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UNIVERSITY OF THE WITWATERSRAND



DOCTORAL THESIS

Two-dimensional turbulent classical and momentumless thermal wakes

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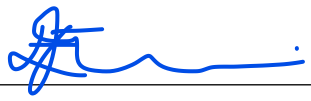
01 August 2023

Declaration of Authorship

I, Erick MUBAI, declare that this thesis titled, "Two-dimensional turbulent classical and momentumless thermal wakes" and the work presented in it are my own. I confirm that:

- This work was done while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed: 

Date: 01 August 2023

UNIVERSITY OF THE WITWATERSRAND

Abstract

Faculty of Science
School of Computer Science and Applied Mathematics

PhD

Two-dimensional turbulent classical and momentumless thermal wakes

by Erick MUBAI

The two-dimensional classical turbulent thermal wake and the two-dimensional momentumless turbulent thermal wake are studied. The governing partial differential equations result from Reynolds averaging the Navier-Stokes, the continuity and energy balance equations. The averaged Navier-Stokes and energy balance equations are closed using the Boussinesq hypothesis and an analogy of Fourier's law of heat conduction. They are further simplified using the boundary layer approximation. This leads to one momentum equation with the continuity equation for an incompressible fluid and one thermal energy equation. The partial differential equations are written in terms of a stream function for the mean velocity deficit that identically satisfies the continuity equation and the mean temperature difference which vanishes on the boundary of the wake. The mixing length model and a model that assumes that the eddy viscosity and eddy thermal conductivity depend on spatial variables only are analysed. We extend the von Kármán similarity hypothesis to thermal wakes and derive a new thermal mixing length. It is shown that the kinematic viscosity and thermal conductivity play an important role in the mathematical analysis of turbulent thermal wakes. We obtain and use conservation laws and associated Lie point symmetries to reduce the governing partial differential equations to ordinary differential equations. As a result we find new analytical solutions for the two-dimensional turbulent thermal classical wake and momentumless wake. When the ordinary differential equations cannot be solved analytically we use a numerical shooting method that uses the two conserved quantities as the targets.

Acknowledgements

I would like first to thank my supervisor, Professor D.P Mason, for his patient support. I would also like to thank the examiners for the suggestions that improved the thesis.

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List of Hypotheses

The following hypotheses will be used in the thesis.

- The Boussinesq hypothesis [3]:

  For an incompressible fluid, the Reynolds stresses $-\rho\overline{v_i' v_j'}$ *are related to the mean velocity gradients by the constitutive equation*

  $$-\rho\overline{v_i' v_j'} = -\frac{2}{3}\rho k\delta_{ij} + 2\mu_T \overline{S}_{ij},$$

  where k is the kinetic energy per unit mass of the fluctuations,

  $$k = \frac{1}{2}\overline{v_i' v_i'},$$

  μ_T is the eddy viscosity and $\overline{S}_{ij}$ is the rate of strain tensor of the mean flow,

  $$\overline{S}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{v}_i}{\partial x_j} + \frac{\partial \overline{v}_j}{\partial x_i}\right)$$

  and the averaging operator is defined by

  $$\overline{f}(t,x,y) = \lim_{N\to\infty} \frac{1}{N}\int_t^{t+N} f(\tau,x,y)d\tau.$$

- The Prandtl hypothesis [4, 5]:

  For large values of the Reynolds number the motions in sections perpendicular to the direction of the flow are mechanically and geometrically similar.

  Prandtl deduced from this hypothesis that in the wake behind a solid body the mixing length is proportional to the breadth of the wake.

- The von Kármán similarity hypothesis [9]:

  Turbulent flow patterns at different points of the flow are similar and differ only in the length and time scales.

Dedicated to Rosa Angelina Masehle Mubai.

Chapter 1

Introduction

1.1 Background and motivation

A thermal wake is fluid flow that is formed behind a heated solid body. Thermal wakes are found in many engineering applications including heat exchangers [1], detection of submarines [2] and cooling of electronic components. As a result, numerical, experimental and theoretical studies that improve our understanding of thermal wakes are essential. In this thesis, we contribute to the theory of thermal wakes by considering the turbulent classical and momentumless thermal wakes.

When a cold fluid passes a stationary heated obstacle at a constant mainstream velocity, the wake behind the obstacle is referred to as the classical thermal wake. The obstacle experiences drag from the wake. However, when the heated obstacle propels itself through the cold fluid the wake behind the obstacle is referred to as the momentumless thermal wake. The force exerted by a momentumless thermal wake on the heated obstacle is zero. In both wakes, the flow is laminar close to the trailing edge of the obstacle and becomes turbulent farther downstream.

In turbulent thermal flows, the Navier-Stokes equation and the energy balance equation are averaged to obtain the governing equations. The flow variables are split into mean values and fluctuations. The averaging yields more unknowns than the number of equations. This is known as the turbulence closure problem [3]. The Boussinesq approximation relates the additional unknowns to the mean flow variables, eddy viscosity and eddy thermal conductivity. The eddy viscosity and eddy thermal conductivity are properties of the flow. Hence, they may depend on the mean velocity and temperature. We consider various models for the eddy viscosity and eddy thermal conductivity.

Prandtl [4] studied a turbulent classical wake. He modelled the eddy viscosity using the concept of a mixing length. He assumed that kinematic viscosity vanishes at the beginning of the mathematical analysis. This assumption seems reasonable since, in most fluids, the kinematic viscosity is small compared to the eddy kinematic viscosity. He also assumed that the mixing length is proportional to the half-width of the wake to determine the mixing length. This assumption is now referred to as Prandtl's hypothesis [4, 5]. Experiments show that wakes become self-similar in the turbulent region [6, 7] (see Figure 1.1 and 1.2). He used the similarity condition to reduce the partial differential equations for the wake to ordinary differential equations. However, he obtained a solution that predicts a wake with a finite width. Experiments show that Prandtl's solution underestimates the width of the classical wake.

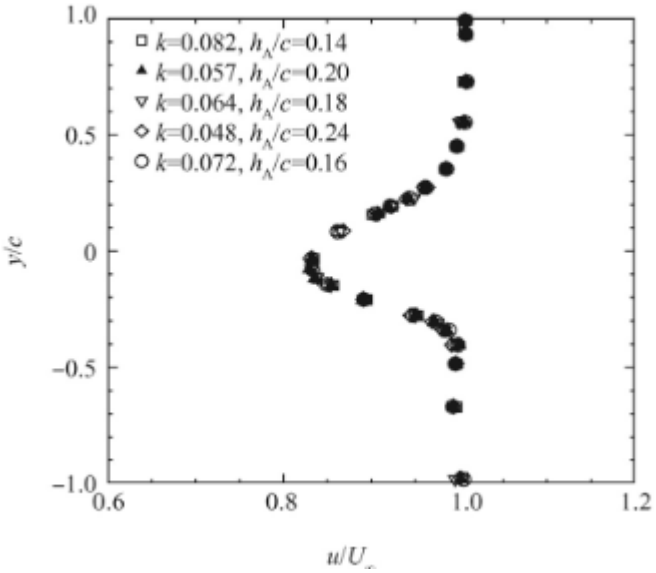

FIGURE 1.1: Dimensionless mean velocity profiles u/U_∞ at various dimensionless distances h_A/c behind a heated cylinder, where U_∞ is the characteristic velocity and c is characteristic distance. The profiles where obtained from experiments. The dimensionless velocity profiles are similar. The image was taken from [6].

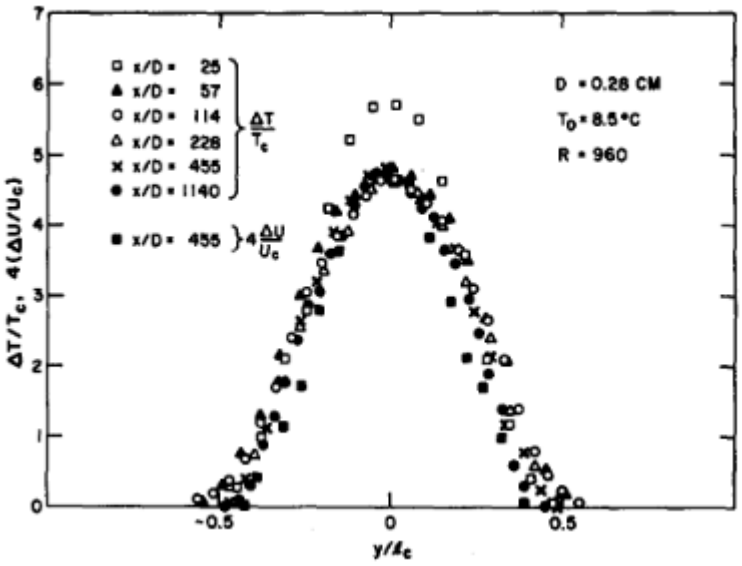

FIGURE 1.2: Dimensionless mean temperature difference profiles $\Delta T/T_c$ at various dimensionless distances x/D behind a heated cylinder, where T_c is the characteristic temperature and D is characteristic distance. The profiles where obtained from experiments. As the dimensionless distance x/D increases the dimensionless temperature profiles become similar. The image was taken from [7].

Hutchinson and Mason [8] studied a classical turbulent wake with the eddy viscosity modelled by Prandtl mixing length hypothesis. They compared the solution obtained when the kinematic viscosity is neglected with the solution obtained when the kinematic viscosity is nonzero. The two solutions were the same in the limit in

which the kinematic viscosity tends to zero in the latter solution. The solution obtained when the kinematic viscosity is neglected corresponds to Prandtl's solution. However, they showed that the solution obtained when the kinematic viscosity is nonzero predicted a boundary larger than the boundary predicted by Prandtl [4]. When the kinematic viscosity was nonzero the mixing length was obtained without imposing the Prandtl hypothesis [4, 5]. We extend their results to the turbulent classical thermal wake.

One of the drawbacks of Prandtl's mixing length hypothesis is that the form of the mixing length is unknown. In [4], Prandtl assumed that the mixing length is constant across the wake. The von Kármán similarity hypothesis that the turbulent flow patterns at different points are similar and differ only in the length and time scales [9] can be used derive the form for the mixing length. Goldstein [10] applied the von Kármán similarity hypothesis to obtain the momentum mixing length for a parallel flow in a symmetrical pipe. He then used the mixing length to find the velocity profiles inside the pipe. He argued that the von Kármán's similarity hypothesis could not be valid near points of vanishing mean velocity gradient or second derivative in mean velocity (points of inflection). Hence he used experimental data at the boundary and centre of the pipe to complete his solution.

Born and Mason [11] used the von Kármán similarity hypothesis to derive the momentum mixing length for flow in a boundary layer. The expression for the mixing length had a second derivative in mean velocity in the denominator. Hence we can expect singularities at points of inflection. Jet and Wake flows have points of inflection between the centreline and the boundary. We extend their results to the turbulent classical thermal wake and discuss the drawbacks of the hypothesis when applied to thermal wakes.

Hutchinson and Mason [12] studied a turbulent momentumless wake with the eddy viscosity dependent on spatial variables. The authors derived conservation laws for the momentumless wake and used them to derive the conserved quantity. The conserved vectors from the conservation laws were used to derive the associated Lie point symmetry. The Lie symmetry was used to reduce the governing partial differential equations for the classical wake to ordinary differential equations for the invariant solution. The ordinary differential equations were completely integrated. We extend the analysis to turbulent momentumless thermal wake. This sort of analysis has also been applied to many other flow problems [8, 13, 14, 15, 16].

Derivation of conservation laws and associated Lie point point symmetries has not been done for a turbulent momentumless wake with eddy viscosity modelled by Prandtl mixing length in the literature. We derive conservation laws and the associated Lie point symmetry for the turbulent momentumless thermal wake. We compare the solution obtained when the kinematic viscosity and thermal conductivity vanish with the solution obtained when the kinematic viscosity and thermal conductivity are nonzero.

The derivation of conservation laws leads to a systematic way of deriving conserved quantities. The conserved vectors from the conservation laws simplify the derivation of a Lie point symmetry by reducing the prolongation by one order. Once the governing partial differential equations are reduced to ordinary differential equations, the double reduction theorem of Sjöberg [17, 18] guarantees that the ordinary differential equations can be integrated at least once.

When the ordinary differential equations cannot be completely integrated, we resort to numerical methods. The boundary conditions of the invariant solution depend on a constant that determines the width of the boundary of the wake, which

is unknown. Hence the boundary conditions are not sufficient to solve the differential equations. We use a shooting method to convert the boundary value problem into an initial value problem. The shooting method "shoots" at conserved quantities for the thermal wake. Mason and Hill [14] used a shooting method that targets one conserved quantity for a jet flow. Mubai and Mason [16] used a shooting method that "shoots" at two conserved quantities for a thermal jet flow. In this thesis, we implement the shooting method of [16].

1.2 Aims and objectives

We have the following aims and objectives for the thesis.

1. To derive the governing partial differential equations for the classical and momentumless thermal wakes.
2. Further explore different models for the eddy viscosity and eddy thermal conductivity in turbulent thermal wakes.
3. Apply Lie point symmetry methods to the classical and momentumless thermal wakes.
4. Study the role played by the kinematic viscosity and thermal conductivity in the mathematical analysis of the thermal wakes.
5. Compare the solution when the kinematic viscosity and thermal conductivity tend to zero with the solution when the kinematic viscosity and thermal conductivity are zero.
6. Use Lie symmetry methods to derive the form for the mixing lengths.
7. Further verify the numerical method described in Mubai and Mason [20] for thermal wakes.

1.3 Structure of the thesis

The thesis is structured in the following manner. In Chapter 2, we review the methods that will be applied to the thermal wakes. In Chapter 3, the governing partial differential equations for the classical thermal wake and momentumless thermal wake are derived using Reynolds averaging and boundary layer theory. We also discuss the Prandtl mixing length hypothesis and its thermal extension. In Chapter 4, we study the classical thermal wake using Lie symmetry methods. In Chapter 5, the extension of von Kármán's similarity hypothesis to the turbulent classical thermal wake is considered. In Chapter 6, we study the momentumless thermal wake using Lie symmetry methods and in Chapter 7, we study the momentumless thermal wake with eddy viscosity and eddy thermal conductivity dependent on spatial variables. Finally, in Chapter 8, we provide conclusions to the thesis.

Chapter 2

Methods: Lie point symmetries, conservation laws, invariant solutions and numerical solutions to ordinary differential equations

2.1 Introduction

The derivation of conservation laws leads to a systematic method of deriving conserved quantities. The conserved vectors from the conservation laws can be used to derive the associated Lie point symmetry. The associated Lie point symmetry can be further used to reduce the governing partial differential equations to ordinary differential equations. The double reduction theorem by Sjöberg [17, 18] guarantees that the ordinary differential equations can be integrated at least once. The momentum partial differential equation for a wake is third order and hence after reduction to an ordinary differential equation, the differential equation needs to be integrated three times. In some cases, we are not able to perform the three integrations and numerical techniques are required to find the solution.

Naz et al [19] compared methods for deriving conservation laws and conserved vectors. In this thesis, we will derive conservation laws using the multiplier and direct methods [20, 21]. Noether [22] was the first to relate symmetries to conservation laws in her well-known Noether theorem. Kara et al [23, 24] gave a formula relating conserved vectors of a differential equation and a Lie-Bäcklund symmetry associated with the conserved vectors. We use this formula to derive the Lie point symmetry associated with the conserved vectors. The conservation laws are integrated to derive conserved quantities.

In this thesis, we use the associated Lie point symmetries to reduce the governing partial differential equations to ordinary differential equations. Some of the ordinary differential equations cannot be solved analytically and a numerical scheme is required to solve them. The ordinary differential equations, boundary conditions and conserved quantities form a boundary value problem. We use a shooting method to convert it to an initial value problem.

Shooting methods are widely known for converting boundary value problems to initial value problems using values at the boundary as targets. However, for wakes and jets, the equation of the boundary is unknown. Hence we have to use conserved quantities as the targets of the shooting method. Mubai and Mason [16] used a shooting method that targets two conserved quantities for a thermal jet flow.

This Chapter is organized in the following manner. In Section 2.2, we discuss conserved vectors and conservation laws. In Section 2.3, we discuss the Lie point

symmetry associated with the conserved vectors. In Section 2.4, we discuss the invariant solutions for governing partial differential equations and in Section 2.5 we discuss the numerical scheme that will be applied in this thesis. We summarize the methods in Section 2.6.

2.2 Conserved vectors and conservation laws

Conservation laws provide a systematic way of deriving conserved quantities for jet and wake flow problems. In this Section, we show how the conservation laws and conserved vectors are found.

Consider the system of two partial differential equations

$$F^i(\underline{x}, \psi, S, \psi_{(1)}, S_{(1)}, \psi_{(2)}, ...) = 0, \qquad i = 1, 2 \tag{2.2.1}$$

for the two unknown functions $\psi(\underline{x})$ and $S(\underline{x})$ where $\psi_{(k)}$ and $S_{(k)}$ are all the k-th order partial derivatives with respect to $\underline{x} = (x_1, x_2, ..., x_n)$. The conserved vector $(T^1, T^2, ..., T^n)$ is defined by

$$D_1 T^1 + D_2 T^2 + \cdots + D_n T^n \Big|_{F^j=0 \text{ for all } j} = 0, \tag{2.2.2}$$

where

$$D_i = \frac{\partial}{\partial x_i} + \psi_i \frac{\partial}{\partial \psi} + S_i \frac{\partial}{\partial S} + \psi_{ij} \frac{\partial}{\partial \psi_j} + S_{ij} \frac{\partial}{\partial S_j} + \cdots . \tag{2.2.3}$$

Equation (2.2.2) is called a conservation law for the system with components (2.2.1).

2.2.1 Multiplier method

The multiplier method assumes that the divergence of the conserved vector can be written as

$$\Lambda_i F^i = D_i T^i, \tag{2.2.4}$$

where the Λ_is are the multipliers and

$$\Lambda_i = \Lambda_i(\underline{x}, \psi, S, \psi_{(1)}, S_{(1)}, \psi_{(2)}, ...). \tag{2.2.5}$$

The standard Euler operators, defined by

$$E_\psi = \frac{\partial}{\partial \psi} + \sum_{s \geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial \psi_{i_1 \cdots i_s}} \tag{2.2.6}$$

and

$$E_S = \frac{\partial}{\partial S} + \sum_{s \geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial S_{i_1 \cdots i_s}}, \tag{2.2.7}$$

annihilate the right-hand side of equation (2.2.4) to give

$$E_\psi \left(\Lambda_i F^i \right) = 0, \tag{2.2.8}$$

$$E_S \left(\Lambda_i F^i \right) = 0. \tag{2.2.9}$$

Equations (2.2.8) and (2.2.9) are determining equations for the multipliers Λ_i. The multipliers are substituted into (2.2.4) to obtain the conserved vector, $(T^1, T^2, ..., T^n)$.

Example 1

Consider the partial differential equation

$$\frac{\partial u}{\partial x} + \ln u \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2}. \tag{2.2.10}$$

When dealing with conservation laws and Lie point symmetries the partial derivatives are written using the subscript notation to indicate that u and its partial derivatives are independent variables. Writing (2.2.10) in subscript notation we obtain

$$u_x + \ln(u)u_y - u_{yy} = 0. \tag{2.2.11}$$

Using the multiplier method we consider a multiplier of the form $\Lambda(x, y, u)$ such that

$$\Lambda(x, y, u)\left(u_x + \ln(u)u_y - u_{yy}\right) = D_x T^1 + D_y T^2, \tag{2.2.12}$$

for all functions $u(x, y)$. Applying the Euler operator

$$E_u = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} - D_y \frac{\partial}{\partial u_y} + D_y^2 \frac{\partial}{\partial u_{yy}} + ... \tag{2.2.13}$$

to (2.2.12) we obtain

$$-2u_{yy}\frac{\partial \Lambda}{\partial u} - \frac{\partial \Lambda}{\partial x} - \ln u \frac{\partial \Lambda}{\partial y} - \frac{\partial^2 \Lambda}{\partial y^2} - 2u_y \frac{\partial^2 \Lambda}{\partial y \partial u} - u_y^2 \frac{\partial^2 \Lambda}{\partial u^2} = 0. \tag{2.2.14}$$

Splitting equation (2.2.14) into coefficients of independent partial derivatives of u and their powers we obtain

$$u_{yy} : \frac{\partial \Lambda}{\partial u} = 0, \tag{2.2.15}$$

$$u_y : \frac{\partial^2 \Lambda}{\partial y \partial u} = 0, \tag{2.2.16}$$

$$u_y^2 : \frac{\partial^2 \Lambda}{\partial u^2} = 0, \tag{2.2.17}$$

$$\text{Remainder} : \frac{\partial \Lambda}{\partial x} + \ln u \frac{\partial \Lambda}{\partial y} + \frac{\partial^2 \Lambda}{\partial y^2} = 0. \tag{2.2.18}$$

By (2.2.15), $\Lambda = \Lambda(x, y)$. Equations (2.2.16) and (2.2.17) are identically satisfied. Splitting (2.2.18) by the coefficient of $\ln u$ and the remainder we obtain

$$\ln u : \frac{\partial \Lambda}{\partial y} = 0, \tag{2.2.19}$$

$$\text{Remainder: } \frac{\partial \Lambda}{\partial x} + \frac{\partial^2 \Lambda}{\partial y^2} = 0. \tag{2.2.20}$$

Equations (2.2.19) to (2.2.20) give

$$\Lambda = c, \tag{2.2.21}$$

where c is a constant. Hence equation (2.2.12) can be written as

$$c\left(u_x + \ln u u_y - u_{yy}\right) = D_x\left(cu\right) + D_y\left(cu \ln u - cu - cu_y\right), \tag{2.2.22}$$

for arbitrary $u(x,y)$. When $u(x,y)$ is a solution of (2.2.11) then (2.2.22) becomes

$$D_x(u) + D_y\left(u\ln u - u - u_y\right)\Big|_{(2.2.11)} = 0. \tag{2.2.23}$$

Hence a conserved vector is

$$\left(u, u\ln u - u - u_y\right). \tag{2.2.24}$$

2.2.2 Conserved vectors: direct method

For the direct method, we have to assume the functional form of the conserved vector components T^i. We then use (2.2.2) to calculate the conserved vector components T^i.

Example 1 continued

To demonstrate the direct method for calculating conserved vectors, we consider (2.2.10).

Assume $T^1 = T^1(u)$ and $T^2 = T^2(u, u_y)$. Then

$$D_xT^1 + D_yT^2\Big|_{(2.2.11)} = 0. \tag{2.2.25}$$

Expanding (2.2.25) we obtain

$$\left(u_x\frac{dT^1}{du} + u_y\frac{\partial T^2}{\partial u} + u_{yy}\frac{\partial T^2}{\partial u_y}\right)\Big|_{(2.2.11)} = 0. \tag{2.2.26}$$

Using (2.2.11) to substitute for u_{yy} in (2.2.26) we obtain

$$u_x\frac{dT^1}{du} + u_y\frac{\partial T^2}{\partial u} + \left(u_x + \ln(u)u_y\right)\frac{\partial T^2}{\partial u_y} = 0. \tag{2.2.27}$$

Splitting (2.2.27) by coefficients of u_x we obtain

$$u_x : \frac{dT^1}{du} + \frac{\partial T^2}{\partial u_y} = 0, \tag{2.2.28}$$

$$\text{Remainder} : \frac{\partial T^2}{\partial u} + \ln u\frac{\partial T^2}{\partial u_y} = 0. \tag{2.2.29}$$

Differentiating (2.2.28) with respect to u_y we obtain

$$\frac{\partial^2 T^2}{\partial u_y^2} = 0. \tag{2.2.30}$$

Therefore

$$T^2 = A(u)u_y + B(u), \tag{2.2.31}$$

where A and B are arbitrary functions of u. Substituting (2.2.31) into (2.2.29) we obtain

$$\frac{dA}{du}u_y + \frac{dB}{du} + \ln uA(u) = 0. \tag{2.2.32}$$

Spitting (2.2.32) by coefficient of u_y and the remainder we obtain

$$u_y : \frac{dA}{du} = 0, \tag{2.2.33}$$

$$\text{Remainder: } \frac{dB}{du} + \ln(u)A = 0. \tag{2.2.34}$$

By (2.2.33) A is a constant. Hence integrating (2.2.34) we obtain

$$B = -Au\ln u + Au + C, \tag{2.2.35}$$

where C is a constant. Now substituting (2.2.31) into (2.2.28) we obtain

$$\frac{dT^1}{du} + A = 0. \tag{2.2.36}$$

Integrating (2.2.36) we obtain

$$T^1 = -Au + D, \tag{2.2.37}$$

where D is a constant. The conserved vector components are

$$T^1 = -Au + D, \qquad T^2 = -Au\ln u + Au + Au_y + C. \tag{2.2.38}$$

We can let C and D be zero since they lead to trivial conserved vectors for which the condition $D_xT^1 + D_yT^2 = 0$ is identically satisfied. Without loss of generality, let $A = -1$. Thus

$$(T^1, T^2) = \left(u, u\ln u - u - u_y\right) \tag{2.2.39}$$

and the conservation law is

$$D_x u + D_y\left(u\ln u - u - u_y\right)\Big|_{(2.2.11)} = 0. \tag{2.2.40}$$

In jets and wakes, we do not know the equation of the boundary and hence the boundary conditions are not sufficient to solve the problem. We need conserved quantities. We obtain the conserved quantities by integrating the conservation laws.

2.3 Associated Lie point symmetries

A vector field operator

$$X = \xi^i(\underline{x}, \psi, S)\frac{\partial}{\partial x_i} + \eta^1(\underline{x}, \psi, S)\frac{\partial}{\partial \psi} + \eta^2(\underline{x}, \psi, S)\frac{\partial}{\partial S} \tag{2.3.1}$$

is said to be associated with the conserved vector $T = (T^1, T^2, ..., T^n)$ provided [23, 24]

$$X^{[\infty]}(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0, \tag{2.3.2}$$

where the prolongation $X^{[\infty]}$ is defined by

$$X^{[\infty]} = \xi^i\frac{\partial}{\partial x_i} + \eta^1\frac{\partial}{\partial \psi} + \eta^2\frac{\partial}{\partial S} + \zeta^\alpha_x\frac{\partial}{\partial u^\alpha_x} + \zeta^\alpha_{xx}\frac{\partial}{\partial u^\alpha_{xx}} + \zeta^\alpha_{xy}\frac{\partial}{\partial u^\alpha_{xy}} + \cdots, \quad \alpha = 1, 2 \tag{2.3.3}$$

and

$$\zeta_i^\alpha = D_i\eta^\alpha - u_k^\alpha D_i\xi^k, \tag{2.3.4}$$

$$\zeta^\alpha_{i_1\cdots i_{s-1}i_s} = D_{i_s}\zeta^\alpha_{i_1\cdots i_{s-1}} - u^\alpha_{ji_1\cdots i_{s-1}} D_{i_s}\xi^j, \tag{2.3.5}$$

$$u^1 = \psi, \qquad u^2 = \overline{S}. \tag{2.3.6}$$

In this thesis, we show that the associated vector field is a Lie point symmetry for the governing partial differential equations of the turbulent thermal wakes considered.

Example 1 continued

We find the vector field operator

$$X = \xi^1(x,y,u)\frac{\partial}{\partial x} + \xi^2(x,y,u)\frac{\partial}{\partial y} + \eta(x,y,u)\frac{\partial}{\partial u} \tag{2.3.7}$$

associated with the conserved vector (2.2.24) and show it is a Lie point symmetry for (2.2.11). When $i = 1$ the association condition (2.3.2) becomes

$$X(T^1) + T^1 D_y\xi^2 - T^2 D_y\xi^1 = 0. \tag{2.3.8}$$

Using the conserved vector (2.2.24), equation (2.3.8) expands to

$$\begin{aligned}\eta + u\frac{\partial\xi^2}{\partial y} + uu_y\frac{\partial\xi^2}{\partial u} - u\ln u\frac{\partial\xi^1}{\partial y} + u\frac{\partial\xi^1}{\partial y} + u_y\frac{\partial\xi^1}{\partial y} - uu_y\ln u\frac{\partial\xi^1}{\partial u}\\ +uu_y\frac{\partial\xi^1}{\partial u} + u_y^2\frac{\partial\xi^1}{\partial u} = 0.\end{aligned} \tag{2.3.9}$$

Splitting equation (2.3.9) by powers of u_y we obtain

$$u_y^2 : \frac{\partial\xi^1}{\partial u} = 0, \tag{2.3.10}$$

$$u_y : u\frac{\partial\xi^2}{\partial u} + \frac{\partial\xi^1}{\partial y} - u\ln u\frac{\partial\xi^1}{\partial u} + u\frac{\partial\xi^1}{\partial u} = 0, \tag{2.3.11}$$

$$\text{Remainder: } \eta + u\frac{\partial\xi^2}{\partial y} + u\frac{\partial\xi^1}{\partial y} - u\ln u\frac{\partial\xi^1}{\partial y} = 0. \tag{2.3.12}$$

From (2.3.10)

$$\xi^1 = \xi^1(x,y). \tag{2.3.13}$$

Equation (2.3.11) becomes

$$u\frac{\partial\xi^2}{\partial u} + \frac{\partial\xi^1}{\partial y} = 0. \tag{2.3.14}$$

The second term of equation (2.3.14) is independent of u. Differentiate (2.3.14) with respect to u. This gives

$$u\frac{\partial\xi^2}{\partial u} = A(x,y), \qquad \frac{\partial\xi^1}{\partial y} = -A(x,y), \tag{2.3.15}$$

where $A(x,y)$ is an arbitrary function of x and y. Integrating the first equation of (2.3.15) we obtain

$$\xi^2(x,y,u) = A(x,y)\ln u + B(x,y), \tag{2.3.16}$$

where $B(x,y)$ is an arbitrary function of x and y. Substituting equation (2.3.16) and the second equation of (2.3.15) into equation (2.3.12) we obtain

$$\eta = -u\ln u\frac{\partial A}{\partial y} - u\frac{\partial B}{\partial y} + uA - u\ln uA. \tag{2.3.17}$$

When $i = 2$ the association condition (2.3.2) becomes

$$X^{[1]}(T^2) + T^2 D_x\xi^1 - T^1 D_x\xi^2 = 0. \tag{2.3.18}$$

Using the conserved vector (2.2.24), equation (2.3.18) expands to

$$\ln u\eta - \zeta_y + u\ln u\frac{\partial\xi^1}{\partial x} - u\frac{\partial\xi}{\partial x} - u_y\frac{\partial\xi^1}{\partial x} - u\frac{\partial\xi^2}{\partial x} - uu_x\frac{\partial\xi^2}{\partial u} = 0. \tag{2.3.19}$$

Using (2.3.4), the prolongation ζ_y is

$$\zeta_y = D_y\eta - u_x D_y\xi^1 - u_y D_y\xi^2. \tag{2.3.20}$$

Expanding the total derivatives in (2.3.20) we obtain

$$\zeta_y = \frac{\partial\eta}{\partial y} + u_y\frac{\partial\eta}{\partial u} - u_x\frac{\partial\xi^1}{\partial y} - u_y\frac{\partial\xi^2}{\partial y} - u_y^2\frac{\partial\xi^2}{\partial u}. \tag{2.3.21}$$

Substituting (2.3.21) into (2.3.19) we obtain

$$\begin{aligned}&\ln(u)\eta - \frac{\partial\eta}{\partial y} - u_y\frac{\partial\eta}{\partial u} + u_x\frac{\partial\xi^1}{\partial y} + u_y\frac{\partial\xi^2}{\partial y} + u_y^2\frac{\partial\xi^2}{\partial u}\\ &+ \left(u\ln u - u - u_y\right)\frac{\partial\xi^1}{\partial x} - u\left(\frac{\partial\xi^2}{\partial x} + u_x\frac{\partial\xi^2}{\partial u}\right) = 0.\end{aligned} \tag{2.3.22}$$

Splitting equation (2.3.22) by powers of u_x and u_y we obtain

$$u_x : \frac{\partial\xi^1}{\partial y} - u\frac{\partial\xi^2}{\partial u} = 0, \tag{2.3.23}$$

$$u_y : \frac{\partial\xi^2}{\partial y} - \frac{\partial\xi^1}{\partial x} - \frac{\partial\eta}{\partial u} = 0, \tag{2.3.24}$$

$$u_y^2 : \frac{\partial\xi^2}{\partial u} = 0, \tag{2.3.25}$$

$$\text{Remainder} : \ln(u)\eta - \frac{\partial\eta}{\partial y} + u\ln u\frac{\partial\xi^1}{\partial x} - u\frac{\partial\xi^1}{\partial x} - u\frac{\partial\xi^2}{\partial x} = 0. \tag{2.3.26}$$

From (2.3.23) and (2.3.25) we see that

$$\xi^1 = \xi^1(x), \qquad \xi^2 = \xi^2(x,y). \tag{2.3.27}$$

Hence $A(x,y) = 0$ and

$$\xi^2 = B(x,y). \tag{2.3.28}$$

Equation (2.3.24) gives

$$2\frac{\partial B}{\partial y}(x,y) - \frac{d\xi^1}{dx}(x) = 0. \tag{2.3.29}$$

Thus

$$\frac{\partial^2 B}{\partial y^2}(x,y) = 0. \tag{2.3.30}$$

Equation (2.3.26) becomes

$$\frac{\partial B}{\partial x} + \ln u \frac{\partial B}{\partial y} + \frac{d\xi^1}{dx} - \ln u \frac{d\xi^1}{dx} = 0. \tag{2.3.31}$$

Splitting equation (2.3.31) by the coefficient of $\ln u$ and the remainder we obtain

$$\ln u : \frac{\partial B}{\partial y} - \frac{d\xi^1}{dx} = 0, \tag{2.3.32}$$

$$\text{Remainder: } \frac{\partial B}{\partial x} + \frac{d\xi^1}{dx} = 0. \tag{2.3.33}$$

Thus from (2.3.29), (2.3.32) and (2.3.33)

$$\frac{\partial B}{\partial x} = 0, \qquad \frac{\partial B}{\partial y} = 0, \qquad \frac{d\xi^1}{dx} = 0. \tag{2.3.34}$$

Hence

$$\xi^1 = c_1, \qquad B = c_2, \tag{2.3.35}$$

where c_1 and c_2 are constants. By (2.3.17)

$$\eta = 0. \tag{2.3.36}$$

The vector field operator then becomes

$$X = c_1\frac{\partial}{\partial x} + c_2\frac{\partial}{\partial y}. \tag{2.3.37}$$

We consider only the general case when $c_1 \neq 0$. The case when $c_1 = 0$ may lead to another reduction, but it is not considered in this example. Hence, the vector field operator (2.3.37) may be divided by c_1 to give

$$X = \frac{\partial}{\partial x} + c_2^*\frac{\partial}{\partial y}, \tag{2.3.38}$$

where $c_2^* = c_2/c_1$.

We verify that the vector field (2.3.38) is a Lie point symmetry of the partial differential equation (2.2.10). Applying the prolongated vector field operator to (2.2.11), subject to (2.2.11) being satisfied, we have

$$X^{[\infty]}\left(u_x + \ln(u)u_y - u_{yy}\right)\Big|_{(2.2.11)} = \left(\zeta_x + \frac{u_y}{u}\eta + \zeta_y \ln u - \zeta_{yy}\right)\Big|_{(2.2.11)} = 0. \tag{2.3.39}$$

Hence, the vector field (2.3.38) is a Lie point symmetry for the partial differential equation (2.2.10) and is associated with the conserved vector (2.2.24).

2.4 Invariant solutions of differential equations

Once the symmetry generator X has been found using (2.3.2), the invariant solutions $\psi = f(\underline{x})$ and $S = g(\underline{x})$ can be obtained using the following invariance conditions:

$$X(\psi - f(\underline{x}))\Big|_{\psi=f(\underline{x})} = 0, \tag{2.4.1}$$

$$X(S - g(\underline{x}))\Big|_{S=g(\underline{x})} = 0. \tag{2.4.2}$$

These equations expand to first-order linear partial differential equations for ψ and S and they are solved for ψ and S by integrating the differential equations of the characteristic curves of the partial differential equations.

Example 1 continued

We continue with the example in the previous two Sections by finding the invariant solution. For this example, the invariant solution $u = f(x, y)$ satisfies

$$X\left(u - f(x,y)\right)\Big|_{u=f(x,y)} = 0. \tag{2.4.3}$$

Equation (2.4.3) expands to

$$\frac{\partial f}{\partial x} + c_2\frac{\partial f}{\partial y} = 0. \tag{2.4.4}$$

The differential equations of the characteristic curves of the partial differential equation (2.4.4) are

$$\frac{dx}{1} = \frac{dy}{c_2} = \frac{df}{0}. \tag{2.4.5}$$

Hence

$$\frac{dy}{dx} = c_1 \tag{2.4.6}$$

and $f = I_2$, where I_2 is a constant. Integrating (2.4.6) gives

$$y - c_1 x = I_1, \tag{2.4.7}$$

where I_1 is a constant. Therefore the general solution of the first order partial differential equation (2.4.4) is

$$I_2 = F(I_1), \tag{2.4.8}$$

where F is an arbitrary function. Since $f = u(x, y)$ it follows that the invariant solution is

$$u(x, y) = F(\xi), \tag{2.4.9}$$

where

$$\xi = y - c_2 x. \tag{2.4.10}$$

Substituting (2.4.9) into the partial differential equation (2.2.10) gives the ordinary differential equation

$$\frac{d^2F}{d\xi^2} + (c_2 - \ln F)\frac{dF}{d\xi} = 0. \tag{2.4.11}$$

Noting that

$$\frac{d}{d\xi}(F\ln F - F) = \ln F\frac{dF}{d\xi}, \tag{2.4.12}$$

equation (2.4.11) becomes

$$\frac{d^2F}{d\xi^2} - \frac{d}{d\xi}(F\ln F) + (1 + c_2)\frac{dF}{d\xi} = 0. \tag{2.4.13}$$

We have reduced the partial differential equation (2.2.10) to the ordinary differential equation (2.4.13) using the method of associated Lie point symmetry. The ordinary differential equation (2.4.13) can be integrated at least once. This demonstrates the double reduction theorem of Sjöberg [17, 18] which states that if a partial differential equation is reduced to an ordinary differential equation using a Lie point symmetry associated with a conserved vector of the partial differential equation then the ordinary differential equation can be integrated at least once.

2.5 Numerical solutions of ordinary differential equations

When the ordinary differential equations for the invariant solution are too nonlinear to be integrated fully we resort to numerical techniques. The ordinary differential equations, boundary conditions and conserved quantities form a boundary value problem. We use a shooting method to convert the boundary value problem to an initial value problem. In this thesis, we implement the shooting method described in [16].

2.5.1 Numerical solutions of initial value problems

An initial value problem consists of a differential equation

$$\frac{d^n y}{dt^n} = \Delta y(t) \tag{2.5.1}$$

together with the initial conditions

$$y(a) = c_0, \qquad \frac{dy}{dt}(a) = c_1, \qquad \cdots \qquad , \frac{d^{n-1}y}{dt^{n-1}}(a) = c_{n-1}, \tag{2.5.2}$$

where Δ is a differential operator of order $n-1$, $t \in [a, b]$ is the independent variable, y is the dependent variable, c_i are constants. We convert the n-th order differential equation (2.5.1) to a system of first-order differential equations by letting

$$y_0 = y, \qquad y_1 = \frac{dy_0}{dt}, \qquad y_2 = \frac{dy_1}{dt}, \qquad \cdots \qquad , y_n = \frac{dy_{n-1}}{dt}. \tag{2.5.3}$$

Hence

$$y_n = \frac{d^n y}{dt^n} = \Delta y(t) = F(y_0, y_1, \cdots, y_{n-1}), \tag{2.5.4}$$

where $F(y_0, y_1, \cdots, y_{n-1})$ is $\Delta y(t)$ written in terms of $y_0, y_1, \cdots, y_{n-1}$. Equations in (2.5.3) and (2.5.4) form a system of first-order differential equations. The initial conditions (2.5.2) become

$$y_0(a) = c_0, \qquad y_1(a) = c_1, \qquad \cdots \qquad , y_{n-1}(a) = c_{n-1}. \tag{2.5.5}$$

The system of first-order differential equations (2.5.3), (2.5.4) with initial conditions (2.5.5) can be solved using methods such as the Euler method and Runge-Kutta methods [25]. In this thesis, we use the Python function scipy.integrate.odeinit [26] to solve the system (2.5.3), (2.5.4) and (2.5.5).

Example 2

Consider the initial value problem

$$\frac{d^2y}{dt^2} + 4y(t) = 0, \qquad y(0) = -2, \qquad \frac{dy}{dt}(0) = 0. \tag{2.5.6}$$

We convert (2.5.6) to a system of first-order differential equations by letting

$$y_0 = y(t), \qquad y_1 = \frac{dy}{dt}, \qquad y_2 = \frac{dy_1}{dt} = \frac{d^2y}{dt^2} = -4y_0. \tag{2.5.7}$$

The initial conditions become

$$y_0(0) = -2, \qquad y_1(0) = 0. \tag{2.5.8}$$

We have the system of ordinary differential equations and initial conditions

$$\frac{d}{dt}\begin{pmatrix} y_0 \\ \\ y_1 \end{pmatrix} = \begin{pmatrix} y_1 \\ \\ -4y_0 \end{pmatrix}, \qquad \begin{pmatrix} y_0 \\ \\ y_1 \end{pmatrix}(0) = \begin{pmatrix} -2 \\ \\ 0 \end{pmatrix}. \tag{2.5.9}$$

We use the Python function scipy.integrate.odeinit to solve the system (2.5.9).

The initial value problem (2.5.6) has an analytical solution, which can be obtained using the ansatz $y(t) = e^{\lambda t}$, where λ is a constant that is determined [27]. The analytical solution for (2.5.6) is

$$y(t) = -2\cos(2t). \tag{2.5.10}$$

Figure 2.1 shows the comparison between the numerical solution with the analytical solution. The two solutions agree.

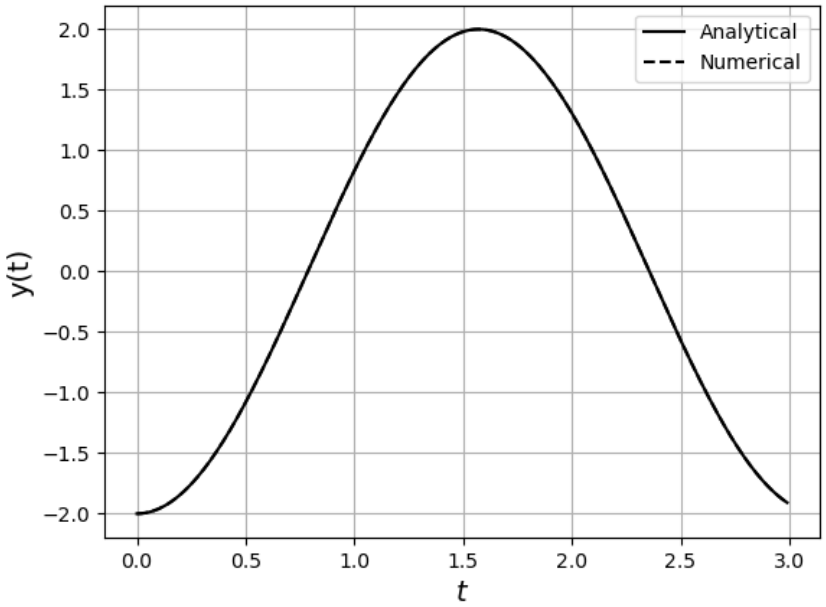

FIGURE 2.1: Numerical and analytical solutions of the initial value problem (2.5.6). The numerical solution is obtained by using the Python function scipy.integrate.odeinit.

2.5.2 Shooting method with conserved quantities as targets

Consider a second-order differential equation

$$\chi\left(\frac{d^2y}{dt^2}, \frac{dy}{dt}, y(t), t\right) = 0 \tag{2.5.11}$$

with an initial condition

$$y(a) = c_1 \tag{2.5.12}$$

and a conserved quantity

$$M = \Omega\left(t, y(t), \frac{dy}{dt}, a, b\right), \tag{2.5.13}$$

where $t \in [a, b]$ and M is a constant independent of t. We convert the boundary value problem to an initial value problem by using a shooting method. The target of the shooting method is the conserved quantity (2.5.13). Suppose

$$\frac{dy}{dt}(a) = \lambda. \tag{2.5.14}$$

We use the shooting method to find the value of λ so that (2.5.11), (2.5.12) and (2.5.14) form an initial value problem. Let

$$\phi(z) = \Omega\left(t, y_z(t), \frac{dy_z}{dt}, a, b\right), \tag{2.5.15}$$

where $y_z(t)$ is the solution of the initial value problem

$$\chi\left(\frac{d^2y}{dt^2}, \frac{dy}{dt}, y(t), t\right) = 0, \qquad y(a) = c_1, \qquad \frac{dy}{dt}(a) = z. \tag{2.5.16}$$

Hence

$$\phi(\lambda) = M. \tag{2.5.17}$$

We solve for λ in (2.5.17) by using a secant method. Suppose z_k is the current best guess for λ. We want to update this guess so that $\phi(z_{k+1})$ is approximately equal to M. That is,

$$\phi(z_k + \delta) \approx M, \tag{2.5.18}$$

where $z_{k+1} = z_k + \delta$. Using a Taylor series expansion to first order and solving for δ we obtain

$$\delta = \frac{M - \phi(z_k)}{\frac{d\phi}{dz}(z_k)}. \tag{2.5.19}$$

Using the secant method to estimate the derivative in (2.5.19) we have

$$\frac{d\phi}{dz}(z_k) = \frac{\phi(z_k) - \phi(z_{k-1})}{z_k - z_{k-1}} \tag{2.5.20}$$

and

$$\delta = \frac{M - \phi(z_k)}{\phi(z_k) - \phi(z_{k-1})}\,(z_k - z_{k-1})\,. \tag{2.5.21}$$

The value of z_k converges to λ as k increases.

We summarize the shooting method above as follows.

1. Guess the initial values z_0 and z_1.
2. Calculate the step δ using equation (2.5.21).
3. Update the current best guess by $z_{k+1} = z_k + \delta$.
4. Check whether $\phi(z_k) = M$ within a specified tolerance. If $\phi(z_k) \neq M$ within a specified tolerance, go to step 2. Otherwise, stop the search and $\lambda = z_k$.

Example 3

Consider the ordinary differential equation for the invariant solution of a two-dimensional turbulent jet flow [28]

$$\frac{d^2F}{d\xi^2} + F\frac{dF}{d\xi} = 0, \tag{2.5.22}$$

with boundary conditions

$$F(0) = 0, \qquad \frac{d^2F}{d\xi^2}(0) = 0, \qquad \frac{dF}{d\xi}(\infty) = 0, \tag{2.5.23}$$

the conserved quantity

$$J = 2\rho \int_0^\infty \left(\frac{dF}{d\xi}\right)^2 d\xi \tag{2.5.24}$$

and ρ is the density of the fluid. Suppose $\frac{J}{2\rho} = 1$.

The second boundary condition in (2.5.23) is identically satisfied when $\xi = 0$ in (2.5.22). To convert the boundary value problem (2.5.22) to (2.5.24) to an initial value

problem, we need to find the initial value of $\frac{dF}{d\xi}$. Let us assume that value is λ and use a shooting method to find it. We have the initial value problem

$$\frac{d^2F}{d\xi^2} + F\frac{dF}{d\xi} = 0, \qquad F(0) = 0, \qquad \frac{dF}{d\xi}(0) = \lambda, \tag{2.5.25}$$

with λ determined using a shooting method. Let

$$\phi(z) = \int_0^\infty \left(\frac{dF_z}{d\xi}\right)^2 d\xi, \tag{2.5.26}$$

where F_z is the solution of the initial value problem

$$\frac{d^2F}{d\xi^2} + F\frac{dF}{d\xi} = 0, \qquad F(0) = 0, \qquad \frac{dF}{d\xi}(0) = z. \tag{2.5.27}$$

The solution of the initial value problem (2.5.25) must satisfy the conserved quantity condition (2.5.24). That is,

$$\phi(\lambda) = 1. \tag{2.5.28}$$

To solve for λ in (2.5.28), we implement the shooting method described above. The steps are as follows:

1. Guess the initial values z_0 and z_1 for λ.
2. Calculate the step δ using
$$\delta = \frac{1 - \phi(z_k)}{\phi(z_k) - \phi(z_{k-1})}\left(z_k - z_{k-1}\right). \tag{2.5.29}$$
3. Update the current best guess by $z_{k+1} = z_k + \delta$.
4. Check whether $\phi(z_k) = 1$ within a specified tolerance. If $\phi(z_k) \neq 1$ within a specified tolerance, go to step 2. Otherwise, stop the search and $\lambda = z_k$.

We determine λ within the tolerance of 10^{-6}. The code for calculating λ is in the Appendix. Table 2.1 shows the values of current best guesses z_k and z_{k-1} and how they estimate the value of the conserved quantity. It can seen that the shooting method in this case converges after three iterations.

k	z_k	z_{k-1}	$\lvert 1 - \phi(z_k)\rvert$
1	1	1.2	0.2394
2	1.2	1.0386	0.0021
3	1.0386	1.0400	7×10^{-5}
4	1.0400	1.0400	2.7139×10^{-8}

TABLE 2.1: Iterations of the shooting method for the system (2.5.22) to (2.5.24) with $\frac{J}{2\rho} = 1$. The initial guesses are $z_0 = 1$ and $z_1 = 1.2$. The tolerance of the shooting method is set to 10^{-6}.

The analytical solution of the boundary value problem (2.5.22) to (2.5.24) is [29, 30]

$$F(\xi) = \left(\frac{3J}{2\rho}\right)^{1/3} \tanh\left(\frac{1}{2}\left(\frac{3J}{2\rho}\right)^{1/3}\xi\right). \tag{2.5.30}$$

Figure 2.2 shows the comparison between the analytical solution (2.5.28) with the numerical solution. The two solutions agree.

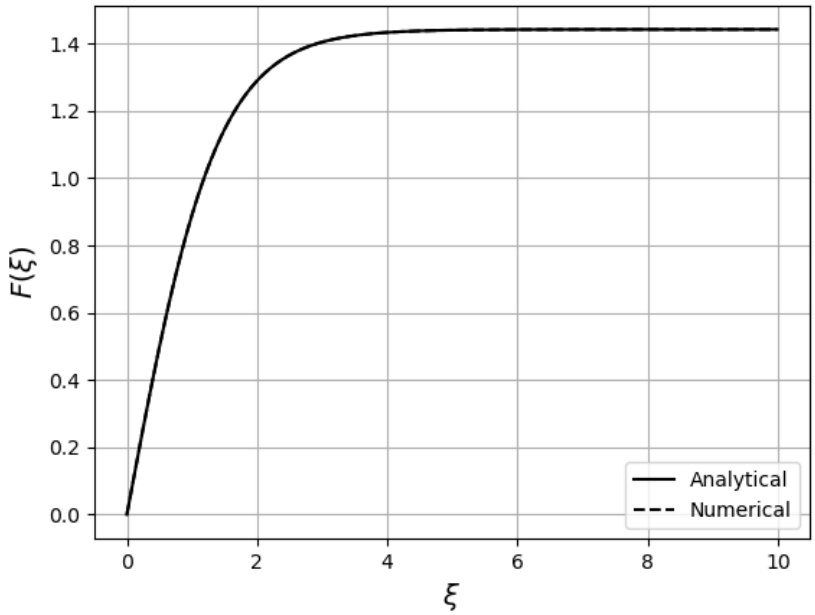

FIGURE 2.2: Numerical and analytical solutions of the boundary value problem (2.5.22) to (2.5.24). The numerical solution is obtained by using a numerical shooting method.

2.6 Summary

We have demonstrated the methods that are applied in this thesis using examples. The partial differential equation (2.2.10) was used to demonstrate the derivation of conserved vectors and conservation laws using the direct and multiplier methods. We used the conserved vectors we derived in Section 2.2 to demonstrate how to derive the associated Lie point symmetry and used this Lie symmetry to derive the invariant solutions. We described the shooting method that uses conserved quantities as targets.

Chapter 3

Mathematical model of the turbulent thermal wake

3.1 Introduction

Turbulent flow looks too irregular to be captured by any mathematical formulation and the Navier-Stokes equation is also too complicated to be solved directly. Reynolds [31] introduced a method of separating the mean flow from the seemingly random turbulent flow. Each variable is split into a mean and a fluctuation and the Navier-Stokes equation together with the energy balance equation are averaged. The fluctuations influence the mean flow variables leading to the turbulence closure problem [3]. The problem is resolved by using the Boussinesq approximation [3] and an analogy to Fourier's law for turbulent heat transfer [32]. These approximations introduce the eddy viscosity and the eddy thermal conductivity which are properties of the flow. This implies that the eddy viscosity and the eddy thermal conductivity may depend on the mean velocity and mean temperature of the fluid. Additional modelling for the eddy viscosity and eddy thermal conductivity is required. We consider Prandtl's mixing length hypothesis with its thermal extension [3, 32, 33] and von Kármán's similarity hypothesis. We also consider the model where the eddy viscosity and eddy thermal conductivity are dependent on spatial variables.

When a fluid flows past a physical boundary it forms a region referred to as the momentum boundary layer where viscosity plays an important role and a region where the effects of viscosity are negligible. If the physical boundary is hot, the fluid can form a region referred to as the thermal boundary layer where thermal conductivity plays an important role. The concept of a boundary layer was first introduced by Ludwig Prandtl [33]. The ratio of the thickness of the thermal boundary layer to the thickness of the momentum boundary layer is referred to as Prandtl number. Boundary layer theory simplifies the governing partial differential equations significantly. It has been demonstrated in the literature that boundary layer theory does not require a physical boundary. Boundary layer theory has been extensively applied to jet and wake problems [13, 12, 14, 15, 16]. In turbulent regions, the momentum and thermal boundary layers are larger than they would be without turbulence. This is because eddy viscosity and eddy thermal conductivity add to the effective viscosity and thermal conductivity. Hence the eddy viscosity and eddy thermal conductivity play an important role when simplifying the governing partial differential equations in the boundary layer.

This Chapter is structured in the following manner. In Section 3.2 we derive the two-dimensional closed Reynolds averaged Navier-Stokes equations and in Section 3.3 we derive the two-dimensional averaged energy balance equation. In Section 3.4 we make the far-wake approximation for the mean velocity deficit and mean temperature difference. In Sections 3.5 and 3.6 we use the boundary layer approximation to

further simplify the mean momentum and energy balance equations. In Sections 3.7 and 3.8 we discuss the concept of mixing lengths for estimating the eddy viscosity and eddy thermal conductivity of the flow. In Sections 3.9 and 3.10 we derive the boundary conditions for the mean velocity deficit and mean temperature difference for a turbulent thermal wake. In Section 3.11 we write the governing partial differential equations in terms of a stream function. In Section 3.12 we give a summary of the model for a classical thermal wake and momentumless thermal wake.

3.2 Averaged momentum equations

Consider a two-dimensional turbulent thermal far-wake. The fluid flows past an obstacle that is symmetrical with a constant mainstream velocity U. When the obstacle does not propel itself the wake behind the obstacle is referred to as a classical wake. When the obstacle propels itself so that the net force on the obstacle is zero the wake behind the obstacle is referred to as a momentumless wake. The fluid in the wake is the same as the fluid in the mainstream flow. The viscous effects are only significant within the boundary layer of the wake and the mainstream flow can be approximated as being inviscid. The x-axis is parallel to the mainstream velocity and passes through the obstacle. The origin of the coordinate system is at the trailing edge of the obstacle. The y-axis is perpendicular to the x-axis as shown in Figure 3.1 for the thermal classical wake and Figure 3.2 for the thermal momentumless wake. We also assume that the fluid is incompressible.

We study the region in the far-wake, where the difference between the mean velocity of the wake and the mainstream velocity U is small. The difference between the temperature and the ambient temperature T_b is also small.

The x and y components of the Navier-Stokes equation are

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}\right)=\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}, \tag{3.2.1}$$

$$\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}\right)=\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}, \tag{3.2.2}$$

where

$$\tau_{xx}=-p+\mu\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_x}{\partial x}\right), \tag{3.2.3}$$

$$\tau_{xy}=\mu\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right), \tag{3.2.4}$$

$$\tau_{yy}=-p+\mu\left(\frac{\partial v_y}{\partial y}+\frac{\partial v_y}{\partial y}\right), \tag{3.2.5}$$

are components of the stress tensor, $v_x = v_x(x,y,t)$, $v_y = v_y(x,y,t)$ are the components of the fluid velocity vector, $p(x,y,t)$ is the pressure in the fluid, ρ and μ are the density and viscosity of the fluid, respectively. Conservation of mass for an incompressible fluid gives

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0. \tag{3.2.6}$$

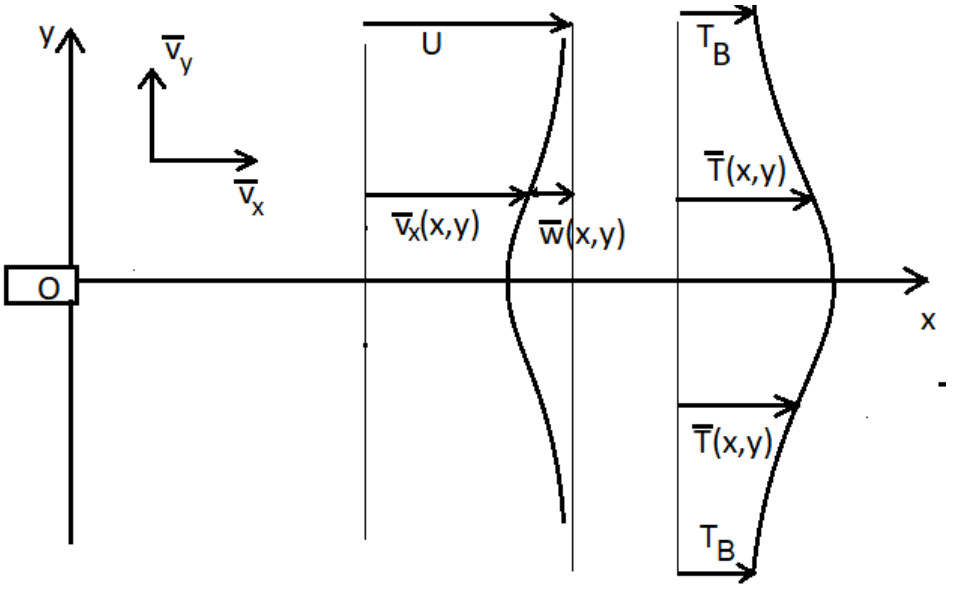

FIGURE 3.1: The mean velocity and the mean temperature profiles of a two-dimensional thermal turbulent classical wake.

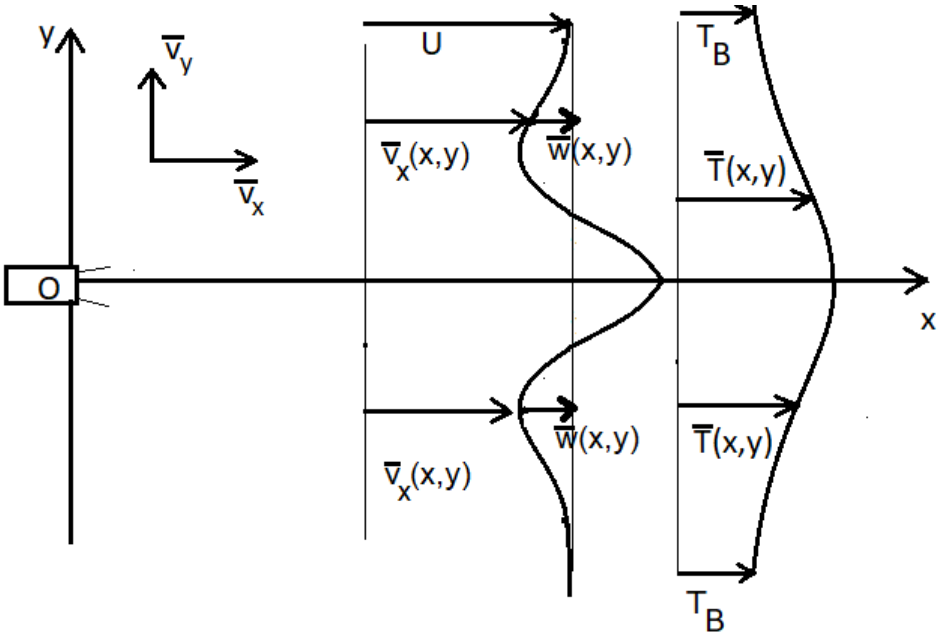

FIGURE 3.2: The mean velocity and the mean temperature profiles of a two-dimensional thermal turbulent momentumless wake.

To perform Reynolds averaging on (3.2.1) to (3.2.6), we write all dependent variables as a sum of the mean and a fluctuation. We assume that the mean velocity and mean temperature flow are independent of time. Such flows are called steady turbulent flows [32]. The flow variables may be written as

$$v_x(t,x,y) = \overline{v}_x(x,y) + v'_x(t,x,y), \tag{3.2.7}$$

$$v_y(t,x,y) = \overline{v}_y(x,y) + v'_y(t,x,y), \tag{3.2.8}$$

$$p(t,x,y) = \overline{p}(x,y) + p'(t,x,y), \tag{3.2.9}$$

$$T(t,x,y) = \overline{T}(x,y) + T'(t,x,y), \tag{3.2.10}$$

where the mean is defined by

$$\overline{f}(t,x,y) = \lim_{N\to\infty} \frac{1}{N} \int_t^{t+N} f(\tau,x,y)d\tau. \tag{3.2.11}$$

The fluctuations are about the mean and model turbulence in the flow. In this thesis, we assume that the mean of the flow variables are independent of time,

$$\lim_{N\to\infty} \frac{f(t+N,x,y)}{N} = 0 \tag{3.2.12}$$

for all flow variables f and $\frac{\partial f}{\partial t}$ is continuous with respect to t. Then the averaging operator has the following properties. The averaging operator is linear:

$$\overline{f+g} = \overline{f} + \overline{g}, \tag{3.2.13}$$

$$\overline{\lambda f} = \lambda \overline{f}, \tag{3.2.14}$$

for all λ independent of t and time-dependent functions f and g. The averaging operator is idempotent:

$$\overline{\overline{f}} = \overline{f}. \tag{3.2.15}$$

The average of the fluctuation variable is zero:

$$\overline{f'} = 0. \tag{3.2.16}$$

The averaging operator commutes with spatial differentiation:

$$\overline{\frac{\partial f}{\partial s}} = \frac{\partial \overline{f}}{\partial s}, \tag{3.2.17}$$

where s can be one of the variables, x, y. The average of the time derivative vanishes:

$$\overline{\frac{\partial f}{\partial t}} = 0. \tag{3.2.18}$$

The averaging operator does not necessarily distribute over multiplication:

$$\overline{fg} \neq \overline{f}\,\overline{g} \tag{3.2.19}$$

in general. These properties are proved in the Appendix. The decomposition of temperature (3.2.10) will be used in Section 3.3. Applying the averaging operator on equations (3.2.1) to (3.2.6) we obtain

$$\overline{v_x \frac{\partial v_x}{\partial x}} + \overline{v_y \frac{\partial v_x}{\partial y}} = \frac{\partial \overline{\tau}_{xx}}{\partial x} + \frac{\partial \overline{\tau}_{xy}}{\partial y}, \tag{3.2.20}$$

$$\overline{v_x \frac{\partial v_y}{\partial x}} + \overline{v_y \frac{\partial v_y}{\partial y}} = \frac{\partial \overline{\tau}_{xy}}{\partial x} + \frac{\partial \overline{\tau}_{yy}}{\partial y} \tag{3.2.21}$$

and

$$\frac{\partial \overline{v}_x}{\partial x} + \frac{\partial \overline{v}_y}{\partial y} = 0, \tag{3.2.22}$$

where

$$\overline{\tau}_{ij} = -\overline{p}\delta_{ij} + \mu\left(\frac{\partial \overline{v}_i}{\partial x_j} + \frac{\partial \overline{v}_j}{\partial x_i}\right). \tag{3.2.23}$$

Substituting (3.2.7) and (3.2.8) into (3.2.20) and (3.2.21) we obtain

$$\overline{\left(\overline{v}_x + v'_x\right)\frac{\partial}{\partial x}\left(\overline{v}_x + v'_x\right)} + \overline{\left(\overline{v}_y + v'_y\right)\frac{\partial}{\partial y}\left(\overline{v}_x + v'_x\right)} = \frac{\partial \overline{\tau}_{xx}}{\partial x} + \frac{\partial \overline{\tau}_{xy}}{\partial y}, \tag{3.2.24}$$

$$\overline{\left(\overline{v}_x + v'_x\right)\frac{\partial}{\partial x}\left(\overline{v}_y + v'_y\right)} + \overline{\left(\overline{v}_y + v'_y\right)\frac{\partial}{\partial y}\left(\overline{v}_y + v'_y\right)} = \frac{\partial \overline{\tau}_{xy}}{\partial x} + \frac{\partial \overline{\tau}_{yy}}{\partial y}. \tag{3.2.25}$$

Expanding the left hand sides of (3.2.24) and (3.2.25) gives

$$\overline{v}_x\frac{\partial \overline{v}_x}{\partial x} + \overline{v}_y\frac{\partial \overline{v}_x}{\partial y} + \overline{v'_x\frac{\partial v'_x}{\partial x}} + \overline{v'_y\frac{\partial v'_x}{\partial y}} = \frac{\partial \overline{\tau}_{xx}}{\partial x} + \frac{\partial \overline{\tau}_{xy}}{\partial y}, \tag{3.2.26}$$

$$\overline{v}_x\frac{\partial \overline{v}_y}{\partial x} + \overline{v}_y\frac{\partial \overline{v}_y}{\partial y} + \overline{v'_x\frac{\partial v'_y}{\partial x}} + \overline{v'_y\frac{\partial v'_y}{\partial y}} = \frac{\partial \overline{\tau}_{xy}}{\partial x} + \frac{\partial \overline{\tau}_{yy}}{\partial y}. \tag{3.2.27}$$

Using the identity for differentiation of a product

$$v'_j\frac{\partial v'_i}{\partial x_k} = \frac{\partial}{\partial x_k}\left(v'_i v'_j\right) - v'_i\frac{\partial v'_j}{\partial x_k} \tag{3.2.28}$$

to expand (3.2.26) and (3.2.27) we have

$$\begin{aligned}\overline{v}_x\frac{\partial \overline{v}_x}{\partial x} + \overline{v}_y\frac{\partial \overline{v}_x}{\partial y} + \frac{\partial}{\partial x}\left(\overline{v'_x v'_x}\right) + \frac{\partial}{\partial y}\left(\overline{v'_x v'_y}\right) - \overline{v'_x\left(\frac{\partial v'_x}{\partial x} + \frac{\partial v'_y}{\partial y}\right)} \\ = \frac{\partial \overline{\tau}_{xx}}{\partial x} + \frac{\partial \overline{\tau}_{xy}}{\partial y},\end{aligned} \tag{3.2.29}$$

$$\begin{aligned}\overline{v}_x\frac{\partial \overline{v}_y}{\partial x} + \overline{v}_y\frac{\partial \overline{v}_y}{\partial y} + \frac{\partial}{\partial x}\left(\overline{v'_x v'_y}\right) + \frac{\partial}{\partial y}\left(\overline{v'_y v'_y}\right) - \overline{v'_y\left(\frac{\partial v'_x}{\partial x} + \frac{\partial v'_y}{\partial y}\right)} \\ = \frac{\partial \overline{\tau}_{xy}}{\partial x} + \frac{\partial \overline{\tau}_{yy}}{\partial y}.\end{aligned} \tag{3.2.30}$$

Subtracting (3.2.22) from (3.2.6) we see that the fluctuating velocity also obeys the continuity equation:

$$\frac{\partial v'_x}{\partial x} + \frac{\partial v'_y}{\partial y} = 0. \tag{3.2.31}$$

Hence (3.2.29) and (3.2.30) simplify to

$$\overline{v}_x\frac{\partial \overline{v}_x}{\partial x} + \overline{v}_y\frac{\partial \overline{v}_x}{\partial y} = \frac{1}{\rho}\frac{\partial \overline{\tau}_{xx}}{\partial x} - \frac{\partial}{\partial x}\left(\overline{v'_x v'_x}\right) + \frac{1}{\rho}\frac{\partial \overline{\tau}_{xy}}{\partial y} - \frac{\partial}{\partial y}\left(\overline{v'_x v'_y}\right), \tag{3.2.32}$$

$$\overline{v}_x\frac{\partial \overline{v}_y}{\partial x} + \overline{v}_y\frac{\partial \overline{v}_y}{\partial y} = \frac{1}{\rho}\frac{\partial \overline{\tau}_{xy}}{\partial x} - \frac{\partial}{\partial x}\left(\overline{v'_x v'_y}\right) + \frac{1}{\rho}\frac{\partial \overline{\tau}_{yy}}{\partial y} - \frac{\partial}{\partial y}\left(\overline{v'_y v'_y}\right). \tag{3.2.33}$$

The Reynolds stresses $-\rho\overline{v'_i v'_j}$ add to the number of unknown variables. Hence there

are more unknown variables than the number of equations. This is known as the turbulence closure problem [3].

In 1877 Boussinesq related, for an incompressible fluid, the Reynolds stresses $-\rho\overline{v_i'v_j'}$ to the mean velocity gradients by the equation

$$-\rho\overline{v_i'v_j'} = -\frac{2}{3}\rho k\delta_{ij} + \mu_T\left(\frac{\partial\overline{v}_i}{\partial x_j} + \frac{\partial\overline{v}_j}{\partial x_i}\right), \tag{3.2.34}$$

where μ_T is referred to as the eddy viscosity, which is a property of the flow, and

$$k = \frac{1}{2}\overline{v_i'v_i'} \tag{3.2.35}$$

is the kinetic energy per unit mass of the fluctuations. Since for an incompressible fluid

$$\frac{\partial\overline{v}_k}{\partial x_k} = 0. \tag{3.2.36}$$

the trace of each side of (3.2.34) are equal as required.

Substituting (3.2.34) into (3.2.32) and (3.2.33) we have

$$\begin{aligned}\overline{v}_x\frac{\partial\overline{v}_x}{\partial x} + \overline{v}_y\frac{\partial\overline{v}_x}{\partial y} = &-\frac{1}{\rho}\frac{\partial}{\partial x}\left(\overline{p} + \frac{2}{3}\rho k\right) + \frac{1}{\rho}\frac{\partial}{\partial x}\left[(\mu+\mu_T)\left(\frac{\partial\overline{v}_x}{\partial x} + \frac{\partial\overline{v}_x}{\partial x}\right)\right] \\ &+\frac{1}{\rho}\frac{\partial}{\partial y}\left[(\mu+\mu_T)\left(\frac{\partial\overline{v}_x}{\partial y} + \frac{\partial\overline{v}_y}{\partial x}\right)\right],\end{aligned} \tag{3.2.37}$$

$$\begin{aligned}\overline{v}_x\frac{\partial\overline{v}_y}{\partial x} + \overline{v}_y\frac{\partial\overline{v}_y}{\partial y} = &-\frac{1}{\rho}\frac{\partial}{\partial y}\left(\overline{p} + \frac{2}{3}\rho k\right) + \frac{1}{\rho}\frac{\partial}{\partial x}\left[(\mu+\mu_T)\left(\frac{\partial\overline{v}_x}{\partial y} + \frac{\partial\overline{v}_y}{\partial x}\right)\right] \\ &+\frac{1}{\rho}\frac{\partial}{\partial y}\left[(\mu+\mu_T)\left(\frac{\partial\overline{v}_y}{\partial y} + \frac{\partial\overline{v}_y}{\partial y}\right)\right],\end{aligned} \tag{3.2.38}$$

where $\overline{p} + \frac{2}{3}\rho k$ is the effective pressure in the mean flow and $\mu + \mu_T$ is the effective viscosity of the mean flow.

3.3 Averaged energy balanced equation

The energy balance equation without viscous terms is

$$\rho c_v\left(\frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial x}\left(\kappa\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right), \tag{3.3.1}$$

where $T = T(t,x,y)$ is the temperature, κ is the thermal conductivity of the fluid and c_v is the specific heat capacity at constant volume. We average the energy balance equation by substituting (3.2.7), (3.2.8) and (3.2.10) into (3.3.1) and then applying the averaging operator to obtain

$$\rho c_v\left[\overline{v}_x\frac{\partial\overline{T}}{\partial x} + \overline{v}_y\frac{\partial\overline{T}}{\partial y} + \overline{v_x'\frac{\partial T'}{\partial x}} + \overline{v_y'\frac{\partial T'}{\partial y}}\right] = \frac{\partial}{\partial x}\left(\kappa\frac{\partial\overline{T}}{\partial x}\right) + \frac{\partial}{\partial y}\left(\kappa\frac{\partial\overline{T}}{\partial y}\right). \tag{3.3.2}$$

Using the identity

$$v_i'\frac{\partial T'}{\partial x_j} = \frac{\partial}{\partial x_j}\left(v_i'T'\right) - T'\frac{\partial v_i'}{\partial x_j} \tag{3.3.3}$$

to expand the third and fourth terms of the left-hand side of equation (3.3.2) we have

$$\begin{aligned}\rho c_v \left[\overline{v}_x \frac{\partial \overline{T}}{\partial x} + \overline{v}_y \frac{\partial \overline{T}}{\partial y} + \frac{\partial}{\partial x}\left(\overline{v_x' T'}\right) + \frac{\partial}{\partial y}\left(\overline{v_y' T'}\right) - \overline{T'\left(\frac{\partial v_x'}{\partial x} + \frac{\partial v_y'}{\partial y}\right)}\right] \\ = \frac{\partial}{\partial x}\left(\kappa \frac{\partial \overline{T}}{\partial x}\right) + \frac{\partial}{\partial y}\left(\kappa \frac{\partial \overline{T}}{\partial y}\right).\end{aligned} \tag{3.3.4}$$

But by (3.2.31) the last term of the left hand side of equation (3.3.4) must vanish, hence (3.3.4) becomes

$$\begin{aligned}\rho c_v \left[\overline{v}_x \frac{\partial \overline{T}}{\partial x} + \overline{v}_y \frac{\partial \overline{T}}{\partial y}\right] = \frac{\partial}{\partial x}\left(\kappa \frac{\partial \overline{T}}{\partial x}\right) - \rho c_v \frac{\partial}{\partial x}\left(\overline{v_x' T'}\right) \\ + \frac{\partial}{\partial y}\left(\kappa \frac{\partial \overline{T}}{\partial y}\right) - \rho c_v \frac{\partial}{\partial y}\left(\overline{v_y' T'}\right).\end{aligned} \tag{3.3.5}$$

The analogy with Fourier's law of heat conduction [32] also applies

$$\rho c_v \overline{v_i' T'} = -\kappa_T \frac{\partial \overline{T}}{\partial x_i}. \tag{3.3.6}$$

Here κ_T is defined as the eddy thermal conductivity, which is a property of the flow. Hence equation (3.3.5) becomes

$$\rho c_v \left[\overline{v}_x \frac{\partial \overline{T}}{\partial x} + \overline{v}_y \frac{\partial \overline{T}}{\partial y}\right] = \frac{\partial}{\partial x}\left[\left(\kappa + \kappa_T\right) \frac{\partial \overline{T}}{\partial x}\right] + \frac{\partial}{\partial y}\left[\left(\kappa + \kappa_T\right) \frac{\partial \overline{T}}{\partial y}\right]. \tag{3.3.7}$$

The sum $\kappa + \kappa_T$ is referred to as the effective thermal conductivity of the mean flow.

3.4 Model of the far-wake

We refer to Figures 3.1 and 3.2 and define the mean velocity deficit, $\overline{w}(x, y)$, by

$$\overline{v}_x(x, y) = U - \overline{w}(x, y). \tag{3.4.1}$$

For the far-wake, we assume that

$$\overline{w} << U, \qquad \overline{v}_y << U. \tag{3.4.2}$$

We neglect terms containing second-order products of $\overline{w}$ and $\overline{v}_y$. Substituting (3.4.1) into (3.2.37) and (3.2.38) and neglecting second-order terms in $\overline{w}$ and $\overline{v}_y$ we obtain

$$\begin{aligned}-U \frac{\partial \overline{w}}{\partial x} = -\frac{1}{\rho}\frac{\partial}{\partial x}\left(\overline{p} + \frac{2}{3}\rho k\right) - \frac{1}{\rho}\frac{\partial}{\partial x}\left[\left(\mu + \mu_T\right)\left(\frac{\partial \overline{w}}{\partial x} + \frac{\partial \overline{w}}{\partial x}\right)\right] \\ + \frac{1}{\rho}\frac{\partial}{\partial y}\left[\left(\mu + \mu_T\right)\left(-\frac{\partial \overline{w}}{\partial y} + \frac{\partial \overline{v}_y}{\partial x}\right)\right],\end{aligned} \tag{3.4.3}$$

$$\begin{aligned}U \frac{\partial \overline{v}_y}{\partial x} = -\frac{1}{\rho}\frac{\partial}{\partial y}\left(\overline{p} + \frac{2}{3}\rho k\right) + \frac{1}{\rho}\frac{\partial}{\partial x}\left[\left(\mu + \mu_T\right)\left(-\frac{\partial \overline{w}}{\partial y} + \frac{\partial \overline{v}_y}{\partial x}\right)\right] \\ + \frac{1}{\rho}\frac{\partial}{\partial y}\left[\left(\mu + \mu_T\right)\left(\frac{\partial \overline{v}_y}{\partial y} + \frac{\partial \overline{v}_y}{\partial y}\right)\right].\end{aligned} \tag{3.4.4}$$

Substituting (3.4.1) into the continuity equation (3.2.22) we have

$$-\frac{\partial \overline{w}}{\partial x}+\frac{\partial \overline{v}_y}{\partial y}=0. \tag{3.4.5}$$

We also assume that in the far-wake the temperature difference

$$\overline{S}(x,y)=\overline{T}(x,y)-T_b, \tag{3.4.6}$$

where T_b is the ambient temperature. The temperature difference, $\overline{S}(x,y)$, is small compared to the ambient temperature T_b. Writing the energy balance equation (3.3.7) in terms of the temperature difference and velocity deficit, $\overline{w}$, we have

$$\begin{aligned}\rho c_v\left[U\frac{\partial \overline{S}}{\partial x}-\overline{w}\frac{\partial \overline{S}}{\partial x}+\overline{v}_y\frac{\partial \overline{S}}{\partial y}\right]&=\frac{\partial}{\partial x}\left[(\kappa+\kappa_T)\frac{\partial \overline{S}}{\partial x}\right]\\&+\frac{\partial}{\partial y}\left[(\kappa+\kappa_T)\frac{\partial \overline{S}}{\partial y}\right].\end{aligned} \tag{3.4.7}$$

Neglecting terms that are second order in $\overline{w}$, $\overline{v}_y$ and $\overline{S}$ we obtain

$$\rho c_v U\frac{\partial \overline{S}}{\partial x}=\frac{\partial}{\partial x}\left[(\kappa+\kappa_T)\frac{\partial \overline{S}}{\partial x}\right]+\frac{\partial}{\partial y}\left[(\kappa+\kappa_T)\frac{\partial \overline{S}}{\partial y}\right]. \tag{3.4.8}$$

3.5 Momentum boundary layer theory

Although jets and wakes do not have a solid boundary, boundary layer theory has been applied to these types of flow in the literature [13, 12, 14, 15, 16]. Boundary layer theory only requires that a variable changes rapidly along one direction compared to a perpendicular direction. For turbulent thermal wakes, the x-component of the mean velocity and the mean temperature difference changes rapidly along the y direction. This implies that the characteristic lengths in the y direction for mean velocity and mean temperature difference are very small compared to the characteristic length in the x direction.

Let U_0 be the characteristic mean velocity in the x direction, L be the characteristic length in the x direction and δ_V be the momentum boundary layer thickness in the y direction. The length L is the distance along the x direction from the obstacle to the point at which the x-component of the mean velocity becomes negligible. To satisfy the continuity equation for the mean flow, the characteristic mean velocity in the y direction must be $\frac{\delta_V}{L}U_0$. In order to ensure that the effective pressure gradient remains important in the averaged momentum equation we balance it with the inertia term and take the characteristic effective pressure to be ρU_0^2. Let ν_0 be the characteristic effective kinematic viscosity.

Hence we non-dimensionalize the momentum equation using the dimensionless variables

$$x^*=\frac{x}{L}, \qquad y^*=\frac{y}{\delta_V}, \tag{3.5.1}$$

$$\nu^*=\frac{\mu+\mu_T}{\rho\nu_0}, \qquad \overline{P}=\frac{\overline{p}+\frac{2}{3}\rho k}{\rho U_0^2}, \qquad \overline{w}^*=\frac{\overline{w}}{U_0}, \qquad \overline{v}_y^*=\frac{L}{\delta_V}\frac{\overline{v}_y}{U_0}. \tag{3.5.2}$$

Substituting these dimensionless variables into (3.4.3) and (3.4.4) we have

$$\begin{aligned} -\frac{\partial \overline{w}^*}{\partial x^*} = &-\frac{\partial \overline{P}^*}{\partial x^*} - \frac{\nu_0}{U_0 L}\frac{\partial}{\partial x^*}\left[\nu^*\left(\frac{\partial \overline{w}^*}{\partial x^*} + \frac{\partial \overline{w}^*}{\partial x^*}\right)\right] \\ &+\frac{\partial}{\partial y^*}\left[\nu^*\left(-\frac{\nu_0}{U_0 L}\frac{L^2}{\delta_V^2}\frac{\partial \overline{w}^*}{\partial y^*} + \frac{\nu_0}{U_0 L}\frac{\partial v_y^*}{\partial x^*}\right)\right], \end{aligned} \tag{3.5.3}$$

$$\begin{aligned} \frac{\delta_V^2}{L^2}\frac{\partial \overline{v}_y^*}{\partial x^*} = &-\frac{\partial \overline{P}^*}{\partial y^*} + \frac{\partial}{\partial x^*}\left[\nu^*\left(-\frac{\nu_0}{U_0 L}\frac{\partial \overline{w}^*}{\partial y^*} + \frac{\nu_0}{U_0 L}\frac{\delta_V^2}{L^2}\frac{\partial \overline{v}_y^*}{\partial x^*}\right)\right] \\ &+\frac{\nu_0}{U_0 L}\frac{\partial}{\partial y^*}\left[\nu^*\left(\frac{\partial \overline{v}_y^*}{\partial y^*} + \frac{\partial \overline{v}_y^*}{\partial y^*}\right)\right]. \end{aligned} \tag{3.5.4}$$

The shear strain gradient along the y-axis is an important component of the dynamics of the fluid flow in wakes and jets and the x-component of the velocity changes rapidly along the y-axis. Therefore the coefficient of the gradient of the mean velocity deficit $\frac{\partial \overline{w}^*}{\partial y^*}$ in equation (3.5.3) has to be order 1. We therefore choose

$$\frac{\delta_V}{L} = \sqrt{\frac{\nu_0}{U_0 L}}. \tag{3.5.5}$$

The dimensionless number

$$Re_t = \frac{U_0 L}{\nu_0} \tag{3.5.6}$$

is the ratio of the inertial force and the effective viscous force. The effective viscous force is the sum of viscous force inherent in the fluid and viscous force generated by turbulence. This ratio is referred to as the turbulent Reynolds number and when ν_0 is only the kinematic viscosity of the fluid, the fluid is laminar and it is referred to simply as the Reynolds number, Re. Clearly

$$Re \geq Re_T. \tag{3.5.7}$$

For a turbulent boundary layer to form we require that

$$\frac{\delta_V}{L} << 1. \tag{3.5.8}$$

Therefore a boundary layer forms when

$$\sqrt{Re_T} >> 1, \tag{3.5.9}$$

by (3.5.5). The condition (3.5.9) is stronger than the well-known condition that $\sqrt{Re} >> 1$ for a laminar boundary layer to form, since $\sqrt{Re}$ is larger than $\sqrt{Re_T}$.

Writing (3.5.3) and (3.5.4) in terms of the turbulent Reynolds number we have

$$\begin{aligned} -\frac{\partial \overline{w}^*}{\partial x^*} = &-\frac{\partial \overline{P}^*}{\partial x^*} - \frac{1}{Re_T}\frac{\partial}{\partial x^*}\left[\nu^*\left(\frac{\partial \overline{w}^*}{\partial x^*} + \frac{\partial \overline{w}^*}{\partial x^*}\right)\right] \\ &+\frac{\partial}{\partial y^*}\left[\nu^*\left(-\frac{\partial \overline{w}^*}{\partial y^*} + \frac{1}{Re_T}\frac{\partial v_y^*}{\partial x^*}\right)\right], \end{aligned} \tag{3.5.10}$$

$$\frac{1}{Re_T}\frac{\partial \overline{v}_y^*}{\partial x^*} = -\frac{\partial \overline{P}^*}{\partial y^*} + \frac{\partial}{\partial x^*}\left[\nu^*\left(-\frac{1}{Re_T}\frac{\partial \overline{w}^*}{\partial y^*} + \frac{1}{Re_T^2}\frac{\partial \overline{v}_y^*}{\partial x^*}\right)\right] + \frac{1}{Re_T}\frac{\partial}{\partial y^*}\left[\nu^*\left(\frac{\partial \overline{v}_y^*}{\partial y^*} + \frac{\partial \overline{v}_y^*}{\partial y^*}\right)\right]. \tag{3.5.11}$$

Neglecting terms of order $\frac{1}{Re_T}$, equations (3.5.10) and (3.5.11) become

$$-\frac{\partial \overline{w}^*}{\partial x^*} = -\frac{\partial \overline{P}^*}{\partial x^*} - \frac{\partial}{\partial y^*}\left[\nu^*\frac{\partial \overline{w}^*}{\partial y^*}\right], \tag{3.5.12}$$

$$0 = \frac{\partial \overline{P}^*}{\partial y^*}. \tag{3.5.13}$$

Rewriting (3.5.12) and (3.5.13) in terms of the original variables we have

$$-U\frac{\partial \overline{w}}{\partial x} = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x} - \frac{\partial}{\partial y}\left[\frac{(\mu+\mu_T)}{\rho}\frac{\partial \overline{w}}{\partial y}\right], \tag{3.5.14}$$

$$0 = \frac{\partial \overline{P}}{\partial y}, \tag{3.5.15}$$

where

$$\overline{P} = \overline{p} + \frac{2}{3}\rho k. \tag{3.5.16}$$

The velocity of the fluid is constant for the inviscid laminar mainstream flow. Therefore from Euler's equation for the mainstream flow outside the boundary layer

$$\frac{d\overline{P}}{dx} = \frac{dp}{dx} = -\rho U\frac{dU}{dx} = 0. \tag{3.5.17}$$

This result is also applicable inside the boundary of the wake since $\overline{P}$ is independent of y, according to (3.5.15). Therefore the effective pressure is constant and

$$U\frac{\partial \overline{w}}{\partial x} = \frac{\partial}{\partial y}\left[\frac{(\mu+\mu_T)}{\rho}\frac{\partial \overline{w}}{\partial y}\right]. \tag{3.5.18}$$

We see that the wake is driven/retarded only by the mean shear strain gradient along the y-direction. The mean velocity deficit satisfies a non-linear diffusion equation, since μ_T may depend on the mean velocity deficit.

3.6 Thermal boundary layer theory

In a turbulent thermal wake, the mean temperature difference changes more rapidly along the y-direction than along the x-direction. Hence we can apply boundary layer theory to the mean temperature difference.

Let δ_θ be the thermal boundary layer thickness in the y-direction. When differentiating the mean temperature difference with respect to y we scale y with δ_θ and when differentiating the mean velocity with respect to y we scale y with δ_V. We assume that the thermal characteristic length for change in the x-direction is the same as that of the momentum characteristic length, L. Let S_0 be the characteristic mean temperature difference and α_0 be the characteristic effective thermal diffusivity.

The effective diffusivity is

$$E_\theta = \frac{\kappa + \kappa_T}{\rho c_v}. \tag{3.6.1}$$

Hence we have the dimensionless variables

$$x^* = \frac{x}{L}, \qquad y^* = \frac{y}{\delta_V} \text{ or } \frac{y}{\delta_\theta}, \tag{3.6.2}$$

$$E_\theta^* = \frac{\kappa + \kappa_T}{\rho c_v \alpha_0}, \qquad \overline{S}^* = \frac{\overline{S}}{S_0}, \qquad \overline{w}^* = \frac{\overline{w}}{U_0}, \qquad \overline{v}_y^* = \frac{L}{\delta_V}\frac{\overline{v}_y}{U_0}. \tag{3.6.3}$$

Substituting the dimensionless variables into (3.4.8) we obtain

$$\frac{\partial S^*}{\partial x^*} = \frac{\alpha_0}{U_0 L}\frac{\partial}{\partial x^*}\left(E_\theta^*\frac{\partial S^*}{\partial x^*}\right) + \frac{\alpha_0}{U_0 L}\frac{L^2}{\delta_\theta^2}\frac{\partial}{\partial y^*}\left(E_\theta^*\frac{\partial S^*}{\partial y^*}\right). \tag{3.6.4}$$

But

$$\frac{\alpha_0}{U_0 L} = \frac{\nu_0}{U_0 L}\frac{\alpha_0}{\nu_0} = \frac{1}{Re_T}\frac{1}{Pr_T}, \tag{3.6.5}$$

where the ratio

$$Pr_T = \frac{\nu_0}{\alpha_0} \tag{3.6.6}$$

is the turbulent Prandtl number. The turbulent Prandtl number is the ratio of the momentum diffusivity to the thermal diffusivity. Hence

$$\frac{\partial S^*}{\partial x^*} = \frac{1}{Re_T}\frac{1}{Pr_T}\frac{\partial}{\partial x^*}\left(E_\theta^*\frac{\partial S^*}{\partial x^*}\right) + \frac{1}{Re_T}\frac{1}{Pr_T}\frac{L^2}{\delta_\theta^2}\frac{\partial}{\partial y^*}\left(E_\theta^*\frac{\partial S^*}{\partial y^*}\right). \tag{3.6.7}$$

The mean temperature difference changes more rapidly along y-axis than along the x-axis. Hence the second term on the right-hand side of (3.6.7) is important and cannot be neglected. Its coefficient must therefore be of order one. We therefore choose

$$\frac{1}{Re_T}\frac{1}{Pr_T}\frac{L^2}{\delta_\theta^2} = 1. \tag{3.6.8}$$

Hence

$$\delta_\theta = \frac{L}{\sqrt{Re_T Pr_T}}. \tag{3.6.9}$$

From (3.5.5) and (3.5.6) we see that

$$\sqrt{Re_T} = \frac{L}{\delta_V}. \tag{3.6.10}$$

Substituting (3.6.10) into (3.6.9) gives

$$\frac{\delta_V}{\delta_\theta} = \sqrt{Pr_T}. \tag{3.6.11}$$

Using (3.6.8), equation (3.6.7) becomes

$$\frac{\partial S^*}{\partial x^*} = \frac{1}{Re_T}\frac{1}{Pr_T}\frac{\partial}{\partial x^*}\left(E_\theta^*\frac{\partial S^*}{\partial x^*}\right) + \frac{\partial}{\partial y^*}\left(E_\theta^*\frac{\partial S^*}{\partial y^*}\right). \tag{3.6.12}$$

If we assume that the ratio δ_V/δ_θ is of order one then Pr_T is also of order one. Therefore neglecting terms of order $\frac{1}{Re_T}$ equation (3.6.12) reduces to

$$\frac{\partial S^*}{\partial x^*} = \frac{\partial}{\partial y^*}\left(E_\theta^* \frac{\partial S^*}{\partial y^*}\right). \tag{3.6.13}$$

Rewriting (3.6.13) in terms of the original dimensional variables we have

$$U\frac{\partial \overline{S}}{\partial x} = \frac{\partial}{\partial y}\left[\frac{\kappa + \kappa_T}{\rho c_v}\frac{\partial \overline{S}}{\partial y}\right]. \tag{3.6.14}$$

3.7 Momentum mixing length

The mixing lengths for momentum transfer and heat transfer have been considered by Schlichting and Gerstern [10]. The idea was however first suggested by Ludwig Prandtl to estimate Reynolds stresses. In this Section, we derive the expression for the eddy viscosity in terms of the momentum mixing length for the turbulent thermal wake.

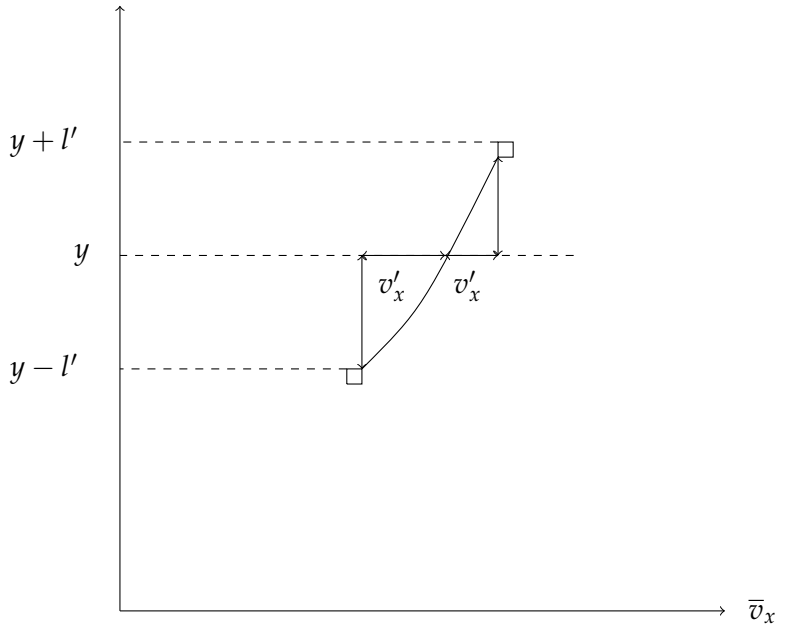

FIGURE 3.3: A fluid lump fluctuating above and below the mean flow before being carried by the mean flow.

Consider the mean flow parallel to the x-axis (Figure 3.3) with

$$\overline{v}_x = U - \overline{w}(x,y), \qquad \overline{v}_y = 0, \qquad \overline{v}_z = 0. \tag{3.7.1}$$

Then

$$\tau_T = \tau_{xy} = -\rho\overline{v'_x v'_y} = \mu_T\left(\frac{\partial \overline{v}_x}{\partial y} + \frac{\partial \overline{v}_y}{\partial x}\right) = -\mu_T\frac{\partial \overline{w}}{\partial y}. \tag{3.7.2}$$

As the fluid lump shown in Figure 3.3 travels from level $y + l'$ to level y it retains its momentum in the x-direction and therefore retains also its velocity in the x-direction. Thus

$$v'_x(x,y) = \overline{v}_x(x, y + l') - \overline{v}_x(x,y). \tag{3.7.3}$$

Expanding $\overline{v}_x(x, y + l')$ using the Taylor series expansion to first order, equation (3.7.3) becomes

$$v'_x(x, y) = l' \frac{\partial \overline{v}_x}{\partial y}(x, y). \tag{3.7.4}$$

Prandtl assumed that v'_y and v'_x are of the same order of magnitude. This assumption is consistent with the conservation of mass equation for the fluctuations,

$$\frac{\partial v'_x}{\partial x} + \frac{\partial v'_y}{\partial y} = 0. \tag{3.7.5}$$

Fluid lumps which arrive at level y downwards from above have $v'_y < 0$ and $v'_x > 0$ while the fluid lumps which arrive at level y upwards from below have $v'_y > 0$ and $v'_x < 0$. Hence v'_x and v'_y always have opposite sign and

$$v'_y(x, y) = -cl' \frac{\partial \overline{v}_x}{\partial y}, \tag{3.7.6}$$

where c is a positive constant. Thus

$$\tau_T = -\rho \overline{v'_x v'_y} = \rho l^2 \left(\frac{\partial \overline{v}_x}{\partial y} \right)^2, \tag{3.7.7}$$

where

$$l^2 = c\overline{l'l'}. \tag{3.7.8}$$

The characteristic length l is referred to as the momentum mixing length. But τ_T must have the opposite sign to $\frac{\partial \overline{v}_x}{\partial y}$ and it is therefore correct to write

$$\tau_T = \rho l^2 \left| \frac{\partial \overline{v}_x}{\partial y} \right| \frac{\partial \overline{v}_x}{\partial y} = -\rho l^2 \left| \frac{\partial \overline{w}}{\partial y} \right| \frac{\partial \overline{w}}{\partial y}. \tag{3.7.9}$$

Comparison of (3.7.9) with (3.7.2) gives

$$\nu_T = \frac{\mu_T}{\rho} = l^2 \left| \frac{\partial \overline{w}}{\partial y} \right|, \tag{3.7.10}$$

where ν_T is the kinematic eddy viscosity. Substituting equation (3.7.10) into (3.5.18) we obtain

$$U \frac{\partial \overline{w}}{\partial x} = \frac{\partial}{\partial y} \left[\left(\nu + l^2 \left| \frac{\partial \overline{w}}{\partial y} \right| \right) \frac{\partial \overline{w}}{\partial y} \right], \tag{3.7.11}$$

where

$$\nu = \frac{\mu}{\rho} \tag{3.7.12}$$

is the kinematic viscosity of the fluid.

3.8 Thermal mixing length

The y-component of the turbulent heat flux vector is, from (3.3.6),

$$h_\theta = -\kappa_T \frac{\partial \overline{S}}{\partial y} = \rho c_v \overline{S' v'_y}. \tag{3.8.1}$$

We use dimensional analysis to obtain $\frac{h_\theta}{\rho c_v}$ by assuming

$$\frac{h_\theta}{\rho c_v} = \left(\frac{|\tau_T|}{\rho}\right)^a \left(\left|\frac{\partial \overline{S}}{\partial y}\right|\right)^b l_\theta^c, \tag{3.8.2}$$

where the exponents a, b and c have to be determined and l_θ is referred to as the thermal mixing length.

Note that $\frac{|\tau_T|}{\rho}$ is proportional to the friction between mean fluid sheets as they move relative to each other and hence contributes to heat transfer in the thermal wake. Also, $\left|\frac{\partial \overline{S}}{\partial y}\right|$ is proportional to the y-component of the mean heat flux and contributes to the turbulent mixing caused by temperature fluctuations.

Let θ be the dimension of temperature, T be the dimension of time and L be the dimension of length. The square brackets indicate the dimension of the quantity enclosed by them. Therefore

$$\left[\frac{h_\theta}{\rho c_v}\right] = \left[\overline{v_y' S'}\right] = \frac{L\theta}{T}, \tag{3.8.3}$$

$$\left[\frac{\tau_T}{\rho}\right] = \left[\overline{v_x' v_y'}\right] = \frac{L^2}{T^2}, \tag{3.8.4}$$

$$\left[\frac{\partial \overline{S}}{\partial y}\right] = \frac{\theta}{L}, \tag{3.8.5}$$

$$[l_\theta] = L. \tag{3.8.6}$$

Substituting these dimensions into (3.8.2) we have

$$L\theta T^{-1} = \left(\frac{L^2}{T^2}\right)^a \left(\frac{\theta}{L}\right)^b L^c. \tag{3.8.7}$$

Equating the exponents of L, T and θ we have

$$L : 1 = 2a - b + c, \tag{3.8.8}$$

$$T : -1 = -2a, \tag{3.8.9}$$

$$\theta : 1 = b, \tag{3.8.10}$$

which gives

$$a = \frac{1}{2}, \qquad b = 1, \qquad c = 1. \tag{3.8.11}$$

Hence we have the relation

$$\frac{h_\theta}{\rho c_v} = C \left(\frac{|\tau_T|}{\rho}\right)^{\frac{1}{2}} \left|\frac{\partial \overline{S}}{\partial y}\right| l_\theta, \tag{3.8.12}$$

where C is a constant. We define $l_\theta^* = Cl_\theta$ and suppress the star. Hence

$$\frac{h_\theta}{\rho c_v} = \left(\frac{|\tau_T|}{\rho}\right)\left|\frac{\partial \overline{S}}{\partial y}\right| l_\theta. \tag{3.8.13}$$

To proceed further consider the upper half of the wake, $y > 0$. The mean temperature difference is greatest at the centreline of the wake and decreases to zero toward the boundary of the wake. Hence

$$\frac{\partial \overline{S}}{\partial y} < 0, \qquad y > 0 \tag{3.8.14}$$

and

$$\frac{h_\theta}{\rho c_v} = -l_\theta \left(\frac{|\tau_T|}{\rho}\right)^{\frac{1}{2}} \frac{\partial \overline{S}}{\partial y}, \qquad y > 0. \tag{3.8.15}$$

But from (3.7.9)

$$\frac{|\tau_T|}{\rho} = l^2 \left|\frac{\partial \overline{w}}{\partial y}\right|^2 \tag{3.8.16}$$

and therefore

$$\frac{h_\theta}{\rho c_v} = -l_\theta l \left|\frac{\partial \overline{w}}{\partial y}\right| \frac{\partial \overline{S}}{\partial y}, \qquad y > 0. \tag{3.8.17}$$

Matching (3.8.17) with (3.8.1) we have

$$\frac{\kappa_T}{\rho c_v} = l_\theta l \left|\frac{\partial \overline{w}}{\partial y}\right|. \tag{3.8.18}$$

Substituting equation (3.8.18) into (3.6.14) we obtain

$$U\frac{\partial \overline{S}}{\partial x} = \frac{\partial}{\partial y}\left[\left(\frac{\kappa}{\rho c_v} + l_\theta l\left|\frac{\partial \overline{w}}{\partial y}\right|\right)\frac{\partial \overline{S}}{\partial y}\right]. \tag{3.8.19}$$

The eddy thermal conductivity, κ_T, is zero when $\frac{\partial \overline{w}}{\partial y}$ is zero, which is zero at the momentum boundary of the wake. Hence κ_T is zero at the momentum boundary of the wake and the thermal boundary should be smaller or equal to the momentum boundary. For a classical wake, the mean thermal and mean momentum profiles share a single boundary.

3.9 Momentum boundary conditions

Since the wake is symmetrical about the centreline, $y = 0$, and we assume that cavities do not form,

$$\overline{v}_y(x, 0) = 0. \tag{3.9.1}$$

The mean velocity deficit is maximum at the centreline, $y = 0$. Hence

$$\frac{\partial \overline{w}}{\partial y}(x, 0) = 0. \tag{3.9.2}$$

The mean velocity deficit $\overline{w}$ vanishes at the boundary of the wake. That is

$$\overline{w}(x, y_B(x)) = 0. \tag{3.9.3}$$

The gradient of the mean velocity deficit in the y-direction vanishes at the boundary because the eddy viscosity vanishes on the boundary of the turbulent wake. That is,

$$\frac{\partial \overline{w}}{\partial y}(x, y_B(x)) = 0. \tag{3.9.4}$$

3.10 Thermal boundary conditions

The temperature difference at the boundary of the wake is zero, that is

$$\overline{S}(x, y_{B_\theta}(x)) = 0, \tag{3.10.1}$$

where $y_{B_\theta}(x)$ is thermal boundary. For a classical wake, the thermal boundary and momentum boundary coincide. The temperature difference is maximum at the centreline of the wake, $y = 0$. Hence

$$\frac{\partial \overline{S}}{\partial y}(x, 0) = 0. \tag{3.10.2}$$

The eddy thermal conductivity of the flow is zero at the boundary. Now the heat flux vector $\underline{h}_\theta$ at the boundary of the wake has components

$$\overline{h}_{\theta x} = -\kappa \frac{\partial \overline{S}}{\partial x}, \tag{3.10.3}$$

$$\overline{h}_{\theta y} = -\kappa \frac{\partial \overline{S}}{\partial y}, \tag{3.10.4}$$

when $\kappa \neq 0$. The heat flux through the boundary of the wake with normal $\underline{n}$ is zero (see Figure 3.4). Therefore

$$y = y_{B_\theta}(x) : -\overline{h}_{\theta x} \sin\alpha + \overline{h}_{\theta y} \cos\alpha = 0 \tag{3.10.5}$$

and hence

$$y = y_{B_\theta}(x) : -\overline{h}_{\theta x} \frac{dy_{B_\theta}}{dx} + \overline{h}_{\theta y} = 0 \tag{3.10.6}$$

since

$$\frac{dy_{B_\theta}}{dx} = \tan\alpha. \tag{3.10.7}$$

After substituting (3.10.3) and (3.10.4), the boundary condition (3.10.6) becomes

$$y = y_{B_\theta}(x) : \kappa \left(\frac{\partial \overline{S}}{\partial x} \frac{dy_{B_\theta}}{dx} - \frac{\partial \overline{S}}{\partial y} \right) = 0. \tag{3.10.8}$$

Writing (3.10.8) in terms of dimensionless variables gives

$$y^* = y^*_{B_\theta}(x^*) : \kappa \left(\frac{\delta_\theta \delta_V}{L^2} \frac{\partial \overline{S}^*}{\partial x^*} \frac{dy^*_{B_\theta}}{dx^*} - \frac{\partial \overline{S}^*}{\partial y^*} \right) = 0, \tag{3.10.9}$$

which since

$$\frac{\delta_\theta \delta_V}{L^2} << 1, \tag{3.10.10}$$

reduces to

$$\kappa \frac{\partial \overline{S}}{\partial y}(x, y_{B_\theta}(x)) = 0. \tag{3.10.11}$$

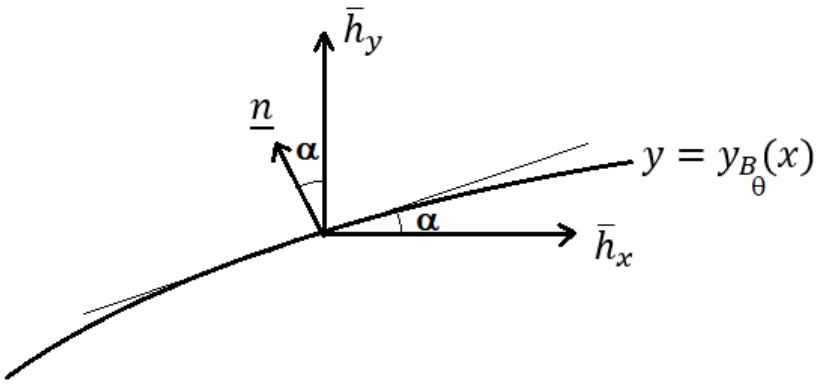

FIGURE 3.4: Schematic diagram of the thermal boundary of the wake.

All boundary conditions derived above are valid for both classical and momentumless wake. They are also valid for any models for μ_T and κ_T.

3.11 Stream function

Since we are dealing with an incompressible fluid we can introduce the stream function ψ defined by

$$\overline{w}(x,y) = \frac{\partial \psi}{\partial y}, \qquad \overline{v}_y(x,y) = \frac{\partial \psi}{\partial x}. \tag{3.11.1}$$

The stream function ψ identically satisfies the continuity equation for mean flow, (3.2.22). Equations (3.5.18) and (3.6.14) become

$$U \frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial}{\partial y}\left[\frac{(\mu + \mu_T)}{\rho}\frac{\partial^2 \psi}{\partial y^2}\right], \tag{3.11.2}$$

$$U \frac{\partial \overline{S}}{\partial x} = \frac{\partial}{\partial y}\left[\frac{\kappa + \kappa_T}{\rho c_v}\frac{\partial \overline{S}}{\partial y}\right]. \tag{3.11.3}$$

Substituting (3.11.1) into (3.7.11) and (3.8.19) we obtain

$$U \frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial}{\partial y}\left[\left(\nu + l^2\left|\frac{\partial^2 \psi}{\partial y^2}\right|\right)\frac{\partial^2 \psi}{\partial y^2}\right], \tag{3.11.4}$$

$$U \frac{\partial \overline{S}}{\partial x} = \frac{\partial}{\partial y}\left[\left(\frac{\kappa}{\rho c_v} + l_\theta l\left|\frac{\partial^2 \psi}{\partial y^2}\right|\right)\frac{\partial \overline{S}}{\partial y}\right]. \tag{3.11.5}$$

Substituting (3.11.1) into the boundary conditions (3.9.1) to (3.9.4) we obtain

$$\frac{\partial \psi}{\partial x}(x,0)=0, \quad \frac{\partial^2 \psi}{\partial y^2}(x,0)=0, \quad \frac{\partial \psi}{\partial y}(x,y_B(x))=0, \quad \frac{\partial^2 \psi}{\partial y^2}(x,y_B(x))=0. \quad (3.11.6)$$

3.12 Summary

In this Chapter, we derived the governing partial differential equation by applying Reynolds averaging to the Navier-Stokes equations and energy balance equation. We resolved the turbulence closure problem by using the Boussinesq approximation and an analogy of Fourier's law of heat conduction. This introduced the eddy viscosity and eddy thermal conductivity. We further simplified the Reynolds averaged equations using boundary layer theory. We derived a model for eddy viscosity and eddy thermal conductivity using the concept of mixing lengths. We dealt with the classical thermal wake and momentumless thermal wake simultaneously.

The governing partial differential equations for a classical and momentumless wake for a general eddy viscosity and eddy conductivity are

$$U\frac{\partial^2 \psi}{\partial x \partial y}=\frac{\partial}{\partial y}\left[\frac{(\mu+\mu_T)}{\rho}\frac{\partial^2 \psi}{\partial y^2}\right], \quad (3.12.1)$$

$$U\frac{\partial \overline{S}}{\partial x}=\frac{\partial}{\partial y}\left[\frac{\kappa+\kappa_T}{\rho c_v}\frac{\partial \overline{S}}{\partial y}\right]. \quad (3.12.2)$$

The governing partial differential equations for a classical and momentumless wake for the eddy viscosity and eddy conductivity modelled by mixing lengths are

$$U\frac{\partial^2 \psi}{\partial x \partial y}=\frac{\partial}{\partial y}\left[\left(\nu+l^2\left|\frac{\partial^2 \psi}{\partial y^2}\right|\right)\frac{\partial^2 \psi}{\partial y^2}\right], \quad (3.12.3)$$

$$U\frac{\partial \overline{S}}{\partial x}=\frac{\partial}{\partial y}\left[\left(\frac{\kappa}{\rho c_v}+l_\theta l\left|\frac{\partial^2 \psi}{\partial y^2}\right|\right)\frac{\partial \overline{S}}{\partial y}\right]. \quad (3.12.4)$$

Boundary conditions for a classical wake are

$$\frac{\partial \psi}{\partial x}(x,0)=0, \quad \frac{\partial^2 \psi}{\partial y^2}(x,0)=0, \quad \frac{\partial \psi}{\partial y}(x,y_B(x))=0, \quad \frac{\partial^2 \psi}{\partial y^2}(x,y_B(x))=0, \quad (3.12.5)$$

$$\frac{\partial \overline{S}}{\partial y}(x,0)=0, \qquad \overline{S}(x,y_B(x))=0, \qquad \kappa\frac{\partial \overline{S}}{\partial y}(x,y_B(x))=0. \quad (3.12.6)$$

For a classical wake, the thermal boundary coincides with the momentum boundary. Boundary conditions for a momentumless wake are

$$\frac{\partial \psi}{\partial x}(x,0)=0, \quad \frac{\partial^2 \psi}{\partial y^2}(x,0)=0, \quad \frac{\partial \psi}{\partial y}(x,y_B(x))=0, \quad \frac{\partial^2 \psi}{\partial y^2}(x,y_B(x))=0, \quad (3.12.7)$$

$$\frac{\partial \overline{S}}{\partial y}(x,0)=0, \qquad \overline{S}(x,y_{B_\theta}(x))=0, \qquad \kappa\frac{\partial \overline{S}}{\partial y}(x,y_{B_\theta}(x))=0. \quad (3.12.8)$$

Chapter 4

Classical thermal wake modelled by mixing lengths

4.1 Introduction

In this Chapter, we use Lie symmetry methods to study the classical thermal wake with turbulent features modelled by the mixing lengths. We derived the governing partial differential equations and boundary conditions for the wake in Chapter 3. The boundary conditions are not enough to solve the governing partial differential equations since the equation of the boundary $y_B(x)$ is unknown. Conserved quantities for the classical thermal wake will be derived and used, in addition to boundary conditions, to obtain $y_B(x)$.

Conserved quantities connect the heated obstacle and the thermal wake behind it. The heated obstacle retards the fluid flow in the wake and supplies the wake with heat. The drag and the heat supplied by the heated obstacle are conserved in the wake. The conserved quantities can be derived by physical arguments. Conservation laws of the governing partial differential equations provide a systematic way of deriving the conserved quantities. We use the multiplier method to find the conservation laws. Conservation laws for a non-thermal classical wake have been derived by Hutchinson and Mason [8]. The results they obtained will be useful for comparison. The momentum flux conserved vector in the conservation law should match the results in [8]. We derive the conserved quantities by integrating the conservation laws across the wake and imposing the boundary conditions. The conserved vectors are used to find the associated Lie point symmetry.

We use the formula relating conserved vectors and the associated Lie point symmetry to find the Lie point symmetry associated with the conserved vectors. Some of the parameters in the associated Lie point symmetry remain unknown after using the formula when the kinematic viscosity and the thermal conductivity both vanish. We use the conserved quantities and Prandtl's hypothesis [4, 5] to determine the unknown parameters. Prandtl's hypothesis states that the momentum mixing length is proportional to the breadth of the wake. We use the associated Lie point symmetry to reduce the governing partial differential equations to ordinary differential equations for the invariant solution. The double reduction theorem by Sjöberg [17, 18] guarantees that the ordinary differential equations can be integrated at least once.

The value of the constant in the equation for the boundary for a classical thermal wake is unknown. Conserved quantities allow us to determine the value of this constant. When the kinematic viscosity and thermal conductivity of the fluid are nonzero, the ordinary differential equations cannot be solved analytically. We apply a numerical scheme to solve the ordinary differential equations. The numerical scheme shoots at the two conserved quantities to convert the boundary value problem into an initial value problem.

This Chapter is organized in the following manner. In Section 4.2, we derive multipliers that are used to find conservation laws in Section 4.3. In Section 4.4, we use the conservation laws to obtain conserved quantities of the classical thermal wake. In Section 4.5, we derive the associated Lie point symmetry and in Section 4.6 we use the associated Lie point symmetry to reduce the governing partial differential equations to ordinary differential equations. In Section 4.7, We summarize and compare the results for nonzero kinematic viscosity and thermal conductivity with the results for vanishing kinematic viscosity and thermal conductivity. In Section 4.8, we derive the analytical solution when the kinematic viscosity and thermal conductivity vanish. In Section 4.9, we derive the equation for lines of constant temperature difference and in Section 4.10 we derive the equations for the streamlines of the mean velocity deficit. In Section 4.11, we describe the numerical solution when the kinematic viscosity and thermal conductivity are nonzero. In Section 4.12, we present the analytical and numerical results. In Section 4.13, we present conclusions.

4.2 Multipliers of the divergence of conserved vectors

For a classical wake,

$$\frac{\partial \overline{w}}{\partial y}(x,y) = \frac{\partial^2 \psi}{\partial y^2}(x,y) \leq 0 \tag{4.2.1}$$

in the upper half of the wake, $0 \leq y \leq y_B(x)$. We assume that the mixing lengths are functions of x. We use the condition that the governing equations admit an associated Lie point symmetry to find the form of these functions. Hence, equations (3.12.3) and (3.12.4) become

$$U\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial}{\partial y}\left[\left(\nu - l^2(x)\frac{\partial^2 \psi}{\partial y^2}\right)\frac{\partial^2 \psi}{\partial y^2}\right], \tag{4.2.2}$$

$$U\frac{\partial \overline{S}}{\partial x} = \frac{\partial}{\partial y}\left[\left(\frac{\kappa}{\rho c_v} - l_\theta(x) l(x)\frac{\partial^2 \psi}{\partial y^2}\right)\frac{\partial \overline{S}}{\partial y}\right] \tag{4.2.3}$$

for the upper half of the wake. Boundary conditions for a classical wake are

$$\frac{\partial \psi}{\partial x}(x,0) = 0, \quad \frac{\partial^2 \psi}{\partial y^2}(x,0) = 0, \quad \frac{\partial \psi}{\partial y}(x,y_B(x)) = 0, \quad \frac{\partial^2 \psi}{\partial y^2}(x,y_B(x)) = 0, \tag{4.2.4}$$

$$\frac{\partial \overline{S}}{\partial y}(x,0) = 0, \qquad \overline{S}(x,y_B(x)) = 0, \qquad \kappa\frac{\partial \overline{S}}{\partial y}(x,y_B(x)) = 0. \tag{4.2.5}$$

When calculating conservation laws and Lie point symmetries we treat x, y, ψ, $\overline{S}$ and all partial derivatives of ψ and $\overline{S}$ with respect to x and y as independent variables. We use the subscript notation to indicate that the partial derivatives are treated as independent variables. Hence, when calculating conservation laws, x, y, ψ, $\overline{S}$, ψ_x, ψ_y, $\overline{S}_x$, $\overline{S}_y$, ψ_{xx} and all higher derivatives of ψ and $\overline{S}$ are independent variables. We use the notation $\frac{\partial \psi}{\partial x}$, $\frac{\partial \psi}{\partial y}$, $\cdots$, $\frac{\partial \overline{S}}{\partial x}$, $\frac{\partial \overline{S}}{\partial y}$, $\cdots$ when we treat $\psi(x,y)$ and $\overline{S}(x,y)$ as functions of the independent variables x and y. Writing the governing equations (4.2.2) and (4.2.3) in terms of the subscript notation we obtain

$$\nu\psi_{yyy} - 2l^2(x)\psi_{yy}\psi_{yyy} - U\psi_{xy} = 0, \tag{4.2.6}$$

$$\frac{\kappa}{\rho c_v}\overline{S}_{yy} - l(x)l_\theta(x)\psi_{yyy}\overline{S}_y - l(x)l_\theta(x)\psi_{yy}\overline{S}_{yy} - U\overline{S}_x = 0. \tag{4.2.7}$$

A conservation law of the system of partial differential equations (4.2.6) and (4.2.7) is defined by

$$D_x T^1 + D_y T^2\Big|_{(4.2.6),(4.2.7)} = 0, \tag{4.2.8}$$

where

$$\begin{aligned} D_x = \frac{\partial}{\partial x} + \psi_x\frac{\partial}{\partial \psi} + \overline{S}_x\frac{\partial}{\partial \overline{S}} + \psi_{xx}\frac{\partial}{\partial \psi_x} + \overline{S}_{xx}\frac{\partial}{\partial \overline{S}_x} + \psi_{xy}\frac{\partial}{\partial \psi_y} \\ + \overline{S}_{xy}\frac{\partial}{\partial \overline{S}_y} + \cdots, \end{aligned} \tag{4.2.9}$$

$$\begin{aligned} D_y = \frac{\partial}{\partial y} + \psi_y\frac{\partial}{\partial \psi} + \overline{S}_y\frac{\partial}{\partial \overline{S}} + \psi_{yx}\frac{\partial}{\partial \psi_x} + \overline{S}_{yx}\frac{\partial}{\partial \overline{S}_x} + \psi_{yy}\frac{\partial}{\partial \psi_y} \\ + \overline{S}_{yy}\frac{\partial}{\partial \overline{S}_y} + \cdots, \end{aligned} \tag{4.2.10}$$

and T^1 and T^2 are components of the conserved vector (T^1, T^2). We will use the multiplier method to obtain conservation laws of the governing equations (4.2.6) and (4.2.7). We first obtain multipliers Λ_1 and Λ_2 that satisfy

$$\begin{aligned} &\Lambda_1\left(\nu\psi_{yyy} - 2l^2(x)\psi_{yy}\psi_{yyy} - U\psi_{xy}\right) \\ &+\Lambda_2\left(\frac{\kappa}{\rho c_v}\overline{S}_{yy} - l(x)l_\theta(x)\psi_{yyy}\overline{S}_y - l(x)l_\theta(x)\psi_{yy}\overline{S}_{yy} - U\overline{S}_x\right) \\ &= D_x T^1 + D_y T^2 \end{aligned} \tag{4.2.11}$$

for arbitrary functions $\psi(x,y)$ and $\overline{S}(x,y)$. The multipliers Λ_1 and Λ_2 can be dependent on x, y, ψ, $\overline{S}$ and all partial derivatives of ψ and $\overline{S}$, but we will limit the dependence to x, y, ψ and $\overline{S}$ and obtain the multipliers that can be computed readily by hand. In order to obtain the determining equations for Λ_1 and Λ_2 we apply the standard Euler operators

$$E_\psi = \frac{\partial}{\partial \psi} - D_x\frac{\partial}{\partial \psi_x} - D_y\frac{\partial}{\partial \psi_y} + D_x^2\frac{\partial}{\partial \psi_{xx}} + D_xD_y\frac{\partial}{\partial \psi_{xy}} + D_y^2\frac{\partial}{\partial \psi_{yy}} - D_y^3\frac{\partial}{\partial \psi_{yyy}} - \cdots, \tag{4.2.12}$$

$$E_{\overline{S}} = \frac{\partial}{\partial \overline{S}} - D_x\frac{\partial}{\partial \overline{S}_x} - D_y\frac{\partial}{\partial \overline{S}_y} + D_x^2\frac{\partial}{\partial \overline{S}_{xx}} + D_xD_y\frac{\partial}{\partial \overline{S}_{xy}} + D_y^2\frac{\partial}{\partial \overline{S}_{yy}} - \cdots, \tag{4.2.13}$$

to (4.2.11). These operators annihilate the divergence of the conserved vector. Hence applying $E_{\overline{S}}$ to (4.2.11) gives

$$\begin{aligned}
&\frac{\partial \Lambda_1}{\partial \overline{S}}\left[\nu\psi_{yyy} - 2l^2(x)\psi_{yy}\psi_{yyy} - U\psi_{xy}\right] \\
&+\frac{\partial \Lambda_2}{\partial \overline{S}}\left[\frac{\kappa}{\rho c_v}\overline{S}_{yy} - l(x)l_\theta(x)\psi_{yyy}\overline{S}_y - l(x)l_\theta(x)\psi_{yy}\overline{S}_{yy} - U\overline{S}_x\right] \\
&+U\frac{\partial \Lambda_2}{\partial x} + U\frac{\partial \Lambda_2}{\partial \psi}\psi_x + U\frac{\partial \Lambda_2}{\partial \overline{S}}\overline{S}_x \\
&+l(x)l_\theta(x)\frac{\partial \Lambda_2}{\partial y}\psi_{yyy} + l(x)l_\theta(x)\frac{\partial \Lambda_2}{\partial \psi}\psi_y\psi_{yyy} + l(x)l_\theta(x)\frac{\partial \Lambda_2}{\partial \overline{S}}\overline{S}_y\psi_{yyy} \\
&+l(x)l_\theta(x)\Lambda_2\psi_{yyyy} + D_y^2\left[\Lambda_2\left(\frac{\kappa}{\rho c_v} - l(x)l_\theta(x)\psi_{yy}\right)\right] = 0.
\end{aligned} \tag{4.2.14}$$

For now, we do not need to expand the last term of (4.2.14). We see that the last term in (4.2.14) does not depend on ψ_x and ψ_{xy}. Since the partial derivatives are treated as independent variables and (4.2.14) is valid for all functions $\psi(x,y)$ and $\overline{S}(x,y)$, we can equate the coefficients of independent partial derivatives of $\psi(x,y)$ and $\overline{S}(x,y)$ to zero. The coefficients of ψ_x and ψ_{xy} give

$$\psi_x : \frac{\partial \Lambda_2}{\partial \psi} = 0, \tag{4.2.15}$$

$$\psi_{xy} : \frac{\partial \Lambda_1}{\partial \overline{S}} = 0. \tag{4.2.16}$$

Therefore

$$\Lambda_1 = \Lambda_1(x,y,\psi), \qquad \Lambda_2 = \Lambda_2(x,y,\overline{S}). \tag{4.2.17}$$

Now expanding the last term of (4.2.14) we obtain

$$\begin{aligned}
&\frac{\kappa}{\rho c_v}\frac{\partial^2 \Lambda_2}{\partial y^2} + U\frac{\partial \Lambda_2}{\partial x} + 2\frac{\kappa}{\rho c_v}\frac{\partial^2 \Lambda_2}{\partial y \partial \psi}\psi_y + \left(2\frac{\kappa}{\rho c_v}\frac{\partial^2 \Lambda_2}{\partial y \partial \overline{S}} + U\frac{\partial \Lambda_2}{\partial \overline{S}}\right)\overline{S}_y - U\frac{\partial \Lambda_2}{\partial \overline{S}}\overline{S}_x \\
&-l(x)l_\theta(x)\frac{\partial^2 \Lambda_2}{\partial y^2}\psi_{yy} + 2\frac{\kappa}{\rho c_v}\frac{\partial \Lambda_2}{\partial \overline{S}}\overline{S}_{yy} - l(x)l_\theta(x)\frac{\partial \Lambda_2}{\partial y}\psi_{yyy} \\
&+\frac{\kappa}{\rho c_v}\frac{\partial^2 \Lambda_2}{\partial \overline{S}^2}\left(\overline{S}_y\right)^2 - 2l(x)l_\theta(x)\frac{\partial^2 \Lambda_2}{\partial y \partial \overline{S}}\overline{S}_y\psi_{yy} - l(x)l_\theta(x)\frac{\partial \Lambda_2}{\partial \overline{S}}\overline{S}_y\psi_{yyy} \\
&-2l(x)l_\theta(x)\frac{\partial \Lambda_2}{\partial \overline{S}}\psi_{yy}\overline{S}_{yy} - l(x)l_\theta(x)\frac{\partial \Lambda_2}{\partial \overline{S}}\psi_{yyy}\overline{S}_y - l(x)l_\theta(x)\frac{\partial^2 \Lambda_2}{\partial \overline{S}^2}\psi_{yy}\left(\overline{S}_y\right)^2 = 0.
\end{aligned} \tag{4.2.18}$$

Equating coefficients of ψ_{yyy}, $\overline{S}_{yy}$ and terms independent of derivatives of ψ and $\overline{S}$ to zero we obtain

$$\psi_{yyy} : \frac{\partial \Lambda_2}{\partial y} = 0, \tag{4.2.19}$$

$$\psi_{yy}\overline{S}_{yy} : \frac{\partial \Lambda_2}{\partial \overline{S}} = 0, \tag{4.2.20}$$

$$1 : \frac{\kappa}{\rho c_v}\frac{\partial^2 \Lambda_2}{\partial y^2} + U\frac{\partial \Lambda_2}{\partial x} = 0. \tag{4.2.21}$$

Hence

$$\Lambda_1 = \Lambda_1(x, y, \psi), \tag{4.2.22}$$

$$\Lambda_2 = c_2, \tag{4.2.23}$$

where c_2 is an arbitrary constant. Solutions (4.2.22) and (4.2.23) identically satisfy (4.2.18).

Now applying E_ψ to (4.2.11) we obtain

$$\begin{gathered}\frac{\partial \Lambda_1}{\partial \psi}\left[\nu\psi_{yyy} - 2l^2(x)\psi_{yy}\psi_{yyy} - U\psi_{xy}\right] \\ -U\frac{\partial^2\Lambda_1}{\partial x\partial y} - U\frac{\partial\Lambda_1}{\partial\psi}\psi_{xy} - U\frac{\partial^2\Lambda_1}{\partial x\partial\psi}\psi_y - U\frac{\partial^2\Lambda_1}{\partial y\partial\psi}\psi_x - U\frac{\partial^2\Lambda_1}{\partial\psi^2}\psi_x\psi_y \\ -2l^2(x)D_y^2\left[\Lambda_1\psi_{yyy}\right] - D_y^3\left[\Lambda_1\left(\nu - 2l^2(x)\psi_{yy}\right)\right] = 0.\end{gathered} \tag{4.2.24}$$

All terms with $\Lambda_2 = c_2$ canceled. We do not need to expand the last two terms of (4.2.24) at this stage. We see that both these terms do not depend on ψ_{xy}. Equating the coefficient of ψ_{xy} to zero we obtain

$$\psi_{xy} : \frac{\partial\Lambda_1}{\partial\psi} = 0. \tag{4.2.25}$$

Therefore

$$\Lambda_1 = \Lambda_1(x, y). \tag{4.2.26}$$

Now expanding the last two terms of (4.2.24) we obtain

$$\begin{gathered}-U\frac{\partial^2\Lambda_1}{\partial x\partial y} - \nu\frac{\partial^3\Lambda_1}{\partial y^3} + 4l^2(x)\frac{\partial^2\Lambda_1}{\partial y^2}\psi_{yyy} + 2l^2(x)\frac{\partial^3\Lambda_1}{\partial y^3}\psi_{yy} \\ +2l^2(x)\frac{\partial\Lambda_1}{\partial y}\psi_{yyyy} = 0.\end{gathered} \tag{4.2.27}$$

Equating the coefficient of ψ_{yyyy} in (4.2.27) to zero we obtain

$$\psi_{yyyy} : \frac{\partial\Lambda_1}{\partial y} = 0. \tag{4.2.28}$$

Hence

$$\Lambda_1 = \Lambda_1(x), \tag{4.2.29}$$

which identically satisfies (4.2.27). Therefore if we look for multipliers of the form $\Lambda_1(x, y, \psi, \overline{S})$ and $\Lambda_2(x, y, \psi, \overline{S})$, then

$$\Lambda_1 = \Lambda_1(x), \qquad \Lambda_2 = c_2, \tag{4.2.30}$$

where c_2 is a constant and $\Lambda_1(x)$ is an arbitrary function of x.

4.3 Conserved vectors

We look for conserved vectors with

$$\Lambda_1 = \Lambda_1(x) = c_1, \tag{4.3.1}$$

where c_1 is a constant. Rewriting (4.2.11) with $\Lambda_1 = c_1$ and $\Lambda_2 = c_2$ we obtain

$$\begin{gathered} c_1\left[\nu\psi_{yyy} - 2l^2(x)\psi_{yy}\psi_{yyy} - U\psi_{xy}\right] + \\ c_2\left[\frac{\kappa}{\rho c_v}\overline{S}_{yy} - l(x)l_\theta(x)\psi_{yyy}\overline{S}_y - l(x)l_\theta(x)\psi_{yy}\overline{S}_{yy} - U\overline{S}_x\right] \\ = D_x\left[-c_2U\overline{S} - c_1U\psi_y\right] + \\ D_y\left[c_1\left(\nu\psi_{yy} - l^2(x)\psi_{yy}^2\right) + c_2\left(\frac{\kappa}{\rho c_v}\overline{S}_y - l(x)l_\theta(x)\psi_{yy}\overline{S}_y\right)\right] \end{gathered} \tag{4.3.2}$$

for arbitrary functions $\psi(x,y)$ and $\overline{S}(x,y)$. When $\psi(x,y)$ and $\overline{S}(x,y)$ satisfy the system of partial differential equations (4.2.6) and (4.2.7), (4.3.2) becomes

$$\begin{gathered} D_x\left[c_2U\overline{S} + c_1U\psi_y\right] + D_y\left[-c_1\left(\nu\psi_{yy} - l^2(x)\psi_{yy}^2\right)\right. \\ \left.-c_2\left(\frac{\kappa}{\rho c_v}\overline{S}_y - l(x)l_\theta(x)\psi_{yy}\overline{S}_y\right)\right]\Big|_{(4.2.6),(4.2.7)} = 0. \end{gathered} \tag{4.3.3}$$

Therefore we obtain the conserved vector

$$T^1 = c_2U\overline{S} + c_1U\psi_y, \tag{4.3.4}$$

$$T^2 = -c_1\left(\nu\psi_{yy} - l^2(x)\psi_{yy}^2\right) - c_2\frac{\kappa}{\rho c_v}\overline{S}_y + c_2l(x)l_\theta(x)\psi_{yy}\overline{S}_y. \tag{4.3.5}$$

When $c_1 = 1$ and $c_2 = 0$ the conserved vector (4.3.4), (4.3.5) becomes

$$\begin{aligned} T^1 &= U\psi_y, \\ T^2 &= -\nu\psi_{yy} + l^2(x)\psi_{yy}^2. \end{aligned} \tag{4.3.6}$$

These are the components of the momentum flux conserved vector $T_M = (T^1, T^2)$. When $c_1 = 0$ and $c_2 = 1$ the conserved vector (4.3.4), (4.3.5) becomes

$$\begin{aligned} T^1 &= U\overline{S}, \\ T^2 &= -\frac{\kappa}{\rho c_v}\overline{S}_y + l(x)l_\theta(x)\psi_{yy}\overline{S}_y. \end{aligned} \tag{4.3.7}$$

These are the components of the thermal flux conserved vector $T_\theta = (T^1, T^2)$. The momentum flux conserved vector and the thermal flux conserved vector satisfy

$$D_xT^1 + D_yT^2\Big|_{(4.2.6),(4.2.7)} = 0. \tag{4.3.8}$$

4.4 Conserved quantities

We now treat $\psi(x,y)$ and $\overline{S}(x,y)$ as dependent variables that satisfy the system of governing partial differential equations (4.2.2), (4.2.3) and boundary conditions (4.2.4) and (4.2.5) in order to obtain conserved quantities. Hence the total derivatives in (4.3.8) become partial derivatives and

$$\frac{\partial T^1}{\partial x} + \frac{\partial T^2}{\partial y} = 0. \tag{4.4.1}$$

4.4.1 Conserved quantity for the momentum flux

The momentum flux conserved vector T_M has components

$$T^1 = U\frac{\partial \psi}{\partial y}, \tag{4.4.2}$$

$$T^2 = -\nu\frac{\partial^2 \psi}{\partial y^2} + l^2(x)\left(\frac{\partial^2 \psi}{\partial y^2}\right)^2. \tag{4.4.3}$$

Substituting (4.4.2) and (4.4.3) into (4.4.1) we obtain

$$U\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left[-\nu\frac{\partial^2 \psi}{\partial y^2} + l^2(x)\left(\frac{\partial^2 \psi}{\partial y^2}\right)^2\right] = 0. \tag{4.4.4}$$

Integrating (4.4.4) with respect to y from the centreline of the wake, $y = 0$, to the boundary of the wake, $y = y_B(x)$, we obtain

$$U\int_0^{y_B(x)} \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) dy - \left[\left(\nu\frac{\partial^2 \psi}{\partial y^2} - l^2(x)\left(\frac{\partial^2 \psi}{\partial y^2}\right)^2\right)\right]\Bigg|_{y=0}^{y=y_B(x)} = 0. \tag{4.4.5}$$

From (4.2.4), the second derivative of ψ with respect to y vanishes at $y = 0$ and $y = y_B(x)$. Hence (4.4.5) reduces to

$$\int_0^{y_B(x)} \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) dy = 0. \tag{4.4.6}$$

We use the theorem for differentiation under the integral sign [34], which states that

$$\frac{d}{dx}\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy = \int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial f}{\partial x}(x,y)dy + f(x,\phi_2(x))\frac{d\phi_2}{dx} - f(x,\phi_1(x))\frac{d\phi_1}{dx}. \tag{4.4.7}$$

Applying the theorem we obtain

$$\frac{d}{dx}\int_0^{y_B(x)} \frac{\partial \psi}{\partial y}dy - \frac{dy_B}{dx}\frac{\partial \psi}{\partial y}(x, y_B(x)) = 0. \tag{4.4.8}$$

Applying the third boundary condition in (4.2.4) the second term of (4.4.8) vanishes and equation (4.4.8) reduces to

$$\frac{d}{dx}\int_0^{y_B(x)} \frac{\partial \psi}{\partial y}dy = 0. \tag{4.4.9}$$

Hence

$$D = 2\rho U\int_0^{y_B(x)} \frac{\partial \psi}{\partial y}(x,y)dy \tag{4.4.10}$$

is a constant independent of x.

4.4.2 Conserved quantity for the thermal flux

The momentum flux conserved vector T_θ has components

$$T^1 = U\overline{S}, \tag{4.4.11}$$

$$T^2 = -\frac{\kappa}{\rho c_v}\frac{\partial \overline{S}}{\partial y} + l(x)l_\theta(x)\frac{\partial^2 \psi}{\partial y^2}\frac{\partial \overline{S}}{\partial y}. \tag{4.4.12}$$

Substituting (4.4.11) and (4.4.12) into (4.4.1) we obtain

$$\frac{\partial}{\partial x}\left[U\overline{S}\right] + \frac{\partial}{\partial y}\left[-\frac{\kappa}{\rho c_v}\frac{\partial \overline{S}}{\partial y} + l(x)l_\theta(x)\frac{\partial^2 \psi}{\partial y^2}\frac{\partial \overline{S}}{\partial y}\right] = 0. \tag{4.4.13}$$

Integrating (4.4.13) with respect to y from the centreline, $y = 0$, to the boundary, $y = y_B(x)$, we obtain

$$U\int_0^{y_B(x)} \frac{\partial \overline{S}}{\partial x}dy + \left[-\frac{\kappa}{\rho c_v}\frac{\partial \overline{S}}{\partial y} + l(x)l_\theta(x)\frac{\partial^2 \psi}{\partial y^2}\frac{\partial \overline{S}}{\partial y}\right]\Bigg|_{y=0}^{y=y_B(x)} = 0. \tag{4.4.14}$$

Applying the boundary conditions (4.2.4) and (4.2.5), the second term of (4.4.14) vanishes. Hence (4.4.14) reduces to

$$\int_0^{y_B(x)} \frac{\partial \overline{S}}{\partial x}dy = 0. \tag{4.4.15}$$

Applying the theorem for differentiation under the integral sign [34] to (4.4.15) we obtain

$$\frac{d}{dx}\int_0^{y_B(x)} \overline{S}(x,y)dy - \frac{dy_B}{dx}\overline{S}(x,y_B(x)) = 0. \tag{4.4.16}$$

Applying the second boundary condition in (4.2.5) the second term of (4.4.16) vanishes. Therefore (4.4.16) reduces to

$$\frac{d}{dx}\int_0^{y_B(x)} \overline{S}(x,y)dy = 0. \tag{4.4.17}$$

Hence

$$K = 2\rho c_v U\int_0^{y_B(x)} \overline{S}(x,y)dy \tag{4.4.18}$$

is a constant independent of x.

The conserved quantity (4.4.18) is satisfied for both $\kappa = 0$ and $\kappa \neq 0$ because the boundary condition that was imposed at $y = y_B(x)$,

$$\kappa\frac{\partial \overline{S}}{\partial y}(x,y_B(x)) = 0, \tag{4.4.19}$$

is satisfied for both $\kappa = 0$ and $\kappa \neq 0$. The derivation of the conserved quantities showed clearly that a conserved quantity depends on both the conserved vector and the boundary conditions. If the wake turns out to be unbounded in the y-direction then the conserved quantities (4.4.10) and (4.4.18) still apply with $y_B(x) = \infty$.

4.5 Associated Lie point symmetry

In this Section, we derive the Lie point symmetry associated with the conserved vectors (4.3.6) and (4.3.7).

A vector field operator

$$X = \xi^1(x,y,\psi,\overline{S})\frac{\partial}{\partial x} + \xi^2(x,y,\psi,\overline{S})\frac{\partial}{\partial y} + \eta^1(x,y,\psi,\overline{S})\frac{\partial}{\partial \psi} + \eta^2(x,y,\psi,\overline{S})\frac{\partial}{\partial \overline{S}} \quad (4.5.1)$$

is associated with the conserved vector (T^1, T^2) provided [23, 24]

$$X^{[\infty]}(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0, \qquad i = 1,2 \quad (4.5.2)$$

where $X^{[\infty]}$ is the prolongation of X up to all orders required to evaluate $X^{[\infty]}(T^1)$ and $X^{[\infty]}(T^2)$. When equation (4.5.2) is expanded for $i = 1,2$ it gives

$$X^{[\infty]}(T^1) + T^1 D_y(\xi^2) - T^2 D_y(\xi^1) = 0, \quad (4.5.3)$$

$$X^{[\infty]}(T^2) + T^2 D_x(\xi^1) - T^1 D_x(\xi^2) = 0. \quad (4.5.4)$$

The conserved vectors (4.3.6) and (4.3.7) contain derivatives up to second order in ψ and first order in $\overline{S}$. Hence we only need to prolongate the vector field operator X up to second-order in ψ and first-order in $\overline{S}$. The prolongated vector field operator is then

$$X^{[\infty]} = \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial y} + \eta^1\frac{\partial}{\partial \psi} + \eta^2\frac{\partial}{\partial \overline{S}} + \zeta^1_y\frac{\partial}{\partial \psi_y} + \zeta^1_{yy}\frac{\partial}{\partial \psi_{yy}} + \zeta^2_y\frac{\partial}{\partial \overline{S}_y}, \quad (4.5.5)$$

where

$$\zeta^\alpha_y = D_y(\eta^\alpha) - u^\alpha_k D_y(\xi^k), \quad (4.5.6)$$

$$\zeta^1_{yy} = D_y(\zeta^1_y) - \psi_{yk} D_y(\xi^k) \quad (4.5.7)$$

and

$$x^1 = x, \qquad x^2 = y, \qquad u^1 = \psi, \qquad u^2 = \overline{S}. \quad (4.5.8)$$

We will show that the vector field operator X is a Lie point symmetry for the partial differential equations (4.2.2) and (4.2.3).

4.5.1 Vector field associated with the momentum flux conserved vector

Expanding equation (4.5.3) with the conserved vector (4.3.6) we obtain

$$-U\zeta^1_y - U\psi_y D_y\xi^2 - \left[\nu\psi_{yy} - l^2(x)\psi^2_{yy}\right] D_y\xi^1 = 0. \quad (4.5.9)$$

By (4.5.6)

$$\zeta^1_y = D_y\eta^1 - \psi_x D_y\xi^1 - \psi_y D_y\xi^2. \quad (4.5.10)$$

Substituting (4.5.10) into (4.5.9) we obtain

$$-UD_y\eta^1 + U\psi_x D_y\xi^1 - \nu\psi_{yy} D_y\xi^1 + l^2(x)\psi^2_{yy} D_y\xi^1 = 0. \quad (4.5.11)$$

Splitting (4.5.11) by powers of ψ_{yy} we obtain

$$\psi^2_{yy} : D_y\xi^1 = 0, \quad (4.5.12)$$

$$\psi_{yy} : \nu D_y\xi^1 = 0, \quad (4.5.13)$$

$$\text{Remainder: } -UD_y\eta^1 + U\psi_x D_y\xi^1 = 0. \tag{4.5.14}$$

Expanding the total derivative in (4.5.12) we obtain

$$\frac{\partial\xi^1}{\partial y} + \psi_y\frac{\partial\xi^1}{\partial\psi} + \overline{S}_y\frac{\partial\xi^1}{\partial\overline{S}} = 0. \tag{4.5.15}$$

Splitting (4.5.15) by powers of ψ_y and $\overline{S}_y$ we obtain

$$\psi_y : \frac{\partial\xi^1}{\partial\psi} = 0, \tag{4.5.16}$$

$$\overline{S}_y : \frac{\partial\xi^1}{\partial\overline{S}} = 0, \tag{4.5.17}$$

$$\text{Remainder: } \frac{\partial\xi^1}{\partial y} = 0. \tag{4.5.18}$$

Hence

$$\xi^1 = \xi^1(x). \tag{4.5.19}$$

Equation (4.5.11) reduces to

$$D_y\eta^1 = 0. \tag{4.5.20}$$

Expanding the derivative $D_y\eta^1$ equation (4.5.20) becomes

$$\frac{\partial\eta^1}{\partial y} + \frac{\partial\eta^1}{\partial\psi}\psi_y + \frac{\partial\eta^1}{\partial\overline{S}}\overline{S}_y = 0. \tag{4.5.21}$$

Splitting (4.5.21) by powers of ψ_y and $\overline{S}_y$ we obtain

$$\psi_y : \frac{\partial\eta^1}{\partial\psi} = 0, \tag{4.5.22}$$

$$\overline{S}_y : \frac{\partial\eta^1}{\partial\overline{S}} = 0, \tag{4.5.23}$$

$$\text{Remainder: } \frac{\partial\eta^1}{\partial y} = 0. \tag{4.5.24}$$

Equations (4.5.22) to (4.5.24) give

$$\eta^1 = a_1(x), \tag{4.5.25}$$

where $a_1(x)$ is an arbitrary function of x. Equations (4.5.19) and (4.5.25) are satisfied for both $\nu = 0$ and $\nu \neq 0$.

Now expanding (4.5.4) with the conserved vector (4.3.6) we obtain

$$\begin{aligned} -\frac{d}{dx}\left(l^2(x)\right)\psi_{yy}^2\xi^1 + \nu\zeta_{yy}^1 - 2l^2(x)\psi_{yy}\zeta_{yy}^1 + \left[\nu\psi_{yy} - l^2(x)\psi_{yy}^2\right]\frac{d\xi^1}{dx} \\ +U\psi_y D_x\xi^2 = 0. \end{aligned} \tag{4.5.26}$$

From (4.5.7)

$$\zeta_{yy}^1 = D_y(\zeta_y^1) - \psi_{yy}D_y(\xi^2). \tag{4.5.27}$$

Substituting (4.5.25) and (4.5.19) into (4.5.10) we obtain

$$\zeta_y^1 = -\psi_y D_y \xi^2. \tag{4.5.28}$$

Substituting (4.5.28) into (4.5.27) we have

$$\zeta_{yy}^1 = -2\psi_{yy} D_y \xi^2 - \psi_y D_y^2 \xi^2. \tag{4.5.29}$$

Substituting (4.5.29) into (4.5.26) we obtain

$$\begin{aligned} &-\frac{d}{dx}\left(l^2(x)\right)\psi_{yy}^2\xi^1 - 2\nu\psi_{yy}D_y\xi^2 - \nu\psi_y D_y^2\xi^2 + 4l^2(x)\psi_{yy}^2 D_y\xi^2 \\ &+2l^2(x)\psi_y\psi_{yy}D_y^2\xi^2 + \left[\nu\psi_{yy} - l^2(x)\psi_{yy}^2\right]\frac{d\xi^1}{dx} + U\psi_y D_x\xi^2 = 0. \end{aligned} \tag{4.5.30}$$

Upon inspection, we see that the last term of (4.5.30) is the only term that contains the derivative products $\psi_x\psi_y$ and $\overline{S}_x\psi_y$. Hence the coefficients of $\psi_x\psi_y$ and $\overline{S}_x\psi_y$ in the last term must vanish. That is,

$$\psi_x\psi_y : \frac{\partial\xi^2}{\partial\psi} = 0, \tag{4.5.31}$$

$$\overline{S}_x\psi_y : \frac{\partial\xi^2}{\partial\overline{S}} = 0. \tag{4.5.32}$$

Hence

$$\xi^2 = \xi^2(x, y). \tag{4.5.33}$$

Equation (4.5.33) implies that the total derivatives D_x and D_y applied to ξ^2 become partial derivatives with respect to x and y and (4.5.30) simplifies to

$$\begin{aligned} &2l^2(x)\frac{\partial^2\xi^2}{\partial y^2}\psi_y\psi_{yy} + \nu\psi_{yy}\left[-2\frac{\partial\xi^2}{\partial y} + \frac{d\xi^1}{dx}\right] + \\ &\psi_{yy}^2\left[-\frac{d}{dx}\left(l^2(x)\right)\xi^1 + 4l^2(x)\frac{\partial\xi^2}{\partial y} - l^2(x)\frac{d\xi^1}{dx}\right] \\ &+\psi_y\left[U\frac{\partial\xi^2}{\partial x} - \nu\frac{\partial^2\xi^2}{\partial y^2}\right] = 0. \end{aligned} \tag{4.5.34}$$

Splitting (4.5.34) by independent powers of partial derivatives of ψ and $\overline{S}$ we obtain

$$\psi_y\psi_{yy} : \frac{\partial^2\xi^2}{\partial y^2} = 0, \tag{4.5.35}$$

$$\psi_y : U\frac{\partial\xi^2}{\partial x} - \nu\frac{\partial^2\xi^2}{\partial y^2} = 0, \tag{4.5.36}$$

$$\psi_{yy} : \nu\left[-2\frac{\partial\xi^2}{\partial y} + \frac{d\xi^1}{dx}\right] = 0, \tag{4.5.37}$$

$$\psi_{yy}^2 : -\frac{d}{dx}\left(l^2(x)\right)\xi^1 + 4l^2(x)\frac{\partial\xi^2}{\partial y} - l^2(x)\frac{d\xi^1}{dx} = 0. \tag{4.5.38}$$

From (4.5.35) and (4.5.36), $\xi^2 = \xi^2(y)$ and

$$\frac{d^2\xi^2}{dy^2} = 0. \tag{4.5.39}$$

Integrating (4.5.39) with respect to y twice we obtain

$$\xi^2(y) = a_2 y + a_3, \tag{4.5.40}$$

where a_2 and a_3 are constants.

In summary, we have the following determining equations.

$$\nu\left[-2a_2 + \frac{d\xi^1}{dx}\right] = 0, \tag{4.5.41}$$

$$-\frac{d}{dx}\left(l^2(x)\right)\xi^1 + 4a_2 l^2(x) - l^2(x)\frac{d\xi^1}{dx} = 0, \tag{4.5.42}$$

$$\xi^1 = \xi^1(x), \tag{4.5.43}$$

$$\xi^2(y) = a_2 y + a_3, \tag{4.5.44}$$

$$\eta^1(x) = a_1(x), \tag{4.5.45}$$

$$\eta^2 = \eta^2(x, y, \psi, \overline{S}). \tag{4.5.46}$$

The vector field operator associated with the conserved vector (4.3.6) is, for $\nu = 0$ and $\nu \neq 0$,

$$X = \xi^1(x)\frac{\partial}{\partial x} + (a_2 y + a_3)\frac{\partial}{\partial y} + a_1(x)\frac{\partial}{\partial \psi} + \eta^2(x, y, \psi, \overline{S})\frac{\partial}{\partial \overline{S}} \tag{4.5.47}$$

provided that (4.5.41) and (4.5.42) are satisfied.

4.5.2 Lie point symmetry associated with the momentum and thermal flux conserved vector

Expanding equation (4.5.3) using the vector field operator (4.5.47) and the conserved vector (4.3.7) we obtain

$$\eta^2 = -a_2\overline{S}. \tag{4.5.48}$$

Now expanding equation (4.5.4) using the vector field operator (4.5.47) and the conserved vector (4.3.7) we obtain

$$\begin{aligned}\frac{\kappa}{\rho c_v}\zeta_y^2 - \frac{d}{dx}\left(l(x)l_\theta(x)\right)\psi_{yy}\overline{S}_y\xi^1 - l(x)l_\theta(x)\overline{S}_y\zeta_{yy}^1 - l(x)l_\theta(x)\psi_{yy}\zeta_y^2 \\ + \left(\frac{\kappa}{\rho c_v}\overline{S}_y - l(x)l_\theta(x)\psi_{yy}\overline{S}_y\right)\frac{d\xi^1}{dx} = 0.\end{aligned} \tag{4.5.49}$$

By (4.5.6) and (4.5.29)

$$\zeta_y^2 = -2a_2\overline{S}_y, \tag{4.5.50}$$

$$\zeta_{yy}^1 = -2a_2\psi_{yy}. \tag{4.5.51}$$

Equation (4.5.49) becomes

$$\begin{aligned}&\frac{\kappa}{\rho c_v}\overline{S}_y\left(\frac{d\xi^1}{dx}-2a_2\right)+\\ &\psi_{yy}\overline{S}_y\left[-\frac{d}{dx}\left(l(x)l_\theta(x)\right)\xi^1+4a_2l(x)l_\theta(x)-\frac{d\xi^1}{dx}l(x)l_\theta(x)\right]=0.\end{aligned} \tag{4.5.52}$$

Splitting (4.5.52) into coefficients of $\overline{S}_y$ and $\psi_{yy}\overline{S}_y$ we obtain

$$\overline{S}_y : \frac{\kappa}{\rho c_v}\left(\frac{d\xi^1}{dx}-2a_2\right)=0, \tag{4.5.53}$$

$$\psi_{yy}\overline{S}_y : -\frac{d}{dx}\left(l(x)l_\theta(x)\right)\xi^1+4a_2l(x)l_\theta(x)-\frac{d\xi^1}{dx}l(x)l_\theta(x)=0. \tag{4.5.54}$$

The determining equations for the vector field operator associated with the conserved vectors (4.3.6) and (4.3.7) are (4.5.41) to (4.5.45) and (4.5.53) to (4.5.54). Equations (4.5.41) and (4.5.53) depend on whether ν and κ are zero or nonzero. We examine two cases: (i) $\nu \neq 0$ and $\kappa \neq 0$, (ii) $\nu = 0$ and $\kappa = 0$. The vector field operator associated to the conserved vectors (4.3.6) and (4.3.7) is

$$X=\xi^1(x)\frac{\partial}{\partial x}+(a_2y+a_3)\frac{\partial}{\partial y}+a_1(x)\frac{\partial}{\partial\psi}-a_2\overline{S}\frac{\partial}{\partial\overline{S}} \tag{4.5.55}$$

provided that equations (4.5.41) to (4.5.45) and (4.5.53) to (4.5.54) are satisfied.

We verify that the vector field operator (4.5.55) is a Lie point symmetry for the partial differential equations (4.2.2) and (4.2.3). Applying the prolongated vector field operator (4.5.55) to (4.2.6), subject to (4.2.6) being satisfied, we have

$$\begin{aligned}X^{[\infty]}\left(\nu\psi_{yyy}-2l^2(x)\psi_{yy}\psi_{yyy}-U\psi_{xy}\right)\Big|_{(4.2.6)}&=\nu\zeta^1_{yyy}-2\frac{d}{dx}\left(l^2(x)\right)\xi^1\psi_{yy}\psi_{yyy}\\ &-2l^2(x)\psi_{yyy}\zeta^1_{yy}-2l^2(x)\psi_{yy}\zeta^1_{yyy}-U\zeta^1_{xy}\Big|_{(4.2.6)}.\end{aligned} \tag{4.5.56}$$

Using the prolongation formulae (2.3.4) and (2.3.5), equation (4.5.56) becomes

$$\begin{aligned}X^{[\infty]}\left(\nu\psi_{yyy}-2l^2(x)\psi_{yy}\psi_{yyy}-U\psi_{xy}\right)\Big|_{(4.2.6)}&=-3\nu a_2\psi_{yyy}-2\frac{d}{dx}\left(l^2(x)\right)\xi^1\psi_{yy}\psi_{yyy}\\ &+10a_2l^2(x)\psi_{yyy}\psi_{yy}+U\left[a_2+\frac{d\xi^1}{dx}\right]\psi_{xy}\Big|_{(4.2.6)}.\end{aligned} \tag{4.5.57}$$

But from (4.2.6),

$$U\psi_{xy}=\nu\psi_{yyy}-2l^2(x)\psi_{yy}\psi_{yyy}. \tag{4.5.58}$$

Substituting (4.5.58) into the right hand side of (4.5.57), equation (4.5.57) simplifies to

$$\begin{aligned} X^{[\infty]}\left(\nu\psi_{yyy}-2l^2(x)\psi_{yy}\psi_{yyy}-U\psi_{xy}\right)\Big|_{(4.2.6)} &= \nu\left[-2a_2+\frac{d\xi^1}{dx}\right]\psi_{yyy} \\ +2\left[-\frac{d}{dx}\left(l^2(x)\right)\xi^1+4a_2l^2(x)-l^2(x)\frac{d\xi^1}{dx}\right]&\psi_{yy}\psi_{yyy}. \end{aligned} \tag{4.5.59}$$

By (4.5.41) and (4.5.42), the right hand side of (4.5.59) vanishes and

$$X^{[\infty]}\left(\nu\psi_{yyy}-2l^2(x)\psi_{yy}\psi_{yyy}-U\psi_{xy}\right)\Big|_{(4.2.6)}=0. \tag{4.5.60}$$

Applying the prolongated vector field operator (4.5.55) to (4.2.7), subject to (4.2.7) being satisfied, we have

$$\begin{aligned} &X^{[\infty]}\left(\frac{\kappa}{\rho c_v}\overline{S}_{yy}-l(x)l_\theta(x)\psi_{yyy}\overline{S}_y-l(x)l_\theta(x)\psi_{yy}\overline{S}_{yy}-U\overline{S}_x\right)\Big|_{(4.2.7)}= \\ &\frac{\kappa}{\rho c_v}\zeta^2_{yy}-\frac{d}{dx}\left(l(x)l_\theta(x)\right)\xi^1\psi_{yyy}\overline{S}_y-l(x)l_\theta(x)\overline{S}_y\zeta^1_{yyy}-l(x)l_\theta(x)\psi_{yyy}\zeta^2_y \\ &-\frac{d}{dx}\left(l(x)l_\theta(x)\right)\xi^1\psi_{yy}\overline{S}_{yy}-l(x)l_\theta(x)\overline{S}_{yy}\zeta^1_{yy}-l(x)l_\theta(x)\psi_{yy}\zeta^2_{yy}-U\zeta^2_x\Big|_{(4.2.7)} \end{aligned} \tag{4.5.61}$$

Using the prolongation formulae (2.3.4) and (2.3.5), equation (4.5.61) becomes

$$\begin{aligned} &X^{[\infty]}\left(\frac{\kappa}{\rho c_v}\overline{S}_{yy}-l(x)l_\theta(x)\psi_{yyy}\overline{S}_y-l(x)l_\theta(x)\psi_{yy}\overline{S}_{yy}-U\overline{S}_x\right)\Big|_{(4.2.7)}= \\ &-3\frac{\kappa}{\rho c_v}a_2\overline{S}_{yy}-\frac{d}{dx}\left(l(x)l_\theta(x)\right)\xi^1\psi_{yyy}\overline{S}_y+5a_2l(x)l_\theta(x)\psi_{yyy}\overline{S}_y \\ &-\frac{d}{dx}\left(l(x)l_\theta(x)\right)\xi^1\psi_{yy}\overline{S}_{yy}+5a_2l(x)l_\theta(x)\psi_{yy}\overline{S}_{yy} \\ &+U\left[a_2+\frac{d\xi^1}{dx}\right]\overline{S}_x\Big|_{(4.2.7)} \end{aligned} \tag{4.5.62}$$

But from (4.2.7),

$$U\overline{S}_x=\frac{\kappa}{\rho c_v}\overline{S}_{yy}-l(x)l_\theta(x)\psi_{yyy}\overline{S}_y-l(x)l_\theta(x)\psi_{yy}\overline{S}_{yy}. \tag{4.5.63}$$

Substituting (4.5.63) into the right hand side of (4.5.62), we obtain

$$\begin{aligned} &X^{[\infty]}\left(\frac{\kappa}{\rho c_v}\overline{S}_{yy}-l(x)l_\theta(x)\psi_{yyy}\overline{S}_y-l(x)l_\theta(x)\psi_{yy}\overline{S}_{yy}-U\overline{S}_x\right)\Big|_{(4.2.7)}= \\ &\frac{\kappa}{\rho c_v}\left[-2a_2+\frac{d\xi^1}{dx}\right] \\ &+\left[-\frac{d}{dx}\left(l(x)l_\theta(x)\right)\xi^1+4a_2l(x)l_\theta(x)-\frac{d\xi^1}{dx}l(x)l_\theta(x)\right]\psi_{yyy}\overline{S}_y \\ &+\left[-\frac{d}{dx}\left(l(x)l_\theta(x)\right)\xi^1+4a_2l(x)l_\theta(x)-\frac{d\xi^1}{dx}l(x)l_\theta(x)\right]\psi_{yy}\overline{S}_{yy}. \end{aligned} \tag{4.5.64}$$

By (4.5.53) and (4.5.54), the right hand side of (4.5.64) vanishes and

$$X^{[\infty]}\left(\frac{\kappa}{\rho c_v}\overline{S}_{yy}-l(x)l_\theta(x)\psi_{yyy}\overline{S}_y-l(x)l_\theta(x)\psi_{yy}\overline{S}_{yy}-U\overline{S}_x\right)\Big|_{(4.2.7)}=0. \tag{4.5.65}$$

Hence, the vector field (4.5.55) is a Lie point symmetry for the partial differential equations (4.2.2) and (4.2.3).

4.5.3 Mixing lengths

In this Subsection, we show that from equations (4.5.42) and (4.5.54) that the momentum and the thermal mixing lengths are proportional.

Making $\frac{d\xi^1}{dx}$ the subject of (4.5.42) we obtain

$$\frac{d\xi^1}{dx}=4a_2-\frac{1}{l^2(x)}\frac{d}{dx}\left(l^2(x)\right)\xi^1. \tag{4.5.66}$$

Substituting (4.5.66) into (4.5.54) we obtain

$$\frac{d}{dx}\left(l(x)l_\theta(x)\right)\xi^1=\frac{l_\theta(x)}{l(x)}\frac{d}{dx}\left(l^2(x)\right)\xi^1. \tag{4.5.67}$$

In this thesis, we consider the general case in which $\xi^1(x)\neq 0$. Diving both sides of (4.5.67) by $\xi^1(x)$ we obtain

$$\frac{d}{dx}\left(l(x)l_\theta(x)\right)=\frac{l_\theta(x)}{l(x)}\frac{d}{dx}\left(l^2(x)\right). \tag{4.5.68}$$

By expanding the derivatives we obtain

$$l_\theta(x)\frac{dl}{dx}=l(x)\frac{dl_\theta}{dx} \tag{4.5.69}$$

and therefore

$$\frac{dl_\theta}{l_\theta}=\frac{dl}{l}. \tag{4.5.70}$$

Hence

$$\ln(l_\theta(x))=\ln(l(x))+C, \tag{4.5.71}$$

where C is a constant. Thus

$$\ln\left(\frac{l_\theta}{l}\right)=C \tag{4.5.72}$$

and

$$l_\theta(x)=Bl(x), \tag{4.5.73}$$

where $B=e^C$.

Equations (4.5.42) and (4.5.73) are equivalent to equations (4.5.42) and (4.5.54). Hence in the following Subsections we consider equations (4.5.42) and (4.5.73) for the mixing lengths. We do not need to consider equation (4.5.54) in addition.

4.5.4 Case $\nu \neq 0$ and $\kappa \neq 0$

When $\nu \neq 0$ and $\kappa \neq 0$ equations (4.5.41) and (4.5.53) become

$$\frac{d\xi^1}{dx} - 2a_2 = 0. \tag{4.5.74}$$

Therefore

$$\xi^1(x) = 2a_2x + a_4, \tag{4.5.75}$$

where a_4 is a constant. Substituting (4.5.75) into (4.5.42) we obtain the variables separable differential equation

$$\frac{1}{l^2(x)}\frac{d}{dx}\left(l^2(x)\right) = \frac{2a_2}{(2a_2x + a_4)}. \tag{4.5.76}$$

In this thesis, we are considering the general case when $\xi^1(x) \neq 0$ and $a_2 \neq 0$. Hence equation (4.5.76) can be written as

$$\frac{1}{l^2(x)}\frac{d}{dx}\left(l^2(x)\right) = \frac{1}{\left(x + \frac{a_4}{2a_2}\right)}. \tag{4.5.77}$$

Integrating (4.5.77) we obtain

$$\ln(l^2(x)) = \ln\left(x + \frac{a_4}{2a_2}\right) + C, \tag{4.5.78}$$

where C is a constant. Therefore

$$l(x) = l_1\left(x + \frac{a_4}{2a_2}\right)^{\frac{1}{2}}, \tag{4.5.79}$$

where $l_1 = e^{C/2}$ and the positive square root is taken because $l(x) > 0$. From (4.5.73)

$$l_\theta(x) = l_2\left(x + \frac{a_4}{2a_2}\right)^{\frac{1}{2}}, \tag{4.5.80}$$

where $l_2 = Bl_1$.

Without loss of generality, we let

$$\xi^{1*}(x) = \frac{\xi^1(x)}{a_2}, \quad \xi^{2*}(y) = \frac{\xi^2(y)}{a_2}, \quad \eta^{1*}(x) = \frac{\eta^1(x)}{a_2}, \quad \eta^{2*}(\overline{S}) = \frac{\eta^2(\overline{S})}{a_2}, \tag{4.5.81}$$

$$a_1^*(x) = \frac{a_1(x)}{a_2}, \qquad a_3^* = \frac{a_3}{a_2}, \qquad a_4^* = \frac{a_4}{a_2}. \tag{4.5.82}$$

To simplify the notation, we omit the asterisks in the following Subsections. Therefore, when $\nu \neq 0$ and $\kappa \neq 0$, the Lie point symmetry associated with the conserved vectors (4.3.6) and (4.3.7) is

$$X = (2x + a_4)\frac{\partial}{\partial x} + (y + a_3)\frac{\partial}{\partial x} + a_1(x)\frac{\partial}{\partial \psi} - \overline{S}\frac{\partial}{\partial \overline{S}}. \tag{4.5.83}$$

The mixing lengths are

$$l(x) = l_1\left(x + \frac{1}{2}a_4\right)^{\frac{1}{2}}, \qquad l_\theta(x) = l_2\left(x + \frac{1}{2}a_4\right)^{\frac{1}{2}}. \tag{4.5.84}$$

4.5.5 Case $\nu = 0$ and $\kappa = 0$

When $\nu = 0$ and $\kappa = 0$ the determining equations for the associated Lie point symmetry and the mixing lengths are (4.5.42) to (4.5.45) and (4.5.73). Again, without loss of generality, we let

$$\xi^{1*}(x) = \frac{\xi^1(x)}{a_2}, \quad \xi^{2*}(y) = \frac{\xi^2(y)}{a_2}, \quad \eta^{1*}(x,\psi) = \frac{\eta^1(x,\psi)}{a_2}, \quad \eta^{2*}(\overline{S}) = \frac{\eta^2(\overline{S})}{a_2}, \tag{4.5.85}$$

$$a_1^*(x) = \frac{a_1(x)}{a_2}, \qquad a_3^* = \frac{a_3}{a_2}. \tag{4.5.86}$$

We omit the asterisks from here on to simplify notation. Hence the determining equations become

$$-\frac{d}{dx}\left(l^2(x)\right)\xi^1 + 4l^2(x) - l^2(x)\frac{d\xi^1}{dx} = 0, \tag{4.5.87}$$

$$l_\theta(x) = Bl(x), \tag{4.5.88}$$

$$\xi^2(y) = y + a_3, \tag{4.5.89}$$

$$\eta^1 = a_1(x), \tag{4.5.90}$$

$$\eta^2(\overline{S}) = -\overline{S}. \tag{4.5.91}$$

When $\nu = 0$ and $\kappa = 0$, the Lie point symmetry associated with the conserved vectors (4.3.6) and (4.3.7) is

$$X = \xi^1(x)\frac{\partial}{\partial x} + (y + a_3)\frac{\partial}{\partial y} + a_1(x)\frac{\partial}{\partial \psi} - \overline{S}\frac{\partial}{\partial \overline{S}} \tag{4.5.92}$$

provided $\xi^1(x)$, $l(x)$ and $l_\theta(x)$ satisfy (4.5.87) and (4.5.88).

The mixing length, $l(x)$, and $\xi^1(x)$ satisfy (4.5.87). In order to obtain $l(x)$ and $\xi^1(x)$ a second equation is required relating $l(x)$ and $\xi^1(x)$. This equation will be derived later in the Chapter.

4.6 Invariant solutions

In this Section, we reduce the governing partial differential equations to ordinary differential equations by finding the invariant solutions for the cases $\nu \neq 0$, $\kappa \neq 0$ and $\nu = 0$, $\kappa = 0$.

4.6.1 Case $\nu \neq 0$ and $\kappa \neq 0$

When $\nu \neq 0$ and $\kappa \neq 0$, the associated Lie point symmetry is

$$X = (2x + a_4)\frac{\partial}{\partial x} + (y + a_3)\frac{\partial}{\partial x} + a_1(x)\frac{\partial}{\partial \psi} - \overline{S}\frac{\partial}{\partial \overline{S}}. \tag{4.6.1}$$

The solution $\psi = f(x, y)$ and $\overline{S} = g(x, y)$ is an invariant of the governing partial differential equations (4.2.2) and (4.2.3) provided

$$X(\psi - f)\Big|_{\psi=f(x,y)} = 0, \tag{4.6.2}$$

$$X(\overline{S} - g)\Big|_{\overline{S}=g(x,y)} = 0. \tag{4.6.3}$$

Expanding equation (4.6.2) we obtain the partial differential equation

$$(2x + a_4)\frac{\partial f}{\partial x} + (y + a_3)\frac{\partial f}{\partial y} = a_1(x). \tag{4.6.4}$$

The differential equations of the characteristic curves of the partial differential equation (4.6.4) are

$$\frac{dx}{2\left(x + \frac{1}{2}a_4\right)} = \frac{dy}{y + a_3} = \frac{df}{a_1(x)}. \tag{4.6.5}$$

Integrating the first and second terms of (4.6.5) we obtain

$$\ln(y + a_3) = \frac{1}{2}\ln\left(x + \frac{1}{2}a_4\right) + C', \tag{4.6.6}$$

where C' is a constant. Exponentiating (4.6.6) we obtain

$$I_1 = \frac{y + a_3}{\left(x + \frac{1}{2}a_4\right)^{\frac{1}{2}}}, \tag{4.6.7}$$

where $I_1 = e^{C'}$ is a constant. Now integrating the equation consisting of the first and last terms of (4.6.5) we obtain

$$f(x, y) - \frac{1}{2}\int^x \frac{a_1(\tau)d\tau}{\left(\tau + \frac{1}{2}a_4\right)} = I_2, \tag{4.6.8}$$

where I_2 is a constant. The general solution of the first order partial differential equation (4.6.4) is

$$I_2 = F(I_1), \tag{4.6.9}$$

where F is an arbitrary function. Since $\psi = f(x, y)$,

$$\psi(x, y) = \frac{1}{2}\int^x \frac{a_1(\tau)}{\left(\tau + \frac{1}{2}a_4\right)} + F(\xi) \tag{4.6.10}$$

where

$$\xi = \frac{y + a_3}{\left(x + \frac{1}{2}a_4\right)^{\frac{1}{2}}}. \tag{4.6.11}$$

Expanding equation (4.6.3) we obtain the partial differential equation

$$(2x + a_4)\frac{\partial g}{\partial x} + (y + a_3)\frac{\partial g}{\partial y} = -g. \tag{4.6.12}$$

The differential equations of the characteristic curves of the partial differential equation (4.6.12) are

$$\frac{dx}{2\left(x + \frac{1}{2}a_4\right)} = \frac{dy}{y + a_3} = -\frac{dg}{g}. \tag{4.6.13}$$

We have already solved the equation consisting of the first and second terms of (4.6.13) and we obtained (4.6.7). Now integrating the first and last terms of (4.6.13) we obtain

$$\ln g + \frac{1}{2}\ln\left(x + \frac{1}{2}a_4\right) = C', \tag{4.6.14}$$

where C' is a constant. Exponentiating (4.6.14) we obtain

$$I_3 = \left(x + \frac{1}{2}a_4\right)^{\frac{1}{2}} g, \tag{4.6.15}$$

where $I_3 = e^{C'}$ is a constant. The general solution of the first order partial differential equation (4.6.12) is

$$I_3 = G(I_1), \tag{4.6.16}$$

where G is an arbitrary function. Therefore, since $\overline{S} = g(x, y)$,

$$\overline{S}(x, y) = \left(x + \frac{1}{2}a_4\right)^{-\frac{1}{2}} G(\xi), \tag{4.6.17}$$

where ξ is defined by (4.6.11). Writing the conserved quantity (4.4.10) in terms of the invariant solution (4.6.10) and (4.6.11) we obtain

$$D = 2\rho U \int_{L(x)}^{\xi_B(x)} \frac{dF}{d\xi} d\xi, \tag{4.6.18}$$

where

$$L(x) = \frac{a_3}{\left(x + \frac{1}{2}a_4\right)^{1/2}}, \qquad \xi_B(x) = \frac{y_B(x) + a_3}{\left(x + \frac{1}{2}a_4\right)^{1/2}}. \tag{4.6.19}$$

Since D is a constant independent of x, $\frac{dD}{dx} = 0$ and

$$\frac{d\xi_B}{dx}\frac{dF}{d\xi}(\xi_B(x)) - \frac{dL}{dx}\frac{dF}{d\xi}(L(x)) = 0. \tag{4.6.20}$$

The mean velocity deficit

$$\overline{w}(x, y) = \frac{\partial \psi}{\partial y} = \frac{1}{\left(x + \frac{1}{2}a_4\right)^{1/2}}\frac{dF}{d\xi} \tag{4.6.21}$$

vanishes at the boundary of the wake $y = y_B(x)$ and is nonzero at the centreline, $y = 0$. At the boundary $\xi = \xi_B(x)$ and at the centreline $\xi = L(x)$. Hence

$$\frac{dF}{d\xi}(\xi_B(x)) = 0 \tag{4.6.22}$$

and

$$\frac{dF}{d\xi}(L(x)) \neq 0. \tag{4.6.23}$$

The first term of (4.6.20) vanishes and equation (4.6.20) reduces to

$$\frac{dL}{dx} = 0. \tag{4.6.24}$$

Therefore, $L(x)$ is a constant and $a_3 = 0$. The upper bound of the integral in (4.6.18) becomes

$$\xi_B(x) = \frac{y_B(x)}{\left(x + \frac{1}{2}a_4\right)^{1/2}}. \tag{4.6.25}$$

The mean velocity gradient

$$\frac{\partial \overline{w}}{\partial y} = \frac{\partial^2 \psi}{\partial y^2} = \frac{1}{\left(x + \frac{1}{2}a_4\right)} \frac{d^2F}{d\xi^2} \tag{4.6.26}$$

and $\frac{d^2F}{d\xi^2}$ vanish at the boundary $\xi = \xi_B(x)$. Assuming that F is analytic at $\xi = \xi_B(x)$ and not a constant, the derivatives of F at $\xi_B(x)$ cannot be all zero. Therefore, there exists an integer $n \geq 3$ such that

$$\frac{d^{n-1}F}{d\xi^{n-1}}(\xi_B(x)) = 0 \tag{4.6.27}$$

and

$$\frac{d^nF}{d\xi^n}(\xi_B(x)) \neq 0. \tag{4.6.28}$$

Differentiating (4.6.27) with respect to x, we have

$$\frac{d^nF}{d\xi^n}(\xi_B(x))\frac{d\xi_B}{dx} = 0. \tag{4.6.29}$$

Therefore $\xi_B(x)$ is a constant and

$$y_B(x) = \xi_B \left(x + \frac{1}{2}a_4\right)^{1/2}. \tag{4.6.30}$$

The conserved quantity (4.6.18) becomes

$$D = 2\rho U \int_0^{\xi_B} \frac{dF}{d\xi} d\xi. \tag{4.6.31}$$

The half-width of the wake $y_B(x)$, (4.6.30), is proportional to the mixing lengths, (4.5.79) and (4.5.80). This is consistent with Prandtl's hypothesis [4, 5] that states that the momentum mixing length is proportional to the momentum boundary of the wake. When the boundary of the wake is infinite $\xi_B = \infty$. Writing the thermal conserved quantity (4.4.18) in terms of the invariant solution (4.6.17) and (4.6.11) we

obtain

$$K = 2\rho c_v U \int_0^{\xi_B} G(\xi) d\xi. \tag{4.6.32}$$

The mean velocity deficit

$$\overline{w}(x,y) = \frac{\partial \psi}{\partial y} = \frac{1}{\left(x + \frac{1}{2}a_4\right)^{\frac{1}{2}}} \frac{dF}{d\xi} \tag{4.6.33}$$

should be finite at all nonnegative values of x except possibly at the obstacle, $x = 0$. Hence the value for a_4 is zero. This is consistent with the fact that the mixing length must vanish close to the obstacle since the flow is laminar there. Since $a_3 = 0$ and $a_4 = 0$ it follows from (4.6.11) that

$$\xi = \frac{y}{x^{1/2}}. \tag{4.6.34}$$

The boundary condition

$$\frac{\partial \psi}{\partial x}(x,0) = 0 \tag{4.6.35}$$

gives

$$a_1(x) - \xi \frac{dF}{d\xi}\bigg|_{\xi=0} = 0. \tag{4.6.36}$$

Since $\frac{dF}{d\xi}(0)$ is finite, because $\overline{w}(x,0)$ given by (4.6.33) is finite, it follows that $a_1(x) = 0$. Writing boundary conditions (4.2.4) and (4.2.5) in terms of the invariant solution (4.6.10), (4.6.17) and (4.6.11) we obtain

$$\frac{d^2F}{d\xi^2}(0) = 0, \qquad \frac{dF}{d\xi}(\xi_B) = 0, \qquad \frac{d^2F}{d\xi^2}(\xi_B) = 0, \tag{4.6.37}$$

$$\frac{dG}{d\xi}(0) = 0, \qquad G(\xi_B) = 0, \qquad \frac{dG}{d\xi}(\xi_B) = 0. \tag{4.6.38}$$

Substituting the invariant solution (4.6.10), (4.6.17), (4.6.11) and the mixing lengths (4.5.79) and (4.5.80) into the governing partial differential equations (4.2.2) and (4.2.3), all with $a_1 = a_3 = a_4 = 0$, we obtain

$$\frac{d}{d\xi}\left[\left(\nu - l_1^2 \frac{d^2F}{d\xi^2}\right)\frac{d^2F}{d\xi^2}\right] + \frac{U}{2}\frac{d}{d\xi}\left(\xi \frac{dF}{d\xi}\right) = 0, \tag{4.6.39}$$

$$\frac{d}{d\xi}\left[\left(\frac{\kappa}{\rho c_v} - l_1 l_2 \frac{d^2F}{d\xi^2}\right)\frac{dG}{d\xi}\right] + \frac{U}{2}\frac{d}{d\xi}(\xi G) = 0. \tag{4.6.40}$$

The differential equations (4.6.39) and (4.6.40) can be integrated at least once. This demonstrates the double reduction theorem by Sjöberg [17, 18]. Equation (4.6.39) agrees with the ordinary differential equation for the invariant solution obtained by Hutchinson and Mason [8].

4.6.2 Case $\nu = 0$ and $\kappa = 0$

When $\nu = 0$ and $\kappa = 0$ the associated Lie point symmetry is, (4.5.92),

$$X = \xi^1(x)\frac{\partial}{\partial x} + (y + a_3)\frac{\partial}{\partial y} + a_1(x)\frac{\partial}{\partial \psi} - \overline{S}\frac{\partial}{\partial \overline{S}}. \tag{4.6.41}$$

There are two determining equations, (4.5.87) and (4.5.88), for the three unknowns $\xi^1(x)$, $l(x)$ and $l_\theta(x)$. The third equation is obtained by making use of Prandtl's hypothesis that the half-width of the wake is proportional to the mixing lengths.

An invariant solution $\psi = f(x, y)$ for the governing partial differential equations (4.2.2) and (4.2.3) satisfies

$$X\left(\psi - f(x,y)\right)\Big|_{\psi=f(x,y)} = 0. \tag{4.6.42}$$

Expanding equation (4.6.42) we obtain the partial differential equation

$$\xi^1(x)\frac{\partial f}{\partial x} + (y + a_3)\frac{\partial f}{\partial y} = a_1(x). \tag{4.6.43}$$

The differential equations of the characteristic curves of the partial differential equation (4.6.43) are

$$\frac{dx}{\xi^1(x)} = \frac{dy}{y + a_3} = \frac{df}{a_1(x)}. \tag{4.6.44}$$

Integrating the first and second terms of (4.6.44) we obtain

$$C' + \int^x \frac{d\tau}{\xi^1(\tau)} = \ln(y + a_3), \tag{4.6.45}$$

where C' is a constant. Exponentiating (4.6.45) we obtain

$$I_1 = (y + a_3)\exp\left(-\int^x \frac{d\tau}{\xi^1(\tau)}\right), \tag{4.6.46}$$

where $I_1 = e^{C'}$ is a constant. Integrating the equation consisting of the first and last terms of (4.6.44) we obtain

$$f - \int^x \frac{a_1(\tau)d\tau}{\xi^1(\tau)} = I_2, \tag{4.6.47}$$

where I_2 is a constant. The general solution for the first order partial differential equation (4.6.43) is

$$I_2 = F(I_1), \tag{4.6.48}$$

where F is an arbitrary function. Therefore, since $\psi = f(x, y)$, the invariant solution is

$$\psi(x,y) = F(\xi) + \int^x \frac{a_1(\tau)d\tau}{\xi^1(\tau)}, \tag{4.6.49}$$

where

$$\xi = (y + a_3)\exp\left(-\int^x \frac{d\tau}{\xi^1(\tau)}\right). \tag{4.6.50}$$

Substituting the invariant solution (4.6.49) and (4.6.50) into the conserved quantity (4.4.10) we obtain

$$D = 2\rho U \int_{L(x)}^{\xi_B(x)} \frac{dF}{d\xi} d\xi, \tag{4.6.51}$$

where

$$\xi_B(x) = (y_B(x) + a_3) \exp\left(- \int^x \frac{d\tau}{\xi^1(\tau)} \right), \tag{4.6.52}$$

$$L(x) = a_3 \exp\left(- \int^x \frac{d\tau}{\xi^1(\tau)} \right). \tag{4.6.53}$$

Since the conserved quantity D is independent of x, $\frac{dD}{dx} = 0$ and

$$\frac{d\xi_B}{dx} \frac{dF}{d\xi}(\xi_B(x)) - \frac{dL}{dx} \frac{dF}{d\xi}(L(x)) = 0. \tag{4.6.54}$$

The mean velocity deficit

$$\overline{w}(x, y) = \exp\left(- \int^x \frac{d\tau}{\xi^1(\tau)} \right) \frac{dF}{d\xi} \tag{4.6.55}$$

vanishes at the boundary of the wake $y = y_B(x)$ and is nonzero at the centreline, $y = 0$. At the boundary $\xi = \xi_B(x)$ and at the centreline $\xi = L(x)$. Hence

$$\frac{dF}{d\xi}(\xi_B(x)) = 0 \tag{4.6.56}$$

and

$$\frac{dF}{d\xi}(L(x)) \neq 0. \tag{4.6.57}$$

The first term of (4.6.54) vanishes and equation (4.6.54) reduces to

$$\frac{dL}{dx} = 0. \tag{4.6.58}$$

Therefore,

$$\frac{a_3}{\xi^1(x)} \exp\left(- \int^x \frac{d\tau}{\xi^1(\tau)} \right) = 0 \tag{4.6.59}$$

and $a_3 = 0$. The upper bound of the integral in (4.6.51) becomes

$$\xi_B(x) = y_B(x) \exp\left(- \int^x \frac{d\tau}{\xi^1(\tau)} \right). \tag{4.6.60}$$

The mean velocity gradient

$$\frac{\partial \overline{w}}{\partial y} = \frac{\partial^2 \psi}{\partial y^2} = \exp\left(-2 \int^x \frac{d\tau}{\xi^1(\tau)} \right) \frac{d^2 F}{d\xi^2} \tag{4.6.61}$$

and $\frac{d^2F}{d\xi^2}$ vanish at the boundary $\xi = \xi_B(x)$. Assuming that F is analytic at $\xi = \xi_B(x)$ and not a constant, the derivatives of F at $\xi_B(x)$ cannot be all zero. Therefore, there exists an integer $n \geq 3$ such that

$$\frac{d^{n-1}F}{d\xi^{n-1}}(\xi_B(x)) = 0 \tag{4.6.62}$$

and

$$\frac{d^n F}{d\xi^n}(\xi_B(x)) \neq 0. \tag{4.6.63}$$

Differentiating (4.6.62) with respect to x, we have

$$\frac{d^n F}{d\xi^n}(\xi_B(x))\frac{d\xi_B}{dx} = 0. \tag{4.6.64}$$

Hence $\xi_B(x)$ is a constant and

$$y_B(x) = \xi_B \exp\left(\int^x \frac{d\tau}{\xi^1(\tau)}\right). \tag{4.6.65}$$

By Prandtl's hypothesis, the mixing length $l(x)$ is proportional to the half-width of the wake $y_B(x)$. Therefore

$$l(x) = l_1 \exp\left(\int^x \frac{d\tau}{\xi^1(\tau)}\right), \tag{4.6.66}$$

where l_1 is a constant. Substituting (4.6.66) into (4.5.87) we obtain

$$2 - \frac{d\xi^1}{dx} = 0. \tag{4.6.67}$$

Hence

$$\xi^1(x) = 2x + a_4, \tag{4.6.68}$$

where a_4 is a constant. Substituting (4.6.68) into (4.6.66) we obtain

$$l(x) = l_1\left(x + \frac{1}{2}a_4\right)^{\frac{1}{2}}, \tag{4.6.69}$$

where the constant of integration is incorporated in l_1. By (4.5.88) the thermal mixing length is

$$l_\theta(x) = l_2\left(x + \frac{1}{2}a_4\right)^{\frac{1}{2}}, \tag{4.6.70}$$

where $l_2 = Bl_1$ is a constant. Hence the invariant solution (4.6.49) and (4.6.50), by (4.6.68), becomes

$$\psi(x,y) = \int^x \frac{a_1(\tau)d\tau}{2\tau + a_4} + F(\xi), \tag{4.6.71}$$

where, since $a_3 = 0$,

$$\xi = y\exp\left(-\int^x \frac{d\tau}{\xi^1(\tau)}\right) = \frac{y}{\left(x + \frac{1}{2}a_4\right)^{1/2}}. \tag{4.6.72}$$

The mean velocity deficit is

$$\overline{w}(x,y) = \frac{\partial\psi}{\partial y} = \frac{1}{\left(x + \frac{1}{2}a_4\right)^{\frac{1}{2}}}\frac{dF}{d\xi}. \tag{4.6.73}$$

Again, since $\overline{w}(x,y)$ must be finite at all nonnegative values of x except possibly at $x = 0$ it follows that $a_4 = 0$. Therefore the associated Lie point symmetry (4.6.41)

becomes

$$X = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + a_1(x)\frac{\partial}{\partial \psi} - \overline{S}\frac{\partial}{\partial \overline{S}}. \tag{4.6.74}$$

An invariant solution $\overline{S} = g(x, y)$ to the governing partial differential equations (4.2.2) and (4.2.3) satisfies

$$X\left(\overline{S} - g(x, y)\right)\Big|_{\overline{S}=g(x,y)} = 0, \tag{4.6.75}$$

where X is given by (4.6.74). Expanding (4.6.75) we obtain the partial differential equation

$$2x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = -g. \tag{4.6.76}$$

The differential equations of the characteristic curves of the partial differential equation (4.6.76) are

$$\frac{dx}{2x} = \frac{dy}{y} = -\frac{dg}{g}. \tag{4.6.77}$$

Integrating the first and second terms of (4.6.77) we obtain

$$C' + \frac{1}{2}\ln x = \ln y, \tag{4.6.78}$$

where C' is a constant. Exponentiating (4.6.78) we obtain

$$I_1 = \frac{y}{x^{\frac{1}{2}}}, \tag{4.6.79}$$

where $I_1 = e^{C'}$ is a constant. Integrating the first and last terms of (4.6.77) we obtain

$$-\ln g = \frac{1}{2}\ln x + C', \tag{4.6.80}$$

where C' is a constant. Exponentiating (4.6.80) we obtain

$$g x^{\frac{1}{2}} = I_2, \tag{4.6.81}$$

where $I_2 = e^{-C'}$ is a constant. The general solution of the first order partial differential equation (4.6.76) is

$$I_2 = G(I_1). \tag{4.6.82}$$

Since $\overline{S} = g(x, y)$, the invariant solution for the mean temperature difference is

$$\overline{S}(x, y) = x^{-\frac{1}{2}} G(\xi), \tag{4.6.83}$$

where

$$\xi = \frac{y}{x^{\frac{1}{2}}}. \tag{4.6.84}$$

Substituting the invariant solution (4.6.49), (4.6.83) and (4.6.84) into the conserved quantities (4.4.10) and (4.4.18) we obtain

$$D = 2\rho U \int_0^{\xi_B} \frac{dF}{d\xi} d\xi, \tag{4.6.85}$$

$$K = 2\rho c_v U \int_0^{\xi_B} G(\xi) d\xi, \tag{4.6.86}$$

where, from (4.6.65),

$$\xi_B = y_B(x) \exp\left(- \int^x \frac{d\tau}{\xi^1(\tau)} \right) = \frac{y_B(x)}{x^{1/2}}. \tag{4.6.87}$$

Again, the boundary condition

$$\frac{\partial \psi}{\partial x}(x, 0) = 0 \tag{4.6.88}$$

gives $a_1(x) = 0$. Writing the remaining boundary conditions (4.2.4) and (4.2.5) in terms of the invariant solution (4.6.49), with $a_1(x) = 0$, and (4.6.83) we obtain

$$\frac{d^2F}{d\xi^2}(0) = 0, \qquad \frac{dF}{d\xi}(\xi_B) = 0, \qquad \frac{d^2F}{d\xi^2}(\xi_B) = 0, \tag{4.6.89}$$

$$\frac{dG}{d\xi}(0) = 0, \qquad G(\xi_B) = 0. \tag{4.6.90}$$

Substituting the invariant solution (4.6.71) with $a_1(x) = 0$, (4.6.83), (4.6.84) and the mixing lengths (4.6.69), (4.6.70) with $a_4 = 0$ into the governing partial differential equations (4.2.2) and (4.2.3), with $\nu = 0$ and $\kappa = 0$, we obtain

$$l_1^2 \frac{d}{d\xi} \left[\left(\frac{d^2F}{d\xi^2} \right)^2 \right] - \frac{U}{2} \frac{d}{d\xi} \left(\xi \frac{dF}{d\xi} \right) = 0, \tag{4.6.91}$$

$$l_1 l_2 \frac{d}{d\xi} \left[\frac{d^2F}{d\xi^2} \frac{dG}{d\xi} \right] - \frac{U}{2} \frac{d}{d\xi} (\xi G) = 0. \tag{4.6.92}$$

The differential equations (4.6.91) and (4.6.92) can be integrated at least once. This demonstrates the double reduction theorem by Sjöberg [17, 18].

4.7 Summary of results

We summarize and compare the results for $\nu \neq 0$ and $\kappa \neq 0$ with the results for $\nu = 0$ and $\kappa = 0$.

(i) The case $\nu \neq 0$ and $\kappa \neq 0$

Lie point symmetry:

$$X = 2x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - \overline{S} \frac{\partial}{\partial \overline{S}} \tag{4.7.1}$$

Fluid variables:

$$\psi(x, y) = F(\xi), \qquad \overline{S}(x, y) = x^{-1/2} G(\xi), \qquad \xi = \frac{y}{x^{1/2}}, \tag{4.7.2}$$

$$\overline{w}(x,y) = \frac{\partial \psi}{\partial y} = \frac{1}{x^{1/2}} \frac{dF}{d\xi}, \qquad \overline{v}_y(x,y) = \frac{\partial \psi}{\partial x} = -\frac{1}{2x} \xi \frac{dF}{d\xi}. \tag{4.7.3}$$

Ordinary differential equations:

$$\frac{d}{d\xi} \left[\left(\nu - l_1^2 \frac{d^2F}{d\xi^2} \right) \frac{d^2F}{d\xi^2} \right] + \frac{1}{2} U \frac{d}{d\xi} \left(\xi \frac{dF}{d\xi} \right) = 0 \tag{4.7.4}$$

$$\frac{d}{d\xi} \left[\left(\frac{\kappa}{\rho c_v} - l_1 l_2 \frac{d^2F}{d\xi^2} \right) \frac{dG}{d\xi} \right] + \frac{1}{2} U \frac{d}{d\xi} \left(\xi G \right) = 0 \tag{4.7.5}$$

Boundary conditions:

$$\frac{d^2F}{d\xi^2}(0) = 0, \qquad \frac{dF}{d\xi}(\xi_B) = 0, \qquad \frac{d^2F}{d\xi^2}(\xi_B) = 0 \tag{4.7.6}$$

$$\frac{dG}{d\xi}(0) = 0, \qquad G(\xi_B) = 0, \qquad \frac{dG}{d\xi}(\xi_B) = 0 \tag{4.7.7}$$

Conserved quantities:

$$D = 2\rho U \int_0^{\xi_B} \frac{dF}{d\xi} d\xi, \qquad K = 2\rho c_v U \int_0^{\xi_B} G(\xi) d\xi \tag{4.7.8}$$

Mixing lengths:

$$l(x) = l_1 x^{1/2}, \qquad l_\theta(x) = l_2 x^{1/2} \tag{4.7.9}$$

Boundary:

$$y_B(x) = \xi_B x^{1/2} \tag{4.7.10}$$

(ii) The case $\nu = 0$ and $\kappa = 0$

Lie point symmetry:

$$X = 2x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - \overline{S} \frac{\partial}{\partial \overline{S}} \tag{4.7.11}$$

Fluid variables:

$$\psi(x,y) = F(\xi), \qquad \overline{S}(x,y) = x^{-1/2} G(\xi), \qquad \xi = \frac{y}{x^{1/2}}, \tag{4.7.12}$$

$$\overline{w}(x,y) = \frac{\partial \psi}{\partial y} = \frac{1}{x^{1/2}} \frac{dF}{d\xi}, \qquad \overline{v}_y(x,y) = \frac{\partial \psi}{\partial x} = -\frac{1}{2x} \xi \frac{dF}{d\xi}. \tag{4.7.13}$$

Ordinary differential equations:

$$l_1^2 \frac{d}{d\xi}\left[\left(\frac{d^2F}{d\xi^2}\right)^2\right] - \frac{U}{2}\frac{d}{d\xi}\left(\xi\frac{dF}{d\xi}\right) = 0 \tag{4.7.14}$$

$$l_1 l_2 \frac{d}{d\xi}\left[\frac{d^2F}{d\xi^2}\frac{dG}{d\xi}\right] - \frac{U}{2}\frac{d}{d\xi}(\xi G) = 0 \tag{4.7.15}$$

Boundary conditions:

$$\frac{d^2F}{d\xi^2}(0) = 0, \qquad \frac{dF}{d\xi}(\xi_B) = 0, \qquad \frac{d^2F}{d\xi^2}(\xi_B) = 0 \tag{4.7.16}$$

$$\frac{dG}{d\xi}(0) = 0, \qquad G(\xi_B) = 0 \tag{4.7.17}$$

Conserved quantities:

$$D = 2\rho U \int_0^{\xi_B} \frac{dF}{d\xi} d\xi, \qquad K = 2\rho c_v U \int_0^{\xi_B} G(\xi) d\xi \tag{4.7.18}$$

Mixing lengths:

$$l(x) = l_1 x^{1/2}, \qquad l_\theta(x) = l_2 x^{1/2} \tag{4.7.19}$$

Boundary:

$$y_B(x) = \xi_B x^{1/2} \tag{4.7.20}$$

We see that the conserved quantity D and the fluid variables $\overline{w}(x,y)$ and $\overline{v}_y(x,y)$ depend on $\frac{dF}{d\xi}$ and not $F(\xi)$. In order to calculate the fluid variables it will therefore be sufficient to solve for $\frac{dF}{d\xi}$. However to derive the streamlines defined by $\psi(x,y) =$ constant, $F(\xi)$ will be required.

The derivation of the result for $\nu = 0$ and $\kappa = 0$ was at times different from that for $\nu \neq 0$ and $\kappa \neq 0$. For example, the Prandtl hypothesis was required for $\nu = 0$ and $\kappa = 0$ to determine the Lie point symmetry and the boundary of the wake. However, the final results for the two cases are essentially the same. The only difference being the ordinary differential equations and the extra boundary condition, $\frac{dG}{d\xi}(\xi_B) = 0$, when $\kappa \neq 0$.

4.8 Analytical solution for $\nu = 0$ and $\kappa = 0$

In this Section, we derive an analytical solution for the turbulent classical thermal wake for $\nu = 0$ and $\kappa = 0$.

The differential equations for the invariant solution when $\nu = 0$ and $\kappa = 0$ are, (4.7.14) and (4.7.15),

$$l_1^2 \frac{d}{d\xi}\left[\left(\frac{d^2F}{d\xi^2}\right)^2\right] - \frac{U}{2}\frac{d}{d\xi}\left(\xi\frac{dF}{d\xi}\right) = 0, \tag{4.8.1}$$

$$l_1 l_2 \frac{d}{d\xi}\left[\frac{d^2F}{d\xi^2}\frac{dG}{d\xi}\right] - \frac{U}{2}\frac{d}{d\xi}(\xi G) = 0. \tag{4.8.2}$$

Integrating (4.8.1) we obtain

$$l_1^2\left(\frac{d^2F}{d\xi^2}\right)^2 - \frac{U}{2}\xi\frac{dF}{d\xi} = C, \tag{4.8.3}$$

where C is a constant. Applying the first boundary condition in (4.6.89), we obtain $C = 0$. Equation (4.8.3) becomes

$$\left(\frac{d^2F}{d\xi^2}\right)^2 = \frac{U}{2l_1^2}\xi\frac{dF}{d\xi}. \tag{4.8.4}$$

Taking the square root of equation (4.8.4) we obtain

$$\frac{d^2F}{d\xi^2} = \pm\frac{1}{l_1}\left(\frac{U}{2}\right)^{1/2}\xi^{\frac{1}{2}}\left(\frac{dF}{d\xi}\right)^{\frac{1}{2}}. \tag{4.8.5}$$

The second derivative of F is real since

$$\overline{w}(x,y) = \frac{1}{x^{1/2}}\frac{dF}{d\xi} > 0. \tag{4.8.6}$$

For a classical wake, the mean velocity gradient along the y-axis is negative in the upper half of the wake. Hence

$$\frac{\partial\overline{w}}{\partial y} = \frac{1}{x}\frac{d^2F}{d\xi^2} < 0 \tag{4.8.7}$$

and we take the minus sign in (4.8.5). Equation (4.8.5) is a first-order differential equation in $\frac{dF}{d\xi}$ and hence it is variables separable. Letting $H = \frac{dF}{d\xi}$, we have

$$H^{-\frac{1}{2}}\frac{dr}{d\xi} = -\frac{1}{l_1}\sqrt{\frac{U}{2}}\xi^{\frac{1}{2}}. \tag{4.8.8}$$

Integrating (4.8.8) we obtain

$$H^{\frac{1}{2}} = P - \frac{1}{3l_1}\sqrt{\frac{U}{2}}\xi^{3/2}, \tag{4.8.9}$$

where P is a constant. Therefore

$$\frac{dF}{d\xi} = \left(P - \frac{1}{3l_1}\sqrt{\frac{U}{2}}\xi^{3/2}\right)^2. \tag{4.8.10}$$

Applying the second boundary condition in (4.6.89) we obtain

$$P = \frac{1}{3l_1}\sqrt{\frac{U}{2}}\xi_B^{3/2} \tag{4.8.11}$$

and

$$\frac{dF}{d\xi} = \frac{U}{18l_1^2}\left(\xi_B^{3/2} - \xi^{3/2}\right)^2. \tag{4.8.12}$$

We have observed that to calculate the conserved quantity D and the fluid variables $\overline{w}(x,y)$ and $\overline{v}_y(x,y)$ it is sufficient to derive $\frac{dF}{d\xi}$ and $F(\xi)$ does not need to be obtained. The second derivative $\frac{d^2F}{d\xi^2}$ is important for physical variables such as the shear stress τ_{xy}. Substituting (4.8.12) into (4.8.5) with the minus sign we obtain

$$\frac{d^2F}{d\xi^2} = -\frac{U}{6l_1^2}\xi^{1/2}\left(\xi_B^{3/2} - \xi^{3/2}\right). \tag{4.8.13}$$

Now integrating (4.8.2) once we obtain

$$l_1 l_2 \frac{d^2F}{d\xi^2}\frac{dG}{d\xi} = \frac{U}{2}\xi G + C, \tag{4.8.14}$$

where C is a constant. Applying the first boundary condition in (4.6.89) we obtain $C = 0$. Hence

$$\frac{d^2F}{d\xi^2}\frac{dG}{d\xi} = \frac{1}{l_1 l_2}\frac{U}{2}\xi G. \tag{4.8.15}$$

Substituting (4.8.13) into (4.8.15) we obtain

$$\frac{dG}{d\xi} = -3\frac{l_1}{l_2}\frac{\xi^{1/2}G}{\xi_B^{3/2} - \xi^{3/2}}. \tag{4.8.16}$$

Separating variables ξ and G we have

$$\frac{dG}{G} = -3\frac{l_1}{l_2}\frac{\xi^{1/2}d\xi}{\xi_B^{3/2} - \xi^{3/2}}. \tag{4.8.17}$$

Integrating (4.8.17) by making the change of variable $\eta = \xi^{3/2}$ we obtain

$$G(\xi) = C\left(\xi_B^{3/2} - \xi^{3/2}\right)^{2\frac{l_1}{l_2}}, \tag{4.8.18}$$

where C is a constant. The boundary condition $G(\xi_B) = 0$ is identically satisfied. Since

$$C = G(0)\xi_B^{-3\frac{l_1}{l_2}}. \tag{4.8.19}$$

We write (4.8.18) as

$$G(\xi) = G(0)\xi_B^{-3\frac{l_1}{l_2}}\left(\xi_B^{3/2} - \xi^{3/2}\right)^{2\frac{l_1}{l_2}}. \tag{4.8.20}$$

The constant $G(0)$ cannot be obtained from the boundary conditions. It will be determined using the conserved quantity (4.6.86).

Substituting (4.8.12) into the conserved quantity (4.6.85) we obtain

$$D = \frac{\rho U^2}{9l_1^2}\int_0^{\xi_B}\left(\xi_B^{3/2} - \xi^{3/2}\right)^2 d\xi. \tag{4.8.21}$$

Evaluating the integral and solving for ξ_B in (4.8.21) we obtain

$$D = \frac{\rho U^2 \xi_B^4}{20l_1^2} \tag{4.8.22}$$

and therefore

$$\xi_B = \left(20\frac{l_1}{\rho U^2}D\right)^{\frac{1}{4}}. \tag{4.8.23}$$

We see that ξ_B is finite and proportional to $D^{1/4}$.

Substituting (4.8.20) into the conserved quantity (4.6.86) we obtain

$$\frac{K\xi_B^{3\frac{l_1}{l_2}}}{2\rho c_v U} = G(0)\int_0^{\xi_B}\left(\xi_B^{3/2}-\xi^{3/2}\right)^{2\frac{l_1}{l_2}}d\xi. \tag{4.8.24}$$

We let

$$\xi = \xi_B f \tag{4.8.25}$$

in the integral in (4.8.24) to obtain

$$\frac{K}{2\rho c_v U} = G(0)\xi_B\int_0^1\left(1-f^{3/2}\right)^{2\frac{l_1}{l_2}}df. \tag{4.8.26}$$

Making another substitution $x = f^{3/2}$ in the integral we obtain

$$\int_0^1\left(1-f^{3/2}\right)^{2\frac{l_1}{l_2}}df = \frac{2}{3}\int_0^1 x^{-1/3}(1-x)^{2\frac{l_1}{l_2}}dx = \frac{2}{3}B\left(\frac{2}{3},1+2\frac{l_1}{l_2}\right), \tag{4.8.27}$$

where

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, \qquad m>0, n>0, \tag{4.8.28}$$

is the Beta function [34]. We use the properties

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \tag{4.8.29}$$

$$\Gamma(n) = (n-1)\Gamma(n-1), \qquad n>1, \tag{4.8.30}$$

where $\Gamma(n)$ is the Gamma function,

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx, \qquad n>0. \tag{4.8.31}$$

Thus

$$B\left(\frac{2}{3},1+2\frac{l_1}{l_2}\right) = \frac{\frac{l_1}{l_2}}{\left(\frac{1}{3}+\frac{l_1}{l_2}\right)}\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(2\frac{l_1}{l_2}\right)}{\Gamma\left(2\left(\frac{1}{3}+\frac{l_1}{l_2}\right)\right)}. \tag{4.8.32}$$

Therefore equation (4.8.24) gives

$$G(0) = \frac{3K}{4\rho c_v U\xi_B}\frac{\left(\frac{1}{3}+\frac{l_1}{l_2}\right)}{\frac{l_1}{l_2}}\frac{\Gamma\left(2\left(\frac{1}{3}+\frac{l_1}{l_2}\right)\right)}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(2\frac{l_1}{l_2}\right)}. \tag{4.8.33}$$

The physical variables become

$$\overline{w}(x,y) = \frac{1}{x^{1/2}}\frac{dF}{d\xi} = \frac{U}{18l_1^2x^{1/2}}\left(\xi_B^{3/2} - \xi^{3/2}\right), \tag{4.8.34}$$

$$\overline{v}_y(x,y) = -\frac{1}{2x}\frac{dF}{d\xi} = -\frac{U}{36l_1^2x}\left(\xi_B^{3/2} - \xi^{3/2}\right), \tag{4.8.35}$$

$$\overline{S}(x,y) = x^{-1/2}G(\xi) = x^{-1/2}G(0)\xi_B^{-3\frac{l_1}{l_2}}\left(\xi_B^{3/2} - \xi^{3/2}\right)^{2\frac{l_1}{l_2}}, \tag{4.8.36}$$

where ξ_B is determined in terms of D by (4.8.23).

4.9 Lines of constant mean temperature difference for $\nu = 0$ and $\kappa = 0$

The analytical solution for $\nu = 0$ and $\kappa = 0$ for the mean temperature difference leads directly to the analytical solution for the lines of constant mean temperature difference.

The lines of constant mean temperature difference are defined by

$$\overline{S}(x,y) = C, \tag{4.9.1}$$

where the constant C takes a range of discrete values. Hence, by (4.8.36), on a line of constant mean temperature difference

$$G(0)\xi_B^{-3\frac{l_1}{l_2}}\left(\xi_B^{3/2} - \xi^{3/2}\right)^{2\frac{l_1}{l_2}} = Cx^{1/2}. \tag{4.9.2}$$

Making ξ the subject of (4.9.2) we obtain

$$\xi = \xi_B\left[1 - \left(\frac{C}{G(0)}\right)^{\frac{l_2}{2l_1}} x^{\frac{l_2}{4l_1}}\right]^{2/3}. \tag{4.9.3}$$

Writing (4.9.3) in terms of x and y we obtain, since $\xi = y/x^{1/2}$,

$$y = \xi_B\left[1 - \left(\frac{C}{G(0)}\right)^{\frac{l_2}{2l_1}} x^{\frac{l_2}{4l_1}}\right]^{2/3} x^{1/2}. \tag{4.9.4}$$

For y to be real it is necessary that

$$1 - \left(\frac{C}{G(0)}\right)^{\frac{l_2}{2l_1}} x^{\frac{l_2}{4l_1}} \geq 0, \tag{4.9.5}$$

that is, that

$$x \leq \left(\frac{G(0)}{C}\right)^2. \tag{4.9.6}$$

The range of x on the line of constant mean temperature difference is therefore

$$0 \leq x \leq \left(\frac{G(0)}{C}\right)^2. \tag{4.9.7}$$

When $C = 0$ the mean temperature difference $\overline{S}(x, y) = 0$ and (4.9.4) reduces to the boundary of the wake

$$y = \xi_B x^{1/2}. \tag{4.9.8}$$

Also from (4.9.4),

$$\begin{aligned}\frac{dy}{dx} = -\frac{l_2}{6l_1}\left(\frac{C}{G(0)}\right)^{\frac{l_2}{2l_1}} \xi_B \left[1 - \left(\frac{C}{G(0)}\right)^{\frac{l_2}{2l_1}} x^{\frac{l_2}{4l_1}}\right]^{-\frac{1}{3}} x^{\frac{l_2}{4l_1}-\frac{1}{2}} \\ + \frac{1}{2}\xi_B \left[1 - \left(\frac{C}{G(0)}\right)^{\frac{l_2}{2l_1}} x^{\frac{l_2}{4l_1}}\right]^{\frac{2}{3}} x^{-\frac{1}{2}}.\end{aligned} \tag{4.9.9}$$

Thus

$$\frac{dy}{dx} \to -\infty \qquad \text{as} \qquad x \to \left(\frac{G(0)}{C}\right)^2 \tag{4.9.10}$$

and

$$\frac{dy}{dx} \to \infty \qquad \text{as} \qquad x \to 0. \tag{4.9.11}$$

When $\frac{l_2}{l_1} < 2$ the first term in (4.9.9) tends to $-\infty$ as $x \to 0$, but it always tends to $-\infty$ slower than the second term tends to ∞.

For a given value of $C > 0$ the lines of constant mean temperature difference for $y > 0$ start at $x = 0$ with gradient ∞ and extend to $x = (G(0)/C)^2$ where the gradient is $-\infty$. When $C = 0$ the line reduces to the boundary of the wake.

4.10 Classical wake streamlines of the mean velocity deficit for $\nu = 0$ and $\kappa = 0$

The stream function $\psi(x, y)$ for the mean velocity deficit $\overline{w}(x, y)$ and $\overline{v}_y(x, y)$ satisfies (3.11.1). Hence if vector $\underline{A} = (-\overline{w}, \overline{v}_y)$ then

$$(\underline{A} \cdot \underline{\nabla})\, \psi = 0 \tag{4.10.1}$$

and ψ is constant along curves with tangent vector $\underline{A}$ at each point. They are the streamlines, as seen by an observer moving with the mainstream velocity U, of the mean velocity deficit directed towards the obstacle. The curves are defined by

$$\psi(x, y) = k_1, \tag{4.10.2}$$

where the constant k_1 is given a range of discrete values. Now from (4.7.12) with $a_1(x) = 0$,

$$\psi(x, y) = F(\xi) \tag{4.10.3}$$

and from (4.8.12)

$$\frac{dF}{d\xi} = \frac{U}{18l_1^2}\left(\xi_B^3 - 2\xi_B^{3/2}\xi^{3/2} + \xi^3\right). \tag{4.10.4}$$

Hence

$$F(\xi) = \frac{U}{72l_1^2}\xi\left[\xi^3 - \frac{16}{5}\xi_B^{3/2}\xi^{3/2} + 4\xi_B^3\right], \tag{4.10.5}$$

where the constant of integration is incorporated in k_1. Let $u = \frac{\xi}{\xi_B}$. Then

$$u^4 - \frac{16}{5}u^{5/2} + 4u = k_2, \tag{4.10.6}$$

where

$$k_2 = \frac{72l_1^2}{U\xi_B^4}k_1. \tag{4.10.7}$$

The boundary of the wake $\xi = \xi_B$ and the centreline $\xi = 0$ are streamlines. When $\xi = \xi_B$, $u = 1$ and $k_2 = \frac{9}{5}$ while when $\xi = 0$, $u = 0$ and $k_2 = 0$. Figure 4.1 shows the plot for $f(u) = u^4 - \frac{16}{5}u^{5/2} + 4u$ with u between 0 and 1. It can seen that $f(u)$ is an increasing function for u between 0 and 1. Hence for each value of k_2 there is only one value for u that is between 0 and 1 and $0 \leq k_2 \leq \frac{9}{5}$.

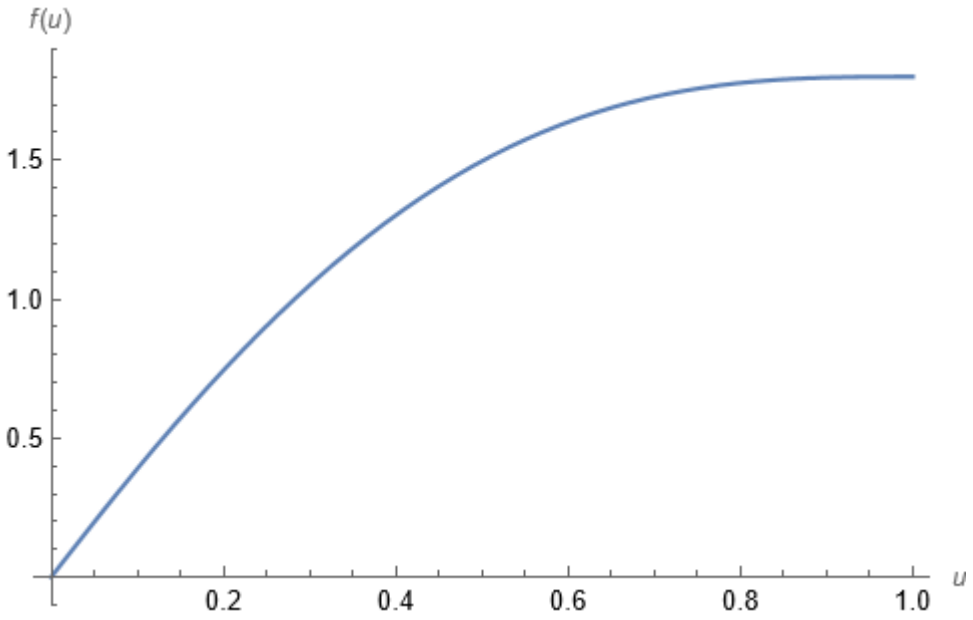

FIGURE 4.1: The plot for $f(u) = u^4 - \frac{16}{5}u^{5/2} + 4u$. The values of u are between 0 and 1.

The streamlines can be obtained as follows. Choose k_2 in the range $0 \leq k_2 \leq \frac{9}{5}$. Solve equation (4.10.6) for u. Then the equation of the streamline for this value of k_2 is given by

$$y = u\xi_B x^{1/2}, \qquad 0 \leq x \leq \infty. \tag{4.10.8}$$

When $u = 1$, equation (4.10.8) reduces to the equation of the boundary $y = \xi_B x^{1/2}$ and when $u = 0$, (4.10.8) reduces to the centreline $y = 0$.

By giving k_2 a range of discrete values in the range $0 \leq k_2 \leq \frac{9}{5}$ the streamlines can be plotted as shown in Figure 4.7.

4.11 Numerical solution when $\nu \neq 0$ and $\kappa \neq 0$

In this Section, we formulate the numerical solution when $\nu \neq 0$ and $\kappa \neq 0$.

When $\nu \neq 0$ and $\kappa \neq 0$ the ordinary differential equations for F and G are, (4.7.4) and (4.7.5),

$$\frac{d}{d\xi}\left[\left(\nu - l_1^2\frac{d^2F}{d\xi^2}\right)\frac{d^2F}{d\xi^2}\right] + \frac{U}{2}\frac{d}{d\xi}\left(\xi\frac{dF}{d\xi}\right) = 0, \tag{4.11.1}$$

$$\frac{d}{d\xi}\left[\left(\frac{\kappa}{\rho c_v} - l_1 l_2\frac{d^2F}{d\xi^2}\right)\frac{dG}{d\xi}\right] + \frac{U}{2}\frac{d}{d\xi}\left(\xi G\right) = 0. \tag{4.11.2}$$

Integrating (4.11.1) we obtain

$$\left(\nu - l_1^2\frac{d^2F}{d\xi^2}\right)\frac{d^2F}{d\xi^2} + \frac{U}{2}\xi\frac{dF}{d\xi} = C, \tag{4.11.3}$$

where C is a constant. Applying boundary conditions (4.6.37) at $\xi = 0$ we obtain $C = 0$. Equation (4.11.3) becomes

$$l_1^2\left(\frac{d^2F}{d\xi^2}\right)^2 - \nu\frac{d^2F}{d\xi^2} - \frac{U}{2}\xi\frac{dF}{d\xi} = 0. \tag{4.11.4}$$

Equation (4.11.4) is quadratic in $\frac{d^2F}{d\xi^2}$. We use the quadratic formula to make $\frac{d^2F}{d\xi^2}$ the subject of (4.11.4) to obtain

$$\frac{d^2F}{d\xi^2} = \frac{\nu}{2l_1^2} \pm \sqrt{\left(\frac{\nu}{2l_1^2}\right)^2 + \frac{U}{2l_1^2}\xi\frac{dF}{d\xi}}. \tag{4.11.5}$$

But

$$\frac{\partial\overline{w}}{\partial y} = \frac{1}{x}\frac{d^2F}{d\xi^2} < 0 \tag{4.11.6}$$

in the upper half, $y > 0$, of the classical turbulent thermal wake. Hence we take the minus sign in (4.11.5) and

$$\frac{d^2F}{d\xi^2} = \frac{\nu}{2l_1^2} - \sqrt{\left(\frac{\nu}{2l_1^2}\right)^2 + \frac{U}{2l_1^2}\xi\frac{dF}{d\xi}}. \tag{4.11.7}$$

Now integrating (4.11.2) we obtain

$$\left(\frac{\kappa}{\rho c_v} - l_1 l_2\frac{d^2F}{d\xi^2}\right)\frac{dG}{d\xi} + \frac{U}{2}\xi G = C, \tag{4.11.8}$$

where C is a constant. Applying boundary conditions (4.6.38) at $\xi = 0$ we obtain $C = 0$. Making $\frac{dG}{d\xi}$ the subject in (4.11.8), with $C = 0$, we obtain

$$\frac{dG}{d\xi} = -\frac{U}{2}\frac{\xi G}{\left(\frac{\kappa}{\rho c_v} - l_1 l_2\frac{d^2F}{d\xi^2}\right)}. \tag{4.11.9}$$

The boundary conditions (4.6.37) and (4.6.38) are not sufficient to fully solve the ordinary differential equations (4.11.7) and (4.11.9) since the value of ξ_B is unknown. Hence the conserved quantities (4.6.31) and (4.6.32) are required to complete the solution of the problem. Finding the numerical solution to a boundary value problem

with conserved quantities is not a trivial task. Hence we convert the boundary value problem to an initial value problem using a shooting method to find the unknown initial values.

Since the physical variables do not depend on F, but on its derivatives, we can set $F(0) = 0$ without affecting the physical variables. Since $\psi(x, y) = F(\xi)$, this is equivalent to choosing the arbitrary additive constant in the stream function. Hence F can be approximated by

$$F(\xi) \approx A\xi^m \qquad \text{as} \qquad \xi \to 0, \tag{4.11.10}$$

where A and m are constants. But

$$\overline{w}(x, y) = \frac{1}{x^{1/2}} \frac{dF}{d\xi} = \frac{m}{x^{1/2}} A\xi^{m-1} > 0 \qquad \text{and finite as} \qquad \xi \to 0. \tag{4.11.11}$$

Hence $m = 1$ and $A > 0$. The approximation becomes

$$F(\xi) = A\xi \qquad \text{as} \qquad \xi \to 0 \tag{4.11.12}$$

and

$$F(0) = 0, \qquad \frac{dF}{d\xi}(0) = A. \tag{4.11.13}$$

Equation (4.11.12) identically satisfies the first boundary condition in (4.6.38). Since $\frac{dG}{d\xi}(0) = 0$ we approximate G by

$$G(\xi) \approx B + C\xi^n \qquad \text{as} \qquad \xi \to 0, \tag{4.11.14}$$

where B, C are nonzero constants and since $G(0)$ is finite, $n > 0$. But

$$\frac{dG}{d\xi} = Cn\xi^{n-1} = 0 \qquad \text{as} \qquad \xi \to 0. \tag{4.11.15}$$

Therefore $n > 1$ and

$$G(0) = B. \tag{4.11.16}$$

We now have the following initial value problem.
Differential equations:

$$\frac{d^2F}{d\xi^2} = \frac{\nu}{2l_1^2} - \sqrt{\left(\frac{\nu}{2l_1^2}\right)^2 + \frac{U}{2l_1^2}\xi\frac{dF}{d\xi}}, \tag{4.11.17}$$

$$\frac{dG}{d\xi} = -\frac{U}{2}\frac{\xi G}{\frac{\kappa}{\rho c_v} - l_1 l_2 \frac{d^2F}{d\xi^2}}. \tag{4.11.18}$$

Initial conditions:

$$F(0) = 0, \qquad \frac{dF}{d\xi}(0) = A, \qquad G(0) = B. \tag{4.11.19}$$

The values of A and B will be determined using a shooting method. The targets for the shooting method are

$$D = 2\rho U \int_0^{\xi_B} \frac{dF}{d\xi} d\xi, \qquad K = 2\rho c_v U \int_0^{\xi_B} G(\xi) d\xi. \tag{4.11.20}$$

The values of ν, $\frac{\kappa}{\rho c_v}$, l_1, l_2, U, D and K are prescribed.

We use the shooting method in Mubai and Mason [16] to determine the unknown values of A and B. In the shooting method, we first guess the values of A and B and use the Python function scipy.integrate.odeint to find the corresponding values of $\frac{dF}{d\xi}$ and G. We use the values obtained for $\frac{dF}{d\xi}$ and G to calculate the values of the conserved quantities D and K using (4.11.20). We check whether the calculated values of the conserved quantities match the prescribed values for D and K within a specified tolerance. If they do not match, we update the initial values A and B so that the calculated conserved quantities become closer to the prescribed values of D and K.

We define

$$D^*(A,B) = 2\rho U \int_0^{\xi_B} \frac{dF}{d\xi}(\xi; A, B) d\xi, \qquad K^*(A,B) = 2\rho c_v U \int_0^{\xi_B} G(\xi; A, B) d\xi, \tag{4.11.21}$$

where $\frac{dF}{d\xi}(\xi; A, B)$ and $G(\xi; A, B)$ are values of $\frac{dF}{d\xi}$ and G when the initial conditions are

$$F(0) = 0, \qquad \frac{dF}{d\xi}(0) = A, \qquad G(0) = B. \tag{4.11.22}$$

We seek to find initial values A and B such that $D^*(A,B) = D$ and $K^*(A,B) = K$, where D and K are the prescribed values for the conserved quantities.

Suppose A_k and B_k are the current best guesses for the values of A and B. To update A_k and B_k we need steps δ_1 and δ_2 such that

$$D - D^*(A_k + \delta_1, B_k + \delta_2) \approx 0, \qquad K - K^*(A_k + \delta_1, B_k + \delta_2) \approx 0. \tag{4.11.23}$$

Using a Taylor series expansion to first order to expand the equations in (4.11.23) and writing them in a matrix format we obtain

$$\begin{pmatrix} \frac{\partial D^*}{\partial A}(A_k, B_k) & \frac{\partial D^*}{\partial B}(A_k, B_k) \\ \\ \frac{\partial K^*}{\partial A}(A_k, B_k) & \frac{\partial K^*}{\partial B}(A_k, B_k) \end{pmatrix} \begin{pmatrix} \delta_1 \\ \\ \delta_2 \end{pmatrix} = \begin{pmatrix} D - D^*(A_k, B_k) \\ \\ K - K^*(A_k, B_k) \end{pmatrix}. \tag{4.11.24}$$

We use a secant method to evaluate the partial derivatives in (4.11.24). For example,

$$\frac{\partial D^*}{\partial A}(A_k, B_k) = \frac{D^*(A_k, B_k) - D^*(A_{k-1}, B_k)}{A_k - A_{k-1}}. \tag{4.11.25}$$

We update guesses for A and B by adding steps δ_1 and δ_2 to the current guesses. That is,

$$A_{k+1} = A_k + \delta_1, \qquad B_{k+1} = B_k + \delta_2. \tag{4.11.26}$$

We summarise the shooting method as follows.

1. Choose (A_0, B_0) and (A_1, B_1) as initial guess.

2. Set $k = 1$.

3. Calculate the values of $\frac{dF}{d\xi}(\xi, A_k, B_k)$, $G(\xi, A_k, B_k)$, $\frac{dF}{d\xi}(\xi, A_{k-1}, B_{k-1})$ and $G(\xi, A_{k-1}, B_{k-1})$ for differential equations (4.11.17) and (4.11.18) and initial conditions (4.11.19) using the Python function scipy.integrate.odeint.

4. Calculate derivatives $\frac{\partial D^*}{\partial A}$, $\frac{\partial D^*}{\partial B}$, $\frac{\partial K^*}{\partial A}$ and $\frac{\partial K^*}{\partial B}$ using the secant method (4.11.25).

5. Calculate steps δ_1 and δ_2 using equation (4.11.24).

6. Update the values of A and B by setting $A_k = A_{k-1}$, $B_k = B_{k-1}$, $A_{k+1} = A_k + \delta_1$ and $B_{k+1} = B_k + \delta_2$.

7. If the error $|D - D^*(A_{k+1}, B_{k+1})| + |K - K^*(A_{k+1}, B_{k+1})| < \epsilon$, where ϵ is the tolerance of the shooting method, we stop. Otherwise, we increment k by 1 and continue to step 3.

Once we obtain the values of A and B we use the Python function scipy.integrate.odeint to solve the initial value problem (4.11.17) to (4.11.19).

Table 4.1 shows the shooting method with initial guesses $(A_0, B_0) = (0.1, 0.1)$ and $(A_1, B_1) = (0.2, 0.2)$ for the system (4.11.17) to (4.11.20). We see that the shooting method converges after three iterations.

k	A_k	B_k	$D^*(A_k, B_k)$	$K^*(A_k, B_k)$	$\lvert D^*(A_k, B_k) - D\rvert + \lvert K^*(A_k, B_k) - K\rvert$
0	0.1	0.1	0.36004	0.36004	1.27991
1	0.2	0.2	0.73064	0.73064	0.53872
2	0.27268	0.27375	1.00620	1.01013	0.01633
3	0.27105	0.27102	0.99994	0.99984	0.00021
4	0.27106	0.27106	1.00000	1.00000	6.96178×10^{-08}

TABLE 4.1: Iterations of the shooting method for the system (4.11.17) to (4.11.20) with $D = 1$, $K = 1$, $\rho = 1$, $\nu = 1$ and $\frac{\kappa}{\rho c_v} = 1$. The initial guesses are $(A_0, B_0) = (0.1, 0.1)$ and $(A_1, B_1) = (0.2, 0.2)$. The tolerance of the shooting method is set to 10^{-7}.

4.12 Results and discussion

In this Section, we present the analytical and numerical results for the classical turbulent thermal wake. We show the role played by the kinematic viscosity and thermal conductivity in the analysis of the wake. For all plots in this Section, $l_1 = 1$, $l_2 = 1$, $D = 1$, $K = 1$, $\rho = 1$, $c_v = 1$, $U = 1$. From these values, the calculated values of ξ_B and $G(0)$ are 0.5254 and 2.1147, respectively.

Figure 4.2 shows the plots for the mean velocity deficit profiles when $\nu = 0$ at $x = 1$, $x = 1.5$ and $x = 2$. The graphs are plotted using the analytical solution (4.8.34). We see that as the distance from the obstacle, x, increases the mean velocity deficit at the centreline, $\overline{w}(x, 0)$, decreases and the boundary of the wake increases. This shows that the turbulent diffusion increases with the distance from the obstacle, x. The flow becomes more turbulent as the distance from the obstacle increases. Turbulence increases the momentum diffusion in the mean flow. We also see that the boundary of the wake is finite.

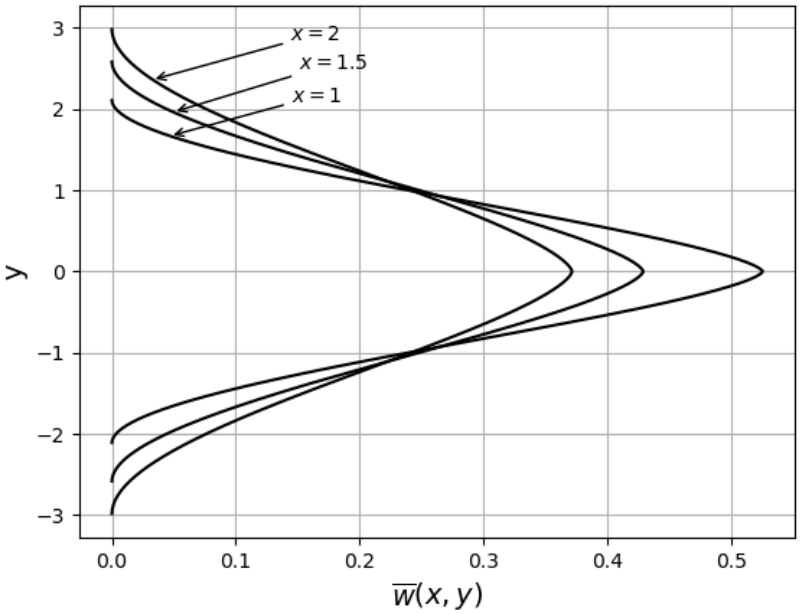

FIGURE 4.2: The mean velocity deficit $\overline{w}(x,y)$ for $\nu = 0$ plotted against y at $x = 1$, $x = 1.5$ and $x = 2$ using the exact analytical solution (4.8.34).

Figure 4.3 shows the plots for the mean temperature difference profiles when $\nu = 0$ and $\kappa = 0$ at $x = 1$, $x = 1.5$ and $x = 2$. The graphs are plotted using the analytical solution (4.8.36). We see that as the distance from the obstacle, x, increases the mean temperature difference at the centreline, $\overline{S}(x,0)$, decreases and the boundary of the wake increases. This shows that turbulent thermal diffusion increases with the distance from the obstacle, x. The flow becomes more turbulent as the distance from the obstacle increases and the turbulence increases the thermal diffusion in the mean flow. We see that the thermal boundary is equal to the momentum boundary in Figure 4.2.

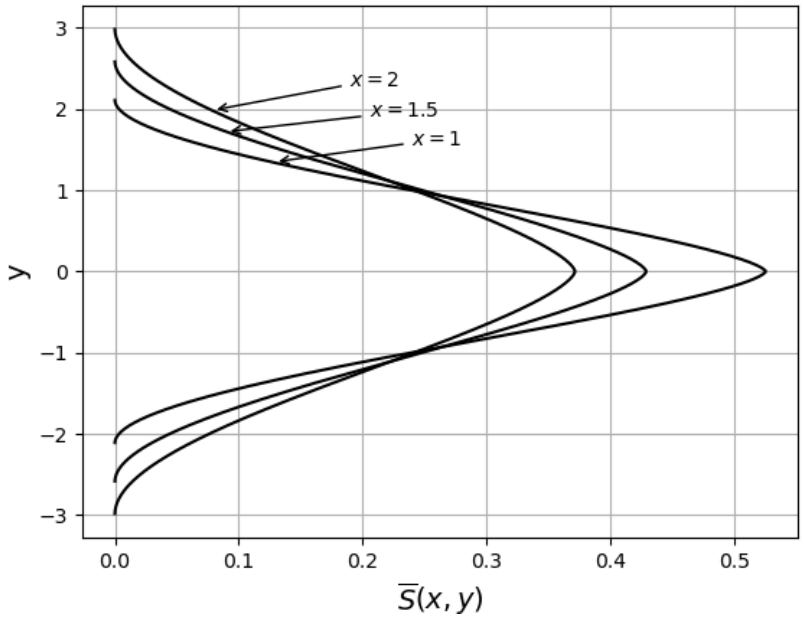

FIGURE 4.3: The mean temperature difference $\overline{S}(x, y)$ for $\nu = 0$ and $\kappa = 0$ plotted against y at $x = 1$, $x = 1.5$ and $x = 2$ using the exact analytical solution (4.8.36).

Figure 4.4 shows the plots for the mean velocity deficit profile at $x = 1$ for various values of kinematic viscosity. The graph for $\nu = 0$ is plotted using the analytical solution (4.8.34) and the graphs for $\nu \neq 0$ are plotted using

$$\overline{w}(x, y) = \frac{1}{x^{1/2}} \frac{dF}{d\xi}, \tag{4.12.1}$$

where $\frac{dF}{d\xi}$ is obtained using the shooting method numerical scheme described in Section 4.11. We see that, as the kinematic viscosity increases, the mean velocity deficit at the centreline, $\overline{w}(1, 0)$, decreases and the profile becomes broader along the y-axis. We also see that as the kinematic viscosity tends to zero in the numerical solution the mean velocity deficit becomes closer to the analytical solution for the mean velocity deficit when the kinematic viscosity vanishes. Hence, the numerical solution for the mean velocity deficit is confirmed using the analytical solution (4.8.34). Since any numerical scheme is limited by its tolerance we cannot definitively discern whether the boundary is finite using the numerical results when $\nu \neq 0$. We believe for physical reasons that the solution extends to $y = \pm\infty$ whenever $\nu \neq 0$.

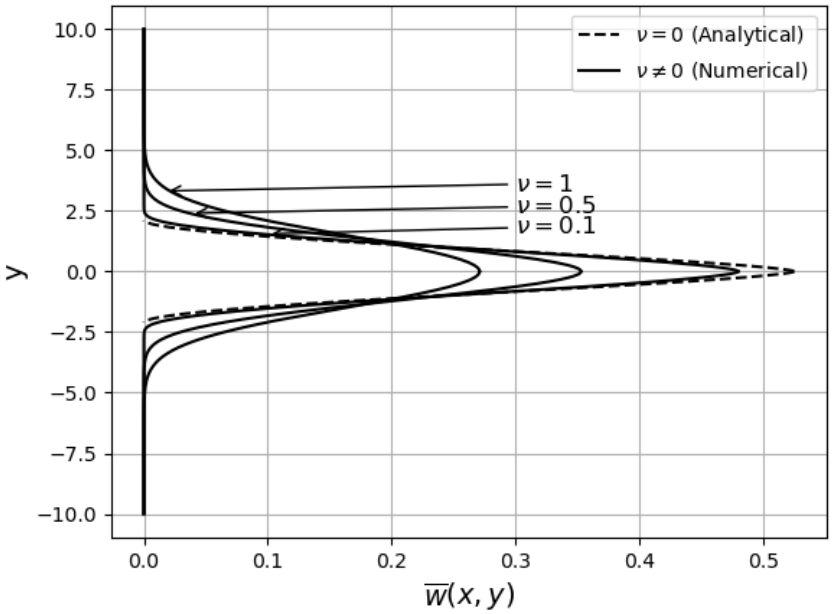

FIGURE 4.4: The mean velocity deficit $\overline{w}(x, y)$ plotted against y for $\nu = 0$, $\nu = 0.1$, $\nu = 0.5$ and $\nu = 1$.

Figure 4.5 shows the plots for the mean temperature difference profile at $x = 1$ for a range of values of kinematic viscosity and thermal conductivity. The graph for $\nu = 0$ and $\kappa = 0$ is plotted using the analytical solution (4.8.36) and the graphs for $\nu \neq 0$ and $\kappa \neq 0$ are plotted using

$$\overline{S}(x, y) = \frac{1}{x^{1/2}} G(\xi), \tag{4.12.2}$$

where $G(\xi)$ is obtained using the shooting method numerical scheme described in Section 4.11. We see that, as the kinematic viscosity and thermal conductivity increase, the mean temperature difference at the centreline, $\overline{S}(1, 0)$, decreases and the profile becomes broader along the y-axis. We also see that as the kinematic viscosity and the thermal conductivity tend to zero in the numerical solution the mean temperature difference becomes closer to the mean temperature difference obtained using the analytical solution when the kinematic viscosity and the thermal conductivity vanish. Hence, the numerical solution for the mean temperature difference is confirmed using the analytical solution (4.8.36). Since any numerical scheme is limited by its tolerance we cannot definitively discern whether the thermal boundary is finite using the numerical results when $\nu \neq 0$ and $\kappa \neq 0$. However, we believe for physical reasons that when $\nu \neq 0$ and $\kappa \neq 0$ the solution for the mean temperature difference extends to $y = \pm\infty$.

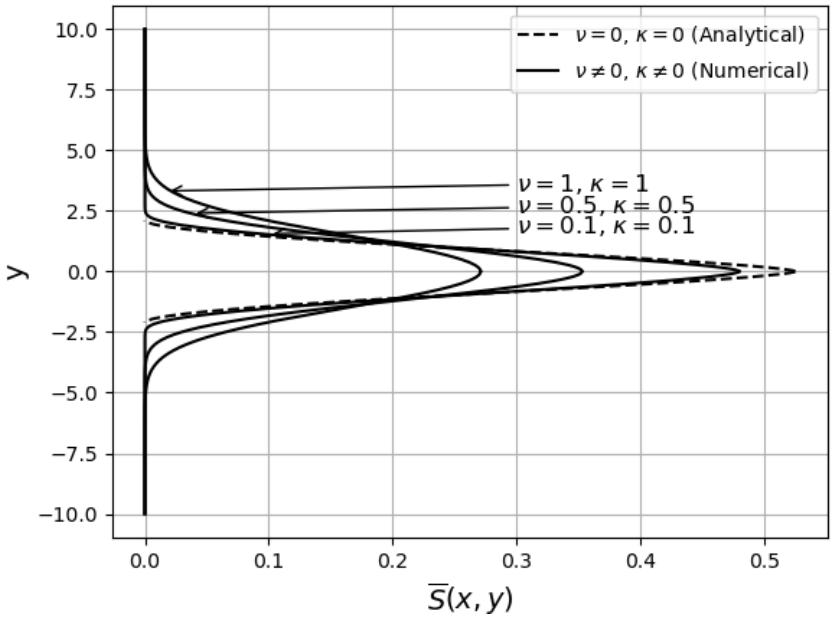

FIGURE 4.5: The mean temperature difference $\overline{S}(x, y)$ plotted against y for $\nu = 0$, $\kappa = 0$; $\nu = 0.1$, $\kappa = 0.1$; $\nu = 0.5$, $\kappa = 0.5$ and $\nu = 1$, $\kappa = 1$.

We saw in Section 4.9 that the analytical solution (4.8.36) leads directly to the equation of the lines of constant mean temperature difference when $\nu = 0$ and $\kappa = 0$. Figure 4.6 shows a plot of lines of constant mean temperature difference when $\nu = 0$ and $\kappa = 0$. The curves are obtained from (4.9.4),

$$y = \pm\xi_B \left[1 - \left(\frac{C}{G(0)}\right)^{\frac{l_2}{2l_1}} x^{\frac{l_2}{4l_1}}\right]^{2/3} x^{1/2}, \tag{4.12.3}$$

where the $+$ and the $-$ signs describe the lines in the upper and lower values of the wake. The range of the curves is

$$0 \leq x \leq \left(\frac{G(0)}{C}\right)^2 \tag{4.12.4}$$

and are obtained by giving C a range of discrete positive values. When $C = 0$, equation (4.12.3) reduces to the equation of the boundary of the wake,

$$y = \pm\xi_B x^{1/2}. \tag{4.12.5}$$

The lines start at $x = 0$ with a gradient of infinite magnitude and end at $x = (G(0)/C)^2$ also with gradient of infinite magnitude. The quantities ξ_B and $G(0)$ are given by (4.8.23) and (4.8.33). Figure 4.6 also shows the relation between the lines of constant mean temperature difference and the boundary of the wake. We see that the mean temperature difference is greater closer to the heated obstacle and decreases as the distance along the centreline increases. The mean temperature difference decreases as the lines of constant mean temperature difference get closer to the boundary of the thermal wake. The lines of constant mean temperature difference are closer to each other near the obstacle and more spread out as the lines

approach the boundary of the thermal wake since the mean heat flux is greater near the heated obstacle and vanishes at the boundary of the thermal wake.

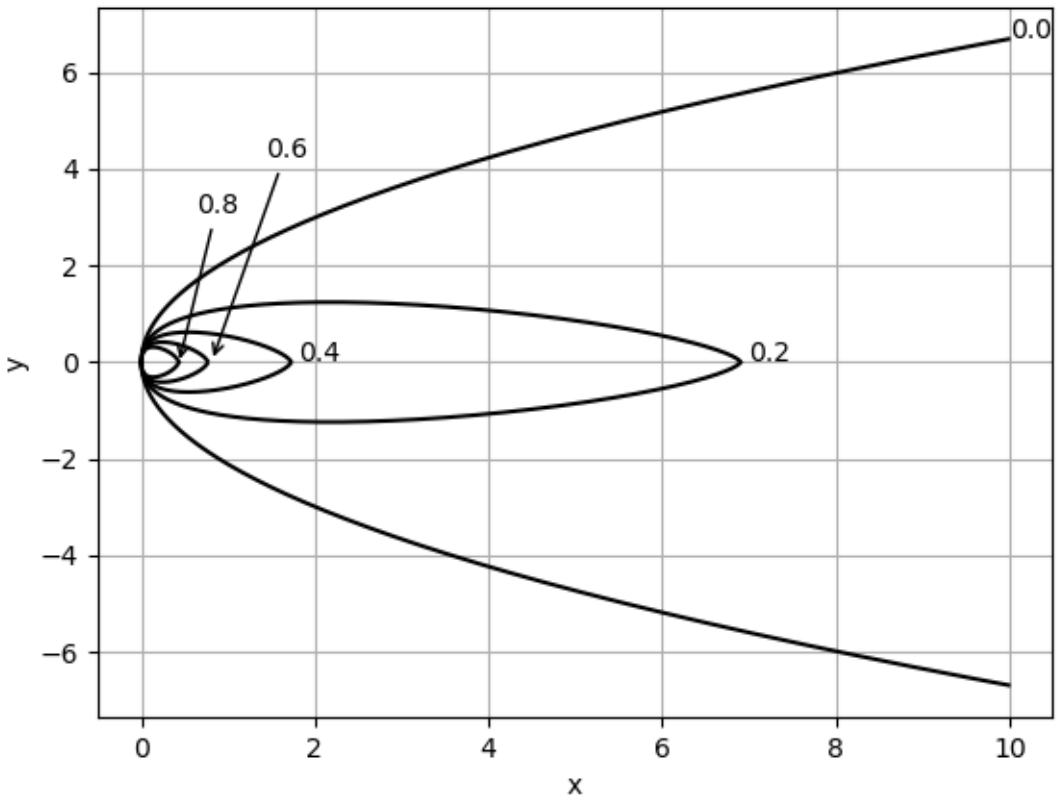

FIGURE 4.6: The lines of constant mean temperature difference for $\nu = 0$ and $\kappa = 0$. On the boundary of the wake the mean temperature difference is zero. The curves are labelled by values of the constant mean temperature difference C.

We saw in Section 4.10 that the analytical solution (4.7.12) leads to the equation of the streamlines when $\nu = 0$. Figure 4.7 shows the plot for the streamlines for the mean velocity deficit when $\nu = 0$. The curves are obtained from (4.10.8),

$$y_B(x) = \pm u \xi_B x^{1/2}, \tag{4.12.6}$$

where u is calculated from (4.10.6),

$$u^4 - \frac{16}{5} u^{5/2} + 4u = k_2 \tag{4.12.7}$$

for each discrete value of k_2. The $+$ and the $-$ signs describe the lines in the upper and lower values of the wake. When $u = 0$, equation (4.12.6) reduces to the equation of the centreline, $y = 0$, and when $u = 1$, equation (4.12.6) becomes the boundary of the wake,

$$y = \pm \xi_B x^{1/2}. \tag{4.12.8}$$

The value of u should be between 0 and 1. To ensure this, discrete values of k_2 are chosen in the range $0 \leq k_2 \leq \frac{9}{5}$. The curves are tangential to the mean velocity

deficit $(-\overline{w}, \overline{v}_z)$ as seen by an observer moving with the mainstream velocity U in the $+x$-direction. For the same interval of k_2 the streamlines are closer together near the centre line than near the boundary of the wake because the mean velocity deficit is greater near the centre line and vanishes on the boundary. From (4.8.35)

$$\overline{v}_y = -\frac{U}{36l_1^2 x}\left(\xi_B^{3/2} - \xi^{3/2}\right) < 0 \tag{4.12.9}$$

and the distance of the streamline from the centre line decreases as x decreases towards the obstacle.

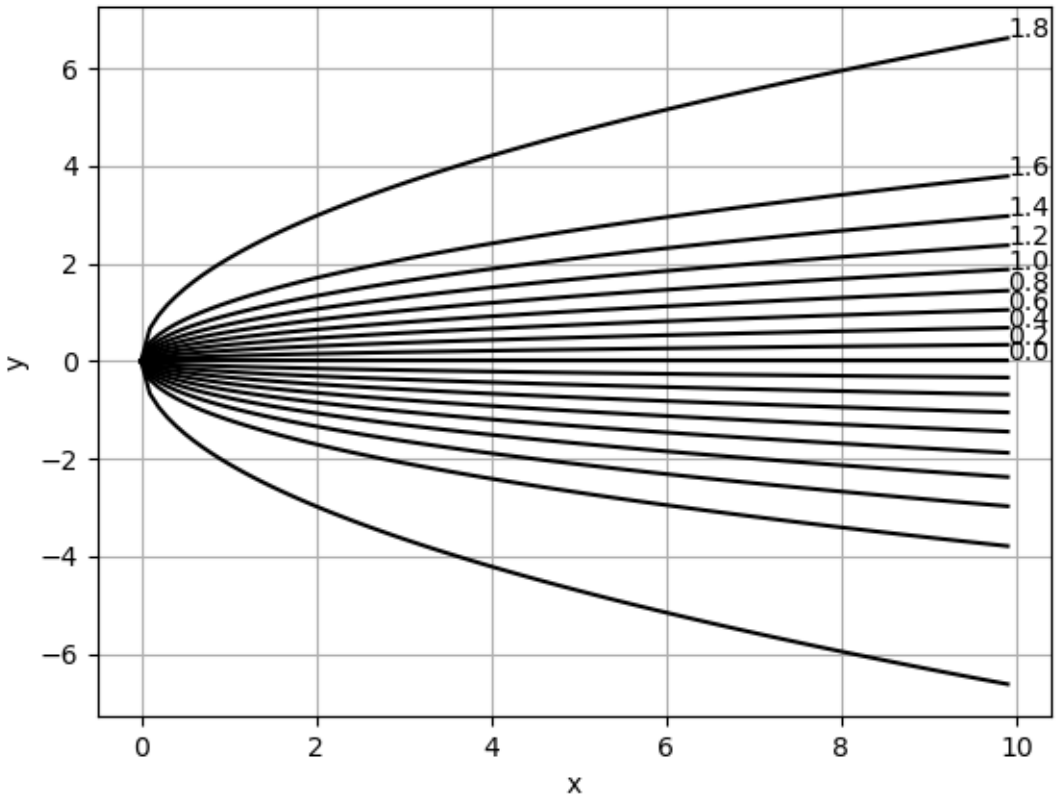

FIGURE 4.7: Streamlines for the mean velocity deficit of turbulent thermal classical wake when $\nu = 0$ plotted using (4.10.6) and (4.10.8) with $\xi_B = 0.5254$ and values of k_2 in the range $0 \leq k_2 \leq 1.8$.

4.13 Conclusions

In this Chapter, we considered multipliers Λ_1 and Λ_2 that are functions of $x, y, \psi, \overline{S}$ to derive conservation laws for the classical turbulent thermal wake and found that Λ_1 is an arbitrary function of x and Λ_2 is a constant. We also found that it was sufficient to take Λ_1 a constant to derive the momentum flux conserved vector and the thermal flux conserved vector for the classical turbulent thermal wake. The conserved vectors were used to derive the associated Lie point symmetry. The conservation laws were integrated to find, with the boundary conditions, conserved quantities for the classical turbulent thermal wake. The conserved quantity for the momentum flux

agrees with the conserved quantity found by Hutchinson and Mason [8]. We showed that for all values of the kinematic viscosity and thermal conductivity the thermal mixing length is proportional to the momentum mixing length. When the kinematic viscosity and thermal conductivity are nonzero the associated Lie point symmetry was fully determined and we used the Lie point symmetry to reduce the governing partial differential equations to ordinary differential equations. When the kinematic viscosity and the thermal conductivity vanish the associated Lie point symmetry could not be determined fully and the momentum mixing length remained an arbitrary function of x. An additional equation was required to determine the Lie point symmetry and the mixing lengths. We used the Prandtl hypothesis to derive the additional equation. We used the Lie point symmetry to reduce the governing partial differential equations to ordinary differential equations. When the kinematic viscosity and thermal conductivity vanish the ordinary differential equations were solved analytically. When the kinematic viscosity and thermal conductivity are nonzero the ordinary differential equations had to be solved numerically. The analytical solution for vanishing kinematic viscosity led to the equation of the streamlines of the mean velocity deficit. The streamlines start at the trailing edge of the obstacle and are different from the streamlines of the mean flow for the two-dimensional turbulent jet which start at the boundary of the jet due to entrainment of the fluid at the boundary [16]. The analytical solution for vanishing kinematic viscosity and thermal conductivity led to the equation for lines of constant mean temperature difference for the classical turbulent thermal wake which are similar to the lines of constant mean temperature difference for a two-dimensional turbulent thermal jet. We saw that the numerical solution as the kinematic viscosity and thermal conductivity tend to zero matched the analytical solution (with the Prandtl hypothesis) for vanishing kinematic viscosity and thermal conductivity, which checked the accuracy of the numerical method. For the two-dimensional turbulent thermal jet the numerical solution for nonzero kinematic viscosity and thermal conductivity did not tend to the analytical solution for vanishing kinematic viscosity and thermal conductivity with the Prandtl hypothesis as the kinematic viscosity and thermal conductivity tend to zero. We plotted the mean velocity deficit and mean temperature difference profiles for different values of x. We saw that as the distance from the obstacle, x, increases the effective momentum and thermal diffusion increase.

We showed that the kinematic viscosity and thermal conductivity play a role in the mathematical analysis. When the kinematic viscosity and thermal conductivity are nonzero we derived the mixing length without imposing the Prandtl hypothesis. However, the solution when the kinematic viscosity and thermal conductivity tend to zero matches the solution when the kinematic viscosity and thermal conductivity vanish. The boundary of the wake is finite when the kinematic viscosity and thermal conductivity vanish, but is infinite when the kinematic viscosity and thermal conductivity are nonzero. This is because for sufficiently large values of y, the kinematic viscosity and thermal conductivity will become larger than the eddy kinematic viscosity and eddy thermal conductivity and the wake will behave increasingly like a laminar wake which extends to $y = \infty$.

Chapter 5

The von Kármán similarity hypothesis applied to the classical thermal wake

5.1 Introduction

In Chapter 4 the mixing length was obtained during the Lie symmetry analysis. Very few expressions for mixing lengths for turbulent flows have been derived from physical or symmetry arguments. A mixing length that has been derived is the von Kármán mixing length. In this Chapter we will extend the von Kármám mixing length to the turbulent thermal wake.

Goldstein [10] applied the von Kármán similarity hypothesis to derive the momentum mixing length for parallel flow. The von Kármán similarity hypothesis states [9]:
The turbulent flow patterns at different points are similar and differ only in the length and time scales.
He pointed out that the von Kármán similarity hypothesis does not yield accurate results near points where the mean velocity gradient or the second derivative of the mean velocity vanish. The similarity hypothesis also fails to yield accurate results in a viscous boundary layer. For parallel mean flow in a symmetrical pipe, the von Kármán similarity hypothesis produces inaccurate results close to the boundary of the pipe, where there is a viscous boundary layer, and the centre of the pipe, where the mean velocity gradient vanishes.

Goldstein [10] avoided using boundary conditions at the boundary and centre of the pipe. Instead, he used experimental data for points outside the viscous boundary layer of the pipe to determine the unknown constants in his solution.

Born and Mason [11] derived the momentum mixing length for boundary layer flow using the von Kármán similarity hypothesis. We extend their results to the classical thermal wake. Their calculations for the momentum mixing length are included in this Chapter so that the Chapter is self-contained.

The mean velocity gradient vanishes at the boundary and centreline of the classical thermal wake. There is a point of inflection for the mean velocity profile between the centreline and the boundary of the classical thermal wake. We expect the von Kármán similarity hypothesis to produce results that are not useful at these three points.

We use a scaling transformation to derive the momentum and thermal mixing lengths from the von Kármán similarity hypothesis. We reduce the governing partial

differential equations to ordinary differential equations. We argue that the boundary conditions derived in Chapter 3 are not useful for the governing equations derived from the von Kármán similarity hypothesis. We propose a possible numerical scheme that remedies the problem.

This Chapter is structured in the following manner. In Section 5.2, we derive equations for velocity and temperature fluctuations. In Section 5.3, we use boundary layer theory to simplify the equations for velocity and temperature fluctuations. In Section 5.4, we derive the momentum and thermal mixing lengths using the von Kármán similarity hypothesis. In Section 5.5, we derive the conserved quantities for the classical turbulent thermal wake. In Section 5.6, we use a scaling transformation to reduce the governing partial differential equations to ordinary differential equations. In Section 5.7, we discuss the applicability of the ordinary differential equations with the boundary conditions for classical thermal wakes. In Section 5.8, we provide conclusions for the Chapter.

5.2 Equations for fluctuations

In this Section, we derive the equations governing the momentum and thermal fluctuations in the turbulent boundary layer of the thermal wake.

5.2.1 Equations for momentum fluctuations

Consider the x and y components of the Navier-Stokes equation,

$$\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\nu\frac{\partial v_x}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu\frac{\partial v_x}{\partial y}\right), \tag{5.2.1}$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\nu\frac{\partial v_y}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu\frac{\partial v_y}{\partial y}\right), \tag{5.2.2}$$

and the continuity equation

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0, \tag{5.2.3}$$

where $v_x = v_x(t,x,y)$, $v_y = v_y(t,x,y)$ are the components of the fluid velocity vector, $p(t,x,y)$ is the pressure in the fluid, ρ and ν are the density and kinematic viscosity of the fluid, respectively. For a real fluid the kinematic viscosity, ν, is very small compared to the eddy viscosity, ν_T. In the turbulent boundary layer of a jet or wake with constant mainstream velocity, the pressure gradients vanish and equation (5.2.2) is identically satisfied. Hence we neglect the kinematic viscosity, the pressure gradient in (5.2.1) and we focus on equation (5.2.1), which becomes

$$\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} = 0, \tag{5.2.4}$$

and the continuity equation, (5.2.3). We split the flow into a mean and fluctuations for velocity components. That is,

$$v_x(t,x,y) = \overline{v}_x(x,y) + v_x'(t,x,y), \qquad v_y(t,x,y) = \overline{v}_y(x,y) + v_y'(t,x,y), \tag{5.2.5}$$

where the operator $\overline{(\cdot)}$ is defined by (3.2.11). Substituting (5.2.5) into (5.2.4) and (5.2.3) we obtain

$$\frac{\partial v_x'}{\partial t} + \overline{v}_x\frac{\partial \overline{v}_x}{\partial x} + \overline{v}_x\frac{\partial v_x'}{\partial x} + v_x'\frac{\partial \overline{v}_x}{\partial x} + v_x'\frac{\partial v_x'}{\partial x} + \overline{v}_y\frac{\partial \overline{v}_x}{\partial y} + \overline{v}_y\frac{\partial v_x'}{\partial y} + v_y'\frac{\partial \overline{v}_x}{\partial y} + v_y'\frac{\partial v_x'}{\partial y} = 0, \tag{5.2.6}$$

$$\frac{\partial \overline{v}_x}{\partial x} + \frac{\partial \overline{v}_y}{\partial y} + \frac{\partial v_x'}{\partial x} + \frac{\partial v_y'}{\partial y} = 0. \tag{5.2.7}$$

Averaging (5.2.7) we obtain

$$\frac{\partial \overline{v}_x}{\partial x} + \frac{\partial \overline{v}_y}{\partial y} = 0. \tag{5.2.8}$$

Hence the first two terms of (5.2.7) vanish and equation (5.2.7) becomes

$$\frac{\partial v_x'}{\partial x} + \frac{\partial v_y'}{\partial y} = 0. \tag{5.2.9}$$

5.2.2 Equation for temperature fluctuations

Consider the energy balance equation,

$$\rho c_v\left[\frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y}\right] = \frac{\partial}{\partial x}\left(\kappa\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right), \tag{5.2.10}$$

where $T = T(t,x,y)$ is the temperature, κ is the thermal conductivity of the fluid and c_v is the specific heat capacity at constant volume. For real fluids, thermal conductivity, κ, is very small compared to the eddy thermal conductivity, κ_T. Hence we neglect κ and equation (5.2.10) becomes

$$\frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} = 0. \tag{5.2.11}$$

We split the flow into mean and fluctuations for temperature and velocity components. That is,

$$T(t,x,y) = \overline{T}(x,y) + T'(t,x,y), \tag{5.2.12}$$

$$v_x(t,x,y) = \overline{v}_x(x,y) + v_x'(t,x,y), \qquad v_y(t,x,y) = \overline{v}_y(x,y) + v_y'(t,x,y). \tag{5.2.13}$$

Substituting (5.2.12) and (5.2.13) into (5.2.11) we obtain

$$\frac{\partial T'}{\partial t} + \overline{v}_x\frac{\partial \overline{T}}{\partial x} + \overline{v}_x\frac{\partial T'}{\partial x} + v_x'\frac{\partial \overline{T}}{\partial x} + v_x'\frac{\partial T'}{\partial x} + \overline{v}_y\frac{\partial \overline{T}}{\partial y} + \overline{v}_y\frac{\partial T'}{\partial y} + v_y'\frac{\partial \overline{T}}{\partial y} + v_y'\frac{\partial T'}{\partial y} = 0. \tag{5.2.14}$$

5.3 Boundary layer approximation for equations for fluctuations

In this Section, we apply boundary layer approximations to equations (5.2.6) and (5.2.14). This has been done already for (5.2.6) by Born and Mason [11]. We include the analysis here for comparison and completeness.

5.3.1 Boundary layer approximations for the momentum equation for fluctuations

The mixing length for the fluctuation flow is confined by the boundary of the wake and the characteristic time for the fluctuation flow is small compared to the characteristic time for the mean flow. Hence, we use different length and time scales for these flows.

Let U_0 be the characteristic mean velocity in the x direction, B be the characteristic fluctuation velocity in the x direction, L be the characteristic length for the mean flow in the x direction, l be the characteristic length in the x direction for the fluctuation velocity and δ_V be the momentum boundary layer thickness for the mean flow in the y direction. We assume that the fluctuation velocity in the y direction is the same order of magnitude as the fluctuation velocity in the x direction and the characteristic length for fluctuations in the x direction is in the same order of magnitude as the characteristic length for fluctuations in the y direction. To satisfy the continuity equation (5.2.8) for the mean flow, the characteristic mean velocity in the y direction must be $\frac{\delta_V}{L}U_0$.

Hence, we non-dimensionalize the momentum equations for fluctuations using the dimensionless variables

$$x^* = \frac{x}{L}, \qquad y^* = \frac{y}{\delta_V} \tag{5.3.1}$$

when differentiating mean velocity components,

$$x^* = \frac{x}{l}, \qquad y^* = \frac{y}{l}, \qquad t^* = \frac{B}{l}t \tag{5.3.2}$$

when differentiating fluctuation velocity components and

$$\overline{v}_x^* = \frac{\overline{v}_x}{U_0}, \qquad \overline{v}_y^* = \frac{L}{\delta_V}\frac{\overline{v}_y}{U_0}, \qquad v_x'^* = \frac{v_x'}{B} \qquad v_y'^* = \frac{v_y'}{B}. \tag{5.3.3}$$

Substituting dimensionless variables (5.3.1) to (5.3.3) into (5.2.6) and multiplying by l/B^2 we obtain

$$\begin{aligned}\frac{\partial v_x'^*}{\partial t^*} + \frac{l}{L}\frac{U_0^2}{B^2}\overline{v}_x^*\frac{\partial \overline{v}_x^*}{\partial x^*} + \frac{U_0}{B}\overline{v}_x^*\frac{\partial v_x'^*}{\partial x^*} + \frac{U_0}{B}\frac{l}{L}v_x'^*\frac{\partial \overline{v}_x^*}{\partial x^*} + v_x'^*\frac{\partial v_x'^*}{\partial x^*} + \frac{l}{L}\frac{U_0^2}{B^2}\overline{v}_y^*\frac{\partial \overline{v}_x^*}{\partial y^*} \\ + \frac{U_0}{B}\frac{\delta_V}{L}\overline{v}_y^*\frac{\partial v_x'^*}{\partial y^*} + \frac{U_0}{B}\frac{l}{\delta_V}v_y'^*\frac{\partial \overline{v}_x^*}{\partial y^*} + v_y'^*\frac{\partial v_x'^*}{\partial x^*} = 0.\end{aligned} \tag{5.3.4}$$

The characteristic time for the fluctuation flow is very small compared to the characteristic time for the mean flow. Hence

$$\frac{l}{B} << \frac{L}{U_0}. \tag{5.3.5}$$

We assume that the characteristic fluctuation velocity, B, is in the same order of magnitude as the characteristic x component of the mean velocity, U. Therefore

$$\frac{l}{L} << 1. \tag{5.3.6}$$

The mixing length of the fluctuation flow is confined by the boundary of the wake. We assume that the characteristic length of the fluctuation flow, l, is in the same order

as the characteristic length in the y direction for the mean flow, δ_V. In boundary layer theory, δ_V / L is assumed to be small and, hence, l/L is also small. Neglecting terms that become small when l/L and δ_V / L are small in (5.3.4), we obtain

$$\frac{\partial v_x'^*}{\partial t^*} + \frac{U_0}{B}\overline{v}_x^* \frac{\partial v_x'^*}{\partial x^*} + \frac{U_0}{B}\frac{l}{\delta_V} v_y'^* \frac{\partial \overline{v}_x^*}{\partial y^*} + v_x'^* \frac{\partial v_x'^*}{\partial x^*} + v_y'^* \frac{\partial v_x'^*}{\partial x^*} = 0. \tag{5.3.7}$$

Writing (5.3.7) in terms of the original variables, we obtain

$$\frac{\partial v_x'}{\partial t} + \overline{v}_x \frac{\partial v_x'}{\partial x} + v_y' \frac{\partial \overline{v}_x}{\partial y} + v_x' \frac{\partial v_x'}{\partial x} + v_y' \frac{\partial v_x'}{\partial y} = 0. \tag{5.3.8}$$

5.3.2 Boundary layer approximation for the thermal equation for fluctuations

The mixing length for the thermal fluctuations is also confined by the boundary of the wake and the characteristic time for the thermal fluctuation flow is small compared to the characteristic time for the mean flow. We introduce additional characteristic dimensionless variables to non-dimensionalize the thermal equation (5.2.14).

Let δ_T be the thermal boundary layer thickness in the y direction, l_θ be the characteristic length for thermal fluctuations and T_0 be the characteristic mean temperature. We assume that the characteristic length for mean temperature change in the x-direction is the same as the momentum characteristic length L, the characteristic length for thermal fluctuations in the x direction is the same order of magnitude as the characteristic length for thermal fluctuations in the y direction and the characteristic temperature fluctuation is of the same order of magnitude as the mean temperature.

Hence, the additional dimensionless variables for the thermal equation (5.2.14) are

$$x^* = \frac{x}{L}, \qquad y^* = \frac{y}{\delta_T}, \tag{5.3.9}$$

when differentiating the mean temperature,

$$x^* = \frac{x}{l_\theta}, \qquad y^* = \frac{y}{l_\theta}, \qquad t^* = \frac{B}{l_\theta} t, \tag{5.3.10}$$

when differentiating the fluctuation temperature and

$$\overline{T}^* = \frac{\overline{T}}{T_0}, \qquad T'^* = \frac{T'}{T_0}. \tag{5.3.11}$$

Equation (5.2.14) is linear in temperature and hence the characteristic temperature T_0 does not play a role in the mathematical analysis. Substituting dimensionless variables (5.3.1) to (5.3.3) and (5.3.9) to (5.3.11) into (5.2.14) we obtain

$$\begin{aligned}\frac{\partial T'^*}{\partial t^*} + \frac{l_\theta}{L}\overline{v}_x^* \frac{\partial \overline{T}^*}{\partial x^*} + \overline{v}_x^* \frac{\partial T'^*}{\partial x^*} + \frac{l_\theta}{L}\frac{B}{U_0} v_x'^* \frac{\partial \overline{T}^*}{\partial x^*} + \frac{B}{U_0} v_x'^* \frac{\partial \overline{T}^*}{\partial x^*} + \frac{\delta_V}{\delta_T}\frac{l_\theta}{L}\overline{v}_y^* \frac{\partial \overline{T}^*}{\partial y^*} \\ + \frac{\delta_V}{L}\overline{v}_y^* \frac{\partial T'^*}{\partial y^*} + \frac{B}{U_0}\frac{l_\theta}{\delta_T} v_y'^* \frac{\partial T'^*}{\partial y^*} + \frac{B}{U_0} v_y'^* \frac{\partial T'^*}{\partial y^*} = 0.\end{aligned} \tag{5.3.12}$$

The mixing length for the thermal fluctuation flow is confined by the thermal boundary of the wake. We assume that the characteristic length for thermal fluctuations, l_θ,

is the same order of magnitude as the thermal characteristic length δ_T. In boundary layer theory, $\frac{\delta_T}{L}$ and $\frac{\delta_V}{L}$ are assumed to be small when the Prandtl number is of order one. Hence

$$\frac{l_\theta}{L} << 1. \tag{5.3.13}$$

Neglecting terms that become small when δ_V/L and l_θ/L are small in (5.3.12) we obtain

$$\frac{\partial T'^*}{\partial t^*} + \overline{v}_x^* \frac{\partial T'^*}{\partial x^*} + \frac{B}{U_0} v_x'^* \frac{\partial \overline{T}^*}{\partial x^*} + \frac{B}{U_0}\frac{l_\theta}{\delta_T} v_y'^* \frac{\partial T'^*}{\partial y^*} + \frac{B}{U_0} v_y'^* \frac{\partial T'^*}{\partial y^*} = 0. \tag{5.3.14}$$

Writing (5.3.14) in terms of original variables we obtain

$$\frac{\partial T'}{\partial t} + \overline{v}_x \frac{\partial T'}{\partial x} + v_x' \frac{\partial \overline{T}}{\partial x} + v_y' \frac{\partial \overline{T}}{\partial y} + v_y' \frac{\partial T'}{\partial y} = 0. \tag{5.3.15}$$

5.4 Mixing lengths

In this Section, we derive the momentum and thermal mixing lengths using the von Kármán similarity hypothesis. The derivation of the momentum length for boundary layer flow has been done by Born and Mason [11] and is included for comparison and completeness.

5.4.1 The von Kármán similarity hypothesis: momentum mixing length

Since the fluctuation velocity components satisfy the continuity equation (5.2.9) we can introduce a stream function ψ' for the fluctuation velocity components. Let

$$v_x'(t,x,y) = \frac{\partial \psi'}{\partial y}(t,x,y), \qquad v_y'(t,x,y) = -\frac{\partial \psi'}{\partial x}(t,x,y). \tag{5.4.1}$$

Substituting (5.4.1) into (5.3.8) we obtain

$$\frac{\partial^2 \psi'}{\partial t \partial y} + \overline{v}_x(x,y)\frac{\partial^2 \psi'}{\partial x \partial y} - \frac{\partial \overline{v}_x}{\partial y}(x,y)\frac{\partial \psi'}{\partial x} + \frac{\partial \psi'}{\partial y}\frac{\partial^2 \psi'}{\partial x \partial y} - \frac{\partial \psi'}{\partial x}\frac{\partial^2 \psi'}{\partial y^2} = 0. \tag{5.4.2}$$

Consider position (x,y) in the neighbourhood of a point (x_0,y_0) at a distance l from (x_0,y_0). Expanding $\overline{v}_x(x,y)$ and $\frac{\partial \overline{v}_x}{\partial y}$ about (x_0,y_0) to first order we have

$$\overline{v}_x(x,y) = \overline{v}_x(x_0,y_0) + (x-x_0)\frac{\partial \overline{v}_x}{\partial x}(x_0,y_0) + (y-y_0)\frac{\partial \overline{v}_x}{\partial y}(x_0,y_0) + O(l^2), \tag{5.4.3}$$

$$\frac{\partial \overline{v}_x}{\partial y}(x,y) = \frac{\partial \overline{v}_x}{\partial y}(x_0,y_0) + (x-x_0)\frac{\partial^2 \overline{v}_x}{\partial y \partial x}(x_0,y_0) + (y-y_0)\frac{\partial^2 \overline{v}_x}{\partial y^2}(x_0,y_0) + O(l^2). \tag{5.4.4}$$

Substituting (5.4.3) and (5.4.4) into (5.4.2) we obtain

$$\begin{aligned}\frac{\partial^2\psi'}{\partial t\partial y}+\left[\overline{v}_x(x_0,y_0)+(x-x_0)\frac{\partial\overline{v}_x}{\partial x}(x_0,y_0)+(y-y_0)\frac{\partial\overline{v}_x}{\partial y}(x_0,y_0)\right]\frac{\partial^2\psi'}{\partial x\partial y}\\ -\left[\frac{\partial\overline{v}_x}{\partial y}(x_0,y_0)+(x-x_0)\frac{\partial^2\overline{v}_x}{\partial y\partial x}(x_0,y_0)+(y-y_0)\frac{\partial^2\overline{v}_x}{\partial y^2}(x_0,y_0)\right]\frac{\partial\psi'}{\partial x}\\ +\frac{\partial\psi'}{\partial y}\frac{\partial^2\psi'}{\partial x\partial y}-\frac{\partial\psi'}{\partial x}\frac{\partial^2\psi'}{\partial y^2}=0.\end{aligned}\tag{5.4.5}$$

We transform (5.4.5) to a coordinate system that moves with the mean fluid velocity at (x_0, y_0). That is, we make the transformation

$$x^* = x - \overline{v}_x(x_0,y_0)t,\tag{5.4.6}$$

$$y^* = y - \overline{v}_y(x_0,y_0)t.\tag{5.4.7}$$

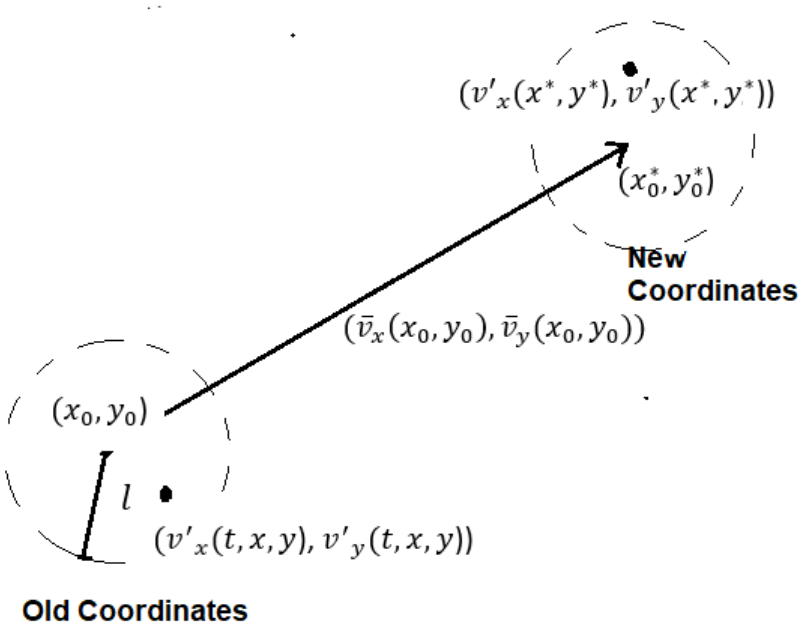

FIGURE 5.1: Transformation (5.4.6) and (5.4.7) is moving the coordinate system with the mean flow. Fluctuations are assumed to be steady in the new coordinates.

Note that

$$x^* - x_0^* = (x - \overline{v}_x(x_0,y_0)t) - (x_0 - \overline{v}_x(x_0,y_0)t) = x - x_0\tag{5.4.8}$$

and similarly

$$y^* - y_0^* = y - y_0.\tag{5.4.9}$$

The transformation preserves length between two points. It is assumed that the fluctuations are steady with respect to the transformed coordinates. Thus

$$v'_x(t,x,y) = v'_x(x^*,y^*),\quad v'_y(t,x,y) = v'_y(x^*,y^*),\quad \psi'(t,x,y) = \psi'(x^*,y^*).\tag{5.4.10}$$

Substituting the transformation (5.4.6) and (5.4.7) into (5.4.5) we obtain

$$-\overline{v}_y(x_0,y_0)\frac{\partial^2\psi'}{\partial y^{*2}}+\left[(x^*-x_0^*)\frac{\partial\overline{v}_x}{\partial x}(x_0,y_0)+(y^*-y_0^*)\frac{\partial\overline{v}_x}{\partial y}(x_0,y_0)+\frac{\partial\psi'}{\partial y^*}\right]\frac{\partial^2\psi'}{\partial x^*\partial y^*}$$
$$-\left[\frac{\partial\overline{v}_x}{\partial y}(x_0,y_0)+(x^*-x_0^*)\frac{\partial^2\overline{v}_x}{\partial y\partial x}(x_0,y_0)+(y^*-y_0^*)\frac{\partial^2\overline{v}_x}{\partial y^2}(x_0,y_0)\right]\frac{\partial\psi'}{\partial x^*}$$
$$-\frac{\partial\psi'}{\partial x^*}\frac{\partial^2\psi'}{\partial y^{*2}}=0. \tag{5.4.11}$$

We substitute the dimensionless variables

$$\xi=\frac{(x^*-x_0^*)}{l},\qquad\qquad \eta=\frac{(y^*-y_0^*)}{l}, \tag{5.4.12}$$

$$\widetilde{x}=\frac{x}{L},\qquad\qquad \widetilde{y}=\frac{y}{\delta_V}, \tag{5.4.13}$$

$$f'(\xi,\eta)=\frac{\psi'}{Bl},\qquad \widetilde{v}_x=\frac{\overline{v}_x}{U_0},\qquad \widetilde{v}_y=\frac{L}{\delta_V}\frac{\overline{v}_y}{U_0}, \tag{5.4.14}$$

into (5.4.11) and multiply by $\frac{\delta_V}{BU_0}$ to obtain

$$-\frac{\delta_V}{l}\frac{\delta_V}{L}\widetilde{v}_y(x_0,y_0)\frac{\partial^2 f'}{\partial\eta^2}+\left[\frac{\delta_V}{L}\xi\frac{\partial\widetilde{v}_x}{\partial\widetilde{x}}(x_0,y_0)+\eta\frac{\partial\widetilde{v}_x}{\partial\widetilde{x}}(x_0,y_0)+\frac{B}{U_0}\frac{\delta_V}{l}\frac{\partial f'}{\partial\eta}\right]\frac{\partial^2 f'}{\partial\xi\partial\eta}$$
$$-\left[\frac{\partial\widetilde{v}_x}{\partial\widetilde{y}}(x_0,y_0)+\frac{l}{L}\xi\frac{\partial^2\widetilde{v}_x}{\partial\widetilde{x}\partial\widetilde{y}}(x_0,y_0)+\frac{l}{\delta_V}\eta\frac{\partial^2\widetilde{v}_x}{\partial\widetilde{y}^2}(x_0,y_0)+\frac{B}{U_0}\frac{\delta_V}{l}\frac{\partial^2 f'}{\partial\eta^2}\right]\frac{\partial f'}{\partial\xi}=0. \tag{5.4.15}$$

Neglecting terms that become small when δ_V/L and l/L are small in (5.4.15) we obtain

$$\eta\frac{\partial\widetilde{v}_x}{\partial\widetilde{y}}(x_0,y_0)\frac{\partial^2 f'}{\partial\xi\partial\eta}+\frac{B}{U_0}\frac{\delta_V}{l}\frac{\partial f'}{\partial\eta}\frac{\partial^2 f'}{\partial\xi\partial\eta}-\frac{\partial\widetilde{v}_x}{\partial\widetilde{y}}(x_0,y_0)\frac{\partial f'}{\partial\xi}-\frac{l}{\delta_V}\eta\frac{\partial^2\widetilde{v}_x}{\partial\widetilde{y}^2}(x_0,y_0)\frac{\partial f'}{\partial\xi}$$
$$+\frac{B}{U}\frac{\delta_V}{l}\frac{\partial^2 f'}{\partial\eta^2}\frac{\partial f'}{\partial\xi}=0. \tag{5.4.16}$$

Substituting the original variables x, y and $\overline{v}_x$ using (5.4.13) and (5.4.14) into (5.4.16) and multiplying by $\frac{UL}{B\delta_V}$ we obtain

$$\frac{l}{B}\frac{\partial\overline{v}_x}{\partial y}(x_0,y_0)\frac{\partial^2 f'}{\partial\xi\partial\eta}+\frac{\partial f'}{\partial\eta}\frac{\partial^2 f'}{\partial\xi\partial\eta}-\frac{l}{B}\frac{\partial\overline{v}_x}{\partial y}(x_0,y_0)\frac{\partial f'}{\partial\xi}-\frac{l^2}{B}\frac{\partial^2\overline{v}_x}{\partial y^2}(x_0,y_0)\eta\frac{\partial f'}{\partial\xi}$$
$$+\frac{\partial^2 f'}{\partial\eta^2}\frac{\partial f'}{\partial\xi}=0. \tag{5.4.17}$$

By the von Kármán similarity hypothesis that the turbulence flow patterns at different points are similar and differ only in the length and time scales,

$$c_1=\frac{l^2}{B}\frac{\partial^2\overline{v}_x}{\partial y^2}(x_0,y_0),\qquad\qquad c_2=\frac{l}{B}\frac{\partial\overline{v}_x}{\partial y}(x_0,y_0), \tag{5.4.18}$$

where c_1 and c_2 are constants. If we assume that c_1 and c_2 are non-zero, that is

$$\frac{\partial^2 \overline{v}_x}{\partial y^2}(x_0, y_0) \neq 0, \qquad \frac{\partial \overline{v}_x}{\partial y}(x_0, y_0) \neq 0, \tag{5.4.19}$$

then their ratios must also be constant. Hence

$$c_3 = \frac{c_1}{c_2} = l \frac{\frac{\partial^2 \overline{v}_x}{\partial y^2}(x_0, y_0)}{\frac{\partial \overline{v}_x}{\partial y}(x_0, y_0)}, \tag{5.4.20}$$

where c_3 is a constant, and the mixing length becomes

$$l = c_3 \frac{\frac{\partial \overline{v}_x}{\partial y}(x_0, y_0)}{\frac{\partial^2 \overline{v}_x}{\partial y^2}(x_0, y_0)}. \tag{5.4.21}$$

We require that the momentum mixing length be positive. Therefore the momentum mixing length at any point (x, y) is

$$l = l_1 \left| \frac{\frac{\partial \overline{v}_x}{\partial y}(x, y)}{\frac{\partial^2 \overline{v}_x}{\partial y^2}(x, y)} \right| = l_1 \left| \frac{\frac{\partial^2 \psi}{\partial y^2}(x, y)}{\frac{\partial^3 \psi}{\partial y^3}(x, y)} \right|, \tag{5.4.22}$$

where l_1 is a constant. Substituting (5.4.18) into (5.4.17) we obtain

$$\frac{\partial f}{\partial \xi} \frac{\partial^2 f}{\partial \eta^2} + \left(c_2 + \frac{df}{d\eta} \right) \frac{\partial^2 f}{\partial \xi \partial \eta} - (c_1 \eta + c_2) \frac{\partial f}{\partial \xi} = 0. \tag{5.4.23}$$

5.4.2 The von Kármán similarity hypothesis: thermal mixing length

Again, for the thermal equation (5.3.15) we let

$$v'_x(t, x, y) = \frac{\partial \psi'}{\partial y}(t, x, y), \qquad v'_y(t, x, y) = -\frac{\partial \psi'}{\partial x}(t, x, y) \tag{5.4.24}$$

since the fluctuation velocity satisfies the continuity equation (5.2.9). Equation (5.3.15) becomes

$$\frac{\partial T'}{\partial t} + \overline{v}_x \frac{\partial T'}{\partial x} + \frac{\partial \psi'}{\partial y} \frac{\partial \overline{T}}{\partial x} - \frac{\partial \psi'}{\partial x} \frac{\partial \overline{T}}{\partial y} - \frac{\partial \psi'}{\partial x} \frac{\partial T'}{\partial y} = 0. \tag{5.4.25}$$

Consider position (x, y) in the neighbourhood of a point (x_0, y_0). When considering momentum variables

$$(x - x_0)^2 + (y - y_0)^2 = O(l^2) \tag{5.4.26}$$

and when considering the thermal variables

$$(x - x_0)^2 + (y - y_0)^2 = O(l_\theta^2). \tag{5.4.27}$$

Expanding mean velocity and temperature gradients about point (x_0, y_0) we have

$$\overline{v}_x(x, y) = \overline{v}_x(x_0, y_0) + (x - x_0) \frac{\partial \overline{v}_x}{\partial x}(x_0, y_0) + (y - y_0) \frac{\partial \overline{v}_x}{\partial y}(x_0, y_0) + O(l^2), \tag{5.4.28}$$

$$\frac{\partial \overline{T}}{\partial x}(x,y) = \frac{\partial \overline{T}}{\partial x}(x_0,y_0) + (x-x_0)\frac{\partial^2 \overline{T}}{\partial x^2}(x_0,y_0) + (y-y_0)\frac{\partial^2 \overline{T}}{\partial x \partial y}(x_0,y_0) + O(l_\theta^2), \tag{5.4.29}$$

$$\frac{\partial \overline{T}}{\partial y}(x,y) = \frac{\partial \overline{T}}{\partial y}(x_0,y_0) + (x-x_0)\frac{\partial^2 \overline{T}}{\partial x \partial y}(x_0,y_0) + (y-y_0)\frac{\partial^2 \overline{T}}{\partial y^2}(x_0,y_0) + O(l_\theta^2). \tag{5.4.30}$$

Substituting (5.4.28) to (5.4.30) into (5.4.25) we obtain

$$\begin{aligned} &\frac{\partial T'}{\partial t} + \left[\overline{v}_x(x_0,y_0) + (x-x_0)\frac{\partial \overline{v}_x}{\partial x}(x_0,y_0) + (y-y_0)\frac{\partial \overline{v}_x}{\partial y}(x_0,y_0)\right]\frac{\partial T'}{\partial x} \\ &+ \left[\frac{\partial \overline{T}}{\partial x}(x_0,y_0) + (x-x_0)\frac{\partial^2 \overline{T}}{\partial x^2}(x_0,y_0) + (y-y_0)\frac{\partial^2 \overline{T}}{\partial x \partial y}(x_0,y_0)\right]\frac{\partial \psi'}{\partial y} \\ &- \left[\frac{\partial \overline{T}}{\partial y}(x_0,y_0) + (x-x_0)\frac{\partial^2 \overline{T}}{\partial x \partial y}(x_0,y_0) + (y-y_0)\frac{\partial^2 \overline{T}}{\partial y^2}(x_0,y_0)\right]\frac{\partial \psi'}{\partial x} \\ &- \frac{\partial \psi'}{\partial x}\frac{\partial T'}{\partial y} = 0. \end{aligned} \tag{5.4.31}$$

We make the transformation

$$x^* = x - \overline{v}_x(x_0,y_0)t, \qquad y^* = y - \overline{v}_y(x_0,y_0)t. \tag{5.4.32}$$

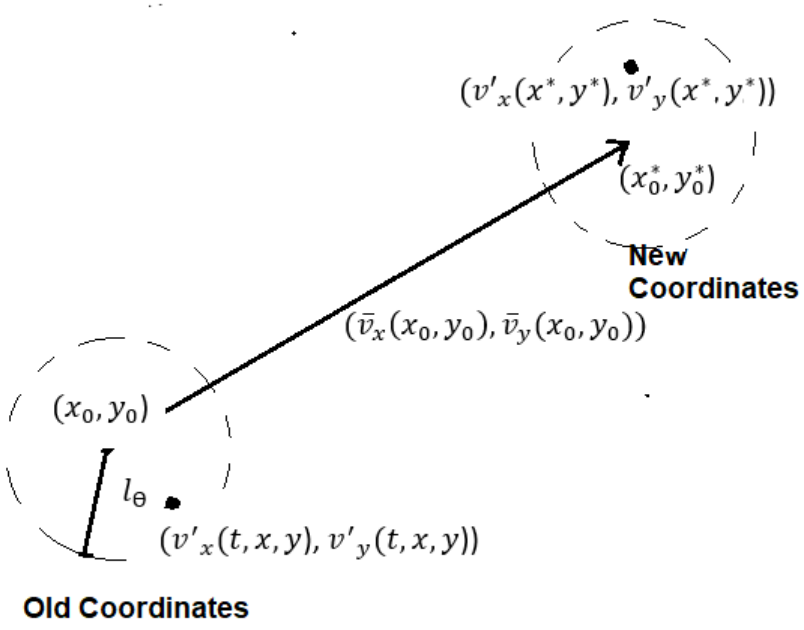

FIGURE 5.2: Transformation (5.4.32) is moving the coordinate system with the mean flow. Fluctuations are assumed to be steady in the new coordinates.

As before, we note that

$$x^* - x_0^* = x - x_0, \qquad y^* - y_0^* = y - y_0. \tag{5.4.33}$$

It is assumed that the fluctuation is steady with respect to the coordinate system (x^*, y^*). Then

$$v'_x(t,x,y) = v'_x(x^*,y^*), \qquad v'_y(t,x,y) = v'_y(x^*,y^*), \tag{5.4.34}$$

$$\psi'(t,x,y) = \psi'(x^*,y^*), \qquad T'(t,x,y) = T'(x^*,y^*). \tag{5.4.35}$$

Substituting the transformation (5.4.32) into (5.4.31) we obtain

$$\begin{aligned} &-\overline{v}_y(x_0,y_0)\frac{\partial T'}{\partial y^*} + \left[(x^*-x_0^*)\frac{\partial \overline{v}_x}{\partial x}(x_0,y_0) + (y^*-y_0^*)\frac{\partial \overline{v}_x}{\partial y}(x_0,y_0)\right]\frac{\partial T'}{\partial x^*} \\ &+ \left[\frac{\partial \overline{T}}{\partial x}(x_0,y_0) + (x^*-x_0^*)\frac{\partial^2 \overline{T}}{\partial x^2}(x_0,y_0) + (y^*-y_0^*)\frac{\partial^2 \overline{T}}{\partial x \partial y}(x_0,y_0)\right]\frac{\partial \psi'}{\partial y^*} \\ &- \left[\frac{\partial \overline{T}}{\partial y}(x_0,y_0) + (x^*-x_0^*)\frac{\partial^2 \overline{T}}{\partial x \partial y}(x_0,y_0) + (y^*-y_0^*)\frac{\partial^2 \overline{T}}{\partial y^2}(x_0,y_0)\right]\frac{\partial \psi'}{\partial x^*} \\ &\qquad -\frac{\partial \psi'}{\partial x^*}\frac{\partial T'}{\partial y^*} = 0. \end{aligned} \tag{5.4.36}$$

We introduce the following dimensionless variables. In the expansion of the momentum variables

$$\xi = \frac{(x^* - x_0^*)}{l}, \qquad \eta = \frac{(y^* - y_0^*)}{l}. \tag{5.4.37}$$

In the expansion of the thermal variables

$$\xi = \frac{(x^* - x_0^*)}{l_\theta}, \qquad \eta = \frac{(y^* - y_0^*)}{l_\theta}. \tag{5.4.38}$$

When differentiating mean velocity components

$$\widetilde{x} = \frac{x}{L}, \qquad \widetilde{y} = \frac{y}{\delta_V}. \tag{5.4.39}$$

When differentiating the mean temperature

$$\widetilde{x} = \frac{x}{L}, \qquad \widetilde{y} = \frac{y}{\delta_T}. \tag{5.4.40}$$

Momentum dimensionless variables are

$$f' = \frac{\psi'}{Bl}, \qquad \widetilde{v}_x = \frac{\overline{v}_x}{U_0}, \qquad \widetilde{v}_y = \frac{L}{\delta_V}\frac{\overline{v}_y}{U_0}. \tag{5.4.41}$$

Thermal dimensionless variables are

$$g' = \frac{T'}{T_0}, \qquad \widetilde{T} = \frac{\overline{T}}{T_0}. \tag{5.4.42}$$

Substituting dimensionless variables (5.4.37) to (5.4.42) into (5.4.36) we obtain

$$\begin{aligned}
&-\frac{\delta_V}{L}\widetilde{v}_y(x_0,y_0)\frac{\partial g'}{\partial \xi}+\left[\frac{l}{L}\xi\frac{\partial \widetilde{v}_x}{\partial \widetilde{x}}(x_0,y_0)+\frac{l}{\delta_V}\eta\frac{\partial \widetilde{v}_x}{\partial \widetilde{y}}(x_0,y_0)\right]\frac{\partial g'}{\partial \xi}\\
&+\left[\frac{B}{U_0}\frac{l_\theta}{L}\frac{\partial \widetilde{T}}{\partial \widetilde{x}}(x_0,y_0)+\frac{B}{U_0}\frac{l_\theta^2}{L^2}\xi\frac{\partial^2 \widetilde{T}}{\partial \widetilde{x}^2}(x_0,y_0)+\frac{B}{U_0}\frac{l_\theta}{\delta_T}\frac{l_\theta}{L}\eta\frac{\partial^2 \widetilde{T}}{\partial \widetilde{x}\partial \widetilde{y}}(x_0,y_0)\right]\frac{\partial f'}{\partial \eta}\\
&-\left[\frac{B}{U_0}\frac{l_\theta}{\delta_T}\frac{\partial \widetilde{T}}{\partial \widetilde{y}}(x_0,y_0)+\frac{B}{U_0}\frac{l_\theta}{\delta_T}\frac{l_\theta}{L}\xi\frac{\partial^2 \widetilde{T}}{\partial \widetilde{x}\partial \widetilde{y}}(x_0,y_0)+\frac{B}{U_0}\frac{l_\theta^2}{\delta_T^2}\eta\frac{\partial^2 \widetilde{T}}{\partial \widetilde{y}^2}(x_0,y_0)\right]\frac{\partial f'}{\partial \xi}\\
&-\frac{B}{U_0}\frac{\partial f'}{\partial \xi}\frac{\partial g'}{\partial \eta}=0.
\end{aligned}\tag{5.4.43}$$

Neglecting terms that are small when δ_V/L, l/L and l_θ/L are small equation (5.4.43) becomes

$$\begin{aligned}
&\frac{l}{\delta_V}\eta\frac{\partial \widetilde{v}_x}{\partial \widetilde{y}}(x_0,y_0)\frac{\partial g'}{\partial \xi}-\frac{B}{U_0}\frac{l_\theta}{\delta_T}\frac{\partial \widetilde{T}}{\partial \widetilde{y}}(x_0,y_0)\frac{\partial f'}{\partial \xi}-\frac{B}{U_0}\frac{l_\theta^2}{\delta_T^2}\frac{\partial^2 \widetilde{T}}{\partial \widetilde{y}^2}(x_0,y_0)\eta\frac{\partial f'}{\partial \xi}\\
&-\frac{B}{U_0}\frac{\partial f'}{\partial \xi}\frac{\partial g'}{\partial \eta}=0.
\end{aligned}\tag{5.4.44}$$

Substituting the original variables x, y and T' using (5.4.39) to (5.4.42) into (5.4.44) we obtain

$$\frac{l}{B}\frac{\partial \overline{v}_x}{\partial y}(x_0,y_0)\eta\frac{\partial T'}{\partial \xi}-l_\theta\frac{\partial \overline{T}}{\partial y}(x_0,y_0)\frac{\partial f'}{\partial \xi}-l_\theta^2\frac{\partial^2 \overline{T}}{\partial y^2}(x_0,y_0)\eta\frac{\partial f'}{\partial \xi}-\frac{\partial f'}{\partial \xi}\frac{\partial T'}{\partial \eta}=0.\tag{5.4.45}$$

By the von Kármán similarity hypothesis that the turbulence flow patterns at different points are similar and differ only in the length and time scales. Hence

$$\frac{l}{B}\frac{\partial \overline{v}_x}{\partial y}(x_0,y_0)=c_2,\tag{5.4.46}$$

$$l_\theta\frac{\partial \overline{T}}{\partial y}(x_0,y_0)=c_4,\tag{5.4.47}$$

$$l_\theta^2\frac{\partial^2 \overline{T}}{\partial y^2}(x_0,y_0)=c_5,\tag{5.4.48}$$

where c_2, c_4 and c_5 are constants. Equation (5.4.46) agrees with (5.4.18). If we assume that c_4 and c_5 are non-zero, that is

$$\frac{\partial^2 \overline{T}}{\partial y^2}(x_0,y_0)\neq 0,\qquad\qquad \frac{\partial \overline{T}}{\partial y}(x_0,y_0)\neq 0,\tag{5.4.49}$$

then their ratios must also be constant. Hence

$$\frac{c_4}{c_5}=l_\theta\frac{\frac{\partial^2 \overline{T}}{\partial y^2}(x_0,y_0)}{\frac{\partial \overline{T}}{\partial y}(x_0,y_0)},\tag{5.4.50}$$

and the mixing length becomes

$$l_\theta = \frac{c_4}{c_5} \frac{\frac{\partial \overline{T}}{\partial y}(x_0, y_0)}{\frac{\partial^2 \overline{T}}{\partial y^2}(x_0, y_0)}. \tag{5.4.51}$$

From (5.4.46) and (5.4.51)

$$B = \frac{c_5}{c_2 c_4} \frac{\frac{\partial \overline{T}}{\partial y}(x_0, y_0)}{\frac{\partial^2 \overline{T}}{\partial y^2}(x_0, y_0)} \frac{\partial \overline{v}_x}{\partial y}(x_0, y_0). \tag{5.4.52}$$

Equation (5.4.45) becomes

$$c_2 \eta \frac{\partial T'}{\partial \xi} - c_4 \frac{\partial f'}{\partial \xi} - c_5 \eta \frac{\partial f'}{\partial \xi} - \frac{\partial f'}{\partial \xi} \frac{\partial g'}{\partial \eta} = 0 \tag{5.4.53}$$

and simplifies to

$$\left(c_4 + c_5 \eta + \frac{\partial T'}{\partial \eta} \right) \frac{\partial f'}{\partial \xi} - c_2 \eta \frac{\partial T'}{\partial \xi} = 0. \tag{5.4.54}$$

We require that the thermal mixing length be positive. Therefore the thermal mixing length at any point (x, y) is

$$l_\theta = l_2 \left| \frac{\frac{\partial \overline{T}}{\partial y}(x, y)}{\frac{\partial^2 \overline{T}}{\partial y^2}(x, y)} \right| = l_2 \left| \frac{\frac{\partial \overline{S}}{\partial y}(x, y)}{\frac{\partial^2 \overline{S}}{\partial y^2}(x, y)} \right|, \tag{5.4.55}$$

where l_2 is a constant.

Substituting the mixing lengths (5.4.22) and (5.4.55) into (3.12.3) and (3.12.4) we obtain

$$U \frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial}{\partial y} \left[\left(\nu + l_1^2 \frac{\left(\frac{\partial^2 \psi}{\partial y^2} \right)^2 \left| \frac{\partial^2 \psi}{\partial y^2} \right|}{\left(\frac{\partial^3 \psi}{\partial y^3} \right)^2} \right) \frac{\partial^2 \psi}{\partial y^2} \right], \tag{5.4.56}$$

$$U \frac{\partial \overline{S}}{\partial x} = \frac{\partial}{\partial y} \left[\left(\frac{\kappa}{\rho c_v} + l_1 l_2 \frac{\frac{\partial^2 \psi}{\partial y^2} \left| \frac{\partial^2 \psi}{\partial y^2} \right| \frac{\partial \overline{S}}{\partial y}}{\left| \frac{\partial^3 \psi}{\partial y^3} \right| \left| \frac{\partial^2 \overline{S}}{\partial y^2} \right|} \right) \frac{\partial \overline{S}}{\partial y} \right]. \tag{5.4.57}$$

In the upper half of a classical wake, $0 \leq y \leq y_B(x)$,

$$\frac{\partial^2 \psi}{\partial y^2} \leq 0. \tag{5.4.58}$$

In the upper half of the wake, equations (5.4.56) and (5.4.57) become

$$U \frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial}{\partial y} \left[\left(\nu - l_1^2 \frac{\left(\frac{\partial^2 \psi}{\partial y^2} \right)^3}{\left(\frac{\partial^3 \psi}{\partial y^3} \right)^2} \right) \frac{\partial^2 \psi}{\partial y^2} \right], \tag{5.4.59}$$

$$U \frac{\partial \overline{S}}{\partial x} = \frac{\partial}{\partial y} \left[\left(\frac{\kappa}{\rho c_v} - l_1 l_2 \frac{\left(\frac{\partial^2 \psi}{\partial y^2} \right)^2 \frac{\partial \overline{S}}{\partial y}}{\left| \frac{\partial^3 \psi}{\partial y^3} \right| \left| \frac{\partial^2 \overline{S}}{\partial y^2} \right|} \right) \frac{\partial \overline{S}}{\partial y} \right]. \tag{5.4.60}$$

5.5 Conserved quantities

5.5.1 Momentum flux conserved quantity

Integrating the governing partial differential equation (5.4.59) with respect to y from the centreline of the wake, $y = 0$, to the boundary, $y = y_B(x)$, we obtain

$$U \int_0^{y_B(x)} \frac{\partial^2 \psi}{\partial x \partial y} dy = \left[\left(\nu - l_1^2 \frac{\left(\frac{\partial^2 \psi}{\partial y^2} \right)^3}{\left(\frac{\partial^3 \psi}{\partial y^3} \right)^2} \right) \frac{\partial^2 \psi}{\partial y^2} \right] \Bigg|_{y=0}^{y=y_B(x)} = 0. \tag{5.5.1}$$

Applying the second and fourth boundary conditions in (3.12.5) the right hand side of (5.5.1) vanishes and equation (5.5.1) reduces to

$$\int_0^{y_B(x)} \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial y} \right) dy = 0. \tag{5.5.2}$$

By the theorem for differentiation under the integral sign [19] equation (5.5.2) becomes

$$\frac{d}{dx} \int_0^{y_B(x)} \frac{\partial \psi}{\partial y} dy - \frac{dy_B}{dx} \frac{\partial \psi}{\partial y}(x, y_B(x)) = 0. \tag{5.5.3}$$

Applying the third boundary condition in (3.12.5) equation (5.5.3) reduces to

$$\frac{d}{dx} \int_0^{y_B(x)} \frac{\partial \psi}{\partial y}(x, y) dy = 0. \tag{5.5.4}$$

Hence

$$D = 2\rho U \int_0^{y_B(x)} \frac{\partial \psi}{\partial y}(x, y) dy \tag{5.5.5}$$

is a constant independent of x.

5.5.2 Thermal flux conserved quantity

Integrating the governing partial differential equation (5.4.60) with respect to y from the centreline, $y = 0$, to the boundary, $y = y_B(x)$, we obtain

$$U \int_0^{y_B(x)} \frac{\partial \overline{S}}{\partial x} dy = \left[\left(\frac{\kappa}{\rho c_v} - l_1 l_2 \frac{\left(\frac{\partial^2 \psi}{\partial y^2} \right)^2 \frac{\partial \overline{S}}{\partial y}}{\left| \frac{\partial^3 \psi}{\partial y^3} \right| \left| \frac{\partial^2 \overline{S}}{\partial y^2} \right|} \right) \frac{\partial \overline{S}}{\partial y} \right] \Bigg|_{y=0}^{y=y_B(x)}. \tag{5.5.6}$$

Applying the second and fourth boundary conditions in (3.12.5) and the first and third boundary condition in (3.12.6) equation (5.5.6) reduces to

$$U \int_0^{y_B(x)} \frac{\partial \overline{S}}{\partial x}(x, y) dy = 0. \tag{5.5.7}$$

By the theorem for differentiation under the integral sign [19] equation (5.5.7) becomes

$$U \frac{d}{dx} \int_0^{y_B(x)} \overline{S}(x, y) dy - U \frac{dy_B}{dx} \overline{S}(x, y_B(x)) = 0. \tag{5.5.8}$$

Applying the second boundary condition in (3.12.6) the second term in (5.5.8) vanishes and therefore (5.5.8) reduces to

$$U\frac{d}{dx}\int_0^{y_B(x)}\overline{S}(x,y)dy=0. \tag{5.5.9}$$

Hence

$$K=2\rho c_v U\int_0^{y_B(x)}\overline{S}(x,y)dy \tag{5.5.10}$$

is a constant independent of x.

5.6 Scaling transformations

In this Section, we derive the scaling transformation admitted by the governing partial differential equations (5.4.59) and (5.4.60). Let

$$\widetilde{x}=\lambda^a x, \qquad \widetilde{y}=\lambda^b y, \tag{5.6.1}$$

$$\widetilde{\psi}=\lambda^c\psi, \qquad \widetilde{S}=\lambda^k\overline{S}, \tag{5.6.2}$$

where a, b, c, k and λ are constants. Substituting (5.6.1) and (5.6.2) into the governing partial differential equations (5.4.59) and (5.4.60) we obtain

$$U\frac{\partial^2\widetilde{\psi}}{\partial\widetilde{x}\partial\widetilde{y}}=\lambda^{-a+2b}\frac{\partial}{\partial\widetilde{y}}\left[\left(\nu-l_1^2\lambda^{-c}\frac{\left(\frac{\partial^2\widetilde{\psi}}{\partial\widetilde{y}^2}\right)^3}{\left(\frac{\partial^3\widetilde{\psi}}{\partial\widetilde{y}^3}\right)^2}\right)\frac{\partial^2\widetilde{\psi}}{\partial\widetilde{y}^2}\right], \tag{5.6.3}$$

$$U\frac{\partial\widetilde{S}}{\partial\widetilde{x}}=\lambda^{-a+2b}\frac{\partial}{\partial\widetilde{y}}\left[\left(\frac{\kappa}{\rho c_v}-l_1l_2\lambda^{-c}\frac{\left(\frac{\partial^2\widetilde{\psi}}{\partial\widetilde{y}^2}\right)^2\frac{\partial\widetilde{S}}{\partial\widetilde{y}}}{\left|\frac{\partial^3\widetilde{\psi}}{\partial\widetilde{y}^3}\right|\left|\frac{\partial^2\widetilde{S}}{\partial\widetilde{y}^2}\right|}\right)\frac{\partial\widetilde{S}}{\partial\widetilde{y}}\right]. \tag{5.6.4}$$

For the scaling transformation (5.6.1) and (5.6.2) to be invariant on the governing partial differential equations (5.4.59) and (5.4.60) the exponents of λ in (5.6.3) and (5.6.4) must vanish. Hence

$$a=2b, \qquad c=0 \tag{5.6.5}$$

and k remains an arbitrary constant. Therefore the governing partial differential equations (5.4.59) and (5.4.60) are invariant under the scaling transformation

$$\widetilde{x}=\lambda^{2b}x, \qquad \widetilde{y}=\lambda^b y, \qquad \widetilde{\psi}=\psi, \qquad \widetilde{S}=\lambda^k\overline{S}. \tag{5.6.6}$$

Suppose that the solution of (5.4.59) is

$$\psi=f(x,y). \tag{5.6.7}$$

Then the solution of (5.6.3) under the invariant transformation (5.6.6) is

$$\psi=f(\lambda^{2b}x,\lambda^b y). \tag{5.6.8}$$

Hence since $\psi = f(x,y)$

$$f(x,y) = f(\lambda^{2b}x, \lambda^{b}y). \tag{5.6.9}$$

Differentiating equation (5.6.9) with respect to λ we obtain

$$0 = 2b\lambda^{2b-1}x\frac{\partial f}{\partial \widetilde{x}} + by\lambda^{b-1}\frac{\partial f}{\partial \widetilde{y}}. \tag{5.6.10}$$

Letting λ tend to one in (5.6.10) we obtain the partial differential equation

$$2bx\frac{\partial f}{\partial x} + by\frac{\partial f}{\partial y} = 0. \tag{5.6.11}$$

The differential equations of the characteristic curves of the partial differential equation (5.6.11) are

$$\frac{dx}{2x} = \frac{dy}{y} = \frac{df}{0}. \tag{5.6.12}$$

Integrating the equation consisting of the first and second terms of (5.6.12) we obtain

$$\ln y - \frac{1}{2}\ln x = C, \tag{5.6.13}$$

where C is a constant. Exponentiating (5.6.13) we obtain

$$I_1 = \frac{y}{x^{1/2}}, \tag{5.6.14}$$

where $I_1 = e^C$ is a constant. From (5.6.12)

$$f = I_2, \tag{5.6.15}$$

where I_2 is a constant. The general solution of the first order partial differential equation (5.6.11) is

$$I_2 = F(I_1), \tag{5.6.16}$$

where F is an arbitrary function. Since $f = \psi(x,y)$,

$$\psi(x,y) = F(\xi), \tag{5.6.17}$$

where

$$\xi = \frac{y}{x^{1/2}}. \tag{5.6.18}$$

Suppose that the solution of (5.4.60) is

$$\overline{S} = g(x,y). \tag{5.6.19}$$

Then the solution of (5.6.4) under the invariant transformation (5.6.6) is

$$\lambda^k\overline{S} = g(\lambda^{2b}x, \lambda^{b}y) \tag{5.6.20}$$

and therefore since $\overline{S} = g(x,y)$,

$$\lambda^k g(x,y) = g(\lambda^{2b}x, \lambda^{b}x). \tag{5.6.21}$$

Differentiating (5.6.21) with respect to λ we obtain

$$k\lambda^{k-1}g = 2b\lambda^{2b-1}x\frac{\partial g}{\partial \widetilde{x}} + b\lambda^{b-1}y\frac{\partial g}{\partial \widetilde{y}}. \tag{5.6.22}$$

Letting λ tend to one in (5.6.22) we obtain the partial differential equation

$$2bx\frac{\partial g}{\partial x} + by\frac{\partial g}{\partial y} = kg. \tag{5.6.23}$$

The differential equations of the characteristic curves of the partial differential equation (5.6.23) are

$$\frac{dx}{2bx} = \frac{dy}{by} = \frac{dg}{kg}. \tag{5.6.24}$$

We have already solved the equation consisting of the first and second terms of (5.6.24) and we obtained (5.6.14). Integrating the equation consisting of the first and last terms of (5.6.24) we obtain

$$\ln g - \frac{\alpha}{2}\ln x = C, \tag{5.6.25}$$

where C is a constant and

$$\alpha = \frac{k}{b}. \tag{5.6.26}$$

Exponentiating (5.6.25) we obtain

$$\frac{g}{x^{\alpha/2}} = I_3, \tag{5.6.27}$$

where $I_3 = e^C$ is a constant. The general solution of the first order partial differential equation (5.6.23) is

$$I_3 = G(I_1), \tag{5.6.28}$$

where G is an arbitrary function. Since $g = \overline{S}(x,y)$,

$$\overline{S}(x,y) = x^{\alpha/2}G(\xi). \tag{5.6.29}$$

Substituting the similarity solution (5.6.17), (5.6.18) and (5.6.29) into the governing partial differential equations (5.4.59) and (5.4.60) we obtain

$$\frac{d}{d\xi}\left[\left(\nu - l_1^2\frac{\left(\frac{d^2F}{d\xi^2}\right)^3}{\left(\frac{d^3F}{d\xi^3}\right)^2}\right)\frac{d^2F}{d\xi^2}\right] + \frac{U}{2}\frac{d}{d\xi}\left(\xi\frac{dF}{d\xi}\right) = 0, \tag{5.6.30}$$

$$\frac{d}{d\xi}\left[\left(\frac{\kappa}{\rho c_v} - l_1 l_2\frac{\left(\frac{d^2F}{d\xi^2}\right)^2\frac{dG}{d\xi}}{\left|\frac{d^3F}{d\xi^3}\right|\left|\frac{d^2G}{d\xi^2}\right|}\right)\frac{dG}{d\xi}\right] + \frac{U}{2}\left[\xi\frac{dG}{d\xi} - \alpha G\right] = 0. \tag{5.6.31}$$

Substituting the similarity solution (5.6.17), (5.6.18) and (5.6.29) into the conserved quantity (5.5.10) we obtain

$$K = 2\rho c_v U x^{\frac{1}{2}(\alpha+1)}\int_0^{\xi_B(x)} G(\xi)d\xi, \tag{5.6.32}$$

where

$$\xi_B(x) = \frac{y_B(x)}{x^{1/2}}. \tag{5.6.33}$$

Since K is a constant independent of x, $\frac{dK}{dx} = 0$ and

$$\frac{1}{2}(\alpha+1)\, x^{\frac{1}{2}(\alpha-1)} \int_0^{\xi_B(x)} G(\xi)d\xi + x^{\frac{1}{2}(\alpha+1)} \frac{d\xi_B}{dx} G(\xi_B) = 0. \tag{5.6.34}$$

At the boundary of the wake $\xi = \xi_B(x)$ and

$$S(x, y_B(x)) = x^{\alpha/2} G(\xi_B(x)) = 0. \tag{5.6.35}$$

Hence, the second term of (5.6.34) vanishes and equation (5.6.34) reduces to

$$(\alpha+1) \int_0^{\xi_B(x)} G(\xi)d\xi = 0. \tag{5.6.36}$$

Assuming that the conserved quantity K is nonzero, we have $\alpha = -1$. Suppose G is analytic at $\xi = \xi_B(x)$ and not a constant. Then not all derivatives vanish and there exists an integer $n \geq 1$ such that

$$\frac{d^{n-1}G}{d\xi^{n-1}}(\xi_B(x)) = 0 \tag{5.6.37}$$

and

$$\frac{d^n G}{d\xi^n}(\xi_B(x)) \neq 0. \tag{5.6.38}$$

Differentiating (5.6.37) with respect to x, we have

$$\frac{d^n G}{d\xi^n}(\xi_B(x)) \frac{d\xi_B}{dx} = 0. \tag{5.6.39}$$

Therefore, $\xi_B(x)$ is a constant and

$$y_B(x) = \xi_B x^{1/2}. \tag{5.6.40}$$

The power law $x^{1/2}$ is the same as found in Chapter 4 for a turbulent classical wake with mixing lengths $l(x)$ and $l_\theta(x)$ depending on x. Also

$$K = 2\rho c_v \int_0^{\xi_B} G(\xi)d\xi. \tag{5.6.41}$$

Equation (5.6.31) becomes

$$\frac{d}{d\xi}\left[\left(\frac{\kappa}{\rho c_v} - l_1 l_2 \frac{\left(\frac{d^2F}{d\xi^2}\right)^2 \frac{dG}{d\xi}}{\left|\frac{d^3F}{d\xi^3}\right| \left|\frac{d^2G}{d\xi^2}\right|}\right)\frac{dG}{d\xi}\right] + \frac{U}{2}\frac{d}{d\xi}(\xi G) = 0. \tag{5.6.42}$$

Substituting the similarity solution (5.6.17), (5.6.18) and (5.6.29) into the conserved quantity (5.5.5) we obtain

$$D = 2\rho U \int_0^{\xi_B} \frac{dF}{d\xi} d\xi. \tag{5.6.43}$$

Substituting the similarity solution (5.6.17), (5.6.18) and (5.6.29) into the boundary conditions (3.12.5) and (3.12.6) we obtain

$$\frac{d^2F}{d\xi^2}(0) = 0, \qquad \frac{dF}{d\xi}(\xi_B) = 0, \qquad \frac{d^2F}{d\xi^2}(\xi_B) = 0, \tag{5.6.44}$$

$$\frac{dG}{d\xi}(0) = 0, \qquad G(\xi_B) = 0, \qquad \kappa\frac{dG}{d\xi}(\xi_B) = 0. \tag{5.6.45}$$

The ordinary differential equations (5.6.30) and (5.6.42) can be integrated analytically once. This is another example of the double reduction theorem of Sjöberg [17, 18]. The parameter α was obtained from the conserved quantity (5.6.32) and not from a boundary condition. Integrating (5.6.30) and (5.6.42) once and imposing the boundary conditions (5.6.44) and (5.6.45) either at $\xi = 0$ or $\xi = \xi_B$ gives

$$\left[\nu - l_1^2\frac{\left(\frac{d^2F}{d\xi^2}\right)^3}{\left(\frac{d^3F}{d\xi^3}\right)^2}\right]\frac{d^2F}{d\xi^2} + \frac{U}{2}\xi\frac{dF}{d\xi} = 0, \tag{5.6.46}$$

$$\left[\frac{\kappa}{\rho c_v} - l_1 l_2\frac{\frac{d^2F}{d\xi^2}\frac{dG}{d\xi}}{\left|\frac{d^3F}{d\xi^3}\right|\left|\frac{d^2G}{d\xi^2}\right|}\right]\frac{dG}{d\xi} + \frac{U}{2}\xi G = 0. \tag{5.6.47}$$

Further analytical integration has not been possible and a numerical solution is required to solve equations (5.6.46) and (5.6.47).

The value of the constant ξ_B is calculated from (5.6.41) and (5.6.43) during the process of solving the problem. We have therefore not determined if ξ_B is finite or infinite and if the wake has a finite boundary or it extends to $y = \pm\infty$.

5.7 Applicability of the von Kármán hypothesis to classical thermal wakes

Since the von Kármán similarity hypothesis cannot be applied at the centreline and boundary of the classical thermal wake because the mean velocity gradient vanishes at both points. The boundary conditions (5.6.44) and (5.6.45) are therefore not useful when using the hypothesis. We cannot integrate the differential equations (5.6.46) and (5.6.47) near $\xi = 0$, $\xi = \xi_B$ and the point inflection between 0 and ξ_B. Hence to obtain the solution for the classical thermal wake we need more data points. There are two regions where the von Kármán similarity hypothesis may apply:

(i) A subset of the region between the centreline and the point of inflection.

(ii) A subset of the region between the point of inflection and the boundary of the wake.

Goldstein [10] used experimental data in regions where the von Kármán similarity hypothesis is valid to solve for the unknown constants of the parallel flow through a symmetrical pipe. We do not have access to experimental data for the classical thermal wake that we can use to fill in the missing data points in the numerical scheme to solve the differential equations (5.6.46) and (5.6.47).

Another option would be to first assume that the eddy viscosity and eddy thermal conductivity are constant. The solution obtained from the assumption to calculate the updated mixing lengths using (5.4.22) and (5.4.55) and the solution corresponding to the mixing lengths. The process could be repeated until the solution converges.

We will not investigate the numerical solution of (5.6.46) and (5.6.47) in this thesis.

5.8 Conclusions

In this Chapter, we derived the equations governing velocity and temperature fluctuations in a classical thermal wake and simplified the equations using boundary layer theory. We used the von Kármán similarity hypothesis to obtain the momentum and thermal mixing lengths. The scale for temperature cancelled during the analysis leaving only the scale for length and time (velocity) in agreement with the von Kármán similarity hypothesis. We reduced the governing partial differential equations to ordinary differential equations using a scaling transformation. Both ordinary differential equations could integrated once because of the Sjöberg double reduction theorem [17, 18]. Since the mean velocity gradient vanishes at the boundary and centreline of the classical thermal wake the boundary conditions could not assist us to solve the ordinary differential equations (5.6.46) and (5.6.47). One option would be to use experimental data as Goldstein [10] did for mean parallel flow and another option would be to use an iterative method based on the mixing lengths to find the mean velocity and mean temperature difference profiles.

The Lie group analysis in Chapter 4 and the scaling transformation in this Chapter gave the same equation for the boundary of the turbulent thermal wake for momentum and thermal mixing lengths depending on x and for the von Kármán mixing lengths. While we were able to show that the boundary of the wake is finite for $\nu = 0$ and $\kappa = 0$ when the mixing lengths depend on x we were not able to do that for the von Kármán mixing lengths because an analytical or numerical solution has not yet been derived.

Chapter 6

Momentumless wake with the Prandtl mixing length model

6.1 Introduction

The wake behind a self-propelled body is referred to as a momentumless wake. The drag force exerted by a momentumless wake on the self-propelled body is zero. Chapters 6 and 7 will be concerned with the momentumless thermal wake produced when a cold fluid passes a heated self-propelled body. In this Chapter, Prandtl's mixing length model will be used for the eddy viscosity while in the next Chapter the eddy viscosity will be modelled as a function of x.

In Chapter 4, we assumed that the multipliers for the conservation laws of the governing partial differential equations are functions of x, y, ψ and $\overline{S}$ so that they can be calculated readily by hand. We saw that under these assumptions Λ_1 remained an arbitrary function of x and Λ_2 is a constant. These multipliers cannot generate conservation laws for a momentumless thermal wake. Hence to derive the multipliers for a momentumless thermal wake, we have to assume that the multipliers depend on derivatives of ψ and $\overline{S}$. These multipliers cannot be calculated readily by hand and a software package such as GeM in Maple is required [35]. Here we use a direct method to calculate the conservation laws for the momentumless thermal wake. Conserved vectors for a non-thermal momentumless wake have been calculated in [12] using a direct method. We extend the results to the momentumless thermal wake. The conservation laws are integrated across the wake, with boundary conditions, to obtain conserved quantities.

Application of Lie point symmetry analysis to a turbulent momentumless wakes with eddy viscosity modelled by the Prandtl mixing length model is new. The mean velocity deficit gradient, $\frac{\partial \overline{w}}{\partial y}$, changes sign between the centreline and the boundary of the wake. Hence we cannot remove the absolute value sign in the governing equations for the momentumless thermal wake by considering only the upper half of the wake. Therefore we have to consider two regions, the region where $\frac{\partial \overline{w}}{\partial y} > 0$ and the region where $\frac{\partial \overline{w}}{\partial y} < 0$. We handle the two cases simultaneously by introducing the plus or minus sign to replace the absolute value in the governing partial differential equations for the turbulent momentumless thermal wake.

We use the conserved vectors for the momentumless thermal wake to derive the associated Lie point symmetry. We then use the associated Lie point symmetry to reduce the governing partial differential equations to ordinary differential equations. The ordinary differential equations cannot be solved analytically for all values of the kinematic viscosity and thermal conductivity. We find a numerical solution to the ordinary differential equations by applying a shooting method that uses the

conserved quantities as targets to convert the boundary value problem to an initial value problem.

We compare the results obtained when the kinematic viscosity and thermal conductivity are nonzero with the results when the kinematic viscosity and thermal conductivity vanish. We also examine the role played by the kinematic viscosity and thermal conductivity in the mathematical analysis of the momentumless thermal wake.

This Chapter is organized in the following manner. In Section 6.2, we derive the conservation laws for the mean flow of the turbulent momentumless thermal wake and in Section 6.3 we integrate the conservation laws to obtain conserved quantities. In Section 6.4, we derive the associated Lie point symmetry and in Section 6.5 we use the Lie symmetry to reduce the governing partial differential equations to ordinary differential equations. In Section 6.6, we summarize the results of the Lie symmetry analysis. In Section 6.7, we present the formulation for the numerical solution when the kinematic viscosity and thermal conductivity are nonzero. In Section 6.8, we present the formulation for the numerical solution when the kinematic viscosity and thermal conductivity tend to zero. In Section 6.9, we present the formulation for the numerical solution when the kinematic viscosity and thermal conductivity vanish. In Section 6.10, we present the results for the numerical solutions derived in Sections 6.7 to 6.9 for the momentumless thermal wake and in Section 6.11 we provide conclusions.

6.2 Conservation laws

The governing partial differential equations for a turbulent momentumless thermal wake are, (3.12.3) and (3.12.4),

$$U\frac{\partial^2\psi}{\partial x\partial y} = \frac{\partial}{\partial y}\left[\left(\nu + l^2\left|\frac{\partial^2\psi}{\partial y^2}\right|\right)\frac{\partial^2\psi}{\partial y^2}\right], \tag{6.2.1}$$

$$U\frac{\partial\overline{S}}{\partial x} = \frac{\partial}{\partial y}\left[\left(\frac{\kappa}{\rho c_v} + l_\theta l\left|\frac{\partial^2\psi}{\partial y^2}\right|\right)\frac{\partial\overline{S}}{\partial y}\right]. \tag{6.2.2}$$

In the upper half of the wake, the mean velocity gradient $\frac{\partial^2\psi}{\partial y^2}$ is positive near the centre line and negative close to the boundary. Hence we put the plus or minus sign when we remove the absolute value sign in equations (6.2.1) and (6.2.2). We carry through the plus or minus sign in the calculations for the conserved vectors and the associated Lie point symmetry. We assume the mixing lengths are functions of x only.

Sign convention for upper half plane:

- The plus or minus sign, $\pm$, means that when $\frac{\partial^2\psi}{\partial y^2} \geq 0$ we take the plus sign, +, and when $\frac{\partial^2\psi}{\partial y^2} < 0$ we take the minus sign, -.
- The minus or plus sign, $\mp$, means that when $\frac{\partial^2\psi}{\partial y^2} \geq 0$ we take the minus sign, -, and when $\frac{\partial^2\psi}{\partial y^2} < 0$ we take the plus sign, +.

Rewriting (6.2.1) and (6.2.2) we have

$$U\frac{\partial^2\psi}{\partial x\partial y} = \frac{\partial}{\partial y}\left[\left(\nu \pm l^2(x)\frac{\partial^2\psi}{\partial y^2}\right)\frac{\partial^2\psi}{\partial y^2}\right], \tag{6.2.3}$$

$$U\frac{\partial\overline{S}}{\partial x} = \frac{\partial}{\partial y}\left[\left(\frac{\kappa}{\rho c_v} \pm l_\theta(x)l(x)\frac{\partial^2\psi}{\partial y^2}\right)\frac{\partial\overline{S}}{\partial y}\right]. \tag{6.2.4}$$

Writing the governing partial differential equations (6.2.3) and (6.2.4) in terms of the subscript notation we have

$$U\psi_{xy} = \nu\psi_{yyy} \pm 2l^2(x)\psi_{yy}\psi_{yyy} \tag{6.2.5}$$

$$U\overline{S}_x = \frac{\kappa}{\rho c_v}\overline{S}_{yy} \pm l(x)l_\theta(x)\psi_{yyy}\overline{S}_y \pm l(x)l_\theta(x)\psi_{yy}\overline{S}_{yy} \tag{6.2.6}$$

In this Section we use a direct method to obtain the conserved vectors for the governing partial differential equations (6.2.5) and (6.2.6). Assume that the conserved vector components are of the form

$$T^1 = ay^2\psi_y + A(\overline{S}), \qquad T^2 = T^2(y,\psi,\psi_y,\psi_{yy},\overline{S}_y), \tag{6.2.7}$$

where a is an arbitrary constant and A is an arbitrary function of $\overline{S}$. The forms for the conserved vector components T^1 and T^2 are motivated by the calculation of the conserved vectors for the momentumless wake in [12] and the experience with the classical thermal wake in Chapter 4. The conserved vector components T^1 and T^2 satisfy the conservation law

$$D_xT^1 + D_yT^2\Big|_{(6.2.5),(6.2.6)} = 0, \tag{6.2.8}$$

where the total derivatives are defined by

$$D_x = \frac{\partial}{\partial x} + \psi_x\frac{\partial}{\partial\psi} + \overline{S}_x\frac{\partial}{\partial\overline{S}} + \psi_{xx}\frac{\partial}{\partial\psi_x} + \psi_{yx}\frac{\partial}{\partial\psi_y} + \overline{S}_{xx}\frac{\partial}{\partial\overline{S}_x} + \overline{S}_{xy}\frac{\partial}{\partial\overline{S}_y} + \cdots, \tag{6.2.9}$$

$$D_y = \frac{\partial}{\partial y} + \psi_y\frac{\partial}{\partial\psi} + \overline{S}_y\frac{\partial}{\partial\overline{S}} + \psi_{xy}\frac{\partial}{\partial\psi_x} + \psi_{yy}\frac{\partial}{\partial\psi_y} + \overline{S}_{xy}\frac{\partial}{\partial\overline{S}_x} + \overline{S}_{yy}\frac{\partial}{\partial\overline{S}_y} + \cdots. \tag{6.2.10}$$

Expanding (6.2.8) using (6.2.9) and (6.2.10) for total derivatives we obtain

$$ay^2\psi_{xy} + \overline{S}_x\frac{dA}{d\overline{S}} + \frac{\partial T^2}{\partial y} + \psi_y\frac{\partial T^2}{\partial\psi} + \psi_{yy}\frac{\partial T^2}{\partial\psi_y} + \psi_{yyy}\frac{\partial T^2}{\partial\psi_{yy}} + \overline{S}_{yy}\frac{\partial T^2}{\partial\overline{S}_y}\Big|_{(6.2.5),(6.2.6)} = 0. \tag{6.2.11}$$

Substituting ψ_{xy} and $\overline{S}_x$ from (6.2.5) and (6.2.6) into (6.2.11) we obtain

$$\begin{aligned} &ay^2\left[\frac{\nu}{U}\psi_{yyy} \pm 2\frac{l^2(x)}{U}\psi_{yy}\psi_{yyy}\right] \\ &+\frac{dA}{d\overline{S}}\left[\frac{\kappa}{\rho c_v U}\overline{S}_{yy} \pm \frac{l(x)l_\theta(x)}{U}\psi_{yyy}\overline{S}_y \pm \frac{l(x)l_\theta(x)}{U}\psi_{yy}\overline{S}_{yy}\right] \\ &+\frac{\partial T^2}{\partial y} + \psi_y\frac{\partial T^2}{\partial\psi} + \psi_{yy}\frac{\partial T^2}{\partial\psi_y} + \psi_{yyy}\frac{\partial T^2}{\partial\psi_{yy}} + \overline{S}_{yy}\frac{\partial T^2}{\partial\overline{S}_y} = 0. \end{aligned} \tag{6.2.12}$$

But T^2 is independent of ψ_{yyy}. The coefficient of ψ_{yyy} in (6.2.12) must therefore be equal to zero. Collecting terms with ψ_{yyy} in (6.2.12) we obtain

$$\frac{\partial T^2}{\partial \psi_{yy}} = -a\frac{\nu}{U}y^2 \mp 2a\frac{l^2(x)}{U}y^2\psi_{yy} \mp \frac{dA}{d\overline{S}}\frac{l(x)l_\theta(x)}{U}\overline{S}_y. \tag{6.2.13}$$

Integrating (6.2.13) with respect to ψ_{yy} we obtain

$$T^2 = -a\frac{\nu}{U}y^2\psi_{yy} \mp a\frac{l^2(x)}{U}y^2\psi_{yy}^2 \mp \frac{dA}{d\overline{S}}\frac{l(x)l_\theta(x)}{U}\psi_{yy}\overline{S}_y + B(y,\psi,\psi_y,\overline{S}_y), \tag{6.2.14}$$

where B is an arbitrary function. Substituting (6.2.14) into the remainder of (6.2.12) we obtain

$$\begin{aligned}\frac{\kappa}{\rho c_v U}\frac{dA}{d\overline{S}}\overline{S}_{yy} - 2a\frac{\nu}{U}y\psi_{yy} \mp 2a\frac{l^2(x)}{U}y\psi_{yy}^2 + \frac{\partial B}{\partial y} + \psi_y\frac{\partial B}{\partial \psi} + \\ \psi_{yy}\frac{\partial B}{\partial \psi_y} + \overline{S}_{yy}\frac{\partial B}{\partial \overline{S}_y} = 0.\end{aligned} \tag{6.2.15}$$

In order for equation (6.2.15) to be satisfied exactly the coefficient of ψ_{yy}^2 must vanish. This implies that the momentum mixing length $l(x)$ must vanish. However the momentum mixing length cannot vanish since we are considering turbulent flow. In the far-wake approximation $l^2(x)\psi_{yy}^2$ is second-order in the velocity deficit $\overline{w}$ and can be neglected. We separate (6.2.15) into coefficients of independent powers of ψ_{yy} and $\overline{S}_{yy}$, except ψ_{yy}^2, so that (6.2.15) is satisfied to first-order in $\overline{w}$. Hence

$$\psi_{yy} : \frac{\partial B}{\partial \psi_y} - 2a\frac{\nu}{U}y = 0, \tag{6.2.16}$$

$$\overline{S}_{yy} : \frac{\kappa}{\rho c_v U}\frac{dA}{d\overline{S}} + \frac{\partial B}{\partial \overline{S}_y} = 0, \tag{6.2.17}$$

$$\text{Remainder: } \frac{\partial B}{\partial y} + \psi_y\frac{\partial B}{\partial \psi} = 0. \tag{6.2.18}$$

Integrating (6.2.16) with respect to ψ_y we obtain

$$B(y,\psi,\psi_y,\overline{S}_y) = 2a\frac{\nu}{U}y\psi_y + C(y,\psi,\overline{S}_y), \tag{6.2.19}$$

where C is an arbitrary function. Substituting (6.2.19) into (6.2.18) we obtain

$$\psi_y\left(2a\frac{\nu}{U} + \frac{\partial C}{\partial \psi}\right) + \frac{\partial C}{\partial y} = 0. \tag{6.2.20}$$

Separating (6.2.20) by the coefficient of ψ_y and the remainder we obtain

$$\psi_y : \frac{\partial C}{\partial \psi} + 2a\frac{\nu}{U} = 0, \tag{6.2.21}$$

$$\text{Remainder: } \frac{\partial C}{\partial y} = 0. \tag{6.2.22}$$

Integrating (6.2.21) with respect to ψ, and using (6.2.22), we obtain

$$C(\psi, \overline{S}_y) = -2a\frac{\nu}{U}\psi + H(\overline{S}_y), \tag{6.2.23}$$

where $H(\overline{S}_y)$ is an arbitrary function of $\overline{S}_y$. Therefore

$$B(y, \psi, \psi_y, \overline{S}_y) = 2a\frac{\nu}{U}y\psi_y - 2a\frac{\nu}{U}\psi + H(\overline{S}_y). \tag{6.2.24}$$

Substituting (6.2.24) into (6.2.17) we obtain

$$\frac{dH}{d\overline{S}_y} = -\frac{\kappa}{\rho c_v U}\frac{dA}{d\overline{S}}. \tag{6.2.25}$$

In summary,

$$T^1 = ay^2\psi_y + A(\overline{S}), \tag{6.2.26}$$

$$T^2 = -ay^2\left[\frac{\nu}{U}\psi_{yy} \pm \frac{l^2(x)}{U}\psi_{yy}^2\right] + 2a\frac{\nu}{U}\left[y\psi_y - \psi\right] \mp \frac{dA}{d\overline{S}}\frac{l(x)l_\theta(x)}{U}\psi_{yy}\overline{S}_y + H(\overline{S}_y), \tag{6.2.27}$$

satisfy (6.2.8) up to first-order in $\overline{w}$ provided

$$\frac{dH}{d\overline{S}_y} = -\frac{\kappa}{\rho c_v U}\frac{dA}{d\overline{S}}. \tag{6.2.28}$$

The variables $\overline{S}$ and $\overline{S}_y$ are treated as independent variables when deriving conservation laws. Hence, by the technique of separation of variables, each side of (6.2.28) must be a constant. Thus

$$\frac{dH}{d\overline{S}_y} = \alpha, \qquad -\frac{\kappa}{\rho c_v}\frac{dA}{d\overline{S}} = \alpha \tag{6.2.29}$$

and therefore

$$H(\overline{S}_y) = \alpha\overline{S}_y + \beta, \tag{6.2.30}$$

$$A(\overline{S}) = -\frac{\rho c_v}{\kappa}\alpha\overline{S} + \gamma, \tag{6.2.31}$$

where α, β and γ are constants. We write

$$b = -\frac{\rho c_v}{\kappa}\alpha. \tag{6.2.32}$$

Thus

$$A(\overline{S}) = bU\overline{S} + \gamma, \tag{6.2.33}$$

$$H(\overline{S}_y) = -b\frac{\kappa}{\rho c_v}\overline{S}_y + \beta \tag{6.2.34}$$

and (6.2.26) and (6.2.27) become

$$T^1 = ay^2\psi_y + bU\overline{S} + \gamma, \tag{6.2.35}$$

$$T^2 = -ay^2\left[\frac{\nu}{U}\psi_{yy} \pm \frac{l^2(x)}{U}\psi_{yy}^2\right] + 2a\frac{\nu}{U}\left[y\psi_y - \psi\right] \mp bl(x)l_\theta(x)\psi_{yy}\overline{S}_y - \frac{\kappa}{\rho c_v}b\overline{S}_y + \beta. \tag{6.2.36}$$

Since $T^1 = \gamma$, $T^2 = \beta$ is a trivial conserved vector we set $\gamma = \beta = 0$. When $a = 1$ and $b = 0$, the conserved vector components (6.2.35), (6.2.36) become

$$\begin{aligned} T^1 &= y^2\psi_y, \\ T^2 &= -y^2\left[\frac{\nu}{U}\psi_{yy} \pm \frac{l^2(x)}{U}\psi_{yy}^2\right] + 2\frac{\nu}{U}\left[y\psi_y - \psi\right]. \end{aligned} \tag{6.2.37}$$

These are the components of the momentum flux conserved vector $T_M = (T^1, T^2)$. When $a = 0$ and $b = 1$, the conserved vector components (6.2.35), (6.2.36) become

$$\begin{aligned} T^1 &= U\overline{S}, \\ T^2 &= \mp l(x)l_\theta(x)\psi_{yy}\overline{S}_y - \frac{\kappa}{\rho c_v}\overline{S}_y. \end{aligned} \tag{6.2.38}$$

These are the components of the thermal flux conserved vector $T_\theta = (T^1, T^2)$. The momentum flux conserved vector and the thermal flux conserved vector satisfy

$$D_x T^1 + D_y T^2\Big|_{(6.2.5),(6.2.6)} = 0 \tag{6.2.39}$$

to first-order in $\overline{w}$.

We can compare the results for the momentumless wake with the classical wake. Equations (6.2.35) and (6.2.36) compare with (4.3.4) and (4.3.5) for the classical wake while (6.2.37) compares with (4.3.6) and (6.2.38) with (4.3.7) for the classical wake. The thermal flux conserved vector (6.2.38) is the same as (4.3.7) for the classical wake, but there are significant differences in the momentum flux conserved vector (6.2.38) and (4.3.6).

6.3 Conserved quantities

We now treat $\psi(x, y)$ and $\overline{S}(x, y)$ as dependent variables of the independent variables x and y that satisfy the system of governing partial differential equations (6.2.3), (6.2.4) and boundary conditions (3.12.7) and (3.12.8) in order to obtain conserved quantities. The total derivatives in (6.2.39) become partial derivatives and

$$\frac{\partial T^1}{\partial x} + \frac{\partial T^2}{\partial y} = 0. \tag{6.3.1}$$

6.3.1 Conserved quantity for the momentum flux

The momentum flux conserved vector T_M has components

$$T^1 = y^2\frac{\partial \psi}{\partial y}, \tag{6.3.2}$$

$$T^2 = -y^2\left[\frac{\nu}{U}\frac{\partial^2\psi}{\partial y^2} \pm \frac{l^2(x)}{U}\left(\frac{\partial^2\psi}{\partial y^2}\right)^2\right] + 2\frac{\nu}{U}\left[y\frac{\partial\psi}{\partial y} - \psi\right]. \tag{6.3.3}$$

Substituting (6.3.2) and (6.3.3) into (6.3.1) we obtain

$$\frac{\partial}{\partial x}\left[y^2\frac{\partial\psi}{\partial y}\right]+\frac{\partial}{\partial y}\left[-y^2\left(\frac{\nu}{U}\frac{\partial^2\psi}{\partial y^2}\pm\frac{l^2(x)}{U}\left(\frac{\partial^2\psi}{\partial y^2}\right)^2\right)\right]+ 2\frac{\nu}{U}\frac{\partial}{\partial y}\left[y\frac{\partial\psi}{\partial y}-\psi\right]=0. \tag{6.3.4}$$

Integrating (6.3.4) with respect to y from the centreline of the wake, $y=0$, to the momentum boundary, $y=y_B(x)$, we have

$$\int_0^{y_B(x)}\frac{\partial}{\partial x}\left[y^2\frac{\partial\psi}{\partial y}\right]dy+\left[-y^2\left(\frac{\nu}{U}\frac{\partial^2\psi}{\partial y^2}\pm\frac{l^2(x)}{U}\left(\frac{\partial^2\psi}{\partial y^2}\right)^2\right)\right]\Bigg|_{y=0}^{y=y_B(x)} +2\frac{\nu}{U}\left[y\frac{\partial\psi}{\partial y}-\psi\right]\Bigg|_{y=0}^{y=y_B(x)}=0. \tag{6.3.5}$$

Applying the boundary conditions (3.12.7) equation (6.3.5) reduces to

$$\int_0^{y_B(x)}\frac{\partial}{\partial x}\left[y^2\frac{\partial\psi}{\partial y}\right]dy-2\frac{\nu}{U}\psi(x,y)\Big|_{y=0}^{y=y_B(x)}=0. \tag{6.3.6}$$

But for a momentumless wake the drag vanishes. That is, from (4.4.10),

$$D=2\rho U\int_0^{y_B(x)}\frac{\partial\psi}{\partial y}dy=2\rho U\psi(x,y)\Big|_{y=0}^{y=y_B(x)}=0 \tag{6.3.7}$$

for both $\nu=0$ and $\nu\neq0$. Hence equation (6.3.6) becomes

$$\int_0^{y_B(x)}\frac{\partial}{\partial x}\left[y^2\frac{\partial\psi}{\partial y}\right]dy=0. \tag{6.3.8}$$

By the theorem for differentiation under the integral sign [34] equation (6.3.8) becomes

$$\frac{d}{dx}\int_0^{y_B(x)}y^2\frac{\partial\psi}{\partial y}dy-\frac{dy_B}{dx}y_B^2(x)\frac{\partial\psi}{\partial y}(x,y_B(x))=0. \tag{6.3.9}$$

Applying boundary conditions (3.12.7) the first derivative of ψ with respect to y at $y=y_B(x)$ vanishes. Hence

$$\frac{d}{dx}\int_0^{y_B(x)}y^2\frac{\partial\psi}{\partial y}(x,y)dy=0. \tag{6.3.10}$$

Therefore we obtain the conserved quantity

$$J=2\rho U\int_0^{y_B(x)}y^2\frac{\partial\psi}{\partial y}(x,y)dy, \tag{6.3.11}$$

that is, J is a constant independent of x.

6.3.2 Conserved quantity of the thermal flux

In this chapter the eddy viscosity is modelled by Prandtl's mixing length model (3.7.10),

$$\nu_T = l^2 \left| \frac{\partial \overline{w}}{\partial y} \right| = l^2 \left| \frac{\partial^2 \psi}{\partial y^2} \right| \tag{6.3.12}$$

and the eddy thermal conductivity is given by (3.8.18),

$$\kappa_T = \rho c_v l_\theta l \left| \frac{\partial \overline{w}}{\partial y} \right| = \rho c_v l_\theta l \left| \frac{\partial^2 \psi}{\partial y^2} \right|. \tag{6.3.13}$$

The eddy kinematic viscosity ν_T and the eddy thermal conductivity κ_T both vanish at $y = y_B(x)$ where

$$\frac{\partial^2 \psi}{\partial y^2}(x, y_B(x)) = 0. \tag{6.3.14}$$

For $y > y_B(x)$ the flow is laminar. The boundary of the mean thermal momentumless wake is therefore the same as the boundary of the mean momentum momentumless wake and is $y = y_B(x)$. For the momentumless wake described by Prandtl's mixing length model the boundary of the momentum and thermal wakes are the same.

The thermal flux conserved vector T_θ has components

$$T^1 = U\overline{S}, \tag{6.3.15}$$

$$T^2 = \mp l(x) l_\theta(x) \frac{\partial^2 \psi}{\partial y^2} \frac{\partial \overline{S}}{\partial y} - \frac{\kappa}{\rho c_v} \frac{\partial \overline{S}}{\partial y}. \tag{6.3.16}$$

Substituting (6.3.15) and (6.3.16) into (6.3.1) we obtain

$$U \frac{\partial \overline{S}}{\partial x} + \frac{\partial}{\partial y} \left[\mp l(x) l_\theta(x) \frac{\partial^2 \psi}{\partial y^2} \frac{\partial \overline{S}}{\partial y} - \frac{\kappa}{\rho c_v} \frac{\partial \overline{S}}{\partial y} \right] = 0. \tag{6.3.17}$$

Integrating (6.3.17) with respect to y from the centreline, $y = 0$, to the thermal boundary, $y = y_B(x)$, we obtain

$$U \int_0^{y_B(x)} \frac{\partial \overline{S}}{\partial x} dy + \left[\mp l(x) l_\theta(x) \frac{\partial^2 \psi}{\partial y^2} \frac{\partial \overline{S}}{\partial y} - \frac{\kappa}{\rho c_v} \frac{\partial \overline{S}}{\partial y} \right] \Bigg|_{y=0}^{y=y_B(x)} = 0. \tag{6.3.18}$$

Applying the boundary conditions (3.12.7) and (3.12.8) the second term in (6.3.18) vanishes for both $\kappa = 0$ and $\kappa \neq 0$. Equation (6.3.18) reduces to

$$\int_0^{y_B(x)} \frac{\partial \overline{S}}{\partial x}(x, y) dy = 0. \tag{6.3.19}$$

By the theorem for differentiation under the integral sign [34] equation (6.3.19) becomes

$$\frac{d}{dx} \int_0^{y_B(x)} \overline{S}(x, y) dy - \frac{dy_B}{dx} \overline{S}(x, y_B(x)) = 0. \tag{6.3.20}$$

Applying the second boundary condition in (3.12.8) the second term of (6.3.20) vanishes. Therefore (6.3.20) becomes

$$\frac{d}{dx}\int_0^{y_B(x)} \overline{S}(x,y)dy = 0 \tag{6.3.21}$$

and

$$K = 2\rho c_v U \int_0^{y_B(x)} \overline{S}(x,y)dy \tag{6.3.22}$$

is a constant independent of x.

6.4 Associated Lie point symmetry

In this Section, we derive the Lie point symmetry associated with the conserved vectors (6.2.37) and (6.2.38).

A vector field operator

$$X = \xi^1(x,y,\psi,\overline{S})\frac{\partial}{\partial x} + \xi^2(x,y,\psi,\overline{S})\frac{\partial}{\partial y} + \eta^1(x,y,\psi,\overline{S})\frac{\partial}{\partial \psi} + \eta^2(x,y,\psi,\overline{S})\frac{\partial}{\partial \overline{S}} \tag{6.4.1}$$

is associated with the conserved vector (T^1, T^2) provided

$$X^{[\infty]}(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0, \qquad i = 1,2 \tag{6.4.2}$$

where $X^{[\infty]}$ is the prolongation of X up to all orders required to evaluate $X^{[\infty]}(T^1)$ and $X^{[\infty]}(T^2)$. When equation (6.4.2) is expanded for $i = 1,2$ it gives

$$X^{[\infty]}(T^1) + T^1 D_y(\xi^2) - T^2 D_y(\xi^1) = 0, \tag{6.4.3}$$

$$X^{[\infty]}(T^2) + T^2 D_x(\xi^1) - T^1 D_x(\xi^2) = 0. \tag{6.4.4}$$

The conserved vectors (6.2.37) and (6.2.38) contain derivatives up to second order in ψ and first order in $\overline{S}$. Hence we only need to prolongate the vector field operator X up to the second order in ψ, first order in $\overline{S}$ and

$$X^{[\infty]} = \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial y} + \eta^1\frac{\partial}{\partial \psi} + \eta^2\frac{\partial}{\partial \overline{S}} + \zeta^1_y\frac{\partial}{\partial \psi_y} + \zeta^1_{yy}\frac{\partial}{\partial \psi_{yy}} + \zeta^2_y\frac{\partial}{\partial \overline{S}_y}. \tag{6.4.5}$$

The first-order prolongations are

$$\zeta^\alpha_y = D_y(\eta^\alpha) - u^\alpha_k D_y(\xi^k), \tag{6.4.6}$$

the second order prolongation is

$$\zeta^1_{yy} = D_y(\zeta^1_y) - \psi_{yk} D_y(\xi^k) \tag{6.4.7}$$

and

$$x^1 = x, \qquad x^2 = y, \qquad u^1 = \psi, \qquad u^2 = \overline{S}. \tag{6.4.8}$$

6.4.1 Vector field associated with the momentum flux conserved vector

The momentum flux conserved vector is, (6.2.37),

$$\begin{aligned} T^1 &= y^2\psi_y, \\ T^2 &= -y^2\left[\frac{\nu}{U}\psi_{yy} \pm \frac{l^2(x)}{U}\psi_{yy}^2\right] + 2\frac{\nu}{U}\left[y\psi_y - \psi\right]. \end{aligned} \tag{6.4.9}$$

Expanding equation (6.4.3) with the conserved vector components T^1 and T^2 in (6.4.9) we obtain

$$\begin{aligned} 2y\psi_y\xi^2 + y^2\zeta_y^1 + y^2\psi_y D_y\xi^2 + y^2\left[\frac{\nu}{U}\psi_{yy} \pm \frac{l^2(x)}{U}\psi_{yy}^2\right] D_y\xi^1 \\ -2\frac{\nu}{U}\left[y\psi_y - \psi\right] D_y\xi^1 = 0. \end{aligned} \tag{6.4.10}$$

By (6.4.6)

$$\zeta_y^1 = D_y\eta^1 - \psi_x D_y\xi^1 - \psi_y D_y\xi^2. \tag{6.4.11}$$

Substituting (6.4.11) into (6.4.10) we obtain

$$\begin{aligned} 2y\psi_y\xi^2 + y^2\left[D_y\eta^1 - \psi_x D_y\xi^1 - \psi_y D_y\xi^2\right] + y^2\psi_y D_y\xi^2 \\ +y^2\left[\frac{\nu}{U}\psi_{yy} \pm \frac{l^2(x)}{U}\psi_{yy}^2\right] D_y\xi^1 - 2\frac{\nu}{U}\left[y\psi_y - \psi\right] D_y\xi^1 = 0 \end{aligned} \tag{6.4.12}$$

Splitting (6.4.12) by powers of ψ_{yy} we obtain

$$\psi_{yy}^2 : D_y\xi^1 = \frac{\partial\xi^1}{\partial y} + \psi_y\frac{\partial\xi^1}{\partial\psi} + \overline{S}_y\frac{\partial\xi^1}{\partial\overline{S}} = 0, \tag{6.4.13}$$

$$\psi_{yy} : \nu D_y\xi^1 = 0, \tag{6.4.14}$$

$$\text{Remainder: } 2y\psi_y\xi^2 + y^2\left[D_y\eta^1 - \psi_x D_y\xi^1\right] - 2\frac{\nu}{U}\left[y\psi_y - \psi\right] D_y\xi^1 = 0. \tag{6.4.15}$$

Splitting (6.4.13) by ψ_y and $\overline{S}_y$ we obtain

$$\psi_y : \frac{\partial\xi^1}{\partial\psi} = 0, \tag{6.4.16}$$

$$\overline{S}_y : \frac{\partial\xi^1}{\partial\overline{S}} = 0, \tag{6.4.17}$$

$$\text{Remainder: } \frac{\partial\xi^1}{\partial y} = 0. \tag{6.4.18}$$

Hence

$$\xi^1 = \xi^1(x). \tag{6.4.19}$$

Equation (6.4.15) reduces to

$$2y\psi_y\xi^2 + y^2 D_y\eta^1 = 0. \tag{6.4.20}$$

Expanding the total derivatives in (6.4.20) we obtain

$$2y\psi_y\xi^2 + y^2\frac{\partial\eta^1}{\partial y} + y^2\psi_y\frac{\partial\eta^1}{\partial\psi} + y^2\overline{S}_y\frac{\partial\eta^1}{\partial\overline{S}} = 0 \tag{6.4.21}$$

Splitting (6.4.21) by ψ_y and $\overline{S}_y$ we obtain

$$\psi_y : y\frac{\partial\eta^1}{\partial\psi} + 2\xi^2 = 0, \tag{6.4.22}$$

$$\overline{S}_y : \frac{\partial\eta^1}{\partial\overline{S}} = 0, \tag{6.4.23}$$

$$\text{Remainder: } \frac{\partial\eta^1}{\partial y} = 0. \tag{6.4.24}$$

Equations (6.4.23) and (6.4.24) give

$$\eta^1 = \eta^1(x,\psi). \tag{6.4.25}$$

Now expanding (6.4.4) with the conserved vector components T^1 and T^2 in (6.4.9) we obtain

$$\begin{gathered} 2y\left[\frac{\nu}{U}\psi_{yy} \pm \frac{l^2(x)}{U}\psi_{yy}^2\right]\xi^2 \pm \frac{y^2}{U}\frac{d}{dx}\left(l^2(x)\right)\psi_{yy}^2\xi^1 \\ +y^2\left[\frac{\nu}{U} \pm 2\frac{l^2(x)}{U}\psi_{yy}\right]\zeta^1_{yy} + y^2\left[\frac{\nu}{U}\psi_{yy} \pm \frac{l^2(x)}{U}\psi_{yy}^2\right]\frac{d\xi^1}{dx} + \\ -2\frac{\nu}{U}\xi^2\psi_y - 2\frac{\nu}{U}y\zeta^1_y + 2\frac{\nu}{U}\eta^1 - 2\frac{\nu}{U}\left[y\psi_y - \psi\right]\frac{d\xi^1}{dx} + y^2\psi_y D_x\xi^2 = 0. \end{gathered} \tag{6.4.26}$$

By (6.4.7) and (6.4.11)

$$\zeta^1_y = D_y\eta^1 - \psi_y D_y\xi^2, \tag{6.4.27}$$

$$\zeta^1_{yy} = D_y^2\eta^1 - \psi_y D_y^2\xi^2 - 2\psi_{yy}D_y\xi^2. \tag{6.4.28}$$

Substituting (6.4.28) and (6.4.27) into (6.4.26) we obtain

$$\begin{gathered} 2y\left[\frac{\nu}{U}\psi_{yy} \pm \frac{l^2(x)}{U}\psi_{yy}^2\right]\xi^2 \pm \frac{y^2}{U}\frac{d}{dx}\left(l^2(x)\right)\psi_{yy}^2\xi^1 \\ +y^2\left[\frac{\nu}{U} \pm 2\frac{l^2(x)}{U}\psi_{yy}\right]\left(D_y^2\eta^1 - \psi_y D_y^2\xi^2 - 2\psi_{yy}D_y\xi^2\right) \\ +y^2\left[\frac{\nu}{U}\psi_{yy} \pm \frac{l^2(x)}{U}\psi_{yy}^2\right]\frac{d\xi^1}{dx} - 2\frac{\nu}{U}y\left[D_y\eta^1 - \psi_y D_y\xi^2\right] - 2\frac{\nu}{U}\xi^2\psi_y + 2\frac{\nu}{U}\eta^1 \\ -2\frac{\nu}{U}\left[y\psi_y - \psi\right]\frac{d\xi^1}{dx} + y^2\psi_y D_x\xi^2 = 0. \end{gathered} \tag{6.4.29}$$

Upon inspection, we see that the last term of (6.4.29) is the only term that depends on $\psi_x\psi_y$ and $\overline{S}_x\psi_y$. When expanding last term of (6.4.29) the coefficients of $\psi_x\psi_y$ and $\overline{S}_x\psi_y$ must be equal to zero. Hence

$$\psi_x\psi_y : \frac{\partial\xi^2}{\partial\psi} = 0, \tag{6.4.30}$$

$$\overline{S}_x\psi_y : \frac{\partial \xi^2}{\partial \overline{S}} = 0 \tag{6.4.31}$$

and therefore

$$\xi^2 = \xi^2(x, y). \tag{6.4.32}$$

Differentiating (6.4.22) with respect to ψ we obtain

$$\frac{\partial^2 \eta^1}{\partial \psi^2} = 0. \tag{6.4.33}$$

Hence

$$\eta^1(x, \psi) = a_1(x)\psi + a_2(x), \tag{6.4.34}$$

where a_1 and a_2 are arbitrary functions of x. Substituting (6.4.34) into (6.4.22) we obtain

$$\xi^2(x, y) = -\frac{1}{2}a_1(x)y. \tag{6.4.35}$$

Equation (6.4.29) reduces to

$$\begin{aligned}&\pm\psi_{yy}^2\frac{y^2}{U}\left[\frac{d}{dx}\left(l^2(x)\right)\xi^1 + 3l^2(x)a_1(x) + \frac{d\xi^1}{dx}l^2(x)\right] + \frac{\nu}{U}y^2\psi_{yy}\left[\frac{d\xi^1}{dx} + a_1(x)\right]\\ &+2\psi\frac{\nu}{U}\left[\frac{d\xi^1}{dx} + a_1(x)\right] + 2\frac{\nu}{U}a_2(x) - 2\frac{\nu}{U}y\psi_y\left[\frac{d\xi^1}{dx} + a_1(x)\right] + \frac{1}{2}y^3\psi_y\frac{da_1}{dx} = 0.\end{aligned} \tag{6.4.36}$$

Splitting (6.4.36) by independent powers of partial derivatives of ψ we obtain

$$\psi_y : -2\frac{\nu}{U}y\left[\frac{d\xi^1}{dx} + a_1(x)\right] + \frac{1}{2}y^3\frac{da_1}{dx} = 0, \tag{6.4.37}$$

$$\psi_{yy}^2 : \frac{d}{dx}\left(l^2(x)\right)\xi^1 + 3l^2(x)a_1(x) + \frac{d\xi^1}{dx}l^2(x) = 0, \tag{6.4.38}$$

$$\psi_{yy} : \nu\left[\frac{d\xi^1}{dx} + a_1(x)\right] = 0, \tag{6.4.39}$$

$$\text{Remainder: } 2\psi\frac{\nu}{U}\left[\frac{d\xi^1}{dx} + a_1(x)\right] + 2\frac{\nu}{U}a_2(x) = 0. \tag{6.4.40}$$

Using (6.4.39) equations (6.4.37) and (6.4.40) give

$$a_1(x) = a_1 \tag{6.4.41}$$

and

$$\nu a_2(x) = 0, \tag{6.4.42}$$

where a_1 is a constant.

In summary, we have

$$\xi^1 = \xi^1(x), \quad \xi^2(y) = -\frac{1}{2}a_1 y, \quad \eta^1 = a_1\psi + a_2(x), \quad \eta^2 = \eta^2(x, y, \psi, \overline{S}), \tag{6.4.43}$$

$$\frac{d}{dx}\left(l^2(x)\right)\xi^1 + 3l^2(x)a_1 + \frac{d\xi^1}{dx}l^2(x) = 0, \tag{6.4.44}$$

$$\nu\left[\frac{d\xi^1}{dx}+a_1\right]=0, \qquad \nu a_2(x)=0. \tag{6.4.45}$$

The associated vector field operator is

$$X=\xi^1(x)\frac{\partial}{\partial x}-\frac{1}{2}a_1 y\frac{\partial}{\partial y}+\left(a_1\psi+a_2(x)\right)\frac{\partial}{\partial \psi}+\eta^2(x,y,\psi,\overline{S})\frac{\partial}{\partial \overline{S}}. \tag{6.4.46}$$

6.4.2 Lie point symmetry associated with the momentum flux and thermal flux conserved vector

The thermal flux conserved vector is

$$\begin{aligned} T^1&=U\overline{S},\\ T^2&=\mp l(x)l_\theta(x)\psi_{yy}\overline{S}_y-\frac{\kappa}{\rho c_v}\overline{S}_y. \end{aligned} \tag{6.4.47}$$

Expanding equation (6.4.3) with the conserved vector components T^1 and T^2 in (6.4.47) and the vector field operator (6.4.46) we obtain

$$\eta^2=\frac{1}{2}a_1\overline{S}. \tag{6.4.48}$$

Expanding equation (6.4.4) with the conserved vector components T^1 and T^2 in (6.4.47) and the vector field operator (6.4.46) we obtain

$$\begin{aligned} \pm\frac{d}{dx}\left(l(x)l_\theta(x)\right)\psi_{yy}\overline{S}_y\xi^1\pm l(x)l_\theta(x)\overline{S}_y\zeta^1_{yy}\pm l(x)l_\theta(x)\psi_{yy}\zeta^2_y\\ +\frac{\kappa}{\rho c_v}\zeta^2_y+\left[\frac{\kappa}{\rho c_v}\overline{S}_y\pm l(x)l_\theta(x)\psi_{yy}\overline{S}_y\right]\frac{d\xi^1}{dx}=0. \end{aligned} \tag{6.4.49}$$

By (6.4.6) and (6.4.28)

$$\zeta^2_y=a_1\overline{S}_y, \qquad \zeta^1_{yy}=2a_1\psi_{yy}. \tag{6.4.50}$$

Substituting (6.4.50) into (6.4.49) we obtain

$$\begin{aligned} &\frac{\kappa}{\rho c_v}\overline{S}_y\left(\frac{d\xi^1}{dx}+a_1\right)\\ &\pm\psi_{yy}\overline{S}_y\left[\frac{d}{dx}\left(l(x)l_\theta(x)\right)\xi^1+3a_1l(x)l_\theta(x)+l(x)l_\theta(x)\frac{d\xi^1}{dx}\right]=0. \end{aligned} \tag{6.4.51}$$

Splitting (6.4.51) into coefficients of $\overline{S}_y$ and $\psi_{yy}\overline{S}_y$ we obtain

$$\overline{S}_y:\frac{\kappa}{\rho c_v}\left(\frac{d\xi^1}{dx}+a_1\right)=0, \tag{6.4.52}$$

$$\psi_{yy}\overline{S}_y:\frac{d}{dx}\left(l(x)l_\theta(x)\right)\xi^1+3a_1l(x)l_\theta(x)+l(x)l_\theta(x)\frac{d\xi^1}{dx}=0. \tag{6.4.53}$$

We verify that the vector field operator (6.4.46) is a Lie point symmetry of the partial differential equations (6.2.1) and (6.2.2). Applying the prolongated vector

field operator (6.4.46) to (6.2.3), subject to (6.2.3) being satisfied, we have

$$X^{[\infty]}\left(\nu\psi_{yyy}\pm 2l^2(x)\psi_{yy}\psi_{yyy}-U\psi_{xy}\right)\Big|_{(6.2.3)}=\nu\zeta^1_{yyy}\pm 2\frac{d}{dx}\left(l^2(x)\right)\xi^1\psi_{yy}\psi_{yyy}$$
$$\pm 2l^2(x)\psi_{yyy}\zeta^1_{yy}\pm 2l^2(x)\psi_{yy}\zeta^1_{yyy}-U\zeta^1_{xy}\Big|_{(6.2.3)}. \tag{6.4.54}$$

Using the prolongation formulae (2.3.4) and (2.3.5), equation (6.4.54) becomes

$$X^{[\infty]}\left(\nu\psi_{yyy}\pm 2l^2(x)\psi_{yy}\psi_{yyy}-U\psi_{xy}\right)\Big|_{(6.2.3)}=\frac{5}{2}\nu a_1\psi_{yyy}\pm 2\frac{d}{dx}\left(l^2(x)\right)\xi^1\psi_{yy}\psi_{yyy}$$
$$\pm 9a_1l^2(x)\psi_{yyy}\psi_{yy}-U\left[\frac{3}{2}a_1-\frac{d\xi^1}{dx}\right]\psi_{xy}\Big|_{(6.2.3)}. \tag{6.4.55}$$

But from (6.2.3),

$$U\psi_{xy}=\nu\psi_{yyy}-2l^2(x)\psi_{yy}\psi_{yyy}. \tag{6.4.56}$$

Substituting (6.4.56) into (6.4.55), equation (6.4.55) simplifies to

$$X^{[\infty]}\left(\nu\psi_{yyy}-2l^2(x)\psi_{yy}\psi_{yyy}-U\psi_{xy}\right)\Big|_{(6.2.3)}=\nu\left[a_1+\frac{d\xi^1}{dx}\right]\psi_{yyy}$$
$$\pm 2\left[\frac{d}{dx}\left(l^2(x)\right)\xi^1+3a_2l^2(x)+l^2(x)\frac{d\xi^1}{dx}\right]\psi_{yy}\psi_{yyy}. \tag{6.4.57}$$

By (6.4.44) and (6.4.45), the right hand side of (6.4.57) vanishes and

$$X^{[\infty]}\left(\nu\psi_{yyy}-2l^2(x)\psi_{yy}\psi_{yyy}-U\psi_{xy}\right)\Big|_{(6.2.3)}=0. \tag{6.4.58}$$

Applying the prolongated vector field operator (6.4.46) to (6.2.4), subject to (6.2.4) being satisfied, we have

$$X^{[\infty]}\left(\frac{\kappa}{\rho c_v}\overline{S}_{yy}\pm l(x)l_\theta(x)\psi_{yyy}\overline{S}_y\pm l(x)l_\theta(x)\psi_{yy}\overline{S}_{yy}-U\overline{S}_x\right)\Big|_{(6.2.4)}=$$
$$\frac{\kappa}{\rho c_v}\zeta^2_{yy}\pm\frac{d}{dx}\left(l(x)l_\theta(x)\right)\xi^1\psi_{yyy}\overline{S}_y\pm l(x)l_\theta(x)\overline{S}_y\zeta^1_{yyy}\pm l(x)l_\theta(x)\psi_{yyy}\zeta^2_y \tag{6.4.59}$$
$$\pm\frac{d}{dx}\left(l(x)l_\theta(x)\right)\xi^1\psi_{yy}\overline{S}_{yy}\pm l(x)l_\theta(x)\overline{S}_{yy}\zeta^1_{yy}\pm l(x)l_\theta(x)\psi_{yy}\zeta^2_{yy}-U\zeta^2_x\Big|_{(6.2.4)}$$

Using the prolongation formulae (2.3.4) and (2.3.5), equation (6.4.59) becomes

$$X^{[\infty]}\left(\frac{\kappa}{\rho c_v}\overline{S}_{yy}\pm l(x)l_\theta(x)\psi_{yyy}\overline{S}_y\pm l(x)l_\theta(x)\psi_{yy}\overline{S}_{yy}-U\overline{S}_x\right)\Big|_{(6.2.4)}=$$
$$\frac{3}{2}a_1\frac{\kappa}{\rho c_v}\overline{S}_{yy}\pm\frac{d}{dx}\left(l(x)l_\theta(x)\right)\xi^1\psi_{yyy}\overline{S}_y\pm\frac{7}{2}a_1l(x)l_\theta(x)\psi_{yyy}\overline{S}_y$$
$$\pm\frac{d}{dx}\left(l(x)l_\theta(x)\right)\xi^1\psi_{yy}\overline{S}_{yy}\pm\frac{7}{2}a_1l(x)l_\theta(x)\psi_{yy}\overline{S}_{yy} \tag{6.4.60}$$
$$-U\left[\frac{1}{2}a_1-\frac{d\xi^1}{dx}\right]\overline{S}_x\Big|_{(6.2.4)}$$

Substituting (6.2.4) into the right hand side of (4.5.62), we obtain

$$\begin{aligned} X^{[\infty]}\left(\frac{\kappa}{\rho c_v}\overline{S}_{yy} \pm l(x)l_\theta(x)\psi_{yyy}\overline{S}_y \pm l(x)l_\theta(x)\psi_{yy}\overline{S}_{yy} - U\overline{S}_x\right)\Big|_{(6.2.4)} = \\ \frac{\kappa}{\rho c_v}\left[a_1 + \frac{d\xi^1}{dx}\right] \\ \pm\left[\frac{d}{dx}\left(l(x)l_\theta(x)\right)\xi^1 + 3a_1 l(x)l_\theta(x) + \frac{d\xi^1}{dx}l(x)l_\theta(x)\right]\psi_{yyy}\overline{S}_y \\ \pm\left[\frac{d}{dx}\left(l(x)l_\theta(x)\right)\xi^1 + 3a_2 l(x)l_\theta(x) + \frac{d\xi^1}{dx}l(x)l_\theta(x)\right]\psi_{yy}\overline{S}_{yy}. \end{aligned} \tag{6.4.61}$$

By (6.4.52) and (6.4.53), the right hand side of (6.4.61) vanishes and

$$X^{[\infty]}\left(\frac{\kappa}{\rho c_v}\overline{S}_{yy} \pm l(x)l_\theta(x)\psi_{yyy}\overline{S}_y \pm l(x)l_\theta(x)\psi_{yy}\overline{S}_{yy} - U\overline{S}_x\right)\Big|_{(6.2.4)} = 0. \tag{6.4.62}$$

Hence, the vector field (6.4.46) is a Lie point symmetry of the partial differential equations (6.2.1) and (6.2.2).

6.4.3 Mixing lengths

In this Subsection, we show that from equations (6.4.44) and (6.4.53) that the momentum and the thermal mixing lengths are proportional.

Making $\frac{d\xi^1}{dx}$ the subject of (6.4.44) we obtain

$$\frac{d\xi^1}{dx} = -\frac{1}{l^2(x)}\frac{d}{dx}\left(l^2(x)\right)\xi^1 - 3a_1. \tag{6.4.63}$$

Substituting (6.4.63) into (6.4.53) we obtain

$$\frac{1}{l(x)l_\theta(x)}\frac{d}{dx}\left(l(x)l_\theta(x)\right)\xi^1 = \frac{1}{l^2(x)}\frac{d}{dx}\left(l^2(x)\right)\xi^1. \tag{6.4.64}$$

We assume that $\xi^1(x) \neq 0$. Diving both sides of (6.4.64) by $\xi^1(x)$ and integrating the resulting equation we obtain

$$\ln(l(x)l_\theta(x)) = \ln(l^2(x)) + C, \tag{6.4.65}$$

where C is a constant. Therefore

$$l_\theta(x) = Bl(x), \tag{6.4.66}$$

where $B = e^C$ is a constant. That is, the thermal mixing length is proportional to the momentum mixing length. Equations (6.4.44) and (6.4.66) are equivalent to equations (6.4.44) and (6.4.53). Hence we only need to consider equations (6.4.44) and (6.4.66) for the mixing lengths.

In summary, we have

$$\xi^1 = \xi^1(x), \qquad \xi^2(y) = -\frac{1}{2}a_1 y, \qquad \eta^1 = a_1\psi + a_2(x), \qquad \eta^2 = \frac{1}{2}a_1\overline{S}, \tag{6.4.67}$$

$$\frac{d}{dx}\left(l^2(x)\right)\xi^1 + 3a_1 l^2(x) + \frac{d\xi^1}{dx}l^2(x) = 0, \tag{6.4.68}$$

$$l_\theta(x) = Bl(x), \tag{6.4.69}$$

$$\nu\left[\frac{d\xi^1}{dx} + a_1\right] = 0, \qquad \frac{\kappa}{\rho c_v}\left[\frac{d\xi^1}{dx} + a_1\right] = 0, \qquad \nu a_2(x) = 0. \tag{6.4.70}$$

6.4.4 Case $\nu \neq 0$ and $\kappa \neq 0$

When $\nu \neq 0$ and $\kappa \neq 0$ equations in (6.4.70) give

$$\frac{d\xi^1}{dx} = -a_1, \tag{6.4.71}$$

$$a_2(x) = 0. \tag{6.4.72}$$

Hence

$$\xi^1 = -a_1 x + a_3, \tag{6.4.73}$$

where a_3 is a constant. We consider the general case in which $a_1 \neq 0$. Substituting (6.4.73) into (6.4.68) we obtain the variables separable differential equation

$$\frac{1}{l^2(x)}\frac{d}{dx}\left(l^2(x)\right) = \frac{2}{(x - a_3/a_1)}. \tag{6.4.74}$$

Integrating (6.4.74) we obtain

$$\ln(l^2(x)) = \ln\left(x - \frac{a_3}{a_1}\right)^2 + C, \tag{6.4.75}$$

where C is a constant. Hence

$$l(x) = l_1\left(x - \frac{a_3}{a_1}\right), \tag{6.4.76}$$

where $l_1 = e^{C/2}$ is a positive constant and the positive square root is taken because $l(x) > 0$. The value of $\frac{a_3}{a_1}$ has to be nonpositive so that $l(x) > 0$ for all nonnegative values of x. Equation (6.4.69) gives

$$l_\theta(x) = l_2\left(x - \frac{a_3}{a_1}\right), \tag{6.4.77}$$

where $l_2 = Bl_1$ is a positive constant.

Without loss of generality let

$$a_3^* = \frac{a_3}{a_1} \tag{6.4.78}$$

and divide the associated Lie point symmetry by a_1. To simplify the notation we omit the asterisks from here on. Therefore when $\nu \neq 0$ and $\kappa \neq 0$ the Lie point

symmetry associated with the conserved vectors (6.4.9) and (6.4.47) is

$$X = (-x + a_3)\frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial y} + \psi\frac{\partial}{\partial \psi} + \frac{1}{2}\overline{S}\frac{\partial}{\partial \overline{S}}. \tag{6.4.79}$$

The mixing lengths are

$$l(x) = l_1(x - a_3), \qquad l_\theta(x) = l_2(x - a_3). \tag{6.4.80}$$

Since $l(x) \geq 0$ and $l_\theta(x) \geq 0$ for all $x \geq 0$ it is necessary that $a_3 \leq 0$.

6.4.5 Case $\nu = 0$ and $\kappa = 0$

When $\nu = 0$ and $\kappa = 0$ the Lie point symmetry associated with the conserved vectors (6.4.9) and (6.4.47) is

$$X = \xi^1(x)\frac{\partial}{\partial x} - \frac{1}{2}a_1 y\frac{\partial}{\partial y} + (a_1\psi + a_2(x))\frac{\partial}{\partial \psi} + \frac{1}{2}a_1\overline{S}\frac{\partial}{\partial \overline{S}}. \tag{6.4.81}$$

The mixing lengths satisfy

$$\frac{d}{dx}\left(l^2(x)\right)\xi^1 + 3l^2(x)a_1 + \frac{d\xi^1}{dx}l^2(x) = 0, \tag{6.4.82}$$

$$l_\theta(x) = Bl(x). \tag{6.4.83}$$

Again, we consider the general case $a_1 \neq 0$. Without loss of generality we let

$$\xi^{1*}(x) = \frac{\xi^1}{a_1}, \qquad a_2^*(x) = \frac{a_2(x)}{a_1}. \tag{6.4.84}$$

and divide the associated Lie point symmetry by a_1. Hence, when $\nu = 0$ and $\kappa = 0$, the Lie point symmetry associated with the conserved vectors (6.4.9) and (6.4.47) is

$$X = \xi^{1*}(x)\frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial y} + (\psi + a_2^*(x))\frac{\partial}{\partial \psi} + \frac{1}{2}\overline{S}\frac{\partial}{\partial \overline{S}}. \tag{6.4.85}$$

Equation (6.4.68) becomes

$$\frac{d}{dx}\left(l^2(x)\right)\xi^{1*}(x) + 3l^2(x) + \frac{d\xi^{1*}}{dx}l^2(x) = 0. \tag{6.4.86}$$

To simplify notation, we drop the asterisks from here on.

Equation (6.4.86) relates $\xi^1(x)$ and $l(x)$. However, an additional equation is required to determine $\xi^1(x)$ and $l(x)$. This equation will be derived later in the Chapter.

6.5 Invariant solution

In this Section, we reduce the governing partial differential equations to ordinary differential equations by deriving the invariant solutions for cases $\nu \neq 0$, $\kappa \neq 0$ and $\nu = 0$, $\kappa = 0$.

6.5.1 Case $\nu \neq 0$ and $\kappa \neq 0$

When $\nu \neq 0$ and $\kappa \neq 0$ the Lie point symmetry associated with the conserved vectors (6.4.9) and (6.4.47) is

$$X = (-x + a_3)\frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial y} + \psi\frac{\partial}{\partial \psi} + \frac{1}{2}\overline{S}\frac{\partial}{\partial \overline{S}}. \tag{6.5.1}$$

The solution $\psi = f(x,y)$ and $\overline{S} = g(x,y)$ is an invariant of the governing partial differential equations (6.2.3) and (6.2.4) provided

$$X(\psi - f)\Big|_{\psi=f(x,y)} = 0, \tag{6.5.2}$$

$$X(\overline{S} - g)\Big|_{\overline{S}=g(x,y)} = 0, \tag{6.5.3}$$

where X is given by (6.5.1). Expanding equation (6.5.2) using the Lie point symmetry generator (6.5.1) we obtain the first order partial differential equation

$$(-x + a_3)\frac{\partial f}{\partial x} - \frac{1}{2}y\frac{\partial f}{\partial y} = f. \tag{6.5.4}$$

The differential equations of the characteristic curves of the partial differential equation (6.5.4) are

$$\frac{dx}{(-x + a_3)} = -2\frac{dy}{y} = \frac{df}{f}. \tag{6.5.5}$$

Integrating the first and second terms of (6.5.5) we obtain

$$\ln y - \frac{1}{2}\ln(x - a_3) = C', \tag{6.5.6}$$

where C' is a constant. Exponentiating (6.5.6) we obtain

$$I_1 = \frac{y}{(x - a_3)^{\frac{1}{2}}}, \tag{6.5.7}$$

where $I_1 = e^{C'}$ is a constant. Now integrating the first and last terms of (6.5.5) we obtain

$$\ln f + \ln(x - a_3) = C, \tag{6.5.8}$$

where C is a constant. Therefore

$$I_2 = (x - a_3)f, \tag{6.5.9}$$

where $I_2 = e^{C}$ is a constant. The general solution of the first order partial differential equation (6.5.4) is

$$I_2 = F(I_1), \tag{6.5.10}$$

where F is an arbitrary function. Since $f = \psi(x,y)$,

$$\psi(x,y) = \frac{F(\xi)}{(x - a_3)}, \tag{6.5.11}$$

where

$$\xi = \frac{y}{(x-a_3)^{\frac{1}{2}}}. \tag{6.5.12}$$

Expanding equation (6.5.3) we obtain the partial differential equation

$$(-x+a_3)\frac{\partial g}{\partial x} - \frac{1}{2}y\frac{\partial g}{\partial y} = \frac{1}{2}g. \tag{6.5.13}$$

The differential equations of the characteristic curves of the partial differential equation (4.6.9) are

$$-\frac{dx}{2(x-a_3)} = -\frac{dy}{y} = \frac{dg}{g}. \tag{6.5.14}$$

We have already solved the differential equation consisting of the first and second terms of (6.5.14) and we obtained (6.5.7). Integrating the differential equation consisting of the first and last terms of (6.5.14) we obtain

$$\ln g + \frac{1}{2}\ln(x-a_3) = C', \tag{6.5.15}$$

where C' is a constant. Therefore

$$(x-a_3)^{1/2} g = I_3, \tag{6.5.16}$$

where $I_3 = e^{C'}$ is a constant. The general solution of the first order partial differential equation (6.5.13) is

$$I_3 = G(I_1), \tag{6.5.17}$$

where G is an arbitrary function. Since $g = \overline{S}(x,y)$,

$$\overline{S}(x,y) = (x-a_3)^{-\frac{1}{2}} G(\xi). \tag{6.5.18}$$

The velocity deficit

$$\overline{w}(x,y) = \frac{\partial \psi}{\partial y} = \frac{1}{(x-a_3)^{3/2}}\frac{dF}{d\xi} \tag{6.5.19}$$

should be finite for $x \geq 0$ except at $x = 0$. Hence $a_3 = 0$ and

$$\psi(x,y) = \frac{1}{x}F(\xi), \qquad \overline{S}(x,y) = \frac{1}{x^{1/2}}G(\xi), \tag{6.5.20}$$

where

$$\xi = \frac{y}{x^{1/2}}. \tag{6.5.21}$$

Writing the conserved quantity (6.3.11) in terms of the invariant solution (6.5.20) and (6.5.21) we obtain

$$J = 2\rho U \int_0^{\xi_B(x)} \xi^2 \frac{dF}{d\xi} d\xi, \tag{6.5.22}$$

where

$$\xi_B(x) = \frac{y_B(x)}{x^{1/2}}. \tag{6.5.23}$$

The mean velocity deficit

$$\overline{w}(x,y) = \frac{\partial \psi}{\partial y} = \frac{1}{x^{3/2}} \frac{dF}{d\xi} \tag{6.5.24}$$

and $\frac{dF}{d\xi}$ vanish at the boundary $\xi = \xi_B(x)$. The mean velocity gradient

$$\frac{\partial \overline{w}}{\partial y} = \frac{\partial^2 \psi}{\partial y^2} = \frac{1}{x^2} \frac{d^2F}{d\xi^2} \tag{6.5.25}$$

and $\frac{d^2F}{d\xi^2}$ also vanish at the boundary $\xi = \xi_B(x)$. Assuming that F is analytic at $\xi = \xi_B(x)$ and not a constant, the derivatives of F at $\xi_B(x)$ cannot be all zero. Therefore, there exists an integer $n \geq 3$ such that

$$\frac{d^{n-1}F}{d\xi^{n-1}}(\xi_B(x)) = 0 \tag{6.5.26}$$

and

$$\frac{d^nF}{d\xi^n}(\xi_B(x)) \neq 0. \tag{6.5.27}$$

Differentiating (6.5.26) with respect to x, we have

$$\frac{d^nF}{d\xi^n}(\xi_B(x)) \frac{d\xi_B}{dx} = 0. \tag{6.5.28}$$

Therefore, $\xi_B(x)$ is a constant and

$$y_B(x) = \xi_B x^{1/2}. \tag{6.5.29}$$

The conserved quantity (6.5.22) becomes

$$J = 2\rho U \int_0^{\xi_B} \xi^2 \frac{dF}{d\xi} d\xi. \tag{6.5.30}$$

Writing the conserved quantity (6.3.22) in terms of the invariant solution (6.5.20), (6.5.18) with $a_3 = 0$ and (6.5.21) we obtain

$$K = 2\rho c_v U \int_0^{\xi_B} G(\xi) d\xi. \tag{6.5.31}$$

Writing the boundary conditions (3.12.7) and (3.12.8) in terms of the invariant solution (6.5.20), (6.5.18) with $a_3 = 0$ and (6.5.21) we obtain

$$F(0) = 0, \qquad \frac{d^2F}{d\xi^2}(0) = 0, \qquad \frac{dG}{d\xi}(0) = 0, \tag{6.5.32}$$

$$\frac{dF}{d\xi}(\xi_B) = 0, \qquad \frac{d^2F}{d\xi^2}(\xi_B) = 0, \qquad \frac{dG}{d\xi}(\xi_B) = 0, \qquad G(\xi_B) = 0. \tag{6.5.33}$$

Substituting the invariant solution (6.5.20), (6.5.18), (6.5.21) and the mixing lengths (6.4.80) into the governing partial differential equations (6.2.3) and (6.2.4) we obtain

$$-\frac{U}{2}\frac{d}{d\xi}\left(2F+\xi\frac{dF}{d\xi}\right)=\frac{d}{d\xi}\left[\left(\nu\pm l_1^2\frac{d^2F}{d\xi^2}\right)\frac{d^2F}{d\xi^2}\right], \tag{6.5.34}$$

$$-\frac{U}{2}\frac{d}{d\xi}\left(\xi G\right)=\frac{d}{d\xi}\left[\left(\frac{\kappa}{\rho c_v}\pm l_1 l_2\frac{d^2F}{d\xi^2}\right)\frac{dG}{d\xi}\right]. \tag{6.5.35}$$

Sign convention for upper half plane:

- The plus or minus sign, $\pm$, means that when $\frac{d^2F}{d\xi^2}\geq 0$ we take the plus sign, +, and when $\frac{d^2F}{d\xi^2}<0$ we take the minus sign, -.
- The minus or plus sign, $\mp$, means that when $\frac{d^2F}{d\xi^2}\geq 0$ we take the minus sign, -, and when $\frac{d^2F}{d\xi^2}<0$ we take the plus sign, +.

Differential equations (6.5.34) and (6.5.35) can be integrated at least once. This demonstrates the double reduction theorem by Sjöberg [17, 18].

6.5.2 Case $\nu=0$ and $\kappa=0$

When $\nu=0$ and $\kappa=0$ the Lie point symmetry associated with the conserved vectors (6.4.9) and (6.4.47) is

$$X=\xi^1(x)\frac{\partial}{\partial x}-\frac{1}{2}y\frac{\partial}{\partial y}+\left(\psi+a_2(x)\right)\frac{\partial}{\partial\psi}+\frac{1}{2}\overline{S}\frac{\partial}{\partial\overline{S}}. \tag{6.5.36}$$

An invariant solution $\psi=f(x,y)$ satisfies

$$X\left(\psi-f(x,y)\right)\Big|_{\psi=f(x,y)}=0. \tag{6.5.37}$$

Expanding equation (6.5.37) using the Lie point symmetry generator (6.5.36) we obtain the partial differential equation

$$\xi^1(x)\frac{\partial}{\partial x}-\frac{1}{2}y\frac{\partial}{\partial y}=f+a_2(x). \tag{6.5.38}$$

The differential equations of the characteristic curves of the partial differential equation (6.5.38) are

$$\frac{dx}{\xi^1(x)}=-2\frac{dy}{y}=\frac{df}{f+a_2(x)}. \tag{6.5.39}$$

Integrating the differential equation consisting of the first and second terms of (6.5.39) we obtain

$$\ln y+\frac{1}{2}\int^x\frac{d\tau}{\xi^1(\tau)}=C', \tag{6.5.40}$$

where C' is a constant. Therefore

$$I_1=y\exp\left(\frac{1}{2}\int^x\frac{d\tau}{\xi^1(\tau)}\right), \tag{6.5.41}$$

where $I_1 = e^{C'}$ is a constant. The equation consisting of the first and last terms of (6.5.39) gives the first-order differential equation

$$\frac{df}{dx} - \frac{1}{\xi^1(x)} f = \frac{a_2(x)}{\xi^1(x)}. \tag{6.5.42}$$

The integrating factor of the first order differential equation (6.5.42) is

$$I.F = \exp\left(-\int^x \frac{d\tau}{\xi^1(\tau)}\right). \tag{6.5.43}$$

Multiplying (6.5.42) by the integrating factor we obtain

$$\begin{aligned} \exp\left(-\int^x \frac{d\tau}{\xi^1(\tau)}\right)\frac{df}{dx} - \exp\left(-\int^x \frac{d\tau}{\xi^1(\tau)}\right)\frac{1}{\xi^1(x)} f \\ = \frac{a_2(x)}{\xi^1(x)} \exp\left(-\int^x \frac{d\tau}{\xi^1(\tau)}\right). \end{aligned} \tag{6.5.44}$$

We use the product rule for differentiation to factor the left-hand side of (6.5.44) to obtain

$$\frac{d}{dx}\left[f \exp\left(-\int^x \frac{d\tau}{\xi^1(\tau)}\right)\right] = \frac{a_2(x)}{\xi^1(x)} \exp\left(-\int^x \frac{d\tau}{\xi^1(\tau)}\right). \tag{6.5.45}$$

Integrating (6.5.45) we obtain

$$f \exp\left(-\int^x \frac{d\tau}{\xi^1(\tau)}\right) - \int^x \frac{a_2(\sigma)}{\xi^1(\sigma)} \exp\left(-\int^\sigma \frac{d\tau}{\xi^1(\tau)}\right) d\sigma = I_2, \tag{6.5.46}$$

where I_2 is a constant. The general solution of the first order partial differential equation (6.5.38) is

$$I_2 = F(I_1), \tag{6.5.47}$$

where F is an arbitrary function. Since $f = \psi(x, y)$,

$$\psi(x,y) = \exp\left(\int^x \frac{d\tau}{\xi^1(\tau)}\right)\left[F(\xi) + \int^x \frac{a_2(\sigma)}{\xi^1(\sigma)} \exp\left(-\int^\sigma \frac{d\tau}{\xi^1(\tau)}\right) d\sigma\right], \tag{6.5.48}$$

where

$$\xi = y \exp\left(\frac{1}{2}\int^x \frac{d\tau}{\xi^1(\tau)}\right). \tag{6.5.49}$$

The y-component of the mean velocity vanishes at the centreline. That is,

$$\begin{aligned} \overline{v}_y(x,0) = \frac{\partial \psi}{\partial x}(x,0) = \frac{1}{\xi^1(x)} \exp\left(\int^x \frac{d\tau}{\xi^1(\tau)}\right) \int^x \frac{a_2(\sigma)}{\xi^1(\sigma)} \exp\left(-\int^\sigma \frac{d\tau}{\xi^1(\tau)}\right) d\sigma \\ + \frac{a_2(x)}{\xi^1(x)} \exp\left(\int^x \frac{d\tau}{\xi^1(\tau)}\right) \exp\left(-\int^x \frac{d\tau}{\xi^1(\tau)}\right) \\ + \frac{1}{\xi^1(x)} \exp\left(\int^x \frac{d\tau}{\xi^1(\tau)}\right) F(0) = 0. \end{aligned} \tag{6.5.50}$$

Hence

$$\int^x \frac{a_2(\sigma)}{\xi^1(\sigma)} \exp\left(-\int^\sigma \frac{d\tau}{\xi^1(\tau)}\right) d\sigma + a_2(x) \exp\left(-\int^x \frac{d\tau}{\xi^1(\tau)}\right) + F(0) = 0. \tag{6.5.51}$$

Differentiating (6.5.51) with respect x we obtain

$$\frac{da_2}{dx} = 0. \tag{6.5.52}$$

Therefore a_2 is a constant and

$$\psi(x,y) = \exp\left(\int^x \frac{d\tau}{\xi^1(\tau)}\right)\left[F(\xi) + a_2\int^x \frac{1}{\xi^1(\sigma)}\exp\left(-\int^\sigma \frac{d\tau}{\xi^1(\tau)}\right)d\sigma\right]. \tag{6.5.53}$$

Substituting (6.5.53) and (6.5.49) into the conserved quantity (6.3.11) we obtain

$$J = 2\rho U\int_0^{\xi_B(x)} \xi^2\frac{dF}{d\xi}d\xi, \tag{6.5.54}$$

where

$$\xi_B(x) = y_B(x)\exp\left(\frac{1}{2}\int^x \frac{d\tau}{\xi^1(\tau)}\right). \tag{6.5.55}$$

The mean velocity deficit

$$\overline{w}(x,y) = \frac{\partial\psi}{\partial y} = \exp\left(\frac{3}{2}\int^x \frac{d\tau}{\xi^1(\tau)}\right)\frac{dF}{d\xi} \tag{6.5.56}$$

and $\frac{dF}{d\xi}$ vanish at the boundary $\xi = \xi_B(x)$. The mean velocity gradient

$$\frac{\partial\overline{w}}{\partial y} = \frac{\partial^2\psi}{\partial y^2} = \exp\left(2\int^x \frac{d\tau}{\xi^1(\tau)}\right)\frac{d^2F}{d\xi^2} \tag{6.5.57}$$

and $\frac{d^2F}{d\xi^2}$ also vanish at the boundary $\xi = \xi_B(x)$. Assuming that F is analytic at $\xi = \xi_B(x)$ and not a constant, the derivatives of F at $\xi_B(x)$ cannot be all zero. Therefore, there exists an integer $n \geq 3$ such that

$$\frac{d^{n-1}F}{d\xi^{n-1}}(\xi_B(x)) = 0 \tag{6.5.58}$$

and

$$\frac{d^nF}{d\xi^n}(\xi_B(x)) \neq 0. \tag{6.5.59}$$

Differentiating (6.5.58) with respect to x, we have

$$\frac{d^nF}{d\xi^n}(\xi_B(x))\frac{d\xi_B}{dx} = 0. \tag{6.5.60}$$

Therefore $\xi_B(x)$ is a constant and

$$y_B(x) = \xi_B\exp\left(-\frac{1}{2}\int^x \frac{d\tau}{\xi^1(\tau)}\right). \tag{6.5.61}$$

The Prandtl hypothesis states that the momentum mixing length $l(x)$ is proportional to the half-width of the wake $y_B(x)$. Therefore

$$l(x) = l_1\exp\left(-\frac{1}{2}\int^x \frac{d\tau}{\xi^1(\tau)}\right), \tag{6.5.62}$$

where l_1 is a positive constant. Substituting (6.5.62) into (6.4.86) we obtain

$$\frac{d\xi^1}{dx} = -2. \tag{6.5.63}$$

Hence

$$\xi^1(x) = -2x + a_3, \tag{6.5.64}$$

where a_3 is a constant. Substituting (6.5.64) into (6.5.62) and simplifying the resulting integral we obtain

$$l(x) = l_1 \left(x - \frac{1}{2}a_3\right)^{\frac{1}{4}}. \tag{6.5.65}$$

The constant of integration is absorbed into l_1. Therefore, by (6.4.66), the thermal mixing length becomes

$$l_\theta(x) = l_2 \left(x - \frac{1}{2}a_3\right)^{\frac{1}{4}}, \tag{6.5.66}$$

where $l_2 = Bl_1$ is a constant. Substituting (6.5.64) into (6.5.61) and simplifying the resulting integral we obtain

$$y_B(x) = \xi_B \left(x - \frac{1}{2}a_3\right)^{\frac{1}{4}}. \tag{6.5.67}$$

Substituting (6.5.64) into (6.5.53), (6.5.49) and simplifying the resulting integrals we obtain

$$\psi(x,y) = -a_2 + \left(x - \frac{1}{2}a_3\right)^{-1/2} F(\xi), \tag{6.5.68}$$

where

$$\xi = \frac{y}{\left(x - \frac{1}{2}a_3\right)^{\frac{1}{4}}}. \tag{6.5.69}$$

The mean velocity deficit is

$$\overline{w} = \frac{\partial \psi}{\partial y} = \frac{1}{\left(x - \frac{1}{2}a_3\right)^{\frac{3}{4}}} \frac{dF}{d\xi}. \tag{6.5.70}$$

Again, since $\overline{w}$ must be finite at all nonnegative values of x except at $x = 0$ it follows that $a_3 = 0$. Therefore, the associated Lie point symmetry (6.5.36) becomes

$$X = -2x\frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial y} + (\psi + a_2)\frac{\partial}{\partial \psi} + \frac{1}{2}\overline{S}\frac{\partial}{\partial \overline{S}} \tag{6.5.71}$$

and stream function becomes

$$\psi(x,y) = a_2 + x^{-1/2}F(\xi), \tag{6.5.72}$$

where

$$\xi = \frac{y}{x^{\frac{1}{4}}}. \tag{6.5.73}$$

Now we find the temperature component of the invariant solution, $\overline{S} = g(x, y)$. The invariant solution $\overline{S} = g(x, y)$ satisfies

$$X\left(\overline{S} - g(x, y)\right)\Big|_{\overline{S}=g(x,y)} = 0, \tag{6.5.74}$$

where X is defined by (6.5.71). Expanding (6.5.74) we obtain the partial differential equation

$$2x\frac{\partial g}{\partial x} + \frac{1}{2}y\frac{\partial g}{\partial y} = -\frac{1}{2}g. \tag{6.5.75}$$

The differential equations of the characteristic curves of the partial differential equation (6.5.75) are

$$\frac{dx}{2x} = 2\frac{dy}{y} = -\frac{2dg}{g}. \tag{6.5.76}$$

Integrating the first and second terms of (6.5.76) we obtain

$$\ln y - \frac{1}{4}\ln x = C', \tag{6.5.77}$$

where C' is a constant. Therefore

$$I_1 = \frac{y}{x^{1/4}}, \tag{6.5.78}$$

where $I_1 = e^{C'}$ is a constant. Integrating the first and last terms of (6.5.76) we obtain

$$\ln g + \frac{1}{4}\ln x = C', \tag{6.5.79}$$

where C' is a constant. Therefore

$$x^{1/4}g = I_2, \tag{6.5.80}$$

where $I_2 = e^{C'}$ is a constant. The general solution of the first order partial differential equation (6.5.75) is

$$I_2 = G(I_1). \tag{6.5.81}$$

Therefore the invariant solution for the mean temperature difference is

$$\overline{S}(x, y) = x^{-1/4}G(\xi), \tag{6.5.82}$$

where

$$\xi = \frac{y}{x^{1/4}}. \tag{6.5.83}$$

The physical variables $\overline{w}$ and $\overline{v}_y$ depend on the partial derivatives of ψ and a_2 is an additive constant in (6.5.72) for ψ. Hence, without affecting the physical variables $\overline{w}$, $\overline{v}_y$ and $\overline{S}$, we can let $a_2 = 0$.

Substituting the invariant solution (6.5.72), with $a_2 = 0$, and (6.5.83) into the conserved quantity (6.3.11) we obtain

$$J = 2\rho U\int_0^{\xi_B}\xi^2\frac{dF}{d\xi}d\xi, \tag{6.5.84}$$

where

$$\xi_B = \frac{y_B(x)}{x^{1/4}}. \tag{6.5.85}$$

Substituting the invariant solution (6.5.72), (6.5.82) and (6.5.83) into the conserved quantity (6.3.22) we obtain

$$K = 2\rho c_v U \int_0^{\xi_B} G(\xi) d\xi. \tag{6.5.86}$$

Writing the boundary conditions (3.12.7) and (3.12.8) in terms of the invariant solution (6.5.72), (6.5.82) and (6.5.83) we obtain

$$F(0) = 0, \qquad \frac{d^2F}{d\xi^2}(0) = 0, \qquad \frac{dF}{d\xi}(\xi_B) = 0, \qquad \frac{d^2F}{d\xi^2}(\xi_B) = 0, \tag{6.5.87}$$

$$\frac{dG}{d\xi}(0) = 0, \qquad G(\xi_B) = 0. \tag{6.5.88}$$

Substituting the invariant solution (6.5.72), (6.5.82), (6.5.83) and the mixing lengths (6.5.65), (6.5.66), with $a_3 = 0$, into the partial differential equations (6.2.3) and (6.2.4), with $\nu = 0$ and $\kappa = 0$, we obtain

$$-\frac{U}{4}\frac{d}{d\xi}\left[2F + \xi\frac{dF}{d\xi}\right] = l_1^2 \frac{d}{d\xi}\left[\pm\left(\frac{d^2F}{d\xi^2}\right)^2\right], \tag{6.5.89}$$

$$-\frac{U}{4}\frac{d}{d\xi}\left(\xi G\right) = l_1 l_2 \frac{d}{d\xi}\left[\pm\frac{d^2F}{d\xi^2}\frac{dG}{d\xi}\right]. \tag{6.5.90}$$

Sign convention for upper half plane:

- The plus or minus sign, $\pm$, means that when $\frac{d^2F}{d\xi^2} \geq 0$ we take the plus sign, +, and when $\frac{d^2F}{d\xi^2} < 0$ we take the minus sign, -.
- The minus or plus sign, $\mp$, means that when $\frac{d^2F}{d\xi^2} \geq 0$ we take the minus sign, -, and when $\frac{d^2F}{d\xi^2} < 0$ we take the plus sign, +.

Differential equations (6.5.89) and (6.5.90) can be integrated at least once. This demonstrates the double reduction theorem by Sjöberg [17, 18].

6.6 Summary of results

We summarize and compare the results for $\nu \neq 0$ and $\kappa \neq 0$ with the results for $\nu = 0$ and $\kappa = 0$.

(i) The case $\nu \neq 0$ and $\kappa \neq 0$

Lie point symmetry:

$$X = -x\frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial y} + \psi\frac{\partial}{\partial \psi} + \frac{1}{2}\overline{S}\frac{\partial}{\partial \overline{S}}. \tag{6.6.1}$$

Fluid variables:

$$\psi(x,y) = \frac{1}{x}F(\xi), \qquad \overline{S}(x,y) = \frac{1}{x^{1/2}}G(\xi), \qquad \xi = \frac{y}{x^{1/2}} \tag{6.6.2}$$

Ordinary differential equations:

$$-\frac{U}{2}\frac{d}{d\xi}\left(2F + \xi\frac{dF}{d\xi}\right) = \frac{d}{d\xi}\left[\left(\nu \pm l_1^2\frac{d^2F}{d\xi^2}\right)\frac{d^2F}{d\xi^2}\right], \tag{6.6.3}$$

$$-\frac{U}{2}\frac{d}{d\xi}(\xi G) = \frac{d}{d\xi}\left[\left(\frac{\kappa}{\rho c_v} \pm l_1 l_2\frac{d^2F}{d\xi^2}\right)\frac{dG}{d\xi}\right] \tag{6.6.4}$$

Boundary conditions:

$$F(0) = 0, \qquad \frac{d^2F}{d\xi^2}(0) = 0, \qquad \frac{dF}{d\xi}(\xi_B) = 0, \qquad \frac{d^2F}{d\xi^2}(\xi_B) = 0 \tag{6.6.5}$$

$$\frac{dG}{d\xi}(0) = 0, \qquad G(\xi_B) = 0, \qquad \frac{dG}{d\xi}(\xi_B) = 0 \tag{6.6.6}$$

Conserved quantities:

$$J = 2\rho U\int_0^{\xi_B}\xi^2\frac{dF}{d\xi}d\xi, \qquad K = 2\rho c_v U\int_0^{\xi_B}G(\xi)d\xi \tag{6.6.7}$$

Mixing lengths:

$$l(x) = l_1 x, \qquad l_\theta(x) = l_2 x \tag{6.6.8}$$

Boundaries:

$$y_B(x) = \xi_B x^{1/2}, \qquad y_{B_\theta} = \xi_B x^{1/2} \tag{6.6.9}$$

(ii) The case $\nu = 0$ and $\kappa = 0$

Lie point symmetry:

$$X = -2x\frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial y} + \psi\frac{\partial}{\partial\psi} + \frac{1}{2}\overline{S}\frac{\partial}{\partial\overline{S}} \tag{6.6.10}$$

Fluid variables:

$$\psi(x,y) = \frac{1}{x^{1/2}}F(\xi), \qquad \overline{S}(x,y) = \frac{1}{x^{1/4}}G(\xi), \qquad \xi = \frac{y}{x^{1/4}} \tag{6.6.11}$$

Ordinary differential equations:

$$-\frac{U}{4}\frac{d}{d\xi}\left[2F + \xi\frac{dF}{d\xi}\right] = l_1^2\frac{d}{d\xi}\left[\pm\left(\frac{d^2F}{d\xi^2}\right)^2\right], \tag{6.6.12}$$

$$-\frac{U}{4}\frac{d}{d\xi}(\xi G) = l_1 l_2\frac{d}{d\xi}\left[\pm\frac{d^2F}{d\xi^2}\frac{dG}{d\xi}\right] \tag{6.6.13}$$

Boundary conditions:

$$F(0) = 0, \qquad \frac{d^2F}{d\xi^2}(0) = 0, \qquad \frac{dF}{d\xi}(\xi_B) = 0, \qquad \frac{d^2F}{d\xi^2}(\xi_B) = 0 \tag{6.6.14}$$

$$\frac{dG}{d\xi}(0) = 0, \qquad G(\xi_B) = 0 \tag{6.6.15}$$

Conserved quantities:

$$J = 2\rho U \int_0^{\xi_B} \xi^2 \frac{dF}{d\xi} d\xi, \qquad K = 2\rho c_v U \int_0^{\xi_B} G(\xi) d\xi \tag{6.6.16}$$

Mixing lengths:

$$l(x) = l_1 x^{1/4}, \qquad l_\theta(x) = l_2 x^{1/4} \tag{6.6.17}$$

Boundaries:

$$y_B(x) = \xi_B x^{1/4}, \qquad y_B = \xi_B x^{1/4} \tag{6.6.18}$$

When $\nu = 0$ and $\kappa = 0$, the Prandtl hypothesis was required to determine the Lie point symmetry, the mixing lengths and the half-width of the wake. The final result for this case is different from the result derived when $\nu \neq 0$ and $\kappa \neq 0$. Prandtl's hypothesis that the momentum mixing length $l(x)$ is proportional to the half-width of the wake $y_B(x)$ is not satisfied when $\nu \neq 0$ and $\kappa \neq 0$. This was not the case for the classical thermal wake. When $\nu \neq 0$ and $\kappa \neq 0$, the mixing lengths are obtained by mathematical analysis. We have shown in this Chapter that ν and κ play an important role in the mathematical analysis.

6.7 Numerical solution for $\nu \neq 0$ and $\kappa \neq 0$

In this Section, we derive the numerical solution for the momentumless thermal wake for $\nu \neq 0$ and $\kappa \neq 0$.

Consider first (6.6.3) for the stream function. Integrating (6.6.3) we obtain

$$-\frac{U}{2}\left(2F + \xi\frac{dF}{d\xi}\right) = \left(\nu \pm l_1^2 \frac{d^2F}{d\xi^2}\right)\frac{d^2F}{d\xi^2} + C, \tag{6.7.1}$$

where C is a constant. Applying boundary conditions (6.5.32) at $\xi = 0$ we find that $C = 0$. Equation (6.7.1) becomes

$$-\frac{U}{2}\left(2F + \xi\frac{dF}{d\xi}\right) = \left(\nu \pm l_1^2 \frac{d^2F}{d\xi^2}\right)\frac{d^2F}{d\xi^2}. \tag{6.7.2}$$

Applying boundary conditions (6.5.33) at $\xi = \xi_B$ in (6.7.2) we see that $F(\xi_B) = 0$ and therefore $\psi(x, y_B(x)) = 0$. The drag is

$$D = 2\rho U \int_0^{y_B(x)} \frac{\partial \psi}{\partial y} dy = \psi(x, y_B(x)) - \psi(x, 0) = \frac{1}{x}F(\xi_B) - \frac{1}{x}F(0) = 0, \tag{6.7.3}$$

which confirms that the wake is momentumless. Note that

$$\nu \pm l_1^2 \frac{d^2F}{d\xi^2} = \nu + l_1^2 \left| \frac{d^2F}{d\xi^2} \right| \geq 0. \tag{6.7.4}$$

Hence, when $\frac{d^2F}{d\xi^2} > 0$

$$2F(\xi) + \xi \frac{dF}{d\xi} < 0 \tag{6.7.5}$$

and when $\frac{d^2F}{d\xi^2} < 0$

$$2F(\xi) + \xi \frac{dF}{d\xi} > 0. \tag{6.7.6}$$

The converses of these statements also hold. We expect $\frac{d^2F}{d\xi^2}$ to be positive close to the centreline of the wake and negative close to the boundary of the wake. We analyze both cases.

Case: $\frac{d^2F}{d\xi^2} > 0$

When $\frac{d^2F}{d\xi^2} > 0$ equation (6.7.2) becomes

$$-\frac{U}{2}\left(2F + \xi \frac{dF}{d\xi}\right) = \left(\nu + l_1^2 \frac{d^2F}{d\xi^2}\right) \frac{d^2F}{d\xi^2} \tag{6.7.7}$$

and

$$2F(\xi) + \xi \frac{dF}{d\xi} < 0. \tag{6.7.8}$$

Expanding equation (6.7.7) we obtain

$$l_1^2 \left(\frac{d^2F}{d\xi^2}\right)^2 + \nu \frac{d^2F}{d\xi^2} + \frac{U}{2}\left(2F + \xi \frac{dF}{d\xi}\right) = 0. \tag{6.7.9}$$

Equation (6.7.9) is a quadratic equation in $\frac{d^2F}{d\xi^2}$. We use the formula for the roots of a quadratic equation to obtain

$$\frac{d^2F}{d\xi^2} = -\frac{\nu}{2l_1^2} \pm \sqrt{\left(\frac{\nu}{2l_1^2}\right)^2 - \frac{U}{2l_1^2}\left(2F(\xi) + \xi \frac{dF}{d\xi}\right)}. \tag{6.7.10}$$

But

$$\frac{d^2F}{d\xi^2} > 0. \tag{6.7.11}$$

Hence we take the plus sign in (6.7.10) and

$$\frac{d^2F}{d\xi^2} = -\frac{\nu}{2l_1^2} + \sqrt{\left(\frac{\nu}{2l_1^2}\right)^2 - \frac{U}{2l_1^2}\left(2F(\xi) + \xi \frac{dF}{d\xi}\right)}. \tag{6.7.12}$$

Case: $\frac{d^2F}{d\xi^2} < 0$

When $\frac{d^2F}{d\xi^2} < 0$ equation (6.7.2) becomes

$$-\frac{U}{2}\left(2F + \xi\frac{dF}{d\xi}\right) = \left(\nu - l_1^2\frac{d^2F}{d\xi^2}\right)\frac{d^2F}{d\xi^2} \tag{6.7.13}$$

and

$$2F(\xi) + \xi\frac{dF}{d\xi} > 0. \tag{6.7.14}$$

Expanding equation (6.7.13) we obtain

$$l_1^2\left(\frac{d^2F}{d\xi^2}\right)^2 - \nu\frac{d^2F}{d\xi^2} - \frac{U}{2}\left(2F + \xi\frac{dF}{d\xi}\right) = 0. \tag{6.7.15}$$

Equation (6.7.15) is a quadratic equation in $\frac{d^2F}{d\xi^2}$. We use the formula for the roots of a quadratic equation to obtain

$$\frac{d^2F}{d\xi^2} = \frac{\nu}{2l_1^2} \pm \sqrt{\left(\frac{\nu}{2l_1^2}\right)^2 + \frac{U}{2l_1^2}\left(2F(\xi) + \xi\frac{dF}{d\xi}\right)}. \tag{6.7.16}$$

But

$$\frac{d^2F}{d\xi^2} < 0. \tag{6.7.17}$$

Hence we take the minus sign in (6.7.16) and

$$\frac{d^2F}{d\xi^2} = \frac{\nu}{2l_1^2} - \sqrt{\left(\frac{\nu}{2l_1^2}\right)^2 + \frac{U}{2l_1^2}\left(2F(\xi) + \xi\frac{dF}{d\xi}\right)}. \tag{6.7.18}$$

Consider next (6.6.4) for the temperature difference. Integrating (6.6.4) we obtain

$$-\frac{U}{2}\xi G = \left(\frac{\kappa}{\rho c_v} + l_1 l_2\left|\frac{d^2F}{d\xi^2}\right|\right)\frac{dG}{d\xi} + C, \tag{6.7.19}$$

where C is a constant. Using boundary conditions (6.5.32) at $\xi = 0$ we find that $C = 0$. Making $\frac{dG}{d\xi}$ the subject in (6.7.19), with $C = 0$, we obtain

$$\frac{dG}{d\xi} = -\frac{U\xi G}{2\left(\frac{\kappa}{\rho c_v} + l_1 l_2\left|\frac{d^2F}{d\xi^2}\right|\right)}. \tag{6.7.20}$$

As for the classical wake, the boundary conditions (6.5.32) and (6.5.33) are not sufficient to fully solve the ordinary differential equations (6.7.12), (6.7.18) and (6.7.20) since the value of ξ_B is unknown. Hence the conserved quantities (6.5.30) and (6.5.31) are needed to complete the solution of the problem. We convert the boundary value problem to an initial value problem using a shooting method to find the unknown initial values.

Since $F(0) = 0$, F can be approximated by

$$F(\xi) \approx A\xi^m \qquad \text{as} \qquad \xi \to 0, \tag{6.7.21}$$

where A and $m > 0$ are constants. But

$$\overline{w}(x,y) = \frac{1}{x^{3/2}}\frac{dF}{d\xi} = \frac{m}{x^{3/2}}A\xi^{m-1} < 0 \qquad \text{and finite as} \qquad \xi \to 0. \tag{6.7.22}$$

Hence $m = 1$ and $A < 0$. The asymptotic behaviour becomes

$$F(\xi) = A\xi \qquad \text{as} \qquad \xi \to 0 \tag{6.7.23}$$

and

$$F(0) = 0, \qquad \frac{dF}{d\xi}(0) = A. \tag{6.7.24}$$

Equation (6.7.23) identically satisfies the second boundary condition in (6.5.32). Since $\frac{dG}{d\xi}(0) = 0$ we approximate G by

$$G(\xi) \approx B + C\xi^n \qquad \text{as} \qquad \xi \to 0, \tag{6.7.25}$$

where B, C and since $G(0)$ is finite, $n > 0$. But

$$\frac{dG}{d\xi} = Cn\xi^{n-1} = 0 \qquad \text{as} \qquad \xi \to 0 \tag{6.7.26}$$

provided $n > 1$ and

$$G(0) = B. \tag{6.7.27}$$

We now have the following initial value problem.
Differential equations:

$$\frac{d^2F}{d\xi^2} = -\frac{\nu}{2l_1^2} + \sqrt{\left(\frac{\nu}{2l_1^2}\right)^2 - \frac{U}{2l_1^2}\left(2F(\xi) + \xi\frac{dF}{d\xi}\right)} \text{ when } 2F + \xi\frac{dF}{d\xi} < 0, \tag{6.7.28}$$

$$\frac{d^2F}{d\xi^2} = \frac{\nu}{2l_1^2} - \sqrt{\left(\frac{\nu}{2l_1^2}\right)^2 + \frac{U}{2l_1^2}\left(2F(\xi) + \xi\frac{dF}{d\xi}\right)} \text{ when } 2F + \xi\frac{dF}{d\xi} > 0, \tag{6.7.29}$$

$$\frac{dG}{d\xi} = -\frac{U\xi G}{2\left(\frac{\kappa}{\rho c_v} + l_1 l_2\left|\frac{d^2F}{d\xi^2}\right|\right)}. \tag{6.7.30}$$

Initial conditions:

$$F(0) = 0, \qquad \frac{dF}{d\xi}(0) = A, \qquad G(0) = B. \tag{6.7.31}$$

The initial values A and B are calculated using a shooting method. The targets of the shooting method are

$$J = 2\rho U\int_0^{\xi_B}\xi^2\frac{dF}{d\xi}d\xi, \qquad K = 2\rho c_v U\int_0^{\xi_B}G(\xi)d\xi, \tag{6.7.32}$$

with values of ν, $\frac{\kappa}{\rho c_v}$, ρ, l_1, l_2, U, J and K prescribed.

We use a shooting method to determine the unknown values of A and B. In the shooting method, we first guess values of A and B and use the Python function scipy.integrate.odeint to find the corresponding values of $\frac{dF}{d\xi}$ and G. We use the

values obtained for $\frac{dF}{d\xi}$ and G to calculate the values of the conserved quantities J and K using (6.7.32). We check whether the calculated values of the conserved quantities match the prescribed values for J and K within the specified tolerance. If they do not match, we update the initial values A and B so that the calculated conserved quantities become closer to the prescribed values of J and K.

We define

$$J^*(A,B) = 2\rho U \int_0^{\xi_B} \xi^2 \frac{dF}{d\xi}(\xi;A,B)d\xi, \qquad K^*(A,B) = 2\rho c_v U \int_0^{\xi_B} G(\xi;A,B)d\xi, \tag{6.7.33}$$

where $\frac{dF}{d\xi}(\xi;A,B)$ and $G(\xi;A,B)$ are found when the initial conditions are

$$F(0) = 0, \qquad \frac{dF}{d\xi}(0) = A, \qquad G(0) = B. \tag{6.7.34}$$

We seek to find initial values A and B such that $J^*(A,B) = J$ and $K^*(A,B) = K$, where J and K are the prescribed values for the conserved quantities.

Suppose A_k and B_k are the current best guesses for the values of A and B. To update A_k and B_k we need steps δ_1 and δ_2 such that

$$J - J^*(A_k + \delta_1, B_k + \delta_2) \approx 0, \qquad K - K^*(A_k + \delta_1, B_k + \delta_2) \approx 0. \tag{6.7.35}$$

Using a Taylor series expansion to first order to expand the equations in (6.7.35) and writing them in a matrix format we obtain

$$\begin{pmatrix} \frac{\partial J^*}{\partial A}(A_k,B_k) & \frac{\partial J^*}{\partial B}(A_k,B_k) \\ \frac{\partial K^*}{\partial A}(A_k,B_k) & \frac{\partial K^*}{\partial B}(A_k,B_k) \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} J - J^*(A_k,B_k) \\ K - K^*(A_k,B_k) \end{pmatrix}. \tag{6.7.36}$$

We use a secant method to evaluate the partial derivatives in (6.7.36). For example,

$$\frac{\partial J^*}{\partial A}(A_k,B_k) = \frac{J^*(A_k,B_k) - J^*(A_{k-1},B_k)}{A_k - A_{k-1}}. \tag{6.7.37}$$

We update guesses for A and B by adding steps δ_1 and δ_2 to the current guesses. That is,

$$A_{k+1} = A_k + \delta_1, \qquad B_{k+1} = B_k + \delta_2. \tag{6.7.38}$$

We summarise the shooting method as follows.

1. Choose (A_0, B_0) and (A_1, B_1) as initial guess.
2. Set $k = 1$.
3. Calculate the values of $\frac{dF}{d\xi}(\xi, A_k, B_k)$, $G(\xi, A_k, B_k)$, $\frac{dF}{d\xi}(\xi, A_{k-1}, B_{k-1})$ and $G(\xi, A_{k-1}, B_{k-1})$ for differential equations (6.7.28) to (6.7.30) and initial conditions (6.7.31) using the Python function scipy.integrate.odeint.
4. Calculate derivatives $\frac{\partial J^*}{\partial A}$, $\frac{\partial J^*}{\partial B}$, $\frac{\partial K^*}{\partial A}$ and $\frac{\partial K^*}{\partial B}$ using the secant method (6.7.37).
5. Calculate steps δ_1 and δ_2 using equation (6.7.36).
6. Update the values of A and B by setting $A_k = A_{k-1}$, $B_k = B_{k-1}$, $A_{k+1} = A_k + \delta_1$ and $B_{k+1} = B_k + \delta_2$.

7. If the error $|J - J^*(A_{k+1}, B_{k+1})| + |K - K^*(A_{k+1}, B_{k+1})| < \epsilon$, where ϵ is the tolerance of the shooting method, we stop. Otherwise, we increment k by 1 and continue to step 3.

Once we obtain the values of A and B we will use the Python function scipy.integrate.odeint to solve the initial value problem (6.7.28) to (6.7.31).

6.8 Numerical solution for $\nu \to 0$ and $\kappa \to 0$

In this Section, we let $\nu \to 0$ and $\kappa \to 0$ in the solution for $\nu \neq 0$ and $\kappa \neq 0$ so that we can compare it with the solution obtained in Section 6.7 when $\nu = 0$ and $\kappa = 0$. Letting $\nu \to 0$ and $\kappa \to 0$ in the differential equations (6.7.28) to (6.7.30)

$$\frac{d^2F}{d\xi^2} = \sqrt{-\frac{U}{2l_1^2}\left(2F(\xi) + \xi\frac{dF}{d\xi}\right)} \text{ when } 2F + \xi\frac{dF}{d\xi} < 0, \tag{6.8.1}$$

$$\frac{d^2F}{d\xi^2} = -\sqrt{\frac{U}{2l_1^2}\left(2F(\xi) + \xi\frac{dF}{d\xi}\right)} \text{ when } 2F + \xi\frac{dF}{d\xi} > 0, \tag{6.8.2}$$

$$\frac{dG}{d\xi} = -\frac{U\xi G}{2l_1 l_2\left|\frac{d^2F}{d\xi^2}\right|}. \tag{6.8.3}$$

Since $\frac{d^2F}{d\xi^2}$ vanishes at $\xi = 0$ there is a singularity for $\frac{dG}{d\xi}$ at $\xi = 0$. Hence we shift the initial conditions to $\xi = \delta$, where δ is very small. From Section 6.7 we see that

$$F(\xi) = A\xi, \qquad G(\xi) = B + C\xi^n, \qquad n > 1, \qquad \text{as } \xi \to 0, \tag{6.8.4}$$

where A, B, C and n are constants. Hence approximating initial conditions at $\xi = \delta$ to first-order in δ we obtain

$$F(\delta) = A\delta, \qquad \frac{dF}{d\xi}(\delta) = A, \qquad G(\delta) = B. \tag{6.8.5}$$

We use the shooting described in Section 6.7 to find the unknown constants A and B. The targets for the shooting method are

$$J = 2\rho U\int_0^{\xi_B} \xi^2\frac{dF}{d\xi}d\xi, \qquad K = 2\rho c_v U\int_0^{\xi_B} G(\xi)d\xi. \tag{6.8.6}$$

The values of δ, ν, ρ, $\frac{\kappa}{\rho c_v}$, l_1, l_2, U, J and K are prescribed. Once we find the values of A and B we solve the initial value problem (6.8.1) to (6.8.3) and (6.8.5) using Python function scipy.integrate.odeint.

6.9 Numerical solution for $\nu = 0$ and $\kappa = 0$

In this Section, we derive the numerical solution for the momentumless thermal wake for $\nu = 0$ and $\kappa = 0$.

Consider first equation (6.6.12) for the stream function. Integrating (6.6.12) we obtain

$$-\frac{U}{4}\left(2F + \xi\frac{dF}{d\xi}\right) = \pm l_1^2\frac{d^2F}{d\xi^2}\frac{d^2F}{d\xi^2} + C, \tag{6.9.1}$$

where C is a constant. Applying boundary conditions (6.5.87) at $\xi = 0$ we find that $C = 0$. Equation (6.9.1) becomes

$$-\frac{U}{4}\left(2F + \xi\frac{dF}{d\xi}\right) = \pm l_1^2 \frac{d^2F}{d\xi^2}\frac{d^2F}{d\xi^2}. \tag{6.9.2}$$

Note that

$$\pm l_1^2 \frac{d^2F}{d\xi^2} = l_1^2 \left|\frac{d^2F}{d\xi^2}\right| \geq 0. \tag{6.9.3}$$

Hence, when $\frac{d^2F}{d\xi^2} > 0$

$$2F(\xi) + \xi\frac{dF}{d\xi} < 0 \tag{6.9.4}$$

and when $\frac{d^2F}{d\xi^2} < 0$

$$2F(\xi) + \xi\frac{dF}{d\xi} > 0. \tag{6.9.5}$$

The converses of these statements also hold. We expect the $\frac{d^2F}{d\xi^2}$ to be positive close to the centreline of the wake and negative close to the boundary of the wake. We analyze both cases.

Case: $\frac{d^2F}{d\xi^2} > 0$

When $\frac{d^2F}{d\xi^2} > 0$ equation (6.9.2) becomes

$$-\frac{U}{4}\left(2F + \xi\frac{dF}{d\xi}\right) = l_1^2\left(\frac{d^2F}{d\xi^2}\right)^2 \tag{6.9.6}$$

and

$$2F(\xi) + \xi\frac{dF}{d\xi} < 0. \tag{6.9.7}$$

Taking square root in (6.9.6) we obtain

$$\frac{d^2F}{d\xi^2} = \pm\sqrt{-\frac{U}{4l_1^2}\left(2F(\xi) + \xi\frac{dF}{d\xi}\right)}. \tag{6.9.8}$$

But

$$\frac{d^2F}{d\xi^2} > 0. \tag{6.9.9}$$

Hence we take the plus sign in (6.9.8) and

$$\frac{d^2F}{d\xi^2} = \sqrt{-\frac{U}{4l_1^2}\left(2F(\xi) + \xi\frac{dF}{d\xi}\right)}. \tag{6.9.10}$$

Case: $\frac{d^2F}{d\xi^2} < 0$

When $\frac{d^2F}{d\xi^2} < 0$ equation (6.9.2) becomes

$$\frac{U}{4}\left(2F + \xi\frac{dF}{d\xi}\right) = l_1^2\left(\frac{d^2F}{d\xi^2}\right)^2 \tag{6.9.11}$$

and

$$2F(\xi) + \xi\frac{dF}{d\xi} > 0. \tag{6.9.12}$$

Taking square root in (6.9.12) we obtain

$$\frac{d^2F}{d\xi^2} = \pm\sqrt{\frac{U}{4l_1^2}\left(2F(\xi) + \xi\frac{dF}{d\xi}\right)}. \tag{6.9.13}$$

But

$$\frac{d^2F}{d\xi^2} < 0. \tag{6.9.14}$$

Hence we take the minus sign in (6.9.13) and

$$\frac{d^2F}{d\xi^2} = -\sqrt{\frac{U}{4l_1^2}\left(2F(\xi) + \xi\frac{dF}{d\xi}\right)}. \tag{6.9.15}$$

Consider next equation (6.6.13) for the mean temperature difference. Integrating (6.6.13) we obtain

$$-\frac{U}{4}\xi G = l_1 l_2\left|\frac{d^2F}{d\xi^2}\right|\frac{dG}{d\xi} + C, \tag{6.9.16}$$

where C is a constant. Using boundary conditions (6.5.88) at $\xi = 0$ we find that $C = 0$. Making $\frac{dG}{d\xi}$ the subject in (6.9.16), with $C = 0$, we obtain

$$\frac{dG}{d\xi} = -\frac{U\xi G}{4l_1 l_2\left|\frac{d^2F}{d\xi^2}\right|}. \tag{6.9.17}$$

Again, the boundary conditions (6.5.87) and (6.5.88) are not sufficient to fully solve the ordinary differential equations (6.9.10), (6.9.15) and (6.9.17) since the value of ξ_B is unknown. Hence the conserved quantities (6.5.84) and (6.5.86) are needed to complete the solution of the problem. We use the shooting method described in Section 6.7 to convert the boundary value problem to an initial value problem.

Since $F(0) = 0$, F can be approximated by

$$F(\xi) \approx A\xi^m \qquad \text{as} \qquad \xi \to 0, \tag{6.9.18}$$

where A and $m > 0$ are constants. But

$$\overline{w}(x,y) = \frac{1}{x^{3/4}}\frac{dF}{d\xi} = \frac{m}{x^{3/4}}A\xi^{m-1} < 0 \qquad \text{and finite as} \qquad \xi \to 0. \tag{6.9.19}$$

Hence $m = 1$ and $A < 0$. The asymptotic behaviour becomes

$$F(\xi) = A\xi \qquad \text{as} \qquad \xi \to 0. \tag{6.9.20}$$

Equation (6.9.17) has a singularity at $\xi = 0$. Hence we shift the initial value for ξ to δ, where δ is small. We have the initial conditions

$$F(\delta) = A\delta, \qquad \frac{dF}{d\xi}(\delta) = A. \tag{6.9.21}$$

Since $\frac{dG}{d\xi}(0) = 0$ we approximate G by

$$G(\xi) \approx B + C\xi^n \qquad \text{as} \qquad \xi \to 0, \tag{6.9.22}$$

where B, C and since $G(0)$ is finite, $n > 0$. But

$$\frac{dG}{d\xi} = Cn\xi^{n-1} = 0 \qquad \text{as} \qquad \xi \to 0 \tag{6.9.23}$$

provided $n > 1$. Approximating the initial value to first-order we have

$$G(\delta) = B. \tag{6.9.24}$$

We now have the following initial value problem.
Differential equations:

$$\frac{d^2F}{d\xi^2} = \sqrt{-\frac{U}{4l_1^2}\left(2F(\xi) + \xi\frac{dF}{d\xi}\right)} \text{ when } 2F + \xi\frac{dF}{d\xi} < 0, \tag{6.9.25}$$

$$\frac{d^2F}{d\xi^2} = -\sqrt{\frac{U}{4l_1^2}\left(2F(\xi) + \xi\frac{dF}{d\xi}\right)} \text{ when } 2F + \xi\frac{dF}{d\xi} > 0, \tag{6.9.26}$$

$$\frac{dG}{d\xi} = -\frac{U\xi G}{4l_1 l_2\left|\frac{d^2F}{d\xi^2}\right|}. \tag{6.9.27}$$

Initial conditions:

$$F(\delta) = A\delta, \qquad \frac{dF}{d\xi}(\delta) = A, \qquad G(\delta) = B. \tag{6.9.28}$$

The targets of the shooting method are

$$J = 2\rho U\int_0^{\xi_B}\xi^2\frac{dF}{d\xi}d\xi, \qquad K = 2\rho c_v U\int_0^{\xi_B}G(\xi)d\xi \tag{6.9.29}$$

with values of δ, ν, $\frac{\kappa}{\rho c_v}$, ρ, l_1, l_2, U, J and K prescribed. We use the shooting method described in Section 6.7 to calculate the unknown constants A and B. We then solve the resulting initial value problem (6.9.25) to (6.9.28) using the Python function scipy.integrate.odeint.

6.10 Results and discussion

In this Section, we present numerical results for the momentumless turbulent thermal wake. We show the role played by the kinematic viscosity and thermal conductivity in the analysis of the momentumless turbulent thermal wake. For all plots in this Section, $l_1 = 1$, $l_2 = 1$, $J = 1$, $K = 1$, $\rho = 1$, $c_v = 1$, $U = 1$.

Figure 6.1 shows the plots for the x-component of the mean velocity profiles when $\nu = 0.01$ at $x = 1$, $x = 1.5$ and $x = 2$. The graphs are plotted using

$$\overline{v}_x(x,y) = U - \frac{1}{x^{3/2}}\frac{dF}{d\xi}, \tag{6.10.1}$$

where $\frac{dF}{d\xi}$ is obtained from the numerical solution described in Section 6.7. We see that as the distance from the obstacle, x, increases the x-component of the mean velocity at the centreline, $\overline{v}_x(x,0)$, decreases and the curves become narrower. This shows that the turbulent kinematic viscosity increases with the distance from the obstacle, x. We saw this behaviour with the classical turbulent thermal wake in Chapter 4. This makes physical sense since the flow becomes turbulent as the distance from the obstacle increases and the momentum diffusion in the mean flow increases with turbulence.

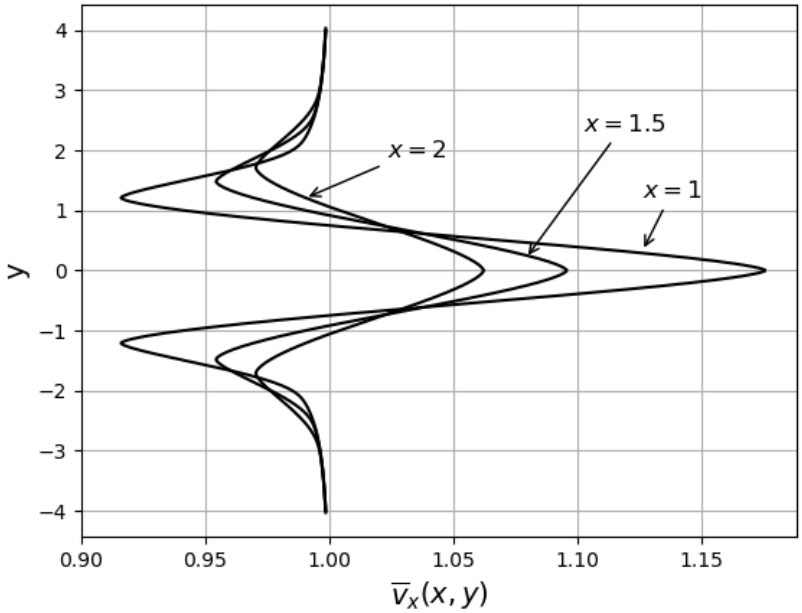

FIGURE 6.1: The x-component of the mean velocity $\overline{v}_x(x,y)$ plotted against y at $x = 1$, $x = 1.5$ and $x = 2$ when $\nu = 0.01$.

Figure 6.2 shows the plots for the mean temperature difference profiles when $\nu = 0.01$ and $\kappa = 0.01$ at $x = 1$, $x = 1.5$ and $x = 2$. The graphs are plotted using

$$\overline{S}(x,y) = \frac{1}{x^{1/2}}G(\xi), \tag{6.10.2}$$

where $G(\xi)$ is obtained from the numerical solution described in Section 6.7. We see that as the distance from the obstacle, x, increases the mean temperature difference at the centreline, $\overline{S}(x,0)$, decreases. This shows that turbulent thermal diffusion increases with the distance from the obstacle, x. This behaviour was observed for the classical turbulent thermal wake. The flow becomes more turbulent as the distance from the obstacle increases and the thermal diffusion in the mean flow increases with turbulence.

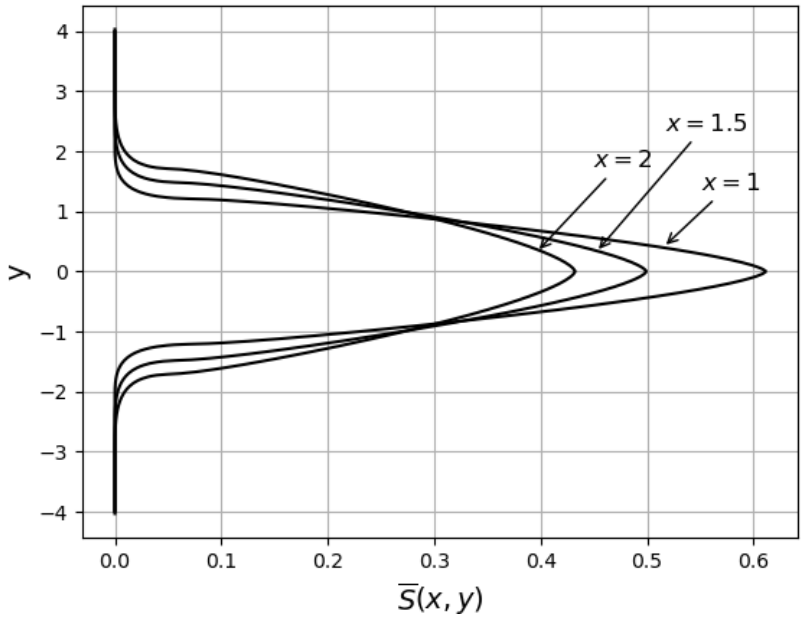

FIGURE 6.2: The mean temperature difference $\overline{S}(x, y)$ plotted against y at $x = 1$, $x = 1.5$ and $x = 2$ when $\nu = 0.01$ and $\kappa = 0.01$.

Figure 6.3 shows the plots for the mean velocity deficit profile at $x = 1$ for various values of kinematic viscosity. The graphs are plotted using

$$\overline{v}_x(x, y) = U - \frac{1}{x^{3/2}} \frac{dF}{d\xi}, \tag{6.10.3}$$

where $\frac{dF}{d\xi}$ is obtained using the numerical scheme described in Section 6.7. We see that as the kinematic viscosity increases the mean velocity at the centreline, $\overline{v}_x(1, 0)$, decreases and the profile becomes narrower along the x-axis. Since any numerical scheme is limited by its tolerance we cannot definitively discern whether the momentum boundary is finite using the numerical results.

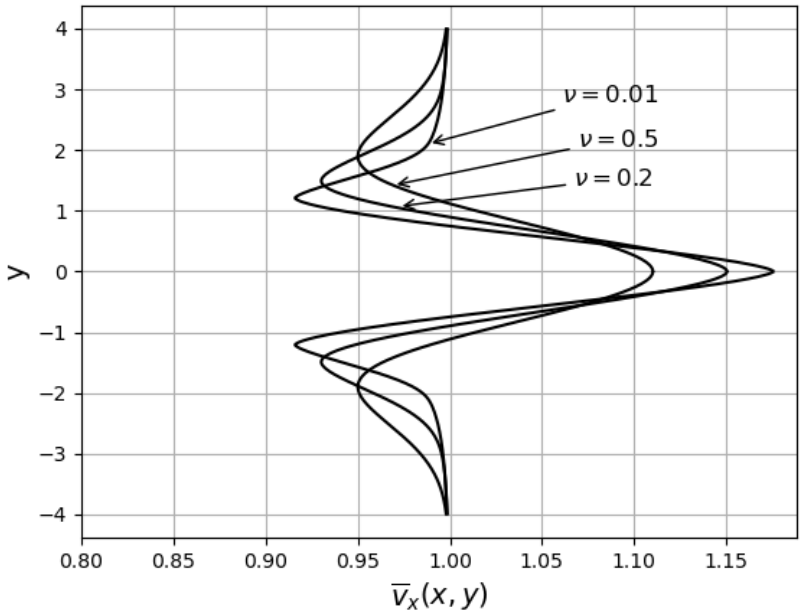

FIGURE 6.3: The x-component of the mean velocity $\overline{v}_x(x, y)$ plotted against y for $\nu = 0.01$, $\nu = 0.2$ and $\nu = 0.5$ at $x = 1$.

Figure 6.4 shows the plots for the mean temperature difference profile at $x = 1$ for various values of kinematic viscosity and thermal conductivity. The graphs are plotted using

$$\overline{S}(x, y) = \frac{1}{x^{1/2}} G(\xi), \tag{6.10.4}$$

where $G(\xi)$ is obtained using the numerical scheme described in Section 6.7. We see that as the kinematic viscosity and thermal conductivity increase the mean temperature difference at the centreline, $\overline{S}(1, 0)$, decreases. Since any numerical scheme is limited by its tolerance we cannot definitively discern whether the thermal boundary is finite using the numerical results.

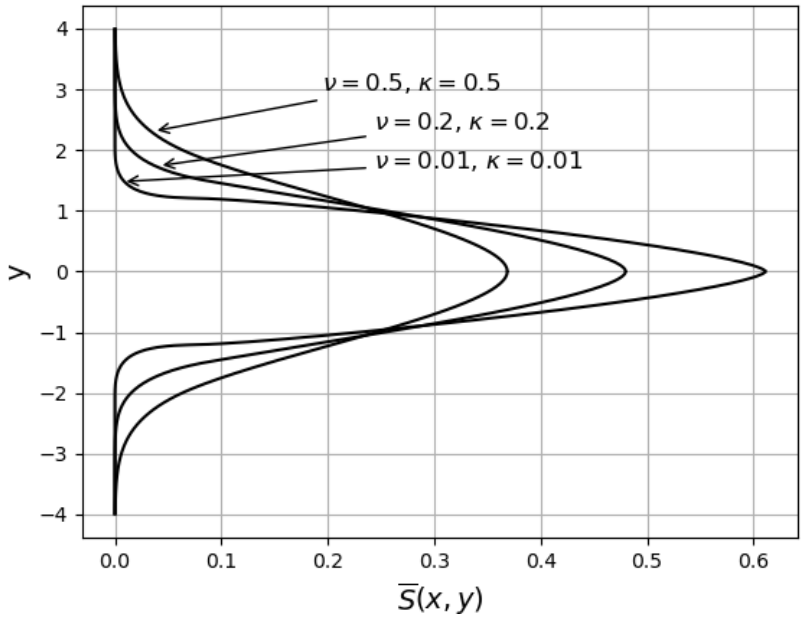

FIGURE 6.4: The mean temperature difference $\overline{S}(x, y)$ plotted against y for $\nu = 0.01$, $\kappa = 0.01$; $\nu = 0.2$, $\kappa = 0.2$ and $\nu = 0.5$, $\kappa = 0.5$ at $x = 1$.

Figure 6.5 shows the comparison of the numerical results for the x-component of the mean velocity for ν tending to zero in the solution for $\nu \neq 0$ with the numerical results for the x-component of the mean velocity when ν vanishes. The plot for the numerical solution for ν tending to zero in the solution for $\nu \neq 0$ is obtained from

$$\overline{v}_x(x, y) = U - \frac{1}{x^{3/2}} \frac{dF}{d\xi}, \tag{6.10.5}$$

where $\frac{dF}{d\xi}$ is obtained from the numerical solution described in Section 6.8. The plot for the numerical solution for vanishing ν is obtained from

$$\overline{v}_x(x, y) = U - \frac{1}{x^{3/4}} \frac{dF}{d\xi}, \tag{6.10.6}$$

where $\frac{dF}{d\xi}$ is obtained from the numerical solution described in Section 6.9. The numerical solutions differ close to the centreline of the wake but begin to converge when $y \approx 1.5$.

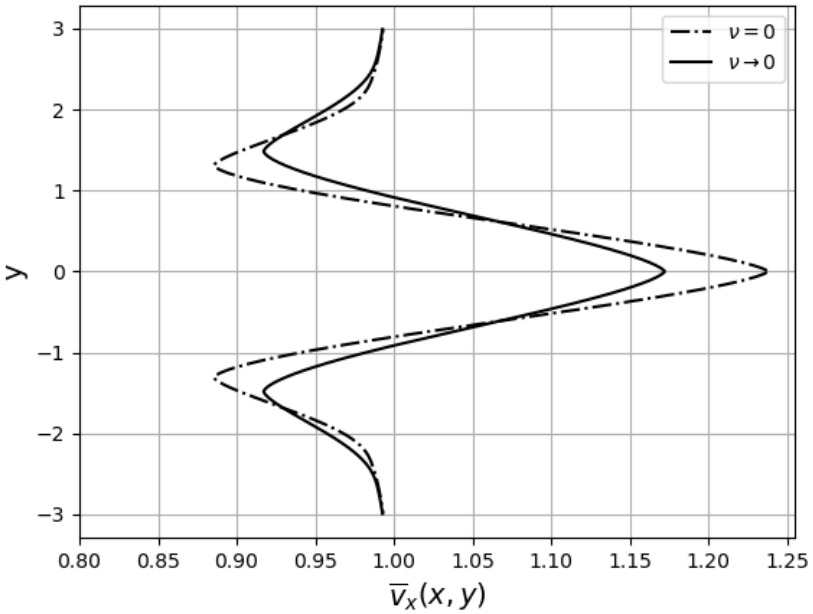

FIGURE 6.5: Graphs comparing the numerical results for the x-component of the mean velocity for ν tending to zero in the solution for $\nu \neq 0$ with the numerical results for the x-component of the mean velocity when ν vanishes. The profiles are plotted at $x = 1$.

Figure 6.6 shows the comparison of the numerical results for the mean temperature difference for ν and κ tending to zero in the solution for $\nu \neq 0$ and $\kappa \neq 0$ with the numerical results for the mean temperature difference when ν and κ vanish. The plot for the numerical solution for ν and κ tending to zero in the solution for $\nu \neq 0$ and $\kappa \neq 0$ is obtained from

$$\overline{S}(x, y) = \frac{1}{x^{1/2}} G(\xi), \tag{6.10.7}$$

where $G(\xi)$ is obtained from the numerical solution described in Section 6.9. The plot for the numerical solution for vanishing ν and κ is obtained from

$$\overline{S}(x, y) = \frac{1}{x^{1/4}} G(\xi), \tag{6.10.8}$$

where $\frac{dF}{d\xi}$ is obtained from the numerical solution described in Section 6.10. The numerical solutions differ close to the centreline of the wake due to the different dependence on x but begin to converge when $y \approx 0.5$.

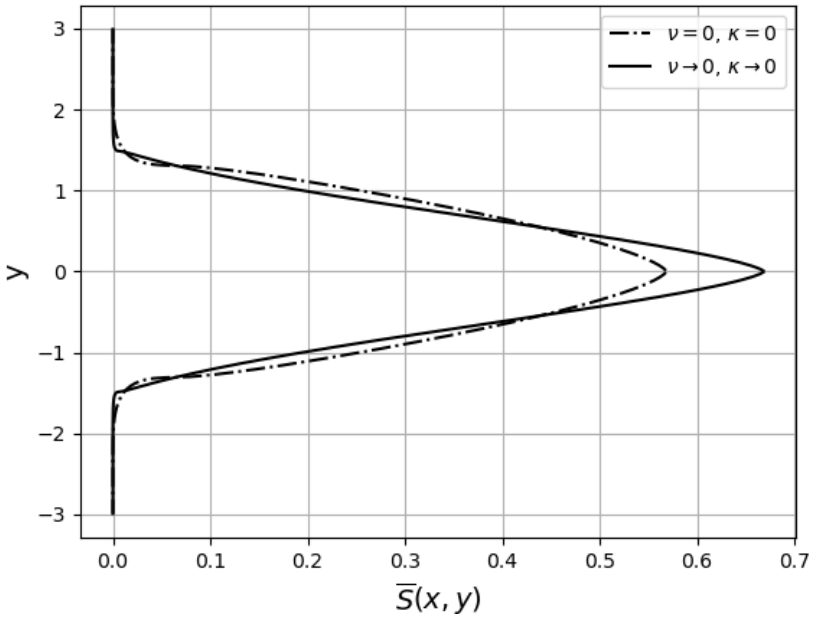

FIGURE 6.6: Graphs comparing the numerical results for the mean temperature difference for ν and κ tending to zero in the solution for $\nu \neq 0$ and $\kappa \neq 0$ with the numerical results for the mean temperature difference when ν and κ vanish. The profiles are plotted at $x = 1$.

6.11 Conclusions

The defining property of the momentumless wake is that the drag on a self-propelled body is zero. We derived the conservation laws for the momentumless thermal wake with eddy viscosity and eddy thermal conductivity modelled by the mixing lengths. The conserved vectors and conservation laws were derived using a direct method. We integrated the conservation laws across the wake, with boundary conditions, to obtain conserved quantities and used the conserved vectors to derive the associated Lie point symmetry. The Lie point symmetry was used to reduce the governing partial differential equations to ordinary differential equations. The ordinary differential equations obtained when the kinematic viscosity and thermal conductivity tend to zero differed from the ordinary differential equations obtained when the kinematic viscosity and thermal conductivity vanish. We compared the numerical solutions for the mean velocity and mean temperature difference when the kinematic viscosity and thermal conductivity tend to zero and when the kinematic viscosity and thermal conductivity vanish. The results differed significantly close to the centreline of the wake, but began to agree close to the effective boundary of the wake. The effective momentum and thermal diffusions increase with the distance from the obstacle. This is expected since turbulence increases as we move farther from the obstacle. The momentum boundary and thermal boundary appear to be infinite for all nonzero values of the kinematic viscosity and thermal conductivity. When $\nu = 0$ and $\kappa = 0$ the momentum and thermal boundaries are the same since Prandtl's mixing length for eddy viscosity was used and appear to be finite. The numerical solution however cannot be used to discern whether the boundaries are finite, since it is limited by its tolerance.

The kinematic viscosity and thermal conductivity play a significant role in the mathematical analysis for the momentumless thermal wake that is modelled by the mixing lengths. When the kinematic viscosity and thermal conductivity are nonzero the mixing lengths were derived without imposing Prandtl's hypothesis. However, the graphs show that when $\nu \neq 0$ and $\kappa \neq 0$ the effective width of the wakes as $\nu \to 0$ and $\kappa \to 0$ are approximately equal to the half-width of the wake when $\nu = 0$ and $\kappa = 0$ derived assuming Prandtl's hypothesis.

Chapter 7

Momentumless wake with eddy viscosity and conductivity dependent on spatial variables

7.1 Introduction

Hutchinson and Mason [12] used Lie point symmetry methods to analyze a wake behind a self-propelled body. They considered an eddy viscosity that dependent on spatial variables x and y. In this Chapter, we extend their results to the momentumless thermal wake.

In Chapter 6 we successfully applied a direct method to obtain conservation laws and conserved vectors for a momentumless thermal wake. Here, we apply a direct method to obtain conservation laws and conserved vectors for the momentumless thermal wake under consideration. We integrate the conservation laws across the wake flow, with boundary conditions, to obtain conserved quantities. We use the conserved vectors to derive the associated Lie point symmetry and use the Lie symmetry to reduce the governing partial differential equations to ordinary differential equations. The ordinary differential equations can be solved analytically. We show that when the eddy thermal diffusivity of the flow is dependent on x only, the effective thermal diffusivity is proportional to the effective kinematic viscosity.

This Chapter is organized in the following manner. In Section 7.2, we derive the conservation laws and conserved vectors for the momentumless thermal wake. In Section 7.3, the conservation laws are integrated across the wake flow to obtain the conserved quantities for the momentumless thermal wake. In Section 7.4, the Lie point symmetry associated with the momentum and thermal flux conserved vectors are derived. In Section 7.5, we use the Lie symmetry to reduce the governing partial differential equations to ordinary differential equations for the invariant solution and in Section 7.6, the results of the Lie symmetry analysis are summarized. In Section 7.7, we derive the analytical solution for a general eddy kinematic viscosity and eddy thermal diffusivity and in Section 7.8, the analytical solution for a power-law eddy kinematic viscosity and a power-law eddy thermal diffusivity are obtained. In Section 7.9, we present the results for the momentumless thermal wake. Finally, in Section 7.10, we provide conclusions for the Chapter.

7.2 Conservation laws

In this Chapter, we assume that the effective kinematic viscosity

$$E = \frac{1}{\rho}\left(\mu + \mu_T\right) = \nu + \nu_T \tag{7.2.1}$$

and the effective diffusivity

$$E_\theta = \frac{1}{\rho c_v}\left(\kappa + \kappa_T\right) \tag{7.2.2}$$

depend on the spatial variables x and y. The partial differential equations (3.12.1) and (3.12.2) governing the mean velocity deficit and mean temperature difference profiles for the classical and momentumless thermal wakes become

$$U\frac{\partial^2\psi}{\partial x\partial y} = \frac{\partial}{\partial y}\left[E(x,y)\frac{\partial^2\psi}{\partial y^2}\right], \tag{7.2.3}$$

$$U\frac{\partial\overline{S}}{\partial x} = \frac{\partial}{\partial y}\left[E_\theta(x,y)\frac{\partial\overline{S}}{\partial y}\right]. \tag{7.2.4}$$

Rewriting the governing partial differential equations (7.2.3) and (7.2.4) in terms of the subscript notation we have

$$U\psi_{xy} = \frac{\partial E}{\partial y}(x,y)\psi_{yy} + E(x,y)\psi_{yyy}, \tag{7.2.5}$$

$$U\overline{S}_x = \frac{\partial E_\theta}{\partial y}(x,y)\overline{S}_y + E_\theta(x,y)\overline{S}_{yy}. \tag{7.2.6}$$

We use the direct method to find the conserved vectors of the governing partial differential equations (7.2.5) and (7.2.6). Assume that the conserved vectors are of the form

$$T^1 = T^1(y,\psi,\psi_y,\overline{S}), \qquad T^2 = T^2(x,y,\psi,\psi_y,\psi_{yy},\overline{S}_y). \tag{7.2.7}$$

The form for the conserved vector components T^1 and T^2 are motivated by the calculation of conserved vectors in [12] and the experience from calculating conserved vectors in Chapter 6. The conserved vectors satisfy the conservation law

$$D_xT^1 + D_yT^2\Big|_{(7.2.5),(7.2.6)} = 0, \tag{7.2.8}$$

where the total derivatives are defined by

$$D_x = \frac{\partial}{\partial x} + \psi_x\frac{\partial}{\partial\psi} + \overline{S}_x\frac{\partial}{\partial\overline{S}} + \psi_{xx}\frac{\partial}{\partial\psi_x} + \psi_{yx}\frac{\partial}{\partial\psi_y} + \overline{S}_{xx}\frac{\partial}{\partial\overline{S}_x} + \overline{S}_{xy}\frac{\partial}{\partial\overline{S}_y} + \cdots, \tag{7.2.9}$$

$$D_y = \frac{\partial}{\partial y} + \psi_y\frac{\partial}{\partial\psi} + \overline{S}_y\frac{\partial}{\partial\overline{S}} + \psi_{xy}\frac{\partial}{\partial\psi_x} + \psi_{yy}\frac{\partial}{\partial\psi_y} + \overline{S}_{xy}\frac{\partial}{\partial\overline{S}_x} + \overline{S}_{yy}\frac{\partial}{\partial\overline{S}_y} + \cdots. \tag{7.2.10}$$

Expanding (7.2.8) using (7.2.9) and (7.2.10) we obtain

$$\psi_x \frac{\partial T^1}{\partial \psi} + \psi_{xy} \frac{\partial T^1}{\partial \psi_y} + \overline{S}_x \frac{\partial T^1}{\partial \overline{S}} + \frac{\partial T^2}{\partial y} + \psi_y \frac{\partial T^2}{\partial \psi} + \psi_{yy} \frac{\partial T^2}{\partial \psi_y} + \psi_{yyy} \frac{\partial T^2}{\partial \psi_{yy}} + \overline{S}_{yy} \frac{\partial T^2}{\partial \overline{S}_y} \Bigg|_{(7.2.5),(7.2.6)} = 0. \tag{7.2.11}$$

Substituting ψ_{xy} and $\overline{S}_x$ from (7.2.5) and (7.2.6) into (7.2.11) we obtain

$$\begin{gathered} \psi_x \frac{\partial T^1}{\partial \psi} + \frac{1}{U}\left(\frac{\partial E}{\partial y}(x,y)\psi_{yy} + E(x,y)\psi_{yyy}\right)\frac{\partial T^1}{\partial \psi_y} + \\ \frac{1}{U}\left(\frac{\partial E_\theta}{\partial y}(x,y)\overline{S}_y + E_\theta(x,y)\overline{S}_{yy}\right)\frac{\partial T^1}{\partial \overline{S}} + \\ \frac{\partial T^2}{\partial y} + \psi_y \frac{\partial T^2}{\partial \psi} + \psi_{yy}\frac{\partial T^2}{\partial \psi_y} + \psi_{yyy}\frac{\partial T^2}{\partial \psi_{yy}} + \overline{S}_{yy}\frac{\partial T^2}{\partial \overline{S}_y} = 0. \end{gathered} \tag{7.2.12}$$

Splitting equation (7.2.12) by powers of ψ_{yyy} we obtain

$$\psi_{yyy} : \frac{E(x,y)}{U}\frac{\partial T^1}{\partial \psi_y} + \frac{\partial T^2}{\partial \psi_{yy}} = 0, \tag{7.2.13}$$

$$\text{Remainder: } \psi_x \frac{\partial T^1}{\partial \psi} + \frac{1}{U}\frac{\partial E}{\partial y}(x,y)\psi_{yy}\frac{\partial T^1}{\partial \psi_y} + \frac{1}{U}\left(\frac{\partial E_\theta}{\partial y}(x,y)\overline{S}_y + E_\theta(x,y)\overline{S}_{yy}\right)\frac{\partial T^1}{\partial \overline{S}} + \frac{\partial T^2}{\partial y} + \psi_y \frac{\partial T^2}{\partial \psi} + \psi_{yy}\frac{\partial T^2}{\partial \psi_y} + \overline{S}_{yy}\frac{\partial T^2}{\partial \overline{S}_y} = 0. \tag{7.2.14}$$

Differentiating (7.2.13) with respect to ψ_{yy} we obtain

$$\frac{\partial^2 T^2}{\partial \psi_{yy}^2} = 0. \tag{7.2.15}$$

Therefore

$$T^2 = A(x,y,\psi,\psi_y,\overline{S}_y)\psi_{yy} + B(x,y,\psi,\psi_y,\overline{S}_y), \tag{7.2.16}$$

where A and B are arbitrary functions of x, y, ψ, ψ_y and $\overline{S}_y$. Substituting (7.2.16) into (7.2.14) we obtain

$$\begin{gathered} \psi_x \frac{\partial T^1}{\partial \psi} + \frac{1}{U}\frac{\partial E}{\partial y}(x,y)\psi_{yy}\frac{\partial T^1}{\partial \psi_y} + \frac{1}{U}\left(\frac{\partial E_\theta}{\partial y}(x,y)\overline{S}_y + E_\theta \overline{S}_{yy}\right)\frac{\partial T^1}{\partial \overline{S}} + \frac{\partial A}{\partial y}\psi_{yy} + \frac{\partial B}{\partial y} \\ + \psi_y\left(\frac{\partial A}{\partial \psi} + \frac{\partial B}{\partial \psi}\right) + \psi_{yy}\left(\frac{\partial A}{\partial \psi_y}\psi_{yy} + \frac{\partial B}{\partial \psi_y}\right) + \overline{S}_{yy}\left(\frac{\partial A}{\partial \overline{S}_y}\psi_{yy} + \frac{\partial B}{\partial \overline{S}_y}\right) = 0. \end{gathered} \tag{7.2.17}$$

The functions in (7.2.17) do not depend on ψ_x, ψ_{yy} and $\overline{S}_{yy}$. Hence, separating (7.2.17) by independent powers of ψ_x, ψ_{yy} and $\overline{S}_{yy}$ we obtain

$$\psi_x : \frac{\partial T^1}{\partial \psi} = 0, \tag{7.2.18}$$

$$\psi_{yy}^2 : \frac{\partial A}{\partial \psi_y} = 0, \tag{7.2.19}$$

$$\psi_{yy}\overline{S}_{yy} : \frac{\partial A}{\partial \overline{S}_y} = 0, \tag{7.2.20}$$

$$\psi_{yy} : \frac{1}{U}\frac{\partial E}{\partial y}(x,y)\frac{\partial T^1}{\partial \psi_y} + \frac{\partial A}{\partial y} + \psi_y \frac{\partial A}{\partial \psi} + \frac{\partial B}{\partial \psi_y} = 0, \tag{7.2.21}$$

$$\overline{S}_{yy} : \frac{E_\theta(x,y)}{U}\frac{\partial T^1}{\partial \overline{S}} + \frac{\partial B}{\partial \overline{S}_y} = 0 \tag{7.2.22}$$

$$\text{Remainder: } \frac{1}{U}\frac{\partial E_\theta}{\partial y}(x,y)\frac{\partial T^1}{\partial \overline{S}}\overline{S}_y + \frac{\partial B}{\partial y} + \psi_y \frac{\partial B}{\partial \psi} = 0. \tag{7.2.23}$$

By (7.2.19) and (7.2.20)

$$A = A(x,y,\psi) \tag{7.2.24}$$

and T^2 becomes

$$T^2 = A(x,y,\psi)\psi_{yy} + B(x,y,\psi,\psi_y,\overline{S}_y). \tag{7.2.25}$$

Substituting (7.2.25) into (7.2.13) we obtain

$$\frac{\partial T^1}{\partial \psi_y} = -\frac{U}{E(x,y)}A(x,y,\psi). \tag{7.2.26}$$

But T^1 is independent of ψ by (7.2.18). Hence differentiating (7.2.26) with respect to ψ we see that

$$A = A(x,y). \tag{7.2.27}$$

Integration (7.2.26) with respect to ψ_y we obtain

$$T^1(y,\psi_y,\overline{S}) = -\frac{U}{E(x,y)}A(x,y)\psi_y + C(y,\overline{S}), \tag{7.2.28}$$

where C is an arbitrary function of y and $\overline{S}$. Since T^1 is independent of x

$$\frac{\partial}{\partial x}\left(\frac{A(x,y)}{E(x,y)}\right) = 0. \tag{7.2.29}$$

Substituting (7.2.28) into (7.2.21) we obtain

$$-\frac{1}{E(x,y)}\frac{\partial E}{\partial y}(x,y)A(x,y) + \frac{\partial A}{\partial y} = -\frac{\partial B}{\partial \psi_y}. \tag{7.2.30}$$

Multiplying (7.2.30) by $\frac{1}{E(x,y)}$ we obtain

$$-\frac{1}{E^2(x,y)}\frac{\partial E}{\partial y}(x,y)A(x,y) + \frac{1}{E(x,y)}\frac{\partial A}{\partial y} = -\frac{1}{E(x,y)}\frac{\partial B}{\partial \psi_y} \tag{7.2.31}$$

and this simplifies to

$$\frac{\partial}{\partial y}\left(\frac{A(x,y)}{E(x,y)}\right) = -\frac{1}{E(x,y)}\frac{\partial B}{\partial \psi_y}. \tag{7.2.32}$$

Substituting (7.2.28) into (7.2.22) we obtain

$$\frac{\partial C}{\partial \overline{S}}(y,\overline{S}) = -\frac{U}{E_\theta(x,y)}\frac{\partial B}{\partial \overline{S}_y}(x,y,\psi,\psi_y,\overline{S}_y). \tag{7.2.33}$$

Equation (7.2.33) is consistent only when the left hand and right hand sides are equal to a function of y. That is,

$$\frac{\partial C}{\partial \overline{S}} = F(y), \qquad \frac{\partial B}{\partial \overline{S}_y} = -\frac{E_\theta(x,y)}{U}F(y), \tag{7.2.34}$$

where F is an arbitrary function of y. Integrating both equations in (7.2.34) we obtain

$$C(y,\overline{S}) = F(y)\overline{S} + G(y), \tag{7.2.35}$$

$$B(x,y,\psi,\psi_y,\overline{S}_y) = -\frac{E_\theta(x,y)}{U}F(y)\overline{S}_y + H(x,y,\psi,\psi_y), \tag{7.2.36}$$

where G is an arbitrary function of y and H is an arbitrary function of x, y, ψ and ψ_y. Substituting (7.2.28) into (7.2.23) we obtain

$$\frac{1}{U}\overline{S}_y\frac{\partial E_\theta}{\partial y}(x,y)\frac{\partial C}{\partial \overline{S}}(y,\overline{S}) + \frac{\partial B}{\partial y} + \psi_y\frac{\partial B}{\partial \psi} = 0. \tag{7.2.37}$$

Substituting (7.2.35) and (7.2.36) into (7.2.37) we obtain

$$\frac{1}{U}\frac{\partial E_\theta}{\partial y}(x,y)F(y)\overline{S}_y - \frac{1}{U}\frac{\partial}{\partial y}\left(E_\theta(x,y)F(y)\right)\overline{S}_y + \frac{\partial H}{\partial y} + \psi_y\frac{\partial H}{\partial \psi} = 0. \tag{7.2.38}$$

Since H is independent of $\overline{S}_y$ the coefficient of $\overline{S}_y$ in (7.2.38) must equal to zero. Hence

$$\frac{\partial E_\theta}{\partial y}(x,y)F(y) - \frac{\partial}{\partial y}\left(E_\theta(x,y)F(y)\right) = 0 \tag{7.2.39}$$

and this simplifies to

$$-\frac{dF}{dy}E_\theta(x,y) = 0. \tag{7.2.40}$$

But in thermal turbulent flow, the effective thermal diffusivity, $E_\theta(x,y)$, does not vanish except at the thermal boundary of the wake. Hence $\frac{dF}{dy}$ must vanish and

$$F = c_2, \tag{7.2.41}$$

where c_2 is a constant. The remainder of (7.2.38) is

$$\frac{\partial H}{\partial y} + \psi_y\frac{\partial H}{\partial \psi} = 0. \tag{7.2.42}$$

Substituting (7.2.36) into (7.2.32) we obtain

$$\frac{\partial}{\partial y}\left(\frac{A(x,y)}{E(x,y)}\right) = -\frac{1}{E(x,y)}\frac{\partial H}{\partial \psi_y}. \tag{7.2.43}$$

Integrating (7.2.43) with respect to ψ_y we obtain

$$H(x,y,\psi,\psi_y) = -E(x,y)\frac{\partial}{\partial y}\left(\frac{A(x,y)}{E(x,y)}\right)\psi_y + g(x,y,\psi), \tag{7.2.44}$$

where g is an arbitrary function of x, y and ψ. Substituting (7.2.44) into (7.2.42) we obtain

$$-\frac{\partial}{\partial y}\left[E(x,y)\frac{\partial}{\partial y}\left(\frac{A(x,y)}{E(x,y)}\right)\right]\psi_y+\frac{\partial g}{\partial y}+\psi_y\frac{\partial g}{\partial \psi}=0. \tag{7.2.45}$$

Splitting (7.2.45) by the coefficient of ψ_y and the remainder we obtain

$$\psi_y:\frac{\partial g}{\partial \psi}-\frac{\partial}{\partial y}\left[E(x,y)\frac{\partial}{\partial y}\left(\frac{A(x,y)}{E(x,y)}\right)\right]=0, \tag{7.2.46}$$

$$\text{Remainder: } \frac{\partial g}{\partial y}=0. \tag{7.2.47}$$

Equation (7.2.47) gives

$$g=g(x,\psi). \tag{7.2.48}$$

Differentiating (7.2.46) with respect to y we obtain

$$\frac{\partial^2}{\partial y^2}\left[E(x,y)\frac{\partial}{\partial y}\left(\frac{A(x,y)}{E(x,y)}\right)\right]=0. \tag{7.2.49}$$

Integrating (7.2.46) with respect to ψ we obtain

$$g(x,\psi)=\frac{\partial}{\partial y}\left[E(x,y)\frac{\partial}{\partial y}\left(\frac{A(x,y)}{E(x,y)}\right)\right]\psi+b(x), \tag{7.2.50}$$

where b is an arbitrary function of x.

In summary, we have

$$T^1=-\frac{U}{E(x,y)}A(x,y)\psi_y+c_2\overline{S}+G(y), \tag{7.2.51}$$

$$\begin{aligned}T^2=A(x,y)\psi_{yy}-E(x,y)\frac{\partial}{\partial y}\left(\frac{A(x,y)}{E(x,y)}\right)\psi_y+\frac{\partial}{\partial y}\left[E(x,y)\frac{\partial}{\partial y}\left(\frac{A(x,y)}{E(x,y)}\right)\right]\psi\\ +b(x)-c_2\frac{E_\theta(x,y)}{U}\overline{S}_y,\end{aligned} \tag{7.2.52}$$

provided that

$$\frac{\partial^2}{\partial y^2}\left[E(x,y)\frac{\partial}{\partial y}\left(\frac{A(x,y)}{E(x,y)}\right)\right]=0 \tag{7.2.53}$$

and

$$\frac{\partial}{\partial x}\left(\frac{A(x,y)}{E(x,y)}\right)=0. \tag{7.2.54}$$

The conserved vector components T^1 and T^2 in (7.2.51) and (7.2.52) satisfy equation (7.2.8) provided that equations (7.2.53) and (7.2.54) are satisfied.

From (7.2.54)

$$\frac{A(x,y)}{E(x,y)}=-f(y) \tag{7.2.55}$$

and hence

$$A(x,y)=-f(y)E(x,y). \tag{7.2.56}$$

The conserved vector components (7.2.51) and (7.2.52) become

$$T^1 = Uf(y)\psi_y + c\overline{S} + G(y), \tag{7.2.57}$$

$$T^2 = -f(y)E(x,y)\psi_{yy} + E(x,y)\frac{df}{dy}\psi_y - \frac{\partial}{\partial y}\left[E(x,y)\frac{df}{dy}\right]\psi - c\frac{E_\theta(x,y)}{U}\overline{S}_y + b(x), \tag{7.2.58}$$

provided that

$$\frac{\partial^2}{\partial y^2}\left[E(x,y)\frac{df}{dy}\right] = 0. \tag{7.2.59}$$

The arbitrary functions $G(y)$ and $b(x)$ give trivial conserved vectors. Hence we let $G(y)$ and $b(x)$ vanish without loss of generality. When $f(y)$ is a constant we obtain a conserved vector for a classical wake and when $f(y)$ is proportional to y^2 we obtain a conserved vector for a momentumless wake (see [12]). Hence we let

$$f(y) = c_1 y^2, \tag{7.2.60}$$

where c_1 is a constant. Substituting (7.2.60) into (7.2.59) we obtain

$$\frac{\partial^2}{\partial y^2}\left[yE(x,y)\right] = 0. \tag{7.2.61}$$

Integrating (7.2.61) twice with respect to y and solving for $E(x,y)$ we obtain

$$E(x,y) = e_1(x) + \frac{e_2(x)}{y}, \tag{7.2.62}$$

where e_1 and e_2 are arbitrary functions of x. But the effective kinematic viscosity $E(x,y)$ should be finite at $y = 0$. Hence $e_2(x) = 0$ and

$$E(x,y) = e_1(x) = e(x). \tag{7.2.63}$$

We drop the subscript in $e_1(x)$ to simplify notation when performing calculations going forward. Substituting (7.2.60) and (7.2.63) into (7.2.51) and (7.2.52) we obtain

$$T^1 = c_1 U y^2 \psi_y + c_2 \overline{S}, \tag{7.2.64}$$

$$T^2 = -c_1 e(x) y^2 \psi_{yy} + 2c_1 e(x) y\psi_y - 2c_1 e(x)\psi - c_2\frac{E_\theta(x,y)}{U}\overline{S}_y. \tag{7.2.65}$$

When $c_1 = 1$ and $c_2 = 0$ we obtain the momentum flux conserved vector $T_M = (T^1, T^2)$ with components

$$\begin{aligned} T^1 &= Uy^2\psi_y, \\ T^2 &= -e(x)y^2\psi_{yy} + 2e(x)y\psi_y - 2e(x)\psi. \end{aligned} \tag{7.2.66}$$

When $c_1 = 0$ and $c_2 = 1$ we obtain the thermal flux conserved vector $T_\theta = (T^1, T^2)$ with components

$$\begin{aligned} T^1 &= \overline{S}, \\ T^2 &= -\frac{E_\theta(x,y)}{U}\overline{S}_y. \end{aligned} \tag{7.2.67}$$

7.3 Conserved quantities

We now treat $\psi(x,y)$ and $\overline{S}(x,y)$ as dependent variables that satisfy the system of governing partial differential equations (7.2.3), (7.2.4) and boundary conditions (3.12.7) to (3.12.8). The total derivatives in (7.2.8) become partial derivatives and

$$\frac{\partial T^1}{\partial x} + \frac{\partial T^2}{\partial y} = 0. \tag{7.3.1}$$

We integrate (7.3.1) with respect to y from the centre line to the boundary of the wake.

7.3.1 Conserved quantity for the momentum flux

The momentum flux conserved vector T_M has components, (7.2.66),

$$T^1 = Uy^2\frac{\partial \psi}{\partial y}, \tag{7.3.2}$$

$$T^2 = -e(x)y^2\frac{\partial^2 \psi}{\partial y^2} + 2e(x)y\frac{\partial \psi}{\partial y} - 2e(x)\psi. \tag{7.3.3}$$

Substituting (7.3.2) and (7.3.3) into (7.3.1) we obtain

$$U\frac{\partial}{\partial x}\left[y^2\frac{\partial \psi}{\partial y}\right] + \frac{\partial}{\partial y}\left[-e(x)y^2\frac{\partial^2 \psi}{\partial y^2} + 2e(x)y\frac{\partial \psi}{\partial y} - 2e(x)\psi\right] = 0. \tag{7.3.4}$$

Integrating (7.3.4) with respect to y from the centre line to the boundary of the wake $y = y_B(x)$ we obtain

$$U\int_0^{y_B(x)} \frac{\partial}{\partial x}\left[y^2\frac{\partial \psi}{\partial y}\right] dy + \left[-e(x)y^2\frac{\partial^2 \psi}{\partial y^2} + 2e(x)y\frac{\partial \psi}{\partial y} - 2e(x)\psi\right]\Bigg|_{y=0}^{y=y_B(x)} = 0. \tag{7.3.5}$$

The mean velocity deficit $\overline{w}(x,y)$ and mean velocity deficit gradient $\frac{\partial \overline{w}}{\partial y}$ vanish at the momentum boundary of the wake. Hence

$$\overline{w}(x, y_B(x)) = \frac{\partial \psi}{\partial y}(x, y_B(x)) = 0 \tag{7.3.6}$$

and

$$\frac{\partial \overline{w}}{\partial y}(x, y_B(x)) = \frac{\partial^2 \psi}{\partial y^2}(x, y_B(x)) = 0. \tag{7.3.7}$$

Equation (7.3.5) reduces to

$$U\int_0^{y_B(x)} \frac{\partial}{\partial x}\left[y^2\frac{\partial \psi}{\partial y}\right] dy - 2e(x)\psi(x,y)\Big|_0^{y_B(x)} = 0. \tag{7.3.8}$$

But for a momentumless wake, the drag is zero. That is,

$$D = 2\rho U\int_0^{y_B(x)} \frac{\partial \psi}{\partial y} dy = 2\rho U\psi(x,y)\Big|_0^{y_B(x)} = 0. \tag{7.3.9}$$

Hence equation (7.3.8) becomes

$$\int_0^{y_B(x)} \frac{\partial}{\partial x}\left[y^2\frac{\partial \psi}{\partial y}\right] dy = 0. \tag{7.3.10}$$

Using the theorem for differentiation under the integral sign [34] equation (7.3.10) becomes

$$\frac{d}{dx}\int_0^{y_B(x)} y^2\frac{\partial \psi}{\partial y}dy - \frac{dy_B}{dx}y_B^2(x)\frac{\partial \psi}{\partial y}(x, y_B(x)) = 0. \tag{7.3.11}$$

Applying the boundary condition (7.3.6) the second term in (7.3.11) vanishes. Equation (7.3.11) reduces to

$$\frac{d}{dx}\int_0^{y_B(x)} y^2\frac{\partial \psi}{\partial y}dy = 0. \tag{7.3.12}$$

Therefore

$$J = 2\rho U\int_0^{y_B(x)} y^2\frac{\partial \psi}{\partial y}dy, \tag{7.3.13}$$

is a constant independent of x.

7.3.2 Conserved quantity for the thermal flux

The thermal flux conserved vector T_θ has components, (7.2.67),

$$T^1 = \overline{S}, \tag{7.3.14}$$

$$T^2 = -\frac{E_\theta(x,y)}{U}\frac{\partial \overline{S}}{\partial y}. \tag{7.3.15}$$

Substituting (7.3.14) and (7.3.15) into (7.3.1) we obtain

$$\frac{\partial \overline{S}}{\partial x} + \frac{\partial}{\partial y}\left[-\frac{E_\theta(x,y)}{U}\frac{\partial \overline{S}}{\partial y}\right] = 0. \tag{7.3.16}$$

Integrating (7.3.16) with respect to y from the centre line, $y = 0$, to the thermal boundary, $y = y_{B_\theta}(x)$, we obtain

$$\int_0^{y_{B_\theta}(x)} \frac{\partial \overline{S}}{\partial x}(x,y)dy + \left[-\frac{E_\theta(x,y)}{U}\frac{\partial \overline{S}}{\partial y}\right]\Bigg|_{y=0}^{y=y_{B_\theta}(x)} = 0. \tag{7.3.17}$$

At the thermal boundary of a momentumless wake the effective diffusivity $E_\theta(x,y)$ is $\frac{\kappa}{\rho c_v}$. Hence

$$\left[-\frac{E_\theta(x,y)}{U}\frac{\partial \overline{S}}{\partial y}\right]\Bigg|_{y=0}^{y=y_{B_\theta}(x)} = -\frac{\kappa}{\rho c_v}\frac{\partial \overline{S}}{\partial y}(x, y_{B_\theta}(x)) + E_\theta(x,0)\frac{\partial \overline{S}}{\partial y}(x,0). \tag{7.3.18}$$

Hence, by the boundary conditions in (3.12.8), the right hand side of (7.3.18) vanishes and the second term in (7.3.17) also vanishes. Equation (7.3.17) reduces to

$$\int_0^{y_{B_\theta}(x)} \frac{\partial \overline{S}}{\partial x}(x,y)dy = 0. \tag{7.3.19}$$

Applying the theorem of differentiation under the integral sign [34] to (7.3.19) we obtain

$$\frac{d}{dx}\int_0^{y_{B_\theta}(x)}\overline{S}(x,y)dy-\frac{dy_{B_\theta}}{dx}\overline{S}(x,y_{B_\theta}(x))=0. \tag{7.3.20}$$

Applying the second boundary condition in (3.12.8) the second term in (7.3.20) vanishes. Therefore (7.3.20) reduces to

$$\frac{d}{dx}\int_0^{y_{B_\theta}(x)}\overline{S}(x,y)dy=0 \tag{7.3.21}$$

and

$$K=2\rho c_v U\int_0^{y_{B_\theta}(x)}\overline{S}(x,y)dy \tag{7.3.22}$$

is a constant independent of x.

If the momentum boundary or the thermal boundary of the wake turns to be infinite then the conserved quantities (7.3.13) and (7.3.22) still apply with $y_B(x)=\infty$ or $y_{B_\theta}(x)=\infty$.

7.4 Associated Lie point symmetry

A vector field operator

$$X=\xi^1(x,y,\psi,\overline{S})\frac{\partial}{\partial x}+\xi^2(x,y,\psi,\overline{S})\frac{\partial}{\partial y}+\eta^1(x,y,\psi,\overline{S})\frac{\partial}{\partial\psi}+\eta^2(x,y,\psi,\overline{S})\frac{\partial}{\partial\overline{S}} \tag{7.4.1}$$

is associated with the conserved (T^1,T^2) provided

$$X^{[\infty]}(T^i)+T^iD_k(\xi^k)-T^kD_k(\xi^i)=0,\qquad\qquad i=1,2 \tag{7.4.2}$$

where $X^{[\infty]}$ is the prolongation of X up to all orders required to evaluate $X^{[\infty]}(T^1)$ and $X^{[\infty]}(T^2)$. When equation (7.4.2) is expanded for $i=1,2$ it gives

$$X^{[\infty]}(T^1)+T^1D_y(\xi^2)-T^2D_y(\xi^1)=0, \tag{7.4.3}$$

$$X^{[\infty]}(T^2)+T^2D_x(\xi^1)-T^1D_x(\xi^2)=0, \tag{7.4.4}$$

The conserved vectors (7.2.66) and (7.2.67) contain derivatives up to second order in ψ and first order in $\overline{S}$. Hence we only need to prolongate the Lie point symmetry X up to second order in ψ, first order in $\overline{S}$ and

$$X^{[\infty]}=\xi^1\frac{\partial}{\partial x}+\xi^2\frac{\partial}{\partial y}+\eta^1\frac{\partial}{\partial\psi}+\eta^2\frac{\partial}{\partial\overline{S}}+\zeta^1_y\frac{\partial}{\partial\psi_y}+\zeta^1_{yy}\frac{\partial}{\partial\psi_{yy}}+\zeta^2_y\frac{\partial}{\partial\overline{S}_y}. \tag{7.4.5}$$

The first order prolongations are

$$\zeta^\alpha_y=D_y(\eta^\alpha)-u^\alpha_kD_y(\xi^k), \tag{7.4.6}$$

the second order prolongation is

$$\zeta^1_{yy}=D_y(\zeta^1_y)-\psi_{yk}D_y(\xi^k) \tag{7.4.7}$$

and

$$x^1 = x, \qquad x^2 = y, \qquad u^1 = \psi, \qquad u^2 = \overline{S}.$$

7.4.1 Vector field associated with the momentum flux conserved vector

Expanding equation (7.4.3) with the conserved vector components T^1 and T^2 in (7.2.66) we obtain

$$\begin{gathered} 2yU\xi^2\psi_y + Uy^2\zeta^1_y + Uy^2\psi_y D_y\xi^2 \\ - \left[-e(x)y^2\psi_{yy} + 2e(x)y\psi_y - 2e(x)\psi\right] D_y\xi^1 = 0 \end{gathered} \tag{7.4.8}$$

By (7.4.6)

$$\zeta^1_y = D_y\eta^1 - \psi_x D_y\xi^1 - \psi_y D_y\xi^2. \tag{7.4.9}$$

Substituting (7.4.9) into (7.4.8) we obtain

$$\begin{gathered} 2yU\xi^2\psi_y + Uy^2\left[D_y\eta^1 - \psi_x D_y\xi^1 - \psi_y D_y\xi^2\right] + Uy^2\psi_y D_y\xi^2 - \\ \left[-e(x)y^2\psi_{yy} + 2e(x)y\psi_y - 2e(x)\psi\right] D_y\xi^1 = 0 \end{gathered} \tag{7.4.10}$$

Splitting (7.4.10) by independent powers of ψ_{yy} we obtain

$$\psi_{yy} : e(x)D_y\xi^1 = 0, \tag{7.4.11}$$

$$\begin{gathered} \text{Remainder: } 2yU\psi_y\xi^2 + Uy^2D_y\eta^1 - Uy^2\psi_x D_y\xi^1 - 2e(x)y\psi_y D_y\xi^1 \\ +2e(x)\psi D_y\xi^1 = 0. \end{gathered} \tag{7.4.12}$$

For turbulent flow the effective kinematic viscosity $e(x)$ never vanishes except at the boundary of the wake. Hence equation (7.4.11) becomes

$$D_y\xi^1 = \frac{\partial\xi^1}{\partial y} + \psi_y\frac{\partial\xi^1}{\partial\psi} + \overline{S}_y\frac{\partial\xi^1}{\partial\overline{S}} = 0. \tag{7.4.13}$$

Splitting (7.4.13) by independent powers of ψ_y and $\overline{S}_y$ we obtain

$$\psi_y : \frac{\partial\xi^1}{\partial\psi} = 0, \tag{7.4.14}$$

$$\overline{S}_y : \frac{\partial\xi^1}{\partial\overline{S}} = 0, \tag{7.4.15}$$

$$\text{Remainder: } \frac{\partial\xi^1}{\partial y} = 0. \tag{7.4.16}$$

Hence

$$\xi^1 = \xi^1(x). \tag{7.4.17}$$

Equation (7.4.12) reduces to

$$2y\psi_y\xi^2 + y^2D_y\eta^1 = 0. \tag{7.4.18}$$

Expanding the total derivative D_y in (7.4.18) we obtain

$$2y\psi_y\xi^2 + y^2\frac{\partial\eta^1}{\partial y} + y^2\psi_y\frac{\partial\eta^1}{\partial\psi} + y^2\overline{S}_y\frac{\partial\eta^1}{\partial\overline{S}} = 0 \tag{7.4.19}$$

Splitting (7.4.19) by independent powers of ψ_y and $\overline{S}_y$ we obtain

$$\psi_y : y\frac{\partial\eta^1}{\partial\psi} + 2\xi^2 = 0, \tag{7.4.20}$$

$$\overline{S}_y : \frac{\partial\eta^1}{\partial\overline{S}} = 0, \tag{7.4.21}$$

$$\text{Remainder: } \frac{\partial\eta^1}{\partial y} = 0. \tag{7.4.22}$$

Equations (7.4.21) and (7.4.22) give

$$\eta^1 = \eta^1(x,\psi). \tag{7.4.23}$$

Now expanding (7.4.4) with the conserved vector components T^1 and T^2 in (7.2.66) we obtain

$$\begin{aligned} &-\frac{de}{dx}y^2\xi^1\psi_{yy} + 2\frac{de}{dx}y\xi^1\psi_y - 2\frac{de}{dx}\xi^1\psi - 2e(x)y\xi^2\psi_{yy} \\ &+2e(x)\xi^2\psi_y - 2e(x)\eta^1 + 2e(x)y\zeta^1_y - e(x)y^2\zeta^1_{yy} \\ &+\left[-e(x)y^2\psi_{yy} + 2e(x)y\psi_y - 2e(x)\psi\right]\frac{d\xi^1}{dx} - Uy^2\psi_y D_x\xi^2 = 0. \end{aligned} \tag{7.4.24}$$

By (7.4.6)

$$\zeta^1_y = D_y\eta^1 - \psi_y D_y\xi^2 \tag{7.4.25}$$

and by (7.4.7)

$$\zeta^1_{yy} = D_y^2\eta^1 - \psi_y D_y^2\xi^2 - 2\psi_{yy}D_y\xi^2. \tag{7.4.26}$$

Substituting (7.4.25) and (7.4.26) into (7.4.24) we obtain

$$\begin{aligned} &-\frac{de}{dx}y^2\xi^1\psi_{yy} + 2\frac{de}{dx}y\xi^1\psi_y - 2\frac{de}{dx}\xi^1\psi - 2e(x)y\xi^2\psi_{yy} \\ &+2e(x)\xi^2\psi_y - 2e(x)\eta^1 + 2e(x)y\left[D_y\eta^1 - \psi_y D_y\xi^2\right] \\ &-e(x)y^2\left[D_y^2\eta^1 - \psi_y D_y^2\xi^2 - 2\psi_{yy}D_y\xi^2\right] \\ &+\left[-e(x)y^2\psi_{yy} + 2e(x)y\psi_y - 2e(x)\psi\right]\frac{d\xi^1}{dx} - Uy^2\psi_y D_x\xi^2 = 0. \end{aligned} \tag{7.4.27}$$

Upon inspection, we see that the last term of (7.4.27) is the only term that depends on $\psi_x\psi_y$ and $\overline{S}_x\psi_y$. When expanding the last term of (7.4.27) the coefficients of $\psi_x\psi_y$ and $\overline{S}_x\psi_y$ must be equal to zero. Hence

$$\psi_x\psi_y : \frac{\partial\xi^2}{\partial\psi} = 0, \tag{7.4.28}$$

$$\overline{S}_x\psi_y : \frac{\partial\xi^2}{\partial\overline{S}} = 0 \tag{7.4.29}$$

and therefore

$$\xi^2 = \xi^2(x, y). \tag{7.4.30}$$

Differentiating (7.4.20) with respect to ψ we obtain

$$\frac{\partial^2 \eta^1}{\partial \psi^2}(x, \psi) = 0. \tag{7.4.31}$$

Hence

$$\eta^1(x, \psi) = a_1(x)\psi + a_2(x), \tag{7.4.32}$$

where a_1 and a_2 are arbitrary functions of x. Substituting (7.4.32) into (7.4.20) we obtain

$$\xi^2 = -\frac{1}{2}a_1(x)y. \tag{7.4.33}$$

Equation (7.4.27) becomes

$$\begin{aligned} &-\frac{de}{dx}y^2\xi^1\psi_{yy} + 2\frac{de}{dx}y\xi^1\psi_y - 2\frac{de}{dx}\xi^1\psi + e(x)y^2a_1(x)\psi_{yy} \\ &-e(x)a_1(x)y\psi_y - 2e(x)\left[a_1(x)\psi + a_2(x)\right] + 3e(x)a_1(x)y\psi_y - 2e(x)y^2a_1(x) \\ &+\left[-e(x)y^2\psi_{yy} + 2e(x)y\psi_y - 2e(x)\psi\right]\frac{d\xi^1}{dx} + \frac{1}{2}Uy^3\psi_y\frac{da_1}{dx} = 0. \end{aligned} \tag{7.4.34}$$

Splitting (7.4.34) by independent powers of derivatives of ψ we obtain

$$\psi_y : 2y\frac{de}{dx}\xi^1(x) + 2ye(x)a_1(x) + 2ye(x)\frac{d\xi^1}{dx} + \frac{1}{2}Uy^3\frac{da_1}{dx} = 0, \tag{7.4.35}$$

$$\psi_{yy} : \frac{de}{dx}\xi^1(x) + e(x)a_1(x) + e(x)\frac{d\xi^1}{dx} = 0, \tag{7.4.36}$$

$$\text{Remainder: } \psi\left[\xi^1(x)\frac{de}{dx} + a_1(x)e(x) + e(x)\frac{d\xi^1}{dx}\right] + e(x)a_2(x) = 0. \tag{7.4.37}$$

Splitting (7.4.35) by independent powers of y we obtain

$$y^3 : \frac{da_1}{dx} = 0 \tag{7.4.38}$$

and

$$y : \frac{de}{dx}\xi^1(x) + e(x)a_1(x) + e(x)\frac{d\xi^1}{dx} = 0, \tag{7.4.39}$$

which is the same as (7.4.36). Therefore

$$a_1(x) = a_1, \tag{7.4.40}$$

where a_1 is a constant. By (7.4.36) and (7.4.37)

$$e(x)a_2(x) = 0. \tag{7.4.41}$$

Hence $a_2(x) = 0$ since $e(x) \neq 0$ for a turbulent wake. Equation (7.4.36) becomes

$$\frac{de}{dx}\xi^1(x) + e(x)\frac{d\xi^1}{dx} = -a_1e(x) \tag{7.4.42}$$

and therefore

$$\frac{d}{dx}\left(e(x)\xi^1(x)\right) = -a_1 e(x). \tag{7.4.43}$$

Integrating (7.4.43) and solving for ξ^1 we obtain

$$\xi^1(x) = -\frac{1}{e(x)}\left[a_1\int^x e(\tau)d\tau + a_3\right], \tag{7.4.44}$$

where a_3 is a constant.

In summary, the vector field associated with the conserved vector (7.2.66) is

$$X = -\frac{1}{e(x)}\left[a_1\int^x e(\tau)d\tau + a_3\right]\frac{\partial}{\partial x} - \frac{1}{2}a_1 y\frac{\partial}{\partial y} + a_1\psi\frac{\partial}{\partial\psi} + \eta^2(x,y,\psi,\overline{S})\frac{\partial}{\partial\overline{S}}. \tag{7.4.45}$$

7.4.2 Lie point symmetry associated with the momentum and thermal flux conserved vector

Expanding equation (7.4.3) with the conserved vector components T^1 and T^2 in (7.2.67) and the vector field operator (7.4.45) we obtain

$$\eta^2 = \frac{1}{2}a_1\overline{S}. \tag{7.4.46}$$

Expanding equation (7.4.4) with the conserved vector components T^1 and T^2 in (7.2.67) and the vector field operator (7.4.45), with (7.4.46), we obtain

$$-\frac{1}{U}\frac{\partial E_\theta}{\partial x}(x,y)\xi^1(x)\overline{S}_y - \frac{1}{U}\frac{\partial E_\theta}{\partial y}(x,y)\xi^2\overline{S}_y - \frac{E_\theta(x,y)}{U}\zeta^2_y - \frac{E_\theta(x,y)}{U}\overline{S}_y\frac{d\xi^1}{dx} = 0. \tag{7.4.47}$$

By (7.4.6)

$$\zeta^2_y = D_y\eta^2 - \overline{S}_x D_y\xi^1 - \overline{S}_y D_y\xi^2 = a_1\overline{S}_y. \tag{7.4.48}$$

Substituting (7.4.48) into (7.4.47) we obtain

$$-\frac{\partial E_\theta}{\partial x}(x,y)\xi^1(x) + \frac{1}{2}a_1 y\frac{\partial E_\theta}{\partial y}(x,y) - a_1 E_\theta(x,y) - E_\theta(x,y)\frac{d\xi^1}{dx} = 0. \tag{7.4.49}$$

Equation (7.4.49) can have solutions for $E_\theta(x,y)$ dependent on x and y. This might be subject of future studies. Some of the solutions for $E_\theta(x,y)$ might not even make physical sense. One of the physical requirements for $E_\theta(x,y)$ is that it must be finite as $y \to y_{B_\theta}(x)$. For this thesis, we consider E_θ that is a function of x and independent of y. Then (7.4.49) becomes

$$\frac{dE_\theta}{dx}\xi^1 + a_1 E_\theta(x) + E_\theta(x)\frac{d\xi^1}{dx} = 0. \tag{7.4.50}$$

From (7.4.42)

$$\frac{d\xi^1}{dx} = -a_1 - \frac{\xi^1(x)}{e(x)}\frac{de}{dx}. \tag{7.4.51}$$

Substituting (7.4.51) into (7.4.50) we obtain

$$\frac{1}{E_\theta(x)}\frac{dE_\theta}{dx} = \frac{1}{e(x)}\frac{de}{dx}. \tag{7.4.52}$$

Integrating (7.4.52) we obtain

$$\ln E_\theta(x) = \ln e(x) + C, \tag{7.4.53}$$

where C is a constant. Hence

$$E_\theta(x) = \alpha e(x), \tag{7.4.54}$$

where $\alpha = e^C$ is a positive constant. That is, the effective kinematic viscosity is proportional to the effective thermal diffusivity.

The vector field associated with the conserved vectors (7.2.66) and (7.2.67) is

$$X = -\frac{1}{e(x)}\left[a_1\int^x e(\tau)d\tau + a_3\right]\frac{\partial}{\partial x} - \frac{1}{2}a_1 y\frac{\partial}{\partial y} + a_1\psi\frac{\partial}{\partial \psi} + \frac{1}{2}a_1\overline{S}\frac{\partial}{\partial \overline{S}}. \tag{7.4.55}$$

In this thesis, we consider the general case $a_1 \neq 0$. The case $a_1 = 0$ might lead to more invariant solutions, but that is beyond the scope of the thesis. Without loss of generality we divide the Lie symmetry (7.4.55) by a_1. Hence we have

$$X = -\frac{1}{e(x)}\left[\int^x e(\tau)d\tau + a_3^*\right]\frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial y} + \psi\frac{\partial}{\partial \psi} + \frac{1}{2}\overline{S}\frac{\partial}{\partial \overline{S}}, \tag{7.4.56}$$

where $a_3^* = a_3/a_1$. To simplify notation, we omit the asterisks from here on.

We verify that the vector field operator (7.4.56) is a Lie point symmetry of the partial differential equations (7.2.3) and (7.2.4). Applying the prolongated vector field operator (7.4.56) to (7.2.5), subject to (7.2.5) being satisfied and $E(x,y) = e(x)$, we have

$$\begin{aligned} X^{[\infty]}\left(\frac{\partial E}{\partial y}(x,y)\psi_{yy} + E(x,y)\psi_{yyy} - U\psi_{xy}\right)\Big|_{(7.2.5)} &= e(x)\zeta^1_{yyy} \\ &+ \frac{de}{dx}\xi^1\psi_{yyy} - U\zeta^1_{xy}\Big|_{(7.2.5)}. \end{aligned} \tag{7.4.57}$$

Using the prolongation formulae (2.3.4) and (2.3.5), equation (7.4.57) becomes

$$\begin{aligned} X^{[\infty]}\left(\frac{\partial E}{\partial y}(x,y)\psi_{yy} + E(x,y)\psi_{yyy} - U\psi_{xy}\right)\Big|_{(7.2.5)} &= \frac{5}{2}e(x)\psi_{yyy} \\ &+ \frac{de}{dx}\xi^1\psi_{yyy} - U\left[\frac{3}{2} - \frac{d\xi^1}{dx}\right]\psi_{xy}\Big|_{(7.2.5)}. \end{aligned} \tag{7.4.58}$$

Substituting (7.2.5) into the right hand side of (7.4.58), equation (7.4.58) simplifies to

$$\begin{aligned} X^{[\infty]}\left(\frac{\partial E}{\partial y}(x,y)\psi_{yy} + E(x,y)\psi_{yyy} - U\psi_{xy}\right)\Big|_{(7.2.5)} &= e(x) \\ &+ \frac{d}{dx}\left(e(x)\xi^1(x)\right). \end{aligned} \tag{7.4.59}$$

Substituting $\xi^1(x)$ given by (7.4.44) into the right hand side of (7.4.59), we obtain

$$X^{[\infty]}\left(\frac{\partial E}{\partial y}(x,y)\psi_{yy} + E(x,y)\psi_{yyy} - U\psi_{xy}\right)\Big|_{(7.2.5)} = 0. \tag{7.4.60}$$

Applying the prolongated vector field operator (7.4.56) to (7.2.6), subject to (7.2.6) being satisfied and $E_\theta(x,y) = \alpha e(x)$, we have

$$X^{[\infty]}\left(\frac{\partial E_\theta}{\partial y}(x,y)\overline{S}_y + E_\theta(x,y)\overline{S}_{yy} - U\overline{S}_x\right)\Big|_{(7.2.6)} = \alpha e(x)\zeta^2_{yy} + \alpha\frac{de}{dx}\xi^1\overline{S}_{yy} - U\zeta^2_x\Big|_{(7.2.6)}. \tag{7.4.61}$$

Using the prolongation formulae (2.3.4) and (2.3.5), equation (7.4.61) becomes

$$X^{[\infty]}\left(\frac{\partial E_\theta}{\partial y}(x,y)\overline{S}_y + E_\theta(x,y)\overline{S}_{yy} - U\overline{S}_x\right)\Big|_{(7.2.6)} = \frac{3}{2}\alpha e(x)\overline{S}_{yy} + \alpha\frac{de}{dx}\xi^1\overline{S}_{yy} - U\left[\frac{1}{2} - \frac{d\xi^1}{dx}\right]\overline{S}_x\Big|_{(7.2.6)}. \tag{7.4.62}$$

Substituting (7.2.6) into the right hand side of (7.4.62), equation (7.4.62) simplifies to

$$X^{[\infty]}\left(\frac{\partial E_\theta}{\partial y}(x,y)\overline{S}_y + E_\theta(x,y)\overline{S}_{yy} - U\overline{S}_x\right)\Big|_{(7.2.6)} = \alpha\left[e(x) + \frac{d}{dx}\left(e(x)\xi^1\right)\right]\overline{S}_{yy}. \tag{7.4.63}$$

Substituting $\xi^1(x)$ given by (7.4.44) into the right hand side of (7.4.63), we obtain

$$X^{[\infty]}\left(\frac{\partial E_\theta}{\partial y}(x,y)\overline{S}_y + E_\theta(x,y)\overline{S}_{yy} - U\overline{S}_x\right)\Big|_{(7.2.6)} = 0. \tag{7.4.64}$$

Hence, the vector field operator (7.4.56) is a Lie point symmetry of the partial differential equations (7.2.3) and (7.2.4).

7.5 Invariant solution

The solution $\psi = f(x,y)$ and $\overline{S} = g(x,y)$ is an invariant of the governing partial differential equations (7.2.3) and (7.2.4) provided

$$X(\psi - f)\Big|_{\psi=f(x,y)} = 0, \tag{7.5.1}$$

$$X(\overline{S} - g)\Big|_{\overline{S}=g(x,y)} = 0, \tag{7.5.2}$$

where X is given by (7.4.56). Expanding equation (7.5.1) we obtain the partial differential equation

$$\left[-\frac{1}{e(x)}\left(\int^x e(\tau)d\tau + a_3\right)\right]\frac{\partial f}{\partial x} - \frac{1}{2}y\frac{\partial f}{\partial y} = f. \tag{7.5.3}$$

The differential equations of the characteristic curves of the partial differential equation (7.5.3) are

$$-\frac{e(x)dx}{\int^x e(\tau)d\tau + a_3} = -2\frac{dy}{y} = \frac{df}{f}. \tag{7.5.4}$$

Integrating the first and second terms of (7.5.4) we obtain

$$\ln y - \frac{1}{2}\ln\left(\int^x e(\tau)d\tau + a_3\right) = C, \tag{7.5.5}$$

where C is a constant. Exponentiating (7.5.5) we obtain

$$I_1 = \frac{y}{\left(\int^x e(\tau)d\tau + a_3\right)^{\frac{1}{2}}} \tag{7.5.6}$$

where $I_1 = e^C$ is a constant. Integrating the first and last terms of (7.5.4) we obtain

$$\ln f + \ln\left(\int^x e(\tau)d\tau + a_3\right) = C, \tag{7.5.7}$$

where C is a constant. Therefore

$$I_2 = \left(\int^x e(\tau)d\tau + a_3\right) f, \tag{7.5.8}$$

where $I_2 = e^C$ is a positive constant. The general solution of the first order partial differential equation (7.5.3) is

$$I_2 = F(I_1), \tag{7.5.9}$$

where F is an arbitrary function. Since $f = \psi(x, y)$,

$$\psi(x, y) = \frac{F(\xi)}{\int^x e(\tau)d\tau + a_3}, \tag{7.5.10}$$

where

$$\xi = \frac{y}{\left(\int^x e(\tau)d\tau + a_3\right)^{\frac{1}{2}}}. \tag{7.5.11}$$

Expanding equation (7.5.2) we obtain the partial differential equation

$$\left[-\frac{1}{e(x)}\left(\int^x e(\tau)d\tau + a_3\right)\right]\frac{\partial g}{\partial x} - \frac{1}{2}y\frac{\partial g}{\partial y} = \frac{1}{2}g. \tag{7.5.12}$$

The differential equations of the characteristic curves of the partial differential equation (7.5.12) are

$$-\frac{e(x)dx}{\int^x e(\tau)d\tau + a_3} = -2\frac{dy}{y} = 2\frac{dg}{g}. \tag{7.5.13}$$

We have already solved the equation consisting of the first and second terms of (7.5.13) and we obtained (7.5.6). Now integrating the equation consisting of the first and last terms of (7.5.13) we obtain

$$\ln g + \frac{1}{2}\ln\left(\int^x e(\tau)d\tau + a_3\right) = C, \tag{7.5.14}$$

where C is a constant. Therefore

$$I_3 = \left(\int^x e(\tau)d\tau + a_3\right)^{1/2} g, \tag{7.5.15}$$

where $I_3 = e^C$ is a constant. The general solution of the first order partial differential equation (7.5.12) is

$$I_3 = G(I_1), \tag{7.5.16}$$

where G is an arbitrary function. Since $g = \overline{S}(x,y)$,

$$\overline{S}(x,y) = \left(\int^x e(\tau)d\tau + a_3\right)^{-\frac{1}{2}} G(\xi). \tag{7.5.17}$$

Writing the conserved quantity (7.3.13) in terms of the invariant solution (7.5.10) and (7.5.11) we obtain

$$J = 2\rho U \int_0^{\xi_B(x)} \xi^2 \frac{dF}{d\xi} d\xi, \tag{7.5.18}$$

where

$$\xi_B(x) = \frac{y_B(x)}{\left(\int^x e(\tau)d\tau + a_3\right)^{1/2}}. \tag{7.5.19}$$

Since J is a constant independent of x, $\xi_B(x)$ is a constant and

$$y_B(x) = \xi_B \left(\int^x e(\tau)d\tau + a_3\right)^{1/2}. \tag{7.5.20}$$

Therefore

$$J = 2\rho U \int_0^{\xi_B} \xi^2 \frac{dF}{d\xi} d\xi. \tag{7.5.21}$$

Writing the conserved quantity (7.3.22) in terms of the invariant solution (7.5.10), (7.5.17) and (7.5.11) we obtain

$$K = 2\rho c_v U \int_0^{\xi_{B_\theta}(x)} G(\xi) d\xi, \tag{7.5.22}$$

where

$$\xi_{B_\theta}(x) = \frac{y_{B_\theta}(x)}{\left(\int^x e(\tau)d\tau + a_3\right)^{1/2}}. \tag{7.5.23}$$

Since K is a constant independent of x, $\xi_{B_\theta}(x)$ is a constant and

$$y_{B_\theta}(x) = \xi_{B_\theta} \left(\int^x e(\tau)d\tau + a_3\right)^{1/2}. \tag{7.5.24}$$

Therefore

$$K = 2\rho c_v U \int_0^{\xi_{B_\theta}} G(\xi) d\xi. \tag{7.5.25}$$

We have not yet established if ξ_B and ξ_{B_θ} are finite or infinite and if ξ_B and ξ_{B_θ} are equal.

Writing boundary conditions (3.12.7) and (3.12.8) in terms of the invariant solution (7.5.10), (7.5.17) and (7.5.11) we obtain

$$F(0) = 0, \qquad \frac{d^2F}{d\xi^2}(0) = 0, \qquad \frac{dG}{d\xi}(0) = 0, \tag{7.5.26}$$

$$\frac{dF}{d\xi}(\xi_B) = 0, \qquad \frac{d^2F}{d\xi^2}(\xi_B) = 0, \qquad \kappa\frac{dG}{d\xi}(\xi_{B_\theta}) = 0, \qquad G(\xi_{B_\theta}) = 0. \tag{7.5.27}$$

Writing the governing partial differential equation (7.2.3) in terms of the invariant solution (7.5.10) and (7.5.11) and using from (7.2.63), $E(x,y) = e(x)$, we obtain

$$\frac{d^3F}{d\xi^3} + \frac{U}{2}\frac{d}{d\xi}\left(2F + \xi\frac{dF}{d\xi}\right) = 0, \tag{7.5.28}$$

Writing the governing partial differential equation (7.2.4) in terms of the invariant solution (7.5.10), (7.5.17), (7.5.11) and using, (7.4.54),

$$E_\theta(x) = \alpha e(x) \tag{7.5.29}$$

for the eddy diffusivity, we obtain

$$\alpha\frac{d^2G}{d\xi^2} + \frac{U}{2}\frac{d}{d\xi}(\xi G) = 0. \tag{7.5.30}$$

The differential equations (7.5.28) and (7.5.30) can integrated at least once. This demonstrates the double reduction theorem by Sjöberg [17, 18]. Equations (7.5.28) and (7.5.30) compare with (6.6.3) and (6.6.4) in the mixing length model.

7.6 Summary of results

We summarize the results obtained in this Chapter so far.
Lie point symmetry:

$$X = -\frac{1}{e(x)}\left[\int^x e(\tau)d\tau + a_3\right]\frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial y} + \psi\frac{\partial}{\partial\psi} + \frac{1}{2}\overline{S}\frac{\partial}{\partial\overline{S}}. \tag{7.6.1}$$

Fluid variables:

$$\psi(x,y) = \frac{F(\xi)}{\int^x e(\tau)d\tau + a_3}, \qquad \xi = \frac{y}{\left(\int^x e(\tau)d\tau + a_3\right)^{\frac{1}{2}}}, \tag{7.6.2}$$

$$\overline{w}(x,y) = \frac{\partial\psi}{\partial y} = \frac{1}{\left(\int^x e(\tau)d\tau + a_3\right)^{3/2}}\frac{dF}{d\xi}, \tag{7.6.3}$$

$$\overline{v}_y(x,y) = \frac{\partial\psi}{\partial x} = -\frac{e(x)}{\left(\int^x e(\tau)d\tau + a_3\right)^2}\left[F(\xi) + \frac{1}{2}\xi\frac{dF}{d\xi}\right], \tag{7.6.4}$$

$$\overline{S}(x,y) = \left(\int^x e(\tau)d\tau + a_3\right)^{-\frac{1}{2}} G(\xi). \tag{7.6.5}$$

Ordinary differential equations:

$$\frac{d^3F}{d\xi^3} + \frac{U}{2}\frac{d}{d\xi}\left(2F + \xi\frac{dF}{d\xi}\right) = 0, \tag{7.6.6}$$

$$\alpha\frac{d^2G}{d\xi^2} + \frac{U}{2}\frac{d}{d\xi}(\xi G) = 0. \tag{7.6.7}$$

Boundary conditions:

$$F(0) = 0, \qquad \frac{d^2F}{d\xi^2}(0) = 0, \qquad \frac{dG}{d\xi}(0) = 0, \tag{7.6.8}$$

$$\frac{dF}{d\xi}(\xi_B) = 0, \qquad \frac{d^2F}{d\xi^2}(\xi_B) = 0, \qquad \kappa\frac{dG}{d\xi}(\xi_{B_\theta}) = 0, \qquad G(\xi_{B_\theta}) = 0. \tag{7.6.9}$$

Conserved quantities:

$$J = 2\rho U\int_0^{\xi_B}\xi^2\frac{dF}{d\xi}d\xi, \qquad K = 2\rho c_v U\int_0^{\xi_{B_\theta}} G(\xi)d\xi. \tag{7.6.10}$$

Effective kinematic viscosity and thermal diffusivity:

$$E(x,y) = e(x), \qquad E_\theta(x) = \alpha e(x). \tag{7.6.11}$$

Boundaries:

$$y_B(x) = \xi_B\left(\int^x e(\tau)d\tau + a_3\right)^{1/2}, \quad y_{B_\theta}(x) = \xi_{B_\theta}\left(\int^x e(\tau)d\tau + a_3\right)^{1/2}. \tag{7.6.12}$$

The mathematical analysis was independent of whether ν and κ are zero or nonzero. The results (7.6.1) to (7.6.12) compare with the results for the mixing length model, (6.6.1) to (6.6.9) for $\nu \neq 0$, $\kappa \neq 0$ and (6.6.10) to (6.6.18) for $\nu = 0$, $\kappa = 0$.

7.7 Analytical solution

The differential equations for the invariant solution are, (7.6.6) and (7.6.7),

$$\frac{d^3F}{d\xi^3} + \frac{U}{2}\frac{d}{d\xi}\left[2F + \xi\frac{dF}{d\xi}\right] = 0, \tag{7.7.1}$$

$$\alpha\frac{d^2G}{d\xi^2} + \frac{U}{2}\frac{d}{d\xi}(\xi G) = 0. \tag{7.7.2}$$

Integrating (7.7.1) we obtain

$$\frac{d^2F}{d\xi^2} + \frac{U}{2}\left[2F + \xi\frac{dF}{d\xi}\right] = C, \tag{7.7.3}$$

where C is a constant. Applying the boundary conditions (7.6.8) at $\xi = 0$ we obtain $C = 0$. Equation (7.7.3) becomes

$$\frac{d^2F}{d\xi^2} + \frac{U}{2}\left[2F + \xi\frac{dF}{d\xi}\right] = 0. \tag{7.7.4}$$

Multiplying equation (7.7.4) by ξ so that we can integrate the second term of the equation we obtain

$$\xi\frac{d^2F}{d\xi^2} + \frac{U}{2}\left[2\xi F + \xi^2\frac{dF}{d\xi}\right] = 0. \tag{7.7.5}$$

In order to integrate the first term of (7.7.5) we add and subtract $\frac{dF}{d\xi}$. Hence

$$\xi\frac{d^2F}{d\xi^2} + \frac{dF}{d\xi} - \frac{dF}{d\xi} + \frac{U}{2}\left[2\xi F + \xi^2\frac{dF}{d\xi}\right] = 0 \tag{7.7.6}$$

and using the product rule for differentiation, equation (7.7.6) becomes

$$\frac{d}{d\xi}\left(\xi\frac{dF}{d\xi}\right) - \frac{dF}{d\xi} + \frac{U}{2}\frac{d}{d\xi}\left(\xi^2F\right) = 0. \tag{7.7.7}$$

Integrating (7.7.7) we obtain

$$\xi\frac{dF}{d\xi} - F + \frac{U}{2}\xi^2F = C, \tag{7.7.8}$$

where C is a constant. Applying the boundary conditions (7.6.8) at $\xi = 0$ we obtain $C = 0$. Therefore equation (7.7.8) becomes

$$\xi\frac{dF}{d\xi} - F + \frac{U}{2}\xi^2F = 0. \tag{7.7.9}$$

Separating variables in (7.7.9) we obtain

$$\frac{dF}{F} = \left(-\frac{U}{2}\xi + \frac{1}{\xi}\right)d\xi. \tag{7.7.10}$$

Integrating (7.7.10) we obtain

$$\ln F = -\frac{U}{4}\xi^2 + \ln\xi + C, \tag{7.7.11}$$

where C is a constant. Exponentiating (7.7.11) we obtain

$$F(\xi) = M\xi\exp\left(-\frac{U}{4}\xi^2\right), \tag{7.7.12}$$

where $M = e^C$ is a constant. Using (7.6.3) we obtain

$$\begin{aligned}\overline{w}(x,y) &= \frac{1}{\left(\int^x e(\tau)d\tau + a_3\right)^{3/2}}\frac{dF}{d\xi} \\ &= \frac{M}{\left(\int^x e(\tau)d\tau + a_3\right)^{3/2}}\left(1 - \frac{U}{2}\xi^2\right)\exp\left(-\frac{U}{4}\xi^2\right).\end{aligned} \tag{7.7.13}$$

The result (7.7.13) was obtained in [12] for the non-thermal momentumless wake. We included calculations for this result so that the Chapter is self contained. We see that the mean velocity deficit $\overline{w}(x,y)$ vanishes when $\xi = \pm\sqrt{U/2}$ and again when $\xi = \pm\infty$. Therefore the momentum boundary of the wake, $y_B(x)$, is infinite.

The constant M cannot be obtained from the boundary conditions (7.6.8) and (7.6.9) on $F(\xi)$ which are identically satisfied. The constant M is obtained from the conserved quantity J given by (7.6.10). Substituting (7.7.12) into (7.6.10) with $\xi_B = \infty$ we obtain

$$J = 2M\rho U \int_0^\infty \left(1 - \frac{U}{2}\xi^2\right) \xi^2 \exp\left(-\frac{U}{4}\xi^2\right) d\xi. \tag{7.7.14}$$

Making the substitution $\tau = \frac{U}{4}\xi^2$ in the integral equation (7.7.14) becomes

$$J = \frac{8M\rho}{\sqrt{U}} \int_0^\infty (1 - 2\tau)\, \tau^{1/2} \exp(-\tau)\, d\tau. \tag{7.7.15}$$

Expanding equation (7.7.15) we obtain

$$J = \frac{8M\rho}{\sqrt{U}} \left[\int_0^\infty \tau^{1/2} \exp(-\tau)\, d\tau - 2\int_0^\infty \tau^{3/2} \exp(-\tau)\, d\tau\right]. \tag{7.7.16}$$

Using the definition [34]

$$\Gamma(n) = \int_0^\infty \tau^{n-1} e^{-\tau} d\tau \tag{7.7.17}$$

for the Gamma function $\Gamma(n)$, equation (7.7.16) becomes

$$J = \frac{8M\rho}{\sqrt{U}} \left[\Gamma\left(\frac{3}{2}\right) - 2\Gamma\left(\frac{5}{2}\right)\right]. \tag{7.7.18}$$

Using the identity [34]

$$\Gamma(n) = (n-1)\Gamma(n), \qquad n > 1, \tag{7.7.19}$$

equation (7.7.18) simplifies to

$$J = -\frac{8M\rho}{\sqrt{U}} \Gamma\left(\frac{1}{2}\right). \tag{7.7.20}$$

But [34]

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{7.7.21}$$

Hence

$$M = -\frac{J}{8\rho} \left(\frac{U}{\pi}\right)^{1/2} \tag{7.7.22}$$

and equation (7.7.12) becomes

$$F(\xi) = -\frac{J}{8\rho} \left(\frac{U}{\pi}\right)^{1/2} \xi \exp\left(-\frac{U}{4}\xi^2\right). \tag{7.7.23}$$

Next integrating (7.7.2) for $G(\xi)$ we obtain

$$\alpha \frac{dG}{d\xi} + \frac{U}{2}\xi G = C, \tag{7.7.24}$$

where C is a constant. Applying the boundary conditions (7.6.8) at $\xi = 0$ we obtain $C = 0$. Hence

$$\alpha \frac{dG}{d\xi} + \frac{U}{2}\xi G = 0. \tag{7.7.25}$$

Separating variables in (7.7.25) we obtain

$$\frac{dG}{G} = -\frac{U}{2\alpha}\xi d\xi \tag{7.7.26}$$

and integrating (7.7.26) gives

$$\ln G = -\frac{U}{4\alpha}\xi^2 + C, \tag{7.7.27}$$

where C is a constant. Exponentiating (7.7.27) yields

$$G(\xi) = G(0) \exp\left(-\frac{U}{4\alpha}\xi^2\right). \tag{7.7.28}$$

Substituting (7.7.28) and (7.6.2) for ξ into (7.6.5) we obtain

$$\overline{S}(x,y) = \frac{1}{\left(\int^x e(\tau)d\tau + a_3\right)^{1/2}} G(\xi) = \frac{G(0)}{\left(\int^x e(\tau)d\tau + a_3\right)^{1/2}} \exp\left(-\frac{U}{4\alpha}\xi^2\right). \tag{7.7.29}$$

We see that the mean temperature difference $\overline{S}(x,y)$ vanishes when $\xi = \pm\infty$. Therefore the thermal boundary parameter ξ_{B_θ} is infinite.

The constant $G(0)$ in (7.7.28) cannot be obtained from the boundary conditions (7.6.8) and (7.6.9) which are identically satisfied by the solution (7.7.28). It is obtained from the conserved quantity K given by (7.6.10). Substituting (7.7.28) into (7.6.10) with $\xi_{B_\theta} = \infty$ we obtain

$$K = 2\rho c_v U G(0) \int_0^\infty \exp\left(-\frac{U}{4\alpha}\xi^2\right) d\xi. \tag{7.7.30}$$

Making the substitution $\tau = \frac{U}{4\alpha}\xi^2$ in the integral equation (7.7.30) becomes

$$K = 2\rho c_v \left(\alpha U\right)^{1/2} G(0) \int_0^\infty \tau^{-1/2} e^{-\tau} d\tau \tag{7.7.31}$$

and since

$$\int_0^\infty \tau^{-1/2} e^{-\tau} d\tau = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{7.7.32}$$

it follows that

$$G(0) = \frac{K}{2\rho c_v (\alpha \pi U)^{1/2}}. \tag{7.7.33}$$

Hence

$$G(\xi) = \frac{K}{2\rho c_v (\alpha \pi U)^{1/2}} \exp\left(-\frac{U}{4\alpha}\xi^2\right). \tag{7.7.34}$$

The eddy kinematic viscosity affects the variables $F(\xi)$ and $G(\xi)$ through ξ by (7.6.2).

7.8 Analytical solution - Eddy kinematic viscosity and eddy thermal conductivity as power-laws

In this Section, we consider the effective kinematic viscosity that is a power law in x.

Suppose

$$E(x) = e(x) = \nu + \nu_{T0}x^{\beta} \tag{7.8.1}$$

and hence

$$E_{\theta}(x) = \alpha e(x) = \alpha\nu + \alpha\nu_{T0}x^{\beta}, \tag{7.8.2}$$

where β is a constant, ν is the kinematic viscosity of the fluid and $\nu_{T0}x^{\beta}$ is the eddy kinematic viscosity of the flow. From (7.7.13), the mean velocity deficit vanishes when $\xi = \pm\sqrt{\frac{2}{U}}$ and $\xi = \pm\infty$. Let $H(x)$ be the y-coordinate when the mean velocity deficit vanishes and changes direction. This happens when $\xi = \sqrt{\frac{2}{U}}$. Hence

$$H(x) = \sqrt{\frac{2}{U}}\left(\int^{x} e(\tau)d\tau + a_3\right)^{1/2} = \begin{cases} \sqrt{\frac{2}{U}}\left[x\left(\nu + \frac{\nu_{T0}}{\beta+1}x^{\beta}\right) + a_3\right]^{1/2}, & \beta \neq -1 \\ \sqrt{\frac{2}{U}}\left[\nu x + \nu_{T0}\ln x + a_3\right]^{1/2}, & \beta = -1. \end{cases} \tag{7.8.3}$$

We expect $H(x) \to 0$ as $x \to 0$. This can only happen when $a_3 = 0$ and $\beta > -1$. Substituting (7.8.1) into (7.5.11) we obtain, for $\beta > -1$,

$$\xi = \frac{y}{\left[x\left(\nu + \frac{\nu_{T0}}{\beta+1}x^{\beta}\right)\right]^{1/2}}. \tag{7.8.4}$$

In summary, the results for a power-law effective kinematic viscosity, with $\beta > -1$, are

$$F(\xi) = -\frac{J}{8\rho}\left(\frac{U}{\pi}\right)^{1/2}\xi\exp\left(-\frac{U}{4}\xi^2\right), \tag{7.8.5}$$

$$G(\xi) = \frac{K}{2\rho c_v(\alpha\pi U)^{1/2}}\exp\left(-\frac{U}{4\alpha}\xi^2\right), \tag{7.8.6}$$

where

$$\xi = \frac{y}{\left[x\left(\nu + \frac{\nu_{T0}}{\beta+1}x^{\beta}\right)\right]^{1/2}}. \tag{7.8.7}$$

The physical variables are

$$\overline{w}(x,y) = -\frac{J}{8\rho}\left(\frac{U}{\pi}\right)^{1/2}\frac{\left(1-\frac{U}{2}\xi^2\right)}{\left[x\left(\nu + \frac{\nu_{T0}}{\beta+1}x^{\beta}\right)\right]^{3/2}}\exp\left(-\frac{U}{4}\xi^2\right), \tag{7.8.8}$$

$$\overline{v}_x(x,y) = U - \overline{w}(x,y) = U + \frac{J}{8\rho}\left(\frac{U}{\pi}\right)^{1/2}\frac{\left(1-\frac{U}{2}\xi^2\right)}{\left[x\left(\nu + \frac{\nu_{T0}}{\beta+1}x^{\beta}\right)\right]^{3/2}}\exp\left(-\frac{U}{4}\xi^2\right), \tag{7.8.9}$$

$$\overline{v}_y(x,y) = \frac{J}{16\rho}\left(\frac{U}{\pi}\right)^{1/2}\frac{\xi\left(3-\frac{U}{2}\xi^2\right)\left(\nu + \nu_{T0}x^{\beta}\right)}{\left[x\left(\nu + \frac{\nu_{T0}}{\beta+1}x^{\beta}\right)\right]^{2}}\exp\left(-\frac{U}{4}\xi^2\right), \tag{7.8.10}$$

$$\overline{S}(x,y) = \frac{K}{2\rho c_v(\alpha\pi U)^{1/2}\left[x\left(\nu + \frac{\nu_{T0}}{\beta+1}x^{\beta}\right)\right]^{1/2}} \exp\left(-\frac{U}{4\alpha}\xi^2\right). \tag{7.8.11}$$

7.9 Results and discussion

In this Section, we plot the analytical results obtained in Section 7.8. For all the results presented in this Section, $J = 1$, $K = 1$, $\rho = 1$, $c_v = 1$ and $U = 1$.

Figure 7.1 shows the graphs for the x-component of the mean velocity profiles when $\nu = 0.5$, $\nu_{T0} = 0.5$ and $\beta = -0.5$ at $x = 1$, $x = 1.5$ and $x = 2$. The profiles are plotted using equation (7.8.9). We see that as the distance from the obstacle, x, increases the x-component of the mean velocity at the centreline, $\overline{v}_x(x,0)$, decreases and the effective width of the of the wake increases. The graphs are consistent with (7.8.3), which shows that the effective width of the wake increases with an increase in x for all values of $\beta > -1$.

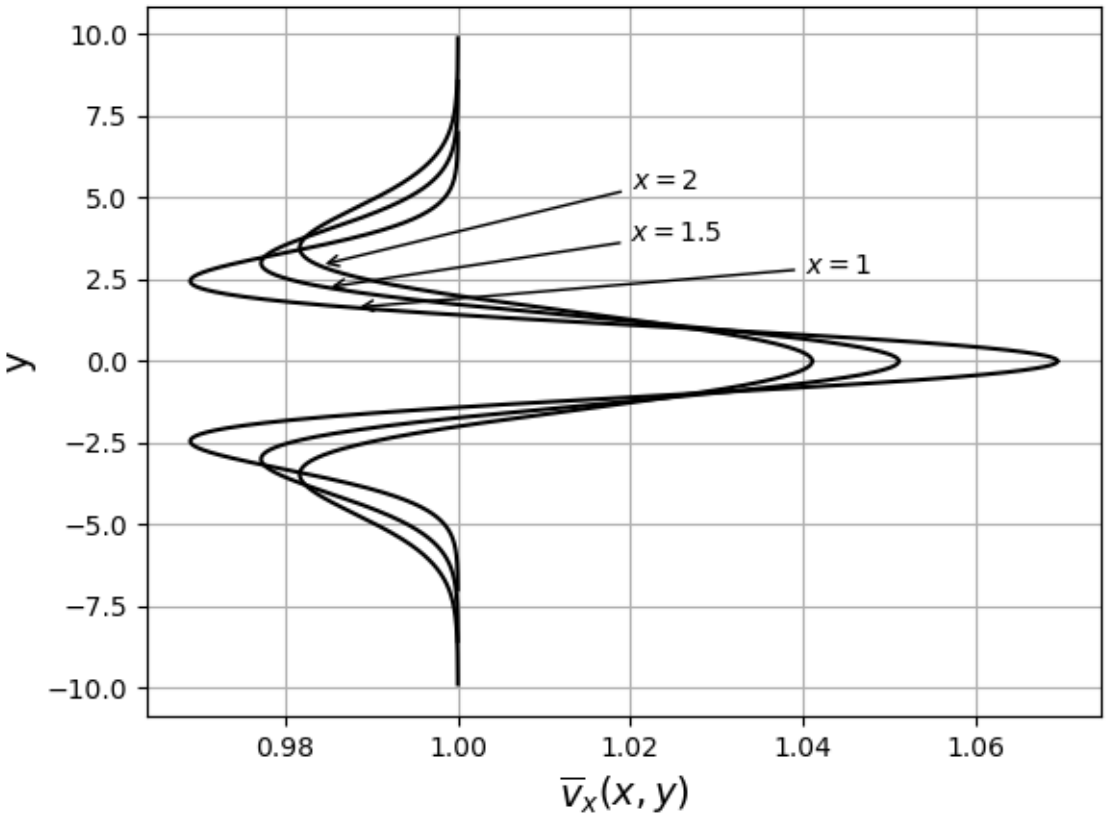

FIGURE 7.1: The profiles of the x-component of the mean velocity $\overline{v}_x(x,y)$ with $\nu = 0.5$, $\nu_{T0} = 0.5$ and $\beta = -0.5$ plotted at a range of values of x.

Figure 7.2 shows the profiles for the mean temperature difference when $\nu = 0.5$, $\nu_{T0} = 0.5$, $\alpha = 1$ and $\beta = -0.5$ at $x = 1$, $x = 1.5$ and $x = 2$. The profiles are plotted using equation (7.8.11). We see that as the distance from the obstacle, x, increases the mean temperature difference at the centreline, $\overline{S}(x, 0)$, decreases and the profiles become broader along the y-axis.

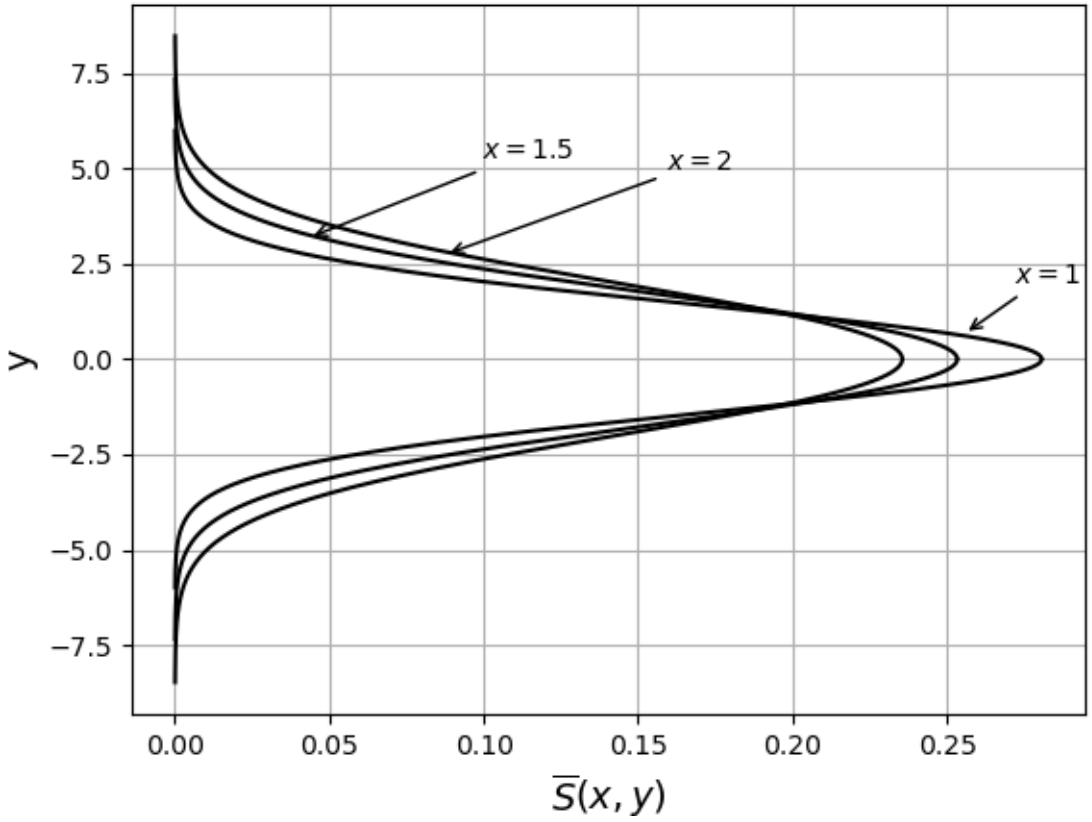

FIGURE 7.2: The profiles of the mean temperature difference $\overline{S}(x, y)$ with $\nu = 0.5$, $\nu_{T0} = 0.5$, $\alpha = 1$ and $\beta = -0.5$ plotted at a range of values of x.

Figure 7.3 shows the graphs for the x-component of the mean velocity profiles when $\nu = 0.5$, $\nu_{T0} = 0.5$ and $\beta = 0.5$ at $x = 1$, $x = 1.5$ and $x = 2$. The profiles are plotted using equation (7.8.9). We see that as the distance from the obstacle, x, increases the x-component of the mean velocity at the centreline, $\overline{v}_x(x, 0)$, decreases and the curves become broader along the y-axis. This suggests that the effective momentum diffusion increases with the distance, x, from the obstacle. This can also be seen in (7.8.1). The results make physical sense since the fluid flow becomes more turbulent as the distance from the trailing edge of the obstacle increases. The turbulence increases the momentum diffusion of the mean flow. The area on the right between the curves and the $x = 1$ line represents the jet generated by the propeller of the obstacle. The area on the left between the curves and the $x = 1$ line represents the drag on the obstacle. The areas of these regions are equal for the momentumless thermal wake.

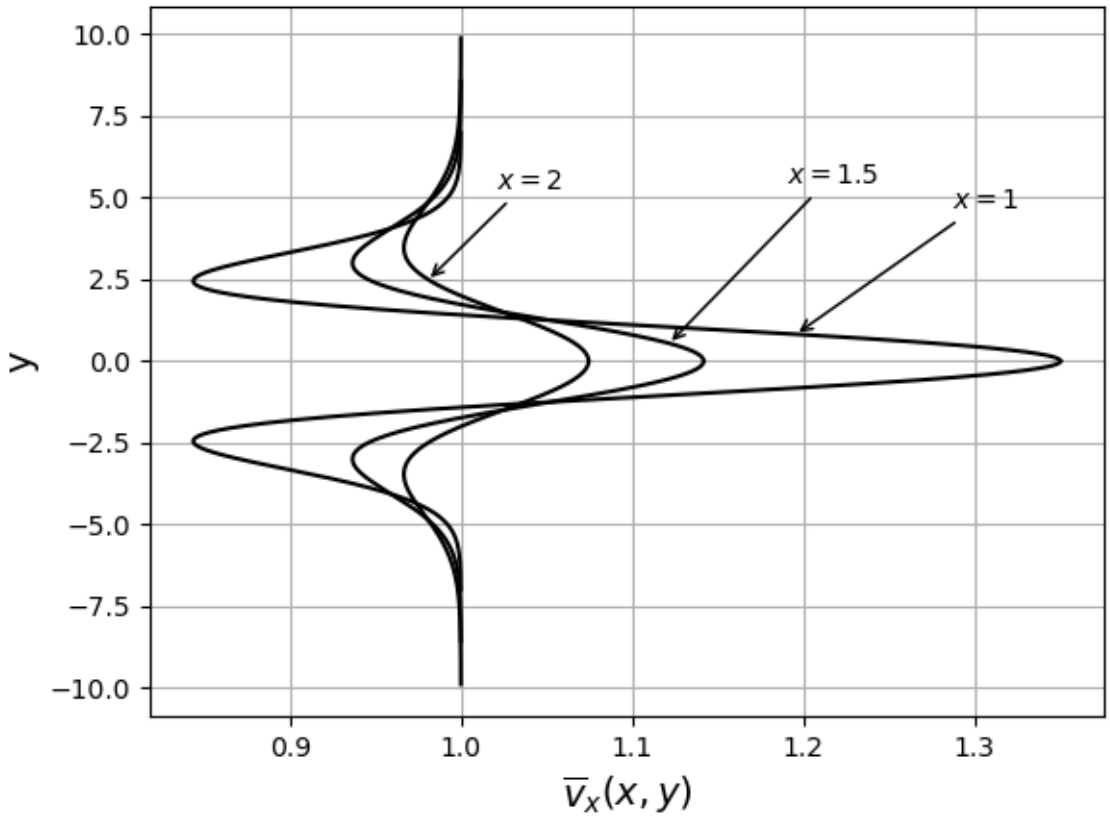

FIGURE 7.3: The profiles of the x-component of the mean velocity $\overline{v}_x(x, y)$ with $\nu = 0.5$, $\nu_{T0} = 0.5$ and $\beta = 0.5$ plotted at a range of values of x.

Figure 7.4 shows the profiles for the mean temperature difference when $\nu = 0.5$, $\nu_{T0} = 0.5$, $\alpha = 1$ and $\beta = 0.5$ at $x = 1$, $x = 1.5$ and $x = 2$. The profiles are plotted using equation (7.8.11). We see that as the distance from the obstacle, x, increases the mean temperature difference at the centreline, $\overline{S}(x, 0)$, decreases and the curves become broader along the y-axis. This suggests that the effective thermal diffusion increases with the distance, x, from the obstacle. This can also be seen in (7.8.2). The results make physical sense since the fluid flow becomes more turbulent as the distance x from the obstacle increases. The turbulence increases the thermal diffusion of the mean temperature difference.

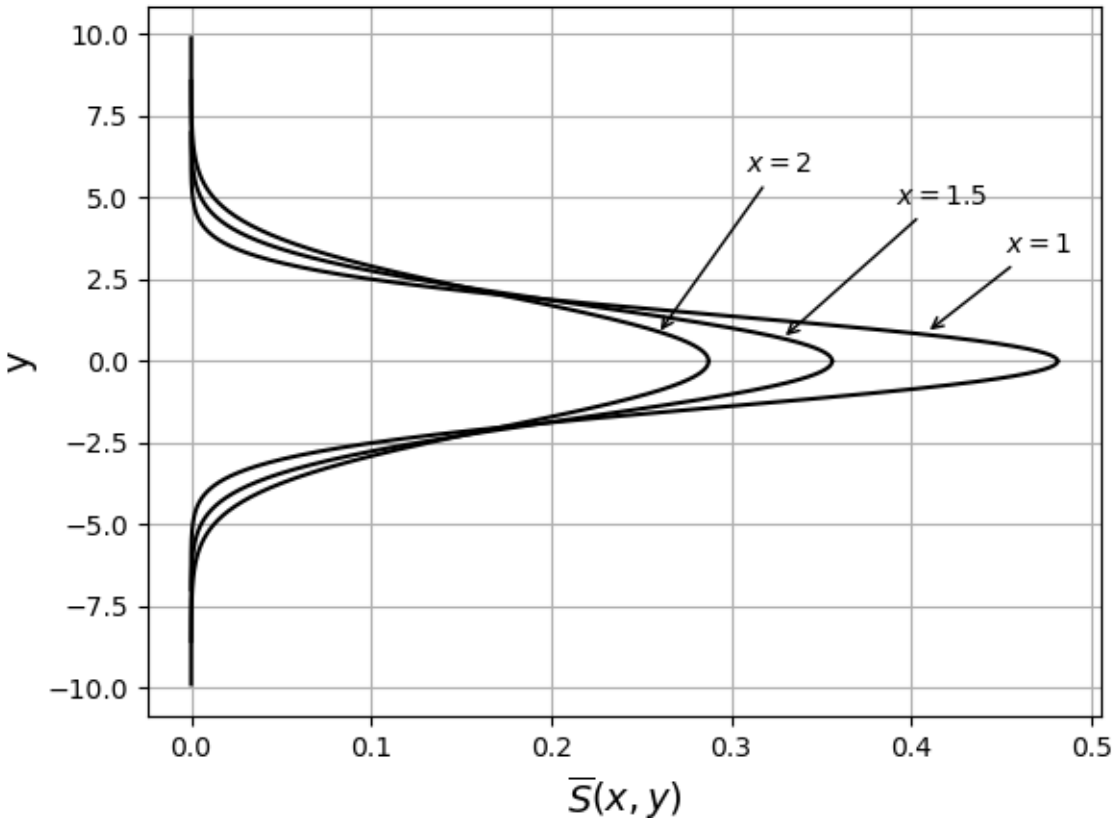

FIGURE 7.4: The profiles of the mean temperature difference $\overline{S}(x, y)$ with $\nu = 0.5$, $\nu_{T0} = 0.5$, $\alpha = 1$ and $\beta = 0.5$ plotted at a range of values of x.

Figure 7.5 shows the profiles of the x-component of the mean velocity for a range of values of β at $x = 0.75$. We see that as the value of β increases the x-component of the mean velocity at the centreline, $\overline{v}_x(x, 0)$, increases and the profiles become broader along the y-axis. However the width to height ratio decreases as β increases which suggests that the effective momentum diffusion decreases with an increase of β. It can also be seen in (7.8.1) that when $0 < x < 1$, as the value of β increases, the effective kinematic viscosity $E(x)$ decreases.

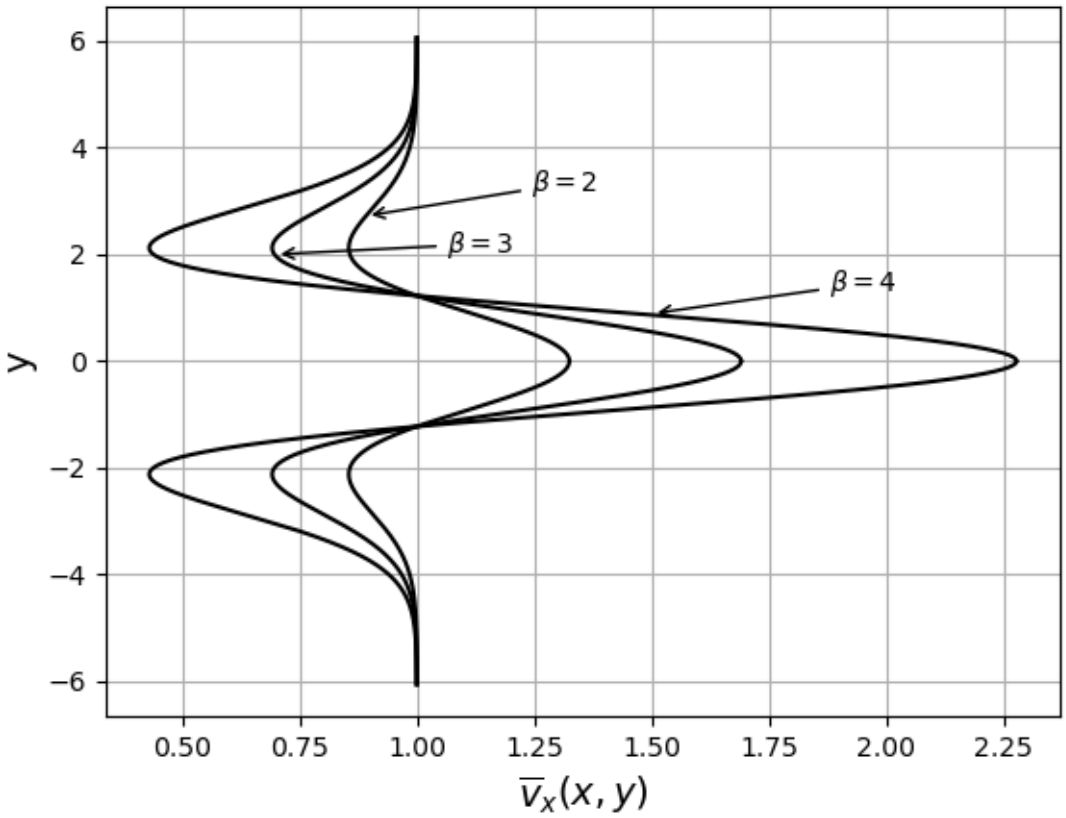

FIGURE 7.5: The profiles of the x-component of the mean velocity $\overline{v}_x(x, y)$ with $\nu = 0.5$ and $\nu_{T0} = 0.5$ plotted for a range of values of β when $x = 0.75$.

Figure 7.6 shows the profiles of the mean temperature difference for a range of values of β when $x = 0.75$. We see that as the values of β increases the mean temperature difference at the centreline, $\overline{S}(x, 0)$, increases and the profiles become slightly broader along the y-axis. However the ratio of the width to the height decreases as β increases which suggests that the effective thermal diffusion decreases with the increase of β. It can also be seen in (7.8.2) that when $0 < x < 1$, as the value of β increases, the effective thermal $E_\theta(x)$ decreases.

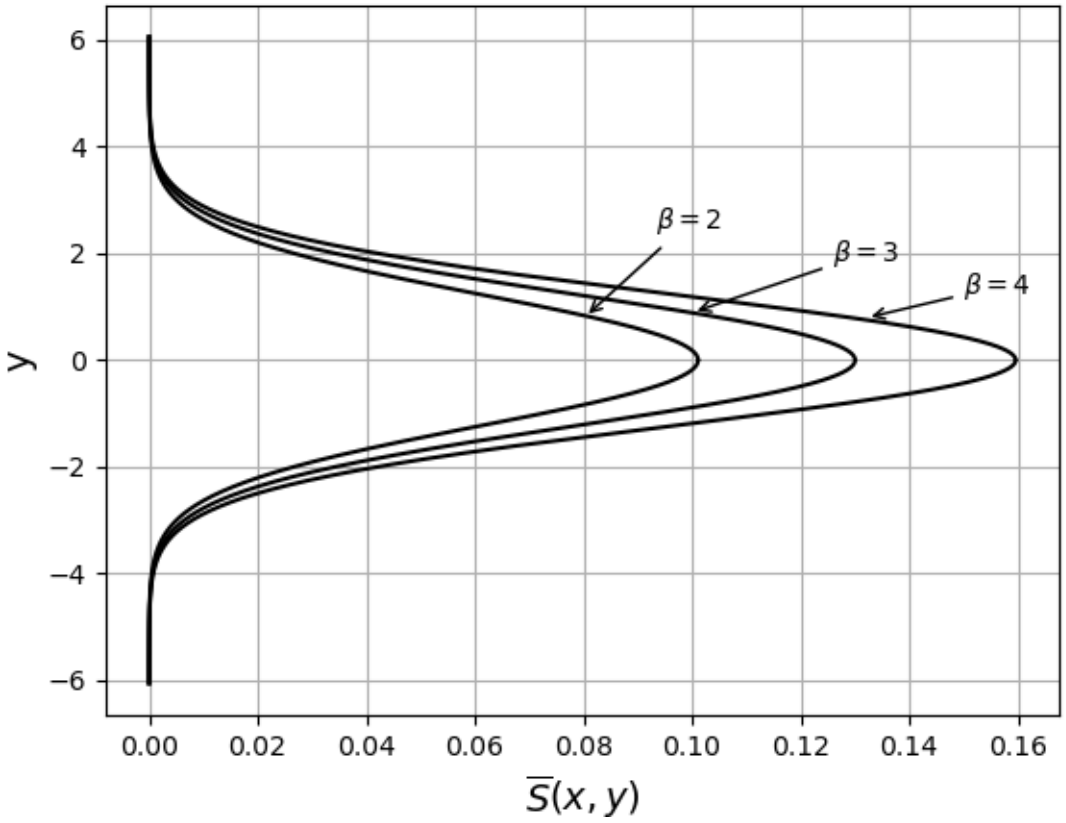

FIGURE 7.6: The profiles of the mean temperature difference $\overline{S}(x, y)$ with $\nu = 0.5$, $\nu_{T0} = 0.5$ and $\alpha = 1$ plotted for a range of values of β when $x = 0.75$.

Figure 7.7 shows the profiles for the x-component of the mean velocity for a range of values of β when $x = 1.25$. We see that as the value of β increases the x-component of the mean velocity at the centreline, $\overline{v}_x(x, 0)$, decreases and the profiles become slightly broader along the y-axis. The ratio of the width to the height increases as β increases which suggests that the turbulent kinematic viscosity increases with an increase of β. It can also be seen in (7.8.1) that when $x > 1$, as the value of β increases, the effective kinematic viscosity $E(x)$ increases.

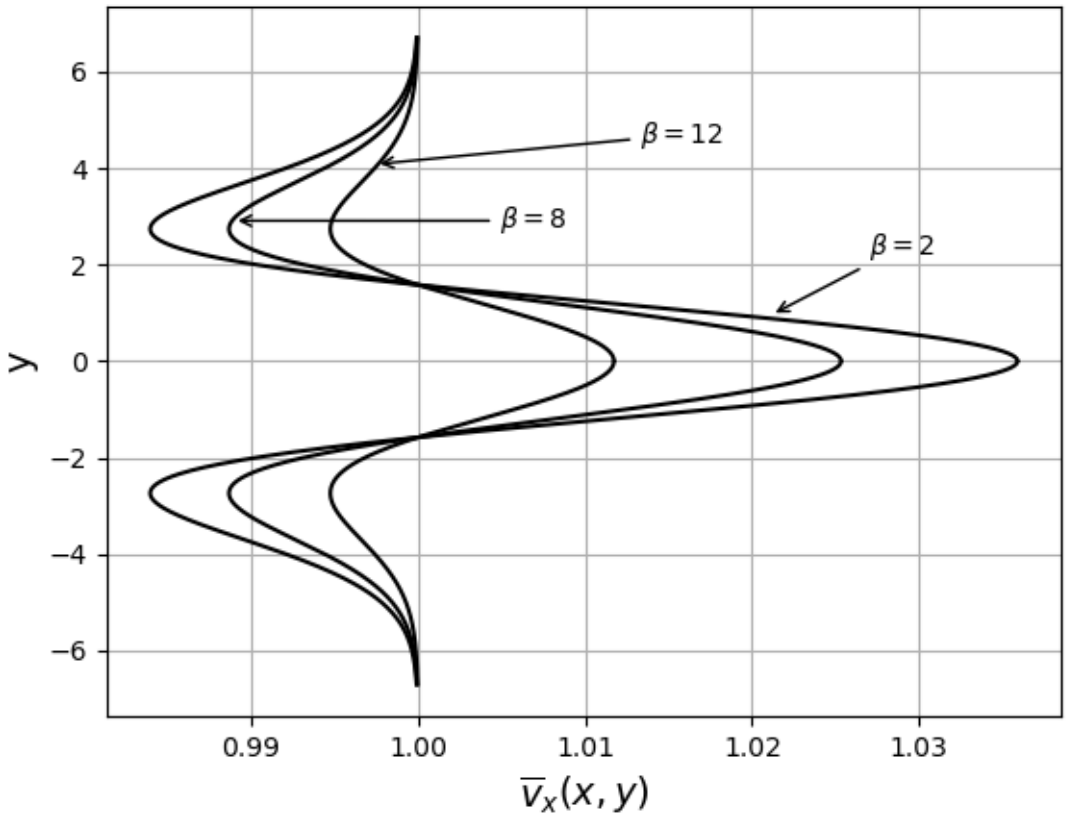

FIGURE 7.7: The profiles of the x-component of the mean velocity $\overline{v}_x(x, y)$ with $\nu = 0.5$ and $\nu_{T0} = 0.5$ plotted for a range of values of β when $x = 1.25$.

Figure 7.8 shows the profiles of the mean temperature difference for a range of values of β when $x = 1.25$. We see that as the value of β increases the mean temperature difference at the centreline, $\overline{S}(x, 0)$, decreases and the profiles become narrower along the y-axis. However the ratio of the width to height increases as β increases which suggests that the turbulent thermal diffusion increases with an increase of β. It can also be seen in (7.8.2) that when $x > 1$, as the value of β increases, the effective thermal diffusion $E_\theta(x)$ increases.

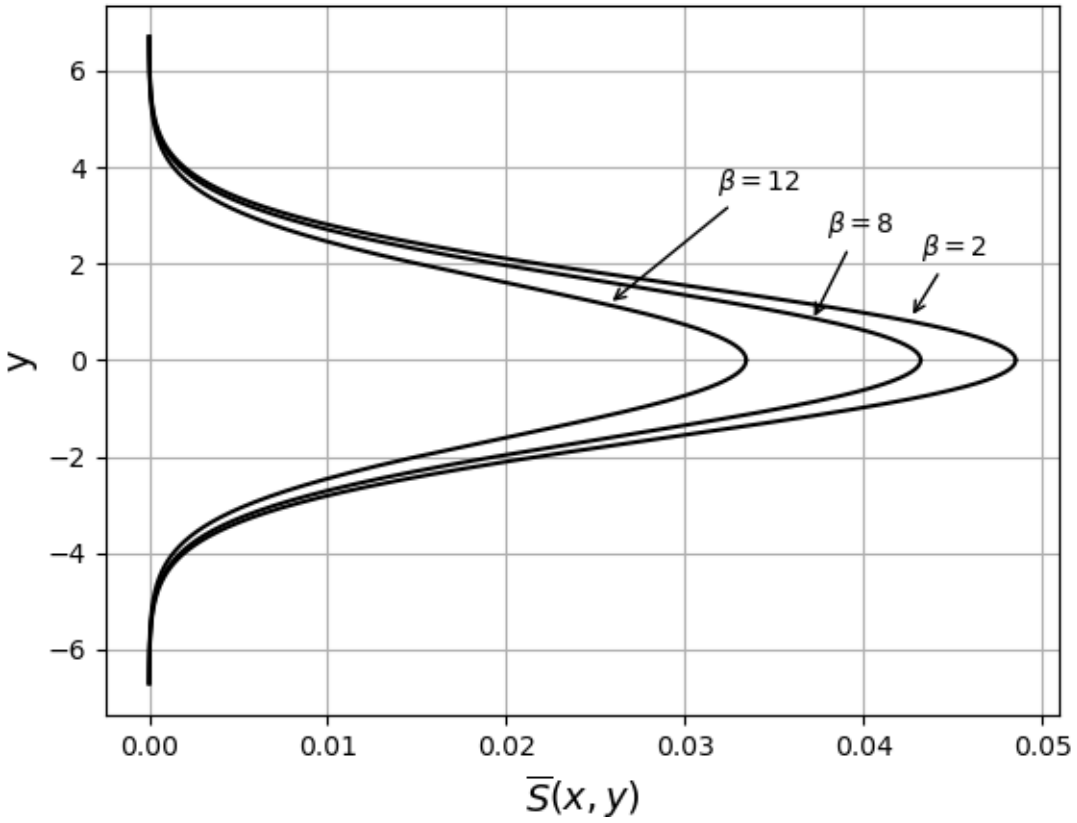

FIGURE 7.8: The profiles of the mean temperature difference $\overline{S}(x, y)$ with $\nu = 0.5$, $\nu_{T0} = 0.5$ and $\alpha = 1$ plotted for a range of values of β when $x = 1.25$.

Figure 7.9 shows the x-component of the mean velocity plotted against y at $x = 1$ for a range of values of ν, with ν_{T0} kept fixed at $\nu_{T0} = 0.5$. We see that as the kinematic viscosity, ν, increases the x-component of the mean velocity at the centreline, $\overline{v}_x(x, 0)$, decreases. The ratio of the width to the height increases as ν increases due to increased diffusion. This behaviour is expected when ν increases.

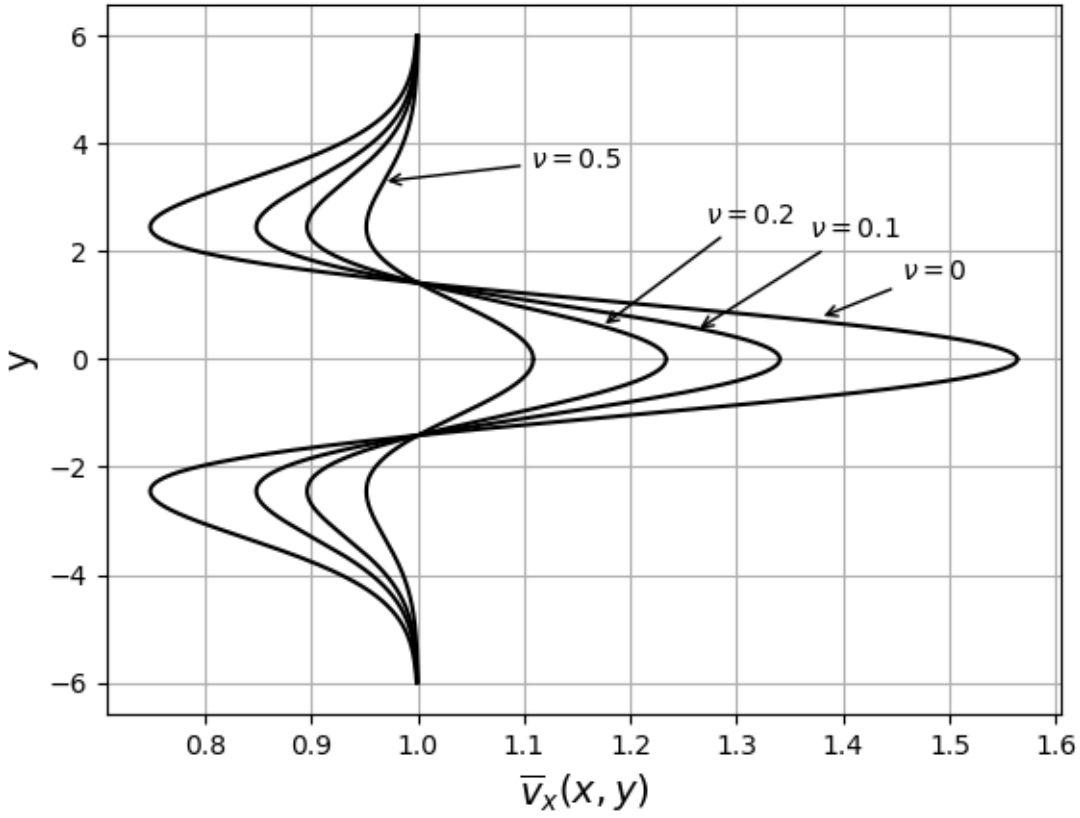

FIGURE 7.9: The x-component of the mean velocity $\overline{v}_x(x, y)$ plotted against y with $\nu_{T0} = 0.5$ and $\beta = -0.5$ for $\nu = 0$, $\nu = 0.1$, $\nu = 0.2$ and $\nu = 0.5$ at $x = 1$.

Figure 7.10 shows the mean temperature difference plotted against y for varying values of ν and $\kappa = \rho c_v \alpha \nu$. We see that as the kinematic viscosity, ν, and the thermal conductivity, $\kappa = \rho c_v \alpha \nu$, increase the mean temperature difference at the centreline, $\overline{S}(x, 0)$, decreases. The ratio of the width to the height of the profile increases. This is expected when the effective momentum diffusion and the thermal diffusion increase.

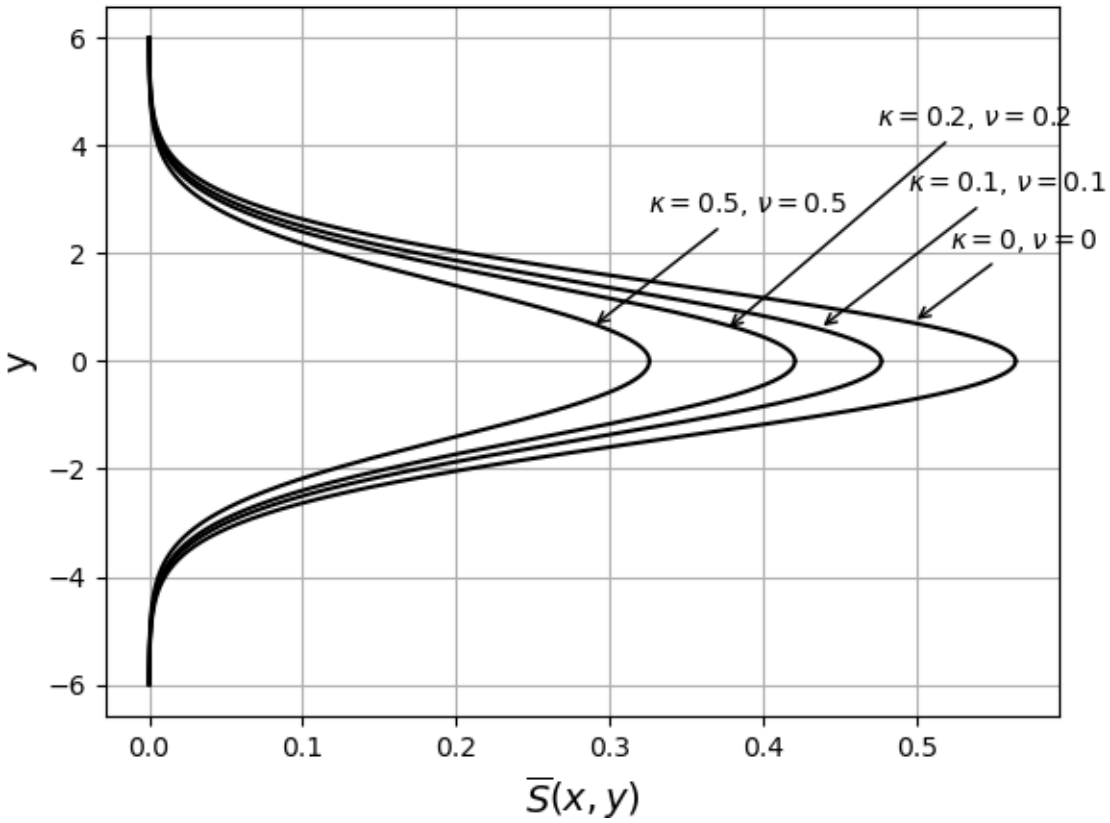

FIGURE 7.10: The mean temperature difference $\overline{S}(x, y)$ plotted against y with $\nu_{T0} = 0.5$, $\alpha = 1$ and $\beta = -0.5$ for $\nu = 0$, $\kappa = 0$; $\nu = 0.1$, $\kappa = 0.1$; $\nu = 0.2$, $\kappa = 0.2$ and $\nu = 0.5$, $\kappa = 0.5$ at $x = 1$. The thermal conductivity is $\kappa = \rho c_v \alpha \nu$.

7.10 Conclusions

In this Chapter, we derived the conservation laws and conserved vectors for the momentumless thermal wake with eddy viscosity and eddy thermal conductivity dependent on spatial variables. Using the fact that the physical properties of the momentumless thermal wake are bounded on the centre line we showed that the effective kinematic viscosity has to be independent of y. We derived the conserved quantities by integrating the conservation laws across the mean flow, with boundary conditions. We used the conserved vectors to derive the associated Lie point symmetry. We showed that when the effective thermal diffusivity is a function of x only, the effective kinematic viscosity is proportional to the effective thermal diffusivity. We used the associated Lie point symmetry to reduce the governing partial differential equations to two ordinary differential equations. We derived the analytical

solution of the ordinary differential equations and considered a power law effective kinematic viscosity. The mean temperature difference and velocity deficit profiles were plotted for different values of the exponent β. When $0 < x < 1$ and $\beta > 0$ the momentum and thermal diffusions increase with a decrease in the exponent and when $x > 1$ and $\beta > 0$ the momentum and thermal diffusions increase with an increase in the exponent. We also investigated the effect of the kinematic viscosity and the thermal conductivity on the mean velocity and mean temperature difference profiles. The diffusions of momentum and temperature difference increase with an increase in kinematic viscosity and thermal conductivity. We showed that the momentum and thermal boundaries of the momentumless thermal wake are infinite, which is consistent with the results for the mixing length model.

The assumption that the eddy viscosity and eddy thermal conductivity only depend on spatial variables simplified the analysis of the momentumless thermal wake significantly. The mathematical analysis did not depend on the values of the kinematic viscosity or thermal conductivity. The model however is less realistic than the mixing length model, since the turbulent flow in the model is independent of the mean velocity. The eddy kinematic viscosity is constant across the wake flow and does not vanish as $y \to \infty$. It is for this reason that the momentum and thermal wakes extend to $y = \pm\infty$.

Chapter 8

Conclusions

In this thesis, we studied the classical thermal wake and the momentumless thermal wake. We considered various models for the eddy viscosity and the eddy thermal conductivity of the thermal wake.

For the classical thermal wake with eddy viscosity and eddy thermal conductivity modelled by mixing lengths, the governing partial differential equations admit an associated Lie point symmetry provided that the mixing lengths are proportional. When the kinematic viscosity and the thermal conductivity are nonzero the governing partial differential equations admit a Lie point symmetry provided that the momentum and thermal mixing lengths are proportional to $x^{1/2}$. When the kinematic viscosity and thermal conductivity are zero one of the coefficients of the associated Lie point symmetry remained an arbitrary function. We imposed the Prandtl hypothesis that the momentum mixing length is proportional to the width of the wake to determine the unknown coefficient. The solution for vanishing kinematic viscosity and thermal conductivity matched the solution obtained by letting the kinematic viscosity and thermal conductivity tend to zero in the analytical solution for nonzero kinematic viscosity and thermal conductivity. The ordinary differential equations for the invariant solution can be integrated completely when the kinematic viscosity and thermal conductivity vanish. However, when the kinematic viscosity and thermal conductivity are nonzero the ordinary differential equations could not be integrated completely and we used the numerical scheme described by Mubai and Mason [16]. The shooting method in the numerical scheme used the two conserved quantities of the classical thermal wake as targets. The analytical solution led to the equation of lines of constant temperature difference. When the kinematic viscosity and thermal conductivity are zero the boundary of the thermal wake is finite. When the kinematic viscosity and thermal conductivity are nonzero the boundary of the thermal wake appears to be infinite from the numerical results. We verified the numerical scheme of Mubai and Mason [16] by showing that, as the kinematic viscosity and thermal conductivity tend to zero, the numerical solution matches the analytical solution.

We considered the von Kármán similarity hypothesis to derive the momentum and thermal mixing lengths. The partial differential equations, with the mixing lengths derived from the von Kármán similarity hypothesis, were reduced to ordinary differential equations using a scaling transformation. The ordinary differential equations could not be solved since the boundary conditions were not useful. They were in regions where the von Kármán similarity hypothesis is not valid. We made suggestions on how the ordinary differential equations could be solved in future research.

The kinematic viscosity and thermal conductivity played an important role in the mathematical analysis of the classical thermal wake. When the kinematic viscosity and thermal conductivity are nonzero we derived the mixing lengths without

imposing Prandtl's hypothesis. When the kinematic viscosity and thermal conductivity vanish we had to impose the Prandtl hypothesis. The solution obtained by letting the kinematic viscosity and thermal conductivity tend to zero matches the solution obtained when the kinematic viscosity and thermal conductivity vanish.

For a momentumless thermal wake with eddy viscosity and eddy conductivity modelled by mixing lengths, the governing partial differential equations admit an associated Lie point symmetry provided the thermal mixing length is proportional to the momentum mixing length. When the kinematic viscosity and thermal conductivity are nonzero the mixing lengths are proportional to x. When the kinematic viscosity and thermal conductivity vanish one of the coefficients in the associated Lie point symmetry remained an arbitrary function. We imposed the Prandtl hypothesis that the momentum mixing length is proportional to the width of the momentum wake to obtain the unknown coefficient and found that the mixing lengths are proportional to $x^{1/4}$. Hence the solution for vanishing kinematic viscosity and thermal conductivity does not match the solution obtained by letting the kinematic viscosity and thermal conductivity tend to zero. We prefer the solution obtained by letting the kinematic viscosity and thermal conductivity tend to zero since it transitions from the solution for nonzero kinematic viscosity and thermal conductivity smoothly. We used the associated Lie point symmetry to reduce the two governing partial differential equations to ordinary differential equations for the invariant solution. The ordinary differential equations cannot be integrated completely for all values of kinematic viscosity and thermal conductivity. We used the shooting method described by Mubai and Mason [16]. The shooting method uses the two conserved quantities for the momentumless thermal wake as targets. The momentum and thermal boundaries of the wake appear to be infinite from the numerical results. The solution for vanishing kinematic viscosity and thermal conductivity differs from the solution obtained by letting the kinematic viscosity and the thermal conductivity tend to zero. The two solutions converge as the distance from the centreline increases.

For a momentumless thermal wake with eddy viscosity and thermal conductivity dependent on spatial variables, the two governing partial differential equations admit an associated Lie point symmetry provided that the effective thermal diffusivity is proportional to the effective kinematic viscosity. We showed that the eddy kinematic viscosity is independent on the spatial variable perpendicular to the mainstream flow. We assumed that the effective thermal diffusivity is independent of the spatial variable perpendicular to the mainstream flow in order to fully determine the associated Lie point symmetry. We used the associated Lie point symmetry to reduce the governing partial differential equations to ordinary differential equations for the invariant solution. The ordinary differential equations were integrated completely. We considered a power-law eddy viscosity and hence a power-law eddy thermal conductivity. We examined the effect of the exponent of the power-law β on the mean velocity and mean temperature difference profiles. When $x > 1$ and $\beta > 0$, the effective kinematic viscosity and effective thermal diffusivity increase with an increase in the exponent. The momentum and thermal boundaries are infinite for the thermal wake for all values of kinematic viscosity and thermal conductivity. The solution for vanishing kinematic viscosity and thermal conductivity matches the solution obtained by letting the kinematic viscosity and the thermal conductivity tend to zero. This is because the derivation of the associated Lie point symmetry did not depend on the values of the kinematic viscosity and thermal conductivity.

The kinematic viscosity and thermal conductivity played an important role in the analysis of the momentumless thermal wake when the eddy viscosity and eddy

thermal conductivity are modelled by mixing lengths. However, the kinematic viscosity and thermal conductivity did not play a role in the mathematical analysis of the momentumless thermal wake when the eddy viscosity and eddy thermal conductivity are dependent only on spatial variables. The mixing length model is more reliable since the eddy viscosity depends on the mean velocity gradient.

For a momentumless thermal wake with eddy viscosity and eddy thermal conductivity modelled using mixing lengths, the solution obtained by letting kinematic viscosity and thermal conductivity tend to zero does not match the solution for vanishing kinematic viscosity and thermal conductivity. In future work, the solutions could be compared with experimental data to determine which solution is more accurate.

The ordinary differential equations for the invariant solution for a classical thermal wake with eddy viscosity and eddy thermal conductivity modelled using the von Kármán similarity hypothesis could not be solved because we did not have useful boundary conditions. A numerical scheme that implements the von Kármán similarity hypothesis in regions of the thermal wake where the hypothesis is valid, and have eddy viscosity and eddy thermal conductivity constant in regions where the hypothesis is not valid, may lead to a solution for the classical thermal wake.

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Appendix A

Averaging operator properties

In this Appendix, we prove some of the properties of the averaging operator $\overline{(\cdot)}$ defined by

$$\overline{f}(t,x,y) = \lim_{N\to 0} \frac{1}{N} \int_t^{t+N} f(\tau,x,y)d\tau. \tag{A.0.1}$$

We assume that dependent variables become independent of time t after the averaging operator is applied on them. All partial derivatives are assumed to be continuous and we assume that

$$\lim_{N\to\infty} \frac{f(t+N,x,y)}{N} = 0 \tag{A.0.2}$$

for all dependent variables $f(t,x,y)$. In order to commute limits with spatial derivatives, we assume that

$$\frac{1}{N}\int_t^{t+N} f(\tau)d\tau \quad \text{and} \quad \frac{\partial}{\partial s}\left(\frac{1}{N}\int_t^{t+N} f(\tau)d\tau\right)$$

converge uniformly as $N \to \infty$, where s is a spatial variable.

(i) The averaging operator is linear:

$$\overline{f+g} = \overline{f} + \overline{g}, \tag{A.0.3}$$

$$\overline{\lambda f} = \lambda \overline{f}, \tag{A.0.4}$$

for all λ independent of t and time-dependent functions f and g.

Proof. Using the definition of the averaging operator on $f+g$, we have

$$\overline{f+g} = \lim_{N\to\infty} \frac{1}{N}\int_t^{t+N} (f(\tau)+g(\tau))\,d\tau \tag{A.0.5}$$

$$= \lim_{N\to\infty} \left(\frac{1}{N}\int_t^{t+N} f(\tau)d\tau + \frac{1}{N}\int_t^{t+N} g(\tau)d\tau\right). \tag{A.0.6}$$

The averages $\overline{f}$ and $\overline{g}$ exist. Hence, the limits

$$\lim_{N\to\infty} \frac{1}{N}\int_t^{t+N} f(\tau)d\tau \quad \text{and} \quad \lim_{N\to\infty} \frac{1}{N}\int_t^{t+N} g(\tau)d\tau$$

exist and

$$\overline{f+g} = \lim_{N\to\infty} \frac{1}{N} \int_t^{t+N} f(\tau)d\tau + \lim_{N\to\infty} \frac{1}{N} \int_t^{t+N} g(\tau)d\tau \tag{A.0.7}$$

$$= \overline{f} + \overline{g}. \tag{A.0.8}$$

Now using the definition of the averaging operator on λf, we have

$$\overline{\lambda f} = \lim_{N\to\infty} \frac{1}{N} \int_t^{t+N} \lambda f(\tau)d\tau = \lim_{N\to\infty} \left(\lambda \frac{1}{N} \int_t^{t+N} f(\tau)d\tau \right). \tag{A.0.9}$$

The limit $\lim_{N\to\infty} \frac{1}{N} \int_t^{t+N} f(\tau)d\tau$ exists. Hence, we can pull λ out of the limit:

$$\overline{\lambda f} = \lambda \lim_{N\to\infty} \frac{1}{N} \int_t^{t+N} f(\tau)d\tau = \lambda \overline{f}. \tag{A.0.10}$$

□

(ii) The averaging operator is idempotent:

$$\overline{\overline{f}} = \overline{f}. \tag{A.0.11}$$

Proof. We assumed that the mean $\overline{f}$ is independent of t. Applying the averaging operator on the mean $\overline{f}$, we have

$$\overline{\overline{f}} = \lim_{N\to\infty} \frac{1}{N} \int_t^{t+N} \overline{f}(x,y)d\tau \tag{A.0.12}$$

$$= \lim_{N\to\infty} \overline{f}(x,y) \frac{1}{N} \int_t^{t+N} 1 d\tau \tag{A.0.13}$$

$$= \lim_{N\to\infty} \overline{f}(x,y) \tag{A.0.14}$$

$$= \overline{f}. \tag{A.0.15}$$

□

(iii) If the dependent variable f is written as a sum of the mean and a fluctuation, the average of the fluctuation variable is zero:

$$\overline{f'} = 0. \tag{A.0.16}$$

Proof. Splitting the dependent variable f into the mean and a fluctuation, we have

$$f = \overline{f} + f'. \tag{A.0.17}$$

Applying the averaging operator on (A.0.17), we obtain

$$\overline{f} = \overline{\overline{f} + f'} \tag{A.0.18}$$

$$= \overline{\overline{f}} + \overline{f'} \qquad \overline{(\cdot)} \text{ is linear} \tag{A.0.19}$$

$$= \overline{f} + \overline{f'} \qquad \overline{(\cdot)} \text{ is idempotent.} \tag{A.0.20}$$

Hence $\overline{f'} = 0$. □

(iv) The averaging operator commutes with spatial differentiation:

$$\overline{\frac{\partial f}{\partial s}} = \frac{\partial \overline{f}}{\partial s}, \tag{A.0.21}$$

where s can be one of the variables, x, y.

Proof. Applying the averaging operator on the derivative $\frac{\partial f}{\partial s}$, we have

$$\overline{\frac{\partial f}{\partial s}} = \lim_{N\to\infty} \frac{1}{N} \int_t^{t+N} \frac{\partial f}{\partial s}(\tau, x, y) d\tau \tag{A.0.22}$$

$$= \lim_{N\to\infty} \frac{\partial}{\partial s} \left(\frac{1}{N} \int_t^{t+N} f(\tau, x, y) d\tau \right). \tag{A.0.23}$$

We assume that $\frac{1}{N}\int_t^{t+N} f(\tau)d\tau$ and $\frac{\partial}{\partial s}\left(\frac{1}{N}\int_t^{t+N} f(\tau)d\tau\right)$ converge uniformly as $N \to \infty$. Hence

$$\overline{\frac{\partial f}{\partial s}} = \frac{\partial}{\partial s} \left(\lim_{N\to\infty} \frac{1}{N} \int_t^{t+N} f(\tau, x, y) d\tau \right) = \frac{\partial \overline{f}}{\partial s}. \tag{A.0.24}$$

□

(v) The average of the time derivative vanishes:

$$\overline{\frac{\partial f}{\partial t}} = 0. \tag{A.0.25}$$

Proof. Applying the averaging operator on the time partial derivative $\frac{\partial f}{\partial t}$, we have

$$\overline{\frac{\partial f}{\partial t}} = \lim_{N\to\infty} \frac{1}{N} \int_t^{t+N} \frac{\partial f}{\partial \tau} d\tau \tag{A.0.26}$$

$$= \lim_{N\to\infty} \left(\frac{f(t+N, x, y)}{N} - \frac{f(t, x, y)}{N} \right) \tag{A.0.27}$$

$$= 0 \tag{A.0.28}$$

by assumption (A.0.2). □

(vi) The averaging operator does not necessarily distribute over multiplication:

$$\overline{fg} \neq \overline{f}\,\overline{g} \tag{A.0.29}$$

in general.

Proof. Suppose $f(t) = \cos t$ and $g(t) = \cos t$. Then

$$\overline{f} = \overline{g} = \lim_{N\to\infty} \frac{1}{N} \int_t^{t+N} \cos\tau d\tau \tag{A.0.30}$$

$$= \lim_{N\to\infty} \left(\frac{\sin(t+N)}{N} - \frac{\sin t}{N} \right) \tag{A.0.31}$$

$$= 0 \tag{A.0.32}$$

and

$$\overline{fg} = \lim_{N\to\infty} \frac{1}{N} \int_t^{t+N} \cos^2 \tau d\tau. \tag{A.0.33}$$

Using the trigonometric identity

$$\cos^2 \theta = \frac{1}{2} \cos 2\theta + \frac{1}{2}, \tag{A.0.34}$$

equation (A.0.33) becomes

$$\overline{fg} = \lim_{N\to\infty} \frac{1}{N} \int_t^{t+N} \left(\frac{1}{2} \cos 2\tau + \frac{1}{2} \right) d\tau \tag{A.0.35}$$

$$= \lim_{N\to\infty} \left(\frac{\sin 2(t+N)}{4N} + \frac{t+N}{2N} - \frac{\sin 2t}{4N} - \frac{t}{2N} \right) \tag{A.0.36}$$

$$= \frac{1}{2}. \tag{A.0.37}$$

□

Appendix B

Codes

B.1 Chapter 2 - Codes

In this Section, we present the codes that were used to generate results in Chapter 2.

Chapter 2 - Example 2 code to solve the initial value problem

$$\frac{d^2y}{dt^2} + 4y(t) = 0, \qquad y(0) = -2, \qquad \frac{dy}{dt}(0) = 0. \tag{B.1.1}$$

```
from scipy.integrate import odeint
import numpy as np
import matplotlib.pyplot as plt

def DiffEqn(y,x):
    return [y[1], -4*y[0]] # Defining the differential equations

Initial = [-2, 0] #Defining the initial conditions

x = np.arange(0, 3, 0.01)

sln = odeint(DiffEqn, Initial, x) # Numerical solution

Exact = -2*np.cos(2*x)

plt.grid()
plt.xlabel(r'$t$', fontsize=14)
plt.ylabel(r'y(t)', fontsize=14)

plt.plot(x, sln[:,0], 'k', label = 'Analytical') # plot y(t) obtained
    from the numerical solution
plt.plot(x, Exact, 'k--', label = 'Numerical') # plot Exact solution
plt.legend()
plt.show()
```

LISTING B.1: Python code for initial value problem.

Chapter 2 - Example 3 code to solve the boundary value problem

$$\frac{d^2F}{d\xi^2} + F\frac{dF}{d\xi} = 0, \tag{B.1.2}$$

with boundary conditions

$$F(0) = 0, \qquad \frac{d^2F}{d\xi^2}(0) = 0, \qquad \frac{dF}{d\xi}(\infty) = 0 \tag{B.1.3}$$

and the conserved quantity

$$1 = \int_0^\infty \left(\frac{dF}{d\xi}\right)^2 d\xi. \tag{B.1.4}$$

```
from scipy.integrate import odeint, simps
import numpy as np
import matplotlib.pyplot as plt

inf = 100 # Large enough number to be considered infinite
deltax= 0.01 # interval length
ConsQ = 1 # Conserved Quantity Constant
z0 = 1 # initial guesses
z1 = 1.2

def DiffEqn(y,x):
    return [y[1], -y[0]*y[1]] # Defining the differential equations

def CalcConsQ(z): # Calculate conserved quantity given initial
    conditions
    initial = [0,z]
    x = np.arange(0,inf, deltax)
    sln = odeint(DiffEqn, initial, x)

    return simps(sln[:,1]**2, x)

def ComputeStep(z0, z1):
    return (ConsQ-CalcConsQ(z1))/(CalcConsQ(z1)-CalcConsQ(z0))*(z1-z0)

def CalcLambda(z0, z1):
    while np.abs(ConsQ - CalcConsQ(z1))>10**(-6):
        tempz = z1
        step = ComputeStep(z0, z1)
        z1 = z1+step
        z0 = tempz

    return z1

Initial = [0, CalcLambda(z0, z1)]
# initial conditions after lambda is obtained

x = np.arange(0, 10, deltax)

sln = odeint(DiffEqn, Initial, x) # Numerical solution

Exact = (3*ConsQ)**(1/3)*np.tanh(0.5*(3*ConsQ)**(1/3)*x)

plt.grid()
plt.xlabel(r'$\xi$', fontsize=14)
plt.ylabel(r'$F(\xi)$', fontsize=14)

```

```
46 plt.plot(x, sln[:,0], 'k', label = 'Analytical') # plot numerical
       solution
47 plt.plot(x, Exact, 'k--', label = 'Numerical') # plot Exact solution
48 plt.legend()
49 plt.show()
```

LISTING B.2: Python code for the shooting method.

B.2 Classical thermal wake: numerical shooting method

In this Section, we present the code that was used to find the numerical solution for the classical thermal wake. The code solves the following initial value problem. Differential equations:

$$\frac{d^2F}{d\xi^2} = \frac{\nu}{2l_1^2} - \sqrt{\left(\frac{\nu}{2l_1^2}\right)^2 + \frac{U}{2l_1^2}\xi\frac{dF}{d\xi}}, \tag{B.2.1}$$

$$\frac{dG}{d\xi} = -\frac{U}{2}\frac{\xi G}{\frac{\kappa}{\rho c_v} - l_1 l_2 \frac{d^2F}{d\xi^2}}. \tag{B.2.2}$$

Initial conditions:

$$F(0) = 0, \qquad \frac{dF}{d\xi}(0) = A, \qquad G(0) = B. \tag{B.2.3}$$

The targets for the shooting method are

$$D = 2\rho U \int_0^{\xi_B} \frac{dF}{d\xi} d\xi, \qquad K = 2\rho c_v U \int_0^{\xi_B} G(\xi) d\xi. \tag{B.2.4}$$

The values of ν, $\frac{\kappa}{\rho c_v}$, l_1, l_2, U, D and K are prescribed.

```
from numpy import linalg
from scipy.integrate import odeint, simps
import numpy as np
from xml.dom import minidom

################################################################
# Prescribed constants
l = 1
l0 = 1
nu = 0
cv = 1
k = 0.01
rho = 1
consD = 1
consK = 1
U = 1
inf = 100
deltax = 0.01
################################################################
# Defining the differential equations
def SimilaritySln(y,x):
   xprime = nu/(2*l**2)-np.sqrt((nu/(2*l**2))**2+(U*x*y[1])/(2*l**2))
   return [y[1], xprime, -U/2*x*y[2]/(k/(rho*cv) - l*l0*xprime)]

################################################################
# Calculating J given values of A and B

def CalcD(A,B):
    initial = [0, A, B]
    x = np.arange(0,inf,0.01) # Infinity = 100
    sln = odeint(SimilaritySln, initial, x)
    sqrdDer = []
    [m, n] = np.shape(sln)
    last = 0

```

```
    for i in range(m):
      if isinstance(sln[i,1], (int, float)):
         # This is to check that the solution value is real/unreal values are excluded
          last = last + 1
          sqrdDer.append(sln[i,1])

    D = 2*rho*U*simps(sqrdDer, x[:last])
    # Calulate the conserved quantity up to well-defined values
    return D
#################################################################
# Calculating K given values of A and B
def CalcK(A,B):
    initial = [0, A, B]
    x = np.arange(0,inf,0.01)
    sln = odeint(SimilaritySln, initial, x)
    sqrdKDer = []
    [m, n] = np.shape(sln)
    last = 0

    for i in range(m):
       if isinstance(sln[i,1], (int, float)) and isinstance(sln[i,2], (int, float)):
          # This is to check that the solution value is real/unreal values are excluded
          last = last + 1
          sqrdKDer.append(sln[i,2])

    K = 2*rho*cv*U*simps(sqrdKDer, x[:last])
    # Calulate the conserved quantity up to well-defined values
    return K
#################################################################
# Calculating steps delta_1 and delta_2 given A_{k-1}, B_{k-1} and A_k, B_k
def ComputeSteps(A1,B1,A2,B2):
    JacobianMatrix = [[(CalcD(A2,B1)-CalcD(A1,B1))/(A2-A1), 0],\
                [(CalcK(A2,B1)-CalcK(A1,B1))/(A2-A1),(CalcK(A1,B2)-CalcK(A1,B1))/(B2-B1)]]
    rhs =[consD-CalcD(A2,B2),consK-CalcK(A2,B2)]
    [epsilon1, epsilon2] = linalg.solve(JacobianMatrix, rhs)
    return [epsilon1, epsilon2]
#################################################################
# Calculating the initial conditions using the shooting method
def CalculateAB(A1, B1, A2, B2):
   while np.abs(consD-CalcD(A2,B2))+np.abs(consK-CalcK(A2,B2))>10**(-7):
      tempA = A2
      tempB = B2
      [step1, step2]=ComputeSteps(A1,B1,A2,B2)
      [A2,B2]=[A2+step1,B2+step2]
      A1 = tempA
      B1 = tempB
      output = [A2,B2]

   return [A2,B2]

#################################################################
# Calculate the numerical solution
[A, B] = CalculateAB(A1, B1, A2, B2)
initial = [0, A, B]
x = np.arange(0, inf,0.01)
```

```
sln = odeint(SimilaritySln, initial, x)
```

LISTING B.3: Python code for the shooting method for the classical thermal wake problem.

B.3 Momentumless thermal wake: numerical shooting method

In this Section, we present the codes that were used to find the numerical solution for the momentumless thermal wake. The codes solve the following initial value problem. Differential equations:

$$\frac{d^2F}{d\xi^2} = -\frac{\nu}{2l_1^2} + \sqrt{\left(\frac{\nu}{2l_1^2}\right)^2 - \frac{U}{2l_1^2}\left(2F(\xi) + \xi\frac{dF}{d\xi}\right)} \text{ when } 2F + \xi\frac{dF}{d\xi} < 0, \tag{B.3.1}$$

$$\frac{d^2F}{d\xi^2} = \frac{\nu}{2l_1^2} - \sqrt{\left(\frac{\nu}{2l_1^2}\right)^2 + \frac{U}{2l_1^2}\left(2F(\xi) + \xi\frac{dF}{d\xi}\right)} \text{ when } 2F + \xi\frac{dF}{d\xi} > 0, \tag{B.3.2}$$

$$\frac{dG}{d\xi} = -\frac{U\xi G}{2\left(\frac{\kappa}{\rho c_v} + l_1 l_2 \left|\frac{d^2F}{d\xi^2}\right|\right)}. \tag{B.3.3}$$

Initial conditions:

$$F(0) = 0, \qquad \frac{dF}{d\xi}(0) = A, \qquad G(0) = B. \tag{B.3.4}$$

The targets of the shooting method are

$$J = 2\rho U\int_0^{\xi_B} \xi^2\frac{dF}{d\xi}d\xi, \qquad K = 2\rho c_v U\int_0^{\xi_B} G(\xi)d\xi, \tag{B.3.5}$$

with values of ν, $\frac{\kappa}{\rho c_v}$, ρ, l_1, l_2, U, J and K prescribed.

```
from numpy import linalg
from scipy.integrate import odeint, simps
import numpy as np

############################################################
class ThermalWakeSimilarySolution:

    def __init__(self, l_1, l_2, nu, kappa, rho, cv, J, K, U, inf, delx,
     A1,\
                 A2, B1, B2):
        self.momMixL = l_1 #All constants that will be required by the
    numerical code
        self.theMixL = l_2
        self.viscosity = nu
        self.conductivity = kappa
        self.density = rho
        self.Tcoefficient = cv
        self.momConservQ = J
        self.therConservQ = K
        self.mainVelocity = U
        self.boundary = inf
        self.deltax = delx
        self.AinitialGuess = [A1, A2]
```

```
22        self.BinitialGuess = [B1, B2]
23        self.AcurrentBestGuess = (A1+A2)/2
24        self.BcurrentBestGuess = (B1+B2)/2
25        self.similaritySln = []
26
27    def changeForFandG(self, y, x): # This is use to define the
      differential Eqn
28        sign_indicator = 2*y[0]+x*y[1] # Determining the sign of the
      second
29        #derivative of  F
30
31        if sign_indicator <= 0:
32            second_der = (-self.viscosity+np.sqrt(self.viscosity**2-2*self
      .mainVelocity*self.momMixL**2*(2*y[0]+x*y[1])))/(2*self.momMixL**2)
33
34        elif sign_indicator > 0:
35            second_der = (self.viscosity-np.sqrt(self.viscosity**2+2*self.
      mainVelocity*self.momMixL**2*(2*y[0]+x*y[1])))/(2*self.momMixL**2)
36        else:
37            second_der = None
38
39        changeG = -self.mainVelocity/2*x*y[2]/(self.conductivity/(self.
      density*self.Tcoefficient)+self.momMixL*self.theMixL*np.abs(
      second_der))
40        return [y[1], second_der, changeG]
41
42    def ComputeJ(self, A, B): # Compute the J conserved quantity
43        initial = [0, A, B]
44        x = np.arange(0, self.boundary, self.deltax)
45        sln = odeint(self.changeForFandG, initial, x)
46        integrand = []
47        itr = 0
48
49        for yi in sln:
50            integrand.append(x[itr]**2*yi[1])
51            itr += 1
52
53        return 2*self.density*self.mainVelocity*simps(integrand, x)
54
55    def ComputeK(self, A, B): # Compute the K conserved quantity
56        initial = [0, A, B]
57        x = np.arange(0, self.boundary, self.deltax)
58        sln = odeint(self.changeForFandG, initial, x)
59        integrand = sln[:,2]
60
61        return 2*self.density*self.mainVelocity*self.Tcoefficient*simps(
      integrand, x)
62
63
64    def ComputeSteps(self, A1,B1,A2,B2): # Compute the steps
65        JacobianMatrix = [[(self.ComputeJ(A2,B1)-self.ComputeJ(A1,B1))/(
      A2-A1), 0],\
66                          [(self.ComputeK(A2,B1)-self.ComputeK(A1,B1))/(
      A2-A1),(self.ComputeK(A1,B2)-self.ComputeK(A1,B1))/(B2-B1)]]
67        rhs =[self.momConservQ-self.ComputeJ(A2,B2),self.therConservQ-
      self.ComputeK(A2,B2)]
68        [epsilon1, epsilon2] = linalg.solve(JacobianMatrix, rhs)
69        return [epsilon1, epsilon2]
70
71    def UpdateBestAB(self): # Updating the guesses of A and B
72        [A1, A2] = self.AinitialGuess
73        [B1, B2] = self.BinitialGuess
74
```

```
75        while np.abs(self.momConservQ-self.ComputeJ(A2,B2))+np.abs(self.
    therConservQ-self.ComputeK(A2,B2))> 10**(-7):
76            tempA = A2
77            tempB = B2
78            [step1, step2]=self.ComputeSteps(A1,B1,A2,B2)
79            [A2,B2]=[A2+step1,B2+step2]
80            A1 = tempA
81            B1 = tempB
82
83        self.AcurrentBestGuess = A2
84        self.BcurrentBestGuess = B2
85
86    def updateSimilaritySol(self): # Obtain the current similarity
    solution based
87        #on the current values of A and B
88        self.UpdateBestAB()
89        initial = [0, self.AcurrentBestGuess, self.BcurrentBestGuess]
90        x = np.arange(0, self.boundary, self.deltax)
91        self.similaritySln = odeint(self.changeForFandG, initial, x)
92
93    def getSimilaritySol(self): # Returns the similarity solution
94        return self.similaritySln
```

LISTING B.4: Python code for the shooting method for the momentumless thermal wake problem (Part I).

```
1  import numpy as np
2  import SelfPropelledWake
3  ####################################################
4  # Constants reguired to solve the numerical problem
5  L = 1
6  Lt = 1
7  nu = 0.01
8  cv = 1
9  kappa = 0.01
10 rho = 1
11 consD = 1
12 consK = 1
13 U = 1
14 Infinity = 100
15 ####################################################
16 # The numerical solution F(\xi), F'(\xi) and G(\xi)
17 WakeSln = SelfPropelledWake.ThermalWakeSimilarySolution(L, Lt, nu,
    kappa, rho, cv, consD, consK, U, Infinity,\
18                         0.01, -0.7, -1.0, 0.5, 0.9)
19 WakeSln.updateSimilaritySol()
20 sln = WakeSln.getSimilaritySol()
```

LISTING B.5: Python code for the shooting method for the momentumless thermal wake problem (Part II).