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2014
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Invariances, Conservation laws and Conserved quantities of the two-dimensional nonlinear Schrödinger-type equation

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A dissertation submitted to the Faculty of Science, University of the Witwatersrand, in fulfillment of the requirements for the degree of Master of Science.

Johannesburg, 2014.

DECLARATION

I declare that the contents of this dissertation are original except where due references have been made. It is being submitted for the degree of Master of Science at the University of the Witwatersrand in Johannesburg. It has not been submitted before for any degree to any other institution.

_____

S.Lepule

This _____ day of _____ 2014.

Acknowledgments

I would firstly like to thank my supervisors Prof. Kara and Prof. Love for guiding me and supporting me while compiling this dissertation, I really could not have done this without them.

I would also like to thank the School of Mathematics for helping me with funding for my Masters degree, I was really lost before they came along.

Lastly, I would like to thank my family and friends for their support and encouragement.

Abstract

Symmetries and conservation laws of partial differential equations (pdes) have been instrumental in giving new approaches for reducing pdes. In this dissertation, we study the symmetries and conservation laws of the two-dimensional Schrödinger-type equation and the Benney-Luke equation, we use these quantities in the Double Reduction method which is used as a way to reduce the equations into a workable pdes or even an ordinary differential equations. The symmetries, conservation laws and multipliers will be determined though different approaches. Some of the reductions of the Schrödinger equation produced some famous differential equations that have been dealt with in detail in many texts.

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Introduction

In this dissertation we will be investigating soliton-type equations, in particular, the nonlinear multidimensional Schrödinger-type equation and the Benney-Luke equation. These equations are characterized by their ability to govern the motion of solitons which will be defined in detail. The main objective of the dissertation is to use the double reduction method to reduce the given equations [18].

The Benney-Luke equation is a well-known Sobolev-type equation, which is used in long wave asymptotic approximation of shallow water waves. The term Sobolev equation is used in Russian literature to refer to any equation with spatial derivatives on the highest order time derivatives [7].

Soliton theory is an attractive field of present day research in nonlinear physics and mathematics [14]. An important component in soliton theory is the nonlinear Schrödinger equation (NLSE) and its variants which appear in a vast spectrum of problems (see [1, 11]). A **soliton** is a pulse-like nonlinear wave (solitary wave) which emerges from a collision with a similar pulse having unchanged shape and speed [17].The solitary wave or soliton was discovered by John Scott Russel in August 1834, he then pioneered the study of the wave and its equation of motion.

The double reduction method will be employed to reduce the Schrödinger equation [2] down to its final solution, this method involves the extensive use of symmetries, conservation laws and multipliers to reduce multidimensional equations into first or second order equations or even ordinary differential equations (odes), which are simpler to handle.

Aims

The aims of the dissertation are as follows:

- investigate the symmetries, conservation laws, conserved vectors and solutions of a NLSE and other soliton equations.
- extend the NSLE to three independent variables (x, y, t), i.e. two space variables and the time component as above. Thus we will be investigating the two-dimensional nonlinear Schrödinger-type equation which is given by the equation, [2, 16]
$$i\frac{\partial \phi}{\partial t} = -\frac{1}{2m}\nabla^2\phi + g\left|\phi\right|^2\phi. \tag{0.0.1}$$
- use the multiplier method for systems in order to determine the conservation laws of the NLSE.
- investigate the solutions of the two-dimensional NLSE using the double reduction method given in [4].

This dissertation is structured as follows.

In the first chapter, we state definitions of concepts that will be required to perform the calculations that are vital in this dissertation.

In the second chapter, we provide an illustrative example through the reduction of the Benney-Luke equation. We will compute the symmetries, conserved vectors and multipliers which will be then used to reduce the partial differential equation (pde) in the double reduction method.

In the third chapter, we will apply the same technique we did in the second chapter to the nonlinear multidimensional Schrödinger equation.

Chapter 1

Definitions and Notation

This chapter contains definitions, notations and theorems required to analyze the equations in this dissertation.

A function $f(x, u, u_{(1)}, \ldots, u_{(r)})$ of a finite number of variables is called a differential function of order r [9, 19, 20].

We denote $\mathcal{A}$ to be the universal vector space of differential functions .

Consider an rth-order system of pdes of independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$

$$G^\mu(x, u, u_{(1)}, \ldots, u_{(r)}) = 0, \qquad \mu = 1, \ldots, \bar{m}, \tag{1.0.1}$$

where $u_{(1)}, u_{(2)}, \ldots, u_{(r)}$ denote the collection of all first, second, ..., rth-order partial derivatives, that is, $u_i^\alpha = D_i(u^\alpha)$, $u_{ij}^\alpha = D_j D_i(u^\alpha)$, ...respectively, with the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \ldots, \qquad i = 1, \ldots, n, \tag{1.0.2}$$

where the summation convention is used whenever appropriate. A current $\Phi = (\Phi^1, \ldots, \Phi^n)$ is conserved if it satisfies

$$D_i \Phi^i = 0 \tag{1.0.3}$$

along the solutions of (1.0.1). It can be shown that every admitted conservation law arises from multipliers $Q_\mu(x, u, u_{(1)}, \ldots)$ such that

$$Q_\mu G^\mu = D_i \Phi^i \tag{1.0.4}$$

holds identically (i.e., off the solution space) for some current Φ^i [6]. The conserved vector is then obtained by the homotopy operator [8, 12].

Definition 1. The *Euler* operator, for each α, is defined by

$$\mathcal{E} = \frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \ldots D_{i_s} \frac{\partial}{\partial u_{i_1 \ldots i_s}^\alpha}, \qquad \alpha = 1, \ldots, m. \tag{1.0.5}$$

The Euler operator is also sometimes referred to as the Euler-Lagrange operator [5].

Definition 2. The Lie-Bäcklund or generalised operator is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \qquad \xi^i, \eta^\alpha \in \mathcal{A} \tag{1.0.6}$$

The operator is an abbreviated form of the following infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \varsigma^\alpha_{i_1,\ldots,i_s} \frac{\partial}{\partial u^\alpha_{i_1,\ldots,i_s}}, \tag{1.0.7}$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} \varsigma^\alpha_i &= D_i(W^\alpha) + \xi^j u^\alpha_{ij}, \\ \varsigma^\alpha_{i_1,\ldots,i_s} &= D_{i1} \ldots D_{is}(W^\alpha) + \xi^j u^\alpha_{ji_1,\ldots,i_s}, \qquad s > 1. \end{aligned} \tag{1.0.8}$$

In (1.0.8), W^α is the Lie characteristic function given by

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j. \tag{1.0.9}$$

One can write the Lie-Bäcklund or generalised operator (1.0.7) in the characteristic form

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} D_{i1} \ldots D_{is}(W^\alpha) \frac{\partial}{\partial u^\alpha_{i1\ldots i_s}}. \tag{1.0.10}$$

We determine the conserved flows [23, 24] by first constructing the multipliers Q_μ which are obtained by noting that the Euler operator annihilates total divergences, i.e., the defining equation would be

$$\frac{\delta}{\delta u^\alpha}[Q_\mu G^\mu] = 0. \tag{1.0.11}$$

Noether's Theorem

The relationship between conservation laws and symmetries of differential equations is a very popular topic which has sparked a lot of interest which eventually lead to interesting concepts like Noether's Theorem which was first proved by Emmy Noether,[13] in 1915, and later published in 1918.

Stated below is the formal statement of Noether's Theorem taken from a book by D.E.Neuenschwander, entitled *Emmy Noether's Wonderful Theorem.*

Theorem 1.0.1. ***Noether's Theorem****[13] If the functional*

$$J = \int_a^b \mathcal{L}(t, q^\mu, \dot{q}^\mu)dt$$

is an extremal, and if under the infinitesimal transformation

$$t' = t + \epsilon\tau + ...$$
$$q^{\mu'} = q^\mu + \epsilon\zeta^\mu + ...$$

the functional is invariant according to the definition (allowing for the inhomogeneous case)

$$\mathcal{L}'\frac{dt'}{dt} - \mathcal{L} = \epsilon\frac{dF}{dt} + O(\epsilon^3), wheres > 1,$$

then the following conservation law holds:

$$p_\mu\zeta^\mu - H\tau - F = const.$$

Main Theorems

The following theorems will form the basis of much of this dissertation.

The first theorem will help us to determine whether there is a an association between the symmetry X and a conserved vector $\mathbf{T}$.

Theorem 1.0.2. *[4] Suppose that X is any Lie-Bäcklund symmetry of a system of differential equations and T_i, $i = 1, ..., n$, are the components of the conserved vector of the system of differential equations. Then*

$$T^{*i} = [T^i, X] = X(T^i) + T^i D_j \eta^j - T^j D_j \eta^i, \ i = 1, ..n \tag{1.0.12}$$

constitute the components of a conserved vector of a system of differential equations.

Theorem 1.0.3. *[4] Suppose that $D_i T^i = 0$ is a conservation law of a PDE system. Then, under a contact transformation, there exists functions $\tilde{T}^i$ such that $J\ D_i T^i = \tilde{D}_i \tilde{T}^i$, where $\tilde{T}^i$ is given as*

$$. \begin{pmatrix} \tilde{T}^1 \\ \tilde{T}^2 \\ \vdots \\ \tilde{T}^n \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix}, \quad J \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix} = A^T \begin{pmatrix} \tilde{T}^1 \\ \tilde{T}^2 \\ \vdots \\ \tilde{T}^n \end{pmatrix} \tag{1.0.13}$$

in which

$$A = \begin{pmatrix} \tilde{D}_1 x_1 & \tilde{D}_1 x_2 & \cdots & \tilde{D}_1 x_n \\ \tilde{D}_2 x_1 & \tilde{D}_2 x_2 & \cdots & \tilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_n x_1 & \tilde{D}_n x_2 & \cdots & \tilde{D}_n x_n \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} D_1 \tilde{x_1} & D_1 \tilde{x_2} & \cdots & D_1 \tilde{x_n} \\ D_2 \tilde{x_1} & D_2 \tilde{x_2} & \cdots & D_2 \tilde{x_n} \\ \vdots & \vdots & \vdots & \vdots \\ D_n \tilde{x_1} & D_n \tilde{x_2} & \cdots & D_n \tilde{x_n} \end{pmatrix} \tag{1.0.14}$$

and $J = \det(A)$.

Theorem 1.0.4. ***(fundamental theorem on double reduction)*** *[4] Suppose that* $D_i T^i = 0$ *is a conservation law of a system of partial differential equations. Then under a similarity transformation of a symmetry* X *of the form (.....) for the pde, there exist functions* $\tilde{T}^i$ *such that* X *is still a symmetry for the pde* $\tilde{D}_i \tilde{T}^i = 0$ *and*

$$\begin{pmatrix} X\tilde{T}^1 \\ X\tilde{T}^2 \\ \vdots \\ X\tilde{T}^n \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} [T^1, X] \\ [T^2, X] \\ \vdots \\ [T^n, X] \end{pmatrix}, \tag{1.0.15}$$

where

$$A = \begin{pmatrix} \tilde{D}_1 x_1 & \tilde{D}_1 x_2 & \cdots & \tilde{D}_1 x_n \\ \tilde{D}_2 x_1 & \tilde{D}_2 x_2 & \cdots & \tilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_n x_1 & \tilde{D}_n x_2 & \cdots & \tilde{D}_n x_n \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} D_1 \tilde{x_1} & D_1 \tilde{x_2} & \cdots & D_1 \tilde{x_n} \\ D_2 \tilde{x_1} & D_2 \tilde{x_2} & \cdots & D_2 \tilde{x_n} \\ \vdots & \vdots & \vdots & \vdots \\ D_n \tilde{x_1} & D_n \tilde{x_2} & \cdots & D_n \tilde{x_n} \end{pmatrix} \tag{1.0.16}$$

and $J = \det(A)$.

Our original system is equivalent to

$$sys_1 = \begin{cases} {q_1}^1 G^1 + {q_1}^2 G^2 = 0, \\ {q_1}^1 G^1 - {q_1}^2 G^2 = 0. \end{cases} \tag{1.0.17}$$

This system can be rewritten as

$$\begin{aligned} D_t {T_1}^t + D_x {T_1}^x + D_y {T_1}^y &= 0, \\ {q_1}^1 G^1 - {q_1}^2 G^2 &= 0. \end{aligned} \tag{1.0.18}$$

Chapter 2

The Benney-Luke equation

In this chapter we will present an illustrative example via a version of the Benney-Luke equation.

2.1 Introduction

The Benney-Luke equation is quite an established equation and has been researched quite extensively over a long period of time. The Stability, Cauchy Problem , Existence and Analyticity of solutions have been researched widely [22].

The Benney-Luke equation is used to model the evolution of long water waves with small amplitudes, in particular the equation below models three-dimensional water waves with surface tension [15].

The Benney-Luke equation reduces to the Kadomtesev-Petviashvil (KP) equation for waveforms propagating predominantly in one direction, slowly evolving in time and having weak transverse variation. Many works have been published about the KP equations which are obtained under the assumption of quasi-two dimensionality and undirectional propagation [15].

The Benney-Luke equation is given by,

$$u_{tt} - u_{xx} - \epsilon u_{yy} + \alpha\epsilon(u_{xxxx} + \epsilon u_{yyyy}) + \epsilon[\partial_t(u_x^2 + \epsilon u_y^2) + u_t(u_{xx} + \epsilon u_{yy})] = 0. \quad (2.1.1)$$

The work covered in this chapter has been submitted for publication, see [21].

2.2 Symmetries and Conservation Laws

The Lie point symmetry generators of (2.1.1) are

$$\begin{aligned} X_1 &= \partial_x, \\ X_2 &= \partial_y, \\ X_3 &= \partial_t, \\ X_4 &= 1. \end{aligned}$$

We consider a linear combination of X_1, X_2 and X_3, viz.,

$$X = c\partial_x + m\partial_y + \partial_t. \quad (2.2.2)$$

We have chosen a linear combination of X_1, X_2 and X_3 above because we aim to obtain a non trivial solution to the equation (2.1.1).

The multipliers and corresponding conserved vectors, which were computed using the methods presented in Chapter 1 namely Definition 2, Equation (1.0.6) and Equation (1.0.10) are,

(i) $Q_1 = 1$

$$\begin{aligned} T_1^x &= (-1 + \epsilon u_t)\, u_x + \alpha\epsilon u_{xxx}, \\ T_1^y &= \epsilon\left((-1 + \epsilon u_t)\, u_y + \alpha\epsilon u_{yyy}\right), \\ T_1^t &= \tfrac{1}{2}\left(2u_t + \epsilon\left(\epsilon u_y^2 + u_x^2\right)\right). \end{aligned}$$

(ii) $Q_2 = u_x$

$$\begin{aligned} T_2^x &= \tfrac{1}{6}\left(\left(-3 + 4\epsilon u_t\right) u_x^2 + u\left(3u_{tt} + \epsilon\left(4\epsilon u_y u_{yt} + \left(-3 + 2\epsilon u_t\right) u_{yy} + 3\alpha\epsilon u_{yyyy} + 2u_x u_{xt}\right)\right)\right) \\ &\quad + \tfrac{1}{6}\left(-3\alpha\epsilon u_{xx}^2 + 6\alpha\epsilon u_x u_{xxx}\right), \\ T_2^y &= \tfrac{1}{6}\epsilon\left(3\alpha\epsilon u_{yyy} u_x + 3u u_{xy} - 2\epsilon u u_t u_{xy} - 3\alpha\epsilon u_{yy} u_{xy}\right) \\ &\quad + \tfrac{1}{6}\left(u_y\left(\left(-3 + 4\epsilon u_t\right) u_x - 2\epsilon u u_{xt} + 3\alpha\epsilon u_{xyy}\right) - 3\alpha\epsilon u u_{xyyy}\right), \\ T_2^t &= \tfrac{1}{6}\left(3u_t u_x + 2\epsilon^2 u_y^2 u_x + 2\epsilon u_x^3 - 3u u_{xt} - 2\epsilon^2 u u_y u_{xy} - 2\epsilon u u_x u_{xx}\right). \end{aligned}$$

(iii) $Q_3 = u_y$

$$\begin{aligned} T_3^x &= \tfrac{1}{6}\left(-3\alpha\epsilon u_{xy} u_{xx} + 3\alpha\epsilon u_x u_{xxy}\right) \\ &\quad + \tfrac{1}{6}\left(u_y\left(\left(-3 + 4\epsilon u_t\right) u_x + 3\alpha\epsilon u_{xxx}\right) - u\left(2\epsilon u_{yt} u_x + \left(-3 + 2\epsilon u_t\right) u_{xy} + 3\alpha\epsilon u_{xxxy}\right)\right), \\ T_3^y &= \tfrac{1}{6}\left(\epsilon\left(\left(-3 + 4\epsilon u_t\right) {u_y}^2 - 3\alpha\epsilon u_{yy}^2 + 6\alpha\epsilon u_y u_{yyy}\right)\right) \\ &\quad + \tfrac{1}{6}\left(u\left(3u_{tt} + 2\epsilon^2 u_y u_{yt} + 4\epsilon u_x u_{xt} - 3u_{xx} + 2\epsilon u_t u_{xx} + 3\alpha\epsilon u_{xxxx}\right)\right), \\ T_3^t &= \tfrac{1}{6}\left(3u_t u_y + 2\epsilon^2 u_y^3 + 2\epsilon u_y\left(-\epsilon u u_{yy} + u_x^2\right) - u\left(3u_{yt} + 2\epsilon u_x u_{xy}\right)\right). \end{aligned}$$

(iv) $Q_4 = u_t$

$$\begin{aligned}
T_4^x &= \tfrac{1}{6}\left(4\epsilon u_t^2 u_x - 3\alpha\epsilon u_{xt}u_{xx} + 3\alpha\epsilon u_x u_{xxt} + u_t\left(-3u_x - 2\epsilon u u_{xt} + 3\alpha\epsilon u_{xxx}\right)\right) \\
&\quad + \tfrac{1}{6}\left(u\left(-2\epsilon u_{tt}u_x + 3u_{xt} - 3\alpha\epsilon u_{xxxt}\right)\right), \\
T_4^y &= \tfrac{1}{6}\epsilon\left(4\epsilon u_t^2 u_y - 3\alpha\epsilon u_{yt}u_{yy} + 3\alpha\epsilon u_y u_{yyt} + u_t\left(-3u_y - 2\epsilon u u_{yt} + 3\alpha\epsilon u_{yyy}\right)\right) \\
&\quad + \tfrac{1}{6}\epsilon\left(u\left(-2\epsilon u_{tt}u_y + 3u_{yt} - 3\alpha\epsilon u_{yyyt}\right)\right), \\
T_4^t &= \tfrac{1}{6}\left(3u_t^2 + 2\epsilon u_t\left(\epsilon u_y^2 + u_x^2 + u\left(\epsilon u_{yy} + u_{xx}\right)\right)\right) \\
&\quad + \tfrac{1}{6}\left(u\left(2\epsilon^2 u_y u_{yt} - 3\epsilon u_{yy} + 3\alpha\epsilon\epsilon u_{yyyy} + 2\epsilon u_x u_{xt} - 3u_{xx} + 3\alpha\epsilon u_{xxxx}\right)\right).
\end{aligned}$$

2.3 Double reduction

The association between X and T_1 will be investigated by substituting the relevant information into the association matrix in Theorem 1.0.2 from Chapter 1.

We get,

$$T^* = X\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} - \begin{pmatrix} D_t\tau & D_x\tau & D_y\tau \\ D_t\xi & D_x\xi & D_y\xi \\ D_t\chi & D_x\chi & D_y\chi \end{pmatrix}\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} + (0)\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} \tag{2.3.3}$$

$$= X\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} + (0)\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.3.4}$$

Therefore X is associated with T_1.

Since an association exists, a solution can be found by applying the double reduction

method. Firstly, transform X to its canonical form $Y = \frac{\partial}{\partial s}$ in (r, s, w, p). Then the generator is of the form

$$Y = 0\frac{\partial}{\partial r} + \frac{\partial}{\partial s} + 0\frac{\partial}{\partial w} + 0\frac{\partial}{\partial p}.$$

Without loss of generality, we choose $X(r) = 0$, $X(s) = 1$, $X(w) = 0$ and $X(p) = 0$, for which we obtain the following invariance condition,

$$\frac{dx}{c} = \frac{dy}{m} = \frac{dt}{1} = \frac{du}{0} = \frac{ds}{1} = \frac{dr}{0} = \frac{dw}{0} = \frac{dp}{0}. \tag{2.3.5}$$

We get the invariants

$$\begin{aligned} b_1 &= x - ct, \\ b_2 &= y - mt, \\ b_3 &= u, \\ b_4 &= r, \\ b_5 &= p, \\ b_6 &= w, \\ b_7 &= s - t. \end{aligned}$$

A suitable combination of the above equations yields the canonical coordinates,

$$\begin{aligned} b_6 = b_3 &\Rightarrow w = u, \\ b_7 = 0 &\Rightarrow s = t, \\ b_4 = b_1 &\Rightarrow r = x - cs, \\ b_5 = b_2 &\Rightarrow p = y - ms, \end{aligned}$$

where $w = w(r, p)$.

The inverse canonical coordinates are given by

$$t = s, \qquad x = r + cs, \qquad y = p + ms, \qquad u = w. \tag{2.3.6}$$

The $\mathbf{A}$ and $\mathbf{A}^{-1}$ matrices are constructed using the equations in (2.3.6) above,

$$\mathbf{A} = \begin{pmatrix} D_r t & D_r x & D_r y \\ D_s t & D_s x & D_s y \\ D_p t & D_p x & D_p y \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & c & m \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{A}^{-1} = \begin{pmatrix} D_t r & D_t s & D_t p \\ D_x r & D_x s & D_x p \\ D_y r & D_y s & D_y p \end{pmatrix} = \begin{pmatrix} -c & 1 & -m \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and $\mathbf{J} = det(\mathbf{A}) = -1$.

The first, second and third derivatives of u in terms of the new dependent variable w are,

$$\begin{aligned} u_x &= w_r, \\ u_y &= w_p, \\ u_t &= -cw_r + mw_p, \\ u_{xxx} &= w_{rrr}, \\ u_{yyy} &= w_{ppp}. \end{aligned} \tag{2.3.7}$$

The reduced conserved form is given below,

$$\begin{pmatrix} T^r \\ T^s \\ T^p \end{pmatrix} = \mathbf{J}(\mathbf{A}^{-1})^T \begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix}. \tag{2.3.8}$$

Now by substituting (2.3.6) and (2.3.7) into (2.3.8) we obtain,

$$\begin{pmatrix} T^r \\ T^s \\ T^p \end{pmatrix} = \begin{pmatrix} (1+c^2)w_r + cmw_p + \frac{1}{2}\epsilon^2 c w_p^2 + \frac{3}{2}c\epsilon w_r^2 - m\epsilon w_p w_r - \alpha\epsilon w_{rrr} \\ cw_r - mw_p - \frac{1}{2}\epsilon^2 w_p^2 - \frac{1}{2}\epsilon w_r^2 \\ -cmw_r + (\epsilon + m^2)w_p - \frac{1}{2}\epsilon^2 m w_p^2 + \frac{1}{2}\epsilon m w_r^2 + c\epsilon^2 w_p w_r - \alpha\epsilon^2 w_{ppp} \end{pmatrix},$$

where the reduced conserved form is also given by $D_r T_1^r = 0$.

The second step of the double reduction method represented by $T^r = k$ which is given as

$$(1-c^2)w_r + cmw_p + \frac{1}{2}\epsilon^2 c w_p^2 + \frac{3}{2}c\epsilon w_r^2 - m\epsilon w_p w_r - \alpha\epsilon w_{rrr} = k \tag{2.3.9}$$

where $k \in \mathbb{R}$ is a constant. We will now solve equation (2.3.9) using its symmetries.

It is clear that $Y_1 = \partial_p$ and $Y_2 = \partial_r$ are symmetries of (2.3.9). We will consider a linear combination of the two say,

$$Y = \partial_r + \beta\partial_p.$$

We do this so we do not obtain a trivial solution to equation (2.1.1).

Whose invariant condition are given by

$$\frac{dr}{1} = \frac{dp}{\beta} = \frac{dw}{0}.$$

That is, let

$$z = p - \beta r, \tag{2.3.10}$$

where w=w(z).

The first, second and third order derivatives of w are,

$$\begin{aligned} w_r &= \quad w'(-\beta), \\ w_{rr} &= \quad w''(-\beta)^2, \\ w_{rrr} &= \quad w'''(-\beta)^3, \\ w_p &= \quad w'(1). \end{aligned} \tag{2.3.11}$$

Substituting equations (2.3.11) into (2.3.9) we obtain,

$$(\beta c^2 - \beta + cm)w' + (\frac{1}{2}\epsilon^2 c + \frac{3}{2}c\epsilon\beta^2 + m\epsilon\beta)w'^2 + \alpha\epsilon\beta^3 w''' = k. \tag{2.3.12}$$

The above computation has now allowed us to transform the pde (2.3.9) into an ode as seen in equation (2.3.12) above.

To further reduce the order of (2.3.12) from a third order ode to a second order ode, let $q = w'$. Thus (2.3.12) becomes

$$(\beta c^2 - \beta + cm)q + (\frac{1}{2}\epsilon^2 c + \frac{3}{2}c\epsilon\beta^2 + m\epsilon\beta)q^2 + \alpha\epsilon\beta^3 q'' = k. \tag{2.3.13}$$

When computing the final solution to equation (2.3.13) we obtain a tedious solution. An analysis of the coefficients of q and its derivatives will be made below.

Let the coefficient of q^2 equal zero, that is

$$\frac{1}{2}\epsilon^2 c + \frac{3}{2}c\epsilon\beta^2 + m\epsilon\beta = 0. \tag{2.3.14}$$

Solve equation (2.3.14) for both ϵ and β:

For β:

$$\beta = \frac{-m \pm \sqrt{m^2\epsilon^2 - 4\frac{3}{2}c\epsilon\frac{1}{2}\epsilon^2 c}}{2\frac{3}{2}c\epsilon}, \qquad c \neq 0, \epsilon \neq 0 \tag{2.3.15}$$
$$= \frac{-m\epsilon \pm \epsilon\sqrt{m^2 - 3\epsilon c^2}}{3c\epsilon}.$$

For ϵ:

$$\epsilon = \frac{\frac{3}{2}\beta^2 c + m\beta}{-\frac{1}{2}c}, \qquad c \neq 0 \tag{2.3.16}$$
$$= -3\beta^2 - \frac{2}{c}m\beta.$$

Substitute the value obtained for ϵ in (2.3.16) into equation (2.3.13),
(ϵ was chosen for its simplicity as compared to β) to obtain

$$-\alpha\beta^4(3\beta + \frac{2}{c}m)q'' + (-\beta + \beta c^2 + cm)q = 0. \tag{2.3.17}$$

Note that (2.3.17) ha the same form of equation as that of a simple harmonic operator.

When solving the ode in (2.3.17) we obtain a solution in z,

$$q = -\frac{ck}{-c^2m + c\beta - c^3\beta} + c_1\, exp\left(\frac{\sqrt{c}\; z\sqrt{cm - \beta + c^2\beta}}{\sqrt{\alpha}\beta^2\sqrt{2m + 3c\beta}}\right) + c_2\, exp\left(-\frac{\sqrt{c}\; z\sqrt{cm - \beta + c^2\beta}}{\sqrt{\alpha}\beta^2\sqrt{2m + 3c\beta}}\right). \tag{2.3.18}$$

Since the solution to (2.3.17) must reside in the null space of the operator, it follows that the constant term in (2.3.18) is zero, which implies that $k = 0$. Hence

$$q = c_1 Exp(A) + c_2 Exp(-A) \tag{2.3.19}$$

where $A = \dfrac{\sqrt{c}\sqrt{cm - \beta + c^2\beta}}{\sqrt{\alpha}\beta^2\sqrt{2m + 3c\beta}}$.

If $A > 0$ then the answer can be written in terms of *cosh* and *sinh*. If $A < 0$ then the solution can be expressed in terms of *cos* and *sin* functions.

$$u = \frac{k(p - \beta r)}{cm - \beta + c^2\beta} + \frac{exp\left(\frac{\sqrt{c}\ (p-\beta r)\sqrt{cm-\beta+c^2\beta}}{\sqrt{\alpha}\beta^2\sqrt{2m+3c\beta}}\right) c_1\sqrt{\alpha}\beta^2\sqrt{2m + 3c\beta}}{\sqrt{c}\sqrt{cm - \beta + c^2\beta}} \tag{2.3.20}$$

$$- \frac{exp\left(-\frac{\sqrt{c}\ (p-\beta r)\sqrt{cm-\beta+c^2\beta}}{\sqrt{\alpha}\beta^2\sqrt{2m+3c\beta}}\right) c_2\sqrt{\alpha}\beta^2\sqrt{2m + 3c\beta}}{\sqrt{c}\sqrt{cm - \beta + c^2\beta}}.$$

Note that if k=0, then the constant term vanishes. As above, the answer can be expressed in terms of the trigonometric functions *cosine* and *sine* or the hypobolic functions *cosh* and *sinh*, depending on the sign of $A = \frac{\sqrt{c}\sqrt{cm-\beta+c^2\beta}}{\sqrt{\alpha}\beta^2\sqrt{2m_3c\beta}}$.

2.4 Conclusion

In this chapter we were able to solve the Benney-Like equation and obtain a non-trivial solution. The symmetries of the Benney-Luke equation were obtained and because of their trivial form we decided to consider a linear combination of the symmetries hence giving us equation (2.2.2).

Next we utilised Theorem 1.0.4, the Fundamental Theorem on Double Reduction, to reduce the Benney-Luke equation into an ode that would be easier to solve. We then reduced the third order ode in equation (2.3.12) which was i.t.o. w into a second order ode i.t.o. q which resulted in equation (2.3.13). We let the coefficient of q^2 in equation (2.3.13) equal zero and this resulted in equation (2.3.17) which is an equation in the same form as a simple harmonic operator. We were then able to solve the ode and found that the solution was in terms of the trigonometric functions *cosine* and *sine* or the hypobolic functions *cosh* and *sinh* depending on the sign of A.

Chapter 3

Two-dimensional Schrödinger-type equation

3.1 Introduction and Background

In this chapter we consider the two-dimensional NLSE

$$i\frac{\partial q}{\partial t} = -\frac{1}{2m}\nabla^2 q + g\left|q\right|^2 q, \tag{3.1.1}$$

where $g = 1$ and $m = 1$.

The equation in (3.1.1) above comes from [16], which is a paper about two-dimensional dark solitions.

The Schrödinger equation is one of the cornerstone of quantum physics, which de-

scribes what a system of quantum objects such as atoms and subatomic particles will do in the future based on it's current state [25].

The nonlinear Schrödinger equation has been studied extensively but the behavior of its symmetries, conserved vectors and solutions given an increase in it's dimensions has not been studied yet. In this chapter we will compute the symmetries and conservation laws of (3.1.1) and use them to reduce (3.1.1) through the double reduction method.

To create a system of equations, substitute $q = u + iv$ into equation (3.1.1) and recall that $u = u(x, y, t)$, $v = v(x, y, t)$. Then separate into real and imaginary parts to obtain

$$\begin{aligned} G^1 &= v_t - \tfrac{1}{2}(u_{xx} + u_{yy}) + u(u^2 + v^2) = 0, \\ G^2 &= -u_t - \tfrac{1}{2}(v_{xx} + v_{yy}) + v(u^2 + v^2) = 0. \end{aligned} \tag{3.1.2}$$

3.2 Symmetries, conserved vectors and the Double Reduction method

The symmetries and conservation laws were computed by using the Lagrangian method and Noether's theorem.

The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{2}vu_t + \frac{1}{2}uv_t + \frac{1}{4}(u_x^2 + v_x^2 + v_y^2 + u_y^2) + \frac{1}{4}(u^2 + v^2)^2. \tag{3.2.3}$$

3.2.1 Case 1: $X_1 = \partial_t + k(u\partial_v - v\partial_u)$

The first prolongation of X_1 which is represented below,

$$X_1^{[1]} = \partial_t + k(u\partial_v - v\partial_u + u_t\partial_{v_t} + u_x\partial_{v_x} + u_y\partial_{v_y} - v_t\partial_{u_t} - v_x\partial_{u_x} - v_y\partial_{u_y}).$$

Through the use of Noether's theorem we get the following conserved vector,

$$\begin{pmatrix} T_1^t \\ T_1^x \\ T_1^y \end{pmatrix} = \begin{pmatrix} -\frac{1}{4}u_x^2 - \frac{1}{4}v_x^2 - \frac{1}{4}u_y^2 - \frac{1}{4}v_y^2 - \frac{1}{4}u^4 - \frac{1}{2}u^2v^2 - \frac{1}{4}v^4 \\ \frac{1}{2}v_tv_x + \frac{1}{2}u_tu_x \\ \frac{1}{2}v_tv_y + \frac{1}{2}u_tu_y \end{pmatrix}. \tag{3.2.4}$$

The association between X_1 and T_1 will be investigated by substituting the relevant information into the association matrix in Theorem 1.0.2 from Chapter 1.

$$T_1^* = X_1 \begin{pmatrix} T_1^t \\ T_1^x \\ T_1^y \end{pmatrix} = \begin{pmatrix} k[u(-u^2v - v^3) - v(-u^3 - v^2u) + u_x(-\frac{1}{2}v_x) + u_y(-\frac{1}{2}v_y) - v_x(-\frac{1}{2}u_x) \\ -v_y(-\frac{1}{2}u_y)] \\ \\ k[u_t(\frac{1}{2}v_x) + u_x(\frac{1}{2}v_t) - v_t(\frac{1}{2}u_x) - v_x(\frac{1}{2}u_t)] \\ \\ k[u_t(\frac{1}{2}v_y) + u_y(\frac{1}{2}v_t) - v_t(\frac{1}{2}u_y) - v_y(\frac{1}{2}u_t)] \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.2.5}$$

Therefore, X_1 and T_1 are associated.

We now proceed to solve the system in (3.1.2) by the method of Double Reduction.

Double reduction

Firstly, transform X_1 into its canonical form $Y_1 = \dfrac{\partial}{\partial s}$ in (r, s, l, p, w), then the generator is of the form

$$Y_1 = 0\frac{\partial}{\partial r} + \frac{\partial}{\partial s} + 0\frac{\partial}{\partial p} + 0\frac{\partial}{\partial w} + 0\frac{\partial}{\partial l}, \tag{3.2.6}$$

where $X(r) = 0$, $X(s) = 1$, $X(p) = 0$, $X(w) = 0$ and $X(l) = 0$.

From (3.2.6) above we obtain the following invariance condition,

$$\frac{dr}{0} = \frac{ds}{1} = \frac{dp}{0} = \frac{dw}{0} = \frac{dl}{0} = \frac{dt}{1} = \frac{dx}{0} = \frac{dy}{0} = \frac{du}{-kv} = \frac{dv}{ku}. \tag{3.2.7}$$

We get the invariants,

$$
\begin{aligned}
b_1 &= r, \\
b_2 &= w, \\
b_3 &= p, \\
b_4 &= l, \\
b_5 &= x, \\
b_6 &= y, \\
\frac{ds}{1} &= \frac{dt}{1} \Rightarrow s - b_7 = t, \\
\frac{du}{-kv} &= \frac{dv}{ku} \Rightarrow u^2 - b_8 = -v^2, \\
\frac{dv}{ku} &= \frac{dt}{1} \Rightarrow \arctan\left(\frac{v}{u}\right) - kt = b_9.
\end{aligned}
\tag{3.2.8}
$$

A suitable combination of the equations in (3.2.8) yields,

$$
\begin{aligned}
b_1 = b_5 \quad &\Rightarrow \quad r = x, \\
b_4 = b_6 \quad &\Rightarrow \quad l = y, \\
\sqrt{b_8} = b_2 \quad &\Rightarrow \quad w = \sqrt{u^2 + v^2}, \\
b_7 = 0 \quad &\Rightarrow \quad s = t, \\
b_3 = b_9 \quad &\Rightarrow \quad p = \arctan\left(\frac{v}{u}\right) - kt.
\end{aligned}
\tag{3.2.9}
$$

So the canonical coordinates are,

$$
r = x, \qquad l = y, \qquad s = t, \qquad w = \sqrt{u^2 + v^2}, \qquad p = \arctan\left(\frac{v}{u}\right) - kt, \tag{3.2.10}
$$

where $p = p(r, l)$ and $w = w(r, l)$.

Using (3.2.10) above we compute $\mathbf{A}$ and $(\mathbf{A}^{-1})^T$,

$$\mathbf{A} = \begin{pmatrix} D_s t & D_s x & D_s y \\ D_r t & D_r x & D_r y \\ D_l t & D_l x & D_l y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\mathbf{A}^{-1})^T \tag{3.2.11}$$

and $\mathbf{J}$=det($\mathbf{A}$)=1.

The inverse canonical coordinates are presented below,

$$x = r, \qquad y = l, \qquad t = s, \qquad u = w(r,l)\cos(p(r,l)+ks), \qquad v = w(r,l)\sin(p(r,l)+ks). \tag{3.2.12}$$

The first and second partial derivatives of u and v in terms of the new dependent variables w and p are,

$$\begin{aligned} u_x = &\quad w_r\cos(p+ks) - wp_r\sin(p+ks), \\ u_{xx} = &\quad w_{rr}\cos(p+ks) - 2w_r p_r\sin(p+ks) - wp_r^2\cos(p+ks) - wp_{rr}\sin(p+ks), \\ u_y = &\quad w_l\cos(p+ks) - wp_l\sin(p+ks), \\ u_{yy} = &\quad w_{ll}\cos(p+ks) - 2w_l p_l\sin(p+ks) - wp_l^2\cos(p+ks) - wp_{ll}\sin(p+ks), \\ u_t = &\quad -kw\sin(p+ks). \end{aligned} \tag{3.2.13}$$

$$
\begin{aligned}
v_x = &\quad w_r \sin(p+ks) + wp_r \cos(p+ks),\\
v_{xx} = &\quad w_{rr} \sin(p+ks) + 2w_r p_r \cos(p+ks) + wp_{rr} \cos(p+ks) - wp_r^2 \sin(p+ks),\\
v_y = &\quad w_l \sin(p+ks) + wp_l \cos(p+ks),\\
v_{yy} = &\quad w_{ll} \sin(p+ks) + 2w_l p_l \cos(p+ks) + wp_{ll} \cos(p+ks) - wp_l^2 \sin(p+ks),\\
v_t = &\quad kw \cos(p+ks).
\end{aligned}
\tag{3.2.14}
$$

The reduced conserved form is given below,

$$
\begin{pmatrix} T_1^s \\ T_1^r \\ T_1^l \end{pmatrix} = \mathbf{J}(\mathbf{A}^{-1})^T \begin{pmatrix} T_1^t \\ T_1^x \\ T_1^y \end{pmatrix}. \tag{3.2.15}
$$

Substituting (3.2.11), (3.2.12), (3.2.13) and (3.2.14) into (3.2.15) we obtain,

$$
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix}
-\frac{1}{4}(w_r \cos(p+ks) - wp_r \sin(p+ks))^2 - \frac{1}{4}(w_r \sin(p+ks) + wp_r \cos(p+ks))^2 \\
-\frac{1}{4}(w_l \cos(p+ks) - wp_l \sin(p+ks))^2 - \frac{1}{4}(w_l \sin(p+ks) - wp_l \sin(p+ks))^2 \\
-\frac{1}{4}w^4 \cos^4(p+ks) - \frac{1}{2}w^4 \cos^2(p+ks)\sin^2(p+ks) - \frac{1}{4}w^4 \sin^4(p+ks) \\
\\
\frac{1}{2}kw \cos(p+ks)[w_r \sin(p+ks) + wp_r \cos(p+ks)] \\
-\frac{1}{2}kw \sin(p+ks)[w_r \cos(p+ks) - wp_r \sin(p+ks)] \\
\\
\frac{1}{2}kw \cos(p+ks)[w_l \sin(p+ks) + wp_l \cos(p+ks)] \\
-\frac{1}{2}kw \sin(p+ks)[w_l \cos(p+ks) - wp_l \sin(p+ks)]
\end{pmatrix}
$$

$$= \begin{pmatrix} -\frac{1}{4}w_r^2 - \frac{1}{4}w^2p_r^2 - \frac{1}{4}w_l^2 - \frac{1}{4}w^2p_l^2 - \frac{1}{4}(w^2\cos^2(p+ks) + w^2\sin^2(p+ks))^2 \\ \frac{1}{2}kw^2p_r \\ \frac{1}{2}kw^2p_l \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{4}w_r^2 - \frac{1}{4}w^2p_r^2 - \frac{1}{4}w_l^2 - \frac{1}{4}w^2p_l^2 - \frac{1}{4}w^2 \\ \frac{1}{2}kw^2p_r \\ \frac{1}{2}kw^2p_l \end{pmatrix}, \tag{3.2.16}$$

where the reduced conserved vectors are given by $D_rT_1^r = 0$ and $D_lT_1^l = 0$, for the second and third entry in (3.2.16) respectively.

The second step of the double reduction method is given by the equations $T_1^l = j$ and $T_1^l = f$ which are represented below,

$$\frac{1}{2}kw^2p_l = j \tag{3.2.17}$$

and

$$\frac{1}{2}kw^2p_r = f. \tag{3.2.18}$$

Differentiate (3.2.17) and (3.2.18) in terms of l and r respectively, which yields

$$\frac{1}{2}k(2ww_lp_l + w^2p_{ll}) = 0 \qquad \Rightarrow \qquad kww_lp_l + \frac{1}{2}kw^2p_{ll} = 0 \tag{3.2.19}$$

and

$$\frac{1}{2}k(2ww_rp_r + w^2p_{rr}) = 0 \qquad \Rightarrow \qquad kww_rp_r + \frac{1}{2}kw^2p_{rr} = 0. \tag{3.2.20}$$

Compute the multipliers of (3.1.2), namely Q_1 and Q_2, using the following formula

$$Q_1(G^1) + Q_2(G^2) = D_xT^x + D_yT^y + D_tT^t. \tag{3.2.21}$$

Expanding the right-hand side (RHS) of (3.2.21) and substituting in the relevant entries from the matrix given in (3.2.4) we get,

$$\begin{aligned}
RHS &= D_t(-\frac{1}{4}u_x^2 - \frac{1}{4}v_x^2 - \frac{1}{4}u_y^2 - \frac{1}{4}v_y^2 - \frac{1}{4}u^4 - \frac{1}{2}u^2v^2 - \frac{1}{4}v^4) \\
&\quad + D_x(\frac{1}{2}v_tv_x + \frac{1}{2}u_tu_x) + D_y(\frac{1}{2}v_tv_y + \frac{1}{2}u_tu_y) \\
&= \frac{1}{2}u_t(u_{xx} + u_{yy}) - u_tu(u^2 + v^2) - u_tv_t + \frac{1}{2}v_t(v_{xx} + v_{yy}) - v_tv(u^2 + v^2) + v_tu_t \\
&= u_t[\frac{1}{2}(u_{xx} + u_{yy}) - u(u^2 + v^2) - v_t] + v_t[\frac{1}{2}(v_{xx} + v_{yy}) - v(u^2 + v^2) + u_t] \\
&= Q_1(G^1) + Q_2(G^2).
\end{aligned}$$

Therefore,

$$\begin{aligned}
Q_1 &= u_t, \\
Q_2 &= v_t.
\end{aligned} \tag{3.2.22}$$

Substitute (3.2.9), (3.2.13) and (3.2.22) into $Q_1G^1 - Q_2G^2 = 0$ from Theorem 1.0.4.

$$
\begin{aligned}
&- \quad kw\sin(p+ks)[\frac{1}{2}(w_{rr}\cos(p+ks) - 2w_r p_r \sin(p+ks) - wp_r^2\cos(p+ks) \\
&- \quad wp_{rr}\sin(p+ks) + w_{ll}\cos(p+ks) \\
&- \quad 2w_l p_l \sin(p+ks) - wp_l^2\cos(p+ks) - wp_{ll}\sin(p+ks)) \\
&- \quad w\cos(p+ks)[w^2\cos^2(p+ks) + w^2\sin^2(p+ks)] - kw\cos(p+ks)] \\
&- \quad kw\cos(p+ks)[\frac{1}{2}(w_{rr}\sin(p+ks) + 2w_r p_r\cos(p+ks) + wp_{rr}\cos(p+ks) \\
&- \quad wp_r^2\sin(p+ks) + w_{ll}\sin(p+ks) + 2p_l w_l\cos(p+ks) \\
&+ \quad wp_{ll}\cos(p+ks) - wp_l^2\sin(p+ks)) \\
&- \quad w\sin(p+ks)[w^2\cos^2(p+ks) + w^2\sin^2(p+ks)] - kw\sin(p+ks)] = 0
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&- \quad kww_{rr}\sin(p+ks)\cos(p+ks) + kw^2p_r^2\sin(p+ks)\cos(p+ks) \\
&- \quad kww_{ll}\cos(p+ks) + kw^2p_l^2\sin(p+ks)\cos(p+ks) + 2k^2w^2\sin(p+ks)\cos(p+ks) \\
&+ \quad 2kw^4\cos(p+ks)\sin^3(p+ks) + 2kw^4\sin(p+ks)\cos^3(p+ks) + kww_r p_r\sin^2(p+ks) \\
&+ \quad \frac{1}{2}kw^2p_{rr}\sin^2(p+ks) \\
&+ \quad kww_l p_l\sin^2(p+ks) + \frac{1}{2}kw^2p_{ll} - kww_r p_r\cos^2(p+ks) \\
&- \quad \frac{1}{2}kw^2p_{rr} - kww_l p_l\cos^2(p+ks) \\
&- \quad \frac{1}{2}kw^2p_{ll}\cos^2(p+ks) = 0
\end{aligned}
$$

and this, in turn, implies that

$$\begin{aligned} & \sin(p+ks)\cos(p+ks)(-kww_{rr}+kw^2p_r^2-kww_{ll}+kw^2p_l^2+2k^2w^2) \\ - \quad & (-kww_rp_r-\frac{1}{2}kw^2p_{rr}-kww_lp_l-\frac{1}{2}kw^2p_{ll})[\cos^2(p+ks)-\sin^2(p+ks)]=0 \end{aligned}$$

and hence

$$\begin{aligned} & kw\sin(p+ks)\cos(p+ks)[wp_l^2+2kw+2w^3-w_{rr}-wp_r^2-w_{ll}] \\ & -kw\cos 2(p+ks)[-w_rp_r-\frac{1}{2}wp_{rr}-w_lp_l-\frac{1}{2}wp_{ll}]=0. \end{aligned} \tag{3.2.23}$$

Substituting (3.2.19) and (3.2.20) into (3.2.23) above we obtain the following result,

$$kw\sin(p+ks)\cos(p+ks)[wp_l^2+2kw+2w^3-w_{rr}-wp_r^2-w_{ll}]=0. \tag{3.2.24}$$

Since k, w, $\sin(p+ks)$ and $\cos(p+ks)$ in (3.2.24) cannot equal zero, we let

$$wp_l^2+2kw+2w^3-w_{rr}-wp_r^2-w_{ll}=0. \tag{3.2.25}$$

Let p be a constant, ie. $p=A$ where $A\in\mathbb{R}$ such that $p_r=p_l=0$.

So equation (3.2.25) now becomes,

$$2kw+2w^3-w_{rr}-w_{ll}=0. \tag{3.2.26}$$

The equation above in (3.2.26) is a special case of the famous Klein-Gordon equation.

The Klein-Gordon equation was first proposed by Oskar Klein and Walter Gordon, [3], in 1926 in an attempt to describe relativistic electrons, which was later proved false. This equation is basically the relativistic version of the Schrödinger equation or the Schrödinger equation for a quantum state.

Let $\alpha = r - cl$, where $c \in \mathbb{R}$.

The partial derivatives of w now become

$$\begin{aligned} w_r &= w_\alpha, \\ w_l &= -cw_\alpha, \\ w_{rr} &= w_{\alpha\alpha}, \\ w_{ll} &= c^2 w_{\alpha\alpha}. \end{aligned} \tag{3.2.27}$$

Let $w_\alpha = w'$ and $w_{\alpha\alpha} = w''$.

Substituting (3.2.27) into (3.2.26) the equation now becomes,

$$2kw + 2w^3 - w'' - c^2 w'' = 0, \tag{3.2.28}$$

which implies that,

$$w'' - \frac{2}{1+c^2}(kw + w^3) = 0. \tag{3.2.29}$$

We use the Euler-Lagrange equation from Chapter 1 , which is stated below

$$\mathcal{E} = \frac{\partial}{\partial w} - D_\alpha \frac{\partial}{\partial w'}. \tag{3.2.30}$$

We know that $\mathcal{E}(\mathcal{L}) = 0$ on the solutions of (3.1.2), so we can compute $\mathcal{L}$ by inverse.

The Lagrangian of equation (3.2.29) above is given below,

$$\mathcal{L} = \frac{1}{2}w'^2 + \frac{2}{1+c^2}\left(\frac{k}{2}w^2 + \frac{1}{4}w^4\right). \tag{3.2.31}$$

Firstly we substitute the equation (3.2.31) into the different terms of the Euler-Lagrange equation, (3.2.30) above which gives us,

$$\frac{\partial \mathcal{L}}{\partial w'} = w' \qquad D_\alpha\left(\frac{\partial \mathcal{L}}{\partial w'}\right) = w'' \qquad -\frac{\partial \mathcal{L}}{\partial w} = -\left[\frac{2}{1+c^2}(kw + w^3)\right].$$

When we add these two terms above we get the original equation of motion, i.e. equation (3.2.29) so we have now verified that the Lagrangian in equation (3.2.31) is the Lagrangian for equation (3.2.29).

One of the symmetries of equation (3.2.29) is $\bar{X}_2 = \dfrac{\partial}{\partial \alpha}$.
Since $\bar{X}_2 = \dfrac{\partial}{\partial \alpha}$ is also a Noether symmetry we can use Noether's Theorem, [13]. We will use the formula below [10],

$$X^{[1]}\ \mathcal{L} + \mathcal{L}\ \frac{d\xi}{d\alpha} = \frac{df}{d\alpha}. \tag{3.2.32}$$

By using formula (3.2.32) above we can deduce that $f = 0$, where f is the gauge

quantity.

Next compute the conserved quantity, I, in this case since we only have one independent variable, α. The formula for I is given by,

$$I = \mathcal{L}\xi + (\eta - w'\xi)\frac{\partial \mathcal{L}}{\partial w'} - f. \tag{3.2.33}$$

We know that $\xi = 1$, $\eta = 0$ and $f = 0$ from the information given above, therefore

$$I = -\frac{1}{2}w'^2 + \frac{2}{1+c^2}\left(\frac{k}{2}w^2 + \frac{1}{4}w^4\right). \tag{3.2.34}$$

By the Double Reduction method, $\dfrac{dI}{d\alpha} = 0$ so therefore

$$-\frac{1}{2}w'^2 + \frac{2}{1+c^2}\left(\frac{k}{2}w^2 + \frac{1}{4}w^4\right) = k_1, \tag{3.2.35}$$

where $k_1 \in \mathbb{R}$.

Let k_1 in equation (3.2.35) above equal zero (for simplicity of our solution), i.e. $k_1 = 0$ which yields,

$$-\frac{1}{2}w'^2 + \frac{2}{1+c^2}\left(\frac{k}{2}w^2 + \frac{1}{4}w^4\right) = 0, \tag{3.2.36}$$

which implies that,

$$\frac{dw}{w\sqrt{\frac{k}{2}+\frac{1}{4}w^2}} = \frac{2}{\sqrt{1+c^2}}\, d\alpha. \tag{3.2.37}$$

To solve the above equation (3.2.37) we will use *Mathematica* for ease of computation. So by integrating both sides of (3.2.37) respectively we get the final solution below,

$$\frac{\sqrt{2}\left[\log w - \log(2k - \sqrt{2k}\sqrt{2k+w^2})\right]}{\sqrt{k}} = \frac{2}{\sqrt{1+c^2}}\, \alpha + K^*, \tag{3.2.38}$$

where K^* is the integration constant.

3.2.2 Case 2: $X_2 = -y\partial_x + x\partial_y$

A Lie-point symmetry generator of (3.1.1) is $X_2 = -y\partial_x + x\partial_y$. The first prolongation is represented below,

$$X_2^{[1]} = -y\partial_x + x\partial_y - u_y\frac{\partial}{\partial u_x} + u_x\frac{\partial}{\partial u_y} - v_y\frac{\partial}{\partial v_x} + v_x\frac{\partial}{\partial v_y}. \tag{3.2.39}$$

The symmetry X_2 yields the following conserved vector T_2,

$$\begin{pmatrix} T_2^t \\ T_2^x \\ T_2^y \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}uxv_y - \frac{1}{2}uyv_x - \frac{1}{2}vxu_y + \frac{1}{2}vyu_x \\ -\frac{1}{4}yu_x^2 - \frac{1}{4}yv_x^2 + \frac{1}{4}yu_y^2 + \frac{1}{4}yv_y^2 - \frac{1}{2}yvu_t + \frac{1}{2}yuv_t \\ +\frac{1}{4}yu^4 + \frac{1}{2}yu^2v^2 + \frac{1}{4}yv^4 + \frac{1}{2}xv_xv_y + \frac{1}{2}xu_xu_y \\ -\frac{1}{4}xu_x^2 - \frac{1}{4}xv_x^2 + \frac{1}{4}xu_y^2 + \frac{1}{4}xv_y^2 + \frac{1}{2}xvu_t - \frac{1}{2}xuv_t \\ -\frac{1}{4}xu^4 - \frac{1}{2}xu^2v^2 - \frac{1}{4}xv^4 - \frac{1}{2}yv_xv_y - \frac{1}{2}yu_xu_y \end{pmatrix}. \tag{3.2.40}$$

The association between X_2 and T_2 will be investigated by substituting the relevant information into the association matrix in Theorem 1.0.1 in Chapter1.

$$T_2^* = X_2\begin{pmatrix} T_2^t \\ T_2^x \\ T_2^y \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} T_2^t \\ T_2^x \\ T_2^y \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.2.41}$$

We have proven that X_2 and T_2 are associated so we may now proceed with the double reduction method.

Double Reduction

Firstly, transform X_2 into it's canonical form $Y_2 = \dfrac{\partial}{\partial s}$ in (r, s, p, w, l). Then the generator is of the form

$$Y_2 = 0\frac{\partial}{\partial r} + \frac{\partial}{\partial s} + 0\frac{\partial}{\partial p} + 0\frac{\partial}{\partial w} + 0\frac{\partial}{\partial l}, \tag{3.2.42}$$

where $X(r) = X(p) = X(w) = X(l) = 0$ and $X(s) = 1$.

From (3.2.42) we obtain the following invariance condition,

$$\frac{dr}{0} = \frac{ds}{1} = \frac{dp}{0} = \frac{dw}{0} = \frac{dl}{0} = \frac{dt}{0} = \frac{dx}{-y} = \frac{dy}{x} = \frac{du}{0} = \frac{dv}{0}. \tag{3.2.43}$$

Solve (3.2.43) by method of invariance and obtain the invariants,

$$
\begin{aligned}
r &= b_1, \\
p &= b_2, \\
w &= b_3, \\
l &= b_4, \\
u &= b_5, \\
v &= b_6, \\
\frac{ds}{1} &= \frac{dy}{x} \Rightarrow xs = y + b_7, \\
t &= b_8, \\
\frac{dx}{-y} &= \frac{dy}{x} \Rightarrow x^2 + y^2 = b_9.
\end{aligned}
\tag{3.2.44}
$$

By choosing different combinations of the equations in (3.2.44) above we obtain,

$$
\begin{aligned}
b_1 = b_8 \quad &\Rightarrow \quad r = t, \\
b_2 = b_5 \quad &\Rightarrow \quad p = u, \\
b_3 = b_6 \quad &\Rightarrow \quad w = v, \\
b_7 = 0 \quad &\Rightarrow \quad s = \frac{y}{x}, \\
b_4 = b_9 \quad &\Rightarrow \quad l = x^2 + y^2.
\end{aligned}
$$

The canonical coordinates are,

$$
r = t, \qquad s = \frac{y}{x}, \qquad l = x^2 + y^2, \qquad p = u, \qquad w = v. \tag{3.2.45}
$$

where $p = p(r, l)$ and $w = w(r, l)$.

We use $l = x^2 + y^2$ from (3.2.45) to reduce (3.1.2) to a one-dimensional pde.

The partial derivatives of (3.1.2) are recalculated with $l = x^2 + y^2$ and are presented below,

$$\begin{aligned} u_x &= 2xu_l, \\ u_{xx} &= 4x^2u_{ll} + 2u_l, \\ u_y &= 2yu_l, \\ u_{yy} &= 4y^2u_{yy} + 2u_l, \\ \\ v_x &= 2xv_l, \\ v_{xx} &= 4x^2v_{ll} + 2v_l, \\ v_y &= 2yv_l, \\ v_{yy} &= 4y^2v_{ll} + 2v_l. \end{aligned} \tag{3.2.46}$$

Substitute the partial derivatives in (3.2.46) into (3.1.2) and we obtain,

$$\begin{aligned} -v_t &= -2(lu_{ll} + u_l) + u(u^2 + v^2), \\ u_t &= -2(lv_{ll} + v_l) + v(u^2 + v^2). \end{aligned} \tag{3.2.47}$$

The Lagrangian of (3.2.47) can be computed and verified with the use of the Euler-Lagrange equation in Chapter 1.

The Lagrangian is given as,

$$\mathcal{L} = -\frac{1}{2}vu_t + \frac{1}{2}uv_t + lu_l^2 + lv_l^2 + \frac{1}{4}(u^2 + v^2)^2. \tag{3.2.48}$$

Using the Lagrangian in equation (3.2.48) above we will now construct a symmetry

of (3.2.47) which is given below,

$$X_3 = 2l\partial_l + 2t\partial_t - u\partial_u - v\partial_v. \tag{3.2.49}$$

The multipliers and corresponding conserved vectors are given below,

(i) $Q_1 = v_t$ and $Q_2 = u_t$.

$$\begin{aligned} T^l &= l\left(-u_t u_l - v_t v_l + u u_{lt} + v v_{lt}\right), \\ T^t &= \tfrac{1}{4}\left(u^4 + 2u^2v^2 - 4u\left(u_l + l u_{ll}\right) + v\left(v^3 - 4\left(v_l + l v_{ll}\right)\right)\right). \end{aligned}$$

(ii) $Q_1 = u$ and $Q_2 = -v$.

$$\begin{aligned} T^l &= 2l\left(v u_l - u v_l\right), \\ T^t &= \tfrac{1}{2}\left(-u^2 - v^2\right). \end{aligned}$$

(iii) $Q_1 = l v_l + t v_t + \frac{1}{2}v$ and $Q_2 = \frac{1}{2}u + l u_l + t u_t$.

$$\begin{aligned} T^l &= \tfrac{1}{4}l\left(u^4 + 2u^2v^2 + v^4 - 4\left(t u_t u_l + l u_l^2 + t v_t v_l + l v_l^2\right) + 2u\left(v_t + 2t u_{lt}\right) - 2v\left(u_t - 2t v_{lt}\right)\right), \\ T^t &= \tfrac{1}{4}\left(t u^4 + 2t u^2 v^2 - 2u\left(2t u_l + l\left(v_l + 2t u_{ll}\right)\right) + v\left(t v^3 + 2l u_l - 4t\left(v_l + l v_{ll}\right)\right)\right). \end{aligned}$$

(iv) $Q_1 = ltv_l - \frac{1}{4}lu + \frac{1}{2}tv + \frac{1}{2}t^2v_t$ and $Q_2 = \frac{1}{2}tu + ltu_l + \frac{1}{4}lv + \frac{1}{2}t^2u_t$.

$$\begin{aligned} T^l = \; & \tfrac{1}{4}l\left(tu^4 + 2tu^2v^2 + tv^4 - 2t\left(tu_tu_l + 2lu_l{}^2 + tv_tv_l + 2lv_l{}^2\right) + 2u\left(tv_t + lv_l + t^2u_{lt}\right)\right) \\ & + \tfrac{1}{4}l\left(2v\left(-tu_t - lu_l + t^2v_{lt}\right)\right), \\ T^t = \; & \tfrac{1}{8}\left(t^2u^4 + u^2\left(l + 2t^2v^2\right) - 4tu\left(tu_l + l\left(v_l + tu_{ll}\right)\right) + v\left(lv + t^2v^3 - 4t\left(-lu_l + tv_l + tlv_{ll}\right)\right)\right). \end{aligned}$$

We choose to use multiplier and conserved vector combination (ii). Since the conserved vector in (ii) was constructed from the symmetry (3.2.49) we may assume that they are associated automatically so showing the calculation is not necessary. Now we can proceed with the Double Reduction.

The canonical form of X_3 in (3.2.49) is given below,

$$Y_3 = 0\frac{\partial}{\partial f} + \frac{\partial}{\partial g} + 0\frac{\partial}{\partial m} + 0\frac{\partial}{\partial n}, \tag{3.2.50}$$

where $X(f) = X(m) = X(n) = 0$ and $X(g) = 1$ and from which we obtain the following invariance condition,

$$\frac{df}{0} = \frac{dg}{1} = \frac{dm}{0} = \frac{dn}{0} = \frac{dl}{2l} = \frac{dt}{2t} = \frac{du}{-u} = \frac{dv}{-v}. \tag{3.2.51}$$

Solve (3.2.50) by the method of invariance and we obtain,

$$
\begin{aligned}
&\frac{df}{0} && \Rightarrow && f = b_1, \\
&\frac{dg}{1} = \frac{dt}{2t} && \Rightarrow && g = \tfrac{1}{2}\ln t + b_2, \\
&\frac{dm}{0} && \Rightarrow && m = b_3, \\
&\frac{dn}{0} && \Rightarrow && n = b_4, \\
&\frac{dt}{2t} = \frac{dl}{2l} && \Rightarrow && \ln t = \ln l + \ln b_5 \quad \Rightarrow \quad t = l\ b_5, \\
&\frac{du}{-u} = \frac{dv}{v} && \Rightarrow && \ln u = \ln v + b_6 \quad \Rightarrow \quad u = v\ b_6, \\
&\frac{dl}{2l} = \frac{du}{-u} && \Rightarrow && \tfrac{1}{2}\ln l = -\ln u + b_7 \quad \Rightarrow \quad \sqrt{l} = \frac{b_7}{u}.
\end{aligned}
\tag{3.2.52}
$$

By choosing different combinations of the equations in (3.2.52) we obtain,

$$
\begin{aligned}
b_1 = b_5 \quad &\Rightarrow \quad f = \frac{t}{l}, \\
b_3 = b_6 \quad &\Rightarrow \quad m = \frac{u}{v}, \\
b_4 = b_7 \quad &\Rightarrow \quad n = \sqrt{l}\ u, \\
b_2 = 0 \quad &\Rightarrow \quad g = \ln(\sqrt{t}).
\end{aligned}
\tag{3.2.53}
$$

So the canonical coordinates are,

$$f = \frac{t}{l}, \qquad g = \ln(\sqrt{t}), \qquad m = \frac{u}{v}, \qquad n = \sqrt{l}\ u, \tag{3.2.54}$$

where $n = n(f)$ and $m = m(f)$.

The inverse canonical coordinates are presented below,

$$t = e^{2g}, \qquad l = \frac{e^{2g}}{f}, \qquad u = \frac{n\sqrt{f}}{e^{g}}, \qquad v = \frac{n\sqrt{f}}{me^{g}}. \tag{3.2.55}$$

Using the inverse canonical coordinates in (3.2.55) we compute $\mathbf{A}$ and $(\mathbf{A}^{-1})^T$ below,

$$\mathbf{A} = \begin{pmatrix} D_f l & D_f t \\ D_g l & D_g t \end{pmatrix} = \begin{pmatrix} -\dfrac{e^{2g}}{f^2} & 0 \\ \dfrac{2e^{2g}}{f} & 2e^{2g} \end{pmatrix}, \tag{3.2.56}$$

$$(\mathbf{A}^{-1})^T = \begin{pmatrix} -\dfrac{f^2}{e^{2g}} & \dfrac{f}{e^{2g}} \\ 0 & \dfrac{1}{2e^{2g}} \end{pmatrix} \tag{3.2.57}$$

and $\mathbf{J} = det(\mathbf{A}) = -\dfrac{2e^{4g}}{f^2}$.

The partial derivatives of u and v in terms of the new independent variables g and f are,

$$u_l = \frac{-2n_f f^{\frac{5}{2}} - nf^{\frac{3}{2}}}{2e^{3g}},$$

$$u_{ll} = \frac{4n_{ff} f^{\frac{9}{2}} + 12n_f f^{\frac{7}{2}} + 3nf^{\frac{5}{2}}}{4e^{5g}}, \tag{3.2.58}$$

$$u_t = \frac{n_f f^{\frac{3}{2}}}{e^{3g}}.$$

$$v_l = -\frac{2n_f f^{\frac{5}{2}} + nf^{\frac{3}{2}}}{2me^{3g}} + \frac{2m_f n f^{\frac{5}{2}}}{2m^2 e^{3g}},$$

$$v_{ll} = \frac{2n_{ff} f^{\frac{9}{2}} + 6n_f f^{\frac{7}{2}} + 3nf^{\frac{5}{2}}}{2me^{5g}} - \frac{2nm_f f^{\frac{7}{2}} + nm_{ff} f^{\frac{9}{2}}}{m^2 e^{5g}} + \frac{2m_f^2 n f^{\frac{9}{2}}}{m^3 e^{5g}}, \tag{3.2.59}$$

$$v_t = \frac{n_f f^{\frac{3}{2}}}{me^{3g}} - \frac{nm_f f^{\frac{3}{2}}}{m^2 e^{3g}}.$$

The reduced conserved form from Theorem 1.0.4 in Chapter1 is given below,

$$\begin{pmatrix} T_2^f \\ T_2^g \end{pmatrix} = \mathbf{J}(\mathbf{A}^{-1})^T \begin{pmatrix} T_2^l \\ T_2^t \end{pmatrix}. \tag{3.2.60}$$

Therefore,

$$\begin{pmatrix} T_2^f \\ T_2^g \end{pmatrix} = -\frac{2e^{4g}}{f^2} \begin{pmatrix} -\frac{f^2}{e^{2g}} & \frac{f}{e^{2g}} \\ 0 & \frac{1}{2e^{2g}} \end{pmatrix} \begin{pmatrix} \frac{2e^{2g}}{f}\left[\frac{-2nn_f f^3 - n^2 f^2}{2me^{4g}} + \frac{2nmn_f f^3 + n^2 m f^2 - 2n^2 m_f f^3}{2m^2 e^{4g}}\right] \\ \frac{1}{2}\left(-\frac{n^2 f}{e^{2g}} - \frac{n^2 f}{m^2 e^{2g}}\right) \end{pmatrix}$$

$$= \begin{pmatrix} n^2 + \dfrac{n^2}{m^2} - \dfrac{4n^2 m_f f^2}{m^2} \\ \\ \dfrac{n^2}{2f} + \dfrac{n^2}{2fm^2} \end{pmatrix} \tag{3.2.61}$$

where the conserved form is also given by $D_f T^f = 0$.

The second step of the double reduction is given as

$$\frac{n^2 m^2 + n^2 - 4n^2 m_f f^2}{m^2} = K_2, \tag{3.2.62}$$

where $K_2 \in \mathbb{R}$.

Let $K_2 = 0$, we will now simplify (3.2.62) which is represented below,

$$m^2 + 1 - 4f^2 m_f = 0. \tag{3.2.63}$$

Equation (3.2.63) above is an ode, so we can proceed and solve for m,

$$4f^2 m_f = m^2 + 1,$$

$$\frac{dy}{m^2 + 1} = \frac{df}{4f^2},$$

$$\arctan(m) = -\frac{4}{f} + \alpha.$$

Then the solution of m is given below,

$$m = \tan\left(\alpha - \frac{4}{f}\right), \tag{3.2.64}$$

where α is an integration constant.

The second equation of (1.0.17) from Theorem 1.0.4 in Chapter 1 is stated below,

$$Q_1(de^1) - Q_2(de^2) = 0,$$

which can be written as

$$-v[v_t - 2(lu_{ll} + u_l) + u(u^2 + v^2)] - u[-u_t - 2(lv_{ll} + v_l) + v(u^2 + v^2)] = 0. \tag{3.2.65}$$

In this case $Q_1 = -v$ and $Q_2 = u$.

Substituting in the relevant entries from (3.2.55) and (3.2.58) into (3.2.65) above we obtain,

$$-\frac{n\sqrt{f}}{me^g}\left[\frac{-4n_{ff}f^{\frac{7}{2}} - 12n_f f^{\frac{5}{2}} - 3nf^{\frac{3}{2}} + 4n_f f^{\frac{5}{2}} + 2nf^{\frac{3}{2}} + 2n^3 f^{\frac{3}{2}}}{2e^{3g}} + \frac{n_f f^{\frac{3}{2}}}{me^{3g}} + \frac{n^3 f^{\frac{3}{2}} - nm_f f^{\frac{3}{2}}}{m^2 e^{3g}}\right]$$

$$-\frac{n\sqrt{f}}{e^g}\left[-\frac{n_f f^{\frac{3}{2}}}{e^{3g}} + \frac{2n_f f^{\frac{5}{2}} + nf^{\frac{3}{2}} + n^3 f^{\frac{3}{2}} - 2n_{ff}f^{\frac{7}{2}} - 6n_f f^{\frac{5}{2}} - 3nf^{\frac{3}{2}}}{me^{3g}}\right]$$

$$-\frac{n\sqrt{f}}{e^g}\left[\frac{4nm_f f^{\frac{5}{2}}+2nm_{ff}f^{\frac{7}{2}}-2nm_f f^{\frac{5}{2}}}{m^2e^{3g}}+\frac{n^3f^{\frac{3}{2}}-4nm_f^2f^{\frac{7}{2}}}{m^3e^{3g}}\right]=0$$

which implies that,

$$\frac{nm_f^n f^2+8nm^2n_{ff}f^4+16nm^2n_f f^3-4n^4m^2f^2+5n^2m^2f^2-2n^2mm_{ff}f^4}{m^3}$$
$$+\frac{4n^2m_f^2f^4+n^2m_f f^2-2n^2mm_f f^3-nmn_f f^2-2n^4f^2}{m^3}=0. \qquad (3.2.66)$$

Substitute (3.2.64), the solution of m, into (3.2.66), to solve for n, which would yield a 2nd order differential equation. Thereafter we proceed to solve for u and v by substituting the canonical and inverse canonical coordinates.

3.3 Concluding Remarks

In this chapter we have used the double reduction method of solving multidimensional equations to find a suitable solution for equation (3.1.1).

The level of difficulty has increased quite substantially with the introduction of a second dependent variable, v. A lot of manipulation was required to finally come out with a suitable solution for equation (3.1.1).

In Case 1 the symmetries and conserved vectors were computed and their association was tested and it resulted in the null vector, thusdeeeming X_1 and T^1 to be associated. Therefore, we could continue to solve equation (3.1.1) using the double reduction method. We were able to reduce the Schrödinger equation to a special case of the famous Klein-Gordon equation due to our choice of trigonometric canonical coordinates. Thereafter, we used the lagrangian to reduce (3.2.29), through some more workings we were finally able to deduce a solution for (3.1.1).

In Case 2, the computation was rendered more complex with our choice of a rotational canonical coordinate l in (3.2.45). We were able to calculate the symmetry in (3.2.49) through the use of the lagrangian which gave rise to three sets of multipliers and conserved vectors, we then went through the steps of the double reduction method and we were able find a solution which was actually a very complex differential equation. through some careful substitution a final solution in terms of u and v can be deduced.

Chapter 4

Conclusion

In this dissertation, we dealt with the construction of solutions of the Benney-Luke and nonlinear Schrödinger equations using the Double Reduction method. This method was previously used on one-dimensional equations . To investigate further we extended the number of independent variables from just two, namely (x, t), to three independent variables those being (x, y, t), i.e a two-dimensional equation. We were able to reduce some of our equations into special cases of some famous differential equations.

First we had to compute the symmetries, conservation laws and multipliers for each equation through the approaches mentioned in Chapter 1. When using these methods we would get more than one of each of these components above, so we would have to choose one that would be more convenient to us. We then continued to use the double reduction method to find the solutions.

In Chapter 2, the Benney-Luke equation was investigated. The double reduction method was used and we were able to reduce our equation into the form of a simple harmonic operator.

Chapter 3 was divided into two cases, in the first case the following symmetry was used; $X_1 = \partial_t + k(u\partial_v - v\partial_u)$. We were able to reduce our equation in this case into a special case of the famous Klein-Gordon equation which made further computation of the equations trivial. In the second case the following symmetry was used; $X_2 = -y\partial_x + x\partial_y$ and were able to get a complex equation in terms of n and m.

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