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**Model Development for Reagent Dosing Control of a Flotation Bank Using
Model Predictive Control**

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Date

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University of Witwatersrand, Johannesburg, in fulfilment of the requirements for
the degree of Master of Science in Engineering

DECLARATION

I declare that the research project, **Model Development for Reagent Dosing Control of a Flotation Bank Using Model Predictive Control**, is my own work and that each source of information used has been acknowledged by means of a complete reference. This thesis has not been submitted before for any other research project, degree or examination at any university.

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ABSTRACT

Flotation reagents are one of the highest costs in the concentrating stage of mining operations. Overuse of reagents can increase these costs. The inflation of reagent costs is one way that profitability is decreased. Alternatively, the inaccurate dosage of reagents can result in poor mineral recovery and grade, which decreases the value of the final concentration product. In addition, the overuse of flotation reagents leads to the excess reagents being carried away in the wastewater, which is an environmental hazard. Model Predictive Control is a technology that can be utilised to optimise reagent dosage. Research has showcased that the current models employed for reagent dosing MPC strategies are linear models. These linear models fail to explain the nonlinear effect of reagent dosage on the grade for a full operating range. Nonlinear models are required to incorporate the nonlinear behaviour of reagents.

The performance of a parametric nonlinear auto-regressive with exogenous outputs (NLarx) model was compared to the performance of two linear parametric models, namely a state space model and a linear, Autoregressive Exogenous Inputs, ARX, model. Data collected from the Mt Isa Lead Zinc Concentrator site was used to train and validate the models that represent a rougher bank of the Zinc circuit. The inputs used in model development consisted of reagent flow rates, rougher feed mineral compositions, total airflow rate and the pulp level. Whereas the model outputs were the rougher bank's Zinc and Pb concentrate grade. The model development results show that the state space models fit the training data better than the NLarx model. However, the NLarx models fit the testing data better than the linear models, providing an improved prediction of unseen data.

The linear ARX model and the nonlinear ARX model were used to create linear and nonlinear MPC controllers. These controllers were compared to one another in a simulated environment. The control objective of simulations was to maintain the Zn % in Conc and Pb % in Conc readings at the respective setpoints. Two sets of simulations were used to assess the controller's performance. The first set of simulations was used to evaluate the controller's setpoint tracking performance. The Zn % in Conc and Pb % in Conc setpoints were either stepped up or stepped down in the simulations. The second set of simulations was used to assess the disturbance rejection of the controller. The disturbance variables were altered in these simulations. The simulation results showcase that the nonlinear controller performed better than the linear controller with setpoint tracking. The nonlinear controller executes better setpoint tracking than the linear controller for Zn % in Conc and Pb % in Conc readings when the Pb % in Conc setpoint change is implemented in the simulation. Similarly, the nonlinear controller performs better than the linear controller with setpoint tracking when the Zn % in Conc setpoint is changed. Overall, the nonlinear controller performed better than the linear controller with setpoint tracking.

The disturbance rejection results showcased that the nonlinear controller performs better than the linear controller. The nonlinear controller was able to maintain the Zn % in Conc and Pb % in Conc at the setpoint for all simulations when the disturbance variable values were changed. However, the linear controller was not able to maintain the Zn % in Conc and Pb % in Conc at the setpoint for all scenarios.

DEDICATION

This study is dedicated to God Almighty. His Grace and Wisdom throughout this journey have been a source of strength. I dedicate this study to my parents, whose sacrifices, both big and small, allowed me to focus on my studies. To my family and friends for the unwavering support that each and every one of them gave me throughout my research journey. Finally, to my partner and biggest cheerleader, Joshua, thank you for the encouragement, patience and energy throughout this entire journey. Through every challenge, your unwavering commitment to our shared goals has been a constant source of motivation.

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LIST OF SYMBOLS

γ	Surface Tension
W	Work
θ	Angle
β	Regression Coefficient
R_{∞}	Maximum Recovery
R	Recovery
M	Mass

NOMENCLATURE

ADP	Adaptive Dynamic Programming
AFC	Advanced Flotation Control
AIC	Akaike Information Criterion
ANN	Artificial Neural Networks
APC	Advanced Process Control
ARX	Autoregressive Exogenous Input
AUD	Australian dollar
BRB	Belief Rule Base
CDF	Cumulative Distribution Function
CV,	Controlled Variable
CVA	Canonical variates analysis
DV	Disturbance Variable
S-GCN	Stacked Graph Convolutional Network
LSSVM	Least Squares Support Vector Machine
MAE	Mean Absolute Error
MIBC	Methyl isobutyl carbinol
MIM	Mount Isa Mines
MLS	Multi-Output Least Square
MPC	Model Predictive Control
MRMR	Maximal-Relevance-Minimal-Redundancy
MV	Manipulated Variable
NLARX	Nonlinear Autoregressive with Exogenous Input
NASH	Naphthalene Sulphate
NNARX	Neural Network Autoregressive with Exogenous Input
NRMSE	Normalised Root Mean Squared Error
OFC	Optimising Flotation Control
PAX	Potassium Amyl Xanthate
PDF	Probability Density Function
PGM	Platinum Group Metals
PI	Proportional Integral
PID	Proportional Integral Derivative
RBR	Rule-Based Reasoning
RMSE	Root Mean Squared Error
SAG	Semi-Autogenous Grinding
SEX	Sodium Ethyl Xanthate
SIBX	Sodium Isobutyl Xanthate
SIPX	Sodium Isopropyl Xanthate
SISO	Single Input Single Output
SS	State Space
SSGMM	Semi-Supervised Gaussian Mixture Model
SVD	Singular Value Decomposition
SVM	Support Vector Machine
SVR	Support Vector Regressor
SWBSD	Shape-Weighted Bubble Size Distribution
XRF	X-ray Fluorescence

1. CHAPTER 1: INTRODUCTION

1.1. Background

Froth flotation is a process used in the mineral processing sector to separate valuable and nonvaluable minerals. Furthermore, froth flotation is a highly complex process with many non-linear relationships between input and output variables. (Le Roux & Craig, 2019). PID controllers are conventionally used as regulatory controllers to maintain the controlled variables in a selected range. However, the froth flotation process contains unmeasurable disturbances that alter the optimal operating conditions. However, PID controllers occasionally perform inadequately in response to the disturbances. Thus, PID controllers in froth flotation require the assistance of advanced process control strategies for the commercialisation or scaling up of plants. (Quintanilla et al., 2021).

An advanced process control strategy used in froth flotation is Model Predictive Control (MPC). MPC employs predictive controllers with dynamic linear or non-linear process models to control multiple variables and combat disturbance interaction (Bergh & Yianatos, 2011). MPC is a form of model-based control.

Implementing MPC in the industry requires the development of simple, dynamic models in addition to accurately determining process constraints and minimising instrumentation constraints (Bergh & Yianatos, 2011). The simple models that must be developed for MPC industry implementation require considering the primary variables such as pulp level, aeration rates, and reagent dosage. These primary variables are altered to improve plant performance and profitability by maintaining controlled variables such as concentrate grade, mineral recovery, mass pull and re-circulating load at setpoints and optimising these controlled variables (Laurila et al., 2002). Models for flotation control have been studied amply for research purposes and have been implemented on simulated systems. However, there are a few mentions of implementing these models in industry (Bergh & Yianatos, 2011). These will be discussed in the proceeding literature review. Quintanilla et al. (2021) state that pulp level and aeration rate variables have been included in the research for linear model development for MPC strategies and have been successfully implemented in the industry. However, reagent addition and froth wash water have not been extensively modelled in research (Quintanilla et al., 2021). Older models of froth flotation include reagent addition to a small degree (Perez-Correa et al., 1998), (Casali et al., 2002), (Hodouin et al., 2000). More recently, a large proportion of reagent addition models in froth flotation control have been used indirectly in machine vision flotation control (Bergh & Yianatos, 2011), (Li et al., 2020), (Zhu et al., 2014), (Jahedsaravani et al., 2016). The industrial implementation of model-based control strategies, especially MPC, is often difficult as the interaction between variables in flotation is perplexing, and accurate variable measurements are often

limited. The industrial implementation of MPC, which included pulp level and airflow rate, is more prominent than the inclusion of reagent dosage. There are minimal publications of level and airflow rate model-based control strategies in the industry, which include the FloatStar controller developed by Mintek (Singh et al., 2003) and an optimising grade controller (Muller et al., 2010).

The literature review below will present the steps taken in modelling for process control, including pulp level, aeration rates, and reagent dosing in model development. However, the manipulation of reagent dosing is an area that requires further investigation in MPC, as there are more aspects to consider when automatic control implementation is considered. This can include considering reagents previously not modelled, such as activators. Other factors include the manipulation of concentration, varying dosing points, and the effect on different types of ores. Further developing reagent dosage control strategies is essential, as reagent dosing is the most vital factor to focus on in achieving steady-state operation in flotation plants. (Xu et al., 2012). The froth depth can give a good indication of the concentrate grade, as the drainage of the entrained gangue and the collapsing of the froth are dependent on the froth depth. The froth structure is heavily reliant on the reagent dosage. Thus, accurate reagent dosage is necessary to prevent the froth from collapsing and achieve the desired concentrate grades.

The cost of reagents is one of the most significant contributors to running costs in a flotation plant. The percentage contribution of reagents per tonne of ore increases as the capacity of the plant increases (Wills & Finch, 2015). Metallurgists typically determine the setpoints for reagent addition control after analysing the chemical composition of streams in the flotation plant. Reagent dosage is not significantly varied in a stabilised plant. The reagent dosage setpoints are potentially altered when feed condition changes occur. Significant consideration should be given to reagent addition setpoints, as a surplus of reagents can result in waste and decrease profitability. Meanwhile, a deficit of reagents can result in a lower grade or recovery and reduced profitability.

There is a need to reduce reagent costs to reduce plant running costs. Models have the potential to be developed with technical parameters (slurry flow rates, slurry properties, froth properties, etc.) and economic parameters (mineral or metal prices, circuit costs, etc.) that can be varied to achieve the optimal concentrate grade and recovery that corresponds with the most profitable point on the grade-recovery curve. Developing an MPC strategy that focuses on manipulating reagent addition can result in reagent cost minimisation and reduced reagent wastage by using the model to determine the adequate reagent dosage. Such a model can direct the savings in reagent costs towards exploring and mining activities.

An MPC strategy that included reagent dosing was implemented at Glencore's Mt Isa Concentrator Zinc Rougher bank. The control objectives of the strategy were to maintain the collector dosage and split between cells within a range, maintain the activator and depressant dosages within a range and maintain the bank concentrate grade at a range (Price et al., 2018). The controller utilised a linear dynamic step response

model developed in AspenTech DMC Plus. The model inputs included the airflow rates to each cell, the levels per pair of cells, the collector and activator flow rates to the cells, the feed density, the feed flow rate, and the Zn amount in the feed. The introduction of the MPC strategy led to an improvement in reagent dosing, where the standard deviation decreased for the collector and activator. The percentage improvement for the collector and activator was 41% and 32% for the standard deviation when using MPC. The strategy also reduced reagent consumption, estimated to reduce the cost by AUD 1M per annum. The grade and recovery benefits included 82% of the concentrate grade maintained above the low limit and 60% of the tails grade maintained below the high limit (Price et al., 2018).

The reagent savings are possible as the limits are continually tested to find the position of minimum activator and collector dosage for maximum recovery and grade. One of the main challenges faced by the MPC strategy is that the models used were linear, and the MPC functioned at a nominal operating point. Thus, a significant change in reagent dosage resulted in non-linear behaviour. This was accommodated by changing the operating range of the controller to a range where the process behaved linearly. Special attention was given to reagent dosing of the collector and activator, as a poor choice in operating ranges could result in poor recovery or grade. The insufficient dosage of collector and activator can result in a drop in recovery as the controller gains are much lower than the process gains, and the lower limit of the range is continually changed. The overdosing of the activator can result in the collapse of froth and low recovery. An overdose in the collector can result in a lower concentrate grade, as the collector can also carry excess gangue to the surface. Therefore, choosing an optimal range was necessary. The prevention of activator overdose takes a more significant priority than the overdosing of the collector. Thus, the range for the activator is smaller than the collector's (Price et al., 2018).

Therein lies the opportunity to develop a nonlinear model representing the process's entire operating range and encompassing the reagents' nonlinear behaviour. If the nonlinear model accounts for the reagents' nonlinear behaviour, the controller range would not need to be continually changed. Furthermore, underdosing or overdosing of reagents could be prevented.

Model development in an MPC implementation is an extensive process that can consume 90 % of the development cost and time. In contrast to the linear case, there is currently no recognised method for developing a nonlinear model using a plant test. Moreover, MPC strategies that utilise nonlinear reagent dosing models are limited in research and industry.

1.2. Problem Statement

Flotation reagents are the highest running costs in the concentrating stage of mining operations. Overusing or underusing reagents inflate these costs. Moreover, overusing or underusing reagents affects the froth structure and can result in concentrate grades and recovery values that do not meet product specifications, consequently decreasing profitability. Overusing flotation reagents also creates wastage. These excess reagents are carried away with wastewater in higher concentrations and present an environmental hazard.

Model predictive control is a technology that can be employed to ensure accurate reagent dosing and optimise reagent use such that profitability is increased and reagent wastage is decreased. Current linear empirical models utilised in flotation MPC strategies are sufficient. The relationship between flotation reagents and the concentrate grade is highly nonlinear. Thus, the inclusion of reagent dosing in these models necessitates the controller to continuously change the operating range to a range where the behaviour is linear. Selecting a reagent dosage range that is too large or too small can result in underdosing or overdosing of reagents. Non-linear models for flotation are required to incorporate non-linear behaviour, but do not currently exist.

1.3. Research Objective

- To develop a data-driven model that incorporates the effect of reagent dosing on the concentrate grade for a Zinc flotation bank.
- To validate the model using flotation bank data.
- To demonstrate on a simulated system that the MPC with non-linear reagent models performs better than one with linear models

1.4. Organisation of thesis

This dissertation is structured as follows. Chapter 2 provides a basic overview of flotation. Additionally, Chapter 2 consists of a literature survey that explores research conducted on flotation models with reagent dosing and the implementation of these models in model-based control strategies and model predictive control strategies. Chapter 3 examines the methodology used to frame this research and explores the flotation dataset and model structures utilised in model identification. Furthermore, the development of

the linear and nonlinear MPCs and the testing of the MPCs on a simulated system are covered in Chapter 3. The results obtained in the model identification are given in Chapter 4. The control performance of the linear and nonlinear MPC results in a simulated system is also showcased in Chapter 4. Chapter 5 presents the analysis and interpretation of the findings about the research objectives and the literature. Additionally, it concludes with research implications and recommendations.

2. CHAPTER 2: LITERATURE SURVEY

This Chapter briefly describes the froth flotation process and the functionality of common flotation reagents. The principles of the flotation process control are also defined in this Chapter. The remainder of the Chapter provides an overview of flotation models with reagent dosing that have been developed and the implementation of these models in model-based control strategies and model predictive control.

2.1. Froth Flotation

2.1.1. Process Description

Froth flotation is a procedure used in Minerals Processing that utilises the difference in surface physicochemical properties of minerals to separate and concentrate the valuable mineral/s from the unwanted minerals and particles known as gangue (Cao et al., 2018). Flotation commonly occurs after the crushing and grinding of the ore. The crushing and grinding processing of the ore is responsible for the size reduction of ore minerals to ensure sufficient exposure and liberation of valuable minerals from gangue. The liberation of valuable minerals from gangue is necessary for froth flotation and will be further explained in the following paragraphs. The crushed and milled ore product is fed into the flotation stage as a slurry, a mixture of water and the ore. Flotation can occur in either a flotation cell or a column (Wills & Finch, 2015). This study focuses on froth flotation in a cell; hence, the overview of the process will be described in a flotation cell scenario.

Froth flotation begins with the addition of slurry to the cell. The slurry, called pulp in the cell, is then agitated with a vertical mechanical stirrer. Air bubbles are introduced into the pulp via an airflow that travels through a vertical agitator. The main principle behind separation in flotation is the minerals' surface relationship with water. The two relationships a particle could have with water are hydrophobic and hydrophilic. A hydrophobic particle tends to repel water, whereas a hydrophilic particle tends to be readily wetted by or attracted to water (Mondal et al., 2021). Flotation aims to drive the attachment of hydrophobic minerals to air bubbles in the pulp that will carry the mineral to the surface, whilst hydrophilic minerals remain within the slurry. Agitation via the mechanical stirrer ensures the bubble-collision rate is increased. The combined air bubble and particle density must be less than the pulp to ensure the mineral reaches the surface. Furthermore, the adhesion force between the mineral and the bubble must be greater than the mineral's weight to prevent the mineral from bubble detachment. Therefore, the particles are recommended to fall within an optimal size range. Crushing and milling ensure that the particle falls within the optimal size range and ensure that the particle liberation from unwanted minerals increases, which additionally

allows for increased particle surface area available for bubble attachment. The forces that play a role in particle-bubble attachment are $\gamma_{s/a}$, $\gamma_{s/w}$ and $\gamma_{w/a}$ which are the surface tensions between solid-air, solid-water, and water-air surfaces, respectively. The angle between the particle surface and the air bubble is known as the contact angle, θ and is a reasonable degree of hydrophobicity of the mineral (Chau et al., 2009). The three forces shown in bubble attachment are depicted in Figure 2-1.

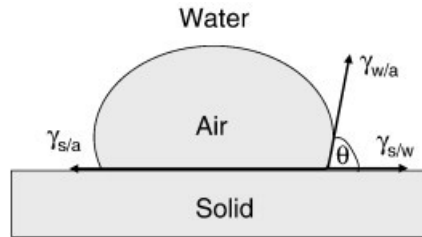


Figure 2-1, Depiction of contact angle and surface tension forces acting on the particle (Chau et al., 2009)

A separate solid-air interface from the solid-water and water-air interfaces is required to break bubble attachment. The work, known as the work of adhesion $W_{s/a}$, required to create a separate solid-air interface is greater than the sum of the solid-water and water-air surface tensions, as shown in Equation 1 (Wills & Finch, 2015).

$$W_{s/a} = \gamma_{w/a} + \gamma_{s/w} - \gamma_{s/a} \quad (1)$$

As $\gamma_{s/w}$ cannot be measured experimentally, the contact angle can be used to determine the work of adhesion as shown in Equation 2 (Wills & Finch, 2015).

$$W_{s/a} = \gamma_{w/a}(1 - \cos\theta) \quad (2)$$

The larger the contact angle, the more significant the work required to separate the solid-air interface. Thus, mineral particles with a larger contact angle tend to be more hydrophobic than mineral particles with a smaller contact angle, as the work required to break bubble attachment is higher (Chau et al., 2009).

The minerals attached to the air bubbles at the surface form a mineral-rich froth. The froth at the top of the surface must be stable to ensure the bubbles do not burst and the target mineral/s are not dropped into the pulp. Once the froth is stable, it is scraped off, and the collection of these minerals and water is known as the concentrate. The slurry remaining in the cell is known as the tailings. The valuable mineral/s are typically preferred to be hydrophobic and collected as a concentrate at the surface. In contrast, the gangue is desired to be hydrophilic and remain in the slurry as tailings (Wills and Finch, 2015). However, the unwanted minerals may naturally occur as hydrophobic. Figure 2-2 illustrates a froth flotation process occurring in a froth flotation cell.

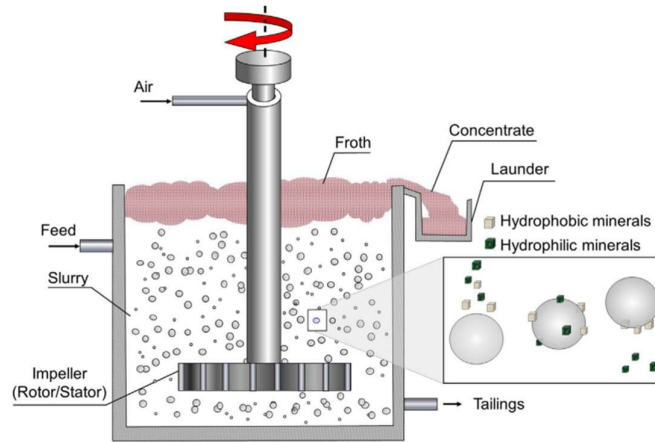


Figure 2-2, A mechanically agitated flotation cell (Wills and Finch, 2015)

The gangue and other unwanted minerals are collected as concentrates in specific cases, and the resulting tailings are rich in valuable minerals. This process is known as reverse flotation. Reagents are added to the slurry during flotation to make the differences in the surface properties more pronounced and facilitate froth formation (Wills & Finch, 2015). The reagent types and functions will be discussed in the proceeding chapter.

Ideally, froth flotation would result in the collection of only valuable minerals; however, processes such as entrainment and entrapment lead to the collection of gangue in the concentrate. Entrainment occurs when ultrafine mineral particles, either valuable minerals or gangue, suspended in water at the top of the cell, enter the froth. Once in the froth, these suspended particles may be collected in the concentrate. Entrapment arises when mineral particles become trapped between hydrophobic minerals attached to air bubbles in the froth (Wang et al., 2021). Other variables affect the outcome of the flotation process in addition to entrainment and entrapment. Klimpel (1984) Categorised the variables that influence each in the froth flotation into physical, chemical and machine-derived variables. These variables are seen in Table 2-1.

Table 2-1 Flotation System Interrelated Variables

Physical/ Operational	Chemical	Machine-Derived
Feed Rate	Collectors	Cell Design
Mineralogy	Frothers	Agitation
Particle Size and liberation	Activators	Air Flow
Pulp Density	Depressants	Cell Bank Configuration
Temperature	pH	Cell Bank Control

The degree of separation of minerals in froth flotation is described using the terms recovery and grade. Grade is the amount of the mineral contained in the various streams in the flotation stage, namely the feed, concentrate, and tailings stream, and is commonly conveyed in parts per million (ppm) or grams per ton (g/t). Recovery is the amount of the total mineral in the ore collected in the concentrate and is presented as a percentage (Wills and Finch, 2015). The equation used to obtain recovery is shown in Equation 3.

$$R = \frac{Cc}{Ff} \quad (3)$$

Where R is the recovery of the mineral, Cc is the mass of metal in concentrate and Ff is the mass of metal in the feed. Ideal separation in froth flotation is accomplished in various stages, as numerous variables affect the flotation process. Thus, numerous stages are required to achieve the intended recovery and grade of the concentrate. The first set of stages typically focuses on maximising the recovery of the target mineral/s, known as the recovery stages. After that, the concentrate produced from the recovery stages is further processed in flotation cells to maximise the target mineral/s grade, known as the cleaner and recleaner stages. Froth flotation can typically occur in a batch setup or a continuous setup. A continuous flotation setup is known as a flotation circuit. Flotation cells are connected to form banks in flotation circuits. Each bank is responsible for achieving a specific goal. Rougher banks are used to maximise the recovery of the target mineral. A second type of bank, the cleaner bank, is utilised to maximise the target mineral's grade. A re-cleaner bank concentrates the target mineral further and improves the target mineral grade. A fourth type of bank is known as a scavenger bank and is used to recover target minerals from the

rougher and cleaner tails that were not recovered the first time. Flotation circuits consist of different amounts and configurations of rougher, cleaner and recleaner banks (Mondal et al., 2021). A typical configuration of flotation banks can consist of a rougher bank followed by cleaner and scavenger banks, as shown in Figure 2-3. The concentrate from the rougher banks is sent to the cleaner banks, and the tails from the rougher banks are sent to the scavenger bank. The tails from the cleaner banks can be sent to a scavenger bank to recover target minerals from the cleaner tails. A re-cleaner bank can be added after the cleaner bank to concentrate the target mineral further. The configuration shown in Figure 2-3 is one of many flotation bank configurations.

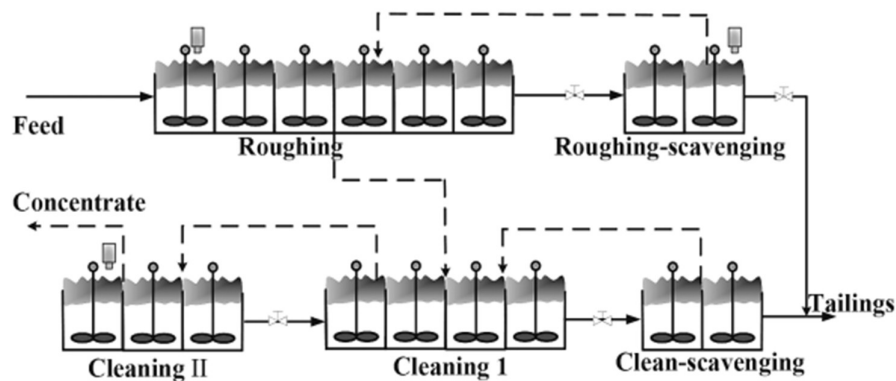


Figure 2-3 Flotation bank configuration (Binfang et al., 2015)

Each flotation bank configuration is tailored to the final product's grade and recovery requirements. Thus, specific configurations can exclude banks such as re-cleaner and scavenger banks.

2.1.2. Froth Flotation Reagents

Flotation reagents with distinct functions can be classified into collectors, frothers, and modifiers.

Collectors

Many minerals are not naturally hydrophobic; thus, a collector is added to the flotation pulp to facilitate mineral attachment to the air bubble. Collectors are organic chemical compounds that are added to the pulp to form a hydrophobic monolayer of non-polar hydrocarbons through adsorption on selected minerals (Kawatra, 2011). The purpose of this hydrophobic layer is to ensure that the target mineral attaches to the air bubble by increasing the contact angle and reducing the stability of the hydrated layer on the minerals (Bulatovic, 2007).

Collectors can be classified into two broad categories that being ionising and non-ionising collectors (Bulatovic, 2007).

Ionising collectors

When added to the pulp, ionising collectors are soluble heteropolar compounds that dissociate into anions and cations. The cation or anion that renders the water-repellent action is called the active repellent ion, whereas the corresponding anion or cation is the non-active ion (Bulatovic, 2007). The active repellent ion in ionising collectors is categorised by containing a non-polar hydrophobic hydrocarbon radical. The hydrocarbon radical cannot exist independently and is attached to a polar connector group known as the solidophilic group. The solidophilic group attaches to the mineral surface through adsorption, and the non-polar radical is orientated toward the water phase, resulting in a hydrophobic mineral surface (Wills & Finch, 2015).

Two types of adsorptions can occur, namely, chemisorption and physisorption. Chemisorption involves the attachment of collector ions to the mineral surface via a chemical reaction. Collectors that use chemisorption are highly selective as the irreversible chemical bonds they form with the mineral surface are specific. Physisorption utilises the reversible attachment to the surface of the mineral via weak van der Waals forces or electrostatic forces. Physisorption is less selective than chemisorption, as the collector can attach to any mineral surface that has the opposite charge (Kawatra, 2011).

Ionising collectors can be further categorised into anionic and cationic collectors (Kawatra, 2011).

Anionic collectors dissociate in the slurry to form an active repellent ion with a negatively charged solidophilic group (Bulatovic, 2007). Anionic collectors can attach to positively charged mineral surfaces. The anionic collectors can further be classified according to differences in non-polar functional groups. These categories are oxhydryl and sulfhydryl collectors. These collectors are generally weak acids or salts of acid (Kawatra, 2011).

Sulfhydryl collectors are mostly used for the flotation of sulphide minerals. Sulfhydryl collectors are generally thiols that contain a -SH group and an organic radical. Sulfhydryl collectors are classified according to how the sulfhydryl group is connected to the organic radical. The sulfhydryl can be directly linked to a carbon atom in the organic radical. The sulfhydryl group can also be connected to the organic radical through oxygen, sulphur, nitrogen or phosphorus (Bulatovic, 2007).

Xanthates are the most popular collectors used for the flotation of sulphur minerals and sometimes oxide minerals. This sulfhydryl is formed when a sulphur atom replaces the double-bonded oxygen atom in carbonic acid. In addition, the oxygen atom in one hydroxyl group is replaced by a sulphur bonded to Hydrogen, Sodium or Potassium. Lastly, the hydrogen in the second hydroxyl group is replaced by an alkyl group (Bulatovic, 2007). Alkyl groups in Xanthates consist of ethyl, isopropyl, amyl and isobutyl (Bulatovic, 2007). Figure 2-4 illustrates the formation of Xanthates from Carbonic Acid where Me represents either H^+ , Na^+ or K^+ .



Figure 2-4, the formation of Xanthates from Carbonic Acid (Wills & Finch, 2015)

Xanthates tend to form unstable Xanthic acid when exposed to moisture. The unstable Xanthic acid dissociates into carbon disulfide and an alcohol. The stability and dissociation of Xanthates depend on the slurry's pH. Xanthates are the most stable in alkaline conditions. The dissociation of xanthates decreases with a decrease in pH; therefore, xanthates are stable in alkaline conditions compared to acidic conditions. In acidic conditions, xanthates with shorter hydrocarbon chains dissociate faster than longer hydrocarbon chains. (Bulatovic, 2007). Xanthates are commonly used with regulators as they are associated with other sulphide minerals in the slurry, and their selectivity can be decreased. One of the other common downfalls of Xanthate is that it dissociates into CS₂, which is a potential environmental hazard (Wills & Finch, 2015).

Xanthates are commonly used in the flotation of sulphide ores and copper, zinc and lead oxide ores. The flotation of base metals regularly involves alkyl xanthate. Examples of xanthate collectors used in industry are potassium amyl xanthate (PAX), sodium ethyl xanthate (SEX), sodium isobutyl xanthate (SIBX) and sodium isopropyl xanthate (SIPX) (Ma et al., 2017).

Oxhydryl collectors are primarily used in industry for non-sulphide flotation, such as oxide flotation (B. A. Wills & Finch, 2015). Oxhydryl collectors are categorised by the organic and sulfo-acid ions within the collector compound. Oxhydryl collectors are less selective than Sulphydryl collectors as physisorption is used. Oleic acids and sodium oleic are commonly used in industry due to the extensive research that was undertaken with these collectors (Bulatovic, 2007).

Cationic collectors dissociate in slurry to form an active repellent ion with a positively charged solidophilic group (Kawatra, 2011). Cationic collectors attach to negatively charged mineral surfaces. A positively charged amine group characterises cationic collectors.

Non-ionizing Collectors

Non-ionising collectors are nonpolar hydrocarbon liquid collectors that do not contain a solidophilic group and do not dissociate in water to form ions. Such reagents attach to minerals by adhesion as opposed to ionising reagents that form a layer over minerals by absorption. These collectors are used with target minerals such as coal and graphite, which are naturally hydrophobic (Bulatovic, 2007).

Frothers

Frothers are reagents added to the froth flotation process to form a froth zone at the top of a flotation cell. Frothers also produce, stabilise, and maintain small bubbles in the float cell. The formation of smaller bubbles aids in the increase of the collision rate of minerals and air bubbles by increasing the surface area and number of bubbles for collisions (Wills & Finch, 2015). Furthermore, frothers are used to reduce bubble rise velocity, thus increasing the residence time of the bubbles in the pulp and increasing the collection of minerals. Other functions of frothers include altering the film between bubbles and solid particles for better attachment and interacting with collectors to increase particle-bubble attachment. The structure of a frother consists of a hydrophilic polar group and a hydrophobic hydrocarbon radical, which allows the hydrophilic side to be oriented towards the water and the hydrophobic side towards the air bubble. The frother creates an envelope around the air bubble and lowers the surface tension of the water around the air bubble to prevent the collapse of the air bubbles (Bulatovic, 2007).

There are many classifications of frothers found in the literature (Drzymala & Kowalczyk, 2018). One such classification is that of the behaviour of frothers at different pH values, with the three groups of frothers categorised as acidic, basic and neutral. Typical examples of acidic frothers used in mineral processing are phenols and alkyl-lauryl sulfonates. Basic frothers consist of pyridine frothers used in base metal processing. Neutral frothers can perform optimally in both acidic and basic mediums. These frothers are the most used in mineral processing. More specifically, the processing of base metals, oxidic minerals and industrial minerals. These frothers can be categorised into the following groups: cyclic alcohols, aliphatic alcohols, alkoxy paraffins and glycols. Frothers are commonly used as blends of frothers from different groups to accommodate a broad particle size range and to control bubble size reduction and froth stability properties (Wills & Finch, 2015).

Modifiers

Modifiers or regulators are reagents added to the flotation pulp to facilitate the collector's action by either increasing or decreasing the tendency of a mineral to repel water. Thus, the selectivity of target minerals for attachment to air bubbles is enabled. There are three common types of modifiers used in flotation, namely activators, depressants, and pH control reagents (Bulatovic, 2007).

Activators

Activators are added to flotation systems to assist the adsorption of collectors onto the mineral surface. Activators are commonly inorganic soluble salts that dissociate in the pulp. The dissociated ions interact with the mineral surface and render a stable compound for collector attachment. Two activators commonly found in sulphide mineral flotation are salts that ionise into Cu ions in solution and Sodium Sulphides. Copper sulfate, CuSO_4 , is typically used to provide Cu ions in the pulp for sphalerite, pyrite, and pyrrhotite

flotation, and other lead-zinc and copper-zinc ores. Sodium sulphide is generally used to activate oxidised lead, zinc and copper minerals by a process known as "sulfidization" (Wills & Finch, 2015).

Depressants

Depressants are added to the system to reduce the adsorption of collectors onto unwanted minerals and ensure selective flotation of the target mineral/s. This selectivity is achieved by rendering the unwanted minerals' surface hydrophobic via one of the following methods: adsorption of hydrophilic species onto the surface, removal of collector activating species, removal of adsorbed collector species and blockage of potential collector adsorption sites (Bulatovic, 2007).

2.1.3. Zinc flotation reagents

Sphalerite is the sulphide mineral from which the zinc is recovered in this study. Sphalerite is not naturally hydrophobic and requires modification before collector addition. CuSO_4 is typically added to replace zinc ions on the surface with copper ions to activate the mineral surface for enhanced collector attachment. Xanthates are commonly used for sulphide mineral flotation. More specifically, a lower molecular weight xanthate is used for zinc flotation. However, the floatability of sphalerite and pyrite is similar, and the depression of pyrite is required. Sulphites are commonly added as pyrite depressing agents. Lime is added to raise the pH to allow pyrite depression. A weak alcohol frother type is recommended to maximise selectivity for sphalerite flotation.

The collector used at the Mount Isa lead zinc concentrator was Sodium Ethyl Xanthate (SEX), which is commonly used in copper flotation; however, with the addition of activator ions such as copper ions, this collector can be used for zinc flotation. Methyl isobutyl carbinol (MIBC) is a weak alcohol frother classified as an aliphatic alcohol and was used in the rougher bank. Naphthalene sulphate is a depressant from the sulfite group and is used in alkaline conditions.

2.2. Process Control in Froth Flotation

2.2.1. Process Control Variables and Layers in Froth Flotation

Froth Flotation is a highly complex process with a vast number of variables that have non-linear relationships. The complexity of the flotation process renders a perplexing control process. Process control involves variables that can be classified into four categories. These categories are manipulated, disturbance, controlled, and internal state variables (Quintanilla et al., 2021). Manipulated variables and disturbances can further be classified into the broader category of input variables, whereas controlled and internal state

variables are categorised into output variables. The controlled variable is the process parameter being controlled. The internal state variable describes the state of the model. Manipulated variables are variables that can be altered to adjust the internal state of the model. Finally, disturbances are variables that cannot be changed (Quintanilla et al., 2021). The classification of flotation process variables into the process control variable categories is dependent on the layer of control (Cilliers & Shean, 2011).

Process control can be described as a hierarchy of 4 interconnected levels (Cilliers & Shean, 2011). The first level is instrumentation, which involves selecting, designing, and maintaining instruments based on the process requirements and required functionality. The second tier, base-level or regulatory control, involves controlling primary variables. Base-level control is usually accomplished using Proportional Integral Derivative controllers in Single Input, Single Output control loops. The third tier of process control can be termed Advanced Flotation Control (AFC), stabilising control that rejects input disturbances and maintains the plant at a steady state. AFC achieves stabilisation by providing the setpoints for the base layer control. Expert control systems are recommended for use in conjunction with AFC in flotation, as the large number of disturbances in the froth flotation process ensues in poor PID controller performance (Cilliers & Shean, 2011). The expert control systems provide the setpoints for the base layer PID control. The fourth tier is known as Optimising Flotation Control (OFC). OFC is used to maximise the profitability and optimise the performance of the process (Cilliers & Shean, 2011).

The base layer control in a flotation plant requires the control of primary variables of a single cell or multiple cells in a flotation bank. The typical variables used in base layer control of the flotation plant are shown in Table 2-2.

Table 2-2, Common Base Layer Variables Found in Flotation ((Oosthuizen et al., 2017)

Manipulated variables	Controlled Variables
Tailings slurry outlet valve position	Pulp level in cells
Airflow valve position	Airflow rate
Reagent addition rate	Reagent Concentration in pulp
Pulp pH	Acid or lime addition rates
Pulp Eh	Nitrogen or air addition rates

The third tier, Advanced Process Control, APC layer, in the flotation control hierarchy, aims to reduce disturbance impact and keep the plant steady by maintaining variables such as the mass pulls, recycle loads,

stream grade and recovery of each bank at their desired setpoint. These variables are maintained by manipulating the base layer-controlled variable. Consequently, APC provides the setpoints for the base layer-controlled variables that would satisfy the APC-controlled variable control objectives. The APC-manipulated variables in this study are pulp level, air flow rate and reagent concentration in pulp (Jakhu, 1998). Expert control systems are utilised with APC to automate operator decision-making and accommodate highly non-linear systems. Types of expert control systems used in APC include Model-based methods, Artificial Neural Networks (ANN), Inductive Machine Learning and Fuzzy Logic.

Model-based methods utilise the models that relate one or more manipulated variables to controlled variables to aid in the computation of the control action carried out by the controller. Model-based control will be discussed in detail in the proceeding sections.

ANNs are machine learning algorithms and consist of a network of core processing nodes that use input information to predict an output. The nodes are organised into three layers: the input, hidden, and output. The inputs are assigned to nodes in the input layer. The nodes in one layer are connected to nodes in another layer through channels with assigned weights. The input values in the first layer are multiplied by the weights of the channels. The output from the first layer becomes the input for the next layer. However, the nodes in the proceeding layer are only activated if the weighted sum of previous inputs is above a certain threshold. An inactivated neuron does not pass inputs to the next layer. The process of multiplying the previous node input by the respective weight is continued for all the hidden layers. At the output layer, the node with the highest weighted sum of prior inputs is used as the input. The weights of ANNs are determined during model training. Inductive machine learning uses training models to develop a set of classification rules. These rules will classify a system based on the input information fed, similar to a decision tree (Kalyani et al., 2008).

Fuzzy logic is a system that mirrors human reasoning. However, the truth of a statement is not expressed in typical Boolean values but rather in a degree of truth between 0 and 1. The inputs undergo fuzzification, where each input is classified into fuzzy sets according to the membership functions. The result is fuzzy variables that indicate the degree to which the input belongs to each fuzzy set. The fuzzy variables proceed to an inference engine where the relevant rules (determined by operators) are applied to the fuzzy input to produce a fuzzy output. The fuzzy output undergoes de-fuzzification, creating an explicit output value (Quintanilla et al., 2021).

Optimisation control can occur in the fourth tier once the flotation plant has stabilised at the APC layer. This layer maximises financial stability by reducing operating costs whilst achieving metallurgical efficiency and remaining within constraints. Optimisation control can also include an environmental aspect, where factors such as energy savings and water usage can be included in the optimisation algorithm. The

optimal operating point is found by altering the operating point on the grade recovery curve orthogonally on the grade recovery curve seen in Figure 2-5.

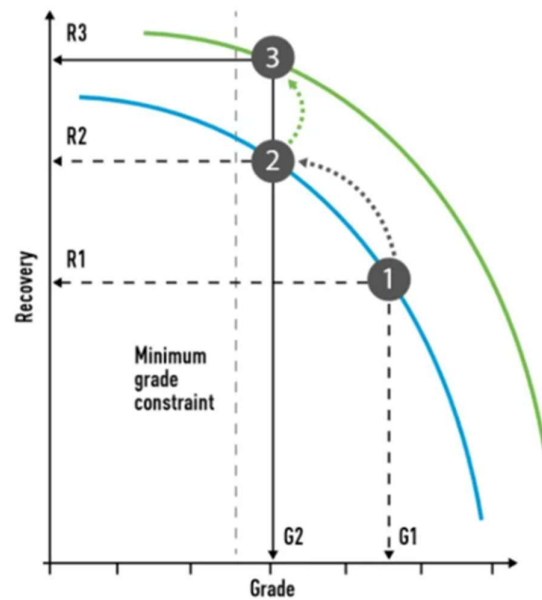


Figure 2-5 Grade vs Recovery curve showing the optimising control objectives (Mishra, 2023)

Firstly, the stabilisation of controlled variables is achieved in the base layer. After that, recovery must be increased while avoiding the minimum grade. Finally, the operating point is moved orthogonally to increase grade and recovery. Financial optimisation must account for different feed grades. Therefore, multiple-grade recovery curves are used for optimisation. Apart from the variables mentioned above, other flotation variables are considered in the froth flotation control. These variables are shown in Table 2-3.

Table 2-3 Froth Flotation Variables

Froth	Ore and feed	Slurry
Bubble Size Distribution	Size Distribution	Density
Bubble Speed	Density	Solids Content
Stability	Mineralogy	Retention time
Froth Depth	Flowrate	Volumetric flowrate
	Shape	
	degree of liberation	

The proceeding sections discuss the instrumentation and control systems commonly used for Pulp Level, Air Flow Rate, Slurry Flow Rate, Elemental Assaying, and Reagent Addition.

Pulp Level

Instrumentation used in pulp level measurement is commonly floats paired with a target plate and an ultrasonic transmitter. The instrument measures the time taken for sound waves to reach the plate target and return to the transmitter (Wills and Finch, 2015). Alternative pulp level measurement instrumentation includes floats paired with angle arms and captive angle transmitters, conductivity probes that measure the differences in electrical conductivities, differential pressure gauges that measure the static head imposed by the pulp, reflex radars, and image-based technologies (Cilliers & Shean, 2011). Dart and pinch valves regulate flow rates that share a relationship with the pulp level. The base layer control of pulp level is achieved by altering the bank's tailings flow rate by manipulating the slurry outlet valve position. Feedback Proportional Integral, PI, control loops maintain the pulp level at a setpoint. Feedforward control is used in conjunction with feedback control to manage upstream disturbances, such as varying feed flows to each bank (Cilliers & Shean, 2011).

Air flowrate

Air flow rate measurement instrumentation includes using thermal gas mass flow sensors that measure the cooling rate of air, where the air flow rate is dependent on the cooling rate. Additionally, air flow rate measurement devices include differential pressure meters. Firstly, a differential pressure metre paired with a venturi tube measures the airflow by measuring the pressure drop caused by an induced restriction to the airflow. Secondly, a differential pressure metre, Pitot tube or annubar tube uses the difference in static pressure and internal pipe pressure to measure air flow rate (Cilliers & Shean, 2011).

The air flow rate in base layer control consists of using feedback PI or PID, Proportional Integral Derivative control loops to maintain the total air flow rate to a bank by manipulating the airflow valve position. The air flow rate to each cell in a bank is controlled by manually adjusting butterfly valves (Wills and Finch, 2015).

Slurry Flowrate

Slurry flow rates are measured using flow meters such as magnetic flow meters that apply Faraday's principle of induction, where a pipe is coiled with an electromagnet and has electrodes on either side. The electric current that flows through this pipe is measured while the slurry flows through. Additional slurry flow rate measurements include density meters (Cilliers & Shean, 2011). The measurement and monitoring of the slurry flow rate is vital in flotation control systems as the slurry flow rate has a direct effect on the

pulp level in the cells and is used in mass balance calculations that are required for stabilising control (Cilliers & Shean, 2011).

Elemental Assaying

Large-scale flotation circuits commonly use X-ray fluorescence (XRF) analyses for on-line elemental analysis of streams. XRF utilises high-energy X-ray radiation to excite elements in the stream and measures the specific fluorescent radiation emitted by each element. The sample time can extend from 15 seconds to 1 minute. The online XRF analyser system is centralised with a single probe that analyses the results from different points. The number of samples that can be analysed simultaneously depends on the analyser's capacity. Certain probes can only account for a single sample; thus, analysers have sampling cycles with delays between 15 s and 1 minute. These cycles range from 10 - 20 minutes. The accuracy of XRF ranges between 1 to 6 %, and element detection ranges from 3 ppm to 30 ppm. Laser-induced breakdown spectroscopy is a method used to aid XRF in providing higher frequency analyses and detecting lighter elements (Kejonen et al., 2018). Studies have been conducted to investigate the use of froth surface image analysis as a premise for determining the species grade in streams (Moolman et al., 1994).

Reagent Addition

Reagent addition instrumentation commonly involves using on-off dosing systems with valves or metering pumps that could be applied for small dosages (Wills and Finch, 2015). Reagent addition flow rates can be measured by magnetic flow meters. Other reagent addition flow rate measuring devices include positive displacement pumps and Coriolis Flow meters that determine the mass flow through inertia (Jakhu, 1998).

2.2.2. Process Control of Reagent Addition

The automatic control of the various flotation reagents typically consists of control loops with feedback PI/D controllers and optional feedforward controllers for disturbance control.

Automatic base layer control of collector addition rate commonly uses feedforward ratio control. The control action is based on a linear ratio of the target mineral in the feed to the flotation feed. The trend observed with collector dosage is that an increase in collector dosage results in an increase in target mineral recovery, where gangue recovery is also increased, as seen in Figure 2-6 (Wills and Finch, 2015). However, the recovery plateaus after a specific collector dosage and recovery remains the same or starts to decrease slightly after that. The gangue recovery continues to increase after the plateau is reached and potentially results in a decrease in selectivity (Wills and Finch, 2015).

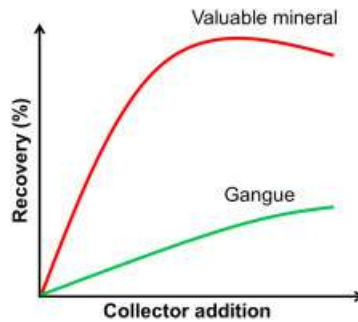


Figure 2-6, collector addition effect on recovery (Wills & Finch, 2015)

The collector addition rate at the beginning of the plateau is recommended for the setpoint in base layer control, as the recovery is at the highest value. However, the interaction between the collector addition rate and the recovery changes with the ore type and interaction with other reagents. Thus, semiautomatic control is recommended to assist in ore variability, where the operator, in a feedback-like manner, determines the setpoint based on the feed grade and analysis of the XRF assays and supplies the setpoint to the controller (Cilliers & Shean, 2011). Feedback control can also administer collector addition control, where the tailings assay data can provide insight into the amount of collector that needs to be added to achieve the desired grade. Typically, the rougher grade can be controlled by manipulating the collector amount. Consequently, the rougher grade influences the cleaner concentrate grade (Wills & Finch, 2015).

The automatic control of frothers has been ineffective, as the flotation system is susceptible to slight changes in the frother addition rate (Wills & Finch, 2015). For example, large unstable bubbles resulting in a low target mineral recovery occur with a frother dosage rate that is too low. At the same time, increased recovery with a low grade brought about by a higher mass pull and entrainment can occur with a high frother addition rate. In addition, mine water composition, feed contamination and ore variability can affect the frother's actions. Therefore, semiautomatic control, such as the operator supplying a setpoint to the frother controller, is recommended. An alternative method, where the ratio of the frother to the solids and water feed rates can also be used (Wills & Finch, 2015).

The setpoints of reagent addition rates can be determined after observation of on-stream analyser measurements of the chemical composition of the streams. The setpoints can also be determined by observation of the froth. The surface features of froth give a good indication of the flotation performance. Features such as froth texture, colour, froth velocity, bubble size distribution and bubble shape, as well as froth stability, can be used to classify froths and predict flotation performance. The relationship between froth features and flotation performance has produced a large body of research and prompted the use of machine vision that captures and analyses froth images to understand froth behaviour. Furthermore, machine vision has been studied to overcome the shortfalls brought about by disturbances such as feed variability in automatic control. The froth property, bubble size, is closely linked to reagent dosing and has

been used in reagent control strategies. Several control strategies for reagent dosing were based on bubble size. This includes the use of a feedback controller based on the relationship between reagent flow velocity and the bubble size of a zinc flotation plant (Hyötyniemi et al., 2001).

Similarly, Kaartinen et al. (2006) used expert controllers that mimic operator decision-making in a feedback control loop of a zinc flotation process. The relationship between froth colour, bubble size distribution, froth speed, bubble collapse rate, bubble load and the concentrate and tailings grade was used in the decision-making of the rule-based controller (Kaartinen et al., 2006). Similarly, a control strategy that used the probability density function (PDF) of the bubble size distribution for reagent dosing control. The PDF was obtained using the kernel function. The control problem aims to minimise the difference between the observed PDF and the objective PDF using an objective function that includes the reagent dosages as inputs. The flotation process was optimised using a Differential Evolution (DE) algorithm that aims to minimise reagent costs (Y. Xie et al., 2016).

The proceeding sections show the models and control strategies that use the relationship between bubble size and reagent dosing.

2.3. Modelling of Flotation Process

Several types of models can be used for control in froth flotation. There are many ways to classify the types of models. However, for this review, these models can be classified into two major categories: empirical and phenomenological (Quintanilla et al., 2021). Empirical models are multivariable models developed from collected plant data and represent the correlation between input and output plant variables through mathematical equations. The relationship between the variables and supporting parameters is determined using statistical methods. The mathematical relationships do not represent the scientific phenomena occurring in the process. Data collection can be obtained through either offline or online methods. The offline collection involves collecting the accumulated data after a specified period, and the online collection of data involves instant data collection as the process occurs (Jovanovic & Miljanović, 2015). Empirical models do not follow a specific format and are designed customarily for each plant and the corresponding ore in a particular range of operating conditions (Quintanilla et al., 2021). Therefore, there is no baseline model on which empirical models are based. A vast majority of models, each with different variations of variables, have been developed for flotation. To simplify the review, only the models and model-based strategies that included reagent dosage as a variable were explored in detail.

An empirical model was developed using data from the processing of North Waziristan copper. The model was used to predict the recovery and grade via regression, which used input variables such as the type and

dosage of the collector, depressant dosage, pulp pH, frother dosage, sulfidizer dosage, pulp density, and conditioning time. The regression structure is shown in Equation 4.

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \dots \beta_7 X_7 + \epsilon \quad (4)$$

Where $Y = Y_R = Y_G$ refers to the recovery and grade of copper, X_i refers to the independent variables, β_i regression coefficients and ϵ refers to the variation caused by unaccounted variables. The regression coefficients were determined using the principle of least squares, which minimises the sum of errors. The sum errors are equivalent to the sum of differences between the measured, Y values and predicted values. The study concluded that the collector dosage, depressant dosage, sulfidizer dosage and pulp density showed the greatest significance in prediction. The grade model indicated that the collector dosage, depressant dosage and conditioning time were the most significant (Ali, 2007). Similarly, Lusambo and Mulenga (2018), also investigated the modelling of the copper flotation system of the Kansanshi Sulphide Ore using non-linear regression. The model used inputs such as feed particle size, feed flow rate, feed density, feed grade, bubble velocity, airflow rate and froth level with the inclusion of SAG (Semi-Autogenous Grinding) mill parameters to predict the copper recovery. The model structure is shown in Equation 5.

$$R_{Cu} = C_0 + C_1 QJ + C_2 W_1^{1.3} + C_3 W_2^{2.5} + C_4 W_3^{3.7} + C_5 v^{-0.5} + C_6 a^{2.8} f^{8.6} + C_7 [\exp(-C_w)] [1 - \exp\left(-\frac{TCu - C_8}{C_9}\right)] \quad (5)$$

Where C_0 to C_9 refers to model coefficients, Q refers to the SAG mill throughput, J refers to the ball mill load, W_i refers to the feed size fraction, v refers to the bubble velocity, a refers to the air flow rate, f refers to the Froth level, C_w refers to feed density and TCu refers to the total copper grade. The model is simplified by removing specific coefficients and parameters, which add little accuracy to the prediction. The simplified model accounted for 90 % of the recovery value (Lusambo & Mulenga, 2018). Another regression-type model was developed by Vazifeh et al. (2010) that related collector and depressant dosages to the recovery in a Copper rougher circuit. The model was developed using Design Expert software and is shown in Equation 6.

$$y_r = 82.47 + 0.94X_1 + 1.25X_2 + 1.23X_3 + 2.44X_4 - 1.55X_1X_2 - 1.7X_1X_4 - 1.58X_2X_4 - 1.49X_3X_4 + 1.74X_1X_2X_4 \quad (6)$$

Where X_1, X_2, X_3 and X_4 represent the levels of reagent dosages in a factorial design and y_r is the predicted recovery. A model to predict copper grade was also developed and is shown in Equation 7.

$$y_r = 0.46 + 0.014X_3 + 0.031X_4 - 0.012X_1X_2 - 0.012X_2X_4 - 0.01X_1X_2X_3 + 0.015X_1X_3X_4 + 1.74X_2X_3X_4 \quad (7)$$

The models determined the reagent dosages that maximise the copper recovery. The models had high correlation coefficients of 0.96 and 0.89 for the recovery and grade models, respectively (Vazifeh et al., 2010).

Monyake et al. (2023) investigated the use of two types of data-driven models, namely Random Forests and Artificial Neural Networks (ANN), to compare the performance of synthesised chitosan-polyacrylamide co-polymers (C-PAMs) that depress pyrite (FeS₂) along with sphalerite (ZnS) depression using Zinc Sulfate whilst concentrating galena (PbS) and chalcopyrite (CuFeS₂) contained in the Mississippi Valley Type Ore. The model inputs included copolymer compositions, reagent dosages, flotation time, slurry pH, and impeller speed, with the model outputs being the base metal grades and recoveries. The models were compared using the statistical measurements of Root Mean Squared Error (RMSE) and the overall coefficient of determination (R^2). The comparison showcased that the Random Forrest model had a greater prediction performance than the ANN model, with a lower RMSE and higher R^2 . The poorer prediction performance of the ANN when compared to the Random Forest was attributed to shortcomings of local optimisation in the ANN model in predicting outputs of a highly non-linear process (Monyake et al., 2023). Similarly, Kalyani et al. (2008) developed an ANN-based process model trained using an error backpropagation algorithm to represent the froth flotation of coal fines. The input variables consisted of the collector dosage, frother dosage, and pulp density, with the coal yield percentage as the output variable. The model presented good generalisation and predicting capability as the training and testing Root Mean Squared Error (RMSE) had comparable values at 0.03 and 0.04, respectively. Furthermore, the model had high coefficients of correlation values at 0.994 and 0.997 for the training and testing sets, respectively (Kalyani et al., 2008). A comparative study of expert system models was also conducted by Jovanović et al. (2022). Two models based on Fuzzy logic and three artificial neural network models were compared based on the prediction capabilities of the copper grade and recovery of final concentrate in an industrial copper circuit. The input variables were the copper in the ore feed, collector dosage in the rougher and the scavenger circuit, frother dosage and slurry pH in the rougher circuit. Both Artificial Neural Network models displayed the highest prediction accuracy with R^2 values at 0.98 and 0.99 (Jovanović et al., 2022). However, Cook et al. (2020) developed a hybrid random forest model that performed better than an artificial neural network that used reagent dosages as input variables. The hybrid random forest model used a firefly algorithm that predicts the optimal number of trees and leaves for each tree. The grade and recovery of galena and chalcopyrite were used as the output variables, and the collector's

dosage, frother dosage, pyrite depressant's dosage, impeller speed, air flow rate, flotation time, and pulp's pH. The hybrid model random forest model was also compared to a support vector machine, a decision tree model and an unmodified random forest model. The decision tree models had the lowest R^2 values at 0.67 and 0.48 for the Pb and Cu grades, respectively, whereas the Hybrid model R^2 values were 0.99 and 0.92 for the Pb and Cu grades, respectively (Cook et al., 2020). Other implementations of expert systems in the modelling of froth flotation include studies into a novel stacked graph convolutional network (S-GCN) used to improve the model interpretability of a potassium chloride flotation system (Wang et al., 2023) and the development of a radial basis Function Neural Network to represent the bauxite froth flotation process (Xiaoli et al., 2016).

Various models have been developed with machine vision of the froth surface, including reagent addition. A froth image classification method based on a maximal-relevance-minimal-redundancy (MRMR)-semi-supervised Gaussian mixture model (SSGMM) model was used to select the optimal reagent dosage condition of a coal flotation process. The MRMR screens out the optimal froth image that supplies the most froth feature images. The SSGMM is a machine-learning tool that classifies froth images based on their features and links these classes to the different reagent conditions. This method assisted in reagent dosage selection, reducing reagent consumption (Cao et al., 2022). Zhang et al. (2020) also used froth image texture data of the lead-zinc flotation process to develop models for classification purposes. Firstly, an isolation forest model was used to detect faults of the nature of froth overloaded with mineral particles and a low number of fine particles attached to the froth. Secondly, a spline regression model was used to determine the root cause of the faults based on texture information. The root causes are related to the reagent dosages of the frother, activator and collector. The average detection accuracy for each of the root causes is 90.70%, 88.64%, and 83.33%, respectively (Zhang et al., 2020). Ai et al. (2018) took a union statistical approach to relate the froth texture features, bubble size and shape to reagent dosages. A multi-output least square support vector regressor model was used on a gold antimony process to reduce reagent changing. The model successfully lowered manual reagent change from 41.04 to 20.89 % (Ai et al., 2018). The bubble size (CDF) and reagent dosage relationship were explored using XGBoost by Tang et al. (2020). The XGBoost or extreme gradient boosting algorithm combines the weak learners of a group of decision trees to correct the errors of previous models. The model improved the stabilisation of the zinc flotation process compared to the manual determination of reagent dosages (Tang et al., 2020).

Empirical models can be used to represent sub-systems of the flotation process. Tsatouhas et al. (2006) developed a model that related the froth recovery to the froth retention time and the froth life using an exponential equation. Plant data was used to fit the data to produce parameters that describe particle transport in froth transfer and froth stability (Tsatouhas et al., 2006). Another factor that is indicative of the froth flotation performance is entrainment. Savassi et al. (1998) modelled the degree of entrainment using a partition curve that showed the relationship between the degree of entrainment and particle size. Two

empirical parameters were used in the computation: entrainment and drainage parameters. The entrainment parameter refers to the particle size for which the degree of entrainment is 20%, and the drainage parameter is linked to the preferential drainage of coarse particles (Savassi et al., 1998). Similarly, Yianatos et al. (2010) explored an entrainment model that related a dimensionless entrainment factor E_{Fi} per size class to a mean particle size parameter, the particle size class "i", and a drainage parameter that is linked to the mineral characteristics and the cell operating conditions. The entrainment factor used a ratio of mineral gangue recovered per size class by entrainment to the water recovery. The model was successfully validated against a rougher bank of a copper flotation circuit (Yianatos & Contreras, 2010). The overall flotation rate constant (k) represents the flotation kinetics and is strongly related to the bubble surface area flux (S_b). The bubble surface area flux is the total surface area of bubbles per unit cross-sectional area of flotation cell per unit of time and is correlated with the Sauter mean bubble size (m) and the superficial gas velocity (m/sec). Gorain et al. (1999) developed an empirical model to predict the bubble surface area flux that could accommodate various cell sizes and flotation scale-ups. Air flow rates and Impeller speeds were proposed as inputs, as these variables are strongly related to the bubble size and superficial gas velocity. The model was validated using plant data from concentrator flotation cells with different sizes and designs in South Africa and Australia, and was able to predict bubble surface area flux (S_b) at high confidence levels (Gorain et al., 1999).

Phenomenological models have also been developed for predictive control; however, the relationships between dependent and independent variables are based on the physics of flotation. Phenomenological models used for flotation control can be further classified into kinetic, population balance, and hydraulic-based modelling. Kinetic modelling involves modelling the flotation process as a chemical reaction. The model represents the transfer of particles out of the pulp phase and considers two collisions between mineral particles and between the hydrophobic particles and air bubbles. Kinetic models can be altered to include the froth phase by incorporating material transfer into and out of the froth phase. The model uses a flotation rate constant, K , and can be expressed as first-order, second-order, and non-integral kinetics. Hydraulic models are based on continuity equations that explore mass and flow conservation in a flotation cell. These models are developed by performing mass or volumetric balances around each flotation cell. Finally, the last type of phenomenological model used for control is the population-based model, which is based on four phases of particles in the flotation process: free in the pulp phase, attached to bubbles in the pulp phase, free in the froth phase, and attached in the froth phase (Quintanilla et al., 2021). Table 2-4 illustrates the baseline equations from which the models are developed.

Table 2-4 Baseline equations of Phenomenological Models

Type of Model	Equation Model was Derived from	Equation Definition
Population Balance Model	$\frac{d}{dt}(V\psi_j) = Q_F\psi_{j,IN} - Q_{ulpout}\psi_{j,OUT} - \sum_{i=1}^k k_{ji}\psi_j V$	Where ψ_j is the concentration of mineral particles, V is the volume of the cell, Q_F and Q_{ulpout} is the volumetric flow rate in and out of the cell, respectively, and k_{ji} is the particle transfer rate between states.
Hydraulic Model	$\iint_{c,s} \rho(\vec{v} \cdot \vec{n})dA + \frac{\partial}{\partial t} \iiint_{c,v} \rho dV = 0$	Where $\iint_{c,s} \rho(\vec{v} \cdot \vec{n})dA$ is the rate of increase of mass and $\frac{\partial}{\partial t} \iiint_{c,v} \rho dV$ is the net convection in
Kinetic Model	$\frac{dN}{dt} = -K_n N^n N_b^m$	Where K is the flotation rate constant, N is the particle concentration, N_b is the bubble concentration, n and m are the kinetic orders.

Hu et al. (2022) modelled the inclusion of reagent effects using the baseline kinetic model equation extension. The classical first-order kinetic equation was modified to represent the recovery of concentrate, as shown in Equation 8, of an aphanitic (microcrystalline) graphite flotation system. Batch experiments were performed with various types and dosages of collectors and frothers to fit the parameters.

$$R = R_{\infty,1}(1 - e^{Kt}) \quad (8)$$

Where R is the recovery, R_{∞} , refers to the maximum recovery and K refers to the flotation rate constant. Batch experiments were performed with various types and dosages of collectors and frothers to determine R_{∞} and K by curve fitting. The first-order kinetic model extensions included rectangular distribution, mixed factor format, and inclusion of the gas/solid adsorption model. A classical second-order kinetic model and a second-order kinetic model with rectangular distribution were also compared. The classical first-order model provided the highest fitting accuracy with a R^2 of 0.998% (Hu et al., 2022).

Perez-Correa et al. (1998) developed a hybrid model approach that included the effect of reagent addition on metallurgical performance. They derived a non-linear dynamic model of a copper plant that included a rougher bank and three cleaner banks from mass balances. Firstly, a single cell was modelled, assuming the process was split into two phases: pulp and froth. Equations 9 and 10 represent the material transfer between phases using a component mass balance.

$$\frac{dM_p^{(i)}}{dt} = - \left[K_p^{(i)} + \frac{q_t}{V_p} \right] M_p^{(i)} + q_a X_a^{(i)} + K_e^{(i)} M_e^{(i)} \quad (9)$$

$$\frac{dM_e^{(i)}}{dt} = - \left[K_e^{(i)} + \frac{q_c}{V_e} \right] M_e^{(i)} + K_p^{(i)} M_p^{(i)} \quad (10)$$

Where $M_p^{(i)}$ is the total mass of mineralogical class i in the pulp, $K_p^{(i)}$ is the flotation rate of mineralogical class i , q_t is the tailings flow rate, V_p is the pulp volume, q_a is the feed flow rate, $X_a^{(i)}$ is the concentration of mineralogical class i in the feed, $K_e^{(i)}$ is the drainage rate, $M_e^{(i)}$ is the froth mass, q_c is the concentrate flow rate, V_e is the froth mass. Two mineralogical classes were investigated: rich minerals, mainly chalcopryrite, and poor minerals, mainly gangue. The flotation rate and drainage rate were empirically modelled using first-order flotation kinetics. The effect of collector and frother addition rates was included in the modelling of the flotation and drainage rate and is represented in Equations 11 and 12.

$$K_p^{(i)} = a^{(i)} Q_{col}^3 + b^{(i)} Q_{col}^2 + c^{(i)} Q_{col} + K_p 0^{(i)}; \quad (i = 1, 2) \quad (11)$$

$$K_e^{(i)} = K_e 0^{(i)} + \beta_1^{(i)} Q_{col} + \beta_2^{(i)} Q_f \quad (12)$$

Where a, b, c, K_0, β_1 and β_2 are parameters that are empirically determined from plant data (Perez-Correa et al., 1998).

Putz and Cipriano (2015) developed this model to include more mineralogical classes i of the ore in each flotation cell j and granulometry class k , as shown in Equations 13 and 14 (Putz & Cipriano, 2015).

$$K_{col}^{ijk} = a_1^{ijk} Q_{col}^1 + a_2^{ijk} Q_{col}^2 + a_3^{ijk} Q_{col}^3 + k_{col,0}^{ijk} \quad (13)$$

$$K_e^{ijk} = k_{e,0}^{ijk} + b_1^{ijk} Q_{col} + b_2^{ijk} Q_f \quad (14)$$

Similarly, Karamoozian and Mehrabadi (2020) developed a kinetic model based on mineral classes to predict cerussite's recovery. The three cases were fast, medium, and slow floatable minerals. The structure of the model is shown in Equation 15.

$$R = R_{\infty} [Z_1(1 - e^{k_1 t}) + Z_2(1 - e^{k_2 t}) + Z_3(1 - e^{k_3 t})] \quad (15)$$

Where R refers to the recovery, R_{∞} is the ultimate recovery, Z_1, Z_2 and Z_3 are the fast floatable, medium floatable, and slow floatable mineral fractions, respectively, k_1, k_2 and k_3 are the fast, medium, and slow kinetic constants. The rate constants were obtained through 64 single-cell batch experimental tests that varied six inputs: particle size, solid content, Na₂S dosage, collector dosage, and collector type. The model presented R^2 values greater than 0.8. After that, a central composition model that explored the surface response of input variables to find the values that resulted in the optimal output was successfully used to find the ultimate recovery (Karamoozian and Mehrabadi, 2020). Chipise et al. (2022) also altered a first-order kinetic model to include different mineral classes. The Kelsall model was modified to include a non-floating mineral class and was used to optimise reagent dosage in a PGM scavenger bank. Experimental data used for the fitting were collected, and different dosages of the collector and depressant were considered to obtain flotation parameters for each group of reagent dosages. The modified Kelsall model is shown in Equation 16.

$$R = 1 - I_{PF} \cdot e^{-kf} - I_{PS} \cdot e^{-ks} - I_{PN} \quad (16)$$

Where R is the recovery of PGMs, I_{PF} is the fraction of fast-floating PGMs, kf is the fast-floating rate constant, I_{PS} is the fraction of slow-floating PGMs, ks is the slow-floating rate constant, I_{PN} is the fraction of non-floating PGMs. The modified model was compared to three different first-order kinetic models. The modified model displayed the greatest fit with R^2 values greater than 0.99 (Chipise et al., 2022).

Venter et al. (2022) extended existing grade and recovery models developed by Oosthuizen et al. (2021) to include the effect of frother dosage on metallurgical performance. The original model was developed by performing a mass balance of minerals around a flotation cell and is shown in Equation 17.

$$\dot{M}_{C_k}^{i,j} = K^{i,j} M_k^{i,j} S_{b_k} \alpha_k + Ent_{Frac}^{i,j} \frac{M_k^{i,j}}{L_k A_k} Q_{C_k} \quad (17)$$

Where $\dot{M}_{C_k}^{i,j}$ is the mass flow rate of minerals, i , of mineral class, j , out of the cell k , K is the flotation rate constant, M_k is the mineral mass in the cell S_{b_k} is the bubble surface area flux, α_k is the air recovery, $Ent_{Frac}^{i,j}$ is an existing entrainment factor model, L_k is the cell level, A_k is the cell cross-sectional area, Q_{C_k} is the volumetric flow rate. The air recovery was determined using the model shown in Equation 18.

$$\frac{d\alpha_k}{dt} = \frac{K_{\alpha J_g} (J_{gk} - K_{0,\alpha J_g})^2}{\lambda_{air_k}} + \frac{K_{\alpha h_f} (h_{f,k} - K_{0,\alpha h_f})^2 - \alpha_k}{\lambda_{air_k}} \quad (18)$$

Where J_{gk} is the superficial gas velocity, λ_{air_k} is the froth residence time, $h_{f,k}$ is the froth height, $K_{\alpha J_g}$ and $K_{\alpha h_f}$ are parameters that are fitted using steady-state experimental data, $K_{0,\alpha J_g}$ and $K_{0,\alpha h_f}$ are empirical parameters determined from online data. A Sobol sensitivity analysis was conducted to determine which manipulated variable provided the most significant contribution of variance to the total controlled variable variance. $K_{0,\alpha J_g}$ and $K_{0,\alpha h_f}$ had the highest sensitivity indexes for grade and recovery. Two frother model expansions were developed that related frother concentration in the pulp to the two parameters and are shown in Equations 19 and 20.

$$K_{0,\alpha h_f} = K_{CF} C_k^f + K_{CF,0} \quad (19)$$

$$\frac{d\alpha_k}{dt} = \frac{K_{\alpha J_g} (J_{gk} - K_{0,\alpha J_g})^2}{\lambda_{air_k}} + \frac{K_{\alpha h_f} (h_{f,k} - K_{0,\alpha h_f})^2 - \alpha_k}{\lambda_{air_k}} + \frac{K_{F\alpha} (C_k^f - K_{F\alpha,0})}{\lambda_{air_k}} \quad (20)$$

The altered models were tested in a simulation that used initial values in (Oosthuizen et al., 2021) for the manipulated variables. The second model expansion predicted the peak air recovery better than the first model expansion, resulting in better recovery predictions (Venter et al., 2022).

The phenomenological models developed thus far do not represent the entire flotation process. In particular, the froth phase was not modelled as extensively as the pulp phase for controlling flotation circuits. Therefore, froth stability factors such as bubble bursting were commonly excluded from these models (Cilliers & Shean, 2011).

2.4. Non-linear dynamic models

In section 2.3, the classification of models depended on the structure; however, the classification of models can also depend on the relationship of the model output with time. This classification consists of two model types, namely static, also known as steady-state models and dynamic models. Static models represent a system at a specified time, and the outputs of static models are computed exclusively by the current inputs of the system and are determined instantly. Therefore, the relationship between the input and outputs is not dependent on the time of the system. Meanwhile, dynamic models represent a system over a time range, and the outputs of dynamic models are computed based on initial conditions and past and current inputs. Furthermore, the output response to the input does not occur instantly. Thus, the relationship between the inputs is dependent on the time. Static models are typically, but not exclusively, represented by algebraic equations; however, dynamic models are represented by differential or difference equations (Marlin, 2000).

Flotation is a highly dynamic and complex process that requires a dynamic model to represent the process. Complicated dynamics, such as the pulp and froth phase interaction and many disturbances, render a difficult modelling task. Static models are typically modelled assuming a steady state. They may be suitable for design, analysis and simulation purposes. In contrast, dynamic models are suitable for control purposes as these models accommodate the prediction of future plant responses and are not restricted to a specified time, as with static models. Furthermore, dynamic models allow for the accumulation of material in the flotation cell and lag times (Marlin, 2000).

Dynamic models can further be classified into linear and nonlinear dynamic models. The relationship between the manipulated and controlled variables is described with a linear function in linear models, whereas nonlinear models use a nonlinear function. Nonlinear models can be linearised to form a linear representation of the system at an operating point. A common method of linearisation used in model development is the Taylor series expansion of a nonlinear function. Linear representations of nonlinear systems can be used for control purposes. However, the operating ranges are limited as the system behaviour can only be assumed linear for a small region. The narrow operating ranges are often accommodated by switching to a linear representation at a different operating point (Marlin, 2000).

Developing a simple and robust dynamic model is critical to MPC implementation. Furthermore, the model type used in flotation control should be able to handle the complex dynamics of the multiphase system with innate instability.

2.5. Model-Based Control of Froth Flotation

The development of vigorous and automated control strategies includes several challenges brought about by the complexity of the flotation circuit dynamics. The main challenges of implementing model-based control strategies and model predictive control in the industry are the non-linearity and the large number of variables included in the flotation process. As seen above, there have been attempts to develop linear models and non-linear models that include the effect of reagent dosing on metallurgical performance to represent flotation. The linear models for aeration rates and level control have been studied extensively, and successful implementations have been used for flotation control in the industry (Quintanilla et al., 2021). Models for pulp level and aeration rate that represent the flotation process using non-linear relationships have been studied and still require industrial development to be included. In relation to airflow and pulp level model-based strategies and MPC strategies, there is a significantly smaller amount of research that explores the inclusion of reagent dosage strategies with the industrial implementation, following a similar trend (Quintanilla et al., 2021). This section explores the model-based control strategies for flotation found in the literature. A separate section explores the model predictive control strategies. Model-based control strategies include control strategies that use models without an MPC controller and an objective function to minimise.

Li et al. (2020) developed a model using a hydraulic structure to represent an antimony flotation process incorporating reagent dosing. The mass balances were performed around four phases in a flotation cell: the free mineral particles in the pulp phase, mineral particles attached to air bubbles in the pulp phase, free mineral particles in the froth phase and mineral particles attached to air bubbles in the froth phase. The equations were combined to represent the pulp phase and the froth phase. The tailings and concentrate grade, along with reagent doses, were included in the model structures shown in Equations 21 and 22.

$$\frac{dR_1}{dt} = - \left[\frac{(r_1 Q_A + Q_A m_1 O_1 O_2)}{(1 + m_1 O_1 O_2) V_{LP} V_{BP}} + \frac{Q_T}{(1 + m_1 O_1 O_2) V_{LP}} \right] R_1 + \frac{Q^T}{V_{LP}^2} + \frac{r_2 Q_A (m_3 O_3 + m_4 O_4 + m_5) V_{LF}}{(1 + m_1 O_1 O_2) V_{LP}} \frac{Q_T Q_C}{V_{LP}} R_2 \quad (21)$$

$$\frac{dR_2}{dt} = \left[\frac{\left(r_1 Q_A + Q_A m_1 O_1 O_2 \frac{V_{LP}}{V_{BP}} \right) Q_C}{(1 + m_1 O_1 O_2) V_{LF}} \frac{Q_C}{Q_T} \right] R_1 - \frac{r_2 Q_A (m_3 O_3 + m_4 O_4 + m_5)}{(1 + m_2 O_3 O_4) V_{LP}} R_2 - \frac{Q_T}{V_{LP}} R_2 \quad (22)$$

Where R_1 and R_2 represents the tailings and concentrate grade, respectively, pH value O_1 , collector flow rate O_2 , activator flow rate O_3 and frother flow rate O_4 , Q_A is the airflow rate, Q_C is the concentrate flow rate, Q_T is the tailings flowrate, V_{LP} is the pulp volume in the pulp phase, V_{BP} is the gas volume in the froth phase, V_{LF} is the pulp volume in the froth phase, $m_1, m_2, m_3, m_4, m_5, r_1$ and r_2 are parameters that need to

be identified. The model was re-arranged to a state space model format, and an internal model that rejected external disturbances was added. The optimal controller was found in a simulation run by using a value iteration, VI, based adaptive dynamic programming (ADP) method that minimised the error between the tailings and concentrate grade and reference grade (Li et al., 2020).

As mentioned in the previous sections, bubble size and reagent dosing control are closely linked. Several model control strategies that used froth images for reagent dosing control were based on bubble size. A study utilised the bubble size as a basis for control by taking froth images to develop a probability density function (PDF) of bubble size. A multi-output least square support vector machine (MLS-SVM) mapping was used to form a relationship between the PDF and the reagent dosage. A reagent dosage predictive control structure was developed using the MLS-SVM, a feedback correction unit, and a reagent dosage optimisation unit (Zhu et al., 2014). Cao et al. (2019) also proposed a reagent dosing tracking control strategy that used a PDF based on froth size. However, the strategy also included a distributed-machine vision predictive model for reagent compensation control. These two strategies were combined to form a hybrid control strategy (Cao et al., 2019). Li and Gui (2018) developed a control strategy based on the distribution of bubble size changes with a change in reagent dosing. A cumulative distribution function (CDF) was used to represent the time series distribution of froth images. The CDF and reagent dosing relationship was developed using a partial least-squares regression based on spline transformation. The strategy included a correction control loop to optimise flotation in the gold antimony process (Li & Gui, 2018).

The combination of expert system models and froth images was also used in reagent control strategies. Jahedsaravani et al. (2016) used neural networks to relate the process variables pH, air-flow rate, and frother dosage to frother characteristics, bubble size distribution, colour, velocity and stability. A three-layer, feedforward perceptron neural network was modelled to relate the froth characteristics to the metallurgical parameters, grade and recovery. A fuzzy controller was used with the models for stabilisation of copper grade and recovery in a batch set-up (Jahedsaravani et al., 2016). Monyake et al. (2023) utilised two models that accounted for varied ore feed properties and unmeasurable properties that make reagent addition control difficult. Firstly, a pre-setting reagent dosing model uses the Belief Rule Base (BRB) method, which uses feed conditions and evidential reasoning to set the reagent dosages. A feedback reagent compensation model that uses rule-based reasoning (RBR) on froth characteristics is added to the pre-setting model to compute the optimal reagent dosage. The models were tested on a simulation of the gold-antimony flotation process and were compared to two other model-based strategies: least squares support vector machine (LSSVM) and artificial neural network (ANN). The two-step model approach performed the best with the lowest mean absolute error (MAE) (Monyake et al., 2023).

The implementation rate of model-based control strategies in industry is still low despite the immense effort into modelling the flotation system. The lack of successful implementation can be attributed to a lack of or

poor quality of measurement instrumentation, a lack of robust dynamic models that can function over a wide range of operating conditions and poor regulatory control. More specifically, the attention to models that include reagent dosages as manipulated variables is scarce in comparison to those that include pulp level and airflow rate (Bergh & Yianatos, 2011).

2.6. Model Predictive Control in Flotation

Model predictive control is an advanced process control strategy that uses a prediction model of the plant to compute the optimal control input over a defined horizon period (Bergh & Yianatos, 2011). An MPC controller consists of 2 components: a prediction model and an optimiser with an objective function. The prediction model represents the dynamics of the plant and must be developed in a manner that balances accuracy and simplicity, as the degree of complexity influences the computation speed. The MPC controller uses the prediction model to predict the future response of the plant and uses an objective function to choose the optimal control inputs that will minimise the error between the reference and predicted controlled variable response path. The objective function is also used to minimise the change in manipulated variables. The optimiser computes the optimal control input by minimising the objective function. A generalised format of the objective function is shown in Equation 23.

$$J(z_k) = J_y(z_k) + J_u(z_k) + J_{\Delta u}(z_k) + J_\varepsilon(z_k) \quad (23)$$

Where $J(z_k)$ is the sum of four terms that relate to the controller's performance. $J_y(z_k)$ is a measure of the output reference tracking, which influences plant output behaviour in relation to the reference values and is expressed in Equation 24.

$$J_y(z_k) = \sum_{j=1}^{n_y} \sum_{i=1}^p \left\{ \frac{w_{i,j}^y}{s_j^y} [r_j(k+i|k) - y_j(k+i|k)] \right\}^2 \quad (24)$$

Where, k is the control interval, p is the prediction horizon, n_y is the number of plant output variables, $y_j(k+i|k)$ is the predicted value of j th plant output at the i th prediction horizon step, s_j^y is the scale factor for the j th plant output, $r_j(k+i|k)$ is the reference value for the j th plant output, $w_{i,j}^y$ is the tuning weight.

$J_u(z_k)$ is a measure of the manipulated variable tracking, which ensures the manipulated variables remain within a specific value or range. The full expression is shown in Equation 25.

$$J_u(z_k) = \sum_{j=1}^{n_u} \sum_{i=0}^{p-1} \left\{ \frac{w_{i,j}^u}{s_j^u} [u_j(k+i|k) - u_{j,target}(k+i|k)] \right\}^2 \quad (25)$$

Where, n_u , is the number of manipulated variables, $w_{i,j}^u$ is the tuning weight for the j th manipulated variable at the i th prediction horizon step, s_j^u is the scale factor for the j th manipulated variable, $u_j(k+i|k)$ is the predicted value at the j th manipulated variable at the i th prediction horizon step, $u_{j,target}(k+i|k)$ is the target value for the j th manipulated variable at the i th prediction horizon step.

$J_{\Delta u}(z_k)$ is a measure of the manipulated variable suppression for manipulated variables that need small movements and is expressed in Equation 26.

$$J_{\Delta u}(z_k) = \sum_{j=1}^{n_u} \sum_{i=0}^{p-1} \left\{ \frac{w_{i,j}^{\Delta u}}{s_j^u} [u_j(k+i|k) - u_j(k+i-1|k)] \right\}^2 \quad (26)$$

Where, $w_{i,j}^{\Delta u}$ is the tuning weight for the j th manipulated variable movement at the i th prediction horizon step. $J_c(z_k)$, is a measure relating to constraint violation and is expressed in Equation 27.

$$J_\varepsilon(z_k) = \rho_\varepsilon \varepsilon_k^2 \quad (27)$$

Where, ε_k is a slack variable that quantifies the worst-case constraint violation and ρ_ε is the tuning weight for the constraint violation.

The prediction horizon is the number of future predicted timesteps that the optimiser computes, and the control horizon is the number of control moves that result in the predicted output (Quintanilla et al., 2021). The objective function is not exclusive to the form seen in Equation 23. However, this objective function is used for the MPC controllers developed in the study,

The type of model predictive controller used in the control strategy depends on the system dynamics and constraints. Linear time-invariant model predictive controllers can be used for systems represented by linear models with linear constraints and a linear cost function. Linear MPC strategies solve an optimisation problem where the local optimum is the global optimum. Linear MPC controllers, such as Adaptive and Gain-scheduled, can be used on systems represented by non-linear models. Adaptive MPC uses a linearised model computed based on the operating conditions. The linear model is updated when there is a change in operating conditions. The objective function is kept constant throughout the different operating ranges as the constraints are kept constant. Gain-scheduled MPC is employed when the constraints are not kept constant and uses a similar approach to adaptive, where the linear model is updated based on the operating conditions. However, the objective function varies with the change in operating conditions to accommodate a change in constraints. A non-linear model predictive controller uses a nonlinear model to predict control

output. Non-linear MPCs allow for non-linear constraints and a non-linear cost function; however, more computational effort is required than linear MPCs.

The implementation of MPC for reagent dosing control has been briefly explored. Moreover, the linear models focused on a limited number of ores. Brooks and Koorts (2017) developed an MPC strategy using a linear model for reagent dosing control for a zinc rougher bank at the Glencore Mt Isa site in Australia. As mentioned in the introduction, this MPC strategy forms the basis of the Master's project. A linear dynamic model was used to control the dosage of ethyl xanthate, CuSO_4 , and the naphthalene sulphate and maintain the concentrate's grades (% Zn) across the banks. The manipulated variables in the model were the aeration rates, reagent dosages for each cell, and pulp levels for each pair of cells (Brooks & Koorts, 2017).

The model used for the MPC strategy was developed using unit step responses, where the response of the controlled variable from the unit step of a manipulated variable or a disturbance was collected. The step responses were used to fit models with the structure developed by Garcia et al. (1989), shown in Equation 28.

$$y(k) = \sum_{i=1}^{n-1} H_i \Delta u(k-i) + H_n \Delta u(k-n) \quad (28)$$

Where $y(k)$ is output at present time k , u is an input, H_i are step response coefficients, and n is the number of coefficients (Garcia et al., 1989). The model predictive control strategy further includes a quadratic optimisation program that was minimised and is shown by Equations 29 to 32, which uses the predicted output responses from Equation 28.

$$\Delta u(k) \dots \Delta u(k+m-1) \sum_{i=1}^p \|\hat{y}(k+l|k) - r(k+l)\|_{r_l}^2 + \|\Delta u(k+l-1)\|_{B_l}^2 \quad (29)$$

$$\hat{y}(k+l|k) = \sum_{i=1}^l H_i \Delta u(k+l-i) + \sum_{i=l+1}^{n-1} H_i \Delta u(k+l-i) + H_n u(k+l-n) + \hat{d}(k+l|k) \quad (30)$$

$$\hat{d}(k+l|k) = \hat{d}(k|k) = y_m(k) - \sum_{i=1}^{n-1} H_i \Delta u(k-i) + H_n u(k+l-n) \quad (31)$$

$$\sum_{l=1}^p C_{y1}^i \hat{y}(k+1|k) + C_{uf}^j u(k+l-1) + C^j \leq 0 \quad j = 1, n_c \quad (32)$$

Where $\hat{y}(k+l|k)$ is the predicted value of y at the time $k+l$, $\hat{d}(k+l|k)$ is the predicted disturbance value at the time $k+l$, $y_m(k)$ is the measurement of y at time k , n_c is the number of constraints, p is the

horizon length, and m is the number of manipulated variables that move in the future. The quadratic program can find the optimum steady-state operating point of the system, the optimum dynamic path of the CVs, and the changes in manipulated variables that need to be made to reach these optimum ranges. The automated step responses of the rougher bank are shown in Figure 2-7.

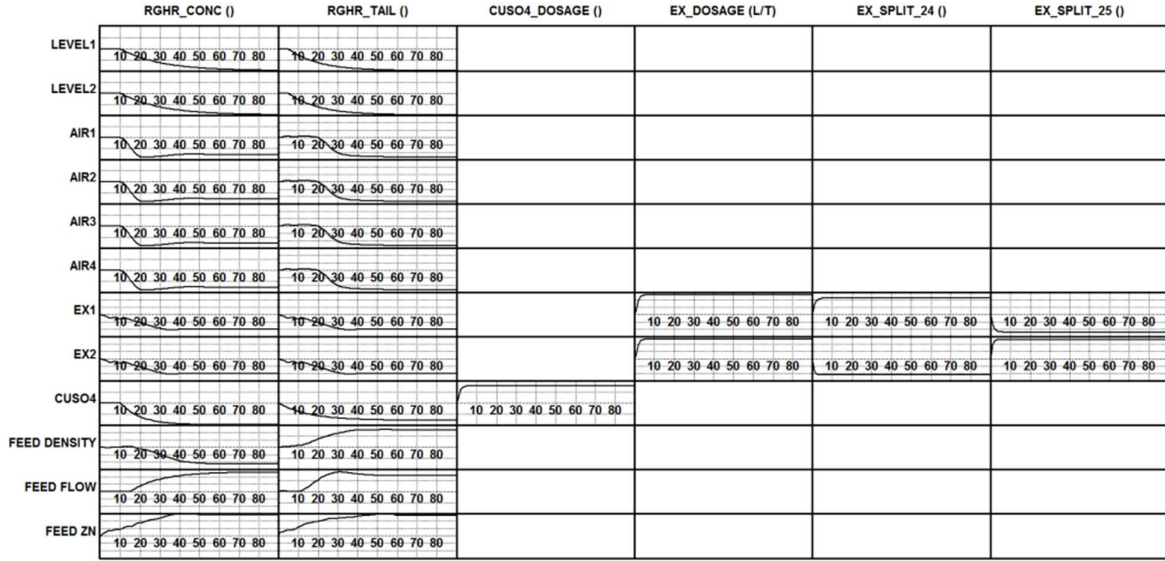


Figure 2-7 Step response matrix for the rougher bank

An MPC strategy implemented at an entry-level in the industry was a model applied to the zinc flotation circuit for Garpenberg in Boliden, Sweden (Lundh et al., 2009). A model based on historical data and one based on first principles were developed and compared. Both models were used to relate the manipulative variables of cell level, air supply, and reagent addition and froth level to the concentrate recovery and grade. On-line X-ray analysers were used to determine the different metal compositions. The empirical model used is a state-space model shown in Equations 33 and 34.

$$x[k+1] = Ax[k] + B_u u[k] + B_d d[k] \quad (33)$$

$$y[k] = Cx[k] \quad (34)$$

Where $u[k]$ and $y[k]$ are the manipulated and output variables at time step k , respectively. $x[k]$ and $d[k]$ are the states and measured disturbance variables, respectively. A , B_u , B_d and C are parameter matrices that were developed using grey-box identification with the known parameters determined from historical time series data (Lundh et al., 2009). The model, based on historical data, rendered the most successful result with a 1% increase in zinc recovery.

A different empirical model used in an MPC strategy for reagent control and breaking into the industry was developed by Zhang et al. (2021). The MPC strategy was applied to a zinc-lead flotation circuit, and the

model was used to predict the effect of changes in reagent conditions on the froth structure. The model is implemented with a Neural Network autoregressive with exogenous input (NNARX) that ensures the model is adaptive to patterns that deviate from previously collected training data. The model is overall described in Equation 35.

$$\bar{y}(k+1) = \mathcal{M}(u(k), \dots, u(k-n_u), x(k), \dots, x(k-n_x), y(k), \dots, y(k-n_y)) \quad (35)$$

Where k is the time step and $\mathcal{M}$ is the NNARX dynamic model. u, x and y are the control inputs, feed inputs and process outputs, respectively. $n_u = 2$, $n_x = 2$ and $n_y = 2$ are past control inputs, past feed inputs and past process outputs, respectively. $u \in \mathbb{R}^3$ are the control inputs, including changes in frother, collector and activator dosages. $x \in \mathbb{R}^2$ are the feed conditions, such as lead and zinc grades. $y \in \mathbb{R}^2$ are the process outputs that represent the froth structure, such as froth structure texture, froth velocity, froth colour and bubble size distribution. The froth structure qualities are captured on computer-based images (Zhang et al., 2021). The strategy was compared to a manual control strategy and a fuzzy expert control system that emulates operators' manual observation and control changes. The MPC strategy presented the smallest standard deviation of the control result from the setpoint and the smallest maximum absolute error (Zhang et al., 2021).

Model predictive control strategies have also incorporated machine vision.

Ai et al. (2019) utilised froth images of an antimony flotation to relate the bubble size and shape to the reagent dosages of the collectors, frothers and activators. Firstly, a shape-weighted bubble size distribution (SWBSD) was developed from the bubble size and shape extracted from the froth images. The SWBSD classifies the system's condition and has a particular grade and recovery per SWBSD. A non-linear least squares support vector regressor (LS-SVR) model was constructed to relate the SWBSD to the reagent dosages. The structure of the LS-SVR is shown in Equation 36.

$$\hat{x}(k+1) = \omega \varphi(x(k), u(k)) + c \quad (36)$$

Where, $x(k)$ is the SWBSD at the time step k , $u(k)$ is the reagent dosages at the time step k . $\hat{x}(k+1)$ is the predicted SWBSD weight vector at the time $k+1$, ω is a diagonal matrix, $(x(k), u(k))$ is a non-linear mapping function and is shown in Equation 37.

$$(x(k), u(k)) = [\varphi_1(x(k), u(k)), \varphi_2(x(k), u(k)), \dots, \varphi_i(x(k), u(k))]^T \quad (37)$$

The mapping function transforms the input data into a hyperdimensional space and computes the output using a radial basis function. $c = [c_1, c_2, \dots, c_i]$ is a bias vector. The predictive control strategy used the model to compute the optimal reagent dosages and was tested on an industrial antimony plant. The control strategy improved reference tracking of the concentrate grade, where an RMSE of 11.2% with manual

control was decreased to 4.83% with the predictive control strategy (Ai et al., 2019). Zhu et al. (2014) utilised a similar approach to reagent dosing control with machine vision. A probability density function was developed from the bubble size. The strategy also used the LS-SVR model to relate a density function to reagent dosages. The approach was tested on a copper flotation plant. The performance of the control strategy was measured by comparing the number of eligible grades in manual control and model predictive control. The MPC strategy produced a higher amount of eligible grade (Zhu et al., 2014).

The industrial application of MPCs within the flotation industry has mostly used linear models, with a few exceptions, including the application of nonlinear models. Successfully implementing MPC depends on dependent dynamic models, reliable measurements, and reasonable regulatory control. Advancements in non-linear state estimation and models that incorporate the interaction between mining, comminution and flotation processes are required to increase adaptability to variation in feed conditions and improve robustness.(Bergh & Yianatos, 2011).

2.7. Model Selection

As discussed in the previous sections, numerous classifications of models can be used to represent the flotation process. The main classification used in this dissertation is phenomenological and empirical models. Each class of model includes a set of advantages and disadvantages. The choice of the model used in the dissertation was based on simplicity, cost, and data availability.

Several challenges are brought about by fundamental modelling (phenomenological models) of flotation. Firstly, key variables that describe the dynamics of the plant are occasionally complex to measure. This could result from a lack of reliable instrumentation or the complexity of the process. Phenomenological models have been used to represent both the pulp and froth phases by assuming perfect mixing or plug flow conditions. However, the froth phase displays complex behaviour and is not accurately represented by models that assume perfect mixing and plug flow. Dynamic froth phase models are available but are not simple enough for control purposes. Thus, there is a lack of simplified phenomenological models that represent the entire flotation process. Complex phenomenological models are suitable for control and analysis purposes and are rarely used for industrial control purposes.

With their simplicity and reduced time consumption, empirical models are a cost-effective choice for process modelling. They are also less cost-intensive than phenomenological models, making them a financially sound option for many projects. However, in complex processes like flotation, the high number of disturbances can challenge the performance of empirical models, particularly when faced with large fluctuations in disturbances such as changes in the ore feed composition. These changes in conditions,

brought about by disturbances, can alter the operating ranges of the plant, posing a challenge for the empirical models developed from the original conditions and operating ranges.

Both types of models require a sampling campaign to obtain data for MPC implementation. After implementing the MPC strategy, certain variables are updated from manual sampling rather than online measurements. The difference in measurement frequency can result in poor model accuracy.

Empirical models were chosen for this study as the available dataset for the Mt Isa excluded variables that would be required to extend existing phenomenological models to include the effect of reagent addition. Collecting data for the additional variables would require an extensive sampling campaign. Furthermore, the models used in the previous MPC strategy implemented at the Mt Isa site were linear empirical models derived from plant step tests. This provided a baseline for selecting variables and unit step response times for each variable that can be used to infer if the developed models correctly capture the rougher bank's dynamics. Additionally, the development of empirical models is less costly than the development of phenomenological models, and the model is simple enough to incorporate into an MPC strategy without extensive computational effort. Finally, extreme changes in feed can be accommodated with the development of a new empirical model, as the cost, time and effort to create a new empirical model is still significantly lower than the development of a phenomenological model that covers a wide range of ore compositions. The various types of empirical models that were considered for this dissertation are described in the following subsection.

2.7.1. Nonlinear Model Types

There are a vast number of nonlinear empirical models that can be used to represent systems. Many nonlinear model structures were historically based on the Volterra series. The Volterra series is an extension of the linear impulse response models, which is a linear convolution integral model (Billings, 2013). The discrete time case model structure is shown in Equation 38.

$$\begin{aligned}
 y(k) = & h_0 + \sum_{m_1=1}^M h_1(m_1)u(k-m_1) + \sum_{m_1=1}^M \sum_{m_2=1}^M h_2(m_1m_2)u(k-m_1)u(k-m_2) \\
 & + \sum_{m_1=1}^M \sum_{m_2=1}^M \sum_{m_3=1}^M h_3(m_1m_2m_3)u(k-m_1)u(k-m_2)u(k-m_3) + \dots \\
 & + \sum_{m_1=1}^M \sum_{m_2=1}^M \dots \sum_{m_n=1}^M h_n(m_1m_2 \dots m_n) u(k-m_1)u(k-m_2) \dots u(k-m_n)
 \end{aligned} \tag{38}$$

Where, $y(k)$ is the measured output, $u(k)$ is the measured input at time k , $h_n(m_1m_2 \dots m_n)$ represents the n th-order Volterra kernel and M is the model horizon. h_1 is a first-order kernel that represents the linear response of the system and is comparable to the impulse response model. h_2 is the second-order kernel that

represents the quadratic relationship between the variables. The higher-order kernels represent more complex nonlinear relationships between the variables (Ljung, 1987). Nonparametric methods, such as correlation and separable function methods, are commonly used for identifying Volterra series models and block-oriented models. Several disadvantages are associated with Volterra series models. Firstly, the Gaussian White noise may be required as an input for identification and may not be available with historical data. Secondly, the initial guess of the order is a complex process, as neglecting higher-order kernels can compromise model accuracy. Thirdly, higher-order kernels consist of a large number of terms which need to be identified. Thus, large datasets are required for the identification of models with a good estimate. The nonlinear kernels can be identified using other nonlinear model structures (He & Yang, 2021).

Another type of nonlinear empirical model is block-oriented models. These models consist of the Hammerstein and Wiener models. The Hammerstein models consist of static nonlinear blocks in series with a linear block. The Wiener model consists of a linear block followed by a static nonlinearity. The nonlinear block transforms the inputs using a nonlinear function, whereas the linear block transforms the inputs using a linear transfer function. The Hammerstein and Wiener models can be combined to form Hammerstein-Wiener models. The structure of these models is shown in Figure 2-8.

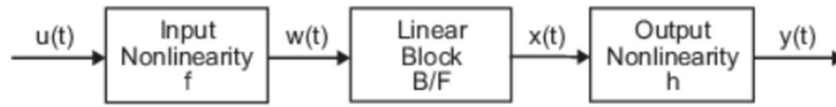


Figure 2-8 Hammerstein- Wiener Model structure (Ljung, 2014)

In Figure 2-8, f is the nonlinear function that transforms input data $u(t)$ such that $w(t) = f(u(t))$. $w(t)$ is the output of the nonlinear block. B/F is a linear transfer function that transforms $w(t)$ as $x(t) = (B/F)w(t)$. B and F are polynomials expressed in the time-shift operator q^{-1} . h is a nonlinear function that maps the output of the linear block $x(t)$ to the system output $y(t)$ as $y(t) = h(x(t))$ (Ljung, 2014). The nonlinear block is referred to as static as only the input at the current timestep is used in the function. The disadvantage of using these models is that the model structure assumes that the nonlinearities in the system are non-dynamic. This can lead to non-generic models, which may yield poor estimates for unseen data (Ljung, 1987).

The Nonlinear Autoregressive Moving Average Model with Exogenous Inputs (NARMAX) model was developed to overcome some of the shortcomings of the Volterra series (He & Yang, 2021). NARMAX models are parametric, and the structure is formatted as the nonlinear transformation of lagged inputs, outputs and noise sequences, also known as past prediction errors. The development of NARMAX models for systems with complex nonlinearities has fewer terms that need to be identified when compared to high-order Volterra series models, as NARMAX models include the lagged outputs in the structure. Thus,

NARMAX models require less computational effort than Volterra series models (He & Yang, 2021). The structure of an NARMAX model is expressed in Equation 39.

$$y(k) = f[(y(k-1), y(k-2), \dots, y(k-n), u(k), u(k-1), \dots, u(k-m), e(k-1), \dots, e(k-p))] + e(k) \quad (39)$$

Where $y(k)$ and $u(k)$ are the system output and input, respectively. $e(k-1), \dots, e(k-p)$ are the past noise sequences which capture the effect of previous disturbances on the system. n , m and p are the maximum lags for the system output, input, and past noise sequences respectively (He & Yang, 2021). Parametric models assume a distribution and use a fixed number of parameters to represent the distribution of the data. Whereas nonparametric models rely on the data to determine the distribution and trends between the variables, which can result in overfitting. Thus, parametric models are preferred for representing unseen data (Imam et al., 2024).

Another common dynamic empirical model that can capture complex nonlinearities of a system is neural networks. Several artificial neural networks were explored in the literature survey that were used to represent nonlinearities in the flotation system. The common types of neural networks that were used for flotation representation were the multilayer perceptron (MLP) and recurrent neural networks (RNN). Within the RNN category, the Long Short-Term Memory network (LSTM) and the Gated Recurrent Unit network (GRU) were frequently employed (Jovanović et al., 2022). Neural networks can produce high-quality prediction accuracy models. However, there is a trade-off between understanding the structure of the model and the interoperability of the nonlinearity effect on model performance. Furthermore, there is an increased risk of overfitting (Gu et al., 2019).

The choice of model structures for this work was dependent on ensuring a controlled experiment to compare the performance of linear and nonlinear controllers. Thus, the structures of the linear and nonlinear controllers must be analogously similar to ensure that the only significant difference is the representation of nonlinearity.

The literature review showcases that there are many developments of flotation empirical models that use reagent dosing to predict grade and recovery, and a small percentage of models that are nonlinear. The regression models outperform the neural network models in most cases. Furthermore, the review showcases that applications of MPC in reagent dosing control are relatively limited, and those that do exist typically employ regression-based models. This could be attributed to the lack of visibility and complexity with the utilisation of neural network models, and suggests that regression models are favourable in the implementation of flotation models within MPC. However, these models are generally linear and considered sufficient to meet control objectives. Literature showcases the exploration of nonlinear MPC for reagent dosing control; however, the potential benefits of using nonlinear models instead of linear

models are not explored. There has been no direct comparison of linear and nonlinear MPC strategies for reagent dosing control; therefore, the reliability of nonlinear models remains an area for further investigation.

The NARMAX model was chosen as a basis for the nonlinear model to be used in nonlinear MPC development. The reasons for choosing a NARMAX from the other empirical options available include the transparency of the model structure and the ability to interpret and select the most impactful terms. Additionally, earlier versions of parametric models, such as the Volterra series models, represent the noise as a Gaussian white noise variable input. This representation is restrictive, as the predicted models may be biased. However, the development of NARMAX models accounts for noise dynamics and requires the noise residuals in the output to be considered when estimating the model coefficients.

The NARMAX model structure was simplified for this dissertation to ensure a reduced number of parameters that need to be computed and to reduce computational effort. This simplified structure is known as the Nonlinear Autoregressive with Exogenous Inputs, NLarx model. The NLarx model is similar to the NARMAX model; however, the noise sequences are excluded from the structure. The NLarx model structure will be discussed further in section 3.2.2. A simpler structure also simplifies the analysis of the model.

The selection of linear model structures for this dissertation was considered in conjunction with the nonlinear model. The linear model structure used in the original MPC strategy was a finite impulse model. As mentioned above, the nonlinear model and linear model must be analogous to ensure a controlled comparison. Thus, the Autoregressive with Exogenous Inputs, ARX, model was chosen as this structure is the linear counterpart of the NLarx model structure. In addition to structural similarity with the NLarx model, the ARX was preferred over the finite impulse model as the number of parameters will be reduced, allowing for a concise model with a lower order (Juricek et al., 2002).

Along with ARX models, state-space models are commonly considered for control-oriented purposes. Earlier MPC control strategies frequently utilised input-output models such as polynomial ARX models. However, there was a large shift towards the use of state space as the structure can be used for MIMO control and address input correlation more effectively. Furthermore, the structure can be analysed to provide insight into stability, observability, and controllability. However, input-output models remain widely used as most systems are incapable of obtaining a full set of measurements to represent the entire state of the system that can be used in state space models (Wu, 2022). Additionally, there is extra computational effort and tuning complexity that occurs if an observer is used in conjunction with the MPC controller. The observer is used to estimate the full state of the system as the MPC algorithm is running. The simple structure of the ARX structure allows for less computational effort and control adaptability when there is a change in plant dynamics. The state space model was also chosen as the second linear model to be identified

in this dissertation. The ARX model will be compared to the state space model to assess if a simpler model structure compromises the model prediction quality.

3. CHAPTER 3: MODEL DEVELOPMENT

This Chapter focuses on the formulation and evaluation of linear and nonlinear empirical models. The models were developed in the MATLAB System Identification App using the Mt Isa Concentrator Dataset. Firstly, the structure and pre-processing of the model data will be explained. Three datasets were produced from data processing and used in the model development. These datasets were the reagent dataset, the total trend dataset and the Hybrid reagent and total trend dataset. The models were built using the State Space, ARX and NLarx model structures and had the Zn % and Pb % in concentrate as model outputs. Three sets of models were generated. The first model set was produced using the reagent dataset, which consisted of the reagent dosages and the Zn and Pb compositions in the rougher feed as inputs. These were referred to as Reagent Models. The second set of models was developed using the total trend dataset, which consisted of the inputs used in the reagent dataset and the Total Airflow rate and Level. These were referred to as Total Trend ARX Models. The third set of models developed was a combination of the models from the first and second sets, referred to as the Hybrid NLarx models.

The results obtained during the identification of empirical linear and nonlinear flotation models are also presented in this chapter. The model results consist of plots that compare the models to training and testing datasets and step response plots. This chapter also explores the reagent sensitivity of the collector and depressant dosage on Zn % in concentrate in the reagent models.

3.1. Model Data and Data Pre-processing

3.1.1. Process and Original MPC Design Overview

The data used for this study were collected from a lead-zinc concentrator at the Mount Isa Mines (MIM) site, a Glencore operation in Queensland, Australia. The concentrator processes an ore containing the zinc mineral, sphalerite, and lead mineral, amongst other minerals. Firstly, the ore is processed in grinding circuits containing rod and ball mills. The milling products are sent to the lead circuit, where Galena is floated through a series of roughers, SAG milling and cleaners (Price et al., 2018).

The tails from the lead circuit are fed to the zinc circuit for sphalerite flotation. The zinc flotation feed is first sent to a rougher bank to maximise zinc recovery. After that, the rougher concentrate is sent to a re-cleaner, where the resulting concentrate is the final zinc concentrate. The tails from the rougher are sent to the scavenger, where the scavenger recovers the remaining sphalerite. The concentrate from the scavenger is combined with the re-cleaner tails and sent to a cyclone, where the ultrafine particles or overflow are sent to a series of cleaners to produce a second zinc concentrate. The underflow from the cyclone is sent to an

Isa mill circuit for re-grinding, followed by a series of cleaners to produce a third concentrate. The zinc flotation circuit is seen in Figure 3-1.

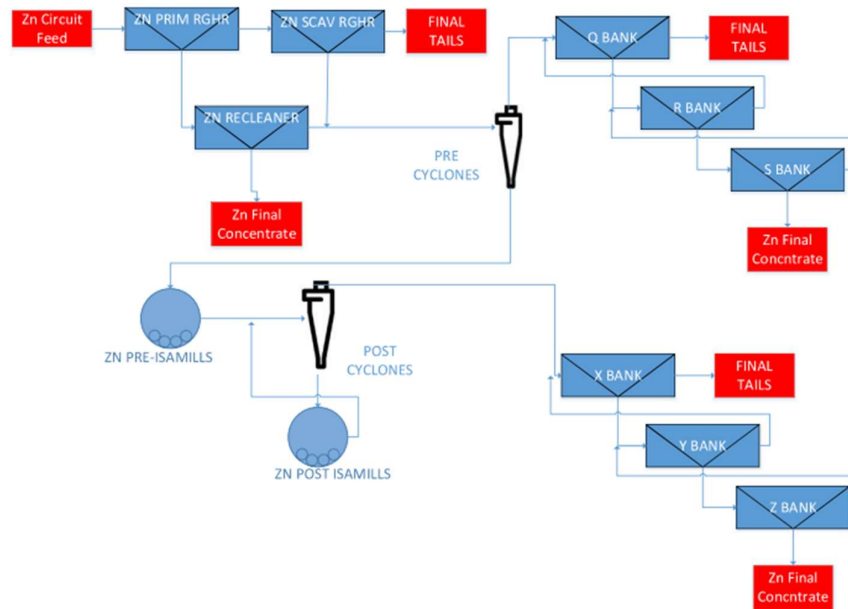


Figure 3-1 Zinc flotation circuit (Price et al., 2018)

The zinc rougher bank is the system that will be modelled in this dissertation. The rougher bank consists of 4 cells in series, as shown in Figure 3-2.

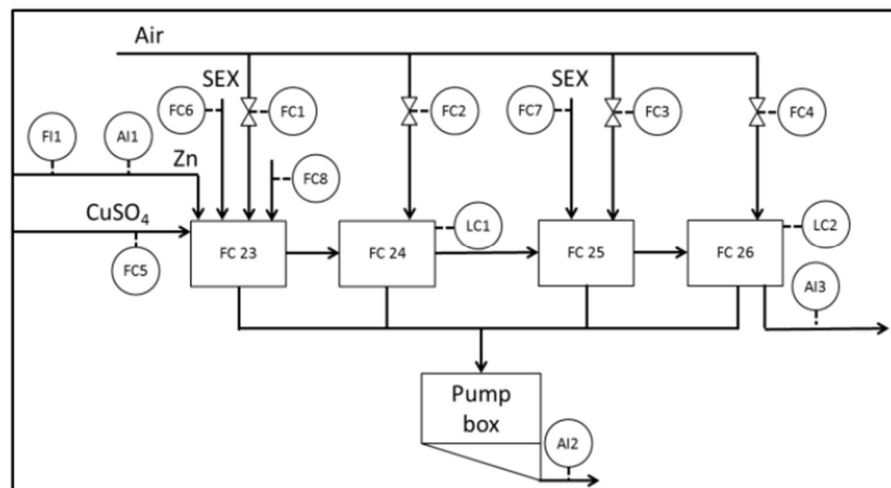


Figure 3-2 Zinc rougher bank process schematic (Brooks & Koorts, 2017)

The rougher cells are labelled FC23, FC24, FC25 and FC26. The reagents added upstream to facilitate separation were Copper sulphate (activator) and naphthalene sulphate (depressant). Methyl isobutyl carbinol (frother) was also added upstream. Sodium ethyl xanthate (SEX), a collector, is added to cells FC23 and FC25.

Levels are controlled per pair of cells via a dart valve, while air varies per cell. Composition measurements by X-ray fluorescence (XRF) are available for all feed, concentrate and tailings streams. LC1 and LC2 are PID controllers that influence the level of pairs of cells, and the liquid level is measured using an Ultrasonic measuring element. FC1 to FC4 are PID controllers for airflow rates, and a pitot measuring element is used to measure the airflow rate. FC5 to FC8 are reagent flow PID controllers. The SEX and CuSO_4 reagent flow rates are measured using a Coriolis measuring element, and the naphthalene sulphate flow rate is measured using a magnetic measuring element. FI1 is the volumetric feed flow rate measured using a magnetic flow meter. Analysers AI1 to AI3 measure zinc, lead, iron, and copper percentages and percentage solids in the feed, concentrate and tails, respectively and were used to prepare assays for performance analysis (Price et al., 2018).

Before APC implementation, the concentrate metallurgical performance results were analysed daily and compared to recovery and grade targets. Metallurgists determined the airflow, level and reagent dosing setpoints based on the zinc grade assays, and these setpoints were changed accordingly to meet targets. The setpoints for level and airflow were maintained by the controller using a wide operating range that was rarely changed. However, the reagent dosage setpoints had smaller ranges as the plant performance is highly sensitive to reagent dosages. The reagent flow rates were frequently changed to meet the desired grade and recovery of the bank.

An APC strategy was implemented on the Zinc flotation circuit (Brooks & Koorts, 2017). This APC strategy consisted of an MPC controller where the control objectives were:

- Control the overall collector (SEX) dosage and split between the cells
- Control activator (CuSO_4) and depressant (NS) dosage
- Maintain bank concentrate grades in a range

The optimisation objective of the MPC controller was to maximise the recovery.

The controlled variables from the Rougher Bank section for this implementation were as follows:

- Rougher Concentrate Zn Grade (%Zn)
- Rougher Tails Zn Grade (%Zn)
- Collector Dosage (L/T)
- CuSO_4 Dosage (L/T)

- Collector Dosage Split between cells FC 23 and FC 24
- Collector Dosage Split between cells FC 25 and FC 26

Whereas the manipulated variables were:

- Liquid Level in cell FC 24 (cm)
- Liquid Level in cell FC 26 (cm)
- Air flow rates to the cell FC 24, FC 25, FC 25 and FC 26 (m³/hr)
- Collector (SEX) dosing rate in cells FC 23 and FC 25 (l/min)
- Activator dosing rate (CuSO₄) in cell FC 23 (l/min)
- Depressant dosing rate (NASH) (l/min)

The disturbance variables were:

- Rougher Feed Density (t/m³)
- Rougher Feed Flowrate (l/s)
- Rougher Feed Zn Grade (% Zn)

These variables were used to create unit step response or finite impulse response models. The linear models are shown in Figure 3-3.

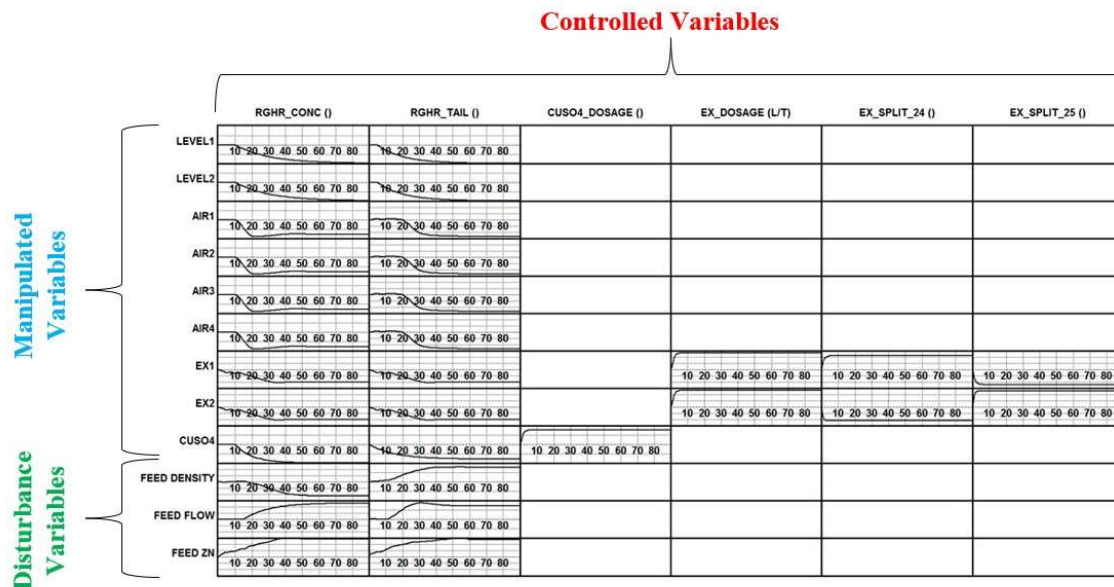


Figure 3-3 Mt Isa Zinc Rougher Bank Unit Sep Test Models (Brooks & Koorts, 2017)

The models were used by an MPC engine in AspenTech DMCPlus to predict the controller action that would produce the desired output.

The algorithm employed by MPC in DMCPlus is described in the section 2.6.

3.1.2. Data Collected

As part of the MPC project conducted at MIM, an extensive dataset was collected, containing both normal operational data and some periods in which planned steps were executed. For initial model development, a smaller dataset with a shorter period was extracted from the larger dataset. The data ranges in the new dataset have been scaled between zero and one hundred for confidentiality reasons. The extracted dataset is shown in Figure 3-4.

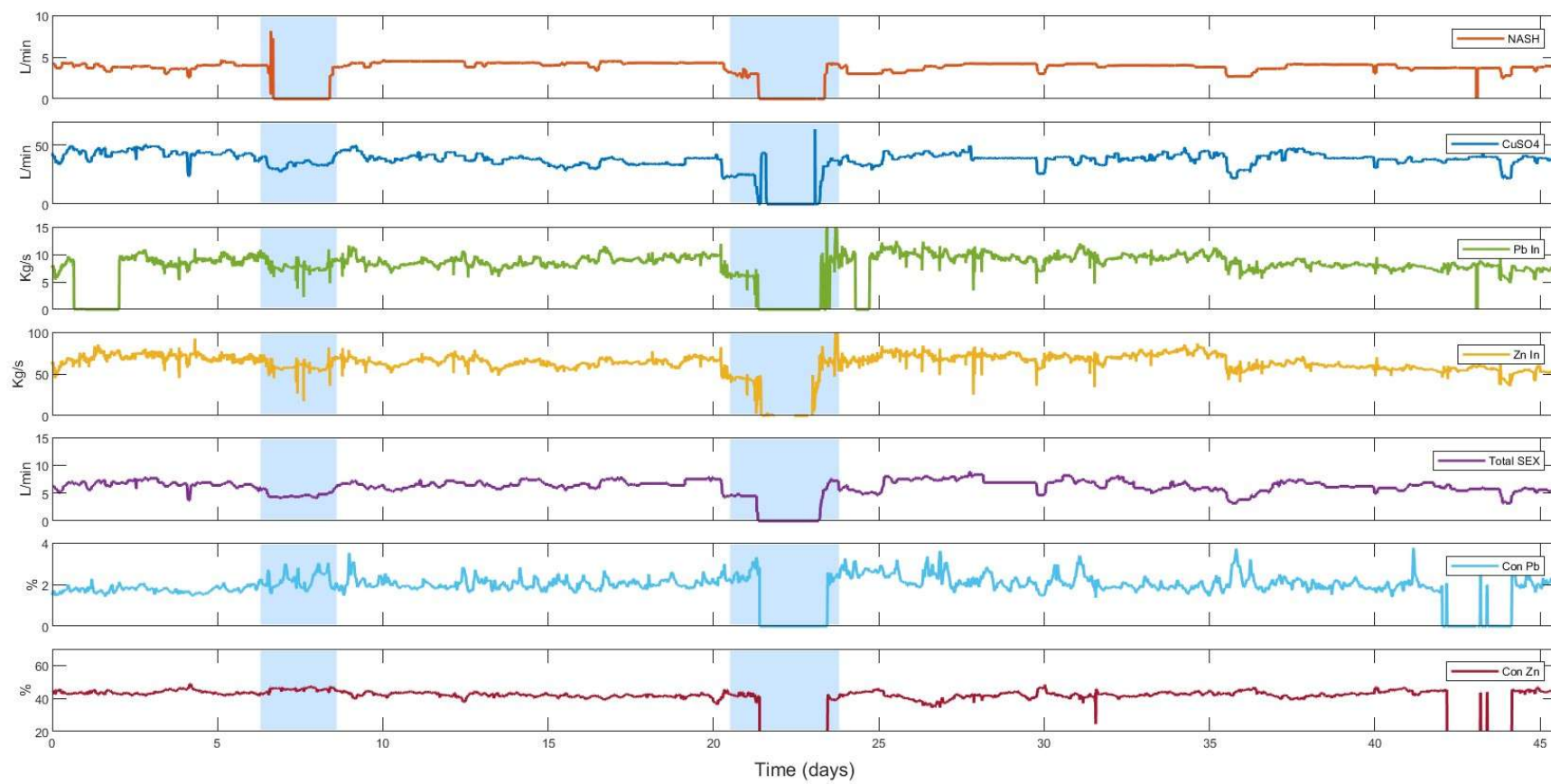


Figure 3-4, Reagent training and testing dataset

The data shown in Figure 3-4 was normalised, and the ranges were excluded from the y-axis for confidentiality purposes. The dataset consists of a total of 45 days of normal operation. Within the 45 days, periods of unsuitable data were excluded from further analysis as these periods included plant upsets and analyser failures. The unsuitable periods of data are shaded in blue. The airflow rates, level and reagent flow rates were recorded in the dataset with a sampling rate of 60s.

This period in the dataset was chosen due to the minimal variations in the setpoints of the level and air controllers. The significant reduction in the variation of the level airflow rate setpoints simplified the analysis and enabled a focused examination of how variations in reagent dosing directly impact the grades of zinc and lead. This smaller dataset will be referred to as the reagent dataset. The reagent dataset was divided into two sets: a training set and a test set. The training set is utilised during the learning process of model development. The data range of the training set extends from the start reagent dataset and ends at day 20 (start the second blue-shaded block). The test dataset is used to assess the model's performance with unseen data. The range of the test dataset starts from day 24 (the end of the second blue block) and ends at day 45.

A second set of models was required to represent the plant, which will be used in the simulation to compare the linear and non-linear MPCs. The dynamics of air and level changes were required to ensure that plant models closely mimicked the actual plant behaviour. Thus, the first dataset was extended to include data within a larger period and train a secondary set of models. Furthermore, the total air flow rates and level trends were included in the second dataset. This extended dataset consisted of a series of step tests performed on the airflow rates and pulp level. A total of 65 days of normal operation were collected. Similar to the reagent dataset, the data consisted of periods of unsuitable data that included plant upsets and analyser failures. These periods were removed from the dataset, resulting in a dataset of approximately 59 days. This dataset will be referred to as the total trend dataset and is shown in Figure 3-5.

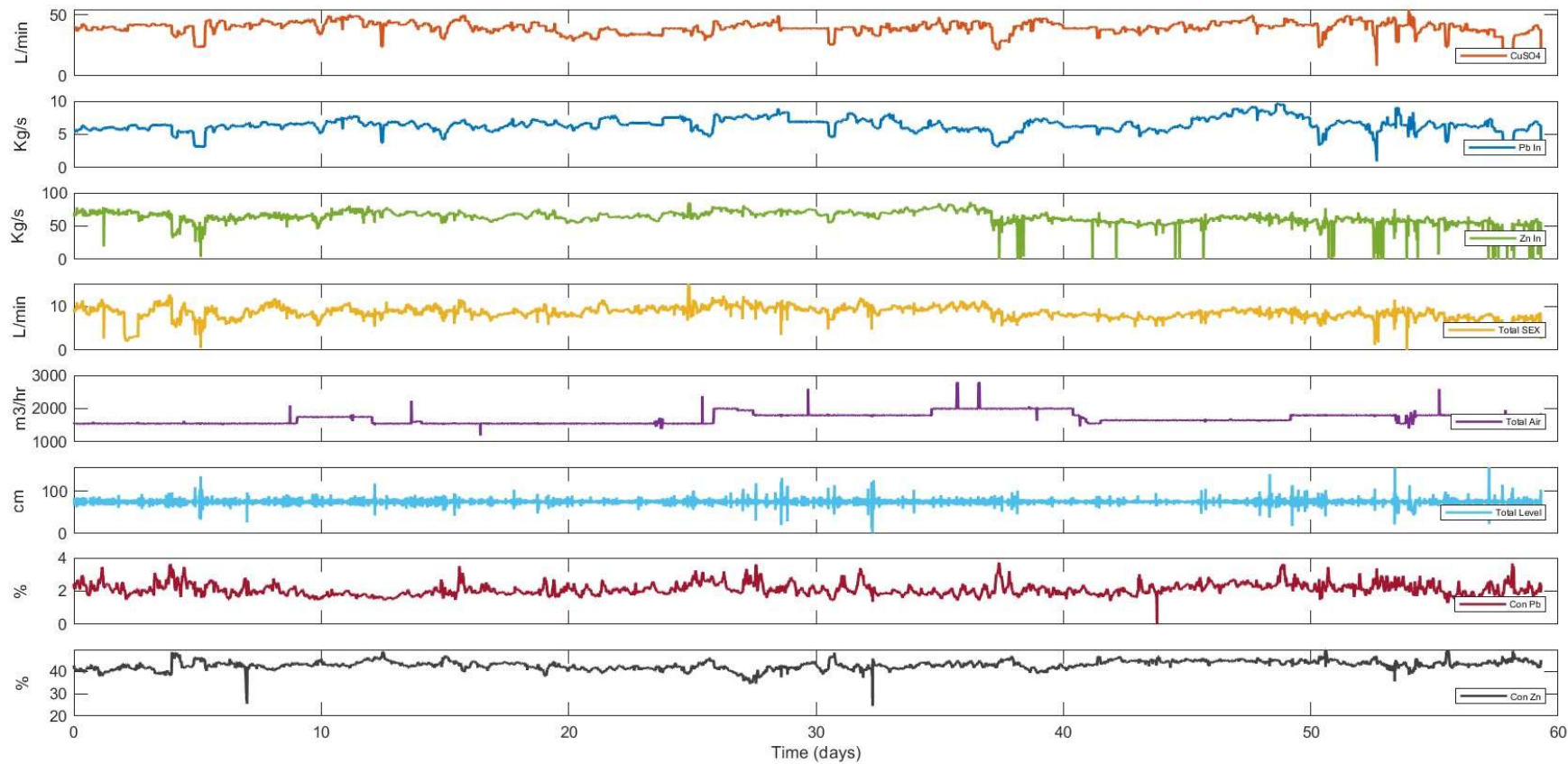


Figure 3-5 Total trend training and testing dataset

The dataset was split into two to create a training and a test dataset. The training data set consisted of approximately the first 39.5 days, and the test dataset consisted of approximately the last 19.5 days.

The rougher variables that were collected are shown in both datasets shown Table 3-1.

Table 3-1 Modelling Dataset Variables

Reagent Dataset		Total Trend Dataset	
Variable	Units	Variable	Units
NASH	l/min	NASH	l/min
CuSO4	l/min	CuSO4	l/min
Total_SEX	l/min	Total_SEX	l/min
Zn_In (Zn Flowrate In)	t/hr	Zn_In (Zn Flowrate In)	t/hr
Pb_In (Lead Flowrate In)	t/hr	Pb_In (Lead Flowrate In)	t/hr
Zn_Conc (Zinc grade in concentrate)	%	Zn_Conc (Zinc grade in concentrate)	%
Pb_Conc (Lead grade in concentrate)	%	Pb_Conc (Lead grade in concentrate)	%
		Total_Level	cm
		Total_Air	m3/hr

A third dataset, consisting of the first 25 days of the total trend dataset, was created and labelled the total trend and reagent dataset. This dataset was used to test the Hybrid NLarx models, which will be discussed in the following sections.

3.1.3. Data Pre-processing

The datasets shown in section 3.1.2 had undergone pre-processing.

Firstly, a centralised XRF is used to conduct online elemental analysis and operates by sampling multiple streams. Therefore, only a single stream can be analysed at a time, and a particular measurement may occasionally be unavailable. In these cases, the analyser reading is interpolated to prevent multiple breaks in the data.

The Zn in the feed stream data recorded by the XRF was noisy compared to other variables in the dataset. This variable was despiked and smoothed using an exponential filter. The exponential filter uses a

combination of weighted past predictions and the current value for the variable to predict the smoothed value.

After that, the feed flow, feed density and feed metal percentages were further analysed. From a modelling perspective, these variables could all be individual inputs as they are independent. However, from a physical point of view, these three variables may be combined to form a metal flow into the plant for both Zn and Pb. Combining these input variables into a single variable, referred to as feature extraction, allowed for a simplified model structure and quicker computational speed, as a smaller dataset without redundant data is utilised for model training. The formulation of the Zn and Pb feed flow rates is shown in Equation 40.

$$M_{in} = F\rho X \quad (40)$$

Where M_{in} is the mass flow of metal into the circuit (t/hr), F is the volume flow of slurry into the circuit (m³/hr) ρ is the density of the slurry (t/m³), X is the mass fraction of metal in the slurry.

Lastly, the variables used as inputs to the models were analysed to determine if any pairs of highly correlated variables existed. There are two SEX flow rates into the rougher bank, which are adjusted together. The two SEX flow rates are highly correlated as they are adjusted together. The high correlation of the two variables can make the interoperability of the model results difficult. Thus, the flow rates were added to form a single variable labelled Total_SEX. The pulp level per pair of cells and the airflow rates to the cells were also adjusted together. Therefore, the two pulp level measurements were combined into one variable labelled Total Level, and the 4 airflow rates were combined into a single variable labelled Total_Air.

3.2. Model Development

The literature survey showcased fundamentally based models that included reagent behaviour on metallurgical performance. After analysis of these models, the approach chosen was to derive empirical models based on the data's appropriate feature selection. This methodology is used for almost all industrial MPC work; its advantages are simplicity and cost, while the main disadvantage is that the results are not generalisable for cases where the feed ore grade is largely varied. The system identification toolbox in MATLAB was utilised to develop a series of multiple input-single-output linear and non-linear models of reagent behaviour. MATLAB was used as the modelling platform, as the software had a simple user interface to prepare the models and a wide variety of empirical model structures. Two parametric linear models were employed: state-space (SS) and autoregressive with exogenous inputs (ARX). The non-linear model used was a non-linear ARX (NLarx) model.

3.2.1. Model Development Environment

The models were developed using MATLAB's System Identification App. MATLAB is user-friendly and consists of a control system toolbox that makes it easier to pair the internal models with the controllers compared to other modelling tools. The system identification toolbox was used as it provided various dynamic empirical model structures to develop the models. The system identification toolbox allowed time series analysis and the choice between discrete and continuous time modelling. The types of model structures included in the app include transfer functions, state space, process models, polynomial models, nonlinear ARX models and Hammerstein-Weiner models. The System Identification interface is shown in Figure 3-6.

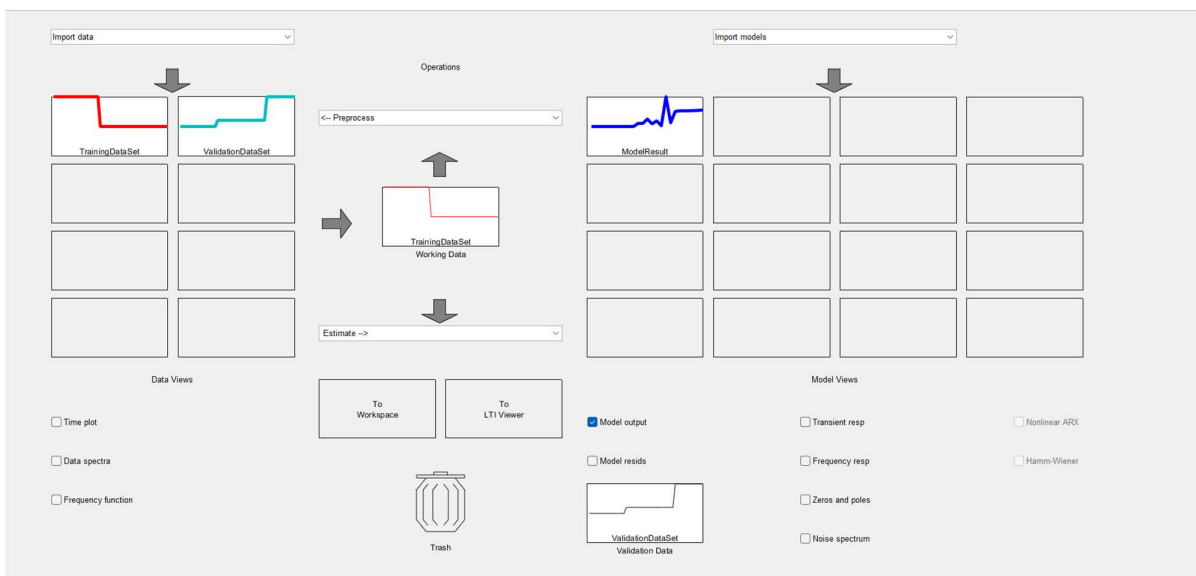


Figure 3-6 System Identification App

The training and test datasets are imported from the MATLAB workspace into the App as time series objects with a sampling rate of 60 seconds. Once in the App, the dataset plots can be viewed by selecting the time plot feature when the desired dataset is highlighted. The datasets used for training the model are dragged to the working data block.

The type of model that can be estimated is chosen from the drop-down list labelled estimate. Each model structure has a unique set of meta-parameters that requires the user to allocate values for those meta-parameters. The app provides a set of recommended values for certain meta-parameters that optimise curve fitting with the selected dataset. The model is estimated based on the given meta-parameters and is stored in the model views section.

A model can be selected from the model view section, and the model output is evaluated using the dataset in the validation data. The model output checkbox is selected to compare the model outputs to the measured

dataset trends. The model output is plotted over the dataset values in the model output window. The model output for the reagent and total trend training and test datasets was visualised in the Model Output window.

3.2.2. Model Structures

State Space

The state space models represent a system via state variables used in first-order differential (continuous time) or discrete-time equations. The state variables are grouped to form state vectors that represent the state of the system and summarise the system's history. The state space structure for a discrete-time model is represented by Equations 41 and 42.

$$x(k + 1) = Ax(k) + Bu(k) + Ke(k) \quad (41)$$

$$y(k) = Cx(k) + Du(k) + e(k) \quad (42)$$

Where one sampling interval of 60 seconds is used for each time instant k , $x(k)$ is the state vector at the time k of length n where n corresponds to the number of state variables. Further, $u(k)$, $e(k)$ and $y(k)$ are the input, white-noise disturbance and output vectors, respectively, at the time k . The input vector, $u(k)$, has a length m corresponding to the number of inputs and the output vector, $y(k)$, has a length p corresponding to the number of outputs. The remaining elements are the state transition matrix, $A \in \mathbb{R}^{n \times n}$, the input matrix, $B \in \mathbb{R}^{n \times m}$, the output matrix, $C \in \mathbb{R}^{p \times n}$, the feed-forward matrix, $D \in \mathbb{R}^{p \times m}$ and the disturbance matrix, K . The parameter matrices are A, B, C, D and K with dimensions that are based on the chosen model order. The model order corresponds to the number of states and corresponds to the state vector length, n . Free parameterisation determines the values within the parameter matrices during model estimation. Thus, all parameter values are chosen automatically based on the selected estimation method. The Noniterative subspace method, N4SID, was used as the estimation method.

Additional information on how the N4SID method is used to find the model order and parameter matrices is found in Appendix A.

The state space model structure above can be used for a multiple-input, multiple-output system. However, the function that determines the optimal meta-parameters for the models to follow (ARX and NLarx) cannot be used for multiple-output systems. Thus, multiple-input, single-output State Space models were developed for each output.

Autoregressive with exogenous inputs (ARX)

The ARX is a regression polynomial model representing the relationship between inputs, outputs and noise and is derived from transfer functions. The model is a combination of an autoregressive and an exogenous model. The structure of a Single Input, Single Output ARX model is defined by Equation 43.

$$A(q)y(k) = B(q)u(k - nk) + e(k) \quad (43)$$

Where $u(k)$ and $e(k)$ represent the output and the noise at the time k . u is the input and nk is the input delay. $A(q)$ and $B(q)$ are polynomials that contain parameters that represent the system dynamics and are defined by Equations 44 and 45.

$$A(q) = 1 + a_1q^{-1} + \dots + a_{na}q^{-na} \quad (44)$$

$$B(q) = b_j + b_{j+1}q^{-1} + \dots + b_{nb}q^{-nb+1} \quad (45)$$

Where q^{-1} is a delay operator ($q^{-1}u(k) = u(k - 1)$), the variables a_i and b_i with $i = 1, \dots, na$ and $j = 1, \dots, nb$ are the internal parameters representing the system dynamics. na refers to the number of poles and nb refers to the number of zeros plus one. na , nb and nk are used to determine how the inputs and outputs are delayed (Rachad et al., 2014). Equation 43 can be expanded to Equation 46 using the delay operators.

$$y(k) + a_1y(k - 1) + \dots + a_{na}y(k - na) = b_1u(k - nk) + \dots + b_{nb}u(k - nk - nb + 1) + e(k) \quad (46)$$

The model structure can be amended to a form where the current output is calculated as a summation of the delayed inputs and outputs, also known as the linear regressors and is shown in Equation 47.

$$y(k) = \sum_{i=1}^{na} a_i y(k - i) + \sum_{j=1}^{nb} b_j u(k - j - nk + 1) + e(k) \quad (47)$$

The values of na , nb and nk are meta parameters. The optimal meta parameters were determined by comparing the fit of models with different combinations of meta parameter values. The process of comparing the meta-parameter values is described in the model meta-parameters section.

The models developed were multiple-input models. The ARX structure shown above can be amended to include multiple inputs and is shown in Equation 48.

$$A(q)y(k) = B_1(q)u_1(k - nk_1) + B_2(q)u_2(k - nk_2) + \dots + B_{nu}(q)u_{nu}(k - nk_{nu}) \quad (48)$$

Where nu is the n -th input. The output value is computed via a summation of the input variables and their corresponding delays, in addition to the output's delays. Each input variable has a unique set of meta-parameters nb and nk , that are dependent on the system's input dynamics.

Nonlinear Autoregressive with Exogenous Inputs (NLarx)

A NLarx model is an amended ARX model. The ARX model represents the linear mapping of inputs and outputs as a summation of regressors shown in Equation 46. However, NLarx models use regressors in non-linear mapping to estimate a result, as shown in the block diagram shown in Figure 3-7.

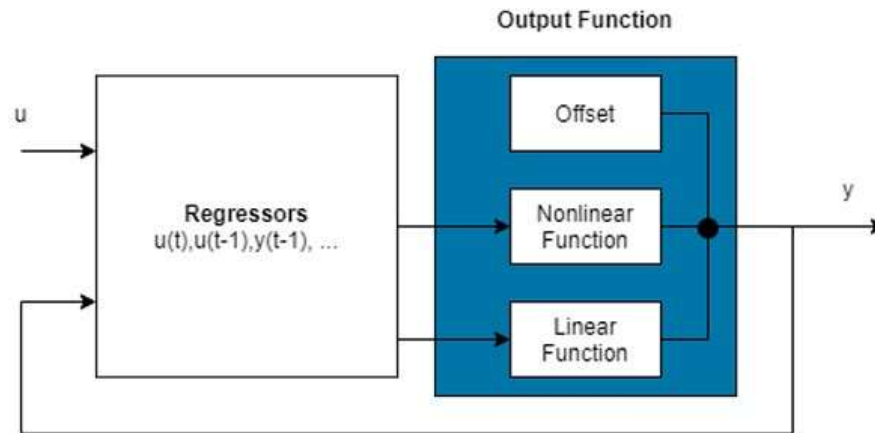


Figure 3-7 NLarx Non-linear mapping (Ljung, 2014)

The non-linear mapping is carried out in two steps (Ljung, 2014):

1. Determine the regressors from the current input and past output data: In NLarx models, custom regressors can be computed in addition to the linear regressors from ARX models. Custom regressors are non-linear functions of linear regressors, typically polynomial functions, periodic functions, or the multiplication of linear regressors with one another.
2. Assign regressors to output blocks: Non-linear mapping occurs by assigning regressors to a linear function, a non-linear function, or a combination of both in the output blocks shown in Figure 3-7 when computing the output. Subsets of regressors can be developed to specify which regressors report solely to the linear function and which regressors report solely to non-linear functions. A model may require only custom regressors assigned to the non-linear function and linear regressors assigned to both linear and non-linear functions. Non-linear estimators include the wavelet, sigmoid, tree partition, and multi-layered neural networks (Ljung, 2014).

The structure of a SISO NLarx model is shown in Equation 49:

$$y(x) = L^T(x - r) + d + g(Q(x - r)) \quad (49)$$

Where $y(x)$ is the output estimated from non-linear and linear functions, x is a vector of regressors, r is the mean of the regressors, and d is a scalar offset. $L^T(x - r)$ is the linear output portion calculated as a weighted sum of linear regressors and is a compact format of the linear ARX structure in Equation 46. $g(Q(x - r))$ represents the output of the non-linear function where g is the non-linear estimator Q is a projection matrix to ensure the result is well-conditioned. In this equation, x represents the regressors for the input and output variables. The linear function and the nonlinear function do not use the regressors directly. The regressors are first centred around the mean of the regressors. Thus, the inputs used for each function are represented as $(x - r)$ (Aldemir et al., 2015).

There are various types of nonlinear basis functions that are recommended for use in NLarx models. These nonlinear functions include the following categories: Polynomial Basis Function, Radial Basis Function (RBF), Wavelet Basis Function, Sigmoid Basis Function, Piecewise Linear Basis Function, Spline Basis Function, Fuzzy Basis Function, etc. (Sjoberg et al., 1995). The available nonlinear functions in MATLAB that can be used within the NLarx structure include Wavelet Networks, Sigmoid Networks, Tree Partition, Neural Network, Custom Network, Gaussian Process, Regression Tree Ensemble and Support Vector Machine. As an initial approach, the Wavelet Networks, Sigmoid Networks and the Tree Partition (Radial Basis Function) nonlinear functions were considered as options for the NLarx model identification. These functions correspond with basis functions recommended for use and are simple for interpretation. The Wavelet Network and Tree Partition functions contain wavelet unit functions and unit tree nodes, respectively, that utilise a radial combination of inputs, whereas the Sigmoid Network contains sigmoid unit functions that employ a ridge combination of inputs. The model performance for each nonlinear structure was evaluated to determine the nonlinear function that was used for the remainder of the model development process. Models were trained for a variation of unit amounts. A range of 1 to 15 units was explored for the Wavelet and Sigmoid Network NLarx models. Whereas a range of 1 to 1000 units with intervals of 100 units was used for the Tree Partition NLarx models. A larger unit amount was used for the Tree Partition function, as a difference in model performance is only observed at higher unit amounts. The models were trained using the reagent dataset. The performance was evaluated by calculating the Normalised Root Mean Squared Error, NRMSE, between the measured and predicted data for each model. The Sigmoid Network models produced the lowest NRMSE values; thus, this nonlinear function was chosen for further identification. The nonlinear function comparison function results are further discussed in the model comparison results section. The sigmoid network with 10 units produced the lowest NRMSE value; therefore, the sigmoid network was used as a non-linear estimator for further NLarx model development. The nonlinear sigmoid function is the summation of the dilated and translated sigmoid functions of the regressor matrix and is shown in Equation 50.

$$g(x) = \sum_{k=1}^n s_k f(X^T Q b_i + c_i) \quad (50)$$

Where $s_1, s_2 \dots s_n$ are scalar weights called output coefficients, $\kappa(s) = (e^s + 1)^{-1}$ are the sigmoid functions, X^T is the transpose of the regressor matrix, Q is the projection matrix, $b_1, b_2, \dots, b_n$ is a vector of dilation coefficients, $c_1, c_2, \dots, c_n$ are translations and n refers to the number of units in the sigmoid network. b_i and c_i consist of scalar values obtained during the model's training and are responsible for scaling and dilating the input used in the sigmoid function. $(X^T Q b_i + c_i)$ is responsible for the linear transformation of the regressor matrix into a scalar that represents the output value for each sigmoid unit. The sigmoid units are then added together to produce a nonlinear function output value. The scalar weights indicate the contribution and amplitude of each sigmoid unit to the overall nonlinear function output value.

The estimation method used to determine the parameters in the model structure was a combination of the following methods: Subspace Gauss-Newton least squares search, Adaptive subspace Gauss-Newton search, Levenberg-Marquardt least-squares search, and the Steepest descent least squares search. The search method alters every iteration, and the choice is based on the search method that finds the first descent direction to minimise the sum of squared errors between the model trend and the actual data. The parameters that were estimated using these iterative methods include: the scalar weight of the linear function and the scalar weights, projection matrix, dilation coefficients and translations of the sigmoid function. The estimated parameter values can be found in Appendix B.

As a multiple-input model needs to be developed, Equation 49 was amended to include regressors for each input and output variable mapped to the linear and non-linear functions. The regressors were combined into the vector $(x - r)$. Furthermore, the regressors used for the NLarx model were linear regressors and custom regressors were not chosen. The meta-parameters from the ARX model were used for the NLarx model development to ensure the structure of the NLarx is analogous to the ARX model. The value of the meta-parameters was used to determine the amount and the delay of the regressors.

3.2.3. Models Synthesis

Model development took place in three stages.

The first stage was used to compare the model fits for each model structure, where the influence of reagent addition on the Zn_Conc and Pb_Conc outputs was solely considered. These models were developed using the state space, ARX and NLarx model structures. The models in this phase were sub-categorised into two sets. The first set of models excluded NASH, and the second set included NASH. Each set includes two models per structure, i.e. 2 state space models, 2 ARX models and 2 NLarx models. The two models per structure include a model to determine the Zn_Conc output and a model to determine the Pb_Conc. A total

of 12 models were developed in this stage. The models were trained and tested using the reagent dataset. The models with NASH were compared to the models without NASH. The models with the best fit were used in phase 3 of model development. The best fit was determined using the normalised root mean squared error and the correlation coefficient.

The second stage was used to develop models that will be used to represent the plant in the simulated system. The models developed in the second stage included the effect of total air flow rates and the combined level measurement of the flotation cells, as the original MPC implementation included these variables as inputs to the models and were proven to have a linear relationship with the Zinc and Lead concentrate grades. Secondly, the effect of reagent dosages on the Zinc and Lead concentrate grade in the presence of the effects of Air flow rates and pulp levels is explored as flotation cell control systems operate with these manipulated variables in conjunction with reagent addition and are amongst the key variables that are considered in froth flotation modelling. Furthermore, these measurements are available in the original dataset. Two ARX models were developed using the additional Total air flow rates and the combined Level measurements. This consisted of a model to determine the Zn_Conc output and a model to determine the Pb_Conc. ARX model structure was chosen as the model structure, which is the baseline for the NLarx models and can be combined with pre-determined NLarx models to form new hybrid models. These models were trained and tested using the Total_Trend Dataset.

The third stage was used to develop two non-linear hybrid models that will be compared to the linear ARX models developed in phase 2. The NLarx model from the set with the best fit in Phase 1 was amended using the ARX model from Phase 2. The NLarx model structure consists of both linear and non-linear mapping of regressors, as mentioned in 3.2.2. The linear mapping of regressors in NLarx models is a weighted summation of the regressors and is identical to the ARX model structure. The Total airflow rate and combined Level measurement regressors and their respective weights from the ARX model in phase 2 were added to the NLarx linear mapping structure. The hybrid NLarx model did not use non-linear mapping for the Total_Air and Total_Level variables, as the relationship between the concentrate grades and the inputs, Total Air flow rate and combined Level measurement, has been sufficiently represented using a linear model in the original MPC implementation. However, the relationship between the concentrate grades and the reagent flow rate contains regions where the relationship cannot be represented linearly. These models were tested against the reagent and total trend dataset.

A summary of the models developed is shown in Table 3-2.

Table 3-2 Model Development Summary

	No NASH Models			NASH Models		
	Inputs	Zinc MISO Models	Lead MISO Models	Input	Zinc MISO Models	Lead MISO Models
Reagent Dataset	<u>Inputs:</u>	Zinc Concentrate No NASH Reagent Model (State Space)	Lead Concentrate No NASH Reagent Model (State Space)	<u>Inputs:</u>	Zinc Concentrate NASH Reagent Model (State Space)	Lead Concentrate NASH Reagent Model (State Space)
	CuSO4			NASH		
	SEX			CuSO4		
	Zn_In	Zinc Concentrate No NASH Reagent Model (ARX)	Lead Concentrate No NASH Reagent Model (ARX)	SEX	Zinc Concentrate NASH Reagent Model (ARX)	Lead Concentrate NASH Reagent Model (ARX)
	Pb_In	Zinc Concentrate No NASH Reagent Model (NLarx)	Lead Concentrate No NASH Reagent Model (NLarx)	Zn_In		
				Pb_In	Zinc Concentrate NASH Reagent Model (NLarx)	Lead Concentrate NASH Reagent Model (NLarx)
Total Trend Dataset	<u>Inputs:</u>	Zinc Concentrate Total Trend Model (ARX)	Lead Concentrate Total Trend Model (ARX)			
	CuSO4					
	SEX					
Hybrid Reagent and Total Trend Dataset	Zn_In	Zinc Concentrate Hybrid Model (NLarx)	Lead Concentrate Hybrid Model (NLarx)			
	Pb_In					
	Total_Level					
	Total_Air					

3.2.4. Model Meta-Parameters

Each model structure requires pre-defined meta parameters during identification. The State Space model requires the optimal model order, which was determined by assessing the performance of different state space models for a range of orders using the N4SID method. The contribution of each state variable to input-output behaviour is calculated for each model order. The log of singular values for the covariance matrix is used as a representation of the state variable's contribution to the input-output variable. The model order that has the highest log of singular values was used for further identification.

The optimal ARX model meta parameters na , nb and nk , were determined by calculating the output variance for different combinations of meta parameter values. Unexplained output variance is the portion of the output not represented by the model. The sum of the squared errors was also calculated and analysed to determine the values of meta parameters that would produce the best fit. A least-squares estimation problem was solved with an overdetermined set of linear equations using QR decomposition. Additional information on how the least-squares estimation method is used to find parameters can be found in Appendix A. The optimal combination of meta-parameters is chosen by considering values that produce the lowest unexplained output variance and the best fit.

The optimal meta-parameters used in the Reagent models and Total Trend models are shown in Table 3-3 and Table 3-4.

Table 3-3 Reagent Model Meta-Parameters

No NASH Models		
	Pb_Conc	Zn_Conc
State Space; n = model order	$n=3$	$n=3$
ARX; $na, nb = [...]$, $nk = [...]$	$na=9$ $nb = [2 \ 1 \ 1 \ 3]$ $nk = [7 \ 10 \ 10 \ 8]$	$na=4$ $nb = [5 \ 10 \ 2 \ 6]$ $nk = [5 \ 1 \ 9 \ 8]$
NLarx; n = Number Sigmoid Unit	ARX meta-parameter $n=10$	ARX meta-parameter $n=10$
NASH Models		
	Pb_Conc	Zn_Conc
State Space; na	$na=3$	$na=3$
ARX; $na, nb = [...]$, $nk = [...]$	$na=9$ $nb = [1 \ 2 \ 1 \ 1 \ 3]$ $nk = [10 \ 7 \ 10 \ 10 \ 8]$	$na=4$ $nb = [2 \ 5 \ 10 \ 2 \ 6]$ $nk = [1 \ 5 \ 1 \ 9 \ 8]$
NLarx; n = Number Sigmoid Unit	ARX meta-parameter $n=10$	ARX meta-parameter $n=10$

Table 3-4 Total Trend Model Meta-Parameters

	Pb_Conc	Zn_Conc
ARX; $na, nb = [...]$, $nk = [...]$	$na = 2$ $nb = [2\ 2\ 2\ 2\ 2]$ $nk = [1\ 1\ 1\ 1\ 1]$	$na = 2$ $nb = [2\ 2\ 2\ 2\ 2]$ $nk = [1\ 1\ 1\ 1\ 1]$

3.2.5. Model Comparison

The Model Curve Fitting results were compared using the Normalised Root Mean Squared Error (NRMSE) and the correlation coefficient, R^2 . The formulas used to calculate these statistics are shown in Appendix B.

3.2.6. Step Response Plots

The input variable step response plots for all the models mentioned in 3.2.3 were constructed using the Simulink setup found in Appendix B. Each input variable was stepped up by one unit from the steady state value.

3.3. Nonlinear Function Comparison Results

The model performance of each nonlinear function for a variation of unit amounts was assessed. The identification of the models with the different nonlinear functions was completed for the NASH and No NASH variables. The unit amount that produced the lowest NRMSE for each nonlinear structure is shown in Table 3-5.

Table 3-5 NLarx Nonlinear Function Comparison

Model	Sigmoid Network		Wavelet Network		Tree Partition	
	Num Units	NRMSE	Num Units	NRMSE	Num Units	NRMSE
Zn No NASH Model	10	4.43	9	7.92	300	4.92
Pb No NASH Model	7	16.84	8	15.80	300	8.13
Zn NASH Model	10	6.86	12	7.61	300	6.89
Pb NASH Model	4	15.76	11	16.36	300	7.12

The NRMSE is lowest for the Zn models when the sigmoid network is utilised. However, the tree partition function produces the lowest NRMSE for the Pb models. The sigmoid network was used for the remainder of the nonlinear model development, as the Zn model accuracy is prioritised. Furthermore, the computational effort to execute the tree partition model with a low NRMSE is high as the number of units exceeds 100. The full set of NRMSE values is shown in Appendix B. The remaining nonlinear basis functions mentioned in the Model Structure section were not explored for further analysis. Although these functions could potentially improve the model fit, the sigmoid function is sufficiently accurate for achieving the objectives of this study.

3.4. Model Comparison Results

3.4.1. Zinc and Lead Concentrate Reagent Model Results

The results of fitting the reagent dataset using ARX, state space and non-linear ARX models are shown in Figure 3-8 to Figure 3-11 and were previously published in IFAC-PapersOnLine (Brooks et al., 2023). The values were omitted from the plots for confidentiality reasons. Figure 3-8 and Figure 3-9 illustrate the curve fitting results of the train and test data of the reagent dataset, respectively, where NASH was excluded. Figure 3-10 and Figure 3-11 represent the train and test data sets where NASH was included.

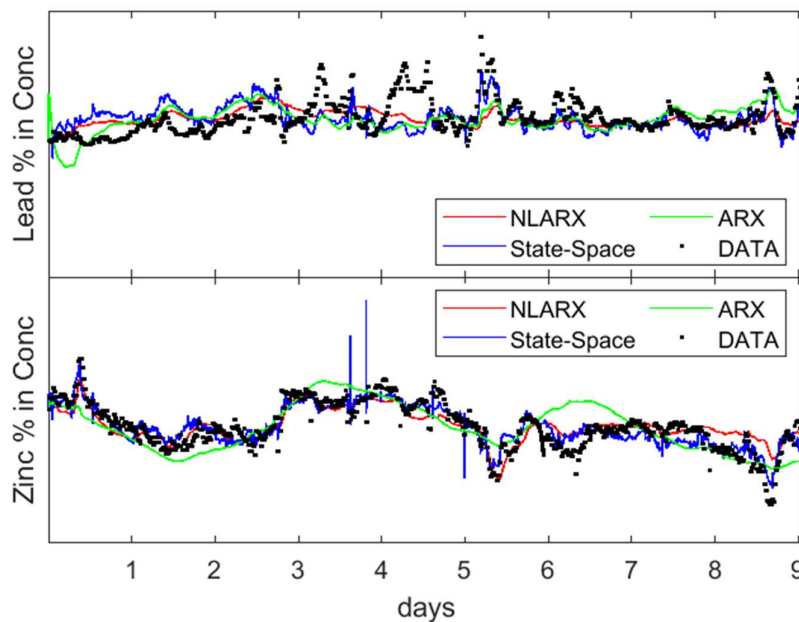


Figure 3-8 Models compared to training data, excluding NASH

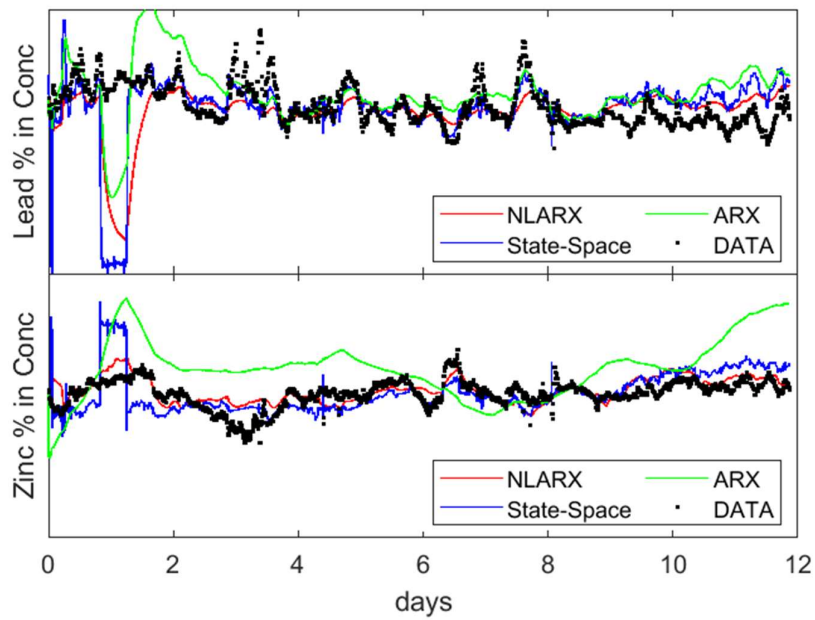


Figure 3-9 Models compared to test data, excluding NASH

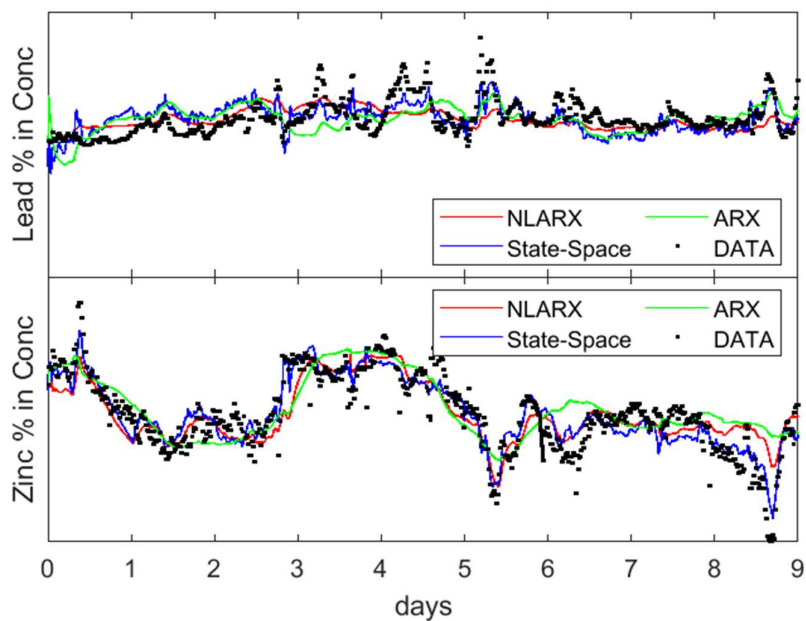


Figure 3-10 Models compared to training data, including NASH

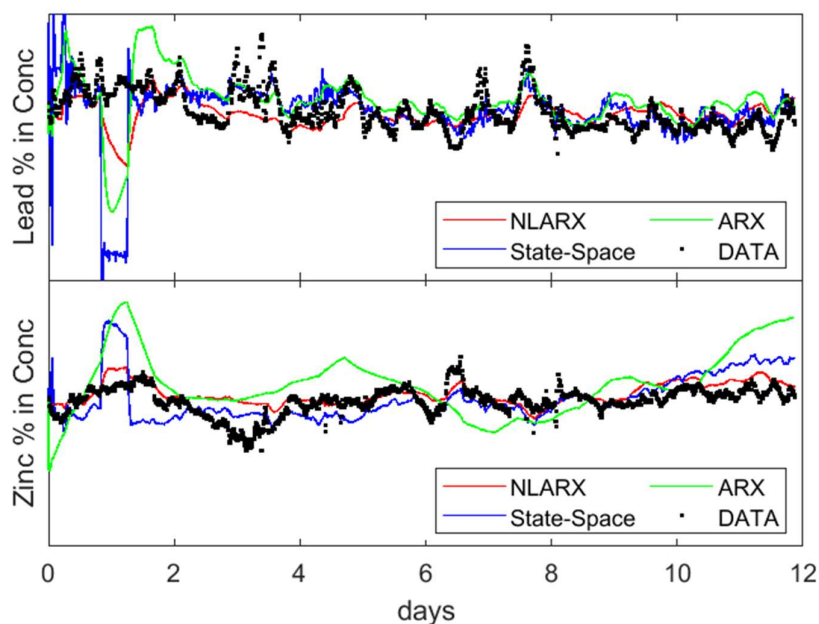


Figure 3-11 Models compared to test data, including NASH

The NRMSE and the correlation coefficient R^2 of the curve fitting for the reagent dataset is shown in Table 3-6.

Table 3-6 NRMSE and correlation coefficient R^2 results for the reagent dataset Models

Variable	Without NASH model		With NASH model	
	NRMSE R^2 Train	NRMSE R^2 Test	NRMSE R^2 Train	NRMSE R^2 Test
Con.Zn (SS)	1.9% 0.77	6.6% 0.22	1.8% 0.80	7.3% 0.17
Con.Zn (ARX)	3.3% 0.45	12.9% 0.05	2.9% 0.51	11.2% 0.12
Con.Zn (NLarx)	2.3% 0.69	4.4% 0.38	2.3% 0.67	4.9% 0.35
Con.Pb (SS)	17.0% 0.09	16.3% 0.00	13.4% 0.37	15.5% 0.02
Con.Pb (ARX)	17.7% 0.07	24.0% 0.06	16.8% 0.12	19.5% 0.09
Con.Pb (NLarx)	15.9% 0.10	19.6% 0.01	15.1% 0.20	15.5% 0.06

Table 3-6 illustrates that the fits for the Zinc % in Conc in the training set are all good for Models excluding and including NASH, where the NRMSE ranges between two and three percent. The models excluding and including NASH training sets' NRMSE values for the Pb % in Conc fits are greater than the Zn% in Conc fits NRMSE values. The training set NRMSE values for the Pb % in Conc fits range from 13% to 17%.

The test sets for the models excluding and including NASH exhibit a similar trend, where the NRMSE values for the Pb % in Conc fits are greater than the Zn% in Conc fits NRMSE values. The Pb % in Conc fits is poorer, possibly because lead is present in smaller concentrations than the zinc in the concentrate; thus, the error in the measurement is significant.

The test set NRMSE values are generally higher than the training set NRMSE values, which is expected as the model structures fit against unseen data when using the test data. The Zn % in Conc NRMSE values are lowest when the state space models are fitted against the training set, when excluding and including NASH. However, the Zn % in Conc NRMSE is lowest when the NLarx models are fitted against the test dataset when excluding and including NASH. Therefore, NLarx models are superior at modelling unseen data. This is further supported by the correlation coefficient values for test data Zn % in Conc fits, as the correlation coefficient is highest with the NLarx models. Thus, a larger proportion of the variation in the Zn % in Conc is attributed to the variation of input variables when the NLarx model is fitted against unseen data. The NLarx models illustrate the poorest Zn % in Conc fits as the NRMSE are the highest, and the correlation coefficients are the lowest compared to the state space and the NLarx models.

The Pb % in Conc curve fitting NRMSE and correlation coefficient results do not provide a clear distinction as to which model structure has the best fit. This is evident as the NLarx model has the best fit with the lowest NRMSE and highest correlation coefficient on the training dataset without NASH. However, the steady-state model has the lowest NRMSE with the test dataset. This contradicts the correlation coefficient, as the steady-state model has the lowest correlation coefficient compared to other model structures. Thus, 0 % of the variation in the Pb % in Conc is attributed to the variation in the input variables. Furthermore, the ARX model structure has the largest NRMSE and correlation coefficient with the test dataset. The models that include NASH do not contain similar trends for Pb % in Conc fits when comparing the models' structures. The Steady-state model has the best fit with the lowest NRMSE and highest correlation coefficient on the training dataset with NASH. The NRMSE values for the NLARX and the steady state for the test data set are comparable. As there was no similar trend in the statistics for the Pb % in Conc fits when comparing the model structures between the models that included NASH and the models that excluded NASH, the Zn % in Conc fits statistics were used as a basis for deciding which model structure performed the best and whether the inputs to the model should include NASH.

When the difference between the NRMSE values of the training dataset and the test dataset for the Zn % in Conc is compared, the models that exclude NASH have a smaller difference than those that include NASH. The models without NASH have less overfitting as the RMSE increases slightly when moving from the training to the test dataset. Thus, the model structures without NASH were explored for the remainder of the research. Overall, the NLARX structure performs the best with unseen data, as this structure has the lowest RMSE values and the highest correlation coefficients with test datasets. Thus, the NLARX structure

without NASH was explored further in a simulation where the NLARX structure was used in a nonlinear MPC.

The Pb % in Conc and Zn % in Conc step response plots for the reagent dataset are shown in Figure 3-12 and Figure 3-13.

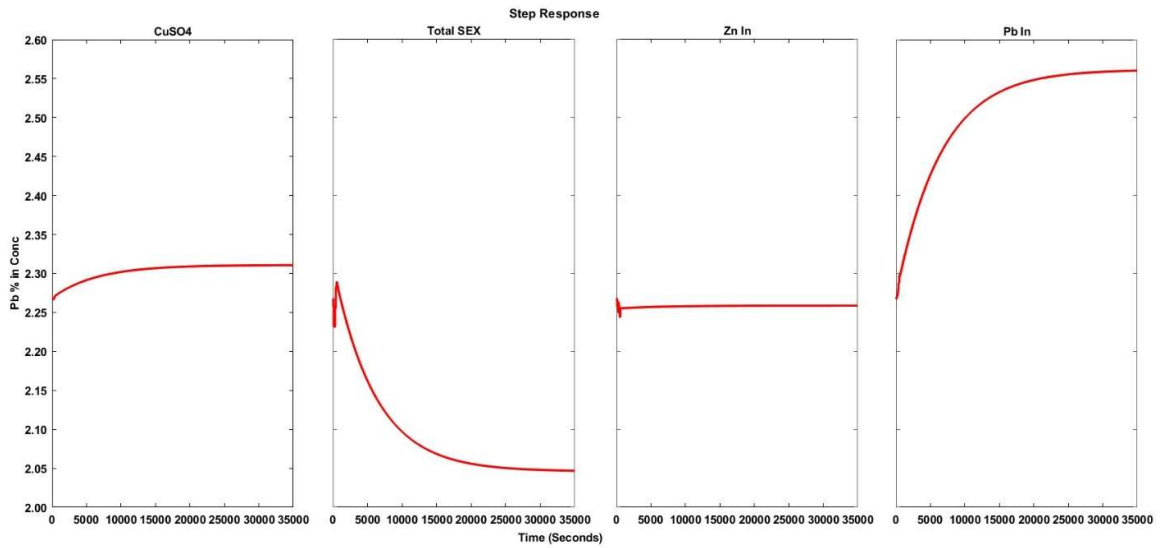


Figure 3-12 Pb % in Conc step response plots

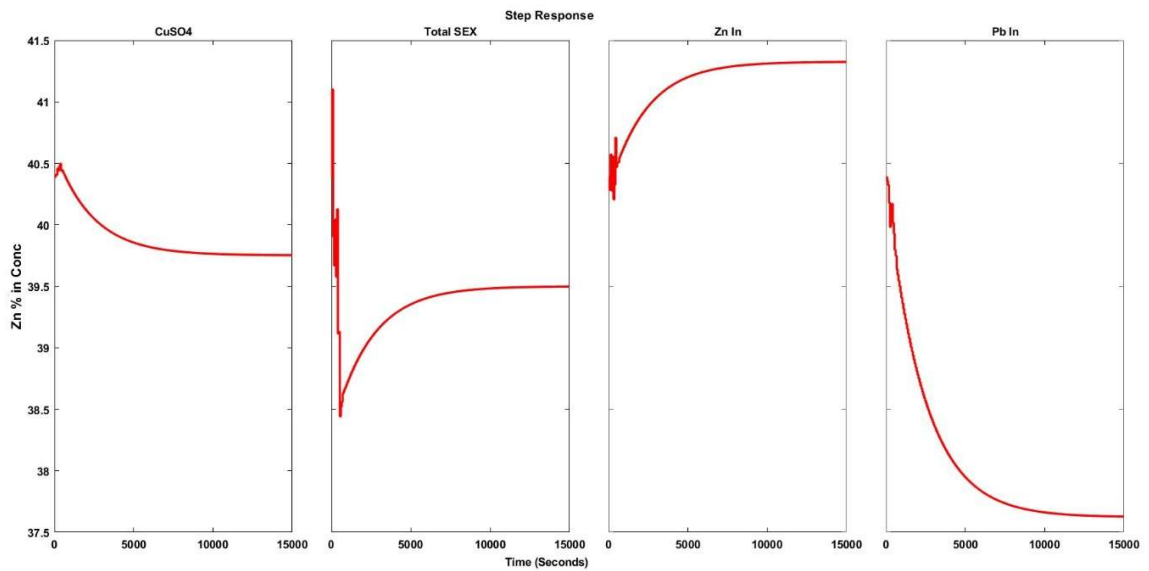


Figure 3-13 Zn % in Conc Response Plots

The step response times for each input variable in the Pb % in Conc model are different. The longest step response time for the Pb % in Conc models was the Total SEX and Zn In step responses at approximately 25,000s. The step response times of the input variables in the Zn % in Conc models were similar. The step

response times vary between 10000s and 15000s for each input variable. The step response times were used to determine simulation scenario setpoint changes and disturbance injection times.

3.4.2. Zinc and Lead Concentrate Total Trend Model (ARX) Results

An additional ARX model was developed using the additional input variables, Total Level and Total Airflow rate, as mentioned in 3.2.3. The results of fitting the total trend dataset using an ARX structure are shown in Figure 3-14 and Figure 3-15. The values were omitted from the plots for confidentiality reasons.

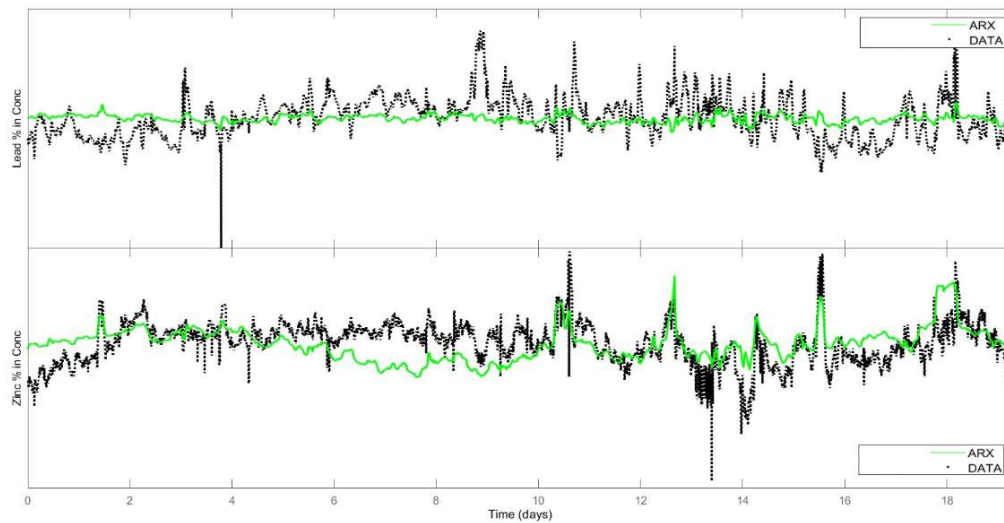


Figure 3-14 Model compared to the total trend training dataset

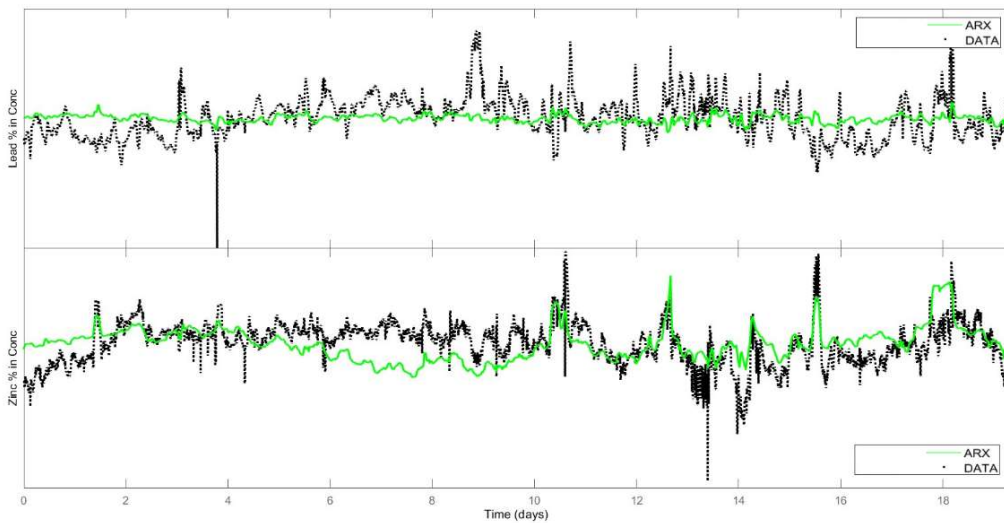


Figure 3-15 Model compared to the total trend testing dataset

The NRMSE and the correlation coefficient R^2 of the curve fitting for the total trend dataset is shown in Table 3-7.

Table 3-7 NRMSE and correlation coefficient R^2 results for Total Trend Dataset Models

Variable	RMSE/Average	RMSE/Average
	R^2 Train (%)	R^2 Test (%)
Con.Zn (ARX)	4.05 0.23	3.18 0.15
Con.Pb (ARX)	15.33 0.20	14.81 0.09

The ARX RMSE values for the test data in the Reagent models were 12.9% and 24 % for Zn % in Conc and Pb % in Conc, respectively. Thus, Table 3-7 illustrates that the ARX model curve fitting improved by including Total Level and Total Airflow rate as the test data RMSE values for the Zn % in Conc and Pb% in Conc are lower than the Reagent models. The correlation coefficient for the total trend test data is also larger than that of the reagent model test data. Thus, the Total Trend ARX model performs better than the Reagent Models with unseen data. Therefore, the Total Trend ARX model was used as a model for the linear MPC simulation.

The Pb % in Conc and Zn % in Conc step response plots for the Total Trend Model are shown in

Figure 3-16.

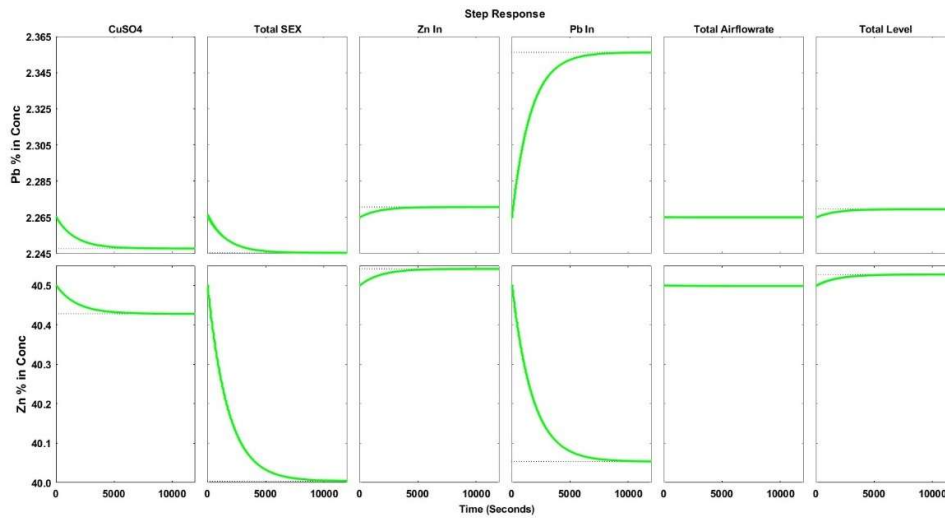


Figure 3-16 Total Trend Step Response Plots

All the response times for the Zn % in Conc and Pb % in Conc from the Total Trend models vary between 5000s and 10000s.

3.4.3. Zinc and Lead Concentrate Hybrid Models Results

The results of fitting the reagent and total trend dataset using the Hybrid NLARX structure are shown in Figure 3-17. The values were omitted from the plots for confidentiality reasons.

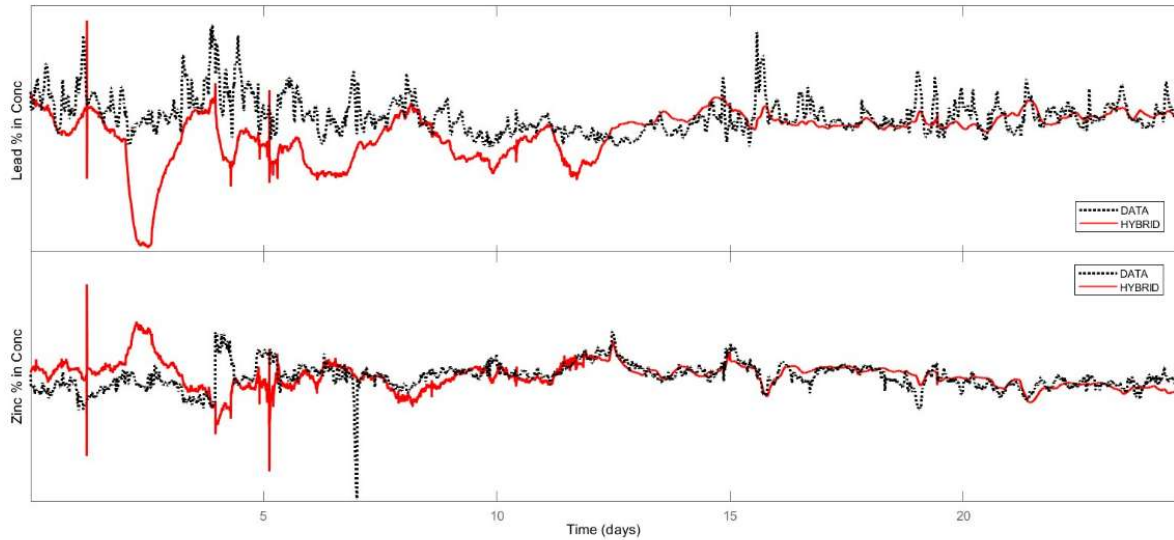


Figure 3-17 Hybrid Models compared to Reagent and Total Trend Dataset

The NRMSE and the correlation coefficient R^2 of the curve fitting for the Hybrid NLarx Model is shown in Table 3-8.

Table 3-8 NRMSE and correlation coefficient R^2 results for Reagent and Total Trend Dataset Models

Variable	RMSE/Average
	R^2
Con.Zn	4.2
	0.28
Con.Pb	20.4
	0.14

The NRMSE and correlation coefficient for the Hybrid NLarx model were used to assess whether the addition of the linear ARX model to the NLarx model declines the performance of the NLarx model. The Hybrid NLarx model combines the Reagent NLarx model and the Total Trend ARX model. The NLarx model was not trained using the Reagent and total trend dataset. Thus, the Reagent and total trend dataset is considered unseen data for the Hybrid NLarx model. Thus, the Reagent NLarx model test data statistic was used as a basis for comparison. The Zn % in Conc RMSE value for the Hybrid model, 4.2 %, is slightly lower than the NLarx model value, 4.4%. Moreover, the correlation coefficient is slightly lower for the Hybrid model than the NLarx model. The Pb % in Conc RMSE value, 20.4, is slightly higher for the Hybrid model than the NLarx model, 19.6%. The correlation coefficient for the Hybrid model is significantly higher than the NLarx model value.

In conclusion, the hybrid models perform comparably to the reagent NLarx model regarding the Zn % in Conc. However, the Pb %in Conc performance is better with the Hybrid model than the NLarx model, as the Pb % in Conc variation is explained by a larger proportion of the Input Variable's variation.

The Pb % in Conc and Zn % in Conc step response plots for the Hybrid Model results are shown in Figure 3-18 and Figure 3-19 respectively.

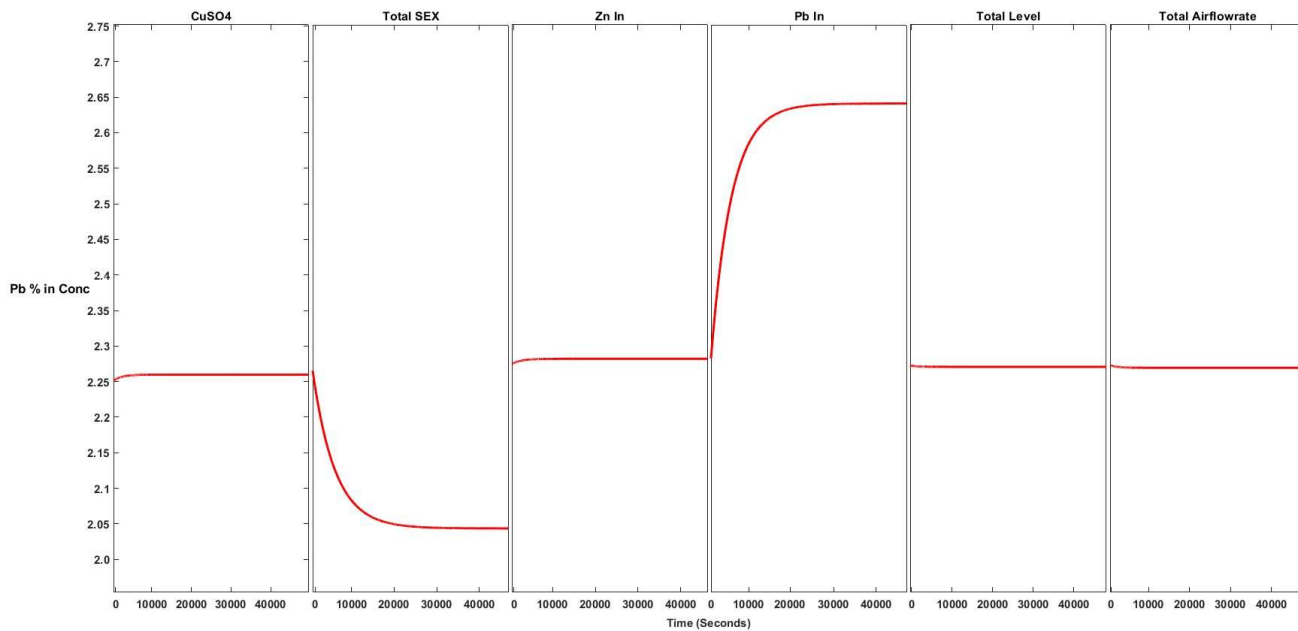


Figure 3-18 Hybrid Model Pb % in Conc Step Response Plots

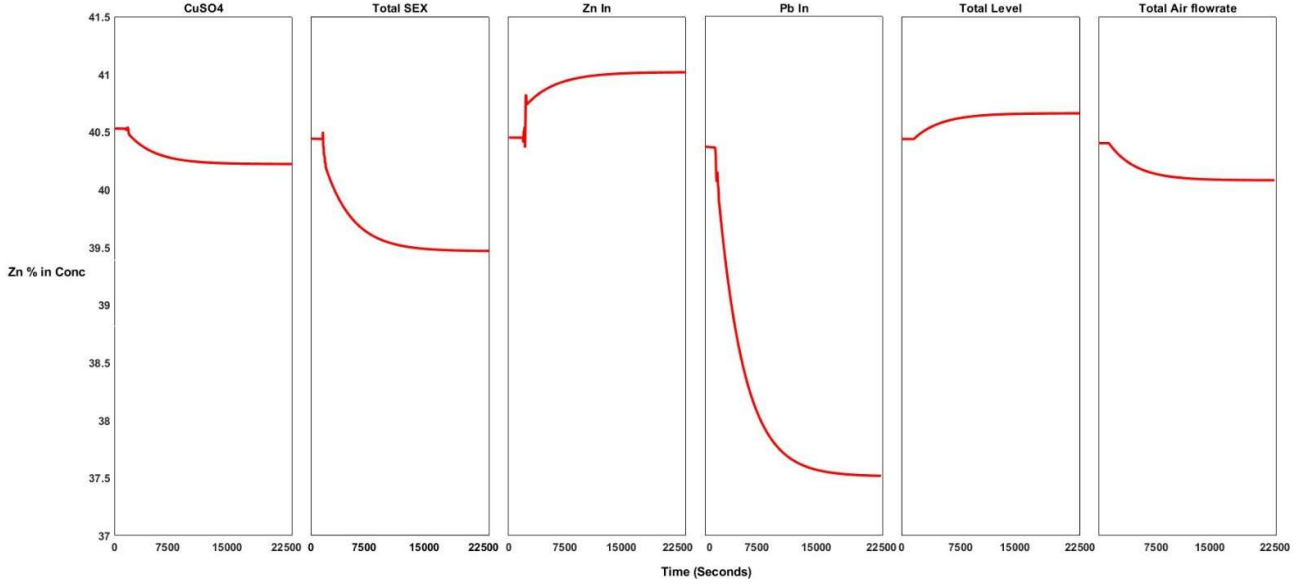


Figure 3-19 Hybrid Model Zn % in Conc Step Response Plots

The step response times for the Pb % in Conc and Zn % in Conc Hybrid models are equivalent to the Reagent Model step response times. This is expected as the Hybrid models are an amended form of the reagent models with the addition of Total Level and Total Airflow rate as input variables. The step response times for Total Level and Total Airflow rate in the Hybrid Pb % in Conc models fall between 5000 s and 10000 s. Whereas, the step response times for the Zn % in Conc models are longer and fall between 7500 s and 15000 s.

3.5. Reagent Sensitivity Results

The degree of non-linearity of the NLarx model structures was determined by varying one input regressor over a range while holding all other input regressors at their mean value and plotting the total NLarx response output, the nonlinear portion of the NLarx response output and the linear portion of the NLarx response output. As mentioned in section 3.2.2, the NLarx structure consists of two portions and is recapped in Equation 51.

$$y(x) = L^T(x - r) + d + g(Q(x - r)) \quad (51)$$

The linear portion of the NLarx structure is represented by $L^T(x - r)$ and was used to plot the linear portion of the NLarx response output and $g(Q(x - r))$ represents the nonlinear portion of the NLarx response output. The three response outputs were utilised in the reagent sensitivity plots to assess how adding a

nonlinear function to the model enhances the representation of the impact of reagent dosages on the zinc% % in the concentrate. The reagent sensitivity plotting was performed for the CuSO₄ and Total SEX dosage, with zinc in the concentrate as output. The CuSO₄ was varied over a range of 0 to 100 l/min, with an interval of 1 l/min and the Total SEX was varied over a range of 0 to 20 l/min, with an interval of 0.2 l/min. The mean values of the regressors can be found in Appendix B. The reagent sensitivity plots for the CuSO₄ and Total SEX dosage are shown in Figure 3-20 and Figure 3-21 respectively.

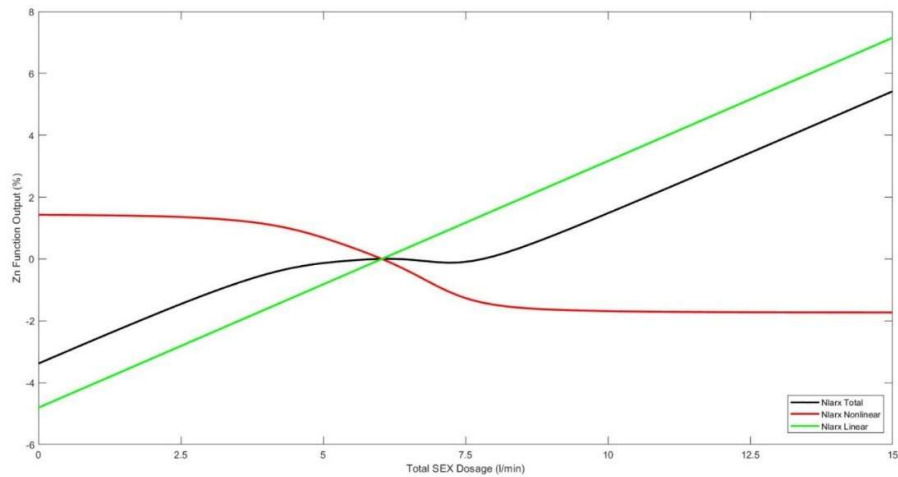


Figure 3-20 Zn % in Conc Reagent Sensitivity Plot for CuSO₄ Dosage

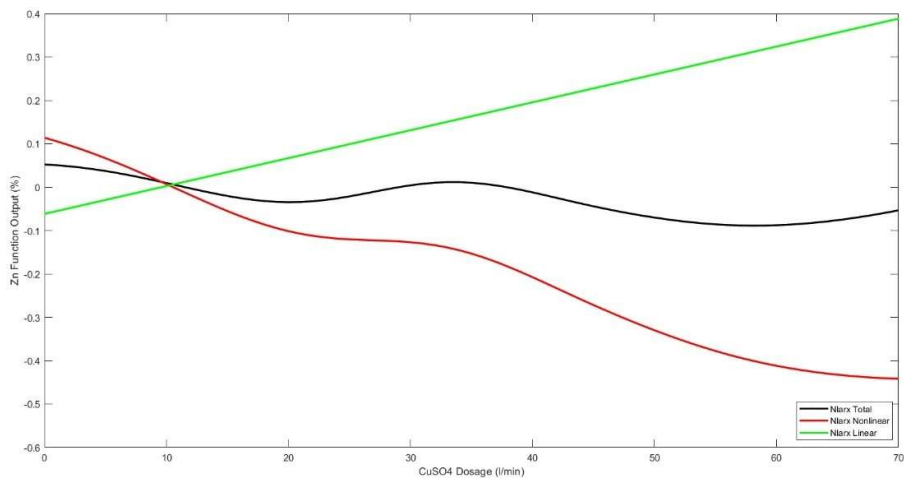


Figure 3-21 Zn % in Conc Reagent Sensitivity for Total SEX Dosage

The output response shown in the reagent sensitivity plots is the deviation of Zn % in Conc from the steady state value of 40.53 %.

In Figure 3-20, the linear response of the CuSO₄ NLarx model depicts a positive constant gain between the CuSO₄ dosage and Zn % in Conc. However, the nonlinear response of the CuSO₄ NLarx model showcases an alternating gain.

Similarly, the linear response of the Total SEX NLarx model depicts a positive constant gain between the Total SEX dosage and Zn % in Conc. The nonlinear response displays a slightly negative gain between a dosage of 0 and 4 (l/min). Thereafter, the gain magnitude is increased between a dosage of 4 and 8 (l/min). The gain magnitude starts to decrease again between a dosage of 8 and 15 (l/min). After a dosage of 15 (l/min), the gain is 0, and the Zn nonlinear function output remains constant. The addition of the two output responses indicates an initial increase in the Zn grade with the increase in collector dosage. At 4 (l/min), the positive gain in the Zn grade starts to decrease. Around 6 (l/min), there is a slight negative gain in Zn grade. After 7.5 (l/min), the Zn grade is positive again, and the Zn grade increases with an increase in the collector dosage.

Overall, the reagent sensitivity plots illustrate contrasting linear and nonlinear responses of the Total SEX and CuSO₄ dosage. Consequently, the total NLarx response for both reagents is significantly different from the linear response, indicating that the NLarx model structure can incorporate the nonlinear behaviour of the reagents. The nonlinear output functions in both reagent sensitivity plots display characteristic logistic sigmoid behaviour. The sigmoid function saturates for small or large inputs, as seen in Figure 3-22.

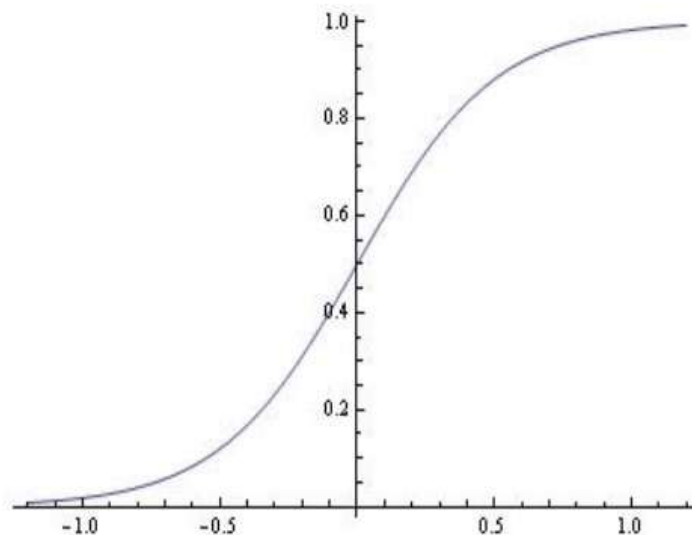


Figure 3-22 Logistic sigmoid function

The reagent sensitivity plots illustrate this saturation in the Zn nonlinear function output for low and high reagent dosages. The input used in the sigmoid function of the NLarx model is the linear transformation of the regressors. The increase or decrease of one of the regressors results in a direct proportional change of the scalar input used in the sigmoid function. Thus, the varying reagent dosage rate is expected to render saturation behaviour at high and low reagent dosing rates.

The nonlinear function output trends were compared to the real-world relationship between Zn Grade and Reagent dosages. Xie et al., (2016) conducted a series of sphalerite flotation experiments to determine the effect of collector dosage on the recovery of Zn. The collector and activator used in the experiment were sodium isobutyl xanthate (SIBX) and CuSO₄, respectively. SEX and SIBX are both xanthate collectors used for Zn flotation and share a similar structure. SIBX has a longer carbon chain. Thus, it is a stronger collector than SEX. However, the selectivity remains alike. The comparable structure exhibited by the two collectors provided a basis for inferring a systematic relationship between zinc recovery and the dosage of SEX. The flotation tests' results that showcase the effect of SIBX dosage on the Zn recovery are shown in Figure 3-23.

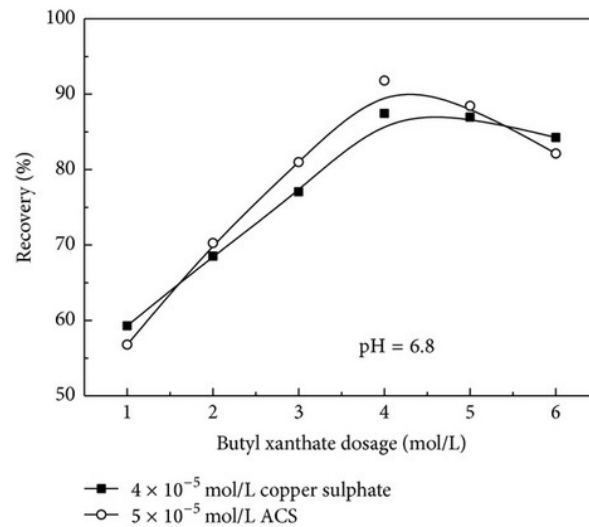


Figure 3-23 The effect of SIBX dosage on the recovery of Zn (Xie et al., 2016)

The recovery and collector dosage values were not used directly for analysis, as recovery values may vary for different ores. However, the trend directions were studied further. The figure showcases that an increase in the collector dosage results in an initial increase in the Zn recovery. At a certain dosage, the recovery reaches a maximum and either increases or starts to decrease. The Zn grade trend can be inferred from a sphalerite recovery grade curve, such as the curve shown Figure 3-24.

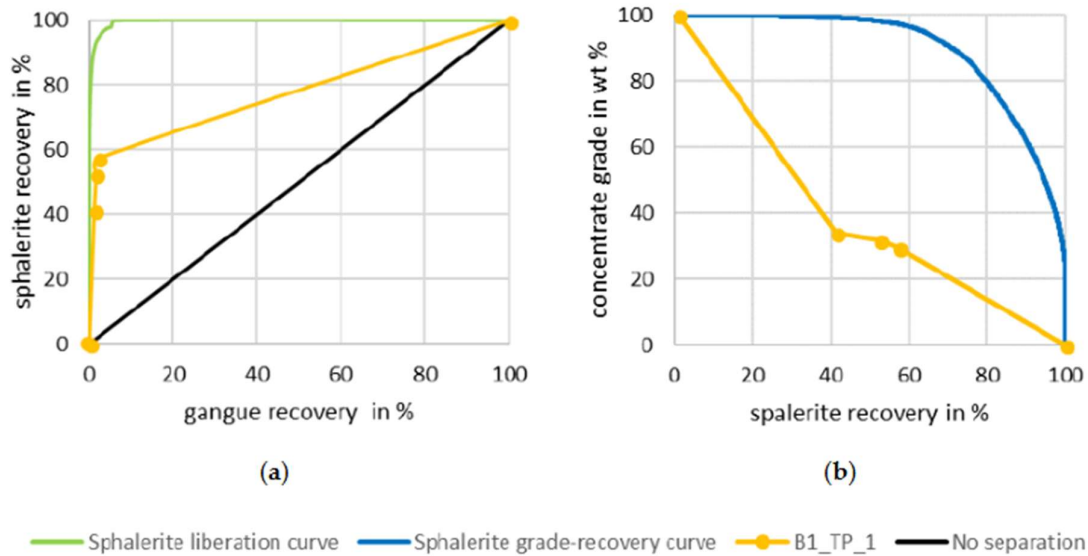


Figure 3-24 Sphalerite grade recovery curve (Babel et al., 2018)

Figure 3-24 depicts that the low recovery range (0-40%) yields a constant or slight negative gain in the Zn grade. At a medium recovery range (40-90%), the negative gain increases for the grade. A recovery higher than 90% can drastically decrease the grade. However, achieving extremely high recoveries of Zn is ideal and not a common occurrence. The Zn recovery is expected to reach a plateau, and the recovery of gangue minerals is also expected to decrease. This decrease in the recovery of gangue minerals results in a lower gain decrease in the Zn grade. The behaviour of the Zn grade in response to the recovery can be used to explain the grade behaviour when the SIBX dosage is increased in the sphalerite flotation tests. Initially, an increase in the collector dosage results in an increase in Zn recovery, which results in a constant or slight decrease in the Zn grade. The Zn recovery eventually plateaus, and the Zn grade starts to decrease further. The behaviour in the Zn grade mimics the behaviour of the nonlinear function output curve in Figure 3-21.

The relationship between the activator dosage and the Zn grade was also explored in the same study performed by Xie et al. (2016). The activator dosage was varied, and the corresponding recovery was recorded. These results are shown in Figure 3-25 (Xie et al., 2016).

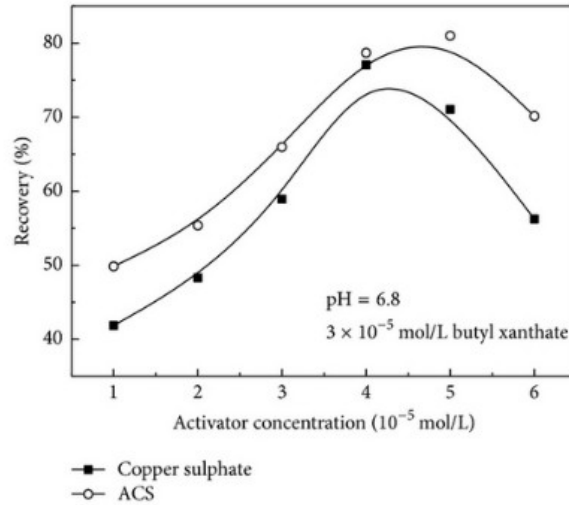


Figure 3-25 Effect of Activator Dosage on Zn Recovery (Xie et al., 2016)

The utilisation of the sphalerite grade recovery curve along with Figure 3-25 infers that an initial increase in the activator dosage results in a negative gain of the Zn grade. Thereafter, the recovery reaches a maximum value and starts to decrease. A further increase in the activator dosage yields a positive gain in the Zn grade. A direct comparison of the grade and activator curve with the CuSO_4 nonlinear function output curve in Figure 3-20 is not feasible due to a difference in reagent dosage values. However, a key observation is the fluctuating behaviour in the Zn grade when the activator is varied. This is reflected in the trends of the CuSO_4 nonlinear function output.

These results indicate that the linear function does not truly capture the dynamics of the relationship between reagent dosages and grade. The incorporation of a nonlinear function along with a linear function in a model facilitates a precise representation of zinc (Zn) grade behaviour in relation to reagent dosages. This approach explains why the nonlinear model provides a lower NRMSE with unseen data. Furthermore, the saturation behaviour of the nonlinear function outputs in response to the dosage rates mimics characteristics of the logistic sigmoid function. Some of the complexities and variabilities inherent in the relationship between these variables can be acknowledged within this nonlinear function.

The models developed in this section were used in the development of MPC controllers. The Zn and Pb total trend models were used to represent the internal plant of linear MPC controller, and the hybrid models were used to represent the internal plant of nonlinear MPC controllers. These two controllers were further tested against an external plant in a simulation and will be discussed in the following section.

4. CHAPTER 4: SIMULATED SYSTEM DEVELOPMENT

The best-performing linear and nonlinear model was used to represent the internal plant of the linear MPC and nonlinear MPC, respectively. Thereafter, the best-performing nonlinear model was used to represent the external plant within a simulated system. The simulated system is used to assess the performance of the controllers against the external plant and was created using Simulink. The main features of the simulated system consisted of an MPC controller block and a Plant block.

The results obtained during the identification of empirical linear and nonlinear flotation models described in Chapter 3 are presented in this chapter. These include the Reagent, Total Trend ARX and Hybrid NLarx models. The model results consist of plots that compare the models to training and testing datasets and step response plots. This chapter also explores the reagent sensitivity of the collector and depressant dosage on Zn % in concentrate in the reagent models. Lastly, the linear MPC and nonlinear MPC were compared, and the control performance of each is presented in this chapter. The controller's ability to adapt to setpoint changes and reject disturbances was assessed. The Reagent model results were previously published in the IFAC World Congress Journal. (Brooks et al., 2023).

4.1. Controller design

This dissertation aims to address the reagent control challenges encountered in the work of Brooks and Koorts (2017). The controller's reagent dosage range was continuously changed to ensure the performance is linear within the chosen range. The MPC controller had the control objective to maintain the bank concentrate grades in range. The incorrect choice of a range can result in overdosing or underdosing of reagents, which produces a concentrate grade that does not remain within the target ranges. Additionally, choosing an incorrect dosing range for reagents can affect the controller's ability to handle process disturbances. For example, the controller's adjustment of the MVs to combat the effect of disturbances may be restricted if the reagent dosing range is too small.

Two controllers were developed for this dissertation: a linear MPC controller, which uses a linear model within the cost function minimisation to calculate the control actions, and a nonlinear MPC controller, which uses a nonlinear model within the cost function minimisation to calculate the control actions. The nonlinear controller is theorised to accommodate the nonlinear behaviour between the reagent and the concentrate grade such that the continuous change of the reagent ranges is not required. To test this theory, the two controllers are compared to determine if a nonlinear MPC controller is superior at maintaining the concentrate grade.

The two grades considered as controlled variables in the controller design are the Zn % and Pb % in concentrate. The Zn % in concentrate is considered as the sphalerite (zinc mineral), which is the main mineral that is collected in the rougher bank concentrate. The Pb% in concentrate was considered as galena (lead mineral), which has a strong association with sphalerite. Furthermore, the amount of Pb in the zinc concentration product is vital as the zinc concentration product must meet standard feed requirements for a smelting operation (Sinclair, 2005).

The control objectives for the linear and nonlinear MPC in this dissertation were created to address the challenges mentioned above and are in order of importance as follows.

- Control Objective (1) – Maintain the Zn % in concentrate at a setpoint
- Control Objective (2) - Minimise Zn % in concentrate setpoint offset when disturbances are introduced
- Control Objective (3) - Maintain the Pb % in concentrate at a setpoint
- Control Objective (4) - Minimise Pb % in concentrate setpoint offset when disturbances are introduced

The controlled, manipulated and disturbance variables for the MPC controller developed by Brooks and Koorts (2017) are shown in Figure 3-3 along with the models that were utilised in the objective function.

The original variables shown in Figure 3-3 were used in models that were able to successfully predict the zinc grade in concentrate, thus they were used as a basis for choosing variables in the controller design of the MPC controllers in this dissertation. The MPC design variables chosen for this study are shown in Table 4-1.

Table 4-1 MPC Design Variables

Controlled Variables	Manipulated Variables	Disturbance Variables
Zn % in rougher concentrate	Combined collector (Total SEX) dosing rate	Zinc Feed Flow Rate (Zn In)
Pb % in rougher concentrate	CuSO ₄ dosing rate	Lead Feed Flow Rate (Pb In)
	Combined airflow rate (Total Airflow rate)	
	Combined level (Total Level)	

The Zn % in rougher concentrate is included as a controlled variable as it is a measure that can be used to determine if control objective (1) and control objective (2) are met. The other controlled variables from the original variables were not considered, as the variables do not provide additional information that can be used for achieving the control objective. The Pb % in rougher concentrate is used as a controlled variable as it is a measure that can be used to determine if control objectives (3) and (4) are met. The control of the Zn % in rougher concentrate is prioritised.

The airflow rates and pulp level are variables that are frequently used to control the concentrate mineral grade in flotation cells (Cilliers & Shean, 2011). Thus, these variables are included as manipulated variables. The reagent dosing rates, collector and activator are also included as manipulated variables. As, this dissertation will explore how modelling the nonlinear behaviour of reagents dosing rates can improve the MPC controller performance. In the original variable set, the airflow rate, level and reagent dosing rates were sufficient for predicting the Zn % in concentrate, thus no other manipulated variables were considered. As mentioned in section 3.1.3, feature extraction was performed on certain variables. Firstly, the two collector dosing rates were combined into a single variable. Similarly, the four airflow rates were combined into a single variable, and the two levels were combined into a single variable.

The disturbance variables from the original variables were combined, as these variables represent characteristics of the rougher feed and can be used to calculate a zinc feed flow. The metal feed flow is calculated using Equation 40 from section 3.1.3. The same calculation is used to calculate a lead feed flow.

The structure of the objective function was kept constant for both controllers and was described using Equations 24 to 27. The following controller design parameters are required for the objective function and were estimated for the controllers:

- Controller sample time
- Nominal Plant Input
- Nominal Plant Output
- Prediction Horizon
- Control Horizon
- Manipulated Variable lower bound
- Manipulated Variable upper bound
- Manipulated Variable rate of change lower bound
- Manipulated Variable rate of change upper bound
- Manipulated Variable rate tuning weights
- Output variable tuning weights

The values for sample time, nominal values, and manipulated variable upper and lower bounds were determined based on the structure of the reagent dataset and total trend datasets. A controller sample time of 60s was used as the data was sampled from the flotation bank at a rate of 60s. The nominal values were equal to the steady state values, which will be discussed in the Simulation Scenarios section. The upper and

lower bounds were selected based on the dataset's minimum and maximum variable values, whilst considering physical constraints. The remainder of the controller parameters were first assessed for the linear controller. The prediction horizon and the control horizon are typically kept constant when determining the remaining control parameters. Thus, the prediction horizon and the control horizon were determined first. Three sets of prediction horizon and control horizon values were chosen, and the controller performance for each set was analysed to determine which set would be used for the entirety of the controller parameter analysis. The sets consisted of small, medium and large prediction and control horizons. These sets are shown in Table 4-2.

Table 4-2 Prediction Horizon and Control Horizon sets

	Prediction Horizon	Control Horizon
Small	4	2
Medium	40	20
Large	80	40

The controller performance for each set was assessed for a scenario where the Zn % in concentrate was stepped up by 0.5 % of the steady state value and a scenario where the Pb % in concentrate was stepped up by 0.5 % of the steady state value. The MPC designer app was used as a visual tool to analyse the performance for each scenario. The MV rate of change limits were set to infinite, and the MV rate of change of weights and CV weights were all set to 0 when analysing the different sets of prediction horizons and control horizons.

Thereafter, the optimal prediction horizon and control horizon were kept constant whilst the MV rate of change bounds and weights were varied. The values for these parameters were chosen to achieve the control objectives whilst considering the following: minimisation of overcorrection, ensuring the movement of manipulated variables remains within physical constraints, obeying general flotation control heuristics and ensuring the settling time corresponds with the step response times of the Hybrid models. The flotation control heuristics include ensuring the pulp level does not vary drastically between each time step, allowing the manipulation of airflow rates over a large range, favouring airflow control over pulp level control and ensuring gradual but significant changes in the reagent dosing rates to avoid underdosing or overdosing. The output variable tuning weights were chosen after the MV rate of change bounds and weights were determined. A larger weight was chosen for the Zn % in Conc output as the primary control objective is to maintain the Zn % in concentrate at a setpoint. Maintaining the Pb % in concentrate is a tertiary objective;

thus, weighting is lower for the Pb % in Conc. The comparison of the controller performance with varying the remainder of the control parameter was also completed using the MPC designer App. A detailed explanation of how these plots were developed is provided in Appendix C.

All the control parameters used for the linear controller were utilised for the nonlinear controller to ensure a controlled comparison.

4.2. Plant Specification

The construction of the linear and the nonlinear MPC requires the specification of internal controller plant models. The objective function utilised by the controllers was explored in Equations 24 to 27. The objective function requires an internal plant model to determine the controller state and predict future plant outputs. The Zinc and Lead total trend ARX models were combined into a single model structure. This ARX model structure was used as the internal plant model for the linear MPC controller. The total trend model structures are shown in Appendix B.

The Zinc and Lead Concentrate Hybrid models were combined into a single model structure. The single Hybrid model structure was utilised as the internal plant model for the nonlinear MPC controller. Both controller internal models were written in the format of a MATLAB function and were called when the controllers were in operation. An external plant was utilised to assess the performance of both controllers. Each controller was tested using an external plant in a simulation. The external plant was developed using the Zinc and Lead Concentrate Hybrid Models. The external plant was also contained within a function and was called during the simulations. The Hybrid Models consist of structures from both the Zinc and Lead Concentrate No NASH Reagent Models (NLarx) and the Zinc and Lead Concentrate Total Trend Model (ARX), as mentioned in section 3.2.3. The construction of the Hybrid model is explored in the preceding sections.

NLarx Section of Hybrid Model

The NLarx section of the Hybrid models is defined in Equation 52 and is included in the NARXHybridFunc function.

$$y_{NlarxTotal} = y_{Lin_{Nlarx}} + y_{Nonlin_{Nlarx}} \quad (52)$$

Where $y_{NlarxTotal}$ is a [2 x 1] output vector of the NLarx prediction model, $y_{Lin_{Nlarx}}$ is a [2 x 1] output vector of linear mapping and $y_{Nonlin_{Nlarx}}$ is a [2 x 1] output vector of the nonlinear mapping. The mapping of regressors is completed for both Pb_Conc and Zn_Conc; thus, there are computations of

$yLin_{Nlarx}$ and $yNonlin_{Nlarx}$ for both CVs, resulting in a $[2 \times 1]$ output matrix. The Linear and Nonlinear mapping is completed for the following inputs: CuSO₄, Total_SEX, Pb_In and Zn_In. The linear mapping output computation was explored in the Model Structures section and is revisited in Equation 53.

$$yLin_{Nlarx} = x_r \times P \times L + d \quad (53)$$

Where $yLin_{Nlarx}$ is the output of the linear mapping, x_r is a regressor matrix that holds the regressors centred around the mean for the multiple inputs and the single output. P , L and d are parameter matrices required for the model computation, which will be explored later in this section. The linear mapping computation for Pb_Conc and Zn_Conc is combined to produce the $[2 \times 1]$ $yLin_{NlarxTotal}$ matrix, where.

$$yLin_{Nlarx}(1) = yLin_{NlarxPb} \quad (54)$$

$$yLin_{Nlarx}(2) = yLin_{NlarxZn} \quad (55)$$

Where $yLin_{NlarxPb}$ and $yLin_{NlarxZn}$ are the outputs of the linear mapping computation for Pb_Conc and Zn_Conc, respectively. The nonlinear mapping computation of the NLarx portion of the Hybrid model is shown in Equation 56.

$$yNonlin_{Nlarx} = yNonlin_{Nlarx} + aVec(k) \times f(fInput) \quad (56)$$

Where $yNonlin_{Nlarx}$ is the output of the nonlinear mapping and $aVec$ is a parameter matrix required by the model structure, which will be explained later in this section. $yNonlin_{Nlarx}$ is calculated within the NARXHybridFunc function through a series of iterations completed using a for loop. The number of iterations is equivalent to the length of $aVec$ and k represents the number of iterations. f is an anonymous function used to describe the sigmoid unit function. The unit sigmoid function is described by Equation 57.

$$f = 1/(\exp(-fInput) + 1) \quad (57)$$

$fInput$ is the input to the sigmoid function and is described by Equation 58.

$$fInput = x_r \times Q \times bMat \times (:,k) + cVec(k) \quad (58)$$

Where x_r is the regressor matrix that holds the regressor centralised around the regressor mean for the input and output variables. Q , $bMat$ and $cVec$ are parameter matrices required by the model structure, which will be explored later in this section. The Nonlinear mapping computation for Pb_Conc and Zn_Conc is added to the $[2 \times 1]$ $yNonlin_{NlarxTotal}$ matrix represented in Equations 59 and 60.

$$yNonlin_{Nlarx}(1) = yNonlin_{NlarxPb} \quad (59)$$

$$yNonlin_{Nlarx}(2) = yNonlin_{NlarxZn} \quad (60)$$

Where, $y_{Nonlin_{NlarxPb}}$ and $y_{Nonlin_{NlarxZn}}$ are the outputs of the nonlinear mapping computation for Pb_Conc and Zn_Conc, respectively.

ARX Section of the Hybrid Model

The Hybrid model includes a section defined by the Linear ARX structure. The Linear ARX structure considers the linear summation of the Total Air flow rate and Total Level inputs and is described by Equation 61.

$$y_{ARX} = x_r \times L_{ARX} \quad (61)$$

Where y_{ARX} is the output of linear mapping for the ARX portion of the Hybrid model, x_r is a regressor matrix containing regressors of the Total Airflow rate and the Total Level, and L_{ARX} is the linear ARX coefficient parameter matrix. The Linear ARX portion of the Hybrid model includes Linear mapping for the Total Airflow rate and the Total Level regressors individually. The outputs from the linear mapping of the Total Airflow rate and the Total Level were combined and are showcased in Equation 62.

$$y_{ARXPb} = y_{ARXPbTotAir} + y_{ARXPbTotLevel} \quad (62)$$

Where y_{ARXPb} is the linear ARX output for Pb_Conc, $y_{ARXPbTotAir}$ is the output of the linear mapping computation for the Total Airflow rate and $y_{ARXPbTotLevel}$ is the output of the linear mapping computation for the Total Level. A similar computation is performed for Zn_Conc, and the output is labelled y_{ARXZn} . The total output of the Linear ARX portion is a [2 x 1] matrix labelled as $y_{ARXTotal}$ and is computed by adding the linear mapping outputs of Pb_Conc and Zn_Conc, which is shown in Equation 63.

$$y_{ARXTotal} = y_{ARXPb} + y_{ARXZn} \quad (63)$$

Where $y_{ARXTotal}$ is the total linear ARX output.

Overall Hybrid model structure

The final output of the NARXHybridFunc Function is a [2 x 1] matrix labelled as x_k and is computed by the summation of the total output from the NLarx portion of the Hybrid model and the total ARX portion of the Hybrid model. This summation is shown in Equation 64.

$$x_k = y_{ARXTotal} + y_{NlarxTotal} \quad (64)$$

x_k is a state vector. The number of states is equivalent to that of the outputs in the hybrid model. Thus, the state vector is identical to the output vector.

Hybrid Model Parameter Matrices

The parameter matrices utilised by the Hybrid Model structures are explored in Table 4-3.

Table 4-3 Hybrid Model Parameter Matrices

Parameter Matrix	Parameter Matrix Name	Definition	Simulation Label
x_r	Regressor matrix	These matrices consist of current, past MV, and past CV values. The table showcases the meta parameters used to determine the number of past regressors for each MV. Each regressor matrix should contain empty rows to house the current values of MVs that the controller computed. The Pb regressor matrix is a [1 x47] matrix. The Zn regressor matrix is a [1 x46] matrix. These matrices are defined in the workspace.	Pb_regressor_matrix Zn_regressor_matrix
P	Linear Subspace	A [47 x 47] progression matrix. These matrices are defined in the workspace.	pPb pZn
L	Linear Coefficient	A [47 x 1] matrix consists of weights assigned to each regressor. These matrices are defined within the NARXHybridFunc.	LPb LZn
d	Output Offset	An output offset is defined in the NARXHybridFunc as a scalar.	dPb dZn
aVec	Output Coefficient	A [10 x 1] vector of scalar weights. These vectors are defined in the workspace.	aVecPb aVecZn
Q	Nonlinear Subspace	A [47 x 47] projection matrix for Pb_Conc and [46 x 46] projection matrix for Zn_Conc. These matrices transform x_r to a lower-dimensional space and are defined in the workspace.	QPb QZn
bMat	Dilation Matrix	A [47 x 10] dilation matrix for Pb_Conc and [46 x 10] dilation matrix for Zn_Conc containing dilation coefficients. These matrices are defined in the workspace.	bMatPb bMatZn
cVec	Translation Matrix	A [1x 10] containing translation scalars. These matrices are defined in the workspace.	cVecPb cVecZn
$x_{r_TotalAir_TotalLevel}$	Linear ARX regressor matrix	These matrices consist of past and current Total Airflow rate and Total Level values defined within the workspace.	Initial_AirLev Pb_regressor_matrix Initial_AirLev Zn_regressor_matrix
L_{ARX}	Linear ARX Coefficient Matrix	This is a [1 x 2] matrix consisting of weights assigned to the Total airflow rate and Total Level regressors during linear mapping. These matrices are defined within the NARXHybridFunc.	LPbARX LZnARX

There are parameter matrices for both Pb_Conc and Zn_Conc outputs. The regressor matrices for Pb_Conc and Zn_Conc are structured according to the Model Meta-Parameters specified in Table 3-3 and Table 3-4. There are columns within the regressor matrix that are empty in the first iteration and act as placeholders for the current MV values, which are computed by the controller. The MV values from the controller enter the plant and fill the empty columns for the current iteration. After that, the columns are replaced by the current MV values from the controller iteration by iteration. In addition, the first column of the regressor matrices is empty and acts as a placeholder for the CV values from the previous iteration. The regressor matrices of the linear portion also contain empty columns for current MV values in the first iteration and are replaced afterwards by the new MV values for each iteration. However, these matrices do not contain columns with previous CV values, as the CV regressors are already mapped within the NLarx portion of the hybrid model. The full specification of the regressor matrices and the values of each parameter matrix are shown in Appendix B.

The plant structure is included in a function and is called into the simulations. The function used to define the plant is labelled the NARXHybridFunc. The code that defines this function is shown in Appendix C, and the process is further described under the Simulated System Development section.

4.3. Simulated System Development

The models in section 3.2 were used to represent the internal plants of the MPC controllers. The linear MPC internal plant was represented using the Total Trend ARX models, and the nonlinear MPC internal plant was represented using the Hybrid models. Simulated systems were developed to compare the performance of linear MPCs with nonlinear MPCs against an external plant. The control objectives were used as metrics to assess controller performance. The setpoint tracking of the controller was analysed to determine whether the first and third control objective was met. Secondly, the disturbance rejection of the controller was analysed to determine if the second and fourth control objective was achieved. The external plant uses Hybrid Models to represent the flotation rougher bank in linear and nonlinear MPC simulations. The simulations were developed using Simulink®. Two simulations were built where the MPCs controlled an external plant. The variables used in the simulation are shown in Table 4-4.

Table 4-4 Simulation Variables

Simulation Variables	
Variable	Control System Variable
CuSO4	MV
Total_SEX	MV
Zn_In (Zn Flowrate In)	DV
Pb_In (Lead Flowrate In)	DV
Zn_Conc (Zinc grade in concentrate)	CV
Pb_Conc (Lead grade in concentrate)	CV
Total_Level	MV
Total_Air	MV

The simulation starts from a steady state. After that, the Pb % in Conc and the Zn % in Conc (the CVs in this control system) are either stepped up or down. The various simulation scenarios that were run are described under the Simulation Scenarios below. The steady-state values for the variables are found in Table 4-5.

Table 4-5 Steady State Values for Simulation Values

Simulation Variables	
Variable	Steady State Value
CuSO4	47.99
Total_SEX	5.24
Zn_In (Zn Flowrate In)	69.29
Pb_In (Lead Flowrate In)	9.56
Zn_Conc (Zinc grade in concentrate)	40.53
Pb_Conc (Lead grade in concentrate)	2.27
Total_Level	65.78
Total_Air	1554.45

The Hybrid models require Pb and Zn Regressor Matrices, as mentioned in 4.2. Initial Regressor Matrices are required for the Hybrid Model computations at the start of each simulation. The Initial Regressor Matrices consist of the steady state values for the CV, MV and DV regressors. The regressor values change once the Pb % in Conc and the Zn % in Conc setpoints change. The Initial Regressor Matrices are found in Appendix B.

Linear MPC Simulation

The first simulation was developed using a Linear MPC and is shown in Figure 4-1.

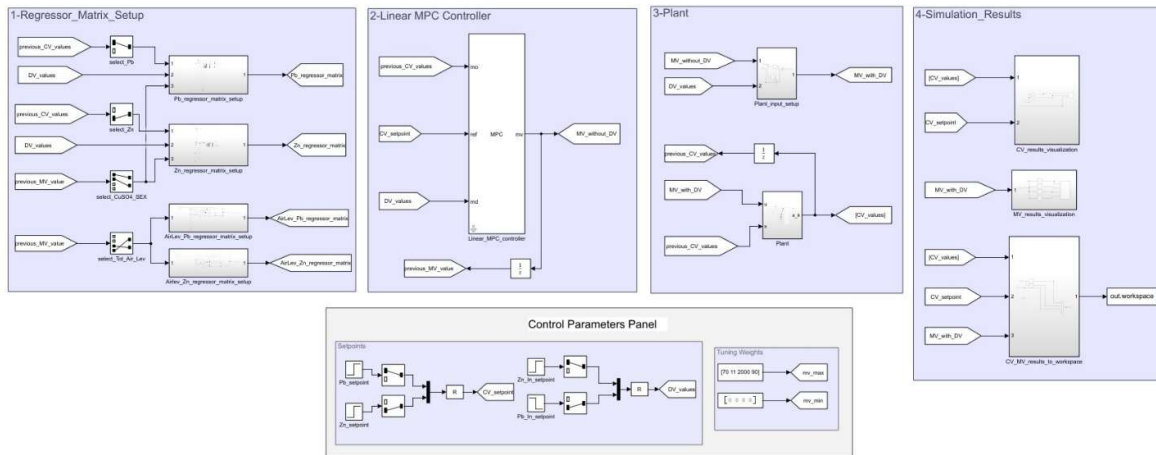


Figure 4-1 Linear MPC Layout

The Linear MPC simulation is split into 5 sections:

1. Regressor Matrix Setup
2. Linear Controller
3. Plant
4. Simulation Results
5. Control Parameters Panel

The Regressor Matrix Setup section is shown in Figure 4-2 is responsible for creating the regressor matrices used in the Hybrid NLarx models. The external plant in the linear MPC simulation utilises the Hybrid NLarx models, so the regressor matrices are required as inputs to the Plant block in the Plant section.

The previous CV and MV values from the previous iteration of the simulation are used to form the regressor matrix as the simulation runs. In addition, the user enters the DV values in the Control Parameter Panel, which are used in the regressor matrix formation. Four regressor matrices are formed.

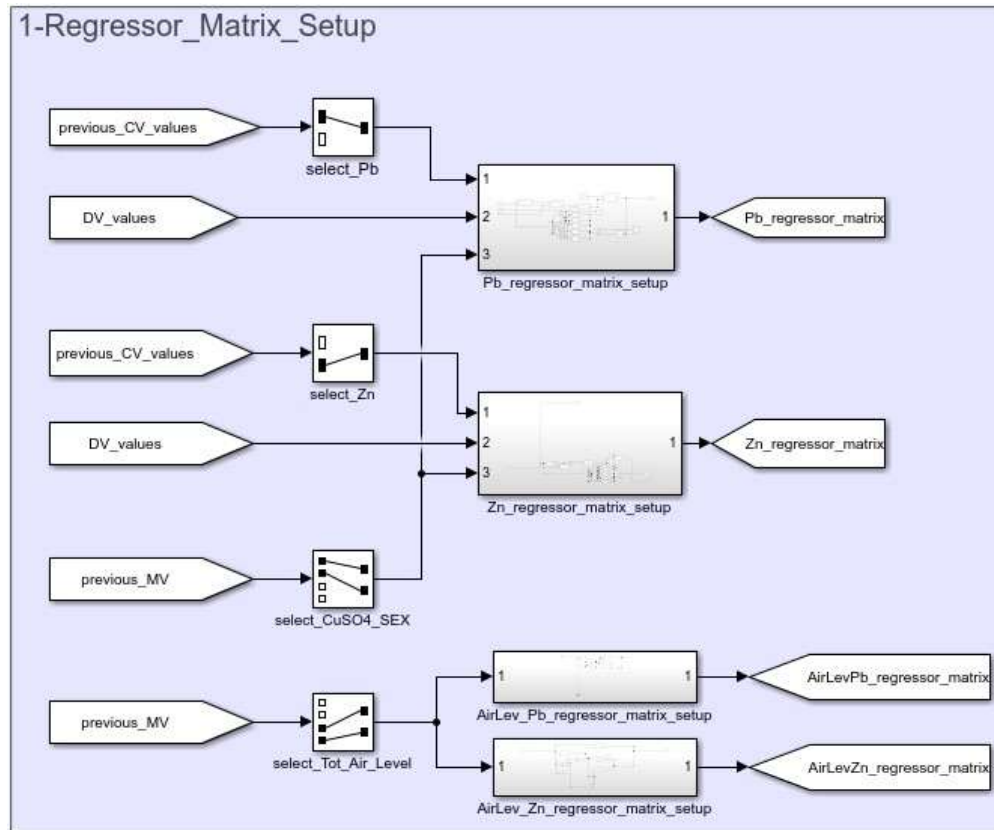


Figure 4-2 Regressor Matrix Setup

The first regressor matrix is the Pb regressor matrix required for the non-linear model section of the Lead Concentrate Hybrid Model. The `Pb_regressor_matrix_setup` block receives the previous Pb Conc grade values produced from the plant, the previous MV values from the MPC controller and the user-defined DV values. The `Pb_regressor_matrix_setup` block uses a series of concatenate, selector and assignment blocks to shift and store the previous CV and MV values from past timesteps. The second regressor matrix is the Zn regressor matrix and is used in the non-linear model section of the Zinc Concentrate Hybrid Model. The Zn regressor matrix is created similarly to the Pb regressor matrix. The nonlinear section of the model requires regressors of the following model input variables: CuSO₄, SEX, Pb_In and Zn_In. The last two regressor matrices are the Air and Level regressor matrices used in the linear section of the Lead and Zinc Concentrate Hybrid Models. These regressor matrices for the Zn and Pb models are named the `AirLevZn_regressor_matrix` and the `AirLevPb_regressor_matrix`. The `AirLev_Zn_regressor_matrix_setup` and the `AirLev_Pb_regressor_matrix_setup` blocks also consist of a series of concatenate, selector and assignment blocks that are used to create Air and Level regressor matrices. These matrices only require regressors from the Total_Air and Total_Level variables.

All the `regressor_matrix_setup` blocks require an initial regressor matrix that holds the initial conditions at the start of the simulation and acts as a placeholder for future iterations. The initial regressor matrices

contain steady-state values of the MVs and CVs at the start of the simulation. A breakdown of the regressor matrices will be showcased later in the Plant section. A complete view of the contents of the regressor_matrix_setup block can be found in Appendix B. The layout of the Linear Controller section is shown in Figure 4-3.

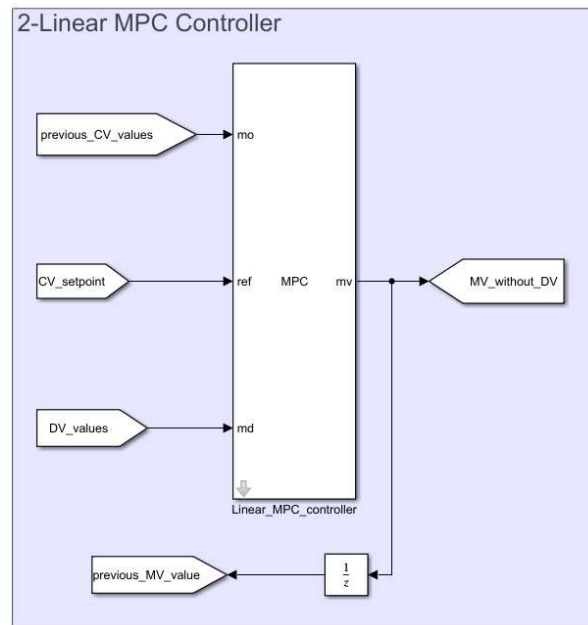


Figure 4-3 Linear MPC Controller Layout

The Linear Controller section consists of a Linear_MPC_controller block. The MPC block receives the following input signals: previous_CV_values, CV setpoint and DV_values. The previous_CV_values signal contains the previous values of Pb % in Conc and Zn % in Conc, which were computed in the previous iteration of the simulation. The CV_setpoint signal contains the CV setpoints' values defined in the Control Parameters Panel. The DV values within the DV_values signals are also specified in the Control Parameters Panel. The Linear_MPC_controller block requires a user-defined model predictive controller object in the workspace to compute the optimal control moves for each iteration. The model predictive controller object is defined using MATLAB's following line of code.

```
mpcobjLin = mpc (Total_Trend_ARX)
```

Nonlinear MPC Simulation

The second simulation was developed using a Nonlinear MPC. The layout of the Nonlinear MPC simulation is shown in Figure 4-4.

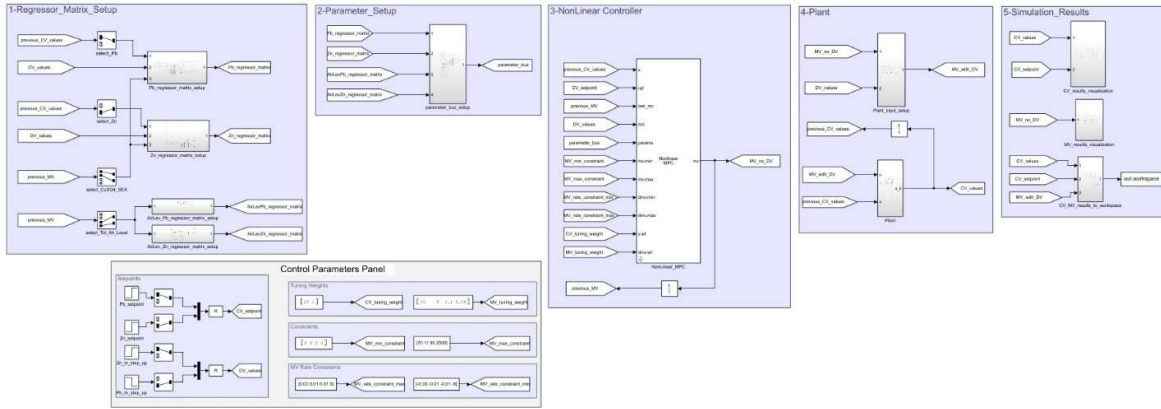


Figure 4-4 Nonlinear MPC Layout

The Nonlinear MPC simulation is split into 6 sections:

1. Regressor Matrix Setup
2. Parameter Setup
3. Nonlinear Controller
4. Plant
5. Simulation Results
6. Control Parameters Panel

The Regressor Matrix Setup section creates the regressor matrices in the Hybrid NLarx model. The Regressor Matrix Setup section in the Nonlinear MPC simulation is identical to that in the linear MPC simulation.

The regressor matrices are sent to the Parameter setup section shown in Figure 4-5. A detailed view of the subsystem within the parameter matrix setup blocks is shown in Appendix C. The regressor matrices are sent to a `parameter_bus_setup` block along with the `Model_parameter_matrices`. The `Model_parameter_matrices` consist of a set of matrices defined in the workspaces and called into Simulink. These are a set of scalar matrices that are used for non-linear mapping within the NLarx model structure. The regressor matrices are also used for non-linear mapping in the NLarx model structure. These `Model_parameter_matrices` will be defined in the discussion of the Nonlinear Controller section. The regressor matrices and the `Model_parameter_matrices` are combined into a bus within the `parameter_bus_setup` block. The bus is required in the Nonlinear Controller section and is used to define the properties of the matrices. The resulting bus is named the `parameter_bus` and is sent to the Nonlinear Controller section.

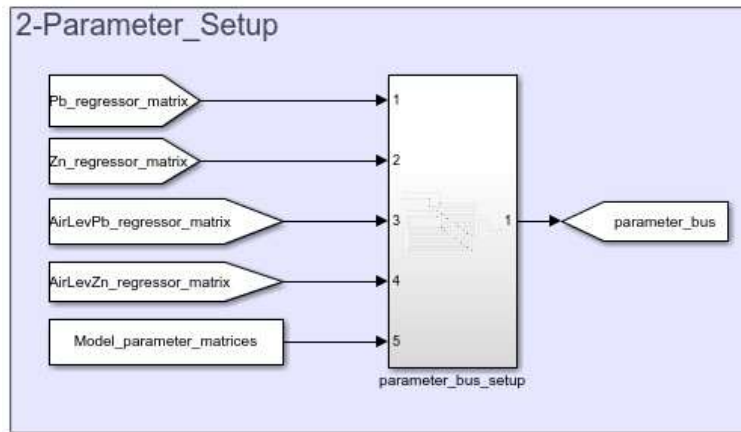


Figure 4-5 Parameter Setup

The Nonlinear Controller section is shown in Figure 4-6 and consists of a Nonlinear MPC Controller block that simulates a nonlinear model predictive controller.

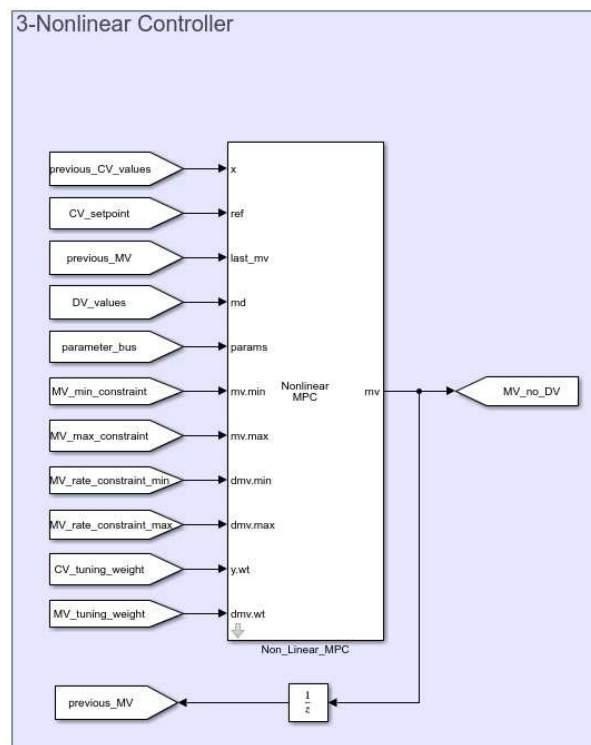


Figure 4-6 Nonlinear Controller Section

The Nonlinear Controller utilises a defined controller object from the workspace to compute the optimal control moves. The controller object contains information about the controller's properties. The properties and assigned values for the properties in this nonlinear controller object are shown in

Table 4-6 .

Table 4-6 Properties of the Nonlinear Controller Object

Properties	Description	Value/Name
Number of Prediction Model States (nx)	Number of states used in the prediction model. Specified as a scalar	2
Number of Prediction Model Outputs (ny)	Number of outputs in the prediction model. Specified as scalar.	2
Manipulated Variables Indices (mvIndex)	Position of manipulated variables in the input vector used in the prediction model. Specified as a vector of scalar values.	[1 2 5 6]
Measured Disturbances indices (mdIndex)	Position of disturbance variables in the input vector used in the prediction model. Specified as a vector of scalar values.	[3 4]
Prediction Model Sample Time (Ts)	Sample time (s) used in the discrete-time model. The rate at which the controller computes the control algorithm, i.e., the duration of each control interval. Specified as a scalar.	60
Prediction Horizon (p)	The number of future time steps or control intervals that the controller predicts during each iteration of the control optimisation algorithm. Specified as a scalar.	40
Control Horizon (m)	The number of future MV moves that the controller uses in the objective function optimisation in each time step. Specified as a scalar.	20
Model	The controller uses the prediction model to make predictions. The model is a structure that consists of the following fields: StateFunction, Output Function, IsContinuousTime and Number of Parameters.	-
State Function	This field specifies the function containing the model structure that will be used for prediction. The series of equations within the model is the state update function. The Output Function can be excluded from this research as the number of outputs and states in the hybrid model are equal; thus, each output value corresponds to a state value.	'NARXHybridFunc'
IsContinuousTime	The type of dynamic model; continuous-time or discrete-time prediction model. Specified as True or False.	False
NumberOfParameters	The number of external parameters that the prediction model requires. These parameters are defined in the workspace. Specified as a scalar.	14

The MATLAB code used to define the Nonlinear Controller object is shown in Appendix C. The Nonlinear Controller block requires a set of inputs produced from either the previous iteration of the simulation run or from the user inserting the input values for controller parameters. These inputs are shown Table 4-7.

Table 4-7 Nonlinear Controller Block Input Values

Signal Name	MATLAB Label	Description	Value/Name
previous_CV_values	x - Model States	A vector containing Pb_conc and Zn_conc values was produced from the plant block in the previous iteration. These values also represent the current model states. These vectors are produced from the plant block in the previous iteration.	Steady-state - [2.265; 40.53]
CV_setpoint	ref - Model output reference values	A vector containing the CV setpoints for the Pb_conc and Zn_conc, respectively. The user specifies the setpoints in the Control Parameters Panel within scope blocks. The values are stepped up or down from the steady values depending on the scenario. These values are specified in the Control Parameters Panel.	Steady-state - [2.265; 40.53]. The CV setpoint change values are described in Table 4-8.
previous_MV	last_mv - Control signals used in the plant at the previous control interval	A vector containing MV values computed by the Nonlinear MPC block in the previous iteration. This vector is produced from the Nonlinear MPC block in the previous iteration.	Steady State = [47.9869; 5.2443; 65.78098; 1554.4513]
DV_values	md - measured disturbances	A vector containing the DV values. The user specifies these values in the Control Parameters Panel within a Step block. The values are stepped up or down from the steady values depending on the scenario. The DV values are specified by the user in the Control Parameters Panel.	Steady State = [69.2939 9.5556]; The DV setpoint change values are described in Table 4-9
parameter_bus	params - Optional parameters	A bus containing the regressor matrices and the Model_parameter_matrices. The prediction model utilises the matrices within this bus. The bus is created in the Parameter_setup section.	The values within the matrices of the parameter_bus are shown in Appendix B.
MV_min_constraint	mv.min - Minimum manipulated variable constraints	A vector containing minimum manipulated variable constraints. These values are specified by the user in the Control Parameters Panel.	[0 0 0 0]
MV_max_constraint	mv.max - Maximum manipulated variable constraints	A vector containing maximum manipulated variable constraints. These values are specified by the user in the Control Parameters Panel.	[70 11 90 2500]
MV_rate_constraint_min	dmv.min - Minimum manipulated variable rate constraints	A vector containing the minimum constraint for the MV increment changes between control intervals. These values are specified by the user in the Control Parameters Panel.	[-0.03 -0.01 -0.01 -9]
MV_rate_constraint_max	dmv.max - Maximum manipulated variable rate constraints	A vector containing the maximum constraint for the MV increment changes between control intervals. These values are specified by the user in the Control Parameters Panel.	[0.03 0.01 0.01 9]
CV_tuning_weight	y.wt - Output variable tuning weights	A vector containing tuning weights for CV reference tracking. These values are specified by the user in the Control Parameters Panel.	[20 1]
MV_tuning_weight	y.wt - Manipulated variable tuning weights	A vector containing tuning weights is used for MV move suppression. These values are specified by the user in the Control Parameters Panel.	[15 8 1.1 0.08]

The controller properties and input values for the linear MPC block were used for the nonlinear MPC block as a basis for controller performance comparison.

The MATLAB code used to define the Nonlinear Controller object is shown in Appendix C.

The MPC nonlinear controller block computes the optimal MV values by solving a nonlinear function. These MV values exit the Nonlinear Controller section and enter the Plant section via a signal labelled MV_without_DV. The Plant section in the Nonlinear control simulation is identical to that in the linear control simulation and utilises the Hybrid NLarx models. The calculated Controlled Variable values from the plant are sent to the Simulation Results section, which is identical to the Simulation Results section in the linear control section.

A Control Parameters Panel was used in the Nonlinear Controller section to specify the Controlled Variable setpoints and the controller parameters. The controller parameters consisted of tuning weights, MV constraints and MV rate constraints. The Control Parameters Panel is shown in Figure 4-7.

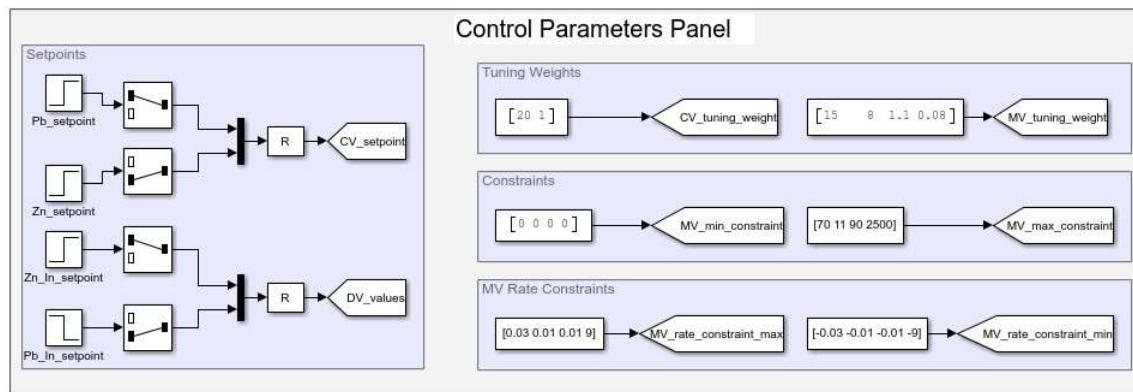


Figure 4-7 Control Parameters Panel for the Nonlinear Control Simulation

Simulation Scenarios

A total of 16 simulations were performed. Each simulation runs for a total of 40000s. After analysis of the step response plots for the Zn % and Pb % in Conc, 40000s was chosen to show the whole dynamics for each change in the controlled variable setpoint and disturbance variable. The first 8 simulations compared the linear and nonlinear MPC controller performance with setpoint tracking. Four scenarios were tested on the linear and nonlinear MPC controller. For each scenario, two setpoint changes occur. The first setpoint change occurs at 2000s. The second setpoint change occurs at 17000s. The Zn % in Conc step response plots indicates that the longest step response time for the input variables is ~ 15000 s. The Zn % in Conc step response time was used as a basis for deciding the simulation times, as the control of Zn % in conc is prioritised. The second setpoint time of 17000s was selected to ensure that the system's dynamic response

for the first setpoint change is captured within the maximum settling time. The description of each scenario is shown in Table 4-8.

Table 4-8 Setpoint Tracking Scenarios

Scenario number	Setpoint Change 1		Setpoint Change 2	
	CV and Setpoint Change Direction	New Setpoint Value	CV and Setpoint Change Direction	New Setpoint Value
1	Zn % in Conc Decreased	38.50	Pb % in Conc Decreased	2.15
2	Zn % in Conc Increased	42.56	Pb % in Conc Increased	2.38
3	Pb % in Conc Increased	2.38	Zn % in Conc Decreased	38.50
4	Pb % in Conc Decreased	2.15	Zn % in Conc Increased	42.56

The last 8 simulations compared the linear and nonlinear MPC controller performance with disturbance rejection. 4 scenarios were tested on the linear and nonlinear MPC controller. For each scenario, there are two changes in the disturbance variable that occur. The first disturbance variable change occurs at 2000s. The second disturbance variable change occurs at 17000s. The description of each scenario is shown in Table 4-9.

Table 4-9 Disturbance Rejection Scenarios

Scenario number	Disturbance Variable Change 1		Disturbance Variable Change 2	
	DV and Change Direction	New Value	DV and Change Direction	New Value
5	Zn In Decreased	65.83	Pb In Increased	10.03
6	Zn In Increased	72.76	Pb In Decreased	9.08
7	Pb In Increased	10.03	Zn In Increased	72.76
8	Pb In Decreased	9.08	Zn In Decreased	65.83

The setpoint tracking simulation scenarios were set up with two CV setpoint changes in a single simulation. The first CV setpoint change is used to examine how the controller performs when a single CV setpoint is changed. After that, the setpoint for the second CV was changed, and the controller performance was assessed for two CV setpoint changes. The CV setpoint change combinations are unique. The first 17000s are used to examine the controller performance when increases and decreases in a single CV setpoint are made. The setpoint changes in the first 17000s are also unique from one another. The Disturbance Rejection scenario combinations were also chosen in a similar manner. The controller property values are shown in Table 4-10.

Table 4-10 Linear Model Predictive Controller Object Properties

Properties	Description	Value/Name
Controller sample time	Controller sample time is specified as a scalar and represented in seconds.	60
Manipulated Variable Indices	The position of manipulated variables in the input vector is used in the prediction model and is specified as a vector of scalar values.	[1 2 5 6]
Disturbance Variable Indices	The position of disturbance variables in the input vector is used in the prediction model and is specified as a vector of scalar values.	[3 4]
Nominal Plant Input	The plant input values are at the nominal operating point of linearising the plant model. The nominal value (steady state) of the manipulated and disturbance variables is included. Specified as a vector of scalar values.	[47.9869; 5.2443; 69.2939; 9.5556; 1554.4513; 65.78098] Default Value = [0; 0; 0; 0; 0; 0]
Nominal Plant Output	The plant output values at the nominal operating point, where the plant model is linearised. The manipulated and disturbance variables' nominal (steady state) values are included. The vector of scalar values is specified.	[2.265; 40.53] Default Value = [0; 0]
Prediction Horizon	The number of future time steps or control intervals that the controller predicts during each iteration of the control optimisation algorithm. Specified as a scalar.	40 Default Value = 10
Control Horizon	The number of future MV moves that the controller uses in the objective function optimisation in each time step. Specified as a scalar.	20 Default Value = 2
Manipulated Variable lower bound	A vector containing minimum manipulated variable constraints.	[0 0 0 0] Default Value = [-Inf; -Inf; -Inf; -Inf]
Manipulated Variable upper bound	A vector containing maximum manipulated variable constraints.	[70 11 90 2500] Default Value = [Inf; Inf; Inf; Inf]
Manipulated Variable rate of change lower bound	A vector containing the minimum constraint for the MV increment changes between control intervals.	[-0.03 -0.01 -0.01 -9] Default Value = [-Inf; -Inf; -Inf; -Inf]
Manipulated Variable rate of change upper bound	A vector containing the maximum constraint for the MV increment changes between control intervals.	[0.03 0.01 0.01 9] Default Value = [Inf; Inf; Inf; Inf]
Manipulated Variable rate tuning weights	A vector containing tuning weights is used for MV move suppression.	[15 8 1.1 0.08] Default Value = [0.1; 0.1; 0.1; 0.1]
Output variable tuning weights	A vector containing tuning weights for CV reference tracking.	[20 1] Default Value = [1; 1]

All the variable indices were chosen based on the order of appearance in the dataset.

The MPC controller block computes the optimal MV values. These MV values exit the Linear Controller section and enter the Plant section via a signal labelled MV_without_DV. The MV values from the current iteration are looped back to the Regressor_Matrix_Setup and are stored as regressors for the next iteration.

The Plant section is shown in Figure 4-8.

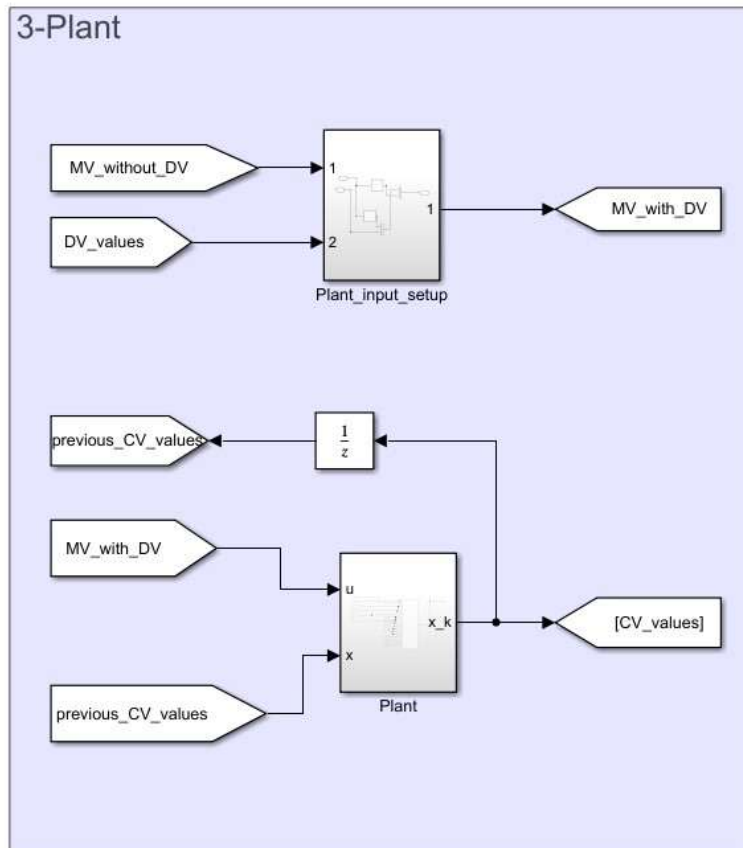


Figure 4-8 Linear Plant Section Layout

In the Plant section, the MV_without_DV signal and the DV_values signal enter the Plant_input_setup block. An MV_with_DV signal is produced from the Plant_input_setup block. The DV_values signal is created in the Control_Parameters Panel using user-defined DV values. The MV_with_DV signal enters the Plant block along with the previous_CV_values signals. The Plant block contains a function block that holds the function NARXHybridFunc, representing the plant. This function block computes the values of the Controlled Variables or Plant Outputs. The function block requires the Model Parameter Matrices that were specified in section 4.2 to complete the computation. The MV_with_DV signal, previous_CV_values signal, and the Model Parameter Matrices are inputs for the function block. The contents of the Plant block are shown in Appendix C. The output of the Plant block is the CV values for the current iteration of the

simulation. The CV values are placed through a unit delay block, which holds the CV values for the current iteration and uses these CV values for the next iteration. This signal is referred to as the previous_CV_values signal. The CV_values signal is sent to the Simulation Results section.

The Simulation Results section is shown. Figure 4-9.

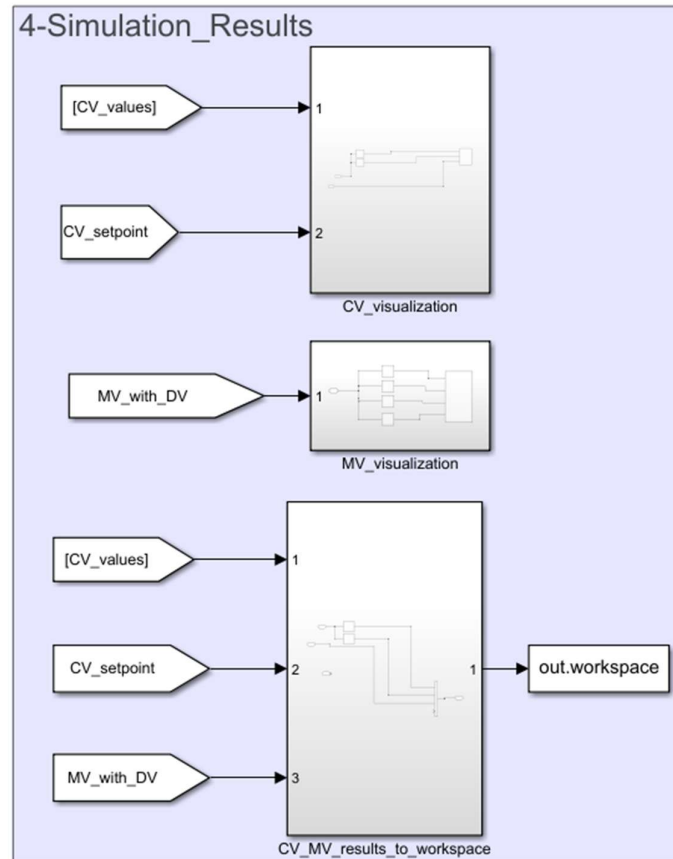


Figure 4-9 Simulation Results Section Layout

The CV_visualization block within the Simulation Results section was used to visualise the CV value readings compared to the CV setpoint values. The CV_visualization block contains scope blocks used to analyse the CV plots. The MV_visualization block was used to visualise the MV readings using scope blocks. The CV_MV_results_to_workspace block exported the simulation results to the MATLAB workspace. The simulation results included the CV value readings, CV setpoints and MV value readings produced at each iteration in the simulation.

A Control Parameters Panel was used in the Linear MPC simulation and is shown in Figure 4-10.

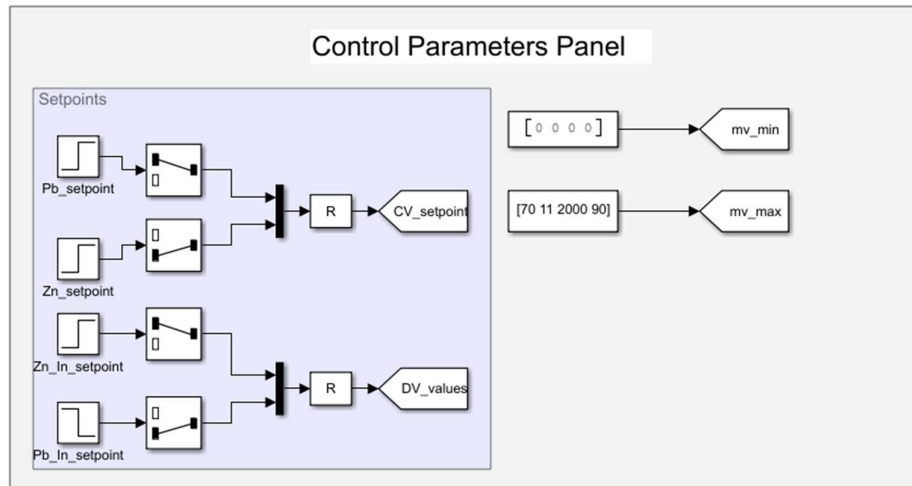


Figure 4-10 Linear MPC Simulation Control Parameters Panel

The Control Parameters Panel was used to specify the Setpoint step change values for the Pb % in Conc and the Zn % in Conc with step change blocks. The time at which the setpoints were stepped up or down was specified within the block. The Zn_In and Pb_In values are also specified in the Control Parameters Panel. Furthermore, the Manipulated Variable lower bound and Upper bound constraints are defined in the Panel. The MV constraints leave the panel through the mv_min and mv_max signals.

4.4. Controller Design Parameters

The controller performance for each scenario using the different sets of prediction horizons and control horizon mentioned in section 4.1 was plotted. The small prediction and control horizon set yielded unfavourable controller behaviour with long response times for both scenarios. The medium prediction and control horizon set showcased reasonable response times that corresponded with the response times of the Hybrid Models and presents a critically damped response. The controlled variable and manipulated variable plots for the medium prediction horizon and control horizon set are shown in

Figure 4-11 to

Figure 4-14. The remainder of the plots are shown in Appendix C.

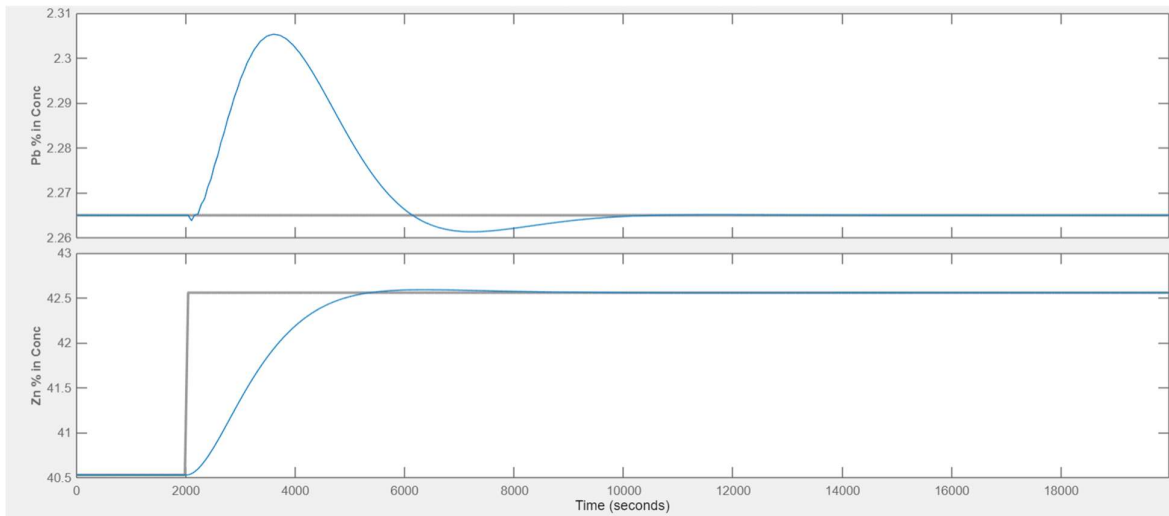


Figure 4-11 Controlled variable plot for a Zn % in Conc setpoint change

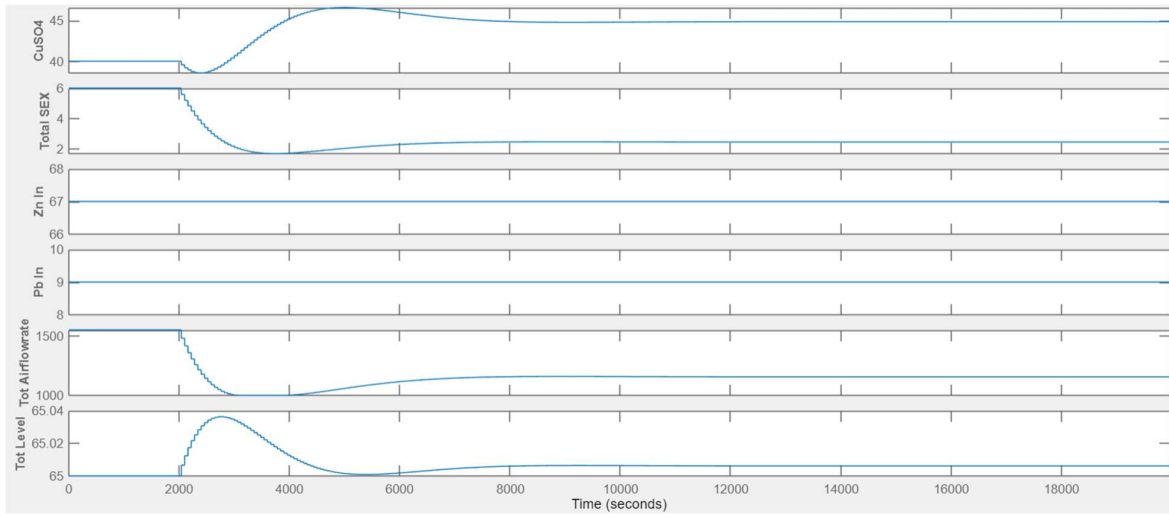


Figure 4-12 Manipulated variable plot for a Zn % in Conc setpoint change

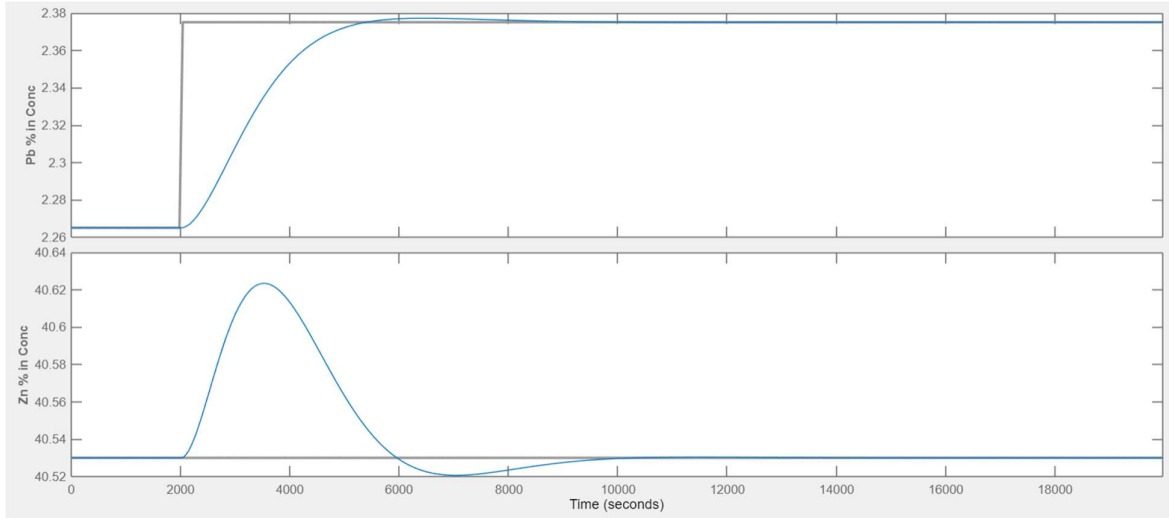


Figure 4-13 Controlled variable plot for a Pb % in Conc setpoint change

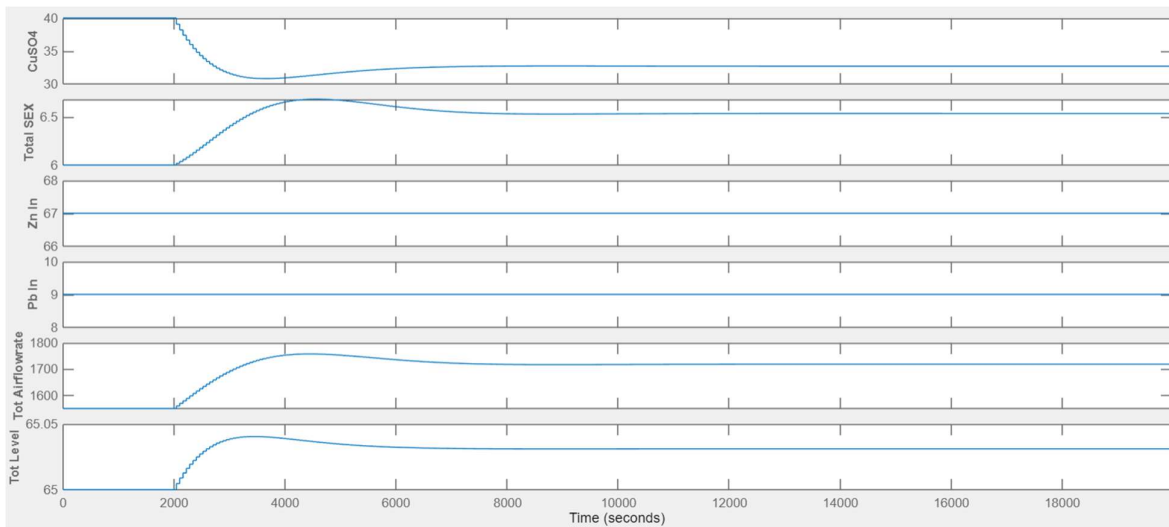


Figure 4-14 Manipulated variable plot for a Zn % in Conc setpoint change

The large prediction and control horizon yielded a similar response to that of the medium set. Thus, the prediction horizon of 40 and control horizon of 20 was chosen for the remainder of the controller parameter analysis, as the performance was satisfactory and to avoid additional computations.

4.5. Simulated System Response

After analysing the model performance results, the Total Trend ARX models and Hybrid NLarx models developed in section 3.2 were used to develop the internal plants of the linear MPC and nonlinear MPC,

respectively. The simulations were designed to assess which controller performs better. The criteria used to compare the performance of the controllers were the efficiency of the controllers in achieving the control objectives. The simulation results for the Setpoint Tracking and Disturbance Rejection performance of the linear and nonlinear MPC are shown in section 4.5.1 and 4.5.2.

4.5.1. Setpoint Tracking

The setpoint tracking capability of the controller was tested using simulations where the CV setpoint changes were implemented. The setpoint tracking results will be used to assess whether the controllers are able to meet control objective 1 and 3. In these simulations, the first CV setpoint change is used to examine how the controller performs when a single setpoint change is implemented. After that, the setpoint for the second CV was changed, and the controller performance was assessed. The setpoint change combinations are unique and were presented in Table 4-8. The results of the controller performance for the setpoint tracking simulations are described in this section. The results consist of Zn % in Conc and Pb % in Conc response plots and the manipulated and disturbance variable plots.

The results for simulation scenario 1 (Zinc % in Conc Setpoint Decreased, followed by a decrease in Lead % in Conc Setpoint) are shown in Figure 4-15 and Figure 4-16.

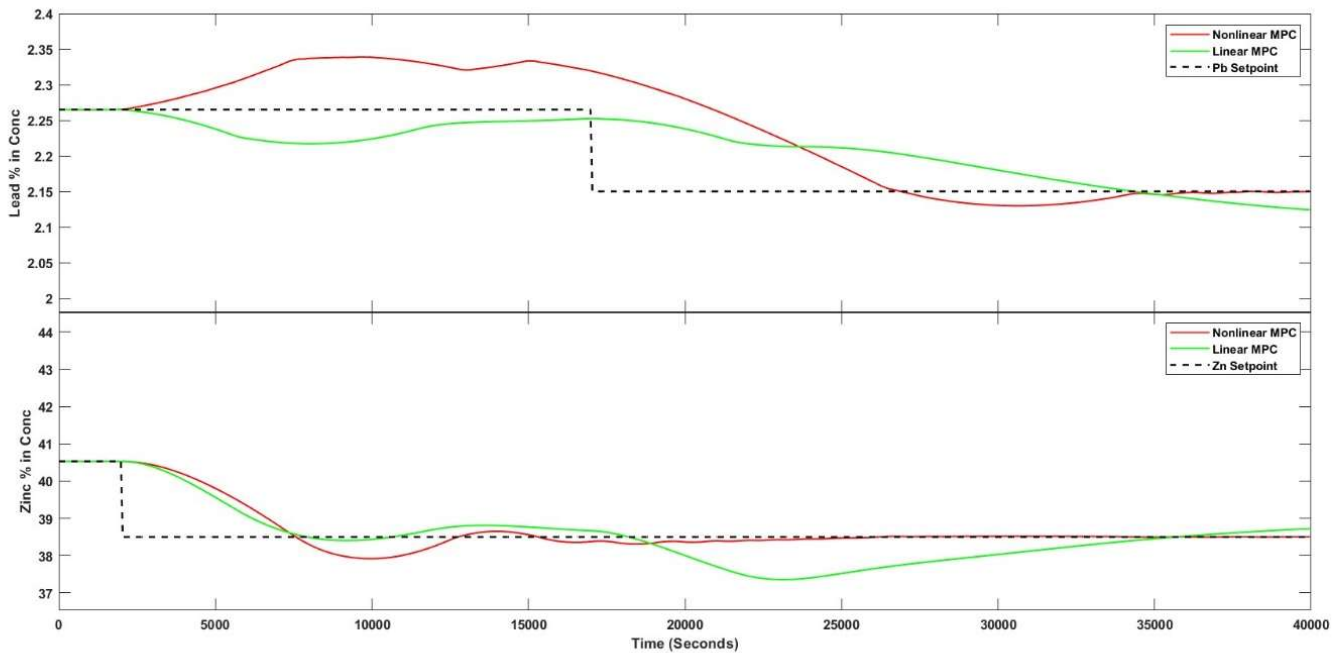


Figure 4-15 Zinc % in Conc and Lead % in Conc Response Plots

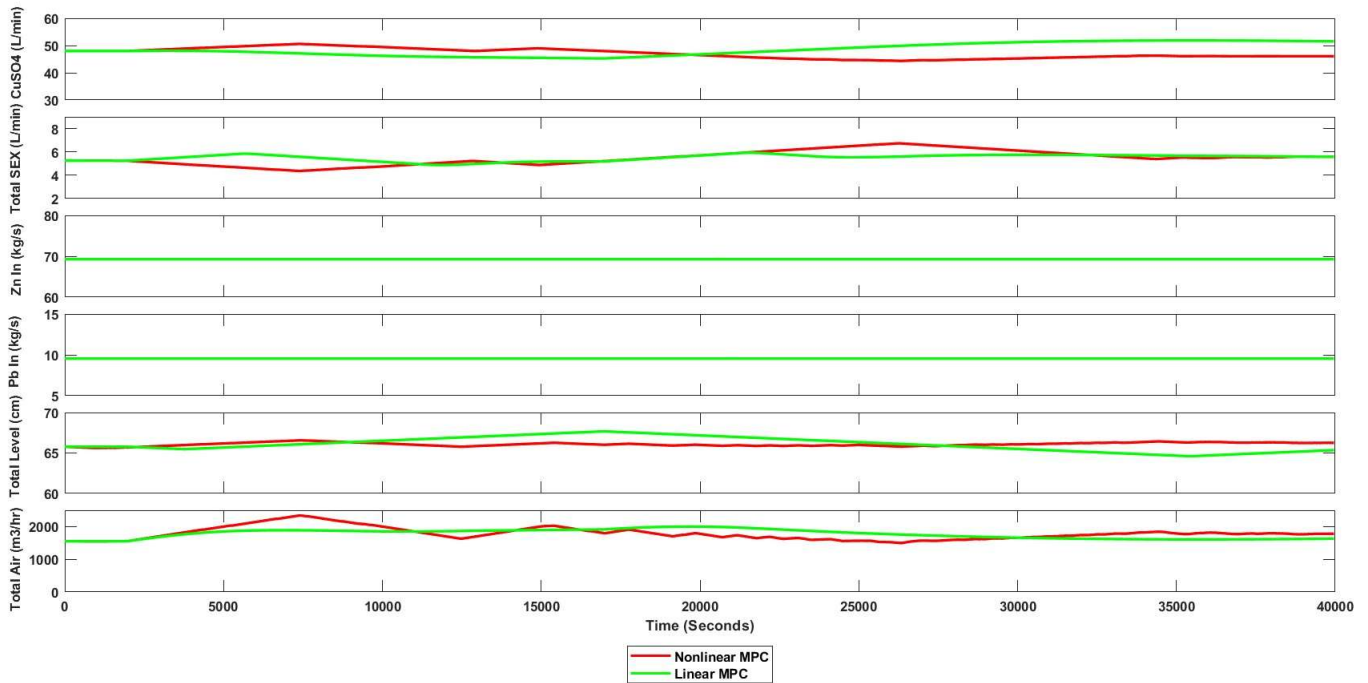


Figure 4-16 Manipulated and Disturbance Variable Response Plots

First 17000 s:

In Figure 4-15, the Pb % in the Conc for both controllers was not able to return to the setpoint once the Zn % in Conc was decreased. The linear MPC controller moved the Pb % in Conc closer to the setpoint. However, the nonlinear MPC moved the Pb % in Conc further from the Pb % in Conc setpoint.

The Zn % in the Conc for both controllers met the new Zn % in Conc setpoint. The Zn % in the Conc using the MPC controller reaches the new setpoint sooner than the nonlinear controller. However, the Zn % in Conc with the linear MPC controller results in a slight offset after reaching the setpoint. The nonlinear controller results in the Zn % in Conc oscillating around the new setpoint. This oscillation indicates that a unstable state was reached, and additional tuning is recommended for the response to return a stable state. However, the results to proceed indicate that a stable state of the Zn % in Conc can be reached when the Pb % in Conc is decreased.

After 17000s:

The Pb % in Conc does not reach the new Pb % in Conc setpoint using the linear MPC controller. Furthermore, the Zn % in Conc does not return to the setpoint using the linear MPC controller. However, the Pb % in Conc does reach the new Pb % in Conc using the nonlinear controller. The nonlinear controller results in an initial small offset with the Zn% in Conc reading when the Pb % in Conc setpoint change is introduced. Thereafter, the Zn % in Conc returns to the setpoint shortly after.

Overall, the nonlinear controller illustrates better Zn % in Conc setpoint tracking than the linear MPC controller when both the Zn % in Conc and Pb % in Conc setpoint decrease. The Pb % in Conc setpoint tracking for the linear controller was better than the nonlinear controller when only the Zinc % in Conc is decreased. However, the Pb % in Conc setpoint tracking for the nonlinear controller improves once the Pb % in Conc is decreased, and the nonlinear controller performs better than the linear controller.

The results for simulation scenario 2 (Zinc % in Conc Setpoint Increased, followed by an Increase in Lead % in Conc Setpoint) are shown in Figure 4-17 and Figure 4-18.

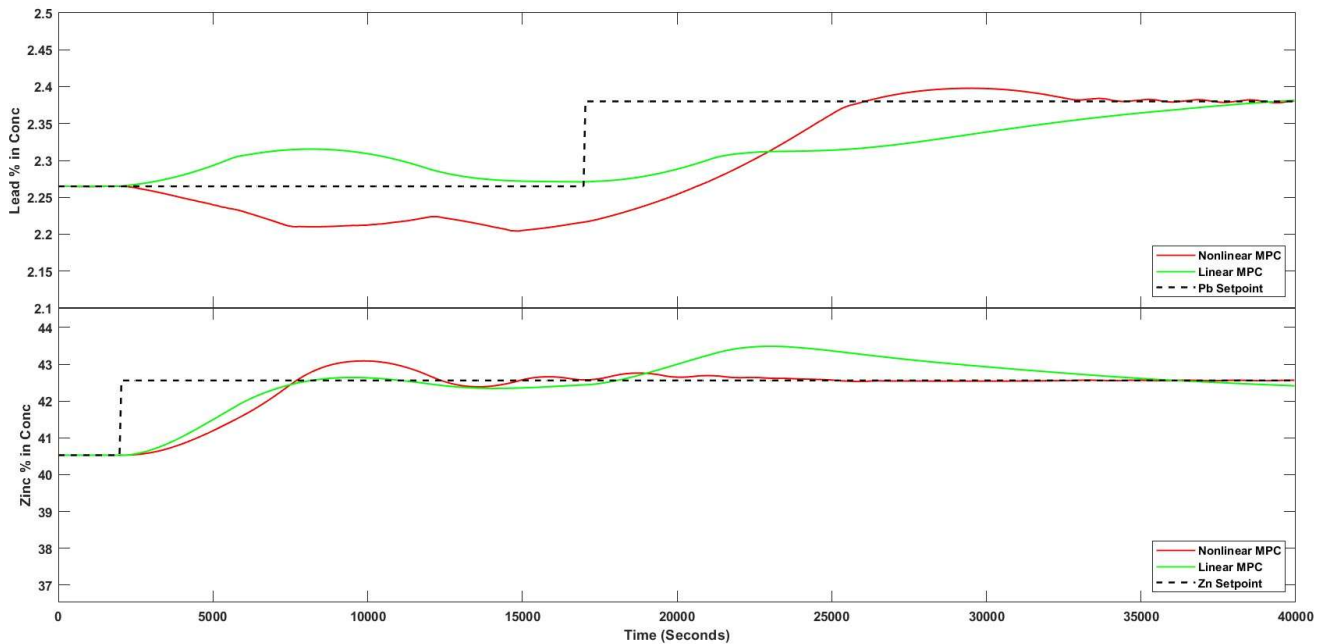


Figure 4-17 Pb % in Conc and Zn % in Conc Response Plots

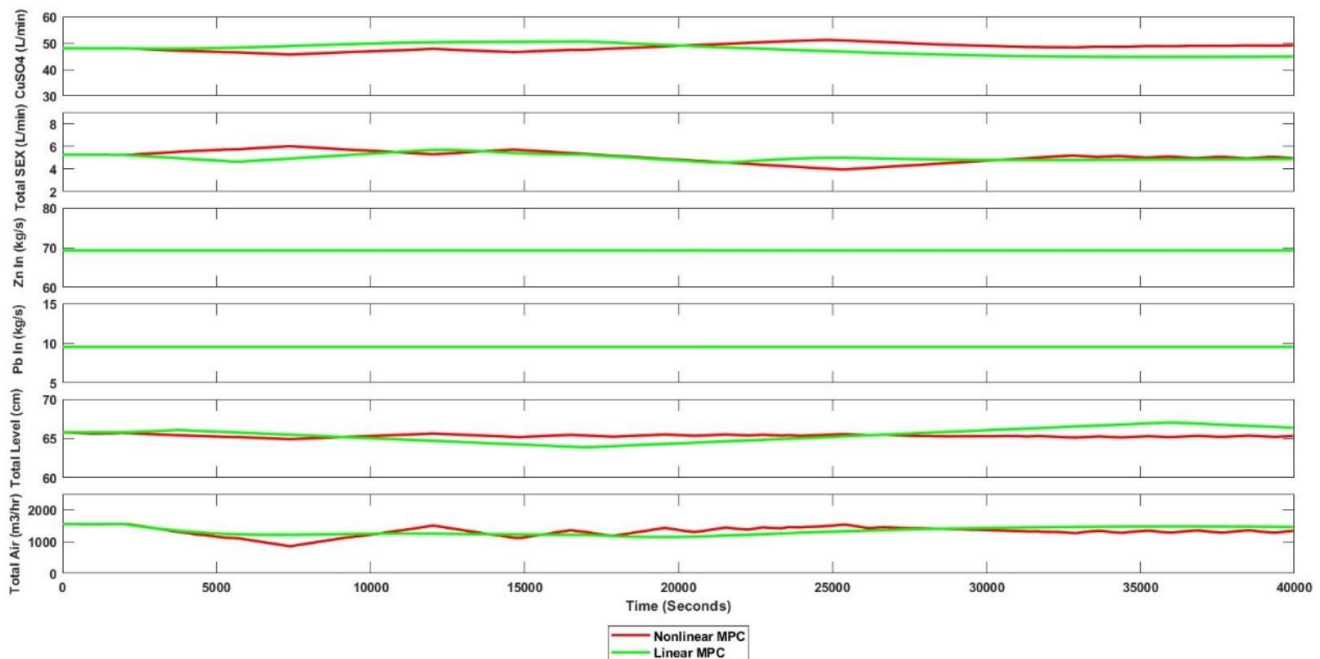


Figure 4-18 Manipulated Variable Response Plots

First 17000 s:

In Figure 4-17, the Pb % in the Conc reading for both controllers was not able to return to the setpoint once the Zn % in Conc was Increased. The linear MPC controller moved the Pb % in Conc closer to the setpoint. However, there was a small difference between the reading and the setpoint. The nonlinear MPC moved the Pb % in Conc further from the Pb % in Conc setpoint.

The Zn % in the Conc reading for both controllers met the new Zn % in Conc setpoint. The Zn % in the Conc reading reaches the new setpoint at a similar time for both controllers. However, the nonlinear controller results in the Zn % in Conc reading oscillating around the new setpoint.

After 17000 s:

The Pb % in Conc does reach the new Pb % in Conc setpoint with both controllers. However, the Pb % in Conc reading reaches the new setpoint sooner with the nonlinear controller. Similarly, the Zn % in Conc reading does reach the new Zn % in Conc setpoint with both controllers. However, the setpoint is reached sooner with the nonlinear controller.

Overall, both controllers illustrate good Zn % in Conc setpoint tracking. The Pb % in Conc setpoint tracking for the linear controller was better than the nonlinear controller when the Zn % in Conc was increased. However, the nonlinear controller can meet the new Pb % in Conc setpoint faster than the linear controller.

The results for simulation scenario 3 (Lead % in Conc Setpoint Increased, followed by a decrease in Zinc % in Conc Setpoint) are shown in Figure 4-19 and Figure 4-20.

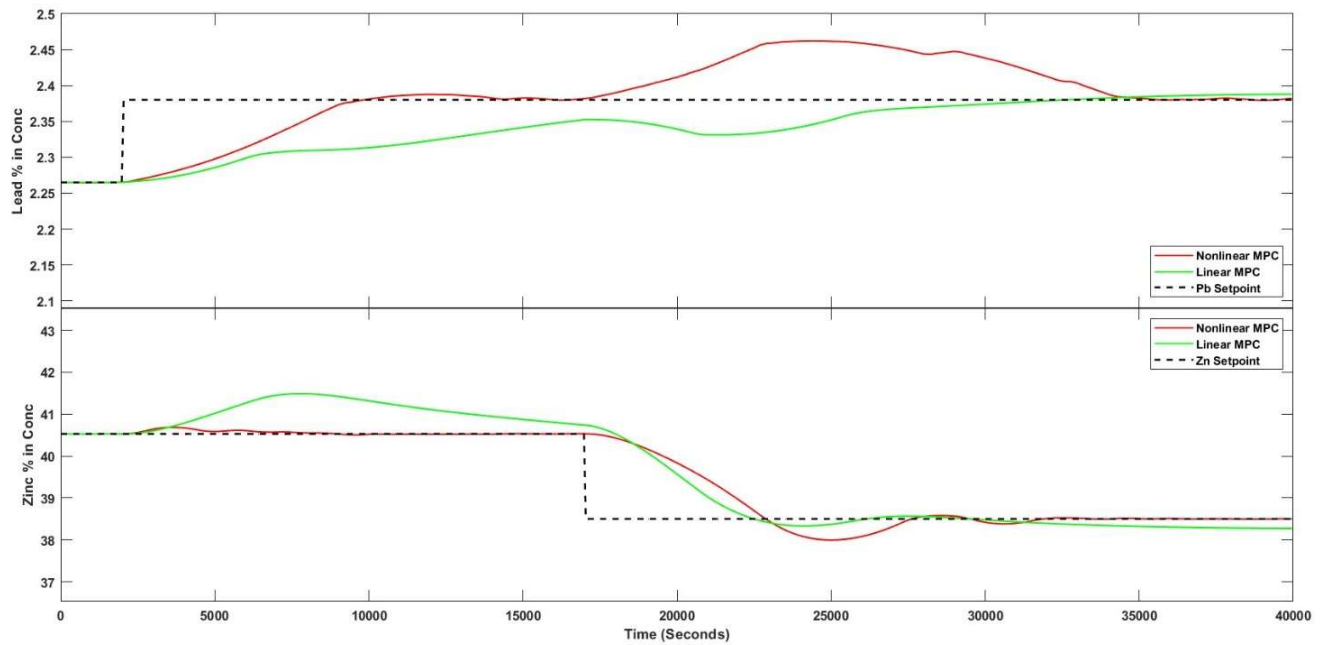


Figure 4-19 Lead % in Conc and Zinc % in Conc Response Plots

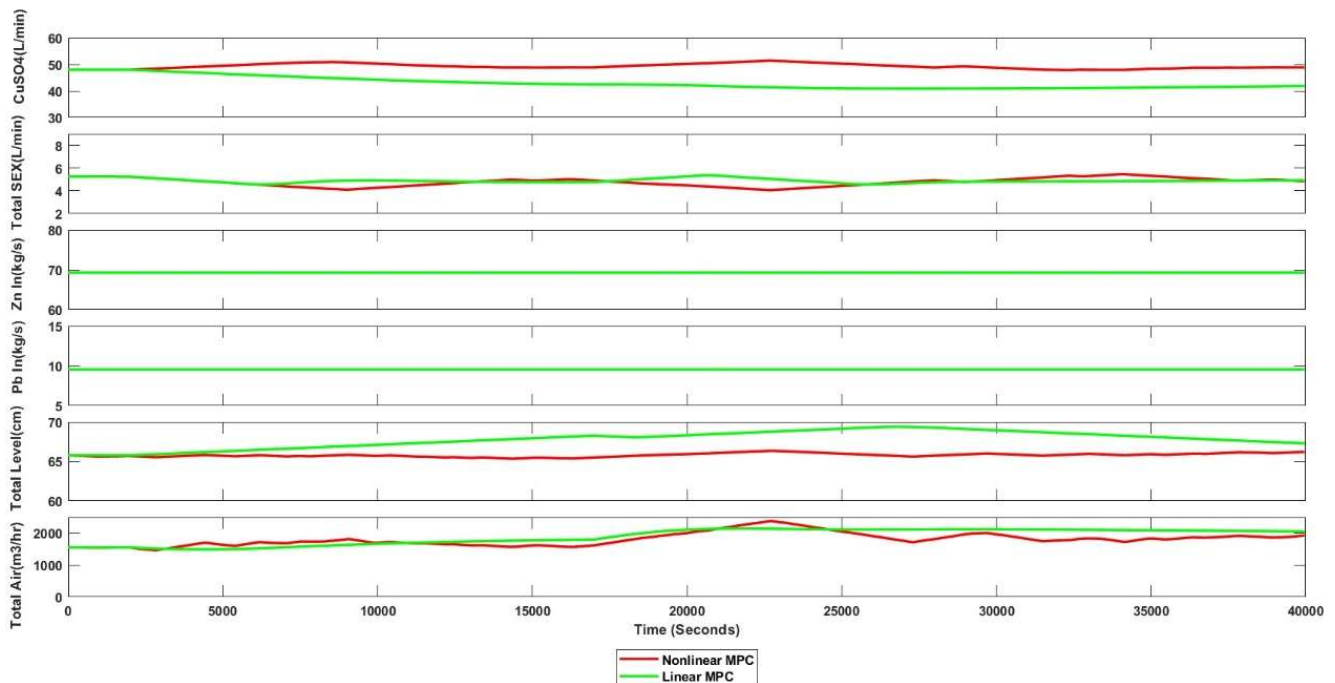


Figure 4-20 Manipulated Variable Response Plots

First 17000 s:

In Figure 4-19, the Pb % in the Conc reading was able to meet the Pb % in Conc setpoint using the nonlinear controller. However, the Pb % in the Conc reading was unable to meet the new Pb % in Conc setpoint using the linear controller. The Zn % in the Conc reading could return to the Zn % in Conc setpoint using the nonlinear controller. However, the Zn % in the Conc reading could not return to the Zn % in Conc setpoint using the linear controller.

After 17000s:

The Pb % in Conc reading does return to the Pb % in Conc setpoint with both controllers. The response time to the setpoint is similar for both controllers. The Zn % in Conc reading does reach the new setpoint with both controllers. However, a slight offset exists between the Zn % in Conc reading and the setpoint. Overall, the nonlinear controller illustrates better Zn % in Conc setpoint tracking than the linear MPC controller with faster return to setpoint response times. The Pb % in Conc setpoint tracking for the nonlinear controller is better than the linear controller when handling a Pb % in Conc setpoint increase. However, it does produce a large error between the reading and the setpoint when the Zn % in Conc is increased. Meanwhile, the linear controller produces a small error between the reading and the setpoint.

The results for simulation scenario 4 (Lead % in Conc Setpoint Decreased, followed by an Increase in Zinc % in Conc Setpoint) are shown Figure 4-21 and Figure 4-22.

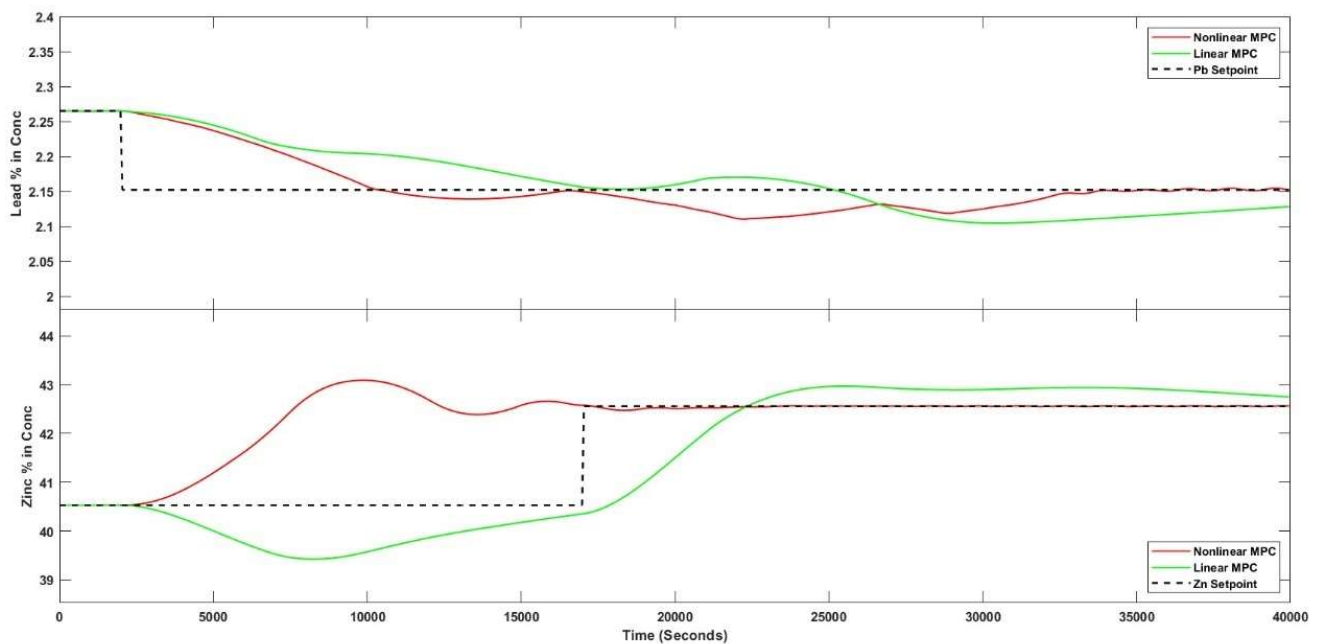


Figure 4-21 Lead % in Conc and Zinc % in Conc response plots

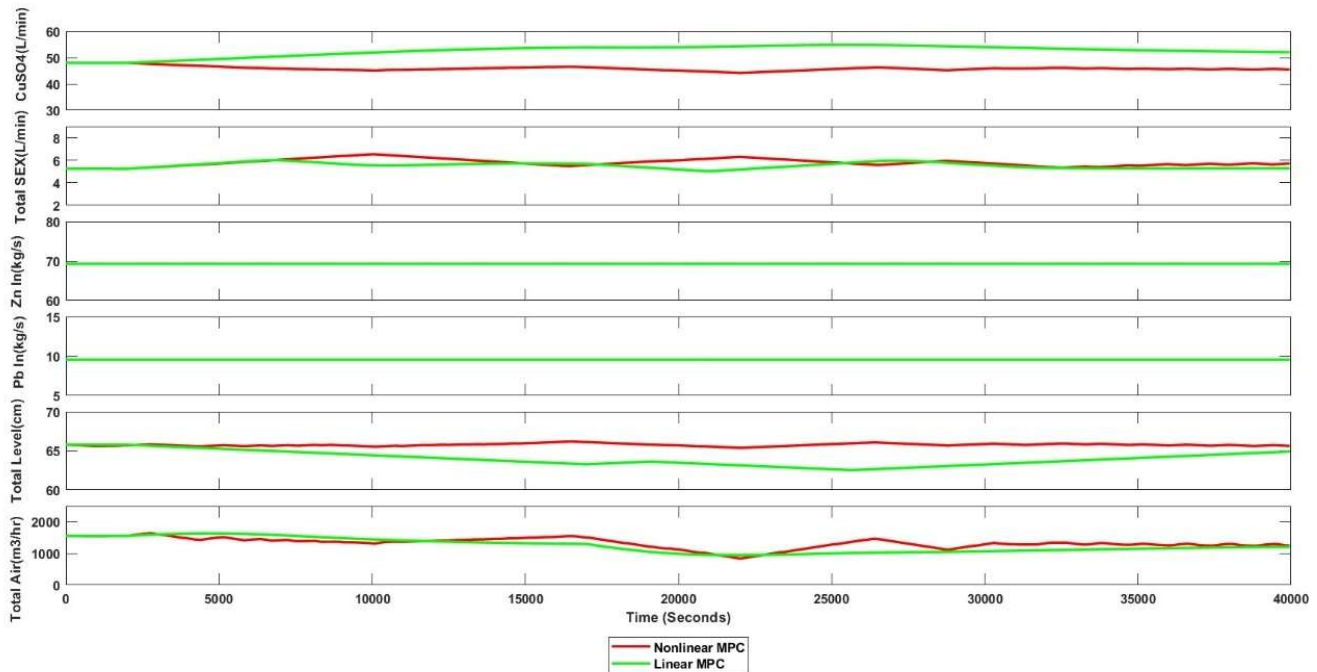


Figure 4-22 Manipulated Variable Input Results

First 17000 s:

In Figure 4-21, the Pb % in the Conc reading was able to meet the new Pb % in Conc setpoint using both controllers. However, the nonlinear controller's Pb % in Conc reading reaches the new setpoint sooner than the linear controller. The Zn % in the Conc reading was not able to return to the Zn % in Conc setpoint using both controllers. However, the linear MPC controller moved the Zn % in Conc closer to the setpoint with a small difference between the reading and the setpoint. The nonlinear MPC moved the Zn % in Conc further from the Zn % in Conc setpoint.

After 17000s:

The Pb % in Conc reading does return to the Pb % in Conc setpoint with the nonlinear controller. However, the Pb % in Conc reading does return to the Pb % in Conc setpoint with the linear controller. The Zn % in Conc reading does reach the new setpoint with the nonlinear controller, whereas the Zn % in Conc reading does not reach the new setpoint with the linear controller.

Overall, the nonlinear controller illustrates better Pb % in Conc setpoint tracking than the linear MPC controller with faster return to setpoint response times. The Zn % in Conc setpoint tracking for the nonlinear controller is better than the linear controller when handling a Zn % in Conc setpoint increase. However, it

does produce a large error between the reading and the setpoint when the Pb % in Conc is decreased. The Normalised Root Mean Squared Error between the Zn % in Conc and Pb % in Conc readings and the setpoints were calculated for each Setpoint Tracking scenario and are shown in Table 4-11. The formula used to evaluate the NRMSE is shown in Appendix B.

Table 4-11 Setpoint Tracking Simulations: NRMSE Results

Simulation	Con Zn		Con Pb	
	Linear MPC	Nonlinear MPC	Linear MPC	Nonlinear MPC
	NRMSE	NRMSE	NRMSE	NRMSE
1 - Zn % in Conc Decreased First Pb % in Conc Decreased	1.73	1.43	2.04	2.87
2 - Zn % in Conc Increased First Pb % in Conc Increased	1.45	1.32	2.07	2.45
3 - Pb % in Conc Increased First Zn % in Conc Decreased	1.62	1.40	2.03	2.06
4 - Pb % in Conc Decreased First Zn % in Conc Increased	2.13	1.74	1.66	1.30

The time the Zn % in Conc and Pb % in Conc reading met the respective setpoints after the Zn % in Conc and Pb % in Conc setpoints were changed was recorded and is shown in Table 4-12.

Table 4-12 Setpoint Tracking Simulations: Response Times

Setpoint Change Simulations	Con Zn		Con Pb	
	Linear MPC	Nonlinear MPC	Linear MPC	Nonlinear MPC
	Response Time (s)	Response Time (s)	Response Time (s)	Response Time (s)
1 - Zn % in Conc Decreased First Pb % in Conc Decreased	7500 0.63 % offset	12500 25000	0.66 % offset 1.16 % offset	2.65 % offset 35000
2 - Zn % in Conc Increased First Pb % in Conc Increased	7500 37500	7500 17000*	15000 39000	2.87 % offset 32500
3 - Pb % in Conc Increased First Zn % in Conc Decreased	0.62 % offset 22500	5000 22500	1.26 % offset 32500	10000 35000
4 - Pb % in Conc Decreased First Zn % in Conc Increased	0.31 % offset 0.59 % offset	10000 32500	17000 2.33 % offset	10000 20000

The NRMSE values for Zn % in Conc in Table 4-11 are lower with the nonlinear controller than the linear controller for all the scenarios. The nonlinear controller NRMSE values for Zn % in Conc range from 1.3 to 1.7 %, and the linear controller NRMSE values for Zn % in Conc range from 1.4 to 2.1%. Furthermore, Table 4-12 illustrates that the Zn % in Conc reading reaches the new setpoint with the nonlinear controller in all scenarios. The Zn % in Conc reading also returns to the setpoint when the Pb% in Conc setpoint is changed in all scenarios.

However, the Zn % in Conc reading only matches the setpoint for both CV setpoint changes in scenario 2 with the linear controller. Whereas the linear controller in scenarios 1,3, and 4 does match the Zn% in Conc reading to the setpoint when either one or both CV setpoints are changed. Thus, the nonlinear controller performs better than the linear controller with Zn % in Conc setpoint tracking.

The NRMSE values for Pb % in Conc in Table 4-11 are lower with the linear controller than the nonlinear controller for all the scenarios. The nonlinear controller NRMSE values for Pb % in Conc range from 1.3 to 2.8 %, and the linear controller NRMSE values for Pb % in Conc range from 1.6 to 2.0%. The NRMSE values are lower for the linear controller than the nonlinear controller; however, Table 4-12 illustrates that the linear controller cannot match the reading to the setpoint at the end of the simulations for most of the scenarios. The nonlinear controller can match the Pb% in Conc reading at the end of the simulation for all scenarios. In scenarios 1 and 2, the nonlinear controller is able to match the Pb% in Conc reading to the setpoint at the end of the simulation.

However, these scenarios showcase that the nonlinear controller cannot maintain the Pb % in Conc reading to the setpoint when a change in the Zn % in Conc setpoint is implemented in isolation. The nonlinear controller can only return the reading to the setpoint once there is a change in the Pb % in Conc setpoint.

Thus, the nonlinear controller performs better than the linear controller with Pb% in Conc setpoint tracking when the Pb% in Conc setpoint is changed. However, the nonlinear controller is not able to maintain the reading at the setpoint when a change in the Zn % in Conc setpoint causes process interaction.

Overall, the results indicate that linear controller and nonlinear controller can achieve control objective 1. However, there are exceptions with the linear controller. For scenarios where the Zn setpoint is changed first, the linear controller can initially meet the new Zn setpoint. However, the controller attempts to move the Pb % in Conc back to the original setpoint which results in an offset in the Zn reading from the setpoint. when maintaining the Pb % in conc and thus causes an offset in the controller reading. This indicates that the controller would be able to keep the Zn % in conc reading at the respective setpoints if there was not a second CV present.

One potential explanation for this phenomenon could be the inadequate performance of the Pb models used in the linear controller development. The controller can make poor control moves if the model is inaccurate. Future work can include reducing the CV weighting of the Pb % in conc to prioritise control of Zn % in conc. Alternatively, the Pb models can be modified to improve model performance. Overall, the nonlinear controller has better setpoint tracking for the Zn % in Conc than the linear controller.

Both controllers show poor setpoint tracking for the Pb % in Conc in specific scenarios. Thus, the controllers are not effective in achieving control objective 3.

4.5.2. Disturbance Rejection Results

The disturbance rejection capability of the controller was tested using simulations where the disturbance variable values were varied. The disturbance rejection results will be used to assess whether the controllers are able to meet control objective 2 and 4. In these simulations, the first DV change is used to examine how the controller performs when a single disturbance change is implemented. After that, the value of the second DV was changed, and the controller performance was assessed for two DV changes. The DV change combinations are unique and were presented in Table 4-9. The controller performance results for the disturbance rejection simulations are presented in this section. The results consist of Zn % in Conc and Pb % in Conc response plots and the manipulated and disturbance variable plots.

An analysis of the system's open-loop response was conducted to evaluate the role of the controller in mitigating disturbances that could disrupt the operation of the flotation bank at its designated setpoints. The open-loop response was developed by adjusting the disturbance variable values in a system without a controller and examining how the external plant's response compares to the setpoints. These results were evaluated for the disturbance rejection simulation scenarios (scenario 5 to 8) and are displayed in this section.

The results for simulation scenario 5 (Zinc In Decreased, followed by an Increase in Lead In) are shown in

Figure 4-23 and Figure 4-24.

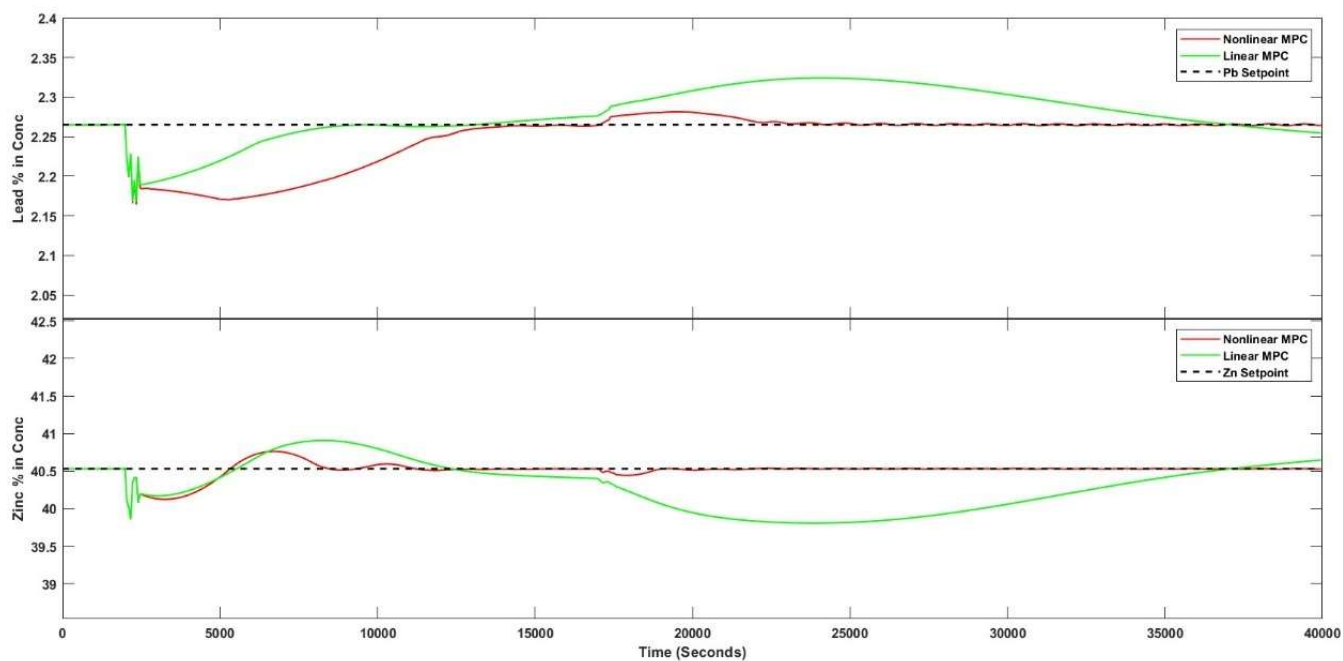


Figure 4-23 Zn % in Conc and Pb % in Conc Response Plots

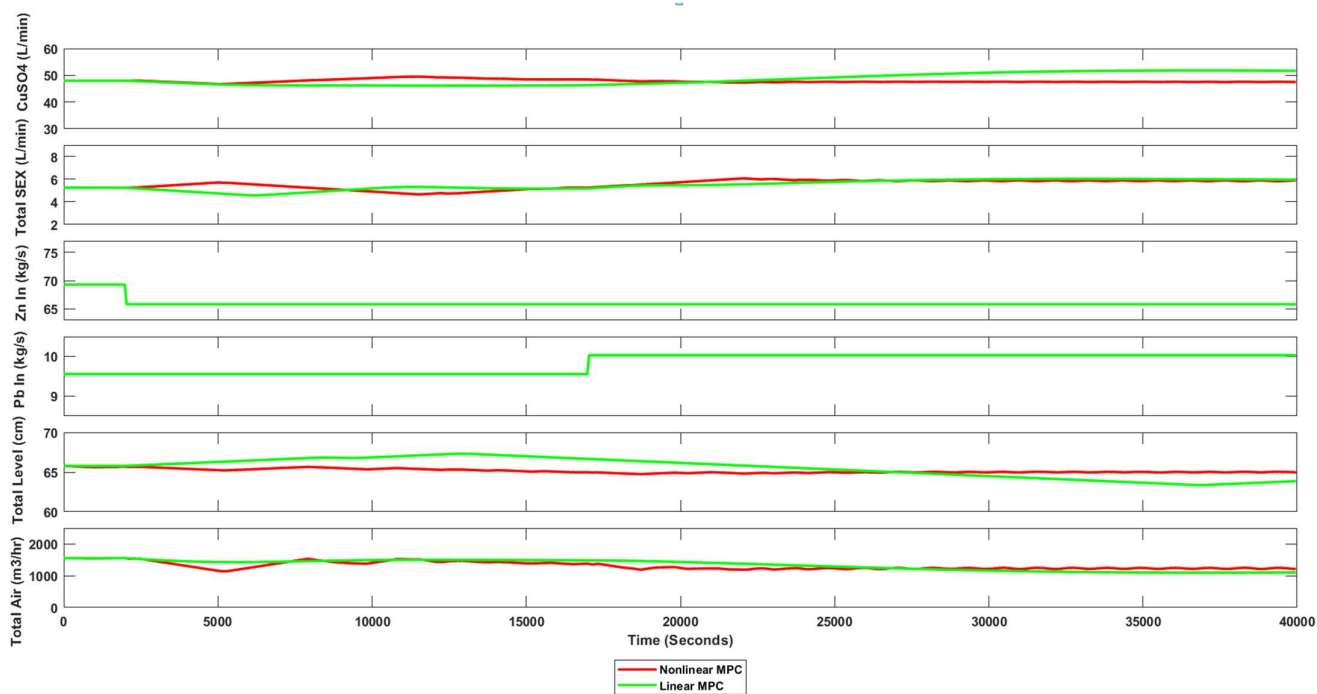


Figure 4-24 Manipulated Variable Input Response Plots

First 17000 s:

In

Figure 4-23, the Pb % in the Conc reading could return to the new Pb % in Conc setpoint using both controllers after the disturbance variable, Zn In, was decreased. However, the linear controller's Pb % in Conc reading returns to the setpoint sooner than the nonlinear controller.

Similarly, the Zn % in the Conc reading was able to return to the Zn % in Conc setpoint using both controllers. However, the nonlinear MPC controller was able to return to the Zn % in Conc sooner than the linear controller. Furthermore, the Zn % in Conc remains at the setpoint for a short duration with the linear controller and then moves past the setpoint, creating a slight offset.

After 17000s:

The Pb % in Conc reading does return to the Pb % in Conc setpoint with the nonlinear controller when the disturbance variable, Pb In, is increased. Additionally, the offset from the setpoint is minimal, and the duration of the offset is short with the nonlinear controller.

The Pb % in Conc reading does not return to the Pb % in Conc setpoint with the linear controller. Similarly, the Zn % in Conc reading does return to the new setpoint with the nonlinear controller, whereas the Zn % in Conc reading does not return to the setpoint with the linear controller. Furthermore, the offset from the setpoint is minimal, and the duration of the offset is short with the nonlinear controller.

Overall, the nonlinear controller illustrates better disturbance rejection than the linear controller with the Zn % in Conc reading when the Zn In and Pb In values are altered. However, the disturbance rejection in the Pb % in Conc reading with the linear controller is better than the nonlinear control when the Zn In is decreased.

However, the nonlinear controller shows better disturbance rejection once the Pb In value increases after the Zn In value increases.

The results for simulation scenario 6 (Zinc In Increased, followed by a Decrease in Lead In) are shown Figure 4-25 and Figure 4-26.

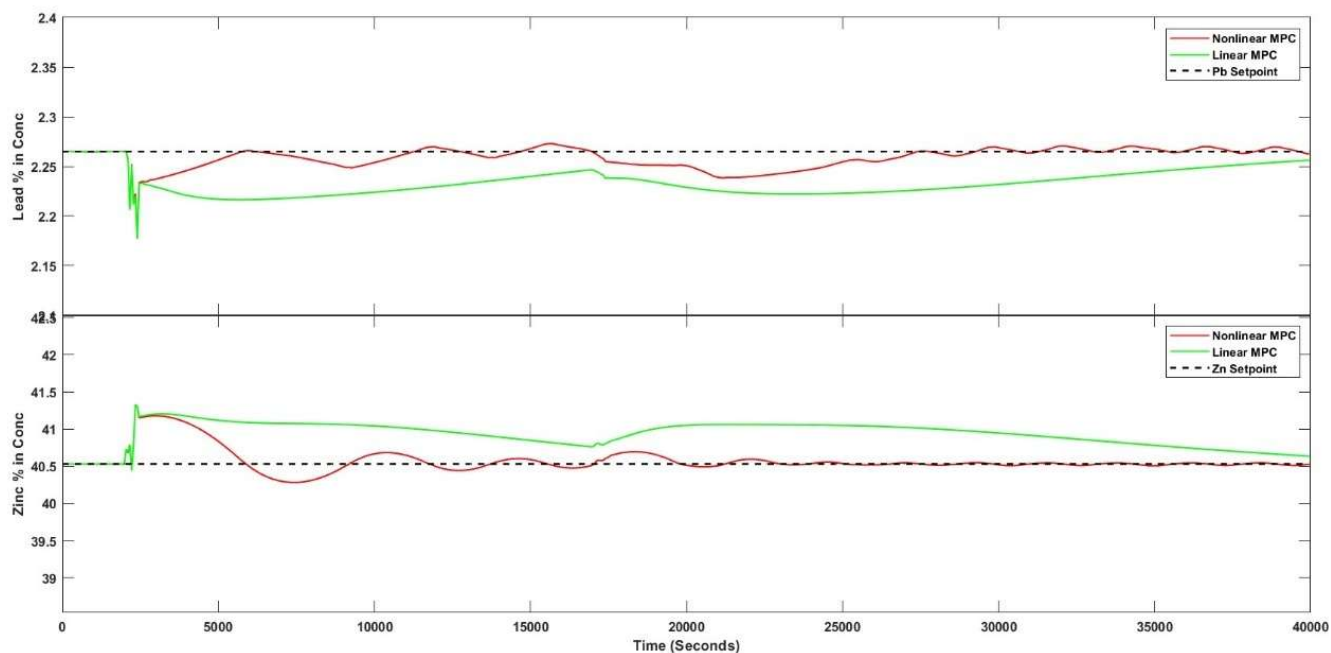


Figure 4-25 Zn % in Conc and Pb % in Conc Response Plots

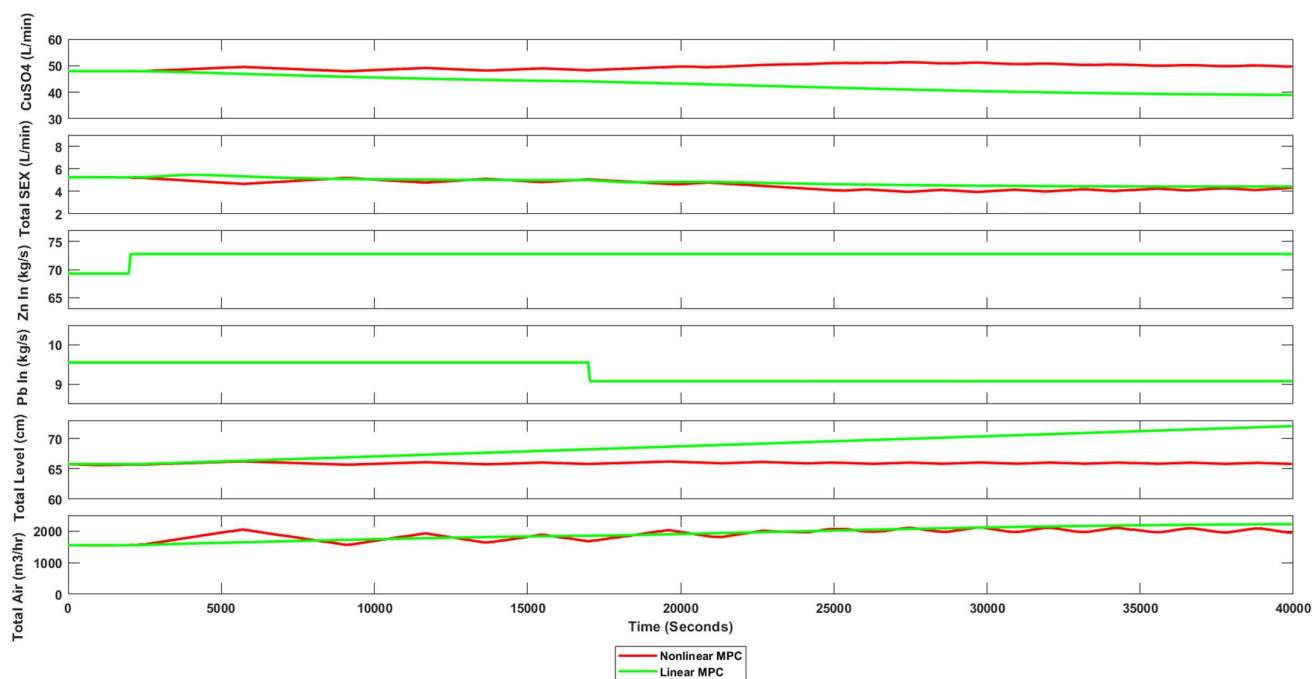


Figure 4-26 Manipulated Variable Input Response Plots

First 17000 s:

In Figure 4-25, the Pb % in the Conc reading was able to return to the new Pb % in Conc setpoint using the nonlinear controller after the disturbance variable, Zn In, was decreased.

However, the reading oscillates around the setpoint with the nonlinear controller. The linear controller's Pb % in Conc reading could not return to the setpoint after the Zn In was decreased.

Similarly, the Zn % in the Conc reading was able to return to the Zn % in Conc setpoint using the nonlinear controller. Once again, the reading oscillates around the setpoint.

However, the linear controller could not return to the Zn % in Conc reading to the setpoint.

After 17000s:

The Pb % in Conc reading does return to the Pb % in Conc setpoint with the nonlinear controller when the disturbance variable, Pb In, is decreased.

However, the reading oscillates around the setpoint with the nonlinear controller. The Pb % in Conc reading does not return to the Pb % in Conc setpoint with the linear controller after the Pb In is decreased.

Similarly, the Zn % in Conc reading does return to the new setpoint with the nonlinear controller, whereas the Zn % in Conc reading does not return to the setpoint with the linear controller.

Overall, the nonlinear controller illustrates better disturbance rejection than the linear controller with the Zn % in Conc reading and Pb% in Conc reading when the Zn In and Pb In values are altered.

However, the nonlinear controller tuning could be altered to reduce the oscillatory behaviour seen with both the Zn % in Conc reading and Pb% in Conc reading.

The results for simulation scenario 7 (Lead In Increased, followed by an Increase in Zinc In) are shown Figure 4-27 and Figure 4-29 Zn % in Conc and Pb % in Conc Response Plots.

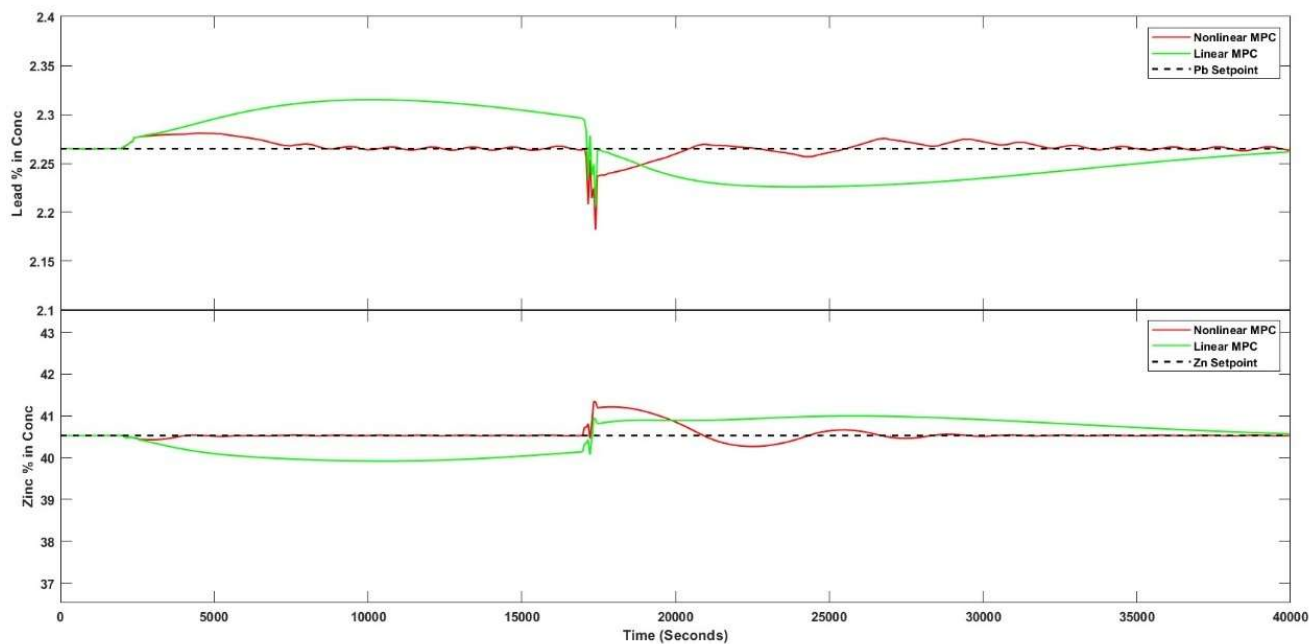


Figure 4-27 Zn % in Conc and Pb % in Conc Response Plots

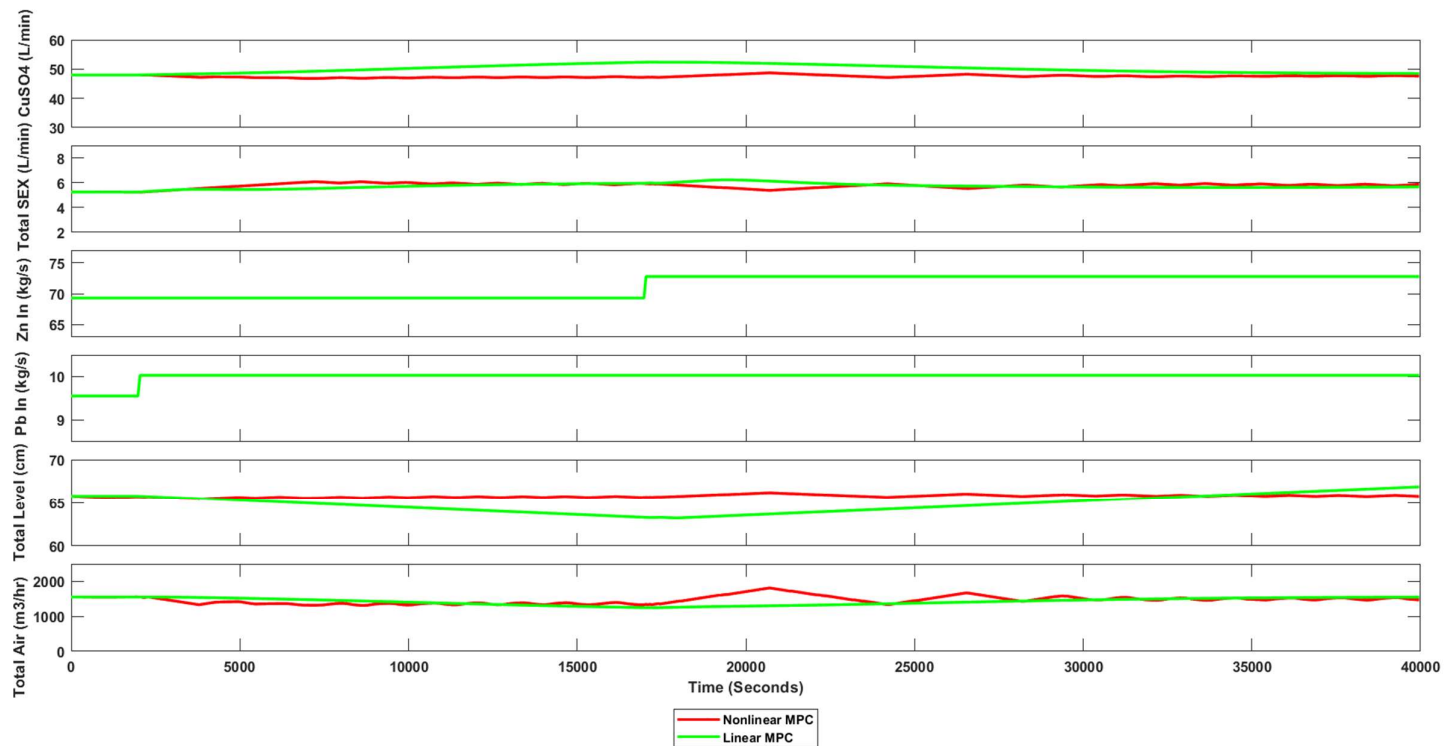


Figure 4-28 Manipulated Variable Input Response Plots

First 17000 s:

In Figure 4-27, the Pb % in the Conc reading was able to return to the new Pb % in Conc setpoint using the nonlinear controller after the disturbance variable, Pb In, was increased.

However, the reading oscillates around the setpoint with the nonlinear controller. The linear controller's Pb % in Conc reading could not return to the setpoint.

Similarly, the Zn % in the Conc reading was able to return to the Zn % in Conc setpoint using the nonlinear controller when the Pb In was increased.

Whereas the linear controller could not return the Zn % in Conc reading to the setpoint. The Zn % in Conc reading reaches a new steady state value but does not match the setpoint.

After 17000s:

The Pb % in Conc reading does return to the Pb % in Conc setpoint with the nonlinear controller when the disturbance variable, Zn In, is decreased.

However, the reading oscillates around the setpoint with the nonlinear controller. The Pb % in Conc reading does return to the Pb % in Conc setpoint with the linear controller after the Zn In, although this only occurs at the end of the simulation.

The Zn % in Conc reading does return to the new setpoint with both controllers; however, the Zn % reading in Conc only reaches the setpoint at the end of the simulation with the linear controller.

Overall, the nonlinear controller illustrates better disturbance rejection than the linear controller, which has faster response times when the Zn In and Pb In values are altered. However, the nonlinear controller tuning could be altered to reduce the oscillatory behaviour seen with Pb% in Conc reading.

The results for simulation scenario 8 (Lead In Decreased, followed by a Decrease in Zinc In) are shown Figure 4-29 and Figure 4-30.

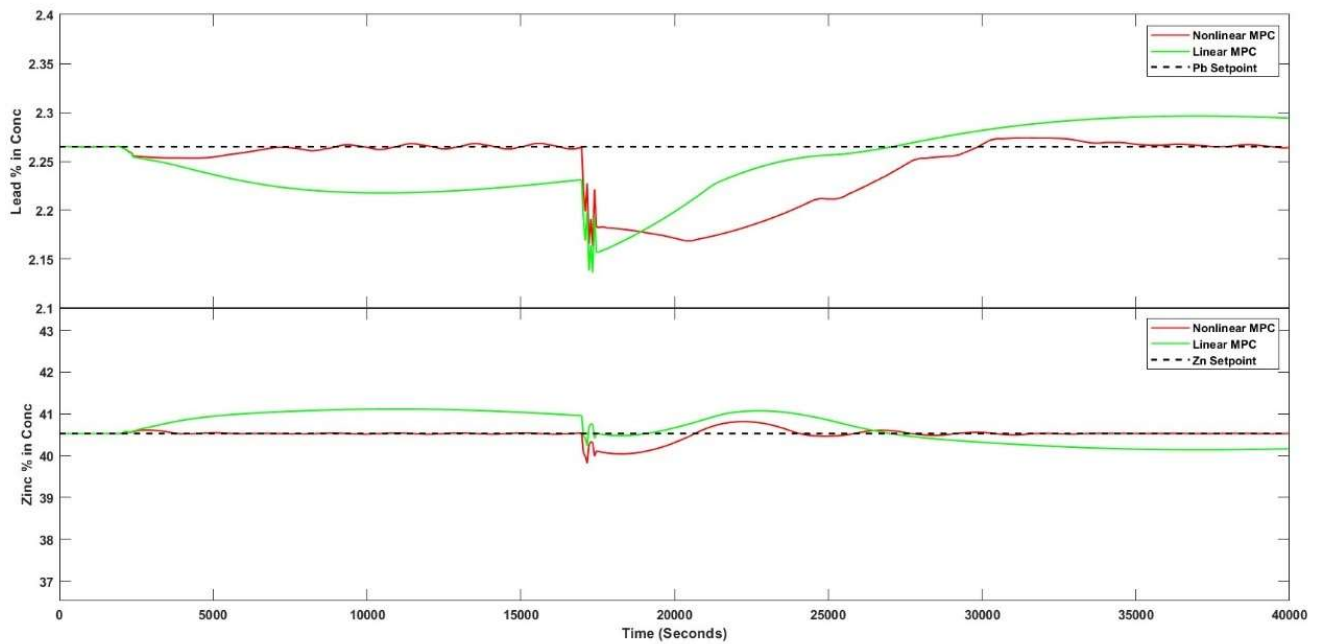


Figure 4-29 Zn % in Conc and Pb % in Conc Response Plots

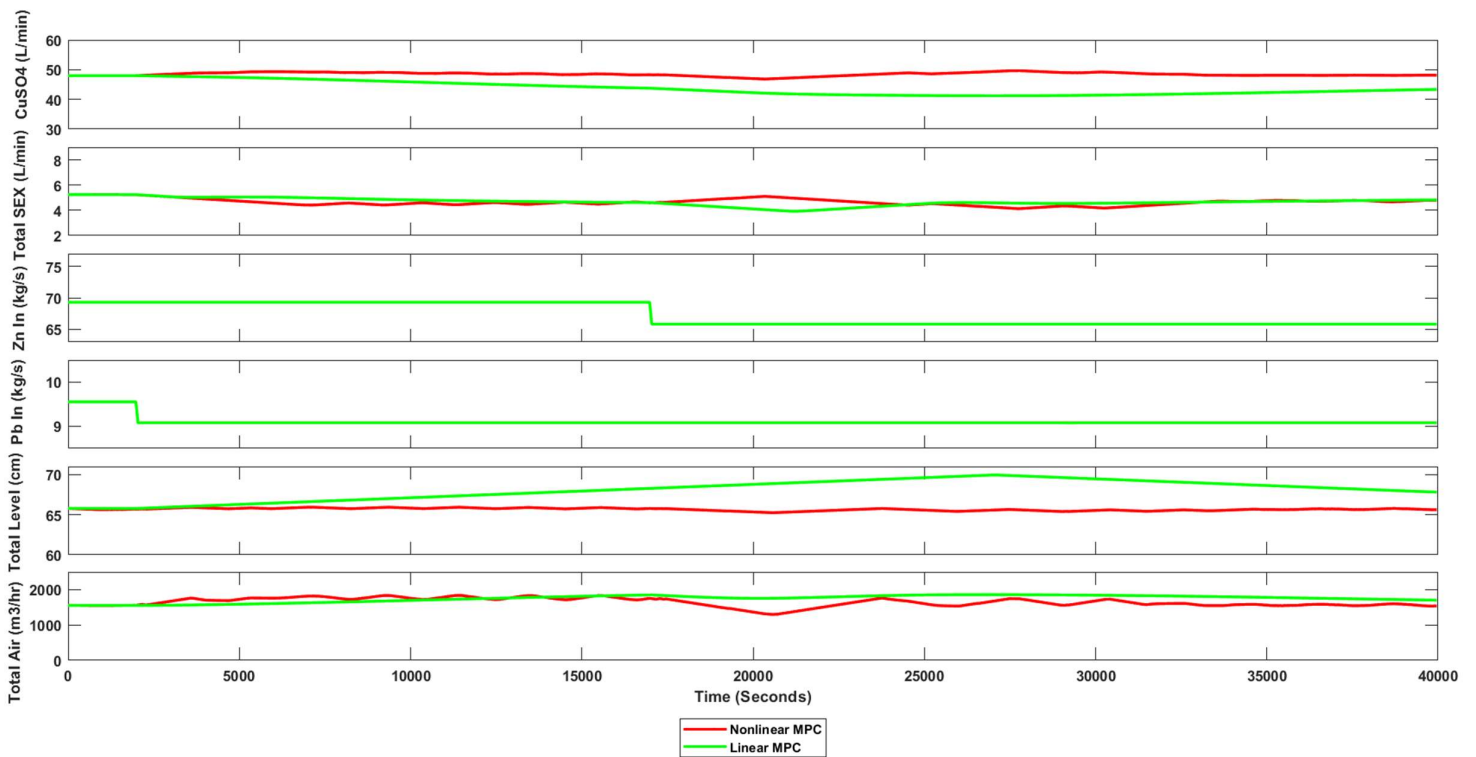


Figure 4-30 Manipulated Variable Input Response Plots

First 17000 s:

In Figure 4-29, the Pb % in the Conc reading was able to return to the new Pb % in Conc setpoint using the nonlinear controller after the disturbance variable, Pb In was decreased. However, the reading oscillates around the setpoint with the nonlinear controller. The linear controller's Pb % in Conc reading could not return to the setpoint.

Similarly, the Zn % in the Conc reading was able to return to the Zn % in Conc setpoint using the nonlinear controller when the Pb In was decreased. Whereas the linear controller could not return the Zn % in Conc reading to the setpoint. The Zn % in Conc reading reaches a new steady state value but does not match the setpoint.

After 17000s:

The Pb % in Conc reading does return to the Pb % in Conc setpoint with the nonlinear controller when the disturbance variable, Zn In, was decreased. The Pb % in Conc reading does not return to the setpoint with the linear controller; however, the Pb % in Conc reading does reach a new steady state value, which does not match the setpoint.

The Zn % in Conc reading does return to the new setpoint with the nonlinear controller. The linear controller was not able to return the Zn % in Conc reading to the setpoint. The Zn % in Conc reading reaches a new steady state value but does not match the setpoint.

Overall, the nonlinear controller illustrates better disturbance rejection when the Zn In and Pb In values are altered. However, the nonlinear controller tuning could be altered to reduce the oscillatory behaviour seen with Pb% in Conc reading.

The Root Mean Squared Error between the Zn % in Conc and Pb % in Conc readings and the setpoints were calculated for each Disturbance Rejection scenario and are shown in Table 4-13.

Table 4-13 Disturbance Rejection Simulations: NRMSE results

Simulation	Con Zn		Con Pb	
	Linear MPC	Nonlinear MPC	Linear MPC	Nonlinear MPC
	NRMSE	NRMSE	NRMSE	NRMSE
5 - Zn In Decreased First Pb In Conc Increased	0.96	0.27	1.54	1.62
6 - Zn In Increased First Pb In Decreased	1.07	0.41	1.49	0.53
7 - Pb In Increased First Zn In Increased	0.99	0.42	1.42	0.38
8 - Pb In Decreased First Zn In Decreased	0.95	0.33	1.72	1.68

The time the Zn % in Conc and Pb % in Conc reading met the respective setpoints after the Zn % in Conc and Pb % in Conc setpoints were changed was recorded and is shown in Table 4-14.

Table 4-14 Disturbance Rejection Simulations: Response Times

Simulation	Con Zn		Con Pb	
	Linear MPC	Nonlinear MPC	Linear MPC	Nonlinear MPC
	Response Time (s)	Response Time (s)	Response Time (s)	Response Time (s)
5- Zn In Decreased First Pb In Increased	0.31 % offset 0.31 % offset	11000 18000	8000 0.66 % offset	12500 22500
6 - Zn In Increased First Pb In Decreased	0.62 % offset 0.31 % offset	6000 24000	1.10 % offset 0.55 % offset	6000 27500
7 - Pb In Increased First Zn In Increased	1.23 % offset 40000	4000 21000	1.66 % offset 40000	7500 22000
8 - Pb In Decreased First Zn In Conc Decreased	1.23% 0.62 % offset	3000 21000	2.21 % offset 1.66 % offset	7500 35000

The NRMSE values for Zn % in Conc in Table 4-13 are lower with the nonlinear controller than with the linear controller for all the disturbance rejection scenarios. The nonlinear controller NRMSE values for Zn % in Conc range from 0.2 to 0.4%, and the linear controller NRMSE values for Zn % in Conc range from 1.4 to 1.7%. Furthermore, Table 4-14 illustrates that the Zn % in Conc reading is maintained at the setpoint with the nonlinear controller when the disturbance variables are changed in all scenarios (5 to 8).

However, the Zn % in Conc reading is not maintained at the setpoint with a linear controller in most disturbance rejection scenarios. The exception is scenario 7, where the Zn % in Conc reading only returns to the setpoint after Zn In is increased. The reading does not return to the setpoint when only Pb In is increased. Therefore, the nonlinear controller performs better than the linear controller with disturbance rejection in the Zn % in Conc reading.

The NRMSE values for Pb % in Conc in Table 4-13 are lower with the nonlinear controller than the linear controller for most disturbance rejection scenarios. These are scenarios 6, 7, and 8. The nonlinear controller NRMSE values for Zn % in Conc range from 0.3 to 1.6 %, and the linear controller NRMSE values for Zn % in Conc range from 0.9 to 1%. In scenario 5, the linear controller NRMSE is lower than the nonlinear controller; however, Table 4-14 showcases that the Pb % in Conc returns to the setpoint when Zn In is decreased and does not return to the setpoint once Pb In is increased. Thus, the nonlinear controller performs better than the linear controller in this scenario.

In all disturbance rejection scenarios, the nonlinear controller can return the Pb % in Conc reading to the setpoint after both Pb In and Zn In are changed; however, the linear controller cannot achieve this in any of the scenarios. Therefore, the nonlinear controller performs better than the linear controller with disturbance rejection in the Pb % in Conc reading.

The open-loop response of the rougher bank plant was evaluated by firstly calculating the gain between the disturbance variables (Zn In and Pb In) and the controlled variables (Zn % in Conc and Pb % in Conc). The calculated gains are shown in Appendix C. The Hybrid step response plants in section 3.4.3. The gain was used to compute the CV setpoint offset for the external plant when DV values are changed without a controller present. These results were compared to CV setpoint offset results of the linear and nonlinear controller in the disturbance rejection scenarios.

The setpoint offsets for the disturbance rejection scenarios are shown Table 4-15.

Table 4-15 CV setpoint offset comparison

	Con Zn			Con Pb		
	No Controller	Linear Controller	Nonlinear Controller	No Controller	Linear Controller	Nonlinear Controller
5- Zn In Decreased First Pb In Increased	-1.26 % offset	-0.31 % offset	No offset	-0.88 % offset	No offset	No offset
	-8.63 % offset	0.31 % offset	No offset	16.74% offset	-0.66 % offset	No offset
6 - Zn In Increased First Pb In Decreased	1.26 % offset	0.62 % offset	No offset	0.88 % offset	-1.10 % offset	No offset
	8.63 % offset	0.31 % offset	No offset	-16.74% offset	-0.55 % offset	No offset
7 - Pb In Increased First Zn In Increased	-7.4% offset	-1.23 % offset	No offset	17.62% offset	1.66 % offset	No offset
	-6.17% offset	No offset	No offset	18.50% offset	No offset	No offset
8 - Pb In Decreased First Zn In Conc Decreased	7.4% offset	1.23 % offset	No offset	-17.62% offset	-2.21 % offset	No offset
	6.17% offset	-0.62 % offset	No offset	-18.50% offset	1.66 % offset	No offset

The results in Table 4-15 indicate that the presence of the linear and nonlinear controller reduces the setpoint offset for all disturbance rejection scenarios. The CV setpoint offset is large when a controller does not intervene. Both controllers are effective and can respond to variations in plant operating conditions. The nonlinear controller is robust and can achieve control objectives 2 and 4 within the maximum settling time. Whereas the linear controller is not successful in achieving these objectives. However, the CV setpoint offset is reduced with the use of the linear controller, and additional tuning of the linear controller could further decrease the CV setpoint offset.

Overall, the nonlinear MPC effectively restores the zinc and lead % in conc to the respective setpoints when disturbance variables fluctuate. In contrast, the linear MPC may have difficulty in returning the CV trends to the respective setpoints, as the linear model may either overpredict or underpredict the reagent dosing rate required. The efficiency of the nonlinear controller with disturbance rejection is advantageous, as the reagent wastage is reduced, and the CVs are maintained at the setpoints without estimating the wrong reagent dosing rates.

5. CONCLUSION

The literature survey has showcased developments of flotation reagent models utilised for research purposes. Empirical models were developed to accurately predict concentrate grades and recovery within simulated environments. Phenomenological models have also rendered successful predictions of concentrate grade and recovery. However, there are still factors that can be enhanced and further modelled, such as the froth phase. There are significantly fewer publications on utilising these models for reagent dosing model-based control strategies, such as MPC. Moreover, the industry implementation of MPC control strategies, including reagent dosing, is minimal. A large proportion of the MPC reagent dosing control strategies employed the variable froth bubble size and machine vision. There were minimal nonlinear model predictive control strategies for reagent dosing control.

Models that were used to represent the relationship between reagent dosing and the Zn % and Pb % in Conc were developed using the MATLAB system identification app. The model development process was simple, as the meta parameters for each structure were determined automatically using an optimisation algorithm. The reagent models illustrated better fits for the training dataset than the testing datasets. By utilising the Zn % in concentrate statistics as the primary performance indicator for the models, it was observed that the state space model provided the best fit with the training datasets and a slight NRMSE difference of 0.4 % between the state space and nonlinear ARX model. However, the nonlinear ARX models provided a best fit for the testing datasets with a decrease of 2.2% in the NRMSE when compared to the state space model. Thus, the NLarx model is better at modelling unseen data when compared to the state space and linear ARX model structures. Furthermore, the reagent sensitivity plots showcase contrasting linear and nonlinear responses; thus, the nonlinearity brought about by the reagents can be captured using the NLarx structure. The models without NASH displayed less overfitting and were used for further analysis in the simulated system development. The ARX reagent models had the poorest fits with both training and test datasets. However, the fit of the ARX models improved when the total air flow rate and the total level were included in the Total Trend Models. The Hybrid NLarx models that included the total air flow rate and total level were comparable to the reagent NLarx models.

The total trend linear ARX models and the hybrid NLarx models were used to develop a linear MPC controller and a nonlinear MPC controller, respectively. The controller's performance was tested in a simulated system developed in Simulink. The development of a simulated system with a nonlinear MPC was a more complex procedure than the development of a simulated system with a linear controller. The linear model was easier to configure in the linear MPC controller as the MPC block could accept the model in the form of a MATLAB model object that was directly transferred from the system identification app to the workspace. The nonlinear MPC block did not have the same functionality for nonlinear models. The nonlinear model structure had to be written as a function, and the function was called in the nonlinear MPC

block. This process was tedious and resulted in a delay in the study. The setpoint tracking and disturbance rejection of these two controllers were evaluated. The nonlinear controller performs better than the linear controller with Zn % in Conc setpoint tracking and Pb % in Conc setpoint tracking when the Pb % in Conc setpoint is changed by 5 %. Furthermore, the nonlinear controller can maintain the Zn % Conc at the setpoint when there is a change in Zn % in Conc setpoint. However, the nonlinear controller is not able to maintain the Pb% in Conc at the setpoint when there is a change in the Zn % in Conc setpoint. The linear controller had poor setpoint tracking with the Zn % in Conc and Pb % in Conc for most scenarios. Therefore, the nonlinear controller performed better than the linear controller with setpoint tracking.

The disturbance rejection results showed that the nonlinear controller could return the Zn % in Conc and Pb % in Conc to the setpoint for all scenarios when the Pb In and Zn In were altered by 5% of the original value. However, the linear controller was not able to return the Zn % in Conc and Pb % in Conc readings to the setpoint for all scenarios. Thus, the nonlinear controller performed better than the linear controller with disturbance rejection.

Overall, a nonlinear model can capture the complex dynamics of reagent behaviour in a flotation bank. The nonlinear MPC that used a nonlinear model to represent the internal plant had better efficiency with the setpoint tracking and disturbance rejection than the linear MPC. The original case study led to savings in reagent costs. The nonlinear MPC's increased efficiency may further enhance these savings.

5.1. Future Work

1. Extending the dataset to further generalise the models and increase robustness. The extension of the dataset consists of data with different feed conditions (ore properties, feed concentrations, etc.) and operating conditions. The potential extension of the dataset could consist of the work by Shardt and Brooks (2018), in which the extended dataset is first segmented. The models obtained in the resulting sub-datasets can be compared.
2. The comparison of linear and nonlinear models to represent reagent effects can be extended to include the flotation of other minerals. The inclusion of other minerals in the analysis can verify whether the responses of various flotation reagents are common.
3. Improving tuning of the nonlinear controller using manual adjustment. Currently, the linear MPC is automatically tuned, and the nonlinear MPC is tuned using the same parameters as the linear MPC to ensure the controller comparison is fair. Tuning the nonlinear MPC without using the existing linear MOC parameters could improve the controller's performance. Additionally, the linear MPC can be tuned without the app to improve performance.
4. Reagent usage optimisation should be included in the linear and nonlinear MPC objective function to assess which controller yields improved reagent reduction.

5. A cost vs benefit analysis of the implementation of nonlinear MPC in the reduction of reagent costs.

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7. APPENDICES

Appendix A - Model Parameter Estimation Methods

N4SID

The N4Horizon is required for the N4SID method and is a row vector with three elements $[r \ sy \ su]$ where r is the number of step-ahead predictors, sy is the number of past outputs and su is the number of past inputs. The Akaike Information Criterion (AIC) algorithm is used to determine elements of the N4Horizon. The N4Horizon requires a N4Weight algorithm. The Canonical Variate Algorithm is the N4Weight option used for singular-value decomposition. The initial states used in the N4SID method are estimated from the training data.

The N4SID method is described below.

Firstly, it is assumed that the state space parameters, $A \in \mathbf{R}^{p \times j}$, $B \in \mathbf{R}^{q \times j}$, $C \in \mathbf{R}^{r \times j}$ and $D \in \mathbf{R}^{r \times j}$. Thereafter, the future and past input data is organised into Hankel matrices as seen in Equation 65 and Equation 66.

$$U_p = \begin{bmatrix} u(0) & u(1) & \cdots & u(j-1) \\ u(1) & u(2) & \cdots & u(j) \\ \vdots & \vdots & \ddots & \vdots \\ u(i-1) & u(i) & \cdots & u(i+j-2) \end{bmatrix} \quad (65)$$

$$U_f = \begin{bmatrix} u(i) & u(i+1) & \cdots & u(i+j-1) \\ u(i+1) & u(i+2) & \cdots & u(i+j) \\ \vdots & \vdots & \ddots & \vdots \\ u(2i-1) & u(2i) & \cdots & u(2i+j-2) \end{bmatrix} \quad (66)$$

The future and past output data are also organised into Hankel matrices and are represented as Y_f and Y_p respectively. The state space model equations are restructured using the Hankel matrices, as shown in Equation 67 to Equation 69.

$$X_f = A^i X_p + \Delta_i U_p \quad (67)$$

$$Y_p = \Gamma_i X_p + H_i U_p \quad (68)$$

$$Y_f = \Gamma_i X_f + H_i U_f \quad (69)$$

Where $X_p \in R^{n \times j}$ and $X_f \in R^{n \times j}$. $\Gamma_i \in R^{li \times n}$ is the observability matrix, $\Delta_i \in R^{n \times mi}$ is the reversed extended controllability matrix and H_i is the block Toeplitz matrix. The order of the model is determined by computing the oblique projection shown in Equation 70.

$$O_i = \frac{Y_f}{u_f Z_p} \quad (70)$$

The oblique projection is multiplied by W_1 and W_2 and Singular Value Decomposition (SVD) was performed on $W_1 O_i W_2$ as shown in Equation 71.

$$W_1 O_i W_2 = U_1 S_1 V_1^T \quad (71)$$

Where S_1 is a matrix of singular values. U_1 is an orthogonal matrix that contains the columns of left singular vectors, Σ_{IOH} and Σ_{OH} are diagonal matrices containing singular values of A and V_1^T is an orthogonal matrix containing the rows of right singular vectors. The singular values represent the importance of the singular vectors. Canonical variates analysis, CVA uses linear combinations of the singular vectors and extracts canonical variates, which are linear combinations that represent the dominant modes of the system dynamics. The model's order n , is equivalent to the number of non-zero singular values in S_1 . W_1 and W_2 are chosen based on the maximisation of the $\Gamma_i X_f$ estimation.

Equation 71 is simplified to Equation 72.

$$O_i = \Gamma_i X_f \quad (72)$$

The extended observability matrix Γ_i is found using Equation 73.

$$\Gamma_i = U_1 (S_1)^{\frac{1}{2}} \quad (73)$$

The state matrix, X_f , is found using Equation 74.

$$X_f = (S_1)^{\frac{1}{2}} (V_{n1})^T \quad (74)$$

The A parameter matrix is determined from the shift structure of Γ_i and is shown in Equation 75.

$$A = \underline{\Gamma_i}^\dagger \bar{\Gamma_i} \quad (75)$$

The parameter matrix, C , is extracted from the first l rows. The parameter matrices, B and D , are derived by multiplying Equation 75 by $\Gamma_i^\perp$ and $U_f^\dagger$ as shown in Equation 76. Equation 76 is simplified to Equation 77 and re-represented in Equation 78.

$$\Gamma_i^\perp Y_f U_f^\dagger = \Gamma_i^\perp \Gamma_i X_f U_f^\dagger + \Gamma_i^\perp H_i U_f U_f^\dagger \quad (76)$$

Where $\Gamma_i^\perp$ is the orthogonal complement of Γ_i . $U_f^\dagger$ is the conjugate transpose of the future inputs' matrix.

$$\Gamma_i^\perp Y_f U_f^\dagger = \Gamma_i^\perp H_i \quad (77)$$

$$M = \Gamma_i^\perp Y_f U_f^\dagger \quad (78)$$

Where $L = \Gamma_i^\perp$ and $M = LH_i$. Matrix B and D is computed using least squares as H_i is composed of matrix B and D as seen in Equation 79.

$$H_i = \begin{bmatrix} D & 0 & \dots & \dots & 0 \\ CB & D & 0 & \dots & 0 \\ CAB & CB & D & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA^{i-2}B & CA^{i-3}B & \dots & CB & D \end{bmatrix} \quad (79)$$

Least Squares Reduction using QR decomposition

The ARX model structure can be re-arranged into a matrix format as shown in Equation 80.

$$D\beta = C \quad (80)$$

Where D is the regressor matrix consisting of both input and output regressors, β is the parameter matrix, and C represents the output matrix. The least-square solution method aims to minimise the sum of squared errors shown in Equation 81.

$$\epsilon = \|D\beta - C\| \quad (81)$$

Thus, the parameter matrix, β_s , that minimises the sum of all squared errors is the least square solution and needs to be determined such that:

$$\|D\beta_s - C\| \leq \|D\beta - C\| \quad (82)$$

The projection of vector C in the column space of D is referred to as C_p and is shown in Equation 83.

$$C_p = Proj_{CS(D)}(C) \quad (83)$$

The C_p vector is the closest vector to C in the column space of D such that:

$$D\beta_s = C_p \quad (84)$$

(84) is solved to find β_s .

The Gram-Schmidt method is applied to $\mathcal{L}$ which is the basis for the column space of D where $D = [\vec{d}_1 \vec{d}_2 \dots \vec{d}_n]$ and $\mathcal{L} = [\vec{d}_1 \vec{d}_2 \dots \vec{d}_n]$.

The Gram-Schmidt method is a QR decomposition method used to decompose a matrix into an orthogonal matrix, Q , and a triangular matrix, R . The resulting QR factorisation is shown in Equation 85.

$$D = [\vec{d}_1 \vec{d}_2 \dots \vec{d}_n] = [\vec{e}_1 \vec{e}_2 \dots \vec{e}_n] \begin{bmatrix} d_1 \cdot e_1 & d_2 \cdot e_1 & \dots & d_n \cdot e_1 \\ 0 & d_2 \cdot e_2 & \dots & d_n \cdot e_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \cdot e_n \end{bmatrix} = QR \quad \mathbb{R}^m \quad (85)$$

Where $Q = [\vec{u}_1 \vec{u}_2 \dots \vec{u}_n]$, $u_{k+1} = a_{k+1} - (a_{k+1} \cdot e_1)e_1 - \dots - (a_{k+1} \cdot e_k)e_k$ and $e_{k+1} = \frac{u_{k+1}}{\|u_{k+1}\|}$

Equation 85 is solved by multiplying D^T to either side, and QR decomposition is applied as shown in Equation 86 and Equation 87.

$$D^T D \beta_s = D^T C_p \quad (86)$$

$$R^T Q^T Q R \beta_s = R^T Q^T C_p \quad (87)$$

The orthonormal decomposition theorem states that $Proj_{CS(D)}(C) = Q Q^T C$, thus Equation 87 is reduced to Equation 88 and β_s is solved using Equation 89.

$$R^T Q^T Q R \beta_s = D \beta_s = C_p = R^T Q^T Q Q^T C \quad (88)$$

$$\beta_s = R^{-1} Q^T C \quad (89)$$

Subspace Gauss-Newton least-squares search

Subspace Gauss-Newton least-squares search minimises the sum of residuals shown in Equation 90.

$$L = \sum r_i^2 \quad (90)$$

Where the residual, r_i , is the difference between the approximation, y , and the actual value, y_i , shown in Equation 91.

$$r_i = y_i - y \quad (91)$$

Where, $y = f(x; \beta)$. The Newton-Rapshon method is used to find the value of β that minimises the sum of squares of, r_i , as shown in Equation 92.

$$\beta_{t+1} = \beta_t - (\nabla_{\beta}^2 L)^{-1} \nabla_{\beta} L \quad (92)$$

Where, β_{t+1} , the approximation at time $t + 1$ and β_t is the approximation from the previous iteration. $\nabla_{\beta} L$ is the gradient of L and $\nabla_{\beta}^2 L$ is a matrix of second partial derivatives of L , the Hessian matrix. Where the gradient is determined using Equation 93 and the Hessian matrix is approximated using Equation 94.

$$\nabla_{\beta} L = 2 \cdot J^T \cdot r \quad (93)$$

$$\nabla_{\beta_j \beta_t}^2 L \approx 2 \cdot J^T \cdot J \quad (94)$$

Where J is the Jacobian matrix. The singular values of the Jacobian matrix below a threshold are discarded in the subspace Gauss-Newton method when computing the gradient direction. The threshold is defined in Equation 95.

$$GnPinConstant \times \epsilon \times \max(\text{size}(J)) \times \text{norm}(J) \quad (95)$$

Where $GnPinConstant$ is a positive scalar with a default value of 1000 in MATLAB. The resulting structure of the Subspace Gauss-Newton method is shown in Equation 96.

$$\beta_{t+1} = \beta_t - (J_t^T \cdot J_t)^{-1} J_t^T \cdot r_t \quad (96)$$

Adaptive Subspace Gauss-Newton least-squares search

This search method uses a similar structure to Equation 96. However, the derivation of the search direction differs as a new threshold is used to reduce the matrix. The eigenvalues in the Jacobian matrix that are less than the threshold defined in Equation 97.

$$\text{gamma} \times \max(sv) \quad (97)$$

Where sv is the singular values of the Hessian matrix and gamma is a scalar value with an initial value of 0.0001. The value of gamma is increased at each iteration by a user-specified factor with a default value

of 2 when there are no values lower than the threshold found within five bisections. The value of *gamma* is decreased by a factor of 2 when there are values found below the threshold without bisections.

Steepest descent least-squares search

This method minimises Equation 90 using an iterative structure shown in Equation 98.

$$\beta_{t+1} = \beta_t - \lambda r_t \quad (98)$$

Where the vector λr_t guides the approximation in the direction of the steepest descent. λ is the step size with a default value of 0.01.

Levenberg-Marquardt least-squares search

This method is a combination of the Subspace Gauss-Newton search and the Steepest decent search with a structure shown in Equation 99.

$$\beta_{t+1} = \beta_t - (\nabla_{\beta}^2 L + \lambda I)^{-1} \nabla_{\beta} L \quad (99)$$

Where the default initial value of λ is 0.01. λ is reduced by a factor of 10 for the next iteration when the value of β_t reduces the output of L otherwise λ is increased by a factor of 10. As λ approaches zero, the method behaviours are similar to the Gauss-Newton method. However, as λ approaches infinity, the method resembles the Steepest Descent method.

Appendix B – Model Development Structures and Matrices

Model Structures

The model structures from model identification are shown below.

Reagent Models

State Space Pb in Conc (No NASH) Model Structure

Continuous-time identified state-space model:

$$\frac{dx}{dt} = A x(t) + B u(t) + K e(t)$$

$$y(t) = C x(t) + D u(t) + e(t)$$

A =

	x1	x2	x3
x1	-0.001153	-0.003576	-0.001373
x2	-0.001849	-0.001572	0.008777
x3	-0.0005084	-0.01089	0.0002004

B =

	u1	u2	u3	u4
x1	1.647e-07	-1.879e-06	1.72e-05	-3.328e-05
x2	1.902e-05	0.0001705	0.000133	3.27e-06
x3	1.004e-05	0.0001354	0.0001247	-0.000184

C =

	x1	x2	x3
y1	14.67	1.08	-3.009

D =

	u1	u2	u3	u4
y1	0	0	0	0

K =

	y1
x1	0.001261
x2	0.0006965
x3	-0.0005049

```
% State space model estimation
Options = n4sidOptions;
Options.Display = 'on';
Options.Focus = 'simulation';
Options.N4Weight = 'CVA';
Options.N4Horizon = [15 0 13];
```

State Space Zn in Conc (No NASH) Model Structure

```

Continuous-time identified state-space model:
dx/dt = A x(t) + B u(t) + K e(t)
y(t) = C x(t) + D u(t) + e(t)

A =
      x1      x2      x3
x1 -5.263e-07 -9.135e-05 0.000171
x2 2.338e-05 -0.004866 0.0296
x3 -2.332e-05 -0.01043 -0.01126

B =
      u1      u2      u3      u4
x1 6.749e-07 5.344e-05 8.047e-05 -3.045e-06
x2 0.0002218 0.008799 0.01096 0.002163
x3 -0.0002015 -0.002837 -0.0008298 -0.003844

C =
      x1      x2      x3
y1 517 0.004291 -0.005083

D =
      u1 u2 u3 u4
y1 0 0 0 0

K =
      y1
x1 3.225e-05
x2 8.697e-07
x3 4.93e-07

% State space model estimation
Options = n4sidOptions;
Options.Focus = 'simulation';
Options.N4Weight = 'CVA';
Options.N4Horizon = [2 2 2];

```

Linear ARX Pb in Conc (No NASH) Model Structure

```

Discrete-time ARX model: A(z)y(t) = B(z)u(t) + e(t)

A(z) = 1 - 1.996 z^-1 + 0.9962 z^-2 - 0.0004346 z^-3 - 0.0004733 z^-4 + 0.0008043 z^-5 - 0.000342 z^-6
      - 0.002219 z^-7 + 0.008014 z^-8 - 0.00576 z^-9

B1(z) = -0.0002838 z^-7 + 0.0002836 z^-8

B2(z) = -1.185e-07 z^-10

B3(z) = 4.908e-07 z^-10

B4(z) = 0.003511 z^-8 - 0.004731 z^-9 + 0.001219 z^-10

```

Linear ARX Zn in Conc (No NASH) Model Structure

Discrete-time ARX model: $A(z)y(t) = B(z)u(t) + e(t)$

$$A(z) = 1 - 2.001 z^{-1} + 1.002 z^{-2} - 0.0008387 z^{-3} + 1.298e-05 z^{-4}$$

$$B1(z) = -0.005599 z^{-5} + 0.009916 z^{-6} - 0.005929 z^{-7} + 0.003918 z^{-8} - 0.002307 z^{-9}$$

$$B2(z) = -0.03087 z^{-1} + 0.01866 z^{-2} + 0.03723 z^{-3} - 0.000357 z^{-4} - 0.0701 z^{-5} + 0.09286 z^{-6} \\ - 0.04101 z^{-7} - 0.0115 z^{-8} - 0.0004554 z^{-9} + 0.005541 z^{-10}$$

$$B3(z) = 0.000235 z^{-9} - 0.0002349 z^{-10}$$

$$B4(z) = -0.006284 z^{-8} + 0.01081 z^{-9} - 0.008584 z^{-10} - 0.0113 z^{-11} + 0.03464 z^{-12} - 0.01928 z^{-13}$$

State Space Pb in Conc (NASH) Model Structure

Continuous-time identified state-space model:

$$\frac{dx}{dt} = A x(t) + B u(t) + K e(t)$$

$$y(t) = C x(t) + D u(t) + e(t)$$

A =

	x1	x2	x3
x1	-5.562e-05	0.0002459	-0.0002984
x2	2.665e-05	-0.0003967	0.001084
x3	0.0001732	-0.001196	-0.0007939

B =

	u1	u2	u3	u4	u5
x1	-1.709e-06	9.323e-07	4.619e-06	-2.861e-06	9.115e-06
x2	2.047e-06	-4.151e-06	-1.116e-05	8.126e-06	-1.759e-05
x3	9.152e-06	5.506e-06	-7.315e-06	5.865e-07	-3.093e-05

C =

	x1	x2	x3
y1	39.41	1.568	-0.5557

D =

	u1	u2	u3	u4	u5
y1	0	0	0	0	0

K =

	y1
x1	0.0003441
x2	0.0002629
x3	-0.0005223

```
% State space model estimation
Options = n4sidOptions;
Options.Focus = 'simulation';
Options.N4Weight = 'CVA';
Options.N4Horizon = [101 2 25];
```

State Space Zn in Conc (NASH) Model Structure

Continuous-time identified state-space model:

$$\frac{dx}{dt} = A x(t) + B u(t) + K e(t)$$

$$y(t) = C x(t) + D u(t) + e(t)$$

A =

	x1	x2	x3
x1	-1.098e-06	6.594e-05	-6.263e-05
x2	-1.531e-06	-0.0004381	6.366e-05
x3	-6.938e-06	-0.003235	-0.01082

B =

	u1	u2	u3	u4	u5
x1	5.19e-06	1.182e-06	-9.768e-06	2.326e-06	-2.377e-06
x2	-4.529e-06	-6.261e-07	8.658e-06	-2.933e-06	7.793e-06
x3	0.0009233	0.0002277	-0.001726	0.0003784	-0.0002068

C =

	x1	x2	x3
y1	441.8	-17.44	-0.03442

D =

	u1	u2	u3	u4	u5
y1	0	0	0	0	0

K =

	y1
x1	3.431e-05
x2	-5.953e-05
x3	-5.621e-05

```
% State space model estimation
Options = n4sidOptions;
Options.Focus = 'simulation';
Options.N4Weight = 'CVA';
Options.N4Horizon = [101 2 25];
```

Linear ARX Pb in Conc (NASH) Model Structure

Discrete-time ARX model: $A(z)y(t) = B(z)u(t) + e(t)$

$$A(z) = 1 - 1.995 z^{-1} + 0.9959 z^{-2} - 0.0004394 z^{-3} - 0.0004832 z^{-4} + 0.0008061 z^{-5} - 0.0003414 z^{-6} \\ - 0.002468 z^{-7} + 0.008636 z^{-8} - 0.006126 z^{-9}$$

$$B1(z) = -1.857e-06 z^{-10}$$

$$B2(z) = -0.0003146 z^{-7} + 0.0003145 z^{-8}$$

$$B3(z) = 2.536e-06 z^{-10}$$

$$B4(z) = 3.689e-07 z^{-10}$$

$$B5(z) = 0.003611 z^{-8} - 0.004791 z^{-9} + 0.001179 z^{-10}$$

Linear ARX Zn in Conc (NASH) Model Structure

Discrete-time ARX model: $A(z)y(t) = B(z)u(t) + e(t)$

$$A(z) = 1 - 2.001 z^{-1} + 1.001 z^{-2} - 0.0007881 z^{-3} - 3.782e-05 z^{-4}$$

$$B1(z) = -0.0008685 z^{-1} + 0.0008691 z^{-2}$$

$$B2(z) = -0.005568 z^{-5} + 0.009898 z^{-6} - 0.00593 z^{-7} + 0.003899 z^{-8} - 0.002298 z^{-9}$$

$$B3(z) = -0.03026 z^{-1} + 0.01771 z^{-2} + 0.03755 z^{-3} - 0.000234 z^{-4} - 0.07096 z^{-5} + 0.09468 z^{-6} \\ - 0.04215 z^{-7} - 0.01152 z^{-8} - 0.0002376 z^{-9} + 0.005423 z^{-10}$$

$$B4(z) = 0.000248 z^{-9} - 0.0002481 z^{-10}$$

$$B5(z) = -0.006157 z^{-8} + 0.0106 z^{-9} - 0.008482 z^{-10} - 0.01132 z^{-11} + 0.03467 z^{-12} - 0.01931 z^{-13}$$

Total Trend ARX Model Structures

Total Trend Linear ARX Pb in Conc (No NASH) Model Structure

Discrete-time ARX model: $A(z)y(t) = B(z)u(t) + e(t)$

$$A(z) = 1 - 0.0594 z^{-1} - 0.8709 z^{-2}$$

$$B1(z) = -0.0006162 z^{-1} - 0.0005885 z^{-2}$$

$$B2(z) = 0.0009651 z^{-1} - 0.002336 z^{-2}$$

$$B3(z) = 0.0002005 z^{-1} + 0.0001967 z^{-2}$$

$$B4(z) = 0.003579 z^{-1} + 0.002777 z^{-2}$$

$$B5(z) = 1.543e-07 z^{-1} - 1.791e-08 z^{-2}$$

$$B6(z) = -4.001e-06 z^{-1} + 0.3165e-08 z^{-2}$$

Total Trend Linear ARX Zn in Conc (No NASH) Model Structure

Discrete-time ARX model: $A(z)y(t) = B(z)u(t) + e(t)$

$$A(z) = 1 - 0.0145 z^{-1} - 0.9198 z^{-2}$$

$$B1(z) = -0.002578 z^{-1} - 0.002145 z^{-2}$$

$$B2(z) = -0.02425 z^{-1} - 0.008326 z^{-2}$$

$$B3(z) = 0.00131 z^{-1} + 0.001451 z^{-2}$$

$$B4(z) = -0.01202 z^{-1} - 0.01731 z^{-2}$$

$$B5(z) = -3.215e-05 z^{-1} - 6.968e-05 z^{-2}$$

$$B6(z) = -0.000612 z^{-1} + 0.002427 z^{-2}$$

NLarx Model Nonlinear Function Comparison Results

		Sigmoid Network	Wavelet Network	Tree Partition	
Model	Num Units	NRMSE	NRMSE	Num Units	NRMSE
Zn No NASH Model	1	6.28	8.54	1	7.61
	2	146.12	8.45	100	5.59
	3	12.56	8.28	200	5.48
	4	6.30	8.99	300	4.92
	5	5.98	8.19	400	4.92
	6	8.40	8.08	500	4.92
	7	14.26	8.01	600	4.92
	8	6.86	7.95	700	4.92
	9	1.93224E+41	7.92	800	4.92
	10	4.43	7.93	900	4.92
	11	5.70	7.93		
	12	2.82E+76	7.93		
	13	2.18E+113	7.93		
	14	19.61	7.93		
	15	11.59	7.97		
Pb No NASH Model	1	18.05	16.79	1	15.42
	2	18.20	16.79	100	11.37
	3	18.36	16.98	200	10.04
	4	18.15	16.48	300	8.13
	5	5.2334E+136	16.51	400	8.13
	6	18.08	16.18	500	8.13
	7	16.84	16.69	600	8.13
	8	21.00	15.80	700	8.13
	9	17.39	16.67	800	8.13
	10	17.24	16.66	900	8.13
	11	16.95	16.79		
	12	16.98	16.19		
	13	17.78	16.57		
	14	16.88	16.85		
	15	17.89	20.71		
Zn NASH Model	1	7.61	7.64	1	7.18
	2	7.65	7.66	100	6.97
	3	9.56	8.88	200	6.88
	4	8.21	7.73	300	6.89
	5	5.70354E+32	7.71	400	6.89
	6	9.65	7.67	500	6.89
	7	7.79413E+58	7.66	600	6.89
	8	6.89	7.65	700	6.89
	9	18.89	9.73	800	6.89
	10	6.86	9.79	900	6.89
	11	8.10	7.66		
	12	6.62E+70	7.61		

	13	8.11	7.63		
	14	10.17	7.64		
	15	6.94	7.65		
Pb NASH Model	1	17.16	17.02	1	14.55
	2	17.89	17.29	100	9.72
	3	17.51	17.13	200	8.59
	4	15.76	17.72	300	7.12
	5	16.55	17.45	400	7.12
	6	18.71	17.77	500	7.12
	7	16.59	17.11	600	7.12
	8	16.85	17.37	700	7.12
	9	18.04	17.37	800	7.12
	10	19.55	17.10	900	7.12
	11	18.42	16.36		
	12	19.81	17.22		
	13	18.57	17.22		
	14	18.59	17.23		
	15	16.85	17.16		

Model and Controller Performance Comparison Equations

The formula used to calculate the NRMSE values that were used for model comparison is shown in Equation 100.

$$RMSE = \sqrt{\frac{\sum_{i=1}^n (x_i - y_i)^2}{n}}$$

$$NRMSE = 100 \times \frac{RMSE}{\bar{y}} \quad (100)$$

For Model Comparison

x_i is the predicted model output, y_i is the actual reading from the datasets, n is the number of data points, in the training or testing datasets, $\bar{y}$ is the mean of the measured dataset and $NRMSE$ is the Normalised Root Mean Squared Error.

For Control Performance Comparison

x_i is the plant output value in the simulation, y_i is the setpoint value of the controlled variable, n is the number of time steps in the simulation and $NRMSE$ is the Normalised Root Mean Squared Error.

The correlation coefficient was calculated using the built-in functions in Excel.

Step Response Setup

The step responses were plotted using the setup illustrated in Figure 7-1.

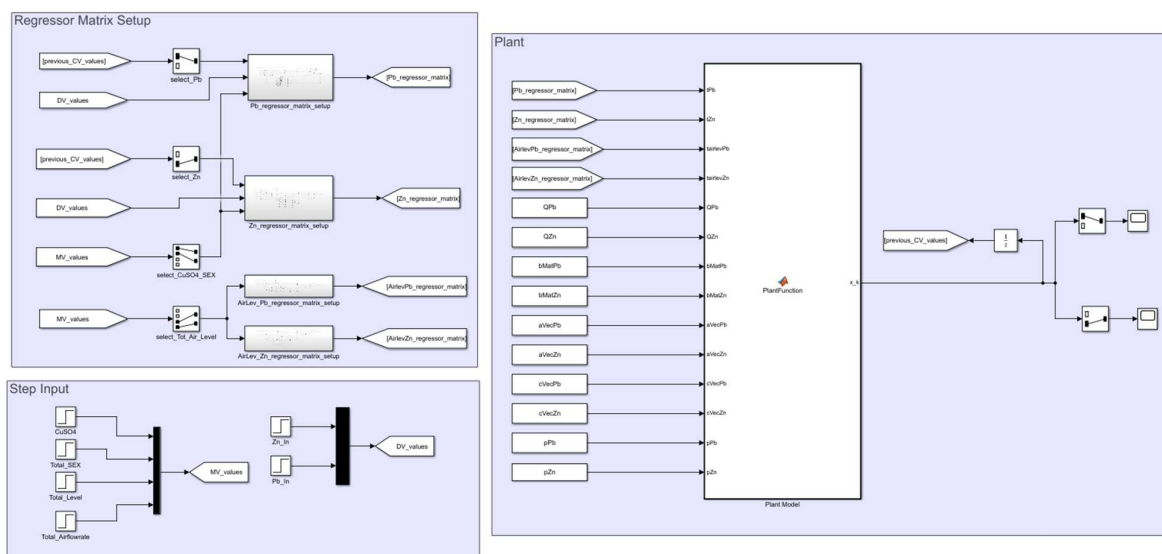


Figure 7-1 Step Response Setup

The Step Response Setup uses a similar Regressor Matrix Setup and Plant sections as those in the nonlinear MPC simulation. The Regressor Matrix Setup section constructs the regressor matrices used in the Plant model. The Plant section contains a function block which contains the plant model. The plant model requires the regressor and parameter matrices as inputs. The regressor matrices contain the current iteration values of the manipulated variables. Thus, the model's manipulated variables do not have to enter the Plant model block in a separate vector. The inputs are stepped up within the Step Input section using step blocks.

Reagent Sensitivity Setup

The Simulink setup used to obtain the Reagent Sensitivity Plots is shown in Figure 7-2.

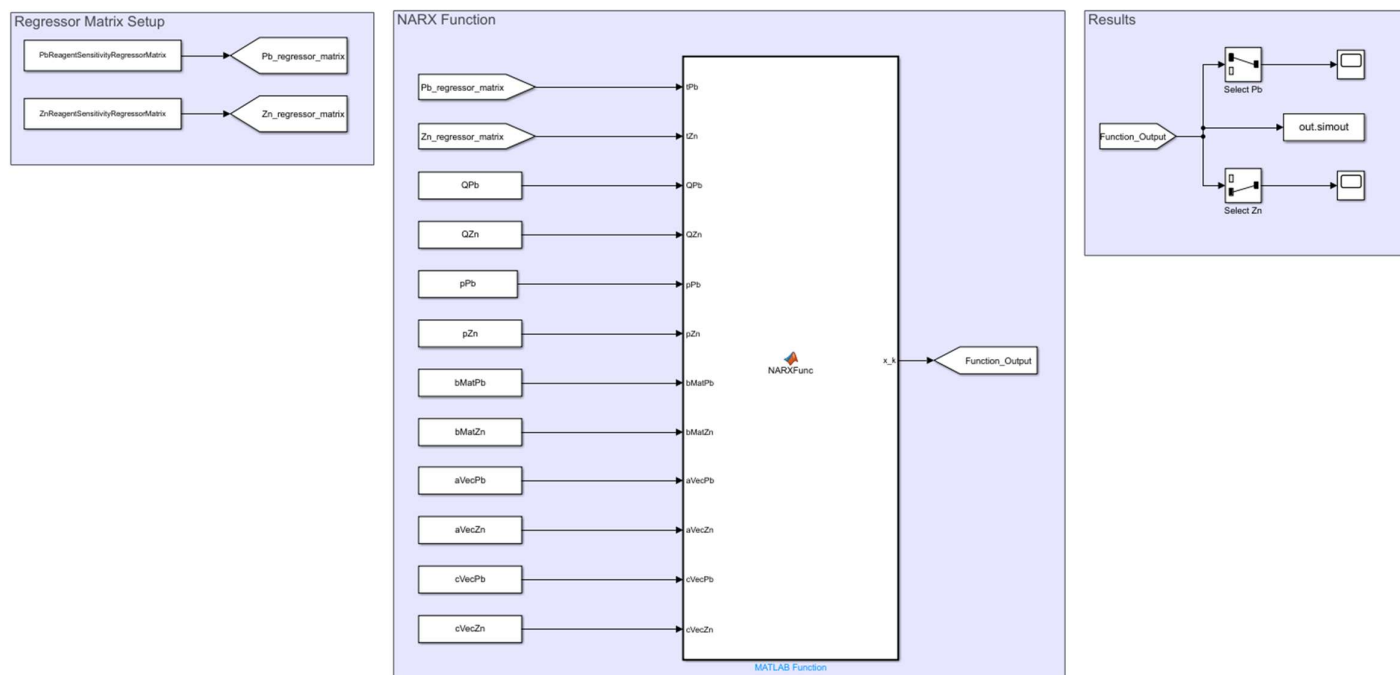


Figure 7-2 Reagent Sensitivity Simulink Setup

The reagent sensitivity plots were created by first setting up MATLAB regressor matrix variables. The variable consists of n rows and m columns. Where n represents the number of iterations or timesteps that the simulation will run, and m represents the number of regressors. Each row of the variable consists of the regressor matrix for each timestep and the value of one of the regressors was varied for each time step while fixing the value of the remaining regressors, i.e. one of the columns within the variable was varied. The values of the regressors were varied within a specified range. The Regressor Matrix Setup section consists of *from workspace* blocks, which read the regressor matrices into the simulation for each timestep. The regressor matrices are sent to the Nlarx function block. The Nlarx function block uses the regressor matrices along with other parameter matrices to calculate the output of the Nlarx function. For the Nlarx linear function output plot, the Nlarx function block was altered. Within the Nlarx function block, the nonlinear portion of the Nlarx function was removed from the block and only the linear portion of the Nlarx function remained.

All the outputs from the Nlarx function section are sent to the Results section. The Results section exports the values to the MATLAB workspace.

Parameter Matrix Specification

The Pb_regressor_matrix and Zn_regressor_matrix consist of the current MV and DV values, past MV and DV values and past CV values. These values are also known as regressors.

The first set of rows in the regressor matrices consists of the CV regressors. After that, the MV and DV regressors are added to the succeeding rows. The MV and DV regressors for the Pb_regressor_matrix and Zn_regressor_matrix were added to the succeeding rows in the following order:

- CuSO4
- Total_SEX
- Zn_In
- Pb_In

The number of regressors for the CVs, Mvs and DVs depends on the values of the model parameters in the NLarx model structures.

The number of CV regressors is equivalent to the value of na. The number of MV and DV values is determined using Equation 101.

$$\text{Number of MV and DV regressors} = nb + nk - 1 \quad (101)$$

The Pb_regressor_matrix has the following structure:

- Column 1 to 9 – Pb_Conc Regressors (9 regressors), where column 1 holds the value of the Pb_Conc from the previous iteration.
- Column 10 to 17 – CuSO4 Regressors (8 regressors), where column 10 holds the value of CuSO4 from the controller in the current iteration.
- Column 18 to 27 – Total_SEX Regressors (10 regressors), where column 18 holds the value of Total_SEX from the controller in the current iteration.
- Column 28 to 37 – Zn_In Regressors (10 regressors), where column 28 holds the value of Zn_In from the controller in the current iteration.
- Column 38 to 47 – Pb_In Regressors (10 regressors), where column 38 holds the value of Pb_In from the controller in the current iteration.

The Zn_regressor_matrix has the following structure:

- Column 1 to 4 – Zn_Conc Regressors (4 regressors); where column 1 holds the value of the Zn_Conc from the previous iteration.
- Columns 5 to 13 – CuSO4 Regressors (9 regressors), where column 5 holds the value of CuSO4 from the controller in the current iteration.
- Columns 14 to 23 – Total_SEX Regressors (10 regressors); Column 14 holds the value of Total_SEX from the controller in the current iteration.
- Columns 24 to 33 – Zn_In Regressors (10 regressors), where column 24 holds the value of Zn_In from the controller in the current iteration.

- Column 34 to 46 – Pb_In Regressors (13 regressors), where column 38 holds the value of Pb_In from the controller in the current iteration.

The AirLevPb_regressor_matrix and AirLevZn_regressor_matrix do not require the past CV values. The number of MV and DV regressors is also determined using Equation 101. The AirLevPb_regressor_matrix has the following structure:

- Columns 1 to 2 – Total Airflow rate Regressors, where column 1 holds the current value of the Total Airflow rate.
- Column 3 to 4 – Total Level Regressors, where column 3 holds the current value of the Total Level.

Model Parameter Matrices

Initial Regressor Matrices

Initial_Pb_regressor_matrix	Initial_Zn_regressor_matrix	Initial_AirLevPb_regressor_matrix	Initial_AirLevZn_regressor_matrix
2.265	40.53	65.780989	65.780989
2.265	40.53	65.780989	65.780989
2.265	40.53	1554.5	1554.5
2.265	40.53	1554.5	1554.5
2.265	47.9869		
2.265	47.9869		
2.265	47.9869		
2.265	47.9869		
2.265	47.9869		
47.9869	47.9869		
47.9869	47.9869		
47.9869	47.9869		
47.9869	47.9869		
47.9869	5.2443		
47.9869	5.2443		
47.9869	5.2443		
47.9869	5.2443		
5.2443	5.2443		
5.2443	5.2443		
5.2443	5.2443		
5.2443	5.2443		
5.2443	5.2443		
5.2443	69.2939		
5.2443	69.2939		
5.2443	69.2939		
5.2443	69.2939		
69.2939	69.2939		
69.2939	69.2939		
69.2939	69.2939		
69.2939	69.2939		
69.2939	69.2939		
69.2939	69.2939		
69.2939	9.5556		
69.2939	9.5556		
69.2939	9.5556		
69.2939	9.5556		
9.5556	9.5556		
9.5556	9.5556		
9.5556	9.5556		
9.5556	9.5556		
9.5556	9.5556		
9.5556	9.5556		
9.5556	9.5556		
9.5556	9.5556		
9.5556	9.5556		

Pb Linear Subspace Matrix

Rows 1 to 48 Columns 1 to 8

0.000197	0.000279	0.012668	0.060275	0.115259364	0.460985	0.027597	-0.00546
0.000198	0.000289	0.012711	0.060278	0.115637205	0.462428	0.027458	-0.0053
0.000199	0.000299	0.012752	0.060265	0.115896607	0.463402	0.027171	-0.00536
0.000199	0.000309	0.012792	0.060236	0.116048304	0.463878	0.026886	-0.00591
0.0002	0.000318	0.01283	0.060191	0.116110354	0.463841	0.026619	-0.0071
0.000201	0.000328	0.012866	0.060129	0.11611059	0.463265	0.026392	-0.00892
0.000201	0.000338	0.0129	0.06005	0.116091424	0.462129	0.026184	-0.01124
0.000202	0.000348	0.012932	0.05995	0.116061168	0.460438	0.025888	-0.01365
0.000203	0.000359	0.012961	0.059832	0.116011961	0.45821	0.025308	-0.0158
-0.01096	0.038081	0.00041	0.015144	-0.014543508	0.033309	-0.90633	0.017459
-0.01096	0.038251	0.000977	0.013963	-0.010501157	0.023875	-0.80671	0.027189
-0.01096	0.038389	0.001518	0.012626	-0.006483057	0.011022	-0.53683	0.028861
-0.01095	0.038501	0.002024	0.011252	-0.002936729	-0.00264	-0.18792	0.020008
-0.01094	0.03859	0.002488	0.009889	9.60589E-05	-0.01587	0.179902	0.000608
-0.01094	0.038656	0.002911	0.008557	0.002638041	-0.02826	0.530666	-0.02346
-0.01093	0.038695	0.003289	0.007281	0.00470447	-0.03864	0.803931	-0.04149
-0.01092	0.0387	0.003606	0.006055	0.0060405	-0.04429	0.907539	-0.04335
-0.00149	-0.0018	0.093287	-0.12296	-0.001176609	0.070848	-0.07328	-0.00708
-0.00149	-0.00179	0.093369	-0.12319	-0.000717219	0.069817	-0.05785	-0.00748
-0.00149	-0.00178	0.093442	-0.12341	-0.000304279	0.068694	-0.03933	-0.00814
-0.00148	-0.00177	0.093505	-0.1236	5.8243E-05	0.067523	-0.01975	-0.0089
-0.00148	-0.00176	0.093557	-0.12377	0.000363166	0.066363	-0.00147	-0.00955
-0.00148	-0.00175	0.093597	-0.12391	0.000610703	0.065267	0.013519	-0.01002
-0.00148	-0.00175	0.093626	-0.12403	0.000804176	0.064249	0.024243	-0.0102
-0.00148	-0.00175	0.093644	-0.12411	0.000940555	0.063295	0.031008	-0.00999
-0.00148	-0.00174	0.093651	-0.12416	0.001017583	0.062364	0.03517	-0.00945
-0.00148	-0.00174	0.093649	-0.12418	0.001039764	0.061409	0.038474	-0.00879
-0.0128	-0.02575	-0.01757	0.013307	-0.488798365	0.227263	0.040813	0.661985
-0.01281	-0.02579	-0.0183	0.011343	-0.411516674	0.201027	0.032212	0.25692
-0.01283	-0.0258	-0.01914	0.007991	-0.310636981	0.160165	0.020573	-0.15776
-0.01283	-0.02576	-0.02002	0.003477	-0.189845257	0.10619	0.006917	-0.49039
-0.01284	-0.02569	-0.02085	-0.00192	-0.056444942	0.043834	-0.00676	-0.65921
-0.01284	-0.02557	-0.02159	-0.00788	0.079869437	-0.02	-0.01753	-0.61846
-0.01284	-0.02542	-0.02216	-0.01408	0.208782535	-0.07789	-0.02229	-0.3741
-0.01284	-0.02522	-0.02252	-0.02019	0.321248278	-0.12364	-0.01944	0.018692
-0.01284	-0.02499	-0.02262	-0.0259	0.411128624	-0.15385	-0.00866	0.474653
-0.01283	-0.02473	-0.02244	-0.03089	0.476274607	-0.16869	0.008637	0.906534
-0.00137	-0.00527	0.080465	0.131777	-0.036281504	-0.12719	0.011304	0.790919
-0.00137	-0.0053	0.080709	0.133782	-0.057504877	-0.11506	0.014328	0.768535
-0.00137	-0.00534	0.08079	0.135156	-0.071500761	-0.1052	0.015571	0.60198
-0.00138	-0.00538	0.080734	0.135896	-0.073844845	-0.10214	0.013984	0.318249
-0.00138	-0.00541	0.08057	0.136015	-0.062135497	-0.10855	0.008088	-0.02014
-0.00139	-0.00544	0.080327	0.135574	-0.036548699	-0.12468	-0.00246	-0.32967
-0.00139	-0.00546	0.08002	0.134604	0.000243355	-0.14818	-0.01462	-0.53776
-0.00139	-0.00547	0.079651	0.133115	0.043363911	-0.17425	-0.02571	-0.60372
-0.00139	-0.00546	0.079228	0.131142	0.086830993	-0.1974	-0.03279	-0.5331
-0.00139	-0.00545	0.078752	0.128727	0.124241422	-0.21216	-0.03517	-0.37196

Rows 1 to 48 Columns 9 to 16

0.002437517	-0.00314	0.033875	-0.00146	0.056694	-0.00336	0.004274	-0.10599353
0.000851542	-0.00301	0.035031	-0.00125	0.037215	-0.00232	-0.00286	-0.09742099
0.000155938	-0.00437	0.031315	-0.00188	0.017091	-0.00359	0.005912	-0.07691931
0.000762783	-0.00622	0.025507	-0.00164	0.003029	-0.00687	-0.00073	-0.03707324
0.002231393	-0.00605	0.020336	-0.00092	-0.00261	-0.00345	-0.00026	0.010131764
0.002868409	-0.00381	0.017942	0.000867	-0.00777	-0.00216	0.000743	0.050688224
0.002547412	-0.00185	0.017339	-0.00094	-0.02285	-0.00396	0.000399	0.081373328
0.001532514	-0.00179	0.017793	-0.00251	-0.04092	0.000324	-0.00072	0.105446504
-5.11101E-05	-0.00186	0.019089	0.00155	-0.05669	0.004117	-0.00404	0.131001734
-1.336796817	1.515972	0.014607	1.85754	-0.05264	2.024969	-1.66809	0.005021157
-0.756135353	-0.1147	-0.04637	-1.48771	0.043026	-2.99501	3.562417	-0.00951722
0.521770097	-1.97203	-0.07444	-1.87132	0.01351	0.234012	-3.33316	-0.00023792
1.546279849	-1.23862	-0.01208	1.509379	-0.01341	2.393654	1.420242	0.009336443
1.552746054	1.222974	0.060726	1.517434	0.045926	-2.39159	1.439586	0.000216529
0.536752653	1.972478	0.05797	-1.8658	0.038616	-0.25314	-3.3385	0.004314372
-0.742239181	0.126421	0.01115	-1.50103	-0.05672	3.000118	3.559635	0.009487556
-1.331438498	-1.50948	-0.00563	1.846519	-0.02127	-2.01042	-1.64502	-0.03110413
0.005341991	0.043069	0.008554	-0.02096	0.002329	0.114849	0.03859	-0.03528066
0.027065962	0.023296	0.006474	-0.03184	0.002892	0.08152	0.057572	-0.02147337
0.043498062	0.01811	0.005906	-0.03125	0.001874	0.063178	0.046907	-0.00996722
0.043869705	0.025416	0.006507	-0.02428	-0.00016	0.052916	0.044395	-0.00076101
0.026132677	0.028395	0.007067	-0.01885	-0.00158	0.028346	0.032966	0.004983913
-0.000459878	0.013985	0.006155	-0.01231	-0.00405	-0.00045	0.007348	0.00377708
-0.023963129	-0.01243	0.00405	0.00825	-0.00685	-0.03129	-0.01888	-0.00037131
-0.037526229	-0.03444	0.002012	0.035064	-0.00659	-0.07296	-0.05068	1.71361E-05
-0.041998352	-0.04433	0.000606	0.047998	-0.00333	-0.10507	-0.07487	0.00725595
-0.042392351	-0.04891	-0.00038	0.047662	-0.00051	-0.1195	-0.08296	0.016359203
-0.014081676	-0.02398	1.188216	-0.01262	-0.06253	-0.00834	-0.0021	2.887619305
0.005040276	-0.00832	0.209915	-0.00859	-0.9504	-0.02222	-0.00555	1.001607541
0.015988338	0.004925	-0.5238	-0.0016	-1.00372	-0.0195	0.005084	-0.39762099
0.016788018	0.008914	-0.73282	0.004501	-0.14009	-0.00278	0.004755	-1.03878626
0.009788309	0.003488	-0.39734	0.00804	0.959677	0.020909	0.000444	-1.35205831
0.000600976	-0.007	0.20781	0.005059	1.42924	0.02381	0.007858	-1.71630682
-0.004742156	-0.01564	0.655085	-0.00557	0.925122	0.008267	0.002597	-1.82665833
-0.006270687	-0.01082	0.592406	-0.006	-0.07793	0.010151	-0.0107	-1.04104235
-0.006815073	0.00867	-0.06899	0.003153	-0.69596	-0.00228	0.006325	0.738431076
-0.009188753	0.033268	-1.12881	0.009695	-0.38044	-0.01345	-0.00661	2.767568769
-0.03227835	-0.02512	1.200549	-0.00226	2.834858	0.064467	0.020912	-0.23009639
-0.015440455	-0.01072	0.673058	0.009194	1.230691	0.014964	-0.00416	-2.08806322
0.000575646	0.019519	-0.35154	0.000888	-0.79625	-0.01907	-0.01965	-3.42017798
0.00842059	0.036503	-1.36824	-0.00611	-1.51723	-0.01213	0.021972	-2.42873969
0.007109972	0.035115	-1.83105	0.01573	-0.49903	0.010309	-0.01855	0.357631575
0.002182579	0.025431	-1.46218	0.030339	1.174548	0.022754	0.009991	2.563139147
-0.001104263	0.002283	-0.44278	0.011765	1.775302	-0.0121	-0.00165	2.572499103
0.000312352	-0.02202	0.696904	-0.01331	0.553129	-0.00144	-0.02497	1.220134218
0.006739693	-0.0344	1.403782	-0.02071	-1.6735	-0.02477	0.027536	0.524738227
0.013624338	-0.02161	1.415407	-0.0198	-3.06909	-0.04088	-0.01432	0.926077344

Rows 1 to 48 Columns 17 to 24

0.002583	0.072469	-0.06479	0.450507	-0.07423	-0.10245	-7.22553	0.343555
-0.00306	0.054194	-0.04062	0.368392	-0.04034	-0.1088	-6.35617	0.432582
-0.00297	0.04022	-0.03577	0.254246	0.01218	-0.11022	-4.72134	0.262647
-5.6E-05	0.038426	-0.02719	0.120096	0.010271	-0.12967	-2.50486	0.138329
-0.00507	0.021496	-0.00591	-0.01178	0.039586	-0.06888	0.018719	0.073579
0.00211	-0.01806	0.026476	-0.13691	0.06869	0.012753	2.531743	-0.16264
-0.005	-0.06411	0.057358	-0.24726	0.042285	0.072508	4.740932	-0.24995
0.00328	-0.08226	0.030229	-0.32691	0.013847	0.134783	6.370394	-0.38577
-5.2E-05	-0.0668	0.027946	-0.37219	0.023318	0.179602	7.224812	-0.45084
0.954778	-0.04849	-0.01101	0.410802	-0.01333	-0.00094	0.007437	-0.02181
-2.45891	0.074158	0.015546	0.1034	0.02649	0.012718	0.020553	0.059788
3.532217	-0.06873	-0.00298	0.273756	0.016718	-0.0151	0.010202	-0.08106
-4.24719	0.066822	0.024323	0.168997	-0.00703	-0.03836	0.013935	0.069463
4.238247	-0.08215	-0.06044	0.139523	0.023842	-0.01276	0.009065	-0.04164
-3.5258	0.102666	0.009819	-0.18869	0.006791	0.011853	-0.01342	0.006852
2.445667	-0.08338	0.04097	0.128009	0.009024	-0.00336	0.00544	0.02074
-0.93791	0.047785	-0.01257	-1.03715	-0.0642	0.04381	-0.0697	-0.00972
0.101536	0.03195	-0.00042	-5.69232	-0.26706	0.281196	-0.48058	-0.00393
0.0811	0.018887	0.000488	-5.06571	-0.23789	0.245344	-0.42209	0.013307
0.064646	0.004429	-0.00412	-4.04718	-0.18539	0.211922	-0.32496	0.010063
0.041771	3.91E-05	-0.0096	-2.68779	-0.13834	0.156411	-0.2007	-0.02191
0.027046	0.001241	-0.0067	-1.09132	-0.07905	0.091602	-0.0576	-0.0024
-0.00667	-0.00584	0.014835	0.636327	0.019625	0.020216	0.088244	0.015386
-0.03251	-0.0142	0.021501	2.378577	0.097495	-0.10256	0.218692	-0.01813
-0.06868	-0.01202	-0.00395	3.991124	0.173699	-0.23356	0.32386	0.001175
-0.09249	-0.00845	-0.01472	5.323117	0.290506	-0.31665	0.391705	0.021046
-0.11145	-0.01601	0.006747	6.230916	0.360049	-0.38624	0.425328	-0.00391
-0.00801	-0.40435	-1.16174	-0.31476	2.790842	-4.15165	0.040888	0.239987
-0.0279	-1.21164	0.912343	-0.00753	0.667277	0.664538	-0.17461	0.80092
-0.01906	-0.32497	0.348595	0.266147	-3.0779	2.837895	-0.2489	-1.38791
0.02851	1.395394	-0.55791	0.355002	-3.63951	3.72766	0.028572	-0.22611
0.041066	1.86257	0.662757	0.193335	-0.65926	2.598871	0.213188	0.474909
0.023061	0.59775	1.464618	-0.08117	0.973042	-1.68797	0.236785	-0.75725
-0.01026	-0.69237	-0.29401	-0.28909	1.798682	-4.43933	0.271551	1.267683
-0.00795	-0.61103	-1.73119	-0.30868	3.218167	-2.88159	0.024782	0.607171
-0.01406	0.016025	-0.4389	-0.10516	1.287763	-0.14352	-0.22797	-1.13494
-0.00771	-0.64032	0.788459	0.306679	-3.35179	3.47179	-0.1356	0.111783
0.060271	3.776597	-3.24567	0.235133	-4.61779	0.006847	0.132093	-3.69278
0.048829	-0.34488	3.670807	0.104505	3.077117	5.229745	0.404823	6.774435
-0.09713	-2.92958	3.730118	-0.04925	0.530707	-0.55298	-0.22164	-4.7303
0.001785	-0.65714	-3.76365	-0.09453	-3.37986	-5.43675	0.269268	-1.2599
0.039274	2.488315	-3.88213	-0.24424	4.976939	0.084126	0.328963	7.006653
0.037417	1.466908	3.069424	-0.2541	5.212641	-0.5963	-0.53915	-6.65575
-0.04917	-2.7051	2.944884	-0.04372	-4.31135	-3.97389	-0.0556	1.749953
-0.08287	-4.24674	-3.89421	0.179545	-1.5789	3.001329	0.027288	4.20191
-0.05208	-0.54023	-2.8266	0.13389	3.012684	4.888614	-0.53451	-7.22378
0.095227	3.733353	4.21015	0.02459	-2.99666	-2.59432	0.171477	3.823682

Rows 1 to 48 Columns 25 to 32

0.0755	0.030164	0.206767	1.527451	-13.5229	-0.07718	-18.3084	0.497191
0.065849	0.04352	0.236363	0.706834	-7.23189	0.15653	-0.0281	0.108966
-0.04934	0.033438	0.182412	-0.29089	2.474012	0.313715	18.28674	-0.43303
0.050711	0.039859	0.101533	-1.06234	11.02551	0.040662	18.29328	-0.51686
-0.05792	-0.017	-0.06134	-1.34932	14.39374	-0.49579	-0.00778	-0.10269
-0.03532	0.03288	-0.22655	-1.05225	11.00388	-0.36646	-18.3058	0.222466
0.029674	-0.0677	-0.21581	-0.28448	2.440366	0.039126	-18.2853	0.122931
-0.11307	-0.06282	-0.15505	0.603789	-7.27321	0.162045	0.045394	-0.02535
0.033292	-0.034	-0.03316	1.197339	-13.5701	0.320206	18.33093	0.170134
0.030875	-0.0261	-0.0092	0.364338	0.069308	-0.03193	-0.0065	0.416388
-0.04043	0.045652	-0.00774	0.20559	0.002	0.028336	-0.00276	0.069163
0.013639	-0.04556	0.006791	0.174936	0.011772	0.008809	-0.00542	-0.05598
0.017468	0.046841	0.030565	-0.01815	-0.0133	-0.00204	0.0047	-0.27989
-0.05856	-0.05878	-0.01699	-0.21022	-0.02121	-0.01101	0.030888	-0.32271
0.070427	0.064383	-0.01497	-0.27181	-0.02084	-0.01487	0.006155	-0.14896
-0.04382	-0.06762	-0.01249	-0.33272	-0.01264	0.020567	-0.00599	0.108351
0.009251	0.042062	0.015404	0.084561	-0.01216	0.00272	-0.0266	0.212499
-0.00237	0.010878	0.20775	-13.2418	-1.49618	-0.01476	-0.58427	-20.1876
0.010644	0.024648	0.161894	-8.06152	-0.8839	-0.12068	-0.01037	-2.75005
-0.00212	0.009263	0.034178	-0.41276	-0.00465	-0.19592	0.409993	16.00946
0.002489	-0.00166	-0.10056	7.227045	0.82384	-0.01643	0.307422	22.09772
0.017108	-0.00852	-0.14655	12.19097	1.312804	0.217669	-0.017	10.75619
0.009848	0.009556	-0.12193	12.67685	1.279297	0.213545	-0.1199	-8.9996
-0.00951	-0.02028	-0.07503	8.462912	0.759266	0.043907	-0.05234	-21.4451
-0.0038	-0.00042	-0.02755	1.023049	-0.0163	0.044513	-0.03084	-16.6291
-0.01096	-0.01537	0.019654	-7.00791	-0.7108	0.054216	-0.04673	1.637626
-0.00581	-0.00501	0.024577	-12.8963	-1.09247	-0.20093	0.131245	19.49903
-0.4012	-0.0235	8.524022	0.124948	0.068125	10.38195	-0.20824	-0.21884
0.822948	0.471935	-5.23386	0.017574	-0.16613	-14.0415	0.083294	0.426469
-0.93518	-0.81304	-9.8867	-0.16671	-0.11911	-9.00098	0.354049	-0.23206
1.069718	0.747827	-1.88704	-0.20311	0.196813	13.49336	-0.096	-0.36131
-0.71431	-1.11247	8.243014	0.013648	0.258883	11.84807	-0.27377	0.532905
-0.20935	1.256464	8.725537	0.250136	-0.04779	-11.2381	-0.02042	0.319274
0.398863	-0.87626	-1.03737	0.176261	-0.22961	-14.0732	0.268964	-0.57008
-0.81791	1.04737	-9.62885	-0.10611	-0.12165	8.194161	0.078822	-0.11796
1.289303	-0.97923	-5.8751	-0.17999	0.105469	14.28411	-0.28598	0.447759
-0.50142	0.279546	8.07151	0.076798	0.044449	-9.84414	0.108397	-0.21941
-2.78609	-2.05035	-3.85782	-0.03248	-0.00513	-4.0716	0.136182	0.068755
7.114536	5.099131	-0.88767	-0.05285	0.019183	1.646105	0.097707	-0.05994
-8.48881	-7.35162	2.641377	0.025918	0.046401	4.640729	-0.08405	0.054853
6.550633	8.743741	3.763004	0.11259	0.022622	0.118771	-0.15918	0.154191
-2.49857	-9.32385	0.159627	0.116848	-0.17011	-4.56364	0.007904	-0.07517
-2.20618	9.441307	-3.57815	0.018027	-0.11255	-0.40568	0.124253	-0.19819
6.640136	-8.78345	-2.93488	-0.10805	0.056972	4.691985	0.086603	0.017716
-8.35318	7.29331	0.808022	-0.07595	0.09638	1.807495	-0.07894	0.065563
7.189606	-5.26173	3.544749	-0.00838	0.182876	-3.59881	-0.02348	-0.09029
-3.17669	2.19262	0.338992	0.020129	-0.07178	-0.31318	-0.11155	0.057166

Rows 1 to 48 Columns 33 to 40

0.420167	20.83125	20.30386	-3.74828	-1.8969	-18.5922	-13.3662	-7.07525
-0.92644	-13.5472	-27.4436	4.335139	2.039272	37.06946	33.83771	20.5017
0.177488	-25.5167	-10.7166	2.74445	2.028665	-18.6477	-38.6386	-31.6966
0.555031	4.723297	31.51492	-3.85749	-3.07054	-17.9267	25.50129	39.14421
-0.20655	27.13299	-0.77487	-4.16198	-0.95068	36.14928	-0.3861	-41.7524
-0.254	4.677447	-30.9047	7.755648	2.987691	-18.1335	-25.1458	39.15534
-0.28186	-25.4907	11.43266	0.436498	0.009276	-17.96	39.13903	-31.7595
0.506859	-13.5146	26.77061	-9.07687	-1.68352	36.07277	-34.8065	20.59148
-0.00721	20.8182	-20.1832	5.625678	0.51742	-18.0807	13.86641	-7.12608
-0.03463	-0.0112	0.06032	-0.0131	0.299056	0.003861	-0.00501	-0.01356
0.010809	0.022344	-0.03895	-0.03224	-0.02551	0.008268	0.00365	0.033339
0.02525	-0.0219	-0.01833	0.072175	-0.16319	-0.00808	0.002194	-0.03853
-0.03552	-0.01858	0.003914	-0.03198	-0.18789	-0.00771	-0.00973	0.036476
0.049942	0.00287	-0.02334	-0.01757	-0.01178	0.02277	0.0297	-0.02428
-0.00361	0.02414	0.021626	0.036986	0.160593	-0.02417	-0.04416	0.01005
-0.03319	0.014228	0.006278	-0.01329	0.150866	0.004373	0.057499	0.003141
0.025265	-0.01324	-0.00901	-0.0011	-0.22408	0.001511	-0.03507	-0.00645
0.538864	0.091939	-1.98543	0.328642	-26.9406	-0.03422	-0.25234	0.150274
0.146773	-0.13651	0.788804	-0.27945	11.81922	-0.14457	0.064007	-0.37497
-0.4768	0.289357	2.585033	-0.53105	32.54123	-0.03576	0.253166	0.242349
-0.7876	0.432575	0.780867	0.19677	9.419676	0.334322	0.280901	-0.05185
-0.31888	-0.43961	-2.17802	0.379038	-26.4261	0.121141	-0.37918	0.205313
0.646403	-0.96779	-1.90331	0.117684	-27.1949	-0.48982	-0.33818	-0.30101
0.677318	-0.05331	0.936437	-0.07962	8.089589	-0.01482	-0.00444	0.038313
-0.03804	1.088612	1.862477	-0.24426	32.27328	0.424045	0.589814	0.082825
-0.26453	0.572989	0.2754	-0.22882	12.74218	0.106783	0.280746	0.172088
-0.10866	-0.86643	-1.15579	0.352735	-26.3489	-0.27043	-0.4969	-0.16495
-10.9095	0.358623	2.241089	10.39427	-0.10226	0.754579	-1.48909	-0.18722
22.1732	-0.74516	-5.80024	-28.1197	0.16332	-1.96621	4.002318	0.455624
-4.12522	0.342535	4.626295	25.22733	0.113205	1.486295	-4.22116	-0.13858
-22.7864	0.281898	1.393167	1.81627	-0.53634	0.694031	1.940883	-0.92131
15.34469	-0.38209	-6.06167	-26.8436	0.601389	-2.45498	0.580326	2.059891
15.81374	0.259824	5.017519	26.95635	0.055211	1.992957	-2.02308	-2.5424
-22.4612	-0.22684	0.479293	-2.09813	-0.63923	0.968432	2.421904	2.11728
-4.72562	-0.1096	-5.50567	-25.1224	0.287668	-4.37038	-2.23153	-1.11556
22.05888	0.503611	5.526478	27.85891	0.216797	4.539847	1.515993	0.26444
-10.3902	-0.27814	-1.91991	-10.0666	-0.158	-1.6468	-0.49525	0.0071
3.842299	-0.25805	-0.79499	-3.15845	0.054423	-0.21967	0.606024	0.020207
-3.91293	0.202499	1.261159	5.18877	0.03644	0.365663	-1.00261	-0.06101
-2.30394	0.212167	-0.39888	-2.04038	-0.06775	-0.146	0.698998	0.047251
4.872554	-0.1441	-0.71916	-2.36185	0.033831	-0.09885	-0.19147	0.035753
0.214274	-0.01631	1.025188	4.754702	-0.06603	0.183451	-0.23894	-0.14611
-4.87623	-0.08898	-0.27645	-2.39418	-0.03016	0.071918	0.488423	0.193458
1.893595	0.077238	-0.59334	-1.9467	0.086556	-0.36004	-0.59985	-0.14718
3.988064	0.162197	0.990829	4.792184	-0.02753	0.543677	0.504809	0.06511
-2.94875	-0.14285	-0.42441	-2.23959	-0.00202	-0.10484	-0.18717	-0.03917
-0.77228	-0.03264	-0.07139	-0.61876	-0.00368	-0.22542	-0.07823	0.035791

Rows 1 to 48 Columns 41 to 48

-1.68734	-0.26728	-1.05825	-0.34887	-0.16097	-0.02061	-0.05685
4.591585	-0.83402	2.17365	0.435849	0.116136	0.0331	0.215921
-5.20736	1.485581	-2.05346	0.570508	0.152023	-0.25478	-0.36067
2.833505	0.556944	1.687964	-0.85696	-0.25172	0.409623	0.445975
0.880906	-1.0229	-1.46283	-0.30966	0.36671	-0.21479	-0.57367
-3.16982	-0.75841	1.602743	0.646151	-0.2508	0.225298	0.673048
2.772562	0.967144	-1.81654	0.178308	-0.05376	-0.33879	-0.72057
-0.5397	0.676257	1.612478	-0.57468	0.205312	0.189547	0.751025
-0.46789	-0.83636	-0.7077	0.268907	-0.14109	-0.03452	-0.36907
-0.04158	-0.30944	-0.03275	-0.17371	-0.14649	0.076928	0.078123
0.072157	0.141819	0.064337	0.145093	0.097115	-0.0576	-0.08653
-0.04996	0.130352	-0.05405	-0.01054	-0.05443	0.03721	0.088774
0.031002	-0.00459	0.031128	-0.05164	0.060505	-0.05457	-0.12075
-0.01952	-0.06038	-0.00304	0.099163	0.059147	-0.0074	0.099651
0.019949	0.021091	-0.00978	-0.00216	-0.11016	0.051043	-0.07828
-0.02877	0.092763	0.014182	-0.07693	0.095951	-0.1015	0.040759
0.013769	-0.01014	-0.01026	0.072253	-0.00057	0.055005	-0.02212
0.857698	32.8391	0.0551	35.81662	35.41666	-29.992	-16.7478
-1.0973	-35.472	-0.43949	-61.9814	-82.3859	83.85939	52.75778
-0.26479	-31.0158	0.831307	4.252383	64.24393	-109.137	-87.759
0.449486	33.76029	-0.34283	60.27971	9.356488	89.81719	114.0577
0.617347	32.72874	-0.33317	-38.0806	-74.6019	-34.3603	-128.004
-0.51305	-32.1208	0.175804	-38.5417	74.32373	-35.0833	127.9293
-0.17272	-34.4955	0.065889	60.05171	-8.86007	90.25137	-113.757
0.060723	30.00753	0.287376	4.991771	-64.9463	-109.23	87.42107
0.176268	35.80305	-0.52364	-62.008	82.31345	83.39036	-52.2964
-0.12454	-32.0284	0.218213	35.23601	-34.8562	-29.5264	16.3963
9.915165	-0.54377	-5.40419	-0.02695	0.103002	-0.07035	0.129204
-30.8211	1.3447	19.26354	0.406361	-0.27944	0.195207	-0.38735
42.48129	-0.97051	-34.2105	-0.82915	0.225586	-0.35894	0.519604
-34.5351	0.075304	45.05357	0.292655	0.107706	0.664966	-0.44378
13.12782	-0.16734	-50.9193	0.680035	-0.22984	-0.8808	0.222887
12.78336	0.459546	50.98231	-0.86306	-0.14727	0.869068	0.128222
-34.3834	0.052929	-45.1677	0.442236	0.554474	-0.87822	-0.50383
42.53792	-0.7024	34.37341	-0.21795	-0.56525	0.979876	0.7668
-30.8178	0.57626	-19.2673	0.193556	0.259385	-0.78241	-0.66119
9.716861	-0.12977	5.295435	-0.07924	-0.03157	0.262538	0.230571
-2.72015	0.026141	1.533279	-0.02645	0.058902	0.014024	0.010721
5.251618	-0.08018	-3.60748	0.051421	-0.09775	0.023774	-0.02836
-5.02689	0.0648	4.894432	-0.11331	-0.02771	-0.0385	0.053514
2.928841	-0.06062	-5.87545	0.096023	0.150791	-0.02126	-0.08141
0.06054	0.02529	6.085853	0.016923	-0.1664	0.087617	0.098315
-3.16211	0.032276	-5.76484	-0.03971	0.046031	-0.14995	-0.11713
5.313608	-0.06126	4.644648	-0.02827	0.075804	0.125711	0.13421
-5.47185	0.120269	-3.14551	0.112185	-0.04655	-0.06023	-0.17445
2.392081	-0.06006	0.869615	-0.11648	-0.031	-0.03861	0.115815
0.438277	-0.00142	0.375594	0.039752	0.041504	0.062757	-0.01161

Zn Linear Subspace Matrix

Row 1 to 47 Column 1 to 8

-0.00045	0.006961	0.094601	0.044859	-0.22859	-0.05745	0.011832	-0.00986
-0.00044	0.00695	0.094672	0.0454	-0.2307	-0.05998	0.008999	9.53E-05
-0.00044	0.006938	0.094678	0.045748	-0.23072	-0.06228	0.005892	0.006544
-0.00044	0.006926	0.094621	0.045886	-0.22863	-0.06406	0.002761	0.008764
0.010578	0.03317	-0.00517	0.013504	0.004809	0.016179	-0.77951	-0.00144
0.010576	0.033324	-0.00543	0.012567	0.005322	0.01191	-0.70959	-0.01398
0.010573	0.033452	-0.00568	0.011498	0.005799	0.007652	-0.52459	-0.02349
0.010569	0.03356	-0.00591	0.010406	0.006277	0.003841	-0.27565	-0.0239
0.010563	0.03365	-0.00612	0.00928	0.006845	0.000631	-0.00341	-0.01299
0.010557	0.033719	-0.00632	0.008143	0.007551	-0.00196	0.2694	0.005167
0.01055	0.033769	-0.0065	0.007024	0.008412	-0.00396	0.52003	0.022719
0.010541	0.033799	-0.00667	0.005946	0.009434	-0.00527	0.707657	0.032217
0.010531	0.0338	-0.00681	0.004898	0.01042	-0.00585	0.780505	0.030757
0.00142	-0.00243	-0.02961	-0.08952	-0.20548	-0.03147	-0.06224	0.00836
0.001419	-0.00242	-0.02964	-0.0897	-0.20563	-0.03203	-0.05032	0.008437
0.001418	-0.00241	-0.02967	-0.08987	-0.20572	-0.03253	-0.03569	0.008752
0.001417	-0.0024	-0.0297	-0.09002	-0.20574	-0.03296	-0.01979	0.009205
0.001416	-0.00239	-0.02972	-0.09016	-0.20568	-0.03331	-0.0041	0.009658
0.001415	-0.00238	-0.02975	-0.09028	-0.20554	-0.03358	0.010022	0.01009
0.001413	-0.00238	-0.02977	-0.09038	-0.20533	-0.03377	0.021385	0.010345
0.001412	-0.00237	-0.02978	-0.09047	-0.20504	-0.03387	0.029506	0.010255
0.001411	-0.00237	-0.0298	-0.09053	-0.20466	-0.03388	0.034768	0.009806
0.001409	-0.00237	-0.02981	-0.09057	-0.20421	-0.0338	0.038251	0.009157
0.012126	-0.0253	0.013494	0.001522	-0.04306	0.493575	0.020479	-0.63935
0.01214	-0.02535	0.013445	-0.00034	-0.02989	0.41866	0.015746	-0.24317
0.012152	-0.02536	0.013376	-0.00328	-0.01449	0.31965	0.009389	0.162151
0.012161	-0.02534	0.013276	-0.00709	0.002463	0.200137	0.002369	0.486776
0.012167	-0.02528	0.013137	-0.01151	0.020047	0.067425	-0.00445	0.650648
0.012171	-0.02518	0.012955	-0.01632	0.037241	-0.06872	-0.00971	0.608826
0.012172	-0.02504	0.012729	-0.0213	0.053054	-0.19774	-0.01179	0.366483
0.01217	-0.02487	0.012461	-0.02626	0.066672	-0.31013	-0.00969	-0.02338
0.012166	-0.02466	0.012154	-0.03108	0.077571	-0.39908	-0.00284	-0.47839
0.012159	-0.02443	0.011815	-0.03565	0.085562	-0.46163	0.008637	-0.9139
0.001296	-0.00552	-0.02352	0.135379	-0.08906	0.036974	0.019755	-0.7958
0.0013	-0.00555	-0.02355	0.137057	-0.0909	0.05872	0.021451	-0.77395
0.001303	-0.00559	-0.02355	0.138158	-0.09137	0.073294	0.021603	-0.61057
0.001307	-0.00562	-0.02355	0.138726	-0.09032	0.076135	0.020398	-0.33211
0.00131	-0.00565	-0.02354	0.138823	-0.08782	0.064686	0.017369	0.0002
0.001313	-0.00568	-0.02353	0.138538	-0.0841	0.038939	0.012239	0.305133
0.001315	-0.0057	-0.02352	0.137929	-0.07942	0.001366	0.005512	0.512425
0.001317	-0.00571	-0.02352	0.137063	-0.07417	-0.04354	-0.00138	0.583055
0.001318	-0.00571	-0.02354	0.135988	-0.06877	-0.08997	-0.00696	0.521551
0.001318	-0.0057	-0.02356	0.134753	-0.06369	-0.13162	-0.01046	0.372311
0.001318	-0.00567	-0.0236	0.13336	-0.05933	-0.16335	-0.01205	0.198256
0.001317	-0.00564	-0.02364	0.131754	-0.0557	-0.18395	-0.01172	0.039125
0.001315	-0.0056	-0.02367	0.129844	-0.05267	-0.19423	-0.00951	-0.08321

Rows 1 to 47 Column 9 to 16

0.006762	-0.02565	0.049162	-0.03481	0.029359	-0.00782	0.553169	-0.11096
0.003522	0.002428	0.02429	-0.02919	-0.00691	0.023917	0.198628	-0.02554
-0.00424	0.014637	-0.0116	0.01785	-0.0095	0.020655	-0.23921	0.007247
-0.00961	0.001277	-0.03849	0.045409	0.013597	-0.03095	-0.5036	0.128012
-1.20201	1.34353	0.031022	-1.58507	0.023104	-1.87506	-0.09148	-1.90021
-0.80174	0.2363	0.035728	0.718352	-0.00943	2.075399	0.080636	3.172871
0.167966	-1.44711	0.007019	2.057151	0.000609	1.140941	0.007222	-1.20143
1.085406	-1.67749	-0.01337	-0.16997	0.033664	-2.19593	-0.04639	-1.60091
1.453101	-0.00925	0.001176	-2.04892	-0.00025	-0.00571	0.082193	3.029484
1.095082	1.667722	0.016034	-0.18576	-0.0842	2.201376	0.009223	-1.58546
0.181687	1.453392	-0.00547	2.053432	-0.07718	-1.12625	-0.07855	-1.21481
-0.79098	-0.22483	-0.04302	0.731879	0.031088	-2.08249	0.050105	3.181261
-1.19812	-1.33877	-0.04322	-1.57578	0.082881	1.864984	-0.00895	-1.88474
-0.0057	0.033877	-0.00284	0.009576	0.003047	-0.06616	0.002961	0.015786
0.015277	0.014023	-0.00146	0.02241	0.003761	-0.04817	0.004367	0.043608
0.033763	0.005177	-0.00082	0.020228	0.003792	-0.0358	0.003714	0.041904
0.041216	0.01254	-0.00074	0.011844	0.003117	-0.03049	0.00202	0.032355
0.03271	0.024087	-0.00108	0.009098	0.002386	-0.02057	0.000645	0.031895
0.011464	0.022442	-0.0016	0.011625	0.003321	-0.00416	-0.00226	0.01714
-0.01349	0.003592	-0.00188	0.00578	0.005929	0.010563	-0.0049	-0.00708
-0.03346	-0.02168	-0.00158	-0.01463	0.008181	0.034043	-0.00484	-0.031
-0.04434	-0.04015	-0.0007	-0.03569	0.008661	0.067145	-0.00299	-0.06266
-0.04742	-0.04871	0.00043	-0.04308	0.007785	0.089565	-0.00178	-0.08413
-0.01216	0.008108	-1.14303	-0.00246	-0.52259	-0.00179	0.55139	-0.00766
0.006021	0.005437	-0.60325	0.004949	0.313852	0.021722	-0.49523	0.004797
0.0189	0.00645	-0.07785	0.012872	0.808775	0.019664	-0.74972	0.017841
0.021001	0.003856	0.282461	0.012788	0.729381	0.004782	-0.0977	0.00718
0.013366	-0.00294	0.385432	0.003812	0.154444	-0.01422	0.77916	-0.01919
0.001527	-0.01137	0.257206	-0.00737	-0.55107	-0.02098	1.03748	-0.01975
-0.00762	-0.01757	0.04428	-0.01159	-0.94137	-0.00923	0.434144	-0.00057
-0.01154	-0.01389	-0.04617	-0.01127	-0.75039	-0.00394	-0.4572	0.000991
-0.01116	0.000542	0.163021	-0.00564	-0.05117	-0.00049	-0.78826	0.015825
-0.01008	0.017452	0.743923	0.007571	0.80113	0.007659	-0.2214	0.003731
-0.03629	-0.01683	0.055672	-0.01212	-1.78087	-0.05691	2.264379	-0.04774
-0.02392	-0.02028	0.185456	-0.02545	-1.15593	-0.02075	0.56456	-0.00723
-0.00545	-0.00468	0.533843	-0.01501	-0.05533	0.023807	-1.49811	0.028242
0.008579	0.009628	0.987239	0.014839	0.905693	0.028235	-2.09971	0.049043
0.011409	0.009451	1.330529	0.024991	1.198503	-0.00855	-0.8175	0.020289
0.004601	-0.0032	1.360982	0.00537	0.667636	-0.02451	1.057814	-0.03357
-0.00224	-0.02042	0.995217	-0.00802	-0.33169	-0.00516	1.635683	-0.02364
-0.00567	-0.03473	0.312885	-0.01419	-1.15231	0.009153	0.312066	-0.02672
-0.00215	-0.03638	-0.47334	-0.01547	-1.26383	0.013603	-1.68387	0.054482
0.005007	-0.01779	-1.13155	-0.0164	-0.59467	0.027073	-2.42392	0.06958
0.01144	0.017512	-1.49906	-0.00409	0.457187	0.023899	-1.15684	-0.0022
0.013333	0.048848	-1.48595	0.021563	1.381047	0.004812	1.189843	-0.03195
0.010571	0.055853	-1.10797	0.040435	1.752466	-0.00454	2.715654	-0.05416

Rows 1 to 47 Column 17 to 24

4.406144	-0.01012	-0.89945	0.111891	0.002186	0.3409973	-3.65728	4.901473
1.839176	-0.06853	-0.27721	0.138235	0.044999	-0.2596954	3.653498	-4.932926
-1.8172	0.028617	0.438614	0.012212	-0.01682	-0.0562899	3.69225	-4.857106
-4.42091	0.05308	0.735635	-0.24852	-0.0307	-0.0174321	-3.71796	4.9087698
-0.03077	-1.38659	0.03101	0.01652	-0.8664	0.0102516	-0.00207	-0.009362
0.116422	3.217629	-0.03731	-0.03033	2.173666	-0.012623	-0.00536	-0.009263
-0.01289	-3.52983	-0.01406	0.004353	-3.12884	-0.0039346	0.004706	-0.008837
-0.04524	2.389836	0.027552	-0.00114	3.84927	-0.024379	0.042958	-0.05494
0.016303	0.014838	-0.00419	0.007116	-4.06868	0.0417686	0.035232	0.0013045
-0.08313	-2.40311	0.013866	-0.02551	3.842381	-0.0032032	-0.0835	0.0881539
0.075936	3.535057	-0.0112	0.015005	-3.11419	-0.0037485	-0.0105	-0.051487
0.06529	-3.21042	-0.00116	0.023173	2.162555	-0.0584785	0.044256	-0.003136
-0.104	1.371249	-0.01837	-0.00219	-0.85109	0.0568682	-0.02525	0.0462557
-0.05513	-0.08747	-0.02914	0.019931	0.019053	0.0092937	-0.09212	0.2213969
-0.04446	-0.06576	-0.01455	0.015758	0.030337	0.0005318	-0.08761	0.2090891
-0.03184	-0.05873	-0.00109	0.014176	0.029974	-0.0032092	-0.06564	0.1655958
-0.01898	-0.03566	0.006732	0.009642	0.028163	0.0049607	-0.0513	0.1144345
-0.00386	-0.02187	0.006608	0.00179	0.016499	0.0074403	-0.03139	0.0458151
0.01072	-0.00474	0.002791	0.00115	0.012949	-0.0097187	-0.01233	-0.034551
0.023131	0.023872	-0.00146	0.005357	-0.00962	-0.0170777	0.021667	-0.098206
0.031558	0.051827	-0.00468	0.001562	-0.02338	-4.047E-05	0.075486	-0.14835
0.037964	0.084888	-0.00358	-0.00993	-0.04705	0.0110193	0.097236	-0.196424
0.043883	0.111737	0.00435	-0.01674	-0.05762	-0.0036475	0.095027	-0.239399
0.30236	-0.00267	2.493617	-1.46999	-0.00814	0.3732429	0.456143	1.7226931
0.271965	0.004355	1.50648	0.242981	-0.00679	-0.9057136	-0.66695	-0.371061
0.082803	0.007743	-0.01095	0.635836	0.010535	-0.1244453	0.181577	-0.95075
-0.20803	0.000348	-1.49766	-0.00856	0.004538	1.1839764	0.617514	-1.092075
-0.42601	0.000278	-2.22289	-0.23873	0.003067	0.5802175	-0.65898	-1.391965
-0.43148	0.009299	-1.99043	0.575148	-0.00378	-0.8916275	-0.83946	-0.277395
-0.2218	0.000254	-1.20295	1.40834	0.006427	-0.6877046	0.579403	1.8310055
0.06778	-0.0146	-0.27353	0.976239	0.001637	0.6763764	0.635089	1.9531146
0.265364	2.98E-05	0.828786	-0.53382	0.01143	0.6017254	-0.3736	-0.053782
0.30238	-0.00348	2.392029	-1.60319	-0.0179	-0.8115251	0.078042	-1.372987
-0.62314	0.012681	-2.40093	-2.22748	0.0238	3.3100171	2.541014	0.6021831
-0.34679	-0.00504	-1.75284	1.275468	-0.0516	-2.1525482	-3.11886	-2.533397
-0.01724	-0.00913	-1.13694	3.673773	0.049019	-3.5620956	-1.99294	-1.114942
0.06394	0.008848	-1.08762	1.816735	-0.00781	1.4886925	3.755204	2.9916785
-0.00527	-0.04417	-0.67833	-1.90696	0.0258	3.5774166	0.93836	1.6773708
0.041079	0.017837	0.804148	-3.02519	-0.02539	-1.0493954	-3.78964	-1.578906
0.177129	0.01465	2.681548	-0.57804	-0.01832	-4.2002804	0.574712	0.1973852
0.327187	-0.00189	3.322112	1.803871	-0.01479	-0.1144198	3.845063	2.4221156
0.432571	0.01571	1.992894	0.817801	0.033552	3.9453648	-1.49896	-1.27826
0.413011	-0.04909	-0.29539	-2.09722	-0.01434	0.916669	-3.47616	-3.270403
0.118059	-0.00451	-1.69811	-2.76777	0.015496	-3.8422499	2.08186	1.5964028
-0.23929	0.04143	-0.84362	0.117928	-0.04493	-1.8140792	2.768791	2.3959889
-0.31542	0.008086	1.074712	3.116798	0.028204	3.5401399	-2.66099	-2.100817

Rows 1 to 47 Column 25 to 32

0.294701	1.562368	1.097705	-1.23111	-4.32221	0.49236	0.046871	0.27182
-0.49261	-1.33672	-1.16024	3.090007	10.40976	-0.90017	0.103877	-0.67341
-0.28356	-1.88169	-0.51362	-3.19367	-10.3888	0.585644	-0.3377	0.712946
0.509074	1.680564	0.588933	1.331606	4.301015	-0.17953	0.184756	-0.30949
0.385868	0.023002	-0.00673	0.021098	0.000463	0.020254	-0.00937	0.02595
0.161853	-0.02716	0.022644	-0.05928	0.002519	-0.0149	0.018918	-0.05535
0.24703	-0.014	-0.01771	0.070564	-0.007	-0.02535	-0.02385	0.062631
0.193377	-0.02675	-0.03679	-0.03472	0.032555	0.060992	0.019992	-0.0749
0.129474	-0.0189	-0.02232	-0.01099	-0.03829	-0.09197	0.004062	0.081034
0.066194	0.002466	0.03135	0.021482	-0.03028	0.084735	-0.03414	-0.07878
-0.22351	0.015774	-0.02185	-0.01835	0.1087	-0.04536	0.045745	0.068858
0.0075	-0.0059	0.027785	0.002439	-0.11174	0.004107	-0.04918	-0.06247
-0.97163	0.047971	0.019143	0.006734	0.043375	0.006782	0.027047	0.034008
-5.90813	0.293917	0.243634	0.019191	-0.0602	0.013007	0.028716	0.013652
-5.23662	0.268006	0.214755	0.006412	-0.02029	0.017636	0.01015	-0.00568
-4.16042	0.210943	0.191622	0.000492	-0.03163	0.004729	0.011581	0.003508
-2.72586	0.164901	0.139664	0.022132	-0.03482	0.000645	0.006466	-0.00206
-1.04701	0.103936	0.081039	-0.00099	-0.00929	0.010808	-0.00926	-0.0031
0.73614	-0.00407	0.026234	-0.01869	0.008832	0.002788	-0.02392	-0.01819
2.497611	-0.10046	-0.08253	0.011229	0.005902	-0.0105	-0.00228	0.005622
4.105631	-0.19842	-0.20053	-0.00609	0.039911	-0.0045	-0.00898	-0.01149
5.419775	-0.32506	-0.26878	-0.03449	0.046911	-0.01557	-0.01079	0.019586
6.329461	-0.40052	-0.33638	-0.00883	0.058819	-0.0158	-0.00854	-0.00099
-0.22944	-3.32565	-3.50325	-0.44573	0.098834	-0.24687	0.350628	-0.32151
-0.0294	-0.4898	0.704013	-0.64131	0.265925	0.645491	-0.77249	-0.06454
0.199798	3.377107	2.342252	1.403099	-0.51593	-0.85887	0.718921	0.941137
0.273999	3.946499	3.126923	0.537135	-0.01964	0.726913	-0.9457	-0.49265
0.135628	0.996811	2.078357	-0.4334	0.235274	-0.22822	1.100352	0.525146
-0.05949	-1.11366	-1.86864	0.30739	-0.36689	-0.57531	-0.50166	-1.28017
-0.18339	-2.52066	-3.66573	-1.14877	0.225866	0.866812	0.284746	0.865019
-0.23207	-3.59755	-1.97715	-0.71112	0.368017	-0.81296	-0.0818	-0.74462
-0.11923	-1.09352	-0.165	1.097251	-0.14216	0.543918	-0.79971	1.399008
0.249608	3.815294	2.928215	0.037506	-0.15019	-0.0594	0.648657	-0.82874
0.200642	4.307371	-0.4851	3.164324	-1.20419	-2.59608	2.1904	1.666165
0.058847	-2.10827	5.263992	-5.36892	2.254951	6.215447	-5.73015	-3.81782
-0.02448	-0.27505	-1.07475	3.428653	-1.57568	-6.60854	7.671445	5.49489
-0.01784	2.162544	-5.01081	1.384617	-0.59052	3.648991	-7.57805	-7.02321
-0.16833	-5.06873	1.421214	-5.96809	2.496046	1.15043	6.239778	7.624765
-0.29004	-4.29503	-1.20317	4.661494	-2.01467	-5.3592	-3.79196	-8.20645
-0.00321	3.521701	-3.91578	-0.15879	0.412023	7.4295	0.215144	8.64948
0.247188	1.221145	4.437724	-4.27968	0.894518	-5.85416	3.155872	-8.35718
0.063247	-1.63115	3.632183	6.410267	-1.26573	1.839453	-5.74299	7.701051
-0.0893	3.21269	-4.14136	-2.67334	0.37167	3.137242	7.679786	-7.18323
0.054713	0.842473	0.563067	-3.34765	0.838703	-6.55426	-7.8581	5.944392
0.086552	-3.21295	3.684061	6.059172	-1.53313	6.406712	6.089222	-4.15839
-0.07235	1.391598	-3.14888	-3.31318	0.909868	-2.87156	-2.54201	1.676073

Rows 1 to 47 Columns 33 to 40

0.34148	0.082489	-0.05016	-0.01519	-0.34957	-0.00785	-0.00936	-0.01157
-0.02402	-0.0412	0.463277	0.013751	0.415188	-0.10815	0.046372	0.077821
-0.38772	-0.0329	-0.27828	-0.00306	0.271215	0.289574	-0.01291	-0.19481
0.068797	0.024033	-0.11995	-0.0064	-0.3401	-0.16622	-0.00778	0.129396
-0.00755	-0.37246	0.029028	-0.41616	0.034611	-0.00328	-0.30141	0.04166
-0.00903	-0.18675	-0.02781	-0.07003	-0.01258	0.040154	0.021854	-0.07079
0.01204	-0.1773	-0.00839	0.056507	-0.02425	-0.06429	0.167762	0.047668
0.023099	0.026295	0.004367	0.279351	0.038128	0.03342	0.184845	-0.02961
-0.01922	0.229745	0.007272	0.319912	-0.05032	0.02387	0.00564	0.021269
-0.01688	0.338009	0.015175	0.141814	-0.00564	-0.03577	-0.17937	-0.02158
-0.00721	0.256327	-0.01861	-0.10127	0.039769	0.003083	-0.13289	0.03252
-0.00378	0.278189	-0.00273	-0.25031	-0.04704	0.046193	0.121984	-0.00858
0.019435	-0.39076	-0.00143	0.041417	0.023036	-0.04535	0.112955	-0.00962
0.219542	13.25698	0.051029	20.18477	-0.52645	0.285202	27.03112	-0.85952
0.174702	8.027291	0.148223	2.7435	-0.14288	0.034526	-11.841	1.081673
0.045634	0.292769	0.198021	-16.0033	0.485939	-0.25278	-32.6423	0.257797
-0.09362	-7.39586	-0.01036	-22.0752	0.772476	-0.4721	-9.41728	-0.40683
-0.14363	-12.3345	-0.25254	-10.7344	0.297911	0.260489	26.57498	-0.63247
-0.12078	-12.7556	-0.23153	9.01025	-0.63824	0.5266	27.31119	0.455031
-0.07788	-8.45494	-0.04064	21.44644	-0.64045	-0.19078	-8.14121	0.188304
-0.04057	-0.91509	-0.02776	16.62551	0.073812	-0.45084	-32.3654	-0.00103
0.002105	7.200033	-0.03862	-1.65351	0.252044	0.080593	-12.7489	-0.1867
0.003183	13.17076	0.212915	-19.545	0.043495	0.184293	26.28657	0.117072
8.496001	-0.12591	-10.3821	0.219635	10.91146	-10.6918	0.195276	-9.96145
-5.30214	-0.03726	14.0473	-0.40244	-22.2134	28.86956	-0.44293	30.9209
-9.85295	0.181916	8.979263	0.203945	4.174724	-25.812	0.182908	-42.5311
-1.80204	0.215561	-13.5421	0.341454	22.8161	-2.02498	0.473941	34.50134
8.307793	-0.03732	-11.8077	-0.49301	-15.4673	27.65418	-0.81789	-13.0566
8.647449	-0.25784	11.30818	-0.31532	-15.7793	-27.6412	0.230991	-12.8836
-1.10759	-0.17665	14.04301	0.524084	22.5344	2.070593	0.572972	34.5486
-9.72274	0.112813	-8.27671	0.133665	4.68363	25.83649	-0.47979	-42.7833
-5.79817	0.185067	-14.3024	-0.41061	-22.0667	-28.6819	-4.3E-06	31.05574
8.150032	-0.07115	9.931177	0.196204	10.41478	10.42044	0.078384	-9.81572
-3.79249	0.019267	4.069956	-0.06922	-3.84349	3.278783	-0.07125	2.745823
-0.94735	0.078483	-1.63544	0.050774	3.938365	-5.3773	-0.00035	-5.28816
2.795857	-0.03155	-4.64671	-0.05259	2.299191	2.121594	0.037865	5.034979
3.607262	-0.09606	-0.07959	-0.14551	-4.90195	2.443513	-0.03633	-2.92108
0.299324	-0.13014	4.560478	0.072483	-0.17261	-4.86753	0.116769	-0.08516
-3.81641	0.013977	0.425058	0.193942	4.91516	2.421529	-0.01132	3.214773
-2.65452	0.097243	-4.75087	-0.01552	-1.901	2.047713	-0.09332	-5.39862
0.619262	0.100321	-1.72372	-0.05871	-4.02007	-4.91924	0.054729	5.556334
3.92383	-0.02242	3.544689	0.064689	2.961482	2.338031	0.000202	-2.47296
-0.2518	0.012784	0.464549	-0.0093	0.785471	0.491174	-0.02797	-0.24052
0.571713	0.006349	-0.18319	-0.03118	0.190577	0.248051	0.010616	-0.3233
0.23592	-0.02477	0.33773	-0.02543	-0.32097	0.037922	0.018844	0.126166
-0.58808	-0.00226	-0.3463	0.006277	0.072744	-0.24863	0.011828	0.04698

Rows to 1 to 47 Columns 41 to 47

-0.0092	-0.10618	-0.00689	-0.00497	-0.06495	-0.04174
-0.03625	0.195719	0.049469	0.083533	0.083073	0.085673
0.073296	-0.1726	-0.07208	-0.19178	0.029055	-0.10297
-0.0213	0.080112	0.039865	0.109296	-0.05353	0.057502
0.311338	0.03229	-0.17365	-0.1477	0.076353	-0.07841
-0.14743	-0.06261	0.145009	0.101437	-0.05551	0.08791
-0.12915	0.052194	-0.01123	-0.05552	0.037297	-0.08972
0.004533	-0.03014	-0.05145	0.062115	-0.05351	0.122018
0.05542	0.0041	0.09971	0.063624	-0.00422	-0.09925
-0.03602	0.006825	-0.00119	-0.09345	0.058662	0.082369
-0.07869	-0.01064	-0.07728	0.080886	-0.10991	-0.04424
-0.07331	0.006276	0.078676	0.087975	0.105551	0.042836
0.092057	0.002643	-0.00755	-0.09848	-0.05539	-0.02315
-32.7933	-0.11148	35.82959	35.44378	-29.975	16.73785
35.44126	0.491362	-61.9694	-82.4828	83.79828	-52.7329
31.00123	-0.78203	4.222812	64.28531	-109.133	87.68815
-33.7279	0.261861	60.2601	9.359452	89.89296	-113.963
-32.6854	0.340389	-38.0447	-74.6702	-34.5084	127.9116
32.18123	-0.14864	-38.5404	74.30476	-34.9966	-127.909
34.50883	-0.0449	60.02635	-8.94656	90.17041	113.7854
-30.0751	-0.3032	4.989538	-64.8241	-109.196	-87.4959
-35.8319	0.505339	-62.0041	82.34869	83.46721	52.40435
31.98077	-0.2106	35.26188	-34.818	-29.5394	-16.4266
0.541529	5.45204	-0.02292	0.093761	-0.06544	-0.12964
-1.3443	-19.3788	0.378692	-0.25722	0.186555	0.393406
0.95326	34.33766	-0.78455	0.211084	-0.34993	-0.5318
-0.01249	-45.1263	0.24777	0.108063	0.658773	0.463082
0.079462	50.95291	0.725486	-0.2233	-0.87496	-0.24716
-0.39543	-51.0054	-0.90739	-0.16292	0.847827	-0.11262
-0.08551	45.17475	0.469075	0.582152	-0.85529	0.500701
0.735718	-34.3406	-0.22119	-0.61041	0.978974	-0.77148
-0.59874	19.27744	0.184249	0.301845	-0.78983	0.666137
0.128566	-5.34365	-0.07343	-0.04454	0.265928	-0.23121
-0.02221	-1.54961	-0.02975	0.06049	0.011854	-0.01279
0.083174	3.623159	0.062577	-0.09734	0.026201	0.031129
-0.06968	-4.90339	-0.11742	-0.0323	-0.04136	-0.05623
0.047291	5.858346	0.096681	0.155788	-0.02034	0.07946
-0.00749	-6.07365	0.015192	-0.17064	0.084461	-0.09483
-0.03691	5.750055	-0.03562	0.053193	-0.13777	0.118196
0.06551	-4.62969	-0.02619	0.057571	0.119095	-0.13912
-0.13318	3.09222	0.105965	-0.01969	-0.063	0.181692
0.05994	-0.81479	-0.10202	-0.06472	-0.04155	-0.12606
0.017337	-0.55537	0.008437	0.132007	0.075452	0.041179
-0.02388	0.33009	0.090131	-0.09462	-0.00482	-0.03488
-0.02534	-0.44233	-0.12095	-0.01305	-0.00989	0.006476
0.057383	0.309131	0.059418	0.026889	-0.00214	0.002733

Output Coefficient Matrices

aVecPb	aVecZn
0.009408	-0.02502
-0.00807	0.726708
0.019784	0.014427
-0.03814	-0.7401
-0.0098	0.398202
0.022023	-0.42741
0.053638	0.059816
0.023376	0.239483
0.069924	-0.40388
0.025608	-0.13664

Translation Matrices

cVecPb	2.479894	-2.10436	-1.48413	-1.1328	0.272654	-0.12167	1.327739	-1.61602	-2.07779	2.417437
cVecZn	2.387995	-3.22942	-1.82365	-0.97728	-0.45046	0.020403	1.117632	-1.7145	-2.52943	2.561853

Dilation Matrices

Pb Dilation Matrix - bMat Pb

-0.07391	0.360234	0.008705	0.162739	0.185786	0.30602	0.049699	-0.20868	-0.03084	0.341708
0.058246	0.008562	-0.0343	-0.06995	0.077069	-0.11882	0.112536	-0.02458	-0.13421	-0.10908
0.060293	-0.11969	0.128108	-0.03702	0.118227	-0.04282	0.07074	0.152331	0.089393	0.216254
-0.12196	0.039104	0.11905	0.053382	0.054998	-0.14271	0.06022	-0.05155	-0.16643	0.006108
-0.0213	0.065214	-0.001	-0.00314	-0.02381	-0.00531	0.036788	-0.00353	-0.02678	0.06716
0.018108	0.220621	0.020831	0.019799	-0.08177	0.101726	0.04859	-0.05241	-0.05842	0.172986
0.023981	-0.07786	0.04865	-0.00972	0.011013	-0.02484	0.036044	0.038593	0.051073	-0.03668
0.033178	-0.00245	-0.02177	-0.01148	-0.05013	0.018828	-0.00276	0.040495	-0.01078	0.010064
-0.02324	-0.02622	-0.01669	0.017108	0.0611	0.002955	-0.01344	-0.00416	-0.00392	-0.05761
-0.04146	0.078641	-0.01428	-0.00351	0.001439	0.016839	0.039154	-0.04898	-0.04635	-0.04146
-0.05239	0.053261	-0.02107	-0.00843	0.010459	0.000882	-0.00431	0.00377	0.007852	0.052813
0.057848	-0.00756	-0.03136	0.007636	-0.00672	-0.06721	-0.01209	0.035177	-0.01974	-0.00731
-0.00973	0.04811	0.018354	0.016167	-0.04666	-0.02747	0.022919	0.035446	-0.00889	-0.00285
0.048897	0.02804	-0.03555	-0.00528	-0.01644	0.000787	-0.0175	0.007056	-0.00889	0.002435
0.008387	-0.05542	0.023519	-0.01495	-0.01213	-0.00393	0.037442	-0.03076	-0.01527	-0.04067
-0.00491	0.049602	-0.00282	0.025899	-0.01886	0.016091	0.001717	-0.00983	0.01774	0.008425
0.02544	-0.02963	-0.02921	-0.02906	0.024046	0.023329	-0.0169	-0.02473	-0.01791	-0.01537
0.056285	-0.04604	-0.01216	-0.03666	0.049028	0.039412	-0.01402	-0.02796	0.025954	0.027716
-0.00646	0.047083	0.008172	-0.00467	-0.01172	-0.00529	0.009861	-0.00101	0.023727	-0.00144
0.017278	-0.01022	0.032006	-0.03358	0.032901	-0.00768	0.022485	0.005773	-0.00061	0.005124
-0.01024	-0.02492	0.001738	-0.01179	-0.02743	-0.00718	-0.0073	-0.01356	-0.00242	0.019511
-0.02139	0.075218	-2.36E-05	0.015307	0.011375	0.016773	0.004216	-0.00064	-0.01478	0.019607
-0.04132	0.022005	-0.00288	-0.0254	0.059977	-0.03625	-0.02285	-0.02598	0.001314	-0.04444
-0.02737	0.014312	0.029377	-0.00522	-0.01089	-0.00273	0.010914	-0.00398	-0.02529	-0.01459
0.037703	0.00345	0.008915	0.019326	-0.04235	0.005726	0.004879	0.021135	0.006855	0.018352
-0.01016	0.002102	0.000361	-0.01463	-0.02703	-0.02888	-0.0027	0.015052	-0.02053	-0.00199
-0.02007	-0.08227	-0.00037	-0.00808	-0.01476	0.040315	-0.01871	0.003363	0.027748	0.021491
-0.00444	0.019602	-0.00665	-0.03009	0.005408	-0.03313	-0.01717	-0.02103	0.011589	-0.02909
0.012639	0.05785	-0.00694	-0.005	0.016118	0.034334	0.000364	-0.04245	-0.02369	-0.02483
-0.00649	0.039756	0.034319	0.006001	0.004672	-0.00108	0.016736	0.000478	-0.02553	-0.02769
-0.00428	-0.06615	0.035902	0.028055	0.011527	0.011592	0.037068	0.050238	-0.0531	0.044975
0.024434	-0.02777	-0.01125	-0.00725	0.020711	-0.00066	-0.00371	0.008239	0.007378	0.013262
-0.02215	0.149182	0.008803	0.029601	-0.0117	-0.03669	0.015511	-0.01586	-0.01999	0.00918
0.002895	0.032457	0.000154	0.026204	-0.00948	-0.00055	-0.01613	-0.00372	-0.00699	0.031589
-0.09496	0.04472	-0.01901	0.009082	-0.03485	-0.03368	-0.01343	-0.03984	-0.0502	-0.03401
0.007395	-0.07618	0.041286	-0.02463	-0.02574	-0.04218	0.011249	-0.00541	-0.00518	-0.02303
0.001193	0.009663	0.024799	-0.02341	-0.00508	-0.00823	0.000828	-0.00809	-0.01471	-0.01178
-0.0281	0.085186	0.040389	0.028768	-0.00158	-0.05709	0.040891	-0.03807	-0.05538	0.004112
-0.02224	0.00755	-0.00824	0.032817	0.019012	-0.02008	-0.01417	0.017154	0.027737	-0.01505
0.029659	-0.03907	0.020342	0.044158	-0.03131	-0.02848	0.031012	0.032238	-0.00936	-0.05113
0.018844	-0.1003	0.024102	-0.02643	0.03697	0.030788	0.022747	0.001933	0.006558	0.00186
0.004076	-0.1065	0.000738	-0.02256	-7.76E-06	-0.0052	-0.00868	0.029387	0.031548	-0.02435
0.030312	0.050352	-0.03723	0.001542	-0.03149	0.016862	-0.0217	-0.00185	0.020964	0.034909
0.00069	-0.02517	-0.01601	0.01218	-0.01625	0.009774	-0.00023	0.014003	0.023292	0.002462
-0.00048	0.024847	0.003754	-0.01421	0.014611	-0.02429	0.015743	0.013228	-0.02342	0.00121
0.008389	-0.02615	0.009615	-0.00475	0.030441	-0.00817	0.007848	0.003606	0.022871	0.022545
0.019016	-0.01465	-0.00347	0.007544	-0.0379	0.013099	-0.00298	0.032466	-0.03071	-0.01028

Zn Dilation Matrix – bMatZn

-0.07422	0.116363	0.143148	0.153744	0.282745	0.30929	0.088389	-0.22203	-0.02015	0.297058
0.086182	0.072981	-0.13537	-0.10909	0.062472	-0.12675	0.145111	-0.02887	-0.14159	-0.12539
0.061556	0.001093	0.159546	-0.04753	0.129045	-0.0574	0.081485	0.175075	0.08671	0.176241
-0.11248	-0.09185	0.145114	0.06187	0.076996	-0.15388	0.061414	-0.04679	-0.18265	0.008616
0.040431	-0.18451	-0.12412	-0.03901	-0.06326	0.016355	0.149771	-0.02515	-0.08108	0.140312
0.006782	-0.05348	-0.00237	-0.00076	-0.01368	0.044058	0.015505	-0.01864	-0.0172	0.037029
0.03904	0.025521	0.045474	-0.00484	0.029492	-0.03716	0.040919	0.046859	0.056094	-0.03121
0.026645	-0.02401	-0.03104	-0.01542	-0.03384	0.025996	-0.00809	0.038871	-0.00803	0.010238
-0.02018	0.048627	-0.01526	0.016068	0.040701	0.007873	-0.01465	-0.00259	-0.00098	-0.04378
-0.04574	0.019082	-0.04875	-0.01054	-0.01146	0.023176	0.057207	-0.05521	-0.04605	-0.04265
-0.03396	0.023985	-0.02606	-0.00774	0.005868	0.005446	0.005707	0.011136	0.004373	0.037942
0.057978	-0.01791	-0.03622	0.000245	0.001885	-0.0677	-0.01105	0.02799	-0.02054	-0.00826
-0.02336	-0.03608	0.024374	0.012731	-0.03438	-0.02275	0.036955	0.042003	-0.0015	0.007187
0.037822	0.016885	-0.03384	-0.01436	-0.03652	-0.00758	-0.02156	0.003061	-0.01264	0.009587
0.01811	0.028676	0.013393	-0.0117	-0.01251	0.000942	0.027636	-0.02101	-0.00869	-0.01895
-0.00473	0.032548	-0.00043	0.055716	0.012435	0.017241	0.000848	-0.03443	0.046272	0.003274
0.013866	0.028708	-0.0379	-0.04049	0.020153	0.018838	-0.02743	-0.03747	-0.02405	-0.02034
0.038121	-0.01199	0.033742	-0.04594	0.045789	0.026055	-0.02117	-0.01557	0.023295	0.033645
-0.01649	-0.00873	0.014861	0.005294	0.012275	-0.0231	0.017118	0.001655	0.01945	-0.00947
0.020409	0.003126	0.01468	-0.02966	0.020517	-0.00272	0.014138	0.011978	-0.00146	0.006369
-0.0046	0.003935	0.016015	-0.0207	-0.02285	-0.01424	-0.02937	-0.02623	-0.00557	0.031403
0.017993	0.012035	-0.00897	0.029702	0.006717	0.017457	0.014623	0.00878	-0.01746	0.026977
-0.01145	-0.00257	-0.00294	-0.01288	0.030182	-0.01918	-0.02239	-0.00881	-0.0022	-0.0234
0.033458	-0.01666	0.033134	-0.00857	-0.00517	0.031109	0.003822	-0.01191	-0.03355	-0.01981
0.030059	-0.02318	0.020794	0.016956	-0.02955	-0.02351	0.022776	0.032308	-0.00238	0.026029
0.023821	-0.00127	-0.00108	-0.02454	-0.01388	-0.00052	-0.02084	0.014902	-0.01605	0.006998
-0.01616	0.017336	0.008108	0.011477	-0.01943	0.018475	-0.01622	0.0113	0.019364	0.014471
0.004946	0.004384	0.005858	-0.02336	-0.01252	-0.01949	-0.01924	-0.01396	0.00735	-0.01911
0.031842	0.003481	-0.01488	-0.01264	-0.01363	0.039717	0.002721	-0.03721	-0.02027	-0.01213
-0.01814	-0.02586	0.019227	-0.00157	0.016048	0.006854	0.005881	-0.01144	-0.01265	-0.01905
-0.00865	0.003523	0.01435	0.002403	0.012872	0.002825	0.019707	0.020051	-0.02015	0.018291
0.001305	0.007441	-0.00035	-0.00675	0.024959	-0.01204	-0.00361	0.007931	0.005005	0.013147
0.002551	-0.01953	-0.01733	0.019984	-0.00366	-0.01779	0.020994	-0.01559	-0.01137	0.001794
0.028216	-0.01908	0.009416	0.020245	-0.01065	0.002475	-0.01022	0.005956	-0.00412	0.020456
-0.02182	0.00193	-0.0127	0.006558	-0.01983	-0.01943	-0.01024	-0.01946	-0.02203	-0.01051
0.022344	-0.01515	0.024132	-0.03276	-0.02679	-0.01252	-0.00217	-0.01239	-0.00162	-0.01086
-7.00E-05	-0.03014	0.023327	-0.02584	-0.00358	0.01178	-0.00881	-0.01532	-0.01554	-0.01406
0.007581	-0.02627	-0.00879	0.007701	0.002522	-0.03399	0.029403	-0.02356	-0.03714	-0.00163
-0.0017	-0.02635	-0.01104	0.020651	0.001901	-0.0076	-0.01255	0.008578	0.018541	-0.00706
0.029553	-0.03652	0.009613	0.027708	-0.02941	-0.01457	0.027062	0.021372	-0.00075	-0.02138
-0.025	0.003357	0.018141	-0.02138	0.024095	0.021625	0.027867	0.014499	0.001206	-0.00518
-0.00269	-0.00993	0.021352	-0.02098	-0.00215	-0.01227	-0.01414	0.029694	0.032103	-0.02044
0.015312	-0.00775	-0.02382	3.07E-06	-0.02203	0.022495	-0.02068	-0.01249	0.014139	0.026286
0.009805	-0.00093	-0.01938	0.011977	-0.01563	0.002678	0.004843	0.019613	0.024392	-0.00094
-0.01289	0.003488	-0.02376	-0.02264	-0.00535	-0.02877	0.021361	0.022227	-0.02544	-0.0008
-0.01212	-0.01592	0.01107	-0.00425	0.021111	-0.00983	0.010624	0.000624	0.025929	0.021113

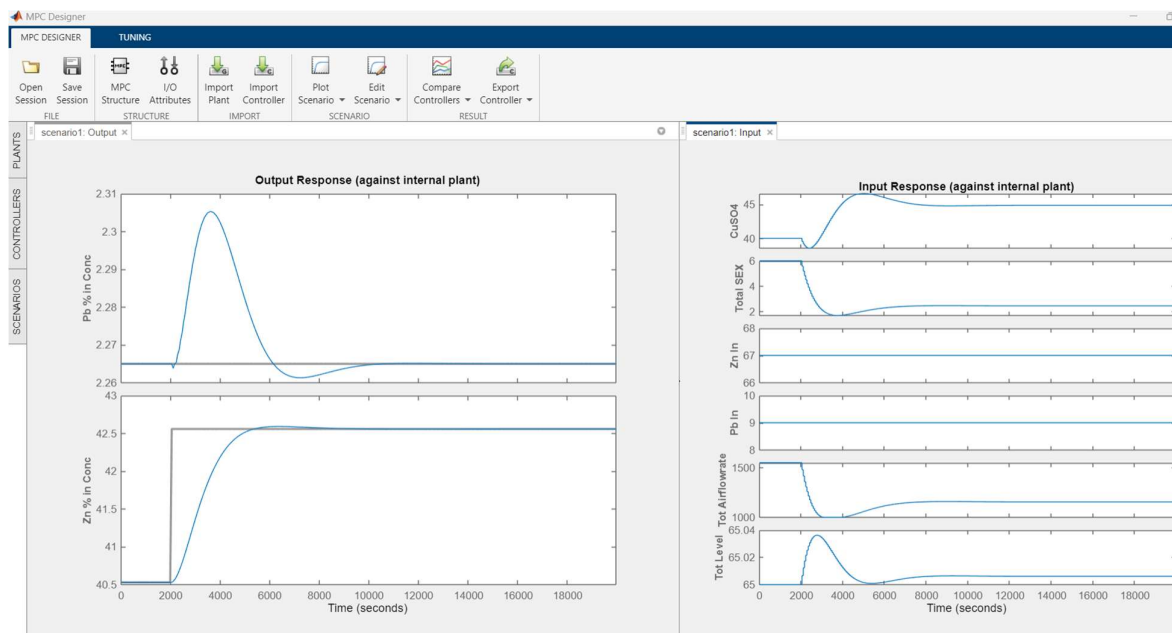
Mean Regressor Matrix

Mean_Pb_Regressor_Matrix	Mean_Zn_Regressor_Matrix
2.008953299	42.91595328
2.008882769	42.9160801
2.008812238	42.91620692
2.008741708	42.91633375
2.008671178	37.68980606
2.008600647	37.68967142
2.008530117	37.68952001
2.008459586	37.6893456
2.008389056	37.68920045
37.6903504	37.68906182
37.69021375	37.68891832
37.69007498	37.68876828
37.68992989	37.68861111
37.68977826	6.256610522
37.68961951	6.256552118
37.68945502	6.256493752
37.68929005	6.256435425
6.256766533	6.256377137
6.256708169	6.256318888
6.256649844	6.256260677
6.256591558	6.256202506
6.256533311	6.256144372
6.256475102	6.256086278
6.256416932	64.47845318
6.256358801	64.47783336
6.256300708	64.47722054
6.256242654	64.47661248
64.4785131	64.47601067
64.47790286	64.47541036
64.47729256	64.47481103
64.47668657	64.47421638
64.47609092	64.47362775
64.47549732	64.47304315
64.47490233	8.726796351
64.47430936	8.72659815
64.47372557	8.726403213
64.4731551	8.726206238
8.726520857	8.72601661
8.726330639	8.725828093
8.726135548	8.725635313
8.725933429	8.72544166
8.725743315	8.725251401
8.725557708	8.725061903
8.725367687	8.724870938
8.725170392	8.724680109
8.72497046	8.724488461
8.724778632	

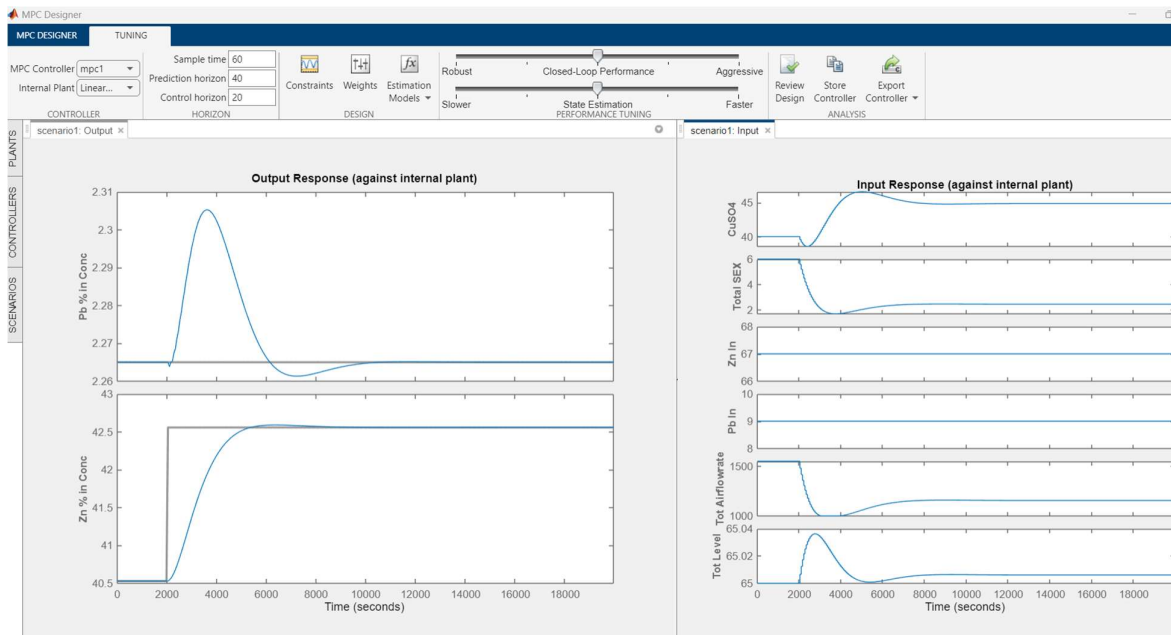
Appendix C - Simulated System Development

Prediction Horizon and Control Horizon Comparison Plots

The app showcases the MV behaviour and CV results for each change in control parameters using a series of plots. The MPC Designer required entering values to set up each scenario, which was used to test the controller performance. The scenario settings required the nominal values for the MVs, DVs and CVs. These nominal values are the same as the steady-state values. The parameter testing scenarios ran for a total of 20000s. The CV setpoint changes were also specified for parameter testing purposes. The Zn Conc setpoint was stepped up by 5% at 2000s in first scenario. The Pb_Conc setpoint was stepped up by 5% at 2000s in the second scenario. Along with the scenario settings, an MPC object and a Plant object were also defined within the app. The MPC object used within the App was `mpcobjLin`, which is the MPC object defined using the `Total_Trend_ARX` model. The MPC object already had the sample time, input variable indices, nominal values, and MV constraint-bound properties defined before entering the App. The Plant was defined using the `idpoly` object holding the `Total_Trend_ARX` model structure. The layout of the scenario setup page within MPC Designer is shown below.



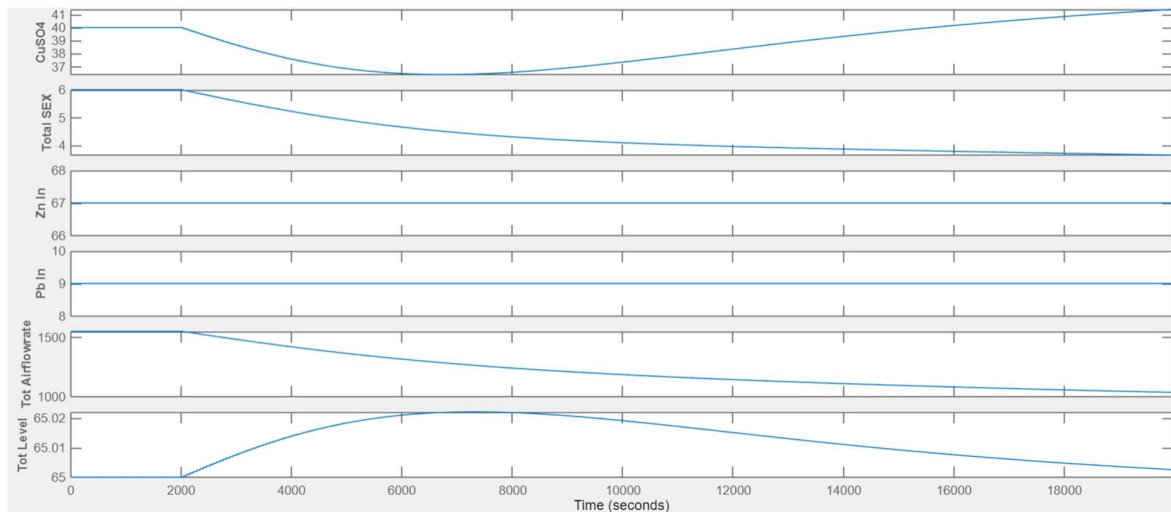
After the MPC structure, plant structure and scenario settings were defined in the App, the controller was tuned using the tuning tab. The MV rate of change bounds, MV rate tuning weights, Output variable tuning weights, and Control and Prediction Horizon parameters were determined using the tuning tab. The App started with the default values for the control parameters specified in Table 4-10. The control parameters were changed to see what values would result in the best control performance. The tuning tab in the MPC designer tab is shown below.



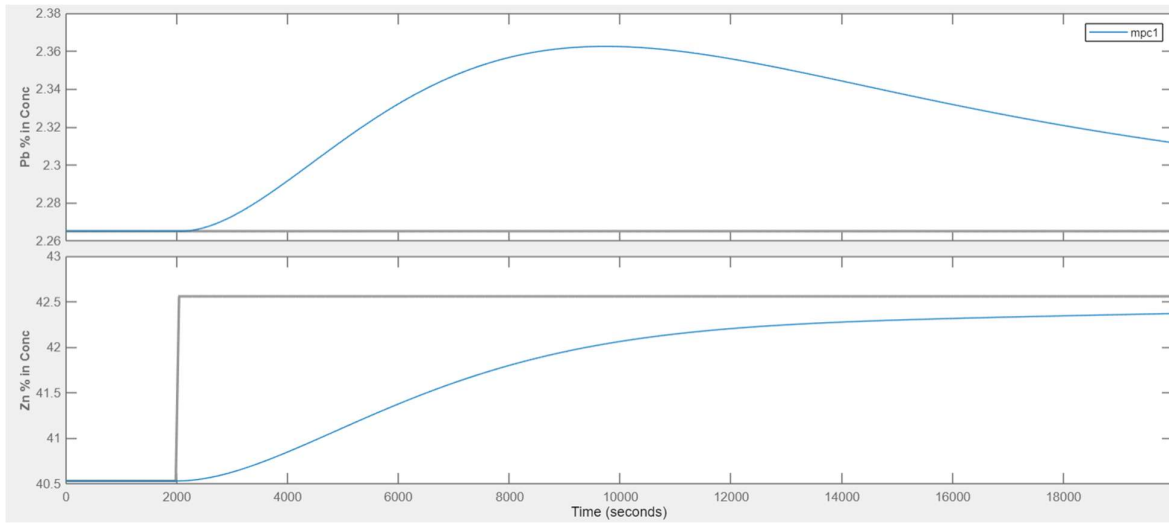
The control performance results are shown in the App via a series of Input and Output Response Plots. The prediction horizon and control horizon sets were tested using this setup. The controller performance for each scenario using the different prediction and control horizon sets are plotted below.

Scenario: Zn Setpoint Change; Prediction Horizon = 4; Control Horizon=2

Input Response (against internal plant)

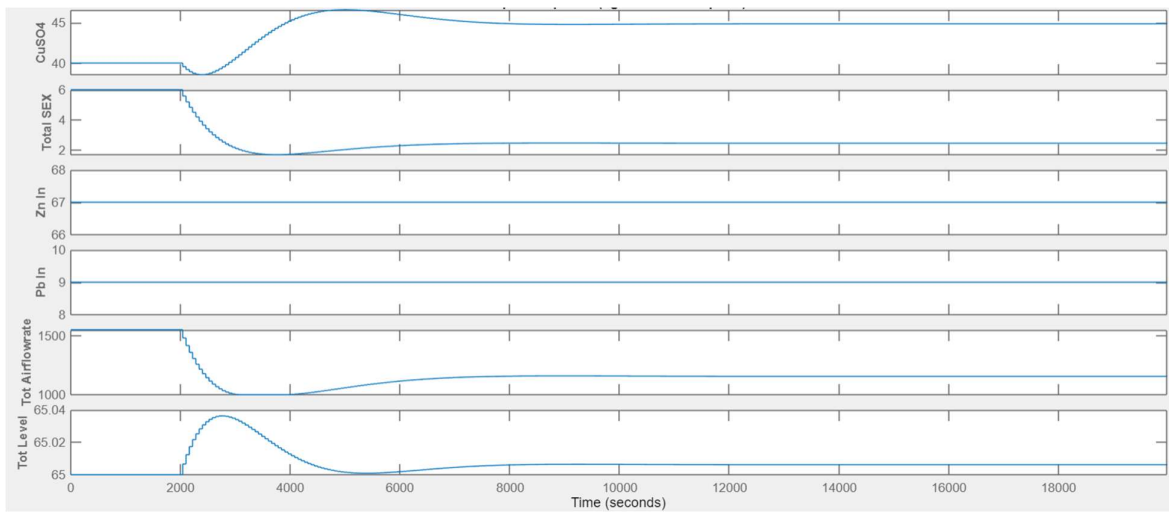


Output Response (against internal plant)

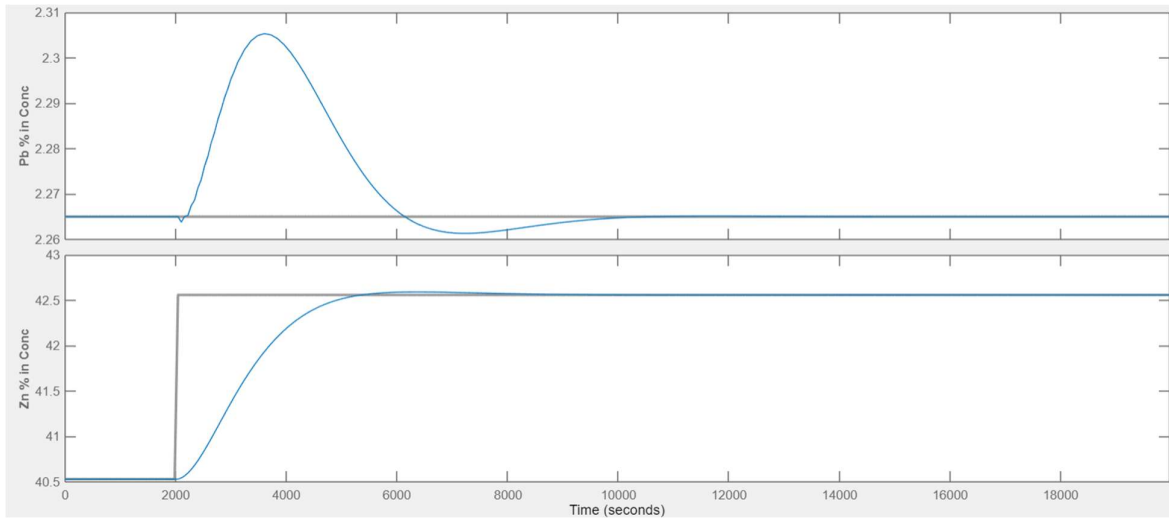


Scenario: Zn Setpoint Change; Prediction Horizon = 40; Control Horizon=20

Input Response (against internal plant)

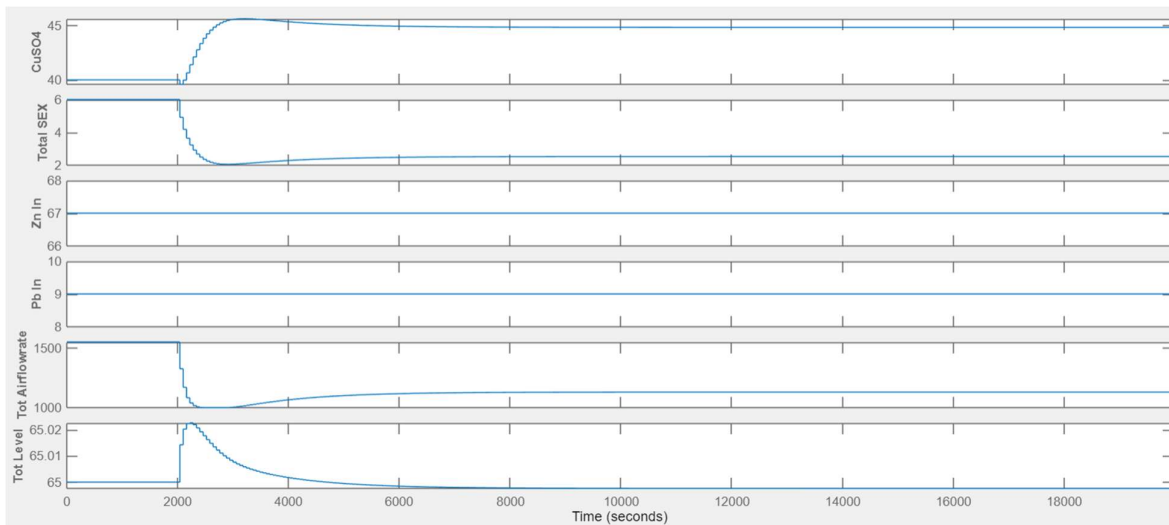


Output Response (against internal plant)

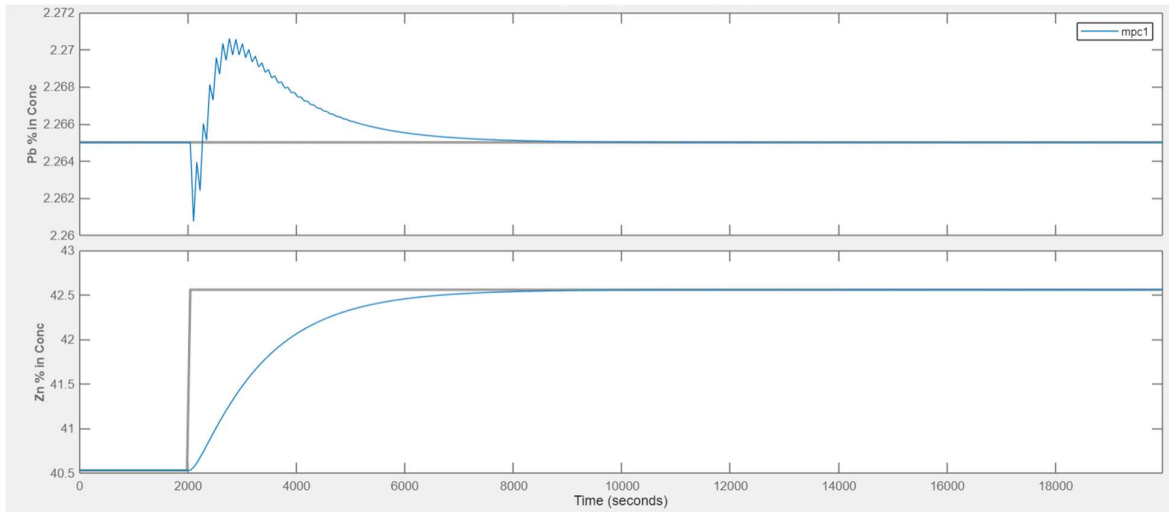


Scenario: Zn Setpoint Change; Prediction Horizon = 80; Control Horizon = 40

Input Response (against internal plant)

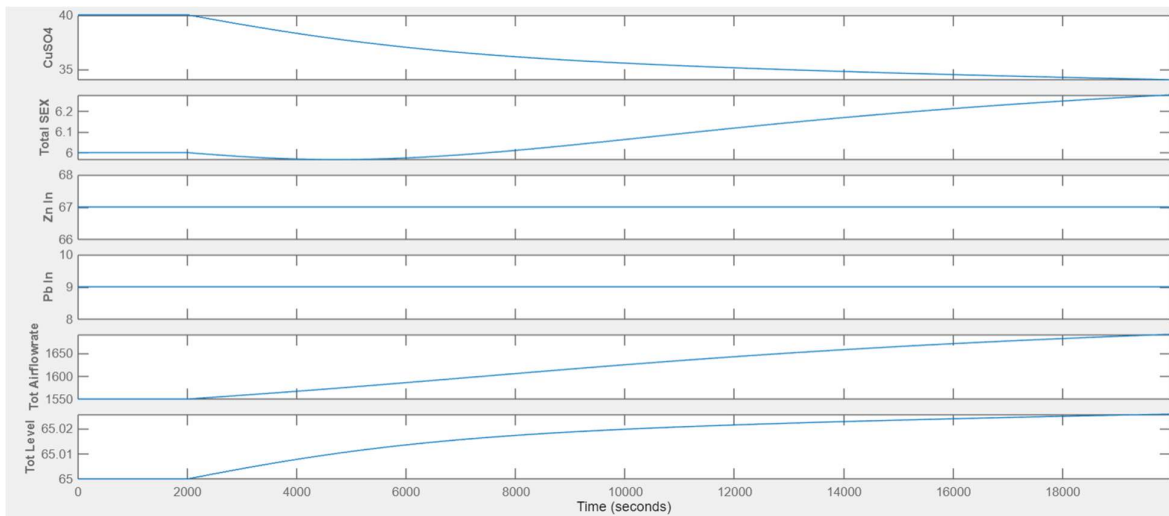


Output Response (against internal plant)

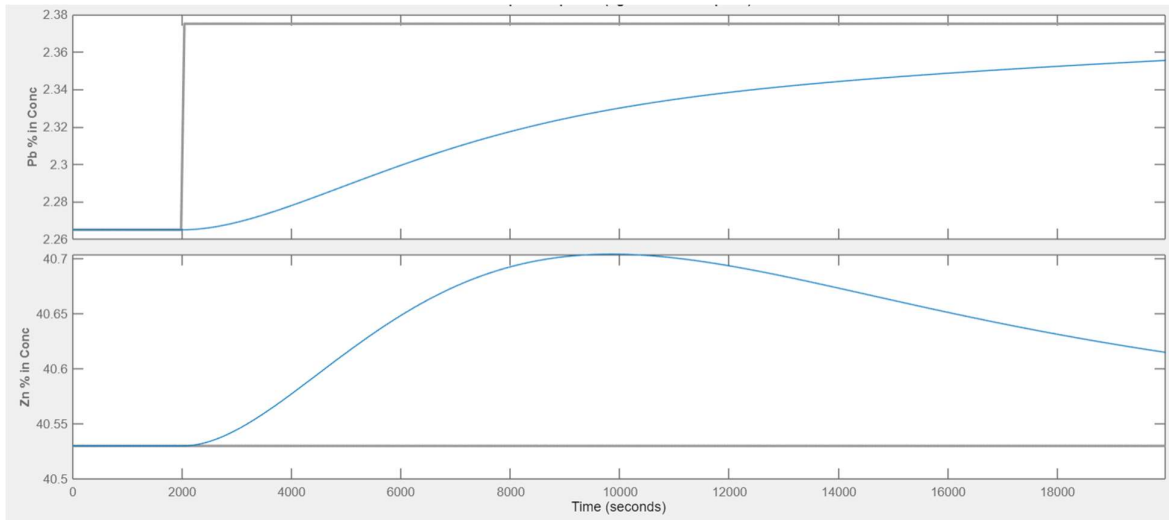


Scenario: Pb Step Response; Prediction Horizon = 4; Controller Horizon = 2

Input Response (against internal plant)

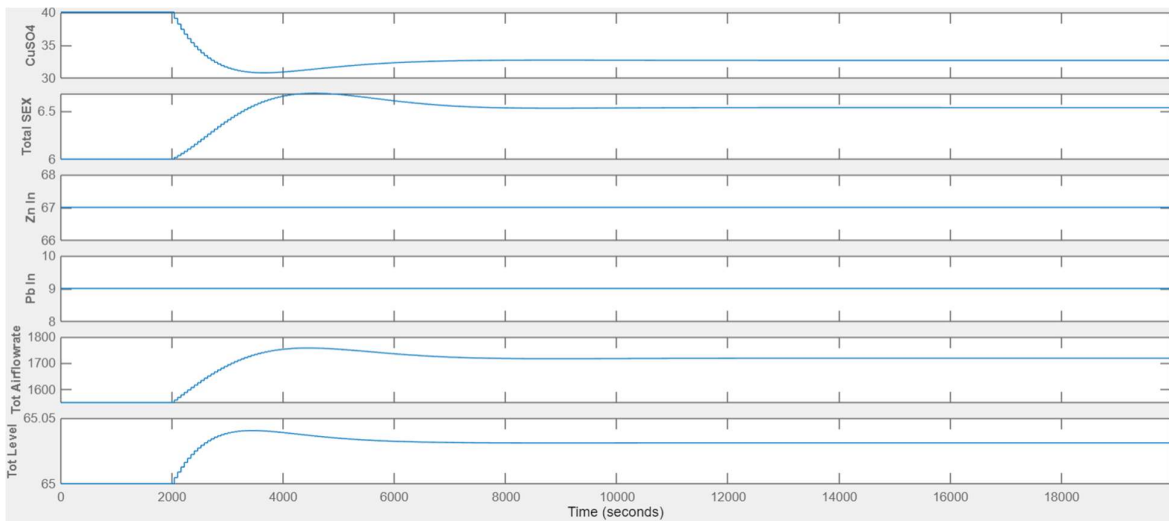


Output Response (against internal plant)

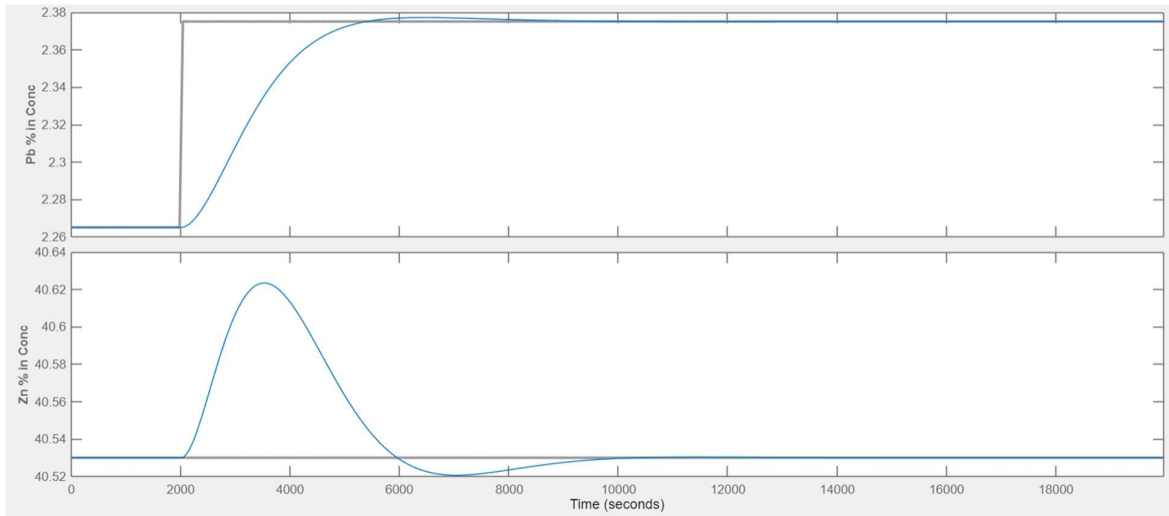


Scenario: Pb Step Response; Prediction Horizon = 40; Controller Horizon = 20

Input Response (against internal plant)

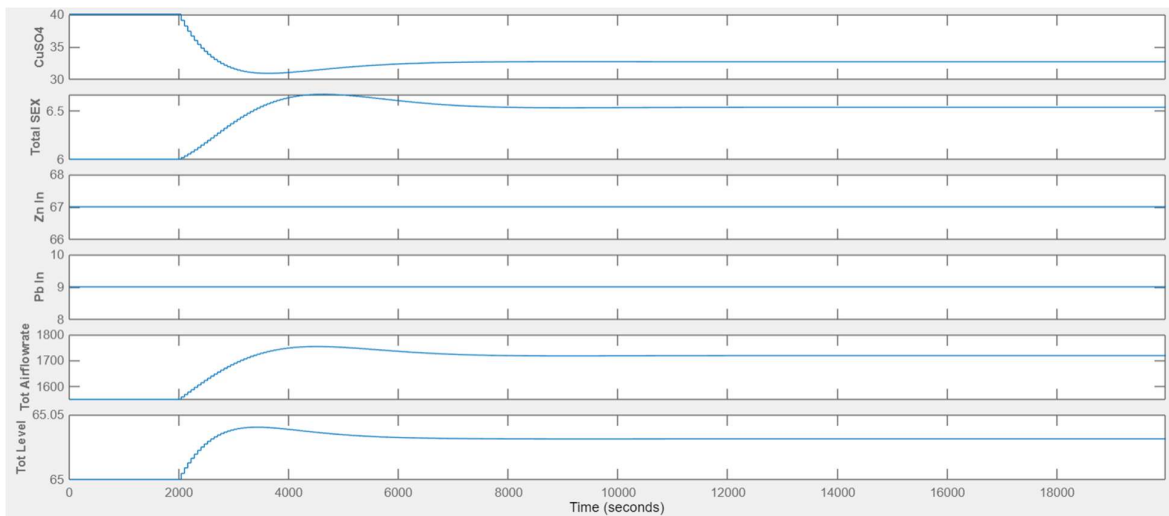


Output Response (against internal plant)

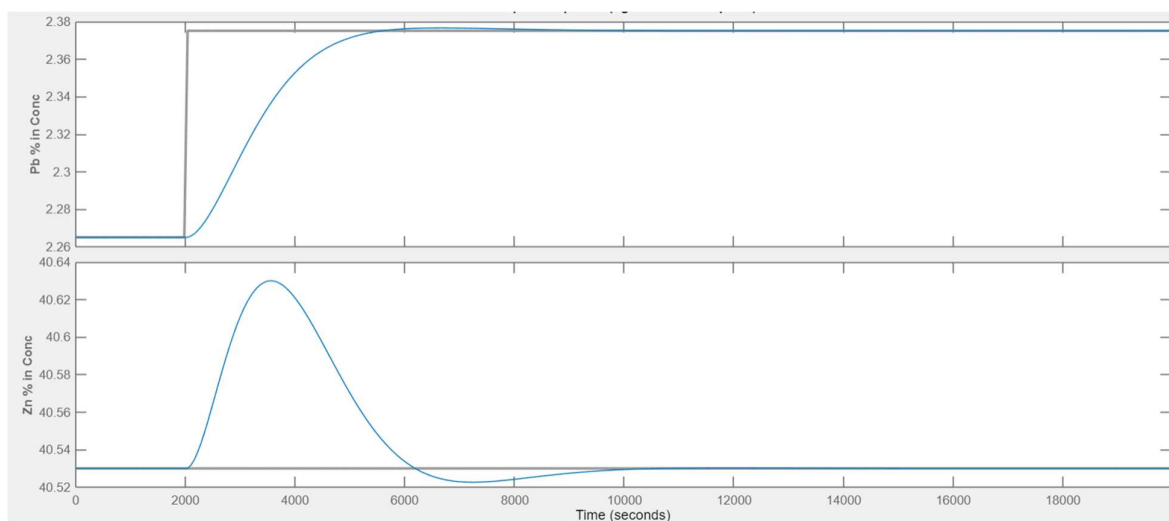


Scenario: Pb Step Response; Prediction Horizon = 80; Controller Horizon = 20

Input Response (against internal plant)



Output Response (against internal plant)



Thereafter, the MV rate of change bounds and weights were varied whilst considering the following: physical limits for each MV, ensuring the increments do not result in oscillatory behaviour, avoiding a long CV response time or a CV response time that does not correspond with the Hybrid model step response times. The tuning tab contains a Closed-Loop Performance slider that automatically varies the parameter values.

Plant Specification Code

The function that is used to define the plant is labelled as the NARXHybridFunc. The code used to define the plant function is shown in Figure 7-3 and Figure 7-4.

```

function x_k = NARXhybridFunc(x,u,tpb,tzn,tairlevPb,tairlevZn,qPb,qZn,bMatPb,bMatZn,avecPb,avecZn,cvecPb,cvecZn,pPb,pZn)

% Pb Output Evaluation

% Assign MVs and Cvs to regressor matrices
tpb(10) = u(1); %CuSO4
tpb(18) = u(2); %Total_SEX
tpb(28) = u(3); %Zn_In
tpb(38) = u(4); %Pb_In
tpb(1) = x(1); %Pb_Conc
tairlevPb(1) = u(5); %Total_level_Pb
tairlevPb(3) = u(6); %Total_airflowrate_Pb

%Normalize Pb regressor matrices
tpbCenter = [2.00895329938514 2.00888276893423 2.00881223848332 2.00874178883242 2.00867117758151 2.0086064713060 2.00853011667969 2.00845958622879 2.00838905577788...
37.6903503972282 37.6902137502440 37.6900749770642 37.6899298887371 37.6897782612727 37.6896195851728 37.6894550204958 37.6892980473356 37.68913653308608...
6.25678816948077 6.25664984447589 6.25659155821784 6.25653331075542 6.25647518194222 6.25641693202225 6.25635880070271 6.25630070808120 6.25624265415772...
64.4785131036502 64.4779028577806 64.4772925585595 64.4766865689050 64.4760909104072 64.4754973218817 64.4749023262737 64.4743093626781 64.4737255733945...
64.4731551024790 8.72652085740777 8.72633863902987 8.72613554806754 8.72593342865509 8.72574331548897 8.72555770803240 8.72536768743900 8.72517039156744...
8.72497846022838 8.72477863166114];

tpbNorm = tpb - tpbCenter;

lpb = [0.0737980345935787 0.0145404193276438 0.0778974880681487 0.165229134708862 0.0906698673555903 0.190834939445601 0.00684923157091638 -0.00130080738760005 0.00101595729491116...
-0.000194763099101206 0.00151636598587353 0.000420648910580234 0.00157978465208433 0.00018821285951901 -0.000207941905721394 -0.00174481536283616 9.55396976103820e-05...
0.0000047611291946 -0.00072745529218699 0.0022840143938336 -0.0004192726556930 -0.000419324952089731 -0.020803482066169 0.00140958609690871 -0.00035442758379505...
0.000156362413745081 -0.000234228083521219 0.00217897839803504 -0.0141481368481653 0.000211463423663159 -0.00937548202574645 5.89495086112313e-05 0.000614395635872658...
0.0056876525883931 0.00514796936375519 -0.00100687137383574 -0.000232715713622464 -0.00270370934966124 -0.00143161553616037 -0.000834202328227837 -0.000941012159487045...
-0.00051765324691288 -0.000490583820635688 -8.92560412823570e-05 -4.80041374055118e-05 -0.000305306388972812 -1.44610953785874e-05];

dpb = -0.0810;

%Pb Linear mapping in NARX portion
ylinPb = (tpbNorm*pPb*lpb) + dpb;

%Pb Nonlinear mapping in NARX portion
f = @(z)1/(exp(-z)+1); % anonymous function for sigmoid unit
yNonlinear1 = 0;
for k = 1:length(avecPb)
    finput = tpbNorm*qPb*bMatPb(:,k)+cvecPb(k);
    yNonlinear1 = yNonlinear1+avecPb(k)*f(finput);
end
yPb = yNonlinear1 + ylinPb + 2.0098;

% Pb Total Airflowrate and Total Level ARX linear mapping
ytotlevPb = -0.000004001*tairlevPb(1) + 0.00000003165*tairlevPb(2);
ytotairPb = -0.0000001543*tairlevPb(3) - 0.00000001791*tairlevPb(4);

```

Figure 7-3 Plant Specification Function Part 1

```

% Zn Output Evaluation

% Assign MVs and Cvs to regressor matrices
tzn(5) = u(1);
tzn(14) = u(2);
tzn(24) = u(3);
tzn(34) = u(4);
tzn(1) = x(2);
tairlevZn(1) = u(5);
tairlevZn(3) = u(6);

%Normalize Zn regressor matrices
tznCenter = [42.9159532773601 42.9160801005565 42.9162069237528 42.9163337469491 37.6898060626770 37.6896714199942 37.6895200131798 37.689345604803 37.6892004461584...
37.6890618197794 37.6889181919282 37.6887682783364 37.6886110708976 6.25661895206385 6.256555211778776 6.25649375226984 6.25643542546129 6.25637713741092...
6.25631888797228 6.25626007738944 6.25620250556478 6.25614437244948 6.25608627809236 64.4784531846139 64.4778333647369 64.477205447623 64.4765814821830...
64.4760106716782 64.475410363409 64.4748110280192 64.4742163765499 64.4736277516353 64.4730431548375 8.7267963511784 8.72659814961437 8.72640321292590...
8.72620623835790 8.72601661031924 8.72582809259983 8.72563931343357 8.72544166001172 8.72525140066387 8.72506190329981 8.72487093786000 8.72468010919652...
8.72448846065606];

tznNorm = tzn - tznCenter;

%Zn Linear mapping in NARX portion
Lzn = [-0.152999961734529 0.361702075642430 1.72682322150668 0.117013265991088 -0.291529783320669 -0.0341119567324934 -0.00417807905762101 0.000266581332443991 -0.00395479913857005...
-0.00335880294623058 0.000331474101902059 -0.00768748848080733 0.00710700246255236 4.66367455102092e-05 0.0124451022526656 0.00584388741816322 0.0054618121088047...
-0.00575230840577551 -0.0151188327209120 -0.00573641444555175 -0.000283593013378619 0.00915727671132087 -0.0278230193827756 0.0340143954866319 0.00547262113553685...
0.00119836032822813 0.00734620930658097 -0.00830297916861955 -0.0115100766279327 0.00388363746792102 -0.0007804415774437759 -0.00334998199180520 0.00481295601631669...
0.00421472852579709 0.0045151976980742 -0.00266182204484300 0.0001321726643270 -0.0023958270417490 0.00427495597173998 0.00575800786758427 -0.00408581745623517...
-0.00429674219301811 0.00602478544114058 0.00417796550085623 -0.00066248670128281 -0.00186678418408273];

dzn = 0.1807;
ylinZn = (tznNorm*pzn*Lzn) + dzn;

%Zn Nonlinear mapping in NARX portion
g = @(o)1/(exp(-o)+1);
yNonlinear2 = 0;
for m = 1:length(avecZn)
    ginput = tznNorm*qzn*bMatZn(:,m)+cvecZn(m);
    yNonlinear2 = yNonlinear2+avecZn(m)*g(ginput);
end
yZn = ylinZn + yNonlinear2 + 42.9158;

%Zn Total Airflowrate and Total Level ARX linear mapping
ytotlevZn = -0.000612*tairlevZn(1) + 0.002427*tairlevZn(2);
ytotairZn = -0.00003215*tairlevZn(3) - 0.00006968*tairlevZn(4);

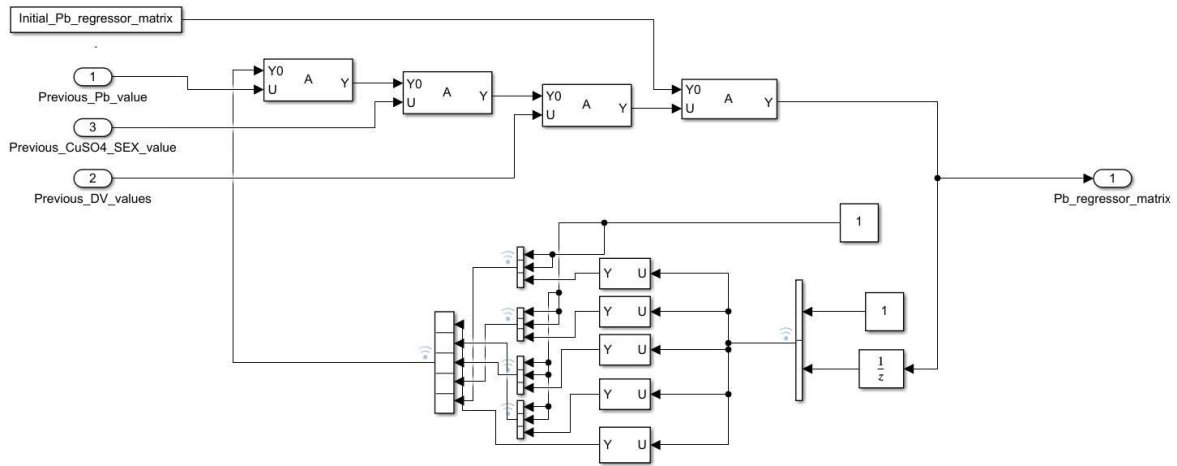
% Compute Output Vector
x_k = zeros(2,1);
x_k(1) = yPb + ytotairPb + ytotlevPb;
x_k(2) = yZn + ytotairZn + ytotlevZn;
end

```

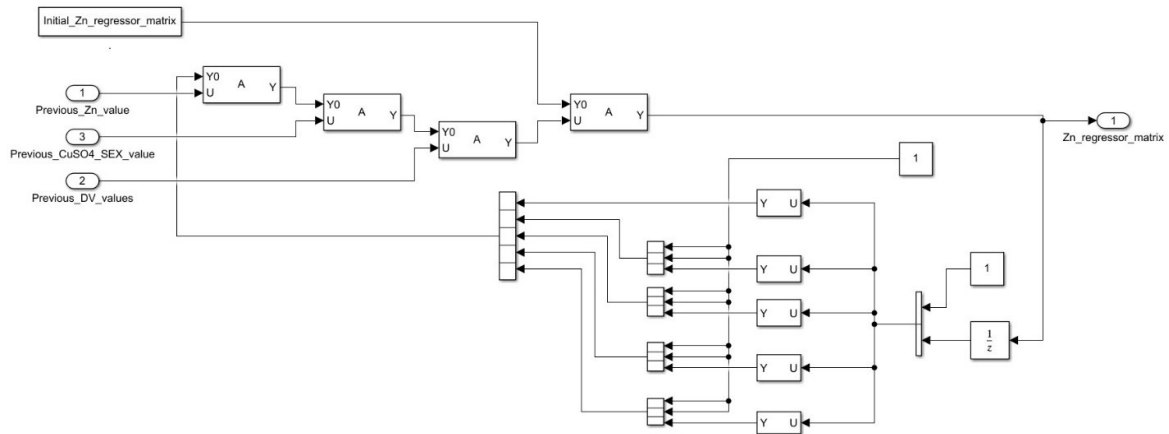
Figure 7-4 Plant Specification Function Part 2

Simulation Blocks

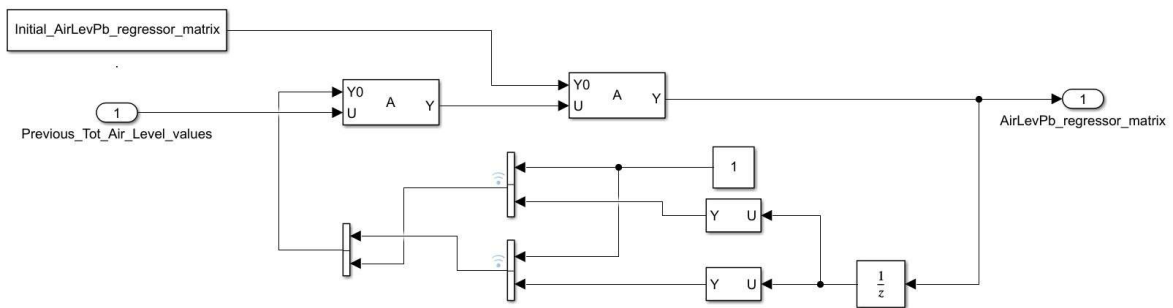
Pb_regressor_matrix



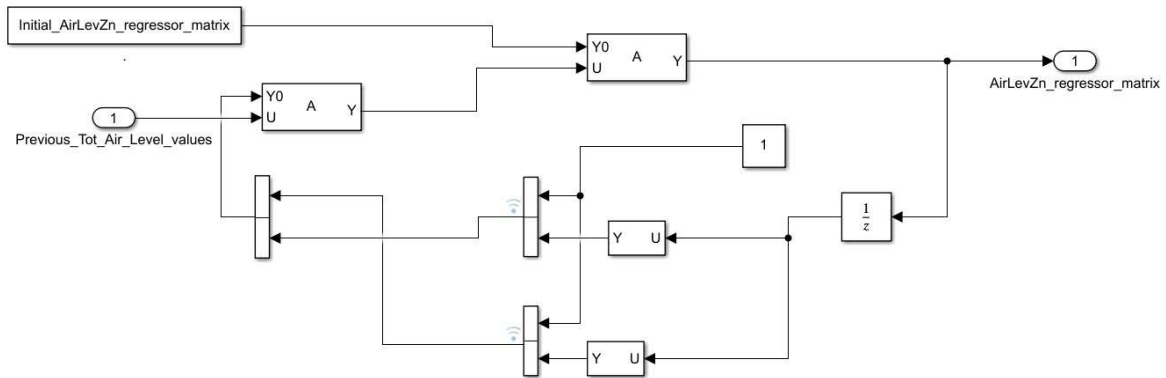
Zn_regressor_matrix



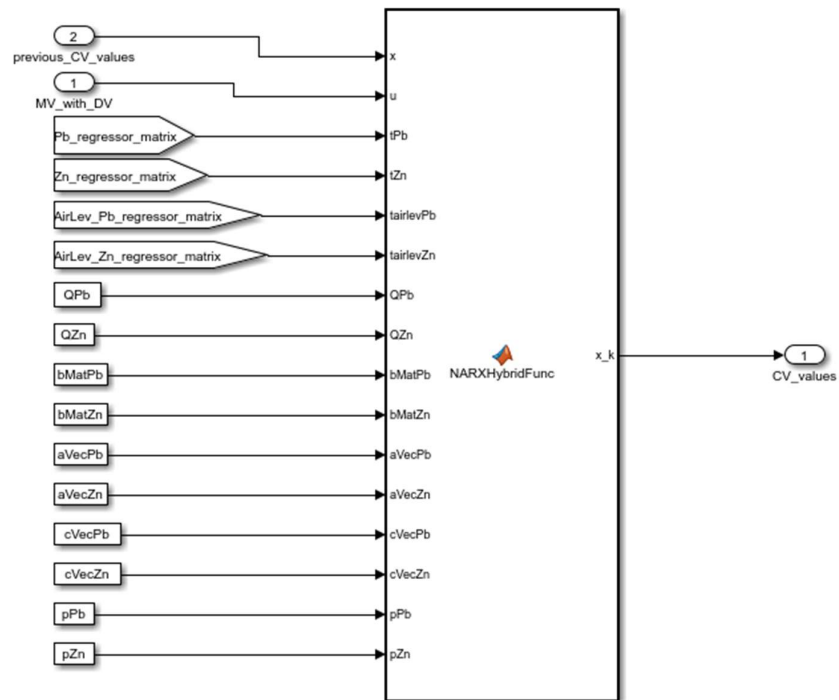
AirLevPb_regressor_matrix



AirLevZn_regressor_matrix



Plant Block



Nonlinear Controller Object Definition Code

```

% Creation of a Nonlinear MPC Controller Object
%2 states and 2 inputs, a total of 6 inputs (2 Mesured Disturbances)
%Nonlinear Controller object labelled as nlobj
nlobj = nlmpc(2,2,'MV',[1 2 5 6],'MD',[3 4]);

%Define Sample Time
Ts = 60;
nlobj.Ts = Ts;

%Define Prediction Horizon
nlobj.PredictionHorizon = 40;

%Define Control Horizon
nlobj.ControlHorizon = 20;
%Define state function
nlobj.Model.StateFcn = 'NARXHybridFunc';

%Declare wether the model is continuous or discrete
nlobj.Model.IsContinuousTime = false;

%Define Number of parameters required by the predition model
nlobj.Model.NumberOfParameters = 14;

% Define the simulink model where the nonlinear controller object will be used
SimulinkNonlinearControllermdl = 'SimulationNARXHybridFunc';

% Creation of a bus object to hold parameters required by the NARXHybridFunc prediction model
%Bus object labelled as myBusObject
createParameterBus(nlobj,[SimulinkNonlinearControllermdl '/SimulationNARXHybridFunc'], ...
    'myBusObject',{tPb,tZn,tairlevPb,tairlevZn,QPb,QZn,bMatPb,bMatZn,aVecPb,aVecZn,cVecPb,cVecZn,pPb,pZn});

```

Open loop gains

Disturbance Variable	Zn % Conc			Pb % in Conc		
	Change in Input Variable	Change in Output Variable	Gain	Change in Input Variable	Change in Output Variable	Gain
Zn In	3.46	0.50	0.14	3.46	0.02	0.01
Pb In	0.48	-3.00	-6.25	0.48	0.40	0.83

