sign arises from the definition of the relative orientations of $\vec{\mu}_l$ and $\vec{L}$.

If could rewrite the above expression as

$$\vec{\mu}_l = -\frac{\mu_B}{\hbar}\vec{L}$$

where the constant $\mu_B = \frac{e\hbar}{2m}$ is called the Bohr magneton and forms the natural unit of measuring the magnetic dipole moment. This is, as was stressed earlier, a very important result, since it connects the theoretical quantity of angular momentum to the observable quantity of magnetic moment. Another important observation is the fact that the magnetic moment to angular momentum ratio does not depend on orbital shape, nor size or period [19].

2.1.2 Angular momentum from SO(3)

Classical group theory provides us with a starting point to make the transition from classical to quantum mechanics.

A group is, fundamentally, an expression of a symmetry [4].

Noether's theorem tells us that, associated with a symmetry of a system, there exists a conserved quantity. The conserved physical quantity in the rotation group is the magnitude of angular momentum. Mathematically, the conserved quantity under operations of the rotation group is the magnitude of the rotated vector. This can be expressed as the invariance of the metric under the transformations

$$R^T g R = g$$

Since we are working in a 3-dimensional flat Euclidean space, the metric g is just the identity matrix I_3. The matrix R as defined by the above relation belongs to the group denoted by O(3) (for "orthogonal") with the special property $\det R = \pm 1$.

We are interested in the special case $\det R = +1$, which defines the special orthogonal group called SO(3), which is the classical group of rotations in 3 dimensions.

For a rotation by an angle θ around the z-axis (the z-axis is arbitrarily chosen and we could easily generalize to any direction), the following matrix can be used:

$$R[U(\theta)] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.5}$$

where the property $\det R = +1$ can be easily verified.

$R[U(\theta)]$ maps the vector (x, y) onto the vector (x', y') in the following fashion:

$$\begin{aligned} x\prime &= x\cos\theta - y\sin\theta \\ y\prime &= x\sin\theta + y\cos\theta \end{aligned}$$

This transformation is continuous, since the parameter θ is continuous.

2.1.3 Generators of SO(3)

Since the action of rotating a vector is continuous, the group SO(3) is a 3 parameter Lie-group. We can hence find the infinitesimal generators of rotation.

If the rotations are infinitesimal, then as $\theta \to d\theta$, $\cos d\theta \to 1$ and $\sin d\theta \to d\theta$ which reduces the above expression to

$$\begin{aligned} x\prime &= x - y\,(d\theta) \\ y\prime &= x\,(d\theta) + y \end{aligned}$$

Upon back-substitution into the rotation matrix and applying the defining relation for the infinitesimal generator L_z , to first order in $d\theta$, we obtain [3]

$$U[R(d\theta)] = I_3 - (d\theta)\,L_z$$

The explicit form of L_z can be determined. Note that L_z was given as an example. The same procedure holds for L_x and L_y.

2.2 Angular Momentum in quantum mechanics

2.2.1 Transition to quantum mechanics

This fact can be used in Q. M. (Quantum Mechanics) after the following considerations:

If we have a basis vector $|x, y\rangle$,which has expectation values $\langle x\rangle$ for the x-coordinate and $\langle y\rangle$ for the y-coordinate then we could rewrite the above equation in terms of expectation values:

$$\begin{aligned} \langle x'\rangle &= \langle x\rangle - d\theta\,\langle y\rangle \\ \langle y'\rangle &= d\theta\,\langle x\rangle + \langle y\rangle \end{aligned}$$

The infinitesimal operator of rotation, which will be denoted $U[R(d\theta)]$ for a rotation by an infinitesimal angle $d\theta$ around the z-axis, changes the basis vector $|x, y\rangle$ in the following way:

$$\langle x, y \mid U\left[R(\theta)\right] \mid \Psi\rangle = \Psi(x + (d\theta)\, y, y - x\,(d\theta)) \tag{2.6}$$

The rotation operator $U^{+}\left[R(\theta)\right]$, which is just the transpose of $U[R(\theta)]$ (since the rotation matrix is real, the transpose and the dagger operations are identical), acts on the basis vector $\langle x, y|$. To find the explicit form of the infinitesimal generator of rotation about the z-axis, which can be used in an infinitely dimensional Hilbert-space, we write down the generator of rotation as previously, with the exception that the generator is now the infinitely dimensional

$$U[R(d\theta)] = I - \frac{i}{\hbar}\,(d\theta)\,L_z$$

Where I is the identity matrix (the operator of zero rotation) and L_z is the generator of rotation to be determined. The constant $\frac{i}{\hbar}$ is introduced in anticipation of what is to follow. We rewrite (2.6):

$$\Big\langle x, y \mid I - \frac{i}{\hbar}\,(d\theta)\,L_z \mid \Psi \;= \Psi(x + (d\theta)\, y, y - x\,(d\theta)) \tag{2.7}$$

$$\Rightarrow \langle x, y \mid I \mid \Psi\rangle - \frac{i}{\hbar}\,(d\theta)\,\langle x, y \mid L_z \mid \Psi\rangle = \Psi(x + (d\theta)\, y, y - x\,(d\theta))$$

$$\Rightarrow \Psi(x, y) - \frac{i}{\hbar}\,(d\theta)\,\langle x, y \mid L_z \mid \Psi\rangle$$

$$= \Psi(x, y) + y\,(d\theta)\,\frac{\partial}{\partial x}\Psi(x, y) + (-x\,(d\theta))\frac{\partial}{\partial y}\Psi(x, y) + ...$$

We have used expansions in $d\theta$ to first order on both sides of (2.7), which reduces to an expression for the infinitesimal generator of rotation in function space:

$$\langle x, y \mid L_z \mid \Psi\rangle = i\hbar[y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}]\Psi(x, y)$$

We are led to the following identification:

$$L_z - coordinate\ basis \rightarrow i\hbar[y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}]$$

We have here arrived at the quantum-mechanical generator of infinitesimal rotation which is what we set out to do. The operator L_z can be identified with the angular momentum operator. The label "coordinate basis" above refers to the fact, that the operator looks this way in a coordinate basis with basis vectors $|x, y\rangle$, as opposed to the "momentum basis", where the linear momentum operators take on a "simple" form, while the position operators take on the form of a differential operator (see below).

The operator L_z takes a value of the wave function $\Psi(x, y, z)$ which assigns a specific value to each point in space (x, y, z), and then reassigns the same value to a rotated point $(x\prime, y\prime, z\prime)$.

If we Fourier transform this operator to momentum space, in which it might be convenient to work at times, we see that

$$L_z - momentum\ basis \longrightarrow i\hbar[p_y\frac{\partial}{\partial p_x} - p_x\frac{\partial}{\partial p_y}]$$

Since the rotations occur in the physical three dimensions, we can generalize to the operators L_y and L_x by simply choosing the rotation to be around the x or y-axis.

An important aspect for the work in this dissertation is the fact that the connection between angular momentum and the magnetic moment is not altered by the transition to quantum mechanics [19].

2.2.2 Finite rotation generators in 3 and ∞ dimensions

The finite rotation operator $U[R(\phi, \hat{n})]$ can be obtained after considering that the rotation group SO(3) is generated by its Lie algebra. The generator of finite rotations is therefore given by [4]:

$$U[R(\phi, \hat{n})] = \lim_{N\to\infty} (I - \frac{i}{\hbar}\frac{\phi}{N}\hat{n}\cdot\vec{L})^N = \exp(-\frac{i}{\hbar}\phi\hat{n}\cdot\vec{L}) \qquad (2.8)$$

where $0 < \phi < 2\pi$ and the plane of rotation defined by the vector $\hat{n}$ is arbitrary.

The $\vec{L}$ can then either be a finite matrix, in which case it acts on a vector (x, y, z) or a differential operator, in which case it acts on the function, or state vector $\Psi(x, y, z)$.

Both forms are representations of generators of infinitesimal rotations of the SO(n) algebra, but they act in different dimensions.

While the matrix (2.5) rotates a given vector in three space around a given axis (z-axis in this case), the differential operator rotates a scalar field $\Psi(x, y, z)$ around the same axis. The labels (x, y, z) of the function $\Psi(x, y, z)$ denote 3 sets of infinite dimensional coordinate systems (x, y and z respectively), in which the basis vectors are infinitesimally removed (the entities x and $x + dx$ would be two basis vectors, for instance) while the components of a canonical two dimensional vector live in a simple two dimensional space.

The two dimensional space features in quantum mechanics in the form of spinors, where a direct product between a spinor and Hilbert space yields a "twice infinite" dimensional state vector, in the sense that we are dealing with two infinite sets. In this case the use of a matrix is appropriate since the wave function has vector characteristics introduced by the spinors. The differential operator must be used in the case when the wave function is a scalar function, since then its variables are continuous and they span an infinite dimensional Hilbert space. There is no way of mapping between a two dimensional and an infinite dimensional Hilbert space. The two operator representations can therefore never yield the same result when acting on the same state vector, since the state vector has to be in either one of the spaces. This fact is exploited when describing spin.

2.2.3 Commutators and eigenvalues

The commutation relations between the infinitesimal Hilbert space generators L_x,L_y and L_z follow the Lie algebra of SO(3). This is expected, since they constitute a different representation of the same algebra [5]:

$$[L_i, L_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} L_k$$

where ϵ_{ijk} is the fully skew-symmetric tensor with $\epsilon_{123} = 1$. These commutation relations can be obtained by explicit construction. The fact that the generators do not commute implies that we cannot construct simultaneous eigenstates of any two of the generators. The generators cannot be diagonalized simultaneously.

Now that we have constructed the operators of rotation, we can solve the eigenvalue problem for a scalar wave function.

The eigenvalues turn out to be [5] :

$$L_z |lm\rangle = m\hbar |lm\rangle \tag{2.9}$$

where $l = 0, 1, 2, 3....$and $m = l, l-1, ..., -l$

We also define the total angular momentum operator squared, which gives the square of the magnitude of the angular momentum vector when acting on an eigenstate, by:

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

This operator does commute with all L_i operators, implying that we can know the eigenvalues of an operator of rotation in any direction and its magnitude at the same time. The eigenvalue problem of this particular operator can be solved to yield:

$$L^2 |lm\rangle = l(l+1) |lm\rangle \tag{2.10}$$

with l defined as in (2.9)

We can now also define the raising and lowering operators:

$$L_{\pm} = L_x \pm iL_y \tag{2.11}$$

$L_{\pm}$ raises and lowers the eigenvalue of L_z by $\hbar$,while leaving the eigenvalue of L^2 alone. The raising/lowering operator commutes with L^2.

A very important feature of both the L^2 and the L_i operators is that they commute with the Hamiltonian of a spherically invariant system. Simultaneous diagonalization can be achieved this way, making a lot of very important physical problems easier to handle.

In this chapter, we have had a look at rotational phenomena, and how these can be set within a mathematical framework in quantum mechanics. We shall see in the next chapter on spin, which was thought to be an internal rotation, how the concepts introduced in this chapter can be used.

Chapter 3

An important quantum label: Spin

It was clear from experiments (e.g. Stern-Gerlach and spectroscopic data) that the orbital angular momentum alone could not account for all the observed facts. The data suggested that stationary particles behaved as if they had angular momentum. This quantum degree of freedom was thought to be internal spin.

The inclusion of spin into quantum mechanics was largely pioneered by E. Wigner [1] and W. Pauli.

Wigner worked mainly in representation theory describing the Poincaré group, while Pauli used the representation of SU(2) to make the physical concept of spin mathematically tractable.

Wigner constructed his argument on spin on semi-classical grounds: he assumed the particle to rotate like a spinning top, which is not an altogether correct analogy. Spin should be thought of more in terms of a quantum number or an asymmetry in the particle itself, or rather in the way it interacts. The "spinning top" description was shown by Dirac [3] to be fallacious. He argued that a particle like the electron would have to rotate at velocities exceeding the speed of light to produce the magnitude of angular momentum measured experimentally.

Spin was later shown by Dirac to be a relativistic effect and we shall see how it emerges from the Poincaré group.

In this chapter, we shall treat spin as a form of angular momentum. We shall lean heavily on the results of the previous chapter. We shall see that the commutation relations remain the same and conclude the chapter by stating the importance of spin lies in the fact that its magnitude is an invariant of the Poincaré group.

3.1 The Description of Spin

3.1.1 Spin as an internal rotation

For now, let us stay with the classical picture, which is intuitively more appealing. It can be abandoned later.

Let us begin by writing down a total angular momentum operator, which includes both spin and orbital angular momentum:

$$\vec{J} = \vec{L} + \vec{S} \tag{3.1}$$

$\vec{J}$ is the total angular momentum, $\vec{L}$ is the orbital angular momentum and $\vec{S}$ is the spin angular momentum. We know [5] that $\vec{J}$ must obey the same commutation relations as the $\{L_i\}$, since the commutation rules reflect the law of combination of general rotations with L_i as a special case treating orbital angular momentum. We can therefore rewrite the above eigenvalue equations (2.9) and (2.10)as:

$$\begin{aligned} J_z \left| jm \right\rangle &= m\hbar \left| jm \right\rangle \\ J^2 \left| jm \right\rangle &= j(j+1) \left| jm \right\rangle \end{aligned}$$

where $j = 0, 1, 2, 3....$and $m = j, j-1, ..., -j$

Of interest to us is the raising/lowering operator of J_i, which is given by analogy to (2.11) as:

$$J_{\pm} = J_z \pm i J_x$$

The eigenvalue problem of this operator yields the following result:

$$J_{\pm} \left| jm \right\rangle = \hbar[(j \mp m)(j \pm m + 1)]^{\frac{1}{2}} \left| j, m \pm 1 \right\rangle$$

If we want to obtain the matrix elements of the J_x , say, in the $|jm\rangle$ basis, then we compute:

$$\langle j'm'|\, J_x \,|jm\rangle = \langle j'm'|\, \frac{J_+ + J_-}{2} \,|jm\rangle$$

which yields the result:

$$= \frac{\hbar}{2}\{\delta_{jj'}\delta_{m',m+1}[(j-m)(j+m+1)]^{\frac{1}{2}} + \delta_{jj'}\delta_{m',m-1}[(j+m)(j-m+1)]^{\frac{1}{2}}\}$$

and when explicitly calculated, the following matrix results :

$$J_x \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{\hbar}{2} & 0 & 0 & 0 & \\ 0 & \frac{\hbar}{2} & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \frac{\hbar}{\sqrt{2}} & 0 & \\ 0 & 0 & 0 & \frac{\hbar}{\sqrt{2}} & 0 & \frac{\hbar}{\sqrt{2}} & \\ 0 & 0 & 0 & 0 & \frac{\hbar}{\sqrt{2}} & 0 & \\ \vdots & & & & & & \ddots \end{pmatrix} \tag{3.2}$$

This is the infinite dimensional matrix representation of J_x.

J_y and J_z can be found in a similar way. The matrix is block-diagonal with zeroes on the main diagonal. The commutation relation $[J_x, J_y] = i\hbar J_z$ is obeyed in each block separately. This representation is irreducible, i. e., it has no invariant subspaces.

So far we have seen that the operator L_i reassigns the scalar value of $\Psi(x, y, z,)$ to a new coordinate (x',y',z'). The spin operator does not act on the scalar value of the wave function but reassigns the components of the wave function into linear combinations of each other. We shall need a matrix representation to accomplish this, since $\vec{S}$ acts on the indices (the spin space) as opposed to the coordinates. The fact that they do act on different parts of the wave function allows us to separate the commutation relations. These relations are the same as for the $\{L_i\}$, since the $\{L_i\}$ are just a special case of the $\{J_i\}$, with $s = 0$, where s is the expectation value of the spin.

So:

$$[J_i, J_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} J_k$$

This can be rewritten, by (3.1), as:

$$[L_i, L_j] + [S_i, S_j] = i\hbar[\sum_{k=1}^{3} \epsilon_{ijk} L_k + \sum_{k=1}^{3} \epsilon_{ijk} S_k]$$

Using the fact that the commutation relations for the $\{L_i\}$ are known, we are led to :

$$[S_i, S_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} S_k \tag{3.3}$$

Now in (3.2) we found general $(2j+1) \times (2j+1)$ matrices in the diagonal blocks of J_x, with $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$,that obeyed these relations, but there is an infinite number of these matrices along the diagonal. To find out which one of this infinite number of matrices is the one that we want (the one that gives the correct measurable eigenvalues), we have to consult experimental data.

3.1.2 Pauli matrices and spinors

Experiment tells us that we measure only two eigenvalues, namely $\pm\frac{\hbar}{2}$. This leads us to conclude that the matrices that we are looking for must have dimensions (2×2). Going back to (3.2) we see that in the case of S_x the matrix looks as follows:

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The other matrices, which are found from the matrix representation of J_y and J_z (not shown, but see for instance [5]), look as follows :

$$S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These are the matrices appropriate for spin-$\frac{1}{2}$ particles, such as the electron, with $l = 0$. The S_i matrices, without the factor $\frac{\hbar}{2}$, are called the Pauli matrices, denoted σ_i, and forms the basis of Su(2). It is easily verified that the commutation relations (3.3) hold for these matrices.

Hence the way to describe a spin $\frac{1}{2}$ particle is through a 2-component wave function called a spinor. The z-direction is singled out, arbitrarily, and the spinors are the eigenvectors of the matrix S_z. The electron spinor looks as follows:

$$\Psi = \begin{pmatrix} \Psi_+(x,y,z) \\ \Psi_-(x,y,z) \end{pmatrix}$$

or, equivalently:

$$\Psi = \Psi_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \Psi_- \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The eigenvalue problem of the S_z operator acting on Ψ will have the form:

$$S_z\,|\Psi\rangle = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \Psi_+(x,y,z) \\ \Psi_-(x,y,z) \end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix} \Psi_+(x,y,z) \\ \Psi_-(x,y,z) \end{pmatrix}$$

where we can see that the eigenvalues $\pm\frac{\hbar}{2}$ are produced for the up/down wave functions.

This, of course, represents a rather "artificial" way of introducing spin into the wave function. Experiment needs to be consulted *a priori*. A much more elegant and insightful way to arrive at the description of spin is provided by the Dirac theory, where spin comes out naturally when relativity is taken into account. Spin can therefore be viewed as a consequence of relativity.

3.1.3 Spin and magnetic moment

In analogy to the discussion on the relationship between angular momentum and magnetic moment, we can also establish a relationship between spin and magnetic moment. We could guess that the relationship is analogous to the angular momentum case, [19]

$$\vec{\mu}_s = g_s \frac{\mu_B}{\hbar}\vec{S}$$

with $g_s = 1$, but this assumption is, in fact, incorrect.

It turns out, from experiment, that $g_s = 2$. This fact can be explained theoretically by a process called Thomas precession, which is a relativistic effect. The Thomas precession effect is a consequence of Lorentz contraction and the simultaneous application of 2 boosts perpendicular to each other and acting on the CMS system. For a derivation of the effect, see for instance [19].

3.1.4 Spin: Quantum mechanics and Groups

Spin and Quantum mechanics

The physical interpretation of the spinor description is, like everything in Q.M., based on probabilities. For a mixture of spin up/down states, the spinor entries reflect the respective probabilities of the particle being in a spin up or down state, with eigenvalues $\pm\frac{\hbar}{2}$. To this end, the spinor has to be normalized. A state polarized in an arbitrary direction can be constructed by applying the same generator of finite rotations that we saw earlier for rotations in 3 dimensions, except that now $\vec{L}$ is substituted by $\vec{\sigma}$, the Pauli matrices, and there is an additional factor of $\frac{1}{2}$, which is a consequence of the fact that there is a 2-1 homomorphism between the SU(2) and SO(3) algebra. The basic need for a factor of $\frac{1}{2}$ arises from the SU(2) algebra dealing with complex numbers. This homomorphism can be proven by an elegant but lengthy argument, which is shown, for instance, in [4].

Spin and Group theory

From group theory [4], we know that SU(2)×SU(2) gives us SO(3,1)$^{\uparrow}$, the Lorentz group which is an infinite Lie group. This group leaves the Minkowski metric of space-time, which we obtain from relativity via the expression $s^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$, invariant.

If we form a direct group product between SO(3,1)$^{\uparrow}$ and T(4), we obtain the Poincaré group.

T(4) is the group of space-time translations and discrete reflections and obeys a four dimensional abelian Lie algebra. The Poincarré group leaves the quantity $s^2 = (x_0 - y_0)^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2 - (x_3 - y_3)^2$ in Minkowski space, invariant.

The Lorentz transformations and the translations do not commute, which is a very important result, as we shall see later. The four generators of infinitesimal translation of T(4), three in space and one in time, all commute.

The Lorentz group itself has six generators of infinitesimal transformation, three of rotation and three boosts. The Poincaré group is therefore a group with 10 generators. This group represents the structure of our space-time, and the postulates of relativity are embedded in it.

Since this group describes the changes that observables undergo when they are subjected to rotations, boosts or translations, the invariants of this group must give us the observables which stay the same under any transformations of the group.

This is a very strong statement.

If these invariants are found, a particle can be described in any frame in terms of them, since they remain unaffected by group operations.

These invariants are also called Casimir operators and they have the important property of commuting with every element of the group.

One of these invariants, the first Casimir operator of the Poincaré group, is found to be the (rest mass)^2 of a particle.

The other one, the second and last Casimir operator of the Poincaré group, was shown by Wigner to be the (magnitude of spin)^2.

He proved that the (magnitude of spin)^2 is not subject to any changes regardless of the transformations. This fact is <u>central</u> to the model that we shall present later. The invariant magnitude squared of spin for a spin-$\frac{1}{2}$ particle is given by the operator S^2 [5]:

$$S^2 = \hbar^2 \begin{pmatrix} \left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right) & 0 \\ 0 & \left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right) \end{pmatrix} = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(Magnitude of spin)^2, as well as (rest mass)^2 can hence be used to describe a particle in the rest frame but it can be translated to any frame since it commutes with the boost. This result is used extensively in physics.

This formalisms derived in this chapter were largely based on intuition and direct experimental evidence. The next chapter provides us with a very elegant way of deriving the spinor formalism. We shall see how spin emerges naturally from the Dirac equation.

Chapter 4

The Dirac equation

Paul Dirac tried to incorporate the theory of relativity into quantum mechanics [3]. He suggested that, to make the Schrödinger equation covariant, the required wave equation be linear in the time as well as in the space derivatives. This must clearly be the case, since, as we saw in the metric for space-time (the quantity kept invariant under operations of the Poincaré group), space and time are treated on an equal footing.

Previous attempts to unite relativity and Q.M. came in the form of the Klein-Gordon equation, which were initially discarded, since the description did not adhere to the probabilistic interpretation of Q.M. due to negative energy solutions.

In this chapter, we shall follow the path taken by Dirac himself in deriving the equation, which bears his name.

We begin with the classical relativistic Hamiltonian, derive the form of a relativistically invariant wave equation and set up the algebra of the coefficients, which turn out to be matrices.

The Dirac equation has great bearing on the concept of spin, since it comes out naturally from the equation. This is not really surprising, since the Dirac equation emerges from the Poincarré group.

We shall have a look at a simple example of how the equation is used. We then point out some of the physically interesting features of the equation and conclude by looking at a popular representation of the Clifford algebra.

4.1 Dirac's Argument

4.1.1 Building the Dirac equation

Analogy with the Schrödinger formalism

The prescriptions to cross from classical mechanics to Q.M., which were applicable to the Schrodinger picture, can be stated as follows [2]:

$$\begin{aligned} H &\rightarrow i\hbar\frac{\partial}{\partial t} \\ \vec{p} &\rightarrow \frac{\hbar}{i}\vec{\nabla} \end{aligned}$$

where H is the classical Hamiltonian and $\vec{p}$ the classical momentum vector.

The operator identification for the linear momentum operator can be deduced from the classical group T(3) of space translations in very much the same way as the angular momentum operator was found earlier from the rotation group SO(3). The time dependence of an evolving wave function is described via the identification of the Hamiltonian H with a time derivative.

The free Schrödinger equation would then arise by applying these substitutions to the classical equation of motion of a free particle:

$$H = \frac{\vec{p}^2}{2m}$$

This is the classical Hamiltonian. To make the transition to special relativity, we have to apply the relativistic relation which describes the total energy of a body:

$$H = \sqrt{\vec{p}^2 + m_0^2}$$

where m_0 is the rest mass of the body, $\vec{p}$ its momentum and c the speed of light.

We can see here already that there is going to be trouble with the square-root. Dirac circumvented this problem by completing the square of the above Hamiltonian. He rewrote the relativistic Hamiltonian as [5]:

$$H = \sqrt{(\vec{\alpha}.\vec{p} + \beta m)^2}$$

or, alternatively, after the Q.M. substitutions have been made [3]:

$$i\frac{\partial\Psi}{\partial t} = \frac{1}{i}(\alpha_1\frac{\partial\Psi}{\partial x_1} + \alpha_2\frac{\partial\Psi}{\partial x_2} + \alpha_3\frac{\partial\Psi}{\partial x_3}) + \beta m\Psi \tag{4.1}$$

where the $\{\alpha_i\}$ and the $\{\beta\}$ had to be determined by comparing coefficients with the completed square expression.

Identification of the coefficients

Initial speculation would tell us that the $\{\alpha_i\}$ and the $\{\beta\}$ could not be c-numbers since the equation would otherwise not be invariant under spatial rotations. The equation had to be a matrix equation. This is verified by the result of the comparison between the coefficients, which yields the conditions for the $\{\alpha_i\}$ and the $\{\beta\}$:

$$\alpha_k\alpha_i + \alpha_i\alpha_k = 2\delta_{ik} \tag{4.2}$$

$$\beta\alpha_i + \alpha_i\beta = 0$$

$$\alpha_i^2 = \beta^2 = 0$$

Hence Ψ was thought to be a vector of the form:

$$\Psi = \begin{bmatrix} \Psi_1 \\ \vdots \\ \Psi_n \end{bmatrix}$$

and the $\{\alpha_i\}$ and the $\{\beta\}$ were $(n \times n)$ matrices.

This has a strong similarity to the 2-component Pauli spinor.

Considering the fact that these matrices had to be even-dimensional to allow for a zero trace and that they could not be the Pauli matrices of which there are only three, the four (4×4) matrices were obtained:

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{4.3}$$

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where the $\{\sigma_i\}$ are the Pauli matrices.

The wave function itself had to be a vector with 4 components.

The Dirac equation could then be expressed in a four dimensional form in order to preserve the symmetry between ct and x^i which is desired in Lorentz covariant systems. (4.1) was rewritten [1]:

$$i(\gamma^0\frac{\partial}{\partial x^0}+\gamma^1\frac{\partial}{\partial x^1}+\gamma^2\frac{\partial}{\partial x^2}+\gamma^3\frac{\partial}{\partial x^3})\Psi-m\Psi=0 \tag{4.4}$$

$$\gamma^0=\beta;\gamma^i=\beta\alpha_i \tag{4.5}$$

$$i=1,2,3$$

The transition to the γ^μ matrices had also the advantage that the commutation relations can be written in a much neater form:

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}I_4$$

4.2 Covariance of the Dirac equation

In order for this equation to be covariant, an operator $S(a)$ needs to be found that relates a wave equation $\Psi(x,y,z)$ observed in a frame O to an observation in O', $\Psi(x',y',z')$ which describes for O' the same physical state. The operator $S(a)$ needs to be invertible. It needs only to depend on the relative velocities of O and O' and on their spatial orientation.

When considering the properties necessary for $S(a)$, it is clear that the transformation must be linear and invertible. Since $S(a)$ acts on a 4-component spinor, it must take on the form of a 4×4 matrix. By explicit construction and substitution into the Dirac equation, the following property can be derived for the transformation $S(a)$[2]:

$$a^{\nu}_{.\mu}\gamma^\mu=S^{-1}(a)\gamma^\nu S(a) \tag{4.6}$$

The 6 independent $\Delta\omega^{\mu\nu}$ generate an infinitesimal proper Lorentz transformation of the form:

$$\begin{aligned} x'^\mu &= a^\mu_{.\nu}x^\nu \\ a^\mu_{.\nu} &= g^\mu_{.\nu}+\Delta\omega^\mu_{.\nu} \end{aligned}$$

where $g^\mu_{.\nu}$ are the entries of the Minkowski metric and $a^\mu_{.\nu}$ are the matrix elements implementing a Lorentz transformation. The entries $\Delta\omega^{\mu\nu}$ are antisymmetric, $\Delta\omega^{\mu\nu}=-\Delta\omega^{\nu\mu}$

Expanding $S(a)$ in powers $\Delta\omega^{\nu\mu}$ to first order (only the linear term was kept) , $S(a)$ was found to be [2]:

$$S(a) = 1 - \frac{i}{4}\sigma_{\mu\nu}\Delta\omega^{\mu\nu}$$

And the inverse of $S(a)$ is:

$$S^{-1}(a) = 1 + \frac{i}{4}\sigma_{\mu\nu}\Delta\omega^{\mu\nu}$$

Substituting the results for $S(a)$ and $S^{-1}(a)$ into (4.6), it is found that a $\sigma_{\mu\nu}$ can be constructed with the correct properties via the commutator:

$$\sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_{\mu}, \gamma_{\nu}\right] \tag{4.7}$$

As mentioned above, the way that the wave function was written recalled a Pauli spinor. The wave function which features in the Dirac theory is called a 4-component Lorentz spinor.

Since the Lorentz rotations form a Lie-group, the infinitesimal proper generator of Lorentz transformation can be exponentiated to give us the generator of finite boosts.

Using the same method as for the derivation of (2.8) [2]

$$S_{finite} = \exp(-\frac{i}{4}\omega\sigma_{\mu\nu}I_n^{\mu\nu})$$

where $I_n^{\mu\nu}$ are the entries of a (4×4) matrix of coefficients for a unit Lorentz rotation about the direction n in Minkowski space and ω is the finite angle of rotation. For a boost in the z-direction, S_{finite} can be constructed as in the following sketch [2]:

We identify

$$\Delta\omega^{\mu\nu} = \Delta\omega I_n^{\mu\nu}$$

where $I_n^{\mu\nu}$ is the infinitesimal generator of the Lorentz group. It can be constructed from the algebra of the Lorentz group (very much the same way as an infinitesimal generator of SO(3) was derived in the chapter on angular momentum) and takes on the form, for a boost in the $z-$direction

$$I_n^{03} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

which has the property

$$\left(I_n^{03}\right)^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $(I_n^{03})^3 = I_n^{03}$.

We can use this in generating a finite rotation with the operator $e^{\omega I_n^{03}}$ in a series expansion. The final result is the matrix

$$\begin{bmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{bmatrix} = \underbrace{\begin{pmatrix} \cosh\omega & 0 & 0 & \sinh\omega \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\omega & 0 & 0 & \cosh\omega \end{pmatrix}}_{S_{finite}} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

The transformation law here acts on the components (x^0, x^1, x^2, x^3) of the vector.

An interesting fact is that a boost which is a "complex rotation" in Minkowski-space, corresponds to a translational motion in three dimensional space. The reason for this is as follows: a rotation in three-dimensional space leaves the quantity $x^2 + y^2 + z^2$ invariant. One of the postulates of special relativity is that, if we define $\tau = ict$, t being time and c the speed of light, the quantity $x^2 + y^2 + z^2 + \tau^2$ remains invariant under operations of the Lorentz group. By analogy we can regard the proper Lorentz transformation as a rotation in (x, y, z, τ)-space [7]

The angular variable ω is related to Lorentz transformations in the same way that θ is to rotations in three-space (see (2.5)). The angle of rotation is connected to the relative velocities of the two frames by the identification:

$$\begin{aligned} \tanh\omega &= \beta \\ \cosh\omega &= \frac{1}{\sqrt{1-\beta^2}} \end{aligned}$$

where the relative velocity of the two frames is $\beta = \frac{v}{c}$.

4.3 Physical Aspects of the Dirac equation

Of special interest to us is the fact that out of the Dirac theory some very important physical aspects arise quite naturally.

The first is that spin is naturally included in this description and can therefore be viewed as a relativistic effect. Secondly, the Lorentz spinor has four components. The upper two components are our classical description of the electron, with spin up and spin down as in the Pauli formalism. What about the lower two?

The lower two components of the spinor describe particles of spin $\frac{1}{2}$ with negative energy also having spin up and spin down [2],[3]. The lower two components of the spinor are also often called the "small" components since they only contribute non-negligibly at relativistic velocities. The anti-particle, which is described by the lower two components, is hence a consequence of a relativistic description. The existence of the negative energy particles in nature is evident in the form of anti-particles.

They were predicted by the Dirac theory and also by the Klein-Gordon equation, the latter describing spinless particles. The later experimental observation of anti-particles served to strengthen the belief in the validity of Dirac's theory, which was largely ignored at the time of publication. Dirac's interpretation of his equation with the use of hole theory shows us that the equation describes more than one particle: it has to take the negative energy solutions, interpreted as the positron, also into account.

At non-relativistic velocities, the Dirac equation reduces to the Pauli equation.

Thirdly, the anomalous gyromagnetic ratio is predicted correctly with $g_s = 2$ and did not have the shortcomings, as in the Pauli theory, of being predicted half as large as it should have been.

The reason for this lies in the Thomas-precession effect being of a relativistic nature. This effect causes the frames of the particle orbiting another particle to rotate with a frequency of one per revolution, thus doubling the effective rotation. Since relativistic effects are naturally built into the Dirac theory, this effect also comes out naturally.

From group theory and the solutions to the Dirac equation for a free particle, it can be shown that the equation describes a spin-$\frac{1}{2}$ particle like an electron. The justification for this is that the second Casimir operator of the Poincaré group, which is (spin magnitude)^2, yields a factor of $\frac{3}{4}$ for the total magnitude of spin squared. We have seen above, in the section on spin, that this corresponds to a spin-$\frac{1}{2}$ particle [6].

4.4 The γ matrices

Since the Dirac theory is relativistic in nature, we would guess that there is a connection between it and the SO(3,1)† group. Let us look at some of the properties of the γ matrices. The anti-commutation relations of the γ_i matrices that we have seen above, define a Clifford algebra as follows [4]:

$$\gamma_\mu\gamma_\nu + \gamma_\mu\gamma_\nu = 2g_{\mu\nu}I_4$$

The I_4 here is the (4×4) unit matrix. It is also obvious that the γ matrices squared will yield $\pm$ the unit matrix, the sign depending on if we work in the Weyl or in the chiral basis. We have been working in the Weyl representation, which allows us to interpret the components of the spinor as probabilities of the particles being in the spin up or down state.

The γ matrices generate the Lorentz group, due to the fact that they are functions of the Pauli matrices. The Pauli matrices in turn form the basis of the SU(2) group and the Lorentz group , SO(3,1)† , is homomorphic to a direct product of the elements of SU(2)$\times$SU(2). Although the γ matrices are not direct products of SU(2)$\times$SU(2), a subtle relationship between the SO(3,1)† and the γ matrices can be assumed.

Other 4×4 matrices often feature in the Dirac theory. These 16 linearly independent matrices can be found by forming products with the γ matrices [2]:

$$\Gamma^S = I_4$$

$$
\begin{aligned}
\Gamma^V_\mu &= \gamma_\mu \\
\Gamma^T_{\mu\nu} &= \sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right] \\
\Gamma^P &= \gamma^5 \equiv \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \\
\Gamma^A_\mu &= \gamma_5\gamma_\mu
\end{aligned}
$$

(S ≡Scalar; V ≡Vector; T ≡Tensor;P ≡Pseudoscalar;A ≡ Axial vector)

These geometric objects represent all the possibilities which can be formed from the Clifford algebra, after invariance principles have been invoked. Their importance lies in the fact that different interaction will be mediated by different geometric objects. this fact will be exploited when discussing the baryon current

Because of their linear independence, any 4×4 matrix can be written in terms of these matrices. From these matrices, the Lorentz transformation properties of the bilinear forms $\overline{\psi}(x)\Gamma^n\psi(x)$ can be deduced, which has great bearing on the covariant current, which has the form of a bilinear covariant vector, with the definition:

$$\overline{\psi} = \psi^\dagger\gamma^0$$

We shall see more of the relativistic current in a later chapter.

In this chapter we saw that the inclusion of relativistic effects gives rise to spin naturally. We have divorced ourselves from the more empirical approach of the previous chapter. The Dirac equation provided new insights into not only spin, but some other important concepts as well. The next chapter will show us a way of tackling some of the problems that arise from a relativistic description of quantum mechanics in connection with spin.

Chapter 5

The Helicity Formalism

We will now turn to the formalism upon which the model is based. The Helicity formalism essentially renders the spin projection quantum number, often denoted by m, onto a chosen quantization axis, the z-axis by convention, invariant under spatial rotation and special Lorentz transformations, i.e. boosts in the z-direction. The spin projection quantum number is not a Lorentz invariant, which can be easily understood by viewing the direction in which the spin is projected as a vector. When boosts are applied to a coordinate system, a contraction of the coordinate axis occurs in the direction of motion or an "opening up" effect of the axes is observed, depending on their spatial orientation. It is clear that a vector in a boosted coordinate system will be altered, and so will be it's projection on any given axis. As we have seen in the section on spin, the magnitude of the spin remains unaffected. The helicity formalism rests on the simple idea of choosing the direction of the boost as the quantization axis. The z-projection will therefore remain the same if boosts are applied.

In the following chapter, we shall start by defining rotations and boosts acting on a state vector and observe what happens to the quantum numbers. We then proceed to build a single particle state in the Helicity basis and consider the effects of discrete operations on the state.

A two particle state is then constructed which will serve as basis for the next chapter.

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5.1 Single particle state in the helicity basis

5.1.1 Boosts and rotations

Let us now see how these ideas are implemented.

A physical state $|jm\rangle$, where j is the spin and m the z-component of spin, may be rotated via the rotation operator. The unitary operator representing a rotation may be written [12]:

$$U\left[R\left(\alpha,\beta,\gamma\right)\right]=e^{-i\alpha J_z}e^{-i\beta J_y}e^{-i\gamma J_z} \tag{5.1}$$

with (α,β,γ) the standard Euler angles and $\vec{J}$ the generators of rotation. The rotation occurs first around the z-axis by α, then the y-axis by β and finally the z-axis again by γ.

The rotation of a state is given by:

$$U\left[R\left(\alpha,\beta,\gamma\right)\right]|jm\rangle=\sum_{m'}|jm'\rangle\, D^j_{m'm}\left(\alpha,\beta,\gamma\right)$$

with $D^j_{m'm}$ the standard rotation matrices as given by [15].

The boosts can be rewritten in the following form, in an arbitrary direction $\vec{p}$:

$$U\left[L\left(\vec{p}\right)\right]=U\left[R\left(\theta,\phi,0\right)\right]U\left[L_z\left(|p|\right)\right]U^{-1}\left[R\left(\theta,\phi,0\right)\right] \tag{5.2}$$

where the boost $U\left[L_z\left(|p|\right)\right]$, along the z-direction, is rotated by the rotation matrix $U\left[R\left(\theta,\phi,0\right)\right]$ and its inverse $U^{-1}\left[R\left(\theta,\phi,0\right)\right]$ in the direction of $\vec{p}$. Hence the state is first rotated onto the z-axis, boosted along this axis and then rotated back to the original direction.

So a canonical boosted state $|jm\rangle$ can be defined as follows:

$$|\vec{p},jm\rangle=U\left[L\left(\vec{p}\right)\right]|jm\rangle$$

or, using (5.2),

$$|\vec{p},jm\rangle=U\left[R\left(\theta,\phi,0\right)\right]U\left[L_Z\left(|p|\right)\right]U^{-1}\left[R\left(\theta,\phi,0\right)\right]|jm\rangle \tag{5.3}$$

It should be stressed that the quantum number m is well-defined in the *rest* frame and ill-defined in the frame where the particle has momentum $\vec{p}$.

The state (5.3) transforms under rotation like the rest state $|jm\rangle$:

$$U\left[R'\right]|\vec{p},jm\rangle=U\left[R'R\right]U\left[L_z\left(|p|\right)\right]U^{-1}\left[R'R\right]U\left[R'\right]|jm\rangle$$

$$= D^j_{m'm}(R')\,|R\vec{p}, jm'\rangle$$

which is a definite advantage, since now the applications to non-relativistic spin formalism may be extended to spin formalism involving relativistic particles with spin, since the rotation matrix $U[R']$ has no effect on the particle momentum.

5.1.2 The single particle state

Let us now define the helicity state of a single particle with spin j and momentum $\vec{p}$. It is clear from the previous discussion that the boost was applied along the quantization axis (z-axis) and then the entire system was rotated in the direction of the momentum. Interchanging the operations (boost $\Leftrightarrow$ rotation), i.e. rotating the system first in the direction of the momentum and then boosting along it does yield the same result. These are the two equivalent ways of approaching the problem.

Let us look at them both. First:

$$|\vec{p}, j\lambda\rangle = U[L(\vec{p})]\,U[R(\theta,\phi,0)]\,|jm\rangle \tag{5.4}$$

where λ is the invariant helicity quantum number.

The operator $U[R(\theta,\phi,0)]$ rotates the state in the direction of the momentum, or, equivalently, the quantization (z-axis) onto the momentum axis. The boost $U[L(\vec{p})]$ as defined in (5.2) boosts the system in the direction of $\vec{p}$ with magnitude $|\vec{p}|$.

Alternatively:

$$|\vec{p}, j\lambda\rangle = U[R(\theta,\phi,0)]\,U[L_z(|p|)]\,|jm\rangle \tag{5.5}$$

where the system is boosted along the z-direction with m already lying on the correct axis, so the boost does not affect the spin projection. The entire system is then rotated in the desired direction $\vec{p}$ by the rotation operator $U[R(\theta,\phi,0)]$.

The equivalence between the two different ways may be seen by expanding (5.4) with the help of (5.2). (5.4) then becomes:

$$|\vec{p}, j\lambda\rangle = U[R(\theta,\phi,0)]\,U[L_Z(|p|)]\,U^{-1}[R(\theta,\phi,0)]\,U[R(\theta,\phi,0)]\,|jm\rangle$$

The rotation operators are unitary, which may be easily understood from the fact that if a state is rotated, the inverse rotation will return the rotated state to the original one, resulting in a zero rotation. The operator of zero rotation is the identity. Hence we can write the unitarity relation:

$$U^{-1}\left[R\left(\theta,\phi,0\right)\right]U\left[R\left(\theta,\phi,0\right)\right]=I$$

which reduces (5.4) to (5.5).

The helicity quantum number λ is consequently invariant under boosts. It is also invariant under rotations, since the quantization axis is rotated together with the entire system.

The quantum label λ is therefore invariant under all the operations of the proper Lorentz group.

5.1.3 Parity and time reversal

What about the operations of time reversal and space inversion (parity), which woul' ' allow us to describe the helicity state under the discrete operations of the Poincarré group?

Let's look at them:

Parity, denoted $\mathfrak{P}$ has the classical correspondence:

$$\mathfrak{P}:\vec{r}\rightarrow-\vec{r}\,;\,\vec{p}\rightarrow-\vec{p}\,;\,\vec{J}\rightarrow\vec{J} \tag{5.6}$$

Time reversal, $\mathfrak{T}$, imposes the classical transformation:

$$\mathfrak{T}:\vec{r}\rightarrow\vec{r}\,;\,\vec{p}\rightarrow-\vec{p}\,;\,\vec{J}\rightarrow-\vec{J} \tag{5.7}$$

Quantum-mechanically, these operators (parity Π and time reversal Υ) have the following effects on a state vector. Parity:

$$\Pi\left|\vec{p},jm\right\rangle=\eta\left|-\vec{p},jm\right\rangle$$

where η is the intrinsic parity of the particle represented by $\left|\vec{p},jm\right\rangle$ assuming values of ±1.

Time reversal:

$$\Upsilon\left|\vec{p},jm\right\rangle=(-1)^{j-m}\left|-\vec{p},j\left(-m\right)\right\rangle$$

The actions of these operators on the helicity states is defined as, which is shown more explicitly in Appendix D, (9.34) to (9.37):

$$\Pi\left|\vec{p},j\lambda\right\rangle=\eta e^{-i\pi j}\left|-\vec{p},j\left(-\lambda\right)\right\rangle$$

and:

$$\Upsilon \left|\vec{p}, j\lambda\right\rangle = e^{-i\pi\lambda} \left|-\vec{p}, j\lambda\right\rangle$$

In the same sense that m is an eigenvalue of J_z, λ is an eigenvalue of $\vec{J}.\vec{p}$, since both the axis on which the eigenvalue is read of as well as the operator are rotated in the same direction.

Hence, under the transformations of parity (5.6)

$$\vec{J}\cdot\vec{p} \to -\vec{J}\cdot\vec{p}$$

and so $\lambda \to -\lambda$, while under the transformation of time reversal (5.7):

$$\vec{J}\cdot\vec{p} \to \left(-\vec{J}\right)\cdot\left(-\vec{p}\right) \to \vec{J}\cdot\vec{p}$$

$\lambda \to \lambda$, i.e. remains invariant.

5.2 Two particle state in the helicity basis

Now we can turn to the construction of two-particle states in the Helicity basis [12].

To pave the way, let us first look at how these states are constructed canonically.

We define the states of two particles, labeled 1 and 2 , having spin s_1 and s_2, quantization axis projection m_1 and m_2 and masses w_1 and w_2 respectively. Let $\vec{p}$ be the momentum of particle 1 specified in terms of the spherical angles (θ, ϕ).

We form the canonical two particle state *in the center of mass frame* as a product wave function:

$$\left|\theta\phi m_1 m_2\right\rangle = a\left\{U\left[L\left(\vec{p}\right)\right]\left|s_1 m_1\right\rangle U\left[L\left(-\vec{p}\right)\right]\left|s_2 m_2\right\rangle\right\} \tag{5.8}$$

where $U\left[L\left(\pm\vec{p}\right)\right]$ is defined as in (5.2) and a is a normalization constant.

We have seen that a state defined in this way, with the particles with definite momentum, transforms under rotation like the rest state, which means that we can couple the spins as if the state was in a rest state:

$$\left|\theta\phi s m_s\right\rangle = \sum_{m_1 m_2}\left(s_1 m_1 s_2 m_2 \mid s m_s\right)\left|\theta\phi m_1 m_2\right\rangle \tag{5.9}$$

where s is the total spin quantum number of the coupled system, m_s it's spin projection and $\left(s_1 m_1 s_2 m_2 \mid s m_s\right)$ the usual Clebsch-Gordon coefficients, coupling the spins of two particles with spin s_1 and s_2 and projection quantum numbers m_1 and m_2 respectively to the total spin s and projection m_s.

If R is a rotation which takes $\Omega = (\theta, \phi)$ to $R' = R\Omega$, it may be shown, by using the relation

$$D^{j_1}_{m_1' m_1} D^{j_2}_{m_2' m_2} = \sum_{j_3 m_3' m_3} (j_1 m_1' j_2 m_2' \mid j_3 m_3') (j_1 m_1 j_2 m_2 \mid j_3 m_3) D^{j_3}_{m_3' m_3}$$

which is a coupling rule that the D-functions obey, that

$$U[R] |\Omega s m_s\rangle = \sum_{m_s} D^{s}_{m_s' m_s}(R) |R' s m_s'\rangle \tag{5.10}$$

The equation (5.10) shows that the rotation matrix $U[R]$ acts on the momentum orientation of particle 1 (and hence on particle 2 as well, via momentum conservation, since the system is described in the two-particle rest frame) and therefore has to rearrange the projection quantum number m_s onto the new z-axis with the help of the rotation matrix $D^{s}_{m_s' m_s}$. The only quantum label which remains unaffected is s.

The total spin magnitude s is invariant under rotation.

A state of fixed orbital angular momentum is constructed from (5.9) :

$$|lmsm_s\rangle = \int d\Omega \, Y^l_m(\Omega) |\Omega s m_s\rangle \tag{5.11}$$

where $d\Omega = d\phi \, d\cos\theta$ and Y^l_m are the spherical harmonics.

Upon investigation of the rotational properties of equation (5.11), one finds that the state of fixed orbital momentum transforms under rotation as a product of the two "rest states" $|lm\rangle$ and $|sm_s\rangle$, so the orbital momentum part is decoupled from the spin part [12].

It follows that one may now construct a state of total angular momentum J and projection M :

$$|JMls\rangle = \sum_{mm_s} (lmsm_s \mid JM) |lmsm_s\rangle \tag{5.12}$$

$$= \sum_{mm_s m_1 m_2} (lmsm_s \mid JM) (s_1 m_1 s_2 m_2 \mid sm_s) \int d\Omega \, Y^l_m(\Omega) |\Omega m_1 m_2\rangle \tag{5.13}$$

It is noteworthy that both l and s are rotational invariants and that the expression essentially reduces to non-relativistic L-S coupling. This can be seen by applying a rotation to the state (5.13), and using properties of the rotation matrices, which are displayed in the appendix of [12], the state (5.13) reduces to

$$U[R]\,|JMls\rangle = D^J_{M'M}\,|JM'ls\rangle$$

where l and s remain untouched by the rotation.

Now we are in a position, on the basis of the previous discussion, to construct two-particle helicity states.

We can write, in analogy with (5.8) :

$$|\theta\phi\lambda_1\lambda_2\rangle = aU[R]\left\{U[L(\vec{p})]\,|s_1m_1\rangle\, U[L(-\vec{p})]\,|s_2m_2\rangle\right\}$$

$$\equiv U[R(\phi,\theta,0)]\,|00\lambda_1\lambda_2\rangle \tag{5.14}$$

where, as before, $|s_im_i\rangle$ is the rest state of the i^{th} particle in the helicity basis.

States of definite angular momentum J and projection quantum number M may be constructed as follows:

$$|JM\lambda_1\lambda_2\rangle = \frac{N_J}{2\pi}\int dR\; D^{J*}_{M\mu}U[R(\phi,\theta,\gamma)]\,|00\lambda_1\lambda_2\rangle \tag{5.15}$$

with N_J a normalization constant and $dR = d\alpha\, d\cos\beta\, d\gamma$. The operator $U[R(\phi,\theta,\gamma)]$ rotates the state vector in the direction defined by the Euler angles (ϕ,θ,γ) , while the operator $D^{J*}_{M\mu}$ is responsible for the change in variables, by rotating the operators accordingly. More clarity on this point may be obtained by consulting Appendix D.

This state is indeed a state of definite angular momentum and the helicity quantum numbers λ_1 and λ_2 are invariant under rotations, both of which may be shown in an argument analogous to the one of the canonical states [12].We can see this as follows:

The rotation matrix $U[R(\phi,\theta,\gamma)]$ in (5.15) may be explicitly written

$$U[R(\phi,\theta,\gamma)]\,|00\lambda_1\lambda_2\rangle$$

$$\begin{aligned} &= U[R(\phi,\theta,0)]\,U[R(0,0,\gamma)]\,|00\lambda_1\lambda_2\rangle \\ &= e^{-i(\lambda_1-\lambda_2)\gamma}\,U[R(\phi,\theta,0)]\,|00\lambda_1\lambda_2\rangle \end{aligned} \tag{5.16}$$

where use has been made of the definitions of the rotation operators as given in (5.1) .

We can see that the last line of (5.16) will introduce a $\delta(\lambda_1 - \lambda_2)$ for the integration over $d\gamma$.

Substituting the relation found in (5.16) into (5.15) and integrating over $d\gamma$, the expression

$$|JM\lambda_1\lambda_2\rangle = N_J \int d\Omega\ D^{J*}_{M\lambda}(\phi, \theta, 0)\, |\phi\theta\lambda_1\lambda_2\rangle$$

with $\lambda = \lambda_1 - \lambda_2$, is obtained.

This expression forms a two-particle state in the helicity description, with λ invariant under rotations and boosts. Our argument will be very similar in the construction of the three-particle state.

The powerful machinery introduced in this chapter will be one of the building blocks of the model presented next

Chapter 6

Model of the Nucleon

Every description of nature has to ultimately stand up to experiment: it must either explain existing data or predict outcomes of new experiments, ideally both.

In the first part of this chapter, some experiments will be outlined which led to the model under consideration. Some of these experiments have been conducted in recent years, and some date back over a decade.

We first look at some of the experiments which suggested that there are some unresolved difficulties concerning spin. It is because of these experiments that the model, which shall be presented in this chapter, was constructed.

We shall then set up the model of the nucleon, leaning heavily on [10] and [13], the two papers in which this particular model was initiated.

We shall construct a three particle state in the helicity basis, endow it with good angular momentum and colour quantum numbers, so that it can eventually be identified as the nucleon.

We shall show the construction of the boost and rotation operators explicitly and deal with the difficulties that arise.

6.1 Experiments which led to the model

Krisch [11] presented data on the comparison between the cross-sections between polarized and anti-polarized p-p scattering at high transverse momentum, where the asymptotically free region is probed. At these high momenta, one can consider diffractive scattering between the constituents. The only interactions may be of a spin-nature. The surprising result was that, for highly relativistic situations, the ratio of the spin-parallel (the spins of the two colliding beams are parallel) to the spin anti-parallel cross-sections increases dramatically.

This can be interpreted as the protons seeing a larger target in collisions of two beams of the same polarization. This is explained in [r12] as a strong spin-spin interaction effect in the two interaction regions each of which has an effective diameter of about $0.3fm$.

The question arises wether the above effect can be explained in terms of spin-dependent quark confining forces or wether it arises from quantum mechanical correlations necessary to produce a spin $\frac{1}{2}$ proton [10].

6.2 Preliminary considerations

6.2.1 Physical introduction

We have now arrived at the point where the model, which will include the previously discussed concepts, is introduced.

Why should we use a relativistic model as opposed to a non-relativistic one?

A non relativistic description of the dynamics of a constituent quark model exists, and, although it gives reasonable results, it is often felt that this description might not be realistic enough. The reason for this belief is that the diameter of the particles (i.e. nucleons) under consideration it is in the order of one Fermi and the Heisenberg uncertainty principle tells us that the constituent quarks would posses average momenta comparable to their effective masses [8]. This immediately tells us that a relativistic description is in order.

The previously discussed Dirac equation seems to fit the bill since the system which we wish to describe is composed of three fermions of spin-$\frac{1}{2}$.

There are, however, some problems with the description of a relativistic three-particle system within the framework of QCD: the fact that the system is not Lorentz covariant can be circumvented by considering quarks on the light-cone [16]. This in turn makes the components transverse to the angular momentum interaction dependent, which can be remedied via a Melosh transformation [r13], but only for a specific form of the interaction potential. An exact, unique form of this potential is not known.

These problems can be avoided by considering only a minimal correlation among the quarks [13]. This is a good enough assumption since, as was stated above, at the momenta under consideration, all but the spin interactions (as well as some others, like isospin and total angular momentum) are effectively negligible in the asymptotically free regime of QCD.

By "minimal correlation", we understand the following in terms of the model: the three-particle system under construction must conform to the quantum state which can eventually be identified with the proton. The system must have a well-defined isospin (T), total angular momentum (J) and parity (p) of the form: $T(J^p) = \frac{1}{2}\left(\frac{1}{2}^+\right)$.

The total 4-momentum P_N^μ of the combined system must be the sum of the 4-momenta of the constituents $p_i^\mu, i = 1,2,3$ and $\mu = 0,1,2,3$. This statement amounts to energy and 3-momentum conservation.

We can write:

$$P_N^\mu = p_1^\mu + p_2^\mu + p_3^\mu \tag{6.1}$$

We can also state, more specifically:

$$\vec{P}_{N(L)} = \sum_i^3 \vec{p}_{i_{\parallel(L)}} \tag{6.2}$$

where $\vec{P}_{N(L)}$ denotes the 3-momentum of the combined 3-particle system (nucleon), $\vec{p}_{i_{\parallel(L)}}$ denotes the momentum of the i^{th} constituent (quark) in the direction of the nucleon momentum (specified by the subscript "$\parallel$") as viewed in the lab frame of reference (indicated by the subscript "L").

It is also clear that

$$\vec{P}_{N_\perp} = \sum_i^3 \vec{p}_{i_\perp(N)} = 0 \tag{6.3}$$

i.e., that the 3-momenta of the constituents, $\vec{p}_{i_\perp(N)}$, in the nucleon rest frame, add to zero.

6.2.2 Outline of the argument

The argument will proceed as follows [10]:

1) A three-particle system is constructed in the helicity basis, using the constraints introduced in the previous chapter.

The system as a whole is described in the rest frame of the nucleon.

As we have seen in the section on the helicity formalism, the system will have the properties of good angular momentum and parity invariant under Lorentz transformations. Furthermore we can use the usual coupling rules for spinors (Clebsch-Gordon coefficients), which was also one of the advantages of using the helicity basis.

2) The algebra of the Lorentz group shows us that application of consecutive boosts do not necessarily commute. We wish to boost the three-particle system as a whole, i.e. the boost is applied to the CMS of the nucleon, as well as the constituents. The constituents are boosted at 90° to the direction of the CMS boost in which case the two boosts do not commute. An appropriate average boost is constructed to overcome this difficulty. The average boost will also introduce an effect similar to Thomas precession, an effect observed in atomic physics which is responsible for the constant of the gyromagnetic ratio to be equal to two to zeroth order.

3) The spatial components of the conserved electromagnetic current are calculated in the "brick wall" or "Breit" frame, with the property $\vec{P}_{N(f)} = -\vec{P}_{N(in)}$,on both the nucleon and quark level. The form factors $G_E(q^2)$ and $G_M(q^2)$ are computed in order to obtain the effective radius and magnetic moment expectation values of the system at $q = 0$.

6.3 Setting up the model

6.3.1 Notation

Let us now look at the notation [10]. A one particle Fermion state with spin j, s and momentum $\vec{p}$ will be denoted by the state vector

$$|\vec{p}, j, s\rangle = a_s^\dagger(\vec{p})|0\rangle \tag{6.4}$$

where $a_s^\dagger(\vec{p})$ is the usual Fock space creation operator and $|0\rangle$ the vacuum state.

The algebra of the Fock space operators is

$$\left\{a_{s'}\left(\vec{k}\right), a_s^\dagger(\vec{p})\right\} = \delta_{ss'}(2\pi)^3 2p_0\delta\left(\vec{k}-\vec{p}\right) \tag{6.5}$$

$$\left\{a_{s'}^\dagger\left(\vec{k}\right), a_s^\dagger(\vec{p})\right\} = 0$$

$$\left\{a_{s'}\left(\vec{k}\right), a_s(\vec{p})\right\} = 0$$

with the Heisenberg field operators

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0} \sum_m \left\{ e^{ipr} v_m(p)\, \mathrm{b}_m^\dagger(\vec{p}) + e^{-ipr} u_m(p)\, \mathrm{a}_m(\vec{p}) \right\} \tag{6.6}$$

including the correct Lorentz invariant phase space element. The states are defined as

$$\langle r, s| = \int \frac{d^3k}{(2\pi)^3\, 2p_0} \left(e^{ikr}\right) \langle 0|\, a_s\left(\vec{k}\right)$$

and

$$|\vec{p}, j, s\rangle = u_s^j(\vec{p})\, a_s^\dagger(\vec{p})\, |0\rangle$$

yielding

$$\langle r, s \mid \vec{p}, j, s\rangle = u_s^j(\vec{p})\, e^{ipr} \tag{6.7}$$

Normalization of the states gives:

$$\langle \vec{p}', j', s' \mid \vec{p}, j, s\rangle = 2p_0 \delta^{jj'} \delta_{ss'} \int d^3r e^{i\vec{r}\cdot(\vec{p}-\vec{p}')}$$

$$\begin{aligned} u_{s'}^{(j')\dagger}(\vec{p})\, u_s^j(\vec{p}) &= 2p_0 \delta^{jj'} \delta_{ss'} \\ \overline{u}_s^{(j')}(\vec{p}')\, u_{s'}^j(\vec{p}) &= 2p_0 \delta^{jj'} \delta_{ss'} \end{aligned} \tag{6.8}$$

and with the Lorentz invariant phase space element

$$\int_{p_0=\infty}^{p_0=+\infty} dp_0 d^3p\ \theta(p_0)\, \delta\left(p^2 - m_0^2\right) \equiv \frac{d^3p}{2p_0}$$

one obtains

$$\int \frac{d^3p}{2p_0} \langle \vec{p}', j', s' \mid \vec{p}, j, s\rangle = \delta^{jj'} \delta_{ss'} \tag{6.9}$$

6.3.2 Three particle state in the helicity basis

We now wish to construct a three particle state. In the section on the helicity formalism we saw that the simplest Ansatz was a product wave function like in (5.8)

$$\prod_{i=1}^{3} |\vec{p}_i, j_i, \lambda_i\rangle = b \prod_{i=1}^{3} \left\{ \sum_{s_i} \left(D^{j}_{s_i\lambda_i}(R_i^o) \right) U\left[L_z(|\vec{p}|)\right] \right\} |j_i, m_{i_o}\rangle \tag{6.10}$$

with the identifications as above, i.e. λ_i the Lorentz invariant spin projection, b a normalization constant, $D^{j}_{s_i\lambda_i}(R^o_i)$ a rotation matrix rotating the z-axis onto the momentum $\vec{p}$ of the individual quarks, $L_z(|\vec{p}|)$ a boost in the z-direction of each quark and $|j_i, m_{i_o}\rangle$ a canonical rest state spinor. The rotation matrix acts on the internal CMS coordinates (Euler angles) of the momentum of each of the three particles. The particles are in the "standard" orientation (See diagram below)

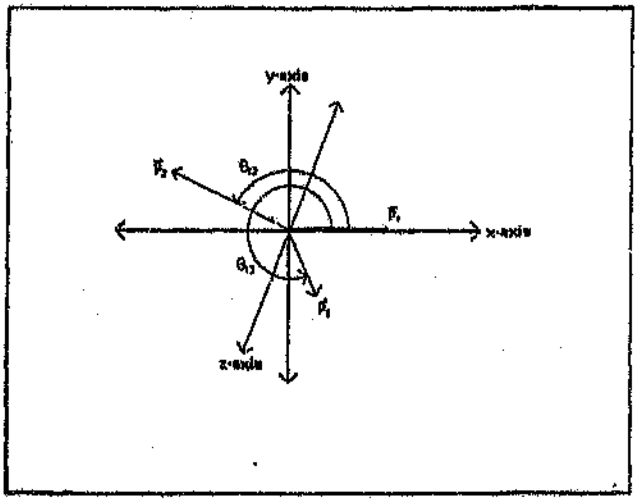


Figure 6.1: The three vectors $\vec{p}_1$,$\vec{p}_2$ and $\vec{p}_3$ in the "standard" orientation, contained within the x,y-plane

The argument for three particle states is essentially identical to the one shown in the previous Chapter for two particles.

One of the differences is the normalization. In the three particle case, the helicity state is normalized by using the restricted three-particle Lorentz-invariant phase space element:

$$dLips(P^2, p_1, p_2, p_3) = (2\pi)^4 \delta^4 (P - p_1 - p_2 - p_3) \prod_{i=1}^{3} \frac{d^3 p_i}{(2\pi)^3 2E_i} \qquad (6.11)$$

with $\vec{P} = 0$ in the CMS, $P^2 = M^2$ and the δ-function tells us that $\sqrt{P^2} = \sum_{i=1}^{3} E_i$.

Equation (6.11) can be rewritten, via a change of coordinates (see Appendix C for detail)

$$dLips(P^2, p_1, p_2, p_3) = \frac{4}{(4\pi)^5} dR dE_1 dE_2 \qquad (6.12)$$

with $\int dR = \int d\alpha d(\cos\beta)\, d\gamma = \frac{1}{2}(4\pi)^2$.

This can be viewed as follows: there are nine degrees of freedom in the expression (6.11) given by the $\vec{p}_i$ and 4 constraints in the form of 4-momentum conservation encoded in the δ-function. So we are left with 5 degrees of freedom given by the angles α, β, γ and the energies of two of the three quarks E_1 and E_2. The angles give the orientation of the normal $\widehat{n} = \widehat{p}_1 \times \widehat{p}_2$.

The three-particle helicity state can then be denoted by:

$$b \prod_{i=1}^{3} |\alpha, \beta, \gamma; E_1, E_2; , j_i, \lambda_i\rangle$$

The state can be normalized in a straightforward fashion and the result is $b = \frac{2}{(4\pi)^{\frac{5}{2}}}$ (see Appendix E)

The state is then endowed with definite angular momentum and the procedure is similar to the one outlined in the section on the helicity formalism, in (5.15). The three degrees of freedom α, β, γ are integrated out to give way to three other degrees of freedom which have a physically more intuitive interpretation, namely $J, S\ (= J_z)$ and μ.

$$|J, S, \mu; E_1, E_2; , j_i, \lambda_i\rangle$$

$$= \sqrt{\frac{2J+1}{2}} \int \frac{dR(\alpha,\beta,\gamma)}{2\pi} D^{J\,*}_{S\mu}(\alpha,\beta,\gamma)\, |\alpha, \beta, \gamma; E_1, E_2; , j_i, \lambda_i\rangle \qquad (6.13)$$

The quantum number μ can be identified as the parity of the system [13] under discussion and can assume values $\mu = \pm\frac{1}{2}$ (See Appendix D for details).

The full SU(4) canonical proton spinor can be written as follows:

$$|p_{n.r}, \uparrow\rangle = N_{n.r} \sum_{a,b,c,s_i,t_i,T,t,J,M} \frac{\epsilon_{abc}}{\sqrt{6}} \delta_{TJ} \left(j_1 s_1 j_2 s_2 \mid Js\right) \left(Jsj_3s_3 \mid \frac{1}{2}\frac{1}{2}\right) \times$$

$$\times \left(T_1 t_1 T_2 t_2 \mid Tt\right) \left(TtT_3t_3 \mid \frac{1}{2}\frac{1}{2}\right) {}^a\chi^{t_1}_{s_1}\, {}^a\chi^{t_2}_{s_2}\, {}^a\chi^{t_3}_{s_3} \qquad (6.14)$$

The quantities in brackets are the usual Clebsch-Gordon coefficients. ${}^a\chi^{t_i}_{s_i}$ denotes a full canonical quark spinor, a referring to the colour quantum number, $t_i = \pm\frac{1}{2}$ to the isospin projection and $s_i = \pm\frac{1}{2}$ to the spin projection. The colour index a will be suppressed.

Using the notation $\chi^{\frac{1}{2}}_{\frac{1}{2}} = u^{\uparrow}, \chi^{\frac{1}{2}}_{-\frac{1}{2}} = u^{\downarrow}, \chi^{-\frac{1}{2}}_{\frac{1}{2}} = d^{\uparrow}, \chi^{-\frac{1}{2}}_{-\frac{1}{2}} = d^{\downarrow}$, for up and down quarks, we arrive at the well-known expressio

$$\begin{aligned} \frac{3}{N_{n.r}} \left| p_{n.r}, \uparrow \right\rangle &= 2\left(u^{\uparrow}d^{\downarrow} + d^{\downarrow}u^{\uparrow}\right)u^{\uparrow} + 2u^{\uparrow}u^{\uparrow}d^{\downarrow} \qquad (6.15) \\ &\quad - \left(d^{\uparrow}u^{\downarrow} - u^{\downarrow}d^{\uparrow}\right)u^{\uparrow} \quad u^{\uparrow}d^{\uparrow}u^{\downarrow} - u^{\uparrow}u^{\downarrow}d^{\uparrow} \\ &\quad - d^{\uparrow}u^{\uparrow}u^{\downarrow} - u^{\downarrow}u^{\uparrow}d^{\uparrow} \end{aligned}$$

$$(6.16)$$

This is the full form of the proton wave function as derived from the group theoretical arguments of SU(4) outlined in the introduction. In rest states, the helicity description reduces to the canonical one.

A helicity spinor can be written in terms of ... onical spinor as

$$\zeta_{\lambda_i} = \sum_{\lambda_i}^{3} \left(D^{\frac{1}{2}}_{s_i \lambda_i} \left(R^o_i \right) \right) \chi_{s_i} \qquad (6.17)$$

and a boosted helicity spinor is written $\left| \vec{p}_i, \lambda_i, t_i \right\rangle$, so that equation (6.14) becomes

$$\left| p^{0\uparrow}; E_1, E_2 \right\rangle = N \sum_{s_i, t_i, T, t, j_{12}, s_{12}} \delta_{T j_{12}} \left(j_1 s_1 j_2 s_2 \mid j_{12} s_{12} \right) \left(j_{12} s j_3 s_3 \mid \frac{1}{2}\frac{1}{2} \right) \times$$

$$\times \left(T_1 t_1 T_2 t_2 \mid Tt \right) \left(Tt T_3 t_3 \mid \frac{1}{2}\frac{1}{2} \right) \prod_{i=1}^{3} \left| p_{i_z}, \lambda_{i_0}, t_i \right\rangle \qquad (6.18)$$

The superscript "0" in $p^{0\uparrow}$ refers to the proton rest frame. It is important to state that the δ-function $\delta_{T j_{12}}$ keeps the symmetry between spin and isospin-spin and is not disturbed by rotations in spin space [13]. A state of well defined angular momentum is constructed from (6.18) as in (6.13).

The result is

$$\left| J = \frac{1}{2}, S = \frac{1}{2}, \mu = \frac{1}{2}; E_1, E_2 \right\rangle$$

$$= N_j \sum_{s_i, t_i, T, t, j_{12}, s_{12}} \delta_{T j_{12}} \left(j_1 s_1 j_2 s_2 \mid j_{12} s_{12} \right) \left(j_{12} s_{12} j_3 s_3 \mid \frac{1}{2}\frac{1}{2} \right) \times$$

$$\times \prod_{i=1}^{3} \left| p_{i_z}, \lambda_{i_0}, t_i \right\rangle \qquad (6.19)$$

with $N_j = \frac{4\pi N_{a,r}}{\sqrt{2(2J+1))}}$, by imposing normalization conditions.

Equation (6.19) will serve us as the "proton" helicity state of well defined angular momentum and parity.

6.4 Boosting the three particle system

We shall now address the question of the boosts [13].

We are dealing with two boosts : the first boost acts on the quarks themselves, since, as was explained earlier, the quarks are moving at relativistic velocities. Three particles will always be confined to a plane. In this case, however, the plane is unique which is a consequence of the second boost, which acts on the entire (CMS) system. The second boost will Lorentz contract the initially (at rest) "spherical" appearance of the nucleon in the direction of motion and make it "flat".

The plane in which the quarks move is therefore defined by the direction of motion of the CMS. This second boost, which seems to complicate matters unnecessarily, is indispensable. Since we will be dealing with a current, we shall need an initial and a final state of the system, with an interaction in between in the form of a collision with a nucleon, which measures the magnetic moment. This is why we use the Breit frame.

The Lie-group structure of the Lorentz group tells us that the generators of boosts do not commute. This fact is closely related to the "parallel transport problem" in general relativity and is also the origin of Thomas-procession-like effects as are observed inside the atom.

The bottom line is that different results are obtained if the boost is first applied to the quarks or to the CMS. Since we do not know which order nature prefers, an averaging procedure is used.

Let us denote by $U\left[L_{(i)}\left(x_i\vec{P}_N\right)\right]$ the boost of the quarks to the CMS velocity. The explicit form of the boost is then given by:

$$U\left[L_{(i)}\left(x_i\vec{P}_N\right)\right] = \sqrt{\frac{E_i + m_{i0}}{2m_{i0}}} \begin{bmatrix} I_2 & \vec{\sigma}.\frac{x_{iR}\vec{P}_N}{E_i+m_{i0}} \\ \vec{\sigma}.\frac{x_{iR}\vec{P}_N}{E_i+m_{i0}} & I_2 \end{bmatrix}$$

where x_i denotes the fraction of longitudinal momentum of the 3-particle state that each quark carries, m_{i0} the non-relativistic mass of the i^{th} quark and $\vec{P}_N$ the momentum of the entire system.

The quarks are individually boosted in the above mentioned plane by the operator $U\left[L_{(i)}\left(\vec{p}_{i\perp(N)}\right)\right]$, which has the form:

$$U\left[L_{(i)}\left(\vec{p}_{i\perp(N)}\right)\right]=\sqrt{\frac{E_{i(N)}+m_{i0}}{2m_{i0}}}\begin{bmatrix} I_2 & \vec{\sigma}.\frac{\vec{p}_{i\perp(N)}}{E_{i(N)}+m_{i0}} \\ \vec{\sigma}.\frac{\vec{p}_{i\perp(N)}}{E_{i(N)}+m_{i0}} & I_2 \end{bmatrix}$$

with m_{i_0} the mass and $E_{i(N)}$ the energy of the i^{th} quark in the CMS. The average boost will then be given by the anti-commutator

$$\frac{1}{2}\left\{U\left[L_{(i)}\left(x_i\vec{P}_N\right)\right],U\left[L_{(i)}\left(\vec{p}_{i\perp(N)}\right)\right]\right\}$$

which can be rewritten in the more convenient form

$$U\left[L_{(i)}\left(x_i\vec{P}_N\right)\right]U\left[L_{(i)}\left(\vec{p}_{i\perp(N)}\right)\right]+\frac{1}{2}\left[U\left[L_{(i)}\left(\vec{p}_{i\perp(N)}\right)\right],U\left[L_{(i)}\left(x_i\vec{P}_N\right)\right]\right] \tag{6.20}$$

For purposes of brevity, which will be handy in the next chapter, we denote:

$$\begin{aligned} e^A &= U\left[L_{(i)}\left(x_i\vec{P}_N\right)\right] \\ e^B &= U\left[L_{(i)}\left(\vec{p}_{i\perp(N)}\right)\right] \\ C &= \left[U\left[L_{(i)}\left(\vec{p}_{i\perp(N)}\right)\right],U\left[L_{(i)}\left(x_i\vec{P}_N\right)\right]\right] \end{aligned} \tag{6.21}$$

so that the expression (6.20) is reduced to

$$e^Ae^B+\frac{1}{2}C \tag{6.22}$$

The commutator is computed to yield

$$\begin{aligned} \frac{1}{2}C &= \frac{1}{2}\left[U\left[L_{(i)}\left(\vec{p}_{i\perp(N)}\right)\right],U\left[L_{(i)}\left(x_i\vec{P}_N\right)\right]\right] \\ &= i\sinh\frac{\alpha_N}{2}\sinh\frac{\alpha}{2}\left(I_2\vec{\sigma}.\left(\widehat{p}_{i\perp_{(N)}}\times\widehat{p}_N\right)\right) \end{aligned}$$

We have now derived all the necessary ingredients for further computation. We have seen in this chapter how the three particle state was set up in the helicity basis and how the boosts were constructed.

We can now turn to calculating the expectation value of the spatial component of the electromagnetic current, and using the results, the magnetic moment expectation value.

Chapter 7

The Current

Now let us discuss the gauge-invariant electromagnetic current, which constitutes the central result of this dissertation.

To obtain an intuitively more appealing picture, we shall start by discussing the current in the Schrödinger picture. We then turn to the relativistic generalization. We shall see that the current forms a 4-vector. We write down the baryon-baryon electromagnetic current and the Gordon decomposition. Using the boosts (6.22) obtained in the previous chapter, we calculate the expectation value of the magnetic moment.

7.1 Derivation of the current

7.1.1 Current in the Schrödinger picture

In Q.M., the quantity that is globally conserved is the total probability of finding the particle anywhere in the universe. The continuity equation, which we will state below, imposes the additional impossibility of the particle suddenly disappearing in one region of space and immediately reappearing in another (in which case the total probability would still be conserved) [5]. These physically and intuitively obvious facts can be cast within a mathematical framework by starting with the Schrödinger equation [5]:

$$i\frac{\partial}{\partial t}\psi = -\frac{1}{2m}\nabla^2\psi + V\psi$$

and it's conjugate :

$$-i\frac{\partial}{\partial t}\psi^* = -\frac{1}{2m}\nabla^2\psi^* + V\psi^*$$

where V is real in order to keep the Hamiltonian hermitian and thus insuring real eigenvalues.

Multiplying the Schrödinger equation by $\psi^*$and it's conjugate equation by ψ and taking the difference gives us the relation:

$$i\frac{\partial}{\partial t}\left(\psi^*\psi\right) = -\frac{1}{2m}\vec{\nabla}.\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)$$

which can be rewritten [5]:

$$\frac{\partial P}{\partial t} = -\vec{\nabla}.\vec{J}$$

where $P = i\left(\psi^*\psi\right)$is the probability, normalized to 1, and $\vec{J} = \frac{1}{2m}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)$ is the *probability current density*, which is in the classical sense the probability flow per unit time per unit area perpendicular to $\vec{J}$.

Now it is clear that a $\vec{J}$ is a vector quantity. Upon substitution for the wave functions with a wave packet, a quantity proportional to $\approx \frac{\vec{p}}{m} = \vec{v}$ is obtained, the velocity. Current can therefore be viewed as the probability (which is a scalar) within a certain volume traveling with a velocity $\vec{v}$ (which is a vector). The label "density" is attached since the current is defined within a certain time and limited to a certain area.

7.1.2 The Dirac current

In order to obtain the current connected with the Dirac theory [2] we start with the Dirac equation, in analogy with the above derivation:

$$i\frac{\partial\Psi}{\partial t} = \frac{1}{i}(\alpha_1\frac{\partial\Psi}{\partial x_1} + \alpha_2\frac{\partial\Psi}{\partial x_2} + \alpha_3\frac{\partial\Psi}{\partial x_3}) + \beta m\Psi$$

and it's conjugate:

$$-i\frac{\partial\Psi^\dagger}{\partial t} = \frac{-1}{i}(\alpha_1\frac{\partial\Psi^\dagger}{\partial x_1} + \alpha_2\frac{\partial\Psi^\dagger}{\partial x_2} + \alpha_3\frac{\partial\Psi^\dagger}{\partial x_3}) + \beta m\Psi^\dagger$$

where use has been made of the properties:

$$\begin{aligned} \alpha_k &= \alpha_k^\dagger \\ \beta &= \beta^\dagger \end{aligned}$$

Again multiplying the Dirac equation by $\Psi^\dagger$ and its conjugate by Ψ, and taking the difference, we obtain [2]:

$$i\frac{\partial}{\partial t}\left(\Psi^\dagger\Psi\right) = \frac{1}{i}\frac{\partial}{\partial x_k}\left(\Psi^\dagger\alpha_k\Psi\right) \tag{7.1}$$

We are led to the identification [5]:

$$P = \Psi^\dagger\Psi \tag{7.2}$$

for the probability density and [5]

$$j^k = \Psi^\dagger\alpha_k\Psi \tag{7.3}$$

for the probability current.

The probability current, as stated here, forms a three vector and it is invariant under spatial rotations. But this is not enough: the continuity equation (7.1) must remain invariant under Lorentz transformations. This is physically obvious, since the total probability must be conserved globally in the Dirac theory and two observers in different frames of reference must be able to agree that this is indeed th. case. We anticipate the probability density and the probability current to form a 4-vector, in order that the covariance of the continuity equation is ensured.

It should be noted that, in analogy with the Schrödinger picture, the operator α_k should be interpreted as the velocity operator in Dirac's theory.

In the notation introduced earlier, the probability current can be rewritten:

$$\begin{aligned} j^\mu(x) &= \Psi^\dagger(x)\,\gamma^0\gamma^\mu\Psi(x) \\ \mu &= 0,1,2,3 \end{aligned} \tag{7.4}$$

which combines the two forms for the probability density and the probability current in (7.3) and (7.2) in the form of a 4-vector, with the probability density being the zeroth component.

It can be shown quite straightforwardly that $j^\mu(x)$ transforms as a 4-vector [5]:

$$j^{\mu'}(x') = a^\mu_{.\nu} j^\mu(x)$$

which makes the continuity equation

$$\frac{\partial j^\mu(x)}{\partial x^\mu} = 0$$

an invariant.

An important extension to the Schrödinger picture is also the fact that with the average current, which is nothing but the expectation value of the velocity operator $\langle \vec{v}_{op} \rangle = c\vec{\alpha}$ (the velocity of light, c, has been included here), the negative energy solutions have to be taken into account as well. This arises from the fact that the velocity operator in the Dirac theory is not a constant in time, since it does not commute with the relativistic Hamiltonian, i.e. $[H, \vec{\alpha}] \neq 0$ (the commutator is the quantum analogue of the classical Poisson brackets. Any operator commuting with the Hamiltonian would have no time-dependence.). This can be seen from the fact that the eigenvalues of the operator $c\vec{\alpha}$ are $\pm c$ [2]. The positive eigenstates of the velocity operator are just the group velocity of the wave packet. If the negative eigenvalue solution for a wave packet are included in the current, cross terms between the negative and positive energy solutions arise which oscillate rapidly in time. This rapid oscillation is called *Zitterbewegung,* and is proportional to the amplitude of the negative energy solutions. As we discussed before, the negative energy solutions only arise at relativistic velocities. This would mean that a wave packet, made of positive energy solutions initially, must be localized in space to the order of its Compton wavelength for it to acquire sufficient momentum (via the uncertainty principle) for the relativistic negative energy solutions to feature non-negligibly.

In the subsequent discussion we shall be dealing with relativistic particles of spin $\frac{1}{2}$, so we shall have to take both particles and antiparticles into account when describing the current.

7.2 The baryon-baryon current

The Dirac current (7.4) does not have the mathematical freedom to describe a particle with internal structure. It describes a point particle, like an electron or a quark. In order to obtain an expression for the current of a particle with structure, we write quite generally the structure of a one particle element [6]

$$\langle p',\beta| j^{\mu}(x) |p,\alpha\rangle = e^{-i(p-p')\cdot x}\overline{u}^{(\beta)}(p')O^{\mu}(p',p)u^{(\alpha)}(p) \tag{7.5}$$

and here $|p,\alpha\rangle = b^{\dagger}_{\alpha}(p)|0\rangle$. p a four momentum vector and α,β a quantum degree of freedom. The operator $O^{\mu}(p',p)$ is to be determined.

It is clear that $O^{\mu}(p',p)$ must be Lorentz invariant. Hence [6]

$$S(\Lambda)\overline{u}^{(\beta)}(p')O^{\mu}(p',p)u^{(\alpha)}(p)S^{-1}(\Lambda) = (\Lambda^{-1})^{\mu}_{\ \nu}\overline{u}^{(\beta)}(\Lambda p')O^{\nu}(\Lambda p',\Lambda p)u^{(\alpha)}(\Lambda p)$$

and [6]

$$\gamma^0\overline{u}^{(\beta)}(p')O^{\mu}(p',p)^{\dagger}u^{(\alpha)}(p)\gamma^0 = \overline{u}^{(\beta)}(p')O^{\mu}(p',p)u^{(\alpha)}(p)$$

which is the Hermiticity constraint.

The spinors $\overline{u}^{(\beta)}(p')$ and $u^{(\alpha)}(p)$ are solutions of the free Dirac equation and hence we can implement the substitution [6]

$$\overline{u}^{(\beta)}(p')O^{\mu}(p',p)u^{(\alpha)}(p) \rightarrow \frac{p^{\mu\prime}\gamma_{\mu}+m}{2m}O^{\mu}(p',p)\frac{p^{\mu}\gamma_{\mu}+m}{2m}$$

where use has been made of the Gordon identity [6]

$$\gamma^{\mu} \rightarrow \frac{1}{2m}\left[(p'+p)^{\mu} + i\sigma^{\mu\nu}(p'-p)_{\nu}\right]$$

Another requirement is that of conservation of current [6]

$$0 = \langle p',\beta| \partial_{\mu}j^{\mu}(x) |p,\alpha\rangle = i\,(p'-p)_{\mu}\langle p',\beta| j^{\mu}(x) |p,\alpha\rangle$$

which is easily seen to be the case from (7.5). This implies that

$$i\,(p'-p)\,O^{\mu}(p',p) = 0$$

Let us define $q = p' - p$. From $p^{2\prime} = p^2 = m^2$, we see that q^2 is the only scalar invariant.

We now have the conditions of Lorentz invariance, conservation of current and hermiticity from which to build the operator $O^{\mu}(p',p)$. So we use the most general decomposition. We know that the operator must have the form of a 4×4 matrix, and since the Clifford algebra forms the basis of all such matrices, we pick out the γ-matrices (or combinations) which have the above mentioned properties. They turn out to be [6]

$$\overline{u}^{(\beta)}(p')O^{\mu}(p',p)u^{(\alpha)}(p)$$

$$= \overline{u}^{(\beta)}(p')\left[F_1\left(q^2\right)\gamma^{\mu} + \frac{iF_2\left(q^2\right)\sigma^{\mu\nu}q_{\nu}}{2m} + \gamma^5\frac{\sigma^{\mu\nu}q_{\nu}}{2m}F_3\left(q^2\right)\right]u^{(\alpha)}(p)$$

The electromagnetic interaction considered here is experimentally found to be invariant under parity operations. The term with the factor $F_3\left(q^2\right)$ turns out, by explicit calculation, not to be invariant under parity, and is therefore set to zero. It may have to be resurrected when considering interactions such as the weak interaction, which can change parity.

Hence the general baryon-baryon Lorentz-invariant electromagnetic current can be written as [14]:

$$J_B^{\mu}\left(q^2\right) =$$

$$u^{\dagger}\left(p_f, s_f\right)e_B\left[F_1\left(q^2\right)\gamma^0\gamma^{\mu} + \frac{iF_2\left(q^2\right)\gamma^0\sigma^{\mu\nu}q_{\nu}}{2m_{0_B}}\right]u\left(p_{in}, s_{in}\right) \tag{7.6}$$

with $q = p_f - p_{in}$, the 4-momentum change, $u^{\dagger}\left(p_f, s_f\right)$ and $u\left(p_{in}, s_{in}\right)$ the final and initial boosted spinors respectively, e_B the baryon electric charge operator and γ^0, γ^{μ} and $\sigma^{\mu\nu}$ defined as in (4.3), (4.5) and (4.7).

The boosted spinors can be easily computed from the explicit form of the boosts:

$$u\left(p,s\right) = U\left[L(\vec{p})\right]u\left(0,s\right) = \sqrt{\frac{E_B + m_{0_B}}{2m_{0_B}}}\begin{bmatrix} I_2 & \frac{\vec{\sigma}.\vec{p}}{E_B+m_{0_B}} \\ \frac{\vec{\sigma}.\vec{p}}{E_B+m_{0_B}} & I_2 \end{bmatrix}\begin{bmatrix} \chi_s \\ 0 \\ 0 \end{bmatrix}$$

where the χ_s $(s = \pm)$ are canonical spinors of the form

$$\chi_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\chi_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

All the calculations will be conducted in the Weyl, or standard representation, with

$$\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

The functions F_1 and F_2 are form factors which can be rewritten in terms of the Sachs form factors [10]:

$$\begin{aligned} G_E &= F_1\left(q^2\right) + \eta F_2\left(q^2\right) \\ G_M &= e_N f_1\left(q^2\right) + \kappa_N e_{(p)} f_2\left(q^2\right) \equiv F_1\left(q^2\right) + F_2\left(q^2\right) \end{aligned} \tag{7.7}$$

$$\eta = \frac{q^2}{4m_{N_0}^2}$$

and κ_N the anomalous magnetic moment.

The expression for the current is evaluated at $|q_N^2| = \left|p_{N_f} - p_{N_{in}}\right|^2 = \left|2\vec{p}_{N_f}\right|^2$, which is a particularity of the Breit frame, in which $\vec{p}_{N_f} = -\vec{p}_{N_{in}}$ and $E_{N_f} = E_{N_{in}}$. This yields for the form factors $F_1(0) = 1$ and $F_2(0) = \tilde{\mu}$, the anomalous baryon magnetic moment.

Equation (7.6) is also referred to as the Gordon decomposition, which expresses the total current as the sum of a convection current, as in the non-relativistic case, and a spin current [2].

The current can be rewritten (see Appendix G) [10]

$$J_B^\mu\left(q^2\right) = \begin{bmatrix} J_B^0\left(q^2\right) \\ \vec{J}_B\left(q^2\right) \end{bmatrix} = \begin{bmatrix} e_B G_E\left(q^2\right) \chi_{s_f}^T \chi_{s_{in}} \\ \frac{ie_B}{2m_{0_B}} G_M\left(q^2\right) \chi_{s_f}^T \left(\vec{\sigma} \times \vec{q}\right) \chi_{s_{in}} \end{bmatrix} \tag{7.8}$$

with the symbols as explained above.

As was pointed out, this is the expression obtained for the current on the baryon level, with the nucleon structure expressed in the form factors.

When looking at the nucleon as an object made up of three individual parts, the expression for the current will change, yet the results for the current expectation values do not.

7.3 Quark level current

On the quark level, the nucleon current takes on the form

$$\langle J, S, \mu; E_1, E_2 | \sum_{i=1}^{3} F_{(i)} \left| J, S, \mu; E_1, E_2 \right\rangle \tag{7.9}$$

with $J = S = \mu = \frac{1}{2}$.

The operator F_i assumes the form:

$$F_{(i)} = e_{(i)}\left[\gamma^0_{(i)}\gamma^\mu_{(i)} + i\frac{F_{2_{(i)}}(q^2)}{2m_{0_{(i)}}}\gamma^0_{(i)}\sigma^{\mu\nu}_{(i)}q_{\nu_{(i)}}\right] \tag{7.10}$$

where the subscript (i) stands to denote the i^{th} quark.

The form factor $F_{1_{(i)}} = 1$ and in the subsequent calculations $F_{2_{(i)}}$ will be set to zero since it is assumed that quarks are point particles and $F_{2_{(i)}}$ is a measure of the internal structure of particles (i.e. here we assume quarks have no internal structure).

The fact that we are summing over $F_{(i)}$ is an indication that the object is made up of three constituents. We shall only consider one term for simplicity. The entire expression, however, will be a sum of three terms and takes the full form

$$\begin{aligned} J^\mu_q &= \sum_{(i)=1}^{3}\left\langle \lambda_{(i)_{final_0}}, t_{(i)_{final}}\right| U^\dagger\left[R\left(\alpha^0_{(i)},\beta^0_{(i)},\gamma^0_{(i)},\right)\right]U^\dagger\left[L_{(i)}\left(\vec{p}_{(i)_{final}}\right)\right] \\ &\times e_{(i)}\gamma^0_{(i)}\gamma^\mu_{(i)}U\left[L_{(i)}\left(\vec{p}_{(i)_{initial}}\right)\right]U\left[R\left(\alpha^0_{(i)},\beta^0_{(i)},\gamma^0_{(i)},\right)\right]\left|\lambda_{(i)_{initial_0}}, t_{(i)_{initial}}\right\rangle \\ &\prod_{(l)\neq(i)}\left\langle \lambda_{(l)_{final_0}}, t_{(l)_{final}}\right| U^\dagger\left[R\left(\alpha^0_{(l)},\beta^0_{(l)},\gamma^0_{(l)},\right)\right]U^\dagger\left[L_{(l)}\left(\vec{p}_{(l)_{final}}\right)\right] \\ &U\left[L_{(l)}\left(\vec{p}_{(l)_{initial}}\right)\right]U\left[R\left(\alpha^0_{(l)},\beta^0_{(l)},\gamma^0_{(l)},\right)\right]\left|\lambda_{(l)_{initial_0}}, t_{(l)_{initial}}\right\rangle \end{aligned} \tag{7.11}$$

where the Clebsch-Gordon coefficients have been left implicit, for the sake of brevity. Since they play no part in the actual computations at this point, they can be left until the end, when they will be restored.

The question of the boosts addressed in the previous chapter plays an important part now.

7.3.1 Boosts and the current

We shall have to use the averaged boost as defined in (6.22).

The expression (7.11) can be reduced to symbolic form

$$J^\mu_q = \zeta^\dagger_{\lambda_{final}}\left[\left(e^{B_{final}}\right)^\dagger\left(e^{A_{final}}\right)^\dagger + \frac{1}{2}C^\dagger_{final}\right] \tag{7.12}$$

$$\times e_k\begin{bmatrix} 0 & \sigma^\mu_{(i)} \\ \sigma^\mu_{(i)} & 0 \end{bmatrix}$$

$$\times \left[e^{A_{initial}} e^{B_{initial}} + \frac{1}{2} C_{initial}\right] \zeta_{\lambda_{initial}}$$

The explicit calculations of the terms in the current are shown in the Appendix A and Appendix B. Since the spin-isospin coupling is left invariant by the rotations, the rotation matrices $\left[R\left(\alpha^0_{(i)}, \beta^0_{(i)}, \gamma^0_{(i)},\right)\right]$ have been set to uni ,. The sum of the resulting terms (also listed in the Appendix) are gauge invariant, i.e. the sum of the terms gives a vector perpendicular to $\vec{q}$.

We saw from (7.8) that the spatial component of the current on the Baryon-level depends on an expression proportional to $(\vec{\sigma} \times \vec{q})$. We shall therefore only consider terms proportional to $(\vec{\sigma} \times \vec{q})$.

7.4 Magnetic moment expectation value

The expression to evaluate is

$$\langle B| \vec{\mu} |B\rangle \mid_{|\vec{q}|^2=0} = \frac{i}{2m_{N_0}} G_{M_{(N)}}(0) \langle N| (\vec{\sigma} \times \vec{q}) |N\rangle \tag{7.13}$$

where

$$\vec{\mu} = \frac{1}{2} \int d^3r \left(\vec{r} \times : \overline{\psi} \vec{F} \psi :\right) \tag{7.14}$$

The left hand side of (7.13) is calculated on the quark level, and the right hand side on the nucleon level. For reference, see for instance [14]. The differential is evaluated at $|\vec{q}|^2 = 0$ since then the collision is "soft" and the outer region of the nucleon is probed, not the inner structure. We can also see that the equation (7.14) is proportional to the classical expression for angular momentum $\vec{r} \times \vec{v}$ where the current takes on the expected interpretation of velocity.

Let us look at the expression on the left hand side of (7.13).

The vector $\langle B|$ is given by the explicit expression

$$\langle B| = N_j \sum_{\lambda_{i_f}, t_{i_f}, T, t, j_{12}, \lambda_f} \delta_{T j_{12}} \{ \prod_{i=1}^{3} \int \frac{d^3p_{i_f}}{(2\pi)^3} \frac{1}{\sqrt{p^0_{i_f}}} \left(f^{t_f}_{\lambda_{i_f}}\left(\vec{p}_{i_f}\right)\right)^* \langle 0| a_{\lambda_{i_f}}\left(\vec{p}_{i_f}\right)$$
$$\times \left(T_1 t_{1_f} T_2 t_{2_f} \mid T t_f\right) \left(T t_f T_3 t_{3_f} \mid \frac{1}{2}\frac{1}{2}\right)^*$$
$$\left(j_1 \lambda_{1_f} j_2 \lambda_{2_f} \mid j_{12} \lambda_f\right) \left(j_{12} \lambda_{f_f} j_3 \lambda_{3_f} \mid \frac{1}{2}\frac{1}{2}\right)^* \} \tag{7.15}$$

while the ket $|B\rangle$ assumes the form

$$|B\rangle = N_j \sum_{\lambda_{i_{in}},t_{i_{in}},T,t,j_{12},\lambda_{in}} \delta_{Tj_{12}} \{ \left(j_1\lambda_{1_f} j_2\lambda_{2_f} \mid j_{12}\lambda_f \right) \left(j_{12}\lambda_{f_f} j_3\lambda_{3_f} \mid \frac{1}{2}\frac{1}{2} \right)$$
$$\times \left(T_1 t_{1_f} T_2 t_{2_f} \mid T t_f \right) \left(T t_f T_3 t_{3_f} \mid \frac{1}{2}\frac{1}{2} \right)$$
$$\prod_{i=1}^{3} \int \frac{d^3 p_{i_{in}}}{(2\pi)^3} \frac{1}{\sqrt{p^0_{i_{in}}}} \left(f^{t_{in}}_{\lambda_{i_{in}}} (\vec{p}_{i_{in}}) \right) a^\dagger_{\lambda_{i_{in}}} (\vec{p}_{i_{in}}) |0\rangle \tag{7.16}$$

where the subscripts "f" and "in" stand for "final " and "initial" respectively.

The ket $|f\rangle$ describes a normalized one electron state of the form:

$$|f\rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2p^0}} \sum_{t,\lambda} \left(f^t_\lambda (\vec{p}) \right) a^\dagger_{\lambda_{i_{in}}} (\vec{p}) |0\rangle$$

which can then, in the non-relativistic limit, be interpreted as Fourrier transformed electron wave functions [14].

The Glebsch-Gordon coefficients, which were written explicitly, mediate the spin coupling, as usual, but can be ignored for the calculations to follow. They will be restored later. The integral represents a relativistic wave-packet.

The expression (7.14) is written out [14]

$$\overline{\psi} =$$

$$N_j \sum_{\lambda'_{i_f},t'_{i_f},T',t',j'_{12},\lambda'_f} \delta_{Tj'_{12}} \{ \prod_{i=1}^{3} [\int \frac{d^3 p'_{i_f}}{(2\pi)^3} \frac{1}{\sqrt{p^0_{i_f}}} \sum_{\lambda'_f} e^{-ip'_f r} \overline{v}_{\lambda'_f} \left(p'_f \right) b^{t'_{i_f}}_{\lambda'_{i_f}} \left(p'_f \right) \tag{7.17}$$
$$+ e^{-ip'_f r} \overline{u}^{t'_{i_f}}_{\lambda'_{i_f}} \left(p'_f \right) a^\dagger_{\lambda'_f} \left(p'_f \right)]$$

$$\times \left(T_1 t'_{1_f} T_2 t'_{2_f} \mid T t'_f \right) \left(T t'_f T_3 t'_{3_f} \mid \frac{1}{2}\frac{1}{2} \right)^*$$

$$\left(j_1 \lambda'_{1_f} j_2 \lambda'_{2_f} \mid j'_{12} \lambda'_f \right) \left(j'_{12} \lambda_{f_f} j_3 \lambda'_{3_f} \mid \frac{1}{2}\frac{1}{2} \right)^* \}$$

and

$$\psi =$$

$$N_j \sum_{\lambda'_{i_{in}}, t'_{i_{in}}, T', t', j'_{12}, \lambda'_{in}} \delta_{Tj'_{12}} \{ \prod_{i=1}^{3} [\int \frac{d^3 p'_{i_{in}}}{(2\pi)^3} \frac{1}{\sqrt{p'^0_{i_{in}}}} \sum_{\lambda'_f} e^{-ip'_{in} r} v_{\lambda'_{in}} \left(p'_{in}\right) b^{\dagger t'_{i_{in}}}_{\lambda'_{i_{in}}} \left(p'_{in}\right)$$

$$+ e^{-ip'_{in} r} u^{t'_{i_{in}}}_{\lambda'_{i_{in}}} \left(p'_{in}\right) a_{\lambda'_{in}} \left(p'_{in}\right)] \tag{7.18}$$

$$\times \left(T_1 t'_{1_{in}} T_2 t'_{2_{in}} \mid T t'_{in}\right) \left(T t'_{in} T_3 t'_{3_{in}} \mid \frac{1}{2}\frac{1}{2}\right)$$

$$\left(j_1 \lambda'_{1_{in}} j_2 \lambda'_{2_{in}} \mid j'_{12} \lambda'_{in}\right) \left(j'_{12} \lambda_{in_f} j_3 \lambda'_{3_{in}} \mid \frac{1}{2}\frac{1}{2}\right) \}$$

$\vec{F} = \vec{\gamma}$. We want to describe a system consisting of particles, as opposed to antiparticles, so we set the coefficients describing the negative energy solutions $v_{\lambda'_{in}} \left(p'_{in}\right) = \overline{v}_{\lambda'_f} \left(p'_f\right) = 0$.

On combining the above terms (7.15),(7.16) , (7.17) (7.18) ,the expression (7.14) takes on the form for a generic term

$$\mu^d = \frac{1}{2} \int d^3 r \left(\vec{r} \times \langle B(\vec{p}_f)| : \overline{\psi} \vec{F} \psi : |B(\vec{p}_{in})\rangle\right)^d$$

$$= \frac{1}{2} \int d^3 r \epsilon^{lbd} r^l e^{-i\vec{q}_1 \cdot \vec{r}} \overline{u}^{t'_{1_f}}_{\lambda'_{1_f}} \left(p'_{1_f}\right) \gamma^b u^{t'_{1_{in}}}_{\lambda'_{1_{in}}} \left(p'_{1_{in}}\right) \tag{7.19}$$

$$\times e^{-i\vec{q}_2 \cdot \vec{r}} \overline{u}^{t'_{2_f}}_{\lambda'_{2_f}} \left(p'_{2_f}\right) u^{t'_{2_{in}}}_{\lambda'_{2_{in}}} \left(p'_{2_{in}}\right) e^{-i\vec{q}_3 \cdot \vec{r}} \overline{u}^{t'_{3_f}}_{\lambda'_{3_f}} \left(p'_{3_f}\right) u^{t'_{3_{in}}}_{\lambda'_{3_{in}}} \left(p'_{3_{in}}\right)$$

$$\times \langle 0| a_{\lambda_{1_f}} a_{\lambda_{2_f}} a_{\lambda_{3_f}} a^{\dagger}_{\lambda'_{3_f}} a^{\dagger}_{\lambda'_{2_f}} a^{\dagger}_{\lambda'_{1_f}} a_{\lambda'_{1_{in}}} a_{\lambda'_{2_{in}}} a_{\lambda'_{3_{in}}} a^{\dagger}_{\lambda'_{3_{in}}} a^{\dagger}_{\lambda'_{2_{in}}} a^{\dagger}_{\lambda'_{1_{in}}} |0\rangle$$

The detailed calculations are shown in Appendix A and Appendix B.

7.5 Non relativistic limit

The form of μ^d can be rewritten as

$$\mu^d = \frac{1}{2} \int d^3 r \left(\vec{r} \times \langle B(\vec{p}_f)| \overline{\psi}_1 \left(p_{1_f}\right) \vec{F}_1 \overline{\psi}_1 \left(p_{1_{in}}\right) \overline{\psi}_2 \left(p_{2_f}\right) \overline{\psi}_2 \left(p_{2_{in}}\right)\right)$$

$$\overline{\psi}_3 \left(p_{3_f}\right) \overline{\psi}_3 \left(p_{3_{in}}\right) + \overline{\psi}_1 \left(p_{1_f}\right) \overline{\psi}_1 \left(p_{1_{in}}\right) \overline{\psi}_2 \left(p_{2_f}\right) \vec{F}_2 \overline{\psi}_2 \left(p_{2_{in}}\right) \overline{\psi}_3 \left(p_{3_f}\right) \overline{\psi}_3 \left(p_{3_{in}}\right)$$

$$+\overline{\psi}_1\left(p_{1_f}\right)\overline{\psi}_1\left(p_{1_{in}}\right)\overline{\psi}_2\left(p_{2_f}\right)\overline{\psi}_2\left(p_{2_{in}}\right)\overline{\psi}_3\left(p_{3_f}\right)\vec{F}_3\overline{\psi}_3\left(p_{3_{in}}\right)|B\left(\vec{p}_{in}\right)\rangle^d$$

the notation of which can be symplified

$$\mu^d=\frac{1}{2}\int d^3r\left(\langle B\left(\vec{p}_f\right)|\sum_{k=1}^{3}\overline{\psi}_k\left(p_{k_f}\right)\vec{F}_k\overline{\psi}_k\left(p_{k_{in}}\right)\prod_{l\neq k}\overline{\psi}_l\left(p_{l_f}\right)\overline{\psi}_l\left(p_{l_{in}}\right)\right)|B\left(\vec{p}_{in}\right)\rangle^d$$

where the spinors represent the composite-boosted spinors

$$\psi_l\left(p_l\right)=\left[e^Ae^B+\frac{1}{2}C\right]\zeta_l$$

This takes on the physically more tractable form:

$$\mu^d=\frac{1}{2}\int d^3r\epsilon^{cbd}r^ce^{-i\vec{q}\cdot\vec{r}}\sum_{k=1}^{3}A_ke_ku_{\lambda_f}\left(\vec{\sigma}\times\vec{q}_k\right)^bu_{\lambda_{in}}+$$

$$B_ke_ku_{\lambda_f}\left[\vec{\sigma}\cdot\left(\vec{p}_{k\perp}\times\vec{q}_k\right)\left(\vec{\sigma}\times\vec{q}_k\right)^b\right]u_{\lambda_{in}}\times$$

$$\left[1-\frac{1}{2}\left(\sqrt{1+\eta}-1\right)\sum_{k\neq l}\left(\frac{E_{l(N)}-m_{l_0}}{2E_{k(N)}}\right)+\left[\frac{1}{2}\left(\sqrt{1+\eta}-1\right)\right]^2\prod_{l\neq k}\left(\frac{E_{l(N)}-m_{l_0}}{2E_{k(N)}}\right)\right]$$

and A_k is given by

$$A_k=\frac{e_k\left(E_{k(N)}+m_{k_0}\right)}{4m_{k_0}E_{k(N)}}$$

and B_k by

$$B_k=\frac{e_k\left(E_{k(N)}^2-m_{k_0}^2\right)}{8m_{k_0}E_{k(N)}\left(E_{k(N)}+m_{k_0}\right)}$$

as can be seen from Appendix B.
We further defin. the quantity $f(\eta)$:

$$f(\eta)=\prod_{l\neq k}\overline{\psi}_l\left(p_{l_f}\right)\overline{\psi}_l\left(p_{l_{in}}\right)=1-\frac{1}{2}\left(\sqrt{1+\eta}-1\right)\sum_{k\neq l}\left(\frac{E_{l(N)}-m_{l_0}}{2E_{k(N)}}\right)+$$

$$\left[\frac{1}{2}\left(\sqrt{1+\eta}-1\right)\right]^2\prod_{l\neq k}\left(\frac{E_{l(N)}-m_{l_0}}{2E_{k(N)}}\right)$$

In the non relativistic limit, the mass energy of the particles becomes very much greater than the kinetic energy, due to the fact that the particles are nearly stationary. The entire energy of the particles is contained within the mass-term ($m_{i_0} \simeq 0.33GeV$).

Hence, $E_{k(N)} \gg |\vec{p}_{k_\perp}|$ and $E_{k(N)} \cong m_{k_0}$. Since the expression is evaluated at $|\vec{q}| = 0$ this implies that $\frac{|\vec{q}^2|}{4M^2_{N_0}} = \eta = 0$ and therefore $f(\eta) = 1$.

The only contribution comes from [17]

$$\int d^3 r e^{-i\vec{q}.\vec{r}} \sum_{k=1}^{3} \{ \frac{e_k}{2m_{k_0}} \varphi_{\lambda_f} \left[\sigma^d \right] \varphi_{\lambda_i} \}$$

The quantization axis was chosen to be the z-axis, so $d = 3$.

By comparing the expressions

$$\frac{1}{2m_{N_0}} G_{M(N)}(0) \langle N | \sigma^3 | N \rangle \Leftrightarrow \sum_{k=1}^{3} \{ \frac{e_k}{2m_{k_0}} \langle N | \sigma^3_{(k)} | N \rangle \}$$

into which the wave function (6.15)is substituted, the result

$$\mu_{proton(n.r.)} = 3\mu_N$$

$$\mu_{neutron(n.r.)} = -2\mu_N$$

is obtained, which leads to the much celebrated relation

$$\frac{\mu_{proton}}{\mu_{neutron}} = -\frac{3}{2}$$

of static QCD.

This brings us to the end of the argument. Some of the mathematically more involved steps are shown in the Appendices.

This chapter outlined the mathematical steps as well as physical concepts used in performing the calculation. A specific result was obtained, which was very encouraging, but much more can still be done. Some of the plans for the future are listed in the next chapter.

Chapter 8

Conclusion and Outlook

8.1 Conclusions

The dissertation was essentially divided into two parts: the first part dealt with fundamental concepts of physics which are of importance and the second part introduced a specific model based on these concepts. A specific calculation was performed (the expectation value of the spatial component of the invariant electromagnetic current) and the magnetic moment expectation value was computed in the non relativistic limit. It is my feeling that both parts were essential in this dissertation, since I believe that a sound understanding of the origins of the contemporary theories is needed in order to fully appreciate modern physics. This understanding does not end at the purely physical or mathematical aspects, but forces the student, of necessity, to comprehend the underlying philosophies. Was not Heisenberg inspired by the works of Wittgenstein?

Quite apart from the historical importance of these developments in physics, they also form the foundation of much that is done today and it is worthwhile, if not necessary, to study the mathematical techniques involved.

It was also necessary to present the previous work on the model so that numerous steps in the computation of the spatial component of the current did not go unjustified.

The calculation of the expectation value of the spatial component of the current itself was a rather lengthy affair, which is the reason why some of the explicit steps were omitted. The calculations were outlined with sufficient detail so that they can be reproduced without much difficulty. More emphasis was given to the more subtle points and difficulties which were encountered, and how they were tackled.

A very encouraging feature of the results were that the ratio $\frac{\mu_{proton}}{\mu_{neutron}} = -\frac{3}{2}$ was recovered in the static limit, which is in accordance with static QCD.

8.2 Outlook

We have proposed a three-level relativistic model of a "nucleon template" and demonstrated the non-relativistic reduction to static QCD. The following issues need to be considered:

Tentative calculations have indicated that there are terms containing the current quark mass$\sim 5MeV$, which leads to mass singularities. This may force us a field theoretical approach in order to consider 1-loop contributions which is beyond the scope of this dissertation.

Calculations shown in Appendix B show that either these terms do not contribute to the expectation value of $\vec{\mu}$ or that they have components in the same direction as $(\vec{\sigma} \times \vec{q})$ and that their contributions should therefore be considered. This is a very recent development and was hence not fully explained in the dissertation, but are quoted in Appendix B.

Another issue is the study of the parity quantum number μ, which may give us insights into the resonance $S_{11}(1535)$ of the proton. We aim to study this resonance in terms of an "Umklapp" process. The overall parity of the three particle system in the ground state is even. Addition of energy to the system will raise one of the quarks to the angular momentum p-state, which is an odd parity state, and hence change overall parity of the system. This has the effect of flipping the normal defining the plane in which the quarks are contained, thus changing μ from $\frac{1}{2}$ to $-\frac{1}{2}$. The energy added to the system will be the energy difference between the even ground state ($938MeV$) and the odd parity partner of the proton at $1535MeV$

Furthermore, some cross sections will have to be computed with the help of the model. To this purpose, the axial vector current will have to be calculated. Work is in progress in this direction.

Overall, it seems that the model is reasonable and has produced results which are confirmed either by experiment or by other theories. Yet there are still many computations to perform and it will be only then that the model can be accepted or rejected with a fair amount of certainty.

Chapter 9

Appendix

9.1 Appendix A

9.1.1 Current

The current operator is computed as follows:

$$(A_f+B_f+E_f,C_f+D_f)\begin{pmatrix} 0 & G \\ G & 0 \end{pmatrix}\begin{pmatrix} A_{in}+B_{in}+E_{in} \\ C_{in}+D_{in} \end{pmatrix}$$

$$=(A_f+B_f+E_f,C_f+D_f)\begin{pmatrix} GC_{in}+GD_{in} \\ GA_{in}+GB_{in}+GE_{in} \end{pmatrix}$$

$$=(A_f+B_f+E_f)(GC_{in}+GD_{in})+(C_f+D_f)(GA_{in}+GB_{in}+GE_{in})$$

$$=A_fGC_{in}+A_fGD_{in}+B_fGC_{in}+B_fGD_{in}+E_fGC_{in}+E_fGD_{in}$$

$$+C_fGA_{in}+C_fGB_{in}+C_fGE_{in}+D_fGA_{in}+D_fGB_{in}+D_fGE_{in}$$

with:

$$G=e_k\vec{\sigma}$$

$$A_{in}=\sqrt{\frac{(E_k+m_{k_0})\left(E_{k(N)}+m_{k_0}\right)}{4m_{k_0}^2}}\sqrt{\frac{m_{k_0}}{E_{k(N)}}}\varphi_{in}$$

$$B_{in}=\sqrt{\frac{(E_k+m_{k_0})\left(E_{k(N)}+m_{k_0}\right)}{4m_{k_0}^2}}\sqrt{\frac{m_{k_0}}{E_{k(N)}}}\frac{(\vec{\sigma}\cdot x_k\vec{p}_{N_{in}})(\vec{\sigma}\cdot\vec{p}_{k_\perp})}{(E_k+m_{k_0})\left(E_{k(N)}+m_{k_0}\right)}\varphi_{in}$$

$$C_{in} = \sqrt{\frac{(E_k + m_{k_0})\left(E_{k(N)} + m_{k_0}\right)}{4m_{k_0}^2}} \sqrt{\frac{m_{k_0}}{E_{k(N)}}} \frac{(\vec{\sigma} \cdot x_k \vec{p}_{N_{in}})}{(E_k + m_{k_0})} \varphi_{in}$$

$$D_{in} = \sqrt{\frac{(E_k + m_{k_0})\left(E_{k(N)} + m_{k_0}\right)}{4m_{k_0}^2}} \sqrt{\frac{m_{k_0}}{E_{k(N)}}} \frac{(\vec{\sigma} \cdot \vec{p}_{k_\perp})}{\left(E_{k(N)} + m_{k_0}\right)} \varphi_{in}$$

$$E_{in} = \sqrt{\frac{m_0}{E_{k(N)}}} \sinh \frac{\alpha_{k(N)}}{2} \sinh \frac{\alpha_k}{2} i(\hat{p}_{k_\perp} \times \hat{p}_{N_{in}}) \cdot \vec{\sigma} \varphi_{in}$$

$$A_f = \sqrt{\frac{(E_k + m_{k_0})\left(E_{k(N)} + m_{k_0}\right)}{4m_{k_0}^2}} \sqrt{\frac{m_{k_0}}{E_{k(N)}}} \varphi_f^T$$

$$B_f = \sqrt{\frac{(E_k + m_{k_0})\left(E_{k(N)} + m_{k_0}\right)}{4m_{k_0}^2}} \sqrt{\frac{m_{k_0}}{E_{k(N)}}} \varphi_f^T \frac{(\vec{\sigma} \cdot \vec{p}_{k_\perp})\left(\vec{\sigma} \cdot x_k \vec{p}_{N_f}\right)}{(E_k + m_{k_0})\left(E_{k(N)} + m_{k_0}\right)}$$

$$C_f = \sqrt{\frac{(E_k + m_{k_0})\left(E_{k(N)} + m_{k_0}\right)}{4m_{k_0}^2}} \sqrt{\frac{m_{k_0}}{E_{k(N)}}} \varphi_f^T \frac{\left(\vec{\sigma} \cdot x_k \vec{p}_{N_f}\right)}{(E_k + m_{k_0})}$$

$$D_f = \sqrt{\frac{(E_k + m_{k_0})\left(E_{k(N)} + m_{k_0}\right)}{4m_{k_0}^2}} \sqrt{\frac{m_{k_0}}{E_{k(N)}}} \varphi_f^T \frac{(\vec{\sigma} \cdot \vec{p}_{k_\perp})}{\left(E_{k(N)} + m_{k_0}\right)}$$

$$E_f = \varphi_f^\top \sqrt{\frac{m_0}{E_{k(N)}}} \sinh \frac{\alpha_{k(N)}}{2} \sinh \frac{\alpha_k}{2} i(\hat{p}_{N_f} \times \hat{p}_{k_\perp}) \cdot \vec{\sigma}$$

$$\frac{\vec{q}}{2} = \vec{p}_{N_f} = -\vec{p}_{N_{in}}$$

9.1.2 Gauge invariant terms

$\underline{A_f G C_{in}}$

$$\frac{-e_k x_k\left(E_{k(N)}+m_{k_0}\right)}{8m_{k_0}E_{k(N)}}\varphi_f^T\left[\sigma^i(\vec{\sigma}\cdot\vec{q})\right]\varphi_{in}$$

$$=\frac{-e_k x_k\left(E_{k(N)}+m_{k_0}\right)}{8m_{k_0}E_{k(N)}}\varphi_f^T\left[\sigma^i\sigma^j q^j\right]\varphi_{in}$$

$$=\frac{-e_k x_k\left(E_{k(N)}+m_{k_0}\right)}{8m_{k_0}E_{k(N)}}\varphi_f^T\left[\left(\delta^{ij}+i\varepsilon^{ijk}\sigma^k\right)q^j\right]\varphi_{in}$$

$$=\frac{-e_k x_k\left(E_{k(N)}+m_{k_0}\right)}{8m_{k_0}E_{k(N)}}\varphi_f^T\left[q^i+i\varepsilon^{ijk}\sigma^k q^j\right]\varphi_{in}$$

$$=\frac{-e_k x_k\left(E_{k(N)}+m_{k_0}\right)}{8m_{k_0}E_{k(N)}}\varphi_f^T\left[q^i\right]\varphi_{in}-\frac{ie_k x_k\left(E_{k(N)}+m_{k_0}\right)}{8m_{k_0}E_{k(N)}}\varphi_f^T\left[\varepsilon^{ijk}\sigma^k q^j\right]\varphi_{in}$$

$$=\frac{-e_k x_k\left(E_{k(N)}+m_{k_0}\right)}{8m_{k_0}E_{k(N)}}\varphi_f^T\left[q^i\right]\varphi_{in}+\frac{ie_k x_k\left(E_{k(N)}+m_{k_0}\right)}{8m_{k_0}E_{k(N)}}\varphi_f^T\left[\varepsilon^{ikj}\sigma^k q^j\right]\varphi_{in}$$

$$=\frac{-e_k x_k\left(E_{k(N)}+m_{k_0}\right)}{8m_{k_0}E_{k(N)}}\varphi_f^T\left[q^i\right]\varphi_{in}+\frac{ie_k x_k\left(E_{k(N)}+m_{k_0}\right)}{8m_{k_0}E_{k(N)}}\varphi_f^T\left[(\vec{\sigma}\times\vec{q})^i\right]\varphi_{in}$$

$\underline{C_f G A_{in}}$

$$\frac{e_k x_k\left(E_{k(N)}+m_{k_0}\right)}{8m_{k_0}E_{k(N)}}\varphi_f^T\left[(\vec{\sigma}\cdot\vec{q})\vec{\sigma}\right]\varphi_{in}$$

$$=\frac{e_k x_k\left(E_{k(N)}+m_{k_0}\right)}{8m_{k_0}E_{k(N)}}\varphi_f^T\left[\sigma^i q^i\sigma^j\right]\varphi_{in}$$

$$=\frac{e_k x_k\left(E_{k(N)}+m_{k_0}\right)}{8m_{k_0}E_{k(N)}}\varphi_f^T\left[\left(\delta^{ij}+i\varepsilon^{ijk}\sigma^k\right)q^i\right]\varphi_{in}$$

$$=\frac{e_k x_k\left(E_{k(N)}+m_{k_0}\right)}{8m_{k_0}E_{k(N)}}\varphi_f^T\left[q^j+i\varepsilon^{ijk}\sigma^k q^i\right]\varphi_{in}$$

$$=\frac{e_k x_k\left(E_{k(N)}+m_{k_0}\right)}{8m_{k_0}E_{k(N)}}\varphi_f^T\left[q^j\right]\varphi_{in}+\frac{e_k x_k\left(E_{k(N)}+m_{k_0}\right)}{8m_{k_0}E_{k(N)}}\varphi_f^T\left[i\varepsilon^{jki}\sigma^k q^i\right]\varphi_{in}$$

$$=\frac{e_k x_k\left(E_{k(N)}+m_{k_0}\right)}{8m_{k_0}E_{k(N)}}\varphi_f^T\left[q^j\right]\varphi_{in}+\frac{ie_k x_k\left(E_{k(N)}+m_{k_0}\right)}{8m_{k_0}E_{k(N)}}\varphi_f^T\left[(\vec{\sigma}\times\vec{q})^i\right]\varphi_{in}$$

Combining the terms $A_f G C_{in} + C_f G A_{in}$:

$$A_f G C_{in} + C_f G A_{in} = \frac{i e_k x_k \left(E_{k(N)} + m_{k_0}\right)}{4 m_{k_0} E_{k(N)}} \varphi_f^T \left[(\vec{\sigma} \times \vec{q})^i \right] \varphi_{in} \tag{9.1}$$

<u>$A_f G D_{in}$</u>

$$\frac{e_k \left(E_k + m_{k_0}\right)}{4 m_{k_0} E_{k(N)}} \varphi_f^T \left[\vec{\sigma} (\vec{\sigma} \cdot \vec{p}_{k_\perp}) \right] \varphi_{in}$$

$$= \frac{e_k \left(E_k + m_{k_0}\right)}{4 m_{k_0} E_{k(N)}} \varphi_f^T \left[\sigma^j \sigma^i p^i_{k_\perp} \right] \varphi_{in}$$

$$= \frac{e_k \left(E_k + m_{k_0}\right)}{4 m_{k_0} E_{k(N)}} \varphi_f^T \left[\left(\delta^{ji} + i \varepsilon^{jik} \sigma^k \right) p^i_{k_\perp} \right] \varphi_{in}$$

$$\frac{e_k \left(E_k + m_{k_0}\right)}{4 m_{k_0} E_{k(N)}} \varphi_f^T \left[p^j_{k_\perp} + i \varepsilon^{jik} \sigma^k p^i_{k_\perp} \right] \varphi_{in}$$

$$= \frac{e_k \left(E_k + m_{k_0}\right)}{4 m_{k_0} E_{k(N)}} \varphi_f^T \left[p^j_{k_\perp} \right] \varphi_{in} - \frac{e_k \left(E_k + m_{k_0}\right)}{4 m_{k_0} E_{k(N)}} \varphi_f^T \left[i \varepsilon^{jki} \sigma^k p^i_{k_\perp} \right]$$

$$= \frac{e_k \left(E_k + m_{k_0}\right)}{4 m_{k_0} E_{k(N)}} \varphi_f^T \left[p^j_{k_\perp} \right] \varphi_{in} - \frac{i e_k \left(E_k + m_{k_0}\right)}{4 m_{k_0} E_{k(N)}} \varphi_f^T \left[(\vec{\sigma} \times \vec{p}_{k_\perp})^j \right]$$

<u>$D_f G A_{in}$</u>

$$\frac{e_k \left(E_k + m_{k_0}\right)}{4 m_{k_0} E_{k(N)}} \varphi_f^T \left[(\vec{\sigma} \cdot \vec{p}_{k_\perp}) \vec{\sigma} \right] \varphi_{in}$$

$$= \frac{e_k \left(E_k + m_{k_0}\right)}{4 m_{k_0} E_{k(N)}} \varphi_f^T \left[\sigma^i p^i_{k_\perp} \sigma^j \right] \varphi_{in}$$

$$= \frac{e_k \left(E_k + m_{k_0}\right)}{4 m_{k_0} E_{k(N)}} \varphi_f^T \left[\left(\delta^{ij} + i \varepsilon^{ijk} \sigma^k \right) p^i_{k_\perp} \right] \varphi_{in}$$

$$\frac{e_k \left(E_k + m_{k_0}\right)}{4 m_{k_0} E_{k(N)}} \varphi_f^T \left[p^j_{k_\perp} + i \varepsilon^{ijk} \sigma^k p^i_{k_\perp} \right] \varphi_{in}$$

$$= \frac{e_k \left(E_k + m_{k_0}\right)}{4 m_{k_0} E_{k(N)}} \varphi_f^T \left[p^j_{k_\perp} \right] \varphi_{in} + \frac{e_k \left(E_k + m_{k_0}\right)}{4 m_{k_0} E_{k(N)}} \varphi_f^T \left[i \varepsilon^{jki} \sigma^k p^i_{k_\perp} \right] \varphi_{in}$$

$$= \frac{e_k \left(E_k + m_{k_0}\right)}{4 m_{k_0} E_{k(N)}} \varphi_f^T \left[p^j_{k_\perp} \right] \varphi_{in} + \frac{i e_k \left(E_k + m_{k_0}\right)}{4 m_{k_0} E_{k(N)}} \varphi_f^T \left[(\vec{\sigma} \times \vec{p}_{k_\perp})^j \right] \varphi_{in}$$

Combining the terms $A_f GD_{in} + D_f GA_{in}$:

$$A_f GD_{in} + D_f GA_{in} = \frac{e_k (E_k + m_{k_0})}{2m_{k_0} E_{k(N)}} \varphi_f^T \left[p_{k\perp}^j\right] \varphi_{in} \tag{9.2}$$

$\underline{B_f GC_{in}}$

$$\frac{-e_k x_k^2}{16m_{k_0} E_{k(N)} (E_k + m_{k_0})} \varphi_f^T \left[(\vec{\sigma} \cdot \vec{p}_{k\perp})(\vec{\sigma} \cdot \vec{q})\vec{\sigma}(\vec{\sigma} \cdot \vec{q})\right] \varphi_{in}$$

$$= \frac{-e_k x_k^2}{16m_{k_0} E_{k(N)} (E_k + m_{k_0})} \varphi_f^T \left[\left(\sigma^l p_{k\perp}^l \sigma^m q^m\right) \left(\sigma^i \sigma^s q^s\right)\right] \varphi_{in}$$

$$= \frac{-e_k x_k^2}{16m_{k_0} E_{k(N)} (E_k + m_{k_0})} \varphi_f^T \left[\left\{\left(\delta^{lm} + i\varepsilon^{lmf} \sigma^f\right) p_{k\perp}^l q^m\right\} \left\{\left(\delta^{is} + i\varepsilon^{isd} \sigma^d\right) q^s\right\}\right] \varphi_{in}$$

$$= \frac{-e_k x_k^2}{16m_{k_0} E_{k(N)} (E_k + m_{k_0})} \varphi_f^T \left[\left(p_{k\perp}^m q^m + i\varepsilon^{lmf} \sigma^f p_{k\perp}^l q^m\right) \left(q^i + i\varepsilon^{isd} \sigma^d q^s\right)\right] \varphi_{in}$$

where $p_{k\perp}^m q^m = 0$. Then

$$\frac{-e_k x_k^2}{16m_{k_0} E_{k(N)} (E_k + m_{k_0})} \varphi_f^T \left[i\varepsilon^{lmf} \sigma^f p_{k\perp}^l q^m \left(q^i + i\varepsilon^{isd} \sigma^d q^s\right)\right] \varphi_{in}$$

$$= \frac{-e_k x_k^2}{16m_{k_0} E_{k(N)} (E_k + m_{k_0})} \varphi_f^T \left[i\varepsilon^{lmf} \sigma^f p_{k\perp}^l q^m q^i + (i^2)\, \varepsilon^{lmf} \sigma^f p_{k\perp}^l q^m \varepsilon^{isd} \sigma^d q^s\right] \varphi_{in}$$

$$= \frac{-e_k x_k^2}{16m_{k_0} E_{k(N)} (E_k + m_{k_0})} \left\{\varphi_f^T \left[i\vec{\sigma} \cdot (\vec{p}_{k\perp} \times \vec{q})\, q^i\right] + \left[(-1)\, \sigma^f \left(\vec{p}_{k\perp} \times \vec{q}\right)^f \left(-\varepsilon^{ids} \sigma^d q^s\right)\right] \varphi_{in}\right\}$$

$$= \frac{-ie_k x_k^2}{16m_{k_0} E_{k(N)} (E_k + m_{k_0})} \varphi_f^T \left[\vec{\sigma} \cdot (\vec{p}_{k\perp} \times \vec{q})\, q^i\right] \varphi_{in}$$

$$- \frac{e_k x_k^2}{16m_{k_0} E_{k(N)} (E_k + m_{k_0})} \varphi_f^T \left[\vec{\sigma} \cdot (\vec{p}_{k\perp} \times \vec{q})\, (\vec{\sigma} \times \vec{q})^i\right] \varphi_{in}$$

$\underline{C_f GB_{in}}$

$$\frac{-e_k x_k^2}{16m_{k_0}E_{k(N)}\left(E_k+m_{k_0}\right)}\varphi_f^T\left[(\vec{\sigma}\cdot\vec{q})\vec{\sigma}(\vec{\sigma}\cdot\vec{q})(\vec{\sigma}\cdot\vec{p}_{k_\perp})\right]\varphi_{in}$$
$$=\frac{-e_k x_k^2}{16m_{k_0}E_{k(N)}\left(E_k+m_{k_0}\right)}\varphi_f^T\left[\left(\sigma^l q^l \sigma^i\right)\left(\sigma^d q^d \sigma^s p_{k_\perp}^s\right)\right]\varphi_{in}$$
$$=\frac{-e_k x_k^2}{16m_{k_0}E_{k(N)}\left(E_k+m_{k_0}\right)}\varphi_f^T\left[\left\{\left(\delta^{li}+i\varepsilon^{fli}\sigma^f\right)q^l\right\}\left\{\left(\delta^{ds}+i\varepsilon^{dsn}\sigma^n\right)p_{k_\perp}^d q^s\right\}\right]\varphi_{in}$$
$$=\frac{-e_k x_k^2}{16m_{k_0}E_{k(N)}\left(E_k+m_{k_0}\right)}\varphi_f^T\left[\left(q^i+i\varepsilon^{ifl}\sigma^f q^l\right)\left(p_{k_\perp}^s q^s+i\varepsilon^{dsn}\sigma^n p_{k_\perp}^s q^d\right)\right]\varphi_{in}$$

where $p_{k_\perp}^s q^s = 0$. Then

$$\frac{-e_k x_k^2}{16m_{k_0}E_{k(N)}\left(E_k+m_{k_0}\right)}\varphi_f^T\left[\left(q^i+i\varepsilon^{ifl}\sigma^f q^l\right)i\varepsilon^{dsn}\sigma^n p_{k_\perp}^s q^d\right]\varphi_{in}$$
$$=\frac{-e_k x_k^2}{16m_{k_0}E_{k(N)}\left(E_k+m_{k_0}\right)}\varphi_f^T\left[i\varepsilon^{dsn}\sigma^n p_{k_\perp}^s q^d q^i+\left(i^2\right)\varepsilon^{ifl}\sigma^f q^l\varepsilon^{dsn}\sigma^n p_{k_\perp}^s q^d\right]\varphi_{in}$$
$$=\frac{-e_k x_k^2}{16m_{k_0}E_{k(N)}\left(E_k+m_{k_0}\right)}\left\{\varphi_f^T\left[-i\vec{\sigma}\cdot\left(\vec{p}_{k_\perp}\times\vec{q}\right)q^i\right]+\left[(-1)\left(\varepsilon^{ifl}\sigma^f q^l\right)\sigma^n(-1)\left(\vec{p}_{k_\perp}\times\vec{q}\right)^n\right]\varphi_{in}\right.$$
$$=\frac{ie_k x_k^2}{16m_{k_0}E_{k(N)}\left(E_k+m_{k_0}\right)}\varphi_f^T\left[\vec{\sigma}\cdot\left(\vec{p}_{k_\perp}\times\vec{q}\right)q^i\right]\varphi_{in}$$
$$-\frac{e_k x_k^2}{16m_{k_0}E_{k(N)}\left(E_k+m_{k_0}\right)}\varphi_f^T\left[(\vec{\sigma}\times\vec{q})^i\vec{\sigma}\cdot\left(\vec{p}_{k_\perp}\times\vec{q}\right]\varphi_{in}$$

$$B_f G C_{in}+C_f G B_{in}=\frac{-e_k x_k^2}{16m_{k_0}E_{k(N)}\left(E_k+m_{k_0}\right)}\varphi_f^T\left\{\vec{\sigma}.\left(\vec{p}_{k_\perp}\times\vec{q}\right),(\vec{\sigma}\times\vec{q})^i\right\}\varphi_{in} \tag{9.3}$$

$\underline{B_f G D_{in}}$

$$\frac{e_k x_k}{8m_{k_0}E_{k(N)}\left(E_{k(N)}+m_{k_0}\right)}\varphi_f^T\left[(\vec{\sigma}\cdot\vec{p}_{k_\perp})(\vec{\sigma}\cdot\vec{q})\vec{\sigma}(\vec{\sigma}\cdot\vec{p}_{k_\perp})\right]\varphi_{in}$$

$$=\frac{e_k x_k}{8m_{k_0}E_{k(N)}\left(E_{k(N)}+m_{k_0}\right)}\left[\left(\sigma^l p^l_{k_\perp}\sigma^m q^m\right)\left(\sigma^i\sigma^d p^d_{k_\perp}\right)\right]\varphi_{in}$$

$$=\frac{e_k x_k}{8m_{k_0}E_{k(N)}\left(E_{k(N)}+m_{k_0}\right)}\varphi_f^T\left[\left(\delta^{lm}+i\varepsilon^{lmf}\sigma^f\right)p^l_{k_\perp}q^m\left(\delta^{id}+i\varepsilon^{ids}\sigma^s\right)p^d_{k_\perp}\right]\varphi_{in}$$

$$=\frac{e_k x_k}{8m_{k_0}E_{k(N)}\left(E_{k(N)}+m_{k_0}\right)}\varphi_f^T\left[i\varepsilon^{lmf}\sigma^f p^l_{k_\perp}q^m\left(\delta^{id}+i\varepsilon^{ids}\sigma^s\right)p^d_{k_\perp}\right]\varphi_{in}$$

$$=\frac{e_k x_k}{8m_{k_0}E_{k(N)}\left(E_{k(N)}+m_{k_0}\right)}\varphi_f^T\left[i\varepsilon^{lmf}\sigma^f p^l_{k_\perp}q^m\left(p^i_{k_\perp}+i\varepsilon^{ids}\sigma^s p^d_{k_\perp}\right)\right]\varphi_{in}$$

$$=\frac{e_k x_k}{8m_{k_0}E_{k(N)}\left(E_{k(N)}+m_{k_0}\right)}\varphi_f^T\left[i\varepsilon^{lmf}\sigma^f p^l_{k_\perp}q^m p^i_{k_\perp}+\left(i^2\right)\varepsilon^{lmf}\sigma^f p^l_{k_\perp}q^m\varepsilon^{ids}\sigma^s p^d_{k_\perp}\right]\varphi_{in}$$

$$=\frac{e_k x_k}{8m_{k_0}E_{k(N)}\left(E_{k(N)}+m_{k_0}\right)}\varphi_f^T\left[i\vec{\sigma}\cdot(\vec{p}_{k_\perp}\times\vec{q})\,p^i_{k_\perp}+(-1)\,\vec{\sigma}.(\vec{p}_{k_\perp}\times\vec{q})\left(-\varepsilon^{isd}\sigma^s p^d_{k_\perp}\right)\right]\varphi_{in}$$

$$=\frac{ie_k x_k}{8m_{k_0}E_{k(N)}\left(E_{k(N)}+m_{k_0}\right)}\varphi_f^T\left[\vec{\sigma}\cdot(\vec{p}_{k_\perp}\times\vec{q})\,p^i_{k_\perp}\right]\varphi_{in}$$

$$+\frac{e_k x_k}{8m_{k_0}E_{k(N)}\left(E_{k(N)}+m_{k_0}\right)}\varphi_f^T\left[\vec{\sigma}\cdot(\vec{p}_{k_\perp}\times\vec{q})\left(\vec{\sigma}\times\vec{p}_{k_\perp}\right)^i\right]\varphi_{in}$$

$\underline{D_f G B_{in}}$

$$\frac{-e_k x_k}{8m_{k_0}E_{k(N)}\left(E_{k(N)}+m_{k_0}\right)}\varphi_f^T\left[(\vec{\sigma}\cdot\vec{p}_{k_\perp})\vec{\sigma}(\vec{\sigma}\cdot\vec{q})(\vec{\sigma}\cdot\vec{p}_{k_\perp})\right]\varphi_{in}$$

$$=\frac{-e_k x_k}{8m_{k_0}E_{k(N)}\left(E_{k(N)}+m_{k_0}\right)}\left[\left(\sigma^l p^l_{k_\perp}\sigma^i\right)\left(\sigma^d q^d \sigma^s p^s_{k_\perp}\right)\right]\varphi_{in}$$

$$=\frac{-e_k x_k}{8m_{k_0}E_{k(N)}\left(E_{k(N)}+m_{k_0}\right)}\varphi_f^T\left[\left(\delta^{li}+i\varepsilon^{lif}\sigma^f\right)p^l_{k_\perp}\left(\delta^{ds}+i\varepsilon^{dsm}\sigma^m\right)q^d p^s_{k_\perp}\right]\varphi_{in}$$

$$=\frac{-e_k x_k}{8m_{k_0}E_{k(N)}\left(E_{k(N)}+m_{k_0}\right)}\varphi_f^T\left[\left(p^i_{k_\perp}+i\varepsilon^{lif}\sigma^f p^l_{k_\perp}\right)i\varepsilon^{dsm}\sigma^m q^d p^s_{k_\perp}\right]\varphi_{in}$$

$$=\frac{-e_k x_k}{8m_{k_0}E_{k(N)}\left(E_{k(N)}+m_{k_0}\right)}\varphi_f^T\left[\left(i\varepsilon^{dsm}\sigma^m q^d p^s_{k_\perp}p^i_{k_\perp}+\left(i^2\right)\varepsilon^{lif}\sigma^f p^l_{k_\perp}\varepsilon^{dsm}\sigma^m q^d p^s_{k_\perp}\right)\right]\varphi_{in}$$

$$=\frac{-e_k x_k}{8m_{k_0}E_{k(N)}\left(E_{k(N)}+m_{k_0}\right)}\varphi_f^T\left[i\vec{\sigma}\cdot\left(\vec{q}\times\vec{p}_{k_\perp}\right)p^i_{k_\perp}-\left(\vec{\sigma}\times\vec{p}_{k_\perp}\right)^i\vec{\sigma}\cdot\left(\vec{q}\times\vec{p}_{k_\perp}\right)\right]\varphi_{in}$$

$$=\frac{ie_k x_k}{8m_{k_0}E_{k(N)}\left(E_{k(N)}+m_{k_0}\right)}\varphi_f^T\left[\vec{\sigma}\cdot\left(\vec{p}_{k_\perp}\times\vec{q}\right)p^i_{k_\perp}\right]\varphi_{in}$$

$$-\frac{e_k x_k}{8m_{k_0}E_{k(N)}\left(E_{k(N)}+m_{k_0}\right)}\varphi_f^T\left[\left(\vec{\sigma}\times\vec{p}_{k_\perp}\right)^i\vec{\sigma}\cdot\left(\vec{p}_{k_\perp}\times\vec{q}\right)\right]\varphi_{in}$$

To combine the terms $B_f G D_{in} + D_f G B_{in}$, use is made of the relation:

$$\vec{\sigma}.\left(\vec{p}_{k_\perp}\times\vec{q}\right)\left(\vec{\sigma}\times\vec{p}_{k_\perp}\right)^i$$

$$=\varepsilon^{lmf}\sigma^f p^l_{k_\perp}q^m\varepsilon^{ids}\sigma^d p^s_{k_\perp}$$

$$=\varepsilon^{ids}p^s_{k_\perp}\left[\sigma^f\sigma^d\right]\varepsilon^{lmf}p^l_{k_\perp}q^m$$

$$=\varepsilon^{id}\ p^s_{k_\perp}\left[2i\varepsilon^{fdk}\sigma^k+\sigma^d\sigma^f\right]\varepsilon^{lmf}p^l_{k_\perp}q^m$$

$$=\varepsilon^{ids}\sigma^d p^s_{k_\perp}\varepsilon^{lmf}\sigma^f p^l_{k_\perp}q^m+2i\varepsilon^{ids}\varepsilon^{fdk}\varepsilon^{lmf}p^s_{k_\perp}\sigma^k p^l_{k_\perp}q^m$$

where the first term in the above expression is equal to:

$$\varepsilon^{ids}\sigma^d p^s_{k_\perp}\varepsilon^{lmf}\sigma^f p^l_{k_\perp}q^m=\left(\vec{\sigma}\times\vec{p}_{k_\perp}\right)^i\vec{\sigma}\cdot\left(\vec{p}_{k_\perp}\times\vec{q}\right)$$

The second term becomes:

$$2i\varepsilon^{ids}\varepsilon^{fdk}\varepsilon^{lmf}p^s_{k_\perp}\sigma^k p^l_{k_\perp}q^m$$

$$= 2i\left(\delta^{if}\delta^{sk} - \delta^{ik}\delta^{sf}\right)p^s_{k_\perp}\sigma^k p^l_{k_\perp}q^m\varepsilon^{lmf}$$

$$= 2ip^s_{k_\perp}\sigma^s p^l_{k_\perp}q^m\varepsilon^{lmi} - 2ip^s_{k_\perp}\sigma^i p^l_{k_\perp}q^m\varepsilon^{lms}$$

$$= 2i\left(\vec{p}_{k_\perp}\cdot\vec{\sigma}\right)\left(\vec{p}_{k_\perp}\times\vec{q}\right)^i - 2i\sigma^i q^m\varepsilon^{lms}p^s_{k_\perp}p^l_{k_\perp}$$

$$= 2i\left(\vec{p}_{k_\perp}\cdot\vec{\sigma}\right)\left(\vec{p}_{k_\perp}\times\vec{q}\right)^i - 2i\sigma^i\left(\vec{p}_{k_\perp}\times\vec{p}_{k_\perp}\right)\cdot\vec{q} = 2i\left(\vec{p}_{k_\perp}\cdot\vec{\sigma}\right)\left(\vec{p}_{k_\perp}\times\vec{q}\right)^i$$

since

$$2i\sigma^i\left(\vec{p}_{k_\perp}\times\vec{p}_{k_\perp}\right)\cdot\vec{q} = 0$$

Combining the terms $B_f G D_{in} + D_f G B_{in}$:

$$B_f G D_{in} + D_f G B_{in} \tag{9.4}$$

$$= \frac{ie_k x_k}{4m_{k_0}E_{k(N)}\left(E_{k(N)}+m_{k_0}\right)}\varphi_f^T\left[\vec{\sigma}\cdot\left(\vec{p}_{k_\perp}\times\vec{q}\right)p^i_{k_\perp}\right]\varphi_{in}$$

$$+\frac{ie_k x_k}{4m_{k_0}E_{k(N)}\left(E_{k(N)}+m_{k_0}\right)}\varphi_f^T\left[\left(\vec{p}_{k_\perp}\cdot\vec{\sigma}\right)\left(\vec{p}_{k_\perp}\times\vec{q}\right)^i\right]\varphi_{in}$$

$\underline{E_f G C_{in}}$

$$i\sqrt{\frac{\left(E_{k(N)}+m_{k_0}\right)}{4E^2_{k(N)}\left(E_k+m_{k_0}\right)}}\sinh\frac{\alpha_{k(N)}}{2}\sinh\frac{\alpha_k}{2}\varphi_f^T\left(\hat{p}_{N_f}\times\hat{p}_{k_\perp}\right)\cdot\vec{\sigma}e_k\vec{\sigma}\left(\vec{\sigma}\cdot x_k\vec{p}_{N_{in}}\right)\varphi_{in}$$

$$= \frac{-ie_k x_k}{2}\sqrt{\frac{\left(E_{k(N)}+m_{k_0}\right)}{4E^2_{k(N)}\left(E_k+m_{k_0}\right)}}\sinh\frac{\alpha_{k(N)}}{2}\sinh\frac{\alpha_k}{2}$$

$$\times\varphi_f^T\left[\left(\hat{p}_{N_f}\times\hat{p}_{k_\perp}\right)^i\left(\vec{\sigma}\cdot\vec{q}\right) + i\varepsilon^{dik}\sigma^k\left(\hat{p}_{N_f}\times\hat{p}_{k_\perp}\right)^d\left(\vec{\sigma}\cdot\vec{q}\right)\right]\varphi_{in}$$

Now,

$$\varepsilon^{dik}\sigma^k\left(\hat{p}_{N_f}\times\hat{p}_{k_\perp}\right)^d\left(\vec{\sigma}\cdot\vec{q}\right)$$

$$= \varepsilon^{dik}\left(\hat{p}_{N_f}\times\hat{p}_{k_\perp}\right)^d q^k + i\varepsilon^{dik}\varepsilon^{klf}\left(\hat{p}_{N_f}\times\hat{p}_{k_\perp}\right)^d\sigma^f q^l$$

$$= -\left[\left(\hat{p}_{N_f}\times\hat{p}_{k_\perp}\right)\times\vec{q}\right]^i + i\sigma^i\left(\hat{p}_{N_f}\times\hat{p}_{k_\perp}\right)\cdot\vec{q} - i\vec{\sigma}\cdot\left(\hat{p}_{N_f}\times\hat{p}_{k_\perp}\right)q^i$$

Then E_fGC_{in} equals $((\hat{p}_{N_f}\times\hat{p}_{k_\perp})\cdot\vec{q}=0)$

$$\tfrac{-ie_kx_k}{2}\sqrt{\tfrac{(E_{k(N)}+m_{k_0})}{4E^2_{k(N)}(E_k+m_{k_0})}}\sinh\tfrac{\alpha_{k(N)}}{2}\sinh\tfrac{\alpha_k}{2}\varphi_f^\top(\hat{p}_{N_f}\times\hat{p}_{k_\perp})^i(\vec{\sigma}\cdot\vec{q})\,\varphi_{in}$$

$$+\tfrac{e_kx_k}{2}\sqrt{\tfrac{(E_{k(N)}+m_{k_0})}{4E^2_{k(N)}(E_k+m_{k_0})}}\sinh\tfrac{\alpha_{k(N)}}{2}\sinh\tfrac{\alpha_k}{2}\varphi_f^\top\left[(\hat{p}_{N_f}\times\hat{p}_{k_\perp})\times\vec{q}\right]^i\varphi_{in}$$

$$\tfrac{-ie_kx_k}{2}\sqrt{\tfrac{(E_{k(N)}+m_{k_0})}{4E^2_{k(N)}(E_k+m_{k_0})}}\sinh\tfrac{\alpha_{k(N)}}{2}\sinh\tfrac{\alpha_k}{2}\varphi_f^\top\vec{\sigma}\cdot(\hat{p}_{N_f}\times\hat{p}_{k_\perp})q^i\varphi_{in}$$

$\underline{C_fGE_{in}}$

$$i\sqrt{\tfrac{(E_{k(N)}+m_{k_0})}{4E^2_{k(N)}(E_k+m_{k_0})}}\sinh\tfrac{\alpha_{k(N)}}{2}\sinh\tfrac{\alpha_k}{2}\varphi_f^\top\left(\vec{\sigma}\cdot x_k\vec{p}_{N_f}\right)e_k\vec{\sigma}(\hat{p}_{k_\perp}\times\hat{p}_{N_{in}})\cdot\vec{\sigma}\varphi_{in}$$

$$=\tfrac{ie_kx_k}{2}\sqrt{\tfrac{(E_{k(N)}+m_{k_0})}{4E^2_{k(N)}(E_k+m_{k_0})}}\sinh\tfrac{\alpha_{k(N)}}{2}\sinh\tfrac{\alpha_k}{2}$$

$$\times\varphi_f^\top\left[q^i\vec{\sigma}\cdot(\hat{p}_{N_f}\times\hat{p}_{k_\perp})+i\varepsilon^{lik}\sigma^k q^l(\hat{p}_{N_f}\times\hat{p}_{k_\perp})^d\sigma^d\right]\varphi_{in}$$

Now,

$$i\varepsilon^{lik}\sigma^k q^l(\hat{p}_{N_f}\times\hat{p}_{k_\perp})^d\sigma^d$$

$$=i\varepsilon^{lid}(\hat{p}_{N_f}\times\hat{p}_{k_\perp})^d q^l-\varepsilon^{lik}\varepsilon^{kdf}q^l(\hat{p}_{N_f}\times\hat{p}_{k_\perp})^d\sigma^f$$

$$=i\left[(\hat{p}_{N_f}\times\hat{p}_{k_\perp})\times\vec{q}\right]^i-\sigma^i(\hat{p}_{N_f}\times\hat{p}_{k_\perp})\cdot\vec{q}+(\vec{q}\cdot\vec{\sigma})(\hat{p}_{N_f}\times\hat{p}_{k_\perp})^i$$

Then C_fGE_{in} equals $((\hat{p}_{N_f}\times\hat{p}_{k_\perp})\cdot\vec{q}=0)$

$$\tfrac{ie_kx_k}{2}\sqrt{\tfrac{(E_{k(N)}+m_{k_0})}{4E^2_{k(N)}(E_k+m_{k_0})}}\sinh\tfrac{\alpha_{k(N)}}{2}\sinh\tfrac{\alpha_k}{2}\varphi_f^\top(\hat{p}_{N_f}\times\hat{p}_{k_\perp})^i(\vec{\sigma}\cdot\vec{q})\,\varphi_{in}$$

$$-\tfrac{e_kx_k}{2}\sqrt{\tfrac{(E_{k(N)}+m_{k_0})}{4E^2_{k(N)}(E_k+m_{k_0})}}\sinh\tfrac{\alpha_{k(N)}}{2}\sinh\tfrac{\alpha_k}{2}\varphi_f^\top\left[(\hat{p}_{N_f}\times\hat{p}_{k_\perp})\times\vec{q}\right]^i\varphi_{in}$$

$$\tfrac{ie_kx_k}{2}\sqrt{\tfrac{(E_{k(N)}+m_{k_0})}{4E^2_{k(N)}(E_k+m_{k_0})}}\sinh\tfrac{\alpha_{k(N)}}{2}\sinh\tfrac{\alpha_k}{2}\varphi_f^\top\vec{\sigma}\cdot(\hat{p}_{N_f}\times\hat{p}_{k_\perp})q^i\varphi_{in}$$

Thus

$$E_f G C_{in} + C_f G E_{in} = 0 \tag{9.5}$$

$\underline{E_f G D_{in}}$

$$ie_k \sqrt{\frac{(E_k + m_{k_0})}{4E_{k(N)}^2 (E_{k(N)} + m_{k_0})}} \sinh \frac{\alpha_{k(N)}}{2} \sinh \frac{\alpha_k}{2} \varphi_f^\top (\hat{p}_{N_f} \times \hat{p}_{k_\perp}) \cdot \vec{\sigma} \sigma^i (\vec{\sigma} \cdot \vec{p}_{k_\perp}) \varphi_{in}$$

$$= ie_k \sqrt{\frac{(E_k + m_{k_0})}{4E_{k(N)}^2 (E_{k(N)} + m_{k_0})}} \sinh \frac{\alpha_{k(N)}}{2} \sinh \frac{\alpha_k}{2}$$

$$\times \varphi_f^\top \left[(\hat{p}_{N_f} \times \hat{p}_{k_\perp}) \cdot \vec{\sigma} p_{k_\perp}^i + i (\hat{p}_{N_f} \times \hat{p}_{k_\perp})^l \sigma^l \varepsilon^{idf} \sigma^f p_{k_\perp}^d \right] \varphi_{in}$$

$$i (\hat{p}_{N_f} \times \hat{p}_{k_\perp})^l \sigma^l \varepsilon^{idf} \sigma^f p_{k_\perp}^d$$

$$= i \left[(\hat{p}_{N_f} \times \hat{p}_{k_\perp})^f \varepsilon^{idf} p_{k_\perp}^d + i (\hat{p}_{N_f} \times \hat{p}_{k_\perp})^l \varepsilon^{idf} \varepsilon^{lfk} \sigma^k p_{k_\perp}^d \right]$$

$$= i \left[\vec{p}_{k_\perp} \times (\hat{p}_{N_f} \times \hat{p}_{k_\perp}) \right] - (\hat{p}_{N_f} \times \hat{p}_{k_\perp})^i (\vec{\sigma} \cdot \vec{p}_{k_\perp})$$

$$+ \left[(\hat{p}_{N_f} \times \hat{p}_{k_\perp}) \cdot \vec{p}_{k_\perp} \right] \sigma^i$$

$E_f G D_{in}$

$$= ie_k \sqrt{\frac{(E_k + m_{k_0})}{4E_{k(N)}^2 (E_{k(N)} + m_{k_0})}} \sinh \frac{\alpha_{k(N)}}{2} \sinh \frac{\alpha_k}{2} \varphi_f^\top \left[(\hat{p}_{N_f} \times \hat{p}_{k_\perp}) \cdot \vec{\sigma} p_{k_\perp}^i \right] \varphi_{in}$$

$$- e_k \sqrt{\frac{(E_k + m_{k_0})}{4E_{k(N)}^2 (E_{k(N)} + m_{k_0})}} \sinh \frac{\alpha_{k(N)}}{2} \sinh \frac{\alpha_k}{2} \varphi_f^\top \left[\vec{p}_{k_\perp} \times (\hat{p}_{N_f} \times \hat{p}_{k_\perp}) \right] \varphi_{in}$$

$$- ie_k \sqrt{\frac{(E_k + m_{k_0})}{4E_{k(N)}^2 (E_{k(N)} + m_{k_0})}} \sinh \frac{\alpha_{k(N)}}{2} \sinh \frac{\alpha_k}{2} \varphi_f^\top \left[(\hat{p}_{N_f} \times \hat{p}_{k_\perp})^i (\vec{\sigma} \cdot \vec{p}_{k_\perp}) \right] \varphi_{in}$$

$\underline{D_f G E_{in}}$

$$ie_k\sqrt{\frac{\left(E_k+m_{k_0}\right)}{4E_{k(N)}^2\left(E_{k(N)}+m_{k_0}\right)}}\sinh\frac{\alpha_{k(N)}}{2}\sinh\frac{\alpha_k}{2}\varphi_f^\top\left[\left(\vec{\sigma}\cdot\vec{p}_{k_\perp}\right)\sigma^i(\hat{p}_{N_f}\times\hat{p}_{k_\perp})\cdot\vec{\sigma}\right]\varphi_{in}$$

$$=ie_k\sqrt{\frac{\left(E_k+m_{k_0}\right)}{4E_{k(N)}^2\left(E_{k(N)}+m_{k_0}\right)}}\sinh\frac{\alpha_{k(N)}}{2}\sinh\frac{\alpha_k}{2}\varphi_f^\top\left[p_{k_\perp}^i\left(\left(\hat{p}_{N_f}\times\hat{p}_{k_\perp}\right)\cdot\vec{\sigma}\right)\right]\varphi_{in}$$

$$-e_k\sqrt{\frac{\left(E_k+m_{k_0}\right)}{4E_{k(N)}^2\left(E_{k(N)}+m_{k_0}\right)}}\sinh\frac{\alpha_{k(N)}}{2}\sinh\frac{\alpha_k}{2}\varphi_f^\top\left[\varepsilon^{lif}\sigma^f p_{k_\perp}^l\left(\left(\hat{p}_{N_f}\times\hat{p}_{k_\perp}\right)^d\sigma^d\right)\right]\varphi_{in}$$

$$\varepsilon^{lif}\sigma^f p_{k_\perp}^l(\hat{p}_{N_f}\times\hat{p}_{k_\perp})^d\sigma^d$$

$$=\varepsilon^{lid}p_{k_\perp}^l(\hat{p}_{N_f}\times\hat{p}_{k_\perp})^d$$

$$+i\varepsilon^{lif}\varepsilon^{fdk}\sigma^k p_{k_\perp}^l(\hat{p}_{N_f}\times\hat{p}_{k_\perp})^d$$

$$=-\left[\vec{p}_{k_\perp}\times\left(\hat{p}_{N_f}\times\hat{p}_{k_\perp}\right)\right]^i$$

$$+i(\vec{\sigma}\cdot\vec{p}_{k_\perp})(\hat{p}_{N_f}\times\hat{p}_{k_\perp})^i-i\sigma^i\vec{p}_{k_\perp}\cdot(\hat{p}_{N_f}\times\hat{p}_{k_\perp})$$

Thus

$$D_f G E_{in}$$

$$=ie_k\sqrt{\frac{\left(E_k+m_{k_0}\right)}{4E_{k(N)}^2\left(E_{k(N)}+m_{k_0}\right)}}\sinh\frac{\alpha_{k(N)}}{2}\sinh\frac{\alpha_k}{2}\varphi_f^\top\left[p_{k_\perp}^i\left(\left(\hat{p}_{N_f}\times\hat{p}_{k_\perp}\right)\cdot\vec{\sigma}\right)\right]\varphi_{in}$$

$$+e_k\sqrt{\frac{\left(E_k+m_{k_0}\right)}{4E_{k(N)}^2\left(E_{k(N)}+m_{k_0}\right)}}\sinh\frac{\alpha_{k(N)}}{2}\sinh\frac{\alpha_k}{2}\varphi_f^\top\left[\vec{p}_{k_\perp}\times\left(\hat{p}_{N_f}\times\hat{p}_{k_\perp}\right)\right]^i\varphi_{in}$$

$$-ie_k\sqrt{\frac{\left(E_k+m_{k_0}\right)}{4E_{k(N)}^2\left(E_{k(N)}+m_{k_0}\right)}}\sinh\frac{\alpha_{k(N)}}{2}\sinh\frac{\alpha_k}{2}\varphi_f^\top\left[(\vec{\sigma}\cdot\vec{p}_{k_\perp})(\hat{p}_{N_f}\times\hat{p}_{k_\perp})^i\right]\varphi_{in}$$

Then

$$E_f G D_{in} + D_f G E_{in}$$

$$ie_k\sqrt{\frac{\left(E_k+m_{k_0}\right)}{E_{k(N)}^2\left(E_{k(N)}+m_{k_0}\right)}}\sinh\frac{\alpha_{k(N)}}{2}\sinh\frac{\alpha_k}{2}\varphi_f^\top\left[p_{k_\perp}^i\left(\hat{p}_{N_f}\times\hat{p}_{k_\perp}\right)\cdot\vec{\sigma}\right)\right]\varphi_{in}$$

$$-ie_k\sqrt{\frac{\left(E_k+m_{k_0}\right)}{E_{k(N)}^2\left(E_{k(N)}+m_{k_0}\right)}}\sinh\frac{\alpha_{k(N)}}{2}\sinh\frac{\alpha_k}{2}\varphi_f^\top\left[(\vec{\sigma}\cdot\vec{p}_{k_\perp})(\hat{p}_{N_f}\times\hat{p}_{k_\perp})^i\right]\varphi_{in} \tag{9.6}$$

9.2 Appendix B

9.2.1 Expectation values

Again, we calculate

$$\frac{1}{2}\int d^3r(\vec{r}\times < B| : \overline{\psi}\vec{\gamma}\psi : |B >)$$

Equation (9.1)

$$A_f GC_{in} + C_f GA_{in} = \frac{ie_k x_k \left(E_{k(N)} + m_{k_0}\right)}{4m_{k_0}E_{k(N)}}\varphi_f^T \left[(\vec{\sigma}\times\vec{q})^i\right]\varphi_{in}$$

$$\frac{1}{2}\int d^3r\varphi_f^T\varepsilon^{dli}r^l\varepsilon^{ifs}\sigma^f q^s\varphi_{in}e^{i\vec{q}_k\cdot\vec{r}}$$

$$= -\frac{i}{2x_k}\int d^3r\varphi_f^T\varepsilon^{dli}\varepsilon^{ifl}\sigma^f\varphi_{in}e^{i\vec{q}_k\cdot\vec{r}}$$

$$= \frac{i}{2x_k}\int d^3r\varphi_f^T\left(\delta^{di}\delta^{if} - \delta^{df}\delta^{ii}\right)\sigma^f\varphi_{in}e^{i\vec{q}_k\cdot\vec{r}}$$

$$= \frac{i}{2x_k}\int d^3r\varphi_f^T(-2)\sigma^d\varphi_{in}e^{i\vec{q}_k\cdot\vec{r}}$$

$$= \frac{-i}{x_k}\int d^3r\varphi_f^T\sigma^d\varphi_{in}e^{i\vec{q}_k\cdot\vec{r}}$$

Thus $A_f GC_{in} + C_f GA_{in}$ gives

$$\sum_{(k)}\frac{e_k\left(E_{k(N)}+m_{k_0}\right)}{4m_{k_0}E_{k(N)}} < B|\,\sigma^3_{(k)}\,|B >)$$

Non relativistically ($E_{k(N)} \rightarrow m_{k_0}$)

$$\sum_{(k)}\frac{e_k}{2m_{k_0}} < B|\,\sigma^3_{(k)}\,|B >) \tag{9.7}$$

and relativistically ($E_{k(N)} \gg m_{k_0}$),

$$\sum_{(k)} \frac{e_k}{4m_{k_0}} < B|\, \sigma^3_{(k)} \,|B > \tag{9.8}$$

Equation (9.2)

$$A_f G D_{in} + D_f G A_{in} = \frac{e_k\left(E_k - m_{k_0}\right)}{2m_{k_0} E_{k(N)}} \varphi_f^T \left[p^j_{k_\perp}\right] \varphi_{in}$$

$$\frac{1}{2} \int d^3 r \varphi_f^T \varepsilon^{dlj} r^l p^j_{k_\perp} \varphi_{in} e^{-ix_k \vec{q}\cdot\vec{r}}$$

$$= \frac{1}{2} \int d^3 r \varphi_f^T \varepsilon^{dlj} r^l p^j_{k_\perp} \varphi_{in} e^{-ix_k \vec{q}\cdot\vec{r}_\parallel} e^{i\vec{p}_{k_\perp}\cdot\vec{r}_\perp} e^{-i\vec{p}_{k_\perp}\cdot\vec{r}_\perp}$$

$$= -i \int d^3 r \varphi_f^T \varepsilon^{dlj} \delta^{lj} \varphi_{in} e^{-ix_k \vec{q}\cdot\vec{r}_\parallel} e^{i\vec{p}_{k_\perp}\cdot\vec{r}_\perp} e^{-i\vec{p}_{k_\perp}\cdot\vec{r}_\perp}$$

$$= 0$$

Equation (9.3)

$$B_f G C_{in} + C_f G B_{in} = \frac{e_k x_k^2}{16 m_{k_0} E_{k(N)} \left(E_k + m_{k_0}\right)} \varphi_f^T \left\{ (\vec{\sigma} \times \vec{q})^i, \vec{\sigma}. \left(\vec{p}_{k_\perp} \times \vec{q}\right) \right\} \varphi_{in}$$

Terms quadratic in q have the following form:

$$\sim \int d^3 r \varphi_f^T r^l q^i q^j \varphi_{in} e^{-ix_k \vec{q}\cdot\vec{r}}$$

$$= \int d^3 r \varphi_f^T r^l q^i \frac{1}{-ix_k} \frac{\partial e^{-ix_k \vec{q}\cdot\vec{r}}}{\partial r^j} \varphi_{in}$$

$$= \frac{1}{ix_k} \int d^3 r \varphi_f^T \delta^{jl} q^i e^{-ix_k \vec{q}\cdot\vec{r}} \varphi_{in}$$

$$= \frac{1}{ix_k} \int d^3 r \varphi_f^T \delta^{jl} \frac{1}{-ix_k} \frac{\partial e^{-ix_k \vec{q}\cdot\vec{r}}}{\partial r^i} \varphi_{in}$$

This is a total derivative, and therefore put to zero.

Equation (9.4)

$$B_f G D_{in} + D_f G B_{in}$$

$$= \frac{ie_k x_k}{4m_{k_0} E_{k(N)} \left(E_{k(N)} + m_{k_0}\right)} \varphi_f^T \left[\vec{\sigma} \cdot (\vec{p}_{k_\perp} \times \vec{q})\, p_{k_\perp}^i\right] \varphi_{in}$$

$$+ \frac{ie_k x_k}{4m_{k_0} E_{k(N)} \left(E_{k(N)} + m_{k_0}\right)} \varphi_f^T \left[(\vec{p}_{k_\perp} \cdot \vec{\sigma})\, (\vec{p}_{k_\perp} \times \vec{q})^i\right] \varphi_{in}$$

$$\tfrac{1}{2} \int d^3 r \varphi_f^T \varepsilon^{dli} r^l \varepsilon^{fsm} \sigma^m p_{k_\perp}^f q^s p_{k_\perp}^i \varphi_{in} e^{-ix_k \vec{q}\cdot\vec{r}}$$

$$= \tfrac{-i}{2x_k} \int d^3 r \varphi_f^T \varepsilon^{dli} \varepsilon^{flm} \sigma^m p_{k_\perp}^f p_{k_\perp}^i \varphi_{in} e^{-ix_k \vec{q}\cdot\vec{r}}$$

$$= \tfrac{-i}{2x_k} \int d^3 r \varphi_f^T \left[(\vec{\sigma} \cdot \vec{p}_{k_\perp}) p_{k_\perp}^d - \sigma^d |\vec{p}_{k_\perp}|^2)\right] \varphi_{in} e^{-ix_k \vec{q}\cdot\vec{r}}$$

Thus (9.4) (first term) gives ($|\vec{p}_{k_\perp}|^2 = E_{k(N)}^2 - m_{k_0}^2$)

$$\sum_{(k)} \frac{e_k (E_{k(N)}^2 - m_{k_0}^2)}{8m_{k_0} E_{k(N)} \left(E_{k(N)} + m_{k_0}\right)} < B|\, \sigma_{(k)}^3 \,|B >$$

Non relativistically this gives zero.
Relativistically:

$$\sum_{(k)} \frac{e_k}{8m_{k_0}} < B|\, \sigma_{(k)}^3 \,|B > \tag{9.9}$$

The second term results in

$$\tfrac{1}{2} \int d^3 r \varphi_f^T \varepsilon^{dli} r^l \sigma^s p_{k_\perp}^s \varepsilon^{ifm} p_{k_\perp}^f q^m \varphi_{in} e^{-ix_k \vec{q}\cdot\vec{r}}$$

$$= \tfrac{-i}{2x_k} \int d^3 r \varphi_f^T \vec{\sigma} \cdot \vec{p}_{k_\perp} 2 p_{k_\perp}^d \varphi_{in} e^{-ix_k \vec{q}\cdot\vec{r}}$$

Since $p_{k_\perp}^3 = 0$, this term contributes nothing to the magnetic moment.

Equation (9.6)

$$E_f G D_{in} + D_f G E_{in}$$

$$ie_k\sqrt{\frac{(E_k+m_{k_0})}{E^2_{k(N)}(E_{k(N)}+m_{k_0})}}\sinh\frac{\alpha_{k(N)}}{2}\sinh\frac{\alpha_k}{2}\varphi_f^T\left[p^i_{k_\perp}\left(\hat{p}_{N_f}\times\hat{p}_{k_\perp}\right)\cdot\vec{\sigma}\right)\Big]\varphi_{in}$$

$$-ie_k\sqrt{\frac{(E_k+m_{k_0})}{E^2_{k(N)}(E_{k(N)}+m_{k_0})}}\sinh\frac{\alpha_{k(N)}}{2}\sinh\frac{\alpha_k}{2}\varphi_f^T\left[(\vec{\sigma}\cdot\vec{p}_{k_\perp})(\hat{p}_{N_f}\times\hat{p}_{k_\perp})^i\right]\varphi_{in}$$

We consider the first term:

$$\frac{1}{2}\int d^3r\varphi_f^T\varepsilon^{dli}r^l\left(\hat{p}_{N_f}\times\hat{p}_{k_\perp}\right)\cdot\vec{\sigma}\Big)p^i_{k_\perp}\varphi_{in}e^{-ix_k\vec{q}\cdot\vec{r}}$$

$$=\frac{1}{2}\int d^3r\varphi_f^T\varepsilon^{dli}r^l\left(\hat{p}_{N_f}\times\hat{p}_{k_\perp}\right)\cdot\vec{\sigma}\Big)p^i_{k_\perp}\varphi_{in}e^{-ix_k\vec{q}\cdot\vec{r}}$$

$$\frac{1}{2}\int d^3r\varphi_f^T\varepsilon^{dli}r^l\left(\hat{p}_{N_f}\times\hat{p}_{k_\perp}\right)\cdot\vec{\sigma}\Big)p^i_{k_\perp}\varphi_{in}e^{-ix_k\vec{q}\cdot\vec{r}_\parallel}e^{i\vec{p}_{k_\perp}\cdot\vec{r}_\perp}e^{-i\vec{p}_{k_\perp}\cdot\vec{r}_\perp}$$

$$=-i\frac{1}{2}\int d^3r\varphi_f^T\varepsilon^{dli}\delta^{li}\left(\hat{p}_{N_f}\times\hat{p}_{k_\perp}\right)\cdot\vec{\sigma}\Big)\varphi_{in}e^{-ix_k\vec{q}\cdot\vec{r}_\parallel}e^{i\vec{p}_{k_\perp}\cdot\vec{r}_\perp}e^{-i\vec{p}_{k_\perp}\cdot\vec{r}_\perp}$$

$$-i\int d^3re^{-i\vec{p}_{k_\perp}\cdot\vec{r}_\perp}\varphi_f^T\left(\hat{p}_{N_f}\times\hat{p}_{k_\perp}\right)\cdot\vec{\sigma}\Big)\varepsilon^{dli}r^l\frac{\partial}{\partial r^i}\left(e^{-ix_k\vec{q}\cdot\vec{r}_\parallel}e^{i\vec{p}_{k_\perp}\cdot\vec{r}_\perp}\right)$$

Thus the contribution is

$$\sum_{(k)}ie_k\sqrt{\frac{(E_k+m_{k_0})}{E^2_{k(N)}\left(E_{k(N)}+m_{k_0}\right)}}\sinh\frac{\alpha_{k(N)}}{2}\sinh\frac{\alpha_k}{2}<B|\left(\hat{p}_{N_f}\times\hat{p}_{k_\perp}\right)\cdot\vec{\sigma}\Big)\left(\vec{r}\times\frac{\vec{\nabla}}{i}\right)^d|B> \tag{9.10}$$

Clearly, non relativistically this is zero.

Relativistically, $E_{k(N)}\to\infty$ so that at $\eta\to 0$, this term vanishes in the relativistic limit.

Let's look at the second term:

The same methods give

$$-\frac{c_k}{2}\sqrt{\frac{\left(E_k+m_{k_0}\right)}{E^2_{k(N)}\left(E_{k(N)}+m_{k_0}\right)}}\sinh\frac{\alpha_{k(N)}}{2}\sinh\frac{\alpha_k}{2} < B|\left(\vec{\sigma}\times\left(\hat{p}_{N_f}\times\hat{p}_{k_\perp}\right)\right)^3|B>$$

$$+\frac{c_k}{2}\sqrt{\frac{\left(E_k+m_{k_0}\right)}{E^2_{k(N)}\left(E_{k(N)}+m_{k_0}\right)}}\sinh\frac{\alpha_{k(N)}}{2}\sinh\frac{\alpha_k}{2} < B|\vec{r}\times\left(\hat{p}_{N_f}\times\hat{p}_{k_\perp}\right)\left(\vec{\sigma}\cdot\frac{\vec{\nabla}}{i}\right)^3|B> \tag{9.11}$$

as contribution to μ^3. For the first term above, clearly $\vec{\sigma} \neq \sigma^3\hat{x}^3$, and the second term vanishes in both the non relativistic (static) and the relativistic limits.

Clearly, since we evaluate μ^3 at $\eta = 0$, (9.10) and (9.11) anyway do not contribute to μ ($\sinh\frac{\alpha_k}{2}\to 0$). However, this is a check that their contribution to the magnetic radius, is zero.

Our extreme non relativistic value for the magnetic moments are thus given by (9.7):

$$\mu_{proton_{(N.R.)}} = 3\mu_N \tag{9.12}$$

$$\mu_{netron_{(N.R.)}} = -2\mu_N$$

9.3 Appendix C

9.3.1 Calculation of dLips

The Lorentz invariant phase space element for a three particle system is given by

$$dLips = (2\pi)^4\,\delta^4\,(P - p_1 - p_2 - p_3)\prod_{i=1}^{3}\frac{d^3p_i}{(2\pi)^3\,2E_i} \tag{9.13}$$

where P is the 4-momentum of the entire system, consisting of 3 particles with 4-momentum p_i each. It is clear that the δ-function is an expression of energy and 3-momentum conservation.

When we move to the center of mass coordinates, the 3-momentum of the entire system reduces to $\vec{P} = 0$. The total energy of a relativistic system can be written as

$$E_{tot}^2 = \vec{P}^2 + M^2 \tag{9.14}$$

where $\vec{P}^2$ is the square of the three momentum of the entire system and M^2 its total mass.

In the center of mass frame, with $\vec{P} = 0$, (9.14) reduces to [13]

$$E_{tot}^2 = M^2 \tag{9.15}$$

$$M = \sqrt{E_{tot}^2} = \sum_{i=1}^{3} E_i$$

The δ-function in (9.13) can be split into the following

$$\delta^4\,(P - p_1 - p_2 - p_3) = \delta\,(E_{tot} - E_1 - E_2 - E_3)\,\delta^3\left(\vec{P} - \vec{p}_1 - \vec{p}_2 - \vec{p}_3\right)$$

which becomes in the center of mass frame, with the help of (9.15)

$$\delta^4\,(P - p_1 - p_2 - p_3) = \delta\,(M - E_1 - E_2 - E_3)\,\delta^3\,(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) \tag{9.16}$$

where $\sum_{i=1}^{3} p_i = 0$ in the CMS frame. Hence (9.13) becomes, on substitution with (9.16)

$$dLips = \delta\,(M - E_1 - E_2 - E_3)\,\delta^3\,(\vec{p}_1 + \vec{p}_2 + \vec{p}_3)\,\frac{1}{8\,(2\pi)^5}\prod_{i=1}^{3}\frac{d^3p_i}{E_i} \tag{9.17}$$

To get rid of one of the integration measures, say d^3p_3, we can make use of [13]

$$\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0 \tag{9.18}$$

This can be seen as follows: on integration over $\prod_{i=1}^{3} \frac{d^3p_i}{E_i}$, which is the only instance in which the δ-function makes sense, all the possible values of the p_i are considered and the only ones which are picked out, i.e. do not result in zero, are the values which are subject to the condition $\sum_{i=1}^{3} p_i = 0$. This fact is mathematically manifested in the function $\delta^3(\vec{p}_1 + \vec{p}_2 + \vec{p}_3)$. When any arbitrary values of $\vec{p}_1$ and $\vec{p}_2$ are picked out, which is the case in an integration, the value of $\vec{p}_3$ follows automatically from the constraint (9.18).

Furthermore, the energy relation for the third particle, namely

$$E_3^2 = \vec{p}_3^{\,2} + m_3^2 \tag{9.19}$$

or

$$E_3 = \sqrt{\vec{p}_3^{\,2} + m_3^2}$$

where only the positive energy solutions are taken into account, can be rewritten, again with the help of (9.18), as

$$E_3 = \sqrt{(\vec{p}_1 + \vec{p}_2)^2 + m_3^2}$$

so that, finally, (9.17) becomes [13]

$$dLips = \delta\left(M - E_1 - E_2 - \sqrt{(\vec{p}_1 + \vec{p}_2)^2 + m_3^2}\right) \frac{1}{8E_3(2\pi)^5} \prod_{i=1}^{2} \frac{d^3p_i}{E_i} \tag{9.20}$$

where the function $\delta^3(\vec{p}_1 + \vec{p}_2 + \vec{p}_3)$ has been eliminated by the implementation of constraint (9.18) and the eradication of the d^3p_3 dependence via integration, which yielded unity as a result.

We use the infinitesimal volume element in spherical coordinates

$$d^3x = r^2 dr \cos\theta d\theta d\phi = r^2 dr d\Omega$$

and analogously

$$d^3p_i = p_i^2 dp_i d\Omega_i$$

to rewrite

$$d^3p_1 d^3p_2 = \vec{p}_1^{\,2} dp_1 d\Omega_1 \vec{p}_2^{\,2} dp_2 d\Omega_2 \tag{9.21}$$

Making use of

$$E_i^2 = \vec{p}_i^2 + m_i^2$$

and differentiating

$$E_i dE_i = |\vec{p}_i|\, dp_i \tag{9.22}$$

since the rest mass, m_i is a constant.
Substituting (9.22) into (9.21) yields [13]

$$\begin{aligned} d^3p_1 d^3p_2 &= \frac{E_1\vec{p}_1^2 dE_1 d\Omega_1 E_2 \vec{p}_2^2 dE_2 d\Omega_2}{|\vec{p}_1|\,|\vec{p}_2|} && (9.23)\\ &= E_1\,|\vec{p}_1|\, dE_1 d\Omega_1 E_2\,|\vec{p}_2|\, dE_2 d\Omega_2 \\ &= E_1 E_2 dE_1 dE_2\,|\vec{p}_2|\,|\vec{p}_1|\, d\Omega_2 d\Omega_1 && (9.24)\end{aligned}$$

We move to Euler angle coordinates via the relation

$$d\Omega_1 d\Omega_2 = d\alpha d\cos\beta d\gamma d\cos\theta_{12} = dRd\cos\theta_{12} \tag{9.25}$$

In this expression, θ_{12} is the angle between $\vec{p}_1$and $\vec{p}_2$ and (α, β, γ) denotes the orientation of the normal to the plane $\vec{n} = \vec{p}_1 \times \vec{p}_2$.
Substituting (9.25) into (9.23) , we obtain [13]

$$d^3p_1 d^3p_2 = E_1 E_2 dE_1 dE_2 dR\,|\vec{p}_2|\,|\vec{p}_1|\, d\cos\theta_{12} \tag{9.26}$$

It is clear that

$$\int d\Omega_1 \int d\Omega_2 = \int dR \int d\cos\theta_{12} = (4\pi)^2$$

The transition from spherical polar coördinates to Euler angles is a well understood transformation.

Starting again from (9.18), it is clear that the vectors $\vec{p}_i$ form a triangle. Using this fact and the cosine rule for triangles

$$\vec{p}_3^2 = \vec{p}_1^2 + \vec{p}_2^2 + 2\,|\vec{p}_1|\,|\vec{p}_2|\cos\theta_{12}$$

in which the length of the vectors $|\vec{p}_1|$ and $|\vec{p}_2|$ are kept constant, we obtain upon differentiation

$$|\vec{p}_3|\, dp_3 = |\vec{p}_1|\,|\vec{p}_2|\, d\cos\theta_{12}$$

Using the relation (9.22) , we can identify [13]

$$E_3 dE_3 = |\vec{p}_1|\,|\vec{p}_2|\, d\cos\theta_{12} \tag{9.27}$$

Substituting (9.27) into (9.26) yields

$$d^3p_1d^3p_2 = E_1E_2E_3dE_1dE_2dE_3dR$$

which finally gives

$$dLips = \frac{1}{8\,(2\pi)^5}\delta\left(M - E_1 - E_2 - E_3\right)dE_1dE_2dE_3dR$$

We eliminate the function $\delta\left(M - E_1 - E_2 - E_3\right)$ out by integrating over dE_3, where the integral is equal to unity. We are left with the expression [12]

$$dLips = \frac{4}{(4\pi)^5}dE_1dE_2dR$$

where $\frac{1}{8(2\pi)^5}$ has been rewritten $\frac{4}{(4\pi)^5}$.
This is the result quoted in (6.12) and used in (6.13).

9.4 Appendix D

9.4.1 The Parity operation

A three particle system with an arbitrary orientation, given by the Euler angles (α, β, γ), is denoted

$$|\alpha, \beta, \gamma, E_i, \lambda_i\rangle = U\left[R\left(\alpha, \beta, \gamma\right)\right]|0, 0, 0, E_i, \lambda_i\rangle \tag{9.28}$$

where E_i is the energy and λ_i the helicity quantum number of the i^{th} particle.

Here again, we write the three particle state in the standard orientation, i.e. with the particles contained within the xy-plane, as a product of three single particle states:

$$|0, 0, 0, E_i, \lambda_i\rangle = b\prod_{i=1}^{3}|p_i, s_i, \lambda_i\rangle$$

We shall now investigate the meaning of the quantum number μ.

We have seen from (6.13) that a state of good total angular momentum J is obtained via [12]

$$\begin{aligned} &|J, S, \mu; E_i, \lambda_i\rangle \\ &= \frac{N_J}{\sqrt{2\pi}}\int dR\left(\alpha, \beta, \gamma\right) D^{J\,*}_{M\mu}\left(\alpha, \beta, \gamma\right)|\alpha, \beta, \gamma; E_i, m_i\rangle \end{aligned} \tag{9.29}$$

where $dR = d\alpha d\cos\beta d\gamma$, the integration measure over the entire space in terms of the Euler angles and the other terms as defined in the chapter on the helicity formalism.

(9.29) becomes

$$\frac{N_J}{\sqrt{2\pi}}\int dR\left(\alpha, \beta, \gamma\right) D^{J\,*}_{M\mu}\left(\alpha, \beta, \gamma\right) U\left[R\left(\alpha, \beta, \gamma\right)\right]|0, 0, 0, E_i, m_i\rangle \tag{9.30}$$

We shall only concentrate on the integration over $d\gamma$ which is where μ comes from.

The relation (9.30) takes on the more specific form [12]

$$\int dR D^{J\,*}_{M\mu}(\alpha,\beta,\gamma)\sum_{m'} e^{-im'_i\alpha} d^J_{m'_i\lambda_i} e^{-i\lambda_i\gamma}\,|0,0,0,E_i,m_i\rangle \tag{9.31}$$

$$\int dR e^{iM\alpha} d^{J\,*}_{M\mu} e^{i\mu\gamma}\sum_{m'} e^{-im'_i\alpha} d^J_{m'_i\lambda_i} e^{-i\lambda_i\gamma}\,|0,0,0,E_i,m_i\rangle \tag{9.32}$$

where the explicit definitions of the operators $D^{J\,*}_{M\mu}(\alpha,\beta,\gamma)$ and $U[R(\alpha,\beta,\gamma)]$ have been used and the eigenvalue problem of $\vec{J}\cdot\vec{n}$ has been evaluated. The constant $\frac{N_J}{\sqrt{2\pi}}$ has been omitted

The exponent $e^{i(\mu-\lambda_i)\gamma}$ can be written in terms of a δ-function, which makes the physical interpretation easier. The eigenvalue of the operator $J_z\gamma$ is λ_i. The exponential picks out the eigenvalue $\lambda_i=\mu$ from the operator $\vec{J}\cdot\vec{n}$. This eigenvalue μ lies along the z-axis. Since $\vec{J}\cdot\vec{n}$ and μ are both rotated by γ via the exponents, it is obvious that μ is rotationally invariant, since any other rotation would again rotate both to the same final state.

Now we are in a position to analyze the effects of the parity operation on the three particle state.

First, we observe how the standard three particle state is affected

$$\Pi\,|0,0,0,E_i,\lambda_i\rangle = b\prod_i \Pi\,|R_i,p_i,s_i,\lambda_i\rangle \tag{9.33}$$

We are now dealing with Euler angles on the left hand side, which give orientation of the normal to the plane in which the particles are contained, but with spherical polar coordinates on the right.

We can rewrite (9.33) as follows

$$\begin{aligned}
\Pi\,|0,0,0,E_i,\lambda_i\rangle &= b\prod_i \Pi U\left[R\left(\phi_i,\frac{\pi}{2},0\right)\right] U\left[(L(p_z)\right]|s_i,m_i\rangle && (9.34)\\
&= b\prod_i \Pi U\left[R\left(\phi_i,\frac{\pi}{2},0\right)\right]|p_{i_z}s_i,m_i\rangle\\
&= b\prod_i\sum_m \Pi D^{s_i}_{m_i\lambda_i}\left(\phi_i,\frac{\pi}{2},0\right)|p_{i_z}s_i,m_i\rangle\\
&= b\prod_i\sum_m \Pi\left|\phi_i,\frac{\pi}{2},p_{i_z},s_i,\lambda_i\right\rangle && (9.35)
\end{aligned}$$

The parity operator Π acts on the rotation operator as follows

$$\Pi D^s_{m\lambda}(\phi,\theta,0) \to D^s_{m\lambda}(\pi+\phi,\pi-\theta,0) = e^{i\pi s} D^s_{m-\lambda}(\phi,\theta,0) \tag{9.36}$$

where the equality in (9.36) is one of the properties of the rotation matrices [12].

Substituting the result (9.36) into the second-last of the equalities of (9.34) gives

$$\begin{aligned}\Pi\left|0,0,0,E_i,\lambda_i\right\rangle &= b\prod_i\sum_m \eta_i e^{i\pi s_i} D^{s_i}_{m_i-\lambda_i}\left(\phi_i,\frac{\pi}{2},0\right)\left|p_{i_z} s_i, m_i\right\rangle \qquad (9.37)\\ &= b\prod_i\sum_m \eta_i e^{i\pi s_i}\left|\pi+\phi_i,\frac{\pi}{2},p_{i_z},s_i,-\lambda_i\right\rangle\end{aligned}$$

We can see by comparison between the last equality of (9.34) with the result of (9.37) that the only change implemented in the state vector by the parity operator was a rotation around the z−axis, i.e. $\phi_i \rightarrow \pi+\phi_i$.

It is on these grounds that we can rewrite (9.37) as

$$\begin{aligned}\Pi\left|0,0,0,E_i,\lambda_i\right\rangle &= b\prod_i\sum_m \eta_i e^{i\pi s_i} U\left[R\left(\pi,0,0\right)\right]\left|0,0,0,E_i,-\lambda_i\right\rangle\\ &= b\prod_i\sum_m \eta_i e^{i\pi s_i} U\left[R\left(0,0,\pi\right)\right]\left|0,0,0,E_i,-\lambda_i\right\rangle\end{aligned}$$

The reason why $U\left[R\left(\pi,0,0\right)\right]=U\left[R\left(0,0,\pi\right)\right]$ in this case is inherent in the fact that both the Euler angles α and γ in the expression $U\left[R\left(\alpha,\beta,\gamma\right)\right]$ are rotations around the z-axis. The rotations do not normally commute, but in this case we are only dealing with a rotation around the z-axis.

So, for a general state $\left|\alpha,\beta,\gamma;E_i,\lambda_i\right\rangle$, the parity operator acts as follows

$$\Pi\left|\alpha,\beta,\gamma;E_i,\lambda_i\right\rangle$$

$$\begin{aligned}&= b\prod_i\sum_m \eta_i e^{i\pi s_i} U\left[R\left(0,0,\pi\right)\right] U\left[R\left(\alpha,\beta,\gamma\right)\right]\left|0,0,0,E_i,-\lambda_i\right\rangle\\ &= b\prod_i\sum_m \eta_i e^{i\pi s_i} U\left[R\left(\alpha,\beta,\gamma+\pi\right)\right]\left|0,0,0,E_i,-\lambda_i\right\rangle\end{aligned}$$

where we can see that $\gamma'=\gamma+\pi$ is the principle operation on the state vector for the angles. We can use this fact in the integral (9.31) :

$$\Pi\left|J,S,\mu;E_i,\lambda_i\right\rangle =$$

$$\int dR' e^{iM\alpha} d^{J\,*}_{M\mu} e^{i\mu(\gamma'-\pi)}\prod_{i=1}^{3}\eta_i e^{-i\pi m_i}\sum_{m'} e^{-im_i'\alpha} d^{J}_{m_i',-\lambda_i} e^{-i(-\lambda_i)\gamma'}\left|0,0,0,E_i,m_i\right\rangle$$

$$\int dR' e^{iM\alpha} d^{J\,*}_{M\mu} e^{i\mu\gamma'} e^{-i\mu\pi} \prod_{i=1}^{3} \eta_i e^{-i\pi m_i} \sum_{m'} e^{-im'_i\alpha} d^{J}_{m'_i,-\lambda_i} e^{i\lambda_i\gamma'} |0,0,0,E_i,m_i\rangle$$

$$\int dR' e^{iM\alpha} d^{J\,*}_{M\mu} e^{i\mu\gamma'} \prod_{i=1}^{3} \eta_i e^{-i\pi m_i - i\mu\pi} \sum_{m'} e^{-im'_i\alpha} d^{J}_{m'_i,-\lambda_i} e^{i\lambda_i\gamma'} |0,0,0,E_i,m_i\rangle$$

$$\prod_{i=1}^{3} \eta_i e^{-i\pi m_i - i\mu\pi} \int dR' D^{J\,*}_{M\mu}(\alpha,\beta,\gamma') |\alpha,\beta,\gamma',E_i,-\lambda_i\rangle$$

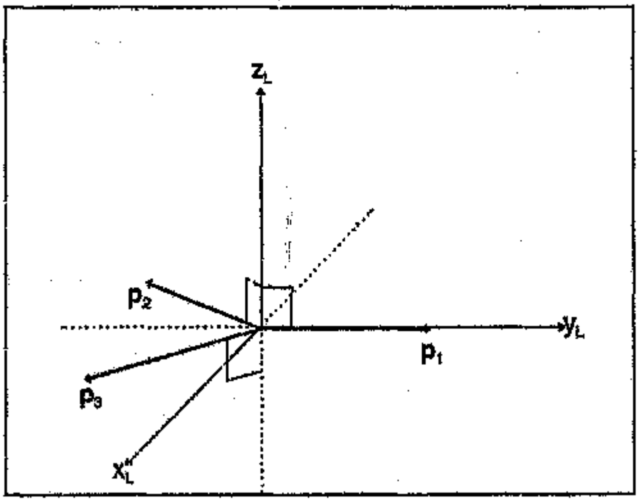
$$\prod_{i=1}^{3} \eta_1\eta_2\eta_3 (-1)^{-m_1-m_2-m_3-\mu} |J,S,\mu;E_i,-\lambda_i\rangle$$

where $dR' = d\alpha d\cos\beta d\gamma'$ and use has been made of the relation $e^{i\theta} = \cos\theta + i\sin\theta$.

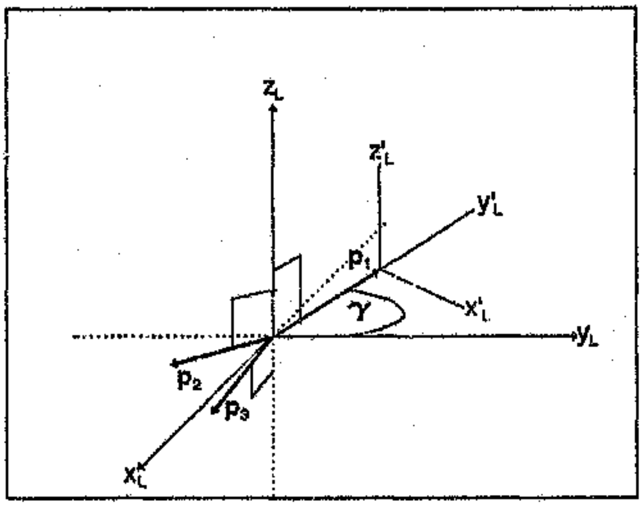
The appearance of μ in the exponent has the effect of reversing the parity, depending on its value, which can be $\mu = \pm\frac{1}{2}$, since it is the angular momentum quantum number and the system is in an overall state of spin projection $\pm\frac{1}{2}$.

Consequently, we can say that under a change of parity, an additional phase factor is picked up, which changes the overall sign of the wave function [13]

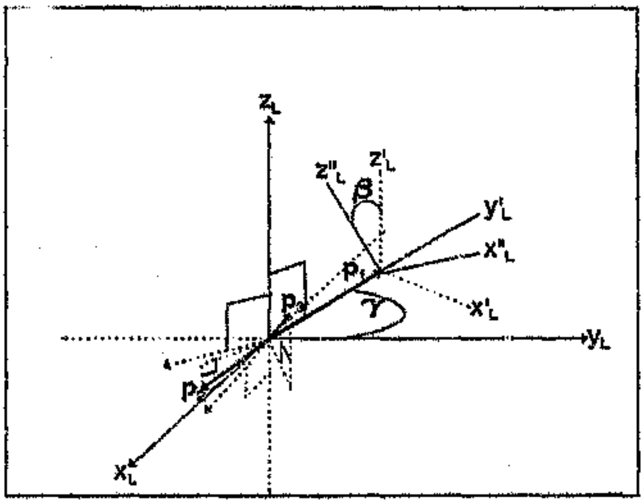
The diagrams below show how the Euler angles implement the parity operation.



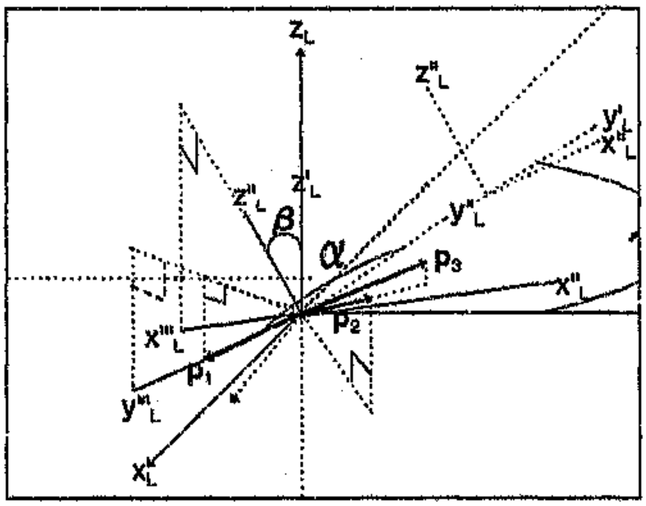
Standard orientation: the particles with momenta $p_i$are contained within the $X_L Y_L$-plane. The subscript "L" implies that the system is in the lab-frame of reference.



The vectors p_i have been rotated by an angle γ around the Z_L-axis. The rotated coordinate system is denoted by $X_L^\prime, Y_L^\prime$ and $Z_L^\prime$, respectively.



A rotation by an angle β has been implemented around the Y_L'-axis into the new coordinate system X_L'', Y_L'' and Z_L''.



The system is rotated around the Z_L''-axis by an angle α. The rotation is complete, with the new coordinate system denoted by X_L''', Y_L''' and Z_L'''.

9.5 Appendix E

9.5.1 Normalization

We want to normalize the state

$$b\prod_{i=1}^{3}\left|\alpha,\beta,\gamma;E_1,E_2;,j_i,\lambda_i\right\rangle$$

The normalization condition is given by

$$\left\langle\alpha',\beta',\gamma';E_1',E_2';,j_i',\lambda_i'\mid\alpha,\beta,\gamma;E_1,E_2;,j_i,\lambda_i\right\rangle \tag{9.38}$$

where the integration goes over $dLips$.
It is clear that (9.38) is equal to [12]

$$\begin{aligned}
&\left\langle\alpha',\beta',\gamma';E_1',E_2';,j_i',\lambda_i'\mid\alpha,\beta,\gamma;E_1,E_2;,j_i,\lambda_i\right\rangle\\
&= \delta\left(\alpha-\alpha'\right)\frac{1}{\cos\beta}\delta\left(\beta-\beta'\right)\delta\left(\gamma-\gamma'\right)\delta\left(E_1-E_1'\right)\delta\left(E_2-E_2'\right)\prod_{i=1}^{3}\delta^{j_i'j_i}\delta^{\lambda_i'\lambda_i}\\
&= \frac{1}{\cos\beta}\delta\left(R-R'\right)\delta\left(E_1-E_1'\right)\delta\left(E_2-E_2'\right)\prod_{i=1}^{3}\delta^{j_i'j_i}\delta^{\lambda_i'\lambda_i}
\end{aligned}$$

The Lorentz invariant phase space element was derived in Appendix C for this particular set of variables and was shown to be

$$dLips=\frac{4}{(4\pi)^5}dE_1dE_2dR$$

Hence, the normalization is formally written

$$b^2=\int\int\int\frac{4}{(4\pi)^5}dE_1dE_2dR\frac{1}{\sin\beta}\delta\left(R-R'\right)\delta\left(E_1-E_1'\right)\delta\left(E_2-E_2'\right)\sum_{j_i}\sum_{\lambda_i}\prod_{i=1}^{3}\delta^{j_i'j_i}\delta^{\lambda_i'\lambda_i}$$

with $dR=d\alpha d\cos\beta d\gamma$.

The factor of $\frac{1}{\sin\beta}$ is introduced to compensate for a $\sin\beta$ appearing in the integrand of the integral over $d\cos\beta$ when a change of variables $\beta'\to-\cos\beta$ is implemented.

All the δ-functions reduce to unity when the integrations over dE_1,dE_2 and dR as well as the sums over j_i and λ_i are carried out.

We are left with

$$b^2 = \frac{4}{(4\pi)^5}$$

or

$$b = \frac{2}{(4\pi)^{\frac{5}{2}}}$$

for the normalization factor, which the result quoted in the chapter "Model of the Nucleon"

9.6 Appendix F

9.6.1 Solution of a free particle

Particle at rest

Let us consider a specific example now, to illustrate how different parts of the wave function behave when operators act on them.

We consider the solution of the Dirac equation of a particle at rest, which is given by [2]:

$$\Psi^r(x) = w^r(0)\exp[-(i\varepsilon_r m_0)t]$$

$$r = 1, 2, 3, 4$$

$$\begin{aligned}\epsilon_r &= +1 \text{ if } r = 1,2\\ \epsilon_r &= -1 \text{ if } r = 3,4\end{aligned}$$

We see that if ϵ_r is negative, then the rest mass energy of the particle, m_0 changes sign, which gives rise to the negative energy interpretation. The spinors in this state have the form:

$$\omega^1(0) = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \omega^2(0) = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \omega^3(0) = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \omega^4(0) = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

9.6.2 Boosted free particle

We shall now see how the Lorentz boost acts on the wave function. If we want to boost the particle to an arbitrary momentum $\vec{p}$, all we need to do is apply the Lorentz boost to the stationary wave function. We shall boost the particle along the x-direction and use the operator that we have found earlier, namely:

$$S = \exp\left(-\frac{i}{2}\varpi\sigma_{01}\right)$$

The spatial part of the wave function, $\exp[-(i\varepsilon_r m)t]$, can be rewritten so that the term in the exponent gives a Lorentz scalar, which will not be affected by changes of frame.

We rewrite it in the following form:

$$\exp[-(i\varepsilon_r p_\mu x^\mu)]$$

This part will not be affected by the transformation and all the energy terms are contained in the spinors.

We need therefore only consider the effect of the boost on the spinor:

$$w^r(p) = \exp(-\frac{i}{2}\omega\sigma_{01})w^r(0)$$

The exponent is expanded in a Maclaurin expansion which yields:

$$w^r(p) = \left(I_4\cosh\frac{\omega}{2} - \alpha_1\sinh\frac{\omega}{2}\right)w^r(0)$$

When explicitly calculated, the matrix below is obtained [2]:

$$w^r(p) = \cosh\frac{\omega}{2}\begin{pmatrix} 1 & 0 & 0 & -\tanh\frac{\omega}{2} \\ 0 & 1 & -\tanh\frac{\omega}{2} & 0 \\ 0 & -\tanh\frac{\omega}{2} & 1 & 0 \\ -\tanh\frac{\omega}{2} & 0 & 0 & 1 \end{pmatrix} w^r(0)$$

We can see that the spinors are changed and when identifying the trigonometric expressions in terms of energy and momentum as [2]:

$$-\tanh\frac{\omega}{2} = \frac{p}{E+m}$$

$$\cosh\frac{\omega}{2} = \sqrt{\frac{E+m}{2m}}$$

We obtain:

$$w^r(p) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 & 0 & 0 & \frac{p}{E+m} \\ 0 & 1 & \frac{p}{E+m} & 0 \\ 0 & \frac{p}{E+m} & 1 & 0 \\ \frac{p}{E+m} & 0 & 0 & 1 \end{pmatrix} w^r(0)$$

It can be seen that the term $\frac{p}{E+m}$ will become very small in the non-relativistic limit, since then the rest energy is very much larger than the momentum. The representation then reduces to the one found by Pauli [2].

9.7 Appendix G

9.7.1 Gordon decomposition

We start with the baryon current (7.6) and derive the Gordon decomposition, using the constraints for the Breit frame (7.8). Although the calculation is quite trivial, it is lengthy.

The baryon current is given by

$$J_B^\mu\left(q^2\right) =$$

$$u^\dagger\left(p_f, s_f\right) e_B \left[F_1\left(q^2\right)\gamma^0\gamma^\mu + \frac{iF_2\left(q^2\right)\gamma^0\sigma^{\mu\nu}q_\nu}{2m_{0_B}}\right] u\left(p_{in}, s_{in}\right)$$

where $u^\dagger\left(p_f, s_f\right)$ and $u\left(p_{in}, s_{in}\right)$ are spinors boosted in a general direction. This can be written schematically

$$U\left[L\left(\vec{p}\right)\right] u\left(0, s\right) = u\left(p, s\right)$$

and the spinors are normalized according to

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