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ORIGINAL PAPER

Extensions and variations of Andrews–Merca identities

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Abstract
Recently, Andrews and Merca have given a new combinatorial interpretation of the total number of even parts in all partitions of n into distinct parts. We generalise this result and consider many more variations of their work. We also highlight some connections with the work of Fu and Tang.

Keywords Partition · Bijection · Generating function · Congruence

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1 Introduction

A partition of a positive integer n is a representation $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_r$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$ and $\sum_{i=1}^{r} \lambda_i = n$. A more compact notation for λ is $(\mu_1^{m_1}, \mu_2^{m_2}, \ldots)$ where $\mu_1 > \mu_2 > \cdots$ and m_i is the multiplicity of the summand μ_i. For instance, $14+14+10+10+7+7+7+1+1+1+1$ can be written as $(14^2, 10^2, 7^3, 1^4)$. The summands in a partition are called *parts*. The union of two partitions λ and β is simply the multiset union $\lambda \cup \beta$ where λ and β are treated as multisets. Partition identities involving various classes of partitions have been studied, for example, see [1, 11]. We recall a popular one: For a positive integer t, the number of partitions of n into parts appearing at most $t-1$ times is equal to the number of partitions of n into parts not divisible by t. Glaisher's bijection [7] establishes a one-to-one correspondence between the aforesaid classes of partitions. Finite versions of this identity were studied in [2, 5, 10].
We shall let ϕ_t denote Glaisher's map, mapping the set of partitions into parts appearing at most $t-1$ times to partitions into parts not divisible by t. This map is described as follows:

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Suppose that $\lambda = (\lambda_1^{c_1}, \lambda_2^{c_2}, \ldots)$ is a partition of n in which each part appears at most $t-1$ times. Write $\lambda_i = t^{r_i} \cdot b_i$ where $t \nmid b_i$ and then map $\lambda_i^{c_i}$ to $(b_i)^{t^{r_i} c_i}$ for each i, where b_i is now a part with multiplicity $t^{r_i} c_i$. We define $\phi_t(\lambda)$ to be the partition

$$\phi_t(\lambda) = \bigcup_{i \geq 1} (b_i)^{t^{r_i} c_i}.$$

Note that the parts of this partition are not multiples of t. The inverse ϕ_t^{-1} is given as follows.
Let $\mu = (\mu_1^{m_1}, \mu_2^{m_2}, \ldots, \mu_r^{m_r})$ be a partition of n in which no part is divisible by t. The notation for μ implies that $\mu_1 > \mu_2 > \cdots$ are parts having multiplicities $m_1, m_2, \ldots$, respectively. For each m_i, we find its t-ary expansion, i.e.

$$m_i = b_{i0} + b_{i1} t + b_{i2} t^2 + b_{i3} t^3 + \cdots + b_{i\ell_i} t^{\ell_i}$$

where $0 \leq b_{ij} \leq t-1$ and $0 \leq j \leq \ell_i$.
We then map $\mu_i^{m_i}$ to $\bigcup_{j=0}^{\ell_i} (t^j \mu_i)^{b_{ij}}$, where $t^j \mu_i$ is now a part with multiplicity b_{ij}. The image of μ, i.e. $\phi_t^{-1}(\mu)$, is given by

$$\bigcup_{i=1}^{r} \bigcup_{j=0}^{\ell_i} (t^j \mu_i)^{b_{ij}}.$$

It is clear that this is a partition of n in which each part appears at most $t-1$ times.

The case of $t = 2$, also known as Euler's "Distinct v.s. Odd" partition theorem, has posed interesting questions and various studies conducted in many directions. Recently, Andrews and Merca [3] considered the following partition functions:

$a(n)$: the total number of even parts in all partitions of n into distinct parts;
$c_e(n)$(resp. $c_o(n)$): the number of partitions of n in which exactly one part is repeated and this part is even (resp. odd);
$b_o(n)$: the number of partitions of n into an odd number of parts in which the set of even parts has only one element;
$b_e(n)$: the number of partitions of n into an even number of parts in which the set of even parts has only one element;
$b(n) = b_o(n) - b_e(n)$ and $c(n) = c_o(n) - c_e(n)$.

They obtained the following result.

Theorem 1 *For all $n \geq 1$,*

$$a(n) = c(n) = (-1)^n b(n).$$

Note that, if $b'(n)$ denotes the number of partitions of n in which the set of even parts is singleton, then $b'(n) = b_o(n) + b_e(n)$. Hence, one of the implications of the theorem is that

$$2 \mid a(n) - b'(n). \tag{1}$$

For related work, see [4, 9].

In this paper, we generalize the identity $a(n) = c(n)$ of Theorem 1 and also generalize (1). We then consider some variations. As a consequence, some connections with the work of Fu and Tang [6] are highlighted and bijective proofs are provided in such cases.
We recall the q-series notation:

For $a, q \in \mathbb{C}$, we have $(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)$ for $n \geq 1$ and $(a;q)_0 = 1$, and also for $|q| < 1$,

$$(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i).$$

Below are some useful q-identities:

$$(q;q)_\infty = \sum_{j=-\infty}^{\infty}(-1)^j q^{\frac{j(3j-1)}{2}} = 1 + \sum_{j=1}^{\infty}(-1)^j q^{\frac{j(3j\pm1)}{2}}, \tag{2}$$

$$(q;q)_\infty^3 = \sum_{j=0}^{\infty}(-1)^j(2j+1)q^{\frac{j(j+1)}{2}}. \tag{3}$$

2 Generalizing the identity $a(n) = c(n)$ of Theorem 1

We first observe the following.

Theorem 2 *Suppose that p and r are non-negative integers such that $p \geq r+2$. Denote by $a_r(n,p)$ the total number of parts congruent to $-r \pmod p$ in partitions of n into distinct parts. Let $g_r(n,p)$ denote the number of partitions in which exactly one part is repeated and the multiplicity of this repeated part is at least $p-r$ and congruent to $-r, -r+1 \pmod p$. Let $g_{r,o}(n,p)$ (resp. $g_{r,e}(n,p)$) denote the number of $g_r(n,p)$-partitions in which the repeated part is odd (resp. even) and let $c_r(n,p) = g_{r,o}(n,p) - g_{r,e}(n,p)$. Then*

$$a_r(n,p) = c_r(n,p).$$

It is important to realize that setting $p=2$ and $r=0$ yields the first equality in Andrews and Merca's theorem, Theorem 1.

Proof Let $F(z,q)$ be the multivariable generating function for $c_r(n,p)$ in which z tracks the repeated part. Then

$$\begin{aligned}
F(z,q) &= \sum_{n=1}^{\infty} z^n (q^{(p-r)n} + q^{(p-r+p)n} + q^{(p-r+2p)n} + \cdots + q^{(p-r+1)n} + q^{(p-r+1+p)n} \\
&\quad + q^{(p-r+1+2p)n} + \cdots) \prod_{j\neq n, j=1}^{\infty}(1+q^j) \\
&= \sum_{n=1}^{\infty} z^n (q^{(p-r)n}(1+q^{pn}+q^{2pn}+q^{3pn}+\cdots) + q^{(p-r+1)n}(1+q^{pn} \\
&\quad + q^{2pn} + q^{3pn} + \cdots)) \prod_{j\neq n, j=1}^{\infty}(1+q^j) \\
&= \sum_{n=1}^{\infty} z^n q^{(p-r)n}(1+q^n)(1+q^{pn}+q^{2pn}+q^{3pn}+\cdots) \prod_{j\neq n, j=1}^{\infty}(1+q^j) \\
&= \sum_{n=1}^{\infty} \frac{z^n q^{(p-r)n}(1+q^n)}{1-q^{pn}} \prod_{j\neq n, j=1}^{\infty}(1+q^j)
\end{aligned}$$

$$= \sum_{n=1}^{\infty} \frac{z^n q^{(p-r)n}}{1-q^{pn}} \prod_{j=1}^{\infty}(1+q^j)$$
$$= (-q;q)_\infty \sum_{n=1}^{\infty} \frac{z^n q^{(p-r)n}}{1-q^{pn}}.$$

Using the fact that $\sum_{n=0}^{\infty}(g_{r,o}(n,p) - g_{r,e}(n,p))q^n = -F(-1,q)$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} g_r(n,p)q^n &= -F(-1,q) \\
&= -(-q;q)_\infty \sum_{n=1}^{\infty} \frac{(-1)^n q^{(p-r)n}}{1-q^{pn}} \\
&= -(-q;q)_\infty \sum_{n=1}^{\infty} (-1)^n q^{-rn} \frac{q^{pn}}{1-q^{pn}} \\
&= -(-q;q)_\infty \sum_{n=1}^{\infty} (-1)^n q^{-rn} \sum_{m=1}^{\infty} q^{pnm} \\
&= -(-q;q)_\infty \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (-1)^n q^{-rn} q^{pnm} \\
&= -(-q;q)_\infty \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (-q^{-r+pm})^n \\
&= -(-q;q)_\infty \sum_{m=1}^{\infty} \frac{-q^{-r+pm}}{1+q^{pm-r}} \\
&= (-q;q)_\infty \sum_{m=1}^{\infty} \frac{q^{-r+pm}}{1+q^{pm-r}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{n=0}^{\infty} a_r(n,p)q^n &= \left(\frac{\partial}{\partial z}\Big|_{z=1} \prod_{n=1}^{\infty}(1+zq^{pn-r})\right) \prod_{i=1, i\neq r}^{p} \prod_{n=1}^{\infty}(1+q^{pn-i}) \\
&= \left(\prod_{i=1, i\neq r}^{p} \prod_{n=1}^{\infty}(1+q^{pn-i})\right) \left(\prod_{n=1}^{\infty}(1+zq^{pn-r})\Big|_{z=1}\right) \\
&\quad \times \left(\frac{\partial}{\partial z}\Big|_{z=1} \sum_{n=1}^{\infty} \log(1+zq^{pn-r})\right) \\
&= \left(\prod_{n=1}^{\infty}(1+q^n)\right) \sum_{n=1}^{\infty} \frac{q^{pn-r}}{1+q^{pn-r}} \\
&= (-q;q)_\infty \sum_{n=1}^{\infty} \frac{q^{-r+pn}}{1+q^{pn-r}}.
\end{aligned}$$

Thus

$$\sum_{n=0}^{\infty} a_r(n,\, p)q^n = \sum_{n=0}^{\infty} c_r(n,\, p)q^n$$

and the theorem follows. □

The above theorem provides a new combinatorial interpretation of the total number of parts congruent to $-r$ modulo p in partitions of n into distinct parts.

In Fu and Tang [6], recall the function $O_{p,j}(n)$ which denotes the number of partitions of n in which there are exactly j different parts congruent to 0 modulo p (possibly repeated).

For $j = 1$, we shall denote this function by $o_p(n)$ which counts partitions of n in which the set of parts congruent to 0 modulo p is singleton. Define $a(n,\, p)$ to be the total number of parts congruent to 0 modulo p in all partitions of n into parts appearing at most $p-1$ times.

Clearly $a(n, 2) = a(n) = a_0(n, 2)$ and $o_2(n) = b'(n)$. We have the following property which extends (1).

Theorem 3 *For all $n \geq 0$,*

$$p \mid a(n,\, p) - o_p(n).$$

Proof The generating function for $o_p(n)$ is given by

$$\sum_{n=0}^{\infty} o_p(n)q^n = \frac{(q^p;\, q^p)_\infty}{(q;\, q)_\infty} \sum_{n=1}^{\infty} \frac{q^{pn}}{1-q^{pn}}$$

and the generating function for $a(n,\, p)$ is given by Herden et al. [8]:

$$\frac{(q^p;\, q^p)_\infty}{(q;\, q)_\infty} \sum_{n=1}^{\infty} \frac{q^{pn} + 2q^{2pn} + 3q^{3pn} + \ldots (p-1)q^{(p-1)pn}}{1 + q^{pn} + q^{2pn} + \ldots + q^{(p-1)pn}}.$$

Recall the identity:

$$\sum_{k=1}^{n} kx^k = \frac{x(1-x^{n+1}) - (n+1)(1-x)x^{n+1}}{(1-x)^2}. \tag{4}$$

Manipulating the generating function, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} a(n,\, p)q^n &= \frac{(q^p;\, q^p)_\infty}{(q;\, q)_\infty} \sum_{n=1}^{\infty} \frac{\sum_{j=1}^{p-1} jq^{pnj}}{\frac{1-q^{p^2n}}{1-q^{pn}}} \\
&= \frac{(q^p;\, q^p)_\infty}{(q;\, q)_\infty} \sum_{n=1}^{\infty} \frac{\frac{q^{pn}(1-q^{p^2n}) - p(1-q^{pn})q^{p^2n}}{(1-q^{pn})^2}}{\frac{1-q^{p^2n}}{1-q^{pn}}} \quad \text{(by (4) with } n = p-1, x = q^{pn}\text{)} \\
&= \frac{(q^p;\, q^p)_\infty}{(q;\, q)_\infty} \sum_{n=1}^{\infty} \frac{1-q^{pn}}{1-q^{p^2n}} \frac{q^{pn}(1-q^{p^2n}) - p(1-q^{pn})q^{p^2n}}{(1-q^{pn})^2} \\
&\equiv \frac{(q^p;\, q^p)_\infty}{(q;\, q)_\infty} \sum_{n=1}^{\infty} \frac{1-q^{pn}}{1-q^{p^2n}} \frac{q^{pn}(1-q^{p^2n})}{(1-q^{pn})^2} \pmod{p}
\end{aligned}$$

$$= \frac{(q^p;q^p)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^{pn}}{1-q^{pn}}$$
$$= \sum_{n=0}^{\infty} o_p(n) q^n.$$

It is notable that

$$\sum_{n=0}^{\infty} a(n,\, p) q^n \equiv \frac{1}{(q;q)_\infty} (q^p;q^p)_\infty \sum_{n=1}^{\infty} \frac{q^{pn}}{1-q^{pn}} \pmod{p}.$$

As the latter factor is a series in q^p, it is immediately true that if $s(pn+j)$ vanishes modulo p for all n where $s(m)$ counts the number of unrestricted partitions of m, so does $a(pn+j,\, p)$.

In particular, applying the Ramanujan's partition congruences: $s(5n+4) \equiv 0 \pmod 5$, $s(7n+5) \equiv 0 \pmod 7$ and $s(11n+6) \equiv 0 \pmod{11}$, we have the following result.

Theorem 4 *For all $n \geq 0$,*

$$a(5n+4, 5) \equiv 0 \pmod 5,$$
$$a(7n+5, 7) \equiv 0 \pmod 7$$

and

$$a(11n+6, 11) \equiv 0 \pmod{11}.$$

Let $o_{p,e}(n)$ (resp. $o_{p,o}(n)$) be the number of $o_p(n)$-partitions in which the number of parts congruent to 0 modulo p is even (resp. odd) and

$$H(z,q) = \frac{(q^p;q^p)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{zq^{pn}}{1-zq^{pn}}.$$

Then

$$\begin{aligned}
\sum_{n=0}^{\infty} o_{p,o}(n) q^n &= \frac{1}{2}\left(H(1,q) - H(-1,q)\right) \\
&= \frac{(q^p;q^p)_\infty}{2(q;q)_\infty} \left(\sum_{n=1}^{\infty} \frac{q^{pn}}{1-q^{pn}} + \sum_{n=1}^{\infty} \frac{q^{pn}}{1+q^{pn}} \right) \\
&= \frac{(q^p;q^p)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^{pn}}{1-q^{2pn}} \\
&= \frac{(q^p;q^p)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} q^{pn} \sum_{m=0}^{\infty} q^{2pnm} \\
&= \frac{(q^p;q^p)_\infty}{(q;q)_\infty} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} q^{(p+2pm)n} \\
&= \frac{(q^p;q^p)_\infty}{(q;q)_\infty} \sum_{m=0}^{\infty} \frac{q^{p+2pm}}{1-q^{2pm+p}} \qquad (5)
\end{aligned}$$

and similarly,

$$\sum_{n=0}^{\infty} o_{p,e}(n)q^n = \frac{1}{2}\left(H(1,q) + H(-1,q)\right)$$
$$= \frac{(q^p;q^p)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^{2pn}}{1-q^{2pn}}. \tag{6}$$

Thus, if for $i = 0, p$, $h_i(n, p)$ is the number of partitions in which parts are congruent to $1, 2, 3, \ldots, p-1$ modulo p or congruent to i modulo $2p$ and the set of parts $\equiv i \pmod{2p}$ is singleton, then

$$o_{p,o}(n) = h_p(n, p) \quad \text{and} \quad o_{p,e}(n) = h_0(n, p).$$

3 Further variations

Consider the following partition functions:

$d_e(n)$: the number of partitions of n in which exactly one even part is repeated and odd parts are unrestricted.
$d_o(n)$: the number of partitions of n in which exactly one odd part is repeated and even parts are unrestricted.

For $i = 0, 2$, $f_i(n)$: the number of partitions of n in which the set of parts congruent to $i \pmod 4$ is singleton. Then we have the following theorem:

Theorem 5 *For $n \geq 1$,*

$$d_e(n) = f_0(n) \quad \textit{and} \quad d_o(n) = f_2(n).$$

Proof Note that

$$\sum_{n=0}^{\infty} d_e(n)q^n = \frac{1}{(q;q^2)_\infty} \sum_{n=1}^{\infty} \frac{q^{2n+2n}}{1-q^{2n}} \prod_{j\neq n,\, j=1}^{\infty} (1+q^{2j})$$
$$= \frac{1}{(q;q^2)_\infty} \sum_{n=1}^{\infty} \frac{q^{4n}(-q^2;q^2)_\infty}{(1-q^{2n})(1+q^{2n})}$$
$$= \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=1}^{\infty} \frac{q^{4n}}{1-q^{4n}}$$
$$= \frac{(-q^2;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty (q^2;q^2)_\infty} \sum_{n=1}^{\infty} \frac{q^{4n}}{1-q^{4n}}$$
$$= \frac{(q^4;q^4)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^{4n}}{1-q^{4n}}$$
$$= \frac{1}{(q;q^4)_\infty (q^2;q^4)_\infty (q^3;q^4)_\infty} \sum_{n=1}^{\infty} (q^{4n} + q^{4n+4n} + q^{4n+4n+4n} + \ldots)$$
$$= \sum_{n=0}^{\infty} f_0(n)q^n.$$

Similarly,

$$\begin{aligned}
\sum_{n=0}^{\infty} d_o(n)q^n &= \frac{1}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{q^{2n-1+2n-1}}{1-q^{2n-1}}\prod_{j\neq n,\, j=1}^{\infty}(1+q^{2j-1}) \\
&= \frac{1}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{q^{4n-2}(-q;q^2)_\infty}{(1-q^{2n-1})(1+q^{2n-1})} \\
&= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{q^{4n-2}}{1-q^{4n-2}} \\
&= \frac{(-q;q^2)_\infty(q;q^2)_\infty}{(q^2;q^2)_\infty(q;q^2)_\infty}\sum_{n=1}^{\infty}\frac{q^{4n-2}}{1-q^{4n-2}} \\
&= \frac{(q^2;q^4)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^{4n-2}}{1-q^{4n-2}} \\
&= \frac{1}{(q;q^4)_\infty(q^3;q^4)_\infty(q^4;q^4)_\infty}\sum_{n=1}^{\infty}(q^{4n-2}+q^{2(4n-2)}+q^{3(4n-2)}+\ldots) \\
&= \sum_{n=0}^{\infty} f_2(n)q^n.
\end{aligned}$$

□

We also note the following bijective proof which is uniform for both identities in the theorem.

The bijection

Let $\lambda = \left(\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_l^{m_l}\right)$ be a partition enumerated by $f_2(n)/f_0(n)$. Then

$$\lambda_i^{m_i} \mapsto \begin{cases} \left(\dfrac{\lambda_i}{2}\right)^{2m_i}, & \lambda_i \equiv 0, 2 \pmod 4; \\ \phi_2^{-1}(\lambda_i^{m_i}), & \lambda_i \equiv 1, 3 \pmod 4. \end{cases}$$

To invert the mapping, let $\mu = \left(\mu_1^{s_1}, \mu_2^{s_2}, \ldots, \mu_l^{s_l}\right)$ be a partition enumerated by $d_o(n)/d_e(n)$. Then

$$\mu_i^{s_i} \mapsto (2\mu_i)^{\lfloor \frac{s_i}{2} \rfloor} \cup \phi_2(\mu_i^{s_i - 2\lfloor \frac{s_i}{2} \rfloor}).$$

Recall that

$$\sum_{n=0}^{\infty} d_e(n)q^n = \frac{(q^4;q^4)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^{4n}}{1-q^{4n}}$$

so that

$$(q;q)_\infty\sum_{n=0}^{\infty} d_e(n)q^n = (q^4;q^4)_\infty\sum_{n=1}^{\infty}\frac{q^{4n}}{1-q^{4n}}. \tag{7}$$

Using (2), we have

$$(q;q)_\infty \sum_{n=0}^{\infty} d_e(n)q^n = \left(1+\sum_{n=1}^{\infty}(-1)^n q^{\frac{n(3n\pm1)}{2}}\right)\sum_{n=0}^{\infty} d_e(n)q^n$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} c_k d_e(n-k)q^n \tag{8}$$

where

$$c_k = \begin{cases}(-1)^j, & \text{if } k=\frac{j(3j\pm1)}{2},\ j\in\mathbb{Z}_{\geq 0};\\ 0, & \text{otherwise.}\end{cases}$$

Note that the right-hand side of (8) is equal to

$$\sum_{n=0}^{\infty}\left[d_e(n)+\sum_{j=1}^{\lfloor\frac{1+\sqrt{24n+1}}{6}\rfloor}(-1)^j\left(d_e\left(n-\frac{j(3j+1)}{2}\right)+d_e\left(n-\frac{j(3j-1)}{2}\right)\right)\right]q^n.$$

Invoking (2) and the identity

$$\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}=\sum_{n=1}^{\infty}\sigma_0(n)q^n$$

where $\sigma_0(n)$ is the number of positive divisors of n, (7) becomes

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2}\cdot\sum_{n=0}^{\infty}d_e(n)q^n=\sum_{n=-\infty}^{\infty}(-1)^n q^{2n(3n+1)}\sum_{n=1}^{\infty}\sigma_0(n)q^{4n}.$$

Hence, we have the following.

Corollary 6 *For all integers $r\geq 0$, let*

$$A(r)=\left\{j\in\mathbb{Z}_{>0}: 2j(3j+1)\equiv r \pmod 4 \text{ and } j\leq\left\lfloor(\sqrt{6r+1}-1)/6\right\rfloor\right\}$$

and

$$B(r)=\left\{j\in\mathbb{Z}_{>0}: 2j(3j-1)\equiv r \pmod 4 \text{ and } j\leq\left\lfloor(1+\sqrt{6r+1})/6\right\rfloor\right\}.$$

Then for a positive integer n, $d_e(n)$ is equal to

$$\begin{cases}\sum\limits_{j=1}^{\lfloor\frac{1+\sqrt{24n+1}}{6}\rfloor}(-1)^{j+1}\left(d_e\left(n-\frac{j(3j+1)}{2}\right)+d_e\left(n-\frac{j(3j-1)}{2}\right)\right), & \text{if } n\not\equiv 0 \pmod 4;\\ \sum\limits_{j=1}^{\lfloor\frac{1+\sqrt{24n+1}}{6}\rfloor}(-1)^{j+1}\left(d_e\left(n-\frac{j(3j+1)}{2}\right)+d_e\left(n-\frac{j(3j-1)}{2}\right)\right)+\gamma(n), & \text{if } n\equiv 0 \pmod 4,\end{cases}$$

where

$$\gamma(n)=\sigma_0(n/4)+\sum_{j\in A(n)}(-1)^j\sigma_0\left(\frac{n-2j(3j+1)}{4}\right)+\sum_{j\in B(n)}(-1)^j\sigma_0\left(\frac{n-2j(3j-1)}{4}\right).$$

In this case, we have a relationship between $d_e(n)$ and $\sigma_0(n)$.

Example 1 Consider $n = 8$.

The $d_e(8)$-partitions are:

$$(4^2), (4, 2^2), (3, 2^2, 1), (2^4), (2^3, 1^2), (2^2, 1^4).$$

We now apply the recurrence. Since $8 \equiv 0 \pmod 4$, we have

$$\begin{aligned}
d_e(8) &= d_e(8-2) + d_e(8-1) - d_e(8-7) - d_e(8-5) + \gamma(8)\\
&= d_e(6) + d_e(7) - d_e(1) - d_e(3) + \gamma(8)\\
&= d_e(4) + 2d_e(5) - d_e(1) + d_e(6) - 2d_e(2) - d_e(1) + \gamma(8)\\
&= d_e(2) + 3d_e(3) + 3d_e(4) + d_e(5) - 3d_e(1) + \gamma(4) + \gamma(8)\\
&= 4d_e(1) + 6d_e(2) + 4d_e(3) + d_e(4) + 4\gamma(4) + \gamma(8)\\
&= 10d_e(1) + 5d_e(2) + d_e(3) + 5\gamma(4) + \gamma(8)\\
&= 6d_e(1) + d_e(2) + 5\gamma(4) + \gamma(8)\\
&= d_e(1) + 5\gamma(4) + \gamma(8)\\
&= 5\gamma(4) + \gamma(8).
\end{aligned}$$

Now, $A(8) = \{1\}$, $B(8) = \{1\}$, $A(4) = \{\}$ and $B(4) = \{1\}$ and so

$$\begin{aligned}
d_e(8) &= 5\gamma(4) + \gamma(8)\\
&= 5[\sigma_0(1) - \sigma_0(0)] + [\sigma_0(2) + \sigma_0(0) - \sigma_0(1)\\
&= \sigma_0(2) + 4\sigma_0(1) - 5\sigma_0(0)\\
&= 2 + 4(1) - 5(0)\\
&= 6.
\end{aligned}$$

We also note that

$$\begin{aligned}
\sum_{n=0}^{\infty} d_o(n)q^n &= \frac{(q^2;q^4)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^{4n-2}}{1-q^{4n-2}}\\
&= \frac{1}{(-q^2;q^2)_\infty (q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^{4n-2}}{1-q^{4n-2}}\\
&\equiv \frac{1}{(q^2;q^2)_\infty (q;q)_\infty} \left(\sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} - \sum_{n=1}^{\infty} \frac{q^{4n}}{1-q^{4n}} \right) \pmod 2\\
&\equiv \frac{1}{(q;q)_\infty^3} \left(\sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} - \sum_{n=1}^{\infty} \frac{q^{4n}}{1-q^{4n}} \right) \pmod 2
\end{aligned}$$

so that

$$\sum_{n=0}^{\infty} d_o(n)q^n \sum_{n=0}^{\infty} q^{n(n+1)/2} \equiv \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} - \sum_{n=1}^{\infty} \frac{q^{4n}}{1-q^{4n}} \pmod 2 \tag{9}$$

where we have used (3) for $(q;q)_\infty^3$.

But the right-hand side of (9) is

$$\sum_{n=1}^{\infty} \sigma_0(n)q^{2n} - \sum_{n=1}^{\infty} \sigma_0(n)q^{4n}$$

$$
\begin{aligned}
&= \sum_{n>1,n\equiv 0 \pmod 4} \sigma_0(n/2)q^n + \sum_{n>1,n\equiv 2 \pmod 4}^{\infty} \sigma_0(n/2)q^n \\
&\quad - \sum_{n>1,n\equiv 0 \pmod 4}^{\infty} \sigma_0(n/4)q^n \\
&= \sum_{n>1,n\equiv 0 \pmod 4} (\sigma_0(n/2) - \sigma_0(n/4))q^n + \sum_{n>1,n\equiv 2 \pmod 4}^{\infty} \sigma_0(n/2)q^n.
\end{aligned}
$$

If $n \equiv 0 \pmod 4$, we can write $n = 2^m b$ for some positive integers $m \geq 2$ and b odd. Then

$$
\begin{aligned}
\sigma_0(n/2) - \sigma_0(n/4) &= \sigma_0(2^m b/2) - \sigma_0(2^m b/4) \\
&= \sigma_0(2^{m-1}b) - \sigma_0(2^{m-2}b) \\
&= \sigma_0(2^{m-1})\sigma_0(b) - \sigma_0(2^{m-2})\sigma_0(b) \quad (\sigma_0 \text{ is multiplicative}) \\
&= \sigma_0(b)(\sigma_0(2^{m-1}) - \sigma_0(2^{m-2})) \\
&= \sigma_0(b)(m - (m-1)) \\
&= \sigma_0(b).
\end{aligned}
$$

On the other hand, if $n \equiv 2 \pmod 4$, we have $n = 2b$ for some odd positive integer b. Thus,

$$\sigma_0(n/2) = \sigma_0(b).$$

Denoting by $v_2(n)$, the 2-adic valuation of n, we obtain the following result.

Corollary 7 *For $n > 0$, we have*

$$
\sum_{j=0}^{\lfloor \frac{\sqrt{8n+1}-1}{2} \rfloor} d_o(n - j(j+1)/2) \equiv \begin{cases} 0 \pmod 2, & \text{if } n \equiv 1 \pmod 2; \\ \sigma_0\left(n/2^{v_2(n)}\right) \pmod 2, & \text{if } n \equiv 0 \pmod 2. \end{cases}
$$

4 Generalizations

The partition function $d_o(n)$ can be generalized as follows: Let k, $p \geq 2$ and $0 \leq r < p$ be integers and define $d_p(n,k,r)$ as the number of partitions of n in which exactly one part $\equiv r \pmod p$ appears at least k times. In this definition, parts not congruent to $r \pmod p$ appear with unrestricted multiplicity. Similary, let $f_p(n,k,r)$ denote the number of partitions of n in which parts $\equiv kr \pmod{pk}$ form a singleton set.

If $r \neq 0$, we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty} d_p(n,k,r)q^n \\
&= \frac{(q^r;q^p)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{k(pn+r)}}{1-q^{pn+r}} \prod_{j\neq n, j=0}^{\infty} (1 + q^{pj+r} + q^{2(pj+r)} + \cdots + q^{(k-1)(pj+r)}) \\
&= \frac{(q^r;q^p)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{k(pn+r)}}{(1-q^{pn+r}) \sum_{i=0}^{k-1} q^{(pn+r)i}} \prod_{j=0}^{\infty} \frac{1-q^{k(pj+r)}}{1-q^{pj+r}}
\end{aligned}
$$

$$= \frac{(q^r;q^p)_\infty (q^{kr};q^{kp})_\infty}{(q^r;q^p)_\infty (q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{pkn+kr}}{1-q^{pkn+kr}}$$
$$= \frac{(q^{kr};q^{kp})_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{pkn+kr}}{1-q^{pkn+kr}}$$
$$= \sum_{n=0}^{\infty} f_p(n,k,r) q^n.$$

If $r = 0$, then

$$\sum_{n=0}^{\infty} d_p(n,k,0) q^n = \frac{(q^p;q^p)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^{kpn}}{1-q^{pn}} \prod_{j\neq n, j=1}^{\infty} (1 + q^{pj} + q^{2pj} + \cdots + q^{(k-1)pj})$$

which simplifies to

$$\frac{(q^{kp};q^{kp})_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^{pkn}}{1-q^{pkn}} = \sum_{n=0}^{\infty} f_p(n,k,0) q^n.$$

Thus, we have:

Theorem 8 *Let $p \geq 2$. For an integer $n \geq 1$,*

$$f_p(n,k,r) = d_p(n,k,r).$$

A bijection (which extends the one for $d_e(n)/f_0(n)$ and $d_o(n)/f_2(n)$), is given as follows: Let $\lambda = \left(\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_l^{m_l}\right)$ be a partition enumerated by $f_p(n,k,r)$. Then

$$\lambda_i^{m_i} \mapsto \begin{cases} \left(\dfrac{\lambda_i}{k}\right)^{km_i}, & \text{if } \lambda_i \equiv 0, k, 2k, \ldots, (p-1)k \pmod{pk}; \\ \phi_k^{-1}(\lambda_i^{m_i}), & \text{otherwise.} \end{cases}$$

From λ, note that there is only one part congruent to $kr \pmod{pk}$ (with unrestricted multiplicity). Under the bijection, this part is mapped to the exactly one part which is congruent to $r \pmod{p}$ repeated at least k times. For other parts of λ that are not congruent to $kr \pmod{kp}$, there are two cases.

Case 1: If the part is divisible by k, it can be shown that such a part must be congruent to $jk \pmod{pk}$ where $j \in \{0, 1, 2, \ldots, p-1\}\backslash\{r\}$. This part is mapped to a part not congruent to $r \pmod{p}$ with multiplicity at least k.

Case 2: If that part of λ is not divisible by k, we apply ϕ_k^{-1}.

Conversely, let $\mu = \left(\mu_1^{s_1}, \mu_2^{s_2}, \ldots, \mu_l^{s_l}\right)$ be a partition enumerated by $d_p(n,k,r)$. Then

$$\mu_i^{s_i} \mapsto (k\mu_i)^{\lfloor \frac{s_i}{k} \rfloor} \cup \phi_k(\mu_i^{s_i - k\lfloor \frac{s_i}{k} \rfloor}) \tag{10}$$

represents the inverse mapping.

Example 2 Consider $n = 9$, $p = 3$, $k = 4$ and $r = 1$.

The $d_3(9,4,1)$-partitions are:

$$(5, 1^4), (4, 1^5), (3, 2, 1^4), (3, 1^6), (2^2, 1^5), (2, 1^7), (1^9).$$

To find the image of (1^9), we perform the transformation:

$$\left\lfloor \frac{9}{4} \right\rfloor = 2 \text{ and } 9 - 4\left\lfloor \frac{9}{4} \right\rfloor = 1.$$

Applying the inverse map in (10) yields

$$1^9 \mapsto (4^2, 1).$$

Similarly, we have

$$\begin{aligned}
(5, 1^4) &\mapsto (5, 4)\\
(4, 1^5) &\mapsto (4, 1^5)\\
(3, 2, 1^4) &\mapsto (4, 3, 2)\\
(3, 1^6) &\mapsto (4, 3, 1^2)\\
(2^2, 1^5) &\mapsto (4, 2^2, 1)\\
(2, 1^7) &\mapsto (4, 2, 1^3).
\end{aligned}$$

4.1 Connection with the work of Fu and Tang

Recall that $o_k(n)$ denotes the number of partitions of n in which the set of parts congruent to 0 modulo k is singleton. Fu and Tang (2017) showed that if $d_k(n)$ is the number of partitions of n where exactly one part is repeated at least k times, then

$$o_k(n) = d_k(n). \tag{11}$$

Setting $k = 4$ in (11), $o_4(n) = d_4(n)$. On the other hand, we have shown that $f_0(n) = d_e(n)$ in Theorem 5. Since $f_0(n) = o_4(n)$, we have the identity

$$d_4(n) = d_e(n). \tag{12}$$

Note that (12) is not 'trivial'. As a preparation for a more generalized result, we describe its bijective proof.

Let λ be enumerated by $d_4(n)$ and j be the part that is repeated at least 4 times. Then $\lambda = \bar{\lambda} \cup j^m$ where m is the multiplicity of j in λ and $\bar{\lambda}$ is the subpartition of λ consisting of parts that are repeated at most 3 times. Write m as

$$m = 4q + i \quad \text{where} \quad 0 \le i \le 3 \text{ and } q \ge 1.$$

One writes $j^m = j^{4q} \cup j^i$ and converts j^{4q} into $(2j)^{2q}$.

Apply Glaisher map ϕ_4 on $\bar{\lambda} \cup j^i$ so that the image $\phi_4(\bar{\lambda} \cup j^i)$ is a partition into parts not divisible by 4.

Decompose the partition as

$$\phi_4(\bar{\lambda} \cup j^i) = \lambda' \cup \lambda''$$

where λ' is the subpartition consisting of odd parts and λ'' is the subpartition consisting of parts $\equiv 2 \pmod 4$. Divide each part of λ'' by 2 and apply ϕ_2^{-1} on the resulting partition.

What follows is a partition into distinct parts. Then multiply every part of this partition by 2 and call the resulting partition β.

Note that

$$(2j)^{2q} \cup \beta \cup \lambda'$$

is a partition enumerated by $d_e(n)$.

The inverse will be demonstrated in the general map. We require the following lemma in the subsequent work.

Lemma 1 *Let p, k and x be positive integers such that $pk \geq 2$. Suppose that $x \not\equiv 0 \pmod{pk}$ and $x \equiv 0 \pmod{p}$. Then $\frac{x}{p} \not\equiv 0 \pmod{k}$.*

Now, setting $r = 0$ in Theorem 8, observe that $d_p(n, k, 0)$ is the number of partitions of n in which exactly one part $\equiv 0 \pmod{p}$ appears at least k times. By the above result of Fu and Tang, the following generalization of (12) follows:

Theorem 9 *For $n \geq 0$,*

$$d_{pk}(n) = d_p(n, k, 0).$$

Our interest in this theorem is in its bijective construction, which extends the given mapping for the special case $k = 2$ and $p = 2$.

Let λ be enumerated by $d_{pk}(n)$ and j be the part that is repeated at least pk times. Then $\lambda = \bar{\lambda} \cup j^m$ where is the multiplicity of j in λ and $\bar{\lambda}$ is the subpartition of λ consisting of parts that are repeated at most $pk - 1$ times. Write m as

$$m = pkq + i \quad \text{where} \quad 0 \leq i \leq pk - 1 \quad \text{and} \quad q \geq 1.$$

One rewrites j^m as $j^m = j^{pkq} \cup j^i$ and converts j^{pkq} into $(pj)^{kq}$.

Apply Glaisher map ϕ_{pk} on $\bar{\lambda} \cup j^i$ so that the image $\phi_{pk}(\bar{\lambda} \cup j^i)$ is a partition into parts not divisible by pk.

Decompose this partition as

$$\phi_{pk}(\bar{\lambda} \cup j^i) = \lambda' \cup \lambda''$$

where λ' is the subpartition consisting of parts $\not\equiv 0 \pmod{p}$ and λ'' is the subpartition consisting of parts $\equiv 0 \pmod{p}$. Divide each part of λ'' by p and note that, by Lemma 1, the resulting parts are not divisible by k. Apply ϕ_k^{-1} on the resulting partition. What follows is a partition into parts that appear at most $k - 1$ times. Then multiply every part of this partition by p and call the resulting partition β.

Note that

$$(pj)^{kq} \cup \beta \cup \lambda'$$

is a partitition enumerated by $d_p(n, k, 0)$.

Example 3 Consider $n = 433$, $p = 3$ and $k = 4$.

Consider the $d_{12}(433)$-partition:

$$(13^{10}, 10^5, 7^{30}, 6^2, 4^5, 1^{11})$$

which is decomposed as:

$$(13^{10}, 10^5, 6^2, 4^5, 1^{11}) \cup (7^{30})$$

where $j = 7$ and $m = 30 = 12 \cdot 2 + 6$. Thus $7^{30} = (7^{24}) \cup (7^6)$ and so

$$(7^{24}) \mapsto (21^8).$$

Applying the Glaisher's map to the remaining parts yields

$$\phi_{12}((13^{10}, 10^5, 7^6, 6^2, 4^5, 1^{11}) = (13^{10}, 10^5, 7^6, 6^2, 4^5, 1^{11}).$$

Note that $6 \equiv 0 \pmod 3$ and thus

$$\beta = 3 \times \phi_4^{-1}\left(\frac{6}{3}, \frac{6}{3}\right) = (6^2).$$

Taking the union of all the image parts gives

$$(21^8, 13^{10}, 10^5, 6^2, 4^5, 1^{11})$$

which is a $d_3(433, 4, 0)$-partition.

The inverse is described as follows:

Let μ be enumerated by $d_p(n, k, 0)$. Decompose the partition as

$$\mu = s^m \cup \beta \cup \lambda'$$

where s is that one part $\equiv 0 \pmod p$ whose multiplicity m is at least k times, β is the subpartition consisting of parts $\equiv 0 \pmod p$ whose multiplicities are $\leq k-1$ and λ' is the subpartition consisting of parts $\not\equiv 0 \pmod p$. Write m as

$$m = kt + f \quad \text{where} \quad 0 \leq f \leq k-1 \quad \text{and} \quad t \geq 1.$$

One rewrites s^m as $s^m = s^{kt} \cup s^f$ and converts s^{kt} into $\left(\frac{s}{p}\right)^{pkt}$.

Divide each part of $\beta \cup s^f$ by p and then apply ϕ_k on the resulting partition. Then multiply every part of the obtained partition by p and denote the resulting partition by μ'. Note that μ' is a partition whose parts are not divisible by pk but divisible by p. Thus the parts of $\lambda' \cup \mu'$ are not divisible by pk. Now compute

$$\mu'' = \phi_{pk}^{-1}(\lambda' \cup \mu')$$

so that μ'' is a partition with parts appearing at most $pk-1$ times. Clearly,

$$\left(\frac{s}{p}\right)^{pkt} \cup \mu''$$

is a partition enumerated by $d_{pk}(n)$.

Reversing the previous example, recall the $d_3(433, 4, 0)$-partition:

$$(21^8, 13^{10}, 10^5, 6^2, 4^5, 1^{11}).$$

We decompose it as follows:

$$(21^8) \cup (6^2) \cup (13^{10}, 10^5, 4^5, 1^{11}).$$

Thus,

$$(21^8) \mapsto (7^{24}),$$
$$3 \times \phi_4\left(\frac{6}{3}, \frac{6}{3}\right) = (6^2).$$

Applying the Glaisher's map to $(13^{10}, 10^5, 4^5, 1^{11}) \cup (6^2)$ gives

$$\phi_{12}^{-1}(13^{10}, 10^5, 6^2, 4^5, 1^{11}) = (13^{10}, 10^5, 7^6, 6^2, 4^5, 1^{11}).$$

Now, taking the union of all the image parts yields

$$(13^{10}, 10^5, 7^{30}, 6^2, 4^5, 1^{11})$$

which is a $d_{12}(433)$-partition.

4.2 A slightly different partition function

For $\alpha \geq k$, let $g(n, \alpha, k, p)$ denote the number of partitions of n in which only one part appears at least α times and its multiplicity is congruent to $\alpha + j$ modulo p where $0 \leq j \leq k-1$ and all other parts appear at most $k-1$ times. Denote by $g_o(n, \alpha, k, p)$ (resp. $g_e(n, \alpha, k, p)$ the number of $g(n, \alpha, k, p)$-partitions in which the part repeated at least α times is odd (resp. even). Then we have:

Theorem 10

$$\sum_{n=0}^{\infty} g_o(n, \alpha, k, p) - g_e(n, \alpha, k, p)q^n = \frac{(q^k; q^k)_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \frac{q^{pn+\alpha}}{1+q^{pn+\alpha}}. \tag{13}$$

Proof If z tracks the part repeated at least α times, observe that

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} g(n, \alpha, k, p)q^n z^m \\
&= \sum_{n=1}^{\infty}\Bigg[z^n \left(\sum_{m=0}^{\infty} q^{(\alpha+mp)n} + \sum_{m=0}^{\infty} q^{(\alpha+1+mp)n} + \cdots + \sum_{m=0}^{\infty} q^{(\alpha+k-1+mp)n} \right) \\
&\quad \times \prod_{j=1, j\neq n}^{\infty} (1 + q^j + q^{2j} + q^{3j} + \cdots + q^{(k-1)j}) \Bigg] \\
&= \sum_{n=1}^{\infty} z^n \sum_{i=0}^{k-1}\sum_{m=0}^{\infty} q^{(\alpha+i+mp)n} \prod_{j=1, j\neq n}^{\infty} (1 + q^j + q^{2j} + q^{3j} + \cdots + q^{(k-1)j}) \\
&= \sum_{n=1}^{\infty} z^n q^{\alpha n} \sum_{i=0}^{k-1} q^{in} \sum_{m=0}^{\infty} q^{mpn} \prod_{j=1, j\neq n}^{\infty} (1 + q^j + q^{2j} + q^{3j} + \cdots + q^{(k-1)j}) \\
&= \sum_{n=1}^{\infty} \frac{z^n q^{\alpha n}}{1-q^{pn}} \sum_{i=0}^{k-1} q^{in} \prod_{j=1, j\neq n}^{\infty} (1 + q^j + q^{2j} + q^{3j} + \cdots + q^{(k-1)j}). \\
&= \sum_{n=1}^{\infty} \frac{z^n q^{\alpha n}}{1-q^{pn}} \prod_{j=1}^{\infty} (1 + q^j + q^{2j} + q^{3j} + \cdots + q^{(k-1)j}). \\
&= \sum_{n=1}^{\infty} \frac{z^n q^{\alpha n}}{1-q^{pn}} \prod_{j=1}^{\infty} \frac{1-q^{kj}}{1-q^j} \\
&= \frac{(q^k; q^k)_\infty}{(q; q)_\infty} \sum_{n=1}^{\infty} \frac{z^n q^{\alpha n}}{1-q^{pn}}
\end{aligned}$$

so that

$$\begin{aligned}
\sum_{n=0}^{\infty} (g_o(n, \alpha, k, p) - g_e(n, \alpha, k, p))q^n &= -\frac{(q^k; q^k)_\infty}{(q; q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n q^{\alpha n}}{1-q^{pn}} \\
&= -\frac{(q^k; q^k)_\infty}{(q; q)_\infty} \sum_{n=1}^{\infty} (-1)^n q^{\alpha n} \sum_{m=0}^{\infty} q^{pnm}
\end{aligned}$$

$$= -\frac{(q^k;q^k)_\infty}{(q;q)_\infty}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(-q^{\alpha+pm})^n$$
$$= -\frac{(q^k;q^k)_\infty}{(q;q)_\infty}\sum_{m=0}^{\infty}\frac{-q^{\alpha+pm}}{1+q^{\alpha+pm}}$$
$$= \frac{(q^k;q^k)_\infty}{(q;q)_\infty}\sum_{m=0}^{\infty}\frac{q^{\alpha+pm}}{1+q^{\alpha+pm}}.$$

□

We have the following relationship between $h(n,\,p)$ and $g(n,\,p,\,p,\,p)$.

Proposition 11 *For all $n \geq 0$,*

$$g_o(n,\,p,\,p,\,p) = h_0(n,\,p) \;\; \textit{and} \;\; g_e(n,\,p,\,p,\,p) = h_p(n,\,p).$$

Proof From the fact that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} g(n,\,p,\,p,\,p)q^n z^m = \frac{(q^p;q^p)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{z^n q^{pn}}{1-q^{pn}},$$

recall that z tracks the part which is repeated at least p times. Then

$$\sum_{n=0}^{\infty} g_o(n,\,p,\,p,\,p)q^n = \frac{(q^p;q^p)_\infty}{(q;q)_\infty}\sum_{n=1,n\equiv 1 \pmod 2}^{\infty}\frac{q^{pn}}{1-q^{pn}}$$
$$= \frac{(q^p;q^p)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^{2pn-p}}{1-q^{2pn-p}}$$
$$= \sum_{n=0}^{\infty} h_0(n,\,p)q^n$$

and on the same note,

$$\sum_{n=0}^{\infty} g_e(n,\,p,\,p,\,p)q^n = \frac{(q^p;q^p)_\infty}{(q;q)_\infty}\sum_{n=1,n\equiv 0 \pmod 2}^{\infty}\frac{q^{pn}}{1-q^{pn}}$$
$$= \frac{(q^p;q^p)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^{2pn}}{1-q^{2pn}}$$
$$= \sum_{n=0}^{\infty} h_p(n,\,p)q^n.$$

□

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