The Emergence of Gravitational Spaces

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Abstract

In this thesis I explore the evidence for whether gravity is an emergent phenomenon. I provide a review of black hole thermodynamics and demonstrate how it provides evidence that gravity is an emergent phenomenon. In addition I provide a review of Jacobson's calculation which shows how the Einstein field equations can be interpreted as a thermodynamic equation of state. I then use Jacobson's work and the Seiberg-Witten map to derive a new gravitational equation of state which shows what gravity on a non-commutative manifold would look like.

Contents

Introduction

Gravity, as a force, was first studied rigorously and mathematically by Newton. For many years the prevailing wisdom was that gravity was one of a few fundamental forces of nature which acted on material bodies in a fixed background geometry [5]. The inverse-square law

$$F = \frac{G_N M_1 M_2}{r^2}$$

worked fairly well in predicting the orbits of planets and in predicting the motions of objects on earth. Nevertheless, this formulation of gravity could not be reconciled with the later discoveries of special relativity and that of a massless photon. Observation confirmed that the Lorentz invariance required by special relativity is indeed in operation in the universe we inhabit. Attempts to make Newton's gravitational field satisfy Lorentz invariance met with failure.

Additionally, Newton's law of gravitation states that a massless photon would not be influenced by the gravitational field of any massive bodies in the universe. In this scheme, therefore, the bending of light by stars and other massive objects would not occur. Einstein's theory of General Relativity which succeeded Newton's Law of Gravitation provided a far better fit to observation and reproduces the inverse-square force law when appropriate limits are taken. It is, therefore, obvious why we consider General Relativity and its tenet of gravity-as-geometry to be the best description of the universe we live in [5].

For all of its successes, however, General Relativity remains a classical theory [5][11]; a theory at odds with the physical assumptions underlying the immensely successful quantum theory. Being classical, General Relativity places no limitations on the accuracy of measurement and does not specify that particles live in a Hilbert space. Unfortunately, attempts to turn the gravitational field of General Relativity into a quantum field, like was done for electromagnetism, have met with failure. Finding a quantum theory of gravity remains one of the biggest challenges in theoretical physics today.

There is, however, evidence to indicate that attempts to quantize classical gravity have met with failure for a specific reason - gravity itself is an emergent phenomenon[8][12][22]. If it is emergent, trying to quantize gravity would be like trying to quantize the classical variable of pressure and doing so is not a reasonable proposition. The first evidence for the emergent nature of gravity comes from the fact that black holes, solutions to the Einstein field equations, satisfy four laws which are extremely similar to the four laws of thermodynamics [14]. Black hole variables thus begin to look like classical thermodynamic variables which appear in the equation of state of a gas.

It was also shown by Jacobson [12], without heavy reliance black hole thermodynamics, that by assuming the proportionality of the area of a null surface to the entropy contained within that surface, that the Einstein field equations themselves are an equation of state. By doing this, spacetime thermodynamics comes full circle - we have an equation of state and variables of a solution to that equation which satisfy four laws. This provides strong evidence that the gravity of General Relativity is the limit, or average of some other theory.

A key difference between spacetime thermodynamics and classical thermodynamics then remains - we know of many equations of state for classical thermodynamics yet there seems to be only one for spacetime thermodynamics. I will show, however, that this need not be the case. It is possible to arrive at new equations of state in classical thermodynamics by changing the assumptions of the microscopic nature of the gas, say, by assuming that the particles are of non-negligible size. Proceeding by analogy and changing some assumptions of General Relativity by saying that if we were able to probe accurately enough, we would find the microscopic structure of spacetime to have a minimum measurable length, we can derive a new gravitational equation of state. By doing this, it is like moving from the ideal gas equation in which the particles are negligible in size to the Van der Waals equation, in which the size of the particles is taken into account.

Part I

General Relativity and Black Holes

1 The Nature of Gravity

1.1 The Newtonian View

The Newtonian formulation of space time is intuitive and straightforward. To do Newtonian mechanics, set up three mutually orthogonal Cartesian axes to measure distances in the three-dimensional universe. These three axes function only as a background, they perform a spatial bookkeeping, always providing us with a measurement of distance in the Newtonian universe. It does not matter where you travel, (or how fast you travel) in the universe, the axes are always the same, always undistorted by whatever physical processes they keep track of [5].

Now imagine that at each point in space there is a perfect clock; each of these perfect clocks will be synchronized with every other clock. The ticking of these perfect clocks is what defines time. The most important part of this construction is the idea that space and time do not influence one another - they are independent. If an observer were to build a clock which never loses time, she could leave earth, travel around the universe and return home to notice that the time on her perfect clock is still perfectly synchronized with whatever perfect clocks were left on earth.

Time, in this formulation, functions as an independent variable. It is always moving forward undeterred, undeterrable, by the machinations of anything in the Newtonian universe. We frequently use it as a parameter when we plot trajectories in our three-dimensional space. No matter what we do decide to do, we cannot ever speed or slow the passage of time or shrink or grow our Cartesian axes. In a nutshell, time and space give us tools to label physical processes but they do not influence those processes and we are never able to influence the time-and-space bookkeeping, no matter what experiments we perform.

In this rigid universe, Newton's laws reign supreme. They tell us all we ever need to know about the motion of masses. An object will move in a straight line until a net force is exerted on it. If this occurs, the acceleration of the object is given by Newton's Second Law:

$$\vec{F} = m\vec{a}. \tag{1}$$

In Newton's scheme, gravity is experienced as a force, governed by the law

$$F = G\frac{Mm}{r^2}, \tag{2}$$

where M and m are the masses of the bodies exerting a force on one another. Any body with mass will set up a force field in all parts of the universe. Any other massive particle in this field will always be attracted to whatever is setting up the field. This field occurs in space but does not influence space or time in any way. We need the spatial bookkeeping of our axes to determine how far apart objects are, but (2) has nothing further to tell us about the axes. Likewise, time is unaffected by (2). We can use universal time to plot the trajectory of the masses, but we can't change the ticking of clocks with (2). Gravity, therefore, acts in a fixed background of space and time.

1.2 Gravity and Special Relativity

The theory of special relativity presents a challenge to the Newtonian view of gravity. Equation (2) states quite unambiguously that the gravitational force is communicated instantaneously between two massive objects; it seems to completely ignore the universal speed limit imposed by special relativity. Nevertheless, maybe we can modify Newton's law of gravitation to work with special relativity without changing the essential character of equation (2). What is meant by this is that we want a theory of gravity which satisfies the equation [8]

$$\nabla^2\phi = 4\pi G\rho_m \tag{3}$$

in the limit $c \to \infty$, while also satisfying:

1. Lorentz covariance - a physical equation which is true in one frame is true in all frames.

2. The fact that mass is always attractive.

3. The equality of gravitational and inertial mass. This is an experimental

fact which inspired the principle of equivalence - no local experiment can distinguish between uniform acceleration and uniform gravitational fields.

For the time being we will work in units where $c \neq 1$; this will change later. In order to build this new theory, we are going to assume that gravity is a scalar field ϕ and try adding this to a particle's Lagrangian. The action for a free, relativistic particle is

$$S = -\int dt\, mc^2 \sqrt{1 - \frac{v^2}{c^2}} = -\int d\tau \left(mc^2\right), \tag{4}$$

where τ is the proper time. The action for a particle moving in our new gravitation field ϕ is

$$S = -\int d\tau \left(mc^2 + \lambda\phi\right). \tag{5}$$

Now vary the action with respect to $x^\mu(\tau)$ to get

$$S = \int d\delta x^\mu u_\mu \left(mc^2 + \lambda\phi\right) + -\lambda \int d\tau \left(\partial_\mu \phi \delta x^\mu\right);$$

where we are using the relation $\delta\phi = [\partial_\mu \phi]\, \delta x^\mu$. Expanding the above gives us

$$\delta S = -\int d\tau \left[\left(\lambda \partial_\mu \phi + \frac{d\left[u^\mu \left(mc^2 + \lambda\phi\right)\right]}{d\tau}\right) \delta x^\mu + \left(mc^2 + \lambda\phi\right) u_\mu \delta x^\mu\right] \tag{6}$$

where u^μ is the particle's 4-velocity. Now assume that the variation vanishes at the endpoints to get:

$$\frac{du^\mu}{d\tau} = -\lambda \frac{\partial^\mu \phi}{\left(mc^2 + \lambda\phi\right)} - \lambda u^\mu u^\nu \frac{\partial_\nu \phi}{\left(mc^2 + \lambda\phi\right)}. \tag{7}$$

Notice that when we take the $c \to \infty$ limit, we get the spatial part of (7) to satisfy [8]

$$m \left(\frac{d\vec{v}}{dt}\right) = -\lambda \nabla \phi \tag{8}$$

which is the equation for a non-relativistic particle moving in the potential $\lambda\phi$. So far this looks promising. We have recovered something which looks like a gravitational field.

Upon further inspection, however, we notice that there is a problem with (7). The trajectory of the particle depends on the mass of the particle - this violates the principle of equivalence. To circumvent this, we need to give the coupling constant the dimensions of mass. So set $\lambda = lm$ where l is some fundamental constant with dimensions of length. For convenience, we also redefine the scalar field by writing $\Phi = l\phi$. Our equations of motion then become

$$\frac{du^\mu}{d\tau} = -\lambda \frac{\partial^\mu \left({}^{\Phi}/_{c^2}\right)}{1 + \left({}^{\Phi}/_{c^2}\right)} - \lambda u^\mu u^\nu \frac{\partial_\nu \left({}^{\Phi}/_{c^2}\right)}{1 + \left({}^{\Phi}/_{c^2}\right)} \tag{9}$$

which neatly demonstrates that all particles, regardless of mass, will follow the same trajectory in some gravitational potential Φ, provided they begin with the same initial conditions. With this coupling, the action for a particle will be

$$S = -mc^2 \int d\tau \left(1 + \frac{\Phi}{c^2}\right). \tag{10}$$

So far so good. It appears that we can have a scalar field in special relativity which obeys the principle of equivalence. Let's now find a Lagrangian for the field itself and see if we can repeat our success. A simple guess for the Lagrangian is

$$L_{field} = -\frac{1}{2l^2}\partial_a\phi\partial^a\phi = -\frac{1}{8\pi Gc}\partial_a\Phi\partial^a\Phi \tag{11}$$

where l is rewritten as $l = \sqrt{4\pi Gc}$ so that we can recover G when we take the non-relativistic limit. By varying (11) we arrive the equations of motion [8]

$$\Box\Phi = 4\pi G\rho_m \tag{12}$$

where

$$\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}.$$

The equation (12) appears to be a perfectly reasonable generalization of Newton's gravitational field to the special-relativistic case. The only immediate problem that we need to deal with is the mass-energy equivalence of matter. To do this, we will couple the field Φ to the stress-energy tensor instead of just the mass of the matter. Bear in mind that the only Lorentz-invariant quantity which can be constructed from T_{ab} is its trace, $T^a{}_a = T$. Let's therefore write (12) as

$$\Box\Phi = -\frac{4\pi G}{c^2}T. \tag{13}$$

This seems to solve the problem of gravity and special relativity completely. We have an equation which is Lorentz invariant and which reduces to classical gravity in the non-relativistic limit, the problems seem to be completely solved!

Before we declare our construction a complete success, let's perform a check. Consider a system of electrons and positrons which interact with, and annihilate each other. Before they interacted, we would have a non-zero mass density which would set up a gravitational field. After they interact, we would have some massless electromagnetic radiation, in the form of photons. Equation (13) should accommodate this because we are no longer using the mass density ρ_m but the trace of the stress-energy tensor, T. The problem arises when we consider the fact that the trace of the stress-energy tensor for the electromagnetic field is always zero. If we accept that (13) is the equation which describes gravity in special relativity, then we will have a scalar field which does not couple to the electromagnetic field.

Our system of positrons and electrons which set up a gravitational field before annihilation will cease to set up a field after they annihilate one another. Furthermore, since this theory will not couple the photon to gravity, the bending of light by a gravitational field will never occur - a direct contradiction with observation. Our attempt to reconcile Newton's gravity with special relativity has failed.

1.3 Gravity as Geometry

Instead of trying to generalise Newton's theory to work with special relativity, general relativity approaches the problem of gravity in another direction [5][9][11]. We will not try to get two existing theories to work together; we will instead demonstrate that special relativity and Newtonian gravity can be recovered from a single, larger theory. By doing this we will not only solve existing problems but also discover new physical phenomena and a radically different way of thinking about time and space.

1.3.1 Manifolds and Metrics

We define a manifold to be a set of points which everywhere "looks like $\mathbb{R}^n$" [11]. By this we mean that around every point there exists a neighbourhood which can be mapped, using a one-to-one mapping, to $\mathbb{R}^n$ - allowing us to always use $\mathbb{R}^n$ as a way to label points in our manifold. The reason we employ the idea of a manifold is we are going to say that every event in spacetime is a point on a manifold. On this manifold we are now going to define the metric tensor, $g_{\mu\nu}$. In the pantheon of the tensor gods, the metric tensor reigns supreme. From it, we can calculate any geometric quantity we would like to. The most important quantity we would want to calculate using $g_{\mu\nu}$ is, of course, the distance between two points.

On our manifold we define the square of the distance between two infinitesimally nearby points to be

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{14}$$

where $g_{\mu\nu}$ is evaluated at one spacetime point. Since the metric tensor is defined at all points on the spacetime manifold, we have a measure for distance no matter where we are in spacetime. General relativity is radically different from special relativity and Newtonian mechanics because the way to measure distance, the metric, changes from point to point in space and time. In the Newtonian scheme, all of space was governed by the flat Euclidean metric at all times. In special relativity the Minkowski metric measures not only distances in space, but also distances in time.

1.3.2 Motion on a Manifold

Motion in the Newtonian universe is simple - everything moves in a straight line unless a net force acts on it. Motion in Einstein's general relativity is not so simple because "straight line" is not a concept which carries over unchanged from Newtonian mechanics. To see this, let's use what we know regarding distance. We are going to make the statement that a particle moving through space will follow a parameterized curve which minimizes distance between two spacetime points. To this end, we will write down the action [8][9][11]

$$S = -m \int \sqrt{-g_{\mu\nu} dx^\mu dx^\nu} \tag{15}$$

and vary it. To do so, we use

$$\delta\left(-g_{\mu\nu}dx^{\mu}dx^{\mu}\right)=-dx^{\mu}dx^{\nu}\left(\partial_{\kappa}g_{\mu\nu}\right)\delta x^{\kappa}-2g_{\mu\nu}dx^{\mu}d\left(\delta x^{\nu}\right) \tag{16}$$

and

$$\delta\sqrt{-g_{\mu\nu}dx^{\mu}dx^{\nu}}=\frac{1}{2}\frac{16}{\sqrt{-g_{\mu\nu}dx^{\mu}dx^{\nu}}}$$

and insert them into (15) to get

$$\delta S=m\int\left[\frac{1}{2}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}\left(\partial_{\kappa}g_{\mu\nu}\right)\delta x^{\kappa}+g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{d\left(\delta x^{\nu}\right)}{d\lambda}\right]d\lambda \tag{17}$$

where we have parameterized our curve by λ. Integrating by parts gives us:

$$\delta S=m\int\left[\frac{1}{2}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}\left(\partial_{\kappa}g_{\mu\nu}\right)\delta x^{\kappa}-\frac{d}{d\lambda}\left(g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\right)\delta x^{\nu}\right]d\lambda+mg_{\mu\nu}\left.\frac{dx^{\mu}}{d\lambda}\delta x^{\nu}\right|_{\lambda_2}^{\lambda_1}. \tag{18}$$

We now demand that the variation δS vanishes for paths with $\delta x^{\mu}=0$, which will give us the equations of motion

$$\frac{d}{d\lambda}\left(g_{\mu\nu}u^{\mu}\right)=\frac{1}{2}\left(\partial_{\kappa}g_{\mu\nu}\right)u^{\mu}u^{\nu} \tag{19}$$

or alternatively

$$g_{\mu\nu}\frac{du^{\mu}}{d\lambda}+u^{\mu}u^{\kappa}\partial_{\kappa}g_{\mu\nu}=\frac{1}{2}u^{\mu}u^{\kappa}\partial_{\nu}g_{\mu\kappa}. \tag{20}$$

We can cast this in a more symmetric form by writing

$$u^{\mu}u^{\kappa}\partial_{\kappa}g_{\mu\nu}=\frac{1}{2}u^{\mu}u^{\nu}\left[\partial_{\nu}g_{\mu\alpha}+\partial_{\mu}g_{\nu\alpha}\right] \tag{21}$$

which will give us,

$$g_{\mu\nu}\frac{du^{\mu}}{d\lambda}=\frac{1}{2}\left(\partial_{\nu}g_{\mu\alpha}-\partial_{\mu}g_{\nu\alpha}-\partial_{\alpha}g_{\nu\mu}\right)u^{\mu}u^{\alpha}. \tag{22}$$

Using the fact that $g^{\alpha\beta}g_{\kappa\beta}=\delta^{\alpha}{}_{\beta}$, we eliminate the $g_{\mu\nu}$ on the left hand side of (22), which gives us the final result [8][11]:

$$\frac{du^{\mu}}{d\lambda}=-\frac{1}{2}g^{\mu\nu}\left(-\partial_{\nu}g_{\alpha\beta}+\partial_{\beta}g_{\alpha\nu}+\partial_{\alpha}g_{\nu\beta}\right)u^{\alpha}u^{\beta}. \tag{23}$$

If we define the Christoffel symbols

$$\Gamma_{\mu\nu\alpha} = \frac{1}{2}\left(-\partial_\nu g_{\alpha\beta} + \partial_\beta g_{\alpha\nu} + \partial_\alpha g_{\nu\beta}\right) \tag{24}$$

then we can say that a massive particle moving in an arbitrary spacetime with the metric $g_{\mu\nu}$ will follow a trajectory which obeys

$$\frac{d}{d\lambda}\frac{dx^\mu}{d\lambda} + \Gamma^\mu{}_{\alpha\beta}u^\alpha u^\beta = 0, \tag{25}$$

which is called the geodesic equation. A geodesic is a generalization of the straight line into a spacetime with any metric. Instead of following a straight line, like in the Newtonian paradigm, particles move along geodesics. If a region of space has a geodesic running through every point in that region, then we say that the region is filled by a geodesic congruence.

1.3.3 The Covariant Derivative

Neither the Christoffel symbols nor the partial derivative obey the tensor transformation law. However, we can use both of them to define the covariant derivative ∇_α :

$$\nabla_\alpha u^\beta = \partial_\alpha u^\beta + \Gamma^\beta{}_{\alpha\kappa}u^\kappa. \tag{26}$$

This is an operator which obeys similar properties to the partial derivative operator, but it is a covariant quantity. From (26), we see that that the Christoffel symbol is what we need to turn the partial derivative into a covariant quantity. We have added the Christoffel symbol to the partial derivative and ended up with something which transforms according to the tensor transformation law, but how are we to interpret this new quantity? Let's work with the Euclidean plane and consider two different coordinate systems - rectilinear and polar [6]:

$$\begin{cases} r = \sqrt{x^2+y^2} & x = r\cos\theta \\ \theta = \arctan\left(\frac{y}{x}\right) & y = r\sin\theta. \end{cases} \tag{27}$$

We can compute the transformations between basis vectors as

$$\vec{e}_r = \frac{\partial x}{\partial r}\vec{e}_x + \frac{\partial y}{\partial r}\vec{e}_y$$

$$= \cos\theta\vec{e}_x + \sin\theta\vec{e}_y \tag{28}$$

and

$$\vec{e}_\theta = \frac{\partial x}{\partial \theta}\vec{e}_x + \frac{\partial y}{\partial \theta}\vec{e}_y$$

$$= -r\sin\theta\vec{e}_x + r\cos\theta\vec{e}_y. \tag{29}$$

The vector fields $\vec{e}_x$ and $\vec{e}_y$ are constant vector fields - they do not vary as one moves around the Euclidean plane. We are going to demonstrate that this is not the case for the basis vectors $\vec{e}_r$ and $\vec{e}_\theta$. As we move around the plane, the polar basis vectors change as

$$\frac{\partial}{\partial r}\vec{e}_r = \frac{\partial}{\partial r}\left(\cos\theta\vec{e}_x + \sin\theta\vec{e}_y\right) = 0 \tag{30}$$

and

$$\frac{\partial}{\partial \theta}\vec{e}_r = \frac{\partial}{\partial r}\left(\cos\theta\vec{e}_x + \sin\theta\vec{e}_y\right)$$

$$= \frac{1}{r}\vec{e}_\theta. \tag{31}$$

Similarly we have

$$\frac{\partial}{\partial r}\vec{e}_\theta = \frac{\partial}{\partial r}\left(-r\sin\theta\vec{e}_x + r\cos\theta\vec{e}_y\right)$$

$$= \frac{1}{r}\vec{e}_\theta \tag{32}$$

as well as

$$\frac{\partial}{\partial \theta}\vec{e}_\theta = -r\cos\theta\vec{e}_x - r\sin\theta\vec{e}_y = -r\vec{e}_r. \tag{33}$$

This demonstrates that if we want to take the derivative of any ordinary vector $\vec{V}$ which has components $\left(V^r, V^\theta\right)$, we will have to write

$$\frac{\partial \vec{V}}{\partial r} = \frac{\partial}{\partial r}\left(V^\alpha \vec{e}_\alpha\right)$$

or for any vector with some arbitrary coordinate x^b [6]:

$$\frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta}\vec{e}_\alpha + V^\alpha \frac{\partial \vec{e_\alpha}}{\partial x^\beta}. \tag{34}$$

The first term on the right hand side in (34) is the term we are used to seeing, it is simply the derivative of a vector component. The second term looks new, it tells us that if we assign some basis vector field to our manifold, then we need to account for the fact that the basis vectors themselves can change as we move around the manifold. The second factor of the second term is exactly what the Christoffel symbol is - it compensates for the twisting and warping of the basis vectors as we move around the manifold.

1.3.4 The Lie Derivative and Killing Vectors

The covariant derivative is not the only derivative we can define on a manifold. To define the Lie derivative on a manifold, we need to define a curve γ, its tangent vector $u^\alpha = {dx^\alpha}/{d\lambda}$ and a vector field A^α. Let there be points P and Q on the curve with coordinate values x^α and $x^\alpha + dx^\alpha$ respectively. We can use these points to define a new coordinate system x'^α by writing [7][9]

$$x'^\alpha = x^\alpha + dx^\alpha = x^\alpha + u^\alpha d\lambda \tag{35}$$

which is nothing but an infinitesimal shift along the curve. If we transform the coordinates in this way, the vector field will transform as

$$A'^\alpha(x') = \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta(x)$$

$$= A^\alpha(x) + \partial_\beta u^\alpha A^\beta(x) d\lambda \tag{36}$$

or in terms of the points P and Q

$$A'^\alpha(Q) = A^\alpha(P) + \partial_\beta u^\alpha A^\beta(P) d\lambda. \tag{37}$$

We can also write the value of A^α at Q as

$$A^\alpha(Q) = A^\alpha(x + dx)$$

$$= A^\alpha(P) + u^\beta \partial_\beta A^\alpha(P) d\lambda. \tag{38}$$

We can use the results in (37) and (38) to define the Lie derivative of the vector field A^α as

$$\pounds_u A^\alpha = \frac{A^\alpha(Q) - A'^\alpha(Q)}{d\lambda} \tag{39}$$

and use earlier results to write this in the form:

$$\pounds_u A^\alpha = u^\beta \partial_\beta A^\alpha - A^\beta \partial_\beta u^\alpha. \tag{40}$$

The Lie derivative needs no Christoffel symbol and no notion of the covariant derivative. We can use the same steps illustrated above to arrive at the Lie derivative for a tensor of any number of indices, for example a tensor with 2 indices is written as [7][9]

$$\pounds_u T^\alpha_{\ \beta} = u^\kappa \partial_\kappa T^\alpha_{\ \beta} - T^\kappa_{\ \beta} \partial_\kappa u^\alpha + T^\alpha_{\ \kappa} \partial_\beta u^\kappa. \tag{41}$$

We can attach an important concept to this derivative. Suppose we take

$$\pounds_u g_{\alpha\beta} = \partial_\beta u_\alpha + \partial_\alpha u_\beta \tag{42}$$

where $g_{\alpha\beta}$ is the metric tensor. If we have that

$$\pounds_u g_{\alpha\beta} = 0 \tag{43}$$

there will be some constants of motion associated with movement along a geodesic. To see this, suppose that v^a is tangent to some geodesic parameterized by t on the manifold. If we compute

$$\frac{d}{dt}(u^\alpha v_\alpha) = v^\beta \nabla_\beta (v^\alpha u_\alpha)$$

$$= \left(v^\beta \nabla_\beta v^\alpha\right) u_\alpha + \left(\nabla_\beta u_\alpha\right) v^\alpha v^\beta = 0$$

where the second term is zero because the factor in brackets is an antisymmetric tensor and the factor outside the brackets is symmetric. This shows that $u^\alpha v_\alpha$ is constant along the geodesic provided that $\pounds_u g_{\alpha\beta} = 0$.

1.3.5 Curvature

If one writes down the Minkowski metric in the form

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{44}$$

or the Euclidean metric in the form

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{45}$$

it is easy to see that the quantity in (24) will be zero for both metrics. However, these forms are correct only if we are using rectilinear coordinates; one can transform to spherical coordinates and, for example, we can rewrite (45) in the form

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 \sin^2 \phi & 0 \\ 0 & 0 & r^2 \end{bmatrix}. \tag{46}$$

This seems like an odd result. At first glance it appears that the Christoffel symbols would be able to provide us with a definition of what it means for a space to be curved. We might've been tempted to define flat space as a space which has zero Christoffel symbols, and a curved space as a space which has non-zero Christoffel symbols. This definition fails because spherical coordinates and rectilinear coordinates both label the same Euclidean space and so we should draw the same conclusion about the space's flatness from either (28) or (29). For a correct definition of what it means for a space to be curved, we will need to use the geodesic equation.

Imagine a tiny closed loop of geodesics living on the manifold. Let the loop be given by the parameterized curve $x(\lambda)$ and let the vector tangent to the curve be given by ${}^{dx^\alpha}\!/_{d\lambda}$. We want to know how this vector changes as we move it around the little loop. To do this, use (25)

$$\frac{d}{d\lambda}\frac{dx^\mu}{d\lambda} = -\Gamma^\mu{}_{\alpha\beta} u^\alpha u^\beta \tag{47}$$

which tells us how the tangent vector changes as we change the parameter λ - exactly what we need. Call the point at which $\lambda = 0$, P. When we move the tangent vector to some new parameter value λ, we will have:

$$u^a(\lambda) = u^a_P - \int\limits_0^\lambda \Gamma^\mu{}_{\alpha\beta} u^\alpha u^\beta d\lambda. \tag{48}$$

Since we have assumed that the loop is very small, we can use a Taylor expansion as an accurate approximation to the behaviour of the functions u^a and $\Gamma^\mu{}_{\alpha\beta}$. To this end, write [6]

$$\Gamma^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\alpha\beta}\Big|_P + \partial_\nu\, \Gamma^\mu{}_{\alpha\beta}\Big|_P [x^\nu(\lambda) - x^\nu_P] + \ldots \tag{49}$$

and

$$u^a(\lambda) = u^a_P - \Gamma^a{}_{\alpha\beta}\big|_P\, u^\alpha_P \left[x^\beta(\lambda) - x^\beta_P\right] + \ldots. \tag{50}$$

After substituting this into (31) we will get:

$$u^a = u^a_P - \Gamma^a{}_{\alpha\beta}\big|_P\, u^\alpha_P \int\limits_0^\lambda \frac{dx^\beta}{d\lambda} d\lambda - \left(\partial_\nu \Gamma^a{}_{\alpha\beta} - \Gamma^a{}_{e\beta}\Gamma^e{}_{\alpha\nu}\right)\big|_P\, u^\alpha_P \int\limits_0^\lambda (x^\nu - x^\nu_P)\, \frac{dx^\beta}{d\lambda} d\lambda. \tag{51}$$

Once we integrate around a closed loop, the second term is going to vanish as well as the term involving x^ν_P. This will leave us with a net change of

$$\Delta u^a = -\left(\partial_\nu \Gamma^a{}_{\alpha\beta} - \Gamma^a{}_{e\beta}\Gamma^e{}_{\alpha\nu}\right)\big|_P\, u^\alpha_P \oint x^\nu dx^\beta \tag{52}$$

which clearly demonstrates that a vector changes when moves around a closed loop, provided that a particular combination of the Christoffel symbols is non-zero. If we interchange the dummy indices ν and β and also use the fact that the integral of $x^\nu dx^\beta + x^\beta dx^\nu$ will vanish when we integrate around a closed loop, then we can write

$$\Delta u^\alpha = -\frac{1}{2}\left(\partial_\kappa \Gamma^\alpha{}_{\beta\delta} - \partial_\delta \Gamma^\alpha{}_{\beta\kappa} + \Gamma^\alpha{}_{\eta\kappa}\Gamma^\eta{}_{\beta\delta} - \Gamma^\alpha{}_{\eta\delta}\Gamma^\eta{}_{\beta\kappa}\right)\Big|_P\, u^\beta_P \oint x^\kappa dx^\delta \tag{53}$$

and therefore define

$$R^\alpha{}_{\beta\kappa\delta} = \partial_\kappa \Gamma^\alpha{}_{\beta\delta} - \partial_\delta \Gamma^\alpha{}_{\beta\kappa} + \Gamma^\alpha{}_{\eta\kappa}\Gamma^\eta{}_{\beta\delta} - \Gamma^\alpha{}_{\eta\delta}\Gamma^\eta{}_{\beta\kappa} \tag{54}$$

and call it the Riemann curvature tensor. We say that a manifold is flat if and only if:

$$R^{\alpha}{}_{\beta\kappa\delta} = 0. \tag{55}$$

Notice that we are not saying that a manifold is flat if the Christoffel symbols are zero, but we are saying that a manifold is flat if a specific combination of the Christoffel symbols and their first derivatives is zero. One could compute the Riemann curvature tensor using the metrics in (28) and (29) and discover that they are both equal to zero - confirming that they both describe a flat manifold.

1.3.6 The Einstein Field Equations

So far we have established how the curvature of spacetime will tell a particle to move, but we have not yet established exactly what will determine the curvature of spacetime. To do this, we will have to write down the Einstein field equations - the equations which tell us exactly how matter influences the geometry of spacetime. While doing this it is important to remember that we are not going to be proving the equality in the field equations. We are going to make a reasonable argument for what form the equations must have and then we are going to write down equations which can be empirically tested. We must bear in mind that even though we have field equations for the spacetime geometry, it is still possible that they may not be correct - there may exist very subtle physical phenomena which we have not yet measured which are captured by some other, more correct, set of field equations.

By saying that matter moves along geodesics, we have described how curvature affects the motion of matter but we have not yet described how the presence of matter will affect the curvature of the manifold. To do this, first note that the energy-momentum tensor will satisfy

$$\nabla_{\beta} T^{\alpha\beta} = 0 \tag{56}$$

so if we want to set some geometrical quantity (call it $G^{\alpha\beta}$) equal to the stress energy tensor, we will require that

$$\nabla_{\alpha} G^{\alpha\beta} = 0. \tag{57}$$

An obvious choice for $G^{\alpha\beta}$ is the metric tensor itself, as it can be shown that

$\nabla_\alpha g^{\alpha\beta} = 0$. This can't be right, however, because in empty space we have that $T^{\alpha\beta} = 0$ but $g^{\alpha\beta} \neq 0$. To fix this problem, first construct the Ricci tensor by taking the trace of the Riemann tensor:

$$R_{\alpha\beta} = R^{\kappa}{}_{\alpha\kappa\beta}. \tag{58}$$

It can be shown that the Ricci tensor is not necessarily divergence free, so it too cannot be what we need for $G^{\alpha\beta}$. Yet if we combine the metric tensor and the Ricci tensor into

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} \tag{59}$$

where $R = R^{\alpha}{}_{\alpha}$, then this combination will be divergence free. If we therefore set

$$G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} = \kappa T^{\alpha\beta} \tag{60}$$

we will have a set of field equations (with some constant κ) which will satisfy the condition in (47). These field equations are theoretically sound and the empirical data which have been collected give strong evidence that they are, at the very least, an accurate description of spacetime.

Note that this heuristic argument isn't the only way to arrive at the Einstein Field Equations. If we write down the action $S = \int d^4x \sqrt{-g}\, R$, then by varying this action and setting the variation δS equal to zero, we get the same equation as that in (60).

2 Black Holes in the Study of Space and Time

The black holes of nature are the most perfect macroscopic objects there are in the universe: the only elements in their construction are our concepts of space and time. And since the general theory of relativity provides only a single unique family of solutions for their descriptions, they are the simplest objects as well.

The unique two-parameter family of solutions which describes the space-time around black holes is the Kerr Family discovered by Roy Kerr in July, 1963. The two parameters are the mass of the black hole and the angular momentum of the black hole. The static solu-

tion, with zero angular momentum, was discovered by Karl Schwarzschild in December, 1915. A study of the black holes of nature is then a study of these solutions.

-S Chandrasekhar [4]

Black holes, are objects which are simply perfect but, as we shall see, not perfectly simple. For the following sections we are going to work in units where $G_N = c = 1$. Take, for example, the Schwarzschild metric,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{61}$$

and consider that all it is, is the metric which solves the Einstein Field Equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \tag{62}$$

for a matter source which is neither charged nor rotating. The metric is a well-known and thoroughly studied piece of mathematics which is interesting in its own right. Since its discovery it has been subjected to all manner of inquiry in order for us to determine what exactly space and time behave like near spherical sources of matter. It can be shown that the singularity at $2M = r$ can be removed using a better coordinate system than the one used above [5][11], but that the singularity at $r = 0$ is a genuine singularity which is a feature of all coordinate systems. The surface at $R = 2M$ is a surface of infinite redshift and that once it has been penetrated by some matter, we can never again observe that matter - any photons bouncing off the matter would remain forever trapped within the black hole. By stopping us from making these measurements, the black hole is hiding all the details about the matter which we would normally be able to acquire which makes it look like the black hole is hiding information.

All these inquiries and results are, of course, perfectly valid and highly interesting physical results. They provide invaluable information which tells us about what happens when matter is yoked to the geometry of time and space. For the sake of this study they are, however, not of the utmost importance. We will be happy to take at face value the mathematics which describes what it will be like to fall into a black hole. We are not going to explore the behaviour of rods and clocks orbiting a spherical body in spacetime. Rather, we are going to explore the nature of black holes as solutions to Einstein's Equations and ask questions about the nature of the theory, instead of what it implies for intrepid galactic explorers.

To see what is meant by 'the nature of the theory', consider the Kerr metric,

$$\begin{aligned} ds^2 = -\left(1 - \tfrac{2Mr}{\rho^2}\right) dt^2 + \tfrac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \\ + \left(r^2 + \alpha^2 + \tfrac{2Mr\alpha^2}{\rho^2} sin^2\theta\right) sin^2\theta d\phi^2 - \tfrac{4Mr\alpha sin^2\theta}{\rho^2} dt d\phi \end{aligned} \tag{63}$$

where

$$\alpha = \frac{J}{M} \tag{64}$$

tells us about the angular momentum of the black hole. As usual, M is the mass of the black hole. The other two symbols ρ and Δ are

$$\rho^2 = r^2 + \alpha^2 \cos^2\theta \tag{65}$$

and

$$\Delta = r^2 - 2Mr + \alpha^2. \tag{66}$$

Suppose we have a Kerr black hole. Let's now dump some matter into it. This will change the mass of the black hole, the size of the black hole and possibly the angular momentum. At this stage we don't know how this change will be effected but we know with certainty that whatever remains after the process will be another Kerr black hole due to the fact that it is a unique solution to the Einstein equations. In this section we are going to study what the mathematics tells us about the change from one black hole to another. What we will discover is that the form of the equations describing the change is identical to the form of equations used for thermodynamics.

This raises the remarkable prospect that we can describe a black hole using thermodynamics because a black hole is a thermodynamic object. Recall that a black hole is, most emphatically, not like a spherical ball of gas with some mass. The gas will also be described by a Kerr metric but it will have no null horizon and we can thus in principle acquire whatever information we want about the gas setting up the gravitational field. To this gas we can assign a temperature, a volume and an entropy. It is important to remember that by carrying out this study of the gas, we are not studying the metric set up by the gas, but rather the gas itself. By contrast a black hole is nothing other than a metric and the parameters which go into that metric; studying the thermodynamics of black holes is therefore studying the thermodynamic behaviour of geometry itself.

A black hole, as opposed to a gas, seems to defy everything we know about thermodynamics. One is a collection of particles, the other is a piece of geometry. The gas radiates, the black hole gives off nothing (classically at least). Entropy is well defined for a gas but we do not know what constitutes a microstate of a black hole (if such a thing even has meaning). It is easy to define the energy of a gas but not easy to define the global energy of a spacetime.

The idea of studying a geometric object as a thermodynamic entity thus seems unintuitive, unproductive and ultimately doomed to failure. Nevertheless we are going to demonstrate that our intuition is not quite correct on the issue and that there is extremely strong evidence that a black hole is thermodynamic. If we can talk about black holes in the language of thermodynamics then the implications will be profound. Recall that statistical physics is a theory which we use to talk about systems which possess a huge number of degrees of freedom - far more than we could possibly accommodate in an explicit computation. A black hole thermodynamics would therefore be evidence, but not proof, that a purely geometric piece of physics has undiscovered degrees of freedom. The spacetime metric could be no more fundamental than pressure, a variable which emerges when we discard a large amount of information about the system under study.

Let's carry the analogy to its limit. We know that in the kinetic theory of gases that pressure is the result of huge numbers of particles colliding with the walls of the container. The thermodynamic variable P therefore incorporates information only about the average momenta of the particles and is not part of a fundamental theory about the behaviour of individual particles. The area of a black hole might play the same role as pressure in the kinetic theory; it might tell us something about the average behaviour of some physical variable we do not yet know about. Remember that the particle nature of matter was not always uncontroversial, in the same way, the continuous nature of space and time, while taken for granted now, might be an illusion. It is black hole thermodynamics which provides the first pieces of evidence that will allow us to start building a convincing case for spacetime being an emergent phenomenon.

3 Some Mathematical Preliminaries

3.1 The Raychaudhuri Equation and the Behaviour of Hypersurfaces

Before we move on to study black holes in detail, we need to develop the mathematics which describes how surfaces in our spacetime will behave. It is not enough to merely know that the horizon of a black hole is a null surface, we have to know how the horizon stretches, shrinks and rotates in response to the movement of matter. By studying the differential geometry of our spacetime, we are going to develop a powerful mathematical toolbox which will allow us to investigate the behaviour of spacetime in sufficient detail to come to conclusions about the behaviour of black holes and causal horizons in general.

It's important to realise that when we study spacetime in this way, we are not assuming that the Einstein Equations hold. Our treatment of the differential geometry will be completely independent of the idea that matter is the source of spacetime curvature. Once we have finished the description of the geometry, we will then be able to add the Einstein Equations into the mathematics and investigate, in detail, the behaviour of the spacetime assuming that it is governed by the field equations in (53). By pursuing this general approach to the study of spacetime, we will be able to study spacetime in greater generality than if we had just assumed the Einstein Equations from the outset.

3.1.1 The Kinematics of Deformable Media

In order to develop some intuition about the behaviour of geodesics in a spacetime, we are first going to study the behaviour of a piece of rubber. We will then demonstrate that there are remarkable similarities between the behaviour of the rubber in absolute Newtonian space and the behaviour of spacetime itself.

For a small displacement $\overrightarrow{x}$ between two points O and P, we can express the time dependence of $\overrightarrow{x}$ as [7][8]:

$$\frac{d}{dt}x^a = B^a{}_b x^b + O(x^2), \tag{67}$$

where $B^a{}_b$ is the tensor which describes the deformation and is determined by the properties of the medium. If the time interval is very short, we can say

$$x^a(t_1) = x^a(t_0) + \Delta x^a(t_0) \tag{68}$$

such that

$$\Delta x^a = B^a{}_b(t_0)x^b(t_0)\Delta t + O(\Delta t^2) \tag{69}$$

which is a fairly accurate approximation.

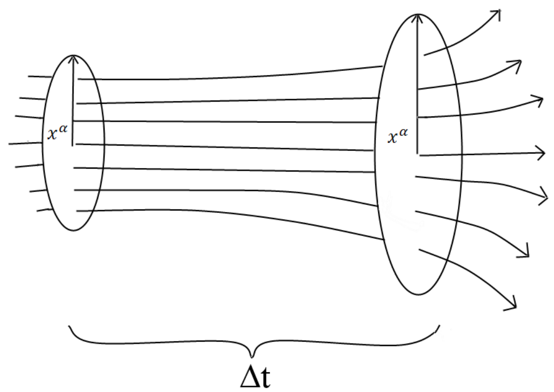

Figure 1: The lines intersecting a given point on the rubber sheet change position relative to another as time elapses. The changes are governed by the matrix $B^a{}_b$.

Now set $x^a(t_0) = r_0(cos\phi, sin\phi)$ in order to use polar coordinates. Let's assume that the matrix representing $B^a{}_b$ is proportional to the identity matrix (the off-diagonal entries are both 0),

$$B^a{}_b = \begin{pmatrix} \frac{1}{2}\theta & 0 \\ 0 & \frac{1}{2}\theta \end{pmatrix}, \tag{70}$$

which will give us a trace of $B^a{}_a = \theta$. The change in x^a is therefore

$$\Delta x^a = \frac{1}{2} r_0 \theta \Delta t(\cos\phi, \sin\phi) \tag{71}$$

which tells us that the area of the circle of radius r_0 has changed to $r_1 = r_0 + \frac{1}{2}\theta r_0 \Delta t$ and that the area change is $\Delta A = A_1 - A_0 = \pi r_0^2 \theta \Delta t$ so we can therefore interpret θ as:

$$\theta = \frac{1}{A_0}\frac{\Delta A}{\Delta t} \tag{72}$$

Therefore we conclude that the trace of the tensor $B^a{}_b$ tells us about how the rubber sheet expands. It is for this reason that we call θ the expansion.

Now let's assume that the tensor $B^a{}_b$ has diagonal as well as off-diagonal entries, but is trace free. For this case we will have:

$$B^a{}_b = \begin{pmatrix} \sigma_+ & \sigma_x \\ \sigma_x & -\sigma_+ \end{pmatrix} \tag{73}$$

The change in the displacement vector is thus $\Delta x^a = r_0 \Delta t(\sigma_+ \cos\phi + \sigma_x \sin\phi, -\sigma_+ \sin\phi + \sigma_x \cos\phi)$ and the parametric equation which describes the radius of the new figure is:

$$r_1(\phi) = r_0(1 + \Delta t \sigma_+ \cos 2\phi + \Delta t \sigma_x \sin 2\phi) \tag{74}$$

This tells us that after applying a symmetric, traceless, $B^a{}_b$, we will have an ellipse oriented at some angle. For this reason, we say that the parameters σ_+ and σ_x are the shear parameters.

Finally assume that the tensor $B^a{}_b$ is traceless and antisymmetric:

$$B^a{}_b = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \tag{75}$$

This will give us a change of $\Delta x^a = r_0 \omega \Delta t(\sin\phi, -\cos\phi)$, with a new displacement vector $x^a(t_1) = r_0(\cos\phi', \sin\phi')$ where $\phi' = \phi - \omega\Delta t$. A tensor of this form will therefore rotate the displacement vector; we call the parameter $\omega$the rotation parameter. These results allow us to break $B^a{}_b$ up into its trace part, its traceless symmetric part and its traceless antisymmetric part and interpret each part in turn:

$$B^a{}_b = \begin{pmatrix} \frac{1}{2}\theta & 0 \\ 0 & \frac{1}{2}\theta \end{pmatrix} + \begin{pmatrix} \sigma_+ & \sigma_x \\ \sigma_x & -\sigma_+ \end{pmatrix} + \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \tag{76}$$

An alternate notation is:

$$B^a{}_b = \frac{1}{2}\theta\delta_{ab} + \sigma_{ab} + \omega_{ab} \tag{77}$$

These results are not too surprising. The deformation of the sheet is governed by a matrix and we can always decompose a matrix into the sum of a symmetric, anti-symmetric and trace part.

3.1.2 The Raychaudhuri Equation for Timelike Geodesics

Let O be a congruence of timelike geodesics defined on our spacetime. Parameterize the geodesics with some parameter λ, such that the vector field of tangents, ξ^α, is always normalized to unit length $\xi^\alpha \xi_\alpha = -1$. Use this to define the tensor field [7][8][11]

$$B_{\alpha\beta} = \nabla_\beta \xi_\alpha \tag{78}$$

which is purely spatial:

$$B_{\alpha\beta}\xi^\alpha = B_{\alpha\beta}\xi^\beta = 0 \tag{79}$$

Now that we have defined $B_{\alpha\beta}$, we will need to interpret it. Consider a one-parameter subset $\gamma_s(\lambda)$ of geodesics in the congruence; s is the variable which selects the geodesic and λ is the variable which tells us where we are on the selected geodesic. Let η^α be the deviation vector from the geodesic $\gamma_0(\lambda)$. The vector η^a therefore represents and infinitesimal spatial displacement from $\gamma_0(\lambda)$ to a nearby geodesic in the congruence. We therefore have

$$\pounds_\xi \eta^\alpha = 0 \tag{80}$$

which tells us

$$\xi^\beta \nabla_\beta \eta^\alpha = \eta^\beta \nabla_\beta \xi^\alpha = B^\alpha{}_\beta \eta^\beta \tag{81}$$

so $B^a{}_b$ is the tensor which tells us how the geodesics in the congruence are stretched and rotated.

Now define the spatial metric

$$h_{\alpha\beta} = g_{\alpha\beta} + \xi_\alpha \xi_\beta \tag{82}$$

which functions as the projector onto the subspace of the tangent space perpendicular to the tangent vectors ξ^α. We say that it is spatial because it is orthogonal to ξ^α: $\xi^\alpha h_{\alpha\beta} = h_{\alpha\beta}\xi^\beta = 0$. It is also effectively three-dimensional and we can see this by thinking about a comoving Lorentz frame at some point

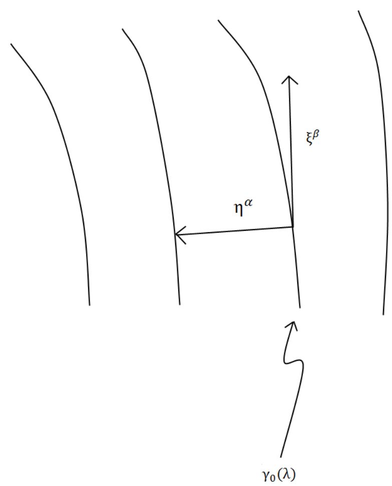

Figure 2: The tangent vector ξ^β for some geodesic $\gamma_0(\lambda)$ is orthogonal to the vector η^α which describes a movement from $\gamma_0(\lambda)$ to a nearby geodesic. As the parameter λ is varied, the change in η^α is governed by $B^\alpha{}_\beta$ - analogous to the behaviour of the rubber sheet.

P in the congruence. In the comoving frame,

$$\xi^\alpha = (-1, 0, 0, 0) \tag{83}$$

$$g_{\alpha\beta} = diag(-1, 1, 1, 1) \tag{84}$$

and therefore the spatial metric is,

$$h_{\alpha\beta} = diag(0, 1, 1, 1). \tag{85}$$

With this in mind we will define the expansion, shear and twist of the congruence by

$$\theta = B^{\alpha\beta} h_{\alpha\beta} \tag{86}$$

$$\sigma_{\alpha\beta} = B_{(\alpha\beta)} - \frac{1}{3}\theta h_{\alpha\beta} \tag{87}$$

$$\omega_{\alpha\beta} = B_{[\alpha\beta]} \tag{88}$$

so, as before, we can decompose the tensor as [7][8][11]:

$$B^{\alpha}{}_{\beta} = \frac{1}{2}\theta\delta_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \tag{89}$$

Now let's study how the deformation tensor $B^{a}{}_{b}$ varies along the geodesics in the congruence. To do this we can write:

$$\xi^{\kappa}\nabla_{\kappa}B_{\alpha\beta} = \xi^{\kappa}\nabla_{\kappa}\nabla_{\beta}\xi_{\alpha} = \xi^{\kappa}\nabla_{\beta}\nabla_{\kappa}\xi_{\alpha} + R_{\kappa\beta\alpha}{}^{\delta}\xi^{\kappa}\xi_{\delta} \tag{90}$$

$$= \nabla_{\beta}(\xi^{\kappa}\nabla_{\kappa}\xi_{\alpha}) - (\nabla_{\beta}\xi^{\kappa})(\nabla_{\kappa}\xi_{\alpha})R_{\kappa\beta\alpha}{}^{\delta}\xi^{\kappa}\xi_{\delta} \tag{91}$$

$$= -B^{\kappa}{}_{\beta}B_{\alpha\kappa} + R_{\kappa\beta\alpha}{}^{\delta}\xi^{\kappa}\xi_{\delta} \tag{92}$$

and then take the trace, giving us [7][8][11]

$$\xi^{\kappa}\nabla_{\kappa}\theta = \frac{d}{d\lambda}\theta = -\frac{1}{3}\theta^{2} - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - \omega^{\alpha\beta}\omega_{\alpha\beta} - R_{\kappa\delta}\xi^{\kappa}\xi^{\delta} \tag{93}$$

which is the Raychaudhuri equation for a timelike congruence of geodesics. Equation (93) describes how the geodesics in the congruence move away from or closer to one another as the value of λ is adjusted.

The Raychaudhuri equation derived above is a result which is entirely independent of any theory of gravity. At no point did we need to invoke the Einstein equation. Since it is a purely geometric result, we are free to use it to reason about spacetime without writing down specifics regarding how geometry is yoked to matter.

We saw that for the deformable medium, that θ could be interpreted as the normalized change in area of a patch of the medium. For the above Raychaudhuri equation, we will now show that θ can be interpreted as the fractional change in the congruence's cross-sectional volume:

$$\theta = \frac{1}{\delta V}\frac{d}{d\lambda}\delta V. \tag{94}$$

In order to see this, begin by selecting a particular geodesic, γ_s, in the congruence. On this geodesic, pick a point P such that $\gamma_s(\lambda_p) = P$. Now construct in a small neighbourhood around P, a small set, $\delta\Sigma(\lambda_p)$, of points P' such that:

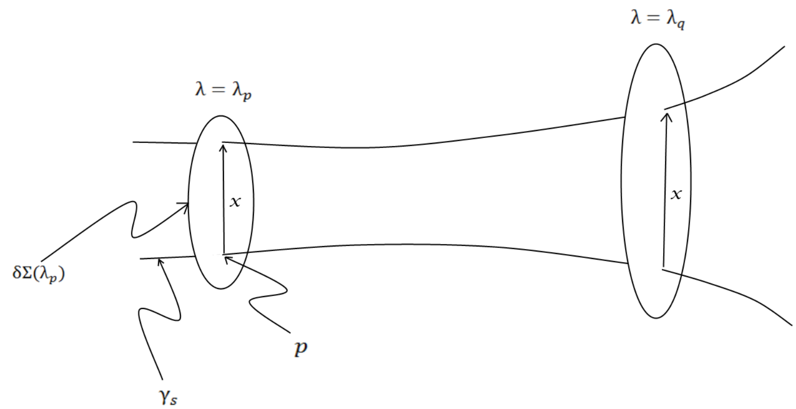

Figure 3: The behaviour of the submanifold $\delta\Sigma(\lambda_p)$ can be deduced by studying the behaviour of geodesics intersecting the submanifold.

1. Through each of these points there passes another geodesic (not γ_s) from the congruence
2. At each point P', $\lambda = \lambda_p$

This set will then give us a small, three-dimensional region; a segment of the hypersurface $\lambda = \lambda_p$. Further assume that the parameterization has been adjusted so that γ_s intersects $\delta\Sigma(\lambda_p)$ orthogonally, which we won't require for the other geodesics in $\delta\Sigma(\lambda_p)$. We are going to call $\delta\Sigma(\lambda_p)$ the congruence's cross section around γ_s at $\lambda = \lambda_p$. What we are interested in is calculating the volume of $\delta\Sigma(\lambda_p)$ and then comparing it to the volume of $\delta\Sigma(\lambda_q)$, which is another hypersurface made up from the same congruence and geodesic γ_s but at a different value of the parameter λ.

Let us introduce coordinates on $\delta\Sigma(\lambda_p)$ by assigning a label $y^a (a = 1, 2, 3)$ to each point P' in the hypersurface. Since a geodesic passes through each point in P', we can use our y^a coordinate system to label the geodesics themselves. Now force each geodesic to keep its label as it moves away from $\delta\Sigma(\lambda_p)$, so that we have a coordinate system in $\delta\Sigma(\lambda_q)$ or, indeed, at any other point along the geodesic γ_s, as shown in figure 3. We have now specified enough to be able to define a coordinate system in our full 4-dimensional space, namely (λ, y^a); so we know which geodesic we are on and how far along that geodesic we are. If

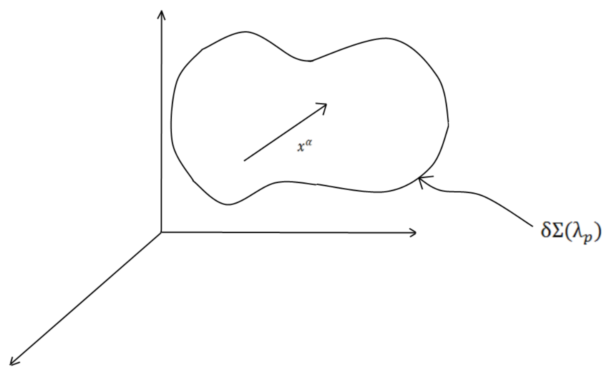

Figure 4: Since displacements along the submanifold are still displacements in the greater manifold, they can still be measured by the metric g_{ab}. By restricting the metric to measuring only displacements in the submanifold, we arrive at the idea of an induced metric.

the original coordinate system in use was x^a, there is a transformation between this original coordinate system and the one which has just been defined [7][8]:

$$x^\alpha = x^\alpha(\lambda, y^a). \tag{95}$$

Since we have specified that y^a be constant along the geodesics, we get

$$\xi^\alpha = \left(\frac{\partial}{\partial\lambda}x^\alpha\right)_{y^a} \tag{96}$$

and can also specify vectors

$$e^\alpha_a = \left(\frac{\partial}{\partial y^a}x^\alpha\right)_\lambda \tag{97}$$

which run tangent to the cross sections. Using these relations, we get that $\pounds_\xi e^\alpha_a = 0$ while on γ_S (but not on other geodesics) we have $\xi_\alpha e^\alpha_a = 0$.

By using (29) we can introduce the three-tensor

$$h_{ab} = g_{\alpha\beta}e^\alpha_a e^\beta_a \tag{98}$$

which is a tensor with respect to transformations of the kind $y^a \to y^{a'}$ but a scalar when the transformation is of the form $x^a \to x^{a'}$. This three-tensor actually gives us a metric tensor on $\delta\Sigma(\lambda_p)$, as in figure 4. To see this, restrict a displacement to the cross section, so that $d\lambda = 0$ and $x^a = x^a(y^a)$, then:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \tag{99}$$

$$= g_{\alpha\beta} \left(\frac{\partial x^\alpha}{\partial y^a} dy^a \right) \left(\frac{\partial x^\beta}{\partial y^b} dy^b \right) \tag{100}$$

$$= \left(g_{\alpha\beta} e^\alpha_a e^\beta_a \right) dy^a dy^b \tag{101}$$

$$= h_{ab} dy^a dy^b. \tag{102}$$

Since γ_s is orthogonal to the cross section, $h_{ab} = h_{\alpha\beta} e^\alpha_a e^\beta_b$ on γ_s where $h_{\alpha\beta} = g_{\alpha\beta} + \xi_\alpha \xi_\beta$ is the transverse metric. By doing this, we have constructed a manifold of lower dimension than the manifold in which the hypersurface lives. Since this new manifold lives within the higher-dimensional manifold, it will inherit the geometry of the higher-dimensional manifold.

The three-dimensional volume element on $\delta\Sigma(\lambda_p)$ is $\delta V = \sqrt{h} d^3 y$. Since the coordinates y^a are comoving, $d^3 y$ does not vary as $\delta\Sigma$ goes to $\delta\Sigma(\lambda_q)$ which tells us that a change in δV is due entirely to a change in $\sqrt{h}$:

$$\frac{1}{\delta V} \frac{d}{d\lambda} \delta V = \frac{1}{\sqrt{h}} \frac{d}{d\lambda} \sqrt{h} = \frac{1}{2} h^{ab} \frac{d}{d\lambda} h_{ab} \tag{103}$$

Let us now calculate the rate of change of this three-metric:

$$\frac{d}{d\lambda} h_{ab} = \xi^\mu \nabla_\mu \left(g_{\alpha\beta} e^\alpha_a e^\beta_b \right)$$

$$= g_{\alpha\beta} \left(\xi^\mu \nabla_\mu e^\alpha_a \right) e^\beta_b + g_{\alpha\beta} e^\alpha_a \left(\xi^\mu \nabla_\mu e^\beta_b \right)$$

$$= g_{\alpha\beta} \left(\nabla_\mu \xi^\alpha e^\mu_a \right) e^\beta_b + g_{\alpha\beta} e^\alpha_a \left(\nabla_\mu \xi^\beta e^\mu_b \right)$$

$$= \nabla_\alpha \xi_\beta e^\alpha_a e^\beta_b + \nabla_\beta \xi_\alpha e^\alpha_a e^\beta_b$$

$$= (B_{\alpha\beta} + B_{\beta\alpha}) e^\alpha_a e^\beta_b.$$

Now multiply by h^{ab} and evaluate this on γ_s:

$$h^{ab}\frac{d}{d\lambda}h_{ab} = (B_{\alpha\beta} + B_{\beta\alpha})h^{ab}e^{\alpha}_{a}e^{\beta}_{b}$$

$$= 2B_{\alpha\beta}h^{\alpha\beta}$$

$$= 2B_{\alpha\beta}g^{\alpha\beta}$$

$$= 2\theta$$

Which shows us that [7][8]:

$$\theta = \frac{1}{\sqrt{h}}\frac{d}{d\lambda}\sqrt{h}$$

and formally establishes that, like in the case of the rubber sheet, the variable θ tells us about the changes in size of the medium.

3.1.3 The Raychaudhuri equation for Null Geodesics

The Raychaudhuri equation for timelike geodesics does not carry over unmodified to the case of null geodesics, even though the geometric setup is the same. We are going to demonstrate that this is the result of the fact that a vector tangent to a null surface is also orthogonal to it. Denote the tangent vector to the null congruence as k^{α} and assume that the geodesics have been affinely parameterized by λ. We now have that a movement along a geodesic in the congruence is given by $dx^{\alpha} = k^{\alpha}d\lambda$. Call the deviation vector η^{α} and let it by orthogonal to, and Lie dragged by, the geodesics. All told, these conditions give us [7][8]:

$$k^{\alpha}k_{\alpha} = 0$$

$$k^{\beta}\nabla_{\beta}k^{\alpha} = 0$$

$$\eta^{\beta}\nabla_{\beta}k^{\alpha} = k^{\beta}\nabla_{\beta}\eta^{\alpha}$$

$$k^{\alpha}\eta_{\alpha} = 0$$

Just like before, we want to know about the behaviour of the deviation vector η^α but, unlike before, the property $k^\alpha \eta_\alpha = 0$ does not remove the component of η^α in the direction of k^α. This is due to the first condition and is thus an unavoidable condition of a null congruence; it is not an artifact of the chosen parameterization or the coordinate system. We now see that we have an additional task - we need to isolate the purely transverse part of η^α.

Unfortunately, we can't employ the same analysis as before. Simply writing

$$h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha k_\beta \tag{104}$$

no longer suffices because $h_{\alpha\beta}k^\beta = k_\alpha$ and this expression, in general, is not 0. Without loss of generality, we can solve this problem by going to a local Lorentz frame at a point P in the congruence and defining the null coordinates:

$$u = t - x$$

$$v = t + x.$$

This new coordinate system will give us a line element, in the local Lorentz frame, of $ds^2 = -dudv + dy^2 + dz^2$. Suppose $k^\alpha$is tangent to curves of $u = constant$. This will give us a transverse line element $ds^2 = dy^2 + dz^2$, which is two-dimensional, not three-dimensional. That the surface is of dimension two is a feature of all null surfaces.

Let's now isolate the transverse part of the metric. Introduce another null vector field, call it N^α and choose it such that $N_\alpha k^\alpha \neq 0$. Since the normalization of a null vector is entirely arbitrary, we will also impose $k^\alpha N_\alpha = -1$. For example, if we set $k_\alpha = -\partial_\alpha u$ in our Lorentz frame, then a choice of $N_\alpha = -\frac{1}{2}\partial_\alpha v$ would suffice for our construction. With this in mind, consider the tensor

$$h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta \tag{105}$$

which is orthogonal to both k^α and N^α. We also have that

$$h_{\alpha\beta} = diag(0, 0, 1, 1)$$

in our local Lorentz frame. This firmly establishes that $h_{\alpha\beta}$ is transverse and two-dimensional and is exactly the transverse metric we need to derive the Raychaudhuri equation for null surfaces. It satisfies:

$$h_{\alpha\beta}k^{\beta} = h_{\alpha\beta}N^{\beta} = 0$$

$$h^{\alpha}{}_{\alpha} = 2.$$

Just like before, we introduce the tensor

$$B_{ab} = \nabla_b k_a \tag{106}$$

which measures the failure of the deviation vector to be parallel transported:

$$k^b\nabla_b\eta^a = \eta^b\nabla_b k^a = B^a{}_b\eta^b \tag{107}$$

Like before, B_{ab} is orthogonal to k^a, but it is not orthogonal to N^a. This implies that (104) has a non-transverse component which we need to remove. We do this by isolating the transverse part of η^a, call it $\tilde{\eta}^a$, by contracting it with our transverse metric:

$$\tilde{\eta}^{\alpha} \equiv h^{\alpha}{}_{\mu}\eta^{\nu} = \eta^{\alpha} + (N_{\mu}\eta^{\mu})\, k^{\alpha}$$

The covariant derivative of $\tilde{\eta}^a$ in the direction of k^{α} represents the relative velocity of two neighbouring geodesics and is given by

$$k^{\beta}\nabla_{\beta}\tilde{\eta}^{\mu} = h^{\mu}{}_{\nu}B^{\nu}{}_{\beta}\eta^{\beta} + k^{\beta}\nabla_{\beta}h^{\mu}{}_{\nu}\eta^{\nu}$$

and upon calculation the second term gives us

$$k^{\beta}\nabla_{\beta}\tilde{\eta}^{\mu} = h^{\mu}{}_{\nu}B^{\nu}{}_{\beta}\eta^{\beta} + \left(k^{\beta}\nabla_{\beta}N_{\nu}\eta^{\nu}\right) k^{\mu}$$

which tells us that $k^{\beta}\nabla_{\beta}\tilde{\eta}^{\mu}$ has a component along k^{μ}. To remove this, we do as before and project using $h^{\alpha}{}_{\mu}$:

$$(\widetilde{k^{\beta}\nabla_{\beta}\tilde{\eta}^{\mu}}) = h^{\alpha}{}_{\mu}\left(k^{\beta}\nabla_{\beta}\tilde{\eta}^{\mu}\right) = h^{\alpha}{}_{\mu}B^{\mu}{}_{\nu}\eta^{\nu}$$

$$= h^{\alpha}{}_{\mu}B^{\mu}{}_{\nu}\tilde{\eta}^{\nu}$$

$$= h^{\alpha}{}_{\mu} h^{\nu}{}_{\beta} B^{\mu}{}_{\nu} \tilde{\eta}^{\beta}$$

Which gives us the transverse components of the relative velocity of the geodesics. We have obtained

$$(\widetilde{k^{\beta} \nabla_{\beta} \tilde{\eta}^{\mu}}) = \tilde{B}^{\alpha}{}_{\beta} \tilde{\eta}^{\beta}$$

where

$$\tilde{B}^{\alpha}{}_{\beta} = h^{\alpha}{}_{\mu} h^{\nu}{}_{\beta} B^{\mu}{}_{\nu}.$$

We can also write the tensor $\tilde{B}^{\alpha}{}_{\beta}$ out more explicitly:

$$\tilde{B}^{\alpha}{}_{\beta} = B_{\alpha\beta} + k_{\alpha} N^{\mu} B_{\mu\beta} + k_{\beta} B_{\alpha\mu} N^{\mu} + k_{\alpha} k_{\beta} B_{\mu\nu} N^{\mu} N^{\nu}.$$

Just like before, we can decompose $\tilde{B}^{\alpha}{}_{\beta}$ into three parts

$$\tilde{B}^{\alpha}{}_{\beta} = \frac{1}{2} \theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

which have the same interpretation as before, namely expansion, shear and rotation. Notice, however, that the numerical factor before θ is now $\frac{1}{2}$ due to the fact that the space is two-dimensional, not three-dimensional like it was for timelike geodesics.

The Raychaudhuri equation for null geodesics now proceeds exactly as before, we get

$$\frac{d\theta}{d\lambda} = -B^{\alpha\beta} B_{\beta\alpha} - R_{\alpha\beta} k^{\alpha} k^{\beta}$$

from the same series of steps and we can check that

$$B^{\alpha\beta} B_{\beta\alpha} = \tilde{B}^{\alpha\beta} \tilde{B}_{\beta\alpha} = \frac{1}{2} \theta^2 + \sigma^{ab} + -\omega^{ab} \omega_{ab}$$

which finally leads us to [7][8]

$$\frac{d}{d\lambda} \theta = -\frac{1}{2} \theta^2 - \sigma^{ab} \sigma_{ab} - \omega^{ab} \omega_{ab} - R_{ab} k^a k^b. \tag{108}$$

We now need to interpret the variable θ, just like we did for the timelike case. To do this, pick a particular null geodesic γ_s from the congruence and on this geodesic pick a point P at some value of the parameter λ_p. Now consider the null curves to which N^α is tangent; let μ be the parameter on these curves and adjust μ so that it is constant on the geodesics in the congruence. Call the auxiliary curve which passes though P, β and let $\mu = \mu_\gamma$ at P. As before, define the cross section $\delta S\left(\lambda_p\right)$ to be the small set of points P' in a neighbourhood of P such that:

1. Through each of these points there passes another geodesic in the congruence and another auxiliary curve
2. At each point in P', we have $\lambda = \lambda_p$ and $\mu = \mu_\gamma$

Further assume that we have correctly adjusted the parameters so that both γ_s and β intersect $\delta S\left(\lambda_p\right)$ orthogonally. Introduce coordinates in $\delta S\left(\lambda_p\right)$ by assigning a label $y^A(A = 2,3)$ to each point in the set P'. Since there is one geodesic through each point in the set, we can use the coordinates y^A to label the geodesics themselves. Now demand that each geodesic keeps its y^A label as it moves away from $\delta S\left(\lambda_p\right)$ so that we have a coordinate system for any other cross section $\delta S\left(\lambda\right)$. By this set up, we have a new coordinate system $\left(\lambda, \mu, y^A\right)$ and there exists a transformation between the original coordinates and the new system: $x^a = x^a\left(\lambda, \mu, y^A\right)$. Since μ and y^A are constant along the geodesics, we obtain:

$$k^\alpha = \left(\frac{\partial x^\alpha}{\partial \lambda}\right)_{\mu, y^A}.$$

We also have that the vectors

$$e^\alpha_A = \left(\frac{\partial x^\alpha}{\partial y^A}\right)_{\lambda, \mu}$$

are tangent to the cross sections. These imply that $\pounds_\xi e^\alpha_A = 0$ and on γ_s: $k_\alpha e^\alpha_A = N_\alpha,\ e^\alpha_A = 0$. The steps from the previous section on timelike congruences can now be carried out with little change. The tensor

$$\sigma_{AB} = g_{\alpha\beta} e^\alpha_A e^\beta_B$$

acts as a tensor on $\delta S\left(\lambda\right)$. Now we see that the cross-sectional area is defined by $\delta A = \sqrt{\sigma} d^2 y$ and that the relation

$$\frac{d}{d\lambda}\sigma_{AB} = \left(B_{\alpha\beta} + B_{\beta\alpha}\right) e^\alpha_A e^\beta_B$$

will give us, upon taking its trace,

$$\theta = \frac{1}{\sqrt{\sigma}} \frac{d}{d\lambda} \sqrt{\sigma}$$

which is equivalent to

$$\theta = \frac{1}{\delta A} \frac{d}{d\lambda} \delta A. \tag{109}$$

This demonstrates that instead of a change of volume, we have a change of area. In the case of null geodesics, we are dealing with a 2-dimensional submanifold as opposed to a 3-dimensional submanifold [7][8].

3.1.4 Integration on Hypersurfaces

For our purposes, a hypersurface is a three-dimensional submanifold of the spacetime manifold. We can define this submanifold in two ways; we can either put a restriction on the coordinates [7],

$$\Phi\left(x^{\alpha}\right) = 0$$

or we can specify a parametric equation

$$x^{\alpha} = x^{\alpha}\left(y^{a}\right)$$

where the y^{a} are coordinates on the hypersurface. The vector $\partial_{a}\Phi$ is normal to the hypersurface Σ because the value of Φ changes only in the direction orthogonal to Σ. Let's now introduce a unit normal n_{α}, define it by

$$n^{\alpha} n_{\alpha} = \epsilon \equiv \begin{cases} -1 & \Sigma \textit{ spacelike} \\ +1 & \Sigma \textit{ timelike} \end{cases} \tag{110}$$

and insist that n^{α} point in the direction of increasing Φ. This implies that $n^{\alpha}\partial_{\alpha}\Phi > 0$ and that n_{α} itself is given by

$$n_{\alpha} = \frac{\epsilon \partial_{\alpha}\Phi}{\left|g^{\mu\nu}\partial_{\mu}\Phi\partial_{\nu}\Phi\right|}$$

but only if the hypersurface is timelike or spacelike. A null hypersurface would result in a denominator of zero. To circumvent this problem, let's call the normal vector $k_{\alpha} = -\partial_{\alpha}\Phi$; the sign is chosen so that the vector k^{α} is future-directed when Φ increases towards the future. Due to the nature of null vectors, we have $k_{\alpha}k^{\alpha} = 0$ which demonstrates that the null vector is actually orthogonal to itself and is therefore tangent to the hypersurface as well. Let's now calculate $\left(\nabla_{\beta}k_{\alpha}\right)k^{\beta}$:

$$\left(\nabla_{\beta}k_{\alpha}\right)k^{\beta} = \left(\nabla_{\beta}\left[\partial_{\alpha}\Phi\right]\right)\partial^{\beta}\Phi$$

$$= \left(\nabla_{\alpha}\left[\nabla_{\beta}\Phi\right]\right)\partial^{\beta}\Phi$$

$$= \frac{1}{2}\left(\nabla_{\alpha}\left[\partial_{\beta}\Phi\partial^{\beta}\Phi\right]\right).$$

Since $\partial_{\beta}\Phi\partial^{\beta}\Phi$ is everywhere zero on Σ,its gradient must be along k_{α} and we get that $\nabla_{\alpha}\left[\partial_{\beta}\Phi\partial^{\beta}\Phi\right] = 2\kappa k_{\alpha}$ (where κ is a scalar) and the normal vector will satisfy

$$\left(\nabla_{\beta}k_{\alpha}\right)k^{\beta} = \kappa k_{\alpha} \tag{111}$$

which is the general form of the geodesic equation. We therefore have that the hypersurface is comprised of ('generated by') null geodesics and the vector $k^{\alpha}$is tangent to these geodesics. Let us parameterize the geodesics by λ so that a small displacement along a geodesic is $dx^{\alpha} = k^{\alpha}d\lambda$.

Let's now install a coordinate system on Σ which is well adapted to the behaviour of the null generators. Let the parameter λ be one of the coordinates and introduce two additional coordinates $\theta^{A}(A = 2,3)$ to label the generators. To describe a point on the hypersurface we need to supply 3 numbers,

$$y^{a} = \left(\lambda, \theta^{A}\right)$$

where the variable θ^{A} tells which geodesic we are on and the parameter λ tells us how far we are along the selected geodesic.

The hypersurface is a manifold in its own right and therefore it is possible to define an intrinsic metric on the hypersurface using the metric of the space in which the hypersurface is embedded. We call this metric the induced metric and we obtain it by restricting the line element to displacements confined to the hypersurface. If we specify the hypersurface by means of the parametric equations $x^{\alpha} = x^{\alpha}\left(y^{\alpha}\right)$, then the vectors

$$e_{a}^{\alpha} = \frac{\partial x^{\alpha}}{\partial y^{a}} \tag{112}$$

are tangent to curves in the hypersurface Σ. For a displacement along a curve in Σ, use the original metric $g_{\alpha\beta}$

$$ds^2_\Sigma = g_{\alpha\beta} dx^\alpha dx^\beta$$

$$= g_{\alpha\beta} \left(\frac{\partial x^\alpha}{\partial y^a} dy^a \right) \left(\frac{\partial x^\beta}{\partial y^b} dy^b \right)$$

$$= g_{\alpha\beta} e^\alpha_a e^\beta_b dy^a dy^b$$

$$= h_{ab} dy^a dy^b \tag{113}$$

to define the induced metric $h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b$. What about the case of a null hypersurface? Using the coordinates we have defined, we have [7]

$$e^\alpha_1 = \left(\frac{\partial x^\alpha}{\partial \lambda} \right)_{\theta^A} \equiv k^\alpha$$

and it follows that the elements of the induced metric are $h_{11} = g_{\alpha\beta} k^\alpha k^\beta = 0$, $h_{1A} = g_{\alpha\beta} k^\alpha e^\beta_A = 0$. This is due to the fact that $e^\alpha_A = (\partial x^\alpha / \partial \theta)_\lambda$ is orthogonal to k^α. So in the null case:

$$ds^2_\Sigma = \sigma_{AB} d\theta^A d\theta^B \tag{114}$$

where

$$\sigma_{AB} = g_{\alpha\beta} e^\alpha_A e^\beta_B$$

and

$$e^\alpha_A = \left(\frac{\partial x^\alpha}{\partial \theta^A} \right)_\lambda .$$

We now have all the geometric tools we need to define the surface element of a null surface of our spacetime. Since we have identified y^1 with λ, the remaining coordinates are constant on the generators of the horizon and $e^\alpha_1 = k^\alpha$. We can now write the directed surface element as

$$d\Sigma_\mu = k^\nu dS_{\mu\nu} d\lambda \tag{115}$$

where the element of two-dimensional surface area $dS_{\mu\nu}$ is

$$dS_{\mu\nu} = \epsilon_{\mu\nu\beta\gamma} e_2^{\beta} e_3^{\gamma} d^2\theta$$

and $\epsilon_{\mu\nu\beta\gamma}$ is the Levi-Civita tensor. An alternate expression for (115) is [7]

$$d\Sigma^{\alpha} = -k^{\alpha}\sqrt{\sigma} d^2\theta d\lambda. \tag{116}$$

Let's now interpret (107) in order to build up some intuition. The left hand side has one up index; it is therefore a vector area and it is directed along k^{α}. The factor $d\lambda$ is a small displacement along the chosen generator and the factor $\sqrt{\sigma} d^2\theta$ is a small piece of cross-sectional area which runs transverse to the vector k^{α}.

4 Black Hole Thermodynamics

4.1 Emergent Physics

Notice that equation (108) is of a form which is extremely similar to that of equation (89), giving the impression that spacetime itself is behaving like a rubber sheet. The differences seem only to account for the fact that (89) is the result of a deformed sheet in a flat, Newtonian spacetime and that (108) describes the intrinsic geometry of a deformed manifold on which a metric tensor has been defined. It is tempting, therefore, to jump to the conclusion that the form of (108) provides evidence that differential geometry, and hence gravity, is something with an emergent interpretation. However, this cannot be the case. To see this, remember that gravity is not a theory of geometry itself, rather it is a theory of how matter couples to geometry and the implications of that coupling.

Investigating whether gravity is an emergent phenomenon is not the same as investigating whether or not results which are purely geometric in nature are the result of some other mathematical theory. A study of emergent gravity must tell us not about geometry, but about the coupling of matter and geometry via some suitable set of field equations. The ultimate goal would be to demonstrate, theoretically, that the empirical results about gravity we know to be true are in fact the limit of some other physical theory describing the relation between matter, space and time. It is for those reasons that we don't consider the similarity in form between (89) and (108) to provide evidence that gravity is emergent

To understand exactly what is meant by describing physics as "emergent", we are going to analyze the way in which fluids are studied. The continuum mechanics description of a fluid uses continuous functions like the mass density $\rho(x,t)$ and the velocity field $v(x,t)$ to describe the physics of a fluid. These functions, however, are only approximations to the underlying physics. Discrete particles, even in incredible quantities, will never have a continuous mass density. We are able to perform meaningful physical calculations with $\rho(x,t)$ because it functions as an excellent approximation. Likewise we know that the particles in a fluid will not all be moving in exactly the same direction at the same time; we decompose their velocities into two parts

$$v = \langle v \rangle + v_r$$

where $\langle v \rangle$ represents the mean velocity of the particle and v_r represents the random component. This decomposition tells us that, on average the particles will be moving in a given direction but that it is entirely possible for some particles to not be moving in the direction of $\langle v \rangle$. But these errant particles will be so small in number that the fluid will be described extremely accurately by only considering the average velocity.

The variables $\rho(x,t)$ and $v(x,t)$ are therefore not fundamental. They arise as a result of averaging over large numbers of particles described by other, more fundamental, physical quantities. After performing this averaging we say that the velocity field *emerges* as a new variable with which we will construct theories and perform calculations. When the claim is made that gravity is emergent, we mean that the geometric variables such as the metric (and therefore distance) are emergent. The claim is that if we knew the underlying theory we would be able to recover variables like g_{ab}, geodesic length and black hole surface area by taking averages and making approximations.

The idea of emergence is linked to the resolution of the physical measurements we make of some system; a physical system may look different if examined with increasing levels of accuracy. When our eyes look at an LCD monitor,we are able to perceive a large spectrum of colours, but the pixels of the monitor are only capable of displaying red, green or blue. It is the aggregate behaviour of large numbers of red, green and blue pixels which we will see as colours and movement on the screen, all the while being indifferent to the changes in colour of the individual pixels. If we were to increase the resolution of our measurement by looking through a magnifying glass at the screen, then we would be able to see the individual pixels.

4.2 The Laws of Black Hole Mechanics

We have defined and developed the necessary mathematical technology to describe the behaviour of null geodesics in a spacetime. Since the generators of a black hole's horizon are null geodesics we have almost all we need to describe the behaviour of a black hole. To complete the preliminaries, let the coordinates on the event horizon be [7]

$$y^a = \left(v, \theta^A\right)$$

where the advanced-time coordinate v is a non-affine parameter for the horizon's generators. As before, θ^A are the coordinates which label the generators. The vectors tangent to the horizon are then

$$\xi^\alpha = \left(\frac{\partial x^\alpha}{\partial v}\right)_{\theta^A} \quad e^\alpha_a \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)_v ,$$

they satisfy $\xi_\alpha e^\alpha_A = 0 = \pounds_\xi e^\alpha_A$ and $\xi^\alpha = t^\alpha + \Omega_H \phi^\alpha$ is a Killing vector. Introduce an auxiliary null vector N^α and normalize it with $N_\alpha \xi^\alpha = -1$, then the spacetime metric in terms of the induced metric on the black hole horizon is

$$g^{\alpha\beta} = -\xi^\alpha N^\beta - N^{\alpha\beta} + \sigma^{AB} e^\alpha_A e^\beta_B$$

where σ^{AB} is the inverse of $\sigma_{AB} = g_{\alpha\beta} e^\alpha_A e^\beta_B$. As before, the vectorial two-surface element is $d\Sigma_\alpha = -\xi_\alpha dS dv$ (we have only changed variable names) and the two-dimensional surface element on a cross section of constant v is $dS_{\alpha\beta} = 2\left[\xi_\alpha N_\beta - N_\alpha \xi_\beta\right] dS$. We will denote this cross-sectional horizon by $\mathcal{H}(v)$.

Finally, once we contract the Einstein Field Equations with null vectors, we can write the Raychaudhuri equation as [7][8][11]:

$$\frac{d\theta}{dv} = \kappa\theta - \frac{1}{2}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - 8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta. \tag{117}$$

We are now in a position to formulate the laws of black hole mechanics.

4.3 The Zeroth Law

The surface gravity of a stationary black hole is uniform over the entire event horizon.

The zeroth law tacitly identifies surface gravity with the temperature of a thermodynamic object. Just like a warm solid at equilibrium is equally hot no matter where one touches it, the surface gravity of a black hole does not vary across the black hole. In order to prove the zeroth law, we need to prove

1. κ is constant along the null geodesics which generate the horizon (it does not depend on the variable by which the generators are parameterized).

2. κ does not vary from generator to generator.

To begin, use the expression [7][11]

$$\kappa^2 = -\frac{1}{2}\nabla^\beta \xi^\alpha \nabla_\beta \xi_\alpha \tag{118}$$

and the identity

$$\nabla_\mu \nabla_\nu \xi_\alpha = R_{\alpha\mu\nu\beta}\xi^\beta \tag{119}$$

which is valid for any killing vector. Now differentiate (46) in the directions tangent to the horizon to get:

$$\partial_\alpha \kappa^2 = 2\kappa \partial_\alpha \kappa = -\nabla^\nu \xi^\mu R_{\mu\nu\alpha\beta}\xi^\beta. \tag{120}$$

Contracting this with ξ^α leads us to $\partial_\alpha \kappa \xi^\alpha = 0$ which demonstrates that the surface gravity is constant on each generator of the black hole horizon, proving 1. To investigate whether or not κ varies from generator to generator, we must now contract (120) with the vectors which tell us about displacements from one generator to another. Doing this yields

$$2\kappa \partial_\alpha \kappa e^\alpha_A = -\nabla^\nu \xi^\mu R_{\mu\nu\alpha\beta}\xi^\beta e^\alpha_A$$

and by showing that this is equal to zero, we will prove 2. To do this, assume that the event horizon is geodesically complete, so it contains a bifurcation two-sphere on which $\xi^\alpha = 0$, this immediately tells us that $\partial_\alpha \kappa e^\alpha_A = 0$ on the sphere. Since $\partial_\alpha \kappa e^\alpha_A$ is constant on the null generators, we immediately have that $\partial_\alpha \kappa e^\alpha_A = 0$ on all cross sections of $v = constant$ on the horizon thus proving that κ is uniform over the entire event horizon [7][11].

How important is assuming the existence of a bifurcation two-sphere? Consider two stationary black holes, identical in every respect in the future of $v = 0$ but different in the past in such a way that only one of them has a bifurcation two-sphere. Our proof applies to the first black hole, but not the other. However, since the spacetimes are identical for $v > 0$, the proof must work for the second black hole regardless of geodesic completeness. This establishes that the zeroth law holds for all stationary black holes.

4.4 The First Law

The changes in mass, angular momentum and surface area (discounting the charge of the black hole) of the black hole are related by

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J \tag{121}$$

4.4.1 The Generalised Smarr Formula

Before we can prove this, we need to generalize Smarr's Formula which relates the mass of the black hole to its angular momentum, angular velocity and surface area. Consider the Komar expressions for the total mass and angular momentum [7][11]

$$M = -\frac{1}{8\pi}\oint_S \nabla^\alpha t^\beta dS_{\alpha\beta} \tag{122}$$

$$J = \frac{1}{16\pi}\oint_S \nabla^\alpha \phi^\beta dS_{\alpha\beta} \tag{123}$$

where the integration occurs over a closed two-surface at infinity. Now consider a spacelike hypersurface Σ extending from the event horizon to spatial infinity. The inner boundary of Σ we will call $\mathcal{H}$, it is a two-dimensional cross-section of the event horizon. The outer boundary of Σ is S, the place where the integration occurs. Now use Gauss' Theorem without the inner boundary. The results tell us that M and J can be expressed as

$$M = M_H + 2\int_\Sigma \left(T_{\alpha\beta} - \frac{1}{2}T g_{\alpha\beta}\right) n^\alpha t^\beta \sqrt{h} d^3y \tag{124}$$

and

$$J = J_H - 2\int_{\Sigma}\left(T_{\alpha\beta} - \frac{1}{2}Tg_{\alpha\beta}\right)n^{\alpha}\phi^{\beta}\sqrt{h}d^3y \tag{125}$$

where M_H and J_H are the mass and angular momentum of the black hole. They are given by expressions (50) and (51) with the integration over $\mathcal{H}$:

$$M = -\frac{1}{8\pi}\oint_{\mathcal{H}} \nabla^{\alpha}t^{\beta}dS_{\alpha\beta} \tag{126}$$

$$J = \frac{1}{16\pi}\oint_{\mathcal{H}} \nabla^{\alpha}\phi^{\beta}dS_{\alpha\beta} \tag{127}$$

The interpretation of (124) and (125) is fairly straightforward, the equations demonstrate that the total mass and the total angular momentum are given by the black hole plus the matter distribution outside the black hole. Now let's use $dS_{\alpha\beta} = 2\left[\xi_{\alpha}N_{\beta} - N_{\alpha}\xi_{\beta}\right]dS$ with expressions (126) and (127) to derive Smarr's formula [7][11]:

$$\begin{aligned}
M_H - 2\Omega_H J_H &= -\frac{1}{8\pi}\oint_{\mathcal{H}} \nabla^{\alpha}\left(t^{\beta} + \Omega_H\phi^{\beta}\right)dS_{\alpha\beta} \\
&= -\frac{1}{8\pi}\oint_{\mathcal{H}} \nabla^{\alpha}\xi^{\beta}dS_{\alpha\beta} \\
&= -\frac{1}{8\pi}\oint_{\mathcal{H}} \xi_{\alpha}\nabla^{\alpha}\xi^{\beta}N_{\beta}dS \\
&= -\frac{1}{4\pi}\oint_{\mathcal{H}} \kappa\xi^{\beta}N_{\beta}dS \\
&= \frac{\kappa}{4\pi}\oint_{\mathcal{H}} dS.
\end{aligned}$$

Note that we have used the Zeroth Law in the last step in order to take κ outside of the integral. This finally gives us [7][11][14]

$$M_H = 2\Omega_H J_H + \frac{\kappa A}{4\pi} \tag{128}$$

which is the generalised Smarr Formula.

4.4.2 Proof of the First Law

Consider a black hole initially in a stationary state. Let's drop a small amount of matter into the black hole so that the mass and angular momentum change slightly. M and J will change according to

$$\delta M = -\int_H T^\alpha_\beta t^\beta d\Sigma_\alpha$$

and

$$\delta J = \int_H T^\alpha_\beta \phi^\beta d\Sigma_\alpha$$

where H is the event horizon. We are only going to work to the first order in the perturbation of $T_{\alpha\beta}$ while keeping t^α, ϕ^α and $d\Sigma_\alpha$at their unperturbed values. Once the process of adding matter to the hole is over, the hole will return to another stationary state which is also a solution of the Einstein Equations. This process is, therefore, a way of moving from one solution of the Einstein Equations to another solution albeit with a different set of parameters.

Let's now substitute $d\Sigma_\alpha = -\xi_\alpha dSdv$ into our expressions for the changes in M and J:

$$\delta M - \Omega_H \delta J = -\int_H T^\alpha_\beta \left(t^\beta + \Omega_H \phi^\beta\right) \xi^\alpha dSdv \tag{129}$$

$$= \int dv \oint_{\mathcal{H}} T_{\beta\alpha} \xi^\beta \xi^\alpha dS. \tag{130}$$

We can now use Raychaudhuri's equation to work out the integral in (58). First we will neglect the quadratic terms since they are higher-order terms and we are only working to first order. We therefore use

$$\frac{d\theta}{dv} = \kappa\theta - 8\pi T_{\alpha\beta}\xi^\alpha \xi^\beta$$

which will give us

$$\delta M - \Omega_H \delta J = -\frac{1}{8\pi}\int dv \oint_{\mathcal{H}} \left(\frac{d\theta}{dv} - \kappa\theta\right) dS$$

$$= -\frac{1}{8\pi}\oint_{\mathcal{H}} \theta dS\Big|_{-\infty}^{\infty} + \frac{\kappa}{8\pi}\int dv \oint_{\mathcal{H}} \theta dS.$$

Before the perturbation, the black hole was stationary so the expansion term θ will vanish. The same is true for after the perturbation, therefore the first term will vanish. Using the definition of θ as the fractional rate of change of area, we get [7][11][14]

$$\begin{aligned}\delta M - \Omega_H \delta J &= \frac{\kappa}{8\pi} \int dv \oint_{\mathcal{H}} \left(\frac{1}{dS} \frac{d}{dv} dS \right) dS \\ &= \frac{\kappa}{8\pi} \oint_{\mathcal{H}} dS \\ &= \frac{\kappa}{8\pi} \delta A\end{aligned}$$

which establishes the first law of black hole mechanics.

4.4.3 A Brief Comparison to the Thermodynamics of Gases

We can draw an analogy between what we have just done and the treatment of an ideal gas. Given an ideal gas, we can say that it satisfies the equation:

$$p_1 V_1 = nRT_1.$$

Let's now change the volume of the container containing the gas according to the first law of thermodynamics:

$$dU = TdS - PdV.$$

We know that while the walls of the container are moving that the system will not be in equilibrium, but we assume that since the change is infinitesimal and occurs extremely slowly we can ignore the non-equilibrium processes and use equilibrium thermodynamics. Once the change in the container has been made, we will still have an ideal gas and it will therefore still satisfy the ideal gas equation

$$p_2 V_2 = nRT_2$$

albeit with different values for the variables. This is the same process we are describing for a black hole. Instead of moving from one solution of the ideal gas equation to another, we are moving from one solution of the Einstein Equations to another. Just like for the gas, we are not effecting a large change; the change will be small enough for it to be adequately described by only considering the first order in $T_{\alpha\beta}$. Like in the case of the gas, we will not pursue a detailed description of the black hole while the change is occurring, we are interested only in the initial and final states.

4.5 The Second Law of Black Hole Mechanics

The surface area of a black hole can never decrease.

By stating the second law of black hole mechanics, we are saying that the quantity which plays the role of entropy is the surface area of a black hole, the constant increase in area is therefore the same thing as the constant increase of entropy in thermodynamics. To prove the second law, consider the focusing theorem [7][11][14]

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - R_{\alpha\beta}k^\alpha k^\beta \leq 0 \tag{131}$$

which is a consequence of the Raychaudhuri equation applied to hypersurface-orthogonal geodesics. In addition, we have that $R_{\alpha\beta}k^{\alpha\beta} \geq 0$ by assuming the null energy condition and the Einstein equations. Upon integration of ${}^{d\theta}/_{d\lambda} \leq -\frac{1}{2}\theta^2$ we get

$$\frac{1}{\theta}(\lambda) \geq \frac{1}{\theta_0} + \frac{\lambda}{2} \tag{132}$$

which tells us that is the congruence is initially converging, then $\theta(\lambda) \to -\infty$ within an affine parameter value of $\lambda \leq {}^2/_{|\theta_0|}$ which signals the occurrence of a caustic. Since the event horizon is generated by null geodesics with no future end points, the generators can never run into caustics. The expansion of the generators must therefore be zero or positive everywhere on the horizon otherwise a contradiction will occur. The fact that θ can never be negative for the null geodesics generating the horizon proves the second law.

4.6 The Third Law of Black Hole Mechanics

If the stress-energy tensor is bounded and satisfies the weak energy condition, the surface gravity of a black hole cannot be reduced to zero in a finite advanced time.

Recall that a black hole of zero surface gravity is an extreme black hole, therefore if $T_{\alpha\beta}$ satisfies the stated conditions the third law can also be interpreted as saying that a black hole cannot become extremal in a finite advanced time. To demonstrate that the third law is a consequence of the condition that for every timelike vector field the matter density seen by the observer on that geodesic$\left(T_{\alpha\beta}k^{\alpha}k^{\beta}\right)$ will be positive, we need to specify a spacetime containing a black hole which has the potential to become extremal at some finite advanced time v. We will choose the charged generalization of the metric [7][11][14]

$$ds^2 = -fdv^2 + 2dvdr + r^2 + r^2 d\Omega^2 \tag{133}$$

where

$$f = 1 - \frac{2m(v)}{r} + \frac{q^2(v)}{r^2}.$$

This metric describes a black hole which has a mass and charge that change with time due to irradiation by dust. To see this, consider the stress-energy tensor

$$T^{\alpha\beta} = T^{\alpha\beta}_{dust} + T^{\alpha\beta}_{em}$$

where

$$T^{\alpha\beta}_{dust} = \rho l^{\alpha} l^{\beta},$$

and

$$\rho = \frac{1}{4\pi r^2}\frac{\partial}{\partial v}\left(m - \frac{q^2}{2r}\right)$$

are the contribution from the dust. The contribution from the electromagnetic field is:

$$T_{em}{}^{\alpha}{}_{\beta} = diag\left(-P, -P, P, P\right), \qquad P = \frac{q^2}{8\pi r^4}. \tag{134}$$

This spacetime will become extremal when $m(v_0) = q(v_0)$ and will produce a violation of the third law if this occurs at some finite value of the advanced time v_0. Suppose our observe is restricted to move in the radial direction only, then $T_{\alpha\beta}u^{\alpha}u^{\beta} = \rho\left(\frac{dv}{d\tau}\right)^2 + P$. Since $\frac{dv}{d\tau}$ is allowed to be arbitrarily large, the weak energy condition requires $\rho > 0$ at the apparent horizon, $r = r_+$ where $r_+ = m + \left(m^2 - q^2\right)^{1/2}$. This implies the following condition:

$$4\pi r_+^3 \rho(r_+) = m\dot{m} - q\dot{q} + \left(m^2 - q^2\right)^{1/2} \dot{m} > 0 \qquad (135)$$

where the overdot implies differentiation with respect to v. Now imagine that the black hole has become extremal at some finite advanced time v_0. This gives us $\Delta(v_0) = 0$ where $\Delta v = m(v) - q(v)$. Since the black hole was not extremal before $v = v_0$ we must have $\Delta(v) > 0$ for $v < v_0$ and $\Delta(v)$ must be decreasing as v approaches v_0. But by (63) we have

$$m(v_0)\,\dot{\Delta}(v_0) > 0$$

which tells us that $\Delta(v)$ must actually be increasing. This contradiction tells us that the weak energy condition prevents the black hole from becoming extremal within a finite advanced time.

4.7 Black Holes and Entropy

Let's perform an incredibly simple thought experiment. Take a container filled with some fluid and dump the contents into a black hole. Right before the fluid is dumped into the hole, we can calculate its entropy. After it is dumped into the black hole, the entropy of the fluid has vanished behind the event horizon. It appears that the black hole has managed to erase the entropy of the fluid from our universe, implying that if we wanted to, we could lower the entropy of the universe simply by throwing matter into black holes.

However, when we throw the matter into the black hole the area of the horizon will increase. Since we have identified horizon area with the entropy of a black hole, it appears that the total entropy of the universe does not decrease. Let's therefore write down the generalised second law of thermodynamics [34]

$$\Delta S_{bh} + \Delta S_{rest\ of\ universe} \geq 0 \qquad (136)$$

which tells us that if we take the entropy of matter together with what we conjecture to be the entropy of a black hole, we will still have a net entropy increase. Black holes therefore cannot reduce the amount of entropy in the universe.

4.8 How Hot is a Black Hole?

A classical black hole, by definition, emits nothing. If nothing escapes, then there can be no thermal emission. If there is no thermal emission, there can be no way to heat up a black hole. If we can't heat a black hole, then it doesn't have a temperature and this renders our attempts to treat a black hole as a thermal object quite pointless. Even if we concede that area measures the entropy of a black hole, the coefficient of entropy in the first law of black hole mechanics is a surface gravity, not a temperature. Unless we can somehow demonstrate that a black hole looks hot (like any other thermodynamic system) then we must accept that the equations we have look like thermodynamic equations but have no deeper connection to thermodynamics.

In order to prove that a black hole looks hot, we need to turn to quantum field theory.

4.8.1 Quantum Field Theory in Curved Spacetime

We can fix the usual Klein-Gordon Lagrangian by inserting a metric determinant to make sure that we are always integrating over the correct region of spacetime and by replacing ordinary derivatives with covariant derivatives. This will give us [2][13]:

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right]. \tag{137}$$

It's possible to expand the field ϕ into a sum of wave modes

$$\phi(x) = \sum_i \left[a_i u_i(x) + a_i^\dagger u_i^*(x)\right]$$

just like we can do in Minkowski space. So far, we have not said anything new. However, let's consider the inner product of a field ψ living in Minkowski space. We define the scalar product

$$(\psi_1, \psi_2) = -i\int \left\{\psi_1(x)\partial_t\psi_2^*(x) - \left[\partial_t\psi_1(x)\right]\psi_2^*(x)\right\} d^{n-1}x \tag{138}$$

$$= -i\int_t \psi_1(x)\overleftrightarrow{\partial_t}\psi_2^*(x) d^{n-1}x$$

where the subscript t denotes a spacelike hyperplane of simultaneity at some instant t (a Cauchy surface). This inner product ensures that the modes are orthogonal:

$$(u_k, u_{k'}) = 0, \quad k \neq k'.$$

Let's now invoke the principle the general covariance, relabel the coordinates in our spacetime and expand our field in terms of new modes

$$\phi(x) = \sum_j \left[\overline{a}_j \bar{u}_j(x) + \bar{a}_j^\dagger \bar{u}_j^*(x)\right]$$

which, like before, will have a vacuum state

$$\overline{a}_j \left|0\right\rangle_n = 0$$

where the n stands for 'new'. The field itself must be completely described by either set of modes, so let's expand the new modes in terms of the old,

$$\bar{u}_j = \sum_i \left[\alpha_{ji} u_i + \beta_{ji} u_i^*\right]$$

conversely we have:

$$u_i = \sum_j \left[\alpha_{ji}^* \overline{u}_j - \beta_{ji} \overline{u}_j^*\right].$$

These relationships are known as Bogolubov Transformations and they tell us how one field expansion is related to another. The matrices α_{ji} and β_{ji} are known as Bogolubov coefficients and we can evaluate them by evaluating the inner products of the field modes

$$\alpha_{ij} = (\overline{u}_i, u_j) \tag{139}$$

and

$$\beta_{ij} = -(\overline{u}_i, u_j^*). \tag{140}$$

Since we also know how to expand fields in terms of creation and annihilation operators, we can rearrange our expressions to express the Bogolubov matrices in terms of creation and annihilation operators. We have

$$a_i = \sum_j \left(\alpha_{ji} \bar{a}_j + \beta_{ji}^* \overline{a}_j^\dagger\right) \tag{141}$$

and

$$\overline{a}_j = \sum_i \left(\alpha_{ji}^* a_i - \beta_{ji}^* a_i^\dagger \right). \tag{142}$$

We have gone over the mathematics of how two mode expansions of the same field are related, but this seems like a purposeless exercise in algebra. Physically, the two equations tell us about the same field so whenever we try to calculate a number to be checked by an experiment, we must get the same answer regardless of what expansion we use. It might be tempting to say that if a field is in the ground state in one description, then it will be in the ground state in all descriptions. Consider, however, what happens when we try to count the number of u_i modes in the ground state of the second expansion [13]:

$$_n \langle 0| a_i^\dagger a_i |0\rangle_n = \sum_j \left|\beta_{ji}^2\right|. \tag{143}$$

This tells us that the vacuum of the second expansion contains u_i modes provided that the right hand side of (134) is not zero.

It seems intuitive that we will always have $\beta_{ji} = 0$, because all descriptions of the same field should agree on what the vacuum state is. To see that this is not the case, consider the Lie derivative of the u_k modes along some timelike vector v. If the u_j are positive frequency modes which satisfy [13]

$$\pounds_v u_j = -i\omega u_j \tag{144}$$

for $\omega > 0$, and the $\overline{u}_k$ modes are linear combinations of the u_j alone, (they then contain only positive frequencies with respect to the vector v) then we have that $\beta_{ij} = 0$. Thus

$$\overline{a}_j |0\rangle_n = 0$$

and

$$a_j |0\rangle_n = 0$$

so the vacuum will be the same for any expansion, either in u_j or $\overline{u}_j$. If this is not true and $\beta_{ji} \neq 0$, then the modes $\overline{u}_j$ will contain a mix of both positive and negative frequency u_j modes - particles will be present.

4.8.2 Hawking's Calculation

In (112), the coefficient of the area, A (and therefore the entropy S) is the surface gravity, κ. The comparison to make between regular thermodynamics and black hole thermodynamics is, therefore, that the role of temperature is played by the surface gravity of the black hole. Classically, any object with some temperature T, will emit particles at a rate proportional to that temperature. By turning to quantum field theory, we can show that a black hole emits particles in a way which depends on κ, demonstrating quite convincingly that surface gravity does play the role of temperature and supplying further evidence to the claim that a black hole is a thermodynamic object [35].

By varying (117), we can arrive at the Klein-Gordon equation for curved space time

$$\frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\right) - m^2\phi = 0 \tag{145}$$

which can be rewritten in terms of the covariant derivative as:

$$\left(\nabla^\mu\partial_\mu - m^2\right)\phi = 0. \tag{146}$$

Now let's define a new inner product for a field ψ in a similar way to (138),

$$(\psi_i, \psi_j) = \int_\Sigma d\Sigma^\mu\sqrt{g_\Sigma}\left[\psi_i^*\nabla_\mu\psi_j - \psi_j\nabla_\mu\psi_i^*\right] \tag{147}$$

where Σ denotes a Cauchy surface and g_Σ is the induced metric on that hypersurface. Just like before, we can specify two different orthogonal bases, $\{\psi_i\}$ and $\{\psi_i'\}$ and write the two in terms of each other by using Bogoliubov coefficients:

$$\psi_i' = \Sigma_j\left[\alpha_{ij}\psi_j + \beta_{ij}\psi_j^*\right]. \tag{148}$$

By using the fact the the complex bases satisfy

$$(\psi_i, \psi_j) = -\left(\psi_i^*, \psi_j^*\right) = \delta_{ij} \tag{149}$$

and

$$\left(\psi_i^*, \psi_j\right) = \left(\psi_i, \psi_j^* = 0\right), \tag{150}$$

we can use (147) to show that the Bogoliubov matrices satisfy:

$$\alpha\alpha^{\dagger} - \beta\beta^{\dagger} = 1 \tag{151}$$

$$\alpha\beta^{T} - \beta\alpha^{T} = 0. \tag{152}$$

Inserting (151) and (152) into (147), we can calculate the inverse of (148) as:

$$\psi_i' = \Sigma_j \left[\alpha_{ij}\psi_j + \beta_{ij}\psi_j^*\right]$$

$$\psi_i' = \Sigma_j \left[\alpha_{ij}\Sigma_k\left(\alpha_{jk}'\psi_k' + \beta_{jk}'\psi'^*_k\right) + \beta_{ij}\Sigma_k\left(\alpha_{jk}'\psi'^*_k + \beta_{jk}'\psi_k'\right)\right]$$

$$= \Sigma_{j,k}\left[\left(\alpha_{ij}\alpha_{jk}' + \beta_{ij}\beta_{jk}'\right)\psi_k' + \left(\alpha_{ij}\beta_{jk}' + \beta_{ij}\alpha_{jk}'\right)\psi'^*_k\right] \tag{153}$$

which gives us the extra conditions

$$\alpha\alpha' + \beta\beta'^* = 1 \tag{154}$$

$$\alpha\beta' - \beta\alpha'^* = 0. \tag{155}$$

Comparing (154) and (155) with (151) and (152), we can see that our conditions are satisfied uniquely when $\alpha' = \alpha^{\dagger}$ and $\beta' = -\beta^{T}$. Thus giving us the further conditions [13]:

$$\alpha^{\dagger}\alpha - \beta^{T}\beta^* = 1 \tag{156}$$

$$\alpha^{\dagger}\beta - \beta^{T}\alpha^* = 0. \tag{157}$$

To prove that particles are created during gravitational collapse, we are going to study the behaviour of a massless Klein-Gordon field during the creation of a black hole. The general solution of the massless scalar wave equation are

$$r^{-1}R_{\omega l}(r)Y_{lm}(\theta, \phi)\, e^{-i\omega t} \tag{158}$$

where Y_{lm} are the usual spherical harmonics and the function $R_{\omega l}(r)$ obeys the differential equation:

$$\frac{d^2 R_{\omega l}}{dr*^2} + \left\{\omega^2 - \left[l(l+1)r^{-2} + 2Mr^{-3}\right]\left[1 - 2Mr^{-1}\right]\right\} R_{\omega l}. \tag{159}$$

Note that for the purposes of simplification, we are using the variable $r* = r + 2M \ln\left|\frac{r}{2M} - 1\right|$. Let's now consider the regions for which r is very large; these regions correspond to past and future null infinity $\mathcal{I}_-$ and $\mathcal{I}_+$. The function $R_{\omega l}$ will then reduce to

$$\frac{d^2 R_{\omega l}}{dr*^2} + \omega^2 R_{\omega l} = 0 \tag{160}$$

which shows us that far from the black hole, this function behaves like:

$$R_{\omega l} \sim e^{\pm i\omega r*}. \tag{161}$$

Thus, in the far past and the far future, our massless solutions look like:

$$r^{-1} Y_{lm}(\theta, \phi)\, e^{-i\omega t \pm i\omega r*}. \tag{162}$$

Now make the substitutions $v = t + r*$ and $u = t - r*$ so that we can arrive at purely ingoing and outgoing solutions f_ω and p_ω:

$$f_\omega \sim \left(r\sqrt{4\pi\omega}\right)^{-1} Y_{lm}(\theta, \phi)\, e^{-i\omega v} \tag{163}$$

$$p_\omega \sim \left(r\sqrt{4\pi\omega}\right)^{-1} Y_{lm}(\theta, \phi)\, e^{-i\omega u}. \tag{164}$$

Since we have not put our black hole into a confined space, our solutions obey the normalization conditions [13]:

$$(p_\omega, p_{\omega'}) = (f_\omega, f_{\omega'}) = \delta(\omega - \omega'). \tag{165}$$

To see how this condition gives us the normalization factor in front of (163) and (164), let's compute the inner product

$$(f_{\omega lm}, f_{\omega' l' m'}) = \int_\Sigma d\Sigma^\mu \sqrt{g_\Sigma}\left[f^*_{\omega lm}\partial_\mu f_{\omega' l' m'} - f_{\omega' l' m'}\partial_\mu f^*_{\omega lm}\right] \tag{166}$$

by noting that the f_ω solve the KG equation at $\mathcal{I}_-$, where $d\Sigma^\mu = drd\phi d\theta\sqrt{-\left(1 - \frac{2M}{r}\right)}$. This will give us:

$$(f_{\omega lm}, f_{\omega' l' m'}) = i \int_r \int_0^{2\pi} \int_0^{\pi} dr d\phi d\theta$$
$$\left[r^2 sin\theta \tfrac{1}{r^2} \tfrac{1}{4\pi\sqrt{\omega\omega'}} Y *_{l'm'} Y_{lm} \left(e^{i\omega(t+r)} \partial_t e^{-i\omega'(t+r)} - e^{i\omega'(t+r)} \partial_t e^{-i\omega(t+r)} \right) \right]$$

$$(f_{\omega lm}, f_{\omega' l' m'}) = \delta_{ll'} \delta_{mm'} \frac{2\omega}{4\pi\omega} 2\pi\delta \left(\omega' - \omega\right). \tag{167}$$

We are now going to study the behaviour of a wave as it moves from $\mathcal{I}_-$, travels through the collapsing spherical mass and on to $\mathcal{I}_+$. Since the collapsing mass will eventually form a black hole, some ingoing rays will not be able to escape and move on to future null infinity. We will therefore study a ray which does escape, but is infinitesimally close to a wave which goes on to form the event horizon of the black hole. Call the ray which goes on to form the event horizon γ and call the ray under consideration α; we will say that the two rays are separated by a distance ϵ, measured along a future-directed null vector n^a, where $n^a l_a = -1$ and l^a is a null vector tangent to γ. Begin the calculation by writing the metric in Kruskal coordinates and setting $d\Omega = 0$, which spherical symmetry enables us to do. Our line element is then [13]

$$ds^2 = \frac{2M}{r} e^{-\frac{r}{2M}} d\tilde{u} d\tilde{v} \tag{168}$$

and all null rays must have either

$$\frac{d\tilde{u}}{d\lambda} = 0 \tag{169}$$

or

$$\frac{d\tilde{v}}{d\lambda} = 0 \tag{170}$$

where λ is any affine parameter. These conditions tell us that for any ingoing null ray, the affine parameter along the curve will always be proportional to u :

$$\lambda = C\tilde{u} = -4MCe^{-\frac{u}{4M}}. \tag{171}$$

Make the choice that the affine parameter goes to 0 on the event horizon. When the null rays intersect γ, we will have $\lambda = -\epsilon$ and therefore $u = 4M\left(\ln 4MC - \ln \epsilon\right)$. Let's now find an expression for ϵ in terms of v. To do this, first say that the time at which α left $\mathcal{I}_-$ is v_0. Any ray that does not fall into the black hole, will therefore have left $\mathcal{I}_-$ some time after v_0 and the distance between α and the ray which continues on to $\mathcal{I}_+$ will be $-\epsilon$. Writing the metric far from the hole in terms of u and v gives us

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dudv \tag{172}$$

from which it follows that the affine parameter is proportional to v and therefore $-\epsilon = D\left(v - v_0\right)$. Combining this with our result for u, we see that the outgoing wavefunction at $\mathcal{I}_+$ will have the form

$$p_\omega(v) \sim \left(\sqrt{4\pi\omega}\right)^{-1} exp\left[-i\omega 4M\left(\ln 4MC - \ln D(v - v_0)\right)\right] \tag{173}$$

when $v < v_0$. We now have all we need to find the Bogoliubov coefficients and therefore count particle numbers at $\mathcal{I}_+$ and $\mathcal{I}_-$.

To find the Bogoliubov coefficients, we write the modes p_ω on the basis $\{f_\omega\}$. Begin by taking the Fourier transform:

$$\tilde{p}_\omega(\omega) = \frac{1}{2\pi}\int dv \left[p_\omega(v)e^{i\omega' v}\right]. \tag{174}$$

It can be shown that by plugging (163) into (174), we can arrive at the inverse Fourier transform [13]

$$p_\omega(v) = \int_0^\infty d\omega' \left[\sqrt{\pi\omega'}\tilde{p}_\omega\left(\omega'\right) f_{\omega'}\left(v\right) + \sqrt{\pi\omega'}\tilde{p}_\omega\left(-\omega'\right) f_{\omega'} * \left(v\right)\right], \tag{175}$$

which we can compare to the expression for the Bogoliubov coefficients:

$$p_\omega = \int_0^\infty d\omega' \left[A_{\omega\omega'} f_{\omega'} + B_{\omega\omega'} f_{\omega'} *\right]. \tag{176}$$

Upon inspection, we see that the Bogoliubov coefficients are

$$A_{\omega\omega'} = \sqrt{\pi\omega'}\tilde{p}_\omega(\omega') \tag{177}$$

and

$$B_{\omega\omega'} = \sqrt{\pi\omega'}\tilde{p}_\omega(-\omega'). \tag{178}$$

To calculate the transform in (164), we use the approximation in (163) to write:

$$\tilde{p}_\omega(\omega) \sim \frac{1}{4\pi\sqrt{\pi\omega}} \int\limits_{-\infty}^{v_0} exp\left[i\omega' v - i\omega 4M\left(\ln\frac{4MC}{D} - \ln\left[v_0 - v\right]\right)\right] dv. \quad (179)$$

In order to evaluate this integral, we will need to perform contour integration in the complex v plane. For $\omega' > 0$, we will close the contour in the upper half of the plane, leaving us with the integration along $v = v_0 + ix$:

$$\tilde{p}_\omega(\omega) \sim \frac{i}{4\pi\sqrt{\pi\omega}} \int_0^\infty exp\left[-\omega' x + i\omega' v_0 - i\omega 4M\left(\ln\frac{4MC}{D} - \ln\left(xe^{-i\pi/2}\right)\right)\right] dx$$

$$= \frac{i}{4\pi\sqrt{\pi\omega}} e^{2M\omega\pi} e^{i\omega' v_0} \left(\frac{4MC}{D}\right)^{-i\omega 4M} \Gamma\left(1 + i\omega 4M\right)\left(\omega'\right)^{-1-i\omega 4M}. \quad (180)$$

For $\omega' < 0$ we integrate as before, but this time close the contour in the lower half of the plane:

$$\tilde{p}_\omega(\omega) \sim \frac{-i}{4\pi\sqrt{\pi\omega}} \int_0^\infty exp\left[\omega' x + i\omega' v_0 - i\omega 4M\left(\ln\frac{4MC}{D} - \ln\left(xe^{-i\pi/2}\right)\right)\right] dx$$

$$= \frac{-i}{4\pi\sqrt{\pi\omega}} e^{2M\omega\pi} e^{i\omega' v_0} \left(\frac{4MC}{D}\right)^{-i\omega 4M} \int_0^\infty e^{\omega' x} x^{i\omega 4M} dx. \quad (181)$$

Comparison of (171) and (170) leads us to conclude that

$$\tilde{p}_\omega(-\omega') = -e^{-4M\omega\pi} e^{-2i\omega' v_0} \tilde{p}_\omega(\omega') \quad for\ \omega' > 0 \quad (182)$$

and we can therefore compare (167), (168), (170) and (172) in order to arrive at the Bogoliubov coefficients

$$A_{\omega\omega'} = \frac{i}{4\pi\sqrt{\omega\omega'}} e^{2M\omega\pi} e^{i\omega' v_0} \left(\frac{4MC}{D}\right)^{-i\omega 4M} \Gamma\left(1 + i\omega 4M\right)^{-i\omega 4M} \quad (183)$$

$$B_{\omega\omega'} = -e^{-4M\omega\pi} e^{-2i\omega' v_0} A_{\omega\omega'}. \quad (184)$$

Let's suppose we now try count the number of modes by using the $B_{\omega\omega'}$ coefficient. To do this, take the continuous analogue of (133), which is the integral:

$$\int_0^\infty |B_{\omega\omega'}|^2 \, d\omega'. \tag{185}$$

Unfortunately, this integral diverges as $(\omega')^{-1/2}$ when ω' gets large, indicating that as the frequency increases, we will eventually end up finding an infinite number of particles. This divergence is due to the fact that we did not impose a restriction limiting the total amount of energy in spacetime to the mass of the black hole. To remedy this, we shouldn't just calculate (175), but rather construct finite wavepackets and calculate the number of particles per frequency interval. This is what Hawking did in his original paper [35].

We are going to perform a simpler calculation and confine the system to a box in order to impose periodic boundary conditions and therefore discretize the modes. We know that the Bogoliubov coefficients satisfy

$$\Sigma_{\omega'} |A_{\omega\omega'}|^2 - \Sigma_{\omega'} |B_{\omega\omega'}|^2 = 1$$

and

$$\Sigma_{\omega'} |B_{\omega\omega'}|^2 = e^{-8M\omega\pi} \Sigma_{\omega'} \Sigma_{\omega'} |A_{\omega\omega'}|^2 .$$

Taken together, these expressions yield

$$\Sigma_{\omega'} |A_{\omega\omega'}|^2 - |B_{\omega\omega'}|^2 = \left(e^{8M\omega\pi} - 1\right) \Sigma_{\omega'} |B_{\omega\omega'}|^2 = 1 \tag{186}$$

and the number of particles counted by the Bogoliubov coefficient $B_{\omega\omega'}$ therefore goes as

$$\Sigma_{\omega'} |B_{\omega\omega'}|^2 = \frac{1}{(e^{8M\omega\pi} - 1)}. \tag{187}$$

The particle flux through a sphere surrounding the black hole is then

$$\Phi = \frac{d\omega}{2\pi} \left(\frac{1}{e^{8M\omega\pi} - 1} \right) \tag{188}$$

and upon comparing this to the Planck spectrum, we realise that the particle flux resembles that of a black body, with a temperature of

$$T_H = \frac{1}{8M\pi}. \tag{189}$$

Given that the surface gravity of a Schwarzschild black hole is given by $\kappa = 1/4M$, we see that the black hole's temperature is given by

$$T_H = \frac{\kappa}{4\pi} \tag{190}$$

which is exactly what one might guess from studying black hole thermodynamics.

Hawking's calculation has one more piece of information to tell us. Write the change of the mass of the black hole as

$$\begin{aligned} \delta M &= \frac{1}{4\,(8M\pi)}\delta A \\ &= \frac{1}{4}T_H \delta A \end{aligned} \tag{191}$$

which allows us to fix a definite constant of proportionality into the entropy-area relationship [13][35] (where we set $\hbar = 1$):

$$S = \frac{1}{4}A. \tag{192}$$

It's important that we go over some of the qualitative aspects of Hawking's calculation in order to understand its relevance and some of the assumptions that went into it.

4.9 Interpreting The Laws of Black Hole Mechanics

We know that when doing regular thermodynamics that the laws with which we work are stunningly insensitive to the underlying microscopic physics.

Like in the case of spacetime as a rubber sheet, it is again tempting to jump to conclusions based on the form of the mathematics. Let's make the statement:

The laws of black hole mechanics have a form which is identical to that of the laws of thermodynamics describing the emergent properties (like pressure) of a gas. This identical form proves that the variables (like area) describing a black hole are the coarse-grained limit of some unknown theory of space and *time. Our black hole parameters are therefore not fundamental and we know that the geometry of spacetime is not a fundamental theory.*

The most obvious problem with the above is that the form of a mathematical equation being identical to the form of another is not in any way a proof of the two equations describing the same phenomenon. The mathematics is form without content, the physics must provide the content and it is not immediately obvious that the physical concepts underlying black hole mechanics and thermodynamics are the same.

Part II

Einstein's Equations as Emergent Physics

5 The Rindler Chart and Causal Horizons

5.1 An Accelerated Coordinate System

Before we move on to studying what a quantum field looks like to an accelerated observer, we are going to develop a natural coordinate system for an observer accelerating in Minkowski space. Note that we are now working in units where $c = 1$.

Set up the usual Minkowski metric [8]:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{193}$$

Suppose that the observer is accelerating in a straight line in our Minkowski space. Without loss of generality, we will simplify the situation by rotating our coordinate system so that observer moves solely along the x axis. This means that we need only transform the coordinates x and t, we can leave the others unchanged. The observer will have relativistic energy

$$E = \sqrt{p^2 + m^2}$$

and relativistic momentum

$$\vec{p} = E\vec{v}.$$

Let's assume that we are initially traveling with the observer so it is in our rest frame. If its acceleration has a constant magnitude g, then after some coordinate time t, its relativistic energy will be:

$$E = \sqrt{(gmt)^2 + m^2}.$$

This allows us to write the velocity as

$$v_x = \frac{p_x}{E} = \frac{gmt}{\sqrt{m^2 + (gmt)^2}}$$

and to therefore find the accelerated detector's position by performing an integration:

$$\int v_x dt = \int \frac{gmt}{\sqrt{m^2 + (gmt)^2}} dt.$$

This will give us

$$x = \frac{1}{gm}\sqrt{m^2 + (gmt)^2}$$

$$x^2 - t^2 = g^{-2}$$

which is a hyperbola in our coordinate system. We can calculate the time elapsed on a clock carried by the observer in the usual way:

$$\tau = \int \sqrt{1 - v^2} dt$$

$$= \int \sqrt{\frac{m^2}{m^2 + (gmt)^2}} dt$$

$$= \frac{1}{g} sinh^{-1}(gt).$$

This allows us to write the equations for the trajectory in parametric form, parameterized by τ

$$gx = cosh(g\tau) \quad gt = sinh(g\tau) \tag{194}$$

with a velocity which has magnitude [8]:

$$v = tanh(g\tau). \tag{195}$$

Now let's consider the above from the observer's perspective. While we are labeling all the points in Minkowski space using t, x, y, z, the observer is going to be labeling the same points using the coordinate system variables T, X, y, z. Our labeling and that of the observer will differ in one key way, the observer will give himself constant coordinates in his own frame. Any object to which he assigns a *constant* value of (T, X) will be an object which is *accelerating* in the (t, x) coordinate system. If the trajectory of the observer is given in the form

$$x = f(\tau)$$

$$t = h(\tau)$$

it can be shown that the transform between our inertial (x, t) and the observer's (X, T) are related by

$$x - ct = f\left(T - \frac{X}{c}\right) - ch\left(T - \frac{X}{c}\right) \tag{196}$$

and

$$x + ct = f\left(T + \frac{X}{c}\right) + ch\left(T + \frac{X}{c}\right). \tag{197}$$

Combining (90), (92) and (93), we can conclude that

$$x - t = \frac{1}{g}e^{[-g(T-X)]} \quad x + t = \frac{1}{g}e^{[g(T+X)]} \tag{198}$$

which finally gives us

$$x = \frac{1}{g}e^{gX}cosh(gT) \tag{199}$$

and

$$t = \frac{1}{g}e^{gX}sinh(gT). \tag{200}$$

The equations (95) and (96) specify the coordinate transformation between ordinary Minkowski space and an accelerated coordinate system. We call this accelerated coordinate system the Rindler Chart [8].

It is important to note that even though the observer is labeling points in flat Minkowski space, his acceleration will cause him to label the space in such a way that it looks curved. When he computes the value of ds^2 for some small line element, he will not get the same answer as the inertial observer (or any inertial observer). To see this, write

$$dt^2 - dx^2 = e^{2gx}\left(dT^2 - dX^2\right)$$

therefore in the accelerated frame we will have:

$$ds^2 = -dt^2 + dx^2 = e^{2gx}(-dt^2 + dx^2).$$

A visual way to confirm that the Rindler Chart indeed represents an accelerated observer is to plot

$$-t^2 + x^2 = \frac{1}{g^2}e^{2gX} = constant$$

on a diagram of the 2D Minkowski plane. Notice that a single, constant value of X will give us a hyperbola; the path of an accelerating observer. Finally note that the Rindler Chart does not cover all of Minkowski space. It partitions the space into separate, causally disconnected, quadrants. The fact that the Rindler Chart does this indicates that an accelerating observer will set up causal horizons in Minkowski space.

5.2 The Unruh Temperature

In developing relativity, Einstein promoted the equivalence of gravitational and inertial mass to a fundamental principle of the theory rather than a measured result. If acceleration looks like gravitational attraction, then it might be true that accelerating observers will register a warm spacetime just like observers will measure black holes to have a non-zero temperature due to the Hawking effect [8][13][36]. Well, it is true. We demonstrate this now.

Let a massless scalar field live in the Minkowski spacetime shared by the inertial and accelerating observers. We, as inertial observers, will say that the wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\phi = 0 \tag{201}$$

has solutions

$$\overline{u}_k = (4\pi\omega)^{-\frac{1}{2}}\, e^{ikx - i\omega t} \tag{202}$$

where the modes have positive frequency with respect to the Killing vector ∂_t :

$$\pounds_{\partial_t}\overline{u}_k = -i\omega\overline{u}_k. \tag{203}$$

The modes which have $k > 0$ are right-moving along $u = constant$, while modes with $k < 0$ are left-moving along $v = constant$. To quantize the field ϕ, we follow the recipe outlined in Section 3.8.

Now take the same field and quantize using the Rindler observer's coordinates. He will say that the massless field obeys the wave equation

$$e^{2gX}\Box\phi = \left(\frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2}\right)\phi = 0 \tag{204}$$

with mode solutions

$$u_k = (4\pi\omega)^{-\frac{1}{2}}e^{ikX\pm i\omega T} \tag{205}$$

where $\omega = |k| > 0$. Notice that (205) includes a positive and a negative sign. The positive sign applies to the region marked L and the negative sign applies to the region marked R. Like before, these modes in the Rindler coordinates satisfy:

$$\pounds_{\pm\partial_T}u_k = -i\omega u_k.$$

Let's now split up the modes in (205) by defining

$$u_k^R \begin{cases} = (4\pi\omega)^{-\frac{1}{2}}e^{ikX-i\omega T} & in\ R \\ = 0 & in\ L \end{cases} \tag{206}$$

and

$$u_k^L \begin{cases} = (4\pi\omega)^{-\frac{1}{2}}e^{ikX+i\omega T} & in\ L \\ = 0 & in\ R. \end{cases} \tag{207}$$

By doing this, we have a set of wave modes which is complete over all of Minkowski space, meaning that the accelerating observer and the inertial observer can meaningfully talk about wave modes at any point in Minkowski space. Both observers can expand the field ϕ in terms of its modes and creation/annihilation operators. The inertial observer will write

$$\phi = \sum\left[a_k\overline{u}_k + a_k^\dagger\overline{u}_k^*\right] \tag{208}$$

while the Rindler observer will write

$$\phi = \sum \left[b_k^{(1)} u_k^L + b_k^{(1)\dagger} u_k^{L*} + b_k^{(2)} u_k^R + b_k^{(2)\dagger} u_k^{R*} \right]. \tag{209}$$

Let's now assume that the two field expansions will have two different vacuum states, $|0_M\rangle$ and $|0_R\rangle$ which satisfy

$$a_k |0_M\rangle = 0$$

and

$$b_k^{(1)} |0_R\rangle = b_k^{(2)} |0_R\rangle = 0.$$

We are now going to test whether or not this assumption is true by computing the Bogolubov transformation between the a and b coefficients.

In order to simplify the calculation, let's make some observations which will simplify the task. First note that the exponent in (206) and (207) changes in sign when we pass from L to R. This means that the functions do not change smoothly at this transition point, rendering them non-analytic. To fix this problem, define some new combinations,

$$u_k^R + e^{-\pi\omega/g} u_k^{L*} \tag{210}$$

and

$$u_{-k}^{R*} + e^{\pi\omega/g} u_k^L \tag{211}$$

which remain analytic when we cross over from L to R. Since the modes above satisfy (89) (i.e they are share the positive frequency analyticity properties of our $\overline{u}_k$), they will also have the vacuum state $|0_M\rangle$. So instead of expanding our field as we did in (205), let's expand it in terms of (210) and (211):

$$\begin{aligned}\phi = \sum \left(2sinh\left(\frac{\pi\omega}{g}\right)\right)^{-1/2} &\left[d_k^{(1)} \left(e^{\pi\omega/2g} u_k^R + e^{-\pi\omega/2g} u_{-k}^{L*}\right)\right. \\ &\left. + d_k^{(2)} \left(e^{-\pi\omega/2g} u_{-k}^{R*} + e^{\pi\omega/2g} u_k^L\right)\right] + h.c\end{aligned} \tag{212}$$

where

$$d_k^{(1)} |0_M\rangle = d_k^{(2)} |0_M\rangle = 0.$$

Now it will be easier to find the Bogolubov coefficients. We take the inner products (ϕ, u_k^R) and (ϕ, u_k^L) where ϕ will first be given by (106) and then (109). We then get

$$b_k^{(1)} = \left(2sinh\left(\frac{\pi\omega}{g}\right)\right)^{-1/2}\left[e^{\pi\omega/2g}d_k^{(2)} + e^{-\pi\omega/2g}d_{-k}^{(1)\dagger}\right] \tag{213}$$

and

$$b_k^{(2)} = \left(2sinh\left(\frac{\pi\omega}{g}\right)\right)^{-1/2}\left[e^{\pi\omega/2g}d_k^{(1)} + e^{-\pi\omega/2g}d_{-k}^{(2)\dagger}\right] \tag{214}$$

which provide the full translation between the vacuum states $|0_M\rangle$ and $|0_R\rangle$. Our Rindler observer will detect particles counted by the number operator $b_k^{(1)\dagger}b_k^{(1)}$. If we contract this number operator with the Minkowski vacuum $|0_M\rangle$ we will get

$$\langle 0_M|\, b_k^{(1)\dagger}b_k^{(1)}\, |0_M\rangle = \frac{e^{-\pi/\omega}}{2sinh\left(\frac{\pi\omega}{g}\right)} = \frac{1}{e^{\frac{2\pi\omega}{g}} - 1} \tag{215}$$

particles with wave number which is precisely the Planck spectrum for radiation at some temperature $T_0 = g/2\pi k_b$ [13][36].

The above result tells us, unambiguously, that simply by accelerating we will measure flat, empty Minkowski space as being 'hot'. After the idea of Hawking radiation, it appeared that the black hole itself was radiating particles, making spacetime look warm. In that case, it would be intuitively appealing to call the hot black hole the thermodynamic object. The above result stands in contrast to that conclusion. It is no longer a black hole which looks hot, it is all of flat, empty spacetime.

5.3 Temperature and Microstructure

We have now managed to prove that an observer accelerating through empty Minkowski space will think that the spacetime is intrinsically 'hot', just like an observer sitting outside a black hole. Classically, temperature tells us about the motion of the particles of a gas - if the particles are moving quickly, then the gas is hot. Slow particles will give us a low temperature. This shows us that the variable temperature is telling us something about the constituents of the gas - its microstructure. The fact that the Unruh and Hawking temperatures are non-zero strongly hint at some hidden spacetime microstructure that we do not yet know about.

6 Jacobson's Idea

Let's review what we have established so far. We have demonstrated that the laws which govern the behaviour of black holes bear a striking similarity to the laws of thermodynamics - laws which are known to describe emergent physics. We also know that an observer hovering above a black hole and an observer accelerating through flat space will both register a non-zero temperature and that for a black hole we can associate the horizon area with entropy. All of these ideas were not assumed, but were deduced from studying quantum field theory in curved space and the behaviour of the solutions to the Einstein equations [12].

We are now going to turn these ideas around and assume the existence of the entropy-area identity and use it with the Unruh temperature to do spacetime thermodynamics without any reference to the Einstein equations. This process will lead to the idea that the Einstein equations have a purely thermodynamic derivation - making them an equation of state, just like the ideal gas law.

6.1 The Einstein Equation of State

Pick a point p in spacetime and make the approximation that the space around p is, locally, Minkowskian. Now choose a small patch of 2-surface which contains p and call this patch O.

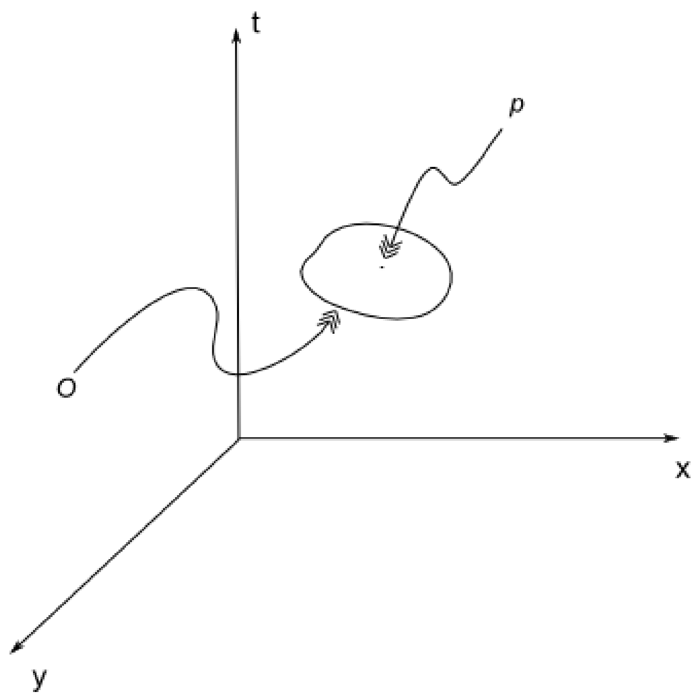

Figure 5: Minkowski space, with two dimensions suppressed, containing O and p.

We need to specify one side of the boundary of the past of O.

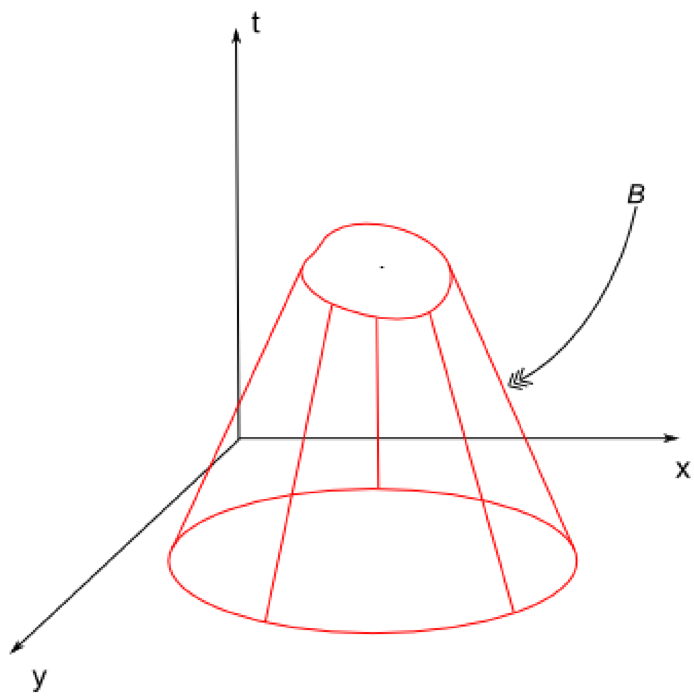

Figure 6: The boundary B of the past of the patch O.

Close to the point p, this boundary is a congruence of null geodesics orthogonal to O. This congruence constitutes the causal horizon with which we will work.

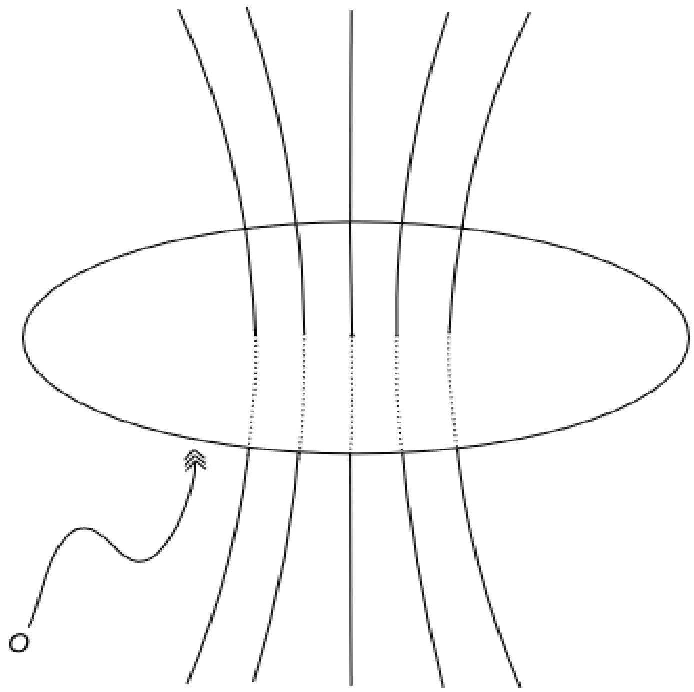

Figure 7: The null geodesics orthogonal to O. We can use the Raychaudhuri equation to study these geodesics and hence the behaviour of the the patch O.

We need to be judicious in our choice of patch as we require that the expansion and shear of the congruence vanish close to p. It is always possible to do this. We need not worry about about the rotation as this vanishes due to the fact that the congruence is normal to our 2-surface patch. By making this construction, we have actually defined a local Rindler horizon around p. There are local Rindler horizons in all null directions around a spacetime point due to the fact an observer can accelerate in any direction.

Since we have an approximately flat region of spacetime near our point p and around the patch O, the spacetime there will have all the usual Poincare symmetries. This means that we will always be able to find an approximate Killing field χ^a generating Lorentz boosts which are orthogonal to O and which vanish at O. Suppose now that we have some stress energy tensor T_{ab} defined in our spacetime. The flow of energy orthogonal to our little patch O will be given by $T_{ab}\chi^a$. So far we have not said anything terribly insightful. Our geometric construction has provided us with a small, approximately flat 2-surface in spacetime, a null congruence which is normal to that patch and some energy which is flowing across that patch.

Let's now choose χ^a to be future pointing to the inside past of our patch O. The energy flux to the past of the patch will then be

$$\delta Q = \int_H T_{ab} \chi^a d\Sigma^b \tag{216}$$

where the integral is over the generators of the inside past horizon of O.

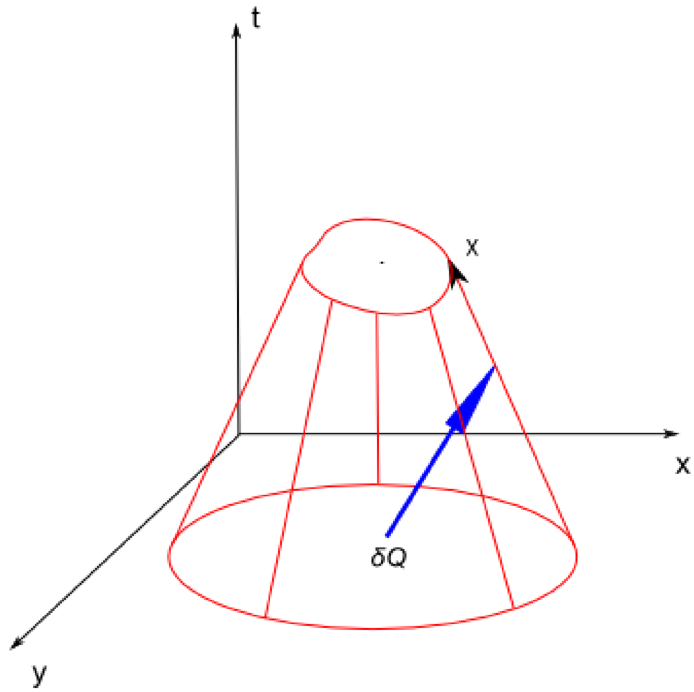

Figure 8: The heat flux across the boundary B. This picture illustrates that heat (in whatever form) is moving across the causal horizon B into an area of spacetime from which a null ray can reach the Rindler observer.

Call the tangent vector to the horizon generators k^a and let it be affinely parameterized by a parameter λ which vanishes at the patch O and is negative to the past of O. We can then say that

$$\chi^a = -\kappa \lambda k^a$$

and that the small patch of directed surface area is

$$d\Sigma^a = k^a d\lambda dA$$

where the dA is an infinitesimal piece of horizon. With all this in mind, we are ready to start doing spacetime thermodynamics. Let's rename the energy flux to heat flux, like can be done in ordinary thermodynamics, and write it as:

$$\delta Q = -\kappa \int_H \lambda T_{ab} k^a k^b d\lambda dA \tag{217}$$

Now let's use the entropy-area rule from black hole thermodynamics and say that the entropy of this heat flux is associated with a small change in the area of the horizon:

$$dS = \eta dA.$$

We have left the constant of proportionality undetermined for now and we will see that the rest of the derivation is insensitive to this. We have already developed the mathematics we need to study the behaviour of the horizon generators. A small patch of cross-sectional area of the null horizon generators is given by:

$$\delta A = \int_H \theta d\lambda dA. \tag{218}$$

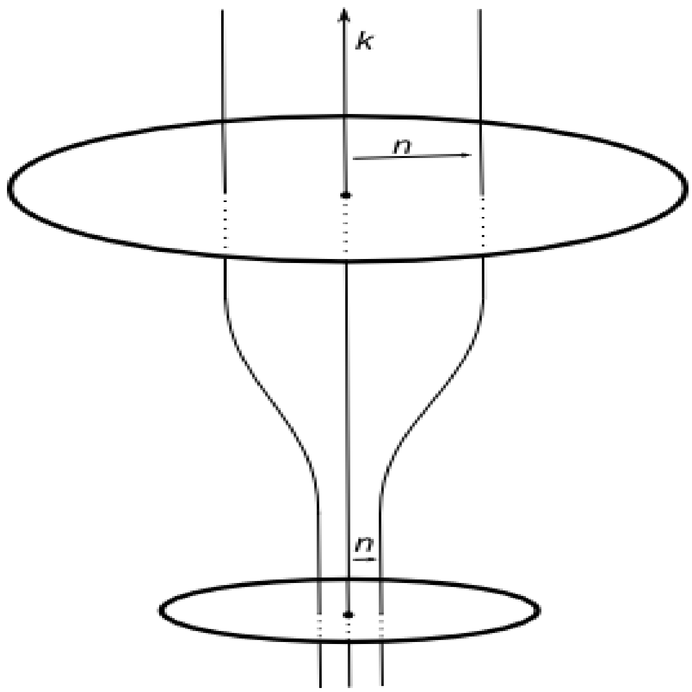

Figure 9: The horizon generators expand as the entropy moves across the horizon.

We now have almost everything we need. Note that the thermodynamic relations we are using are:

$$\delta Q = TdS = \eta \delta A.$$

If we use the Unruh Temperature as our temperature in the above, then we have all we need to set

$$-\kappa \int_H \lambda T_{ab} k^a k^b d\lambda dA = \int_H \theta d\lambda dA \tag{219}$$

and employ the Raychaudhuri equation for null surfaces. Recall that we have chosen our congruence to give us vanishing expansion and shear, so that the Raychaudhuri equation reduces to

$$\frac{d\theta}{d\lambda} = -R_{ab} k^a k^b$$

and upon integration we get $\theta = -\lambda R_{ab} k^a k^b$. This tells us that (4) becomes

$$-\kappa \int_H \lambda T_{ab} k^a k^b d\lambda dA = \int_H -\lambda R_{ab} k^a k^b d\lambda dA \tag{220}$$

and we get to

$$T_{ab} k^a k^b = \left(\frac{\hbar\eta}{2\pi}\right) R_{ab} k^a k^b \tag{221}$$

for all null k^a. Since we have $g_{ab} k^a k^b = 0$ for null k^a, we can add the metric tensor into (6) for free

$$T_{ab} k^a k^b = \left(\frac{\hbar\eta}{2\pi}\right) R_{ab} k^a k^b + f g_{ab} k^a k^b$$

Where f is some undetermined function. What we have derived so far may include the null vector k^a but the expressions are true for *any* null vector, so we can write

$$T_{ab} = \left(\frac{\hbar\eta}{2\pi}\right) R_{ab} + f g_{ab} \tag{222}$$

and use the fact that T_{ab} is divergence-free and the contracted Bianchi Identity to specify $f = -\frac{1}{2}R + \Lambda$, where $\Lambda$is a constant. This gives us [12]:

$$\left(\frac{2\pi}{\hbar\eta}\right) T_{ab} = R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} \tag{223}$$

Why is this result important? Jacobson's calculation brings spacetime thermodynamics full circle. By studying the behaviour of back holes, we were able to infer that the solutions to the Einstein Equations encode the area-entropy relationship. By assuming that the entropy-area relationship exists, we were then able to get back to the Einstein Equations. More importantly, by demonstrating that there exists a thermodynamic derivation of the EFEs, the result demonstrates that the geometrical variables of gravity (and not just the black hole parameters) can be treated as thermodynamic variables.

6.2 Implications for Quantum Gravity

Quantizing the metric g_{ab} as a way of putting together a theory of quantum gravity results in divergences which have hitherto proven resistant to renormalization. If the metric truly is an emergent variable like pressure, then it would be misguided to quantize it. One would not arrive at the quantum theory of the atom by imposing commutation relations on, say, pressure and volume. If spacetime geometry truly is the coarse-grained limit of some other theory, then this might explain why quantizing the metric has not yet met with success - prompting us to abandon plans to quantize the metric and rather look for another quantum theory of gravity which produces the Einstein Field Equations in some limit.

Part III

Extending the Einstein Equation of State

In classical thermodynamics, there is more than a single equation of state. If we change our assumptions about the physics describing the microscopic constituents of the thermodynamic system, we will end up changing the macroscopic, thermodynamic description. For example, by assuming the molecules of a gas have a non-negligible size and that the particle-particle interactions are also non-negligible, we can derive the Van der Waals equation of state

$$\left(p + \frac{a}{v^2}\right)(v - b) = kT \tag{224}$$

where a is the variable which measures attraction between particles and b measures the volume excluded by the particles.

In order to arrive at (224), we needed to make very few assumptions about the microscopic nature of gases. We did not need to study the kinetic theory of gases, statistical mechanics or quantum theory. This tells us that it might be possible to move from the Einstein equation of state to another gravitational equation of state without having a detailed knowledge of the microscopic nature of spacetime.

7 The Case for a Minimum Length Scale

It is tempting to say that, just like particles have a non-zero size, spacetime has a minimum measurable distance. Far from being an idle assumption, there is strong evidence that this is the case. Without resorting to quantum field theory or string theory, we can demonstrate how general relativity and classical optics imply a built-in minimum to the measurable distance of spacetime.

Let's try to measure the position of a particle using a test particle. The test particle has the 4-momentum vector $(\omega, \vec{k})$ and rest mass μ which we will later try take to 0. We shoot the test particle along the x - axis with a velocity

$$v = \frac{k}{\sqrt{\mu^2 + k^2}} \tag{225}$$

where $k^2 = \omega^2 - \mu^2$ and $k = \left|\vec{k}\right|$. What motion does the gravitational particle induce on the test particle as they move closer together? To obtain the metric that the test particle creates, begin by boosting into the rest frame of the particle. If we denote the new boosted coordinate frame using primes, we can write

$$g'_{00} = 1 + 2\phi' \quad g'_{11} = -\frac{1}{g'_{00}} \quad g'_{22} = g'_{33} = -1$$

where

$$\phi = \frac{G\mu}{|x'|}.$$

If we transform back into the original frame using

$$g_{\mu\nu} = \frac{\partial \left(x'\right)^\beta}{\partial x^\mu} \frac{\partial \left(x'\right)^\alpha}{\partial x^\nu} g'_{\beta\alpha} \tag{226}$$

we find that the metric components are

$$g_{00} = \frac{1 + 2\phi}{1 + 2\phi\left(1 - v^2\right)} + 2\phi \quad g_{11} = \frac{-1 + 2\phi v^2}{1 + 2\phi\left(1 - v^2\right)} + 2v^2\phi \tag{227}$$

and

$$g_{01} = g_{10} = \frac{2v\phi}{1 + \phi\left(1 - v^2\right)} - 2v\phi \quad g_{22} = g_{33} = -1 \tag{228}$$

where

$$\phi = \frac{\phi'}{1 - v^2} = -\frac{G\omega}{R}. \tag{229}$$

Note that $R = vt - x$ is the mean distance between the test particle and the particle being measured. In order to prevent an event horizon occurring, we need to have $2\phi < 1$ and therefore

$$-2\phi' = 2\frac{G\omega}{R}\left(1 - v^2\right) < 1. \tag{230}$$

Let's study the regime where $-\phi \gg 1$. Assume that the world line of the measured particle is timelike, so that the infinitesimal interval is given by

$$ds^2 = \left(g_{00} + 2g_{10}u + g_{11}u^2\right) dt^2 > 0 \tag{231}$$

where u is the velocity in the x-direction. If we make the simplification $\alpha = 1 + 2\phi\left(1 - v^2\right)$, then it can be shown that

$$u > \frac{1 + 2\phi\left(1 + \alpha\right)}{1 - 2\phi v^2\left(1 + \alpha\right)} \tag{232}$$

and

$$\frac{u}{1-u} > -\frac{1}{2}\left(1 + 2\phi\right). \tag{233}$$

The time τ (from the point of view of the measured particle) required for the test particle to move a distance R away from the measured particle is at least $\tau \gtrsim {}^{R}/_{(1-u)}$, in which time the measured particle moves a distance

$$L = u\tau \gtrsim R\frac{u}{1-u} \gtrsim \frac{R}{2}\left(-1 - 2\phi\right). \tag{234}$$

Given the limit we are working in, we have that

$$L \gtrsim G\omega. \tag{235}$$

Projection onto the x-axis gives us $\Delta x \gtrsim G\omega \sin\epsilon$. Let us now consider a photon with frequency ω moving along the x direction. If this photon scatters from a particle lying on the x axis, classical optics tells us that the wavelength of the photon sets a limit to the possible resolution Δx :

$$\Delta x \gtrsim \frac{1}{2\pi\omega\sin\epsilon}. \tag{236}$$

This gives us a minimum measurable length of

$$\Delta x \gtrsim \sqrt{G} = l_P. \tag{237}$$

7.1 Imposing a Minimum Length

Since we can't perform any physical measurement to measure any lengths less than l_p, it might make sense to say that spacetime is a lattice; it resembles a net rather than a rubber sheet. Events in spacetime can therefore only be finitely close together and if we want to travel anywhere, we must jump from one point to the next. Unfortunately, simply putting spacetime on a lattice breaks the diffeomorphism invariance of Einstein's gravity. We therefore need a more subtle way to impose a minimal length into the theory.

7.2 Non-commutative Geometry

Although it might not be immediately obvious, non-commutative geometry plays an important role in ordinary quantum mechanics. Classically, we are used to dealing with a phase space of position $\vec{x}$ and momentum $\vec{p}$ and a commutator between the two variables being zero:

$$[\vec{x}, \vec{p}] = 0 \tag{238}$$

If we think of phase space as a manifold, then in quantum mechanics the coordinates on the phase space manifold obey the commutation relation

$$[\hat{x}, \hat{p}] = i\hbar \tag{239}$$

which tells us that the geometry of the phase space manifold is, in fact, non-commutative. We also know from quantum mechanics that it is not a foundation of the theory that the variables for position and momentum must take on only discrete values; the phase space of the quantum world is not a lattice. Position and momentum are continuous variables but we may not measure them to arbitrary accuracy at the same time. This forms a key realization in how a non-commutative geometry is different to a discrete lattice of points - variables may take on continuous values but it may not be possible to perform some measurements to arbitrary accuracy. Imposing the commutator

$$[\vec{x_1}, \vec{x_2}] = i\varepsilon \tag{240}$$

(where ε is some currently unspecified constant) is not the same as simply forcing spacetime onto a lattice so it might be a way to implement some alternate microstructure of spacetime while simultaneously sidestepping the problems associated with placing spacetime on a lattice.

There is an excellent reason to declare that the coordinates on a manifold do not commute. Recall that we can write entropy as

$$S = k \ln \Omega \tag{241}$$

where Ω is the number of states accessible to the system. Since Ω is an integer, the value of S cannot be continuous - it can only take on certain values. If we take the black hole entropy as $S = {}^1\!/\!_4 A$, we are met with a contradiction. In classical geometry, A is continuous, but if we put this into (241) we get that $S \sim \ln A$. Statistically we therefore have that S is not continuous but in black hole thermodynamics it is. Implementing the commutation relation (240) forces areas (such as null horizons) to only take on integer values. By introducing non-commutative geometry, we can reconcile the contradiction between our statistical idea of entropy and entropy in black hole thermodynamics.

7.2.1 The Moyal $\star$-Product

Now that we have decided that we want the coordinates on our manifold to not commute, we would like to find a simple way to work with this non-commutativity. To do this, we are going to replace multiplication of functions living on the manifold with a new product called the Moyal star product. This is an operation which obeys the desirable properties of associativity, bilinearity and the Leibniz product rule [25][26]:

$$(f \star g) \star h = f \star (g \star h)$$

$$af \star bg = abf \star g$$

$$\partial_\mu (f \star g) = (\partial_\mu f) \star g + f \star (\partial_\mu g).$$

We can also expand the $\star$-product as a power series in the non-commutative parameter by writing

$$f(x) \star g(x) = exp\left[\frac{i}{2}\theta^{ij}\frac{\partial}{\partial\alpha^i}\frac{\partial}{\partial\beta^j}\right] f\left(x+\alpha\right) g\left(x+\beta\right) = fg + i\frac{1}{2}\theta^{ij}\partial_i f \partial_j g + O(\theta^2) \tag{242}$$

which means that whenever we want to use the $\star$-product, we simply expand the functions we are multiplying together to the desired accuracy.

Doing non-commutative geometry in this way is highly desirable because we need only work with functions which we already know and their derivatives which we can calculate.

7.3 The Tetrad Formalism and Gravity as a Gauge Theory

We are going to briefly review the tetrad formalism of general relativity in order to bring out a salient point - gravity can be thought of as a gauge theory [1]. In the following section, we will see how this will allow us to implement a minimum length in a fairly subtle and interesting way.

It is common to use the partial derivatives at a point on a manifold to define a set of basis vectors $\hat{e}_{(\mu)} = \partial_\mu$ at that point; call these the 'coordinate basis vectors'. This is not the only set of bases which we are allowed to construct at any given point on the manifold. We have a huge amount of freedom to choose almost any set of bases we want and we are going to exploit that freedom in order to set up a set of orthonormal bases. Let us therefore choose $\hat{e}_{(a)}$ and $\hat{e}_{(b)}$ such that

$$g\left(\hat{e}_{(a)}, \hat{e}_{(b)}\right) = \eta_{ab}.$$

Since any vector can be expressed as a linear combination of the basis vectors, we can say that $\hat{e}_{(\mu)} = e^a_\mu \hat{e}_{(a)}$, where e^a_μ is a matrix relating the vector $\hat{e}_{(\mu)}$ to the vector $\hat{e}_{(a)}$. We call these matrices 'tetrads'. To arrive at the inverse of a tetrad, switch its upper and lower indices. These satisfy

$$e^\mu_a e^a_\nu = \delta^\mu_\nu \qquad e^a_\mu e^\mu_b = \delta^a_b$$

and we can also use them to switch back to the coordinate basis vectors:

$$\hat{e}_{(a)} = e^\mu_a \hat{e}_{(\mu)}.$$

The tetrads and their inverses therefore allow us to switch back and forth between the metric tensor and the Minkowski metric:

$$g_{\mu\nu} e^\mu_a e^\nu_b = \eta_{ab}$$

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}.$$

To set up an orthonormal basis of one forms, we need only state that we wish to preserve the relationship

$$\hat{\theta}^{(a)} \hat{e}_{(b)} = \delta^a_b$$

from which it follows that $\hat{\theta}^{(\mu)} = e^{\mu}_{a}\hat{\theta}^{(a)}$ and $\hat{\theta}^{(a)} = e^{a}_{\mu}\hat{\theta}^{(\mu)}$.

Since we can write any vector as a linear combination of the basis vectors, we can use the tetrads to transform the vector components:

$$V^a = e^a_\mu V^\mu.$$

This indicates that we are using tetrads to switch between Latin and Greek indices. Regardless of the rank of the tensor, we can switch between the two bases by contracting with the required amount of tetrads:

$$V^a{}_b = e^a_\mu V^\mu{}_b = e^\nu_b V^a{}_\nu = e^a_\mu e^\nu_b V^\mu{}_\nu.$$

If we wish to change the orthonormal basis vectors while preserving the orthonormality condition, we will implement changes of the type

$$\hat{e}_{(a)} \to \hat{e}_{(a')} = \Lambda_{a'}{}^{a}(x)\hat{e}_a$$

where the position-dependent transformations $\Lambda_{a'}{}^{a}$ leave the form of the metric the same:

$$\Lambda_{a'}{}^{a}\Lambda_{b'}{}^{b}\eta_{ab} = \eta_{a'b'}.$$

These transformations are the usual Lorentz transformations, only now there is a different, local Lorentz transform for each point on the manifold. By making use of tetrads, it becomes clear that we can not only perform coordinate transforms, but also Lorentz transforms of tensors with Greek indices.

Recall that in the usual formalism, we often dealt with the connection which, although it carried indices, did not transform as a regular tensor. In order to get the tetrad form of the connection, we cannot simply contract the connection with the e^a_μ matrices. In order to work in the tetrad frame, we are going to introduce something called the spin connection to replace the ordinary connection. Each Latin index gets a spin connection:

$$\nabla_\mu X^a{}_b = \partial_\mu X^a{}_b + \omega_\mu{}^a{}_c X^c{}_b - \omega_\mu{}^c{}_b X^a{}_c.$$

The spin connection and regular connection are related by

$$\Gamma^\nu_{\mu\lambda} = e^\nu_a \partial_\mu e^a_\lambda + e^\nu_a e^b_\lambda \omega_\mu{}^a{}_b$$

and

$$\omega_\mu{}^a{}_b = e^a_\nu e^\lambda_b \Gamma^\nu_{\mu\lambda} - e^\lambda_b \partial_\mu e^a_\lambda.$$

The spin connection obeys the transformation law:

$$\omega_\mu{}^{a'}{}_{b'} = \Lambda^{a'}{}_a \Lambda_{b'}{}^b \omega_\mu{}^a{}_b - \Lambda_{b'}{}^c \partial_\mu \Lambda^{a'}{}_c.$$

A key benefit in writing out general relativity in the tetrad formalism is that it allows us to plainly see how the theory is gauge invariant. When it comes to differential geometry, we deal with the tangent space and cotangent space of a manifold. In gauge theories, we have to deal with internal vector spaces which are defined as extra information for each point on the manifold. The combination of the tangent space and the internal vector spaces is called the fiber bundle. We also need a structure group which tells us how to sew together the fibers on overlapping coordinate patches.

Suppose that some field $\phi^A(x^\mu)$ lives in this bundle. If the metric is Lorentzian and we sew the fibers together with rotations, then the field is left unchanged under $SO(3)$ rotations of the form

$$\phi^A(x^\mu) \to \phi^{A'}(x^\mu) = O^{A'}{}_A(x^\mu)\phi^A(x^\mu).$$

A theory invariant under this sort of translation is a gauge theory. If we try to take the partial derivative ∂_μ of our new field, we will gain an extra term due to the dependence of $O^{A'}{}_A$ on position. To combat this, we do as before and introduce a connection which we will call $A_\mu{}^A{}_B$. Under gauge transformations, this will transform as

$$A_\mu{}^{A'}{}_{B'} = O^{A'}{}_A O^B_{B'} A_\mu{}^A{}_B - O^C_{B'} \partial_\mu O^{A'}{}_C$$

which looks exactly like the transformation law for the spin connection. This construction illustrates mathematically just how gravity is a gauge theory.

7.4 The Seiberg-Witten Map

Let's now demonstrate that if we know what a gauge theory looks like a commutative manifold, we will be able to write down what the same theory looks like if it lived on a non-commutative manifold. For R^n with coordinates x^i we impose coordinate non-commutativity by saying that the coordinates obey the algebra [26]:

$$\left[x^i, x^j\right] = i\theta^{ij}$$

where θ is real. We want to use this to deform the algebra of functions living on R^n to a non-commutative algebra such that

$$f \star g = fg + i\frac{1}{2}\theta^{ij}\partial_i f \partial_j g + O(\theta^2). \tag{243}$$

The unique solution to this problem is:

$$f(x)\star g(x) = exp\left[\frac{i}{2}\theta^{ij}\frac{\partial}{\partial\alpha^i}\frac{\partial}{\partial\beta^j}\right] f\left(x+\alpha\right) g\left(x+\beta\right) = fg + i\frac{1}{2}\theta^{ij}\partial_i f\partial_j g + O(\theta^2).$$

If the functions f and g are matrix-valued functions then the star product becomes the tensor product of matrix multiplication with the star product of functions as just defined.

For a commutative gauge theory, we write the gauge transformations and field strength as

$$\delta_\lambda A_i = \partial_i + i\left[\lambda, A_i\right],$$

$$F_{ij} = \partial_i A_j - \partial_j A_i - i\left[A_i, A_j\right]$$

and

$$\delta_\lambda F_{ij} = i\left[\lambda, F_{ij}\right].$$

For a non-commutative gauge theory, we are going to use the same formulae for the gauge transformation law and the field strength, except that the matrix multiplication is defined by the star product. If the gauge parameter is $\hat{\lambda}$ the gauge transformations and field strength of non-commutative Yang-Mills theory are:

$$\hat{\delta}_{\hat{\lambda}}\hat{A}_i = \partial_i\hat{\lambda} + i\hat{\lambda}\star\hat{A}_i - i\hat{A}_i\star\hat{\lambda}$$

$$\hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i - i\hat{A}_i \star \hat{A}_j + i\hat{A}_j \star \hat{A}_i$$

$$\hat{\delta}_{\hat{\lambda}} \hat{F}_{ij} = i\hat{\lambda} \star \hat{F}_{ij} - i\hat{F}_{ij} \star \hat{\lambda}.$$

To the first order in θ, these expressions are:

$$\hat{\delta}_{\hat{\lambda}} \hat{A}_i = \partial_i \hat{\lambda} - \theta^{kl} \partial_k \hat{\lambda} \partial_l \hat{A}_i + O(\theta^2)$$

$$\hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i - \theta^{kl} \partial_k \hat{A}_i \partial_l \hat{A}_j + O(\theta^2)$$

$$\hat{\delta}_{\hat{\lambda}} \hat{F}_{ij} = -\theta^{kl} \partial_k \hat{\lambda} \partial_l \hat{F}_{ij} + O(\theta^2).$$

We are now going to find a mapping from ordinary gauge fields A to non-commutative gauge fields $\hat{A}$ which are local to any finite order in θ. We also require that if two ordinary gauge fields A and A' are equivalent by an ordinary gauge transformation $U = exp(i\lambda)$, then the non-commutative gauge fields $\hat{A}$ and $\hat{A}'$ will be gauge equivalent by a non-commutative gauge transformation $U = exp(i\hat{\lambda})$. Note that $\hat{\lambda}$ depends on A and λ.

Let's now take the gauge fields to be of rank N, so that the gauge parameters are $N \times N$ matrices. We want a mapping between commutative and non-commutative fields such that

$$\hat{A}(A) + \hat{\delta}_{\hat{\lambda}} \hat{A}(A) = \hat{A}(A + \delta_\lambda A) \tag{244}$$

where the variables λ and $\hat{\lambda}$ are infinitesimal. This ensures that if we transform A by λ, then the transformation of $\hat{A}$ by $\hat{\lambda}$ is equivalent. This forces ordinary fields which are gauge-equivalent to be mapped to non-commutative gauge fields which are also gauge-equivalent. Let's work to first order in θ and write $\hat{A} = A + A'(A)$ and $\hat{\lambda}(\lambda, A) = \lambda + \lambda'(\lambda, A)$. Thus, we expand (234) as

$$A'_i(A+\delta_\lambda A) - A'_i(A) - \partial_i \lambda' - i\,[\lambda', A_i] - i\,[\lambda, A'_i] = -\frac{1}{2}\theta^{kl}\,(\partial_k \lambda \partial_l A_i + \partial_l A_i \partial_k \lambda) + O(\theta^2) \tag{245}$$

where all the products appearing in the above are ordinary matrix products. Equation (235) is solved by

$$\hat{A}_i(A) = A_i + A_i'(A) = A_i - \frac{1}{4}\theta^{kl}\left\{A_k, \partial_l A_i + F_{li}\right\} + O(\theta^2) \tag{246}$$

and

$$\hat{\lambda}_i(\lambda, A) = \lambda + \lambda_i'(\lambda, A) = \lambda + \frac{1}{4}\theta^{ij}\left\{\partial_i \lambda, A_j\right\} + O(\theta^2). \tag{247}$$

We can thus write the gauge strength as

$$\hat{F}_{ij} = F_{ij} + \frac{1}{4}\theta^{kl}\left(2\left\{F_{ik}, F_{jl}\right\} - \left\{A_k, D_l F_{ij} + \partial_l F_{ij}\right\}\right) + O(\theta^2). \tag{248}$$

where D_l is the covariant derivative. Equations (236), (237) and (238) tell us what a gauge theory would look like if it were to live on a non-commutative manifold and it does so in terms of what the field looks like on a regular, commutative manifold. We therefore have a new non-commutative theory expressed entirely in terms of expressions and functions we already know.

So far, we have only worked to first order. To work to higher orders in θ, consider mapping the field $\hat{A}(\theta)$ to $\hat{A}(\theta+\delta\theta)$. The only property of the $\star-$product that one needs to check in order to see that (236) and (237) satisfy (234) is

$$\delta\theta^{ij}\frac{\partial}{\partial\theta^{ij}}(f \star g) = \frac{i}{2}\delta\theta^{ij}\frac{\partial f}{\partial x^i} \star \frac{\partial g}{\partial x^j} \tag{249}$$

when $\theta = 0$. Since this is true for any value of θ, we can write down formulas which tell us how $\hat{A}(\theta)$ and $\hat{\lambda}(\theta)$ change when θ is varied. These are:

$$\delta\hat{A}_i(\theta) = \delta\theta^{kl}\frac{\partial}{\partial\theta^{kl}}\hat{A}_i(\theta) = -\frac{1}{4}\delta\theta^{kl}\left[\hat{A}_k \star \left(\partial_l \hat{A}_i + \hat{F}_{li}\right) + \left(\partial_l \hat{A}_i + \hat{F}_{li}\right) \star \hat{A}_k\right] \tag{250}$$

$$\delta\hat{\lambda}(\theta) = \delta\theta^{kl}\frac{\partial}{\partial\theta^{kl}}\hat{\lambda}(\theta) = -\frac{1}{4}\delta\theta^{kl}\left(\partial_k \lambda \star A_l + A_l \star \partial_k \lambda\right) \tag{251}$$

and [26]

$$\begin{gathered}\delta\hat{F}_{ij}(\theta) = \delta\theta^{kl}\tfrac{\partial}{\partial\theta^{kl}}\hat{F}_{ij}(\theta) = \\ -\tfrac{1}{4}\delta\theta^{kl}\Big[2\hat{F}_{ik} \star \hat{F}_{jl} + 2\hat{F}_{jl} \star \hat{F}_{ik} \\ -\hat{A}_k \star \left(\hat{D}_l \hat{F}_{ij} + \partial_l \hat{F}_{ij}\right) - \left(\hat{D}_l \hat{F}_{ij} + \partial_l \hat{F}_{ij}\right) \star \hat{A}_k\Big]\end{gathered} \tag{252}$$

7.4.1 The Seiberg-Witten Map and Gravity

Let's take what we have learned from the Seiberg-Witten map and apply it to the tetrad formalism of gravity. As before, this will tell us what geometric variables would look like were they to live on a non-commutative manifold and do so in terms of what we already know.

The $U(N)$ gauge fields are subject to the conditions $\hat{A}_\mu^\dagger = -\hat{A}_\mu$. This condition is maintained under the transformation [26]:

$$\hat{A}_\mu^g = \hat{g} \star \hat{A}_\mu \star \hat{g}_\star^{-1} - \hat{g} \star \partial_\mu \hat{g}_\star^{-1}$$

where $\hat{g} \star \hat{g}_\star^{-1} = 1 = \hat{g}_\star^{-1} \star \hat{g}$. Introduce the gauge fields $\hat{\omega}_\mu^{AB}$ which are subject to:

$$\hat{\omega}_\mu^{AB\dagger}(x,\theta) = -\hat{\omega}_\mu^{AB}(x,\theta)$$

$$\hat{\omega}_\mu^{AB}(x,\theta)^r \equiv \hat{\omega}_\mu^{AB}(x,-\theta) = -\hat{\omega}_\mu^{AB}(x,\theta).$$

Expanding these fields in terms of θ :

$$\hat{\omega}_\mu^{AB}(x,\theta) = \omega_\mu^{AB} - i\theta^{\nu\rho}\omega_{\mu\nu\rho}^{AB} + \dots$$

The above conditions tell us that the terms in the expansion must satisfy:

$$\omega_\mu^{AB} = -\omega_\mu^{BA}, \quad \omega_{\mu\nu\rho}^{AB} = \omega_{\mu\nu\rho}^{BA}.$$

We assume that these new, deformed fields are related to the old fields via the Seiberg-Witten map. This is defined by the property

$$\hat{\omega}_\mu^{AB}(\omega) + \delta_{\hat{\lambda}}\hat{\omega}_\mu^{AB}(\omega) = \hat{\omega}_\mu^{AB}(\omega + \delta_\lambda \omega)$$

where $\hat{g} = e^{\hat{\lambda}}$ and the infinitesimal transformation of of ω_μ^{AB} is given by

$$\delta_\lambda \omega_\mu^{AB} = \partial_\mu \lambda^{AB} + \omega_\mu^{AC}\lambda^{CB} - \lambda^{AC}\omega_\mu^{CB}.$$

The deformed fields are given by the same expression, but with matrix multiplication replaced by the star product:

$$\delta_{\hat{\lambda}}\hat{\omega}_\mu^{AB} = \partial_\mu \hat{\lambda}^{AB} + \hat{\omega}_\mu^{AC} \star \hat{\lambda}^{CB} - \hat{\lambda}^{AC} \star \hat{\omega}_\mu^{CB}.$$

To solve this equation we write

$$\hat{\omega}_\mu^{AB} = \omega_\mu^{AB} + {\omega'^{AB}}_\mu$$

$$\hat{\lambda}^{AB} = \lambda^{AB} + \lambda'^{AB}$$

where ω' and λ' are both functions of θ. We substitute this into the variational equation to get:

$$\omega_\mu^{AB}(\omega + \delta\omega) - \omega_\mu^{'AB}(\omega)$$

$$= \partial_\mu \lambda^{'AB} + \omega_\mu^{AC} \lambda^{'CB} - \lambda^{'CB} \omega_\mu^{CB} + \omega_\mu^{'AC} \lambda^{CB} - \lambda^{AC} \omega_\mu^{'CB}$$

$$+ \frac{i}{2} \theta^{\nu\rho} \left(\partial_\nu \omega_\mu^{AB} \partial_\rho \lambda^{CB} + \partial_\rho \lambda^{AC} \partial_\nu \omega_\mu^{CB} \right).$$

This is solved to first order in θ by:

$$\hat{\omega}_\mu^{AB} = \omega_\mu^{AB} - \frac{i}{4} \theta^{\nu\rho} \left\{ \omega_\nu, \partial_\rho \omega_\mu + R_{\rho\mu} \right\}^{AB} + O(\theta^2) \tag{253}$$

$$\hat{\lambda}^{AB} = \lambda^{AB} + \frac{i}{4} \theta^{\nu\rho} \left\{ \partial_\nu \lambda, \omega_\rho \right\}^{AB} + O(\theta^2).$$

Using the same logic as in the previous section, we can write the deformed fields to all orders as [26]

$$\delta \hat{\omega}_\mu^{AB} = \frac{i}{4} \theta^{\nu\rho} \left\{ \hat{\omega}_\nu ,_\star \partial_\rho \hat{\omega}_\mu + \hat{R}_{\rho\mu} \right\}^{AB}, \tag{254}$$

while noting that we use the $\star$-product when expanding the anti-commutator where [26]

$$\hat{R}_{\mu\nu}^{AB} = \partial_\mu \hat{\omega}_\nu^{AB} - \partial_\nu \hat{\omega}_\mu^{AB} + \hat{\omega}_\mu^{AC} \star \hat{\omega}_\nu^{CB} - \hat{\omega}_\nu^{AC} \star \hat{\omega}_\mu^{CB}. \tag{255}$$

From now on, we are going to work to second order in the non-commutative parameter. Thus, before we can write (245) entirely in terms of undeformed variables, we need to determine the second-order corrections to the spin connection. To do this, take the first order corrections in (243) and insert them into (244) and then integrate. This gives us a second order correction of [26]

$$\frac{1}{32}\theta^{\nu\rho}\theta^{\kappa\sigma}\left(\{\omega_\kappa, 2\{R_{\sigma\nu}, R_{\mu\rho}\} - \{\omega_\nu, (D_\rho R_{\sigma\mu} + \partial_\rho R_{\sigma\mu})\} - \partial_\sigma\{\omega_\nu, (\partial_\rho\omega_\mu + R_{\rho\mu})\}\}^{AB}\right.$$
$$\left. + [\partial_\nu\omega_\kappa, \partial_\rho(\partial_\sigma\omega_\mu + R_{\sigma\mu})]^{AB} - \{\{\omega_\nu, (\partial_\rho\omega_\kappa + R_{\rho\kappa})\}, (\partial_\sigma\omega_\mu + R_{\sigma\mu})\}^{AB}\right). \tag{256}$$

We need to now determine the deformed tetrad $\hat{e}^a_\mu$ in terms of undeformed variables. To do this, we treat the tetrad as the gauge field of the translational generator of the inhomogeneous Lorentz group. This is done by contracting $SO(4,1)$ to $ISO(3,1)$. If our $SO(4,1)$ gauge field is ω^{AB}_μ, then we define the field strength to be

$$R^{AB}_{\mu\nu} = \partial_\mu\omega^{AB}_\nu + \partial_\nu\omega^{AB}_\mu + \omega^{AC}_\mu\omega^{CB}_\nu + \omega^{AC}_\nu\omega^{CB}_\mu \tag{257}$$

where $A = a, 5$. Now define $\omega^{a5}_\mu = ke^a_\mu$ so that we get

$$R^{ab}_{\mu\nu} = \partial_\mu\omega^{ab}_\nu - \partial_\nu\omega^{ab}_\mu + \omega^{ac}_\mu\omega^{cb}_\nu + k^2\left(e^a_\mu e^b_\nu - e^a_\nu e^b_\mu\right) \tag{258}$$

and

$$R^{a5}_{\mu\nu} = kT^a_{\mu\nu} = k\left(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega^{ac}_\mu e^c_\nu - \omega^{ac}_\nu e^c_\mu\right). \tag{259}$$

Now perform the contraction by taking $k \to 0$ and impose the condition $T^a_{\mu\nu} = 0$ so that we can solve for ω^{ab}_μ in terms of e^a_μ.

For the deformed case, we write $\hat{\omega}^{a5}_\mu = ke^a_\mu$ and $\hat{\omega}^{55}_\mu = k\hat{\phi_\mu}$. We need not impose that $\hat{T}^a_{\mu\nu} = 0$ because ϕ_u drops out when $k \to 0$. This gives us the deformed tetrad to second order as [26]

$$\begin{aligned}
\hat{e}^a_\mu = e^a_\mu - \tfrac{i}{4}\theta^{\nu\rho}\left(\omega^{ac}_\mu\partial_\rho e^c_\mu + \left(\partial_\rho\omega^{ac}_\mu + R^{ac}_{\rho\mu}\right)e^c_\nu\right) \\
+\tfrac{1}{32}\theta^{\nu\rho}\theta^{\kappa\sigma}\left(2\{R_{\sigma\nu}, R_{\mu\rho}\}^{ac}e^c_\kappa - \omega^{ac}_\kappa\left(D_\rho R^{cd}_{\sigma\mu} + \partial_\rho R^{cd}_{\sigma\mu}\right)e^d_\nu\right. \\
-\{\omega_\nu, (D_\rho R_{\sigma\mu} + \partial_\rho R_{\sigma\mu})\}^{ad}e^d_\kappa - \partial_\sigma\{\omega_\nu, (\partial_\rho\omega_\mu + R_{\rho\mu})\}^{ac}e^c_\kappa \\
-\omega^{ac}_\kappa\partial_\sigma\left(\omega^{cd}_\nu\partial_\rho e^d_\mu + \left(\partial_\rho\omega^{cd}_\mu + R^{cd}_{\rho\mu}\right)e^d_\nu\right) + \partial_\nu\omega^{ac}_\kappa\partial_\rho\partial_\sigma e^c_\mu \\
-\partial_\rho\left(\partial_\sigma\omega^{ac}_\mu + R^{ac}_{\sigma\mu}\right)\partial_{nu}e^c_\kappa - \{\omega_\nu, (\partial_\rho\omega_\kappa + R_{\rho\kappa})\}^{ac}\partial_\sigma e^c_\mu \\
\left. - \left(\partial_\sigma\omega^{ac}_\mu + R^{ac}_{\sigma\mu}\right)\left(\omega^{cd}_\nu\partial_\rho e^d_\kappa + \left(\partial_\rho\omega^{cd}_\kappa + R^{cd}_{\rho\kappa}\right)e^d_\nu\right)\right).
\end{aligned} \tag{260}$$

We now have a deformed spin connection entirely in terms of normal geometric variables as well as a deformed tetrad entirely in terms of normal geometric variables. We are now going to see how we can use these 'basic' indexed objects to build up more complex objects such as $\hat{R}^{AB}_{\mu\nu}$ in (255).

We only want terms up to the second order, so let us expand $\hat{R}^{AB}_{\mu\nu}$ as:

$$\hat{R}^{ab}_{\mu\nu} = R^{ab}_{\mu\nu} + i\theta^{\rho\tau}R^{ab}_{\mu\nu\rho\tau} + \theta^{\rho\tau}\theta^{\kappa\sigma}R^{ab}_{\mu\nu\rho\tau\kappa\sigma}. \tag{261}$$

Expanding the first two terms in (245) gives us

$$\begin{array}{c} \partial_\mu\omega^{AB}_\nu - \partial_\mu i\theta^{\rho\tau}\omega^{AB}_{\nu\rho\tau} + \partial_\mu\theta^{\rho\tau}\theta^{\kappa\sigma}\omega^{AB}_{\nu\rho\tau\kappa\sigma} \\ -\partial_\nu\omega^{AB}_\mu + \partial_\nu i\theta^{\rho\tau}\omega^{AB}_{\mu\rho\tau} - \partial_\nu\theta^{\rho\tau}\theta^{\kappa\sigma}\omega^{AB}_{\mu\rho\tau\kappa\sigma} \end{array}. \tag{262}$$

We now need to evaluate the star product

$$\hat{\omega}^{AC}_\mu \star \hat{\omega}^{CB}_\nu \tag{263}$$

and by doing so we need to only switch the μ and ν symbols to get the last term. To second order, the star product is

$$f \star g = fg + \frac{i}{2}\theta^{ij}\partial_i f\partial_j g - \frac{1}{4}\theta^{ij}\theta^{kl}\partial_i\partial_k f\partial_j\partial_l g \tag{264}$$

so the second-order approximation to (253) becomes

$$\hat{\omega}^{AC}_\mu \star \hat{\omega}^{CB}_\nu = \hat{\omega}^{AC}_\mu\hat{\omega}^{CB}_\nu + \frac{i}{2}\theta^{ij}\partial_i\hat{\omega}^{AC}_\mu\partial_j\hat{\omega}^{CB}_\nu - \frac{1}{4}\theta^{ij}\theta^{kl}\partial_i\partial_k\hat{\omega}^{AC}_\mu\partial_j\partial_l\hat{\omega}^{CB}_\nu. \tag{265}$$

It is important to remember that the spin connections in the above are still the deformed versions, we need to expand them first and only then evaluate the star product explicitly. Let's do this term-by-term. From $\hat{\omega}^{AC}_\mu\hat{\omega}^{CB}_\nu$ we get

$$\left(\omega^{AC}_\mu - i\theta^{\rho\tau}\omega^{AC}_{\mu\rho\tau} + \theta^{\rho\tau}\theta^{\kappa\sigma}\omega^{AC}_{\mu\rho\tau\kappa\sigma}\right)\left(\omega^{CB}_\nu - i\theta^{\rho\tau}\omega^{CB}_{\nu\rho\tau} + \theta^{\rho\tau}\theta^{\kappa\sigma}\omega^{CB}_{\nu\rho\tau\kappa\sigma}\right) \tag{266}$$

which expands to 9 terms:

$$\begin{array}{c} \omega^{AC}_\mu\omega^{CB}_\nu - i\theta^{\rho\tau}\omega^{AC}_{\mu\rho\tau}\omega^{CB}_\nu + \theta^{\rho\tau}\theta^{\kappa\sigma}\omega^{AC}_{\mu\rho\tau\kappa\sigma}\omega^{CB}_\nu \\ -i\theta^{\rho\tau}\omega^{AC}_\mu\omega^{CB}_{\nu\rho\tau} - \theta^{\rho\tau}\theta^{\kappa\sigma}\omega^{AC}_{\mu\rho\tau}\omega^{CB}_{\nu\kappa\sigma} - i\theta^{\rho\tau}\theta^{\kappa\sigma}\theta^{\delta\lambda}\omega^{CB}_{\nu\rho\tau}\omega^{AC}_{\mu\kappa\sigma\delta\lambda} \\ +\theta^{\rho\tau}\theta^{\kappa\sigma}\omega^{CB}_{\nu\rho\tau\kappa\sigma}\omega^{AC}_\mu - i\theta^{\rho\tau}\theta^{\kappa\sigma}\theta^{\delta\lambda}\omega^{AC}_{\mu\rho\tau}\omega^{CB}_{\nu\kappa\sigma\delta\lambda} + \theta^{\rho\tau}\theta^{\kappa\sigma}\theta^{\delta\lambda}\theta^{\eta\phi}\omega^{AC}_{\mu\rho\tau\kappa\sigma}\omega^{CB}_{\nu\delta\lambda\eta\phi} \end{array}. \tag{267}$$

Now we repeat the above for the next term in (265),

$$\frac{i}{2}\theta^{ij}\left[\partial_i\left(\omega_\mu^{AC}-i\theta^{\rho\tau}\omega_{\mu\rho\tau}^{AC}+\theta^{\rho\tau}\theta^{\kappa\sigma}\omega_{\mu\rho\tau\kappa\sigma}^{AC}\right)\partial_j\left(\omega_\nu^{CB}-i\theta^{\rho\tau}\omega_{\nu\rho\tau}^{CB}+\theta^{\rho\tau}\theta^{\kappa\sigma}\omega_{\nu\rho\tau\kappa\sigma}^{CB}\right)\right] \tag{268}$$

which also gives us 9 terms

$$\begin{gathered}\tfrac{i}{2}\theta^{ij}\partial_i\omega_\mu^{AC}\partial_j\omega_\nu^{CB}+\tfrac{1}{2}\theta^{ij}\theta^{\rho\tau}\partial_i\omega_\mu^{AC}\partial_j\omega_{\nu\rho\tau}^{CB}+\tfrac{i}{2}\theta^{ij}\partial_i\omega_\mu^{AC}\theta^{\rho\tau}\theta^{\kappa\sigma}\partial_j\omega_{\nu\rho\tau\kappa\sigma}^{CB}\\ +\tfrac{1}{2}\theta^{ij}\theta^{\rho\tau}\partial_i\omega_{\mu\rho\tau}^{AC}\partial_j\omega_\nu^{BC}-\tfrac{i}{2}\theta^{ij}\theta^{\rho\tau}\theta^{\kappa\sigma}\partial_i\omega_{\mu\rho\tau}^{AC}\partial_j\omega_{\nu\kappa\sigma}^{CB}\\ +\tfrac{1}{2}\theta^{ij}\theta^{\rho\tau}\theta^{\kappa\sigma}\theta^{\delta\lambda}\partial_i\omega_{\mu\rho\tau}^{AC}\partial_j\omega_{\nu\kappa\sigma\delta\lambda}^{CB}+\tfrac{i}{2}\theta^{ij}\theta^{\rho\tau}\theta^{\kappa\sigma}\end{gathered} \tag{269}$$

The calculation for the final term in (265) proceeds in the same manner. We get:

$$\begin{gathered}-\tfrac{1}{4}\theta^{ij}\theta^{kl}\left[\partial_i\partial_k\left(\omega_\mu^{AC}-i\theta^{\rho\tau}\omega_{\mu\rho\tau}^{AC}+\theta^{\rho\tau}\theta^{\kappa\sigma}\omega_{\mu\rho\tau\kappa\sigma}^{AC}\right)\partial_j\partial_l\left(\omega_\nu^{CB}-i\theta^{\rho\tau}\omega_{\nu\rho\tau}^{CB}+\theta^{\rho\tau}\theta^{\kappa\sigma}\omega_{\nu\rho\tau\kappa\sigma}^{CB}\right)\right]\\ =-\tfrac{1}{4}\theta^{ij}\theta^{kl}\left[\left(\partial_i\partial_k\omega_\mu^{AC}-i\theta^{\rho\tau}\partial_i\partial_k\omega_{\mu\rho\tau}^{AC}+\theta^{\rho\tau}\theta^{\kappa\sigma}\partial_i\partial_k\omega_{\mu\rho\tau\kappa\sigma}^{AC}\right)\right.\\ \left.\left(\partial_j\partial_l\omega_\nu^{CB}-i\theta^{\rho\tau}\partial_j\partial_l\omega_{\nu\rho\tau}^{CB}+\theta^{\rho\tau}\theta^{\kappa\sigma}\partial_j\partial_l\omega_{\nu\rho\tau\kappa\sigma}^{CB}\right)\right]\\ =-\tfrac{1}{4}\theta^{ij}\theta^{kl}\left[\partial_i\partial_k\omega_\mu^{AC}\partial_j\partial_l\omega_\nu^{CB}-i\theta^{\rho\tau}\partial_i\partial_k\omega_\mu^{AC}\partial_j\partial_l\omega_{\nu\rho\tau}^{CB}+\theta^{\rho\tau}\theta^{\kappa\sigma}\partial_i\partial_k\omega_\mu^{AC}\partial_j\partial_l\omega_{\nu\rho\tau\kappa\sigma}^{CB}\right.\\ -i\theta^{\rho\tau}\partial_i\partial_k\omega_{\mu\rho\tau}^{AC}\partial_j\partial_l\omega_\nu^{CB}+\theta^{\rho\tau}\theta^{\kappa\sigma}\partial_i\partial_k\omega_{\nu\rho\tau}^{CB}\partial_j\partial_l\omega_{\nu\kappa\sigma}^{CB}\\ -i\theta^{\rho\tau}\theta^{\kappa\sigma}\theta^{\mu\lambda}\partial_i\partial_k\omega_{\mu\rho\tau}^{AC}\partial_j\partial_l\omega_{\nu\kappa\sigma\mu\lambda}^{CB}+\theta^{\rho\tau}\theta^{\kappa\sigma}\partial_i\partial_k\omega_{\mu\rho\tau\kappa\sigma}^{AC}\partial_j\partial_l\omega_\nu^{CB}\\ \left.-i\theta^{\rho\tau}\theta^{\kappa\sigma}\theta^{\mu\lambda}\partial_i\partial_k\omega_{\mu\rho\tau\kappa\sigma}^{AC}\partial_j\partial_l\omega_{\nu\mu\lambda}^{CB}+\theta^{\rho\tau}\theta^{\kappa\sigma}\theta^{\mu\lambda}\theta^{\xi\phi}\partial_i\partial_k\ \omega_{\mu\rho\tau\kappa\sigma}^{AC}\partial_j\partial_l\omega_{\nu\mu\lambda\xi\phi}^{CB}\right]\end{gathered} \tag{270}$$

Looking at the terms in equations (267), (269) and (270), we notice that we have begun to pick up terms which run as high as sixth order! Since we are only working to second order, we will neglect those terms. Once we have thrown those terms away, we can begin to match the zeroth, first and second terms to the correct parts in (251). After changing some dummy indices, we get:

$$\begin{gathered}R_{\mu\nu}^{ab}=\partial_\mu\omega_\nu^{ab}+\partial_\nu\omega_\mu^{ab}+\omega_\mu^{ac}\omega_\nu^{cb}+\omega_\nu^{ac}\omega_\mu^{cb}\\ R_{\mu\nu\rho\tau}^{ab}=\partial_\mu\omega_{\nu\rho\tau}^{ab}+\omega_\mu^{ac}\omega_{\nu\rho\tau}^{cb}+\omega_{\mu\rho\tau}^{ac}\omega_\nu^{cb}-\tfrac{1}{2}\partial_\rho\omega_\mu^{ac}\partial_\tau\omega_\nu^{cb}-\mu\leftrightarrow\nu\\ R_{\mu\nu\rho\tau\kappa\sigma}^{ab}=\partial_\mu\omega_{\nu\rho\tau\kappa\sigma}^{ab}+\omega_\mu^{ac}\omega_{\nu\rho\tau\kappa\sigma}^{cb}+\omega_{\mu\rho\tau\kappa\sigma}^{ac}\omega_\nu^{cb}-\omega_{\mu\rho\tau}^{ac}\omega_{\nu\kappa\sigma}^{cb}\\ -\tfrac{1}{4}\partial_\rho\partial_\kappa\omega_\mu^{ac}\partial_\tau\partial_\sigma\omega_\nu^{cb}-\mu\leftrightarrow\nu.\end{gathered} \tag{271}$$

Where we already know the zeroth, first and second order values for ω_μ^{ac} and ω_ν^{cb}. We now have everything we need to compute the mixed tensor in (245) totally in terms of geometric variables we already know. Although it has required much legwork to get to this point, we have managed to demonstrate that it is possible to introduce a minimum length scale without needing to know anything more than we already know from the continuous geometry of a normal manifold.

7.5 The Non-commutative Raychaudhuri Equation

7.5.1 The Deformed Riemann Tensor

From the previous section, we see that we have all we need to write down what geometric quantities would look like if they lived on a non-commutative manifold. Let us therefore take these variables and derive a new expression which tells us what the Raychaudhuri equation would look like if it lived on a non-commutative manifold. To do this, we are going to repeat the derivation of the Raychaudhuri equation but we will replace multiplication with the star product and replace all the geometric variables with their deformed counterparts.

The first step to take is to find an expression for the deformed Riemann tensor which appears in the Raychaudhuri equation. Equation (257) will not suffice because it mixes tetrad indices with coordinate-frame indices whereas we want a Riemann tensor with only coordinate-frame indices. We can accomplish this by taking a cue from (255) and writing

$$R^{\sigma}_{\mu\nu\rho} = \partial_\nu \Gamma^{\sigma}_{\mu\rho} - \partial_\rho \Gamma^{\sigma}_{\mu\nu} + \Gamma^{\sigma}_{\alpha\nu}\Gamma^{\alpha}_{\mu\rho} - \Gamma^{\sigma}_{\alpha\rho}\Gamma^{\alpha}_{\mu\nu} \tag{272}$$

as

$$\hat{R}^{\sigma}_{\mu\nu\rho} = \partial_\nu \hat{\Gamma}^{\sigma}_{\mu\rho} - \partial_\rho \hat{\Gamma}^{\sigma}_{\mu\nu} + \hat{\Gamma}^{\sigma}_{\alpha\nu} \star \hat{\Gamma}^{\alpha}_{\mu\rho} - \hat{\Gamma}^{\sigma}_{\alpha\rho} \star \hat{\Gamma}^{\alpha}_{\mu\nu}. \tag{273}$$

At this point we have expressions for the deformed tetrad and the deformed spin connection, but we do not have an expression for the deformed Christoffel symbol. To find expressions for the Christoffel symbols, we turn to the classical relation

$$\Gamma^{\nu}_{\mu\lambda} = e^{\nu}_{a}\partial_\mu e^{a}_{\lambda} + e^{\nu}_{a} e^{b}_{\lambda}\omega^{a}_{\mu b} \tag{274}$$

and turn it into the deformed relation

$$\hat{\Gamma}^{\nu}_{\mu\lambda} = \hat{e}^{\nu}_{a} \star \partial_\mu \hat{e}^{a}_{\lambda} + \hat{e}^{\nu}_{a} \star \hat{e}^{b}_{\lambda} \star \hat{\omega}^{a}_{\mu b} \tag{275}$$

by adding stars and hats. Since we already know what $\hat{e}_a^\nu$ and $\hat{\omega}_{\mu b}^a$ look like, all that remains is to find the first and second corrections to the Christoffel symbols by evaluating the star products. Begin by expanding to second order:

$$\hat{\Gamma}_{\mu\lambda}^\nu = \Gamma_{\mu\lambda}^\nu - i\theta^{xy}\Gamma_{\mu\lambda xy}^\nu + \theta^{xy}\theta^{pq}\Gamma_{\mu\lambda xypq}^\nu. \tag{276}$$

Then, just as before, we can explicitly evaluate (275) and match the first and second order terms which come out of the calculation to the correct terms in (276). After discarding terms of order three and greater, we get the results:

$$\Gamma_{\mu\lambda}^\nu = e_a^\nu \partial_\mu e_\lambda^a + e_a^\nu e_\lambda^b \omega_{\mu b}^a, \tag{277}$$

$$\Gamma_{\mu\lambda xy}^\nu = \begin{array}{cc} -e_a^\nu \partial_\mu e_{\lambda xy}^a & -e_{axy}^\nu \partial_\mu e_\lambda^a \\ -\frac{1}{2}\partial_x e_a^\nu \partial_y \partial_\mu e_\nu^a & -e_{\lambda xy}^b e_a^\nu \omega_{\mu b}^a \\ +e_{axy}^\nu e_\lambda^b \omega_{\mu b}^a & \frac{1}{2}\partial_x e_a^\nu \partial_y e_\lambda^b \omega_{\mu b}^a \\ +e_a^\nu e_\lambda^b \omega_{\mu bxy}^a & -\partial_x e_a^v e_\lambda^b \partial_y \omega_{\mu b}^a \end{array}, \tag{278}$$

and

$$\Gamma_{\mu\lambda xypq}^\nu = \begin{array}{cc} +e_a^v \partial_\mu e_{\lambda xypq}^a & -e_{axy}^v \partial_\mu e_{\lambda pq}^a \\ +e_{axypq}^v \partial_\mu e_\lambda^a & -\frac{1}{2}\partial_x e_a^v \partial_y \partial_\mu e_{\nu pq}^a \\ -\frac{1}{2}\partial_x e_{apq}^\nu \partial_y \partial_\mu e_\nu^a & -\frac{1}{4}\partial_x \partial_p e_a^v \partial_y \partial_q \partial_\mu e_\nu^a \\ +e_{\lambda xypq}^b e_a^v \omega_{\mu b}^a & -e_{axy}^\nu e_{\lambda pq}^b \omega_{\mu b}^a \\ +e_{axypq}^\nu e_\lambda^b \omega_{\mu b}^a & -\frac{1}{2}\partial_y e_{\lambda pq}^b \partial_x e_a^\nu \omega_{\mu b}^a \\ -\frac{1}{4}\partial_x \partial_p e_a^\nu \partial_y \partial_q e_\lambda^b \omega_{\mu b}^a & +e_{\lambda xy}^b e_a^\nu \omega_{\mu bpq}^a \\ +e_{axy}^\nu e_\lambda^b \omega_{\mu bpq}^a & +\frac{1}{2}\partial_x e_a^\nu \partial_y e_\lambda^b \omega_{\mu bpq}^a \\ +e_a^\nu e_\lambda^b \omega_{\mu bxypq}^a & -\frac{1}{2}\partial_x e_{\lambda pq}^b e_a^\nu \partial_y \omega_{\mu b}^a \\ -\frac{1}{2}\partial_x e_{apq}^\nu e_\lambda^b \partial_y \omega_{\mu b}^a & -\frac{1}{4}\partial_x \partial_p e_a^\nu \partial_q \partial_y e_\lambda^b \omega_{\mu b}^a \\ -\frac{1}{4}\partial_x \partial_p e_a^\nu e_\lambda^b \partial_y \partial_p \omega_{\mu b}^a & \end{array}. \tag{279}$$

Encouragingly, we see that this matches to first order. Every single factor that appears in (277), (278) and (279) has already been calculated. So even though expressing (252) in terms of classical index quantities would be incredibly tedious, it would be totally straightforward.

Now that we know what the Christoffel symbols would look like on a deformed manifold, let's calculate (273). Since we are only working to second order, we are going to expand it as

$$\hat{R}^{\nu}_{\mu\delta\lambda} = R^{\nu}_{\mu\delta\lambda} + i\theta^{\rho\tau} R^{\nu}_{\mu\delta\lambda\rho\tau} + i\theta^{\rho\tau}\theta^{\kappa\sigma} R^{\nu}_{\mu\delta\lambda\rho\tau\kappa\sigma} \tag{280}$$

just like the mixed tensor. By following the exact method we used in the calculations for equations (260) - (270), we can show that the zeroth, first and second order terms in (280) are

$$R^{\nu}_{\mu\delta\lambda} = \partial_\delta \Gamma^{\nu}_{\mu\lambda} - \partial_\mu \Gamma^{\nu}_{\delta\lambda} + \Gamma^{\alpha}_{\mu\lambda}\Gamma^{\nu}_{\alpha\delta} + \mu \leftrightarrow \delta \tag{281}$$

$$\begin{aligned} R^{\nu}_{\mu\delta\lambda\rho\tau} = -\partial_\delta \Gamma^{\nu}_{\mu\lambda\rho\tau} + \partial_\mu \Gamma^{\nu}_{\delta\lambda\rho\tau} - \Gamma^{\alpha}_{\mu\lambda}\Gamma^{\nu}_{\alpha\delta\rho\tau} - \Gamma^{\alpha}_{\mu\lambda\rho\tau}\Gamma^{\nu}_{\alpha\delta} \\ +\tfrac{1}{2}\partial_\rho \Gamma^{\alpha}_{\mu\nu}\partial_\tau \Gamma^{\nu}_{\alpha\delta} + \mu \leftrightarrow \delta \end{aligned} \tag{282}$$

and

$$\begin{aligned} R^{\nu}_{\mu\delta\lambda\rho\tau\kappa\sigma} = \partial_\delta \Gamma^{\nu}_{\mu\lambda\rho\tau\kappa\sigma} - \partial_\mu \Gamma^{\nu}_{\delta\lambda\rho\tau\kappa\sigma} + \Gamma^{\alpha}_{\mu\lambda}\Gamma^{\nu}_{\alpha\delta\rho\tau\kappa\sigma} \\ +\Gamma^{\alpha}_{\mu\lambda\rho\tau}\Gamma^{\nu}_{\alpha\delta\kappa\sigma} + \tfrac{1}{2}\partial_\rho \Gamma^{\alpha}_{\mu\nu}\partial_\tau \Gamma^{\nu}_{\alpha\delta\kappa\sigma} \\ +\tfrac{1}{2}\partial_\rho \Gamma^{\alpha}_{\mu\nu\kappa\sigma}\partial_\tau \Gamma^{\nu}_{\alpha\delta} - \tfrac{1}{4}\partial_\rho\partial_\kappa \Gamma^{\alpha}_{\mu\lambda}\partial_\tau\partial_\sigma \Gamma^{\nu}_{\alpha\delta} + \mu \leftrightarrow \delta. \end{aligned} \tag{283}$$

Computing the terms in the deformed Riemann tensor is a necessary step we needed to take, but one must recall that the Riemann tensor appears in the Raychaudhuri equation due to the fact that covariant derivatives do not commute. Up until this point, we have not explored what a covariant derivative would look like if it operated on a manifold with a minimum length. It is therefore necessary that we develop a prescription for a covariant derivative which reproduces (273) when we calculate its anti-commutator.

7.5.2 Inventing a Deformed Covariant Derivative

Classically, we have

$$\nabla_\nu V^\rho = \partial_\nu V^\rho + \Gamma^{\rho}_{\nu\sigma} V^\sigma \tag{284}$$

and we say that when a covariant derivative acts on a tensor, we pick up one Christoffel symbol for each index; positive for up indices and negative for down indices. I propose that a deformed covariant derivative acts in almost exactly the same way:

$$\hat{\nabla}_\nu V^\rho = \partial_\nu \hat{V}^\rho + \hat{\Gamma}^{\rho}_{\nu\sigma} \star \hat{V}^\sigma \tag{285}$$

All we need do is keep the partial derivative and use the deformed Christoffel symbol with a star product instead of regular multiplication. The true test is to take the star product anti-commutator and see what results. Classically, we compute the anti-commutator as

$$\left[\nabla_\delta \nabla_\mu - \nabla_\mu \nabla_\delta\right] X^v \tag{286}$$

$$\begin{aligned} &= \partial_\delta \left(\nabla_\mu X^v\right) - \Gamma^\alpha_{\delta\mu} \nabla_\alpha X^v + \Gamma^v_{\delta\sigma} \nabla_\mu X^\sigma \\ &-\partial_\mu \left(\nabla_\delta X^v\right) + \Gamma^\alpha_{\mu\delta} \nabla_\alpha X^v - \Gamma^v_{\mu\sigma} \nabla_\delta X^\sigma \end{aligned} \tag{287}$$

$$\begin{aligned} &= \partial_\delta \left(\partial_\mu X^v + \Gamma^v_{\mu\alpha} X^\alpha\right) - \Gamma^\alpha_{\delta\mu} \left(\partial_\alpha X^v + \Gamma^v_{\alpha\sigma} X^\sigma\right) + \Gamma^v_{\delta\sigma} \left(\partial_\mu X^\sigma + \Gamma^\sigma_{\mu\alpha} X^\alpha\right) \\ &-\partial_\mu \left(\partial_\delta X^v + \Gamma^v_{\delta\alpha} X^\alpha\right) + \Gamma^\alpha_{\mu\delta} \left(\partial_\alpha X^v + \Gamma^v_{\alpha\sigma} X^\sigma\right) - \Gamma^v_{\mu\sigma} \left(\partial_\delta X^\sigma + \Gamma^\sigma_{\delta\alpha} X^\alpha\right) \end{aligned} \tag{288}$$

$$\begin{aligned} &= \partial_\delta \partial_\mu X^v + \partial_\delta \Gamma^v_{\mu\alpha} X^\alpha - \Gamma^\alpha_{\delta\mu} \partial_\alpha X^v - \Gamma^\alpha_{\delta\mu} \Gamma^v_{\alpha\sigma} X^\sigma + \Gamma^v_{\delta\sigma} \partial_\mu X^\sigma + \Gamma^v_{\delta\sigma} \Gamma^\sigma_{\mu\alpha} X^\alpha \\ &-\partial_\mu \partial_\delta X^v - \partial_\mu \Gamma^v_{\delta\alpha} X^\alpha + \Gamma^\alpha_{\mu\delta} \partial_\alpha X^v + \Gamma^\alpha_{\mu\delta} \Gamma^v_{\alpha\sigma} X^\sigma - \Gamma^v_{\mu\sigma} \partial_\delta X^\sigma - \Gamma^v_{\mu\sigma} \Gamma^\sigma_{\delta\alpha} X^\alpha \end{aligned} . \tag{289}$$

Now we assume the no-torsion condition $\Gamma^i_{jk} = \Gamma^i_{kj}$ to arrive at the result

$$\left[\partial_\delta \Gamma^v_{\mu\lambda} - \partial_\mu \Gamma^v_{\delta\lambda} + \Gamma^v_{\delta\sigma} \Gamma^\sigma_{\mu\lambda} - \Gamma^v_{\mu\sigma} \Gamma^\sigma_{\delta\lambda}\right] = R^v_{\lambda\delta\mu}. \tag{290}$$

Before we can repeat the above calculation using the star product and show that this prescription for a deformed covariant derivative does indeed reproduce (273), we need to show that the star product is distributive. Namely that if

$$f = (a + b + c) \tag{291}$$

then we will get

$$f \star g = a \star g + b \star g + c \star g. \tag{292}$$

It is fairly obvious that this is true, since

$$f \star g = (a+b+c) \star g = (a+b+c)g + \frac{i}{2} \theta^{\rho\tau} \partial_\rho (a+b+c) \partial_\tau g - \frac{1}{4} \theta^{\rho\tau} \theta^{\kappa\sigma} \partial_\rho \partial_\kappa (a+b+c) \partial_\tau \partial_\sigma g. \tag{293}$$

which gives us

$$
\begin{aligned}
f \star g = ag &+ \tfrac{i}{2}\theta^{\rho\tau}\partial_\rho a \partial_\tau g - \tfrac{1}{4}\theta^{\rho\tau}\theta^{\kappa\sigma}\partial_\rho\partial_\kappa a \partial_\tau\partial_\sigma g \\
&+bg + \tfrac{i}{2}\theta^{\rho\tau}\partial_\rho b \partial_\tau g - \tfrac{1}{4}\theta^{\rho\tau}\theta^{\kappa\sigma}\partial_\rho\partial_\kappa b \partial_\tau\partial_\sigma g \\
&+cg + \tfrac{i}{2}\theta^{\rho\tau}\partial_\rho c \partial_\tau g - \tfrac{1}{4}\theta^{\rho\tau}\theta^{\kappa\sigma}\partial_\rho\partial_\kappa c \partial_\tau\partial_\sigma g.
\end{aligned} \tag{294}
$$

We also know

$$
a \star g = ag + \frac{i}{2}\theta^{\rho\tau}\partial_\rho a \partial_\tau g - \frac{1}{4}\theta^{\rho\tau}\theta^{\kappa\sigma}\partial_\rho\partial_\kappa a \partial_\tau\partial_\sigma g \tag{295}
$$

which means that we can repeat (293) for b and c to prove distributivity.

Let us now redo the calculation for (286), but add star products and deformed Christoffel symbols. Begin with

$$
\left[\hat{\nabla}_\delta\hat{\nabla}_\mu - \hat{\nabla}_\mu\hat{\nabla}_\delta\right] \star X^v \tag{296}
$$

so we get

$$
\begin{aligned}
&= \partial_\delta\left(\hat{\nabla}_\mu X^v\right) - \hat{\Gamma}^\alpha_{\delta\mu} \star \left(\hat{\nabla}_\alpha X^v\right) + \hat{\Gamma}^v_{\delta\sigma} \star \left(\hat{\nabla}_\mu X^\sigma\right) \\
&-\partial_\mu\left(\hat{\nabla}_\delta X^v\right) + \hat{\Gamma}^\alpha_{\mu\delta} \star \left(\hat{\nabla}_\alpha X^v\right) - \hat{\Gamma}^v_{\mu\sigma} \star \left(\hat{\nabla}_\delta X^\sigma\right).
\end{aligned} \tag{297}
$$

Note that I should strictly be writing $\hat{X^v}$ and $\hat{\Gamma}^v_{\alpha\sigma} \star X^\sigma$. I am not going to do this because I am only using the X^v to keep track of indices - nothing more. Now do as before and expand (297):

$$
\begin{aligned}
&= \partial_\delta\left(\partial_\mu X^v + \hat{\Gamma}^v_{\mu\alpha}X^\alpha\right) - \hat{\Gamma}^\alpha_{\delta\mu} \star \left(\partial_\alpha X^v + \hat{\Gamma}^v_{\alpha\sigma}X^\sigma\right) + \hat{\Gamma}^v_{\delta\sigma} \star \left(\partial_\mu X^\sigma + \hat{\Gamma}^\sigma_{\mu\alpha}X^\alpha\right) \\
&-\partial_\mu\left(\partial_\delta X^v + \hat{\Gamma}^v_{\delta\alpha}X^\alpha\right) + \hat{\Gamma}^\alpha_{\mu\delta} \star \left(\partial_\alpha X^v + \hat{\Gamma}^v_{\alpha\sigma}X^\sigma\right) - \hat{\Gamma}^v_{\mu\sigma} \star \left(\partial_\delta X^\sigma + \hat{\Gamma}^\sigma_{\delta\alpha}X^\alpha\right).
\end{aligned} \tag{298}
$$

Now use the distributive property of the $\star$-product:

$$
\begin{aligned}
&= \partial_\delta\partial_\mu X^v + \partial_\delta\hat{\Gamma}^v_{\mu\alpha}X^\alpha - \hat{\Gamma}^\alpha_{\delta\mu} \star \partial_\alpha X^v - \hat{\Gamma}^\alpha_{\delta\mu} \star \hat{\Gamma}^v_{\alpha\sigma}X^\sigma + \hat{\Gamma}^v_{\delta\sigma} \star \partial_\mu X^\sigma + \hat{\Gamma}^v_{\delta\sigma} \star \hat{\Gamma}^\sigma_{\mu\alpha}X^\alpha \\
&-\partial_\mu\partial_\delta X^v + -\partial_\mu\hat{\Gamma}^v_{\delta\alpha}X^\alpha + \hat{\Gamma}^\alpha_{\mu\delta} \star \partial_\alpha X^v + \hat{\Gamma}^\alpha_{\mu\delta} \star \hat{\Gamma}^v_{\alpha\sigma}X^\sigma - \hat{\Gamma}^v_{\mu\sigma} \star \partial_\delta X^\sigma - \hat{\Gamma}^v_{\mu\sigma} \star \hat{\Gamma}^\sigma_{\delta\alpha}X^\alpha.
\end{aligned} \tag{299}
$$

At this stage in the undeformed version, we said that the no-torsion condition exists, that $\Gamma^i_{jk} = \Gamma^i_{kj}$ and a whole bunch of terms drop out to give us what we need. However, if we were to make the statement that $\hat{\Gamma}^i_{jk} = \hat{\Gamma}^i_{kj}$ we would be saying

$$\Gamma^{\nu}_{\mu\lambda} - i\theta^{xy}\Gamma^{\nu}_{\mu\lambda xy} + \theta^{xy}\theta^{pq}\Gamma^{\nu}_{\mu\lambda xypq} = \Gamma^{\nu}_{\lambda\mu} - i\theta^{xy}\Gamma^{\nu}_{\lambda\mu xy} + \theta^{xy}\theta^{pq}\Gamma^{\nu}_{\lambda\mu xypq} \quad (300)$$

which is the assumption we are going to make. By doing this, we can eliminate all the unwanted terms in (299) just like we did in (289) and arrive at

$$\left[\partial_{\delta}\hat{\Gamma}^{v}_{\mu\lambda} - \partial_{\mu}\hat{\Gamma}^{v}_{\delta\lambda} + \hat{\Gamma}^{v}_{\delta\sigma} \star \hat{\Gamma}^{\sigma}_{\mu\lambda} - \hat{\Gamma}^{v}_{\mu\sigma} \star \hat{\Gamma}^{\sigma}_{\delta\lambda}\right] X^{\nu} = \hat{R}^{v}_{\lambda\delta\mu} \quad (301)$$

which demonstrates exactly what we need - the anti-commutator of our deformed covariant derivative gives us the deformed Riemann tensor.

7.5.3 The Leibniz/Product Rule for the Deformed Covariant Derivative

The second property of the covariant derivative required by the Raychaudhuri equation is that the Leibniz rule holds:

$$\nabla_e X^a X^b = (\nabla_e X^a)\, X^b + X^a \left(\nabla_e X^b\right). \quad (302)$$

This may seem trivial, but keep in mind that there is actually a way to show this by knowing that the covariant derivative adds one Christoffel symbol for each index

$$\nabla_e X^a X^b = \partial_e \left(X^a X^b\right) + \Gamma^{a}_{e\mu} X^{\mu} X^b + \Gamma^{b}_{e\mu} X^a X^{\mu} \quad (303)$$

$$= (\partial_e X^a)\, X^b + X^a \left(\partial_e X^b\right) + \Gamma^{a}_{e\mu} X^{\mu} X^b + \Gamma^{b}_{e\mu} X^a X^{\mu}$$

which is the same thing as (302).

Does the same rule hold for a deformed covariant derivative? We noted before that the star product obeys the Leibniz rule for an ordinary partial derivative, but this does not constitute a proof that it will obey a Leibniz rule for our deformed covariant derivative. Although it may seem clear that it does obey a Leibniz rule, let us demonstrate this and remove any doubt. Working in full, we have

$$\hat{\nabla}_e \hat{X}^a \star \hat{X}^b = \partial_e \left(\hat{X}^a \star \hat{X}^b\right) + \hat{\Gamma}^{a}_{e\mu} \star \hat{X}^{\mu} \star \hat{X}^b + \hat{\Gamma}^{b}_{e\mu} \star \hat{X}^a \star \hat{X}^{\mu} \quad (304)$$

which looks like (303). It appears then that our prescription for a covariant derivative does obey a kind of Leibniz rule. The Leibniz rule and (291) are really the only two pieces of information we need to repeat the derivation of Raychaudhuri's equation.

It is not entirely obvious that the term $\partial_e\left(\hat{X}^a \star \hat{X}^b\right)$ will end up looking like the first term of (303). The star-product term should behave that way to first order, but does it behave that way to higher orders? Let us begin:

$$\partial_e\left(\hat{X}^a \star \hat{X}^b\right) = \partial_e\left(\hat{X}^a\hat{X}^b\right) + \partial_e\left[\frac{i}{2}\theta^{\rho\tau}\partial_\rho\hat{X}^a\partial_\tau\hat{X}^b\right] + \partial_e\left[-\frac{1}{4}\theta^{\rho\tau}\theta^{\kappa\sigma}\partial_\rho\partial_\kappa\hat{X}^a\partial_\tau\partial_\sigma\hat{X}^b\right]. \tag{305}$$

These results show that the star product obeys, to first order at least, the Leibniz rule. First order

$$\partial_e\left(\hat{X}^a\hat{X}^b\right) = \left(\partial_e\hat{X}^a\right)\hat{X}^b + \hat{X}^a\left(\partial_e\hat{X}^b\right) \tag{306}$$

and second order:

$$\partial_e\left[\frac{i}{2}\theta^{\rho\tau}\partial_\rho\hat{X}^a\partial_\tau\hat{X}^b\right] = \left[\frac{i}{2}\theta^{\rho\tau}\partial_e\left(\partial_\rho\hat{X}^a\partial_\tau\hat{X}^b\right)\right] \tag{307}$$

The expressions $\partial_\rho\hat{X}^a$ and $\partial_\tau\hat{X}^b$ are simply functions. This means when acted on by ∂_e, we will get:

$$\partial_e\left(\partial_\rho\hat{X}^a\partial_\tau\hat{X}^b\right) = \left(\partial_e\partial_\rho\hat{X}^a\right)\partial_\tau\hat{X}^b + \left(\partial_e\partial_\tau\hat{X}^b\right)\partial_\rho\hat{X}^a. \tag{308}$$

(307) Thus becomes:

$$\frac{i}{2}\theta^{\rho\tau}\left[\left(\partial_e\partial_\rho\hat{X}^a\right)\partial_\tau\hat{X}^b + \partial_\rho\hat{X}^a\left(\partial_e\partial_\tau\hat{X}^b\right)\right]. \tag{309}$$

In a similar way, $\partial_\rho\partial_\kappa\hat{X}^a$ and $\partial_\tau\partial_\sigma\hat{X}^b$ are also functions. In a similar way, we process (305)

$$\partial_e\left[-\frac{1}{4}\theta^{\rho\tau}\theta^{\kappa\sigma}\partial_\rho\partial_\kappa\hat{X}^a\partial_\tau\partial_\sigma\hat{X}^b\right] = \left[-\frac{1}{4}\theta^{\rho\tau}\theta^{\kappa\sigma}\partial_e\left(\partial_\rho\partial_\kappa\hat{X}^a\partial_\tau\partial_\sigma\hat{X}^b\right)\right] \tag{310}$$

and say

$$\partial_e \left(\partial_\rho \partial_\kappa \hat{X}^a \partial_\tau \partial_\sigma \hat{X}^b\right) = \left(\partial_e \partial_\rho \partial_\kappa \hat{X}^a\right) \partial_\tau \partial_\sigma \hat{X}^b + \partial_\rho \partial_\kappa \hat{X}^a \left(\partial_e \partial_\tau \partial_\sigma \hat{X}^b\right). \quad (311)$$

This means that the third term in (305) becomes:

$$\partial_e \left[-\frac{1}{4}\theta^{\rho\tau}\theta^{\kappa\sigma}\partial_\rho \partial_\kappa \hat{X}^a \partial_\tau \partial_\sigma \hat{X}^b\right] = -\frac{1}{4}\theta^{\rho\tau}\theta^{\kappa\sigma}\left[\left(\partial_e \partial_\rho \partial_\kappa \hat{X}^a\right) \partial_\tau \partial_\sigma \hat{X}^b + \partial_\rho \partial_\kappa \hat{X}^a \left(\partial_e \partial_\tau \partial_\sigma \hat{X}^b\right)\right] \quad (312)$$

Now we use the fact that partial derivatives commute and move the ∂_e to next to the vectors, as in $\partial_e \partial_\rho \partial_\kappa \hat{X}^a$ will become $\partial_\rho \partial_\kappa \partial_e \hat{X}^a$, Let's now rewrite the terms in (309) as

$$\left(\partial_\rho \partial_e \hat{X}^a\right) \partial_\tau \hat{X}^b + \left(\partial_\tau \partial_e \hat{X}^b\right) \partial_\rho \hat{X}^a \quad (313)$$

So then (299) becomes:

$$\frac{i}{2}\theta^{\rho\tau}\left[\left(\partial_\rho \partial_e \hat{X}^a\right) \partial_\tau \hat{X}^b\right] + \frac{i}{2}\theta^{\rho\tau}\left[\partial_\rho \hat{X}^a \left(\partial_\tau \partial_e \hat{X}^b\right)\right] \quad (314)$$

Similarly, we have for (302):

$$\begin{aligned} &-\tfrac{1}{4}\theta^{\rho\tau}\theta^{\kappa\sigma}\left[\left(\partial_e \partial_\rho \partial_\kappa \hat{X}^a\right) \partial_\tau \partial_\sigma \hat{X}^b + \partial_\rho \partial_\kappa \hat{X}^a \left(\partial_e \partial_\tau \partial_\sigma \hat{X}^b\right)\right] \\ &= -\tfrac{1}{4}\theta^{\rho\tau}\theta^{\kappa\sigma}\left[\left(\partial_\rho \partial_\kappa \partial_e \hat{X}^a\right) \partial_\tau \partial_\sigma \hat{X}^b\right] - \theta^{\rho\tau}\theta^{\kappa\sigma}\tfrac{1}{4}\left[\partial_\rho \partial_\kappa \hat{X}^a \left(\partial_\tau \partial_\sigma \partial_e \hat{X}^b\right)\right] \end{aligned} \quad (315)$$

Now combine (306), (314) and (315)

$$\begin{aligned} &\left(\partial_e \hat{X}^a\right) \hat{X}^b + \tfrac{i}{2}\theta^{\rho\tau}\left[\left(\partial_\rho \partial_e \hat{X}^a\right) \partial_\tau \hat{X}^b\right] - \tfrac{1}{4}\theta^{\rho\tau}\theta^{\kappa\sigma}\left[\left(\partial_\rho \partial_\kappa \partial_e \hat{X}^a\right) \partial_\tau \partial_\sigma \hat{X}^b\right] \\ &+\hat{X}^a \left(\partial_e \hat{X}^b\right) + \tfrac{i}{2}\theta^{\rho\tau}\left[\partial_\rho \hat{X}^a \left(\partial_\tau \partial_e \hat{X}^b\right)\right] - \tfrac{1}{4}\theta^{\rho\tau}\theta^{\kappa\sigma}\tfrac{1}{4}\left[\partial_\rho \partial_\kappa \hat{X}^a \left(\partial_\tau \partial_\sigma \partial_e \hat{X}^b\right)\right] \end{aligned}. \quad (316)$$

This allows us to conclude:

$$\partial_e \left(\hat{X}^a \star \hat{X}^b\right) = \left(\partial_e \hat{X}^a\right) \star \hat{X}^b + \hat{X}^a \star \left(\partial_e \hat{X}^b\right). \quad (317)$$

Ultimately, we have shown that the star product obeys the Leibniz rule for the deformed covariant derivative:

$$\hat{\nabla}_e \hat{X}^a \star \hat{X}^b = \left(\hat{\nabla}_e \hat{X}^a\right) \star \hat{X}^b + \hat{X}^a \star \left(\hat{\nabla}_e \hat{X}^b\right). \tag{318}$$

7.5.4 Putting The Ingredients Together

We know that we have:

1. A prescription for the deformed covariant derivative.
2. A way to get from equation (296) to (301).
3. A deformed covariant derivative as well as a star product which obey the Leibniz rule.

We now need only a single remaining ingredient necessary to reproduce the Raychaudhuri equation - a prescription for the deformed metric. Classically, the metric and tetrad are related by the fairly simple expression

$$g_{\mu\nu} = \eta_{mn} e^m_\mu e^n_\nu \tag{319}$$

which is readily calculated if one has an explicit expression for the tetrad at all points on the manifold. While it would be nice to use (319) for our deformed work, there remains the possibility that if we use it, we might get imaginary terms appearing in expressions we need to be real. To prevent this, we define the metric as

$$\hat{g}_{\mu\nu} = \frac{1}{2}\eta_{ab}\left(\hat{e}^a_\mu \star \hat{e}^{b\dagger}_\nu + \hat{e}^b_\mu \star \hat{e}^{a\dagger}_\nu\right) \tag{320}$$

which is real by definition. We now have all the ingredients we need to calculate the Raychaudhuri equation; we will repeat the entire process simply by adding stars and hats, knowing that we have explicitly justified each step we are taking. We can feel confident just shunting symbols about knowing that we have justified whatever symbol-shunting it is that we need to do.

Before we begin, there is an important point to make regarding the vectors which appear in the derivation. Since they are deformed versions of the original tensors, we are going to make the substitution $\xi^a \to \hat{\xi^a}$. We will, however at no point calculate the explicit form of the $\hat{\xi^a}$ vectors. This is because the vectors will eventually fall out of the expression for the Einstein Field Equations when we complete Jacobson's equation. There is no need to explicitly calculate an expansion for $\hat{\xi^a}$ when $\hat{\xi^a}$ is not going to appear in the final answer.

Since each step has been justified, there is no need to walk through the entire derivation again. We can do a slightly simplified derivation by writing

$$\hat{B}_{\mu\nu} = \hat{\nabla}_\nu \star \hat{\xi_\mu} = \partial_\nu \hat{\xi_\mu} + \hat{\Gamma}^\alpha_{\nu\mu} \star \hat{\xi_\alpha} \tag{321}$$

and define the deformed expansion, shear and twist as

$$\hat{\vartheta} = \hat{B}^{\mu\nu} \star \hat{h}_{\mu\nu}, \tag{322}$$

$$\hat{\sigma}_{\mu\nu} = \hat{B}_{(\mu\nu)} - \frac{1}{2}\hat{\vartheta} \star \hat{h}_{\mu\nu} \tag{323}$$

and

$$\hat{\omega}_{\mu\nu} = \hat{B}_{[\mu\nu]}. \tag{324}$$

Note that from now on we are going to use the variable ϑ instead of θ for the expansion variable in order to avoid confusing it with the non-commutative parameter. The Raychaudhuri equation is then derived as

$$\hat{\xi}^\alpha \star \hat{\nabla}_\alpha \hat{B}_{\mu\nu} = \hat{\xi}^\alpha \star \hat{\nabla}_\nu \hat{\nabla}_\alpha \hat{\xi}_\mu + \hat{R}^\delta_{\alpha\nu\mu} \star \hat{\xi}^\alpha \star \hat{\xi}_\delta \tag{325}$$

$$= \hat{\nabla}_\nu \left(\hat{\xi}^\alpha \star \hat{\nabla}_\alpha \hat{\xi}_\mu\right) - \left(\hat{\nabla}_\nu \hat{\xi}^\alpha\right) \star \left(\hat{\nabla}_\alpha \hat{\xi}_\mu\right) + \hat{R}^\delta_{\alpha\nu\mu} \star \hat{\xi}^\alpha \star \hat{\xi}_\delta \tag{326}$$

$$= \hat{B}^\alpha_\nu \star \hat{B}_{\mu\alpha} + \hat{R}^\delta_{\alpha\nu\mu} \star \hat{\xi}^\alpha \star \hat{\xi}_\delta. \tag{327}$$

Which allows us to take the trace and arrive at

$$\frac{d}{d\lambda}\hat{\vartheta} = -\frac{1}{2}\hat{\vartheta} \star \hat{\vartheta} - \hat{\sigma}_{\mu\nu} \star \hat{\sigma}^{\mu\nu} + \hat{\omega}_{\mu\nu} \star \hat{\omega}^{\mu\nu} + \hat{R}_{\alpha\mu} \star \hat{\xi}^\alpha \star \hat{\xi}^\mu. \tag{328}$$

Equation (328) is a highly desirable expression. It shows that our "hats and stars" prescription carries over to the Raychaudhuri equation unchanged.

7.6 A New Gravitational Equation of State

We are now going to take the deformed Raychaudhuri equation and use it to repeat Jacobson's calculation. We can skip over the conceptual explanation and start with

$$\delta Q = -\kappa \int_H \lambda \hat{T}_{ab} \star \hat{k}^a \star \hat{k}^b d\lambda dA \tag{329}$$

and feel justified doing this integral, for the reasons outlined in. Now we are going to make use of the deformed Raychaudhuri equation to get

$$-\kappa \int_H \lambda \hat{T}_{ab} \star \hat{k}^a \star \hat{k}^b d\lambda dA = \int_H -\lambda \hat{R}_{ab} \star \hat{k}^a \star \hat{k}^b d\lambda dA. \tag{330}$$

There is an important point to be made about making the approximation of negligible twist and shear while retaining terms which may be even smaller. We must realize that when we set σ and ω equal to zero, we are making a conceptual statement about spacetime thermodynamics. By doing this, we are more closely adhering to the gravitational analog of the equilibrium approach to thermodynamics. Although some leading-order terms of $\hat{\sigma}_{\mu\nu}$ and $\hat{\omega}_{\mu\nu}$ might be larger than the higher-order terms of $\hat{R}_{ab}$, these first-order terms more properly belong in a study of non-equilibrium gravitational thermodynamics [37]. Removing the integrals from (330) gives us

$$\hat{T}_{ab} \star \hat{k}^a \star \hat{k}^b d\lambda dA = \hat{R}_{ab} \star \hat{k}^a \star \hat{k}^b d\lambda dA. \tag{331}$$

and by requiring that our definition for a null surface still holds:

$$\hat{g}_{ab} \star \hat{k}^a \star \hat{k}^b = 0. \tag{332}$$

We recover

$$\hat{T}_{ab} \star \hat{k}^a \star \hat{k}^b d\lambda dA = \left(\hat{R}_{ab} + \hat{g}_{ab}\right) \star \hat{k}^a \star \hat{k}^b d\lambda dA \tag{333}$$

which becomes

$$\hat{T}_{ab} = \hat{R}_{ab} + \hat{g}_{ab}. \tag{334}$$

The Bianchi identities carry through as before, since we are only contracting indices, this leaves us with:

$$\left(\frac{2\pi}{\hbar\eta}\right) \hat{T}_{ab} = \hat{R}_{ab} - \frac{1}{2}\hat{R} \star \hat{g}_{ab} + \Lambda \hat{g}_{ab}. \tag{335}$$

The result in (334) is as pleasing as it is simple, the trouble lies in expanding the expressions to second order in the non-commutative parameter and in interpreting the non-commutative stress-energy tensor.

The first issue is not terribly problematic, we could simply work as we have before and evaluate (334) term-by-term. The Ricci tensor would be expressed as

$$\begin{aligned}\hat{R}_{\mu\lambda} &= \hat{R}^{\nu}_{\mu\nu\lambda} \\ &= \hat{R}^{0}_{\mu 0\lambda} + \hat{R}^{1}_{\mu 1\lambda} + \hat{R}^{2}_{\mu 2\lambda} + \hat{R}^{3}_{\mu 3\lambda,}\end{aligned} \tag{336}$$

from which we could calculate the deformed Ricci scalar. The star product-term would be

$$\hat{R} \star g_{\mu\nu} = \hat{R}\hat{g}_{\mu\nu} + \frac{i}{2}\theta^{ij}\partial_i\hat{R}\partial_j\hat{g}_{\mu\nu} - \frac{1}{4}\theta^{ij}\theta^{kl}\partial_i\partial_k\hat{R}\partial_j\partial_l\hat{g}_{\mu\nu} \tag{337}$$

and we already have a fairly direct expression for the metric in equation (319).

There are two ways to deal with a stress-energy tensor living on a non-commutative space. The first, more rigorous way is to proceed as in [30] and say that $\hat{T}_{\mu\nu}$ is the stress-energy tensor of a massive field living on a non-commutative manifold. If we assume that it is a massive scalar field which is the gravitational source, then we can expand $\hat{T}_{\mu\nu}$ in powers of the non-commutative parameter [30]:

$$\begin{aligned}\hat{T}_{\mu\nu} &= \tfrac{1}{2}\left(\partial_\mu\phi \star \partial_\nu\phi + \partial_\nu\phi \star \partial_\mu\phi\right) - \tfrac{1}{2}\eta_{\mu\nu}\left(\partial_\alpha\phi \star \partial_\alpha\phi - m^2\phi \star \phi\right) \\ &= T_{\mu\nu} + \eta_{\mu\nu}\tfrac{m^2 l^4}{16}\theta^{\alpha\beta}\theta^{\sigma\rho}\partial_\alpha\partial_\sigma\phi\partial_\beta\partial_\rho\phi.\end{aligned} \tag{338}$$

By doing this, we can take an explicit form of the stress-energy tensor, calculate the non-commutative corrections and match them, term-by-term, to the corrections to the geometric variables. This, then, provides a complete description for calculating a full theory of deformed gravity without the need for the approximations in [30].

Another way to approach the problem is to take a conceptually sound but non-rigorous approach to the spacetime thermodynamics. Recall that the statistical concept of entropy does not care whether the physics we are doing is classical or quantum, commutative or non-commutative. The entropy crossing the null horizon in the Rindler spacetime can therefore be of any form ,which means that if we wanted to calculate the area

$$\int_H -\lambda\hat{R}_{ab} \star \hat{k}^a \star \hat{k}^b d\lambda dA \tag{339}$$

or

$$\int_H -\lambda R_{ab} k^a k^b d\lambda dA, \tag{340}$$

we would get the same result, since these are both equal to dS (up to constants) and dS does not care about the nature of the matter which contains the entropy. This approach cannot go as far as the first approach since it does not supply us with any details about $T_{\mu\nu}$, only its contraction $T_{\mu\nu}k^\mu k^\nu$.

8 Discussion

8.1 Relation to a Theory of Quantum Gravity

We have used the Seiberg-Witten map to construct a full theory of gravity living on a non-commutative spacetime. The non-commutative corrections are straightforward, if very tedious, to calculate and if we are given a scalar field, then we can use the deformed Einstein Equations to calculate almost any quantity of interest. What remains to be said is that our calculations do not constitute a theory of quantum gravity. By forcing the manifold's coordinates to be non-commutative and writing out the second-order corrections we have not turned any of the geometric variables into quantum variables. Our metric is still not a quantized metric.

We could use our formulas and methods to calculate, for example, non-commutative corrections to black holes. These corrections may be more accurate than the predictions made by classical gravity, they may even match the predictions made by a true quantum theory of gravity to the first few orders, but it is extremely unlikely they would match to all orders.

8.2 Improving the Calculations

As pointed out in [25], we need to question whether or not our calculations do indeed break diffeomorphism invariance. If it is found that the above calculations do indeed break diffeomorphism invariance, it is still possible to change the work to accommodate it. As was noted in the paper by Chamseddine[25], it is possible to instead write down a different star product, not the Moyal star product, which preserves diffeomorphism invariance on the manifold. What this means for the calculation is not that we will stop making star product substitutions as in (255) and (275); we will in fact make the exact some substitutions. Rather, what is going to change is the form of the corrections to the classical variables.

We will have to start from the beginning to calculate the corrections to the classical variables, but the rest of the work will stay as is. In this way, the algebra we have done where we have not explicitly written down the corrections should remain unchanged. Repeating the derivations of the correction terms will not be trivial, but we also have not lost every bit of work we have done.

Another key point to consider is interpreting the correction terms we have calculated. It is necessary to do the mathematics to arrive at definitive answers for the calculations, but now that the calculations have been done it would be nice to in some way interpret what the corrective terms are telling us about the behaviour of gravity on a non-commutative manifold.

8.3 Further Questions

Recall that in classical General Relativity, the Raychaudhuri equation is used to study the behaviour of hypersurfaces in spacetime; particularly the surfaces of black holes. In order to prove the second law of black hole thermodynamics, we used the Raychaudhuri equation, the Einstein Field Equations and the assumption that an observer in spacetime must always see a positive energy density. Since we now have $\left(\frac{2\pi}{\hbar\eta}\right)\hat{T}_{ab} = \hat{R}_{ab} - \frac{1}{2}\hat{R}\star\hat{g}_{ab} + \Lambda\hat{g}_{ab}$, we will be able to project onto a null surface and use the modified Raychaudhuri equation to get

$$\frac{d}{d\lambda}\hat{\vartheta} = -\frac{1}{2}\hat{\vartheta}\star\hat{\vartheta} - \hat{\sigma}_{\mu\nu}\star\hat{\sigma}^{\mu\nu} + \hat{\omega}_{\mu\nu}\star\hat{\omega}^{\mu\nu} + \hat{R}_{\alpha\mu}\star\hat{\xi}^{\alpha}\star\hat{\xi}^{\mu} \leq 0. \tag{341}$$

As before, we want to integrate, but we will not be integrating $d\theta/d\lambda \leq -\frac{1}{2}\theta^2$, rather it will be $\frac{d}{d\lambda}\hat{\vartheta} \leq -\frac{1}{2}\hat{\vartheta}\star\hat{\vartheta}$. The non-commutative expansion is manifestly not the same thing as the ordinary expansion. It comes with many curvature terms which are also dependent upon the parameter λ. So if we write

$$\frac{1}{\hat{\vartheta}}(\lambda) \geq \frac{1}{\hat{\vartheta}_0} + \frac{\lambda}{2} \tag{342}$$

it is not immediately obvious that we are guaranteed to run into a caustic at the parameter value of $\lambda \leq 2/|\theta_0|$; the same place we run into a caustic in the classical case.

Since the Raychaudhuri equation is used to study the behaviour of hypersurfaces in spacetime like was done above, it would be interesting to study the behaviour of the same surfaces using the non-commutative Raychaudhuri equation instead. By doing this, we will almost certainly get different answers to the classical case.

Appendix A

In this appendix we will explicitly compute the expressions found in (277), (278) and (279). Note that we will eventually end up working with a large amount of dummy indices and more than one non-commutative parameter. Because of this, we need to be careful when forming products such as

$$(i\theta^{xy}e^{\nu}_{axy})(\theta^{xy}\theta^{pq}\partial_{\mu}e^{a}_{\lambda xypq}) \tag{343}$$

where the tensor product between indices is implied and the contraction is always over all spacetime dimensions, even if the index is Latin. We could write (341) as

$$i\theta^{xy}\theta^{xy}\theta^{pq}e^{\nu}_{axy}\partial_{\mu}e^{a}_{\lambda xypq} \tag{344}$$

but this is clumsy and confusing. To avoid this, we will have to relabel the dummy indices which come with the non-commutative parameter. Thus, instead of (342) we can instead replace $x \to j$, $y \to k$ in the first theta; $\theta^{xy} \to \theta^{jk}$. We are, of course, allowed to do this because these are only dummy indices. Following this relabeling, we write

$$i\theta^{jk}\theta^{xy}\theta^{pq}e^{\nu}_{ajk}\partial_{\mu}e^{a}_{\lambda xypq} \tag{345}$$

instead of (342). For what follows, note that we will keep the convention that coordinate frame indices are Greek and tetrad frame indices are Latin, but the dummy indices coming with the non-commutative parameter will always be Latin and will always sum over all spacetime dimensions.

We start off by stating that the Christofell symbol on a non-commutative spacetime, $\hat{\Gamma}^{\nu}_{\mu\lambda}$, can be expanded in terms of the non-commutative parameter,

$$\hat{\Gamma}^{\nu}_{\mu\lambda} = \Gamma^{\nu}_{\mu\lambda} - i\theta^{xy}\Gamma^{\nu}_{\mu\lambda xy} + \theta^{xy}\theta^{pq}\Gamma^{\nu}_{\mu\lambda xypq}. \tag{346}$$

where we are going to use $\hat{e}^{\nu}_{a} \star \partial_{\mu}\hat{e}^{a}_{\lambda} + \hat{e}^{\nu}_{a} \star \hat{e}^{b}_{\lambda} \star \hat{\omega}^{a}_{\mu b}$ to find $\Gamma^{\nu}_{\mu\lambda xy}$ and $\Gamma^{\nu}_{\mu\lambda xypq}$. Start with

$$\hat{\Gamma}^{\nu}_{\mu\lambda} = \hat{e}^{\nu}_{a} \star \partial_{\mu}\hat{e}^{a}_{\lambda} + \hat{e}^{\nu}_{a} \star \hat{e}^{b}_{\lambda} \star \hat{\omega}^{a}_{\mu b} \tag{347}$$

and start the calculation by expanding the star product of the first term on the right hand side:

$$\hat{e}_a^\nu \star \partial_\mu \hat{e}_\lambda^a = \hat{e}_a^\nu \partial_\mu \hat{e}_\lambda^a + \frac{i}{2}\theta^{xy}\partial_x \hat{e}_a^\nu \partial_y \left(\partial_\mu \hat{e}_\lambda^a\right) + \frac{1}{4}\theta^{xy}\theta^{pq}\partial_x\partial_p \hat{e}_a^\nu \partial_y \partial_q \left(\partial_\mu \hat{e}_\lambda^a\right). \quad (348)$$

We still need to expand the non-commutative tetrads

$$\hat{e}_a^\nu = e_a^\nu + i\theta^{xy} e_{axy}^\nu + \theta^{xy}\theta^{pq} e_{axypq}^\nu \quad (349)$$

$$\hat{e}_\nu^a = e_\nu^a + i\theta^{xy} e_{\nu xy}^a + \theta^{xy}\theta^{pq} e_{\nu xypq}^a \quad (350)$$

which we already have expressions for in (260). Now take (347) and (348) and insert them into the first time on the right hand side of (346):

$$\begin{gathered}\hat{e}_a^\nu \partial_\mu \hat{e}_\lambda^a = \left(e_a^\nu + i\theta^{xy} e_{axy}^\nu + \theta^{xy}\theta^{pq} e_{axypq}^\nu\right)\left(\partial_\mu e_\lambda^a + i\theta^{xy}\partial_\mu e_{\lambda xy}^a + \theta^{xy}\theta^{pq}\partial_\mu e_{\lambda xypq}^a\right) \\ = e_a^\nu \partial_\mu e_\lambda^a + i\theta^{xy} e_a^\nu \partial_\mu e_{\lambda xy}^a + \theta^{xy}\theta^{pq} e_a^\nu \partial_\mu e_{\lambda xypq}^a \\ + i\theta^{xy} e_{axy}^\nu \partial_\mu e_\lambda^a - \theta^{xy}\theta^{pq} e_{axy}^\nu \partial_\mu e_{\lambda pq}^a + i\theta^{xy}\theta^{pq}\theta^{jk} e_{axy}^\nu \partial_\mu e_{\lambda pqjk}^a \\ + \theta^{xy}\theta^{pq} e_{axypq}^\nu \partial_\mu e_\lambda^a + i\theta^{xy}\theta^{pq}\theta^{jk} e_{axypq}^\nu \partial_\mu e_{\lambda jk}^a + \theta^{xy}\theta^{pq}\theta^{jk}\theta^{lm} e_{axypq}^\nu \partial_\mu e_{\lambda jklm}^a\end{gathered}. \quad (351)$$

Now do the same for the second term on the right hand side of (346):

$$\begin{gathered}\tfrac{i}{2}\theta^{xy}\partial_x \hat{e}_a^\nu \partial_y \left(\partial_\mu \hat{e}_\lambda^a\right) \\ = \tfrac{i}{2}\theta^{xy}\left[\left(\partial_x e_a^\nu + i\theta^{pq}\partial_x e_{apq}^\nu + \theta^{pq}\theta^{jk}\partial_x e_{apqjk}^\nu\right)\left(\partial_y \partial_\mu e_\lambda^a + i\theta^{pq}\partial_y\partial_\mu e_{\lambda pq}^a + \theta^{pq}\theta^{jk}\partial_y\partial_\mu e_{\lambda pqjk}^a\right)\right] \\ = \tfrac{i}{2}\theta^{xy}\Big[\partial_x e_a^\nu \partial_y\partial_\mu e_\lambda^a + i\theta^{pq}\partial_x e_a^\nu \partial_y \partial_\mu e_{\lambda pq}^a + \theta^{pq}\theta^{jk}\partial_x e_a^\nu \partial_y\partial_\mu e_{\lambda pqjk}^a \\ + i\theta^{pq}\partial_x e_{apq}^\nu \partial_y\partial_\mu e_\lambda^a - \theta^{pq}\theta^{jk}\partial_x e_{apq}^\nu \partial_y\partial_\mu e_{\lambda jk}^a + i\theta^{pq}\theta^{jk}\theta^{lm}\partial_x e_{apq}^\nu \partial_y\partial_\mu e_{\lambda jklm}^a. \\ \theta^{pq}\theta^{jk}\partial_x e_{apqjk}^\nu \partial_y \partial_\mu e_\lambda^a + i\theta^{pq}\theta^{jk}\theta^{lm}\partial_x e_{apqjk}^\nu \partial_y\partial_\mu e_{\lambda lm}^a \\ + \theta^{pq}\theta^{jk}\theta^{lm}\theta^{rs}\partial_x e_{apqjk}^\nu \; \partial_y\partial_\mu e_{\lambda lmrs}^a\Big]\end{gathered} \quad (352)$$

Note that in the above we have kept terms which are of order 3 and higher. We will discard these terms later, but they have been included in the above calculations for completeness. And finally the third term

$$\tfrac{1}{4}\theta^{xy}\theta^{pq}\partial_x\partial_p\hat{e}^{\nu}_a\partial_y\partial_q\left(\partial_\mu\hat{e}^a_\lambda\right) = \tfrac{1}{4}\theta^{xy}\theta^{pq}\left[\left(\partial_x\partial_p e^{\nu}_a + i\theta^{jk}\partial_x\partial_p e^{\nu}_{ajk} + \theta^{jk}\theta^{lm}\partial_x\partial_p e^{\nu}_{ajklm}\right)\right.$$
$$\left.\left(\partial_y\partial_q\partial_\mu e^a_\lambda + i\theta^{jk}\partial_y\partial_q\partial_\mu e^a_{\lambda jk} + \theta^{jk}\theta^{lm}\partial_y\partial_q\partial_\mu e^a_{\lambda jklm}\right)\right] = \tfrac{1}{4}\theta^{xy}\theta^{pq}\left(\partial_x\partial_p e^{\nu}_a\partial_y\partial_q\partial_\mu e^a_\lambda\right) + \ldots \tag{353}$$

where we have truncated after the only term which is second order. All of the other terms are higher than second order and there is no need to continue writing them down as we already have examples of higher order terms in (349) and (350).

We have calculated all we need to calculate from the first term on the right hand side of (345). We now turn to the second term and start by calculating $\hat{e}^{\nu}_a \star \hat{e}^b_\lambda$

$$\hat{e}^{\nu}_a \star \hat{e}^b_\lambda = \hat{e}^{\nu}_a\hat{e}^b_\lambda + \frac{i}{2}\theta^{xy}\partial_x\hat{e}^{\nu}_a\partial_y\hat{e}^b_\lambda - \frac{1}{4}\theta^{xy}\theta^{pq}\partial_x\partial_p\hat{e}^{\nu}_a\partial_y\partial_q\hat{e}^b_\lambda$$

and work term-by-term, as before. We start with

$$\begin{aligned}\hat{e}^{\nu}_a\hat{e}^b_\lambda &= \left(e^{\nu}_a + i\theta^{xy}e^{\nu}_{axy} + \theta^{xy}\theta^{pq}e^{\nu}_{axypq}\right)\left(e^b_\lambda + i\theta^{xy}e^b_{\lambda xy} + \theta^{xy}\theta^{pq}e^b_{\lambda xypq}\right)\\ &= e^{\nu}_a e^b_\lambda + i\theta^{xy}e^{\nu}_a e^b_{\lambda xy} + \theta^{xy}\theta^{pq}e^{\nu}_a e^b_{\lambda xypq} + i\theta^{xy}e^b_\lambda e^{\nu}_{axy}\\ &\quad -\theta^{xy}e^{\nu}_{axy}\theta^{pq}e^b_{\lambda pq} + \theta^{xy}\theta^{pq}e^b_\lambda e^{\nu}_{axypq}\end{aligned} \tag{354}$$

where we have excluded terms higher than second order. We go on to calculate:

$$\begin{aligned}\tfrac{i}{2}\theta^{xy}\partial_x\hat{e}^{\nu}_a\partial_y\hat{e}^b_\lambda &= \tfrac{i}{2}\theta^{xy}\partial_x\left(e^{\nu}_a + i\theta^{pq}e^{\nu}_{apq} + \theta^{pq}\theta^{jk}e^{\nu}_{apqjk}\right)\partial_y\left(e^b_\lambda + i\theta^{pq}e^b_{\lambda pq} + \theta^{pq}\theta^{jk}e^b_{\lambda pqjk}\right)\\ &= \partial_x e^{\nu}_a\partial_y e^b_\lambda + i\theta^{pq}e^{\nu}_{apq}\partial_y e^b_\lambda + i\theta^{pq}\partial_y e^b_{\lambda pq}\partial_x e^{\nu}_a.\end{aligned} \tag{355}$$

Note that we actually only get a single term from

$$-\frac{1}{4}\theta^{xy}\theta^{pq}\partial_x\partial_p\hat{e}^{\nu}_a\partial_y\partial_q\hat{e}^b_\lambda = -\frac{1}{4}\theta^{xy}\theta^{pq}\partial_x\partial_p e^{\nu}_a\partial_y\partial_q e^b_\lambda \tag{356}$$

since this term is already of second order. We still need to calculate the second star product of the second term in (345) and to do this, we are going to collect all the terms in (352), (353) and (354) and call them $\hat{f}^{\nu b}_{a\lambda}$:

$$\begin{gathered}\hat{f}^{\nu b}_{a\lambda} = e^{\nu}_{a}e^{b}_{\lambda} + i\theta^{xy}e^{\nu}_{a}e^{b}_{\lambda xy} + \theta^{xy}\theta^{pq}e^{\nu}_{a}e^{b}_{\lambda xypq} + i\theta^{xy}e^{b}_{\lambda}e^{\nu}_{axy} \\ -\theta^{xy}e^{\nu}_{axy}\theta^{pq}e^{b}_{\lambda pq} + \theta^{xy}\theta^{pq}e^{b}_{\lambda}e^{\nu}_{axypq} + \tfrac{i}{2}\theta^{xy}\partial_x e^{\nu}_{a}\partial_y e^{b}_{\lambda} - \tfrac{1}{2}\theta^{xy}\theta^{pq}\partial_x e^{\nu}_{apq}\partial_y e^{b}_{\lambda} \\ -\tfrac{1}{2}\theta^{xy}\theta^{pq}\partial_y e^{b}_{\lambda pq}\partial_x e^{\nu}_{a} - \tfrac{1}{4}\theta^{xy}\theta^{pq}\partial_x\partial_p e^{\nu}_{a}\partial_y\partial_q e^{b}_{\lambda}.\end{gathered} \tag{357}$$

This means that the final step is to calculate

$$\hat{f}^{\nu b}_{a\lambda} \star \hat{\omega}^{a}_{\mu b} = \hat{f}^{\nu b}_{a\lambda}\hat{\omega}^{a}_{\mu b} + \frac{i}{2}\theta^{xy}\partial_x \hat{f}^{\nu b}_{a\lambda}\partial_y \hat{\omega}^{a}_{\mu b} - \frac{1}{4}\theta^{xy}\theta^{pq}\partial_x\partial_p \hat{f}^{\nu b}_{a\lambda}\partial_y\partial_q \hat{\omega}^{a}_{\mu b} \tag{358}$$

where we are making the expansion

$$\hat{\omega}^{a}_{\mu b} = \omega^{a}_{\mu b} - i\theta^{xy}\omega^{a}_{\mu bxy} + \theta^{xy}\theta^{pq}\omega^{a}_{\mu bxypq}. \tag{359}$$

The first term on the right hand side of (356) is expanded as

$$\begin{gathered}(e^{\nu}_{a}\ e^{b}_{\lambda} + i\theta^{xy}e^{\nu}_{a}e^{b}_{\lambda xy} + \theta^{xy}\theta^{pq}e^{\nu}_{a}e^{b}_{\lambda xypq} + i\theta^{xy}e^{b}_{\lambda}e^{\nu}_{axy} \\ -\theta^{xy}e^{\nu}_{axy}\theta^{pq}e^{b}_{\lambda pq} + \theta^{xy}\theta^{pq}e^{b}_{\lambda}e^{\nu}_{axypq} + \tfrac{i}{2}\theta^{xy}\partial_x e^{\nu}_{a}\partial_y e^{b}_{\lambda} - \tfrac{1}{2}\theta^{xy}\theta^{pq}\partial_x e^{\nu}_{apq}\partial_y e^{b}_{\lambda} \\ -\tfrac{1}{2}\theta^{xy}\theta^{pq}\partial_y e^{b}_{\lambda pq}\partial_x e^{\nu}_{a} - \tfrac{1}{4}\theta^{xy}\theta^{pq}\partial_x\partial_p e^{\nu}_{a}\partial_y\partial_q\ e^{b}_{\lambda})\Big(\omega^{a}_{\mu b} \\ -i\theta^{xy}\omega^{a}_{\mu bxy} + \theta^{xy}\theta^{pq}\ \omega^{a}_{\mu bxypq}\Big)\end{gathered} \tag{360}$$

and all the terms up to second order are

$$\begin{gathered}e^{\nu}_{a}e^{b}_{\lambda}\omega^{a}_{\mu b} + i\theta^{xy}e^{b}_{\lambda xy}e^{\nu}_{a}\omega^{a}_{\mu b} + \theta^{xy}\theta^{pq}e^{b}_{\lambda xypq}e^{\nu}_{a}\omega^{a}_{\mu b} \\ +i\theta^{xy}e^{\nu}_{axy}e^{b}_{\lambda}\omega^{a}_{\mu b} - \theta^{xy}\theta^{pq}e^{b}_{\lambda xy}e^{\nu}_{apq}\omega^{a}_{\mu b} + \theta^{xy}\theta^{pq}e^{b}_{\lambda}e^{\nu}_{axypq}\omega^{a}_{\mu b} \\ +\tfrac{i}{2}\theta^{xy}\partial_x e^{\nu}_{a}\partial_y e^{b}_{\lambda}\omega^{a}_{\mu b} - \tfrac{1}{2}\theta^{xy}\theta^{pq}\partial_x e^{b}_{\lambda xy}\partial_y e^{\nu}_{a}\omega^{a}_{\mu b} - \tfrac{1}{2}\theta^{xy}\theta^{pq}\partial_x e^{\nu}_{axy}\partial_y e^{b}_{\lambda}\omega^{a}_{\mu b} \\ -\tfrac{1}{4}\theta^{xy}\theta^{pq}\partial_x\partial_p e^{\nu}_{a}\partial_y\partial_q e^{b}_{\lambda}\omega^{a}_{\mu b} + i\theta^{xy}e^{\nu}_{a}e^{b}_{\lambda}\omega^{\mu}_{bxy} + \theta^{xy}\theta^{pq}e^{b}_{\lambda xy}e^{\nu}_{a}\omega^{a}_{\mu pq} \\ +\theta^{xy}\theta^{pq}e^{b}_{\lambda}e^{\nu}_{axy}\omega^{a}_{\mu pq} + \tfrac{1}{2}\theta^{xy}\theta^{pq}\partial_x e^{\nu}_{a}\partial_y e^{b}_{\lambda}\omega^{a}_{\mu bxy} \\ +\theta^{xy}\theta^{pq}e^{b}_{\lambda}e^{\nu}_{a}\omega^{a}_{\mu bxypq}.\end{gathered} \tag{361}$$

Since the second term in (356) is already a first-order term, we are going to pick up fewer terms when we expand it:

$$\begin{gathered}\tfrac{i}{2}\theta^{xy}\partial_x e^{\nu}_{a}e^{b}_{\lambda}\partial_y\omega^{a}_{\mu b} - \tfrac{1}{2}\theta^{xy}\theta^{pq}\partial_x e^{b}_{\lambda pq}e^{\nu}_{a}\partial_y\omega^{a}_{\mu b} \\ -\tfrac{1}{2}\theta^{xy}\theta^{pq}\partial_x e^{\nu}_{apq}e^{b}_{\lambda}\partial_y\omega^{a}_{\mu b} - \tfrac{1}{2}\theta^{xy}\theta^{pq}\partial_x\partial_p e^{\nu}_{a}e^{b}_{\lambda}\partial_y\partial_q\omega^{a}_{\mu b}.\end{gathered} \tag{362}$$

The third and final term of (356) is already a second-order term. This means that we will only keep:

$$-\frac{1}{4}\theta^{xy}\theta^{pq}\partial_x\partial_p e^\nu_a e^b_\lambda \partial_y\partial_q \omega^a_{\mu b}. \tag{363}$$

We now have all the terms up to second order from (345). If we now start collecting first order terms and matching them to the first order term on the right hand side of (344) we will get

$$\Gamma^\nu_{\mu\lambda} = e^\nu_a \partial_\mu e^a_\lambda + e^\nu_a e^b_\lambda \omega^a_{\mu b}, \tag{364}$$

as expected. Matching the first order term in (344) to all the collected first order terms we get

$$\Gamma^\nu_{\mu\lambda xy} = \begin{array}{cc} -e^\nu_a \partial_\mu e^a_{\lambda xy} & -e^\nu_{axy}\partial_\mu e^a_\lambda \\ -\frac{1}{2}\partial_x e^\nu_a \partial_y \partial_\mu e^a_\nu & -e^b_{\lambda xy} e^\nu_a \omega^a_{\mu b} \\ +e^\nu_{axy} e^b_\lambda \omega^a_{\mu b} & \frac{1}{2}\partial_x e^\nu_a \partial_y e^b_\lambda \omega^a_{\mu b} \\ +e^\nu_a e^b_\lambda \omega^a_{\mu bxy} & -\partial_x e^v_a e^b_\lambda \partial_y \omega^a_{\mu b} \end{array}, \tag{365}$$

and matching second-order to second-order gives us:

$$\Gamma^\nu_{\mu\lambda xypq} = \begin{array}{cc} +e^v_a \partial_\mu e^a_{\lambda xypq} & -e^v_{axy}\partial_\mu e^a_{\lambda pq} \\ +e^v_{axypq}\partial_\mu e^a_\lambda & -\frac{1}{2}\partial_x e^v_a \partial_y \partial_\mu e^a_{\nu pq} \\ -\frac{1}{2}\partial_x e^\nu_{apq}\partial_y\partial_\mu e^a_\nu & -\frac{1}{4}\partial_x\partial_p e^v_a \partial_y \partial_q \partial_\mu e^a_\nu \\ +e^b_{\lambda xypq} e^v_a \omega^a_{\mu b} & -e^\nu_{axy} e^b_{\lambda pq}\omega^a_{\mu b} \\ +e^\nu_{axypq} e^b_\lambda \omega^a_{\mu b} & -\frac{1}{2}\partial_y e^b_{\lambda pq}\partial_x e^\nu_a \omega^a_{\mu b} \\ -\frac{1}{4}\partial_x\partial_p e^\nu_a \partial_y \partial_q e^b_\lambda \omega^a_{\mu b} & +e^b_{\lambda xy} e^\nu_a \omega^a_{\mu bpq} \\ +e^\nu_{axy} e^b_\lambda \omega^a_{\mu bpq} & +\frac{1}{2}\partial_x e^\nu_a \partial_y e^b_\lambda \omega^a_{\mu bpq} \\ +e^\nu_a e^b_\lambda \omega^a_{\mu bxypq} & -\frac{1}{2}\partial_x e^b_{\lambda pq} e^\nu_a \partial_y \omega^a_{\mu b} \\ -\frac{1}{2}\partial_x e^\nu_{apq} e^b_\lambda \partial_y \omega^a_{\mu b} & -\frac{1}{4}\partial_x\partial_p e^\nu_a \partial_q \partial_y e^b_\lambda \omega^a_{\mu b} \\ -\frac{1}{4}\partial_x\partial_p e^\nu_a e^b_\lambda \partial_y \partial_p \omega^a_{\mu b} & \end{array}. \tag{366}$$

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