CHAPTER 3

THEORETICAL FRAMEWORK AND REVIEW OF RELATED LITERATURE

3.1 Introduction

In articulating a perspective for this study, I focus on three key concepts: "thinking", "understanding" and "knowledge connectedness". The concepts "thinking" and "understanding" have been used interchangeably in the literature relating to mathematical activity. I acknowledge the difficulty of defining fully the terms "thinking" and "understanding". Moreover, it is not the intention of this study to deal with the explicit defining of terms extensively. However, this study is concerned with some indicators of thinking and understanding. The term "thinking" is conceived as the ability to exercise the mind in order to make a decision. On the other hand, the term 'understanding' is conceived as the ability to know or comprehend the nature or the meaning of something (Collins English Dictionary, 1979 and The Oxford English Reference Dictionary,1995). These skills are intertwined and both take place in one's mind. This implies that in order for an individual to come to know the nature or the meaning of something, she should first exercise the mind to make a decision about that something and vice-versa. Dindyal (2003) (quoting Burton, 1984) defined thinking as the means used by humans to improve their understanding of, and exert some control over their environment. Making meaning in mathematics involves making connections among different concepts in mathematics, that is "… the forging of connections across domains [in mathematics]" (Noss and Hoyles, 1996, p. 130). Students' ability to make meaning and establish connections – "bridges" – between concepts is dependent on the nature of the subject matter they are working with and the levels of thinking and understanding that is being demanded in the problem-solving context related to the subject.

The present chapter describes the theoretical framework used in this study. It starts off with two geometric modes: synthetic geometry (Euclidean geometry) and analytic geometry (coordinate geometry). In this section, I suggest that between synthetic and analytic geometry there is the continuity and the complementarity of *modus operandi* (Gascón, 2002).

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Besides, I stress the role of algebraic thinking in geometrical understanding within the interplay of these two geometric modes.

In the section on mathematical understanding I review this notion as a way of gaining insight into algebraic and geometric understanding. The key idea is that in order to understand algebra or geometry, one should be aware of the way these branches of mathematics are constructed. She should develop ‘fluency’ of the language of algebra or geometry by being involved in social processes through discursive practice in particular settings (e.g. school).

In the section on algebraic thinking and geometrical understanding I attempt to explain the cognitive processes which underlie both algebraic thinking and geometrical understanding and I provide a detailed explanation of the elements involved in those processes. A study such as this one that involves geometry investigation might be expected to require extensive reference to van Hiele model of thinking in geometry. As I note later, related studies especially Schoenfeld (1986) and Dindyal (2003) have indicated some key limitations of van Hiele framework. I discuss these limitations later in this chapter and justify why my study was not extensively informed by this framework.

The section on conceptual model for algebraic thinking in geometrical understanding presents the connections between these two cognitive processes and how algebraic thinking might aid geometric understanding in practice in line with Prawat´s framework on transfer and learning.

I also provide a theoretical framework schema and illustrate how it was implemented in this study.

Finally related empirical studies are described focusing on the content, key findings, limitations, contributions, and implications for my study.

3.2 Theoretical Framework

Geometric Modes: Synthetic versus Analytic Geometry

There are claims that Euclidean geometry might itself be a stumbling block for students' understanding and performance in geometry in general. Kline (1972) supports this claim when he asserts that:

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> Descartes was disturbed by the fact that every proof in Euclidean geometry called for some new, often ingenious, approach. He explicitly criticized the geometry of the ancients as being too abstract, and so much tied to figures. (p. 308).

Moreover Descartes criticized synthetic geometry (the geometrical arguments of Euclid and Apollonius) for lacking a general method (strategy). According to Hansen (1998) Descartes achieved this goal of finding a general method in geometry by the introduction of coordinate systems and the creation of analytic geometry (coordinate geometry).

Tall and Vinner (1981) found that iconic processing (visual and spatial imagery) which served students well at an earlier stage might later become an impediment for the development of the formal theory in the student's mind. San (1996) found that the use of iconic support in mathematical problem solving is of paramount importance, wherever students meet a certain concept for the first time. However, San supports the idea that there is a need to move along the continuum to abstraction so that image schemata – a bridge between abstract logical structures and particular concrete images and experience – may become more flexible and abstract. Similar perspectives can be found in Watson *et al* (1993) and Campbell *et al* (1995).

For this reason to promote flexible and abstract image schemata in students' minds it might be important that we foster the use of algebra in geometry thinking. This can facilitate students to move from particular concrete images (Euclidean geometry or synthetic geometry) to abstract logical structures (e.g. analytic geometry). Stillwell (1994) observed that the bridge between Euclidean geometry and abstract algebra is the algebraic geometry (or analytic geometry) invented by Fermat and Descartes, a method devoted to the application of algebra in geometry and geometry in algebra. The ultimate aim was the geometric solution to problems by algebraic methods.

According to Kline (1972) the importance assigned to algebra as a tool for geometry has to do with the possibility to "recognize the typical problems of geometry and to bring together problems that in geometrical form would not appear to be related at all" (p. 314). He asserts that "Algebra brings to geometry the most natural principles of classification and the most natural hierarchy of method" and that "Not only can questions of solvability and geometrical constructability be decided elegantly, quickly, and fully from parallel algebra, but without it

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they cannot be decided at all. Thus, system and structure were transferred from geometry to algebra. (p. 314)

Kline (1972) continues saying:

> For example, in synthetic geometry, to prove that the altitudes of a triangle meet in a point, intersections inside and outside the triangle are considered separately. In analytic geometry they are considered together. (p. 322)

Dindyal (2003) also asserts that "many [geometric] problems are easier to solve using coordinate geometry (analytic geometry), transformations or vectors than using traditional Euclidean geometry" (p. 6). An example to illustrate this idea is as follows. If we are to show that a line which bisects two sides of a triangle is parallel to the third side in the Euclidean geometry, then we can use, for example, the following knowledge schemas: We need to use an auxiliary theorem which states that a line bisecting one side of a triangle and is parallel to a second side bisects the third side of the triangle. Afterwards, it is recommendable to approach the theorem using an indirect proof, where at each step what we say has a meaning in relation to a figure (Lewis, 1964, pp. 325, 326). While in the coordinate geometry context, we can use a direct proof, approaching it, for example, by setting any coordinates for the vertexes of the triangle and using collinear vectors and formulae of midpoint coordinates. It might be simpler to use vector algebra geometrically. Analysing the solutions we can conclude that using traditional Euclidean geometry requires one to be acquainted with previously demonstrated theorems, postulates and axioms. Besides, one needs to make complicated connections and be bound to figures at each step. However, using coordinate geometry requires one to know vector algebra equations, which are simpler to deduce and to recall. Afterwards, one applies them to directly solve the problem. In other words through coordinate geometry we can solve the geometric problems in more elegant, quick, and fuller way than in Euclidean geometry.

Anton and Rorres (1991) corroborate this view stating that vector algebra is a method, which usually leads to simpler and more elegant proofs than the methods studied in a first course in geometry, the same methods that Euclid himself used. Anton and Rorres explain that this is because the basic equations of vector algebra are vector-space axioms and therefore apply to Euclidean space. Thus, any new equation that can be generated from these basic equations constitutes a theorem. It is argued that since it is easier to manipulate equations than to write

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out in words how one is applying axioms, this method results in a certain economy of thought and a cleaner presentation of proofs.

The preceding discussion suggests that there is power in using algebra in working in geometry. In order to use the power of algebra as a tool into geometric understanding, one needs to describe what constitutes algebraic and geometric understanding and how algebraic understanding might enhance geometric understanding. As a way of gaining insight into algebraic and geometric understanding, it is important to examine the notion of mathematical understanding from a general perspective.

Mathematical Understanding

It has been recognized by mathematics education researchers that there are no definite answers to questions about what constitutes “mathematical understanding” (Mousley, 2003). However, one can only notice some “manifestations” of this complex phenomenon. Sierpinska (1994) characterises such manifestations as models of understandings. In order for an individual to be acquainted with the models of understanding, and therefore, to (partially) understand what mathematical understanding is, it seems important to refer to what the individual intends to understand – ‘the object of understanding’, mathematics in this case. According to Devlin (2000), all mathematical facts are proved by deduction from some initial sets of assumptions, or *axioms* (from the Latin *axioma*, meaning ‘a principle’). The axioms are assumed to be, or at least widely accepted within the mathematical community, obvious truths. In other words, all mathematical facts are constructed from the basic ones by proof. Thus, mathematics can metaphorically be considered as a ‘building’. It is from this perspective that understanding in mathematics can be portrayed as cumulative structuring (Mousley, 2003). That is, mathematics is a networked framework of ideas, where understanding of simpler ideas (axioms) forms a foundation or anchor point for higher-level ideas (propositions and theorems). According to Devlin, the axioms are like the foundations of a building. In common language, one can portray mathematical understanding as cumulative structuring by recognizing the need to know how the ‘building’ of mathematics is constructed. Devlin, quoting Galileo, contends: “The great book of nature can be read only by those who know the language in which it is written. And this language is mathematics” (p.10). Mathematics is also considered as a ‘language’. And language is a social tool. Hence, in order for one to know the ‘language’ of mathematics, in other words, to understand mathematics, one must be involved in social processes through discursive practice in

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particular settings. Thus, mathematical understanding may be portrayed as a social process (Mousley, 2003). Within this context, one can develop fluency of the 'language' of mathematics at different levels depending on the different forms of meaning held and developed simultaneously but individually. In this way, understanding may be seen as a form of knowing (Mousley, 2003).

Algebra and geometry may be conceived as two different branches of mathematics. That is why the three models of mathematical understanding are applicable to either algebra or geometry. In order for an individual to understand algebra or geometry, one should be aware of the way these branches of mathematics are constructed. She should develop 'fluency' of the language of algebra or geometry by being involved in social processes through discursive practice in particular settings (e.g. school).

Algebraic Thinking and Geometrical Understanding

In attempting to understand algebraic thinking and geometric understanding, it might be important to understand what is involved in these processes. Charbonneau (1996, p. 15), quoting Michael Mahoney, stated that an algebraic way of thinking involves the following:

> 1) Operational symbolism. 2) The preoccupation with mathematical relations rather than with mathematical objects, which relations determine the structures constituting the subject-matter of modern algebra. The algebraic mode of thinking is based, then, on relational rather than on predicate logic. 3) Freedom from any ontological questions and commitments and, connected with this, abstractness rather than intuitiveness.

These properties of algebraic thinking, i. e. symbolization, mathematical relations and abstraction, make algebra a particularly powerful tool for understanding other domains of mathematics, in this case, geometry. The process of abstraction is particularly complex as it involves several cognitive processes which is worth to clarify. According to von Glasersfeld (1995) as quoted by Steele and Johanning (2004) "abstraction is the process by which the mind selects, organizes, and combine actions, and then records them in memory" (p. 66). Thompson (1994) as quoted by Steele and Johanning (2004) summed up the notion of action within cognitive processes saying that "action (is) not an observable behaviour, but an activity of the mind (which is) tied to the experience of individuals as they interact with the

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physical world around them” (p. 66). Wheeler (1996, p. 322) argued that “Algebra introduces one to a set of tools (representations) – tables, graphs, formulae, equations, arrays, identities, functional relations” that are intricately related and constitute “a substantial technology that can be used to discover and invent things [in any branch of mathematics]”. Kieran (1996) as quoted by Steele and Johanning (2004) defined the term algebraic thinking as “the use of any representations that handle quantitative situations in a relational way” (p. 65). In turn, Kaput and Blanton (2001), Lee (2001), and Mason (1996) as quoted by Irwin and Britt (2005) all argued that developing an awareness of, and applying generality in any mathematical domain is in itself an indicator of algebraic thinking. Dindyal (2003) also pointed out the significance of the above three aspects of algebraic thinking, which concern the use of symbols and algebraic relations; the use of representations and the use of patterns for generalisations (one aspect of abstraction). Greenes and Findell suggested that the big ideas of algebraic thinking involve representation, proportional reasoning, balance, meaning of variable, patterns and functions, and inductive and deductive reasoning (Greenes and Findell, 1998 as quoted by Vennebush, Marquez, and Larsen, 2005). Similarly, Kriegler contended that algebraic thinking combines two very important components: algebraic ideas, including patterns, variables, and functions, with mathematical thinking tools, specifically problem solving, representation (e.g. diagrams, words, tables), and reasoning (Kriegler, 2004 as quoted by Vennebush, Marquez, and Larsen, 2005). Hence, Vennebush *et al* (2005) added saying that “algebraic thinking is used in any activity that combines a mathematical process with one of the big ideas in algebra, such as understanding patterns and functions, representing situations with symbols, using mathematical models, and analysing change (p. 87).” This combination is largely noticed in geometry due to its proper nature.

Stillwell (1998) described geometry as a dual subject in its essence, “First there is the visual, self-contained, synthetic side, which seems intuitively natural; then the algebraic, analytic side, which takes over when intuition fails and integrates geometry into the larger world of mathematics” (p. 105).

Stillwell described geometric intuition as “our imagination (that) leads us to conclusions via steps that ‘look right’ but may not have a purely logical basis” (p. 37). Meanwhile, he asserted that the results seen by intuition should be validated by logic. One of the attempts to validate intuition results is the so-called synthetic geometry. Although in this system, all theorems are derived by pure logic from a list of visually plausible axioms, those theorems

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are close to intuition, that is the steps in a proof imitate the way we see a theorem. In turn Gueudet-Chartier (2006) characterized intuition as a type of cognition that seeks self evidence, immediacy, and certitude. She added saying that models are a central factor of intuition in mathematics. Fischbein (1987) as quoted by Gueudet-Chartier called them intuitive models. She considered geometric intuition the use of models stemming from geometry. Because of the nature of geometry, she stated that a geometric model is always associated with a figural model. In her study the figural model she used corresponded to the use of drawings that mean pictorial representations. Similarly, in my study I associate geometric intuition with visualization and construction of (mental or physical) pictures. Fischbein (1999) corroborated the relation between the synthetic aspect of geometry and intuition stating that "Intuition… means a global, synthetic grasp and interpretation of a situation" (p. 23). Accordingly, in my study I considered the strategies used in the Euclidean Geometry problem solving as were mostly driven by intuition because the theorems are close to intuition.

The dual nature of geometry makes the subject extremely rich. It enables the interplay and interconnection between mathematical language (e.g. algebra) and the language of pictures, between the synthetic approach (where at each step what you say has a meaning in relation to a figure) and the analytic approach (using coordinates to facilitate transfer to a numeric or algebraic framework, which allows blind calculation) Douady (1998, p. 26, 27). According to Douady, a critical aspect necessary for the development of a more complete understanding [of a geometric concept] is to understand the equivalence between all its definitions, to have them all accessed and made available at each moment, to be able to conveniently choose one or the other, and to transfer properties from one framework to another. This is an important proficiency particularly for students studying geometry.

For this proficiency in geometry to be realised, an individual needs to develop three kinds of cognitive processes which fulfill specific epistemological functions. A framework that Duval (1998) presented recognizes that these cognitive processes are closely connected and their synergy is cognitively necessary for proficiency in geometry. These processes are:

- Visualization processes, for example the visual representation of a geometrical statement, or the heuristic exploration of a complex geometrical situation;
- Construction processes (using tools); and

- Reasoning in relationship to discursive processes for extension of knowledge, for proof, for explanation (p.38).

Duval added up saying these processes can be performed separately and independently. However, these three kinds of cognitive processes are closely connected.

Hershkowitz (1998) suggested that the visual reasoning (visualization processes) is much more than an intuitive support for higher level reasoning: it is the backbone of a rigorous proof. She explained (quoting Hanna, 1990) that the visual process includes: 1) a new way of looking at the situation in order to suggest a generalisation, 2) its proof and verification in one process, and 3) an explanation of "why" the generalisation holds. However, Duval claimed that visual reasoning in some cases can be misleading or impossible [due to its subjectivity]. For example a circle can be seen as an ellipse in another plane than the frontal one. Besides, quadrature of the circle is impossible within ruler-and-compass context (proved by the German mathematician Ferdinand Lindemann in 1882 as quoted by Devlin, 2000); consequently, within this context visual reasoning is impossible.

Construction processes depend only on connections between mathematical properties and the technical constrains of the tools used (Duval, 1998).

Reasoning takes place when by experimentation [construction by ruler and compass or geometrical softwares] and inductive generalisation [by visualization processes], one extends her geometrical knowledge about shapes and relations and extends her "vocabulary" of legitimate ways of reasoning. Deductive reasoning [dependent exclusively on the corpus of propositions – definitions, axioms, and theorems] then becomes a vehicle for understanding and explaining why an inductively discovered conjecture might hold (Hershkowitz, 1998).

Another framework describing the development of geometrical thinking that has been the subject of considerable research is the van Hiele model of thinking in geometry (see, for instance, Schoenfeld, 1986). Schoenfeld stated that the foundations of the van Hiele model are in pedagogical "experience" and teaching experiments, and that the justifications offered for it are loose by any rigorous standard and justified theory. Similarly lacking is a detailed explanation of cognitive processes that underlie competent performance in geometry. However, he recognized that this model provides an important starting point for conceptualizing learner growth and understanding in geometry: formal symbolic manipulations in the deductive universe are meaningless for those students who lack a deep intuitive understanding of the properties of the elements in the empirical reference world from which the abstract objects manipulated in the deductive universe are abstracted. This reinforces what was previously mentioned that Euclidean geometry might serve as an introductory subject to geometry to foster a deep intuitive understanding of the properties of the geometric objects. Meanwhile, there might be a need to move along to formal symbolic manipulations in the deductive universe (e. g. analytic geometry) for the development of abstract logical structures (abstract image schemata) in students' mind that may facilitate the development of the formal theory of geometry.

Griffiths (1998) added arguing that "it is misleading to try to place a pupil at one (van Hiele) level (globally), since they can be at different levels with different pieces of work. Moreover, the van Hiele levels seem better applied not to the pupil, but to the path needed for teaching a piece of work" (p. 202). Likewise, Senk (1989) found that students did not think at the same van Hiele level in all areas of geometry content. Accordingly she advised "research that attempts to sort out the overlap between skill or knowledge and the thinking processes that characterize van Hiele levels should use instruments that are content specific" (p. 320).

Dindyal (2003) also recognized that the van Hiele model provides a valuable framework for studying geometric thinking. However, this framework mostly targets Euclidean geometry. That is why studies that have used the van Hiele framework have tended to focus on geometry in a purely Euclidean context at the school level. Dindyal observed that "algebra and algebraic thinking, which has significantly influenced the secondary school geometry curriculum, has not been considered in this [van Hiele] framework" (p. 10).

In general, the main focus of research on the van Hiele theory has been directed at the notion of a hierarchy of five levels of thinking. As modified by Hoffer (1981) quoted by Burger and Shaughnessy (1986) the five van Hiele levels are as follows:

Level 1 (Visualization): The student reasons about basic geometric concepts, such as simple shapes, primarily by means of visual considerations of the concept as a whole without explicit regard to properties of its components.

Level 2 (Analysis): The student reasons about geometric concepts by means of an informal analysis of component parts and attributes. Necessary properties of the concept are

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established. In other words, figures are identified by their mathematical properties. The properties, however, are seen to be independent of one another. They are discovered and generalized from the observation (visualization) of a few examples.

Level 3 (Abstraction): The student logically orders the properties of concepts, forms abstract definitions, and can distinguish between the necessity and sufficiency of a set of properties in determining a concept.

Level 4 (Deduction): The student reasons formally within the context of a mathematical system, complete with undefined terms, axioms, an underlying logical system, definitions, and theorems.

Level 5 (Rigour): The student can compare systems based on different axioms and can study various geometries in the absence of concrete models.

Pegg *et al* (1998) asserted that some researchers such as Burger and Shaughnessy (1986) and Gutiérrez *et al* (1991), had even challenged what van Hiele refers to as one of the most distinctive features of the levels, namely, their discontinuity. As a result two consequences of this theory were observed, namely:

> 1) Learners may be on different van Hiele levels for different concepts, and 2) the van Hiele levels are not particularly suited for fine-grained analyses of a learner's understanding. (p. 280)

From the previous descriptions, links can be established between the van Hiele Theory and the conceptual model adapted in this study (Table 3.1) so as some contributions may be proposed to face some of the shortcomings mentioned earlier. The conceptual model adapted in this study is presented in Fig 3.1.

Table 3.1: Links between van Hiele levels and the conceptual model adapted in the study

Conceptual Model Adapted	**van Hiele Levels**
Synthetic aspect of geometry	Levels 1 and 2
Analytic aspect of geometry	Levels 2, 3, 4, and 5

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According to Stillwell (1998) the synthetic side of geometry is concerned with visualization processes driven by intuition, thus corresponding to van Hiele Level 1 and partly Level 2. Level 2 (partly) to Level 5 seem to characterise the analytic side of geometry as analysis, abstraction, deduction, and rigour are mainly analytic concepts and especially in this case algebraic stances where reasoning plays a key role. Hence, the concern of this study is not to place the learner's understanding according to the van Hiele levels discontinuously. On the contrary the concern is in placing the learner's understanding in terms of the two aspects of geometry, synthetic and analytic being algebra (algebraic thinking) the bridge between the two aspects in the Mozambique context particularly at Universidade Pedagógica. The research in this study is therefore theoretically significant as it explores how algebra may serve as a tool into geometrical understanding, a conceptual relationship that has not received any focused attention in the conceptualization of the van Hiele model of understanding.

Conceptual Model for Algebraic Thinking in Geometrical Understanding

The above discussion points to a critical relationship between algebraic thinking and geometric understanding, which are essential elements of mathematical understanding. Accordingly, the schema below is an attempt to highlight the connections between algebraic thinking and geometrical understanding.

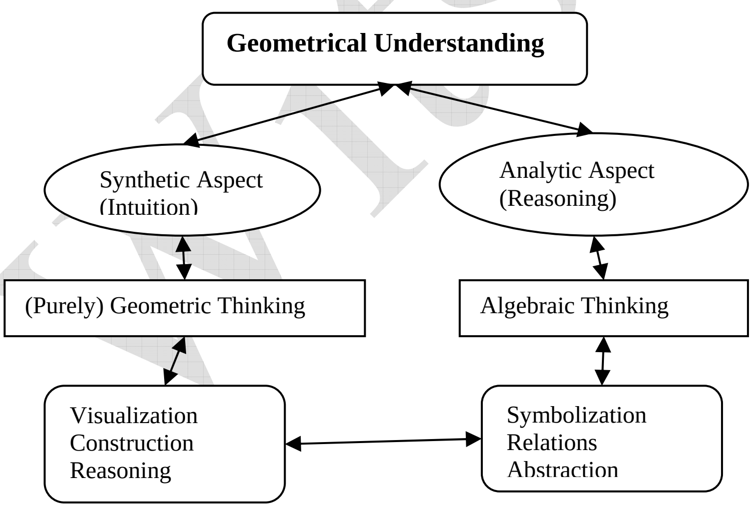

Figure 3.1: Conceptual model for algebraic thinking in geometrical understanding

The connections among cognitive processes are, in general, very complex to describe (Prawat, 1989). This model only attempts to explore one aspect of these connections, specifically, how algebraic thinking might be an aid to geometrical understanding, which portrays a subset of what constitutes the universe of mathematical understanding. The figure shows that the cognitive processes (symbolization, relations, and abstraction), which underlie algebraic thinking, are interconnected. These cognitive processes might jointly be used to aid any of the cognitive processes which constitute geometrical understanding (visualization processes, construction processes, and reasoning), either separately or jointly. As mentioned previously, to be proficient in geometry, in other words, to conceptually understand and perform in geometry, the three cognitive processes which underlie geometric understanding should be acquired and nurtured conjointly. The arrows in two directions mean that both algebraic and geometric cognitive processes are mutually interconnected due to the nature of both disciplines (Charbonneau, 1996). However, the concern of this study is the direction algebra is aiding geometry.

It is interesting to mention that Lakatos (1977, p. 121) as quoted by Gascón (2002) asserted that classically algebra was considered by Descartes as “an analytic art”; that is why, the term “analytic” is used almost as a synonym of “algebraic”. Accordingly, in this study when I use the term analytic strategy I mean algebraic strategy or better a strategy where we can find “its natural justification and interpretation within algebra” (ibid., p. 13). Moreover Charbonneau (1996) asserted that “analysis is at the heart of algebra” (p. 36).

Algebra is viewed as the official language of communication and expression for the most part of mathematics (including geometry) (Noss and Hoyles, 1996). That is, algebra constitutes one of a possible myriad of generalisations expressible in widely different forms. This is the critical issue of this study, which is to investigate to what extent algebra holds this privileged position within mathematics regarding its connections to geometry. Algebraic thinking (a way of abstracting) can be seen as a way of layering meanings on each other, connecting between ways of knowing and seeing, rather than as a way of replacing meaning of each other. My contention is that these meanings become reshaped as learners exploit available tools (in algebra) to move the focus of their attention onto new objects and relationships (for instance in geometry).

The central idea, therefore, is that algebra can be a particularly fruitful domain of ‘situated abstraction’, a setting where we may see the externalized (or internalized) face of mathematical objects and relationships through the window of algebraic thinking and accompanying linguistic and notational structures.

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However, a critical question arises concerning how algebraic thinking might aid geometric understanding in practice. In other words, what is (are) the framework(s), which "studies(y) how ideas get formed and transformed when expressed through different media (in our case algebra and geometry), when actualized in particular contexts, when worked out by individual minds [in the context of problem solving]" (Noss and Hoyles, 1996 quoting Ackermann, 1991, p. 121)? This question relates to the problematic nature of (knowledge) transfer and learning. Prawat (1989) defined transfer as "the ability to draw on or access one's intellectual resources (knowledge, strategy and disposition) in situations where those resources may be relevant" (p. 1). In turn, Greeno *et al* (1996) as quoted by Steele and Johanning (2004) said that transfer is a process of acquiring a schema that relates across situations. A schema is a mechanism in human memory that allows for the storage, synthesis, generalisation, and retrieval of similar experiences (Marshall, 1995 as quoted by Steele and Johanning, 2004). The framework used by Prawat shows that two factors influence students´ ability to access knowledge, strategy, and disposition are *organization* and *awareness*. In the knowledge base category, organization is equivalent to connectedness of key concepts and procedures which provide the glue that holds cognitive structures (or schemas) together. According to Prawat, the adequacy of this structure determines the accessibility or availability of information at a later time. Besides, developing connectedness also requires one to make representational links and deal with her informal knowledge. Developing reflective awareness in students is seen as another way to enhance knowledge accessibility. This awareness takes place when students articulate their own thoughts in different ways and in different settings.

In the strategic category organization "appears to be how to strike the right balance between specificity on the one hand and generalizability on the other" (ibid. p. 33). According to Prawat specific strategies, when accessed, lead to a certain result, although they do not readily transfer to new, potentially relevant situations as general strategies do. However, general strategies are more difficult to access. Reflective awareness in this category, "students should be aware of how factors such as the nature of the material to be learned or the kind of outcome they wish to achieve can influence the sort of strategy they select" (ibid. p. 33).

In the dispositional category organization explains the relationship between dispositions (defined as performance and mastery dispositions) and strategies (or cognitive skills).

Mastery dispositions aim at increasing competence, while performance dispositions intent to do well and thus gain a positive judgment of one's competence. Prawat quoting Borkowski *et al* (1987) asserted that "… affective and motivational components (e.g., self-attributions about achievement) can energize or hinder the use of a strategy or skill on a transfer task" (p. 27). Specifically, Prawat contended that "complex learning strategies such as planning, monitoring, and checking appear to go hand in hand with a mastery orientation of learning. If this is true, the best way to develop reflective awareness at the dispositional level is to render strategic action more familiar, automatic, and generally easier to perform" (p. 33).

In light of the conceptual model suggested and the framework on learning and transfer, instruments were designed to collect appropriate data to answer the research questions of this study. Specifically, geometry tasks were constructed where the subjects were required to use algebraic thinking aiming at seeing whether such thinking could facilitate students' (conceptual) geometric understanding, i. e. visualization, construction, and reasoning processes working in synergy. Students' algebraic and geometric knowledge base, strategies, and disposition were researched in light of *organization* and *awareness* to see whether they could access and utilize their intellectual resources in problem situations where those resources might be relevant.

3.3 Theoretical Framework Schema

This section presents a discussion of how the theoretical framework was implemented in the study. Firstly a schema (Fig 3.2) is provided and later explained in detail.

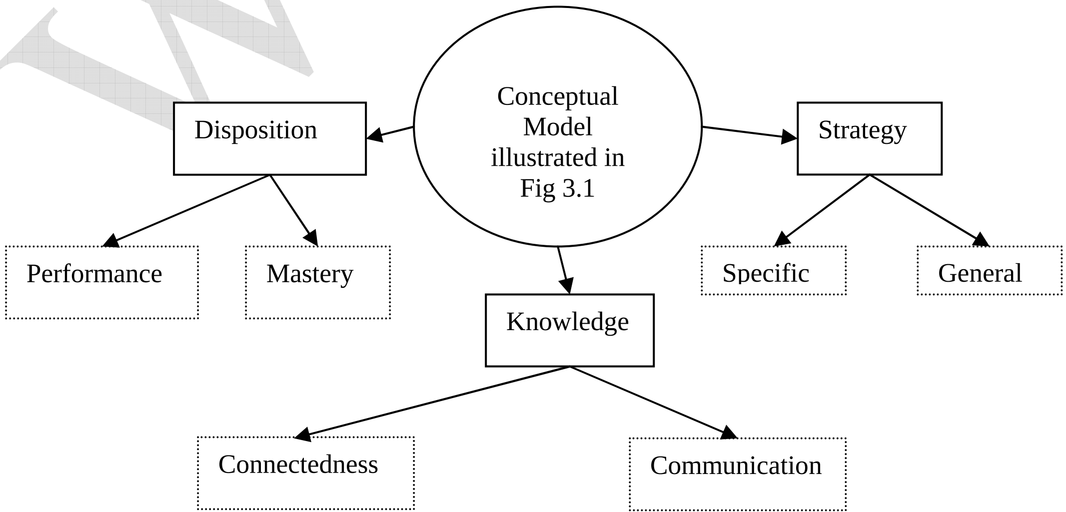

Figure 3.2: Theoretical framework schema

This study investigated how first-year university students at Universidade Pedagógica in Maputo-Mozambique bring their knowledge and thinking of algebra in understanding and working with geometry. The conceptual model for algebraic thinking in geometrical understanding was drawn using the reviewed literature. My interest, therefore, was to find out the extent to which this model was relevant to the context of my study with students from Universidade Pedagógica. Prawat's framework on transfer and learning appeared to be relevant. Student's knowledge and thinking was investigated in terms of connectedness of key concepts and procedures when students articulated or communicated their own thoughts in different ways and in different settings. Students' communication through written responses and interviews represented situations involving meaning making in algebra and geometry, that is, making connections among different concepts in algebra and geometry. This represented "the forging of connections across domains [in my case algebra and geometry]" (Noss and Hoyles, 1996, p. 130).

According to this model, the strategy category consisted of two aspects: specific strategies (in my case synthetic aspect of geometry) and general strategies (in my case analytic aspect of geometry) (Hansen, 1998). Hence the students' strategies were analysed in light of these two aspects. Moreover the students were asked to clarify the sort of strategy they selected in terms of concepts used and/or the outcomes they wished to achieve.

At dispositional category I looked for the way the teaching and learning process was taking place in students themselves and in the classroom whether the orientation was that of mastery or/and of performance. I further researched whether these two types of disposition energized or hindered the use of strategies or skills on the tasks.

3.4 Research Questions and Theoretical Framework Schema

In order to answer the research questions in light of the framework of this study, a schema (Fig 3.3) summarizes how I dealt with the analysis of data accordingly. More details regarding the research questions are presented in Chapter 1, Section 1.2.

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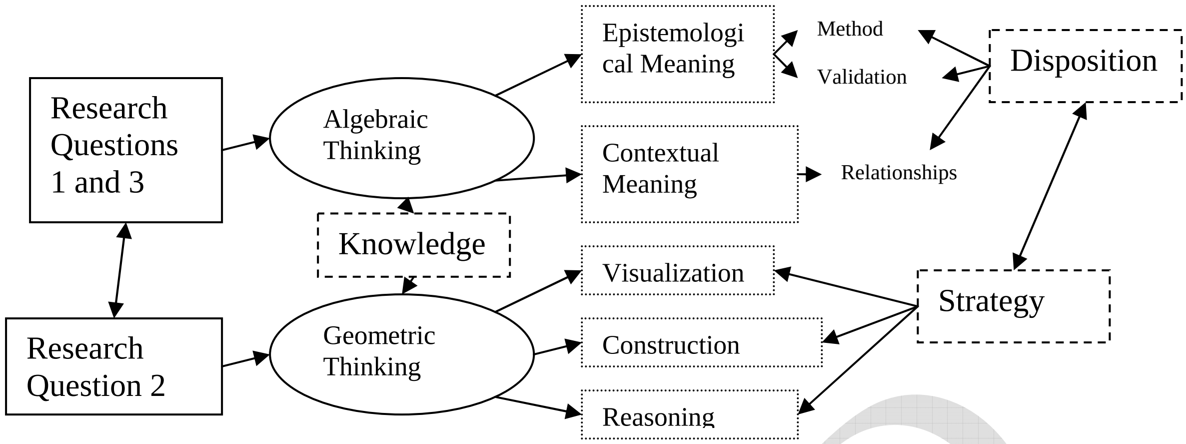

Figure 3.3: Link between research questions and theoretical framework schema

This schema shows that Research Questions 1 and 3 focus on the indicators of algebraic thinking in relation to the meanings students held while solving geometric tasks. In turn, Research Question 2 refers to conceptual geometric understanding portrayed by visualization, construction, and reasoning working in synergy. All three research questions are interconnected as they attempt to explore how students work with geometry problems, what algebraic knowledge, if any they bring to the solution of these problems, and how knowledge of algebra and algebraic ways of thinking promotes or hinders success in geometry problem-solving processes. Knowledge, strategy, and disposition constitute the lenses (analytical tools) through which I attempted to answer the research questions of this study.

3.5 Some Empirical Studies on Geometry and Algebra Teaching

There have been very few empirical studies that have investigated the link between algebraic thinking and geometric understanding (Dindyal, 2003).

In a study related to this research, Schoenfeld (1986) attempted to answer the following questions:

> 1) What does an individual need to understand in order to develop competency in a particular mathematical domain [e. g. in geometry]? 2) What, if anything, will assure that a student (a) becomes fluent in performing the symbolic manipulations in that domain, and (b) comes to understand the deep meaning of the mathematical notions represented by the symbols and the procedures [e.

WitsETD

> g. algebra] that are used to operate on them? (p. 227) [Emphasis added in brackets]

Schoenfeld's study indicated that most students from high school sophomore through college senior were competent at making deduction and doing constructions, but they often compartmentalized their knowledge in inappropriate ways. Consequently, much of their knowledge went unused, and the students' problem solving performance was far weaker than it should be. Schoenfeld said:

> … the students RO and LI first worked the construction problem and then the proofs. As it happens, their conjecture regarding the construction was incorrect, although it looked reasonably accurate, and they accepted it as being correct. (They were not told whether it was correct or not.) In a post mortem, the students worked the proof problems. They showed no signs of making connections between the two, so the investigator (AHS) asked about the relationship. (ibid., p. 244)

In the following interview transcript extract, AHS probed RO in terms of connections between her proofs and empirical solution (constructions).

> **AHS:** Can we look at the construction problem again? Does this one [pointing to the diagram from the proof problem] give you any ideas about that one [pointing to the diagram from the construction problem]?
> **RO:** Oh! I would draw a perpendicular from this [pointing to P]. Would that be right? A line perpendicular to this. Would that be right?
> **AHS:** How do you know whether it's right or not? I could tell you…
> **RO:** [Pointing to the location she hypothesized to be the circle's center] *Put the center of the circle here and then draw the circle.* [Emphasis added]
> **AHS:** … but I won't always be around to answer your questions. And you could try it and see if it looks good, if you wanted. But could you know whether or not the construction ought to work, without trying it? Or do you have to try it out, or ask someone who knows whether it's right?
> **RO:** I would try it myself. (ibid., p. 244)

The problem given to his subjects is presented in Fig 3.4.

WitsETD

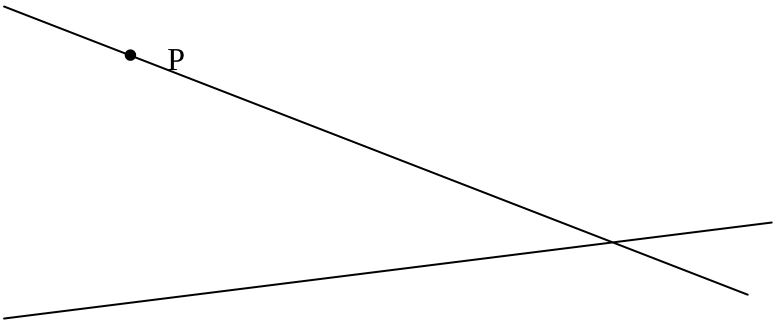

Figure 3.4: You have two intersecting straight lines and a point P on one of them, as in the figure above. Show how to construct, using straightedge and compass, a circle that is tangent to both lines and has the point P as its point of tangency to one of the lines.

Schoenfeld presented the ideas of RO as a model for a vast range of students (more than 100) with the same approach to construction problems and proofs as he pointed out that:

> It is important to understand that the segment of dialogue with the student RO that was quoted earlier does not reflect singular or highly unusual behavior. RO was an average student who had finished a year of high school geometry. The vast majority of that course was devoted to proofs, and she was fairly competent at them. The course also included a unit on constructions, and she was relatively competent at those as well. The fact is, she simply saw little or no connection between the two topics. From her perspective the entities manipulated in geometric proofs were hypothetical objects, and the standard of correctness for a proof was its logical connectedness. On the other hand, the objects that she manipulated in constructions were (obviously) real, and the standard of correctness for a construction was its accuracy. Thus the result of a proof might suggest which procedures might be tried in a construction but did not serve to guarantee that the construction would be correct. (ibid., p. 256)

Hence Schoenfeld found that the empiricist students were (usually) capable of making the appropriate deductive arguments, but they either failed to see the connections or dismissed the proofs as being irrelevant.

He also found that such compartmentalization of knowledge was often a result of inadequacies in mathematics instruction. Schoenfeld observed that unless students learn to take advantage of both deductive (as a means of discovery) and empirical (as a means of development of intuition) approaches to geometry and learn to profit from the interaction of

WitsETD

those two approaches; students will not reap the benefits of their knowledge. Schoenfeld recommended that further research should be carried out to elucidate what it means to “understand” particular mathematical ideas and what it takes to be able to implement mathematical procedures related to them.

Schoenfeld´s work is important for this research because it points to the need to think about how cognitive processes underlying geometric conceptual understanding should be acquired and nurtured as interconnected processes in practice (e. g. in the teaching-learning process, and elaboration of suitable geometric tasks). It raises an important question about what it means to `understand` particular mathematical ideas (e.g. critical algebraic and geometric concepts and their relationships) and what it takes to be able to access and use algebraic thinking and knowledge in order to understand and perform well in geometry.

Lawson and Chinnappan (2000) investigated the relationship between success in problem solving and the quality of the knowledge connections developed by two groups of 18 high-achieving and 18 low-achieving Year 10 students undertaking geometry tasks. Those students were from a private college. In this college, students were streamed into different classes on the basis of their performance in Year 9 and Year 10 mathematics tests. The college curriculum required that all students complete a topic involving trigonometry and geometry during Years 8, 9, and 10. High-achieving students came from the upper two Year 10 streams. The low-achieving students came from the three classes of the lower streams. To study this relationship, they investigated the influence of a range of measures of content knowledge and knowledge organization on students’ problem-solving performance. In this regard specific tasks were designed, *inter alia,* a recall or recognition task which provided evidence only about the student’s knowledge in a discrete form and did not require the student to know relationships between a knowledge component and other related components (Fig 3.5); a graded hinting task which examined the levels of connectedness of knowledge schemas used in solving simple geometry tasks (Table 3.2); an application task which required students to use a schema to develop a sample problem, the solution of which would call upon use of that schema. This task did not, however, require the students to directly relate the target schema to other related schemas (Fig 3.6). And an elaboration task which aimed for investigating how different theorem schemas that were relevant to a particular problem could be related one to the other (Fig 3.7).

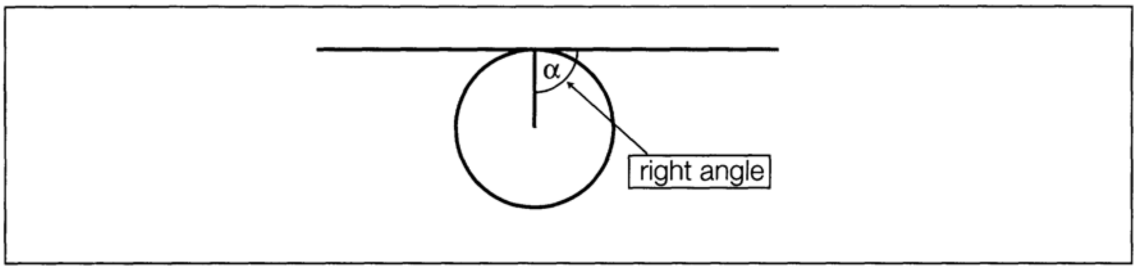

Figure 3.5: Figure used in Recognition Task

Table 3.2 Example of a sequence of hints used in the Hinting Task

Level	Hint
1	What do you notice about triangle ABC?
2	What do you notice about lines AC and BC?
3	What does the question statement tell you about lines AC and BC?
4	Lines AC and BC are of equal length.
5	What can you say about triangle ABC?
6	Triangle ABC is isosceles. Angles BAC and ABC are equal.

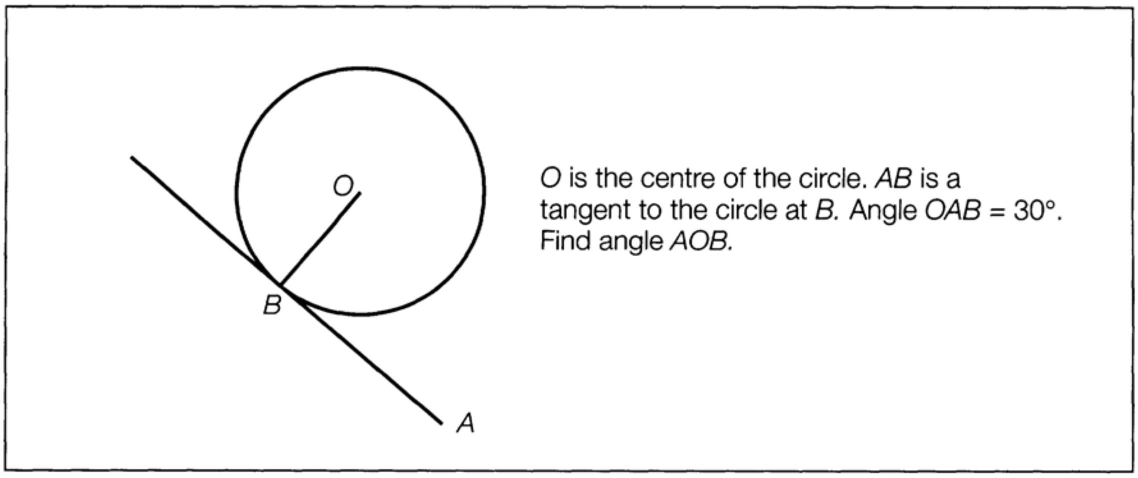

Figure 3.6: Figure used in Application Task

Theorem 1: The perpendicular drawn from the centre of a circle to its chord bisects the chord.
Theorem 2: Pythagoras' theorem.

Figure 3.7: Figure used in Elaboration Task

They were interested in examining the extent to which those sets of content and connectedness indicators would differentiate between those two groups of students who differed in levels of achievement in high school geometry. Of particular interest was investigation on the prediction that the groups would be differentiated by students' performances on the tasks designed to provide information about the quality of knowledge organization. The high-achieving students' performance on the indicators of knowledge organization (content and connectedness indicators) used in their study showed that, compared with the low-achieving group, they (a) were able to retrieve more knowledge spontaneously and (b) could activate or establish more links among given knowledge schemas and related information. Thus, they found out that the organizational quality of students' geometric knowledge was associated with better problem-solving performance. An individual with a high level of organized knowledge could retrieve more knowledge spontaneously and activate or establish more links among given knowledge schemas and related information than would be possible otherwise. Accordingly, Lawson and Chinnappan recommended mathematics educators and classrooms teachers to devise strategies for helping all students to improve the state of connectedness of their knowledge bases. Besides, they should assess students using more open-ended tasks, which require them to retrieve knowledge and use of connections.

Lawson and Chinnapan's (2000) study illuminated my research on "knowledge connectedness" from a concept-mapping perspective. In this sense, "a graded hinting task" was adapted for use in examining the levels of connectedness of knowledge schemas, that is, "the degree to which new nodes of information are connected with one another to form a single well-defined structure" (Chinnappan *et al*, 1999, p. 168). This task indexed the level of the "internal connectedness" of a schema (Lawson and Chinnappan, 2000). They explain that:

WitsETD

> A schema with components that are effectively organized is one for which minimal levels of cueing are required for activation. When a greater level of hinting support is needed for access, we argued that the knowledge schema is either less extensive or less well connected. (p. 31)

As to "external connectedness" (relationships among knowledge schemas), an elaboration task was designed. This task aimed to assess how different knowledge schemas that are relevant to a particular problem can be related, one to the other (ibid., p. 31). A third type of tasks was labelled as free recall task. This task permitted to derive estimates of knowledge that is functionally available to students.

These tasks helped me to identify and profile four types of students: 1) those who accessed knowledge and used it effectively; 2) those who accessed knowledge but used it inappropriately; 3) those who accessed knowledge but could not use it and 4) those that did not access knowledge. These categories were adapted from Lawson and Chinnappan (1994).

Dindyal (2003) investigated six focus secondary students' use of algebraic thinking: the use of symbols and algebraic relations, the use of different forms of representations, and the use of patterns and generalisations in geometry and their related conceptual difficulties. He attempted to find out the nature of the algebraic thinking that those students at high school level used in their geometry course and whether teachers encouraged the use of this type of thinking in the class. Four of the focus students were assigned van Hiele Level 1. Two of them had difficulties in the use of each of three aspects of algebraic thinking. They encountered difficulties when working with variables and unknowns, and writing and solving equations. They also found the use of different representations difficult. The other two were better at using the different forms of algebraic thinking. They could confidently work with variables and unknowns. They had a more positive approach to problem solving and a quest for finding the solution than the other two mentioned earlier. It should be noted that algebra test scores of that group ranged from 5 to 24 out of 30. The other group consisted of two students with the highest van Hiele levels (Level 3 and 4). In general they were much better in using algebraic thinking in geometry than the other focus students. They could work with variables, unknowns, and equations with great confidence. They seemed to be generally more relational or conceptual in their use of algebraic thinking. The algebra test scores of that group were 23 and 27 out of 30. The results showed that the students used algebraic thinking in solving problems in geometry, and accordingly it showed that algebraic thinking has strong

WitsETD

connections to thinking in geometry. The study highlighted the importance of algebra in geometry and it suggested that students had better preparation in algebra prior to joining the geometry class. In addition, it was found that students' use of algebraic thinking was related to their teachers' use of the same in their classes.

Dindyal (2003) recognised that his study had some limitations namely. Firstly, there was limited geometry content in the interview tasks that were used. More advanced work in coordinate geometry were not included in the tasks. Secondly, the study did not make any connections with the students' contextual background that is known to be critically important in the analyses of mathematical learning. It focused on how the different interview tasks were carried out by students rather than on how students with specific characteristics approached the tasks.

Further research is needed, according to Dindyal, regarding three aspects of algebraic thinking: symbols and algebraic relations, representations and patterns and generalisations. In this study, these three aspects of algebraic thinking were also considered in working with geometry tasks by the target students both in the Euclidean and Analytic geometry. Firstly, I analysed how students linked these aspects of algebraic thinking to the geometric concepts they used in those tasks as well as the respective meanings through interviews and written responses. Secondly, I investigated their strategies while solving these tasks in order to find out whether they used specific and/or general strategies and accordingly to see whether those aspects of algebraic thinking contributed for their choice. Thirdly, I was interested in knowing whether their choice of strategies had to do with their dispositions either driven by mastery and/or performance orientation through class observations and interviews.

Dindyal also proposed that there is a need for studies that provide richer information about how algebraic thinking is used by students in geometrical contexts. In addition, he recommended that future studies may use a more varied sample (his study was carried out with only 6 focus students from two different schools). He proposed further research on use of algebraic thinking in geometrical contexts at other levels, in particular at the middle school level [and onwards]. Dindyal (2004) recognised that there is still a strong interest in how students think in geometry. Theoretically, he recommended further research on models or frameworks that address some of the shortcomings of the van Hiele theory or possibly research on a totally new framework.

WitsETD

According to Dindyal's (2003) review, "little or no research has been carried out on the influence of algebra in the geometry curriculum, in particular, algebraic thinking in geometry has not been explored" [enough] (p. 10). The current study attempted to make a contribution to this important and emerging research field. Specifically, this study explored what kinds of meanings tertiary-level students showed for different critical algebraic and geometric concepts that they worked with. Besides, it was concerned with what resources (contextual, structural, and conceptual) promoted or hindered first year university students' use of algebraic thinking in geometry in Mozambique and what their implications were for students' geometric conceptual understanding.

Table 3.3 below is a summary of the major findings and some gaps I found in the key literature regarding algebraic thinking aiding geometrical understanding. Previous research carried out in this area illuminated my study in two aspects, namely (1) algebraic versus geometric thinking and (2) sample. Regarding the first aspect, my study produced appropriate research questions. Accordingly, it replicated some key findings and contributed in some limitations identified in the previous research (Chapter 5). Concerning the second aspect, this study focused on tertiary education which broadened geometry content.

The research questions of this study were:

1. How do first-year university students solve geometry problems? To what extent do they use algebraic knowledge and thinking in solving such problems? What kinds of meanings do students make of different algebraic and geometric concepts involved in problem solving situations?
2. To what extent does algebraic knowledge and thinking aid students' conceptual understanding and problem solving performance in geometry?
3. To what extent is geometric work linked to algebraic thinking in the first year university geometry course at the Universidade Pedagógica in Maputo?

In order to answer these research questions I searched for an appropriate methodology which followed, on one hand, a component of Dindyal´s (2003) approach on the exploration of students´ algebraic thinking in geometry, on the other hand, which included the research methodology, the instruments for data collection, and the approach to data analysis.

WitsETD

Table 3.3 A Summary of some empirical studies related to the current study

Empirical study	Aspect/Content	Key findings	Limitations/Further research	Possible contributions of / Implications for my study
Schoenfeld (1986)	Algebraic vs. geometric thinking	- Lack of interaction between deductive and empirical approaches to geometry hindered understanding. - Instruction contributed to compartmentalization of knowledge.	- What it means to understand particular mathematical ideas? - What it takes to be able to implement mathematical procedures related to those ideas?	-It provided possible answers to those questions related to algebraic thinking and geometrical thinking/ understanding.
	Sample	Students from high school sophomore through college senior		Tertiary levels students
Lawson and Chinnappan (2000)	Algebraic vs. geometric thinking	The organizational quality of students' geometric knowledge was associated with better problem-solving performance.	- It missed to describe the nature of geometric knowledge in detail (see my model of algebraic thinking in geometrical understanding).	- It provided a possible anatomy of the nature of geometrical knowledge/ understanding. -Connectedness of knowledge base was important for my study.
	Sample	Year 10 college students		Tertiary levels students
Dindyal (2003)	Algebraic vs. geometric thinking	- Algebraic thinking had strong connections to thinking in geometry. - Instruction fostered that connection.	- Limited geometry content in the interview tasks - No connections with the students' contextual background - A need of studies that provide richer information about how algebraic thinking is used by students in geometrical contexts - A need of more varied sample - A need of further research on use of algebraic thinking in geometrical contexts at other levels - A need of further research on models or frameworks that address some of the shortcomings of the van Hiele theory or possibly research on a totally new framework	- Broadened geometry content (Euclidean and Analytic geometry) - Contextual background of the participants considered - Tertiary level students as participants - A new approach different to van Hiele approach considered
	Sample	Secondary students		Tertiary level students