Applications of Lie Symmetry Techniques to Models Describing Heat Conduction in Extended Surfaces

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Declaration

I declare that this project is my own, unaided work. It is being submitted in fulfillment of the Degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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Abstract

In this thesis we consider the construction of exact solutions for models describing heat transfer through extended surfaces (fins). The interest in the solutions of the heat transfer in extended surfaces is never ending. Perhaps this is because of the vast application of these surfaces in engineering and industrial processes. Throughout this thesis, we assume that both thermal conductivity and heat transfer are temperature dependent. As such the resulting energy balance equations are nonlinear. We attempt to construct exact solutions for these nonlinear models using the theory of Lie symmetry analysis of differential equations.

Firstly, we perform preliminary group classification of the steady state problem to determine forms of the arbitrary functions appearing in the considered equation for which the principal Lie algebra is extended by one element. Some reductions are performed and invariant solutions that satisfy the Dirichlet boundary condition at one end and the Neumann boundary condition at the other, are constructed.

Secondly, we consider the transient state heat transfer in longitudinal rectangular fins. Here the imposed boundary conditions are the step change in the base temperature and the step change in base heat flow. We employ the local and nonlocal symmetry techniques to analyze the problem at hand. In one case the reduced equation transforms to the tractable Ermakov-Pinney equation. Nonlocal symmetries are admitted when some arbitrary constants appearing in the governing equations are specified. The exact steady state solutions which satisfy the prescribed boundary conditions are constructed.

Since the obtained exact solutions for the transient state satisfy only the zero initial temperature and adiabatic boundary condition at the fin tip, we sought numerical solutions.

Lastly, we considered the one dimensional steady state heat transfer in fins of different profiles. Some transformation linearizes the problem when the thermal conductivity is a differential consequence of the heat transfer coefficient, and exact solutions are determined. Classical Lie point symmetry methods are employed for the problem which is not linearizable. Some reductions are performed and invariant solutions are constructed.

The effects of the thermo-geometric fin parameter and the power law exponent on temperature distribution are studied in all these problems. Furthermore, the fin efficiency and heat flux are analyzed.

Acknowledgments

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Chapter 1

Introduction

1.1 Literature review

1.1.1 Background and motivation

Extended surfaces (fins) play an important role in increasing the efficiency of heating systems. In particular, fins are used in power generators, air conditioning, semiconductors, refrigeration, cooling of computer processor, exothermic reactors and many other devices in which heat is generated and must be transported [1]. Many problems describing heat transfer in fins have been well documented (see e.g. [2, 3]). The literature on this topic is quite sizeable.

The exact solutions satisfying the relevant boundary conditions provide insight into heat transfer processes and may be used as bench marks for the numerical schemes (see e.g. [4]). The analysis of constructed solutions may also assist in the designs of the fins, for example, it is well known that a longer and thicker fins provide higher heat transfer rates than shorter and thinner ones [5].

1.1.2 Recent developments

Considerable effort is given in devising accurate and efficient exact, analytical and approximate schemes for solving differential equations, particularly those arising in heat conduction though the one-dimensional (see e.g. [6, 7, 8, 9, 10, 11]) and the two-dimensional extended surfaces (see e.g. [12, 13, 14, 15, 16, 17, 18, 19, 20, 21]). In one-dimensional cases, the obtained solutions include series solutions [7, 8, 11], homotopy methods [6] and differential transformation method [4, 22]. Few exact solutions exist for one- and two-dimensional problems. In fact, existing solutions are constructed only for constant thermal conductivity and heat transfer coefficient.

Recently Moitsheki *et al.*,[23, 24, 25] constructed the exact solutions of the one-dimensional fin problem given nonlinear thermal conductivity and heat transfer coefficient. Some of this work has been extended in [26] whereby the introduction of the Kirchhoff transformation linearized the one-dimensional fin problem when heat transfer coefficient is a differential consequence of thermal conductivity.

Exact solution for steady two-dimensional fin problems exists only for the linear constant coefficient Laplace equation whereby internal heat generation function (source or sink term) is usually neglected (see e.g. [5]). The fin base temperature is usually assumed to be a constant, but it may be modeled as non-constant function of spatial variable [13].

Symmetry methods in particular group classification, have been used to analyze the one-dimensional fin problems with heat transfer coefficient depending on the spatial variable [27, 28, 29, 30, 31]. However, these analysis excluded real-world applications. In recent years many authors have been interested in the steady state problems [6, 7, 8, 25] describing heat flow in one-dimensional longitudinal rectangular fins. Recently Moitsheki and Harley [32], Moitsheki and Mhlongo [33] and Mhlongo *et al.*, [34, 35] considered analysis of heat transfer in fins of various profiles with both heat transfer coefficient and thermal conductivity being given by temperature dependent.

An accurate transient analysis provided insight into the design of fins that would fail in steady state operations but worked well for some operating period [38]. The transient problem is considered for a fin of arbitrary profile in [39]. However, both thermal conductivity and heat transfer coefficient are considered to be constants.

1.2 Aims and objectives of the thesis

The main objective of the thesis is to analyze fin problems in one-dimensional case when thermal conductivity and heat transfer coefficient are both non-constant and temperature-dependent. Furthermore we aim to analyze the problem subject to various boundary conditions. Also, we aim to investigate heat transfer in fins of different profiles.

Techniques such as local and nonlocal symmetries, equivalent transformations and preliminary group classification are utilized. Symmetry reductions are performed and group invariant solutions are attempted to be constructed. The fin efficiency and the effects on temperature distribution of any parameter that may appear in the dimensionless models are analysed. We will also make use of computer aided procedures, in particular, we will use the freely available software DIMSYM [40] and REDUCE [41].

1.3 Outline of the thesis

The thesis is outlined as follows:

- Chapter 2 will briefly describe the mathematical formulation of heat conduction problem.

- Symmetry techniques for differential equations will be briefly discussed in chapter 3. The concepts, Lie point (local) symmetries, Lie algebras, nonlocal (potential) symmetries, preliminary group classification, invariant solution and optimal system will be discussed.

- Chapter 4 deals with the analysis of a steady non linear one-dimensional heat transfer in fin of a rectangular profile. Part of the results were published in a paper by Moitsheki and Mhlongo [33].

- In chapter 5 we discuss the transient response of longitudinal rectangular fin to step change in base temperature and in base heat flow conditions. Some of the results are published in Mhlongo *et al.*, [34].

- In chapter 6 will look at the analysis of a steady one-dimensional fin problem of different profiles. The results have been submitted to ISI journal for possible publication [35].

- Lastly we provide conclusions in chapter 7.

Chapter 2

Mathematical description

2.1 Introduction

In this chapter we briefly discuss the models representing heat transfer in longitudinal fins of different profiles. The fins considered here are given in terms of the characteristic length may be a contributing factor to why not many exact solutions exist, in particular similarity (invariant) solutions. Mathematical formulation is given in Section 2.2. Section 2.3 briefly discusses various fin profiles. Descriptions of heat transfer coefficient and thermal conductivity follow in section 2.4. Physical boundary conditions are briefly described in Section 2.5. Section 2.6 discusses fin efficiency and heat flux. Dimensionless variables and numbers are described in Section 2.7. Lastly we provide the concluding remarks in section 2.8.

This chapter is mainly on a brief theoretical background on heat transfer in longitudinal fins of different profiles. For a complete theory the reader is referred to text such as [2].

2.2 The energy balance model

We consider a longitudinal one-dimensional fin with a cross sectional area A_c as shown in Fig. 2.1. The perimeter of the fin is denoted by P and the length of fin by L. The fin is attached to a fixed base surface of temperature T_b and extends into a fluid of temperature T_a. The fin profile is given by the function $F(X)$, fin thickness δ depends on the fin profile and the fin thickness at the base is δ_b.

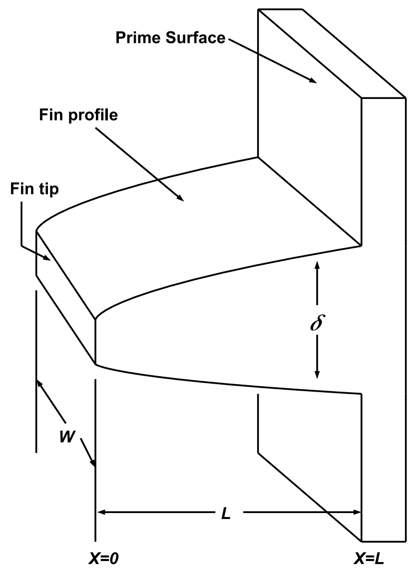

Figure 2.1: Schematic representation of a longitudinal fin of an arbitrary profile.

The energy balance for a longitudinal fin of an arbitrary profile is given by

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial X}\left(\frac{2}{\delta_b} F(X) K(T) \frac{\partial T}{\partial X}\right) - \frac{P}{A_c} H(T)\left(T - T_a\right), \quad 0 < X < L, \tag{2.1}$$

where ρ is the mass density, c is the specific heat, K and H are the non-uniform thermal conductivity and heat transfer coefficient depending on the temperature (see for example [6, 7, 10, 25]), T is the temperature distribution, $F(X)$ is the fin profile, t is time and X is the spatial variable. The thermal conductivity denoted by $K(T)$ is the property of a material's ability to conduct heat which, in many engineering applications, varies with temperature. The heat transfer coefficient is the amount of heat which passes through a unit area of a medium or system in a unit time when the temperature difference between the boundaries of the system is 1 degree. The fin length is measured from the tip to the base as shown in Fig. 2.1.

2.3 Fin profiles

In this section we present some fin profiles considered in this study. Fins with triangular and parabolic profiles contain less material and are more efficient requiring minimum weight.

2.3.1 Applications of fins

It is well known [2] that a straight fin with a concave parabolic profile provides the maximum heat dissipation for a given profile area. However the concave parabolic shape is difficult and costly to manufacture. Thus for most applications the rectangular profile is preferred for the sake of simplicity in the fabrication even though it does not utilize the material most efficiently [36].

The optimization design focuses on finding the shape and dimensions of the fins that would minimize the volume or mass for a given amount of heat dissipation, or alternatively, to maximize the heat dissipation for a given volume or mass. One way to analyze the optimization problem is to select a suitable profile, then to determine the dimensions of the fins and to yield the maximum heat dissipation for a given volume and shape of the fin.

Laor and Kalman [37] analyzed a general optimization problem of convective fins with constant thermal parameters The profile function $F(X)$ for longitudinal fins will take the general form

$$F(X) = \frac{\delta_b}{2}\left(\frac{X}{L}\right)^{\frac{1-2p}{1-p}}. \tag{2.2}$$

Here $p \in \mathbb{R},\ p \neq 1$.

2.3.2 Longitudinal fin of rectangular profile

For the longitudinal fin of rectangular profile shown in Fig. 2.2, the exponent on the general profile of Eq. (2.2) satisfies the geometry when $p = \frac{1}{2}$. The profile function for the fin then becomes [2]

$$F(X) = \frac{\delta_b}{2}. \tag{2.3}$$

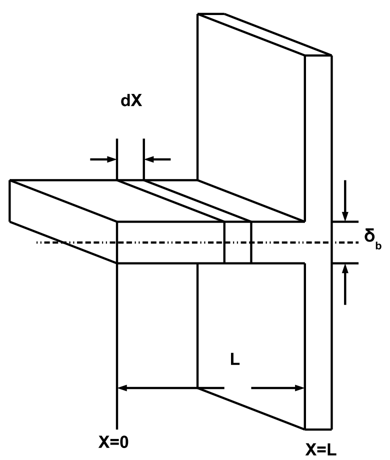

Figure 2.2: Schematic representation of a longitudinal fin of a rectangular profile.

2.3.3 Longitudinal fin of triangular profile

If $p = 0$ we get a longitudinal fin of triangular profile shown in Fig. 2.3. The profile function for the fin then becomes

$$F(X) = \frac{\delta_b}{2} \frac{X}{L}. \tag{2.4}$$

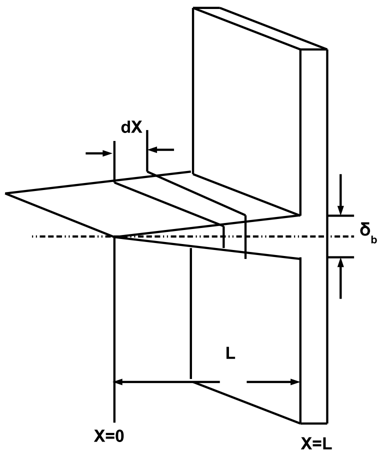

Figure 2.3: Schematic representation of a longitudinal fin of a triangular profile.

2.3.4 Longitudinal fin of concave parabolic profile

If $p \to \infty$ we get a longitudinal fin of concave parabolic profile shown in Fig. 2.4. The profile function for the fin then becomes

$$F(X) = \frac{\delta_b}{2} \left(\frac{X}{L} \right)^2. \tag{2.5}$$

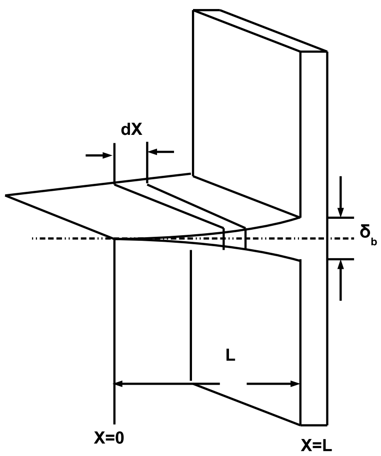

Figure 2.4: Schematic representation of a longitudinal fin of a concave parabolic profile.

2.3.5 Longitudinal fin of convex parabolic profile

The profile functions for longitudinal fin of convex parabolic profile which is shown in Fig. 2.5 is given by Eq. (2.6)

$$F(X) = \frac{\delta_b}{2}\left(\frac{X}{L}\right)^{\frac{1}{2}}. \tag{2.6}$$

The results of exponential fin profile are included although not discussed in details.

2.4 Boundary conditions

We consider a couple of boundary conditions. First throughout this work, we assume that the fin tip is insulated. If the tip is not assumed to be insulated then the problem becomes overdetermined (see also, [47]). This boundary

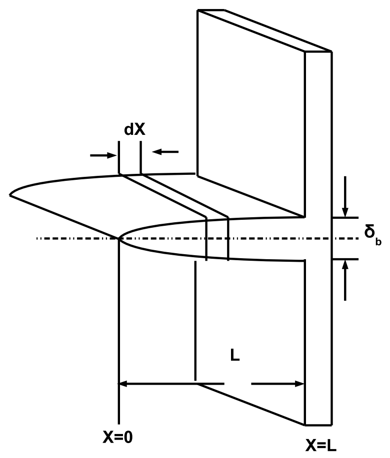

Figure 2.5: Schematic representation of a longitudinal fin of a convex parabolic profile.

condition is realized for sufficiently long fins [46]. In this case we have

$$\frac{\partial T(0,t)}{\partial X} = 0. \tag{2.7}$$

The boundary condition at the base of the fin is given by the step change in base temperature [2]

$$T(L,t) = T_b, \tag{2.8}$$

in one case and by the step change in base heat flow

$$\frac{\partial T(L,t)}{\partial X} = \frac{q_b}{k_a A_c}, \tag{2.9}$$

in the other.

Here q_b is the base heat flux, and other parameters have been defined earlier. The initial fin temperature is assumed to be

$$T(X,0) = 0. \tag{2.10}$$

2.5 Thermal conductivity and heat transfer coefficient

The thermal conductivity of a material may be assumed for other physical phenomena as a constant and as a function of spatial variable for functionally graded fins. However, in many engineering applications it is given as a linear function and expressed as [6, 9, 32]

$$K(T) = k_a[1 + \beta(T - T_a)], \tag{2.11}$$

where β is the thermal conductivity gradient and k_a is thermal conductivity at the ambient temperature.

Also the thermal conductivity of the fin may be assumed to vary nonlinearly with the temperature, that is

$$K(T) = k_a \left(\frac{T - T_a}{T_b - T_a}\right)^m \quad \text{and} \quad k(T) = k_a \left(\frac{k_a A_c (T - T_a)}{q_b L}\right)^m, \tag{2.12}$$

for step change in base temperature and step change in base heat flow conditions, respectively. Indeed, thermal conductivity of some material such as Gellium Nitride (GaN) and Aluminium Nitride (AlN) can be modeled by the power law (see e.g. [42]). Furthermore, the experimental data indicates that the exponent of the power law for these materials is positive for lower temperatures and negative for at higher temperatures [43, 44, 45].

On the other hand, for most industrial applications the heat transfer coefficient may be given by the power law [46]

$$H(T) = h_b \left(\frac{T - T_a}{T_b - T_a}\right)^n, \tag{2.13}$$

given the step change in the base temperature and

$$h(T) = h_b \left(\frac{k_a A_c (T - T_a)}{q_b L}\right)^n \tag{2.14}$$

given the step change in the base heat flow. Here the exponent n and h_b are constants. The constant n may vary between -6.6 and 5. However, in most practical applications it lies between -3 and 3 [46]. If the heat transfer coefficient is given by Equation (2.13) and (2.14), then the hypothetical boundary condition (i.e. insulation) at the tip of the fin is taken into account [46]. Also, the heat transfer through the outermost edge of the fin is negligible compared to that which passes through the side [47]. The exponent n represents laminar film boiling or condensation when $n = -\frac{1}{4}$, laminar natural convection when $n = \frac{1}{4}$, turbulent natural convection when $n = \frac{1}{3}$, nucleate boiling when $n = 2$, radiation when $n = 4$, and $n = 0$ implies a constant heat transfer coefficient. Exact solutions may be constructed for the steady-state one-dimensional differential equation describing temperature distribution in a straight fin when the thermal conductivity is a constant and $n = -1, 0, 1$ and 2 [46]. In this thesis, we attempt to construct exact steady state solutions given nonconstant thermal conductivity.

2.6 Fin efficiency and heat flux

2.6.1 Fin efficiency

The heat transfer rate from a fin is given by Newton's second law of cooling (see e.g. [2])

$$Q = \int_0^L PH(T)(T - T_a)dX. \tag{2.15}$$

Fin efficiency is defined as the ratio of the fin heat transfer rate to the rate that would be if the entire fin were at the base temperature and it is given by

$$\eta = \frac{Q}{Q_{ideal}} = \frac{\int_0^L PH(T)(T - T_a)dX}{Ph_bL(T - T_a)}. \tag{2.16}$$

2.6.2 Heat flux

Heat flux at the base of the fin is given by the Fourier's law (see e.g. [2])

$$q_b = A_c K(T)\frac{dT}{dX}. \tag{2.17}$$

The total heat flux of the fin is given by

$$q = \frac{q_b}{A_c H(T)(T - T_a)}. \tag{2.18}$$

2.7 Non-dimensionalization

We introduce the dimensionless variables and the dimensionless numbers given by

$$x = \frac{X}{L}, \quad \tau = \frac{k_a t}{\rho c L^2}, \quad k = \frac{K}{k_a}, \quad h = \frac{H}{h_b}, M^2 = \frac{P h_b L^2}{A_c k_a}, f(x) = \frac{2}{\delta_b}F(X), \tag{2.19}$$

with step change in base heat flow, the dimensionless temperature becomes [2]

$$\theta = \frac{k_a A_c (T - T_a)}{q_b L}, \tag{2.20}$$

and with the step change base temperature we have

$$\theta = \frac{T - T_a}{T_b - T_a}. \tag{2.21}$$

Hence Eq. (2.1) becomes

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x}\left(f(x)k(\theta)\frac{\partial \theta}{\partial x}\right) - M^2 h(\theta)\theta, \quad 0 \le x \le 1. \tag{2.22}$$

The initial condition is given by

$$\theta(x, 0) = 0, \tag{2.23}$$

the step change in fin base temperature is given by

$$\theta(1, \tau) = 1, \tag{2.24}$$

the step change in fin base heat flux is given by

$$\left.\frac{\partial \theta}{\partial x}\right|_{x=1} = 1 \tag{2.25}$$

and fin tip boundary condition is given by

$$\left.\frac{\partial \theta}{\partial x}\right|_{x=0} = 0. \tag{2.26}$$

Here M is the thermo-geometric fin parameter, θ is the dimensionless temperature, x is the dimensionless spatial variable, $f(x)$ is the dimensionless fin profile, τ is dimensionless time, k is the dimensionless thermal conductivity, h is the dimensionless heat transfer coefficient and h_b is the heat transfer coefficient at the fin base. M is inversely proportional to the length L, and that is why the analysis of constructed solutions assist in the designs of the fins, for example, it is well known that a longer and thicker fins provide higher heat transfer rates than shorter and thinner ones [5]. The non-dimensional heat transfer coefficient and thermal conductivity of the fin are given by

$$h(\theta) = \theta^n, \quad k(\theta) = \theta^m$$

respectively.

The fin efficiency and heat flux in dimensionless variables are given by

$$\eta = \int_0^1 \theta^{n+1} dx \tag{2.27}$$

and

$$q = \frac{1}{Bi} \frac{k(\theta)}{h(\theta)} \frac{d\theta}{dx} \tag{2.28}$$

respectively. Here the dimensionless parameter $Bi = \dfrac{h_b L}{k_a}$ is the Biot number.

2.8 Concluding remarks

In this chapter, we have discussed the mathematical formulation representing heat transfer in longitudinal fins of various profiles. We have taken into consideration, the energy balance equation, the physical boundary conditions and definition of fin efficiency and heat flux. Furthermore, the initial and boundary value problems are given in terms of the non-dimensionless variables and numbers.

Chapter 3

Symmetry analysis of differential equations

3.1 Introduction

In this chapter we shall discuss Lie symmetry techniques for differential equations. A brief historical account and theoretical background are given in Sections 3.1.1 and 3.1.2 respectively. In Section 3.2 we discuss the calculations of classical (local) Lie point symmetries. Calculations of nonlocal (potential) symmetries are discussed in Section 3.3. In Section 3.4 we discuss the equivalence transformation and the notion of preliminary group classification. Furthermore, we present the method for constructing optimal system of subalgebras in Section 3.5. In Sections 3.6 we discuss the construction of the group-invariant solutions for partial differential equations (PDEs). Methods of linearization and reductions of ordinary differential equations (ODEs) are discussed in Section 3.7. Conclusion is given in Section 3.8.

3.1.1 A brief historical account

The theory of groups of transformation for differential equations (DEs) was introduced by Sophus Lie in the latter part of the nineteenth century, as an extension to various specialized methods for solving ODEs. His work was motivated by the lectures given Sylow on Galois theory and Abel's works. As a significant contribution, Lie showed that the order of an ODE could be reduced by one, constructively, if it is invariant under a one-parameter Lie group of point transformations.

Various topics in determining the solutions of ODEs are related to Lie's work including among others, integrating factor, separable equation, homogeneous equation, reduction of order, the methods of undetermined coefficients and variation of parameters for linear equations, solution of the Euler equation, and the use of the Laplace transform. Lie also indicated that for linear PDEs, invariance under a Lie group leads directly to superpositions of solutions in terms of transforms.

3.1.2 A brief theoretical background of Lie symmetry analysis

In brief, a symmetry of a differential equation is an invertible transformation of dependent and independent variables which leave the form of the equation in question unchanged [48, 49, 50, 51, 52, 53]. This point transformation, in Lie's view [54], forms a group that depend on a continuous parameter. The elementary examples of Lie groups include groups of translations, rotations, and scalings.

In his fundamental theorem, Lie showed that groups are characterized by their infinitesimal generators. These infinitesimal generators, can be extended to act on the space of independent and dependent variables, and their derivatives up to any finite order. If the coefficients of governing differential equation (or a system of equations) are functions of independent and/or dependent variables, then the vector fields or symmetries admitted by the equation in question when these coefficients are arbitrary span the *principal Lie algebra* (PLA) (see e.g. [55]). The consequence of the action of the infinitesimal generators on DEs is the reduction in the original equation as follows (but not limited to),

(i) In the case of the first order ODE equations, it reduces to a separable first order ODE.

(ii) A second order 1+1 dimensional PDE may be reduced to a second order ODE.

(iii) A nonlinear second order ODE may be reduced linear second order ODE or to first order ODE or to the ODE with a cubic in the first derivative.

(iv) A PDE with n independent variables can be reduced to one with $n-1$ independent variables.

For further theory and applications of symmetry analysis excellent text such as those of [48, 49, 50, 51, 52, 53, 56, 57, 58] can be used.

3.2 Calculation of Lie point (local) symmetries

3.2.1 One-parameter group of transformations

A set G_T of transformations

$$T_\epsilon : \overline{x}^i = f^i(x,u,\epsilon), \quad \overline{u}^\alpha = \phi^\alpha(x,u,\epsilon), i = 1,2\ldots,n; \ \alpha = 1,2,\ldots,m; \tag{3.1}$$

where ϵ is real parameter which is continuous in the neighborhood $\mathbf{D} \subset \mathbb{R}$ of $\epsilon = 0$ and f^i, ϕ^α are differentiable functions. G_T is a *continuous one-parameter (local) Lie group of transformations* in $\mathbb{R}^{n+m}$ if the following properties hold,

i) *Identity.* If $T_e \in G$ such that $T_e T_\epsilon = T_\epsilon T_e = T_\epsilon$, for any $\epsilon \in \mathbf{D}_* \subset \mathbf{D}$ and $T_\epsilon \in G_T$. T_e is the identity in G_T and e is the identity in $\mathbf{D}$.

ii) *Closure.* If T_ϵ, T_δ are in G_T and ϵ, δ in $\mathbf{D}_* \subset \mathbf{D}$, then $T_\epsilon T_\delta = T_\gamma \in G$, $\gamma = \psi(\epsilon, \delta) \in \mathbf{D}$.

iii) *Inverse element.* For $T_\epsilon, \epsilon \in \mathbf{D}_* \subset \mathbf{D}$, there exists $T_\epsilon^{-1} = T_{\epsilon^{-1}} \in G_T, \epsilon^{-1} \in \mathbf{D}$ such that $T_\epsilon T_{\epsilon^{-1}} = T_{\epsilon^{-1}} T_\epsilon = T_e$

iv) *Associativity.* If $T_\epsilon, T_\delta, T_\gamma$ are in G_T and ϵ, δ, γ in $\mathbf{D}_* \subset \mathbf{D}$, then $(T_\epsilon T_\delta)T_\gamma = T_\epsilon(T_\delta T_\gamma)$.

v) ϵ is a continuous parameter i.e. $\epsilon \in \mathbf{D}_*$, where $\mathbf{D}$ is an interval in $\mathbb{R}$,

vi) φ_i and ϕ_α are analytic,

vii) $\psi(\epsilon, \delta)$ is an analytic function of ϵ and δ.

According to the theory of Lie, the construction of a one-parameter group G_T is equivalent to the determination of the corresponding infinitesimal transformation generated by the infinitesimal generator. One-parameter groups are obtained by their corresponding generator either by Lie equations or by the exponential map.

3.2.2 The invariance criterion

An rth order PDE in s independent variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and one dependent variable u, given by

$$F(\mathbf{x}, u, u^{(1)}, \ldots, u^{(r)}) = 0, \tag{3.2}$$

where $u^{(r)}$, denotes the set of coordinates corresponding to all rth order partial derivatives of u with respect to $x_1, x_2, \ldots, x_n$. That is, a coordinate $u^{(r)}$ is denoted by

$$u_{j_1 j_2 \ldots j_k} = \frac{\partial^r u}{\partial x_{j_1} \partial x_{j_2} \ldots \partial x_{j_r}}, \tag{3.3}$$

with $j_q = 1, 2, \ldots, n$ and $q = 1, 2, \ldots, r$. Note that in case of the ODE, one consider one independent and one dependent variable. To determine the symmetry for the Eq. (3.2), one may seek the transformations of the form

$$\overline{\mathbf{x}}_j = x_j + \epsilon \xi_j(\mathbf{x}, u) + O(\epsilon^2), \tag{3.4}$$

$$\overline{u} = u + \epsilon \eta(\mathbf{x}, u) + O(\epsilon^2) \tag{3.5}$$

called infinitesimal transformations. The coefficients ξ_j and η are the components of the infinitesimal generator acting on the $(\mathbf{x}, u)$ space given by

$$\Gamma = \xi_j(\mathbf{x}, u) \frac{\partial}{\partial x_j} + \eta(\mathbf{x}, u) \frac{\partial}{\partial u}, \tag{3.6}$$

which leaves the DE in question invariant. The action of Γ is extended in the governing equation through the rth prolongation given by

$$\Gamma^{[r]} = \Gamma + \eta_1(\mathbf{x}, u, u^{(1)}) \frac{\partial}{\partial u_j} + \ldots + \eta_{j_1 j_2 \ldots j_r}(\mathbf{x}, u, u^{(1)}, \ldots, u^{(r)}) \frac{\partial}{\partial u_{j_1 j_2 \ldots j_r}}, \tag{3.7}$$

where $r = 1, 2, \ldots$

$$\eta_j = D_j(\eta) - u_k D_j(\xi_k), \; j = 1, 2, \ldots, n; \tag{3.8}$$

$$\eta_{j_1 j_2 \ldots j_r} = D_{jr}(\eta_{j_1 j_2 \ldots j_{r-1}}) - u_{j_1 j_2 \ldots j_{r-1} k} D_{jr}(\xi_k), \tag{3.9}$$

with $j_q = 1, 2, \ldots, n$ and $q = 1, 2, \ldots, r$, $r = 2, 3, \ldots$ and D_j being the total x_j derivative operator defined by

$$D_j = \frac{\partial}{\partial x_j} + u_j \frac{\partial}{\partial u} + \ldots + u_{j_1 j_2 \ldots j_r} \frac{\partial}{\partial u_{j_1 j_2 \ldots j_r}}. \tag{3.10}$$

The invariance criterion for symmetry determination is given by

$$\Gamma^{[r]}\,(F=0)|_{F=0}=0. \tag{3.11}$$

Since the coefficients of Γ do not involve derivatives, we can separate (3.11) with respect to the derivatives of u and solve the resulting overdetermined system of linear homogeneous partial differential equations known as *determining equations.* The calculation are algorithmic and may be were facilitated by a computer software such as REDUCE [41] or YaLie [59].

It may happen that the only solution to the overdetermined system of linear equations is trivial. When the general solution of the determining equations is nontrivial two cases arise: (a) if the general solution contains a finite number, p, of essential arbitrary constants then it corresponds to a p-parameter Lie algebra spanned by the base vectors (3.6); and (b) if the general solution cannot be expressed in terms of a finite number of essential constants, for example when it contains an arbitrary function of independent and/or dependent variables variables, then it corresponds to an infinite-parameter Lie group of transformations of the infinite-dimensional symmetry generator.

Illustrative example

As an illustrative example, we consider a nonlinear ODE

$$\frac{d}{dx}\left[\theta^m\frac{d\theta}{dx}\right]-M^2\theta^{n+1}=0. \tag{3.12}$$

Given $n=-3m-4$, and considering the transformation $y=\theta^{m+1}$, then Eq. (3.12) becomes the Ermakov-Pinney type equation [60]

$$y''=(m+1)M^2y^{-3}. \tag{3.13}$$

The invariance criterion for symmetry determination is given by

$$\Gamma^{[2]} \quad \left(y'' - (m+1)M^2y^{-3}\right)\Big|_{y''=(m+1)M^2y^{-3}} = 0, \tag{3.14}$$

where

$$\Gamma^{[2]} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta_1\frac{\partial}{\partial y'} + \eta_2\frac{\partial}{\partial y''} \tag{3.15}$$

is the second prolongation of symmetry generator and the coefficients η_1 and η_2 are determined from the equations

$$\eta_1 = D_x(\eta) - y'D_x(\xi), \tag{3.16}$$

$$\eta_2 = D_x(\eta_1) - y''D_x(\xi). \tag{3.17}$$

The total derivative D_x is given by

$$D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \dots. \tag{3.18}$$

The resulting determining equations are given by

$$\xi_{yy} = 0, \tag{3.19}$$

$$\eta_{yy} - 2\xi_{xy} = 0, \tag{3.20}$$

$$2y^3\eta_{xy} - y^3\xi_{xx} + 3(m+1)M^2\xi_y = 0, \tag{3.21}$$

$$y^4\eta_{xx} - (m+1)M^2y\eta_y + 2(m+1)M^2y\xi_x - 3(m+1)M^2\eta = 0. \tag{3.22}$$

Solving Eqs. (3.19)-(3.22), yield the symmetry generators

$$\Gamma_1 = \frac{\partial}{\partial x}, \tag{3.23}$$

$$\Gamma_2 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \tag{3.24}$$

$$\Gamma_3 = x\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y}. \tag{3.25}$$

One may easily show that symmetry generators (3.23)-(3.25) span the $SL(2,\mathbb{R}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R},\ ad - bc = 1\}$ Lie algebra (see e.g. [61]).

3.3 Calculation of nonlocal symmetries

A symmetry generator is said to be nonlocal if at least one of the coefficients of the infinitesimal generator depends explicitly on independent and the dependent variables and also on integrals of the dependent variables.

Using nonlocal (potential) symmetry method, it is possible to find further exact solutions that are not obtainable via local symmetries. Furthermore, one may construct solutions for boundary value problems posed for PDEs and linearisation of nonlinear PDEs is also possible via potential symmetry analysis [49].

The method for finding potential symmetries involves writing a given PDE in a conserved form with respect to some choices of its variables. We consider a scalar rth order PDE $R\{\mathbf{x}, u\}$ with s independent variables $\mathbf{x} = (x_1, x_2, \dots, x_s)$ and a single dependent variable u and suppose $R\{\mathbf{x}, u\}$ can be written in a conserved form

$$D_j f^j(\mathbf{x}, y, u^{(1)}, u^{(2)}, \dots, u^{(r-1)}) = 0, \tag{3.26}$$

where D_j is the total derivative operator defined in (3.10). Now, one may introduce $r-1$ auxiliary dependent variables or the potentials $v = (v^1, v^2, \dots, v^{r-1})$ and form an auxiliary system $S\{\mathbf{x}, u, v\}$ namely;

$$\begin{aligned}
f^1(\mathbf{x}, y, u^{(1)}, u^{(2)}, \dots, u^{(r-1)}) &= \frac{\partial}{\partial x_2} v^1; \\
f^k(\mathbf{x}, y, u^{(1)}, u^{(2)}, \dots, u^{(r-1)}) &= (-1)^{(k-1)} \left(\frac{\partial}{\partial x_{k+1}} v^k + \frac{\partial}{\partial x_{k-1}} v^{k-1} \right), \ 1 < k < s; \\
f^s(\mathbf{x}, y, u^{(1)}, u^{(2)}, \dots, u^{(r-1)}) &= (-1)^{(s-1)} \frac{\partial}{\partial x_{s-1}} v^{s-1}.
\end{aligned} \tag{3.27}$$

Local symmetries admitted by (3.27) may induce nonlocal symmetries of $R\{\mathbf{x}, u\}$. The necessary conditions for a PDE written in conserved form, to admit potential symmetries is presented in [62]. In addition [63] provides an association between potential symmetries and reduction methods of order two.

We also note that any solution $u(x), v(x)$ of $S\{\mathbf{x}, u, v\}$ will define a solution of $u(x)$ of $R\{\mathbf{x}, u\}$ because $R\{\mathbf{x}, u\}$ is contained in $S\{\mathbf{x}, u, v\}$. The reader is referred to [49] for detailed outline of this technique.

3.4 Equivalence transformations and preliminary group classification

An *equivalence transformation* of the class of differential equations (3.2) is an isomorphism $F(\mathbf{x}, u, u^{(1)}, \ldots, u^{(r)}) \mapsto F(\overline{\mathbf{x}}, \overline{u}, \overline{u}^{(1)}, \ldots, \overline{u}^{(r)})$, that is, a one-to-one and onto mapping of dependent and independent variables of Eq. (3.2) to those of say, $\overline{F}(\overline{\mathbf{x}}, \overline{u}, \overline{u}^{(1)}, \ldots, \overline{u}^{(r)}) = 0$ in the same family. The said transformation enables one to map solutions of the differential equation to those of an equivalent equation. The concept of equivalence transformation has many advantages. One of them is that, we can develop methods of solving a family of differential equations instead of one [56]. The equivalence transformations defines a set of all point transformations which form a group called *equivalence group* $G_{\mathcal{E}}$. Lie's algorithm is used to calculate equivalence transformations after regarding all arbitrary coefficient functions of the system of governing differential equations to be variables (see e.g. [55, 64]).

In the case when a DE has arbitrary functions, then the invariance criterion leads to the determining equation which may be solved when the functions are considered arbitrary. The symmetry generators admitted by the equation, given arbitrary functions, span the PLA (see e.g. [55, 64]).

One may utilize the equivalence transformation to determine the forms of arbitrary functions which extended the principal Lie algebra by one (see e.g. [55]). The method of searching for the forms of arbitrary functions which increase the PLA by one, is called preliminary group classification. The preliminary group classification does not provide all the forms of arbitrary functions which allow extra symmetries being admitted. One may consider enhanced group classification (see e.g. [65]). We however, we restrict our study to the preliminary group classification.

3.5 Lie algebras

In this section we discuss the notion of Lie algebras. The reader is referred to texts by Bluman *et al.*, [48, 49] for more details. Consider the infinitesimal generators

$$\Gamma_1 = \xi_1^i \frac{\partial}{\partial x^i} + \eta_1 \frac{\partial}{\partial u} \tag{3.28}$$

and

$$\Gamma_2 = \xi_2^i \frac{\partial}{\partial x^i} + \eta_2 \frac{\partial}{\partial u}. \tag{3.29}$$

The Lie bracket or the commutator of Γ_1 and Γ_2 is defined as

$$[\Gamma_1, \Gamma_2] = (\Gamma_1 \xi_2^i - \Gamma_2 \xi_1^i) \frac{\partial}{\partial x^i} + (\Gamma_1 \eta_2 - \Gamma_2 \eta_1) \frac{\partial}{\partial u} = \Gamma_1 \Gamma_2 - \Gamma_2 \Gamma_1. \tag{3.30}$$

The nonzero commutator of any two infinitesimal generator is also, an infinitesimal generator. It follows from (3.30) that the Lie bracket is skew-symmetric that is

$$[\Gamma_1, \Gamma_2] = -[\Gamma_2, \Gamma_1]. \tag{3.31}$$

Furthermore, any three infinitesimal generators $\Gamma_1, \Gamma_2, \Gamma_3$ satisfy the Jacobi's identity

$$[[\Gamma_1, \Gamma_2], \Gamma_3] + [[\Gamma_2, \Gamma_3], \Gamma_1] + [[\Gamma_3, \Gamma_1], \Gamma_2] = 0. \tag{3.32}$$

A Lie algebra $\mathcal{L}$ is a vector space over some vector field $\mathcal{F}$ with an additional law of combination of elements in $\mathcal{L}$ satisfying the properties skew-symmetry and the Jacobi's identity. Furthermore, the following axioms hold;

the commutator is bilinear,

$$[a\Gamma_1 + b\Gamma_2, \Gamma_3] = a[\Gamma_1, \Gamma_3] + b[\Gamma_2, \Gamma_3] \tag{3.33}$$

$$[\Gamma_1, a\Gamma_2 + b\Gamma_3] = a[\Gamma_1, \Gamma_2] + b[\Gamma_1, \Gamma_3] \tag{3.34}$$

where a, b are arbitrary constants. The infinitesimal generators (or base vectors) span the Lie algebra.

It is more convenient to represent the Lie brackets in a commutator table. For example, considering the generators in (3.23)-(3.25), we construct the commutator Table. 3.1.

Table 3.1: Lie Bracket of the admitted symmetry algebra for (3.13)

$[\Gamma_i, \Gamma_j]$	Γ_1	Γ_2	Γ_3
Γ_1	0	$2\Gamma_3$	Γ_1
Γ_2	$-2\Gamma_3$	0	$-\Gamma_2$
Γ_3	$-\Gamma_1$	Γ_2	0

3.6 One dimensional optimal system of subalgebras

Essentially there are two ways of constructing the one dimensional optimal system of subalgebras (see e.g. [66]), one is by Ovsiannikov [56, 67], based on determining the matrix of inner automorphism corresponding to the operators of the adjoint group of a given Lie algebra. The other method is by Olver

[53] whereby the generator is simplified as much as possible by subjecting it to chosen adjoint transformation. Here we adopt and briefly discuss the Olver's method.

Suppose that a PDE of the form (3.2) admits an s dimensional Lie algebra $\mathcal{L}_s$ viz., $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$. Reductions of independent variables by one is possible using any linear combination of base vectors

$$\Gamma = a_1\Gamma_1 + a_2\Gamma_2 + \ldots + a_s\Gamma_s \tag{3.35}$$

In order to ensure that a minimal complete set of reduction is obtained from symmetries admitted by the governing equation, an optimal system [55, 56] is constructed. An optimal system of a Lie algebra is a set of r dimensional subalgebras such that every r dimensional subalgebra is equivalent to a unique element of the set under some element of the adjoint representation;

$$\mathrm{Ad}\left(e^{\epsilon\Gamma_i}\right)\Gamma_j = \sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}\left(\mathrm{Ad}\Gamma_i\right)^n\Gamma_j = \Gamma_j - \epsilon\left[\Gamma_i, \Gamma_j\right] + \frac{\epsilon^2}{2!}\left[\Gamma_i, \left[\Gamma_i, \Gamma_j\right]\right] - \ldots, \tag{3.36}$$

where $[\Gamma_i, \Gamma_j]$ is the commutator of Γ_i and Γ_j. Patera and Winternitz [68] constructed the optimal system of all one dimensional Lie subalgebras arising from three and four dimensional Lie algebras by comparing the Lie algebra with standard classifications previously evaluated. An alternative method developed by Olver [53] involves simplifying as much as possible the generator (3.37) by subjecting it to chosen adjoint transformations.

Illustrative example

Consider the three dimensional Lie algebra spanned by the base vectors (3.23)-(3.25). Their Lie brackets are shown in the commutator Table. 3.1. Reduction of order of Eq. (6.31) by one is possible using any linear combination

$$\Gamma = a_1\Gamma_1 + a_2\Gamma_2 + a_3\Gamma_3 \tag{3.37}$$

where a_1, a_2, a_3 are real constants. We need to simplify as much as possible the coefficients a_1, a_2, a_3 by carefully applying the adjoint maps to Γ. To compute the optimal system we first need to determine the Lie brackets as given for example, in Table. 3.2. Using (3.36) in conjunction with the commutator Table 3.1, for example

$$\mathrm{Ad}(\exp(\epsilon\Gamma_1))\Gamma_3 = \Gamma_3 - \epsilon[\Gamma_1, \Gamma_3] + \frac{\epsilon^2}{2!}[\Gamma_1, [\Gamma_1, \Gamma_3]] - \ldots = \Gamma_3 - \epsilon\Gamma_1$$

we construct the adjoint representation table shown in Table. 3.2.

Table 3.2: Adjoint representation table for (3.13)

$\mathrm{Ad}\,(\exp(\epsilon\Gamma_i))\,\Gamma_j$	Γ_1	Γ_2	Γ_3
Γ_1	Γ_1	$\Gamma_2 - 2\epsilon\Gamma_3 + \epsilon^2\Gamma_1$	$\Gamma_3 - \epsilon\Gamma_1$
Γ_2	$\Gamma_1 + 2\epsilon\Gamma_3 + \epsilon^2\Gamma_2$	Γ_2	$\Gamma_3 + \epsilon\Gamma_2$
Γ_3	$e^{\epsilon}\Gamma_1$	$e^{-\epsilon}\Gamma_2$	Γ_3

Starting with a nonzero vector (3.37) with $a_1 \neq 0$ and rescaling such that $a_1 = 1$, it follows from Table. 3.2 that acting on Γ by $\mathrm{Ad}\left(\exp\left(\frac{1 \pm i\sqrt{3}}{2}\Gamma_2\right)\right)$, one obtains $\Gamma^I = \Gamma_1 + \widetilde{a}_3\Gamma_3$. Acting on Γ^I by $\mathrm{Ad}\,(\exp(c_1\Gamma_3))$ we get $\Gamma_1 + \widetilde{a}_3 e^{-c_1}\Gamma_3$. Depending on the sign of $\widetilde{a}_3$, the coefficient of Γ_3 can be assigned either $+1, -1$, or 0. Next, suppose that $a_1 = 0$, and assume that $a_3 \neq 0$ (say $a_3 = 1$ by rescaling); acting on the remaining vector by $\mathrm{Ad}\,(\exp(-a_2\Gamma_2))$, we obtain Γ_3. No further simplifications are possible. Therefore any one dimensional subalgebra spanned by Γ^I with $a_1 = 0$ and $a_3 \neq 0$ is equivalent to the one spanned by Γ_3. Eventually, we choose $a_2 \neq 0$ (say $a_2 = 1$ by rescaling) and this yield Γ_2, and so no further simplifications are possible. Thus the one dimensional optimal system is $\{\Gamma_1 \pm \Gamma_3, \Gamma_3, \Gamma_2\}$.

3.7 Basis of invariants

In the method of variable reduction by invariants one seeks a compatible invariant solution expressed in the form

$$\mu(\varphi_1, \varphi_2, \ldots, \varphi_n) = 0; \tag{3.38}$$

where $\varphi_1, \varphi_2, \ldots, \varphi_n$ is a complete set of n independent invariants for a one-parameter Lie point transformation group (3.4) and (3.5).

The basis for invariants may be constructed by solving the characteristics equations in Pfaffian form, corresponding to the (3.6) and (3.11)

$$\frac{dx_1}{\xi_1} = \frac{dx_2}{\xi_2} = \ldots = \frac{du}{\eta} \tag{3.39}$$

A one-parameter group of transformations is a classical symmetry of (3.2), provided (3.4) and (3.5) leave (3.2) invariant. Moreover, an invariant solution of (3.2),

$$u = G(\mathbf{x}), \tag{3.40}$$

must satisfy the invariant surface condition (I.S.C.)

$$\sum_j \xi_j(\mathbf{x}, u)\frac{\partial u}{\partial x_j} = \eta(\mathbf{x}, u), \tag{3.41}$$

which follows from

$$\frac{d}{d\epsilon}(\overline{u} - G(\overline{\mathbf{x}})) = 0, \tag{3.42}$$

Illustrative example

Equation

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x}\left[\theta^m \frac{\partial \theta}{\partial x}\right] - M^2\theta^{n+1} \tag{3.43}$$

admits among others symmetry generator

$$\Gamma = \left(\frac{m-n}{2}\right) x\frac{\partial}{\partial x} + \theta\frac{\partial}{\partial \theta} - n\tau\frac{\partial}{\partial \tau}. \tag{3.44}$$

By solving the characteristic equation

$$\frac{d\tau}{-n\tau} = \frac{dx}{\left(\frac{m-n}{2}\right) x} = \frac{d\theta}{\theta} \tag{3.45}$$

we obtain the basis of invariants

$$\gamma = x\tau^{\frac{m-n}{2n}} \qquad \text{and} \qquad \theta = \tau^{-\frac{1}{n}} G(\gamma). \tag{3.46}$$

Substituting (3.46) into the governing equation (3.43), one obtains the second order ODE

$$[G^m G']' + \frac{1}{n}G - \gamma G' - M^2 G^{n+1} = 0. \tag{3.47}$$

Although Eq. (3.47) may not be solved exactly, the application of a symmetry generator have resulted in the reduction which may be easier to solve numerically.

3.8 Methods of linearization and reductions of ODEs.

The second order (in particular nonlinear) ODE admitting two, three or eight dimensional Lie algebra may be integrated completely using two dimensional Lie (sub)algebra. The two admitted symmetries by a second order ODE, may either linearize the original equation or give rise to a second order equation in terms of the canonical coordinates, with cubic as highest degree of the first order derivative. Note that any linear second order ODE admitting eight symmetries is equivalent to the simple equation $y'' = 0$.

Suppose that a given equation admits a non-Abelian two dimensional algebra (subalgebra). That is, suppose the admitted Lie algebra is given by $[\Gamma_1, \Gamma_2] = \lambda_1 \Gamma_1, \quad \lambda_1 \in \mathbb{R}$. Following reduction of the original given equation by Γ_1, Γ_2 in new variables is automatically admitted by the reduced equation. Such symmetries are referred to as *inherited symmetries.*

In the case where the admitted symmetry algebra is one-dimensional or when one considers the one-dimensional subalgebra, the order of equation may be reduced by one using the method of differential invariants discussed below.

3.8.1 Method of differential invariants

This method involves determining invariants from the first prolongation of the given symmetries. That is, suppose one is given a second order ODE, then the order of this equation may be reduced by one upon determining the invariants from the first prolongation of the given symmetry generator. To illustrate, we consider the example below.

Illustrative example

Given symmetry generator Γ_2 in (3.24), the first prolongation is given by

$$\Gamma_2^{[1]} = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y} + (y - xy') \frac{\partial}{\partial y'} \tag{3.48}$$

The basis of invariants may be constructed by solving the equations

$$\frac{dx}{x^2} = \frac{dy}{xy} = \frac{dy'}{y - xy'}. \tag{3.49}$$

The invariants are therefore

$$t = \frac{y}{x}, \quad u = y - xy' \tag{3.50}$$

Writing $u = u(t)$, one obtains

$$u\frac{du}{dt} = \frac{(m+1)M^2}{t^3}, \tag{3.51}$$

which is a variable separable equation. The exact solution is given by

$$\frac{u^2}{2} + \frac{(m+1)M^2}{2t^2} + k_1 = 0. \tag{3.52}$$

In terms of the original variables we obtain

$$y^2(y - xy')^2 + (m+1)M^2x^2 + K_1y^2 = 0. \tag{3.53}$$

where $K_1 = 2k_1$.

3.8.2 Lie's method of canonical coordinates

This method involves reduction of the second order ODE using two dimensional Lie (sub)algebra. Any two dimensional Lie algebra can be transformed using proper choice of basis and canonical variables t and u. Furthermore, (see e.g. [69])

(i) A second order ODE admitting commutating pair of symmetries Γ_1 and Γ_2 that is, $[\Gamma_1, \Gamma_2] = 0$, such that a point transformation $t = \phi(x, y)$ and $u = \psi(x, y)$ which bring the canonical form to

(a)

$$\Gamma_1 = \frac{\partial}{\partial t}, \quad \Gamma_2 = \frac{\partial}{\partial u} \quad \text{and}$$

(b)

$$\Gamma_1 = \frac{\partial}{\partial u}, \quad \Gamma_2 = t\frac{\partial}{\partial u}$$

reduce the original equation into

(a)

$$u'' = f(u') \quad \text{and}$$

(b)

$$u'' = f(u) \text{ respectively.}$$

(ii) A second order ODE admitting non-commutating symmetries Γ_1 and Γ_2 i.e. $[\Gamma_1, \Gamma_2] = \Gamma_1$, such that a point transformation $t = \phi(x, y)$ and $u = \psi(x, y)$ which bring the canonical form to

(a)

$$\Gamma_1 = \frac{\partial}{\partial u}, \quad \Gamma_2 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} \text{ and}$$

(b)

$$\Gamma_1 = \frac{\partial}{\partial u}, \quad \Gamma_2 = u\frac{\partial}{\partial u}$$

reduce the original equation into

(a)

$$u'' = \frac{1}{t}f(u') \text{ and}$$

(b)

$$u'' = f(t)u' \text{ respectively.}$$

Note that (a) is an equation which is at most cubic in the first derivative and (b) is linear. For the detailed account on reductions of ODEs, in particular second order ODEs, by Lie point symmetries the reader is referred to [69].

Illustrative example

As an illustration, we observed that Eq. (3.13) admits a three dimensional Lie algebra spanned by the base vectors given in (3.23)-(3.25). The non commuting pair of symmetries Γ_1 and Γ_3 leads to the canonical variables

$$t = y^2, \qquad u = x + y^2 \tag{3.54}$$

The corresponding canonical forms of Γ_1 and Γ_3 are

$$\Gamma_1^* = \partial_u, \qquad \Gamma_3^* = t\partial_t + u\partial_u \tag{3.55}$$

Writing $u = u(t)$ transforms Eq. (3.13) to

$$u'' = -\frac{1}{2}\frac{u'-1}{t}\left[1 + 4(m+1)M^2(u'-1)^2\right] \tag{3.56}$$

Here the prime denotes the total derivative with respect to t.

Solving the equation Eq. (3.56), we obtain the solution that satisfies the Neumann boundary condition at $x = 0$, and the Dirichlet condition at $x = 1$, namely

$$\theta = \left[1 + (m+1)M^2(x^2-1)\right]^{\frac{1}{2(m+1)}}. \tag{3.57}$$

3.9 Concluding Remarks

In this chapter, a brief outline of Lie symmetry techniques is provided. Both historical and theoretical background of the field of symmetry analysis are discussed. A connection between the one-parameter group of transformations and corresponding infinitesimal transformation is provided. Furthermore we discussed the determination of local and nonlocal symmetries, the notion of equivalence transformations and Lie algebras. Also, we discussed the steps for construction of the optimal systems. The use of symmetries admitted by both the ODEs and PDEs are discussed and illustrated.

Chapter 4

Preliminary group classification of a steady nonlinear one-dimensional fin problem

Some results in this chapter have been published in an Institute for Scientific Information journal of 2013 impact factor 0.6777, as follows;

> R.J. Moitsheki and M.D. Mhlongo, Classical Lie point symmetry analysis of a steady nonlinear one-dimensional fin problem, Journal of Applied Mathematics, Vol. 2012, Article ID 671548, 13 pages.

4.1 Introduction

In this chapter we consider a one-dimensional model describing steady state heat transfer in a longitudinal rectangular fin. Both the heat transfer coefficient and the thermal conductivity are given as arbitrary functions of temperature. Preliminary group classification is performed and cases for which the PLA is increased by one are determined. Realistic cases are then selected and the problem is analyzed. In section 4.2, we provide the mathematical formulation of the problem. Symmetry analysis is performed in Section 4.3. We determine the principal Lie algebra, equivalence transformations and list the cases for which the principal Lie algebra is extended. In section 4.4, we employ symmetry techniques to determine wherever possible, the invariant solutions.

4.2 Mathematical models

Consider a longitudinal rectangular one-dimensional fin as shown in Fig. 2.2. The steady energy balance equation is given by [8]

$$A_c \frac{d}{dX}\left(K(T)\frac{dT}{dX}\right) = PH(T)(T - T_a), \qquad 0 \leq X \leq L, \tag{4.1}$$

and the variables and parameters are defined in chapter 2.

In dimensionless variables equation (4.1) becomes

$$\frac{d}{dx}\left[k(\theta)\frac{d\theta}{dx}\right] - M^2 h(\theta)\theta = 0, \quad 0 \leq x \leq 1, \tag{4.2}$$

and the boundary conditions become

$$\theta(1) = 1 \qquad \text{and} \qquad \theta'(0) = 0. \tag{4.3}$$

Since $h(\theta)$ is an arbitrary function of temperature, we can therefore equate the product $h(\theta)\theta$ to $G(\theta)$. Moitsheki *et al.*, [25] conducted the analysis of Eq. (4.2), wherein the heat transfer coefficient was assumed to be given by the power law function of temperature. In this investigation, both the heat transfer coefficient and thermal conductivity are arbitrary functions of temperature. We employ preliminary group classification techniques to determine the forms which lead to exact solutions. We now consider the governing equation

$$\frac{d}{dx}\left[k(\theta)\frac{d\theta}{dx}\right] - M^2 G(\theta) = 0, \quad 0 \leq x \leq 1. \tag{4.4}$$

We note that equation (4.4) is linearizable provided G is a differential consequence of k. The proof of this statement follows from chain rule [24]. This implies that Eq. (4.4) may be linearizable for any k such that its derivative is G. Also, the linearization of Eq. (4.4) was performed in [70] wherein approximate techniques were employed to solve the problem. In this study we apply Lie symmetry techniques to analyze the problem.

4.3 Symmetry analysis

In the next subsections we construct the equivalence algebra and hence equivalence group of transformations admitted by Eq. (4.4). Furthermore we determine the Lie point symmetries admitted by Eq. (4.4) with arbitrary functions k and G i.e. we seek the principal Lie algebra. Symmetry technique are algorithmic and tedious. Here we utilize the interactive computer software algebra REDUCE [41] to facilitate the calculations.

4.3.1 Equivalence transformations

To determine the equivalence transformation, one may seek the equivalence algebra generated by the vector field

$$\widetilde{\Gamma} = \xi(x,\theta)\partial_x + \eta(x,\theta)\partial_\theta + \eta^1(x,\theta,k,G)\partial_k + \eta^2(x,\theta,k,G)\partial_G. \tag{4.5}$$

The second prolongation is given by

$$\widetilde{\Gamma}^{[2]} = \Gamma + \zeta_x\partial_{\theta'} + \zeta_{xx}\partial_{\theta''} + \mu^1_x\partial_{k_x} + \mu^1_\theta\partial_{k'} + \mu^2_x\partial_{G_x}, \tag{4.6}$$

where

$$
\begin{aligned}
\zeta_x &= D_x(\eta) - \theta' D_x(\xi), \\
\zeta_{xx} &= D_x(\zeta_x) - \theta'' D_x(\xi), \\
\mu_x^1 &= \tilde{D}_x(\eta^1) - k_x \tilde{D}_x(\xi) - k' \tilde{D}_x(\eta), \\
\mu_x^2 &= \tilde{D}_x(\eta^2) - H_x \tilde{D}_x(\xi) - G' \tilde{D}_x(\eta), \\
\mu_\theta^1 &= \tilde{D}_\theta(\eta^1) - k_x \tilde{D}_\theta(\xi) - k' \tilde{D}_\theta(\eta),
\end{aligned} \tag{4.7}
$$

with D_x and $\tilde{D}_x$ being the total derivative operator defined by

$$D_x = \partial_x + \theta' \partial_\theta + \theta'' \partial_{\theta'} + ...,$$

$$\tilde{D}_x = \partial_x + k_x \partial_k + G_x \partial_G + k_{xx} \partial_{k_x} + ... = \partial_x,$$

and

$$\tilde{D}_\theta = \partial_\theta + k' \partial_k + ...$$

respectively. The prime implies differentiation with respect to θ. The invariance surface condition is given by

$$
\begin{aligned}
\widetilde{\Gamma}^{[2]}\left(\frac{d}{dx}\left[k(\theta)\frac{d\theta}{dx}\right] - M^2 G(\theta) = 0\right)\Big|_{\left(\frac{d}{dx}\left[k(\theta)\frac{d\theta}{dx}\right] - M^2 G(\theta) = 0\right)} &= 0, \\
\widetilde{\Gamma}^{[2]}(k_x = 0)|_{k_x = 0} &= 0, \\
\widetilde{\Gamma}^{[2]}(G_x = 0)|_{G_x = 0} &= 0.
\end{aligned} \tag{4.8}
$$

This system of equations yields the infinite dimensional equivalence algebra spanned by the base vectors

$$\widetilde{\Gamma}_1 = \partial_x, \ \widetilde{\Gamma}_2 = x\partial_x - 2G\partial_G, \ \ \widetilde{\Gamma}_3 = u(\theta)\partial_\theta - u'(\theta)k\partial_k, \ \tilde{\Gamma}_4 = v(H)\left(k\partial_k + G\partial_G\right),$$

admitted by Eq. (4.4). Here u and v are arbitrary functions of θ and G, respectively.

4.3.2 Principal Lie algebra

To determine the Principal Lie algebra of the governing equation we seek transformations of the form

$$\begin{aligned}\overline{x} &= x + \epsilon\xi(x,\theta) + O(\epsilon^2)\\ \overline{\theta} &= \theta + \epsilon\eta(x,\theta) + O(\epsilon^2),\end{aligned} \tag{4.9}$$

generated by the vector field

$$\Gamma = \xi(x,\theta)\frac{\partial}{\partial x} + \eta(x,\theta)\frac{\partial}{\partial \theta},$$

which is admitted by the governing equation for any arbitrary functions k and G. We seek invariance in the form

$$\Gamma^{[2]}(\frac{d}{dx}\left[k(\theta)\frac{d\theta}{dx}\right] - M^2G(\theta) = 0)|_{(\frac{d}{dx}\left[k(\theta)\frac{d\theta}{dx}\right] - M^2G(\theta)=0)} = 0$$

Here $\Gamma^{[2]}$ is the second prolongation defined by

$$\Gamma^{[2]} = \Gamma + \eta_1\frac{\partial}{\partial\theta'} + \eta_2\frac{\partial}{\partial\theta''},$$

where

$$\begin{aligned}\eta_1 &= D_x(\eta) - \theta' D_x(\xi)\\ \eta_2 &= D_x(\eta_1) - \theta'' D_x(\xi)\end{aligned} \tag{4.10}$$

with

$$D_x = \frac{\partial}{\partial x} + \theta'\frac{\partial}{\partial\theta} + \theta''\frac{\partial}{\partial\theta'}.$$

The principal Lie algebra is one-dimensional and spanned by space translation. For nontrivial function k and G we obtain the determining equations

(1) $k'\xi_\theta - k\xi_{\theta\theta} = 0,$

(2) $k^2\eta_{\theta\theta} + kk'\eta_\theta + kk''\eta - (k')^2\eta - 2k^2\xi_{x\theta} = 0,$

(3) $2k\eta_{x\theta} + 2k'\eta_x - k\xi_{xx} - 3M^2G\xi_\theta = 0,$

(4) $k^2\eta_{xx} + M^2kH\eta_\theta - M^2kH'\eta + M^2Hk'\eta - 2M^2kH\xi_x = 0.$

The determining equation (1) implies that $\xi = \phi(\theta) + \psi(x)$ and $k = \phi'(\theta)$, where ϕ and ψ are arbitrary functions of θ and x, respectively. The determining equations (2), (3) and (4) become

(2*) $\eta_{\theta\theta}\phi'^2 + \phi''\phi'\eta_\theta + (\phi'''\phi' - \phi''^2)\eta = 0,$

(3*) $2\eta_{x\theta}\phi' + 2\eta_x\phi'' + \phi'\psi'' - 3M^2\phi'G = 0,$

(4*) $\eta_{xx}\phi'^2 + M^2\eta_\theta\phi'G - M^2G'\phi'\eta + M^2\phi''\eta G - 2M^2\phi'\psi'G = 0.$

It appears that full group classification of Eq. (4.4) may be difficult to achieve. Hence, we resort to the preliminary group classification techniques.

4.3.3 Preliminary group classification

We follow the sketch of the preliminary group classification technique as outlined in [55]. We note that the Eq. (4.4) admits an infinite equivalence algebra as given in Section 4.3.1. So we are free to take any finite dimensional subalgebra as large as we desire and use it for preliminary group classification. We choose a five dimensional equivalence algebra spanned by the vectors

$$\tilde{\Gamma}_1 = \partial_x,\ \tilde{\Gamma}_2 = x\partial_x - 2G\partial_G,\ \tilde{\Gamma}_3 = \partial_\theta,\ \tilde{\Gamma}_4 = \theta\partial_\theta - k\partial_k,\ \tilde{\Gamma}_5 = k\partial_k + G\partial_G. \tag{4.11}$$

Recall that k and G are θ dependent. Thus we consider the projections of (4.11) on the space of (θ, k, G). The nonzero projections of operators (4.11) are

$$\mathbf{v_1} = pr(\tilde{\Gamma}_2) = -2H\partial_H,\ \mathbf{v_2} = pr(\tilde{\Gamma}_3) = \partial_\theta,\ \mathbf{v_3} = pr(\tilde{\Gamma}_4) = \theta\partial_\theta - k\partial_k,$$
$$\mathbf{v_4} = pr(\tilde{\Gamma}_5) = k\partial_k + G\partial_G. \tag{4.12}$$

Proposition 1 (see e.g. [55]) Let $\mathcal{L}_r$ be an $r-$dimensional subalgebra of the algebra $\mathcal{L}_4$. Denote by $Z_i,\ i = 1, ..., r$ a basis of $\mathcal{L}_r$ and by W_i the elements of the algebra $\mathcal{L}_5$ such that $Z_i =$ projections of W_i on (θ, k, G). If equations

$$k = \mu(\theta), \qquad G = \varphi(\theta)$$

are invariant with respect to the algebra $\mathcal{L}_r$ then the equation

$$\frac{d}{dx}\left(\mu(\theta)\frac{d\theta}{dx}\right) - M^2\varphi(\theta) = 0 \tag{4.13}$$

admits the operator

$$Z_i = \text{ projection of } W_i \text{ on } (x, \theta).$$

Proposition 2 (see e.g. [55]) Let Eq. (4.13) and equation

$$\frac{d}{dx}\left(\overline{\mu(\theta)}\frac{d\theta}{dx}\right) - M^2\overline{\varphi(\theta)} = 0 \tag{4.14}$$

be constructed according to proposition 1 via subalgebras $\mathcal{L}_r$ and $\overline{\mathcal{L}_r}$, respectively. If $\mathcal{L}_r$ and $\overline{\mathcal{L}_r}$, are similar subalgebras in $\mathcal{L}_5$ the Eq. (4.13) and (4.14) are equivalent with respect to the equivalence group G_5 generated by $\mathcal{L}_r$. This propositions imply that the problem of preliminary group classification of Eq. (4.4) is reduced to the algebraic problem of constructing nonsimilar subalgebras of $\mathcal{L}_4$ or optimal system of subalgebras [55]. We explore methods in [53] to construct the one dimensional optimal system of subalgebras. The set of nonsimilar one dimensional subalgebras is

$$\{\mathbf{v_1} + \alpha\mathbf{v_3} + \beta\mathbf{v_4},\ \mathbf{v_3} \pm \mathbf{v_2} + \alpha\mathbf{v_4},\ \mathbf{v_3} + \alpha\mathbf{v_4},\ \mathbf{v_4} + \alpha\mathbf{v_2}, \mathbf{v_2}\}.$$

Here α and β are arbitrary constants. As an example we apply Proposition 1 to one of the element of the optimal system. since this involves routine calculations of invariants we list the rest of cases in Table 4.1, wherein λ, p

and q are arbitrary constants. Note that the power law k was obtained in [25], therefore we omit this case in this manuscript.

Consider the subalgebra

$$\mathbf{v_2} + \mathbf{v_4} = k\partial_k + G\partial_G + \partial_\theta$$

where without loss of generality we have assumed α to be unity. A basis of invariants is obtained from the equation

$$\frac{dk}{k} = \frac{dG}{G} = \frac{d\theta}{1}$$

and the forms of k and G are

$$k = \mathrm{e}^{\theta} \quad \text{and} \quad G = \mathrm{e}^{\theta}.$$

For simplicity we have allowed both integration constants to vanish. Further cases are listed in Table 4.1. By applying Proposition 1, we obtain the symmetry generator $\Gamma_2 = \partial_\theta$. We shall show in Section 4.4, that for this forms of k and G one may obtain seven more Lie point symmetry generators.

4.4 Symmetry reductions and invariant solutions

The main use of symmetries is to reduced the number of independent variables of the given equation by one. If a partial differential equation (PDE) is reduced to an ordinary differential equation (ODE), one may or may not solve the resulting ODE exactly. If a second order ODE admits a two-dimensional Lie (sub)algebra, then one can use Lie's method of *canonical coordinates* to completely integrate the equation (see e.g. [69]).

Table 4.1: Extensions of the Principal Lie algebra.

Forms		Symmetries
k	G	$\Gamma_1 = \partial_x$.
$e^{p\theta}$	$e^{q\theta}$	$\Gamma_2 = x\partial_x + \frac{2}{p-q}\partial_\theta, \quad p \neq q$.
p	$e^{q\theta}$	$\Gamma_2 = x\partial_x - \frac{2}{q}\theta\partial_\theta$.
$(1+\lambda\theta)$	$(1+\lambda\theta)^p$	$\Gamma_2 = x\partial_x + \frac{2(1+\lambda\theta)}{\lambda(p-2)}\partial_\theta, \quad p \neq -6$.
		$\Gamma_2 = 2\lambda x^2\partial_x + x(1+\lambda\theta)\partial_\theta$, $\Gamma_3 = 2\lambda x\partial_x + (1+\lambda\theta)\partial_\theta, \quad p = -6$

Example 1

As an illustrative example, we consider the case $k = e^{p\theta}$ and $h = \theta^{-1}e^{q\theta}$ where $p \neq q$. In this case Eq. (4.2) admits a non-Abelian two-dimensional Lie algebra spanned by the base vectors listed in Table 1. This non commuting pair of symmetries leads to the canonical variables

$$t = e^{\frac{p-q}{2}\theta}, \qquad \text{and} \qquad u = c_1 e^{\frac{p-q}{2}\theta} + x,$$

where c_1 is an arbitrary constant. We have two cases, the 'particular' canonical variables when $c_1 = 0$ and the 'general' canonical variables given a nonzero c_1, say $c_1 = 1$.

The corresponding canonical forms of Γ_1 and Γ_2 are

$$\Gamma_1^* = \partial_u, \qquad \text{and} \qquad \Gamma_2^* = t\partial_t + u\partial_u.$$

Writing $u = u(t)$ transforms Eq. (4.4) to

$$u'' = \frac{u'}{t}\left[\left(\frac{2p}{p-q} - 1\right) - \left(\frac{p-q}{2}\right) M^2 u'^2\right], \quad p \neq q. \tag{4.15}$$

Here prime is the total derivative with respect to t. Three cases arise.

case i, for $u' = 0$ we obtain the constant solution which is not related to the original problem. Thus we ignore it.

case ii, If the term in the square bracket vanish then we obtain in terms of original variables the exact 'particular' solution

$$\theta = \left(\frac{2}{p-q}\right) \ln\left[\frac{(p-q)M}{\pm\sqrt{2(p+q)}}\left(x - 1 \pm \frac{\sqrt{2(p+q)}}{(p-q)M}\, \mathrm{e}^{(p-q)/2}\right)\right].$$

Note that this exact solution satisfy the boundary only at one end. The Neumann's boundary condition leads to a contradiction since the thermo-geometric fin parameter is a nonzero constant.

case iii, If $u' \neq 0$ and $\left(\frac{2p}{p-q} - 1\right) - \left(\frac{p-q}{2}\right) M^2 u'^2 \neq 0$, we obtain the solution in complicated quadratures and therefore we omit it.

4.4.1 General canonical form

In this case the transformed equations is given by

$$u'' = \frac{(u'-1)}{t}\left[\left(\frac{2p}{p-q} - 1\right) - \left(\frac{p-q}{2}\right) M^2 (u'-1)^2\right], \quad p \neq q. \tag{4.16}$$

Clearly $u' - 1 \rightarrow y'$ reduces Eq. (4.16) to Eq. (4.15). We herein omit further analysis.

Example 2

We consider as an example Eq (4.4) with thermal conductivity given as exponential function of temperature i.e., $k = \mathrm{e}^{p\theta}$ and heat transfer coefficient is given as the product $\theta^{-1}\mathrm{e}^{q\theta}$. Given $p = q$, then equation (4.4) admits a maximal eight dimensional symmetry algebra spanned by the base vectors

$$\left.\begin{aligned}
&\Gamma_1 = \mathrm{e}^{\sqrt{p}Mx+p\theta}\left\{\partial_x + \tfrac{M}{\sqrt{p}}\,\partial_\theta\right\},\\
&\Gamma_2 = \mathrm{e}^{-\sqrt{p}Mx+p\theta}\left\{\partial_x - \tfrac{M}{\sqrt{p}}\,\partial_\theta\right\},\\
&\Gamma_3 = \mathrm{e}^{\sqrt{p}Mx-p\theta}\partial_\theta, \qquad X_4 = \frac{\sqrt{p}\mathrm{e}^{2\sqrt{p}Mx}}{M}\partial_\theta,\\
&\Gamma_5 = \partial_x, \qquad \Gamma_6 = \mathrm{e}^{-\sqrt{p}Mx-p\theta}\left\{\partial_x + \partial_\theta\right\},\\
&\Gamma_7 = \mathrm{e}^{-2\sqrt{p}Mx}\left\{-\tfrac{\sqrt{p}}{M}\,\partial_x + \partial_\theta\right\}, \quad \Gamma_8 = \partial_\theta.
\end{aligned}\right\} \tag{4.17}$$

Equation (4.4) is linearizable or equivalent to $y'' = 0$ (see e.g. [69]). In fact, we note that the point transformation $\mu = \mathrm{e}^{p\theta}, p \in \mathbb{R}$ linearizes Eq (4.4) given $p = q$. Following a simple manipulation we obtain the invariant solutions satisfying the prescribed boundary conditions, namely

$$\theta = \ln\left[\frac{\mathrm{e}^p\cosh\left(M\sqrt{p}\,x\right)}{\cosh\left(M\sqrt{p}\right)}\right]^{\frac{1}{p}}, \qquad p > 0. \tag{4.18}$$

Solution (4.18) is depicted in Figs. 4.1 and 4.2. Note that for $p = 0$ and $p < 0$ we obtain solutions which have no physical significance for heat transfer in fins. Therefore we herein omit such solutions.

The fin efficiency is defined as the ratio of actual heat transfer from the fin surface to the surrounding fluid while the whole fin surface is kept at the same temperature (see e.g. [2]). Given (4.18) fin efficiency (η) is given by

$$\eta = \int_0^1 e^{p\theta}dx = e^p\tanh\left(M\sqrt{p}\right). \tag{4.19}$$

The fin efficiency (4.19) is depicted in Fig. 4.3.

4.5 Concluding remarks

We considered a one dimensional fin model describing steady state heat transfer in longitudinal rectangular fins. Here, the thermal conductivity and heat

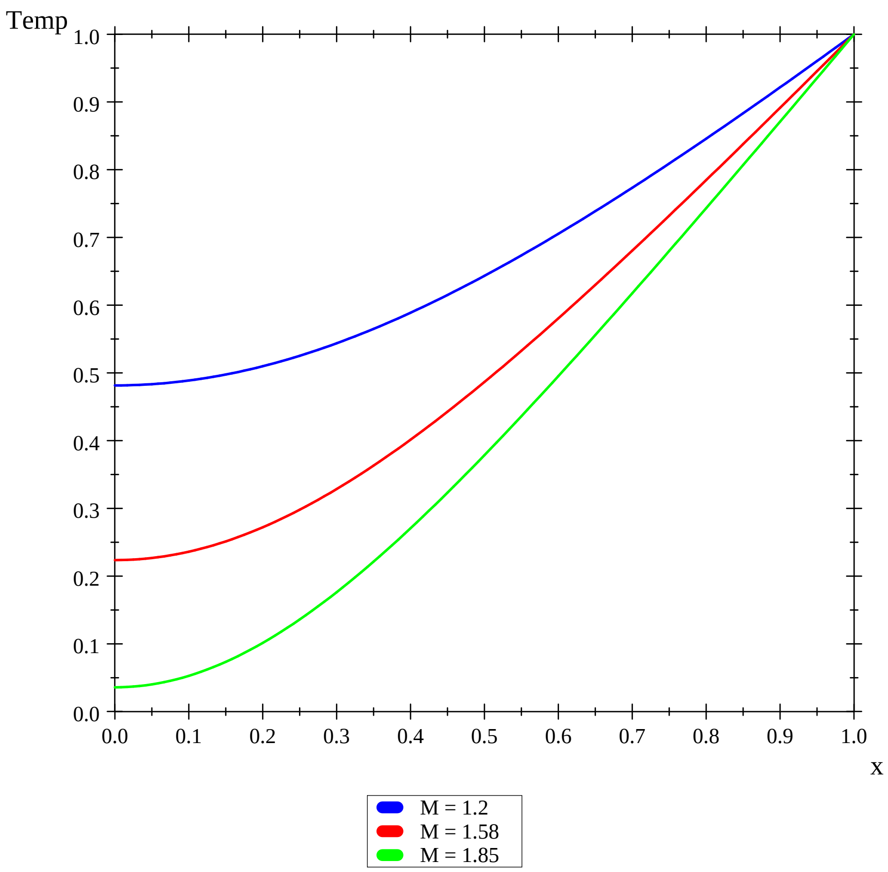

Figure 4.1: Temperature profile in a fin with varying values of the thermo-geometric fin parameter. Here p is fixed at unity.

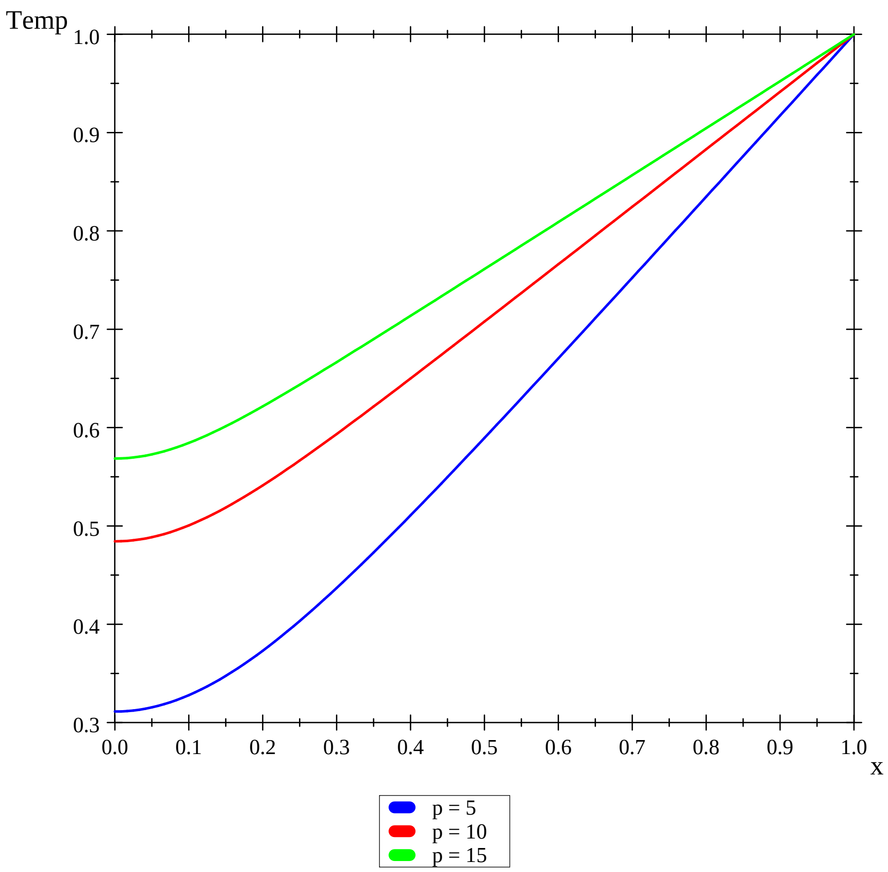

Figure 4.2: Temperature profile in a fin with varying values of the p. Here the thermo-geometric fin parameter is fixed at 1.85.

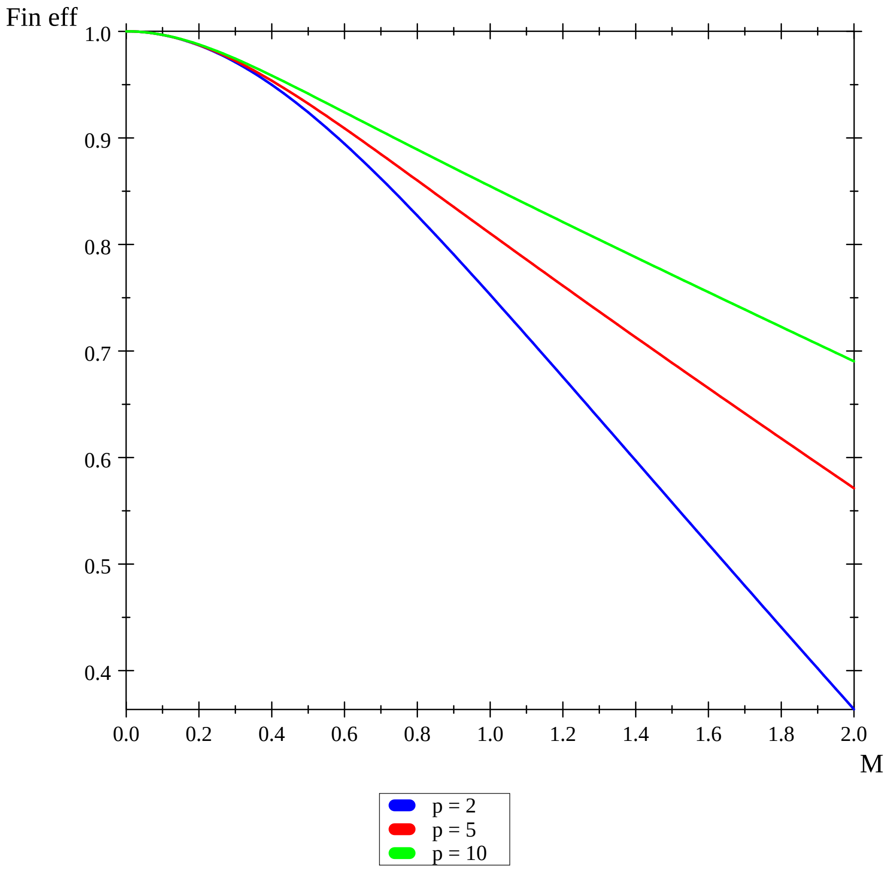

Figure 4.3: Fin efficiency.

transfer coefficient are temperature dependent. As such the considered problem are highly nonlinear. This is a significant improvement to the results presented in the literature (see e.g. [6, 7]). Preliminary group classification led to a number of cases of thermal conductivity and heat transfer coefficient for which extra symmetries are obtained. Exact solutions are constructed when thermal conductivity and heat transfer coefficient increase exponential with temperature. We observed in Fig. 4.1, that temperature inversely proportional to the values of the thermo-geometric fin parameter. Furthermore we observe that for certain values of M, the solution is not physically sound (see also, [71]). One may recall that the thermo-geometric fin parameter de-

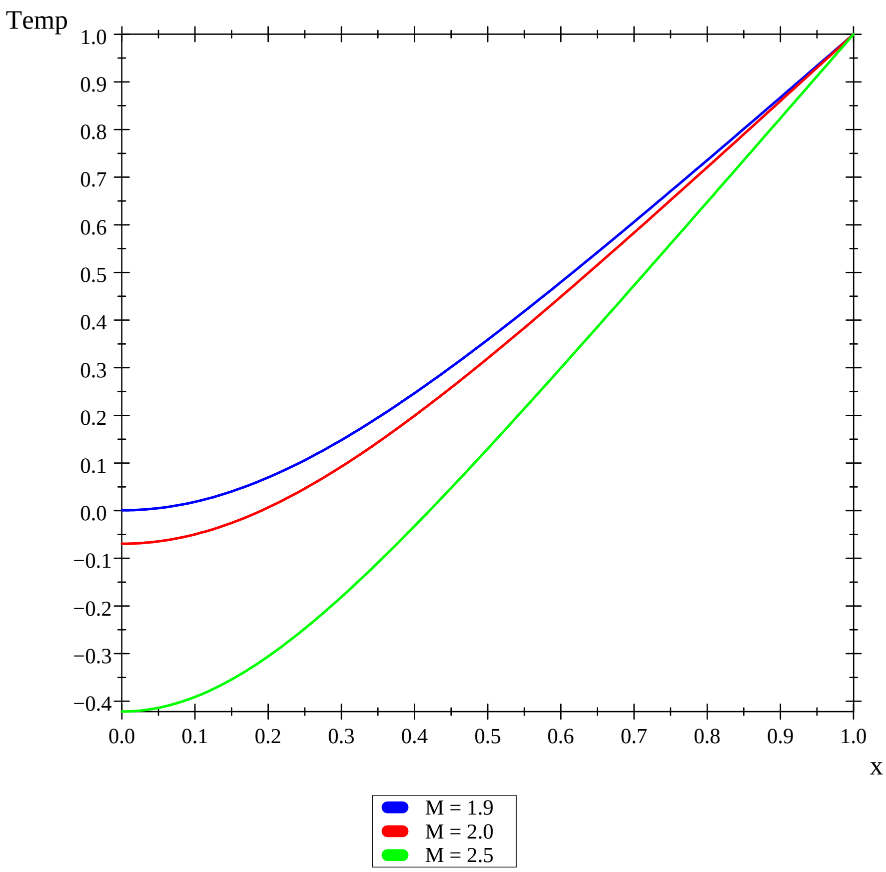

Figure 4.4: Temperature profile in a fin of varying values of the thermo-geometric fin parameter. Here p is fixed at unity.

pends also on heat transfer coefficient at the base of the fin. We notice that the exponential temperature dependent heat transfer coefficient in this paper leads to lower values of M for which the solutions are realistic. That is, the maximum values of M, say M_{max} for which the solutions are physically sound is around 2. We observe in Fig. 4.4, that for as values of M increase beyond 2 the temperature profile becomes negative. This contradicts the rescaling of temperature (the dimensionless temperature). Unlike in [25] and [32] whereby heat transfer is given by a power law, this value is much higher. The reasons behind this observation is studied elsewhere. In Fig. 4.2, temperature increases with increased values of the exponent p. Furthermore, fin efficiency decreases with increased values of the thermo-geometric fin parameter. We observed in Fig. 4.3, that the maximum value of the thermo-geometric fin parameter for which the fin efficiency is realistic in again around 2.

Chapter 5

Transient response of longitudinal rectangular fins to step change in base temperature and in base heat flow conditions

Some results in this chapter have been published in an Institute for Scientific Information journal of 2013 impact factor 2.407, as follows;

> M.D. Mhlongo, R.J. Moitsheki and O.D. Makinde, Transient response of longitudinal rectangular fins to step change in base temperature and in base heat flow conditions, International Journal of Heat and Mass Transfer, 57 (2013) 117-125.

5.1 Introduction

In this chapter we extend the work in [32]. Unlike in [32] wherein thermal conductivity was given as a linear function of temperature, here both the thermal conductivity and the heat transfer coefficient are given as power law temperature dependent, as such the models become highly nonlinear. Furthermore we consider the step change in base heat flux flow conditions. Few exact solutions are known and perhaps this is due to the added difficulty by the nonlinearity. In fact, exact solutions are constructed for fin problems when both thermal conductivity and heat transfer coefficient are constant. In this investigation we attempt to construct the exact solutions given (i) step change in base temperature and (ii) step change in base heat flow conditions. The governing equation admits a number of local point symmetries, and nonlocal symmetries for specific values of the exponents. We construct the steady state solutions which satisfy the boundary conditions. Since the exact solutions for transient state do not satisfy the entire boundary conditions we seek the numerical solutions. Mathematical models are given in Section 5.2. We utilize local and nonlocal symmetry methods in Section 5.3. In Section 5.4, we seek numerical solutions and provide exciting results. Lastly we provide the concluding remarks in Section 5.5.

5.2 Mathematical models

The energy balance for a longitudinal fin of a rectangular profile is a special case of Eq. (2.22) given by

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x}\left(k(\theta)\frac{\partial \theta}{\partial x}\right) - M^2 h(\theta)\theta, \qquad 0 \le x \le 1. \tag{5.1}$$

The use of the power law for the heat transfer coefficient and thermal conductivity, changes the one dimensional transient heat conduction Eq. (5.1) into

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x}\left[\theta^m \frac{\partial \theta}{\partial x}\right] - M^2 \theta^{n+1}, \qquad 0 \le x \le 1. \tag{5.2}$$

The initial condition is given by

$$\theta(x,0) = 0, \tag{5.3}$$

the step change in fin base temperature is given by

$$\theta(1,\tau) = 1, \tag{5.4}$$

the step change in fin base heat flux is given by

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=1} = 1 \tag{5.5}$$

and fin tip boundary condition is given by

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=0} = 0. \tag{5.6}$$

We refer to Eq. (5.2) as the governing equation.

5.3 Classical Lie point symmetry analysis

Determining symmetries for the governing Eq. (5.2), implies seeking transformations of the form

$$\left.\begin{aligned} \overline{x} &= \ x + \epsilon\xi^1(\tau,x,\theta) + O(\epsilon^2) \\ \overline{\tau} &= \ \tau + \epsilon\xi^2(\tau,x,\theta) + O(\epsilon^2) \\ \overline{\theta} &= \ \theta + \epsilon\Theta(\tau,x,\theta) + O(\epsilon^2) \end{aligned}\right\} \tag{5.7}$$

generated by the vector field

$$\Gamma = \xi^1(\tau,x,\theta)\frac{\partial}{\partial x} + \xi^2(\tau,x,\theta)\frac{\partial}{\partial\tau} + \Theta(\tau,x,\theta)\frac{\partial}{\partial\theta}, \tag{5.8}$$

which leave the governing equation invariant. Here ϵ is a Lie group parameter. Note that we seek symmetries that leave a single Eq. (5.2) invariant rather than the entire boundary value problem, and apply the boundary condition to the obtained invariant solutions. It is well known that the dimension of symmetry algebra admitted by the governing equation may reduce if one seeks invariance of the entire BVP (see e.g. [49]). The action of Γ is extended to all the derivatives appearing in the governing equation through second prolongation

$$\Gamma^{[2]} = \Gamma + \zeta^{\tau}\frac{\partial}{\partial \theta_{\tau}} + \zeta^{x}\frac{\partial}{\partial \theta_{x}} + \zeta^{xx}\frac{\partial}{\partial \theta_{xx}}, \tag{5.9}$$

where

$$\begin{aligned}
\zeta^{\tau} &= D_{\tau}(\Theta) - \theta_x D_{\tau}(\xi^1) - \theta_{\tau} D_t(\xi^2),\\
\zeta^{x} &= D_{x}(\Theta) - \theta_x D_{x}(\xi^1) - \theta_{\tau} D_x(\xi^2),\\
\zeta^{xx} &= D_{x}(\zeta_x) - \theta_{xx} D_{x}(\xi^1) - \theta_{x\tau} D_x(\xi^2)
\end{aligned}$$

and, D_x and D_{τ} are the operators of total differentiation with respect to x and τ respectively. The operator Γ is a point symmetry of the governing Eq. (5.2), if

$$\begin{aligned}
\Gamma^{[2]}\left(\frac{\partial \theta}{\partial \tau} - \frac{\partial}{\partial x}\left[\theta^m \frac{\partial \theta}{\partial x}\right] + M^2\theta^{n+1} = 0\right)\bigg|_{\frac{\partial \theta}{\partial \tau} - \frac{\partial}{\partial x}\left[\theta^m \frac{\partial \theta}{\partial x}\right] + M^2\theta^{n+1}=0} &= 0,\\
\Gamma^{[2]}\left(\frac{\partial \theta}{\partial x} = 0\right)\bigg|_{\frac{\partial \theta}{\partial x}=0} &= 0.
\end{aligned} \tag{5.10}$$

Since the coefficients of Γ do not involve derivatives, we can separate (5.10) with respect to the derivatives of θ and solve the resulting overdetermined system of linear homogeneous partial differential equations known as the *determining equations*. Further calculations are omitted at this stage as they were facilitated by a freely available package DIMSYM [40], a subprogram of REDUCE [41].

More often, differential equations arising in real world problem involve one or more functions depending on either the independent variables or on the dependent variables given a system of equations. It is possible by symmetry techniques to determine the cases which allow the equation in question to admit extra symmetries. The exercise of searching for the forms of arbitrary functions that extend the principal Lie algebra is called *group classification*. Here we assume the realistic thermal conductivity and heat transfer coefficient for a fin with rectangular profile.

5.3.1 Local symmetries

Group classification of a class of nonlinear heat equation with a source has been carried out by Dorodnitsyn in the late nineteen seventies and early eighties (see Chapter 10 in [72]). Also, classical symmetry analysis of diffusion equations for thermal energy storage was performed by Moitsheki and Makinde [73]. Here we consider a subclass of Dorodnitsyn class of heat equation arising in heat flow in longitudinal fins. For some case of the exponents of heat transfer coefficient and thermal conductivity the considered equation is transformable to tractable equation Ermakov-Pinney type equation. We determine the general exact analytical solutions. Symmetry analysis reveals a three dimensional Lie algebra being admitted by Eq. (5.2) with $m \neq n$. The algebra is spanned by the base vectors

$$\Gamma_1 = \partial_\tau, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = \frac{1}{m}\left\{\theta\partial_\theta + \frac{m-n}{2}x\partial_x - n\tau\partial_\tau\right\}. \tag{5.11}$$

Following the method in [53], we construct the one-dimensional optimal system of subalgebras and obtain the set

$$\{\Gamma_3,\ \Gamma_2 \pm \Gamma_1,\ \Gamma_2,\ \Gamma_1\}. \tag{5.12}$$

The reductions by these elements of the one-dimensional optimal system are given in Table 5.1, wherein the prime indicates differentiation with respect to the given invariant (similarity variable). We observe that reductions in Table 5.1 lead to general exact analytical solutions provided $m = 0$ and $n = -1$, but this choice renders the original equation linear. Such linear problems are of no interest to us therefore we omit them here. Given $n = m = -4/3$, Eq. (5.2) admits a five dimensional Lie algebra spanned by base vectors (see also see Chapter 10 in [72])

$$\left.\begin{array}{l}\Gamma_1 = \partial_\tau, \quad \Gamma_2 = -\frac{3}{4M^2}\partial_x, \quad \Gamma_3 = 3\theta\partial_\theta + 4\tau\partial_\tau \\ \Gamma_4 = \frac{1}{\sqrt{3}}\left\{-3M\cos\left(\frac{2Mx}{\sqrt{3}}\right)\theta\partial_\theta + \sqrt{3}\sin\left(\frac{2Mx}{\sqrt{3}}\right)\partial_x\right\}, \\ \Gamma_5 = \frac{1}{\sqrt{3}}\left\{3M\sin\left(\frac{2Mx}{\sqrt{3}}\right)\theta\partial_\theta + \sqrt{3}\cos\left(\frac{2Mx}{\sqrt{3}}\right)\partial_x\right\}.\end{array}\right\} \tag{5.13}$$

The one-dimensional optimal system of subalgebra is

$$\{\Gamma_5 + \alpha\Gamma_3,\ \Gamma_3 + \alpha\Gamma_4,\ \Gamma_4 + \alpha\Gamma_1,\ \Gamma_2 \pm \Gamma_1,\ \Gamma_2\}. \tag{5.14}$$

The reductions by these elements of the optimal system are provided in Table 5.2.

Local symmetry reductions

Symmetries, when admitted, may be used to reduced the independent variables of a partial differential equation by one. The reduced equations may or may not be solved. The similarity solutions constructed by symmetries are called invariant (exact) solution. Even when explicit exact solutions may not be constructed, reduction of variable has a number of advantages. For example, the nonlinear boundary value problems which are compatible may be easily solvable by the numerical schemes or the differential equations in fewer variables or lower order have been extensively studied.

Example 1: Steady state solutions

The admitted symmetry generator $\Gamma = \partial_\tau$ leads to the steady state heat transfer, that is, the governing Eq. (5.2) is invariant under time translation. The

Table 5.1: Reductions by elements of the optimal systems (5.12)

Symmetry	Reductions
Γ_3	$\gamma = x\tau^{(m-n)/2n}$; $\theta = \tau^{-1/n}G(\gamma)$ where G satisfies $-\frac{1}{n}G + \gamma G' = [G^m G']' - M^2 G^{n+1}$.
$\Gamma_2 \pm \Gamma_1$	$\gamma = x \pm a\tau$; $\theta = G(\gamma)$, where G satisfies $\pm aG = [G^m G']' - M^2 G^{n+1}$.

Table 5.2: Reductions by elements of the optimal systems (5.14)

Symmetry	Reductions
$\Gamma_2 + \Gamma_1$	$\gamma = x + \frac{3}{4M}\tau$; $\theta = G(\gamma)$ where G satisfies $\left[G^{-4/3}G'\right]' = \frac{3}{4M^2}G' + M^2 G^{-1/3}$.
$\Gamma_4 + \alpha\Gamma_1$, $\alpha \neq 0$	$\gamma = \left[\csc\left(\frac{2Mx}{\sqrt{3}}\right) - \cot\left(\frac{2Mx}{\sqrt{3}}\right)\right] \mathrm{e}^{-\tau/\alpha}$; $\theta = \sin\left(\frac{2Mx}{\sqrt{3}}\right)^{-3/2} G(\gamma)$; where G satisfies $\gamma^2\left[G^{-4/3}G'\right]' + \gamma G^{-4/3}G' + M^2 G^{-1/3} + \frac{\gamma}{\alpha}G' = 0$.
$\Gamma_3 + \alpha\Gamma_4$, $\alpha \neq 0$	$\gamma = \left[\csc\left(\frac{2Mx}{\sqrt{3}}\right) - \cot\left(\frac{2Mx}{\sqrt{3}}\right)\right]^{\sqrt{3}/2\alpha M} \tau^{-1/4}$; $\theta = \sin\left(\frac{2Mx}{\sqrt{3}}\right)^{-3/2}\left[\csc\left(\frac{2Mx}{\sqrt{3}}\right) - \cot\left(\frac{2Mx}{3\sqrt{3}}\right)\right]^{3\sqrt{3}/2\alpha M} G(\gamma)$; $GG'' + \frac{1}{4}\alpha^2\gamma^3 G' G^{7/3} + (3-\alpha^2)\gamma^2 G^2 - \frac{4}{3}G'^2 - \frac{1}{\gamma}GG' = 0$.
$\Gamma_5 + \alpha\Gamma_3$, $\alpha \neq 0$	$\gamma = \left[\sec\left(\frac{2Mx}{\sqrt{3}}\right) + \tan\left(\frac{2Mx}{\sqrt{3}}\right)\right]^{\sqrt{3}/2M} \tau^{-1/4\alpha}$; $\theta = \cos\left(\frac{2Mx}{\sqrt{3}}\right)^{-3/2}\left[\sec\left(\frac{2Mx}{\sqrt{3}}\right) + \tan\left(\frac{2Mx}{\sqrt{3}}\right)\right]^{3\sqrt{3}\alpha/2M} G(\gamma)$; $\alpha\gamma^2 GG'' - \frac{4}{3}\alpha\gamma^2(G')^2 - \alpha(2\alpha-\gamma)GG'$ $+\frac{1}{4}\gamma^{1+4\alpha}G^{7/3}G' - 3\alpha^3 G^2 - 4\alpha M^2 G = 0$.

steady state problem with given $m = n = -4/3$, is given by

$$\frac{d}{dx}\left[\theta^{-4/3}\frac{d\theta}{dx}\right] - M^2\theta^{-1/3} = 0, \tag{5.15}$$

(i) Subject to the step change in base temperature conditions

$$\theta'(0) = 0, \quad \theta(1) = 1 \qquad \text{and} \tag{5.16}$$

(ii) Subject to the step change in base heat flow conditions

$$\theta'(0) = 0, \quad \theta'(1) = 1. \tag{5.17}$$

The exact solution to Eq. (5.15) subject to the conditions (5.16) is

$$\theta = \left\{ \frac{\cos\left(\frac{M}{\sqrt{3}}\right)}{\cos\left(\frac{Mx}{\sqrt{3}}\right)} \right\}^3. \tag{5.18}$$

Solution (5.18) is depicted in Fig. 5.2. The exact analytical solution to Eq. (5.15) subject to (5.17) is given by

$$\theta = \frac{\cos\left(\frac{M}{\sqrt{3}}\right)^4}{\sqrt{3}M \sin\left(\frac{M}{\sqrt{3}}\right)} \cos\left(\frac{Mx}{\sqrt{3}}\right)^{-3}. \tag{5.19}$$

Solution (5.19) is depicted in Fig. 5.1. Note that the exact analytical steady-state solutions may be also obtained when $m = n \neq -4/3$ [25].

Example 2: Solutions for transient heat flow

Symmetry generator Γ_3 in (5.13) leads to the functional form of the exact solution

$$\theta(\tau, x) = \tau^{3/4} H(x),$$

where H satisfies the ordinary differential equation

$$\frac{3}{4}H = \frac{d}{dx}\left(H^{-4/3}\frac{dH}{dx}\right) - M^2 H^{-1/3}. \tag{5.20}$$

The above functional form is a separation of variables, however this has been exposed by symmetry methods. The transformation

$$w = H^{-1/3},$$

reduces equation (5.20) to the tractable Ermakov-Pinney type equation

$$w'' + \frac{M^2}{3}w + \frac{1}{4}w^{-3} = 0. \tag{5.21}$$

Equation (5.21) admits the algebra of $SL(2,\mathbb{R})$ which is isomorphic to the noncompact algebra $so(2,1)$ [61]. Furthermore equation of this type arise in many other areas including for example cosmology, elasticity, quantum mechanics and nonlinear systems (see e.g. [61] and references therein). We omit the symmetry analysis of Eq. (5.21) but directly solve it using method by Pinney [60]. The general exact solution to Eq. (5.21) is given by

$$c_1 w^2 = -\frac{1}{4}y^2 + y^2\left(c_2 + c_1\int\frac{dx}{y^2}\right)^2, \tag{5.22}$$

where $y(x)$ is a solution to the linear equation (see e.g. [74])

$$y'' + \frac{M^2}{3}y = 0.$$

In terms of the original variables we obtain the general exact solutions

$$\theta = \left\{\frac{\sqrt{c_1}\ \tau^{1/4}}{\cos\left(\frac{Mx}{\sqrt{3}}\right)\left[\left(c_2 + \frac{c_1\sqrt{3}}{M}\tan\left(\frac{Mx}{\sqrt{3}}\right)\right)^2 - \frac{1}{4}\right]^{1/2}}\right\}^3 \tag{5.23}$$

and

$$\theta = \left\{\frac{\sqrt{c_1}\tau^{1/4}}{\sin\left(\frac{Mx}{\sqrt{3}}\right)\left[\left(c_2 - \frac{c_1\sqrt{3}}{M}\cot\left(\frac{Mx}{\sqrt{3}}\right)\right)^2 - \frac{1}{4}\right]^{1/2}}\right\}^3. \tag{5.24}$$

Note that these general exact solutions satisfy the zero initial temperature and the no flux flow (adiabatic) boundary condition at the fin tip.

5.3.2 Nonlocal symmetries

In this section we employ nonlocal symmetry methods in an attempt to solve the given heat transfer problem, particularly when $m \neq n$. The symmetry solutions constructed using nonlocal symmetries cannot be obtained via Lie point (local) symmetries. It is common when constructing nonlocal symmetries to first express the governing equation in conserved form as a system of first order differential equations, known as the auxiliary system. A single equation may be expressed in more than one auxiliary system, thus one needs to be careful as nonlocal symmetry bearing auxiliary system may be hidden [75]. Given $n = -1$ and $m = -2$, one may write Eq. (5.2) as a system

$$\left.\begin{aligned} v_x &= \theta, \\ v_\tau &= \theta^{-2}\theta_x - M^2 x. \end{aligned}\right\} \tag{5.25}$$

Here v is the potential variable. This system indicate a globally conserved quantity

$$\int \theta dx$$

with the corresponding flux density $\theta^{-2}\theta_x - M^2 x$. Lie point symmetries admitted by the auxiliary system (5.25) yield nonlocal symmetries of Eq. (5.2) provided that at least one of the infinitesimals depend explicitly on the potential variable [49, 76]. Eq. (5.2) with $n = -1$ and $m = -2$ was analyzed using Lie-B$\ddot{a}$cklund transformation by Dorodnitsyn and Svirschchevskii (see page 137 in [72]). The point symmetries admitted by the auxiliary system (5.25) are

$$\left.\begin{aligned} &\mathbf{v}_1 = \partial_\tau, \quad \mathbf{v}_2 = v\partial_v + 2u\partial_u - x\partial_x + 2\tau\partial_\tau, \quad \mathbf{v}_3 = \partial_v, \\ &\mathbf{v}_4 = \tfrac{1}{M^2}\left\{-M^2\tau\partial_v + \partial_x\right\}, \\ &\mathbf{v}_5 = \tfrac{1}{M^2}\left\{-M^2\tau v\partial_v - \left(\theta^2 + 2M^2\tau\theta\right)\partial_\theta + \left(v + M^2 x\tau\right)\partial_x - M^2\tau^2\partial_\tau\right\}. \end{aligned}\right\} \tag{5.26}$$

Clearly $\mathbf{v}_5$ is a genuine nonlocal symmetry of Eq. (5.2).

Nonlocal symmetry reductions

Here we perform symmetry reductions using the nonlocal symmetry $\mathbf{v}_5$. The basis for invariants may be constructed by solving the characteristic equations in Pfaffian form

$$-\frac{dv}{M^2\tau v} = -\frac{d\theta}{\theta^2 + 2M^2\tau\theta} = \frac{dx}{v + M^2 x\tau} = -\frac{d\tau}{M^2\tau^2}. \tag{5.27}$$

Following a straightforward calculations we obtain the general exact solutions for the system (5.25), namely

$$v + M^2 x\tau = \frac{\sqrt{2}}{c_1}\tanh\left(\frac{v + c_2\tau}{\sqrt{2}c_1\tau}\right), \tag{5.28}$$

$$\theta = \frac{c_1^2 M^2\tau}{\operatorname{sech}^2\left(\frac{v+c_2\tau}{\sqrt{2}c_1\tau}\right) - c_1^2\tau}. \tag{5.29}$$

We note that these general exact solutions are implicit and we therefore omit further analysis at this stage.

5.3.3 Exact results

We have obtained the exact analytical steady-state solutions using the local symmetry techniques and these are depicted in Figs. 5.2 and 5.1. These solutions are obtained for $m = n = -4/3$ may be used as benchmarks for the numerical schemes and provide insight into thermal flow processes in fins. In particular, the exponent $m = -4/3$ for power law thermal conductivity agrees well with the range of GaN model 2 at $k_a = 220$ W/mK, and is close to the AlN at $k_a = 350$ W/mK (see Table 2 in [42]). Furthermore these models compare and are in agreement with the experimental observations [42]. We observe in Figs. 5.2 and 5.1, that temperature decreases with increasing values of the thermo-geometric fin parameter. Note that the thermo-geometric fin parameter $M = (Bi)^{1/2}E$, where $Bi = h_b\delta/k_a$ is the Biot number and

$E = L/\delta$ is the aspect ratio or the extension factor. Evidently, small values of M correspond to the relatively short and thick fins of high conductivity and high values of M correspond to longer and thin fins of poor conductivity [77]. A fin is an excellent dissipator at small values of M. We observe that temperature decreases as the values of the thermo-geometric fin parameter increase. The general exact analytical transient solutions satisfy the initial condition and only the adiabatic boundary condition at the fin tip.

Some of the limitations of applications of Lie symmetry methods include; (a) exact analytical solutions are harder to construct particularly when the spatial variable or boundary is defined by a characteristic length, and (b) one losses a number of admitted symmetries when analyzing the entire boundary value problem rather than a single equation [49]. One may assume semi-infinite fins. On the other hand, symmetry methods may be used to determine the forms of arbitrary functions, appearing in the equation, for which extra symmetries are admitted. One may then select the physically realistic cases.

The advantage of utilizing the Lie symmetry methods, particularly in this problem, is that we were able to construct exact solutions which are invariant under time translation. We observe that the transient numerical solutions approached these obtained exact solution. As such this gave confidence in the choice of the numerical schemes used. Furthermore, if necessary one may manipulate the general exact solutions (5.23) and (5.24) and use these as benchmarks for the numerical schemes.

5.4 Numerical solutions

The transient heat flow problem is harder to solve exactly, particularly when $m \neq n$. Thus we resort to numerical methods to determine solutions. In this section we seek numerical solution for the nonlinear initial boundary value problems (IBVP) (5.1) with (5.3) - (5.6). The given IBVP will give rise to DE (stiff DE) for which certain numerical methods are numerically unstable, unless the step size is taken to be extremely small. The availability of robust stiff ODE integrators gives advantage to the method of lines [78]. We first consider IBVP (5.2) with (5.3), (5.4) and (5.6).

5.4.1 Step change in base temperature

The IBVP (5.2) with (5.3), (5.4) and (5.6) is transformed into a system of ODEs using finite difference for the spatial derivative. The discretization is based on a linear cartesian mesh and uniform grid. A spatial interval $0 \leq x \leq 1$ is partitioned into N equal parts with grid size $\Delta x = 1/N$ and grinds points $x_i = i\Delta x, \quad 0 \leq i \leq N$. The first and second order spatial derivatives in Eq. (5.2) are approximated with second order central difference. Let $\theta_i(\tau)$ be the approximation of $\theta(\tau, x_i)$, then the semi discrete system for the problem becomes

$$\frac{d\theta_i}{d\tau} = \frac{\theta_i^m}{(\Delta x)^2}(\theta_{i+1} - 2\theta_i + \theta_{i-1}) + m\theta^{m-1}\left[\frac{\theta_{i+1} - \theta_{i-1}}{2\Delta x}\right]^2 - M^2\theta_i^{n+1}, \quad (5.30)$$

with initial conditions

$$\theta_i(0) = 0, \ \ 0 \leq i \leq N. \quad (5.31)$$

The equations corresponding to the first and last grid points are modified to incorporate the boundary conditions, that is.,

$$\theta_N = 1, \theta_2 = \theta_0. \quad (5.32)$$

Eqs. (5.30) - (5.32) are first order initial value problem and contain only one independent variable. The in built ODE solver program in Matlab can easily be employed to integrate the set of ordinary differential equations using a fourth order Runge-Kutta iteration scheme. The results are depicted in Figs. 5.3 - 5.7. Figs. 5.8 and 5.9 depict the effects of the parameters n and M on the base heat flux.

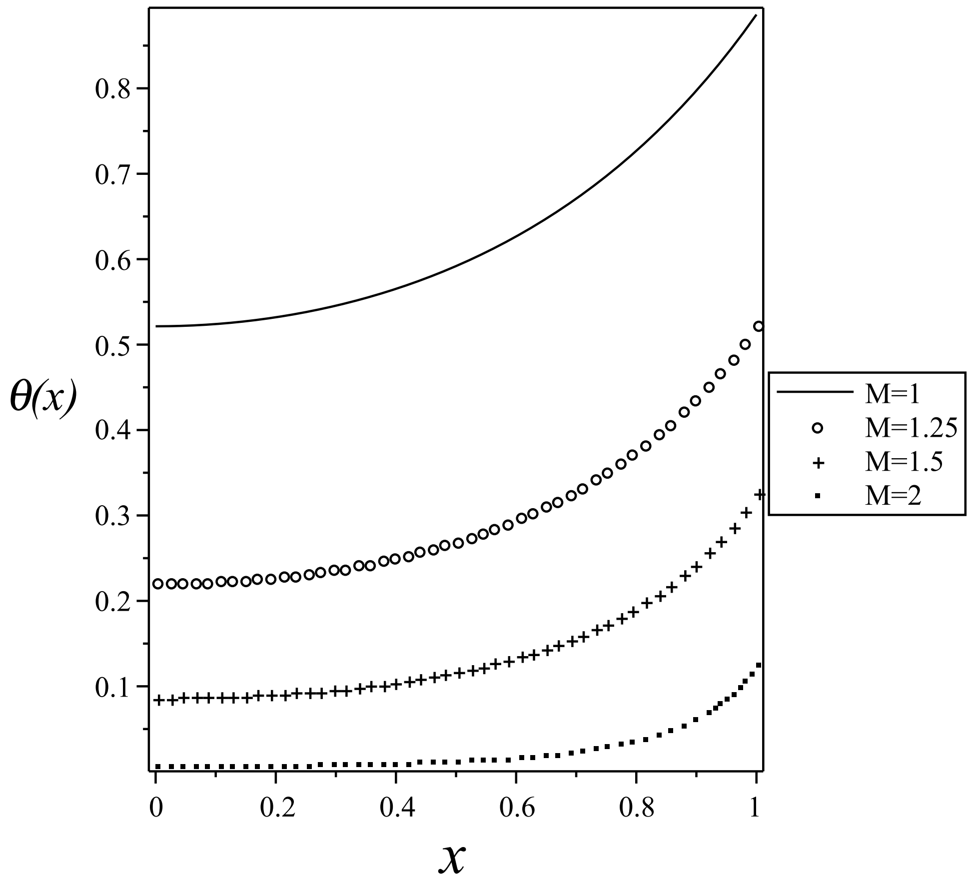

Figure 5.1: Exact steady state temperature profile given step change in heat flux at the fin base.

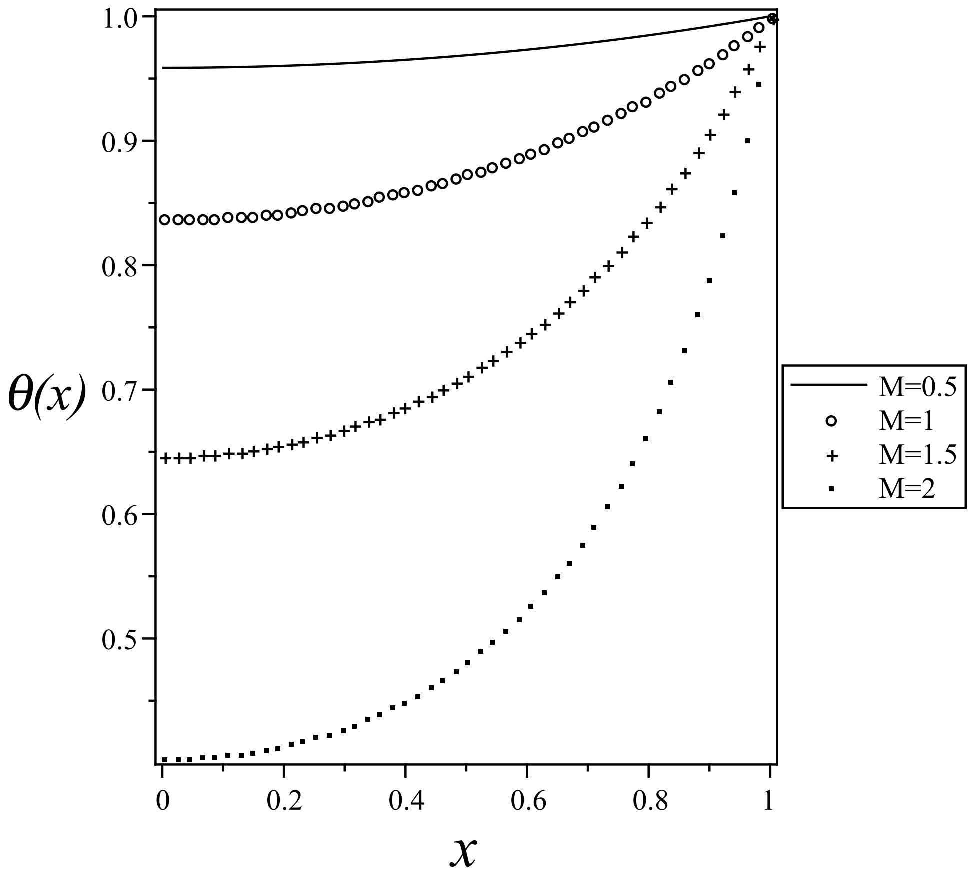

Figure 5.2: Exact steady state temperature profile for step change in base temperature condition.

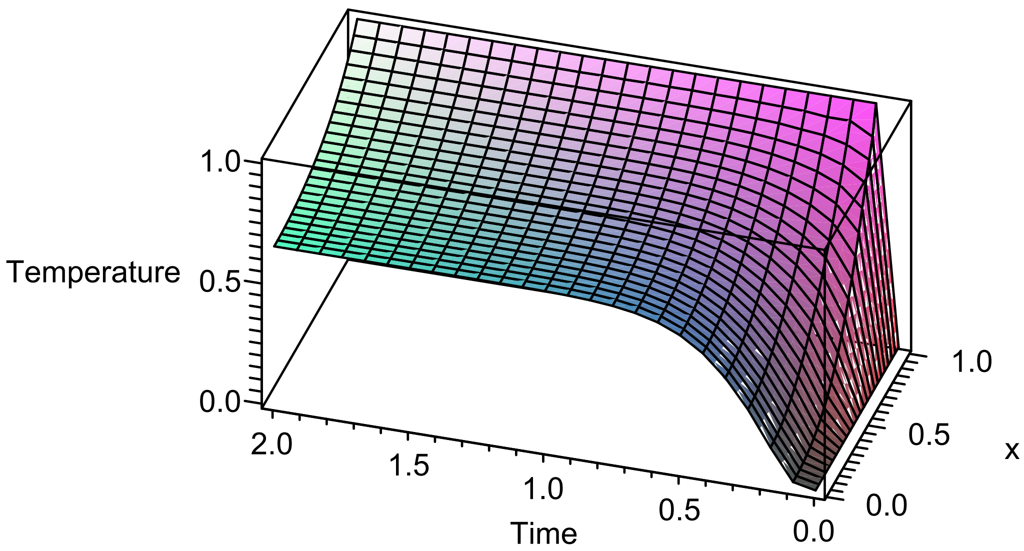

Figure 5.3: Temperature variation with increasing time and space for step change in base temperature condition when $m = 0.1, n = 4, M = 2$.

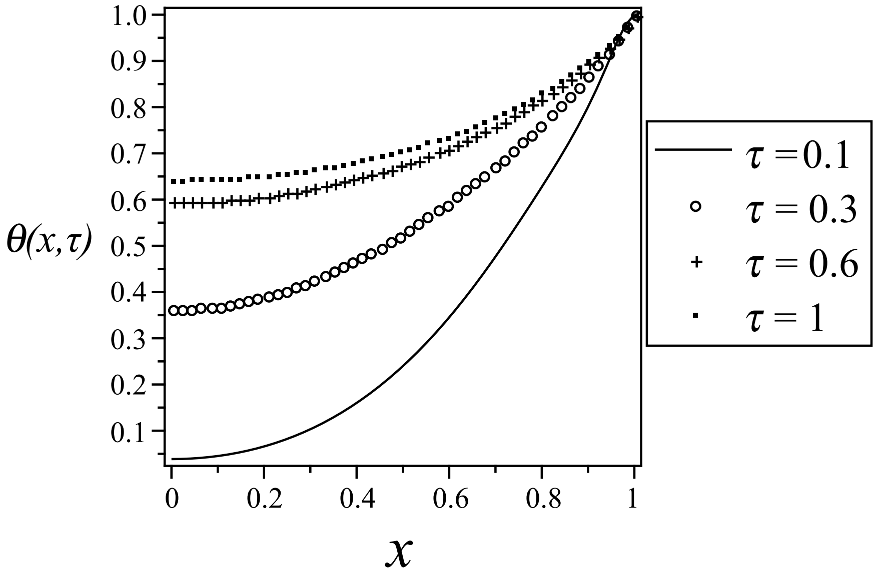

Figure 5.4: Temperature variation with increasing time for step change in base temperature condition when $m = 0.1, n = 4, M = 2$.

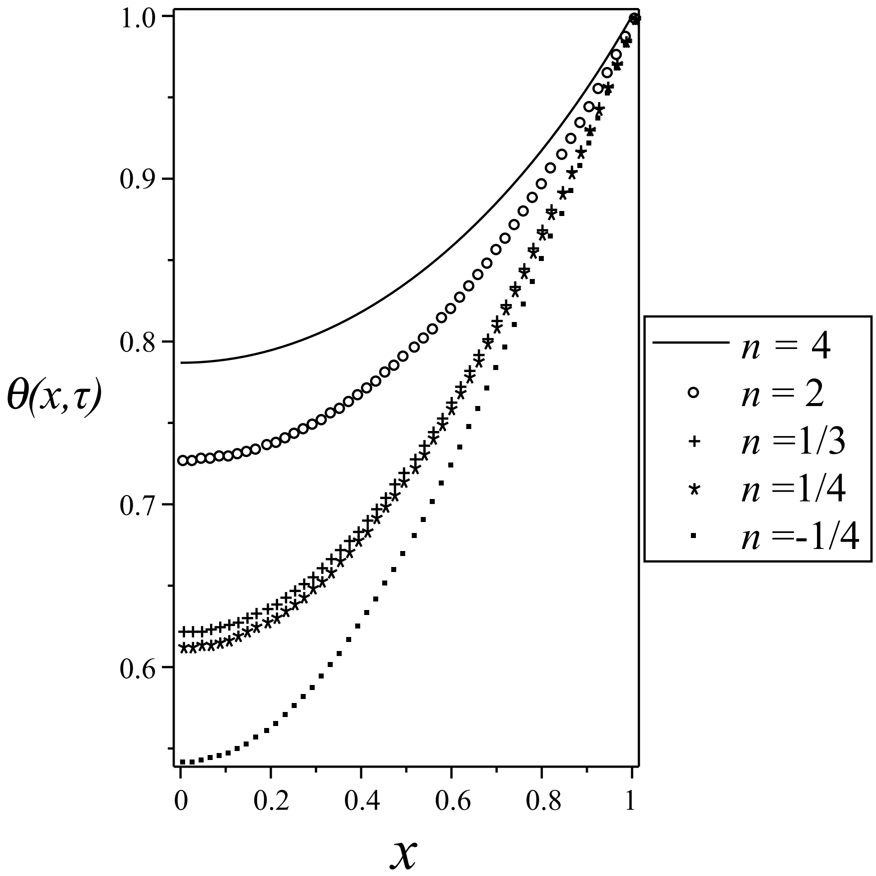

Figure 5.5: Temperature profiles for step change in base temperature condition when $\tau = m = M = 1$.

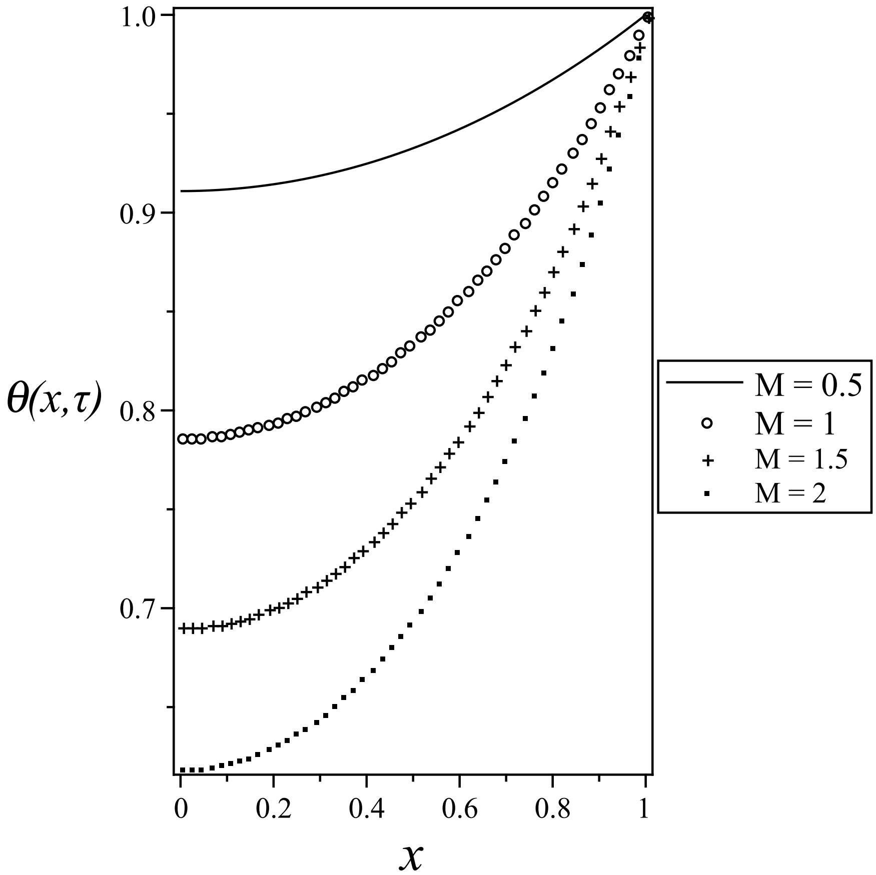

Figure 5.6: Temperature profiles when M for step change in base temperature condition when $\tau = m = 1, n = 4$.

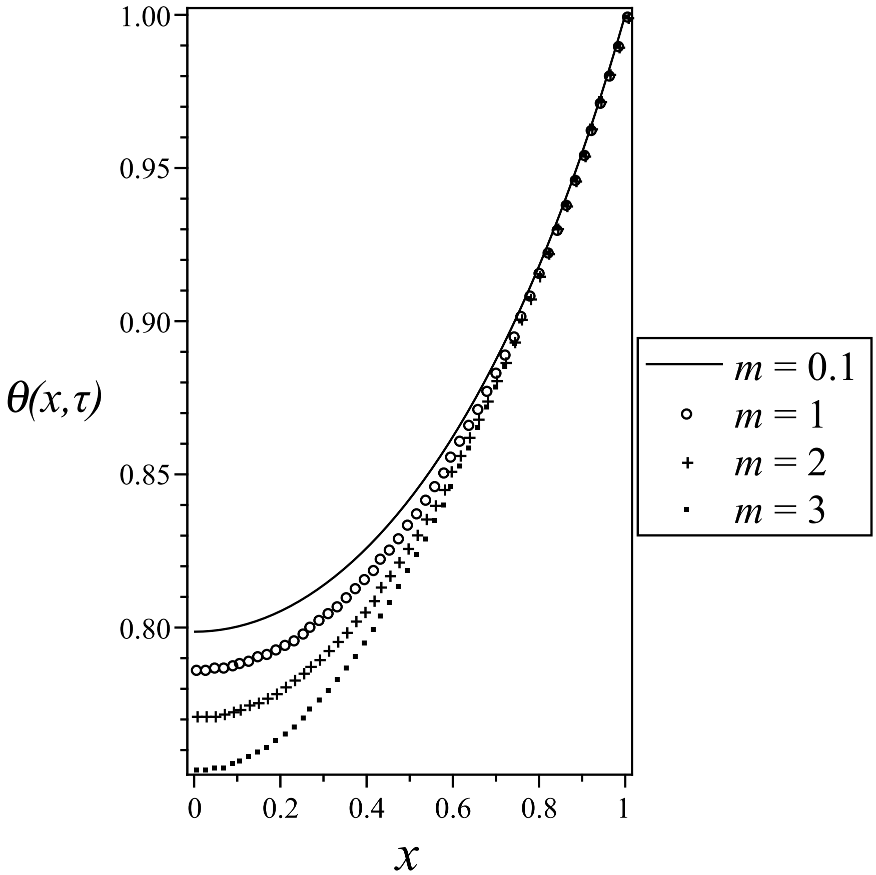

Figure 5.7: Temperature profiles increasing m for step change in base temperature condition when $\tau = M = 1, n = 4$.

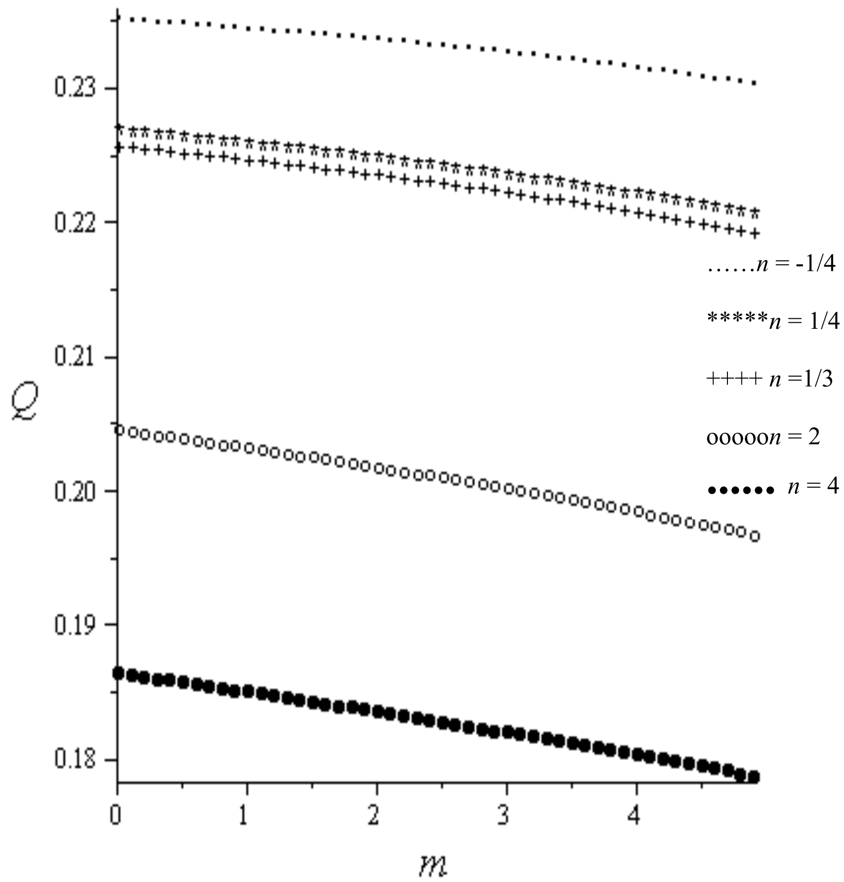

Figure 5.8: Effect of n on the base heat flow when $\tau = 1, M = 0.5$

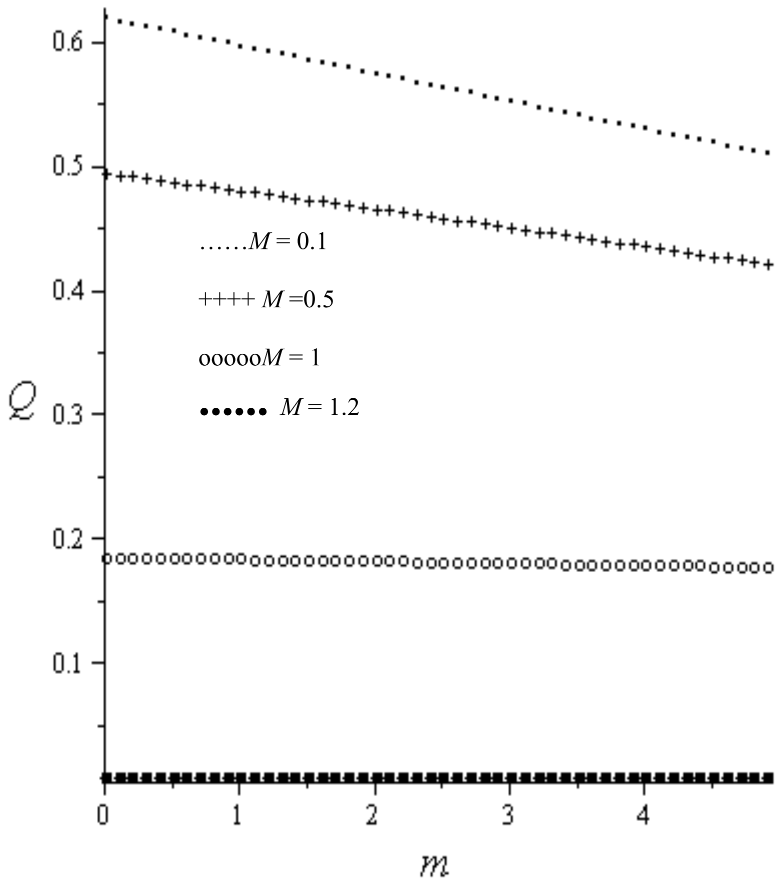

Figure 5.9: Effect of M on the base heat flow when $\tau = 1, n = 4$

5.4.2 Step change in base heat flow

The IBVP (5.1) with (5.3), (5.5) and (5.6) is treated the same way as the IBVP (5.1) with (5.3), (5.4) and (5.6) above. Here the boundary condition become

$$\theta_N = \theta_{N-2} + \frac{2}{N}, \quad \theta_2 = \theta_0. \tag{5.33}$$

The solutions are depicted in Figs. 5.10 - 5.14.

5.4.3 Numerical results

We observed that the problem at hand particulary when considering the transient heat flow, is difficult to solve analytically, thus the problem is solved numerically when the exponents are distinct. Fig. 5.3, depicts the temperature profile given increasing time and space. In Fig. 5.4, we observe that the fin temperature increases with increasing time and approaches the steady state. We observe in Fig. 5.5 that temperature increase with the increasing values of n (recalling that, $n = 4$ represents radiation and $n = -1/4$ represents laminar film boiling state). In Fig. 5.6, we observe that temperature decreases with increasing values of the thermo-geometric fin parameter. The increase in values of the thermal conductivity result in decreasing temperature as displayed in 5.7 where in the base heat flow is fixed. This implies heat is lost rapidly at high thermal conductivity. Figs. 5.8 and 5.9 show the effects of n and M on the base heat flux flow. The heat flow decreases with increasing values of n and M. Figs. 5.10 depicts the heat flow profile at fixed m, n and M. In Fig. 5.11, we observe that the fin temperature increases with increasing time. In Fig. 5.12 we observe that temperature increases with increasing values of n, whilst temperature decreases with increasing values of M. Unlike in Fig. 5.7, temperature increases with increasing thermal conductivity as shown in Fig. 5.14 where the base heat flow changes. This indicates the fin response to different changes in the fin base temperature.

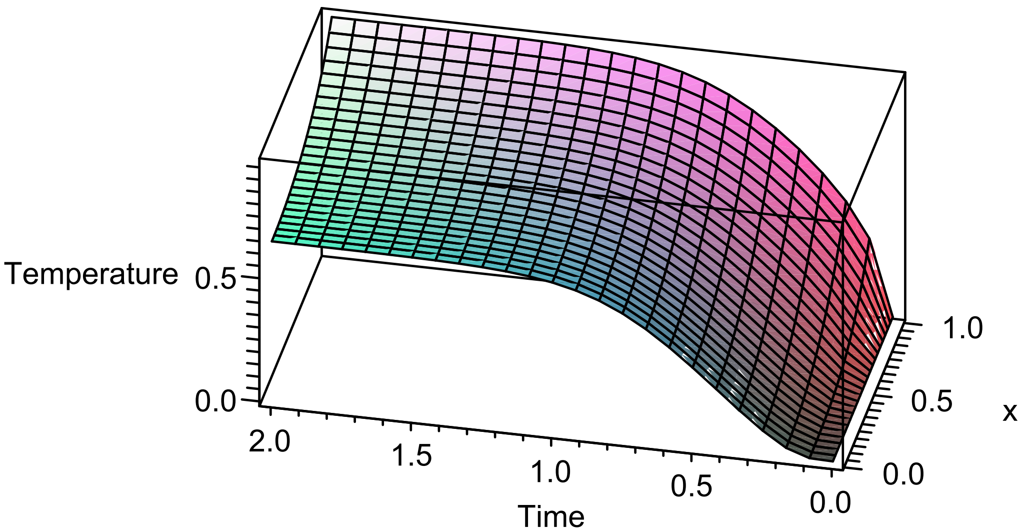

Figure 5.10: Temperature variation with increasing time and space for constant base heat flow condition when $m = 0.1, n = 4, M = 2$.

5.5 Concluding remarks

In this chapter we considered the models describing the heat transfer in longitudinal rectangular fins. We employed symmetry techniques in attempt to solve the resulting IBVP. In particular, we employed the nonlocal and the Lie point (local) symmetry techniques. The governing equation admitted nonlocal

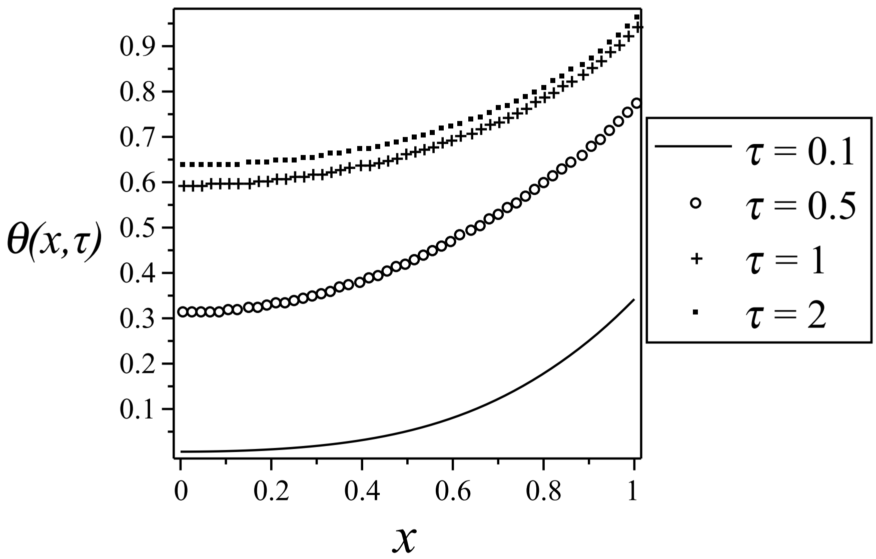

Figure 5.11: Temperature variation with increasing time for constant base heat flow condition when $m = 0.1, n = 4, M = 2$.

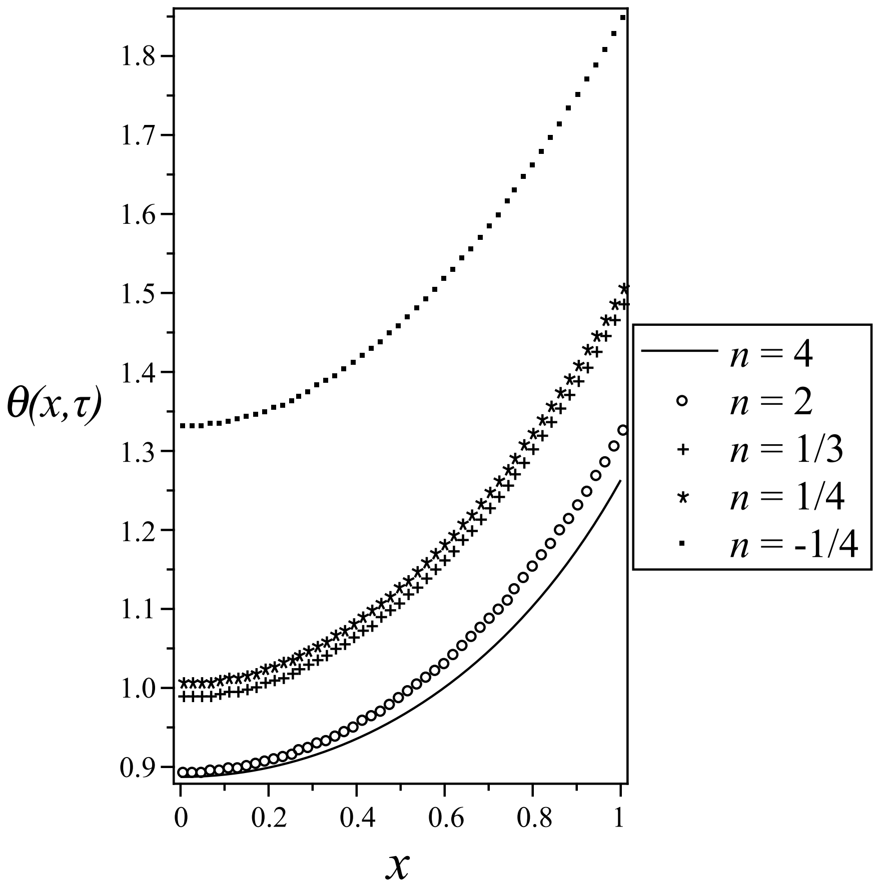

Figure 5.12: Temperature profiles for constant base heat flow condition when $\tau = 2, m = 0.5, M = 1$.

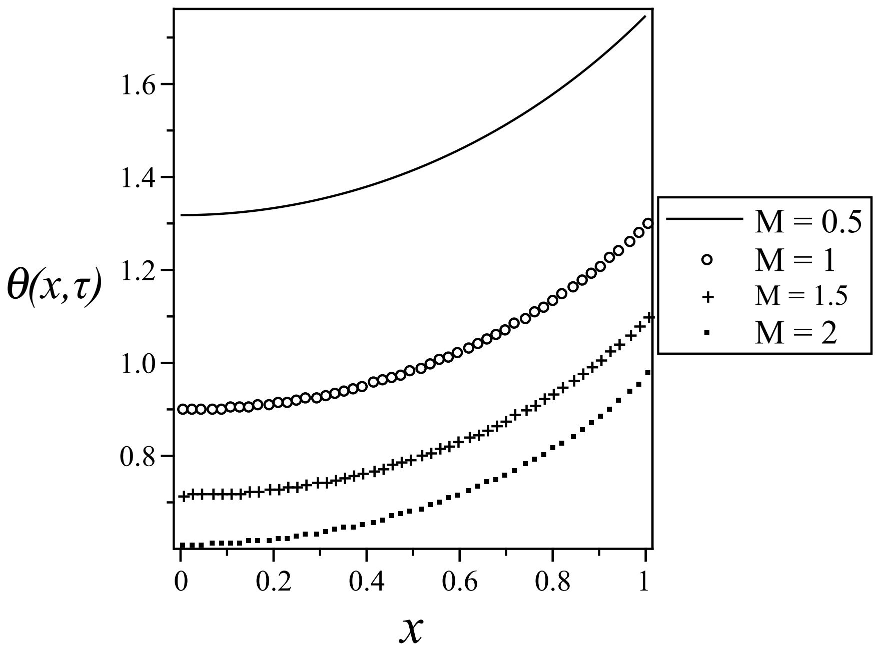

Figure 5.13: Temperature profiles with increasing M for constant base heat flow condition when $\tau = 2, m = 0.5$.

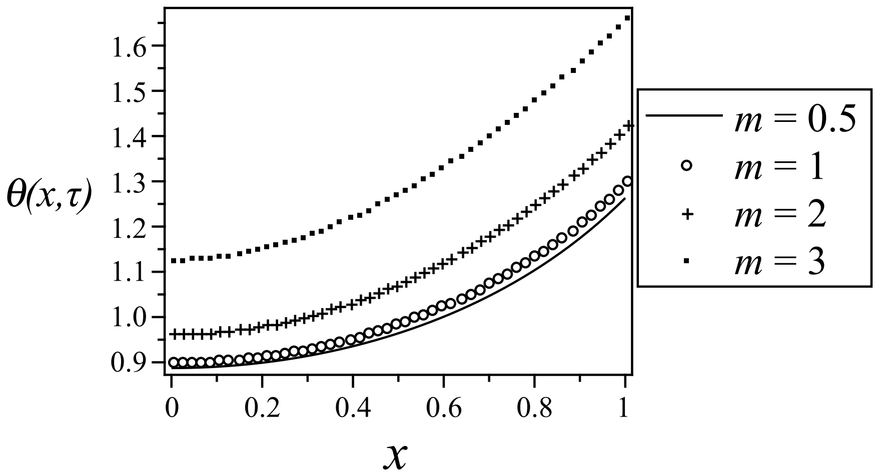

Figure 5.14: Temperature profiles with increasing m for constant base heat flow condition when $\tau = 2, m = 0.5, M = 1$.

symmetries for a special case. The genuine nonlocal symmetry generator led to general exact analytical implicit solutions. On the other hand, Lie point symmetries led to construction of general exact analytical solutions for transient heat flow when $m = n = -4/3$. Furthermore, the time invariance led to some exiting exact analytical solutions for the steady state problem. The case $m = n = -4/3$, was fully exploited since it allowed easier mathematical manipulations and some exact analytical solutions particularly for the steady state problem. Furthermore, the assumption that temperature dependent thermal conductivity may be given by the power law is physically realistic. The convergence of transient numerical solutions to exact analytical steady state is conspicuous. We have observed that the symmetry analysis of the transient problem led to the general exact solutions which satisfied the adiabatic fin tip boundary condition and the zero initial temperature. The problem is even harder when $m \neq n$. However, after gaining confidence in the method of second order central difference and integration by fourth order Runge-Kutta iteration scheme were used to obtain the numerical solutions.

The method of separation of variable may be invoked in attempt to solve Eq. (5.1) when the exponents are the same. The resulting separated ordinary differential equation in spatial variable is transformable to a tractable Ermakov-Pinney equation. Nonlinear superposition principle may be exploited.

Chapter 6

Some exact solutions of nonlinear fin problem for steady heat transfer in longitudinal fin with different profiles

Some results in this chapter have been submitted to an Institute for Scientific Information journal for consideration to be published.

6.1 Introduction

In this chapter, we determine exact solutions of nonlinear fin problem for steady heat transfer in longitudinal fin of various profiles. Here the thermal conductivity is related to temperature by a power law. In section 6.2, we provide the mathematical formulation of the problem. In Section 6.3, we employ some transformation that linearizes the problem and construct exact solutions. We employ Lie point symmetry techniques to determine wherever possible, the invariant solutions of some problems which are not linearizable. In section 6.4 remarks are given in Section 6.6.

6.2 Mathematical models

We consider a longitudinal one-dimensional fin with the fin profile given by the function $F(X)$. The steady-state energy balance for a longitudinal fin is a special case of (2.1) and is given by

$$A_c \frac{d}{dX}\left(F(X)K(T)\frac{dT}{dX}\right) = \frac{\delta_b}{2} PH(T)\left(T - T_a\right), \quad 0 < X < L, \tag{6.1}$$

where parameters are as explained in chapter 1. The prescribed boundary conditions are given by (see e.g. [2])

$$T(L) = T_b, \qquad \text{and} \qquad \left.\frac{dT}{dX}\right|_{X=0} = 0. \tag{6.2}$$

Introducing the dimensionless variables and the dimensionless numbers reduce Eq. (6.1) to

$$\frac{d}{dx}\left(f(x)k(\theta)\frac{d\theta}{dx}\right) = M^2 h(\theta)\theta, \quad 0 < x < 1. \tag{6.3}$$

The dimensionless boundary conditions are given by

$$\theta(1) = 1, \qquad \left.\frac{d\theta}{dx}\right|_{x=0} = 0. \tag{6.4}$$

If we consider the power law for the heat transfer coefficient and thermal conductivity, the one dimensional heat balance equation then becomes

$$\frac{d}{dx}\left[f(x)\theta^m \frac{d\theta}{dx}\right] = M^2 \theta^{n+1}, \quad 0 < x < 1, \tag{6.5}$$

Recently Equation (6.5) has been analyzed using the Differential Transform Methods (DTM) [79]. It was revealed that DTM may not be suitable to solve equations such as (6.5) for given fractional powers of $f(x)$. Here we employ basic integration and Lie point symmetry techniques.

6.3 Exact solutions

In subsections 6.3.1 - 6.3.3 we analyze the governing Equation (6.5), given $m = n$. We then analyze the case $m \neq n$ in subsection 6.3.4. If $m = n$, then Eq. (6.5) is linearizable by a transformation $y = \theta^{n+1}$. Under such a transformation Eq. (6.5) becomes

$$\frac{d}{dx}\left[f(x)\frac{dy}{dx}\right] - (n+1)M^2 y = 0 \tag{6.6}$$

the boundary conditions transform to

$$y(1) = 1, \quad \text{and} \quad y'(0) = 0. \tag{6.7}$$

Eq. (6.6) is analyzed for various situations in the next sections and all the solutions in illustrative examples satisfy both the Dirichlet and Neumann boundary conditions.

6.3.1 Case: $n > -1$ with $m = n \neq -1$.

As an illustration we use two examples, and the rest of the solutions are listed in Tables 6.1 and 6.2.

Example 1

Given $f(x) = \sqrt{x+1}$, Eq. (6.6) becomes

$$(x+1)y'' + \frac{y'}{2} - (n+1)M^2\sqrt{x+1} = 0, \tag{6.8}$$

with solution

$$\theta = \left[(x+1)^{\frac{1}{4}} J_{\frac{1}{3}}\left(\beta\left(\frac{x+1}{2}\right)^{\frac{3}{4}}\right)\left(\frac{YY}{J_{\frac{1}{3}}(\beta)} - YY - 1\right)\right]^{\frac{1}{n+1}}, \tag{6.9}$$

where $\beta = \frac{4}{3}M\sqrt{-n-1}$, $\gamma = 2^{\frac{3}{4}}$ and YY is given by the expression

$$\frac{\frac{1}{2}\left(Y_{\frac{1}{3}}(\frac{\beta}{\gamma}) + \frac{3}{2}\beta Y_{\frac{4}{3}}(\beta)\right) J_{\frac{1}{3}}(\beta\gamma)}{-J_{\frac{1}{3}}(\beta)Y_{\frac{1}{3}}(\beta\gamma) + Y_{\frac{1}{3}}(\beta)J_{\frac{1}{3}}(\beta\gamma) + \frac{3}{2}\beta\left(Y_{\frac{1}{3}}(\beta\gamma)J_{\frac{4}{3}}(\beta) - J_{\frac{1}{3}}(\beta\gamma)Y_{\frac{4}{3}}(\beta)\right)}. \quad (6.10)$$

The efficiency is given by

$$\eta = \int_0^1 \left[(x+1)^{\frac{1}{4}} J_{\frac{1}{3}}\left(\beta\left(\frac{x+1}{2}\right)^{\frac{3}{4}}\right)\left(\frac{YY}{J_{\frac{1}{3}}(\beta)} - YY - 1\right)\right] dx.$$

The temperature distribution along the surface for are depicted in Figures 6.1 and 6.2. The fin efficiency as function of the thermo-geometric parameter is shown in Figure 6.3.

Example 2

In case of $f(x) = (x+1)^3$, the Eq. (6.6) is transformed into

$$(x+1)^3 y'' + 3(x+1)^2 y' - (n+1)M^2 y = 0 \quad (6.11)$$

has a solution in terms of Bessel functions given by

$$\theta = \left[\frac{J_2\left(\frac{\alpha}{\sqrt{x+1}}\right) JJ}{J_2\left(\frac{\alpha}{\sqrt{2}}\right)(x+1)} - \frac{Y_2\left(\frac{\alpha}{\sqrt{x+1}}\right)}{Y_2\left(\frac{\alpha}{\sqrt{2}}\right)(x+1)}(JJ-2)\right]^{\frac{1}{n+1}}, \quad (6.12)$$

where $\alpha = 2M\sqrt{-n-1}$ and

$$JJ = \frac{-2Y_1(\alpha)J_2(\frac{\alpha}{\sqrt{2}})}{Y_2(\frac{\alpha}{\sqrt{2}})J_1(\alpha) - J_2(\frac{\alpha}{\sqrt{2}})Y_1(\alpha)}. \quad (6.13)$$

The efficiency is given by

$$\eta = \int_0^1 \left[\frac{J_2\left(\frac{\alpha}{\sqrt{x+1}}\right) JJ}{J_2\left(\frac{\alpha}{\sqrt{2}}\right)(x+1)} - \frac{Y_2\left(\frac{\alpha}{\sqrt{x+1}}\right)}{Y_2\left(\frac{\alpha}{\sqrt{2}}\right)(x+1)}(JJ-2)\right] dx.$$

The temperature distribution along the surface for this profile is depicted in Figures 6.4 and 6.5. The fin efficiency as function of the thermo-geometric fin parameter is shown in Figure 6.6.

Table 6.1: General solutions for $n > -1$ with $m = n \neq -1$ where $i^2 = -1$

$f(x)$	Solution
x^a	$\theta^{n+1} = c_1\ x^{\frac{1-a}{2}} J_{\frac{1-a}{a-2}} \left(\frac{i2M\sqrt{n+1}\ x^{1-\frac{a}{2}}}{a-2} \right) + c_2\ x^{\frac{1-a}{2}} Y_{\frac{1-a}{a-2}} \left(\frac{i2M\sqrt{n+1}\ x^{1-\frac{a}{2}}}{a-2} \right)$
1	$\theta^{n+1} = c_1 \sinh\left(M\sqrt{n+1}\ x\right) + c_2 \cosh\left(M\sqrt{n+1}\ x\right)$
$x^{\frac{1}{2}}$	$\theta^{n+1} = c_1\ x^{\frac{1}{4}} J_{\frac{1}{3}} \left(\frac{i4M\sqrt{n+1}\ x^{\frac{3}{4}}}{3} \right) + c_2\ x^{\frac{1}{4}} Y_{\frac{1}{3}} \left(\frac{i4M\sqrt{n+1}\ x^{\frac{3}{4}}}{3} \right)$
x	$\theta^{n+1} = c_1 J_0 \left(i2M\sqrt{(n+1)\ x} \right) + c_2 Y_0 \left(i2M\sqrt{(n+1)\ x} \right)$
x^2	$\theta^{n+1} = c_1\ x^{-\frac{1}{2}+\frac{1}{2}\sqrt{1+4M^2(n+1)}} + c_2\ x^{-\frac{1}{2}-\frac{1}{2}\sqrt{1+4M^2(n+1)}}$
x^3	$\theta^{n+1} = \frac{c_1 J_2\left(i2M\sqrt{\frac{(n+1)}{x}}\right) + c_2 Y_2\left(i2M\sqrt{\frac{(n+1)}{x}}\right)}{x}$
e^{ax}	$\theta^{n+1} = c_1\ e^{-\frac{ax}{2}} J_1 \left(\frac{i2M\sqrt{n+1}e^{-\frac{ax}{2}}}{a} \right) + c_2\ e^{-\frac{ax}{2}} Y_1 \left(\frac{i2M\sqrt{n+1}e^{-\frac{ax}{2}}}{a} \right)$
$\sin x$	$\theta^{n+1} = c_1 HeunG\left(2, -(n+1)M^2, 0, 1, \frac{1}{2}, 1, \sin x + 1\right) + \frac{(1-\sin x)^{\frac{3}{4}} \times}{\sqrt{\cos x}}$ $\left[c_2\left(1+\sqrt{1-\cos^2 x}\right)^{\frac{1}{4}}\right] HeunG\left(2, -(n+1)M^2 + \frac{5}{4}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, 1, \sin x + 1\right)$
$\cos x$	$\theta^{n+1} = c_1 HeunG\left(2, -(n+1)M^2, 0, 1, \frac{1}{2}, 1, \cos x + 1\right) + \frac{(1-\cos x)^{\frac{3}{4}}}{\sqrt{\sin x}} \times$ $\left[c_2\left(1+\sqrt{1-\sin^2 x}\right)^{\frac{1}{4}}\right] HeunG\left(2, -(n+1)M^2 + \frac{5}{4}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, 1, \sin x + 1\right)$

Table 6.2: Modified general solutions for $f(\mathcal{X})$ where $\mathcal{X} = x + 1$, $n > -1$ and $m = n \neq -1$ and $i^2 = -1$

$f(\mathcal{X})$	Solution
$\mathcal{X}^a$	$\theta = \left[c_1\ \mathcal{X}^{\frac{1-a}{2}} J_{\frac{1-a}{a-2}} \left(\frac{i2M\sqrt{n+1}\ \mathcal{X}^{1-\frac{a}{2}}}{a-2}\right) + c_2\ \mathcal{X}^{\frac{1-a}{2}} Y_{\frac{1-a}{a-2}} \left(\frac{i2M\sqrt{n+1}\ \mathcal{X}^{1-\frac{a}{2}}}{a-2}\right)\right]^{\frac{1}{n+1}}$
1	$\theta = \left[c_1 \sinh\left(M\sqrt{n+1}\ \mathcal{X}\right) + c_2 \cosh\left(M\sqrt{n+1}\ \mathcal{X}\right)\right]^{\frac{1}{n+1}}$
$\mathcal{X}^{\frac{1}{2}}$	$\theta = \left[c_1\ \mathcal{X}^{\frac{1}{4}} J_{\frac{1}{3}} \left(\frac{i4M\sqrt{n+1}\ \mathcal{X}^{\frac{3}{4}}}{3}\right) + c_2\ \mathcal{X}^{\frac{1}{4}} Y_{\frac{1}{3}} \left(\frac{i4M\sqrt{n+1}\ \mathcal{X}^{\frac{3}{4}}}{3}\right)\right]^{\frac{1}{n+1}}$
$\mathcal{X}$	$\theta = \left[c_1 J_0 \left(i2M\sqrt{(n+1)\ \mathcal{X}}\right) + c_2 Y_0 \left(i2M\sqrt{(n+1)\ \mathcal{X}}\right)\right]^{\frac{1}{n+1}}$
$\mathcal{X}^2$	$\theta = \left[c_1\ \mathcal{X}^{-\frac{1}{2}+\frac{1}{2}\sqrt{1+4M^2(n+1)}} + c_2\ \mathcal{X}^{-\frac{1}{2}-\frac{1}{2}\sqrt{1+4M^2(n+1)}}\right]^{\frac{1}{n+1}}$
$\mathcal{X}^3$	$\theta = \left[\frac{c_1 J_2\left(i2M\sqrt{\frac{(n+1)}{\mathcal{X}}}\right) + c_2 Y_2\left(i2M\sqrt{\frac{(n+1)}{\mathcal{X}}}\right)}{\mathcal{X}}\right]^{\frac{1}{n+1}}$
$e^{a\mathcal{X}}$	$\theta = \left[c_1\ e^{-\frac{a\mathcal{X}}{2}} J_1 \left(\frac{i2M\sqrt{n+1}e^{-\frac{a\mathcal{X}}{2}}}{a}\right) + c_2\ e^{-\frac{a\mathcal{X}}{2}} Y_1 \left(\frac{i2M\sqrt{n+1}e^{-\frac{a\mathcal{X}}{2}}}{a}\right)\right]^{\frac{1}{n+1}}$
$\sin\mathcal{X}$	$\theta^{n+1} = c_1 HeunG\left(2, -(n+1)M^2, 0, 1, \frac{1}{2}, 1, \sin\mathcal{X} + 1\right) + \frac{(1-\sin\mathcal{X})^{\frac{3}{4}}}{\sqrt{\cos\mathcal{X}}} \times$ $\left[c_2\left(1 + \sqrt{1-\cos^2\mathcal{X}}\right)^{\frac{1}{4}}\right] HeunG\left(2, -(n+1)M^2 + \frac{5}{4}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, 1, \sin\mathcal{X} + 1\right)$
$\cos\mathcal{X}$	$\theta^{n+1} = c_1 HeunG\left(2, -(n+1)M^2, 0, 1, \frac{1}{2}, 1, \cos\mathcal{X} + 1\right) + \frac{(1-\cos\mathcal{X})^{\frac{3}{4}}}{\sqrt{\sin\mathcal{X}}} \times$ $\left[c_2\left(1 + \sqrt{1-\sin^2\mathcal{X}}\right)^{\frac{1}{4}}\right] HeunG\left(2, -(n+1)M^2 + \frac{5}{4}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, 1, \sin\mathcal{X} + 1\right)$

6.3.2 Case: $n < -1$ with $m = n \neq -1$.

We shall use two examples as in the previous case, the general solutions are listed in Tables 6.3 and 6.4.

Example 3

Starting with $f(x) = x^2$, Eq. (6.6) in its changed form will be

$$x^2 y'' + 2xy' + (n+1)M^2 y = 0 \tag{6.14}$$

with solution

$$\theta = x^{\frac{1}{2(n+1)}\left(\sqrt{1-4(n+1)M^2}-1\right)}. \tag{6.15}$$

The efficiency is given by

$$\begin{aligned} \eta &= \int_0^1 x^{\frac{1}{2}\left(\sqrt{1-4(n+1)M^2}-1\right)} dx \\ &= \frac{2}{\sqrt{1-4(n+1)M^2}+1+2n}. \end{aligned} \tag{6.16}$$

Example 4

We consider $f(x) = \sqrt{x+1}$ as the second example in this case. This transforms Eq. (6.6) into

$$(x+1)y'' + \frac{y'}{2} + (n+1)M^2\sqrt{x+1}y = 0 \tag{6.17}$$

with solution

$$\theta = \left[(x+1)^{\frac{1}{4}} J_{\frac{1}{3}}\left(\beta\left(\frac{x+1}{2}\right)^{\frac{3}{4}}\right)\left(\frac{YY}{J_{\frac{1}{3}}(\beta)} - YY - 1\right)\right]^{\frac{1}{n+1}}, \tag{6.18}$$

where $\beta = \frac{4}{3}M\sqrt{n+1}$, $\gamma = 2^{\frac{3}{4}}$ and YY given by the expression

$$\frac{\frac{1}{2}\left(Y_{\frac{1}{3}}(\frac{\beta}{\gamma}) + \frac{3}{2}\beta Y_{\frac{4}{3}}(\beta)\right) J_{\frac{1}{3}}(\beta\gamma)}{-J_{\frac{1}{3}}(\beta)Y_{\frac{1}{3}}(\beta\gamma) + Y_{\frac{1}{3}}(\beta)J_{\frac{1}{3}}(\beta\gamma) + \frac{3}{2}\beta\left(Y_{\frac{1}{3}}(\beta\gamma)J_{\frac{4}{3}}(\beta) - J_{\frac{1}{3}}(\beta\gamma)Y_{\frac{4}{3}}(\beta)\right)}. \quad (6.19)$$

The efficiency is given by

$$\eta = \int_0^1 \left[(x+1)^{\frac{1}{4}} J_{\frac{1}{3}}\left(\beta\left(\frac{x+1}{2}\right)^{\frac{3}{4}}\right)\left(\frac{Y}{J_{\frac{1}{3}}(\beta)} - Y - 1\right)\right] dx.$$

Table 6.3: Solution for $n < -1$ with $m = n \neq -1$

$f(x)$	Solution
x^a, $a \neq 2$	$\theta = \left[c_1\ x^{\frac{1-a}{2}} J_{\frac{1-a}{a-2}}\left(\frac{i2M\sqrt{n+1}\ x^{1-\frac{a}{2}}}{a-2}\right) + c_2\ x^{\frac{1-a}{2}} Y_{\frac{1-a}{a-2}}\left(\frac{i2M\sqrt{n+1}\ x^{1-\frac{a}{2}}}{a-2}\right)\right]^{\frac{1}{n+1}}$
1	$\theta = \left[c_1 \sin\left(M\sqrt{-(n+1)}\ x\right) + c_2 \cos\left(M\sqrt{-(n+1)}\ x\right)\right]^{\frac{1}{n+1}}$
$x^{\frac{1}{2}}$	$\theta = \left[c_1\ x^{\frac{1}{4}} J_{\frac{1}{3}}\left(\frac{4M\sqrt{n+1}\ x^{\frac{3}{4}}}{3}\right) + c_2\ x^{\frac{1}{4}} Y_{\frac{1}{3}}\left(\frac{4M\sqrt{n+1}\ x^{\frac{3}{4}}}{3}\right)\right]^{\frac{1}{n+1}}$
x	$\theta = \left[c_1 J_0\left(2M\sqrt{(n+1)\ x}\right) + c_2 Y_0\left(2M\sqrt{(n+1)\ x}\right)\right]^{\frac{1}{n+1}}$
x^2	$\theta = \left[c_1\ x^{-\frac{1}{2}+\frac{1}{2}\sqrt{1-4M^2(n+1)}} + c_2\ x^{-\frac{1}{2}-\frac{1}{2}\sqrt{1-4M^2(n+1)}}\right]^{\frac{1}{n+1}}$
x^3	$\theta = \left[\frac{c_1 J_2\left(2M\sqrt{\frac{(n+1)}{x}}\right) + c_2 Y_2\left(2M\sqrt{\frac{(n+1)}{x}}\right)}{x}\right]^{\frac{1}{n+1}}$
e^{ax}	$\theta = \left[c_1\ e^{-\frac{ax}{2}} J_1\left(\frac{2M\sqrt{n+1}e^{-\frac{ax}{2}}}{a}\right) + c_2\ e^{-\frac{ax}{2}} Y_1\left(\frac{2M\sqrt{n+1}e^{-\frac{ax}{2}}}{a}\right)\right]^{\frac{1}{n+1}}$
$\sin x$	$\theta^{n+1} = c_1 HeunG\left(2, (n+1)M^2, 0, 1, \frac{1}{2}, 1, \sin x + 1\right) + \frac{(1-\sin x)^{\frac{3}{4}}}{\sqrt{\cos x}} \times$ $\left[c_2\left(1+\sqrt{1-\cos^2 x}\right)^{\frac{1}{4}}\right] HeunG\left(2, (n+1)M^2 + \frac{5}{4}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, 1, \sin x + 1\right)$
$\cos x$	$\theta^{n+1} = c_1 HeunG\left(2, (n+1)M^2, 0, 1, \frac{1}{2}, 1, \cos x + 1\right) + \frac{(1-\cos x)^{\frac{3}{4}}}{\sqrt{\sin x}} \times$ $\left[c_2\left(1+\sqrt{1-\sin^2 x}\right)^{\frac{1}{4}}\right] HeunG\left(2, (n+1)M^2 + \frac{5}{4}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, 1, \sin x + 1\right)$

Table 6.4: Modified solution for $f(\mathcal{X})$ where $\mathcal{X} = x+1$, $n < -1$ and $m = n \neq -1$

$f(\mathcal{X})$	Solution
$\mathcal{X}^a$ $a \neq 2$	$\theta = \left[c_1\ \mathcal{X}^{\frac{1-a}{2}} J_{\frac{1-a}{a-2}}\left(\frac{i2M\sqrt{n+1}\ \mathcal{X}^{1-\frac{a}{2}}}{a-2}\right) + c_2\ \mathcal{X}^{\frac{1-a}{2}} Y_{\frac{1-a}{a-2}}\left(\frac{i2M\sqrt{n+1}\ \mathcal{X}^{1-\frac{a}{2}}}{a-2}\right)\right]^{\frac{1}{n+1}}$
1	$\theta = \left[c_1 \sin\left(M\sqrt{n+1}\ \mathcal{X}\right) + c_2 \cos\left(M\sqrt{n+1}\ \mathcal{X}\right)\right]^{\frac{1}{n+1}}$
$\mathcal{X}^{\frac{1}{2}}$	$\theta = \left[c_1\ \mathcal{X}^{\frac{1}{4}} J_{\frac{1}{3}}\left(\frac{4M\sqrt{n+1}\ \mathcal{X}^{\frac{3}{4}}}{3}\right) + c_2\ X^{\frac{1}{4}} Y_{\frac{1}{3}}\left(\frac{4M\sqrt{n+1}\ \mathcal{X}^{\frac{3}{4}}}{3}\right)\right]^{\frac{1}{n+1}}$
$\mathcal{X}$	$\theta = \left[c_1 J_0\left(2M\sqrt{(n+1)\ X}\right) + c_2 Y_0\left(2M\sqrt{(n+1)\ \mathcal{X}}\right)\right]^{\frac{1}{n+1}}$
$\mathcal{X}^2$	$\theta = \left[c_1\ \mathcal{X}^{-\frac{1}{2}+\frac{1}{2}\sqrt{1-4M^2(n+1)}} + c_2\ \mathcal{X}^{-\frac{1}{2}-\frac{1}{2}\sqrt{1-4M^2(n+1)}}\right]^{\frac{1}{n+1}}$
$\mathcal{X}^3$	$\theta = \left[\frac{c_1 J_2\left(2M\sqrt{\frac{(n+1)}{\mathcal{X}}}\right) + c_2 Y_2\left(2M\sqrt{\frac{(n+1)}{\mathcal{X}}}\right)}{\mathcal{X}}\right]^{\frac{1}{n+1}}$
$e^{a\mathcal{X}}$	$\theta = \left[c_1\ e^{-\frac{a\mathcal{X}}{2}} J_1\left(\frac{2M\sqrt{n+1}e^{-\frac{a\mathcal{X}}{2}}}{a}\right) + c_2\ e^{-\frac{a\mathcal{X}}{2}} Y_1\left(\frac{2M\sqrt{n+1}e^{-\frac{a\mathcal{X}}{2}}}{a}\right)\right]^{\frac{1}{n+1}}$
$\sin\mathcal{X}$	$\theta^{n+1} = c_1 HeunG\left(2, (n+1)M^2, 0, 1, \frac{1}{2}, 1, \sin\mathcal{X}+1\right) + \frac{(1-\sin\mathcal{X})^{\frac{3}{4}}}{\sqrt{\cos\mathcal{X}}} \times$ $\left[c_2\left(1+\sqrt{1-\cos^2\mathcal{X}}\right)^{\frac{1}{4}}\right] HeunG\left(2, (n+1)M^2 + \frac{5}{4}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, 1, \sin\mathcal{X}+1\right)$
$\cos\mathcal{X}$	$\theta^{n+1} = c_1 HeunG\left(2, (n+1)M^2, 0, 1, \frac{1}{2}, 1, \cos\mathcal{X}+1\right) + \frac{(1-\cos X)^{\frac{3}{4}}}{\sqrt{\sin\mathcal{X}}} \times$ $\left[c_2\left(1+\sqrt{1-\sin^2\mathcal{X}}\right)^{\frac{1}{4}}\right] HeunG\left(2, (n+1)M^2 + \frac{5}{4}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, 1, \sin\mathcal{X}+1\right)$

6.3.3 Case: $m = n = -1.$

The governing Eq. (6.5) becomes

$$\frac{d}{dx}\left[f(x)\theta^{-1}\frac{d\theta}{dx}\right] = M^2. \tag{6.20}$$

After simplification Eq. (6.20) becomes

$$\frac{d\theta}{\theta} = \left[\frac{M^2x + c_1}{f(x)}\right] dx. \tag{6.21}$$

The general solutions to Eq. (6.21) for various $f(x)$ are given in Tables 6.5 and 6.6.

Example 5

$f(x) = x^3$ is considered as an example, and the solutions is

$$\theta = e^{M^2(1-\frac{1}{x})}, \tag{6.22}$$

and satisfying boundary conditions

$$\theta(1) = 1, \qquad \lim_{x\to 0}\frac{d\theta}{dx} = 0. \tag{6.23}$$

The corresponding fin efficiency is given by

$$\eta = \int_0^1 e^{M^2(1-\frac{1}{x})}dx. \tag{6.24}$$

6.3.4 Case: $m \neq n$

Example 6

We consider $f(x) = 1$ and $n = -3m - 4$ and let $y = \theta^{m+1}$,which transforms the governing Eq. (6.5) into

$$y'' = (m+1)M^2 y^{-3}. \tag{6.25}$$

By direct integration using Polyanin and Zaitsev [74] we get

$$x = \int \left(\frac{2c_1 y^2 - 2(m+1)M^2}{2y^2} \right)^{-\frac{1}{2}} dy + c_2, \tag{6.26}$$

which yields

$$x = \sqrt{\frac{c_1 y^2 - (m+1)M^2}{c_1^2}} + c_2. \tag{6.27}$$

Using the boundary conditions (6.7) we get a solution

$$y = \sqrt{\frac{\frac{1}{2}\left[1+\sqrt{1-4(m+1)M^2}\right]^2 x^2 + 2(m+1)M^2}{1+\sqrt{1-4(m+1)M^2}}}. \tag{6.28}$$

In terms of the original variables we have

$$\theta = \left[\frac{\frac{1}{2}\left[1+\sqrt{1-4(m+1)M^2}\right]^2 x^2 + 2(m+1)M^2}{1+\sqrt{1-4(m+1)M^2}} \right]^{\frac{1}{2(m+1)}}. \tag{6.29}$$

The corresponding fin efficiency is given by

$$\eta = \int_0^1 \left[\frac{\frac{1}{2}\left[1+\sqrt{1-4(m+1)M^2}\right]^2 x^2 + 2(m+1)M^2}{1+\sqrt{1-4(m+1)M^2}} \right]^{\frac{1}{2}} dx. \tag{6.30}$$

6.4 Symmetry reductions and invariant solutions

We follow the procedure in Section 4.3.2, and employ the direct group classification to determine cases for which extra symmetries are admitted by Eq. (6.5). Here we consider problems which are not linearizable. A few cases arise.

6.4.1 Case: $m = n = -1$

For $f(x) = x^a$ and $f(x) = e^{ax}$, the symmetries are listed in Tables 6.7 and 6.8.

6.4.2 Case: $m \neq n, m \neq -1, f(x) = (x+1)^a$

Example 7

As an illustrative example, we consider $f(x) = 1$ and $n = -3m - 4$. The governing Eq. (6.5) becomes

$$y'' = (m+1)M^2 y^{-3}. \tag{6.31}$$

The symmetries were discussed in subsection 3.2.2 and its solution given in subsection 3.8.2. The temperature distribution along the surface for this profile is depicted in Figures 6.7 and 6.8. The fin efficiency as function of the thermo-geometric fin parameter is shown in Figure 6.9.

Example 8

Given $f(x) = x + 1$ and $n \neq m,\ m = -1$, then the the governing Eq. (6.5) becomes

$$y'' = \frac{(n+1)M^2 e^y - y'}{x+1} \tag{6.32}$$

after the substitution $e^y = \theta^{n+1}$. The two dimensional Lie algebra admitted by Eq. (6.32) are listed in Table 6.10. Canonical variables are

$$t = \frac{1}{\sqrt{(x+1)e^y}}, \qquad u = 2 - \ln(x+1) + \frac{1}{\sqrt{(x+1)e^y}}. \tag{6.33}$$

The corresponding canonical forms of X_1 and X_2 are

$$\Gamma_1 = \partial_u, \qquad \Gamma_2 = t\partial_t + u\partial_u. \tag{6.34}$$

By writing $u = u(t)$ Eq. (6.32) is transformed into

$$u'' = -\frac{1}{t}(u'-1)\left[\frac{(n+1)M^2}{2}(u'-1)^2 - 1\right]. \tag{6.35}$$

We solve Eq. (6.35), and obtain the solution that satisfy both the Dirichlet and Neumann boundary conditions.

$$\theta = \left[\frac{2\cosh^2\left(\tanh^{-1}\left(\frac{1}{2M}\sqrt{\frac{2}{n+1}}\right) - M\sqrt{\frac{n+1}{2}}\ln 2\right)}{(x+1)\cosh^2\left(\tanh^{-1}\left(\frac{1}{2M}\sqrt{\frac{2}{n+1}}\right) - M\sqrt{\frac{n+1}{2}}\ln(x+1)\right)}\right]^{\frac{1}{n+1}}$$

The fin efficiency is given by

$$\eta_1 = \int_0^1 \left[\frac{2\cosh^2\left(\tanh^{-1}\left(\frac{1}{2M}\sqrt{\frac{2}{n+1}}\right) - M\sqrt{\frac{n+1}{2}}\ln 2\right)}{(x+1)\cosh^2\left(\tanh^{-1}\left(\frac{1}{2M}\sqrt{\frac{2}{n+1}}\right) - M\sqrt{\frac{n+1}{2}}\ln(x+1)\right)}\right] dx$$

Table 6.5: Solution for $m = n = -1$

$f(x)$	Solution
x^a, $a \neq 2$	$\theta = c_1\ e^{-\frac{x^{1-a}\left[(a-2)c_2+(a-1)M^2 x\right]}{(a-1)(a-2)}}$
1	$\theta = c_1\ e^{\frac{1}{2}x\left(2c_2+M^2 x\right)}$
$x^{\frac{1}{2}}$	$\theta = c_1\ e^{\frac{2}{3}\sqrt{x}\left(3c_2+M^2 x\right)}$
x	$\theta = c_1\ e^{M^2 x} x^{c_2}$
x^2	$\theta = c_1\ x^{M^2} e^{-\frac{c_2}{x}}$
x^3	$\theta = c_1\ e^{-\frac{1}{2}\frac{c_2+2M^2 x}{x^2}}$
e^{ax}	$\theta = c_1\ e^{-\frac{e^{-ax}\left[ac_2+(a+1)\left(1+Ie^{Ix}\right)^{-M^2 x} x\right]}{a^2}}$
$\sin x$	$\theta = c_1\ \left[\frac{1-Ie^{Ix}}{1+Ie^{Ix}}\right]^{M^2 x} e^{IM^2 polylog_2\left(-e^{Ix}\right)-IM^2 polylog_2\left(e^{Ix}\right)-2c_2 \arg\tanh\left(e^{Ix}\right)}$
$\cos x$	$\theta = c_1\ \left[\frac{1-Ie^{Ix}}{1+Ie^{Ix}}\right]^{M^2 x} e^{I\left[M^2 dilog\left(1+Ie^{Ix}\right)-M^2 dilog\left(1+Ie^{Ix}\right)-2c_2 \arctan\left(e^{Ix}\right)\right]}$
$\ln x$	$\theta = c_1 e^{-\left[M^2 Ei_1(-2\ln x)+c_2 Ei_1(-\ln x)\right]}$

Table 6.6: Modified general solution for $f(\mathcal{X})$ where $\mathcal{X} = x + 1$, and $m = n = -1$

$f(\mathcal{X})$	Solution
$\mathcal{X}^a$ $a \neq 2$	$\theta = c_1\ e^{-\frac{\mathcal{X}^{1-a}\left[(a-2)c_2+(a-1)M^2\mathcal{X}\right]}{(a-1)(a-2)}}$
1	$\theta = c_1\ e^{\frac{1}{2}\mathcal{X}\left(2c_2+M^2\mathcal{X}\right)}$
$\mathcal{X}^{\frac{1}{2}}$	$\theta = c_1\ e^{\frac{2}{3}\sqrt{X}\left(3c_2+M^2\mathcal{X}\right)}$
$\mathcal{X}$	$\theta = c_1\ e^{M^2\mathcal{X}}\mathcal{X}^{c_2}$
$\mathcal{X}^2$	$\theta = c_1\ \mathcal{X}^{M^2}e^{-\frac{c_2}{\mathcal{X}}}$
$\mathcal{X}^3$	$\theta = c_1\ e^{-\frac{1}{2}\frac{c_2+2M^2\mathcal{X}}{\mathcal{X}^2}}$
$e^{a\mathcal{X}}$	$\theta = c_1\ e^{-\frac{e^{-a\mathcal{X}}\left[ac_2+(a+1)\left(1+Ie^{I\mathcal{X}}\right)^{-M^2\mathcal{X}}\mathcal{X}\right]}{a^2}}$
$\sin\mathcal{X}$	$\theta = c_1\left[\frac{1-Ie^{I\mathcal{X}}}{1+Ie^{I\mathcal{X}}}\right]^{M^2\mathcal{X}} e^{IM^2 polylog_2\left(-e^{I\mathcal{X}}\right)-IM^2 polylog_2\left(e^{I\mathcal{X}}\right)-2c_2\,\mathrm{arg\,tanh}\left(e^{IX}\right)}$
$\cos\mathcal{X}$	$\theta = c_1\left[\frac{1-Ie^{I\mathcal{X}}}{1+Ie^{I\mathcal{X}}}\right]^{M^2\mathcal{X}} e^{I\left[M^2 dilog\left(1+Ie^{I\mathcal{X}}\right)-M^2 dilog\left(1+Ie^{I\mathcal{X}}\right)-2c_2\arctan\left(e^{I\mathcal{X}}\right)\right]}$
$\ln\mathcal{X}$	$\theta = c_1 e^{-\left[M^2 Ei_1(-2\ln\mathcal{X})+c_2 Ei_1(-\ln\mathcal{X})\right]}$

Table 6.7: Symmetries for $m = n = -1, f(x) = x^a \text{and} f(x) = e^{ax}$

$f(x)$	Symmetries
1	$X_1 = \frac{\partial}{\partial\theta},\ X_2 = 2x^2\frac{\partial}{\partial x} + (2\ln\theta + x^2M^2)\theta\frac{\partial}{\partial\theta} \quad X_3 = x\frac{\partial}{\partial x} + x^2M^2\theta\frac{\partial}{\partial\theta}$ $X_4 = 2(x^2M^2 - 2\ln\theta)x\frac{\partial}{\partial x} + (x^4M^4 - 4\ln^2\theta)\theta\frac{\partial}{\partial\theta} \quad X_5 = (2\ln\theta - x^2M^2)\theta\frac{\partial}{\partial\theta}$ $X_6 = (3x^2M^2 - 2\ln\theta)\frac{\partial}{\partial x} + 2x^3M^4\theta\frac{\partial}{\partial\theta} \quad X_7 = -x\theta\frac{\partial}{\partial\theta},\ X_8 = -\theta\frac{\partial}{\partial\theta}$
$\sqrt{x}$	$X_1 = -\theta\frac{\partial}{\partial\theta},\ X_2 = -4x\sqrt{x}\frac{\partial}{\partial x} - (\frac{8}{3}x^2M^2 - 2\sqrt{x}\ln\theta)\theta\frac{\partial}{\partial\theta} \quad X_3 = x\frac{\partial}{\partial x} + x\sqrt{x}M^2\theta\frac{\partial}{\partial\theta}$ $X_4 = \left(\sqrt{x}\ln\theta - \frac{2}{3}x^2M^2\right)\frac{\partial}{\partial x} + \left(x\ln\theta - \frac{2}{3}x^2\sqrt{x}M^2\right)M^2\theta\frac{\partial}{\partial\theta} \quad X_5 = \frac{(3\sqrt{x}\ln\theta - 2x^2M^2)}{3\sqrt{x}}\theta\frac{\partial}{\partial\theta}$ $X_6 = 2\sqrt{x}\theta\frac{\partial}{\partial\theta},\ X_7 = -\frac{1}{\sqrt{x}}\frac{\partial}{\partial x} - xM^2\theta\frac{\partial}{\partial\theta} \quad X_8 = \left(\frac{2}{3}x\sqrt{x}M^2 - \ln\theta\right)x\frac{\partial}{\partial x}$ $- \left(\frac{1}{2}\ln^2\theta - \frac{4}{9}x^3M^4 + \frac{1}{3}x\sqrt{x}M^2\ln\theta\right)\theta\frac{\partial}{\partial\theta}$
x	$X_1 = -x^2\ln x\frac{\partial}{\partial x} + (xM^2 - x\ln xM^2 - \ln\theta)\,\theta\ln x\frac{\partial}{\partial\theta} \quad X_2 = x\ln x\frac{\partial}{\partial x} + x\ln xM^2\theta\frac{\partial}{\partial\theta}$ $X_3 = (\ln\theta - xM^2)\,x\frac{\partial}{\partial x} + (\ln\theta - M^2x)\,x\theta M^2\frac{\partial}{\partial\theta} \quad X_4 = -x\frac{\partial}{\partial x} - xM^2\theta\frac{\partial}{\partial\theta}$ $X_5 = -\theta\frac{\partial}{\partial\theta},\ X_6 = \theta\ln x\frac{\partial}{\partial\theta},\ X_7 = (\ln\theta - xM^2)\,\theta\frac{\partial}{\partial\theta} \quad X_8 = -(\ln\theta - xM^2)\,x\ln x\frac{\partial}{\partial x}$ $-\theta\left(\ln^2\theta + \ln\theta\ln xM^2x - 2x\ln\theta M^2 - x^2M^4\ln x + x^2M^4\right)\frac{\partial}{\partial\theta}$
x^2	$X_1 = -x\frac{\partial}{\partial x},\ X_2 = -\frac{\partial}{\partial x} + \left(\frac{\ln\theta - \ln xM^2 - M^2}{x}\right)\theta\frac{\partial}{\partial\theta} \quad X_3 = -x^2\frac{\partial}{\partial x} - x\theta M^2\frac{\partial}{\partial\theta}$ $X_4 = x^2(\ln xM^2 - \ln\theta)\frac{\partial}{\partial x} + x\theta(\ln xM^2 - \ln\theta)M^2\frac{\partial}{\partial x} \quad X_5 = (\ln\theta - M^2\ln x)\theta\frac{\partial}{\partial\theta}$ $X_6 = x^2\left(M^2 - \ln x - \ln\theta\right)\frac{\partial}{\partial x} + \theta\left(\ln\theta - M^2\ln x\right)^2\frac{\partial}{\partial\theta} \quad X_7 = -\frac{\theta}{x}\frac{\partial}{\partial\theta},\ X_8 = -\theta\frac{\partial}{\partial\theta}$
e^{ax}	$X_1 = -\theta\frac{\partial}{\partial\theta},\ X_2 = -\frac{\theta}{ae^{ax}}\frac{\partial}{\partial\theta},\ X_3 = -\frac{e^{ax}}{a}\frac{\partial}{\partial x} - \frac{x\theta M^2}{a}\frac{\partial}{\partial\theta} \quad X_4 = \frac{\partial}{\partial x} + (ax+1)M^2\frac{\partial}{\partial\theta}$ $X_5 = \frac{1}{a^2e^{ax}}\frac{\partial}{\partial x} + (a^2e^{ax}\ln x + M^2)\theta\frac{\partial}{\partial\theta} \quad X_6 = \left(\frac{a^2e^{ax}\ln\theta + (ax+1)M^2}{a^2e^{ax}}\right)\theta\frac{\partial}{\partial\theta}$ $X_7 = \left(\frac{ae^{ax}\ln\theta + xM^2}{a^2}\right)\theta\frac{\partial}{\partial x} + x\theta M^2\left(\frac{a^2e^{ax}\ln\theta + xM^2}{a^2e^{ax}}\right)\theta\frac{\partial}{\partial\theta} \quad X_8 = \left(\frac{ae^{ax}\ln\theta + xM^2}{ae^{ax}}\right)\theta\frac{\partial}{\partial x}$ $+ \left(\frac{a^3e^{2ax}\ln^2\theta + a^2xe^{ax}M^2\ln\theta + ae^{ax}M^2\ln\theta + xM^4}{ae^{ax}}\right)\theta\frac{\partial}{\partial\theta}$

Table 6.8: Modified symmetries for $f(\mathcal{X}) = \mathcal{X}^a$, $f(\mathcal{X}) = e^{a\mathcal{X}}$ where $\mathcal{X} = x + 1$ and $m = n = -1$,

Profile $f(\mathcal{X})$	Symmetries
1	$X_1 = \frac{\partial}{\partial\theta}$, $X_2 = 2\mathcal{X}^2\frac{\partial}{\partial x} + (2\ln\theta + \mathcal{X}^2M^2)\theta\frac{\partial}{\partial\theta}$, $X_3 = \mathcal{X}\frac{\partial}{\partial x} + \mathcal{X}^2M^2\theta\frac{\partial}{\partial\theta}$ $X_4 = 2(\mathcal{X}^2M^2 - 2\ln\theta)x\frac{\partial}{\partial x} + (\mathcal{X}^4M^4 - 4\ln^2\theta)\theta\frac{\partial}{\partial\theta}$, $X_5 = (2\ln\theta - \mathcal{X}^2M^2)\theta\frac{\partial}{\partial\theta}$ $X_6 = (3\mathcal{X}^2M^2 - 2\ln\theta)\frac{\partial}{\partial x} + 2\mathcal{X}^3M^4\theta\frac{\partial}{\partial\theta}$, $X_7 = -\mathcal{X}\theta\frac{\partial}{\partial\theta}$, $X_8 = -\theta\frac{\partial}{\partial\theta}$
$\sqrt{\mathcal{X}}$	$X_1 = -\theta\frac{\partial}{\partial\theta}$, $X_2 = -4\mathcal{X}\sqrt{\mathcal{X}}\frac{\partial}{\partial x} - (\frac{8}{3}\mathcal{X}^2M^2 - 2\sqrt{\mathcal{X}}\ln\theta)\theta\frac{\partial}{\partial\theta}$, $X_3 = \left(\sqrt{\mathcal{X}}\ln\theta - \frac{2}{3}\mathcal{X}^2M^2\right)\left(\frac{\partial}{\partial x} + M^2\theta\frac{\partial}{\partial\theta}\right)$, $X_4 = \frac{(3\sqrt{\mathcal{X}}\ln\theta - 2\mathcal{X}^2M^2)}{3\sqrt{\mathcal{X}}}\theta\frac{\partial}{\partial\theta}$ $X_5 = 2\sqrt{\mathcal{X}}\theta\frac{\partial}{\partial\theta}$, $X_6 = -\frac{1}{\sqrt{\mathcal{X}}}\frac{\partial}{\partial x} - XM^2\theta\frac{\partial}{\partial\theta}$, $X_7 = \mathcal{X}\left(\frac{\partial}{\partial x} + \sqrt{\mathcal{X}}M^2\theta\frac{\partial}{\partial\theta}\right)$, $X_8 = \left(\frac{2}{3}\mathcal{X}\sqrt{\mathcal{X}}M^2 - \ln\theta\right)\mathcal{X}\frac{\partial}{\partial x}$ $-\left(\frac{1}{2}\ln^2\theta - \frac{4}{9}\mathcal{X}^3M^4 + \frac{1}{3}\mathcal{X}\sqrt{\mathcal{X}}M^2\ln\theta\right)\theta\frac{\partial}{\partial\theta}$
$\mathcal{X}$	$X_1 = -\mathcal{X}^2\ln\mathcal{X}\frac{\partial}{\partial x} + (\mathcal{X}M^2 - \mathcal{X}\ln\mathcal{X}M^2 - \ln\theta)\,\theta\ln\mathcal{X}\frac{\partial}{\partial\theta}$, $X_2 = (\ln\theta - \mathcal{X}M^2)\,\mathcal{X}\frac{\partial}{\partial x} + (\ln\theta - M^2\mathcal{X})\,\mathcal{X}\theta M^2\frac{\partial}{\partial\theta}$, $X_3 = -\mathcal{X}\frac{\partial}{\partial x} - \mathcal{X}M^2\theta\frac{\partial}{\partial\theta}$ $X_4 = -\theta\frac{\partial}{\partial\theta}$, $X_5 = \theta\ln\mathcal{X}\frac{\partial}{\partial\theta}$, $X_6 = (\ln\theta - \mathcal{X}M^2)\,\theta\frac{\partial}{\partial\theta}$, $X_7 = \mathcal{X}\ln\mathcal{X}\left(\frac{\partial}{\partial x} + M^2\theta\frac{\partial}{\partial\theta}\right)$ $X_8 = -(\ln\theta - \mathcal{X}M^2)\,\mathcal{X}\ln\mathcal{X}\frac{\partial}{\partial x}$ $-\theta\left(\ln^2\theta + \ln\theta\ln\mathcal{X}M^2\mathcal{X} - 2\mathcal{X}\ln\theta M^2 - \mathcal{X}^2M^4\ln\mathcal{X} + \mathcal{X}^2M^4\right)\frac{\partial}{\partial\theta}$
X^2	$X_1 = -\mathcal{X}\frac{\partial}{\partial x}$, $X_2 = -\frac{\partial}{\partial x} + \left(\frac{\ln\theta - \ln\mathcal{X}M^2 - M^2}{x}\right)\theta\frac{\partial}{\partial\theta}$, $X_3 = -X^2\frac{\partial}{\partial x} - \mathcal{X}\theta M^2\frac{\partial}{\partial\theta}$, $X_4 = X^2(\ln\mathcal{X}M^2 - \ln\theta)\frac{\partial}{\partial x} + \mathcal{X}\theta(\ln\mathcal{X}M^2 - \ln\theta)M^2\frac{\partial}{\partial x}$, $X_5 = (\ln\theta - M^2\ln\mathcal{X})\theta\frac{\partial}{\partial\theta}$, $X_6 = X^2\left(M^2 - \ln\mathcal{X} - \ln\theta\right)\frac{\partial}{\partial x} + \theta\left(\ln\theta - M^2\ln\mathcal{X}\right)^2\frac{\partial}{\partial\theta}$, $X_7 = -\frac{\theta}{\mathcal{X}}\frac{\partial}{\partial\theta}$, $X_8 = -\theta\frac{\partial}{\partial\theta}$
$e^{a\mathcal{X}}$	$X_1 = -\theta\frac{\partial}{\partial\theta}$, $X_2 = -\frac{\theta}{ae^{a\mathcal{X}}}\frac{\partial}{\partial\theta}$, $X_3 = -\frac{e^{a\mathcal{X}}}{a}\frac{\partial}{\partial x} - \frac{\mathcal{X}\theta M^2}{a}\frac{\partial}{\partial\theta}$, $X_4 = \frac{\partial}{\partial x} + (a\mathcal{X} + 1)M^2\frac{\partial}{\partial\theta}$, $X_5 = \frac{1}{a^2e^{a\mathcal{X}}}\frac{\partial}{\partial x} + (a^2e^{a\mathcal{X}}\ln\mathcal{X} + M^2)\theta\frac{\partial}{\partial\theta}$, $X_6 = \left(\frac{a^2e^{a\mathcal{X}}\ln\theta + (a\mathcal{X}+1)M^2}{a^2e^{a\mathcal{X}}}\right)\theta\frac{\partial}{\partial\theta}$, $X_7 = \left(\frac{ae^{a\mathcal{X}}\ln\theta + \mathcal{X}M^2}{a^2}\right)\theta\frac{\partial}{\partial x} + \mathcal{X}\theta M^2\left(\frac{a^2e^{aX}\ln\theta + XM^2}{a^2e^{aX}}\right)\theta\frac{\partial}{\partial\theta}$, $X_8 = \left(\frac{ae^{a\mathcal{X}}\ln\theta + \mathcal{X}M^2}{ae^{a\mathcal{X}}}\right)\theta\frac{\partial}{\partial x} + \left(\frac{a^3e^{2a\mathcal{X}}\ln^2\theta + a^2\mathcal{X}e^{a\mathcal{X}}M^2\ln\theta + ae^{a\mathcal{X}}M^2\ln\theta + \mathcal{X}M^4}{ae^{aX}}\right)\theta\frac{\partial}{\partial\theta}$

Table 6.9: Symmetries for $m \neq n, n \neq -1$ various $f(x)$

Fin profile (parameter a)	Parameter n	Symmetries
	$f(x) = x^a$	
arbitrary	n arbitrary	$X_1 = x\frac{\partial}{\partial x} + \frac{a-2}{n-m}y\frac{\partial}{\partial y}$
Rectangular	n arbitrary	$X_1 = \frac{\partial}{\partial x}$, $X_2 = x\frac{\partial}{\partial x} - \frac{2}{n-m}y\frac{\partial}{\partial y}$
$a = 0$	$n = -3m - 4$	$X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial y}$, $X_3 = \frac{1}{m+1}[-2(m+1)x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}]$
Convex parabolic	n arbitrary	$X_1 = \frac{m}{3(m+1)}[2x\frac{\partial}{\partial x} - 3y\frac{\partial}{\partial y}]$
$a = \frac{1}{2}$	$n = -4m - 5$	$X_1 = 4x\sqrt{x}\frac{\partial}{\partial x} + 2\sqrt{x}y\frac{\partial}{\partial y}]$, $X_2 = -2x\frac{\partial}{\partial x} - \frac{3}{5}y\frac{\partial}{\partial y}$
Triangular	n arbitrary	$X_1 = \frac{m}{m+1}[(n-m)x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}]$
$a = 1$	$m = -1$	$X_1 = -x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, $X_2 = (2 - \ln x)x\frac{\partial}{\partial x} + \ln x\frac{\partial}{\partial y}$
Concave parabolic	n arbitrary	$X_1 = -2x\frac{\partial}{\partial x}$
$a = 2$	$n = -m - 2$	$X_1 = \frac{\partial}{\partial x} - \frac{y}{x}\frac{\partial}{\partial y}$, $X_2 = x\frac{\partial}{\partial x}$
Cubic	n arbitrary	$X_1 = -2x\frac{\partial}{\partial x}$
$a = 3$	$n = \frac{-3m-5}{2}$	$X_1 = -\frac{1}{2x}\frac{\partial}{\partial x} + \frac{y}{x^2}\frac{\partial}{\partial y}$, $X_2 = \frac{x}{2}\frac{\partial}{\partial x} - \frac{y}{5}\frac{\partial}{\partial y}$
	$f(x) = e^{ax}$	
exponential	n arbitrary	$X_1 = \frac{m}{a(m+1)}[(m-n)\frac{\partial}{\partial x} - ay\frac{\partial}{\partial y}]$
a arbitrary	$n = -2m - 3$	$X_1 = \frac{e^{ax}}{a^2}\frac{\partial}{\partial x} - \frac{e^{ax}}{a}y\frac{\partial}{\partial y}$, $X_2 = \frac{1}{a}\frac{\partial}{\partial x} - \frac{y}{3}\frac{\partial}{\partial y}$

Table 6.10: Modified symmetries for $f(\mathcal{X})$ where $\mathcal{X} = x+1$ and $m \neq n, n \neq -1$ various

Fin profile (parameter a)	Parameter n	Symmetries
	$f(\mathcal{X}) = \mathcal{X}^a$	
arbitrary	n arbitrary	$X_1 = \mathcal{X}\frac{\partial}{\partial x} + \frac{a-2}{n-m}y\frac{\partial}{\partial y}$
Rectangular	n arbitrary	$X_1 = \frac{\partial}{\partial x}$, $X_2 = \mathcal{X}\frac{\partial}{\partial x} - \frac{2}{n-m}\theta\frac{\partial}{\partial y}$
$a = 0$	$n = -3m - 4$	$X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial y}$,
		$X_3 = \frac{1}{m+1}[-2(m+1)\mathcal{X}\frac{\partial}{\partial x} - \theta\frac{\partial}{\partial y}]$
Convex parabolic	n arbitrary	$X_1 = \frac{m}{3(m+1)}[2\mathcal{X}\frac{\partial}{\partial x} - 3\theta\frac{\partial}{\partial y}]$
$a = \frac{1}{2}$	$n = -4m - 5$	$X_1 = 4\mathcal{X}\sqrt{\mathcal{X}}\frac{\partial}{\partial x} + 2\sqrt{\mathcal{X}}y\frac{\partial}{\partial y}$,
		$X_2 = -2\mathcal{X}\frac{\partial}{\partial x} - \frac{3}{5}y\frac{\partial}{\partial y}$
Triangular	n arbitrary	$X_1 = \frac{m}{m+1}[(n-m)\mathcal{X}\frac{\partial}{\partial x} - \theta\frac{\partial}{\partial y}]$
$a = 1$	$m = -1$	$X_1 = -\mathcal{X}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$,
		$X_2 = (2 - \ln\mathcal{X})\mathcal{X}\frac{\partial}{\partial x} + \ln\mathcal{X}\frac{\partial}{\partial y}$
Concave parabolic	n arbitrary	$X_1 = -2\mathcal{X}\frac{\partial}{\partial x}$
$a = 2$	$n = -m - 2$	$X_1 = \frac{\partial}{\partial x} - \frac{y}{\mathcal{X}}\frac{\partial}{\partial y}$, $X_2 = \mathcal{X}\frac{\partial}{\partial x}$
Cubic	n arbitrary	$X_1 = -2\mathcal{X}\frac{\partial}{\partial x}$
$a = 3$	$n = \frac{-3m-5}{2}$	$X_1 = -\frac{1}{2\mathcal{X}}\frac{\partial}{\partial x} + \frac{y}{\mathcal{X}^2}\frac{\partial}{\partial y}$, $X_2 = \frac{\mathcal{X}}{2}\frac{\partial}{\partial x} - \frac{y}{5}\frac{\partial}{\partial y}$
	$f(\mathcal{X}) = e^{a\mathcal{X}}$	
exponential	n arbitrary	$X_1 = \frac{m}{a(m+1)}[(m-n)\frac{\partial}{\partial x} - ay\frac{\partial}{\partial y}]$
a arbitrary	$n = -2m - 3$	$X_1 = \frac{e^{a\mathcal{X}}}{a^2}\frac{\partial}{\partial x} - \frac{e^{a\mathcal{X}}}{a}y\frac{\partial}{\partial y}$,
		$X_2 = \frac{1}{a}\frac{\partial}{\partial x} - \frac{y}{3}\frac{\partial}{\partial y}$

Table 6.11: Lie Bracket of the admitted symmetry algebra for $m \neq n, n \neq -1$ and various $f(x)$

$f(x) = x^a$,	$a = 0$,	n arbitrary	$f(x) = x^a$,	$a = 0$,	$n = -3m - 4$	
$[X_i, X_j]$	X_1	X_2	$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	X_1	X_1	0	X_1	X_3
X_2	X_1	0	X_2	$-X_1$	0	$2X_2$
			X_3	$-X_3$	$-2X_2$	0
$f(x) = x^a$,	$a = \frac{1}{2}$,	$n = -4m - 5$	$f(x) = x^a$,	$a = 1$,	$m = -1$	
$[X_i, X_j]$	X_1	X_2	$[X_i, X_j]$	X_1	X_2	
X_1	0	X_1	X_1	0	X_1	
X_2	$-X_1$	0	X_2	$-X_1$	0	
$f(x) = x^a$,	$a = 2$,	$n = -m - 2$	$f(x) = e^{ax}$,	any a,	n arbitrary	
$[X_i, X_j]$	X_1	X_2	$[X_i, X_j]$	X_1	X_2	
X_1	0	X_1	X_1	0	X_1	
X_2	$-X_1$	0	X_2	X_1	0	

Table 6.12: The type of second-order equations admitting L_2 for $m \neq n$, $n \neq -1$ various $f(x)$

Fin profile (parameter a)	Parameter n	Canonical form of the equation
	$f(x) = x^a$	
Rectangular	n arbitrary	$u'' = -\frac{(n-m)}{2}\frac{u'}{t}\left[\frac{2(n+m)+4}{(n-m)^2} - M^2 u'^2\right]$
$a = 0$	$n = -3m - 4$	$u'' = -\frac{1}{2}\frac{u'-1}{t}\left[1 + 4(m+1)M^2(u'-1)^2\right]$
Convex parabolic $a = \frac{1}{2}$	$n = -4m - 5$	$u'' = -\frac{u'-1}{t}\left[\frac{3}{5} + 40(m+1)M^2(u'-1)^2\right]$
Triangular $a = 1$	$m = -1$	$u'' = -\frac{u'-1}{t}\left[1 - \frac{(n+1)M^2(u'-1)^2}{2}\right]$
Concave parabolic $a = 2$	$n = -m - 2$	$u'' = -\frac{(u'-1)^3}{t}(m+1)M^2$
Cubic $a = 3$	$n = \frac{-3m-5}{2}$	$u'' = -\frac{5(n+1)M^2}{16}\frac{u'-1}{t}\left[\frac{16}{25(n+1)M^2} - (u'-1)^2\right]$
	$f(x) = e^{ax}$	
exponential a arbitrary	$n = -2m - 3$	$u'' = -\frac{u'-1}{t}\left[\frac{1}{3} + \frac{3(m+1)M^2(u'-1)^2}{2a^4}\right]$

Table 6.13: Reductions arising from Table 6.12

Fin profile (parameter a)	
$f(x)=x^a$	
Rectangular $a=0$ $n=-3m-4$	(i) $u'=0 \Rightarrow u=\text{const}$ (ii) $1+4(m+1)M^2(u'-1)^2=0 \Rightarrow u=\left(1\pm i\frac{1}{2\sqrt{m+1}M}\right)t+c$ (iii) $u''\neq 0 \Rightarrow u=t\pm\frac{1}{\sqrt{m+1}M}\sqrt{c_1t-1}+c_2$
Convex parabolic $a=\frac{1}{2}$ $n=-4m-5$	(i) $u'=0 \Rightarrow u=\text{const}$ (ii) $\frac{3}{5}+40(m+1)M^2(u'-1)^2 \Rightarrow u=\left(1\pm i\frac{1}{10M}\sqrt{\frac{3}{2(m+1)}}\right)t+c$ (iii) $u''\neq 0 \Rightarrow u=t\pm\frac{1}{10M}\sqrt{\frac{3}{2(m+1)}}\ln\left[t+\sqrt{c_1t^2-1}\right]+c_2$
Triangular $a=1$ $m=-1$	(i) $u'=0 \Rightarrow u=\text{const}$ (ii) $1-\frac{(n+1)M^2(u'-1)^2}{2}=0 \Rightarrow u=\left(1\pm\frac{1}{M}\sqrt{\frac{2}{n+1}}\right)t+c$ (iii) $u''\neq 0 \Rightarrow u=t\pm M\sqrt{\frac{n+1}{2}}\arcsin c_1t+c_2$
Concave parabolic $a=2$ $n=-m-2$	(i) $u'=0 \Rightarrow u=\text{const}$ (ii) $u''\neq 0 \Rightarrow u=t\pm\int\frac{1}{\sqrt{c+(m+1)M^2\ln t}}dt$
Cubic $a=3$ $m=\frac{-3m-5}{2}$	(i) $u'=0 \Rightarrow u=\text{const}$ (ii) $\frac{16}{25(n+1)M^2}-(u'-1)^2 \Rightarrow u=\left(1\pm\frac{4}{M\sqrt{n+1}}\right)t+c$ (iii) $u''\neq 0 \Rightarrow u=t\pm\frac{4}{M\sqrt{n+1}}\int\frac{dt}{\sqrt{1-(c_1t)^{\frac{2}{5}}}}+c_2$
$f(x)=e^{ax}$	
exponential a arbitrary $n=-2m-3$	(i) $u'=0 \Rightarrow u=\text{const}$ (ii) $\frac{1}{3}+\frac{3(m+1)M^2(u'-1)^2}{2a^4} \Rightarrow u=\left(1\pm i\frac{a^2}{3M}\sqrt{\frac{2}{m+1}}\right)t+c$ (iii) $u''\neq 0 \Rightarrow u=t\pm\int\frac{a^2}{3M}\sqrt{\frac{2}{m+1}}\frac{1}{\sqrt{(c_1t)^{\frac{2}{3}}-1}}dt+c_2$

Table 6.14: Original variables for $m \neq n$, $n \neq -1$ various $f(x)$

Fin profile (parameter a)	Parameter n	Solution
	$f(x) = x^a$	
Rectangular $a = 0$	$n = -3m - 4$	$\theta = [1 + (m+1)M^2(x^2 - 1)]^{\frac{1}{2(m+1)}}$
Convex parabolic $a = \frac{1}{2}$	$n = -4m - 5$	$\theta = \left[x^{\frac{1}{2}} \left(\frac{1}{c_1} \cosh\left(\pm 5Mc_1\sqrt{\frac{2(m+1)}{3}}\left(1 - \frac{1}{\sqrt{x}}\right)\right)\right)\right]^{\frac{1}{5(m+1)}}$
Triangular $a = 1$	$m = -1$	$\theta = \left[\frac{2\cosh^2\left(\tanh^{-1}\left(\pm\frac{1}{2M}\sqrt{\frac{2}{n+1}}\right) - \pm M\sqrt{\frac{n+1}{2}}\ln 2\right)}{(x+1)\cosh^2\left(\tanh^{-1}\left(\pm\frac{1}{2M}\sqrt{\frac{2}{n+1}}\right) - \pm M\sqrt{\frac{n+1}{2}}\ln(x+1)\right)}\right]^{\frac{1}{n+1}}$
Concave parabolic $a = 2$	$n = -m - 2$	$u = t \pm \int \frac{1}{\sqrt{c+(m+1)M^2\ln t}}dt$
	$f(x) = e^{ax}$	
exponential a arbitrary	$n = -2m - 3$	$\theta = \left[\frac{-9(m+1)M^2(ae^{ax}+c)^2e^{3ax}}{2a^4}\right]^{\frac{1}{3(m+1)}}$

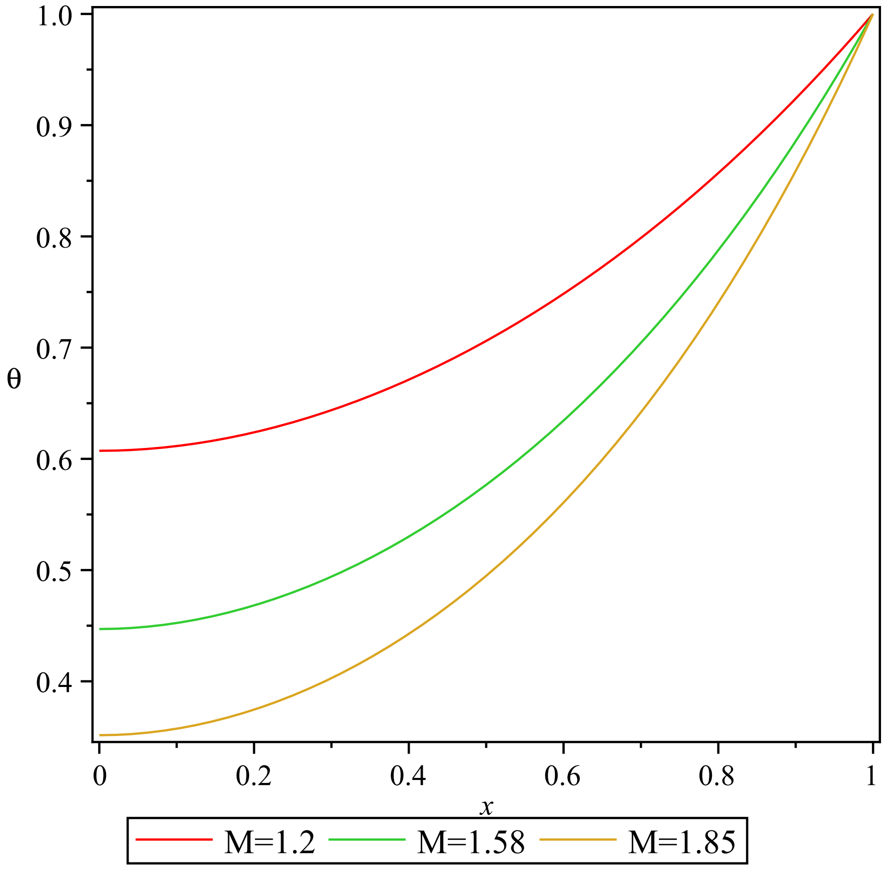

Figure 6.1: Temperature distribution in a fin with a profile $f(x) = \sqrt{x+1}$ given in (6.9) in a fin with varying values of M.

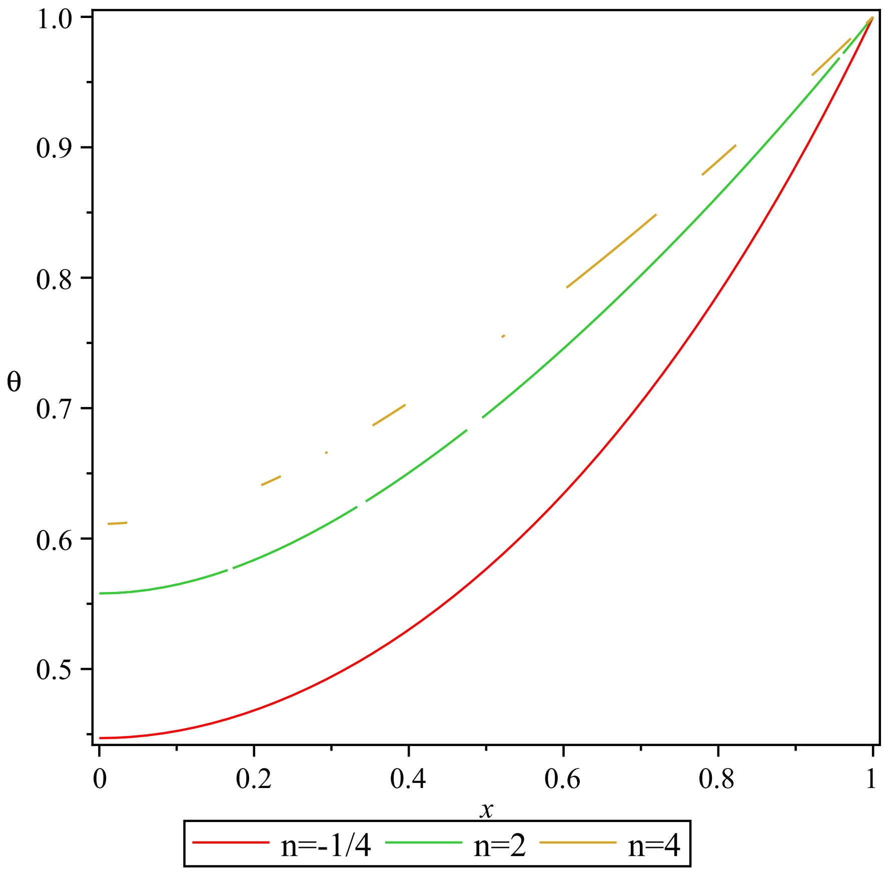

Figure 6.2: Temperature distribution in a fin with a profile $f(x) = \sqrt{x+1}$ given in (6.9) in a fin with varying values of n.

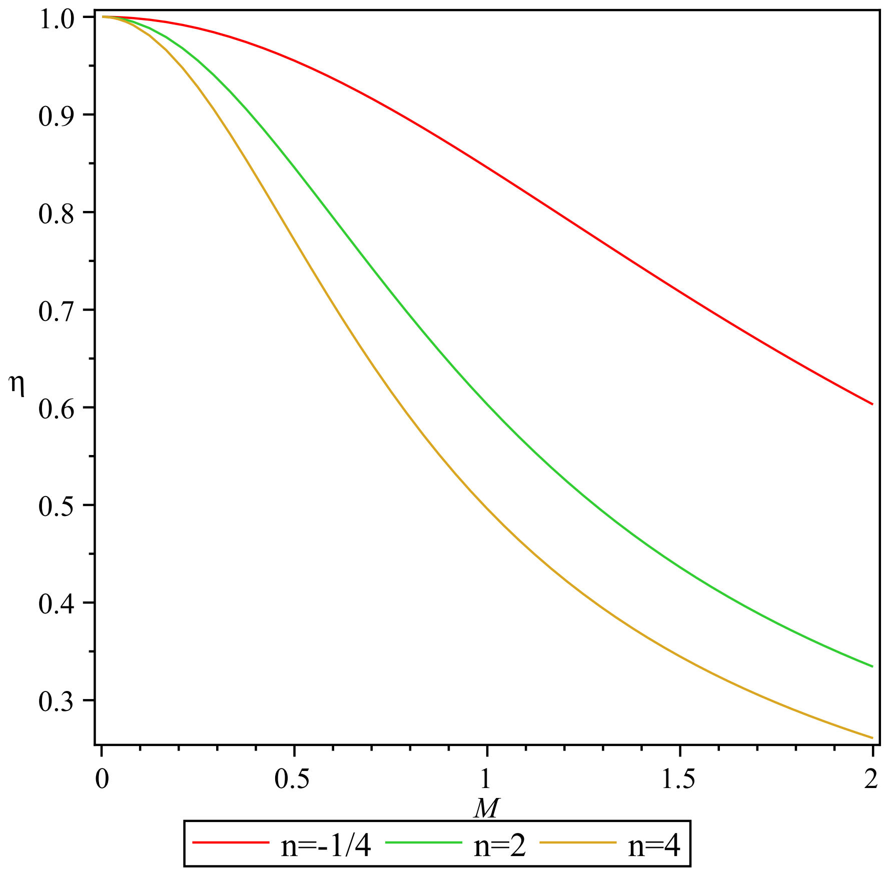

Figure 6.3: Efficiency of a fin with a profile $f(x) = \sqrt{x+1}$ as given in the solution (6.11) with varying values of n.

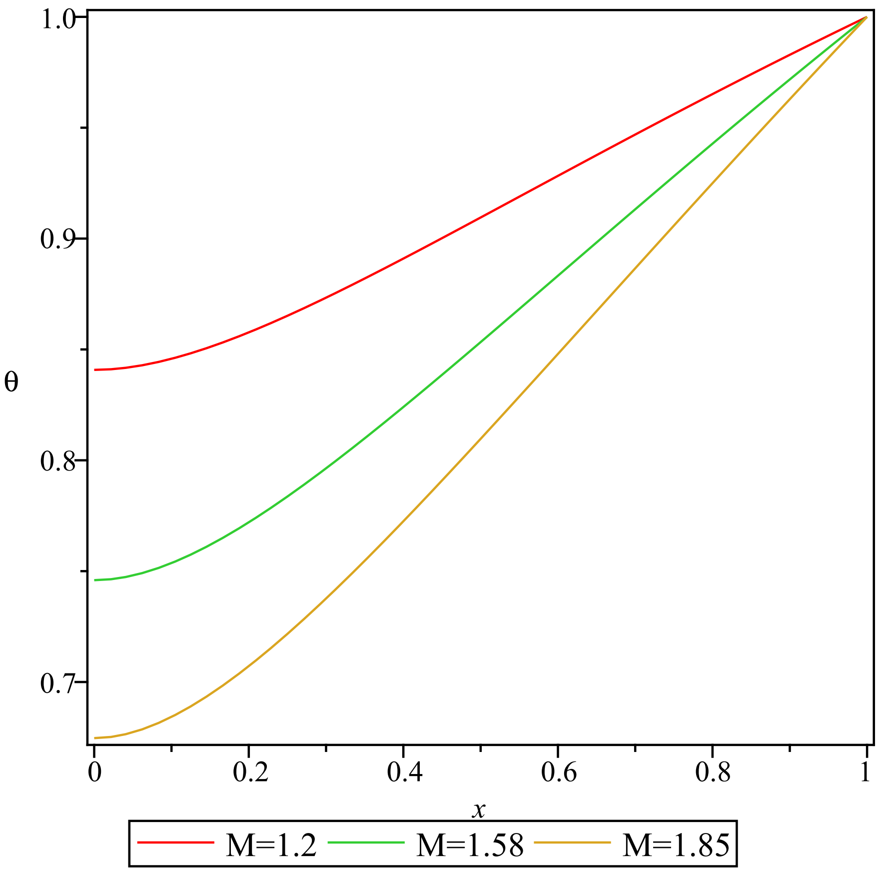

Figure 6.4: Temperature distribution of a fin profile $f(x) = (x+1)^3$ as given in the solution (6.12) in a fin with varying values of the thermogeometric fin parameter. Here, n is fixed at $-\frac{1}{4}$.

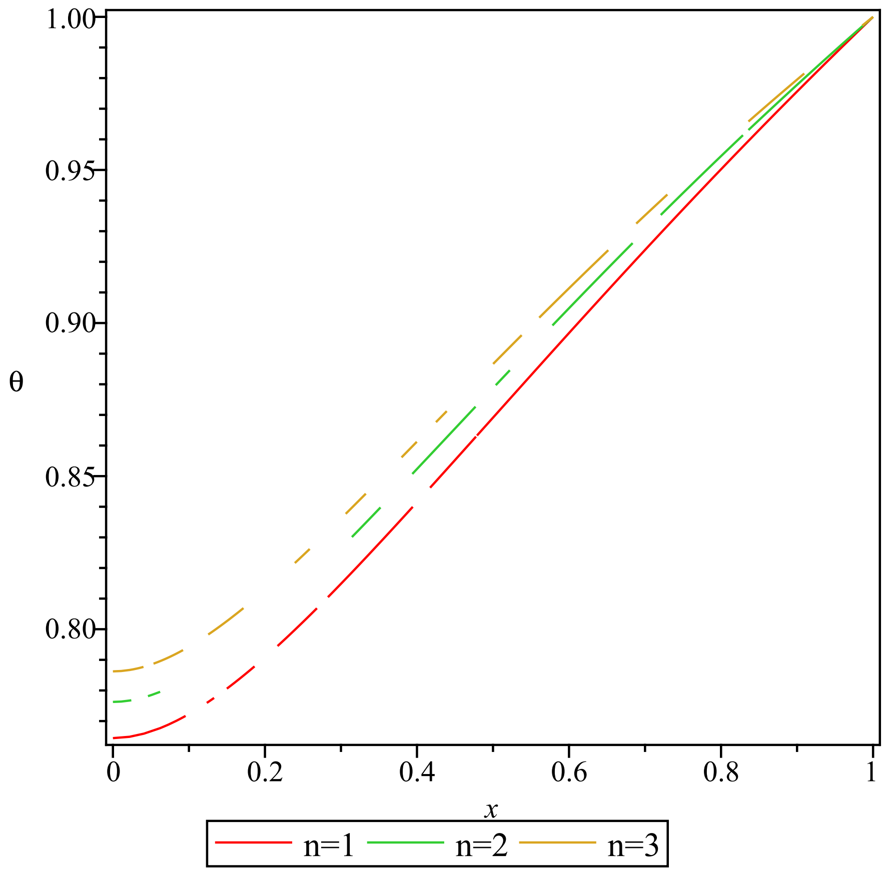

Figure 6.5: Temperature distribution of a fin profile $f(x) = (x+1)^3$ as given in the solution (6.12) in a fin with varying values of n . Here the thermogeometric fin parameter is fixed at 1.58.

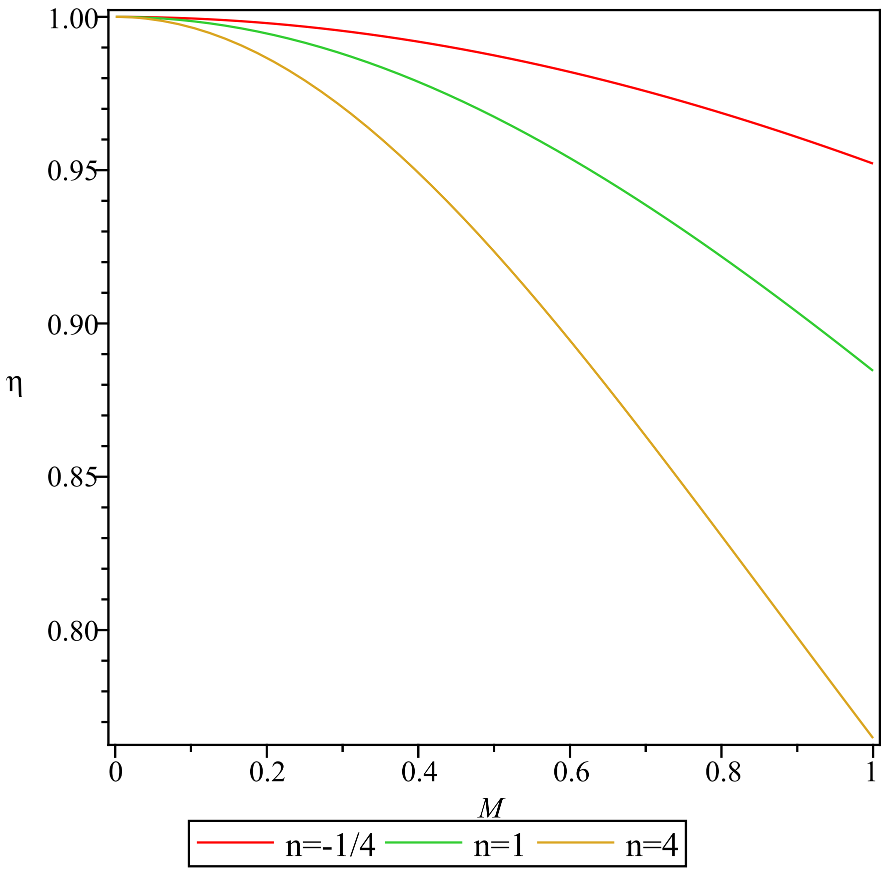

Figure 6.6: Fin efficiency for the profile $f(x) = (x+1)^3$ as given in the solution (6.14) in a fin with varying values of n.

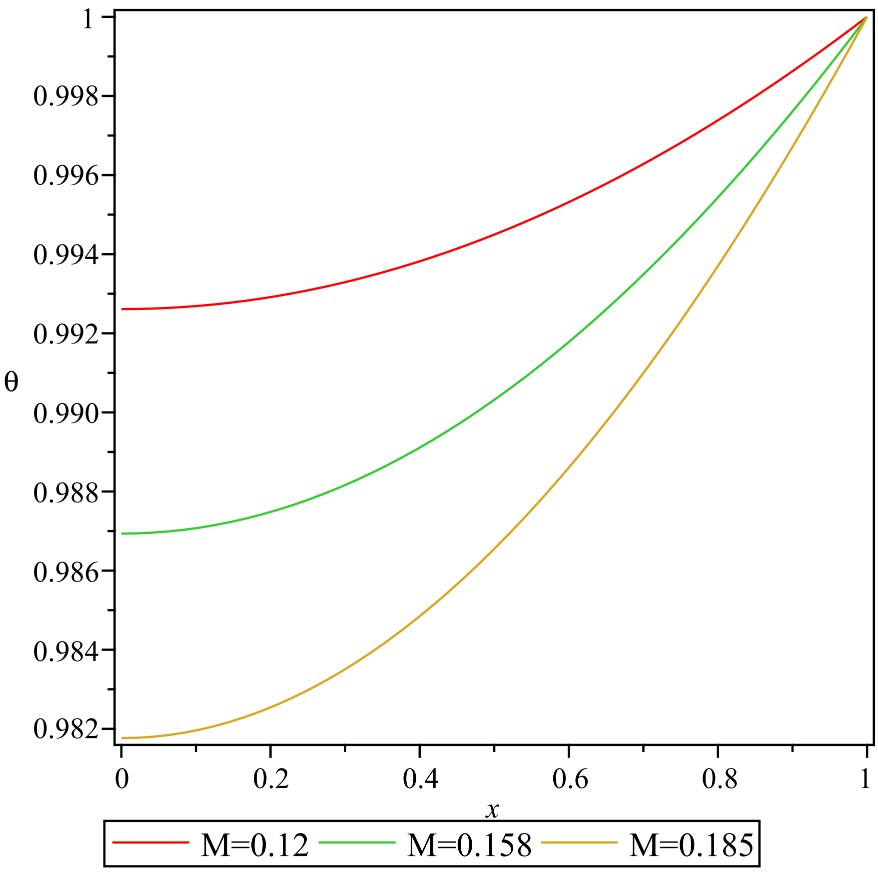

Figure 6.7: Temperature distribution in a fin with a profile $f(x) = 1, n \neq m$ as given in the solution (6.36) with varying values of M.

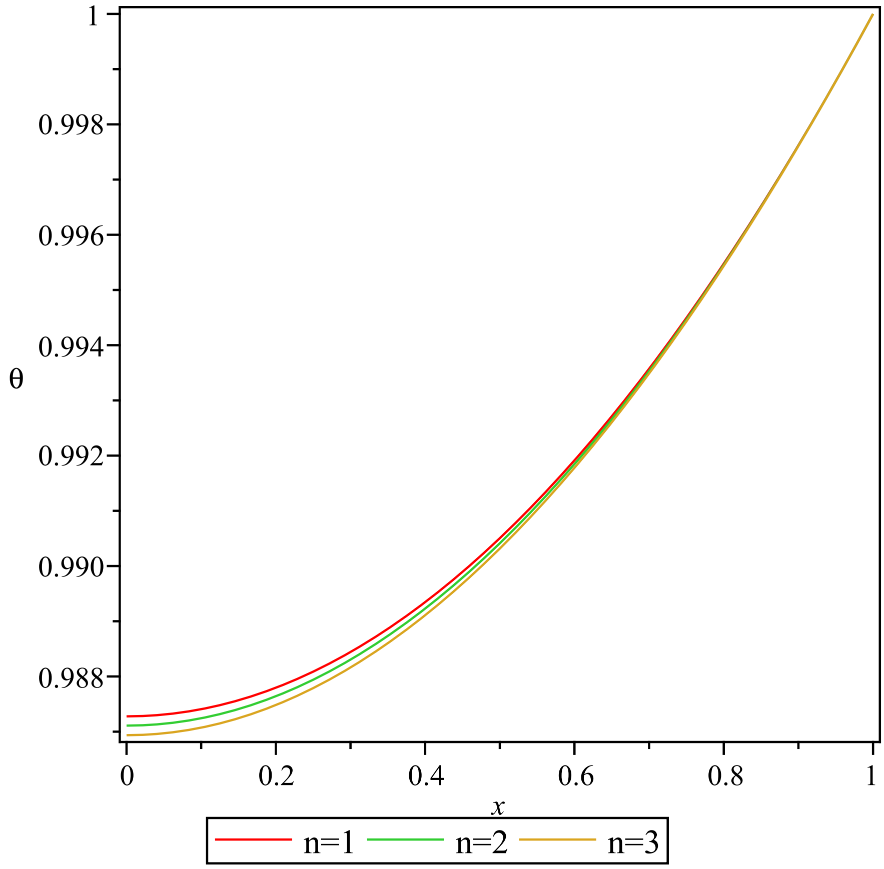

Figure 6.8: Temperature distribution in a fin with a profile $f(x) = 1, n \neq m$ as given in the solution (6.36) with varying values of n.

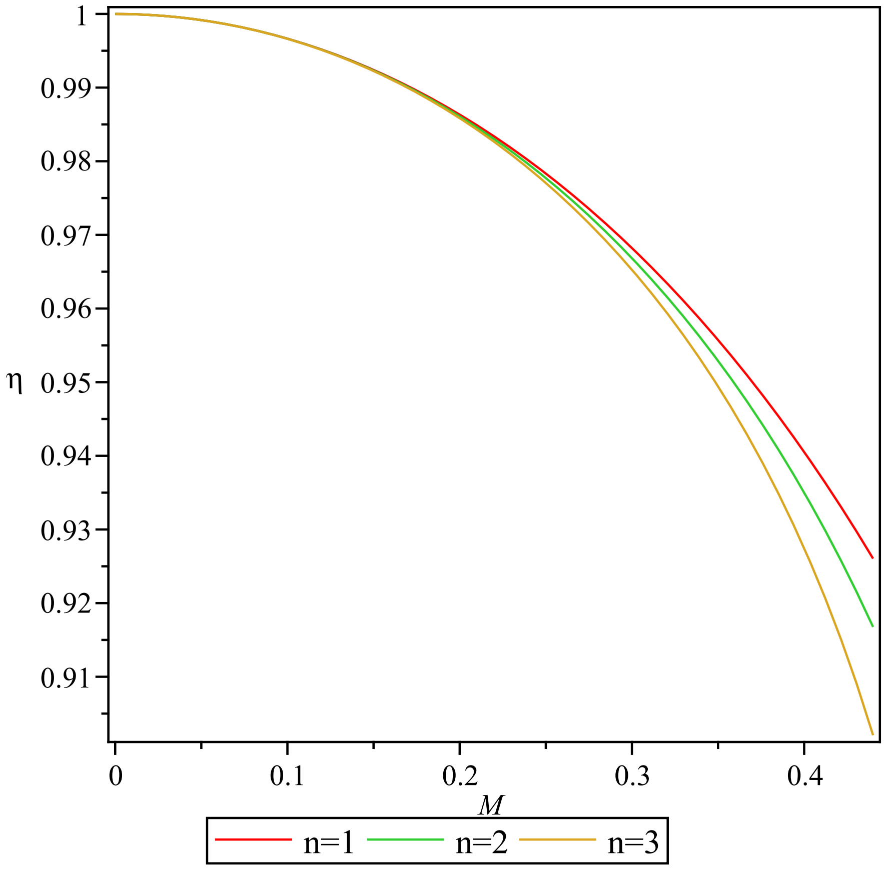

Figure 6.9: Efficiency of a fin with a profile $f(x) = 1, n \neq m$ as given in the solution (6.36) with varying values of n.

6.5 Concluding remarks

We have analyzed the steady-state fin problem using solutions given in Eqs. (6.9) and (6.11) and the results are as expected. We observe in Figure 6.4 that for the case of laminar film boiling or condensation, the temperature is inversely proportional to the thermo-geometric fin parameter. An increase in values of M yielded the decrease in values of temperature. Temperature distribution along the surface was studied for varying values of n between -3 and 3, while M was kept constant. The results depicted in Figure 6.5 shows that the temperature is directly proportional to the parameter n. The fin efficiency as function of the thermo-geometric fin parameter is shown in 6.6. Similar trends can be observed from Figures showing temperature distribution and efficiency for other profiles.

Solutions for $f(x) = x^a$ are furnished in the Table 6.1 most of them do not satisfy one of the boundary conditions. This was modified into $f(x) = (x+1)^a$ as shown in Table 6.2 and the solutions given satisfied both boundary conditions. Symmetries and further analysis of $f(x) \ \epsilon \ \{\sin x, \cos x, \ln x\}$ where ignored in this paper. The solution for $f(x) = e^{ax}$ for $n = m$ is given in [80] therefore we focused on the case where $n \neq m$. Exact solution for fin problem with power law temperature-depended thermal conductivity and heat transfer coefficient were determined. Lie symmetry techniques were used in cases where direct integration was not feasible. Results showing longitudinal fin of of various profiles were presented. The obtained solutions satisfy the physical boundary conditions. Solutions constructed could be used as benchmarks or validation tests for numerical schemes.

Chapter 7

Conclusions

A significant improvement to the study of heat transfer in extended surfaces has been achieved in this thesis. Unlike, the existing knowledge that exact solutions can only be constructed when heat transfer coefficient and thermal conductivity are constant we have actually shown that one may assume these thermal properties to be temperature dependent. Lie symmetry methods revealed the possibility of classifying these properties and construction of exact solutions.

In one case classical Lie point symmetry analysis of a steady nonlinear one dimensional fin problem was considered. Dirichlet boundary conditions were imposed at one end while the Neumann boundary conditions were imposed at the other end. The heat transfer coefficient and thermal conductivity were given as arbitrary functions of temperature. We utilize the preliminary group classification methods to determine the forms of arbitrary functions for which the PLA increased by one. Realistic forms were chosen and invariant solutions were constructed. The effects of thermogeometric fin parameter and the exponent of the thermal properties on temperature were studied. Furthermore, the fin efficiency was analyzed.

As the second step in our investigation we considered the transient response of a longitudinal rectangular fin to step change in base temperature and base heat flow conditions. Heat conductivity and heat transfer coefficient were assumed to be power law temperature dependent. In the analysis both local and nonlocal symmetry techniques were applied to the fin problem. The symmetry reduction yielded the tractable Emmakov-Pinney equation one case. Nonlocal symmetries were admitted when some arbitrary constants appearing in the governing equations were specified. The exact analytical steady state solutions which satisfy the prescribed boundary conditions were constructed. The obtained general exact analytical solutions for the transient state satisfy only zero initial temperature and adiabatic boundary condition at the fin tip, which prompted resorting to numerical solutions. The boundary value problem considered here introduced stiffness. However, we are able to transform it into a system of ODEs using finite difference for the spatial derivative. The built in ODE solver program was used to integrate a set of ODEs using a fourth order Runge-Kutta iterations scheme. Again the effects of thermo-geometric fin parameter and power law exponent on temperature distribution are studied.

Lastly, a one dimensional steady state heat transfer in fins of different profiles was studied. here it is assumed that the base temperature is constant and the fin tip is insulated. The thermal conductivity and heat coefficients were assumed to be temperature dependent which made the resulting differential equation highly nonlinear. An introduction of a point transformation linearizes the problem when the problem is given by the same power law for thermal conductivity and heat transfer coefficient. Classical Lie point symmetry methods were employed in cases where the problem was not linearizable. Some invariant solutions were constructed. The effects of parameters appearing in the model and the fin efficiency were studied.

These solutions may be used as bench marks for the approximate analytical and numerical solutions (see e.g. [22]).

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