

CONSTRAINED SOLUTIONS TO IFPPs BY FINITE-DIMENSIONAL APPROXIMATION

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the degree of Doctor of Philosophy.

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Declaration

I declare that this thesis is my own, unaided work, except where otherwise acknowledged. It is being submitted for the degree of Doctor of Philosophy to the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination to any other university.

Signed this 18th day of January 2024.

A handwritten signature in black ink, appearing to read 'André Martin McDonald', with a stylized, cursive script.

André Martin McDonald

Abstract

The inverse Frobenius-Perron problem (IFPP) refers to the probabilistic modelling problem that requires the design of a discrete-time, one-dimensional dynamical system (i) that is ergodic and (ii) generates a zero-input time response that possesses prescribed statistical metrics as specifications. Systems designed using solutions to the IFPP hold potential as highly efficient and versatile random signal generators as well as solutions to practical radar signal processing problems.

Distinct formulations of the IFPP have appeared in the literature:

- IFPP-I: The system is designed such its zero-input response possesses a prescribed probability density function (PDF).
- IFPP-II: The system is designed such that its zero-input response simultaneously possesses a prescribed PDF and a prescribed power spectral density (PSD).
- IFPP-III: An unknown system is reconstructed from sequences of PDF estimates of its zero-input response, which are derived from measurements of the unknown system.

The work presented in this thesis contributes towards IFPP-II and IFPP-III by developing novel methods that improve upon exiting solutions. These novel methods construct candidate systems that admit an exact finite-dimensional representation for the FPO, which governs the evolution of density functions of the system's response.

The finite dimensionality of the FPO yields practical analytical methods for designing dynamical systems with prescribed statistical metrics as specifications, and for the consistent reconstruction of unknown dynamical systems. However, the structure of these systems imposes restrictions on the statistical metrics that are realisable, and only permit the approximation of arbitrary metrics that may be prescribed.

Examples are presented that demonstrate the realisation of power spectra with prescribable poles and multiple peaks that are adjustable, and the reconstruction of

unknown systems with consistent branch monotonicity and power spectrum mode characteristics, using the novel methods. Although the application-specific design and performance evaluation of random signal generators constructed using the proposed methods are not considered in this work, it is anticipated that these methods will be of interest in applications such as low-complexity random signal generation for Digital Radio Frequency Memory (DRFM) systems.

*In loving memory of my wife Marilize McDonald,
who embarked on this journey with me.
Thank you for your love, support and patience.
I will carry you in my heart forever.*

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List of Symbols

The principal symbols used in this thesis are summarised below, and the first equation in which each symbol appears is given.

S	A map from the unit interval onto itself, <i>Equation (2.1)</i>
$\mathcal{T}$	The time instants at which a dynamical system's state is defined, <i>Equation (2.1)</i>
x_t	The system state at time instant t , <i>Equation (2.1)</i>
Ω	The unit interval $[0,1]$, <i>Equation (2.1)</i>
μ	The Borel measure, <i>Equation (2.8)</i>
ν	A σ -finite measure, <i>Equation (2.9)</i>
$\mathcal{B}$	The Borel σ -algebra, <i>Equation (2.12)</i>
$\mathcal{D}$	The set of density functions, <i>Equation (2.15)</i>
X_t	The random variable representing the system state at time instant t (PDSA framework), <i>Equation (2.16)</i>
P_t	The probability measure induced by random variable X_t (PDSA framework), <i>Equation (2.16)</i>
P	The underlying measure (PDSA framework), <i>Equation (2.16)</i>
f_t	The density of states at time instant t (PDSA framework), <i>Equation (2.21)</i>
$\mathcal{M}_S(t)$	The expected value of the stochastic process $(X_t)_{t \in \mathcal{T}}$ at time instant t (PDSA framework), <i>Equation (2.22)</i>

$\bar{\mathcal{M}}_S(x_1)$	The mean value of the system trajectory starting from the initial state x_1 (PDSA framework), <i>Equation (2.23)</i>
$\mathcal{P}_S$	The Frobenius-Perron operator of the map S , <i>Equation (2.24)</i>
$\mathcal{P}_{S,\lambda}$	The Frobenius-Perron operator of the map S , defined with respect to the measure λ , <i>Equation (2.37)</i>
$\mathcal{U}_S$	The Koopman operator of the map S , <i>Equation (2.40)</i>
f_S^*	The invariant density of the map S , <i>Equation (2.58)</i>
χ_B	The characteristic function of the set B , <i>Equation (2.58)</i>
P_S^*	The invariant measure of the map S , <i>Equation (2.60)</i>
$\mathcal{R}_S(t_1, t_2)$	The ensemble ACF of the stochastic process $(X_t)_{t \in \mathcal{T}}$ (PDSA framework), <i>Equation (2.65)</i>
$\bar{\mathcal{R}}_S(\tau, x_1)$	The time ACF of the trajectory generated by the map S and starting from the initial state x_1 (PDSA framework), <i>Equation (2.66)</i>
$\mathcal{P}_S$	The power spectrum distribution function of the map S (PDSA framework), <i>Equation (2.78)</i>
$\mathcal{P}_{S,C}$	The continuous component of the power spectrum distribution function of the map S (PDSA framework), <i>Equation (2.80)</i>
$\mathcal{P}_{S,I}$	The impulsive component of the power spectrum distribution function of the map S (PDSA framework), <i>Equation (2.80)</i>
$\mathcal{R}_{S,C}$	The continuous component of the autocorrelation function of the map S (PDSA framework), <i>Equation (2.82)</i>
$\mathcal{R}_{S,I}$	The impulsive component of the autocorrelation function of the map S (PDSA framework), <i>Equation (2.82)</i>
$\mathcal{S}_S$	The power spectral density of the map S (PDSA framework), <i>Equation (2.83)</i>
$\mathcal{S}_{S,C}$	The continuous component of the power spectral density of the map S (PDSA framework), <i>Equation (2.83)</i>
$\mathcal{S}_{S,I}$	The impulsive component of the power spectral density of the map S (PDSA framework), <i>Equation (2.83)</i>
$\delta(\cdot)$	The Dirac delta function, <i>Equation (2.89)</i>

$\mathcal{Q}$	A partition of the unit interval, <i>Equation (2.90)</i>
Q_m	An interval of the partition $\mathcal{Q}$, <i>Equation (2.90)</i>
M	The number of intervals in the partition $\mathcal{Q}$, <i>Equation (2.90)</i>
$\Gamma(\mathcal{Q})$	The class of piecewise monotonic maps defined over the partition $\mathcal{Q}$, <i>Equation (2.90)</i>
S_m	The branch function of the map S defined over interval Q_m , <i>Equation (2.90)</i>
$\bar{P}(\mathcal{Q})$	The set of partition points of the partition $\mathcal{Q}$, <i>Equation (2.90)</i>
q_m	A partition point of the partition $\mathcal{Q}$, and an endpoint of the interval $Q_m = (q_m, q_{m+1})$, <i>Equation (2.90)</i>
$N(x)$	The indices of the intervals in $\mathcal{Q}$ that possess an image under S that contains the point x , <i>Equation (2.92)</i>
F_t	The distribution function associated with the random variable X_t (PDSA framework), <i>Equation (2.94)</i>
d_m	A constant from the set $\{0, 1\}$ that indicates the monotonicity of the branch function S_m , <i>Equation (2.100)</i>
Q_k^c	The closure of the interval Q_k , <i>Equation (2.105)</i>
$\mathcal{R}$	A subpartition of the partition $\mathcal{Q}$, <i>Equation (2.106)</i>
$\mathcal{R}^{(m)}$	A partition of the interval Q_m , <i>Equation (2.106)</i>
$R_j^{(m)}$	An interval of the partition $\mathcal{R}^{(m)}$, <i>Equation (2.106)</i>
$c(m)$	The number of intervals in the partition $\mathcal{R}^{(m)}$, <i>Equation (2.106)</i>
$r_j^{(m)}$	A partition point of the partition $\mathcal{R}^{(m)}$, and an endpoint of the interval $R_j^{(m)} = (r_j^{(m)}, r_{j+1}^{(m)})$, <i>Equation (2.106)</i>
$\mathcal{D}_{\mathcal{Q}}^{(\ell)}$	The set of order- ℓ piecewise polynomial density functions, <i>Equation (2.109)</i>
$S_{j,m}$	The branch function of the map S defined over the interval $R_j^{(m)}$, <i>Equation (2.109)</i>

$v_{j,m}$	The slope of the branch function $S_{j,m}$ of the map S defined over the interval $R_j^{(m)}$, <i>Equation (2.110)</i>
$w_{j,m}$	The y -intercept of the branch function $S_{j,m}$ of the map S defined over the interval $R_j^{(m)}$, <i>Equation (2.110)</i>
$\mathbf{f}_t$	The vector representation of the density f_t (PDSA framework), <i>Equation (2.113)</i>
$\mathbf{P}_S^{(\ell)}$	The order- ℓ Frobenius-Perron matrix of the map S , <i>Equation (2.113)</i>
$\mathbf{Q}_S$	A submatrix of the order-one FPM of a semi-Markov map, <i>Equation (2.117)</i>
$\mathbf{R}_S$	A submatrix of the order-one FPM of a semi-Markov map, <i>Equation (2.117)</i>
$T(k, m)$	The indices j of the intervals $R_j^{(m)}$ that are mapped to interval Q_k by the map S , <i>Equation (2.120)</i>
$s_{j,m}$	The sign of the slope of the branch function $S_{j,m}$ of the map S defined over the interval $R_j^{(m)}$, <i>Equation (2.121)</i>
$\mathbf{v}_1^T$	(In general) The left Perron eigenvector of the order-zero FPM of a semi-Markov map, <i>Equation (2.124)</i>
λ_m	An eigenvalue of the order-zero or order-one FPM of a semi-Markov map, <i>Equation (2.125)</i>
$\mathbf{w}_1$	(In general) The right Perron eigenvector of the order-zero FPM of a semi-Markov map, <i>Equation (2.127)</i>
$\tilde{\mathbf{v}}_1^T$	(In general) The left Perron eigenvector of the order-one FPM of a semi-Markov map, <i>Equation (2.133)</i>
$\tilde{\mathbf{w}}_1$	(In general) The right Perron eigenvector of the order-one FPM of a semi-Markov map, <i>Equation (2.133)</i>
c_m	The weight of the power spectrum mode, associated with the eigenvalue λ_m , of a semi-Markov map, <i>Equation (2.134)</i>
$\tilde{\lambda}_m$	A unique eigenvalue from the set of eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_{2M}$ that satisfy $0 < \lambda_m < 1$, <i>Equation (2.146)</i>
M_Z	The number of zeros of the rational power spectrum of the map S , <i>Equation (2.154)</i>

M_P	The number of poles of the rational power spectrum of the map S , <i>Equation (2.154)</i>
η_m	A zero of the rational power spectrum of the map S , <i>Equation (2.154)</i>
κ_m	A pole of the rational power spectrum of the map S , <i>Equation (2.154)</i>
$\Psi_{S,C}(z)$	The bilateral z -transform of the function $\mathcal{R}_{S,C}(\tau)$, <i>Equation (2.249)</i>
$\delta[\cdot]$	The Kronecker delta function, <i>Equation (2.252)</i>
f_P	A prescribed invariant density for the map S , <i>Equation (3.1)</i>
$\mathcal{R}_P$	A prescribed autocorrelation function for the map S , <i>Equation (3.2)</i>
$\mathcal{S}_P$	A prescribed power spectral density for the map S , <i>Equation (3.2)</i>
σ_S^2	The variance of the process generated by an ergodic map S , <i>Equation (3.3)</i>
F_P	The distribution function associated with an invariant density f_P prescribed for the map S , <i>Equation (3.12)</i>
$\tilde{S}$	(In general) A topological conjugate of the map S , <i>Equation (3.20)</i>
h	(In general) A conjugating transformation, <i>Equation (3.20)</i>
u_m	An inter-branch correspondence function that relates branches S_1 and S_m of the map S , <i>Equation (3.47)</i>
Ω'	The interval $[-1, 1]$, <i>Equation (3.100)</i>
$\mathcal{R}_S^*$	The normalised ACF of the map S , <i>Equation (3.112)</i>
$x_{t,k}^n$	The k^{th} state of the ensemble at time instant t of probing experiment n , <i>Equation (3.172)</i>
K	The number of states in each ensemble of the probing experiments, <i>Equation (3.172)</i>
N	The number of probing experiments, <i>Equation (3.172)</i>
T	The number of time instants in each probing experiment, <i>Equation (3.172)</i>

Γ	The gamma function, <i>Equation (4.19)</i>
$\mathcal{T}$	The map $\mathcal{T} : \mathbb{P}_N \rightarrow \mathbb{P}_{N+1}$ associated with the original disaggregation technique, <i>Equation (4.23)</i>
$T(\cdot, \cdot, \cdot)$	The operator for defining the matrix to be inserted using the original disaggregation technique, <i>Equation (4.25)</i>
$\mathcal{U}$	The map $\mathcal{T} : \mathbb{P}_N \rightarrow \mathbb{P}_{N+2}$ associated with the novel disaggregation technique, <i>Equation (4.33)</i>
$U(\cdot, \cdot, \cdot)$	The operator for defining the matrix to be inserted using the novel disaggregation technique, <i>Equation (4.35)</i>
$\mathbf{P}_{(t)}$	The stochastic matrix after t rounds of disaggregation, <i>Equation (4.46)</i>
$\mathbf{Z}^\dagger$	The Moore-Penrose pseudoinverse of the matrix $\mathbf{Z}$, <i>Equation (5.40)</i>
ε_S	The L^1 error between the unknown map S and the reconstructed map $\hat{S}$, <i>Equation (5.56)</i>
$\varepsilon_{f_S^*}$	The L^1 error between the invariant densities of the unknown map S and the reconstructed map $\hat{S}$, <i>Equation (5.56)</i>
$\varepsilon_{\mathcal{S}_S}$	The L^1 error between the power spectral densities of the unknown map S and the reconstructed map $\hat{S}$, <i>Equation (5.56)</i>

Nomenclature

ACF Autocorrelation Function

ACI Absolutely Continuous Invariant

DRFM Digital Radio Frequency Memory

FPM Frobenius-Perron Matrix

FPO Frobenius-Perron Operator

IEP Inverse Eigenvalue Problem

IFPP Inverse Frobenius-Perron Problem

NIEP Non-negative Inverse Eigenvalue Problem

PDF Probability Density Function

PDSA Probabilistic Dynamical Systems Analysis

PSD Power Spectral Density

PSDF Power Spectrum Distribution Function

PWC Piecewise Constant

PWL Piecewise Linear

StIEP Stochastic Inverse Eigenvalue Problem

Chapter 1

INTRODUCTION

1.1 Background and Motivation

In the first half of the 20th century, researchers such as Borel [1], Rényi [2], and Ulam and von Neumann [3] first showed that non-linear deterministic systems with unsophisticated evolution rules may exhibit complex and seemingly random behaviour [4]. The formal analysis of these systems started to gain momentum as the domain of systems engineering emerged at Bell Telephone Laboratories in the early 1940s [5–7] and the functional analytic approach to system analysis was developed in the 1960s by Zames, Sandberg, Porter and others [8–11]. Work carried out at the time demonstrated that a memoryless non-linearity (i.e., a non-linear mapping) configured as a system with a feedback path has the potential to yield *emergent behaviour*. One instance of such emergent behaviour is *chaos*.

Whereas chaotic systems were initially regarded as nothing more than mathematical curiosities, the discovery and characterisation of chaotic behaviour in Smale’s horseshoe map [12] and the Lorenz system [13] in the 1960s introduced *chaos theory* to a wider audience of non-mathematicians. Motivated by these systems and with the advent of modern, powerful computing technology in the 1970s, scientists and engineers scrutinised differential equations in their domains and discovered that chaotic behaviour is more prevalent than initially thought [4]. Today, chaotic systems have been characterised in areas as diverse as chemistry [14], electrical and electronic engineering [15], celestial mechanics [16] and ecology [17].

The 1970s witnessed growing interest in the application of techniques from ergodic theory to study chaotic systems from a *probabilistic perspective* [4, 18, 19]. In contrast

to the geometric techniques used to study periodic and quasi-periodic systems, these analytical techniques describe how state ensembles evolve under the action of the system's evolution rule. More specifically, these techniques characterise the evolution of *probability density functions* (PDFs) of the system's zero-input state response, instead of individual responses. This approach has proven successful in numerous application areas and its maturation has led not only to deeper understanding of how chaotic dynamics emerge in systems, but also to powerful methods for systems analysis and modelling.

It became widely known in the 1970s that dynamical systems with unsophisticated (simple) evolution rules are not excluded from generating a zero-input response with complex dynamics and rich statistical behaviour [20, 21]. These systems hold potential as highly efficient yet versatile *random signal generators*. However, to qualify as a random signal generator for applications such as simulation, the statistical metrics of the system's response, such as its PDF and power spectral density (PSD), must be prescribable. The work presented in this thesis addresses the *probabilistic modelling problem* of designing a dynamical system to possess prescribed statistical metrics. This problem is known as the *inverse Frobenius-Perron problem* (IFPP).

The IFPP requires the design of a discrete-time, one-dimensional dynamical system that is (i) ergodic and (ii) generates a zero-input time response that satisfies prescribed statistical metrics as specifications [4, 21]. Several formulations of this problem have been actively studied and its importance has been demonstrated not only in random signal generation [22–26] but also in the modelling of neural processes of the olfactory system [27], the motion of fluids [28], and more recently, seismic signals measured in the Philippines [29]. Solutions to the IFPP have also been applied to design more robust clogged bits for rock drills [30] and improved radar test waveforms [31].

The original formulation of the IFPP, labelled IFPP-I, prescribes the PDF of the system's zero-input response [4, 21]. Numerous strategies for solving this problem have emerged and matured over the years, and the methods developed are well understood. Although systems designed to solve IFPP-I have been successfully applied as random signal generators [22–26], these systems lack the characteristics required for random signal generation in *Monte-Carlo simulations*. These simulations involve complex phenomena and require the prescription of the statistical characteristics of the random signals, in addition to their PDF. This is evident in Monte-Carlo simulations for the performance evaluation of mobile communications and radar systems.

In mobile communications systems, the fading characteristics of the received signal

have a major impact on system performance [21, 32]. Monte-Carlo performance evaluation in this domain requires random signals with a particular PDF and a PSD selected from several models developed for multipath radio channels. Random signal generators that only permit selection of the PDF cannot be used in this context.

In radar engineering, digital radio frequency memory (DRFM) systems for evaluating the performance of radar systems are gaining popularity [21, 33–37]. DRFM systems operate within the hardware loop and are required to generate realistic random signals that mimic the radar’s environment. These signals must be generated at very high sample rates of up to 5 gigasamples per second. In the radar environment, multipath propagation [38] and clutter [39–41] affect system performance and yield return signals with Doppler spectra of different shapes. DRFM systems rely on low-complexity random signal generators that afford as much freedom as possible to prescribe the signals’ PDF and PSD.

By the 1990s, it became evident that the PDF of the system response is a relatively weak constraint in the context of the IFPP, and that considerable freedom remained to be exploited [42]. From the viewpoint of statistical metrics, specifying the PDF neither uniquely determines the response’s autocorrelation function (ACF) nor its PSD. These observations brought about *generalised* formulations of the IFPP that specify additional statistical metrics as specifications for the system response [4].

Motivated by the promise of more versatile random signal generators that may be designed around solutions to the generalised formulations of the IFPP, the work presented in this thesis contributes towards their solution through novel methods for the construction of dynamical systems with prescribed statistical metrics as specifications. Although the application-specific design and performance evaluation of random signal generators constructed using the proposed methods are not considered in this work, it is anticipated that these methods will find application in this domain.

The review of existing methods and techniques for solving the IFPP, as presented in chapter 3 of this thesis, yielded one journal article [4]. Refer to section 1.4 for a full list of publications that stemmed from the work presented in this thesis.

1.2 Gaps and Novel Contributions

The novel work presented in this thesis addresses two generalised formulations of the IFPP, labelled IFPP-II and IFPP-III. In this section, a brief overview of each

problem formulation is presented, and the gaps in existing methods for solving each problem are identified. The contributions of the novel work that bridges these gaps, as well as the novel methods developed to solve each problem, are described.

1.2.1 IFPP-II

IFPP-II requires the design of an ergodic system to simultaneously satisfy a prescribed response PDF and PSD as specifications [4]. Progress has been made towards solving this problem by generalising the existing solution strategies for IFPP-I to permit the prescription of the response PSD. This yielded an analytical method [43] and a numerical method [42] for system construction. After these methods appeared in the literature, attention shifted to the design of systems with more general structures that afford more design freedom to realise the response PDF and PSD; see, for example, [44].¹

The existing analytical and numerical methods have limitations that prevent their practical application [21]. The analytical method is restricted to realising PSDs with only a single mode [43]. The numerical method relies on a computationally impractical search over a high dimensional space of candidate solutions for a system that meets the specifications [42]. This method is subject to non-convergence and theory on the class of PSDs that are realisable by the method lacks.

To identify the cause of the existing methods' limitations, and to create context for the novel work presented here, the problem of designing a system with a prescribed PSD is briefly considered. The system configuration under consideration yields a general expression for the PSD as a function of the system's Frobenius-Perron operator (FPO), which characterises the evolution of density functions of the system's state from one time instant to the next [4]. Existing methods either explicitly or implicitly aim to design a system to possess a candidate FPO that yields the prescribed PSD.

The design difficulty lies in the complexity of the expressions that relate the PSD of a system selected from the class of candidates to an infinite dimensional FPO and ultimately the structural parameters of the system's evolution rule. Although the author of this thesis conjectures that progress on this front is possible by using techniques from the calculus of variations and perturbation theory, the work in [43] was forced to restrict the candidate systems to those with rudimentary power spectra²

¹The novel work presented here does not consider the design of systems with more general structures, and instead seeks to solve IFPP-II using the original system configuration.

²The terms *power spectrum* and *PSD* are used as synonyms in this thesis.

in order to simplify these expressions and realise an analytical design algorithm. To yield systems with more sophisticated power spectra, the complexity of the expressions led to the reliance on numerical methods to search for a system that implicitly satisfies the expression for the prescribed PSD [42]. However, the space of candidate systems is high dimensional, which renders this approach impractical.

The work presented here developed a novel analytical method for the *approximate* solution of IFPP-II that addresses the above shortcomings. This novel method is restricted to designing systems with a *finite dimensional FPO*. The thesis underlying this approach is that finite dimensionality permits sufficient simplification to admit a practical analytical algorithm for system design. More specifically, the finite dimensionality of the FPO and the properties of the evolution rules selected for the candidate systems yield expressions that relate the response PDF and PSD to a finite collection of structural parameters in a relatively *unsophisticated* and *exact* manner. These structural parameters are analytically selected to satisfy the specifications. However, finite dimensionality also restricts the classes of PDFs and PSDs that are realisable using the method. This implies a trade-off whereby the novel method can only approximate arbitrary statistical metrics prescribed as specifications.

The novel method constructs a system that is configured to use a piecewise linear map from the class of semi-Markov maps as its evolution rule. This permits a strategy whereby a matrix representation of the FPO is designed with characteristics that yield the prescribed statistical metrics. A candidate semi-Markov map is then designed to satisfy this matrix and exclusively realise the metrics implied by the matrix. However, the successful implementation of this strategy to yield an analytical method required the resolution of several design issues using multiple novel contributions. The primary contributions of the work that led to the resolution of the design issues are

- the relaxation of the design specifications for the matrix representation of the FPO in a novel manner that still permits the approximation of the prescribed response PDF,
- the development of a novel method for analytically constructing a stochastic matrix of arbitrary dimensionality and with a prescribable complex-valued eigenspectrum, and
- the definition of a novel subclass of the semi-Markov maps, referred to as the *hat maps*. The hat maps possess structural features designed to prevent the manifestation of unwanted spectral peaks in the system's power spectrum.

The space limitation of this section prevents the enumeration and description of all the novel contributions made towards solving IFPP-II. However, these contributions are listed in Tables 4.2.1 and 4.2.2 on pages 200 and 201.

The novel method designs a system with a response PDF that is piecewise constant and a response PSD that is expressed as a rational function. This method selects the response PDF to approximate prescribed functions that are smooth, and permits the prescription of the power spectrum poles. Although this method does not permit the approximation of arbitrary continuous power spectra, it is demonstrated that the poles may be selected to realise power spectra with multiple peaks, where the centre frequencies and bandwidths of the peaks may be independently adjusted.³ Due to this flexibility and the piecewise linear nature of the systems' evolution rules, it is anticipated the novel method will be relevant in random signal generation applications that trade accuracy of the statistical metrics for greater efficiency, such as DRFM systems [33–37].

Unlike the existing methods [42, 43], the novel method cannot realise arbitrary PDFs prescribed for the system response in an exact fashion. However, it realises a larger class of response PSDs compared to the single-mode power spectra associated with the existing analytical method [43]. Furthermore, in contrast to the existing numerical method [42], the novel method is computationally feasible and entirely analytical, which bestows immunity to numerical problems such as non-convergence.

The novel work that addresses IFPP-II, as presented in this thesis, yielded one accepted journal article [45] and four accepted conference articles [24, 46–48]. A further journal article that describes the novel method was also submitted [21]; this article is currently under peer review. The novel work regarding IFPP-II was also featured in two presentations at international conferences which require submissions to undergo peer review before their acceptance [49, 50].

1.2.2 IFPP-III

IFPP-III is a recent problem formulation that requires the reconstruction of an unknown dynamical system from PDF estimates of the system's zero-input response, which are derived from measurements of the unknown system as time progresses [51–56]. Solutions to this problem are relevant in system modelling applications

³It is shown in chapter 4 that the novel method is restricted with respect to the values that may be prescribed for the poles. This restricts the number of sharp peaks in the power spectrum and the frequencies at which these peaks may be realised.

where it is impossible to isolate and track individual system responses from one time instant to the next. Examples of such applications include particle image velocimetry for flow visualisation [57], flow cytometry [58], the modelling of emulsion polymerisation [59], papermaking systems [60] and communication systems [61].

The original solution to IFPP-III [51, 52] is designed to reconstruct an unknown system configured to use a canonical⁴ semi-Markov map as its evolution rule. Although the solution’s ability to reconstruct the system without tracking the time evolution of individual responses is revolutionary, it has two shortcomings [56]. More specifically, the original solution cannot reconstruct the system with a consistent evolution rule if the unknown map possesses any decreasing branch functions. In this case, the original solution also yields a system with a response PSD that has inconsistent mode characteristics (i.e., mode centre frequencies and bandwidths).

The original solution reconstructs the unknown system to be consistent with a matrix estimate of its FPO, which is derived from the PDF estimates of its response. However, this matrix representation of the unknown FPO, referred to in this work as the *order-zero Frobenius-Perron matrix* (FPM), does not depend on the monotonicity of the unknown map’s branch functions [56]. The original solution is thus unable to identify the monotonicity of the unknown branch functions, and assumes all these functions are increasing during reconstruction. This leads to the inconsistent reconstruction of any canonical semi-Markov map that possesses one or more decreasing branches.

The work presented here developed a novel method for solving IFPP-III that addresses the original solution’s shortcomings. This novel method reconstructs the unknown system to be consistent with a subset of the elements from a *higher-order* matrix representation of its FPO. This *order-one FPM* is chosen for its dependency on the monotonicity of the unknown branch functions. The thesis underlying this strategy is that an estimate of this matrix permits the accurate resolution of branch function monotonicity in canonical semi-Markov maps, thereby admitting an algorithm for the consistent and accurate reconstruction of systems that use these maps.

The novel method’s successful implementation of the above strategy required the resolution of several design issues using multiple novel contributions. The primary contributions of the work that led to the resolution of the design issues are

- the development of a method-of-moments technique for estimating piecewise linear density functions that is novel in the context of IFPP-III,

⁴The term *canonical* refers to additional restrictions imposed on the structure of the map.

- the derivation of estimates for two submatrices of the unknown order-one FPM, which utilise the Moore-Penrose pseudoinverse and which were shown to be optimal in the limit of perfect estimation accuracy, and
- the development of a novel subclass of the semi-Markov maps, labelled the *generalised hat maps*. These maps possess structural features that are chosen in order to satisfy the required elements of any order-one FPM estimate.

The space limitation of this section prevents the enumeration and description of all the novel contributions made towards solving IFPP-III. However, these contributions are listed in Tables 5.2.1 and 5.2.2 on pages 286 and 287.

The novel work yielded two propositions on the characteristics of the novel method for solving IFPP-III. These propositions prove the novel method's consistent reconstruction of any unknown system configured to use a canonical semi-Markov map, and the consistency of the system's power spectrum mode characteristics regardless of the unknown map's canonicity. Both the propositions hold in the limit of perfect estimation accuracy and regardless of the monotonicity of the unknown branches.

The proposition statements were verified by numerically reconstructing examples of systems that are configured to use canonical semi-Markov maps with decreasing branch functions as evolution rules. These examples demonstrated accurate system reconstruction despite imperfect estimation accuracy, which led to the conclusion that the novel method resolves the original solution's shortcomings. However, unlike the original solution, the novel method introduces additional branches into the reconstructed map. This is an additional source of reconstruction error.

The novel method enlarges the class of systems that are reconstructible with consistent characteristics. Due to its greater versatility, it is anticipated that the novel method will find wider applicability than the original solution.

The novel work that addresses IFPP-III, as presented in this thesis, yielded one accepted journal article [56] and featured in one presentation at an international conference which requires submissions to undergo peer review before acceptance [62].

1.3 Structure of the Thesis

The structure of the remainder of this thesis is outlined below.

Chapter 2 lays the mathematical foundation for the work presented in the subsequent chapters. The relevant concepts and techniques from dynamical systems and ergodic theory, amongst other domains, are introduced. A framework that represents the response of the system of interest as a stochastic process is developed. This probabilistic dynamical systems analysis (PDSA) framework permits system analysis from a probabilistic perspective, as required for the development of solutions to the IFPP. Relevant properties of the stochastic process are proved, and general expressions are derived for its statistical metrics as functions of the FPO.

The chapter concludes by defining and characterising the class of piecewise linear semi-Markov maps, which are constructed and configured as systems by the novel methods for solving the IFPP that are presented in the later chapters. The matrix representations of the FPO associated with these systems, and their relevant properties, are derived. Expressions for the statistical metrics of systems that use these maps as evolution rules are derived. These expressions are comparatively unsophisticated functions of the structural parameters of the maps.

Chapter 3 defines the three distinct formulations of the IFPP and identifies multiple strategies from the literature that have been successfully applied to solve the problems. This leads to the classification of existing methods from the literature, their review and comparison, and the identification of gaps in existing work. Several examples are also presented to demonstrate the characteristics of the existing methods.

Chapter 4 presents the novel work that addresses IFPP-II. An incremental approach to the solution of this problem is adopted. This leads to the definition of a relaxed problem formulation that permits the approximation of statistical metrics that may be prescribed for the system. This formulation is labelled IFPP-IIa. Two novel strategies for solving IFPP-IIa are proposed. The design issues associated with the most feasible strategy are identified, and the novel contributions that permit the resolution of these issues are presented. The novel method that implements the strategy to solve IFPP-IIa is described next. Several numerical examples are presented that involve the design of elementary and more sophisticated systems using the novel method. The characteristics and capabilities of the novel method are then compared to those of the existing methods, and conclusions are drawn.

Chapter 5 presents the novel work that addresses IFPP-III. The shortcomings of the original solution to this problem are identified, and two novel solution strategies for overcoming these deficiencies are proposed. The steps of the novel method that implements the most feasible solution strategy are presented and each of the

novel contributions is described. The propositions regarding the characteristics of the novel solution are stated and proved, and two numerical examples that involve the reconstruction of unknown systems using both methods are presented. The numerical results are compared to the propositions and interpreted. Several conclusions regarding the novel method are presented.

1.4 Publications

Tables 1.1 and 1.2 list the peer-reviewed publications that stemmed from the work presented in this thesis.

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- [8] G. Zames. Functional analysis applied to nonlinear feedback systems. *IEEE Trans. Circuit Theor.*, 10(3):392–404, 1963.

Table 1.1: Peer-reviewed publications yielded by the work presented in this thesis (part 1 of 2).

Ref. no.	Type and status	Publication
[4]	Journal, appeared	A. M. McDonald, M. A. van Wyk, and G. Chen. The inverse Frobenius-Perron problem: A survey of solutions to the original problem formulation. <i>AIMS Math.</i> , 6(10):11200–11232, 2021. doi: https://doi.org/10.3934/math.2021650 .
[45]	Journal, appeared	A. M. McDonald, M. A. van Wyk, and G. Chen. A direct solution to the stochastic inverse eigenvalue problem for complex-valued eigenspectra. <i>Linear Algebra Appl.</i> , 658:262–282, 2023. doi: https://doi.org/10.1016/j.laa.2022.11.005 .
[56]	Journal, appeared	A. M. McDonald and M. A. van Wyk. Identification of nonlinear discrete systems from probability density sequences. <i>IEEE Trans. Circuits Syst. I: Regular Papers</i> , 70(2):1–14, 2022. doi: https://doi.org/10.1109/TCSI.2022.3219133
[21]	Journal, under peer review (IEEE Access)	A. M. McDonald and M. A. van Wyk. The construction of ergodic maps with prescribed invariant density and power spectrum poles. <i>TechRxiv</i> , pages 1–14, 2022. doi: https://doi.org/10.36227/techrxiv.21031384.v1 .
[46]	Full conference paper, appeared	A. McDonald and A. van Wyk. Solution of the inverse Frobenius-Perron problem for semi-Markov chaotic maps via recursive Markov state disaggregation. In <i>Proc. 25th IEEE Eur. Signal Process. Conf.</i> , pages 1654–1658, 2017. doi: https://doi.org/10.23919/EUSIPCO.2017.8081480 .
[47]	Full conference paper, appeared	A. McDonald and A. van Wyk. Construction of semi-Markov ergodic maps with selectable spectral characteristics via the solution of the inverse eigenvalue problem. In <i>Proc. 2017 Asia Pacific Signal Inf. Process. Assoc. Annu. Summit Conf</i> , pages 987–993, 2017. doi: https://doi.org/10.1109/APSIPA.2017.8282167 .

Table 1.2: Peer-reviewed publications yielded by the work presented in this thesis (part 2 of 2).

Ref. no.	Type and status	Publication
[24]	Full conference paper, appeared	A. M. McDonald and M. A. van Wyk. Efficient generation of random signals with prescribed probability distribution and spectral bandwidth via ergodic transformations. In Proc. 26th IEEE Eur. Signal Process. Conf., pages 331–335, 2018. doi: https://doi.org/10.23919/EUSIPCO.2018.8553152 .
[48]	Full conference paper, appeared	A. M. McDonald and M. A. van Wyk. A novel approach to solving the generalized inverse Frobenius-Perron problem. In Proc. 2020 IEEE Int. Symp. Circuits Syst., pages 1–5, 2020. doi: https://doi.org/10.1109/ISCAS45731.2020.9181115 .
[50]	Conference talk	M. A. van Wyk and A. M. McDonald. Solving the inverse Frobenius-Perron problem via monotization of random variable transformations. In 12th Chaotic Modelling and Simulation International Conference Book of Abstracts, pages 111–112, 2019. http://www.cmsim.org/images/CHAOS_2019_BoA_Final_version.pdf .
[62]	Conference talk	A. M. McDonald and M. A. van Wyk. Estimation of ergodic maps with unified power spectra from causal sequences of state density functions. In 13th Chaotic Modelling and Simulation International Conference Book of Abstracts, pages 90–91, 2020. http://www.cmsim.org/images/Book_of_Abstracts_CHAOS_2020---.pdf
[49]	Conference talk	A. M. McDonald and M. A. van Wyk. Design of ergodic maps with prescribed invariant densities and multimodal power spectra. In 14th Chaotic Modelling and Simulation International Conference Book of Abstracts, pages 81–82, 2021. http://www.cmsim.org/images/B00K_C_2021_compressed.pdf

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Chapter 2

MATHEMATICAL BACKGROUND

2.1 Introduction

Over the past century, it became known that non-linear deterministic systems can generate a response that evolves in what appears to be an unpredictable and random manner [1–5]. Studying the behaviour of these chaotic systems by observing individual responses over long time intervals reveals exceedingly complex behaviour. The *probabilistic approach* to analysing chaotic systems considers the evolution of PDFs of the system’s zero-input state response, instead of individual responses. This approach has yielded powerful methods for systems analysis and modelling [6, 7].

Many of the concepts and methods for analysing chaotic systems from a probabilistic viewpoint appear in mathematical literature on ergodic theory and are thus inaccessible to non-mathematicians in general. This observation spurred authors such as Boyarsky and Góra [6] and Lasota and Mackey [7] to publish books that equip scientists and engineers who encounter chaotic systems in practice with the relevant tools to analyse these systems. These books took a significant step towards bridging the knowledge gap in the analysis of chaotic systems that exists between a relatively small group of mathematicians and the rest of the scientific community.

The IFPP requires the construction of an ergodic system with prescribed statistical metrics. Although the work presented here does not seek to characterise chaotic systems, the probabilistic techniques that were developed to analyse these systems may be applied to ergodic systems in general and are thus of importance in the context of the IFPP. This chapter draws primarily upon [6–8] to present the definitions, concepts and theory required to address the IFPP. The chapter also defines and

characterises the class of semi-Markov maps. These maps admit matrix solutions to the IFPP and are constructed by the methods presented in subsequent chapters.

The remainder of this chapter is set out as follows. Section 2.2 lists the novel contributions presented in the chapter. The class of dynamical systems that the IFPP requires the construction of is defined in section 2.3, and the concept of chaos is introduced in a brief and informal manner. Section 2.4 establishes a framework for analysing the relevant dynamical systems from a probabilistic viewpoint, introduces the FPO and Koopman operator, and defines the concept of ergodicity. The same section defines the statistical metrics associated with an ergodic system and derives general expressions that relate these metrics to the system's FPO.

Section 2.5 defines the class of piecewise monotonic maps and presents an explicit form for the FPO of systems configured to use these maps as evolution rules. Section 2.6 defines the class of semi-Markov maps, which comprise a subclass of the piecewise monotonic maps. This section also derives matrix representations for the finite dimensional FPO of these maps, and states several relevant properties of these matrices. The section concludes by deriving expressions that relate the statistical metrics of ergodic systems that use semi-Markov maps to the matrix representations of the FPO and ultimately the parameters of the map's branch functions.

2.2 Novel Contributions of this Chapter

The novel contributions of this chapter are enumerated and described in Table 2.1.

2.3 Dynamical Systems and Chaotic Behaviour

Informally speaking, a *deterministic* dynamical system is a system¹ with a state that evolves over time according to a well-defined rule, which is referred to as the *evolution rule* of the system. The *state space* is defined as the set of all possible values that the system's state can assume.

This work only considers dynamical systems with evolution rules that do not change over time and with one-dimensional states that evolve over discrete and equally spaced instants in time. Hence, attention is restricted to *discrete-time* and *one-dimensional*

¹As is common practice in dynamical systems literature, no attempt is made to provide precise definitions of certain notions such as a system and its state.

Table 2.1: Novel contributions presented in chapter 2.

Reference	Publications by author	Description
Sec. 2.4.2, Def. 2.4.4, 2.4.15, Prop. 2.4.6 to 2.4.11	–	The development of the PDSA framework for the probabilistic analysis of dynamical systems, and the specification of conditions for the equality of time and ensemble averages in this framework.
Sec. 2.4.7, Eq. (2.67), Rem. 2.4.5	–	The use of the Koopman operator to derive a general expression for the ACF of an ergodic map.
Sec. 2.5.3, Eq. (2.104)	–	The derivation of an expression for the evolution of distribution functions under iteration of a piecewise monotonic map.
Sec. 2.6.3, Eq. (2.117)	–	The derivation of an expression for the order-one FPM of a semi-Markov map, as counterpart to a similar expression for Markov maps presented in [9].
Sec. 2.6.3, Prop. 2.6.3 to 2.6.11	–	The presentation of proofs that the order-zero and order-one FPMs of semi-Markov maps possess several relevant properties. The presentation of the information in this chapter unifies theory and results scattered throughout the literature.
Sec. 2.6.4, Eq. (2.141), (2.142)	–	The derivation of an expression for the ACF of a semi-Markov map with diagonalisable order-one FPM, as counterpart to a similar expression for Markov maps presented in [9].
Sec. 2.6.5, Eq. (2.152), (2.154)	–	The derivation of an expression for the PSD of a semi-Markov map with diagonalisable order-one FPM, as counterpart to a similar expression for Markov maps presented in [9].

dynamical systems that are *time invariant*. Although the state of a dynamical system may be defined over an abstract state space, in this work the state space is selected as the unit interval of the real line. Any evolution rule defined over the unit interval may be transformed to an evolution rule defined over an arbitrary but compact real interval via topological conjugation with an affine conjugating transformation [8].

A *dynamical system* and the *trajectories* it generates are defined as follows.

Definition 2.3.1 (Dynamical system). A dynamical system comprises a triple $(\mathcal{T}, \Omega, S)$ and a state $x_t \in \Omega$ defined at each time instant $t \in \mathcal{T}$, where $\mathcal{T} \triangleq \{1, 2, \dots\}$. In this definition, $\Omega \triangleq [0, 1]$ denotes the state space of the system and $S : \Omega \rightarrow \Omega$ denotes a surjective map that represents the evolution rule of the system. The state evolves according to the expression

$$x_{t+1} = S(x_t), \quad t \in \mathcal{T}, \quad (2.1)$$

where x_1 is referred to as the *initial state* of the system.

Definition 2.3.2 (Trajectory of a dynamical system). The sequence of states $(x_1, x_2, \dots)$ that emerges as a dynamical system evolves is referred as the *trajectory* of the system starting from the initial state x_1 .

Equation (2.1) implies that

$$x_t = S^{t-1}(x_1), \quad t \in \mathcal{T}, \quad (2.2)$$

where S^t denotes the t -fold composition of S ,

$$S^t(x) \triangleq \begin{cases} x, & \text{if } t = 0, \\ \underbrace{S \circ \dots \circ S}_{S \text{ appears } t \text{ times}}(x), & \text{if } t > 0. \end{cases} \quad (2.3)$$

Figure 2.1 is a diagram of the dynamical system introduced in Definition 2.3.1. The evolution rule of the system is implemented by using the map S (a memoryless non-linearity) to generate the system response in a recursive fashion [10]. The system is driven by an input v_t which determines the initial state of the system. The input is set to zero for the remaining time instants. In this definition,

$$v_t = \begin{cases} v \in \Omega, & \text{if } t = 0, \\ 0, & \text{if } t > 0, \end{cases} \quad (2.4)$$

and

$$x_{t+1} = \begin{cases} 0, & \text{if } t = -1, \\ S(x_t + v_t), & \text{if } t \geq 0. \end{cases} \quad (2.5)$$

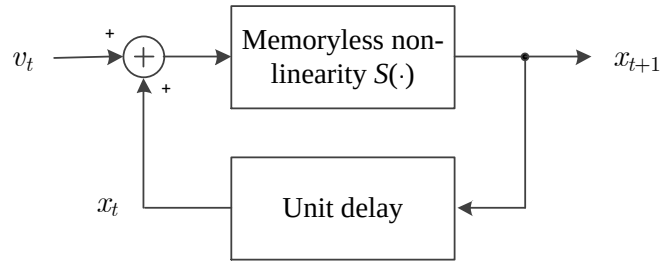


Figure 2.1: Diagram of a recursive system driven by an input v_t . Reproduced with permission from [10].

The initial state of the system is given by $x_1 = S(v_0)$. The zero-input response of the system, or its trajectory, then evolves independently according to $x_{t+1} = S(x_t)$, where $t \in \mathcal{T}$.

Remark 2.3.1. Throughout this thesis, the terms ‘system’ and ‘map’ are frequently interchanged when there is no possibility of confusion, and system-related concepts are attributed to the map itself. For example, the phrase ‘trajectory of the map S ’ is often used as a shortcut for the phrase ‘zero-input response of the system configured to use the map S as evolution rule’. Similarly, a map may be described as exhibiting chaotic behaviour, where reference is actually made to the system configured to use the map. In all such cases, it is implicitly assumed that the system configuration of Fig. 2.1 is used.

The motivation for above approach is to ensure consistency with the publications [8, 11–16] that stemmed from the work presented in this thesis. These publications associate system-related concepts with the map. Furthermore, by following this approach, unnecessarily long statements are avoided.

The *logistic map*, popularised by May [17], is arguably the best known example of a chaotic map. It is used to illustrate various concepts in this chapter.

Example 2.3.1 (The logistic map). The logistic map $S : \Omega \rightarrow \Omega$ is defined by

$$S(x) = \alpha x(1 - x), \quad \alpha \in [1, 4]. \quad (2.6)$$

This work only considers the logistic map with parameter $\alpha = 4$. In this case, the map exhibits chaotic behaviour. This particular map is referred to simply as the logistic map in this manuscript and is plotted in Fig. 2.2.

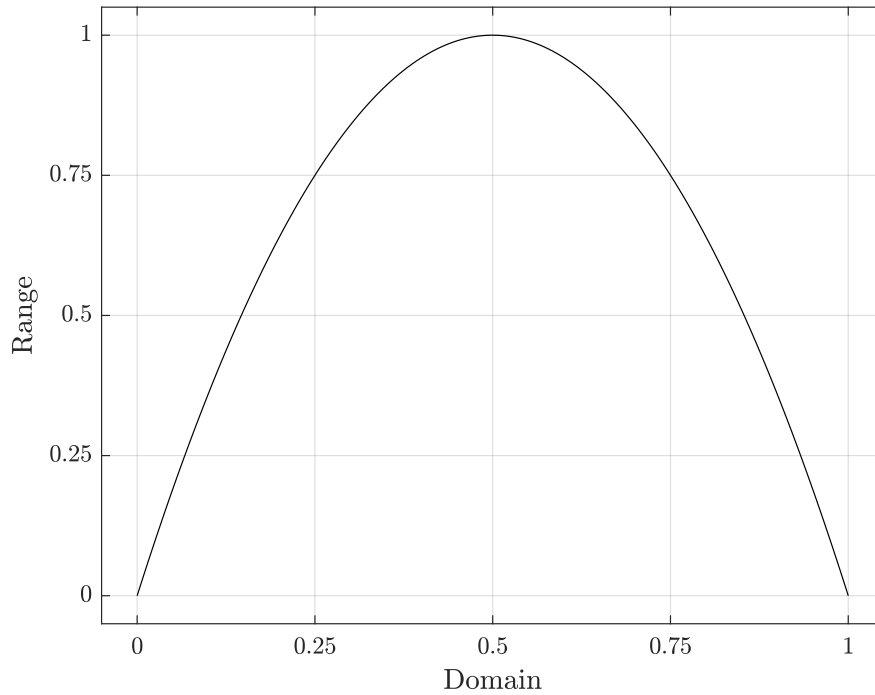


Figure 2.2: The logistic map $S(x) = 4x(1 - x)$.

The logistic map has complicated dynamics, despite its unsophisticated structure. Different trajectories of the logistic map may possess distinct characteristics. This point is illustrated with the three trajectories plotted in Figs. 2.3(a) to 2.3(c). The trajectory of Fig. 2.3(a) evolves in a manner that appears unpredictable, whereas the remaining trajectories settle into regular and predictable behaviour. In particular, the trajectory of Fig. 2.3(b) reaches the point $x = 3/4$, which is the non-zero solution of the expression $S(x) = x$, and then remains fixed. This point is referred to as a *fixed point* of the logistic map. The trajectory of Fig. 2.3(c) becomes periodic, such that the period is equal to six time instants.

Chaos refers to the seemingly unpredictable nature of a system's dynamics [18, 19]. Roughly speaking, a non-linear system is chaotic if it exhibits sensitive dependence on the initial state and has an infinite number of different periodic responses. Formally, the term chaos refers to one of several mathematical definitions which are either topological or measure theoretical in nature (the interested reader is referred to the summary in [19]). As this work is not concerned with the study of chaotic dynamics, these definitions are not investigated further, and a formal and precise definition of a chaotic system is not provided. However, it is important to note that only non-linear systems can exhibit chaotic behaviour, regardless of the definition used.

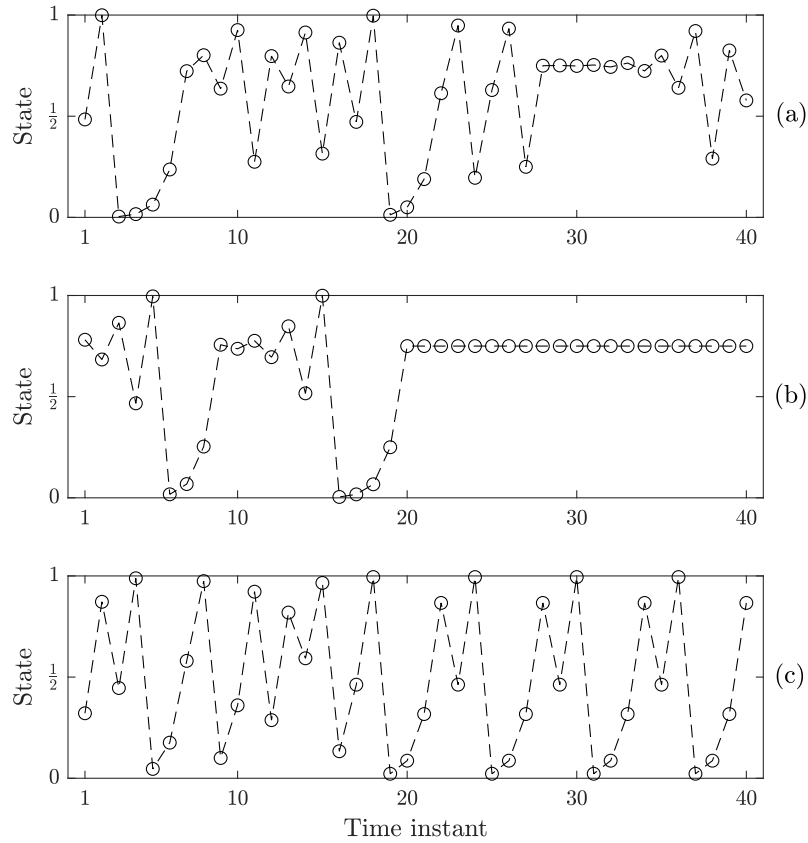


Figure 2.3: Three trajectories of the logistic map with distinct characteristics.

Let the state space Ω be endowed with the distance function

$$d(a, b) \triangleq |a - b|, \quad a, b \in \Omega, \quad (2.7)$$

and consider two distinct initial states that are arbitrarily close to each other in the state space. Sensitive dependence on the initial state implies that the trajectories starting from these initial states diverge locally at an exponential rate [19]. In the case of physical systems with states that cannot be measured with perfect accuracy, sensitive dependence on the initial state implies that accurate prediction of the points on the system's trajectory arbitrarily far into the future is impossible.

From the engineering perspective, chaos may manifest as what appears to be interference or a random noise signal in a system, even though the phenomenon being observed is deterministic in nature and inherent to the system (i.e., it does not originate from a source in the system's environment, but from the system itself) [20]. Chaos also has the characteristic of generating a *density* of system states and may thus be investigated using techniques for the probabilistic analysis of systems. This approach is the topic of the following section.

2.4 Probabilistic Analysis of Systems

This section first introduces the mathematical concepts of measure spaces, function spaces and densities, which are used throughout this work. The probabilistic dynamical systems analysis (PDSA) framework is introduced next. This framework uses random variables to represent the state of a dynamical system at different instants in time, thereby defining a stochastic process. The FPO and its adjoint operator (the Koopman operator) are then defined and their relevant properties are enumerated. The FPO of the map² configured as the system's evolution rule characterises the evolution of the PDFs associated with the random variables of the PDSA framework.

The concepts of an invariant measure and an invariant density of a map are defined. Thereafter, the notion of an ergodic map is defined, and the implications of ergodicity are described. The extension of the Birkhoff individual ergodic theorem, which states that averages computed along almost every trajectory generated by an ergodic map are equal to averages computed with respect to the map's unique invariant density over the state space, is presented. This theorem is shown to imply that the invariant density of an ergodic map configured as the evolution rule of a system coincides with its response PDF. The ACF and PSD³ associated with an ergodic map are defined, and simplified expressions are derived that relate these metrics to the map's FPO.

2.4.1 Measure spaces, function spaces and densities

Using set theory as a starting point, the concept of a *topology* may be defined, and refined to other spaces by introducing more structure [21]. By introducing the notion of distance, a topological space becomes a *metric space*. By further defining an algebra and a norm over the elements of the metric space, it becomes a *normed vector space*. By then defining an inner product, the normed vector space becomes an *inner product space*, as commonly used by engineers and physicists.

Over the past century, *measure theory* became a formal branch of mathematics with the work of Borel, Lebesgue and others [22]. In the context of this work, measure theory permits the computation of statistics over the elements of the spaces introduced above. Again using set theory as a starting point, the concept of a σ -*algebra* of a set

²In chapter 1, the FPO and the concept of ergodicity are associated with the *system* and not the *map*. However, in the remainder of this work, both the FPO and ergodicity are associated with the map, under the implicit assumption that the system configuration of Fig. 2.1 is used.

³Here, the metrics of the system response are attributed to the map; refer to Remark 2.3.1.

is defined. A *measure* is defined on the σ -algebra, which then permits the definition of a *measure space*. Measure spaces then permit the development of the *Lebesgue integral*, which is a centrepiece of measure theory.⁴

The probabilistic analysis of dynamical systems is concerned with the evolution of state density functions over time. The concepts of measure spaces, function spaces and densities are thus required to develop a formal framework for this investigation. Here, the objective is not to develop these concepts from first principles; the reader is instead referred to [7, Ch. 2] for the relevant background and theory. Instead, certain definitions and notation used throughout this work are introduced.

The function spaces and operators introduced in this chapter may be defined over arbitrary measure spaces, subject to some general restrictions. Although this permits abstraction, this work selects the elements of the measure space to allow for the development of solutions to the IFPP. Each measure space is defined to be of the form $(\Omega, \mathcal{B}, \nu)$, where $\Omega \triangleq [0, 1]$ is the state space of the systems under consideration and $\mathcal{B}$ is the Borel σ -algebra on Ω . The theory presented remains valid if *any* measure ν that adheres to the restrictions (to be specified below) is selected. However, all measures considered in this work are required and assumed to be σ -finite.

In this work, the *Borel measure* is denoted by μ . Recall that the Borel measure of a real interval $A \triangleq [a, b]$ is equal to its length; i.e.,

$$\mu(A) = b - a. \quad (2.8)$$

The shorthand notation $L^p \triangleq L^p(\Omega, \mathcal{B}, \nu)$ is used to denote the L^p function space, where $1 \leq p < \infty$. The norm $\|f\|_p$ is defined as

$$\|f\|_p \triangleq \left[\int_{\Omega} |f(x)|^p d\nu(x) \right]^{1/p}, \quad f \in L^p. \quad (2.9)$$

The notation L^∞ denotes the space of functions that are bounded almost everywhere, and the norm $\|f\|_\infty$, where $f \in L^\infty$, is defined as the essential supremum of $|f|$.

Let the function spaces L^p and $L^{p'}$ be adjoint; i.e.,

$$\frac{1}{p} + \frac{1}{p'} = 1, \quad 1 \leq p < \infty. \quad (2.10)$$

In the case where $p = 1$, the space adjoint to L^1 is defined as the space L^∞ . The functional $g(f)$ is then defined as

$$g(f) \equiv \langle f, g \rangle \triangleq \int_{\Omega} f(x)g(x)d\nu(x), \quad f \in L^p, \quad g \in L^{p'}. \quad (2.11)$$

⁴Unless explicitly stated otherwise, all integrals in this chapter are Lebesgue integrals.

Functions are considered equal *almost everywhere* (a.e.) if they cannot be distinguished by calculating integrals. That is, two integrable functions f and g are equal a.e. iff

$$\int_B f(x) d\nu(x) = \int_B g(x) d\nu(x), \quad \forall B \in \mathcal{B}. \quad (2.12)$$

Let ν_f and ν_g denote the measures defined by the left- and right-hand side of (2.12), and suppose $A = \{x \in \Omega : f(x) \neq g(x)\} \in \mathcal{B}$. Then,

$$\begin{aligned} \nu_f(B) = \nu_g(B) \quad \forall B \in \mathcal{B} &\Leftrightarrow f = g \text{ a.e.} \\ &\Leftrightarrow \nu(A) = 0 \end{aligned} \quad (2.13)$$

and

$$\exists B \in \mathcal{B} : \nu_f(B) \neq \nu_g(B) \Leftrightarrow \nu(A) > 0. \quad (2.14)$$

The terms *PDF*, *density function* and *density*, as used in this work, refer to the function space defined below.

Definition 2.4.1 (Probability density function). A PDF is a member of the set

$$\mathcal{D}(\Omega, \mathcal{B}, \nu) \triangleq \{f \in L^1(\Omega, \mathcal{B}, \nu) : f \geq 0 \text{ and } \|f\|_1 = 1\}. \quad (2.15)$$

The shorthand notation $\mathcal{D} \triangleq \mathcal{D}(\Omega, \mathcal{B}, \nu)$ is used in this work.

2.4.2 Probabilistic dynamical systems analysis framework

The PDSA framework defines a stochastic process to represent the time evolution of a dynamical system's state. The evolution of this stochastic process is governed exclusively by the system's deterministic evolution rule. This approach permits the analysis of the system from a probabilistic perspective.

The PDSA framework for analysing the dynamical system $(\mathcal{T}, \Omega, S)$ is developed by first defining an underlying probability space $(\Omega, \mathcal{B}, P)$, where P is an underlying probability measure. A random variable X_1 is then defined on this underlying probability space. This random variable represents the initial state of the system.

To formally define the PDSA framework, recall that a random variable is a $\mathcal{B}$ -measurable function defined on a probability space [23]. A function f is $\mathcal{B}$ -measurable if every $B \in \mathcal{B}$ has a pre-image under f that is also a member of $\mathcal{B}$.

Definition 2.4.2 (Underlying probability space and random variable X_1). Let $(\Omega, \mathcal{B}, P)$ denote an underlying probability space and let $X_1 : \Omega \rightarrow \Omega$ denote a $\mathcal{B}$ -measurable function. The random variable $X_1 \triangleq X_1(\omega)$ is defined on $(\Omega, \mathcal{B}, P)$.

The random variable X_1 induces a probability space.

Definition 2.4.3 (Induced probability space [23]). The random variable $X_1 : \Omega \rightarrow \Omega$ on the probability space $(\Omega, \mathcal{B}, P)$ induces the probability space $(\Omega, \mathcal{B}, P_1)$, where

$$P_1(B) = P(X_1^{-1}(B)), \quad B \in \mathcal{B}, \quad (2.16)$$

denotes the *induced measure*.

The quantity $P_1(B)$ is interpreted as the fraction of system trajectories that are defined to start from an initial state in the set $B \in \mathcal{B}$. The probability measure P_1 is determined by the underlying measure P and the random variable X_1 in (2.16).

The state evolution of the dynamical system $(\mathcal{T}, \Omega, S)$ is given by (2.1). This leads to the definition of the functions

$$X_{t+1} = S \circ X_t, \quad t \in \mathcal{T}. \quad (2.17)$$

These functions represent the state of the dynamical system as evolved from the random variable X_1 . The following proposition is proved.

Proposition 2.4.1 (Random variables $X_2, X_3, \dots$). If $S : \Omega \rightarrow \Omega$ is $\mathcal{B}$ -measurable, each function $X_2, X_3, \dots$ defined by (2.17) is a random variable on $(\Omega, \mathcal{B}, P)$.

Proof. The proposition is proved in section 2.A.1. □

The PDSA framework is defined such that the condition of Proposition 2.4.1 is satisfied, which implies that each X_t , where $t \in \mathcal{T}$, is a random variable on $(\Omega, \mathcal{B}, P)$.

Now, let $(\Omega, \mathcal{B}, P_{t+1})$ denote the probability space induced by X_{t+1} . The induced measure P_{t+1} satisfies

$$\begin{aligned} P_{t+1}(B) &= P(X_{t+1}^{-1}(B)) \\ &= P(X_t^{-1}(S^{-1}(B))) \\ &= P_t(S^{-1}(B)), \quad B \in \mathcal{B}, \end{aligned} \quad (2.18)$$

where (2.17) was used in the second step. This expression characterises the evolution of the probability measures P_t , where $t > 1$, from the initial probability measure P_1 .

Let $(\Omega, \mathcal{B}, \nu)$ denote a measure space. The measure P_t is *absolutely continuous* with respect to the measure ν if

$$\nu(B) = 0 \Rightarrow P_t(B) = 0, \quad B \in \mathcal{B}. \quad (2.19)$$

A $\mathcal{B}$ -measurable function $f : \Omega \rightarrow \Omega$ is *non-singular* with respect to the measure ν if

$$\nu(B) = 0 \Rightarrow \nu(f^{-1}(B)) = 0, \quad B \in \mathcal{B}. \quad (2.20)$$

The following proposition is proved.

Proposition 2.4.2 (Absolute continuity of $P_2, P_3, \dots$). Let $S : \Omega \rightarrow \Omega$ be $\mathcal{B}$ -measurable, and let $(\Omega, \mathcal{B}, \nu)$ denote a measure space. If P_1 is absolutely continuous with respect to ν and S is non-singular with respect to ν , each measure $P_2, P_3, \dots$ is absolutely continuous with respect to ν .

Proof. The proposition is proved in section 2.A.2. □

Suppose that the conditions of Proposition 2.4.2 are satisfied, such that each measure P_t is absolutely continuous with respect to the measure ν . The Radon-Nikodym theorem [7] then implies the existence of a density function $f_t \in \mathcal{D}(\Omega, \mathcal{B}, \nu)$ associated with each probability measure P_t , such that

$$P_t(B) = \int_B f_t(x) d\nu(x), \quad B \in \mathcal{B}. \quad (2.21)$$

The function f_t represents a density of system states at time instant t . This density is unique over sets of non-zero measure.

The sequence of random variables $(X_t)_{t \in \mathcal{T}}$ is interpreted as a stochastic process. This stochastic process, together with its corresponding sequences of probability measures $(P_t)_{t \in \mathcal{T}}$ and densities $(f_t)_{t \in \mathcal{T}}$, are the primary components of the PDSA framework for analysing the dynamical system introduced in Definition 2.3.1. The PDSA framework is formally defined below.

Definition 2.4.4 (Probabilistic dynamical system analysis (PDSA) framework). Let $(\Omega, \mathcal{B}, \nu)$ denote a measure space. Consider a dynamical system $(\mathcal{T}, \Omega, S)$ with evolution rule $S : \Omega \rightarrow \Omega$ that is both $\mathcal{B}$ -measurable and non-singular with respect to ν . Let X_1 denote the random variable on the underlying probability space $(\Omega, \mathcal{B}, P)$ that represents the initial state of the system, and let $(\Omega, \mathcal{B}, P_1)$ denote the probability space induced by X_1 . Let P_1 be absolutely continuous with respect to ν . Propositions 2.4.1 and 2.4.2 then imply the existence of a random variable X_t representing the state of the dynamical system at each time instant $t \in \mathcal{T}$, its induced probability space $(\Omega, \mathcal{B}, P_t)$ and density f_t .

The PDSA framework is defined to comprise

- the measure space $(\Omega, \mathcal{B}, \nu)$,

- the dynamical system $(\mathcal{T}, \Omega, S)$,
- the underlying probability space $(\Omega, \mathcal{B}, P)$,
- the stochastic process $(X_t)_{t \in \mathcal{T}}$,
- the induced probability spaces $(\Omega, \mathcal{B}, P_t)$, where $t \in \mathcal{T}$, and
- the densities f_t , where $t \in \mathcal{T}$.

The expected value of the stochastic process $(X_t)_{t \in \mathcal{T}}$ is defined as follows.

Definition 2.4.5 (Expected value of the stochastic process $(X_t)_{t \in \mathcal{T}}$). Invoke the PDSA framework of Definition 2.4.4. The expected value of the process $(X_t)_{t \in \mathcal{T}}$ is defined as

$$\mathcal{M}_S(t) \triangleq E[X_t] = \int_{\Omega} X_t(\omega) dP(\omega), \quad t \in \mathcal{T}. \quad (2.22)$$

The mean value of a particular sample function of the stochastic process $(X_t)_{t \in \mathcal{T}}$, which is equivalent to the mean value computed along the corresponding trajectory of the system, is defined below.

Definition 2.4.6 (Mean value of a system trajectory). Invoke the PDSA framework of Definition 2.4.4. The mean value of the system trajectory starting from the initial state x_1 is defined as

$$\bar{\mathcal{M}}_S(x_1) \triangleq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} S^k(x_1), \quad x_1 \in \Omega. \quad (2.23)$$

2.4.3 The Frobenius-Perron operator

Equation (2.17) implies that each density f_t of the PDSA framework evolves under repeated iteration of the map S . Rechart [24] studied the Markov operator $\mathcal{P}_S$ that characterises the evolution of densities under repeated iteration of a non-singular transformation S . This operator later became known as the FPO [7].

In the PDSA framework, the FPO maps density functions from one time instant to the next according to the expression

$$f_{t+1} = \mathcal{P}_S(f_t), \quad t \in \mathcal{T}. \quad (2.24)$$

Equations (2.17) and (2.24) reveal that the density associated with a random variable that is transformed into another random variable evolves in accordance with the FPO.

However, the term FPO has special meaning as it is associated with the *repeated iteration* of a non-singular transformation. In particular, the FPO has been studied to determine the asymptotic properties of the sequence of densities $(f_t)_{t \in \mathcal{T}}$ [7].

The FPO is formally defined below.

Definition 2.4.7 (Frobenius-Perron operator (FPO) [7]). Let $(\Omega, \mathcal{B}, \nu)$ denote a measure space and let $S : \Omega \rightarrow \Omega$ denote a map that is non-singular with respect to ν . Let $f_t \in L^1(\Omega, \mathcal{B}, \nu)$. The unique operator $\mathcal{P}_S : L^1 \rightarrow L^1$, defined by

$$\int_B (\mathcal{P}_S f_t)(x) d\nu(x) = \int_{S^{-1}(B)} f_t(x) d\nu(x), \quad B \in \mathcal{B}, \quad (2.25)$$

is referred to as the FPO corresponding to S .

Suppose that the measure space $(\Omega, \mathcal{B}, \mu)$ is selected, where μ denotes the Borel measure, and let $B = [0, x]$. If the densities f_t and $\mathcal{P}_S f_t$ are Riemann integrable, the Lebesgue integrals of (2.25) may be replaced with Riemann integrals to obtain

$$\int_0^x (\mathcal{P}_S f_t)(s) ds = \int_{S^{-1}([0, x])} f_t(s) ds. \quad (2.26)$$

Differentiation on both sides with respect to x then yields

$$(\mathcal{P}_S f_t)(x) = \frac{d}{dx} \int_{S^{-1}([0, x])} f_t(s) ds. \quad (2.27)$$

Equation (2.27) is used to derive an explicit form for the FPO corresponding to the class of piecewise monotonic maps in section 2.5. It is also used to study the evolution of densities under iteration of the logistic map in the following example.

Example 2.4.1 (Density evolution under the logistic map). Consider the logistic map $S(x) = 4x(1 - x)$. It follows that

$$S^{-1}([0, x]) = [0, S_1^{-1}(x)] \cup [S_2^{-1}(x), 1], \quad (2.28)$$

where

$$S_1^{-1}(x) = \frac{1 - \sqrt{1 - x}}{2}, \quad x \in [0, 1] \quad (2.29)$$

and

$$S_2^{-1}(x) = \frac{1 + \sqrt{1 - x}}{2}, \quad x \in [0, 1]. \quad (2.30)$$

Equation (2.27) then yields

$$(\mathcal{P}_S f_t)(x) = f_{t+1}(x) = \frac{f_t\left(\frac{1 - \sqrt{1 - x}}{2}\right) + f_t\left(\frac{1 + \sqrt{1 - x}}{2}\right)}{4\sqrt{1 - x}}, \quad t \in \mathcal{T}. \quad (2.31)$$

The evolution of three initial densities f_1 under iteration of the logistic map is illustrated in the three columns of Fig. 2.4. The initial densities are plotted in the

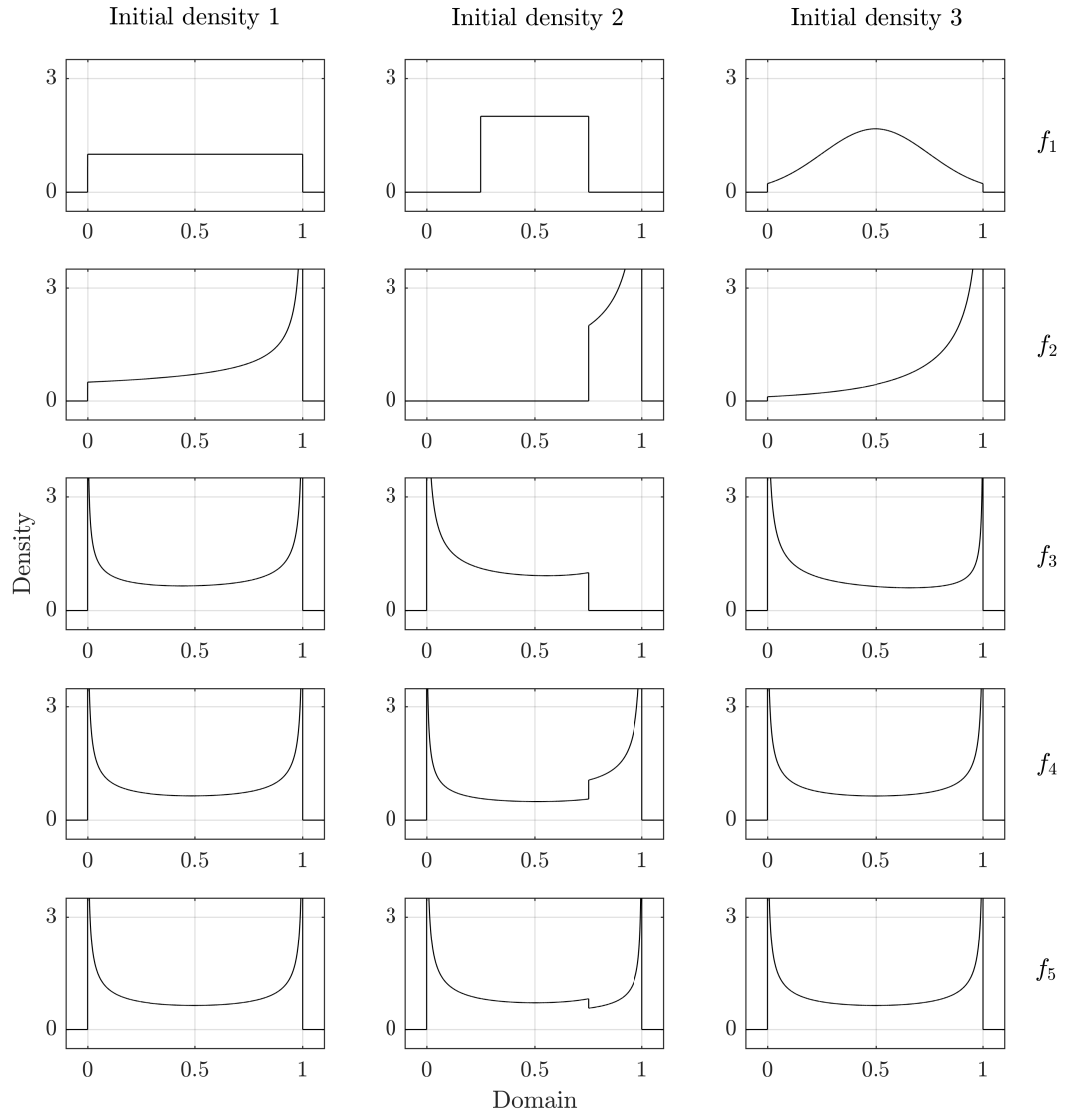


Figure 2.4: Evolution of three initial densities under iteration of the logistic map.

top row of the figure and were selected as (i) a uniform density on $[0, 1]$ (left column), (ii) a uniform density on $[1/4, 3/4]$ (middle column) and (iii) a scaled Gaussian density function restricted to the unit interval (right column). One of the initial densities f_1 as well as the four densities f_2 to f_5 that evolve from it are plotted in each column.

The final row of Fig. 2.4 suggests that the three initial densities appear to converge to the same density under repeated application of the FPO. The convergence of density sequences under the FPO has been studied extensively in the literature. This topic is not investigated here; the interested reader is instead referred to [7] for a summary of the theory and key findings.

The FPO has several important properties [7]. Those properties relevant to this work are stated without proof. A measure space $(\Omega, \mathcal{B}, \nu)$ is assumed in each case.

(FP1) Linearity:

$$\mathcal{P}_S(\alpha f + \beta g) = \alpha \mathcal{P}_S(f) + \beta \mathcal{P}_S(g), \quad f, g \in L^1, \alpha, \beta \in \mathbb{R}. \quad (2.32)$$

(FP2) Markov property:

$$\mathcal{P}_S(f) \geq 0 \text{ a.e.}, \quad \text{if } f \in L^1, f \geq 0 \text{ a.e.} \quad (2.33)$$

and

$$\|\mathcal{P}_S(f)\|_1 = \|f\|_1, \quad \text{if } f \in L^1, f \geq 0 \text{ a.e.} \quad (2.34)$$

This property implies that $\mathcal{P}_S$ maps density functions to density functions, as expected.

(FP3) FPO of a composition of maps: The FPO $\mathcal{P}_{S^t}$ corresponding to the t -fold composition S^t of the map S , as defined by (2.3), satisfies

$$\mathcal{P}_{S^t} = \mathcal{P}_S^t. \quad (2.35)$$

In this expression, $\mathcal{P}_S^0(f) \triangleq f$.

(FP4) Non-expansiveness:

$$\|\mathcal{P}_S^t(f) - \mathcal{P}_S^t(g)\|_1 \leq \|\mathcal{P}_S^{t-1}(f) - \mathcal{P}_S^{t-1}(g)\|_1, \quad f, g \in L^1, f \neq g, t \geq 1. \quad (2.36)$$

This property states that the L^1 distance between two distinct densities f and g cannot increase as the densities evolve under the FPO. Non-expansiveness is implied by the Markov property of the FPO. However, non-expansiveness is not sufficient to guarantee the existence of a unique fixed point of $\mathcal{P}_S$. In contrast, the stronger requirement of contractiveness guarantees, by the Banach fixed point theorem, the existence of a unique fixed point [25].

The following proposition is proved.

Proposition 2.4.3 (Relationship between FPOs defined over two measure spaces). Let $(\Omega, \mathcal{B}, \lambda)$ and $(\Omega, \mathcal{B}, \nu)$ denote two measure spaces, and the map $S : \Omega \rightarrow \Omega$ be non-singular with respect to both λ and ν . Let $\mathcal{P}_{S, \lambda}$ and $\mathcal{P}_{S, \nu}$ denote the FPOs corresponding to S that are defined over $(\Omega, \mathcal{B}, \lambda)$ and $(\Omega, \mathcal{B}, \nu)$, respectively. If λ is absolutely continuous with respect to ν , then

$$g \mathcal{P}_{S, \lambda}^t(f) = \mathcal{P}_{S, \nu}^t(h) \text{ a.e.}, \quad f \in L^1(\Omega, \mathcal{B}, \lambda), t = 0, 1, \dots, \quad (2.37)$$

where

$$h \triangleq f.g \quad (2.38)$$

and g is the Radon-Nikodym derivative of λ with respect to ν ; i.e.,

$$\lambda(B) = \int_B g(x) d\nu(x), \quad B \in \mathcal{B}. \quad (2.39)$$

Proof. The proposition is proved in section 2.A.3. $\square$

2.4.4 The Koopman operator

The Koopman operator [26] corresponding to a map S is defined below.

Definition 2.4.8 (Koopman operator [7]). Let $(\Omega, \mathcal{B}, \nu)$ denote a measure space and let $S : \Omega \rightarrow \Omega$ denote a map that is non-singular with respect to ν . Let $f \in L^\infty(\Omega, \mathcal{B}, \nu)$. The operator $\mathcal{U}_S : L^\infty \rightarrow L^\infty$, defined by

$$(\mathcal{U}_S f)(x) = f(S(x)), \quad (2.40)$$

is referred to as the Koopman operator with respect to S .

Two properties of the Koopman operator that are relevant to this work are stated below. The interested reader is referred to [7] for the corresponding proofs.

(K1) Linearity:

$$\mathcal{U}_S(\alpha f + \beta g) = \alpha \mathcal{U}_S(f) + \beta \mathcal{U}_S(g), \quad f, g \in L^\infty, \alpha, \beta \in \mathbb{R}. \quad (2.41)$$

(K2) Adjoint to the FPO: The Koopman operator is adjoint to the FPO; i.e.,

$$\langle \mathcal{P}_S f, g \rangle = \langle f, \mathcal{U}_S g \rangle, \quad \forall f \in L^1, g \in L^\infty. \quad (2.42)$$

2.4.5 Invariant measures and stationary densities

Invariant measures and *measure preserving maps* are relevant concepts in the context of the IFPP, and are defined below.

Definition 2.4.9 (Invariant measure and measure preserving map [7]). Let $(\Omega, \mathcal{B}, \nu)$ denote a measure space and let $S : \Omega \rightarrow \Omega$ denote a $\mathcal{B}$ -measurable map. If

$$\nu(S^{-1}(B)) = \nu(B), \quad \forall B \in \mathcal{B}, \quad (2.43)$$

the measure ν is referred to as S -invariant, and S is said to preserve the measure ν .

A *fixed point* of the FPO is a function $f \in L^1(\Omega, \mathcal{B}, \nu)$ that satisfies

$$\mathcal{P}_S(f) = f. \quad (2.44)$$

A fixed point of the FPO that is also a density is referred to as a *stationary density* of the FPO. The following theorem identifies a relationship between S -invariant measures and the fixed points of the FPO corresponding to S .

Theorem 2.4.1 (Invariant measures and fixed points of the FPO [7]). *Let $(\Omega, \mathcal{B}, \nu)$ denote a measure space and let $S : \Omega \rightarrow \Omega$ denote a map that is non-singular with respect to ν . Consider a non-negative function $f \in L^1$. The measure ν_f defined by*

$$\nu_f(B) = \int_B f(x) d\nu(x), \quad B \in \mathcal{B}, \quad (2.45)$$

is S -invariant if and only if f is a fixed point of the FPO corresponding to S .

Proof. The reader is referred to [7, Th. 4.1.1]. □

Invoke the PDSA framework introduced in Definition 2.4.4, and suppose that S preserves the measure P_t . Equation (2.43) then implies that $P_t(S^{-1}(B)) = P_t(B)$ for all $B \in \mathcal{B}$. Substituting this expression into (2.18) yields

$$P_{t+1}(B) = P_t(B), \quad \forall B \in \mathcal{B}. \quad (2.46)$$

Hence, if P_t is S -invariant,

$$P_{t+k} = P_t, \quad k \geq 0. \quad (2.47)$$

Now, consider the density f_t associated with the measure P_t . Substitution of (2.21) into (2.47) yields

$$\int_B f_{t+k}(x) d\nu(x) = \int_B f_t(x) d\nu(x), \quad \forall B \in \mathcal{B}, k \geq 0. \quad (2.48)$$

This implies that the densities f_{t+k} and f_t cannot be distinguished using integration. Hence,

$$f_{t+k} = f_t \text{ a.e., } k \geq 0. \quad (2.49)$$

This expression is consistent with f_t being a fixed point of the FPO (Theorem 2.4.1). The density f_t associated with an absolutely continuous and S -invariant probability measure P_t is referred to in this work as an *S -invariant density*.

This work frequently refers to invariant measures and invariant densities of a map S , instead of S -invariant measures and densities. Also, the star superscript is used in this work to indicate an invariant measure or invariant density of the map under

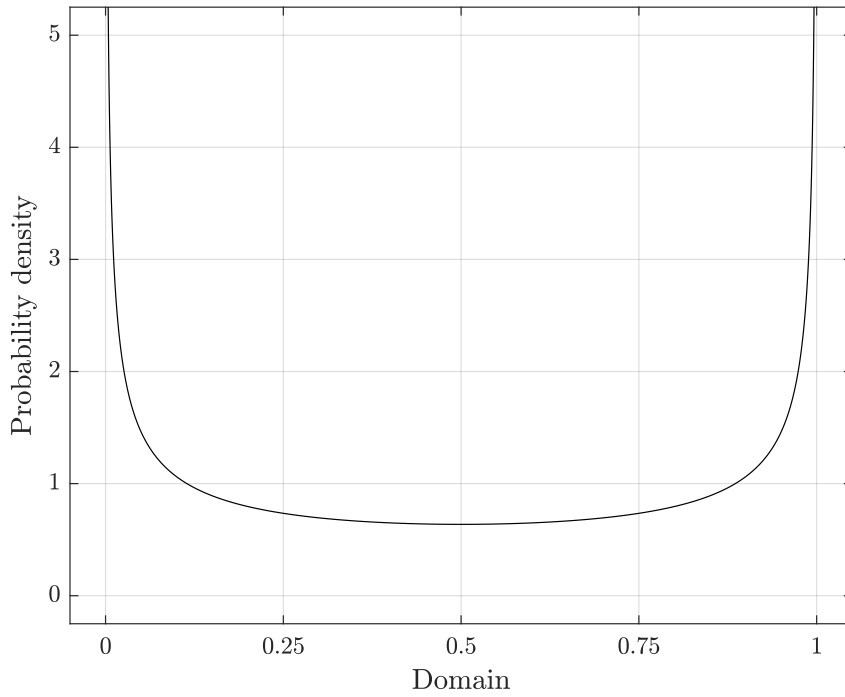


Figure 2.5: The invariant density function of the logistic map.

consideration. In particular, P_S^* and f_S^* are frequently used to denote an invariant measure and invariant density of the map S . The invariant density necessarily satisfies

$$f_S^* = \mathcal{P}_S(f_S^*). \quad (2.50)$$

Example 2.4.2 (Invariant density of the logistic map). Let $(\Omega, \mathcal{B}, \mu)$ denote a measure space, where μ denotes the Borel measure. Let $S : \Omega \rightarrow \Omega$ denote the logistic map $S(x) = 4x(1 - x)$. Consider the density $f \in \mathcal{D}(\Omega, \mathcal{B}, \mu)$ defined by

$$f(x) = \frac{1}{\pi \sqrt{x(1-x)}}. \quad (2.51)$$

This density is plotted in Fig. 2.5.

An expression was derived for the FPO $\mathcal{P}_S$ corresponding to the logistic map in (2.31) of example 2.4.1. Substituting f_t in (2.31) with the density f of (2.51) yields the expression $\mathcal{P}_S f = f$, which proves that $f_S^* = f$ is a stationary density of $\mathcal{P}_S$. Theorem 2.4.1 then implies that f_S^* is associated with an invariant measure⁵ P_S^* of S . Hence, f is an invariant density of the logistic map.

⁵It is shown in Remark 2.4.1 of the following section that this invariant measure and its corresponding invariant density are both unique.

2.4.6 Ergodicity

To define the concept of ergodicity, the notion of an *invariant set* is first introduced.

Definition 2.4.10 (Invariant set [7]). Let $S : \Omega \rightarrow \Omega$. Any set $B \subseteq \Omega$ that satisfies

$$S^{-1}(B) = B \quad (2.52)$$

is referred to as a set invariant under the map S , or an S -invariant set.

Consider a dynamical system $(\mathcal{T}, \Omega, S)$ and assume that B is any S -invariant set. Equation (2.52) implies that S maps B into itself and that no element of $\Omega \setminus B$ is mapped into B . Now consider any trajectory of the system that starts from an initial state $x_1 \in B$. It follows that

$$x_{t+1} = S^t(x_1) \in B, \quad t \geq 0. \quad (2.53)$$

This implies that any trajectory starting from an initial state that is a member of an invariant set can never ‘depart’ from this set. Similarly, consider any trajectory of the system that starts from an initial state $x_1 \notin B$. It follows that

$$x_{t+1} = S^t(x_1) \notin B, \quad t \geq 0. \quad (2.54)$$

Hence, any trajectory starting from an initial state that is not a member of a particular invariant set can never ‘enter’ this set. The existence of an S -invariant set B implies that S may be studied on the sets B and $\Omega \setminus B$ separately.

A measurable set is called *trivial* if it has zero measure.

Definition 2.4.11 (Trivial set [7]). Let $(\Omega, \mathcal{B}, \nu)$ denote a measure space. A set $B \in \mathcal{B}$ is referred to as ν -trivial if

$$\nu(B) = 0. \quad (2.55)$$

The concept of ergodicity is defined as follows.

Definition 2.4.12 (Ergodicity [7]). Let $(\Omega, \mathcal{B}, \nu)$ denote a measure space and let $S : \Omega \rightarrow \Omega$ denote a map that is non-singular with respect to ν . Then S is referred to as ergodic if, for every measurable S -invariant set B , either B or its complement $\Omega \setminus B$ is trivial. That is,

$$S^{-1}(B) = B \Rightarrow \nu(B) = 0 \text{ or } \nu(\Omega \setminus B) = 0, \quad B \in \mathcal{B}. \quad (2.56)$$

Invoke the PDSA framework of Definition 2.4.4. If the map S is ergodic, there exists no measurable set B , with measure $0 < \nu(B) < \nu(\Omega)$, wherein state trajectories are confined over all time instants. Thus, ergodic maps must be studied on the entire state space Ω .

The following theorem regarding the stationary density of the FPO corresponding to an ergodic map is relevant to the formulation of the IFPP.

Theorem 2.4.2 (Stationary density associated with an ergodic map [7]). *Let $(\Omega, \mathcal{B}, \nu)$ denote a measure space and let $S : \Omega \rightarrow \Omega$ denote a map that is non-singular with respect to ν . If S is ergodic, there exists at most one stationary density f_S^* of the FPO $\mathcal{P}_S$ corresponding to S . Furthermore, if there exists a unique stationary density f_S^* of $\mathcal{P}_S$ and $f_S^* > 0$ a.e., then S is ergodic.*

Proof. The reader is referred to [7, Th. 4.2.2]. □

The following propositions on the uniqueness of the invariant probability measure and invariant density of an ergodic map follow immediately from Theorem 2.4.2.

Proposition 2.4.4 (Uniqueness of the invariant probability measure of an ergodic map). *Let $(\Omega, \mathcal{B}, \nu)$ denote a measure space and let $S : \Omega \rightarrow \Omega$ denote a map that is non-singular with respect to ν . Let S be ergodic. If P_S^* is an S -invariant probability measure, then it is unique.*

Proof. The proposition is proved in section 2.A.4. □

Proposition 2.4.5 (Uniqueness of the invariant density of an ergodic map). *Let $(\Omega, \mathcal{B}, \nu)$ denote a measure space and let $S : \Omega \rightarrow \Omega$ denote a map that is non-singular with respect to ν . Let S be ergodic. If f_S^* is an S -invariant density, then it is unique (over subsets of non-zero measure).*

Proof. The proposition is proved in section 2.A.5. □

Remark 2.4.1 (Ergodicity of the logistic map). The logistic map $S(x) = 4x(1 - x)$ is ergodic [7]. Hence, (2.51) defines the unique invariant density of the logistic map, as implied by Proposition 2.4.5. Proposition 2.4.4 implies that the corresponding invariant probability measure is also unique.

Remark 2.4.2. Ergodicity is only one of multiple types of stability that the stationary density may exhibit. Maps that possess stronger forms of stability with respect to the

stationary density, referred to as *mixing* and *exact* maps, have been investigated. The interested reader is referred to [7, Sec. 4.3] for exact definitions and the convergence properties of density sequences generated under iteration of these maps.

Birkhoff [27, 28] presented a theorem that may be used to analyse the asymptotic behaviour of individual system trajectories. This theorem became known as the *Birkhoff individual ergodic theorem*. As the current work is concerned with the evolution of densities and not individual trajectories, this theorem is not stated here. The interested reader is instead referred to [7, Th. 4.2.3].

An extension of the Birkhoff individual ergodic theorem permits computation of the statistics of ergodic systems by equating averages computed along individual system trajectories to averages computed over the state ensemble [7]. This theorem is especially useful when analysing systems that exhibit sensitive dependence on the initial state. Computer simulations that attempt to calculate averages along trajectories of such systems suffer from inaccuracy due to finite precision arithmetic. The extension of the Birkhoff individual ergodic theorem is presented below.

Theorem 2.4.3 (Extension of the Birkhoff individual ergodic theorem [7]). *Let $(\Omega, \mathcal{B}, \nu_S^*)$ denote a finite measure space and let $S : \Omega \rightarrow \Omega$ denote an ergodic map that preserves the measure ν_S^* . Then, for any integrable function h , the average of h along the trajectory of S is equal almost everywhere to the average of h over the state space Ω . Specifically,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} h(S^k(x)) = \frac{1}{\nu_S^*(\Omega)} \int_{\Omega} h(x) d\nu_S^*(x) \text{ a.e.} \quad (2.57)$$

Proof. The reader is referred to [7, Th. 4.2.4] for a proof. $\square$

Let $(\Omega, \mathcal{B}, \nu)$ denote a measure space, and let S denote an ergodic map that possesses the unique invariant probability measure P_S^* . Suppose that P_S^* is absolutely continuous with respect to ν . Equation (2.57) may then be manipulated to obtain

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} h(S^k(x)) = \int_{\Omega} h(x) f_S^*(x) d\nu(x) \text{ a.e.}, \quad (2.58)$$

where f_S^* denotes the unique invariant density of S .

Remark 2.4.3 (PDF of the system's zero-input response). Let $h(x) = \chi_B(x)$ in (2.58), where $\chi_B : \Omega \rightarrow \Omega$ is defined for any set $B \in \mathcal{B}$ by

$$\chi_B(x) \triangleq \begin{cases} 1, & \text{if } x \in B, \\ 0, & \text{if } x \notin B. \end{cases} \quad (2.59)$$

This function is referred to as the *characteristic function* of the set $B \in \mathcal{B}$. Equation (2.58) then implies that the asymptotic fraction of samples from the zero-input system response (or, equivalently, the states from the system trajectory) that fall within B is equal to $P_S^*(B)$, for almost every response (or trajectory). In an asymptotic sense, this implies that the distribution of samples in almost every response is consistent with the invariant density of the ergodic map.

Remark 2.4.4. Remark 2.4.3 reveals an important consideration for the design of a dynamical system as a random signal generator, where the zero-input response of the system is produced and interpreted as a sample function of the random process. To generate a random process with a prescribed PDF along almost every sample function, it is sufficient to configure the system to use an ergodic map that is designed to possess an invariant density that coincides with the prescribed PDF. This design problem is the first formulation of the IFPP, labelled IFPP-I [8].

The following propositions regarding the expected value of the stochastic process and the mean value of trajectories in the PDSA framework are proved.

Proposition 2.4.6 (Expected value of the process in the PDSA framework). Invoke the PDSA framework introduced in Definition 2.4.4. Let S be ergodic and possess the unique invariant probability measure P_S^* . Furthermore, let the underlying measure P be selected as

$$P = P_S^*, \quad (2.60)$$

and let the initial random variable be selected as

$$X_1(\omega) = \omega. \quad (2.61)$$

The expected value $\mathcal{M}_S(t)$ of the process $(X_t)_{t \in \mathcal{T}}$ is then independent of time and given by

$$\mathcal{M}_S(t) \equiv \mathcal{M}_S = \int_{\Omega} x dP_S^*(x), \quad t \in \mathcal{T}. \quad (2.62)$$

Proof. The proposition is proved in section 2.A.6. □

Proposition 2.4.7 (Mean value of trajectories in the PDSA framework). Invoke the PDSA framework introduced in Definition 2.4.4. Let S be ergodic and possess the unique invariant probability measure P_S^* . The mean value of almost every trajectory generated by the system is then given by

$$\bar{\mathcal{M}}_S(x_1) \equiv \bar{\mathcal{M}}_S = \int_{\Omega} x dP_S^*(x), \quad x \in \Omega. \quad (2.63)$$

Now, let the underlying measure and initial random variable be selected as indicated in (2.60) and (2.61). It follows that the mean of almost every trajectory generated by the system is equal to the expected value of the process $(X_t)_{t \in \mathcal{T}}$; i.e.,

$$\bar{\mathcal{M}}_S = \mathcal{M}_S. \quad (2.64)$$

Proof. The proposition is proved in section 2.A.7. $\square$

2.4.7 Autocorrelation function

The autocorrelation function quantifies statistical correlation that exists within a process or trajectory. Two ACFs are defined for the PDSA framework in what follows. The first corresponds to the process $(X_t)_{t \in \mathcal{T}}$ (an ensemble average), and the second to trajectories generated by a map (a time average).

Definition 2.4.13 (Ensemble ACF of the stochastic process $(X_t)_{t \in \mathcal{T}}$). The *ensemble ACF* of the process $(X_t)_{t \in \mathcal{T}}$ is defined as

$$\mathcal{R}_S(t_1, t_2) = E[X_{t_1} X_{t_2}] \triangleq \int_{\Omega} X_{t_1}(\omega) X_{t_2}(\omega) dP(\omega), \quad t_1, t_2 \in \mathcal{T}, \quad (2.65)$$

where P denotes the underlying probability measure.

Definition 2.4.14 (Time ACF of a system trajectory [10, 12]). The time ACF of the trajectory generated by the map S and starting from the initial state x_1 is defined as

$$\bar{\mathcal{R}}_S(\tau, x_1) \triangleq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} S^k(x_1) S^{|\tau|+k}(x_1), \quad \tau \in 0, \pm 1, \pm 2, \dots \quad (2.66)$$

Let the map S be ergodic and possess a unique invariant probability measure P_S^* that is absolutely continuous with respect to the Borel measure μ . Using the extension of the Birkhoff individual ergodic theorem, it is shown in section 2.B that the time ACF $\bar{\mathcal{R}}_S(\tau, x_1)$ is equal to the corresponding average computed over the state ensemble for almost every initial state x_1 of the system trajectory. A general expression that relates the time ACF of almost every system trajectory to an ergodic map's FPO is also derived in section 2.B. This expression is given by

$$\bar{\mathcal{R}}_S(\tau, x_1) \equiv \bar{\mathcal{R}}_S(\tau) = \int_{\Omega} x(\mathcal{P}_S^{|\tau|} g)(x) d\mu(x), \quad \tau = 0, \pm 1, \pm 2, \dots, \quad (2.67)$$

where

$$g(x) \triangleq x f_S^*(x). \quad (2.68)$$

In these expressions, f_S^* denotes the unique invariant density of the map S , and $\mathcal{P}_S$ denotes the FPO corresponding to S and defined with respect to $(\Omega, \mathcal{B}, \mu)$.

Remark 2.4.5. Although (2.67) is not a novel result, to the best of this author's knowledge, its derivation using the Koopman operator (as presented in section 2.B) is novel. The following expression was obtained by Mori et al. [29, Eq. 3.3] for the time ACF of an ergodic map S :

$$\bar{\mathcal{R}}_S(\tau) = \int_{\Omega} A(x)(\mathcal{P}_S^{|\tau|}\tilde{g})(x)d\mu(x), \quad \tau = 0, \pm 1, \pm 2, \dots, \quad (2.69)$$

where

$$\tilde{g}(x) = B(x)f_S^*(x) \quad (2.70)$$

and

$$A(x) = B(x) = x. \quad (2.71)$$

Equation (2.67) is identical to (2.69).

In Propositions 2.4.6, it was proved that the assumption of an ergodic map S and appropriate choices for the underlying measure P and initial random variable X_1 of the PDSA framework yields a process $(X_t)_{t \in \mathcal{T}}$ with an expected value $\mathcal{M}_S(t)$ that becomes independent of time. Under the same assumptions, it was proved in Proposition 2.4.7 that almost all trajectories generated by S share the same mean value $\bar{\mathcal{M}}_S$ and that this mean is equal to the expected value of the process. In the following propositions, analogous results are proved for the ACF.

Proposition 2.4.8 (Ensemble ACF of the process in the PDSA framework). Invoke the PDSA framework introduced in Definition 2.4.4. Let S be ergodic and possess the unique invariant probability measure P_S^* . Furthermore, let the underlying measure and initial random variable be selected according to (2.60) and (2.61). The ensemble ACF $\mathcal{R}_S(t_1, t_2)$ of the process $(X_t)_{t \in \mathcal{T}}$ then only depends on the time difference $\tau \triangleq t_2 - t_1$ and is given by

$$\mathcal{R}_S(t_1, t_2) \equiv \mathcal{R}_S(\tau) = \int_{\Omega} x S^{|\tau|}(x) dP_S^*(x), \quad \tau = t_2 - t_1 = 0, \pm 1, \pm 2, \dots \quad (2.72)$$

Proof. The proposition is proved in section 2.A.8. □

Proposition 2.4.9 (Time ACF of trajectories in the PDSA framework). Invoke the PDSA framework introduced in Definition 2.4.4, and let S be ergodic and possess the unique invariant probability measure P_S^* . The time ACF of almost every trajectory generated by the system is then given by

$$\bar{\mathcal{R}}_S(\tau, x_1) \equiv \bar{\mathcal{R}}_S(\tau) = \int_{\Omega} x S^{|\tau|}(x) dP_S^*(x), \quad \tau = 0, \pm 1, \pm 2, \dots \quad (2.73)$$

Now, let the underlying measure and initial random variable be selected according to (2.60) and (2.61). It follows that the time ACF of almost every trajectory generated by the system is equal to the ensemble ACF of the process $(X_t)_{t \in \mathcal{T}}$; i.e.,

$$\bar{\mathcal{R}}_S(\tau) = \mathcal{R}_S(\tau). \quad (2.74)$$

Proof. The proposition is proved in section 2.A.9. $\square$

Remark 2.4.6. The expression of (2.67) was derived for the time ACF of an ergodic map in section 2.B. This expression then holds for both the ensemble ACF of (2.72) and the time ACF of (2.73); i.e.,

$$\bar{\mathcal{R}}_S(\tau) = \mathcal{R}_S(\tau) = \int_{\Omega} x(\mathcal{P}_S^{|\tau|}g)(x)d\mu(x), \quad \tau = 0, \pm 1, \pm 2, \dots \quad (2.75)$$

The PDSA framework is refined to ensure that the above propositions regarding the statistics of the process $(X_t)_{t \in \mathcal{T}}$ and the system trajectories always apply.

Definition 2.4.15 (Refinement of the PDSA framework). In the remainder of this work, the map S of the PDSA framework (Definition 2.4.4) is specified to be ergodic. Furthermore, the underlying measure is selected as $P = P_S^*$ and the initial random variable is selected as $X_1(\omega) = \omega$, thereby ensuring that $P_1 = P_S^*$. As a consequence, the conditions of Propositions 2.4.6 to 2.4.9 are satisfied, thereby implying

- the expected value $\mathcal{M}_S(t) \equiv \mathcal{M}_S$ of $(X_t)_{t \in \mathcal{T}}$ is independent of time and equal to the mean value $\bar{\mathcal{M}}_S(x_1) = \bar{\mathcal{M}}_S$ of almost all system trajectories, and
- the ensemble ACF $\mathcal{R}_S(t_1, t_2) \equiv \mathcal{R}_S(\tau)$ of $(X_t)_{t \in \mathcal{T}}$ is dependent only on the time difference $\tau \triangleq t_2 - t_1$ and is equal to the time ACF $\bar{\mathcal{R}}_S(\tau, x_1) \equiv \bar{\mathcal{R}}_S(\tau)$ of almost all system trajectories.

2.4.8 Power spectral density

Stated informally, the PSD quantifies how the power of a process or trajectory is distributed over its representation in the frequency domain. In order to define the PSD, the concept of a *wide-sense stationary* stochastic process is defined first.

Definition 2.4.16 (Wide-sense stationary stochastic process [30]). A stochastic process $(X_t)_{t \in \mathcal{T}}$ is wide-sense stationary if its expected value $\mathcal{M}_S(t) \equiv \mathcal{M}_S$ is finite and independent of time, and its ensemble ACF $\mathcal{R}_S(t_1, t_2) \equiv \mathcal{R}_S(\tau)$ is finite and only dependent on the time difference $\tau = t_2 - t_1$.

Proposition 2.4.10 (Process $(X_t)_{t \in \mathcal{T}}$ of the PDSA framework is wide-sense stationary). Invoke the PDSA framework introduced in Definition 2.4.4 and refined in Definition 2.4.15. The process $(X_t)_{t \in \mathcal{T}}$ is wide-sense stationary.

Proof. Refer to Definition 2.4.15 for statements regarding the expected value of the process being independent of time and its ensemble ACF only being dependent on the time difference. The expected value and ensemble ACF satisfy

$$\mathcal{M}_S = \int_{\Omega} x dP_S^*(x) < \int_{\Omega} dP_S^*(x) = P_S^*(\Omega) = 1 \quad (2.76)$$

and

$$\mathcal{R}_S(\tau) = \int_{\Omega} x S^{|\tau|}(x) dP_S^*(x) < \int_{\Omega} dP_S^*(x) = P_S^*(\Omega) = 1. \quad (2.77)$$

Hence, the process $(X_t)_{t \in \mathcal{T}}$ is wide-sense stationary. $\square$

The *power spectrum distribution function* (PSDF) of a wide-sense stationary process arises from the Wiener-Khinchin theorem (see, for instance, [31]). Several definitions of the PSDF appear in the literature. In [30], the PSDF is defined for real valued discrete-time processes, but only over angular frequencies $\omega \in [0, \pi]$. The Wiener-Khinchin theorem presented below was adapted from this reference such that the PSDF is defined over frequencies $\omega \in [-\pi, \pi)$. Also, the theorem presented in [30] relates the PSDF to the *autocovariance function*; here it is related to the ACF.

Theorem 2.4.4 (Wiener-Khinchin theorem, adapted from [30]). *Let $(X_t)_{t \in \mathcal{T}}$ denote a wide-sense stationary process with the ensemble ACF $\mathcal{R}_S(\tau)$. There exists a monotonically increasing function $\mathcal{P}_S : [-\pi, \pi) \rightarrow [0, \mathcal{R}_S(0)]$, referred to as the PSDF of the process, that satisfies*

$$\mathcal{R}_S(\tau) = \int_{-\pi}^{\pi} e^{i\omega\tau} d\mathcal{P}_S(\omega). \quad (2.78)$$

The integral in this equation is a Riemann-Stieltjes integral.

The PSDF $\mathcal{P}_S(\omega)$ characterises the contribution, to the mean-square value of the process, that is accounted for by angular frequencies in the range $[-\pi, \omega)$. The PSDF satisfies $\mathcal{P}_S(-\pi) = 0$ and

$$\lim_{\omega \rightarrow \pi} \mathcal{P}_S(\omega) = \mathcal{R}_S(0). \quad (2.79)$$

Due to its monotonicity, the PSDF $\mathcal{P}_S$ may be decomposed as [30, 31]

$$\mathcal{P}_S(\omega) = \mathcal{P}_{S,C}(\omega) + \mathcal{P}_{S,I}(\omega), \quad (2.80)$$

where $\mathcal{P}_{S,C}$ is a non-decreasing continuous function and $\mathcal{P}_{S,I}$ is a non-decreasing function of discrete steps. The subscripts ‘C’ and ‘I’ in these terms refer to the *continuous* and *impulsive* components of the PSD, to be defined below.

Let $f \in \mathcal{R}(A, \alpha)$ denote the space of functions that are Riemann-Stieltjes integrable over the interval $A = [a, b]$ with respect to the integrator α . The Riemann-Stieltjes integral has the following property [32, Th. 6.12(e)]:

(RS1): If $f \in \mathcal{R}(A, \alpha_1)$ and $f \in \mathcal{R}(A, \alpha_2)$, then $f \in \mathcal{R}(A, \alpha_1 + \alpha_2)$ and

$$\int_a^b f d(\alpha_1 + \alpha_2) = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2. \quad (2.81)$$

Substituting (2.80) into (2.78) and applying property (RS1) yields

$$\mathcal{R}_S(\tau) = \underbrace{\int_{-\pi}^{\pi} e^{i\omega\tau} d\mathcal{P}_{S,C}(\omega)}_{\mathcal{R}_{S,C}(\tau)} + \underbrace{\int_{-\pi}^{\pi} e^{i\omega\tau} d\mathcal{P}_{S,I}(\omega)}_{\mathcal{R}_{S,I}(\tau)}. \quad (2.82)$$

Let the PSD $\mathcal{S}_S$ associated with the process $(X_t)_{t \in \mathcal{T}}$ be defined as

$$\mathcal{S}_S(\omega) = \mathcal{S}_{S,C}(\omega) + \mathcal{S}_{S,I}(\omega), \quad \omega \in [-\pi, \pi), \quad (2.83)$$

where $\mathcal{S}_{S,C}$ and $\mathcal{S}_{S,I}$ denote the continuous and impulsive components of the PSD. As $\mathcal{P}_{S,C}$ is monotone and bounded on the interval $(-\pi, \pi)$ [30], it is differentiable almost everywhere over this interval; this is a consequence of Lebesgue’s theorem on the differentiability of monotone functions [33, Sec. 6.2]. The continuous component $\mathcal{S}_{S,C}$ of the PSD is then defined as

$$\mathcal{S}_{S,C}(\omega) = 2\pi \frac{d\mathcal{P}_{S,C}(\omega)}{d\omega} \text{ a.e.} \quad (2.84)$$

Substitution of this expression into (2.82) reveals that the continuous component $\mathcal{S}_{S,C}$ of the PSD is the discrete-time Fourier transform of $\mathcal{R}_{S,C}$. That is,

$$\mathcal{S}_{S,C}(\omega) = \sum_{\tau=-\infty}^{\infty} \mathcal{R}_{S,C}(\tau) e^{-i\omega\tau} \quad (2.85)$$

and

$$\mathcal{R}_{S,C}(\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{S}_{S,C}(\omega) e^{i\omega\tau} d\omega. \quad (2.86)$$

The impulsive component $\mathcal{R}_{S,I}$ of the ACF satisfies [31, Sec. 6.1.1, Eq. 4]

$$\begin{aligned} \mathcal{R}_{S,I}(\tau) &= \int_{-\pi}^{\pi} e^{i\omega\tau} d\mathcal{P}_{S,I}(\omega) \\ &= \sum_{k=1}^K r_k e^{i\omega_k \tau}. \end{aligned} \quad (2.87)$$

In this expression, $\{\omega_k\}_{k=1}^K$ denotes the angular frequencies at which discrete steps occur in $\mathcal{P}_{S,I}$, and r_k denotes the size of the corresponding step; i.e.,

$$r_k \triangleq \lim_{\omega \rightarrow \omega_k^+} \mathcal{P}_{S,I}(\omega) - \lim_{\omega \rightarrow \omega_k^-} \mathcal{P}_{S,I}(\omega). \quad (2.88)$$

The impulsive component $\mathcal{S}_{S,I}$ of the PSD is defined using Dirac delta functions $\delta(\cdot)$ to coincide with $\mathcal{R}_{S,I}$ in (2.87) as a discrete-time Fourier transform pair. That is,

$$\mathcal{S}_{S,I}(\omega) = 2\pi \sum_{k=1}^K r_k \delta(\omega - \omega_k). \quad (2.89)$$

Proposition 2.4.11 (PSD of the process in the PDSA framework). Invoke the PDSA framework introduced in Definition 2.4.4 and refined in Definition 2.4.15. The PSD $\mathcal{S}_S$ of the process $(X_t)_{t \in \mathcal{T}}$ is defined by (2.83), where its continuous and impulsive components are defined by (2.85) and (2.89), respectively.

Proof. It was proved in Proposition 2.4.10 that the process $(X_t)_{t \in \mathcal{T}}$ is wide-sense stationary. Hence, the Wiener-Khinchin theorem applies (Theorem 2.4.4). $\square$

Remark 2.4.7. Invoke the PDSA framework introduced in Definition 2.4.4 and refined in Definition 2.4.15. As a consequence of the ensemble ACF of the process $(X_t)_{t \in \mathcal{T}}$ being equal to the time ACF of almost all trajectories generated by an ergodic map, in the remainder of this work, reference is made to the ACF *of the map*, without distinguishing between the ensemble and time ACF. The notation $\mathcal{R}_S$ is used to denote this ACF. Furthermore, in the remainder of this work, reference is made to the PSD *of the map*, even though the PSD is associated with the process $(X_t)_{t \in \mathcal{T}}$. The notation $\mathcal{S}_S$ is used to denote this PSD.

Remark 2.4.8. The ACF and PSD of the ergodic map, as defined in Remark 2.4.7, coincide with the ACF and PSD of the zero-input response of the system configured to use the map as evolution rule (i.e., the system of Fig. 2.1). This reveals a further consideration for the design of a system as a random signal generator. To generate a random process with a prescribed ACF or PSD, it is sufficient to configure the system to use an ergodic map that possesses the prescribed ACF or PSD. This design problem is addressed by the second formulation of the IFPP, labelled IFPP-II [8].

2.5 Piecewise Monotonic Maps

The class of piecewise monotonic maps is defined in this section. An explicit representation for the FPO corresponding to these maps is presented. An expression that

characterises the evolution of *distribution functions* under iteration of maps from this class is also derived; this expression is used to develop solutions for the IFPP in the later chapters. The section is concluded by stating a theorem on the existence of absolutely continuous measures that are preserved by piecewise monotonic maps. The material presented in this section appears in [8, 10, 12, 16].

2.5.1 Definition and properties

Let $\Gamma(\mathcal{Q})$ denote the class of piecewise monotonic maps defined over a partition

$$\mathcal{Q} = \{Q_1, Q_2, \dots, Q_M\} \quad (2.90)$$

of the unit interval Ω , where $M > 1$. Here, $Q_m = (q_m, q_{m+1})$ denotes the partition intervals and $\bar{P}(\mathcal{Q}) = \{q_1, q_2, \dots, q_{M+1}\}$ denotes the partition points, which satisfy

$$0 = q_1 < q_2 < \dots < q_{M+1} = 1. \quad (2.91)$$

The class $\Gamma(\mathcal{Q})$ is defined below (an example from this class is plotted in Fig. 2.6).

Definition 2.5.1 (Class $\Gamma(\mathcal{Q})$ [6]). A surjective map $S : \Omega \rightarrow \Omega$ is piecewise monotonic (i.e., $S \in \Gamma(\mathcal{Q})$) if there exists a partition $\mathcal{Q}$ of the unit interval such that (i) each branch function $S_m \triangleq S|_{(q_m, q_{m+1})}$ of the map is a C^r function, for some $r \geq 1$, that can be extended to a C^r function over $[q_m, q_{m+1}]$, and (ii) $|S'_m| > 0$ for $m = 1, 2, \dots, M$, where $S'_m \triangleq dS_m/dx$.

Piecewise monotonic maps possess the following properties.

Proposition 2.5.1. The map $S \in \Gamma(\mathcal{Q})$ is $\mathcal{B}$ -measurable.

Proof. The pre-image $S^{-1}(B)$ of any Borel set $B \in \mathcal{B}$ can be expressed as a finite union of Borel sets, each corresponding to a pre-image $S_m^{-1}(B)$ or a partition point. As the number of partition points are finite, it then follows that $S^{-1}(B) \in \mathcal{B}$. Hence, S is $\mathcal{B}$ -measurable. $\square$

Proposition 2.5.2. The map $S \in \Gamma(\mathcal{Q})$ is non-singular with respect to the Borel measure μ .

Proof. A map S is non-singular with respect to μ if $|S'| > 0$ almost everywhere, where $S' \triangleq dS/dx$ [34]. Definition 2.5.1 requires that $|S'_m| > 0$ for each $m = 1, 2, \dots, M$. This implies that $|S'| > 0$ everywhere except possibly at the partition points $\bar{P}(\mathcal{Q})$. As the number of partition points is finite, $|S'| > 0$ almost everywhere. Hence, S is non-singular. $\square$

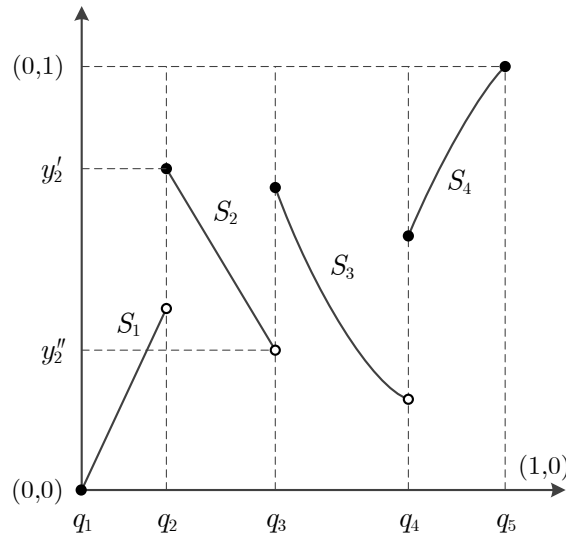


Figure 2.6: An example of a piecewise monotonic map S . Reproduced with permission from [8].

2.5.2 Frobenius-Perron operator

Let $(\Omega, \mathcal{B}, \mu)$ denote a measure space equipped with the Borel measure μ . Furthermore, let $\mathcal{P}_S$ denote the FPO defined over this space and which corresponds to a map $S \in \Gamma(\mathcal{Q})$. An explicit representation for $\mathcal{P}_S$ is given by [6, Equation 4.3.4]

$$(\mathcal{P}_S f_t)(x) = \sum_{m \in N(x)} \frac{f_t \circ S_m^{-1}(x)}{|S'_m \circ S_m^{-1}(x)|}, \quad (2.92)$$

where

$$N(x) \triangleq \{m : x \in S(Q_m)\}. \quad (2.93)$$

The interested reader is referred to [6, Section 4.3] for the derivation of this expression.

2.5.3 Evolution of distribution functions

Invoke the PDSA framework of Definitions 2.4.4 and 2.4.15, and select $\nu = \mu$. Recall that P_t denotes the probability measure induced by the random variable X_t , and that P_t is absolutely continuous with respect to μ . Equation (2.18) states that

$$P_{t+1}(B) = P_t(S^{-1}(B)), \quad B \in \mathcal{B}. \quad (2.94)$$

The *distribution function* $F_t : \Omega \rightarrow [0, 1]$ associated with X_t is defined as [23]

$$F_t(x) \triangleq P_t([0, x]). \quad (2.95)$$

As P_t is absolutely continuous with respect to μ , the distribution function satisfies, for any non-empty interval A ,

$$P_t(A) = F_t(\sup(A)) - F_t(\inf(A)). \quad (2.96)$$

Now, let $S \in \Gamma(\mathcal{Q})$, and define $y'_m \triangleq \sup(S_m)$ and $y''_m \triangleq \inf(S_m)$ (refer to Fig. 2.6). Consider a particular $x \in \Omega$. Three cases are considered:

Case 1: If $x \leq y''_m$, it follows that $[0, x] \cap S(Q_m) = \emptyset$, where $Q_m = (q_m, q_{m+1})$. Hence, $[0, x]$ has no pre-image in Q_m under S_m .

Case 2: If $y''_m < x < y'_m$, then $x \in S(Q_m)$. Also, $[0, x] \cap S(Q_m) = (y''_m, x]$ and $[0, x]$ has a pre-image in Q_m under S_m . This pre-image is given by

$$S_m^{-1}([0, x]) = \begin{cases} (q_m, S_m^{-1}(x)] & \text{if } S'_m > 0, \\ [S_m^{-1}(x), q_{m+1}), & \text{if } S'_m < 0. \end{cases} \quad (2.97)$$

Equation (2.96) then implies

$$P_t(S_m^{-1}([0, x])) = \begin{cases} F_t(S_m^{-1}(x)) - F_t(q_m), & \text{if } S'_m > 0, \\ F_t(q_{m+1}) - F_t(S_m^{-1}(x)), & \text{if } S'_m < 0. \end{cases} \quad (2.98)$$

The latter equation may be simplified as

$$P_t(S_m^{-1}([0, x])) = |F_t \circ S_m^{-1}(x) - F_t(q_{m+d_m})|, \quad (2.99)$$

where

$$d_m = \begin{cases} 0, & \text{if } S'_m > 0, \\ 1, & \text{if } S'_m < 0. \end{cases} \quad (2.100)$$

Case 3: If $x \geq y'_m$, then $S_m^{-1}([0, x]) = Q_m$. Equation (2.96) then implies

$$P_t(S_m^{-1}([0, x])) = P_t((q_m, q_{m+1})) = F_t(q_{m+1}) - F_t(q_m). \quad (2.101)$$

Now, define $N(x)$ as in (2.93), and let

$$M(x) \triangleq \{m : x \geq y'_m\}. \quad (2.102)$$

Consider a particular point $x \in \Omega$. The interval $[0, x]$ has a pre-image under each branch S_m that satisfies the condition of either case 2 and case 3. Equation (2.94) then implies that

$$P_{t+1}([0, x]) = \sum_{m \in N(x)} \underbrace{P_t(S_m^{-1}([0, x]))}_{\text{Case 2}} + \sum_{m \in M(x)} \underbrace{P_t(S_m^{-1}([0, x]))}_{\text{Case 3}} + P_t(C), \quad (2.103)$$

where $C \triangleq \{q \in \bar{P}(\mathcal{Q}) : S(q) \leq x\}$ denotes the partition points that fall within the pre-image of $[0, x]$ under S . Since there is only a finite number of partition points, $\mu(C) = 0$. Hence, the absolute continuity of P_t with respect to μ implies $P_t(C) = 0$. Substitution of (2.95), (2.99) and (2.101) into (2.103) then yields [16]

$$F_{t+1}(x) = \sum_{m \in N(x)} |F_t \circ S_m^{-1}(x) - F_t(q_{m+d_m})| + \sum_{m \in M(x)} F_t(q_{m+1}) - F_t(q_m). \quad (2.104)$$

2.5.4 Existence of absolutely continuous invariant measures

Sufficient conditions for the existence of measures that are preserved by a piecewise monotonic map and absolutely continuous with respect to the Borel measure were derived in [6, 35]. These measures are referred to as *absolutely continuous invariant* (ACI) measures.

Theorem 2.5.1 (Existence of ACI measures [6, 35]). *Let $S \in \Gamma(\mathcal{Q})$. If $|S'_m| \geq \alpha > 1$ over all partition intervals Q_m , and the function $h(\cdot) \triangleq 1/|S'(\cdot)|$ is of bounded variation, where S' is the appropriate one-sided derivative at the partition points $\bar{P}(\mathcal{Q})$, then S admits an ACI measure with a corresponding invariant density of bounded variation.*

Proof. The interested reader is referred to [6, Theorem 5.2.1]. □

2.6 Markov and Semi-Markov Maps

Two subclasses of the piecewise monotonic maps are introduced in this section, namely the *Markov* and *semi-Markov* maps. Investigation of what became known as the Markov maps started with the work of Rényi in the 1950s [6]. Several decades later, work on characterising the Markov maps intensified, with a series of articles by Góra, Boyarsky and their colleagues appearing in the literature [36–42]; the interested reader is referred to [6] for a summary of the key theorems and results. The semi-Markov maps, which generalises the class of Markov maps, was formally introduced in [43]. Several important characteristics of these maps that are relevant to the solution of IFPP-I were presented in the same article.

The structure of Markov and semi-Markov maps admit a finite-dimensional representation of the FPO that is exact (i.e., it characterises the evolution of piecewise

polynomial density functions in an exact fashion). These maps permit matrix solutions to the IFPP, as first demonstrated by Friedman and Boyarsky [38] and Góra and Boyarsky [43]. Due to the comparatively unsophisticated and explicit relationship between the statistical metrics of an ergodic semi-Markov map and a finite collection of its branch parameters, these maps are selected as candidates for construction by the methods developed in the later chapters.

The Markov and semi-Markov map classes are defined formally in this section. The Folklore theorem, which states conditions for the ergodicity of maps from these classes, is then stated without proof. It is shown that the FPO corresponding to a semi-Markov map may be represented as a matrix if attention is restricted to density functions that are defined to be piecewise polynomial over the intervals of the relevant partition. This matrix is referred to as the *Frobenius-Perron matrix* (FPM), and its elements are functions of the map's branch parameters.

Expressions are derived for the order-zero and order-one FPMs, which characterise the evolution of piecewise constant (PWC) and piecewise linear (PWL) density functions under iteration of a semi-Markov map, respectively. Relevant properties of these matrices are derived. The section concludes by deriving simplified expressions that relate the statistical metrics of an ergodic semi-Markov map to these matrices.

2.6.1 Definition

Informally stated, a map $S \in \Gamma(\mathcal{Q})$ is $\mathcal{Q}$ -Markov if the image of each partition interval $Q_m \in \mathcal{Q}$ under S is expressible as a union of intervals from the same partition $\mathcal{Q}$ [8]. A formal definition, which takes into account the fact that the partition intervals Q_m are open, is provided in what follows. The notation $Q_m^c = \text{cl}\{Q_m\}$ is used to denote the closure of the interval Q_m .

Definition 2.6.1 ($\mathcal{Q}$ -Markov map [8, 43]). A map $S \in \Gamma(\mathcal{Q})$ is $\mathcal{Q}$ -Markov if

$$\text{cl}\{S(Q_m)\} = \bigcup_{k \in W(m)} Q_k^c, \quad \forall Q_m \in \mathcal{Q}, \quad (2.105)$$

where $W(m) \subseteq \{1, 2, \dots, M\}$ for $m = 1, 2, \dots, M$.

Semi-Markov maps are defined over a two-level hierarchy of partitions [8, 10, 12]. Let $\mathcal{Q}$ denote a partition of the unit interval as before, and let the non-empty set of open intervals $\mathcal{R}^{(m)} \triangleq \{R_1^{(m)}, R_2^{(m)}, \dots, R_{c(m)}^{(m)}\}$ constitute a partition of the closure of Q_m for each $m = 1, 2, \dots, M$. The unit-interval partition $\mathcal{R} = \{R_j^{(m)}\}$, where

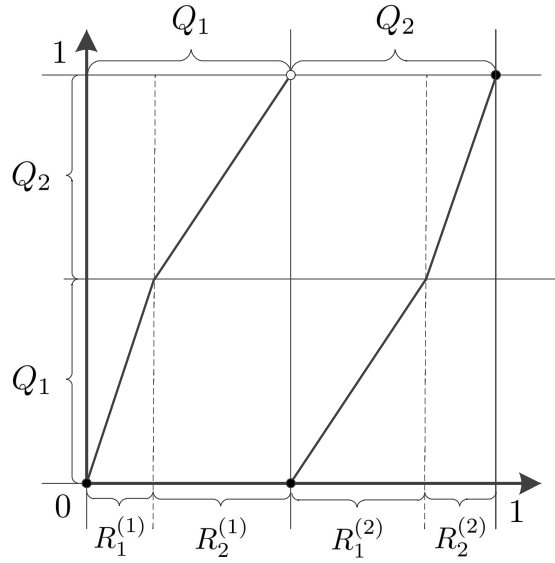


Figure 2.7: An example of an $\mathcal{R}(\mathcal{Q})$ -semi-Markov map. Reproduced with permission from [12] (© [2023] IEEE).

$m = 1, 2, \dots, M$ and $j = 1, 2, \dots, c(m)$, is referred to as a *subpartition* of $\mathcal{Q}$. Each interval of the subpartition is defined to satisfy

$$\mu(R) > 0, \quad R \in \mathcal{R}. \quad (2.106)$$

The class of $\mathcal{R}(\mathcal{Q})$ -semi-Markov maps are defined below. An example of a map from this class is plotted in Fig. 2.7.

Definition 2.6.2 ($\mathcal{R}(\mathcal{Q})$ -semi-Markov map [8, 10, 12, 43]). Let $\mathcal{Q}$ denote a partition of Ω , and let $\mathcal{R}$ denote a subpartition of $\mathcal{Q}$. The map $S \in \Gamma(\mathcal{R})$ is $\mathcal{R}(\mathcal{Q})$ -semi-Markov if the image of each interval $R \in \mathcal{R}$ under S is an interval of the partition $\mathcal{Q}$; i.e.,

$$S(R) \in \mathcal{Q}, \quad R \in \mathcal{R}. \quad (2.107)$$

The partition $\mathcal{Q}$ is referred to as the *Markov partition* of the map.

Remark 2.6.1. Any $\mathcal{Q}$ -Markov map is an $\mathcal{R}(\mathcal{Q})$ -semi-Markov map over an appropriately defined subpartition $\mathcal{R}$ [43]. However, an $\mathcal{R}(\mathcal{Q})$ -semi-Markov map S is not necessarily $\mathcal{Q}$ -Markov: as the closure of each interval $\mathcal{Q}_m \in \mathcal{Q}$ has a partition $\mathcal{R}^{(m)}$ that comprises possibly multiple intervals from the partition $\mathcal{R}$, the map $S \in \Gamma(\mathcal{R})$ is not necessarily piecewise monotonic over $\mathcal{Q}$. It follows that the class of $\mathcal{R}(\mathcal{Q})$ -semi-Markov maps contains the class of $\mathcal{Q}$ -Markov maps as a subset, and is thus more general. *For this reason, the remainder of this work focusses primarily on the class of semi-Markov maps, and not the class of Markov maps.*

The following propositions are proved.

Proposition 2.6.1. Let S denote any $\mathcal{R}(\mathcal{Q})$ -semi-Markov map. Then S is $\mathcal{R}$ -Markov.

Proof. Consider any interval $R \in \mathcal{R}$. As S is $\mathcal{R}(\mathcal{Q})$ -semi-Markov, this interval has an image $Q_m \in \mathcal{Q}$ under S for some $m = 1, 2, \dots, M$. But the closure of Q_m has the partition $\mathcal{R}^{(m)}$. This implies that

$$\text{cl}\{S(R)\} = \bigcup_{k=1}^{c(m)} (R_k^{(m)})^c. \quad (2.108)$$

Hence, S is $\mathcal{R}$ -Markov by Definition 2.6.1. $\square$

Remark 2.6.2. As any $\mathcal{R}(\mathcal{Q})$ -semi-Markov map is $\mathcal{R}$ -Markov (*but not always $\mathcal{Q}$ -Markov*), theory developed for Markov maps apply to any $\mathcal{R}(\mathcal{Q})$ -semi-Markov map on the subpartition $\mathcal{R}$.

Proposition 2.6.2 (A composition of semi-Markov maps is semi-Markov). Let S denote an $\mathcal{R}(\mathcal{Q})$ -semi-Markov map. The t -fold composition of S , denoted by S^t , is a $\mathcal{W}(\mathcal{Q})$ -semi-Markov map, where $\mathcal{W}$ is a subpartition of the partition $\mathcal{Q}$.

Proof. This proposition is proved in section 2.A.10. $\square$

2.6.2 Invariant densities and ergodicity

The following theorem states that any invariant density of an $\mathcal{R}$ -Markov map that is expanding and has linear branch functions is PWC on the intervals of partition $\mathcal{R}$.

Theorem 2.6.1 (Invariant density of Markov maps [39]). *Let $S \in \Gamma(\mathcal{R})$ denote an $\mathcal{R}$ -Markov map that is PWL (i.e., each of its branch functions is linear) and expanding (i.e., $S|_R$ has a slope greater than unity for each $R \in \mathcal{R}$). Any invariant density of S is then PWC on the intervals of $\mathcal{R}$.*

Proof. The interested reader is referred to [39]. $\square$

The counterpart of this theorem for $\mathcal{R}(\mathcal{Q})$ -semi-Markov maps, which states that the invariant density is PWC on the intervals of partition $\mathcal{Q}$, is provided below.

Theorem 2.6.2 (Invariant density of semi-Markov maps [43]). *Let S denote an $\mathcal{R}(\mathcal{Q})$ -semi-Markov map that is PWL and expanding; i.e., $S|_R$ is linear with slope greater than unity for each $R \in \mathcal{R}$. Any invariant density of S is then PWC on partition $\mathcal{Q}$.*

Proof. The interested reader is referred to [43, Th. 2]. $\square$

The Folklore theorem establishes conditions for both the existence and uniqueness of ACI measures for $\mathcal{Q}$ -Markov maps that are piecewise C^2 . This theorem originated from the work of Rényi [3], who considered piecewise onto transformations.

The Folklore theorem refers to the *aperiodicity* property of a map.

Definition 2.6.3 (Aperiodicity of a map [6]). The map $S \in \Gamma(\mathcal{Q})$ is aperiodic if, for each $m = 1, 2, \dots, M$, there exists some $j > 0$ such that $S^j(Q_m^c) = \Omega$.

The Folklore theorem is stated below.

Theorem 2.6.3 (Folklore theorem [6]). *Let $S \in \Gamma(\mathcal{Q})$ denote an aperiodic $\mathcal{Q}$ -Markov map, where each branch function S_m can be extended to a strictly monotonic C^2 function on the closed interval Q_m^c . If S satisfies $|S'_m| \geq 1 + \varepsilon$ for all m and some $\varepsilon > 0$ (i.e., S is piecewise expanding), then it has a unique ACI measure P_S^* . Furthermore, there exists a unique invariant density f_S^* corresponding to P_S^* that can be chosen as a piecewise continuous function with discontinuities only at the partition points $\bar{P}(\mathcal{Q})$, and satisfying $K^{-1} \leq f_S^*(x) \leq K$ for some $K > 0$.*

Proof. The interested reader is referred to [6, Theorem 6.1.1] $\square$

Any $\mathcal{Q}$ -Markov map $S \in \Gamma(\mathcal{Q})$ that satisfies the conditions of the Folklore theorem is necessarily ergodic, as its invariant measure P_S^* is unique (this is a consequence of Theorem 2.4.2).

Remark 2.6.3 (Ergodicity of semi-Markov maps). Proposition 2.6.1 states that any $\mathcal{R}(\mathcal{Q})$ -semi-Markov map $S \in \Gamma(\mathcal{R})$ is $\mathcal{R}$ -Markov. Hence, the Folklore theorem may be applied to confirm the ergodicity of a semi-Markov map.

2.6.3 Matrix representation of the Frobenius-Perron operator

In this section, a matrix representation is derived for the FPO belonging to a PWL semi-Markov map (i.e., a semi-Markov map with linear branch functions). This matrix representation of the FPO is referred to as the map's FPM. The particular FPM derived below characterises the evolution of density functions, under iteration of the map, that are defined to be *piecewise polynomial functions* over the intervals of the map's Markov partition.

Definition 2.6.4 (Order- ℓ piecewise polynomial density function [8, 10, 12]). Let $\mathcal{D}_Q^{(\ell)} \subset \mathcal{D}(\Omega, \mathcal{B}, \mu)$ denote the set of density functions that are defined piecewise over $\mathcal{Q}$ such that $f_t \in \mathcal{D}_Q^{(\ell)}$ implies $f_t|_{Q_m}$ is a polynomial function with degree not exceeding $\ell \in \mathbb{N}_0$ for each $m = 1, 2, \dots, M$. Any density $f \in \mathcal{D}_Q^{(\ell)}$ is referred to as an order- ℓ piecewise polynomial density function.

In the following subsections, expressions are derived for the *order-zero* and *order-one* FPMs of a semi-Markov map, which characterise the evolution of density functions that are *PWC* and *PWL* on the map's Markov partition, respectively. Relevant properties of these matrices are subsequently derived.

Derivation of expressions for the FPM

Consider the PWL $\mathcal{R}(\mathcal{Q})$ -semi-Markov map S defined as

$$S(x) = \sum_{m=1}^M \sum_{j=1}^{c(m)} S_{j,m}(x) \chi_{R_j^{(m)}}(x), \quad (2.109)$$

where the branch functions are defined as

$$S_{j,m}(x) \triangleq S|_{R_j^{(m)}}(x) \triangleq v_{j,m}x + w_{j,m}, \quad m = 1, 2, \dots, M, \quad j = 1, 2, \dots, c(m). \quad (2.110)$$

Let $f_t \in \mathcal{D}_Q^{(\ell)}$ be defined as

$$f_t(x) \triangleq \sum_{m=1}^M \sum_{k=0}^{\ell} [f_{t,k,m} x^k] \chi_{Q_m}(x). \quad (2.111)$$

As $S \in \Gamma(\mathcal{R})$, the explicit form of the FPO in (2.92) applies. Using the above notation, (2.92) may be rewritten as

$$(\mathcal{P}_S f_t)(x) = \sum_{(j,m) \in N(x)} \frac{f_t \circ S_{j,m}^{-1}(x)}{|S'_{j,m} \circ S_{j,m}^{-1}(x)|}, \quad (2.112)$$

where $N(x) \triangleq \{(j, m) : x \in S(R_j^{(m)})\}$.

Substitution of f_t from (2.111) into (2.112) reveals that

$$f_{t+1} = \mathcal{P}_S(f_t) \quad (2.113)$$

is also a member of $\mathcal{D}_Q^{(\ell)}$ [10, 12]. This implies that the evolution of a piecewise polynomial density function $f_t \in \mathcal{D}_Q^{(\ell)}$ under iteration of an $\mathcal{R}(\mathcal{Q})$ -semi-Markov map S can be expressed as the matrix equation

$$\mathbf{f}_{t+1} = \mathbf{P}_S^{(\ell)} \mathbf{f}_t, \quad (2.114)$$

where $\mathbf{f}_t$ denotes the order- ℓ vector representation of the density function f_t and $\mathbf{P}_S^{(\ell)}$ denotes the order- ℓ matrix representation of the FPO $\mathcal{P}_S$. The matrix $\mathbf{P}_S^{(\ell)}$ is referred to as the order- ℓ *Frobenius-Perron matrix* (FPM) of S . The vectors in (2.114) are defined as

$$\mathbf{f}_t \triangleq [\mathbf{f}_{t,\ell}, \mathbf{f}_{t,\ell-1}, \dots, \mathbf{f}_{t,0}]^T, \quad (2.115)$$

where

$$\mathbf{f}_{t,k} \triangleq [f_{t,k,1}, f_{t,k,2}, \dots, f_{t,k,M}], \quad k = 0, 1, \dots, \ell. \quad (2.116)$$

In the case where $\ell = 1$, the order-one FPM $\mathbf{P}_S^{(1)}$ characterises the evolution of PWL density functions $f_t \in \mathcal{D}_Q^{(1)}$. An expression may be derived for the order-one FPM $\mathbf{P}_S^{(1)}$ by substituting (2.110) and (2.111) into (2.112). This expression is given by [10, 12]

$$\mathbf{P}_S^{(1)} = \begin{pmatrix} \mathbf{Q}_S & \mathbf{0}_{M \times M} \\ \mathbf{R}_S & \mathbf{P}_S^{(0)} \end{pmatrix}, \quad (2.117)$$

where $\mathbf{Q}_S \triangleq [q_{k,m}]_{k,m=1}^M$, $\mathbf{R}_S \triangleq [r_{k,m}]_{k,m=1}^M$, and $\mathbf{0}_{M \times M}$ denotes the $M \times M$ matrix containing zeros. The expression reveals that the order-zero FPM $\mathbf{P}_S^{(0)}$ appears as the bottom-right submatrix of the order-one FPM $\mathbf{P}_S^{(1)}$.

The elements of the submatrices $\mathbf{Q}_S$ and $\mathbf{R}_S$ are derived as [10, 12]

$$q_{k,m} = \begin{cases} \sum_{j \in T(k,m)} s_{j,m} \frac{1}{v_{j,m}^2}, & \text{if } |T(k,m)| > 0, \\ 0, & \text{if } |T(k,m)| = 0, \end{cases} \quad (2.118)$$

and

$$r_{k,m} = \begin{cases} - \sum_{j \in T(k,m)} s_{j,m} \frac{w_{j,m}}{v_{j,m}^2}, & \text{if } |T(k,m)| > 0, \\ 0, & \text{if } |T(k,m)| = 0, \end{cases} \quad (2.119)$$

where

$$T(k,m) \triangleq \{j : S(R_j^{(m)}) = Q_k\}. \quad (2.120)$$

In these expressions, $s_{j,m} \triangleq \text{sgn}(v_{j,m})$, where

$$\text{sgn}(x) \triangleq \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases} \quad (2.121)$$

The elements of the order-zero FPM $\mathbf{P}_S^{(0)} \triangleq [p_{k,m}]_{k,m=1}^M$ are derived as [10, 12]

$$p_{k,m} = \begin{cases} \sum_{j \in T(k,m)} |v_{j,m}|^{-1}, & \text{if } |T(k,m)| > 0, \\ 0, & \text{if } |T(k,m)| = 0. \end{cases} \quad (2.122)$$

This expression implies that the elements of the order-zero FPM are dependent only on the *absolute values* of the relevant branch functions' slopes. Stated differently, these matrix elements do not depend on whether each branch is increasing or decreasing.

Remark 2.6.4. Theory regarding the matrix representation of the FPOs belonging to Markov and semi-Markov maps has appeared throughout the literature. Although the theory originates from earlier work, the detailed summary presented in [6] on the order-zero matrix representation of a Markov map's FPO and its properties is cited here. The detailed technical report of [9], which considers the higher-order matrix representation of a Markov map's FPO, is of interest. In this report, a matrix is derived that characterises the evolution of densities under iteration of a $\mathcal{Q}$ -Markov map, where the densities are piecewise polynomial over the intervals of $\mathcal{Q}$.

The matrix representations for the FPO of $\mathcal{Q}$ -Markov maps presented in [6, 9] are not directly applicable to the $\mathcal{R}(\mathcal{Q})$ -semi-Markov maps, as the class of $\mathcal{R}(\mathcal{Q})$ -semi-Markov maps is larger than the class of $\mathcal{Q}$ -Markov maps. The order-zero matrix representation of the FPO belonging to a semi-Markov map, which is equivalent to the order-zero FPM defined in this section, was first presented in [43].

Properties of the order-zero FPM

Column stochastic and *doubly stochastic* matrices are encountered throughout this work. These matrices are defined below.

Definition 2.6.5 (Column stochastic matrix). A square matrix $\mathbf{P} = [p_{k,m}]$ is column stochastic if each of its elements satisfies $0 \leq p_{k,m} \leq 1$ and each of its columns sums to unity.

The abbreviated term *stochastic matrix* is used in this work to refer to a column stochastic matrix.

Definition 2.6.6 (Doubly stochastic matrix). A square matrix is doubly stochastic if it is column stochastic and each of its rows sums to unity.

The order-zero FPM of a PWL $\mathcal{R}(\mathcal{Q})$ -semi-Markov map has several properties that are relevant in the context of the IFPP. In this section, these properties are derived using the definitions and notation introduced in this work. The Markov map counterparts of these properties are dispersed throughout the literature; see, for instance, [6, 9, 38].

Proposition 2.6.3 (The order-zero FPM is column stochastic if $\mathcal{Q}$ is uniform). Let $\mathcal{Q}$ denote the uniform partition of the unit interval Ω ; that is,

$$\mu(Q_m) = \frac{1}{M}, \quad m = 1, 2, \dots, M. \quad (2.123)$$

The corresponding order-zero FPM $\mathbf{P}_S^{(0)}$ is column stochastic.

Proof. This proposition is proved in section 2.A.11, page 79. $\square$

Proposition 2.6.4 (Unity eigenvalue of the order-zero FPM). The order-zero FPM has at least one unity eigenvalue λ_1 . Let the left eigenvector corresponding to this eigenvalue be denoted by $\mathbf{v}_1^T$. The elements of this eigenvector coincide with the lengths of the intervals of partition $\mathcal{Q}$; i.e.,

$$\mathbf{v}_1 = [\mu(Q_1), \mu(Q_2), \dots, \mu(Q_M)]^T. \quad (2.124)$$

Proof. The proposition is proved in section 2.A.11, page 80. $\square$

Observe that the partition $\mathcal{Q}$ need not be uniform for Proposition 2.6.4 to apply.

Proposition 2.6.5 (The order-zero FPM is similar to a stochastic matrix). The order-zero FPM $\mathbf{P}_S^{(0)}$ is similar to a column stochastic matrix. Specifically, there exists a diagonal matrix $\mathbf{D}$ with strictly positive entries along its diagonal such that

$$\mathbf{P}_S^{(0)} = \mathbf{D}\tilde{\mathbf{P}}\mathbf{D}^{-1}, \quad (2.125)$$

where $\tilde{\mathbf{P}}$ is a column stochastic matrix. The matrix $\mathbf{D}$ is given by

$$\mathbf{D} = \text{diag}\{[\mu(Q_1)^{-1}, \mu(Q_2)^{-1}, \dots, \mu(Q_M)^{-1}]\}. \quad (2.126)$$

Proof. The proposition is proved in section 2.A.11, page 80. $\square$

Proposition 2.6.6 (The spectral radius of the order-zero FPM). The spectral radius $\rho \triangleq \max\{|\lambda_1|, |\lambda_2|, \dots, |\lambda_M|\}$ of the order-zero FPM $\mathbf{P}_S^{(0)}$ is equal to unity, where λ_m , $m = 1, 2, \dots, M$, denotes the eigenvalues of $\mathbf{P}_S^{(0)}$.

Proof. The proposition is proved in section 2.A.11 on page 81. $\square$

Proposition 2.6.6 implies that the order-zero FPM can have no eigenvalue larger than unity in absolute value.

Proposition 2.6.7 (Primitivity of the order-zero FPM). The order-zero FPM $\mathbf{P}_S^{(0)}$ is primitive if and only if S is aperiodic.

Proof. The proposition is proved in section 2.A.11, page 82. $\square$

Proposition 2.6.7 implies that the order-zero FPM is primitive if S satisfies the conditions of the Folklore theorem, as this theorem requires S to be aperiodic. In this case, the Perron-Frobenius theorem [44] may be used to characterise the matrix. These characteristics are listed in the following proposition.

Proposition 2.6.8 (Additional characteristics of the order-zero FPM). Let S satisfy the conditions of the Folklore theorem. The Perron-Frobenius theorem implies that the order-zero FPM of S has the following characteristics.

1. The eigenvalue $\lambda_1 = 1$ is a simple root of the characteristic polynomial of the order-zero FPM. Hence, the order-zero FPM has no remaining eigenvalues $\lambda_2, \lambda_3, \dots, \lambda_M$ equal to unity.
2. All remaining eigenvalues $\lambda_2, \lambda_3, \dots, \lambda_M$ of the order-zero FPM satisfy

$$|\lambda_m| < \lambda_1, \quad m = 2, 3, \dots, M. \quad (2.127)$$

The eigenvalue λ_1 is referred to as the *Perron* or *maximal* eigenvalue of the order-zero FPM.

Proposition 2.6.9 (Eigenvector that coincides with the invariant density). Let S satisfy the conditions of the Folklore theorem. Then $\mathbf{P}_S^{(0)}$ possesses a unique (up to a multiplicative constant) right eigenvector $\mathbf{w}_1$ associated with the unity eigenvalue λ_1 , such that $\mathbf{w}_1 = \mathbf{f}_S^*$ is the vector representation of the unique invariant density f_S^* of the map S .

Proof. The proposition is proved in section 2.A.11 on page 84. $\square$

Remark 2.6.5. Proposition 2.6.9 implies an unsophisticated relationship between the order-zero FPM of an ergodic semi-Markov map and its invariant density (and, by extension, the PDF of the zero-input response generated by a system configured to use the map as its evolution rule). In particular, this density coincides with the right Perron eigenvector of the order-zero FPM. As the elements of the order-zero FPM are simple functions of the map's branch slopes, this provides an opportunity to analytically select the invariant density of the map during its design. This approach is followed in the novel methods that are developed in subsequent chapters.

Properties of the order-one FPM

The following propositions regarding the order-one FPM are proved.

Proposition 2.6.10 (Eigenspectrum of the order-one FPM). The eigenspectrum of the order-one FPM $\mathbf{P}_S^{(1)}$ is comprised of the eigenspectrum of the order-zero FPM $\mathbf{P}_S^{(0)}$ and the eigenspectrum of the top-left submatrix $\mathbf{Q}_S$ of the order-one FPM in (2.117).

Proof. The proposition is proved in section 2.A.12 on page 84. $\square$

In the following proposition, a sufficient condition for the eigenvalues of $\mathbf{Q}_S$ to be strictly smaller than unity in absolute value is stated.

Proposition 2.6.11 (Bound on the eigenvalues of $\mathbf{Q}_S$). Let each diagonal element of the order-zero FPM $\mathbf{P}_S^{(0)} = [p_{k,m}]$ satisfy

$$0 < p_{m,m} < 1, \quad m = 1, 2, \dots, M. \quad (2.128)$$

The eigenvalues of the top-left submatrix $\mathbf{Q}_S$ of the order-one FPM in (2.117) then satisfy

$$|\lambda_m| < 1, \quad m = M + 1, M + 2, \dots, 2M. \quad (2.129)$$

Proof. The proposition is proved in section 2.A.12 on page 84. $\square$

Remark 2.6.6. A similar claim to that made by Proposition 2.6.11 was proved in [9, Sec. 3.3] for the higher-order matrix representation of the FPO belonging to Markov maps. Although the result in [9] applies to Markov maps, it is more general in the sense that an order- ℓ matrix representation of the FPO is considered, where $\ell > 1$ is arbitrary. A similar generalisation for the class of semi-Markov maps is not presented here, as it is not relevant to the solution of the IFPP formulations considered.

Remark 2.6.7. Suppose that each diagonal element of the order-zero FPM of a semi-Markov map S satisfies $0 < p_{m,m} < 1$. Propositions 2.6.8, 2.6.10 and 2.6.11 then imply that the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_{2M}$ of the order-one FPM satisfy

$$\lambda_1 = 1 \quad (2.130)$$

and

$$|\lambda_m| < 1, \quad m = 2, 3, \dots, 2M. \quad (2.131)$$

2.6.4 Autocorrelation function

In this section, a simplified expression is derived for the ACF of an ergodic semi-Markov map. This expression relates the ACF to the map's order-one FPM, and thereby the parameters of the map's branch functions.

The general expression for the ACF $\mathcal{R}_S$ of an ergodic map S is given by (2.67),

$$\mathcal{R}_S(\tau) = \int_{\Omega} x(\mathcal{P}_S^{|\tau|} g)(x) d\mu(x), \quad \tau = 0, \pm 1, \pm 2, \dots, \quad (2.132)$$

where $\mathcal{P}_S$ denotes the map's FPO, $g(x) \triangleq x f_S^*(x)$ and f_S^* denotes the map's invariant density. This expression may be simplified to a matrix expression if S is selected as a semi-Markov map (refer also to [12]). This is addressed in the following proposition.

Proposition 2.6.12 (ACF of an ergodic semi-Markov map). Invoke the PDSA framework introduced in Definition 2.4.4 and refined in Definition 2.4.15. Let $S \in \Gamma(\mathcal{R})$ denote a PWL $\mathcal{R}(\mathcal{Q})$ -semi-Markov map that satisfies the conditions of the Folklore theorem (as stated in Theorem 2.6.3), such that the map is ergodic. Let f_S^* denote the map's unique invariant density, which is PWC on the intervals of partition $\mathcal{Q}$, and let $\mathbf{f}_S^*$ denote the vector representation of this density.

Let $\lambda_1, \lambda_2, \dots, \lambda_{2M}$ denote the eigenvalues of the order-one FPM $\mathbf{P}_S^{(1)}$ belonging to S . Furthermore, let $\tilde{\mathbf{v}}_m^T$ and $\tilde{\mathbf{w}}_m$ denote the left and right eigenvectors associated with eigenvalue λ_m of the order-one FPM. The eigenvectors are normalised such that

$$\tilde{\mathbf{v}}_m^H \tilde{\mathbf{w}}_m = 1, \quad m = 1, 2, \dots, 2M, \quad (2.133)$$

where $\tilde{\mathbf{v}}_m^H$ denotes the conjugate transpose of $\tilde{\mathbf{v}}_m$.

If the order-one FPM of S is diagonalisable, the ACF of the map is given by [10, 12]

$$\mathcal{R}_S(\tau) = \sum_{m=1}^{2M} c_m \lambda_m^{|\tau|}, \quad \tau = 0, \pm 1, \pm 2, \dots, \quad (2.134)$$

where

$$c_m = \mathbf{h}^T \tilde{\mathbf{w}}_m \tilde{\mathbf{v}}_m^H \mathbf{g}, \quad m = 1, 2, \dots, 2M \quad (2.135)$$

and $\lambda_m^0 \triangleq 1$ if $\lambda_m = 0$. In these expressions,

$$\mathbf{g} \triangleq [(\mathbf{f}_S^*)^T, \mathbf{0}_{1 \times M}]^T \quad (2.136)$$

and

$$\mathbf{h} \triangleq [\mathbf{h}_3, \mathbf{h}_2]^T, \quad (2.137)$$

where the elements of $\mathbf{h}_n \triangleq [h_{n,1}, h_{n,2}, \dots, h_{n,M}]$ are defined by

$$h_{n,m} \triangleq \frac{q_{m+1}^n - q_m^n}{n}, \quad m = 1, 2, \dots, M, \quad n = 2, 3. \quad (2.138)$$

Here, $\bar{\mathcal{P}}(\mathcal{Q}) = \{q_1, q_2, \dots, q_{M+1}\}$ are the partition points of the partition $\mathcal{Q}$.

Proof. The proposition is proved in section 2.A.13. □

Remark 2.6.8. Equation (2.134) implies that the ACF of an ergodic semi-Markov map is a weighted combination of oscillating and decaying⁶ *temporal modes* that coincide with the eigenvalues λ_m of the map's order-one FPM. The angular frequency of oscillation of each mode coincides with the argument of the eigenvalue, whereas the rate of decay is inversely proportional to the eigenvalue modulus. The weight c_m of the mode depends on the corresponding eigenvectors of the order-one FPM. Equation (2.134) also reveals that the map permits a maximum of $2M$ distinct modes.

As the ACF and PSD are related as a discrete-time Fourier transform pair, it follows that the PSD of the map comprises *power spectrum modes* that also coincide with the eigenvalues of the order-one FPM. The centre frequency of each power spectrum mode coincides with the argument of the eigenvalue, and the bandwidth of the mode is inversely proportional to the modulus of the eigenvalue. The centre frequency and bandwidth of each mode are referred to as the *mode's characteristics*.

Proposition 2.6.10 states that the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_{2M}$ of the order-one FPM $\mathbf{P}_S^{(1)}$ are comprised of the eigenvalues of the *order-zero FPM* $\mathbf{P}_S^{(0)}$ (denoted by $\lambda_1, \lambda_2, \dots, \lambda_M$) and the eigenvalues of the *top-left submatrix* $\mathbf{Q}_S$ of $\mathbf{P}_S^{(1)}$ (denoted by $\lambda_{M+1}, \lambda_{M+2}, \dots, \lambda_{2M}$). Hence, the power spectrum mode characteristics of the map coincide with the eigenvalues of these submatrices.

The ACF $\mathcal{R}_S$ comprises a continuous component $\mathcal{R}_{S,C}$ and an impulsive component $\mathcal{R}_{S,I}$, as shown in (2.82):

$$\mathcal{R}_S(\tau) = \mathcal{R}_{S,C}(\tau) + \mathcal{R}_{S,I}(\tau). \quad (2.139)$$

If a further assumption is made regarding the diagonal elements of the order-zero FPM belonging to the map S , simplified expressions may be derived for these components of the map's ACF.

⁶Strictly speaking, the mode that coincides with the Perron eigenvalue $\lambda_1 = 1$ neither oscillates nor decays.

Proposition 2.6.13 (Components of the ACF of a semi-Markov map). Suppose that the conditions of Proposition 2.6.12 are met, and that the diagonal elements of the order-zero FPM belonging to the map S satisfy

$$0 < p_{m,m} < 1, \quad m = 1, 2, \dots, M. \quad (2.140)$$

The continuous and impulsive components of the map's ACF are then given by

$$\mathcal{R}_{S,C}(\tau) = \sum_{m=2}^{2M} c_m \lambda_m^{|\tau|}, \quad \tau = 0, \pm 1, \pm 2, \dots \quad (2.141)$$

and

$$\mathcal{R}_{S,I}(\tau) = c_1 \lambda_1^{|\tau|} = c_1. \quad (2.142)$$

Proof. Suppose that (2.140) is satisfied. It was shown in Remark 2.6.7 that the eigenvalues of the map's order-one FPM then satisfy

$$|\lambda_m| < \lambda_1 = 1, \quad m = 2, 3, \dots, 2M. \quad (2.143)$$

The expressions for $\mathcal{R}_{S,C}$ and $\mathcal{R}_{S,I}$ then follow by observing that the only term of (2.134) that has the required form for the impulsive component of the ACF, as expressed in (2.87), is the term associated with the unity eigenvalue $\lambda_1 = 1$. Specifically, (2.87) is satisfied with $r_k = c_1$ and $\omega_k = \omega_1 = 0$. $\square$

Remark 2.6.9. The derivation in section 2.A.13 yields the following intermediate expression for the ACF $\mathcal{R}_S$ of an ergodic semi-Markov map [10, 12]:

$$\mathcal{R}_S(\tau) = \mathbf{h}^T (\mathbf{P}_S^{(1)})^{|\tau|} \mathbf{g}, \quad \tau = 0, \pm 1, \pm 2, \dots, \quad (2.144)$$

where $\mathbf{g}$ and $\mathbf{h}$ are given by (2.136) and (2.137), respectively. This expression directly relates the ACF to the order-one FPM of the map. The elements of the order-one FPM are simple functions of the branch parameters of the map; refer to (2.117) to (2.122). The expression for the ACF presented in (2.134), as obtained by further simplifying the above expression, reveals the relationship between the power spectrum modes and the eigenvalues of the (diagonalisable) order-one FPM. These expressions permitted the development of the novel solutions to the generalised formulations of the IFPP, as described in the subsequent chapters.

An expression equivalent to (2.144), but for Markov maps, is provided in [9, Eq. 3.39]. Furthermore, the theory presented in [9] permits computation of the higher-order correlation statistics of Markov maps (see, for example, [9, Eq. 3.38]). Although the definition of the FPM in this work does not prohibit the generalisation of the theory to permit computation of the higher-order statistics of semi-Markov maps, it is not pursued here, as it is not relevant to IFPP formulations considered in this work.

2.6.5 Power spectral density

The PSD $\mathcal{S}_S$ of an ergodic map S is given by (2.83):

$$\mathcal{S}_S(\omega) = \mathcal{S}_{S,C}(\omega) + \mathcal{S}_{S,I}(\omega), \quad \omega \in [-\pi, \pi). \quad (2.145)$$

In this expression, $\mathcal{S}_{S,C}$ denotes the continuous component and $\mathcal{S}_{S,I}$ the impulsive component of the PSD. In this section, expressions are derived for the two components of an ergodic semi-Markov map's PSD (refer also to [12]).

Proposition 2.6.14 (PSD of an ergodic semi-Markov map). Invoke the PDSA framework introduced in Definition 2.4.4 and refined in Definition 2.4.15. Let S denote a PWL $\mathcal{R}(\mathcal{Q})$ -semi-Markov map that satisfies the conditions of the Folklore theorem (Theorem 2.6.3), such that S is ergodic. Let $\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_{M_P}$ denote the *unique* eigenvalues amongst the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_{2M}$ of the map's order-one FPM which satisfy $0 < |\lambda_m| < 1$. Here, $\lambda_1 = 1$ denotes the Perron eigenvalue.

Let the map's order-one FPM be diagonalisable, and let each diagonal element $p_{m,m}$ of the map's order-zero FPM satisfy $0 < p_{m,m} < 1$. The continuous component $\mathcal{S}_{S,C}$ of the map's PSD is then given by [10, 12]

$$\mathcal{S}_{S,C}(\omega) = C + \sum_{m=1}^{M_P} \frac{\tilde{c}_m}{(e^{i\omega} - \tilde{\lambda}_m)(e^{-i\omega} - \tilde{\lambda}_m)}, \quad (2.146)$$

where

$$C \triangleq \sum_{\{k>1:\lambda_k=0\}} c_k \quad (2.147)$$

and

$$\tilde{c}_m \triangleq \sum_{\{k>1:\lambda_k=\tilde{\lambda}_m\}} c_k(1 - \lambda_k^2), \quad m = 1, 2, \dots, M_P. \quad (2.148)$$

The impulsive component $\mathcal{S}_{S,I}$ of the map's PSD is given by

$$\mathcal{S}_{S,I}(\omega) = 2\pi c_1 \delta(\omega), \quad (2.149)$$

where $\delta(\cdot)$ denotes the Dirac delta function. The weights c_m in these expressions are defined in (2.135).

Proof. The proposition is proved in section 2.A.14. □

The z -transform counterpart $\Psi_{S,C}$ of $\mathcal{S}_{S,C}$ in (2.146) was derived in (2.261) of section 2.A.14. This expression is given by

$$\Psi_{S,C}(z) = C + \sum_{m=1}^{M_P} \frac{\tilde{c}_m}{(z - \tilde{\lambda}_m)(z^{-1} - \tilde{\lambda}_m)}, \quad (2.150)$$

where the region of convergence is given by $L < |z| < L^{-1}$, and

$$L \triangleq \max\{0, |\tilde{\lambda}_1|, |\tilde{\lambda}_2|, \dots, |\tilde{\lambda}_{M_P}|\}. \quad (2.151)$$

Equation (2.150) is manipulated to obtain

$$\Psi_{S,C}(z) = \frac{C \prod_{k=1}^{M_P} (z - \tilde{\lambda}_k)(z^{-1} - \tilde{\lambda}_k) + \sum_{m=1}^{M_P} \tilde{c}_m \prod_{k=1, k \neq m}^{M_P} (z - \tilde{\lambda}_k)(z^{-1} - \tilde{\lambda}_k)}{\prod_{m=1}^{M_P} (z - \tilde{\lambda}_m)(z^{-1} - \tilde{\lambda}_m)}. \quad (2.152)$$

The following propositions are proved.

Proposition 2.6.15. If $z = \alpha$ is a root of the denominator of (2.152), where $\alpha \neq 0$, then so is $z = \alpha^{-1}$ [10].

Proof. Substitution of $z = \alpha$ into the denominator of (2.152) and setting the resulting equation to zero yields

$$\prod_{m=1}^{M_P} (\alpha - \tilde{\lambda}_m)(\alpha^{-1} - \tilde{\lambda}_m) = 0. \quad (2.153)$$

Substitution of $z = \alpha^{-1}$ into the denominator of (2.152) then yields the left-hand side of (2.153), which implies that the denominator equals zero. Hence, $z = \alpha^{-1}$ is also a root of the denominator. $\square$

Proposition 2.6.16. If $z = \alpha$ is a root of the numerator of (2.152), where $\alpha \neq 0$, then so is $z = \alpha^{-1}$ [10].

Proof. The proof follows the same approach as the proof of Proposition 2.6.15. $\square$

Propositions 2.6.15 and 2.6.16 imply that the roots of the numerator (and the roots of the denominator) appear in pairs. This implies that (2.152) may be expressed as [10]

$$\tilde{\Psi}_S(z) = K_S H_S(z) H_S(z^{-1}), \quad (2.154)$$

where

$$H_S(z) \triangleq \frac{\prod_{m=1}^{M_Z} (z - \eta_m)}{\prod_{m=1}^{M_P} (z - \kappa_m)}, \quad 0 \leq M_Z \leq M_P \quad (2.155)$$

and

$$\kappa_m = \tilde{\lambda}_m, \quad m = 1, 2, \dots, M_P. \quad (2.156)$$

The numerator and denominator of (2.155) is defined as unity if $M_Z = 0$ or $M_P = 0$, respectively. Here, η_m is defined such that $|\eta_m| \leq 1$, $m = 1, 2, \dots, M_Z$.

For convenience, the region of convergence of (2.154) is restated here. It is defined by

$$L < |z| < L^{-1} \quad (2.157)$$

and

$$L \triangleq \max\{0, |\tilde{\lambda}_1|, |\tilde{\lambda}_2|, \dots, |\tilde{\lambda}_{M_P}|\}. \quad (2.158)$$

Remark 2.6.10. Equations (2.154) and (2.155) reveal that the continuous component of the map's PSD is a rational function. The complex numbers $\kappa_1, \kappa_2, \dots, \kappa_{M_P}$ and $\eta_1, \eta_2, \dots, \eta_{M_Z}$ are referred to as the *poles* and *zeros* of the PSD's continuous component. Equation (2.156) implies that these poles are equal to the unique non-zero and non-unity (in absolute value) eigenvalues of the map's order-one FPM.

The expressions reveal that a semi-Markov map with an $M \times M$ order-zero FPM admits a maximum of $2M - 1$ power spectrum poles, due to the Perron eigenvalue being equal to unity (the corresponding mode manifests in the impulsive component of the PSD). The zeros of the PSD have a more sophisticated relationship with the map's parameters that depend on the mode weights c_m .

As the eigenvalues of the order-one FPM comprise the eigenvalues of the order-zero FPM and submatrix $\mathbf{Q}_S$ of the order-one FPM, it follows that the poles of the map comprise the unique non-zero and non-unity (in absolute value) eigenvalues of the latter two matrices. Furthermore, as the eigenvalues $\lambda_m \in \mathbb{C} \setminus \mathbb{R}$ of a real matrix appear in complex conjugate pairs, so do the corresponding poles.

Remark 2.6.11. As an $\mathcal{R}(\mathcal{Q})$ -semi-Markov map is $\mathcal{R}$ -Markov, the PSD of a Markov map is necessarily also a rational function. The counterpart of (2.152) for Markov maps is derived in [9, Eq. 3.41]. However, the latter expression was not derived under the assumption that the order-one FPM is diagonalisable; hence, it is more general than the expressions derived in this section for semi-Markov maps.

By assuming that the order-one FPM is diagonalisable, the theory developed in this section applies to semi-Markov maps that possess PSDs with non-repeating poles. In later chapters, attention is restricted to designing this subset of the semi-Markov maps, instead of considering the general class of maps that may possess repeating poles. Therefore, the more general expression for the PSD is not developed here.

Remark 2.6.12. Although this chapter does not present examples of semi-Markov maps, their ACFs and PSDs, the expression for the PSD of these maps, as given by (2.152), was numerically verified in Examples 4.8.1 to 4.8.3 in chapter 4. In these examples, semi-Markov maps were constructed using a novel solution to IFPP-II. The

theoretical power spectra of these maps were derived using (2.152) and are plotted as solid lines in Figs. 4.8.7, 4.8.8, 4.8.15, 4.8.16 and 4.8.20.

These above theoretical results were verified by numerically computing trajectories of the semi-Markov maps, and then computing the (discrete-time) short-time Fourier transform of each trajectory. In this approach, each trajectory is divided into shorter non-overlapping segments of 500 states each, and the discrete-time Fourier transform of each segment is computed. The squared magnitudes of the resulting frequency-domain representations of the segments are then averaged to obtain an estimate of the PSD associated with the trajectory. These numerical results are plotted as dots in each of the abovementioned figures.

The dots plotted in each of the figures overlap the solid curves plotted using the expressions derived in this chapter. This implies that the expressions derived for the PSD are correct. As the expressions for the PSD were derived from the expressions for the ACF, this also implies that the expressions for the ACF are valid.

2.7 Summary

This chapter lays the mathematical foundation for the work that follows in the subsequent chapters. The concept of a dynamical system is defined, and the PDSA framework for analysing these systems from a probabilistic viewpoint is introduced. This framework uses a stochastic process to represent the zero-input response of the dynamical system of interest, which is configured to use an ergodic map as the evolution rule. The FPO and Koopman operator are introduced. Invariant measures and densities are defined, and the concept of ergodicity is introduced. It is proved that an ergodic map has a unique invariant density.

The extension of the Birkhoff individual ergodic theorem, which equates averages computed over trajectories to those computed over the ensemble, is stated. This theorem is applied to analyse the stochastic process of the PDSA framework. It is shown that the invariant density of the ergodic map configured as the system's evolution rule coincides with the PDF of the system's zero-input response.

Using the extension to the Birkhoff individual ergodic theorem, a general expression is derived for the ACF of the map as a function of the FPO. It is then proved that the stochastic process of the PDSA framework is wide-sense stationary. The Wiener-Khinchin theorem is subsequently invoked, thereby deriving an expression

for the PSD of the process, and by extension, the zero-input response of the system.

The chapter defines and characterises the class of piecewise monotonic maps. An expression for the FPO of these maps, as well as an expression that characterises the evolution of distribution functions under their iteration, are derived. The focus then shifts to two closely related subclasses of the piecewise monotonic maps, namely the Markov and semi-Markov maps.

Markov and semi-Markov maps possess branch functions that map intervals of a partition onto unions of intervals from the same partition. This characteristic admits an *exact* finite-dimensional representation of the FPO. More specifically, it permits a matrix representation, referred to as the FPM, that exactly governs the evolution of piecewise polynomial density functions under iteration of the map.

It is shown that the semi-Markov maps are a superset of the Markov maps, which leads to their characterisation in lieu of the Markov maps. The Folklore theorem, which states conditions for the ergodicity of the semi-Markov maps, is presented. Expressions are subsequently derived for the order-zero and order-one FPM of a semi-Markov map, which characterise the evolution of densities that are PWC and PWL over the intervals of the Markov partition, respectively.

Several properties of the order-zero and order-one FPM of a semi-Markov map, as relevant to the solution of the IFPP, are proved. It is shown that any order-zero FPM is similar to a stochastic matrix. If the map satisfies the conditions of the Folklore theorem as well as a mild condition regarding the diagonal elements of its order-zero FPM, it is shown that the order-one FPM possesses a single maximal eigenvalue equal to unity. All other eigenvalues of the matrix are strictly smaller in absolute value than this Perron eigenvalue. It is also shown that the left and right Perron eigenvectors coincide with the lengths of the partition's intervals and the vector representation of its invariant density, respectively.

In the remainder of the chapter, expressions are derived for the ACF and PSD of a semi-Markov map. If the order-one FPM of a semi-Markov map is diagonalisable, the ACF is a weighted sum of temporal modes, each coinciding with an eigenvalue of the order-one FPM. Since the PDF and ACF are related as a discrete-time Fourier transform pair, this implies that the PSD comprises power spectrum modes with centre frequencies and bandwidths (i.e., mode characteristics) that coincide with the arguments and the moduli of these eigenvalues.

The expressions derived for the PSD of a semi-Markov map are rational functions.

It is shown that the poles of these power spectra coincide with the unique non-zero and non-unity (in absolute value) eigenvalues of the order-one FPM. The zeros of the PSD are related to the eigenvectors of the map's order-one FPM in a more sophisticated fashion.

2.8 Conclusions

By virtue of admitting an exact finite-dimensional representation of the FPO, the semi-Markov maps permit simplification of the general expressions that relate an arbitrary map's statistical metrics to its infinite dimensional FPO and ultimately its structural parameters. In particular, the expressions for the statistical metrics of these maps are comparatively unsophisticated functions of a finite collection of the map's branch parameters. This increases their potential for successful analytical utilisation in the design problem.

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Appendix

2.A Proofs of Propositions

2.A.1 Proof of Proposition 2.4.1

The proposition is proved using mathematical induction. Assume that S is $\mathcal{B}$ -measurable and that X_1 denotes a random variable on $(\Omega, \mathcal{B}, P)$, as defined by the PDSA framework. Now, assume that X_t , for some $t > 1$, is also a random variable on $(\Omega, \mathcal{B}, P)$. It is proved that

$$X_{t+1} = S \circ X_t \tag{2.159}$$

is a random variable on $(\Omega, \mathcal{B}, P)$.

As X_t is a random variable, it is $\mathcal{B}$ -measurable. Since S is $\mathcal{B}$ -measurable, every $C \in \mathcal{B}$ has a pre-image $B \in \mathcal{B}$ under S . Observing that every $B \in \mathcal{B}$ has a pre-image $A \in \mathcal{B}$ under X_t (this follows from the $\mathcal{B}$ -measurability of X_t), it is concluded from (2.159) that every $C \in \mathcal{B}$ has a pre-image $A \in \mathcal{B}$ under X_{t+1} . Hence, X_{t+1} is $\mathcal{B}$ -measurable, which implies that it is a random variable on $(\Omega, \mathcal{B}, P)$.

2.A.2 Proof of Proposition 2.4.2

In this proof, the phrases *absolutely continuous* and *non-singular* are both assumed to be with respect to the measure ν . The theorem is proved using mathematical induction. Assume that P_1 is absolutely continuous and that S is non-singular, as stated in the proposition. Now, assume that P_t , for some $t > 1$, is also absolutely continuous. It is proved that the measure P_{t+1} is absolutely continuous.

Consider any $B \in \mathcal{B}$ satisfying $\nu(B) = 0$. As S is measurable and non-singular, it follows that $A \triangleq S^{-1}(B) \in \mathcal{B}$ and $\nu(A) = 0$. The absolute continuity of P_t then

implies that $P_t(A) = 0$. In turn, (2.18) implies that

$$P_{t+1}(B) = P_t(S^{-1}(B)) = P_t(A) = 0. \quad (2.160)$$

This implies that P_{t+1} is absolutely continuous.

2.A.3 Proof of Proposition 2.4.3

Substituting $t = 0$ into (2.37) yields $h = f.g$, which coincides with (2.38). Now, consider an arbitrary $t > 0$, and let S^t denote the t -fold composition of S as defined in (2.3). Property (FP3) of the FPO implies that $\mathcal{P}_{S,\lambda}^t = \mathcal{P}_{S^t,\lambda}$ and $\mathcal{P}_{S,\nu}^t = \mathcal{P}_{S^t,\nu}$. Let $f \in L^1(\Omega, \mathcal{B}, \lambda)$. From the definition of the FPO in (2.25),

$$\int_B (\mathcal{P}_{S^t,\lambda} f)(x) d\lambda(x) = \int_{(S^t)^{-1}(B)} f(x) d\lambda(x), \quad B \in \mathcal{B}. \quad (2.161)$$

Suppose that λ is absolutely continuous with respect to ν . Equation (2.39) then implies that

$$\int_B (\mathcal{P}_{S^t,\lambda} f)(x) g(x) d\nu(x) = \int_{(S^t)^{-1}(B)} f(x) g(x) d\nu(x). \quad (2.162)$$

Let $h = f.g$. Equation (2.25) then implies

$$\int_{(S^t)^{-1}(B)} f(x) g(x) d\nu(x) = \int_B (\mathcal{P}_{S^t,\nu} h)(x) d\nu(x). \quad (2.163)$$

Comparing (2.162) and (2.163) reveals that

$$\int_B (\mathcal{P}_{S^t,\lambda} f)(x) g(x) d\nu(x) = \int_B (\mathcal{P}_{S^t,\nu} h)(x) d\nu(x). \quad (2.164)$$

This implies that

$$g \mathcal{P}_{S,\lambda}(f) = \mathcal{P}_{S,\nu}(h) \text{ a.e.} \quad (2.165)$$

Property (FP3) of the FPO then implies that

$$g \mathcal{P}_{S,\lambda}^t(f) = \mathcal{P}_{S,\nu}^t(h) \text{ a.e.,} \quad B \in \mathcal{B}, \quad t = 1, 2, \dots \quad (2.166)$$

thereby completing the proof.

2.A.4 Proof of Proposition 2.4.4

Suppose that S is ergodic and that an S -invariant probability measure P_S^* exists. Let f_S^* denote the unique (over subsets of non-zero measure) density associated with P_S^* . Theorem 2.4.1 then implies that f_S^* is a stationary density of the FPO $\mathcal{P}_S$.

The theorem is proved by contradiction. Suppose that another S -invariant probability measure $\tilde{P}_S^* \neq P_S^*$ exists, and let $\tilde{f}_S^*$ denote the unique density associated with $\tilde{P}_S^*$. Since $\tilde{P}_S^* \neq P_S^*$, (2.14) implies that $\tilde{f}_S^*$ is not almost everywhere equal to f_S^* . Theorem 2.4.1 then implies that $\tilde{f}_S^*$ is a second, distinct stationary density of $\mathcal{P}_S$. However, this contradicts Theorem 2.4.2, which requires the existence of at most one stationary density of $\mathcal{P}_S$. Hence, the S -invariant probability measure P_S^* must be unique.

2.A.5 Proof of Proposition 2.4.5

The theorem is proved by contradiction. Suppose that S is ergodic and that two S -invariant densities f_S^* and $\tilde{f}_S^*$ that are not almost everywhere equal exist. The probability measures P_S^* and $\tilde{P}_S^*$ associated with these densities are both S -invariant. However, (2.14) implies that $P_S^* \neq \tilde{P}_S^*$. This contradicts Proposition 2.4.4, which requires the invariant probability measure of an ergodic map to be unique.

2.A.6 Proof of Proposition 2.4.6

The expected value of the process $(X_t)_{t \in \mathcal{T}}$ was defined in (2.22):

$$\mathcal{M}_S(t) = \int_{\Omega} X_t(\omega) dP(\omega), \quad t \in \mathcal{T}, \quad (2.167)$$

where the initial random variable X_1 is defined on the underlying probability space $(\Omega, \mathcal{B}, P)$. The expected value may equivalently be computed over the induced measure according to the expression

$$\mathcal{M}_S(t) = \int_{\Omega} x dP_t(x), \quad t \in \mathcal{T}. \quad (2.168)$$

Now, select $P = P_S^*$ and $X_1(\omega) = \omega$, as directed in the proposition statement. It follows that the induced measure P_1 is equal to the invariant measure P_S^* . Equation (2.47) then implies that $P_t = P_S^*$ for all $t \in \mathcal{T}$. Substituting P_t with P_S^* in (2.168) then yields

$$\mathcal{M}_S(t) \equiv \mathcal{M}_S = \int_{\Omega} x dP_S^*(x), \quad t \in \mathcal{T}. \quad (2.169)$$

Hence, the expected value of the process is no longer time dependent.

2.A.7 Proof of Proposition 2.4.7

The mean value of a trajectory starting from an initial state x_1 was defined in (2.23):

$$\bar{\mathcal{M}}_S(x_1) \triangleq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} S^k(x_1), \quad x_1 \in \Omega. \quad (2.170)$$

Let $(\Omega, \mathcal{B}, P_S^*)$ denote a measure space, and let $h(x) = I(x) \triangleq x$. It is shown in Proposition 2.B.1 on page 90 that $h \in L^1(\Omega, \mathcal{B}, P_S^*)$ and is thus integrable by definition. Equation (2.57) of Theorem 2.4.3 then implies that

$$\bar{\mathcal{M}}_S(x_1) \equiv \bar{\mathcal{M}}_S = \int_{\Omega} x dP_S^*(x) \quad (2.171)$$

for almost every $x_1 \in \Omega$. Hence, almost all trajectories have the same mean value, which is denoted by $\bar{\mathcal{M}}_S$. Furthermore, as the right-hand sides of (2.171) and (2.62) are equal, the expected value and mean value computed over time are equal; i.e.,

$$\bar{\mathcal{M}}_S = \mathcal{M}_S. \quad (2.172)$$

2.A.8 Proof of Proposition 2.4.8

The ensemble ACF of (2.65) may be defined over the induced measure as

$$\mathcal{R}_S(t_1, t_2) = \begin{cases} \int_{\Omega} x S^{t_2-t_1}(x) dP_{t_1}(x), & \text{if } t_1 \leq t_2, \\ \int_{\Omega} x S^{t_1-t_2}(x) dP_{t_2}(x), & \text{if } t_1 > t_2. \end{cases} \quad (2.173)$$

Let $\tilde{t} \triangleq \min\{t_1, t_2\}$ and $\tau \triangleq t_2 - t_1$. Equation (2.173) may then be rewritten as

$$\mathcal{R}_S(t_1, t_2) \equiv \mathcal{R}_S(\tilde{t}, \tau) = \int_{\Omega} x S^{|\tau|}(x) dP_{\tilde{t}}(x), \quad \tilde{t} \in \mathcal{T}, \tau = 0, \pm 1, \pm 2, \dots \quad (2.174)$$

By selecting $P = P_S^*$ and $X_1(\omega) = \omega$, it follows that $P_1 = P_S^*$. Equation (2.47) then implies that $P_t = P_S^*$ for all $t \in \mathcal{T}$. Equation (2.174) is then rewritten as

$$\mathcal{R}_S(t_1, t_2) \equiv \mathcal{R}_S(\tau) = \int_{\Omega} x S^{|\tau|}(x) dP_S^*(x), \quad \tau = 0, \pm 1, \pm 2, \dots \quad (2.175)$$

This expression implies that the ensemble ACF $\mathcal{R}_S(t_1, t_2)$ of the process $(X_t)_{t \in \mathcal{T}}$ is only dependent on the time difference $\tau = t_2 - t_1$. In this case, the ensemble ACF is denoted by $\mathcal{R}_S(\tau)$.

2.A.9 Proof of Proposition 2.4.9

Using the extension of the Birkhoff individual ergodic theorem, it was proved in (2.267) of section 2.B that the time ACF associated with the trajectories of S may be expressed as an average over the state ensemble:

$$\bar{\mathcal{R}}_S(\tau, x_1) \equiv \bar{\mathcal{R}}_S(\tau) = \int_{\Omega} x S^{|\tau|}(x) dP_S^*(x), \quad \tau = 0, \pm 1, \pm 2, \dots \quad (2.176)$$

The right-hand sides of this expression and (2.72) are identical. Hence, the time ACF equals the ensemble ACF; i.e.,

$$\bar{\mathcal{R}}_S(\tau) = \mathcal{R}_S(\tau). \quad (2.177)$$

2.A.10 Proof of Proposition 2.6.2

The proof requires several propositions, which are provided below.

Proposition 2.A.1 (One-step refinement of the Markov partition). Let S denote an $\mathcal{R}(\mathcal{Q})$ -semi-Markov map, and let

$$\mathcal{W} = S^{-1}(\mathcal{R}) \triangleq \{W = (a, b) : S(W) \in \mathcal{R}\}. \quad (2.178)$$

The map S is $\mathcal{W}(\mathcal{R})$ -semi-Markov.

Proof. Consider any interval $Q_k \in \mathcal{Q}$. Since S is an $\mathcal{R}(\mathcal{Q})$ -semi-Markov map, the pre-image of Q_k under S may be expressed as

$$S^{-1}(Q_k) = \bigcup_{m \in A} \bigcup_{j \in B(m)} R_j^{(m)} \quad (2.179)$$

for some $A \subseteq \{1, 2, \dots, M\}$ and $B(m) \subseteq \{1, 2, \dots, c(m)\}$. Now, $\mathcal{R}^{(k)}$ is a partition of the closure of Q_k . Since each interval $R_j^{(m)}$ in the preceding expression is mapped monotonically to Q_k by

$$S_{j,m} \triangleq S|_{R_j^{(m)}}, \quad (2.180)$$

it follows that the pre-image, under $S_{j,m}$, of the intervals that constitute the partition $\mathcal{R}^{(k)}$ is a partition of the closure of each $R_j^{(m)}$ in (2.179). These statements hold for each Q_k .

As each $R_j^{(m)} \in \mathcal{R}$ is mapped monotonically to some interval Q_k under S , it then follows that the closure of each interval $R_j^{(m)} \in \mathcal{R}$ has a partition

$$\mathcal{W}_j^{(m)} \triangleq \{W_{j,1}^{(m)}, W_{j,2}^{(m)}, \dots, W_{j,b(m,j)}^{(m)}\} \quad (2.181)$$

such that

$$S(W_{j,k}^{(m)}) \in \mathcal{R}, \quad m = 1, 2, \dots, M, \quad j = 1, 2, \dots, c(m), \quad k = 1, 2, \dots, b(m, j). \quad (2.182)$$

Since the closure of each interval $R_j^{(m)} \in \mathcal{R}$ has a partition $\mathcal{W}_j^{(m)}$, it follows that

$$\mathcal{W} \triangleq \{W_{j,k}^{(m)}\} \quad (2.183)$$

is a subpartition of $\mathcal{R}$. The pre-image of each $R \in \mathcal{R}$ under S comprises a number of intervals B_i , where each interval B_i is necessarily an interval of partition $\mathcal{W}$. As the pre-images B_i associated with all intervals $R \in \mathcal{R}$ were used to construct the partition $\mathcal{W}$, it then follows that

$$\mathcal{W} = S^{-1}(\mathcal{R}). \quad (2.184)$$

Now, since each $W \in \mathcal{W}$ is a subinterval of some $R \in \mathcal{R}$, and $S|_R$ is monotonic, the function $S|_W$ is monotonic. Hence, $S \in \Gamma(\mathcal{W})$. Since $\mathcal{W}$ is a subpartition of $\mathcal{R}$, and each interval $W \in \mathcal{W}$ is mapped monotonically to an interval $R \in \mathcal{R}$ under S , it follows that S is $\mathcal{W}(\mathcal{R})$ -semi-Markov. $\square$

Proposition 2.A.2 (Multiple-step refinement of the Markov partition). Let S denote an $\mathcal{R}(\mathcal{Q})$ -semi-Markov map, and let

$$\mathcal{W}_k \triangleq \begin{cases} \mathcal{Q}, & \text{if } k = -1, \\ \mathcal{R}, & \text{if } k = 0, \\ S^{-k}(\mathcal{R}), & \text{if } k = 1, 2, \dots, \end{cases} \quad (2.185)$$

where $S^{-k}(\mathcal{R}) \triangleq S^{-1}(S^{-1}(\dots(S^{-1}(\mathcal{R}))\dots))$ and $S^{-1}(\mathcal{R})$ is defined in (2.178). The map S is $\mathcal{W}_k(\mathcal{W}_{k-1})$ -semi-Markov for all $k = 0, 1, 2, \dots$

Proof. The proof follows from sequential application of Proposition 2.A.1. $\square$

Proposition 2.A.3 (Composition of two semi-Markov maps). Let S_1 denote a $\tilde{\mathcal{W}}(\mathcal{R})$ -semi-Markov map and let S_2 denote an $\mathcal{R}(\mathcal{Q})$ -semi-Markov map. Then the composition $S_2 \circ S_1$ is a $\tilde{\mathcal{W}}(\mathcal{Q})$ -semi-Markov map.

Proof. As $S_1 \in \Gamma(\tilde{\mathcal{W}})$ is $\tilde{\mathcal{W}}(\mathcal{R})$ -semi-Markov, it maps each $W \in \tilde{\mathcal{W}}$ monotonically to some interval $R \in \mathcal{R}$. As $S_2 \in \Gamma(\mathcal{R})$ is $\mathcal{R}(\mathcal{Q})$ -semi-Markov, it maps each $R \in \mathcal{R}$ monotonically to some interval $Q \in \mathcal{Q}$. It follows that the composition $S_2 \circ S_1$ maps each $W \in \tilde{\mathcal{W}}$ monotonically to some interval $Q \in \mathcal{Q}$. Hence $S_2 \circ S_1 \in \Gamma(\tilde{\mathcal{W}})$ is a $\tilde{\mathcal{W}}(\mathcal{Q})$ -semi-Markov map. $\square$

Proposition 2.A.4 (The t -fold composition of a semi-Markov map is semi-Markov). Let S denote an $\mathcal{R}(\mathcal{Q})$ -semi-Markov map. The t -fold composition S^t of the map S is a $\mathcal{W}_{t-1}(\mathcal{Q})$ -semi-Markov map for $t = 1, 2, \dots$, where $\mathcal{W}_t$ is defined in (2.185).

Proof. Let $t = 1$. As $\mathcal{W}_{t-1} = \mathcal{W}_0 = \mathcal{R}$, the map S is required to be $\mathcal{R}(\mathcal{Q})$ -semi-Markov, which agrees with the assumption made in the proposition statement.

Now, let $t > 1$. Define the maps $S_1 = S_2 = \dots = S_t = S$. Proposition 2.A.2 states that S_m is $\mathcal{W}_{m-1}(\mathcal{W}_{m-2})$ -semi-Markov. Proposition 2.A.3 then states that the composition $S^t = S_1 \circ S_2 \circ \dots \circ S_t$ is a $\mathcal{W}_{t-1}(\mathcal{W}_{-1})$ -semi-Markov map. As $\mathcal{W}_{-1} \triangleq \mathcal{Q}$, the proposition is proved. $\square$

Proposition 2.A.4 implies that the t -fold composition of an $\mathcal{R}(\mathcal{Q})$ -semi-Markov map S is a $\mathcal{W}_{t-1}(\mathcal{Q})$ -semi-Markov map, where $\mathcal{W}_{t-1}$ is a subpartition of $\mathcal{Q}$.

2.A.11 Proofs of propositions regarding the order-zero FPM

Proof of Proposition 2.6.3

Let S denote an $\mathcal{R}(\mathcal{Q})$ -semi-Markov map. The slope $v_{j,m}$ of each branch function $S_{j,m}$ satisfies

$$|v_{j,m}| = \frac{\mu(S(R_j^{(m)}))}{\mu(R_j^{(m)})}, \quad (2.186)$$

where $\mu(R_j^{(m)}) > 0$ by (2.106). Equation (2.120) implies that

$$S(R_j^{(m)}) = Q_k, \quad \forall j \in T(k, m). \quad (2.187)$$

Substitution of (2.187) into (2.186) yields

$$|v_{j,m}| = \frac{\mu(Q_k)}{\mu(R_j^{(m)})}, \quad \forall j \in T(k, m). \quad (2.188)$$

Substitution of the latter equation into (2.122) yields

$$p_{k,m} = \begin{cases} \frac{1}{\mu(Q_k)} \sum_{j \in T(k, m)} \mu(R_j^{(m)}), & \text{if } |T(k, m)| > 0, \\ 0, & \text{if } |T(k, m)| = 0. \end{cases} \quad (2.189)$$

As $\mathcal{R}$ is a partition, all intervals $R_j^{(m)} \in \mathcal{R}$ are disjoint. Hence,

$$\sum_{j \in T(k, m)} \mu(R_j^{(m)}) = \mu\left(\bigcup_{j \in T(k, m)} R_j^{(m)}\right). \quad (2.190)$$

Equations (2.189) and (2.190), together with the expression $\mu(Q_k) = 1/M$, imply

$$\begin{aligned} \sum_{k=1}^M p_{k,m} &= M \sum_{k=1}^M \mu\left(\bigcup_{j \in T(k, m)} R_j^{(m)}\right) \\ &= M \mu\left(\bigcup_{k=1}^M \bigcup_{j \in T(k, m)} R_j^{(m)}\right). \end{aligned} \quad (2.191)$$

As the map S is $\mathcal{R}(\mathcal{Q})$ -semi-Markov, each $R_j^{(m)} \in \mathcal{R}^{(m)}$ satisfies $S(R_j^{(m)}) \in \mathcal{Q}$. Hence,

$$\bigcup_{k=1}^M \bigcup_{j \in T(k, m)} R_j^{(m)} = Q_m. \quad (2.192)$$

Substituting the latter equation into (2.191) and then substituting $\mu(Q_m) = 1/M$ into the resulting equation yields

$$\sum_{k=1}^M p_{k,m} = 1, \quad m = 1, 2, \dots, M. \quad (2.193)$$

Now, (2.122) implies that each element $p_{k,m}$ of the order-zero FPM satisfies $p_{k,m} \geq 0$. Equation (2.193) then implies that each element must satisfy $p_{k,m} \leq 1$. Matrix $\mathbf{P}_S^{(0)} = [p_{k,m}]$ then satisfies the conditions of Definition 2.6.5 and is column stochastic.

Proof of Proposition 2.6.4

Let

$$\mathbf{v} = [\mu(Q_1), \mu(Q_2), \dots, \mu(Q_M)]^T \quad (2.194)$$

and define

$$\mathbf{v}^T \mathbf{P}_S^{(0)} \triangleq [\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_M]. \quad (2.195)$$

Equation (2.189) implies that

$$\begin{aligned} \tilde{v}_m &= \sum_{k=1}^M \sum_{j \in T(k,m)} \mu(R_j^{(m)}) \\ &= \mu(Q_m), \quad m = 1, 2, \dots, M. \end{aligned} \quad (2.196)$$

This implies that $\mathbf{v}^T = \mathbf{v}^T \mathbf{P}_S^{(0)}$. Hence, $\mathbf{P}_S^{(0)}$ possesses a unity eigenvalue λ_1 with corresponding left eigenvector $\mathbf{v}_1^T \triangleq \mathbf{v}^T$.

Proof of Proposition 2.6.5

It was shown in Proposition 2.6.4 that $\mathbf{P}_S^{(0)}$ has a unity eigenvalue λ_1 with corresponding left eigenvector $\mathbf{v}_1^T$, where the elements of this vector coincide with the lengths of the intervals in $\mathcal{Q}$. Let $\mathbf{D}$ denote a diagonal matrix with strictly positive entries along its diagonal, to be defined below, and consider the matrix

$$\tilde{\mathbf{P}} = \mathbf{D}^{-1} \mathbf{P}_S^{(0)} \mathbf{D}. \quad (2.197)$$

Note that $\mathbf{D}$ is invertible due to the positivity of its diagonal elements. As $\tilde{\mathbf{P}}$ and $\mathbf{P}_S^{(0)}$ are similar, they share the same eigenvalues [45]. Hence, $\tilde{\mathbf{P}}$ has at least one unity eigenvalue. Let $\tilde{\mathbf{v}}_1^T$ denote the left eigenvector of $\tilde{\mathbf{P}}$ that coincides with this unity eigenvalue. Equation (2.197) then implies that

$$\tilde{\mathbf{v}}_1^T = \mathbf{v}_1^T \mathbf{D}. \quad (2.198)$$

Equation (2.124) then suggests that the selection

$$\mathbf{D} = \text{diag}\{\mu(Q_1)^{-1}, \mu(Q_2)^{-1}, \dots, \mu(Q_M)^{-1}\} \quad (2.199)$$

leads to the left eigenvector $\tilde{\mathbf{v}}_1^T$ with all elements equal to unity (note that each interval Q_m in (2.199) has positive Borel measure). Since $\tilde{\mathbf{v}}_1^T$ is associated with the unity eigenvalue,

$$\tilde{\mathbf{v}}_1^T = \tilde{\mathbf{v}}_1^T \tilde{\mathbf{P}}. \quad (2.200)$$

Let $\tilde{\mathbf{v}}_1 = [\tilde{v}_{1,1}, \tilde{v}_{1,2}, \dots, \tilde{v}_{1,M}]^T$ and $\tilde{\mathbf{P}} = [\tilde{p}_{k,m}]_{k,m=1}^M$. Equation (2.200) implies that

$$\sum_{k=1}^M \tilde{v}_{1,k} \tilde{p}_{k,m} = \tilde{v}_{1,m}, \quad m = 1, 2, \dots, M. \quad (2.201)$$

Since all $\tilde{v}_{1,m} = 1$, the latter equation implies

$$\sum_{k=1}^M \tilde{p}_{k,m} = 1. \quad (2.202)$$

Now, (2.122) implies that each element of the order-zero FPM $\mathbf{P}_S^{(0)}$ is non-negative. Since each element of $\mathbf{D}$ and $\mathbf{D}^{-1}$ is non-negative, (2.197) implies that each element of $\tilde{\mathbf{P}}$ is non-negative. Equation (2.202) then implies that each element of $\tilde{\mathbf{P}} = [\tilde{p}_{k,m}]$ satisfies

$$0 \leq \tilde{p}_{k,m} \leq 1. \quad (2.203)$$

Matrix $\tilde{\mathbf{P}}$ then satisfies the conditions of Definition 2.6.5 and is column stochastic.

Proof of Proposition 2.6.6

Proposition 2.6.5 states that the order-zero FPM $\mathbf{P}_S^{(0)}$ is similar to a column stochastic matrix $\tilde{\mathbf{P}}$. This implies that these matrices share the same eigenvalues and hence the same spectral radius [45].

The proposition is proved by applying the Gershgorin circle theorem [45] to the matrix $\tilde{\mathbf{P}}$. This theorem states that each eigenvalue $\tilde{\lambda}$ of any $M \times M$ matrix $\tilde{\mathbf{P}} = [\tilde{p}_{k,m}]$ (i.e., not necessarily stochastic) lies within at least one of M *Gershgorin discs* in the complex plane, defined by

$$|\tilde{\lambda} - \tilde{p}_{m,m}| \leq r_m, \quad m = 1, 2, \dots, M \quad (2.204)$$

and

$$r_m = \sum_{k=1, k \neq m}^M |\tilde{p}_{k,m}|. \quad (2.205)$$

Since $\tilde{\mathbf{P}}_S^{(0)} = [\tilde{p}_{k,m}]$ is column stochastic, the corresponding Gershgorin discs are defined as

$$|\tilde{\lambda} - \tilde{p}_{m,m}| \leq 1 - \tilde{p}_{m,m}, \quad m = 1, 2, \dots, M. \quad (2.206)$$

These discs all lie within the unit circle and intersect the unit circle at the point $\tilde{\lambda} = 1$. Hence, no eigenvalue $\tilde{\lambda}$ can lie outside the unit circle in the complex plane, which implies that the spectral radius $\tilde{\rho}$ of $\tilde{\mathbf{P}}$ satisfies $\tilde{\rho} \leq 1$. As the spectral radius ρ of the order-zero FPM is equal to $\tilde{\rho}$, it follows that $\rho \leq 1$. However, Proposition 2.6.4 states that $\mathbf{P}_S^{(0)}$ has at least one unity eigenvalue. Hence, $\rho = 1$.

Proof of Proposition 2.6.7

The proposition is proved by observing that the t -fold composition of a semi-Markov map is semi-Markov, and then invoking certain implications that the structural characteristics of these maps have regarding aperiodicity and the order-zero FPM.

It is first proved that aperiodicity of S implies primitivity of the order-zero FPM. Suppose that S is aperiodic (Definition 2.6.3), which implies that there exists some $t(m) > 0$ such that $S^{t(m)}(Q_m^c) = \Omega$ for each $m = 1, 2, \dots, M$. Here, Q_m^c denotes the closure of Q_m . As S is assumed to be surjective, $S(\Omega) = \Omega$ and

$$S^{t(m)}(Q_m^c) = \Omega \Rightarrow S^{t(m)+k}(Q_m^c) = \Omega, \quad \forall k \geq 0. \quad (2.207)$$

Select

$$t = \max_m \{t(m)\}. \quad (2.208)$$

It follows that

$$S^t(Q_m^c) = \Omega, \quad m = 1, 2, \dots, M. \quad (2.209)$$

The map S^t is the t -fold composition of the $\mathcal{R}(\mathcal{Q})$ -semi-Markov map S . Property (FP3) of the FPO implies that the FPO $\mathcal{P}_{S^t}$ of S^t satisfies

$$\mathcal{P}_{S^t} = \mathcal{P}_S^t. \quad (2.210)$$

The $M \times M$ order-zero FPM of S is the matrix representation of $\mathcal{P}_S$ that characterises the evolution of density functions that are PWC over the Markov partition $\mathcal{Q}$. If attention is restricted to these density functions, the right-hand side of (2.210) may be represented as the t -fold product of the order-zero FPM of the map S with itself. This is expressed as

$$\mathcal{P}_{S^t} \sim [\mathbf{P}_S^{(0)}]^t, \quad (2.211)$$

where the symbol $\sim$ denotes that the right-hand side of the expression is the matrix representation of the left-hand side.

Proposition 2.6.2 states that S^t is a $\mathcal{W}(\mathcal{Q})$ -semi-Markov map. Hence, its FPO $\mathcal{P}_{S^t}$ may be represented as an $M \times M$ matrix (i.e., the order-zero FPM $\mathbf{P}_{S^t}^{(0)}$) if attention is restricted to densities that are PWC over its Markov partition $\mathcal{Q}$. Equation (2.211) then implies that

$$\mathbf{P}_{S^t}^{(0)} = [\mathbf{P}_S^{(0)}]^t. \quad (2.212)$$

Consider any $m \in \{1, 2, \dots, M\}$. Since S^t is a $\mathcal{W}(\mathcal{Q})$ -semi-Markov map, it is only permitted to map one or more subintervals of Q_m , each in its entirety and in a monotonic fashion, to any interval $Q_k \in \mathcal{Q}$. Each such subinterval is mapped with a

single branch function. Equation (2.209) then implies that S^t must possess at least M branch functions $S_{k,m}^t$, where $k = 1, 2, \dots, M$, each possessing a domain W_k of non-zero Borel measure that is a subinterval of Q_m and which satisfies

$$S_{k,m}^t(W_k) = Q_k, \quad \forall k = 1, 2, \dots, M. \quad (2.213)$$

Equation (2.122) then implies that column m of the order-zero FPM $\mathbf{P}_{S^t}^{(0)}$ of S^t must possess elements that are strictly positive. As this argument is valid for any $m \in \{1, 2, \dots, M\}$, it follows that all elements of the order-zero FPM $\mathbf{P}_{S^t}^{(0)}$ must be strictly positive. Equation (2.212) then implies that the t -fold product of the order-zero FPM $\mathbf{P}_S^{(0)}$ must be strictly positive. Hence, $\mathbf{P}_S^{(0)}$ is primitive.

It is subsequently proved that the primitivity of the order-zero FPM $\mathbf{P}_S^{(0)}$ of S implies aperiodicity. If $\mathbf{P}_S^{(0)}$ is primitive, there exists some $t > 0$ such that all the elements of its t -fold product are positive. The t -fold product of $\mathbf{P}_S^{(0)}$ coincides with the t -fold composition of the FPOs of S according to the expression

$$\mathcal{P}_S^t \sim [\mathbf{P}_S^{(0)}]^t. \quad (2.214)$$

Equation (2.210) then implies that

$$\mathcal{P}_{S^t} \sim [\mathbf{P}_S^{(0)}]^t. \quad (2.215)$$

Proposition 2.6.2 states that S^t is a $\mathcal{W}(\mathcal{Q})$ -semi-Markov map. Hence, its FPO $\mathcal{P}_{S^t}$ may be represented as an $M \times M$ matrix (i.e., the order-zero FPM $\mathbf{P}_{S^t}^{(0)}$) if attention is restricted to densities that are PWC over its Markov partition $\mathcal{Q}$. It follows that

$$\mathbf{P}_{S^t}^{(0)} = [\mathbf{P}_S^{(0)}]^t. \quad (2.216)$$

Hence, all elements of $\mathbf{P}_{S^t}^{(0)}$ are strictly positive.

Now, consider any $m \in \{1, 2, \dots, M\}$. Equations (2.122) and the positivity of the elements in column m of $\mathbf{P}_{S^t}^{(0)}$ imply that S^t must possess at least M branch functions $S_{k,m}^t$, where $k = 1, 2, \dots, M$, each possessing a domain W_k of non-zero Borel measure that is a subinterval of Q_m and which satisfies

$$S_{k,m}^t(W_k) = Q_k, \quad \forall k = 1, 2, \dots, M. \quad (2.217)$$

This implies that $S^t(Q_m) = \Omega \setminus B$, where B is a set of zero Borel measure that contains at most the finite number of partition points $\bar{P}(\mathcal{Q})$. This argument is valid for any $m \in \{1, 2, \dots, M\}$.

Proof of proposition 2.6.9

Proposition 2.6.8 states that the order-zero FPM has exactly one eigenvalue λ_1 equal to unity (stated equivalently, it is a simple root of the characteristic polynomial of the matrix, which implies that the eigenspace associated with it is one-dimensional). Let $\mathbf{w}_1$ denote the corresponding right eigenvector, which then satisfies

$$\mathbf{w}_1 = \mathbf{P}_S^{(0)} \mathbf{w}_1. \quad (2.218)$$

Comparing the latter equation with (2.114) then implies that $\mathbf{w}_1$ coincides with the unique density function that is PWC over $\mathcal{Q}$ and S -invariant. As the invariant density f_S^* of S is unique and PWC over $\mathcal{Q}$ (a consequence of the Folklore theorem), it then follows that

$$\mathbf{w}_1 \equiv \mathbf{f}_S^*, \quad (2.219)$$

where $\mathbf{f}_S^*$ is the vector representation of the invariant density f_S^* . The Perron-Frobenius theorem guarantees that the elements of $\mathbf{f}_S^*$ are strictly positive, which agrees with the Folklore theorem's requirement that

$$K^{-1} \leq f_S^* \leq K \quad (2.220)$$

almost everywhere for some $K > 0$.

2.A.12 Proofs of propositions regarding the order-one FPM

Proof of proposition 2.6.10

The determinant of a block triangular matrix is the product of the determinants of its diagonal blocks [45]. Equation (2.117) then implies that the characteristic equation $\det(\mathbf{P}_S^{(1)} - \lambda \mathbf{I})$ of the order-one FPM satisfies

$$\det(\mathbf{P}_S^{(1)} - \lambda \mathbf{I}) = \det(\mathbf{P}_S^{(0)} - \lambda \mathbf{I}) \det(\mathbf{Q}_S - \lambda \mathbf{I}). \quad (2.221)$$

The proposition then follows from the observation that the eigenspectrum of a matrix is comprised of the roots of its characteristic equation.

Proof of proposition 2.6.11

Assume the condition of (2.128) holds. Proposition 2.6.5 states that the order-zero FPM $\mathbf{P}_S^{(0)} = [p_{k,m}]$ is similar to a column stochastic matrix $\tilde{\mathbf{P}} = [\tilde{p}_{k,m}]$ such that

$$\mathbf{P}_S^{(0)} = \mathbf{D} \tilde{\mathbf{P}} \mathbf{D}^{-1}, \quad (2.222)$$

where

$$\mathbf{D} = \text{diag}\{\mu(Q_1)^{-1}, \mu(Q_2)^{-1}, \dots, \mu(Q_M)^{-1}\} \quad (2.223)$$

and $\mu(Q_m) > 0$ for all m . Let $d_k \triangleq d_{k,k}$, where $\mathbf{D} \triangleq [d_{k,m}]$. It follows that

$$p_{k,m} = \frac{d_k}{d_m} \tilde{p}_{k,m}, \quad k, m = 1, 2, \dots, M. \quad (2.224)$$

Equations (2.118) and (2.122) imply that the elements of $\mathbf{Q}_S = [q_{k,m}]$ satisfy

$$q_{k,m} = s_{k,m} p_{k,m}^2, \quad (2.225)$$

where $s_{k,m} \in \{-1, 1\}$. Substitution of (2.224) into the latter equation yields

$$q_{k,m} = s_{k,m} \frac{d_k^2}{d_m^2} \tilde{p}_{k,m}^2. \quad (2.226)$$

Now, consider the matrix $\tilde{\mathbf{Q}}_S = [\tilde{q}_{k,m}]$ defined by the similarity transformation

$$\tilde{\mathbf{Q}}_S = \mathbf{D}^{-2} \mathbf{Q}_S \mathbf{D}^2. \quad (2.227)$$

It follows that

$$\tilde{q}_{k,m} = s_{k,m} \tilde{p}_{k,m}^2. \quad (2.228)$$

The Gershgorin circle theorem [45] states that each eigenvalue $\tilde{\lambda}$ of $\tilde{\mathbf{Q}}_S$ must lie within at least one of the Gershgorin discs in the complex plane, defined by

$$|\tilde{\lambda} - \tilde{q}_{m,m}| \leq r_m, \quad m = 1, 2, \dots, M \quad (2.229)$$

and

$$r_m = \sum_{k=1, k \neq m}^M |\tilde{q}_{k,m}|. \quad (2.230)$$

Equations (2.228) and (2.224) imply that

$$\tilde{q}_{m,m} = s_{m,m} p_{m,m}^2, \quad m = 1, 2, \dots, M. \quad (2.231)$$

Equation (2.128) then implies that $0 < |\tilde{q}_{m,m}| < 1$. Hence, the centre of each Gershgorin disc defined by (2.229) lies strictly inside the unit circle in the complex plane and on the real line. Equations (2.228) to (2.230) then imply that the shortest distance between the perimeter of each disc and the unit circle in the complex plane — either the point $(0, 1)$ or $(-1, 0)$ — is given by

$$h_m = 1 - \sum_{k=1}^M \tilde{p}_{k,m}^2, \quad m = 1, 2, \dots, M. \quad (2.232)$$

As $\tilde{\mathbf{P}}$ is column stochastic, $0 \leq \tilde{p}_{k,m} \leq 1$ and

$$\sum_{k=1}^M \tilde{p}_{k,m}^2 \leq \sum_{k=1}^M \tilde{p}_{k,m} = 1. \quad (2.233)$$

Now, (2.224) implies $\tilde{p}_{m,m} = p_{m,m}$ and (2.128) states that $0 < p_{m,m} < 1$. Hence, $\tilde{p}_{m,m}^2 < \tilde{p}_{m,m}$, which implies that

$$\sum_{k=1}^M \tilde{p}_{k,m}^2 < 1. \quad (2.234)$$

Substitution of this expression into (2.232) yields

$$h_m > 0. \quad (2.235)$$

Hence, each of the Gershgorin discs lies *strictly inside* the unit circle in the complex plane, and each of the eigenvalues $\tilde{\lambda}$ of $\tilde{\mathbf{Q}}_S$ satisfies $|\tilde{\lambda}| < 1$. The spectral radius $\tilde{\rho}$ of $\tilde{\mathbf{Q}}_S$ therefore satisfies $\tilde{\rho} < 1$.

As $\mathbf{Q}_S$ and $\tilde{\mathbf{Q}}_S$ are defined as similar matrices in (2.227), their spectral radii are equal [45]. Hence, the spectral radius of $\mathbf{Q}_S$ satisfies $\rho < 1$. This implies (2.129).

2.A.13 Proof of Proposition 2.6.12

The ACF of an ergodic map S is given by (2.67):

$$\mathcal{R}_S(\tau) = \int_{\Omega} x(\mathcal{P}_S^{|\tau|} g)(x) d\mu(x), \quad \tau = 0, \pm 1, \pm 2, \dots, \quad (2.236)$$

where $\mathcal{P}_S$ denotes the map's FPO, $g(x) \triangleq x f_S^*(x)$ and f_S^* denotes the map's invariant density. The right-hand side of (2.236) contains the multiplicative term $\mathcal{P}_S^{|\tau|}(g)$, which may be represented using a matrix expression. As $f_S^* \in \mathcal{D}_Q^{(0)}$, the function $g(x) = x f_S^*(x)$ is a member of $\mathcal{D}_Q^{(1)}$. The order-one FPM $\mathbf{P}_S^{(1)}$ characterises the evolution of densities that belong to $\mathcal{D}_Q^{(1)}$. Hence, the matrix representation of $\mathcal{P}_S^{|\tau|}(g)$ is given by

$$\mathcal{P}_S^{|\tau|}(g) \sim (\mathbf{P}_S^{(1)})^{|\tau|} \mathbf{g}, \quad (2.237)$$

where $(\mathbf{P}_S^{(1)})^0 \triangleq \mathbf{I}$ and

$$\mathbf{g} \triangleq [(\mathbf{f}_S^*)^T, \mathbf{0}_{1 \times M}]^T. \quad (2.238)$$

As $\mathcal{P}_S^{|\tau|}(g)$ is a member of $\mathcal{D}_Q^{(1)}$, it may be expressed as

$$(\mathcal{P}_S^{|\tau|} g)(x) \triangleq \sum_{m=1}^M [\tilde{g}_{1,m}(\tau)x + \tilde{g}_{0,m}(\tau)] \chi_{Q_m}(x). \quad (2.239)$$

The integrand $f(x, \tau) \triangleq x(\mathcal{P}_S^{|\tau|} g)(x)$ of (2.236) is a member of $\mathcal{D}_Q^{(2)}$. As the order-zero coefficient of each polynomial is zero, the integrand may be expressed as

$$f(x, \tau) \triangleq \sum_{m=1}^M [\tilde{g}_{1,m}(\tau)x^2 + \tilde{g}_{0,m}(\tau)x] \chi_{Q_m}(x). \quad (2.240)$$

As f is Riemann integrable, it follows that

$$\begin{aligned}
 \int_{\Omega} f(x, \tau) d\mu(x) &= \int_0^1 \sum_{m=1}^M [\tilde{g}_{1,m}(\tau)x^2 + \tilde{g}_{0,m}(\tau)x] \chi_{Q_m}(x) dx \\
 &= \sum_{m=1}^M \int_{q_m}^{q_{m+1}} [\tilde{g}_{1,m}(\tau)x^2 + \tilde{g}_{0,m}(\tau)x] dx \\
 &= \sum_{m=1}^M \left[\frac{q_{m+1}^3 - q_m^3}{3} \right] \tilde{g}_{1,m}(\tau) + \sum_{m=1}^M \left[\frac{q_{m+1}^2 - q_m^2}{2} \right] \tilde{g}_{0,m}(\tau).
 \end{aligned} \tag{2.241}$$

The integral of the preceding equation may then be expressed as

$$\mathcal{R}_S(\tau) = \int_{\Omega} f(x, \tau) d\mu(x) = \mathbf{h}^T (\mathbf{P}_S^{(1)})^{|\tau|} \mathbf{g}, \quad \tau = 0, \pm 1, \pm 2, \dots, \tag{2.242}$$

where $\mathbf{h} \triangleq [\mathbf{h}_3, \mathbf{h}_2]^T$. The elements of

$$\mathbf{h}_n \triangleq [h_{n,1}, h_{n,2}, \dots, h_{n,M}], \quad n = 2, 3, \tag{2.243}$$

are defined by

$$h_{n,m} \triangleq \frac{q_{m+1}^n - q_m^n}{n}, \quad m = 1, 2, \dots, M, \tag{2.244}$$

where $\bar{\mathcal{P}}(\mathcal{Q}) = \{q_1, q_2, \dots, q_{M+1}\}$ are the partition points of the partition $\mathcal{Q}$. Observe that this equation expresses the ACF of an ergodic $\mathcal{R}(\mathcal{Q})$ -semi-Markov map as a function of its order-one FPM and two vectors that depend on the PWC invariant density of the map and the partition points of $\mathcal{Q}$.

Now, assume that the order-one FPM $\mathbf{P}_S^{(1)}$ is diagonalisable. Using the notation introduced in the statement of Proposition 2.6.12 for the eigenvalues and eigenvectors of the order-one FPM, (2.242) may then be manipulated to obtain

$$\mathcal{R}_S(\tau) = \sum_{m=1}^{2M} c_m \lambda_m^{|\tau|}, \quad \tau = 0, \pm 1, \pm 2, \dots, \tag{2.245}$$

where

$$c_m = \mathbf{h}^T \tilde{\mathbf{w}}_m \tilde{\mathbf{v}}_m^H \mathbf{g}, \quad m = 1, 2, \dots, 2M \tag{2.246}$$

and $\lambda_m^0 \triangleq 1$ if $\lambda_m = 0$.

2.A.14 Proof of Proposition 2.6.14

Let the conditions of the proposition statement be satisfied. The impulsive component $\mathcal{S}_{S,I}$ of the PSD coincides with component $\mathcal{R}_{S,I}$ of the ACF, which is given by (2.142), as a discrete-time Fourier transform pair. Equations (2.142), (2.87) and (2.89) imply

$$\mathcal{S}_{S,I}(\omega) = 2\pi c_1 \delta(\omega), \tag{2.247}$$

where c_1 is defined in (2.135).

The continuous component $\mathcal{S}_{S,C}$ of the PSD was defined in (2.85) as the discrete-time Fourier transform of $\mathcal{R}_{S,C}$;

$$\mathcal{S}_{S,C}(\omega) = \sum_{\tau=-\infty}^{\infty} \mathcal{R}_{S,C}(\tau) e^{-i\omega\tau}, \quad \omega \in [-\pi, \pi). \quad (2.248)$$

To define the poles and zeros of the PSD, as required in a later section, this proof makes use of the bilateral z -transform $\Psi_{S,C}$ of $\mathcal{R}_{S,C}$, which is defined as

$$\Psi_{S,C}(z) \triangleq \sum_{\tau=-\infty}^{\infty} \mathcal{R}_{S,C}(\tau) z^{-\tau}, \quad (2.249)$$

The continuous component $\mathcal{S}_{S,C}$ of the PSD is related to $\Psi_{S,C}$ according to the expression

$$\mathcal{S}_{S,C}(\omega) = \Psi_{S,C}(z)|_{z=e^{i\omega}}, \quad \omega \in [-\pi, \pi). \quad (2.250)$$

The z -transform $\Psi_{S,C}$ of $\mathcal{R}_{S,C}$, where the latter is given by (2.141) as

$$\mathcal{R}_{S,C}(\tau) = \sum_{m=2}^{2M} c_m \lambda_m^{|\tau|}, \quad \tau = 0, \pm 1, \pm 2, \dots, \quad (2.251)$$

is derived next. It was shown in Remark 2.6.7 that $|\lambda_m| < \lambda_1 = 1$ for each $m = 2, 3, \dots, 2M$. Hence, (2.251) may be decomposed as

$$\mathcal{R}_{S,C}(\tau) = \underbrace{\sum_{\{m>1: \lambda_m=0\}} c_m \delta[\tau]}_{\mathcal{R}_{S,C}^{(1)}(\tau)} + \underbrace{\sum_{\{m>1: 0<|\lambda_m|<1\}} c_m \lambda_m^{|\tau|}}_{\mathcal{R}_{S,C}^{(2)}(\tau)}, \quad (2.252)$$

where $\delta[\cdot]$ is the Kronecker delta function (in this work, the Kronecker delta function is distinguished from the Dirac delta function by the use of square brackets). The z -transform $\Psi_{S,C}(z)$ of this function is given by

$$\Psi_{S,C}(z) = \underbrace{\sum_{\tau=-\infty}^{\infty} \mathcal{R}_{S,C}^{(1)}(\tau) z^{-\tau}}_{\Psi_{S,C}^{(1)}(z)} + \underbrace{\sum_{\tau=-\infty}^{\infty} \mathcal{R}_{S,C}^{(2)}(\tau) z^{-\tau}}_{\Psi_{S,C}^{(2)}(z)}. \quad (2.253)$$

The function $\Psi_{S,C}^{(1)}(z)$ is derived as

$$\Psi_{S,C}^{(1)}(z) = \sum_{\{m>1: \lambda_m=0\}} c_m, \quad z \in \mathbb{C}. \quad (2.254)$$

The function $\Psi_{S,C}^{(2)}(z)$ is decomposed as

$$\Psi_{S,C}^{(2)}(z) = \underbrace{\sum_{\tau=0}^{\infty} \mathcal{R}_{S,C}^{(2)}(\tau) z^{-\tau}}_{\Psi_{S,C}^{(2,1)}(z)} + \underbrace{\sum_{\tau=-\infty}^{-1} \mathcal{R}_{S,C}^{(2)}(\tau) z^{-\tau}}_{\Psi_{S,C}^{(2,2)}(z)}. \quad (2.255)$$

The function $\Psi_{S,C}^{(2,1)}(z)$ is derived as

$$\begin{aligned}
\Psi_{S,C}^{(2,1)}(z) &= \sum_{\tau=0}^{\infty} \mathcal{R}_{S,C}^{(2)}(\tau) z^{-\tau} \\
&= \sum_{\tau=0}^{\infty} \left\{ \sum_{\{m>1: 0<|\lambda_m|<1\}} c_m \lambda_m^\tau \right\} z^{-\tau} \\
&= \sum_{\{m>1: 0<|\lambda_m|<1\}} c_m \sum_{\tau=0}^{\infty} \lambda_m^\tau z^{-\tau} \\
&= \sum_{\{m>1: 0<|\lambda_m|<1\}} c_m \left[\frac{1}{1 - \lambda_m z^{-1}} \right], \quad |z| > \bar{L},
\end{aligned} \tag{2.256}$$

where

$$\bar{L} \triangleq \max\{|\lambda_m| : 0 < |\lambda_m| < 1, m > 1\}. \tag{2.257}$$

Similarly, the function $\Psi_{S,C}^{(2,2)}(z)$ is derived as

$$\begin{aligned}
\Psi_{S,C}^{(2,2)}(z) &= \sum_{\tau=-\infty}^{-1} \phi_{S,C}^{(2)}(\tau) z^{-\tau} \\
&= \sum_{\{m>1: 0<|\lambda_m|<1\}} c_m \left[\frac{\lambda_m z}{1 - \lambda_m z} \right], \quad |z| < \bar{L}^{-1}.
\end{aligned} \tag{2.258}$$

Substituting (2.256) and (2.258) into (2.255) and manipulating the resulting expression yields

$$\Psi_{S,C}^{(2)}(z) = \sum_{\{m>1: 0<|\lambda_m|<1\}} c_m \left[\frac{1 - \lambda_m^2}{(z - \lambda_m)(z^{-1} - \lambda_m)} \right], \quad \bar{L} < |z| < \bar{L}^{-1}. \tag{2.259}$$

Substituting (2.254) and (2.259) into (2.253) yields

$$\Psi_{S,C}(z) = \sum_{\{m>1: \lambda_m=0\}} c_m + \sum_{\{m>1: 0<|\lambda_m|<1\}} c_m \left[\frac{1 - \lambda_m^2}{(z - \lambda_m)(z^{-1} - \lambda_m)} \right]. \tag{2.260}$$

Now, let $\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_{M_P}$ denote the *unique* eigenvalues amongst $\lambda_2, \lambda_3, \dots, \lambda_{2M}$ that satisfy $0 < |\lambda_m| < 1$. Equation (2.260) is then rewritten as

$$\Psi_{S,C}(z) = C + \sum_{m=1}^{M_P} \frac{\tilde{c}_m}{(z - \tilde{\lambda}_m)(z^{-1} - \tilde{\lambda}_m)}, \tag{2.261}$$

where

$$C = \sum_{\{k>1: \lambda_k=0\}} c_k \tag{2.262}$$

and

$$\tilde{c}_m \triangleq \sum_{\{k>1: \lambda_k=\tilde{\lambda}_m\}} c_k (1 - \lambda_k^2). \tag{2.263}$$

The region of convergence of (2.261) is defined by $L < |z| < L^{-1}$ and

$$L \triangleq \max\{0, |\tilde{\lambda}_1|, |\tilde{\lambda}_2|, \dots, |\tilde{\lambda}_{M_P}|\}. \tag{2.264}$$

The proof of the proposition is then completed by substituting $z = e^{i\omega}$ in (2.261).

2.B Derivation of a Simplified Expression for the Time ACF

Invoke the PDSA framework of Definition 2.4.4, and let S be ergodic and possess a unique invariant probability measure P_S^* that is absolutely continuous with respect to the Borel measure μ . By assuming ergodicity, the map S is already assumed to be non-singular with respect to μ . Furthermore, since

$$P_S^*(S^{-1}(B)) = P_S^*(B), \quad B \in \mathcal{B}, \quad (2.265)$$

the map S is non-singular with respect to P_S^* . A simplified expression for the time ACF $\bar{\mathcal{R}}_S(\tau, x_1)$ of the system's trajectories, as defined by (2.66), is derived below.

Let $I : \Omega \rightarrow \Omega$ be defined by $I(x) \triangleq x$. Furthermore, let $h_\tau : \Omega \rightarrow \Omega$ be defined by

$$h_\tau(x) \triangleq xS^{|\tau|}(x), \quad \tau = 0, \pm 1, \pm 2, \dots \quad (2.266)$$

It is assumed throughout this work that the map S permits h_τ to be $\mathcal{B}$ -measurable for all finite τ . The following proposition is proved.

Proposition 2.B.1. Let $(\Omega, \mathcal{B}, P_S^*)$ denote a probability space. Function I is integrable and h_τ is integrable for each $\tau = 0, \pm 1, \pm 2, \dots$

Proof. The following lemma is used [7, Remark 2.2.3].

Lemma 2.B.1. Let $(\Omega, \mathcal{B}, \nu)$ denote a measure space. If $f, g : \Omega \rightarrow \mathbb{R}$ are measurable, g is integrable, and $|f| \leq g$, then f is integrable.

Since $I^{-1}(B) = B$ for every $B \in \mathcal{B}$, the function I is $\mathcal{B}$ -measurable. Let $\chi_B : \Omega \rightarrow \Omega$ be defined by (2.59). As χ_Ω is both measurable and integrable and $|I| \leq \chi_\Omega$, Lemma 2.B.1 implies that I is integrable.

Now, consider the function h_τ . This function is assumed to be measurable, as stated at the beginning of this proof. As $|h_\tau| \leq \chi_\Omega$, Lemma 2.B.1 then implies that h_τ is integrable for each $\tau = 0, \pm 1, \pm 2, \dots$ $\square$

Let $(\Omega, \mathcal{B}, P_S^*)$ denote a probability space. As h_τ is integrable by Proposition 2.B.1, the conditions of the extension of the Birkhoff individual ergodic theorem (i.e., Theorem 2.4.3 with $h \triangleq h_\tau$) are satisfied. Equation (2.57) then implies that the time

ACF of the trajectory starting from the initial state x_1 is given by

$$\begin{aligned}\bar{\mathcal{R}}_S(\tau, x_1) &\triangleq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} S^k(x_1) S^{|\tau|+k}(x_1) \\ &= \int_{\Omega} x S^{|\tau|}(x) dP_S^*(x), \quad \tau = 0, \pm 1, \pm 2, \dots,\end{aligned}\tag{2.267}$$

for almost every $x_1 \in \Omega$. The dependency on the initial state x_1 is thus removed from $\bar{\mathcal{R}}_S(\tau, x_1)$, such that $\bar{\mathcal{R}}_S(\tau, x_1) \equiv \bar{\mathcal{R}}_S(\tau)$.

Since I is integrable, it is a member of $L^1(\Omega, \mathcal{B}, P_S^*)$ by definition. The inequality $|S^{|\tau|}| \leq 1$ implies that $S^{|\tau|}$ is bounded and therefore a member of $L^\infty(\Omega, \mathcal{B}, P_S^*)$ for every $\tau \in \mathbb{Z}$. The right-hand side of (2.267) may then be expressed as

$$\bar{\mathcal{R}}_S(\tau) = \langle I, S^{|\tau|} \rangle, \quad \tau = 0, \pm 1, \pm 2, \dots\tag{2.268}$$

Let $\tau = 0$. Equation (2.268) then simplifies to

$$\bar{\mathcal{R}}_S(0) = \langle I, I \rangle.\tag{2.269}$$

Suppose instead that $|\tau| \geq 1$. As S is non-singular with respect to P_S^* , the FPO and Koopman operators corresponding to this map may be defined over the measure space $(\Omega, \mathcal{B}, P_S^*)$. Let $\mathcal{P}_{S, P_S^*}$ and $\mathcal{U}_{S, P_S^*}$ denote the FPO and Koopman operator defined in this manner. Using (2.40), the defining equation of the Koopman operator, (2.268) may be rewritten as

$$\bar{\mathcal{R}}_S(\tau) = \langle I, S^{|\tau|-1} \circ S \rangle = \langle I, \mathcal{U}_{S, P_S^*}(S^{|\tau|-1}) \rangle, \quad |\tau| \geq 1.\tag{2.270}$$

Property (K2) of the Koopman operator states that it is adjoint to the FPO. Equation (2.42), the defining equation for the adjoint operator, then implies that

$$\bar{\mathcal{R}}_S(\tau) = \langle \mathcal{P}_{S, P_S^*}(I), S^{|\tau|-1} \rangle, \quad |\tau| \geq 1.\tag{2.271}$$

Now, suppose that $|\tau| \geq 2$. Using (2.40), (2.271) may be rewritten as

$$\bar{\mathcal{R}}_S(\tau) = \langle \mathcal{P}_{S, P_S^*}(I), S^{|\tau|-2} \circ S \rangle = \langle \mathcal{P}_{S, P_S^*}(I), \mathcal{U}_{S, P_S^*}(S^{|\tau|-2}) \rangle, \quad |\tau| \geq 2.\tag{2.272}$$

Equation (2.42) then implies that

$$\bar{\mathcal{R}}_S(\tau) = \langle \mathcal{P}_{S, P_S^*}^2(I), S^{|\tau|-2} \rangle, \quad |\tau| \geq 2.\tag{2.273}$$

Continuing in this fashion yields the expression

$$\bar{\mathcal{R}}_S(\tau) = \langle \mathcal{P}_{S, P_S^*}^m(I), S^{|\tau|-m} \rangle, \quad |\tau| \geq m, \quad m = 0, 1, \dots\tag{2.274}$$

Selecting $m = |\tau|$ in the expression above yields

$$\bar{\mathcal{R}}_S(\tau) = \langle \mathcal{P}_{S, P_S^*}^{|\tau|}(I), I \rangle, \quad \tau = 0, \pm 1, \pm 2, \dots,\tag{2.275}$$

where $\mathcal{P}_S^0(I) \triangleq I$. This expression implies that

$$\bar{\mathcal{R}}_S(\tau) = \int_{\Omega} x(\mathcal{P}_{S, P_S^*}^{|\tau|} I)(x) dP_S^*(x), \quad \tau = 0, \pm 1, \pm 2, \dots \quad (2.276)$$

As P_S^* is absolutely continuous with respect to the Borel measure μ , (2.276) may be rewritten as

$$\bar{\mathcal{R}}_S(\tau) = \int_{\Omega} x(\mathcal{P}_{S, P_S^*}^{|\tau|} I)(x) f_S^*(x) d\mu(x), \quad \tau = 0, \pm 1, \pm 2, \dots \quad (2.277)$$

where f_S^* is the unique invariant density of the map S .

As S is non-singular with respect to μ , the FPO $\mathcal{P}_{S, \mu}$ corresponding to S may be defined on the measure space $(\Omega, \mathcal{B}, \mu)$. Since P_S^* is absolutely continuous with respect to μ , the conditions of Proposition (2.4.3), which relates the FPOs corresponding to the same map but defined on two distinct measure spaces, are satisfied. This proposition yields

$$f_S^* \mathcal{P}_{S, P_S^*}^{|\tau|}(I) = \mathcal{P}_{S, \mu}^{|\tau|}(g) \text{ a.e., } \tau = 0, \pm 1, \pm 2, \dots, \quad (2.278)$$

where

$$g(x) \triangleq f_S^*(x) I(x) = x f_S^*(x). \quad (2.279)$$

Substitution of (2.278) into (2.277) yields

$$\bar{\mathcal{R}}_S(\tau) = \int_{\Omega} x(\mathcal{P}_{S, \mu}^{|\tau|} g)(x) d\mu(x), \quad \tau = 0, \pm 1, \pm 2, \dots \quad (2.280)$$

Chapter 3

LITERATURE REVIEW

3.1 Introduction

The IFPP is a probabilistic modelling problem that requires the design of a discrete time, one-dimensional dynamical system that is (i) ergodic and (ii) generates a zero-input time response that satisfies prescribed statistical metrics as specifications [1–3]. Several formulations of this problem have been actively studied and its importance has been demonstrated in applications such as the motion of fluids [4], neural processes pertaining to the olfactory system [5] and more recently, seismic signals measured in the Philippines [6]. Solutions to the IFPP have also been applied to design more robust clogged bits for rock drills [7], improved radar test waveforms [8] and highly efficient random number and signal generators [9–13], to mention a few.

Publications that address the IFPP, including those that stemmed from the work presented in this thesis [1, 2, 13–17], do not refer to an underlying dynamical system. Instead, these publications formulate the IFPP as a problem of designing an *ergodic map* with prescribed statistical metrics. For consistency with these works, and in the interest of brevity, the work presented in this thesis also attributes system-related concepts to the map; refer to Remark 2.3.1. However, in all cases, it is implicitly assumed that the map is configured as the system of Fig. 2.1.

The original formulation of the IFPP, labelled IFPP-I, prescribes the invariant density of the map [18–25]. Three solution strategies have emerged for this problem. These are the topological conjugation [18], direct [19] and matrix solution strategies [20–25].

Methods that utilise these strategies have matured over the years and are well understood; some recent examples are described in [26–31]. A recent survey article by the author of this thesis addresses IFPP-I and its solutions [1].

By the 1990s, it became evident that the invariant density of the map is a relatively weak constraint in the context of the IFPP, and that considerable freedom remained to be exploited [32]. From the viewpoint of statistical metrics, specifying the invariant density neither uniquely determines the map’s ACF nor its PSD. These observations brought about *generalised* formulations of the IFPP that specify additional statistical metrics as specifications for the map [1].

The second formulation of the IFPP, labelled IFPP-II, requires the design of an ergodic map to simultaneously satisfy a prescribed invariant density and PSD [1, 3].¹ Progress has been made towards solving this problem by generalising the existing solution strategies for IFPP-I to permit prescription of the ACF. This approach yielded an analytical method [33] and a numerical method [32] for map design. However, these methods are severely restricted with respect to the power spectra realisable or are computationally impractical. After these methods appeared in the literature, attention shifted to the design of systems with more general structures that afford more freedom to meet the specifications; see, for example, [34].

The third problem formulation, labelled IFPP-III, is a more recent problem that requires the reconstruction of an unknown ergodic map from density estimates of state ensembles that evolve under iteration of the map [1, 2, 35–39]. Solutions to this problem are relevant in modelling applications where individual system trajectories cannot be isolated and tracked from one time instant to the next [40–44]. A matrix solution to this problem was proposed and shown to consistently reconstruct members from a subclass of the semi-Markov maps [35, 36]. However, the solution is unable to determine the monotonicity of the unknown map’s branch functions and fails to consistently reconstruct the map if it possesses any decreasing branch functions. This restricts the practical applicability of this solution.

This chapter presents a review of the existing strategies and methods that have been developed to solve the IFPP. Section 3.2 lists the novel contributions presented in the chapter. Section 3.3 formally defines the three formulations of the problem. IFPP-I is then addressed in section 3.4. The three strategies for solving this problem are developed, and existing methods from the literature are classified according to the solution strategy used. This permits their comparison and the identification of gaps

¹The Wiener-Khinchin theorem implies that the ACF may be equivalently prescribed.

in existing work. The characteristics of these methods are illustrated by constructing examples of maps with prescribed invariant densities.

The matrix solution strategy for IFPP-I, which constructs and configures a semi-Markov map as a candidate system, is developed from first principles. It is found that existing methods that implement this strategy do not address the problem of selecting the map's PWC invariant density to approximate an arbitrary invariant density that may be prescribed. To bridge this gap, the chapter develops two novel methods for selecting the invariant density to permit approximation of smooth invariant density functions that satisfy a mild condition. The distinct design requirements that these methods impose become relevant in chapter 4.

Section 3.5 reviews existing methods that have been developed to solve IFPP-II. The analytical and numerical methods from the literature are presented in depth. Examples are used to illustrate these methods and their characteristics. The shortcomings of the methods are discussed, and the cause of these shortcomings is identified.

The investigation into IFPP-II reveals a gap in the literature for a method that uses the matrix strategy to construct a semi-Markov map. The generalisation of a recent matrix method for solving IFPP-I to permit prescription of the power spectrum poles is briefly investigated. An example is presented to illustrate the approach, and several shortcomings are identified.

Section 3.6 presents a review of recent methods developed to solve IFPP-III. The solution of the original formulation of this problem, which requires the reconstruction of a semi-Markov map, is covered in depth. The shortcomings of this method are identified and attributed. Several generalisations of IFPP-III and methods developed to solve these problems are briefly investigated.

A summary of the chapter and several conclusions are provided in section 3.7, together with a list of gaps identified in the IFPP literature.

The material presented in this chapter appears, in part, in [1].

3.2 Novel Contributions of this Chapter

The novel contributions of this chapter are outlined in Tables 3.2.1 and 3.2.2.

Table 3.2.1: Novel contributions presented in chapter 3 (part 1 of 2).

Reference	Publications by author	Description
Sec. 3.3	[1]	The identification of three distinct formulations of the IFPP in the literature, and the classification and comparison of existing methods from the literature on the basis of these problem formulations.
Sec. 3.4	[1]	The identification of three distinct solution strategies for the IFPP, and the classification, review and comparison of existing methods that use each strategy.
Sec. 3.7.4, Tab. 3.7.1	[1]	The identification of numerous gaps in the literature for each solution strategy of IFPP-I, IFPP-II and IFPP-III, as permitted by the above classification of the methods.
Sec. 3.4, Ex. 3.4.1, 3.4.2, 3.4.3	[1]	A collection of examples that demonstrate each solution strategy for IFPP-I by designing maps with the same invariant density, thereby permitting a comparison of the methods' characteristics.
Sec. 3.4.3	–	The derivation of the matrix solution strategy for IFPP-I from first principles, thereby revealing implicit design choices and assumptions of existing methods from the literature. The presentation of the information in this chapter unifies theory and results scattered throughout the literature.
Sec. 3.4.3, p. 121	–	The generalisation of the matrix solution strategy for IFPP-I by addressing the <i>approximation</i> of an arbitrary prescribed invariant density, instead of only addressing the prescription of a PWC invariant density.
Sec. 3.4.3, p. 121, Tab. 3.4.1, Prop. 3.4.1, 3.4.2	–	The development of two methods for approximating an arbitrary prescribed invariant density, the derivation of a bound on their error, a proof that the error converges to zero as the number of partition intervals are increased for one of the methods, and the identification of requirements that these methods impose on the design of the order-zero FPM in the second solution step.

Table 3.2.2: Novel contributions presented in chapter 3 (part 2 of 2).

Section	Publications by author	Description
Sec. 3.4.3, Prop. 3.4.4	–	The formal derivation of conditions for the ergodicity of a semi-Markov map constructed using the matrix solution strategy for IFPP-I.
Sec. 3.5, 3.7	[3]	The identification of the shortcomings of existing methods aimed at solving IFPP-II, and the identification of their cause as the complexity of the expressions for relating the metrics to the map's structural parameters.
Sec. 3.5.3	–	The partial generalisation of the matrix solution for IFPP-I, as presented in [22–25], to permit the prescription of power spectrum poles, and the identification the new solution's shortcomings.
Sec. 3.5.3, Rem. 3.5.16	–	The interpretation of the power spectra reported in [22–25] using the theory developed in chapter 2.
Sec. 3.6.1, Prop. 3.6.2	[2]	The identification, formal proof and attribution of a first deficiency of the original solution for IFPP-III, which concerns the solution's inability to identify the monotonicity of the unknown map's branches.
Sec. 3.6.1, Prop. 3.6.3	[2]	The presentation of a formal proof that confirms the ability of the original solution for IFPP-III to realise an invariant density that is consistent with that of the unknown map.
Sec. 3.6.1, Prop. 3.6.4	[2]	The identification, formal proof and attribution of a second deficiency of the original solution for IFPP-III, which concerns the solution's inability to realise a PSD consistent with that of the unknown map.
Sec. 3.6.2, Rem. 3.6.5	–	The identification of map continuity as a critical requirement for the extended solution of IFPP-III to accurately identify branch monotonicity.

3.3 Formulations of the IFPP

Three formulations of the IFPP, as identified in the literature, are defined below. Each formulation prescribes a distinct collection of statistical metrics for the map.

3.3.1 The original problem formulation: IFPP-I

The original problem formulation (IFPP-I) was first defined and addressed in [18]. Let $f_P \in \mathcal{D}(\Omega, \mathcal{B}, \mu)$ denote a prescribed invariant PDF that satisfies $f_P > 0$ almost everywhere. IFPP-I requires the design of a map $S : \Omega \rightarrow \Omega$ that is ergodic and possesses f_P as its invariant density [1].

More formally, IFPP-I requires the design of a map S that is non-singular with respect to the Borel measure μ [1]. The map is required to be ergodic and preserve a probability measure P_S^* that is absolutely continuous with respect to μ . This measure is to coincide with the prescribed invariant density according to the expression

$$P_S^*(B) = \int_B f_P(x) d\mu(x), \quad B \in \mathcal{B}. \quad (3.1)$$

Observe that the ergodicity of S then implies that P_S^* is unique (Proposition 2.4.4), and that the map possesses the unique invariant density $f_S^* = f_P$ (Proposition 2.4.5).

3.3.2 The second problem formulation: IFPP-II

IFPP-II is a generalisation of IFPP-I that appears in [32, 33]. Let $f_P \in \mathcal{D}(\Omega, \mathcal{B}, \mu)$ denote a prescribed invariant PDF that satisfies $f_P > 0$ almost everywhere. IFPP-II requires the design of a map $S : \Omega \rightarrow \Omega$ that is ergodic and possesses f_P as its invariant density, in an identical fashion as required by IFPP-I [1, 3]. Simultaneously, IFPP-II requires that either the ACF $\mathcal{R}_S$ or PSD $\mathcal{S}_S$ of the map S , as defined in (2.72) and (2.83) to (2.89), equals a prescribed ACF $\mathcal{R}_P$ or prescribed PSD $\mathcal{S}_P$; i.e.,

$$\mathcal{R}_S = \mathcal{R}_P \quad \text{or} \quad \mathcal{S}_S = \mathcal{S}_P. \quad (3.2)$$

Prescription of either the ACF or the PSD is equivalent to prescribing its counterpart. This is motivated by invoking the PDSA framework introduced in Definition 2.4.4 and refined in Definition 2.4.15. In this framework, the stochastic process $(X_t)_{t \in \mathcal{T}}$ generated by the map S is wide-sense stationary, and the Wiener-Khinchin theorem (i.e., Theorem 2.4.4) applies. This theorem relates the ACF and PSD as a discrete-time Fourier transform pair, which then implies that the prescription of the ACF implies the prescription of the PSD (and vice versa).

The conditions that the prescribed ACF or PSD must satisfy are defined below.

Conditions for the prescribed ACF

The function $\mathcal{R}_P : \mathbb{Z} \rightarrow \mathbb{R}$ is required to satisfy the following necessary conditions [45] in order to qualify as an ACF that may be prescribed for the map:²

$$\begin{aligned}\mathcal{R}_P(0) &= \mathcal{M}_S^2 + \sigma_S^2, \\ |\mathcal{R}_P(\tau)| &\leq \mathcal{R}_P(0), \quad \tau = \pm 1, \pm 2, \dots, \\ \mathcal{R}_P(-\tau) &= \mathcal{R}_P(\tau), \quad \tau = 1, 2, \dots\end{aligned}\tag{3.3}$$

In these expressions, $\mathcal{M}_S$ denotes the expected value of the process generated by the map,

$$\mathcal{M}_S = \int_{\Omega} x f_P(x) d\mu(x),\tag{3.4}$$

and σ_S^2 denotes the variance of the process generated by the map,

$$\sigma_S^2 = \int_{\Omega} (x - \mathcal{M}_S)^2 f_P(x) d\mu(x),\tag{3.5}$$

where f_P denotes the prescribed invariant density.

Remark 3.3.1. The conditions of (3.3) are necessary for the function to qualify as a valid ACF for the map, but not sufficient. The derivation of sufficient conditions under which a prescribed invariant density and prescribed ACF are simultaneously realisable as statistical metrics of a map is not undertaken in this work.

Conditions for the prescribed PSD

The function $\mathcal{S}_P = \mathcal{S}_{P,C} + \mathcal{S}_{P,I}$, where $\mathcal{S}_{P,C}$ denotes a continuous component and $\mathcal{S}_{P,I}$ denotes an impulsive component, is required to satisfy the necessary conditions [45] outlined below in order to qualify as a PSD that may be prescribed for the map.

The function $\mathcal{S}_P$ must satisfy

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{S}_P(\omega) d\omega = \mathcal{M}_S^2 + \sigma_S^2,\tag{3.6}$$

where $\mathcal{M}_S$ and σ_S^2 are defined in (3.4) and (3.5).

The continuous function $\mathcal{S}_{P,C} : [-\pi, \pi) \rightarrow [0, \infty)$ must satisfy

$$\mathcal{S}_{P,C}(-\omega) = \mathcal{S}_{P,C}(\omega), \quad \omega \in (-\pi, \pi).\tag{3.7}$$

²These conditions follow from the observation that S generates a real-valued and wide-sense stationary process.

The impulsive function $\mathcal{S}_{P,I}$ is required to have the form

$$\mathcal{S}_{P,I}(\omega) = 2\pi \left[\mathcal{M}_S^2 \delta(\omega) + \sum_{m=1}^M r_m \delta(\omega - \omega_m) \right], \quad \omega \in (-\pi, \pi), \quad (3.8)$$

and satisfy

$$\mathcal{S}_{P,I}(-\omega) = \mathcal{S}_{P,I}(\omega), \quad \omega \in (-\pi, \pi), \quad (3.9)$$

where $r_m \in \mathbb{R}^+$ and $\omega_m \in [-\pi, \pi)$ for every $m = 1, 2, \dots, M$.

Remark 3.3.2. The conditions of (3.6) to (3.9) are necessary for the function to qualify as a valid PSD, but not sufficient. The derivation of sufficient conditions under which a prescribed invariant density and prescribed PSD are simultaneously realisable as statistical metrics of a map is not undertaken in this work.

3.3.3 The third problem formulation: IFPP-III

The third formulation of the IFPP, labelled IFPP-III, recently appeared in the literature [1, 2, 35–39]. Let $S : \Omega \rightarrow \Omega$ denote an unknown $\mathcal{R}(\mathcal{Q})$ -semi-Markov map that is ergodic and possesses the unique invariant density function f_S^* , which is necessarily PWC over the intervals of the map's Markov partition $\mathcal{Q}$ (Theorem 2.6.3). IFPP-III requires the construction of an ergodic map $\hat{S}$ that approximates S as closely as possible.³ Colloquially, this is referred to as *reconstructing* the unknown map S , even though it is understood that $\hat{S}$ may not necessarily be identical to the unknown map.

IFPP-III mandates that the unknown map is to be reconstructed from measurements of its state, which are collected during several experiments wherein the system is probed [2, 36]. In each of these *probing experiments*, an ensemble of states with values drawn from a selected probability density (the *probing density*) is permitted to evolve over time, and all states that constitute the ensemble are measured at several evenly spaced instants in time. However, the problem assumes that it is either impractical or physically impossible to isolate and follow individual state trajectories from one time instant to the next. IFPP-III thus mandates the estimation of the PDF corresponding to each collection of measurements, thereby producing a causal sequence of state density functions for each probing experiment. The unknown map is to be reconstructed from these density sequences.

Remark 3.3.3. Although the above definition of IFPP-III does not explicitly refer to any statistical metrics that are prescribed for the map $\hat{S}$, by requiring that $\hat{S}$

³The literature does not explicitly define a formal metric for the closeness of the approximation.

approximate S as closely as possible, it is implicitly required that $\hat{S}$ possess identical statistical metrics to S . This also implies that $\hat{S}$ is required to possess identical dynamics to the map S . However, convergence of the approximation of an ergodic map is not a sufficient condition for the convergence of the approximation of its invariant density.

3.4 Review of Literature: IFPP-I

Solutions to IFPP-I may generally be divided into three categories with respect to the strategy employed to construct the ergodic map. These are the *topological conjugation*, *direct* and *matrix* solution strategies. Below, each solution strategy is outlined in a separate section. The relevant literature is reviewed and examples that illustrate how IFPP-I is solved using each strategy are provided. A condensed version of the review presented in this section appears in [1].

The examples provided in the following sections (see also [1]) involve the construction of ergodic maps with invariant densities $f_P \in \mathcal{D}(\Omega, \mathcal{B}, \mu)$ from the family of quadratic polynomials

$$f_P(x) = ax^2 + bx + c, \quad (3.10)$$

with coefficients

$$\begin{aligned} a &\triangleq 3(y_1 + y_0) - 6, \\ b &\triangleq -2(y_1 + 2y_0) + 6, \\ c &\triangleq y_0. \end{aligned} \quad (3.11)$$

In these expressions, $y_0 \triangleq f_P(0)$ and $y_1 \triangleq f_P(1)$ are free parameters to be prescribed. The corresponding distribution function $F_P : \Omega \rightarrow [0, 1]$ is given by

$$F_P(x) = \frac{a}{3}x^3 + \frac{b}{2}x^2 + cx, \quad (3.12)$$

and the region of valid parameter pairs (y_0, y_1) is defined by

$$\begin{cases} 0 < y_0 + y_1 < 3 + \sqrt{y_0 y_1}, & \text{if } 0 \leq y_1 \leq 3, \\ 3 - \sqrt{y_0 y_1} < y_0 + y_1 < 3 + \sqrt{y_0 y_1}, & \text{if } 3 < y_1 \leq 4. \end{cases} \quad (3.13)$$

The density functions considered in the examples, with the corresponding parameter pairs (y_0, y_1) , are presented in Figs. 3.4.1 and 3.4.2.

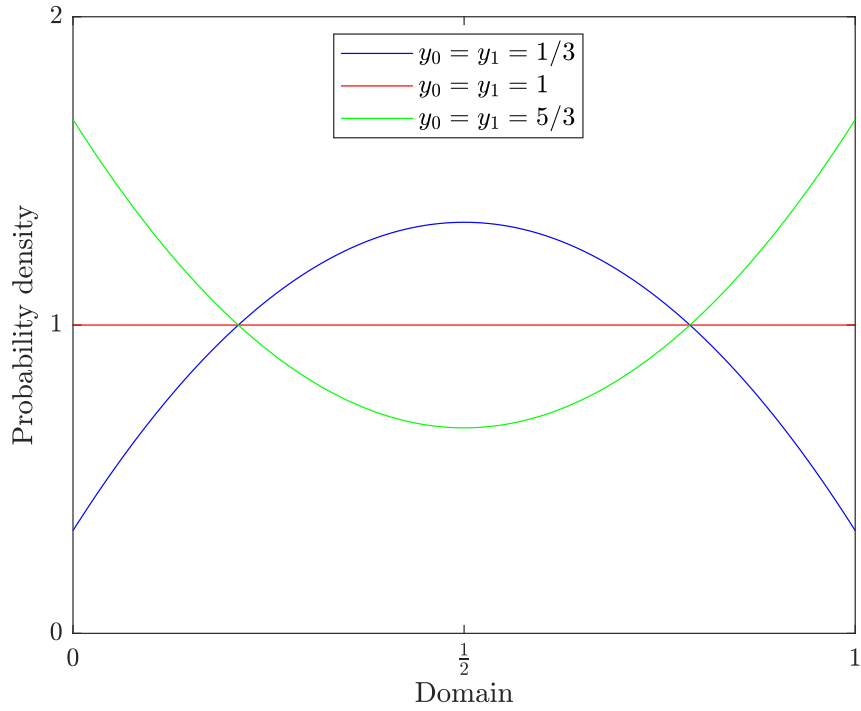


Figure 3.4.1: A subset of the quadratic polynomial density functions considered in the examples for IFPP-I. Reproduced with permission from [1].

3.4.1 Topological conjugation solution strategy

Amongst the three strategies for solving IFPP-I, topological conjugation was the first to appear in the literature [18]. Map construction using topological conjugation is less complicated when compared to the other solution strategies. However, the freedom it grants to manipulate the structural features of the map is more difficult to exploit, and it requires an existing ergodic map with a known invariant density as a starting point.

Outline

Invoke the PDSA framework introduced in Definition 2.4.4 and refined in Definition 2.4.15, and let $\nu = \mu$, where μ denotes the Borel measure. In this framework, the map S is $\mathcal{B}$ -measurable and non-singular with respect to μ . Furthermore, the random variable X_t induces the probability space $(\Omega, \mathcal{B}, P_t)$ and evolves according to

$$X_{t+1} = S \circ X_t, \quad t \in \mathcal{T}. \quad (3.14)$$

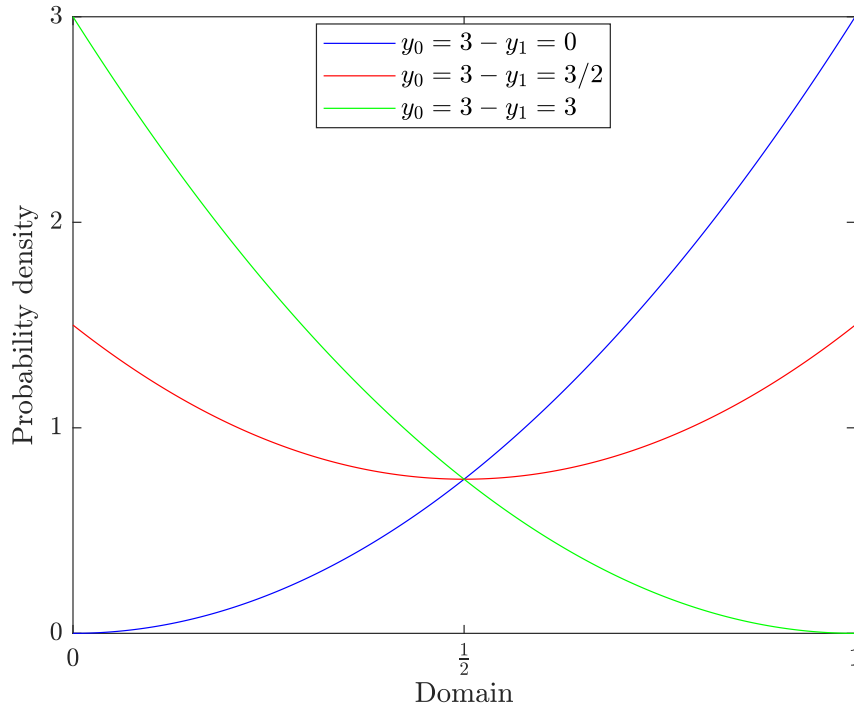


Figure 3.4.2: The remaining polynomial density functions considered in the examples for IFPP-I. Reproduced with permission from [1].

Each probability measure P_t evolves according to the expression

$$P_{t+1}(B) = P_t(S^{-1}(B)), \quad t \in \mathcal{T}, \quad B \in \mathcal{B}, \quad (3.15)$$

is absolutely continuous with respect to the Borel measure μ , and possesses a corresponding density $f_t \in \mathcal{D}(\Omega, \mathcal{B}, \mu)$.

Now, define a homeomorphism $h : \Omega \rightarrow \Omega$ with derivative that satisfies $|h'| > 0$ a.e. Since h is continuous, it is Borel measurable. Hence, each function defined by [1]

$$Y_t = h \circ X_t, \quad t \in \mathcal{T}, \quad (3.16)$$

is a random variable that induces the probability space $(\Omega, \mathcal{B}, \tilde{P}_t)$, where

$$\tilde{P}_t(B) = P_t(h^{-1}(B)), \quad t \in \mathcal{T}, \quad B \in \mathcal{B}. \quad (3.17)$$

As $|h'| > 0$ a.e., the function h is non-singular with respect to μ [46]. This implies that each measure $\tilde{P}_t$ is absolutely continuous with respect to μ . Hence, each Y_t has a corresponding density function $\tilde{f}_t \in \mathcal{D}(\Omega, \mathcal{B}, \mu)$ that satisfies

$$\tilde{P}_t(B) = \int_B \tilde{f}_t(x) d\mu(x), \quad t \in \mathcal{T}, \quad B \in \mathcal{B}. \quad (3.18)$$

The sequence of random variables $(Y_t)_{t \in \mathcal{T}}$ may be interpreted as a stochastic process

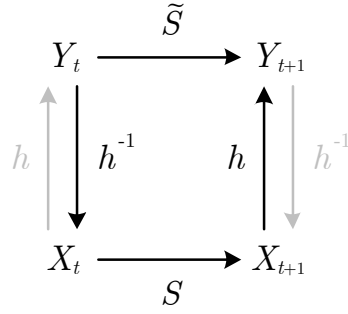


Figure 3.4.3: Functional relationships between random variables that represent the states of two dynamical systems with maps S and $\tilde{S}$. The reverse conjugate relationship is shown in grey. Reproduced with permission from [1].

representing the state of a dynamical system $(\mathcal{T}, \Omega, \tilde{S})$, where $\tilde{S}$ denotes the system's evolution rule. The random variables evolve according to the expression

$$Y_{t+1} = \tilde{S} \circ Y_t, \quad t \in \mathcal{T}. \quad (3.19)$$

With reference to Fig. 3.4.3, the map $\tilde{S}$ satisfies

$$\tilde{S} = h \circ S \circ h^{-1}. \quad (3.20)$$

Definition 3.4.1 (Topologically conjugate maps [1, 18, 47, 48]). Two maps $\tilde{S}$ and S that are related according to (3.20), where h denotes a homeomorphism, are referred to as *topological conjugates*. The function h , which is assumed throughout this work to satisfy $|h'| > 0$ a.e., is referred to as the *conjugating function* or *conjugating transformation*.

Topological conjugation establishes an equivalence relation between maps [1], such that the maps of any class generated by topological conjugation share the same properties of ergodicity, mixing and exactness (refer to [49, Sec. 4.3] for more information regarding the latter two concepts). The property that ergodicity is preserved under topological conjugation is proved in the following theorem.

Theorem 3.4.1 (Topological conjugation preserves ergodicity [1]). *Let S and $\tilde{S}$ be topological conjugates. If S is ergodic with invariant density $f_S^* > 0$ a.e., then $\tilde{S}$ is also ergodic.*

Proof. Let S be ergodic with ACI measure P_S^* and invariant density $f_S^* > 0$ a.e. Equation (3.20) implies that

$$\tilde{S}^{-1}(B) = h \circ S^{-1} \circ h^{-1}(B), \quad B \in \mathcal{B}. \quad (3.21)$$

The measure $P \triangleq P_S^* \circ h^{-1}$ then satisfies

$$\begin{aligned} P(\tilde{S}^{-1}(B)) &= P_S^*(S^{-1} \circ h^{-1}(B)) \\ &= P_S^*(h^{-1}(B)) \\ &= P(B), \quad B \in \mathcal{B}, \end{aligned} \tag{3.22}$$

where the second step follows from the fact that S preserves the measure P_S^* . Equation (3.21) implies that $\tilde{S}$ preserves the measure $P_{\tilde{S}}^* \triangleq P$.

It is proved that $P_{\tilde{S}}^*$ is unique by contradiction. Assume that $\tilde{S}$ preserves a second distinct measure $U_{\tilde{S}}^*$. Then, due to S being the topological conjugate of $\tilde{S}$ with conjugating function⁴ $g \triangleq h^{-1}$, S necessarily preserves the corresponding measure $U_S^* = U_{\tilde{S}}^* \circ g^{-1}$, which is distinct from P_S^* . However, since S is ergodic, its invariant measure must be unique (see Proposition 2.4.4). The assumption that $\tilde{S}$ preserves $U_{\tilde{S}}^*$ then contradicts the original statement that S preserves the measure P_S^* . Hence, $\tilde{S}$ preserves no measure other than P_S^* . Theorem 2.4.2 then implies that $\tilde{S}$ is ergodic. $\square$

Topological conjugation presents a means for solving IFPP-I [1]. Suppose that S is an ergodic map with a known ACI measure P_S^* and an invariant density that satisfies $f_S^* > 0$ a.e. To solve IFPP-I, an ergodic map $\tilde{S}$ that is a topological conjugate of S is designed, such that the invariant density $f_{\tilde{S}}^*$ of $\tilde{S}$ is equal to the prescribed density f_P . This is achieved by deriving a conjugating transformation h such that the invariant measure $P_{\tilde{S}}^* = P_S^* \circ h^{-1}$ corresponds to f_P . To illustrate how the transformation h may be derived, consider the distribution functions F_S^* and $F_{\tilde{S}}^*$ associated with the measures P_S^* and $P_{\tilde{S}}^*$. If h is selected to be increasing, it follows that

$$F_{\tilde{S}}^* = F_S^* \circ h^{-1}. \tag{3.23}$$

Assume that the distribution functions F_S^* and F_P , where F_P is associated with the prescribed density f_P , are both homeomorphisms from Ω onto itself. Selecting $F_{\tilde{S}}^* = F_P$ in (3.23) yields

$$h = F_P^{-1} \circ F_S^*. \tag{3.24}$$

If $|h'| > 0$ a.e., h is non-singular. In this case, the conjugating transformation h of (3.24) yields the ergodic map $\tilde{S}$ of (3.20) with the prescribed invariant density f_P .

Remark 3.4.1. The conjugating transformation of (3.24) is not unique; i.e., there exists multiple distinct conjugating transformations that yield the same invariant density. By permitting multiple conjugating transformations, this solution strategy affords a measure of freedom to choose the structural characteristics of the map.

⁴It is assumed here that $|g'| > 0$ a.e.

However, this approach is harder to implement, as compared to the direct solution strategy. The latter method permits the direct specification of functional relationships that the branches of the map are required to satisfy.

The topological conjugation solution strategy is illustrated in the following example.

Example 3.4.1 (Topological conjugation solution [1]). Ergodic maps $\tilde{S}$ with the invariant density functions plotted in Figs. 3.4.1 and 3.4.2 are constructed. The ergodic maps $S = S_T$ and $S = S_B$, defined by

$$S_T(x) \triangleq \begin{cases} 2x, & \text{if } 0 \leq x < \frac{1}{2}, \\ -2(x-1), & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases} \quad (3.25)$$

and

$$S_B(x) \triangleq \begin{cases} 2x, & \text{if } 0 \leq x < \frac{1}{2}, \\ 2(x - \frac{1}{2}), & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases} \quad (3.26)$$

are considered. These maps are referred to as the *tent map* and *Bernoulli map*, and are plotted in Figs. 3.4.4(c) and 3.4.4(f), respectively.

Both the tent map and Bernoulli map possess the uniform invariant density function, which corresponds to a distribution function $F_S^*(x) = x$. Substitution of (3.24) into (3.20) then yields the map $\tilde{S}_T$, a topological conjugate of the tent map, given by

$$\tilde{S}_T(x) = \begin{cases} F_P^{-1}(2F_P(x)), & \text{if } 0 \leq x < F_P^{-1}(\frac{1}{2}), \\ F_P^{-1}(-2F_P(x) + 2), & \text{if } F_P^{-1}(\frac{1}{2}) \leq x \leq 1, \end{cases} \quad (3.27)$$

and the map $\tilde{S}_B$, a topological conjugate of the Bernoulli map, given by

$$\tilde{S}_B(x) = \begin{cases} F_P^{-1}(2F_P(x)), & \text{if } 0 \leq x < F_P^{-1}(\frac{1}{2}), \\ F_P^{-1}(2F_P(x) - 1), & \text{if } F_P^{-1}(\frac{1}{2}) \leq x \leq 1, \end{cases} \quad (3.28)$$

where F_P is defined in (3.12). The maps $\tilde{S}_T$ and $\tilde{S}_B$ are plotted in Figs. 3.4.4(d, e) and Figs. 3.4.4(g, h) respectively, and the invariant density function of each conjugate map is plotted, in the same colour, at the top of the same column (Figs. 3.4.4(a, b)).

Due to the strict monotonicity and smoothness of F_P and F_P^{-1} , the conjugate maps $\tilde{S}$ have the same number of branches as the underlying map S . Furthermore, the positivity or negativity of each branch slope is preserved under conjugation with $h = F_P^{-1}$. In certain cases, topological conjugation also preserves symmetry.

Definition 3.4.2 (Function symmetry [1]). A function $G : \Omega \rightarrow \Omega$ is referred to as *symmetric* around $x = x_0$ if it satisfies

$$G(x) = G\left(\frac{x_0 - 1}{x_0}x + 1\right), \quad \text{if } 0 \leq x \leq x_0, \quad (3.29)$$

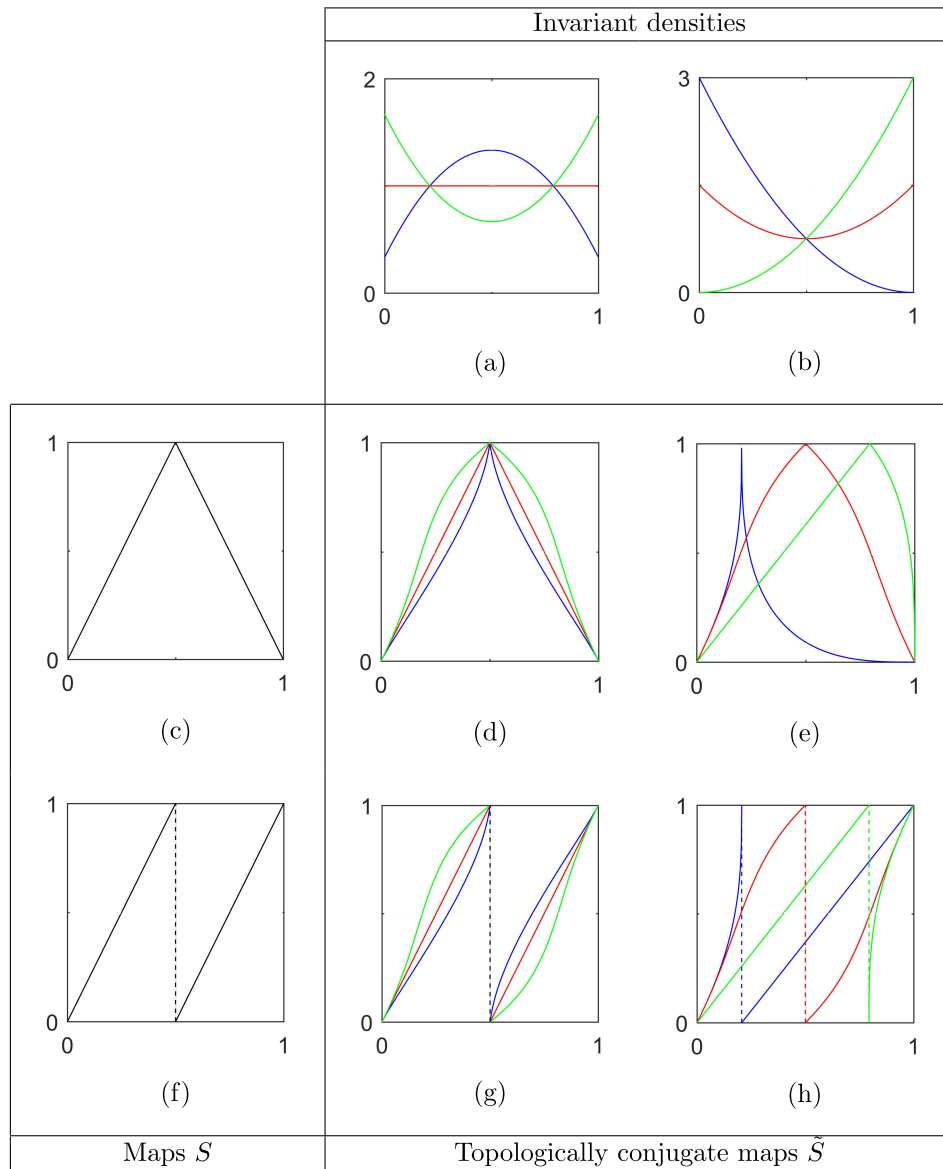


Figure 3.4.4: Topologically conjugate maps $\tilde{S}$ constructed to solve IFPP-I. Reproduced with permission from [1].

antisymmetric around $x = x_0$ if it satisfies

$$G(x) = 1 - G\left(\frac{x_0 - 1}{x_0}x + 1\right), \quad \text{if } 0 \leq x < x_0, \quad (3.30)$$

and *translationally symmetric* around $x = x_0$ if

$$G(x) = G\left(\frac{1 - x_0}{x_0}x + x_0\right), \quad \text{if } 0 \leq x < x_0. \quad (3.31)$$

Functions that are symmetric, antisymmetric or translationally symmetric around $x_0 = 1/2$ are simply referred to as symmetric, antisymmetric or translationally symmetric.

The tent map is an example of a symmetric map, whereas the Bernoulli map is both antisymmetric and translationally symmetric [1]. For the symmetric density functions prescribed in Example 3.4.1, it is found that (i) the corresponding map $\tilde{S}_T$, which is conjugate to the tent map, is also symmetric (Fig. 3.4.4(d)), and (ii) the map $\tilde{S}_B$, which is conjugate to the Bernoulli map, is also antisymmetric (Fig. 3.4.4(g)). In general, any map $\tilde{S}$ with symmetric invariant density function $f_{\tilde{S}}^*$, where $\tilde{S}$ is conjugate to a symmetric (antisymmetric) map S with symmetric invariant density function f_S^* via the conjugating transformation of (3.24), is also symmetric (antisymmetric). However, in the case where the density function $f_{\tilde{S}}^*$ lacks symmetry, the domain of each branch of the map S may change under conjugation (refer to the blue and green maps of Figs. 3.4.4(e,h)).

Remark 3.4.2. Whereas the number of branches of the conjugate map $\tilde{S}$ and each branch function's monotonicity may be selected through the choice of the underlying map S , topological conjugation that uses (3.24) exclusively provides no direct opportunity to select the domain, image or the shape of each branch function [1]. In contrast, the direct solution and matrix solution strategies described in sections 3.4.2 and 3.4.3 provide greater opportunity to select the characteristics of each map branch. This implies that the topological conjugation strategy, when applied in isolation to elementary maps such as the tent and Bernoulli maps, is a less attractive option for solving the more general formulations of the IFPP, where greater freedom regarding the choice of structural characteristics is required.

In contrast to the remaining solution strategies, topological conjugation *guarantees* the ergodicity of the conjugate map $\tilde{S}$, provided that the original map S is ergodic [1]. Also, by using topological conjugation in the manner outlined in this section, the prescribed invariant density is realised *exactly* (i.e., the invariant density of the conjugate map is *equal* to the prescribed density). This is accomplished regardless of the invariant density of map S (provided this density is greater than zero a.e.).

Although the use of topological conjugation on its own is limited, especially when attempting to develop solutions to the more general formulations of the IFPP, it may be combined with the matrix solution strategy to develop more versatile methods [21]. A new method that implicitly combines these strategies and which permits the simultaneous prescription of the map's invariant density (albeit as a piecewise constant density) and the poles of its rational PSD is developed in chapter 4.

Literature

Grossman and Thomae [18] proposed topological conjugation as a strategy for solving IFPP-I [1]. Equivalence classes of ergodic maps that are conjugate to maps from the family of PWL transformations $S_v : \Omega \rightarrow \Omega$ were constructed, where

$$S_v(x) \triangleq (-1)^{[vx]} vx \pmod{1}. \quad (3.32)$$

In this expression, $v \in \mathbb{R}$ and $[\cdot]$ denotes the integer part of its argument. The maps S_v are ergodic for all $v > 1$ and possess a uniform invariant density function for integer $v \geq 2$. Conjugate maps $\tilde{S} = F_P^{-1} \circ S_v \circ F_P$ with the invariant densities listed in Fig. 3.4.5 were derived; for convenience, expressions for the corresponding distribution functions and their inverses are provided in the same figure.

Using topological conjugation to design chaotic noise generators, van Wyk and Steeb [50] derived an expression that may be solved for the required conjugating transformation h [1]. This expression contains both the invariant density f_S^* of the underlying map S and the prescribed invariant density f_P , and is given by

$$h^{-1}(x) = \int_{-\infty}^x \frac{f_P(z)}{f_S^* \circ h^{-1}(z)} dz. \quad (3.33)$$

The latter expression was solved analytically to yield (3.24). A topological conjugate of a Bernoulli-type map with selectable discontinuity (as studied in [51]), and with an exponential invariant density, was constructed.

Berlinet and Biau [52] constructed exact maps with prescribed invariant density functions for the purpose of non-uniform random variate generation [1]. In this approach, map trajectories are used as random variate sequences. Remark 2.4.3 implies that the distribution of the random variates produced using this technique is asymptotically consistent with the invariant density of the map. The maps constructed are topologically conjugate to the maps $S(x) = vx \pmod{1}$ with integer $v > 1$. The conjugate maps preserve the number of branches and the positivity of the branch slopes of the underlying map and may therefore be interpreted as a generalisation of the class of linear congruential random number generators from uniform to non-uniform distributions. The statistics of maps with invariant density corresponding to the Gumbel probability distribution, which has the distribution function $F_P = \exp(-\exp(-x))$, $x \in \mathbb{R}$, were investigated numerically.

Pingel et al. [53] demonstrated a technique for constructing symmetric maps with asymmetric invariant density functions that is related to topological conjugation (see

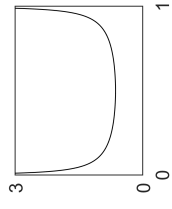
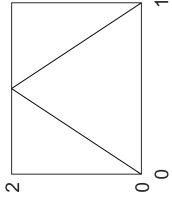
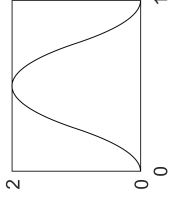
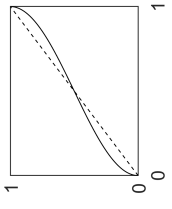
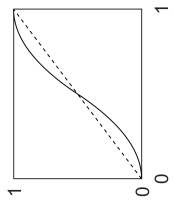
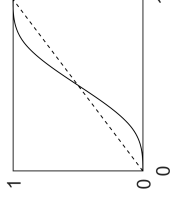
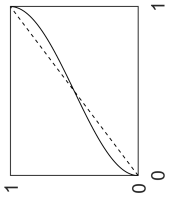
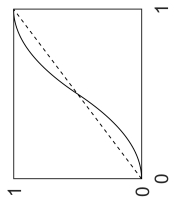
Distribution	Bowl-shaped	Triangular	Quadratic
Density function	$f_*(x) = \frac{1}{\pi\sqrt{x(1-x)}}$. 	$f_*(x) = 2 - 2 - 4x $. 	$f_*(x) = \begin{cases} 4(1 - 1 - 2x)^2, & \text{if } x \in [0, \frac{1}{4}] \cup (\frac{3}{4}, 1], \\ 2 - 4(1 - 2x)^2, & \text{if } x \in [\frac{1}{4}, \frac{3}{4}]. \end{cases}$ 
Distribution function	$F_*(x) = \frac{2}{\pi} \arcsin(\sqrt{x})$. 	$F_*(x) = \begin{cases} 2x^2, & \text{if } x \in [0, \frac{1}{2}], \\ 1 - 2(1 - x)^2, & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$ 	$F_*(x) = \begin{cases} \frac{16}{3}x^3, & \text{if } x \in [0, \frac{1}{4}), \\ \frac{2}{3}(1 - 2x)^3 + 2x - \frac{1}{2}, & \text{if } x \in [\frac{1}{4}, \frac{3}{4}], \\ 1 - \frac{16}{3}(1 - x)^3, & \text{if } x \in (\frac{3}{4}, 1]. \end{cases}$ 
Inverse distribution function	$F_*^{-1}(x) = \sin^2(\frac{\pi}{2}x)$. 	$F_*^{-1}(x) = \begin{cases} \sqrt{\frac{x}{2}}, & \text{if } x \in [0, \frac{1}{2}], \\ 1 - \sqrt{\frac{1-x}{2}}, & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$ 	$F_*^{-1}(x) = \begin{cases} \sqrt[3]{3x/16}, & \text{if } x \in [0, \frac{1}{12}), \\ \frac{1}{2} + \frac{1}{\sqrt{2}} \sin(\frac{1}{3} \tan^{-1}(z)), & \text{if } x \in [\frac{1}{12}, \frac{11}{12}], \\ 1 - \sqrt[3]{3(1-x)/16}, & \text{if } x \in (\frac{11}{12}, 1], \end{cases}$ $z = \frac{3(x - \frac{1}{2})}{\sqrt{2 - 9(x - \frac{1}{2})^2}}.$

Figure 3.4.5: Probability density functions with corresponding distribution functions and their inverses (derived from [18]).

also [30]) [1]. This approach was first considered in [54]. Let $S = S_T$ denote the tent map. Now, consider the map

$$\tilde{S}(x) = U^{-1} \circ S \circ u(x), \quad (3.34)$$

where $u : \Omega \rightarrow \Omega$ is a conjugating transformation that is antisymmetric and satisfies $u' > 0$. In this expression,

$$U(x) = u(x) + v(x), \quad (3.35)$$

where v is symmetric and $v(0) = 0$. The functions u and v may be (uniquely) selected to realise a symmetric map $\tilde{S}$ with a prescribed invariant density f_P . In particular, if U is selected equal to F_P , where F_P is the distribution function corresponding to the prescribed invariant density, and F_P is decomposed into antisymmetric and symmetric components according to

$$u(x) = \frac{F_P(x) + (1 - F_P(1 - x))}{2} \quad (3.36)$$

and

$$v(x) = \frac{F_P(x) - (1 - F_P(1 - x))}{2}, \quad (3.37)$$

then $\tilde{S}$ is symmetric with invariant density function $\tilde{f}_S^* = dU/dx$ that need not be symmetric. However, as the mapping between trajectories of S and $\tilde{S}$ is no longer one-to-one, the uniqueness of the invariant density (and hence the ergodicity of $\tilde{S}$) cannot be guaranteed.

3.4.2 Direct solution strategy

The direct solution strategy permits the construction of a map S that is piecewise monotonic over a selectable partition $\mathcal{Q}$ [1]. This is achieved by solving (2.92) or (2.104) analytically for the unknown map S ; recall that these equations describe the evolution of density and distribution functions (respectively) under iteration of an unknown piecewise monotonic map. The direct solution strategy is outlined in what follows.

Outline

The direct solutions to IFPP-I encountered in the literature [19, 53, 55–62] construct *complete* piecewise monotonic maps, as defined below [1].

Definition 3.4.3 (Complete piecewise monotonic map). If each branch of a piecewise monotonic map $S \in \Gamma(\mathcal{Q})$ maps its domain to the full codomain of the map, it is referred to as complete. Formally, $S \in \Gamma(\mathcal{Q})$ is complete if

$$\text{cl}(S(Q_m)) = \Omega, \quad \forall m = 1, 2, \dots, M. \quad (3.38)$$

Let $S \in \Gamma(\mathcal{Q})$, where $\mathcal{Q} = \{Q_1, Q_2, \dots, Q_M\}$. Either (2.92) or (2.104) may be solved using the direct solution strategy. These equations are repeated here for convenience. Equation (2.92) is given by

$$(\mathcal{P}_S f_t)(x) = \sum_{m \in N(x)} \frac{f_t \circ S_m^{-1}(x)}{|S'_m \circ S_m^{-1}(x)|} \quad (3.39)$$

and (2.104) is given by

$$F_{t+1}(x) = \sum_{m \in N(x)} |F_t \circ S_m^{-1}(x) - F_t(q_{m+d_m})| + \sum_{m \in M(x)} F_t(q_{m+1}) - F_t(q_m), \quad (3.40)$$

where

$$\begin{aligned} N(x) &\triangleq \{m : x \in S(Q_m)\}, \\ M(x) &\triangleq \{m : x \geq \sup(Q_m)\}, \end{aligned} \quad (3.41)$$

and

$$d_m \triangleq \begin{cases} 0, & \text{if } S'_m > 0, \\ 1, & \text{if } S'_m < 0. \end{cases} \quad (3.42)$$

The partition points of $\mathcal{Q}$ are denoted by $\overline{P}(\mathcal{Q}) = \{q_1, q_2, \dots, q_{M+1}\}$ in (3.40).

The solution of (3.40) for the unknown map S is considered below [1]; equivalent expressions that involve density functions are provided towards the end of the section. Let the map S in (3.40) be complete. This equation may then be simplified as

$$F_{t+1}(y) = \sum_{m=1}^M |F_t \circ S_m^{-1}(y) - F_t(q_{m+d_m})|. \quad (3.43)$$

Assume that S possesses an ACI measure P_S^* with the corresponding distribution function F_S^* . Substituting $y = S_1(x)$ and setting $F_{t+1} = F_t = F_S^*$ in the latter expression yields

$$F_S^* \circ S_1(x) = \sum_{m=1}^M |F_S^* \circ S_m^{-1} \circ S_1(x) - F_S^*(q_{m+d_m})|, \quad \text{if } q_1 < x < q_2. \quad (3.44)$$

This expression implicitly defines a map S that preserves the measure P_S^* . Now, consider the case where

$$P_S^*(B) = 0 \Leftrightarrow \mu(B) = 0, \quad B \in \mathcal{B}, \quad (3.45)$$

such that F_S^* is monotonically increasing and continuous, and therefore invertible. Equation (3.44) may then be manipulated to yield

$$S_1(x) = (F_S^*)^{-1} \left(\sum_{m=1}^M |F_S^* \circ S_m^{-1} \circ S_1(x) - F_S^*(q_{m+d_m})| \right), \quad \text{if } q_1 < x < q_2. \quad (3.46)$$

This expression for the first branch S_1 of the map S is dependent on the remaining branches of the map, as well as the partition points of $\mathcal{Q}$ and the constants $d_1, d_2, \dots, d_M$.

The functions $u_m : Q_1 \rightarrow Q_m$, defined by

$$u_m(x) \triangleq \begin{cases} x, & \text{if } m = 1, \\ S_m^{-1} \circ S_1(x), & \text{if } m = 2, 3, \dots, M, \end{cases} \quad (3.47)$$

appear in (3.46). The expression

$$S_m \circ u_m(x) = S_1(x), \quad \text{if } q_1 < x < q_2, \quad (3.48)$$

reveals that each function u_m defines an implicit relationship between branches S_1 and S_m . This is illustrated in Fig. 3.4.6. These functions are central to the direct solution strategy and are referred to as *inter-branch correspondence functions*.

Definition 3.4.4 (Inter-branch correspondence function [1]). The functions u_m defined by (3.47) are referred to as inter-branch correspondence functions.

Consider the case where the first branch S_1 is selected to be monotonically increasing, such that $d_1 = 0$. To solve IFPP-I using the direct solution strategy, the partition $\mathcal{Q}$ is selected and each of the inter-branch correspondence functions u_m is specified, thereby also specifying the value of d_m for all $m > 1$ (i.e., the function u_m implicitly determines whether the slope of S_m is positive or negative) [1]. Equation (3.46) is then solved for the first branch S_1 . In particular, selecting $F_S^* = F_P$ in (3.46) yields

$$S_1(x) = F_P^{-1} \left(\sum_{m=1}^M |F_P \circ u_m(x) - F_P(q_{m+d_m})| \right), \quad \text{if } q_1 < x < q_2, \quad (3.49)$$

where $d_1 = 0$. The remaining branches are then given by

$$S_m(x) = S_1 \circ u_m^{-1}(x), \quad \text{if } q_m < x < q_{m+1}, \quad m = 2, 3, \dots, M. \quad (3.50)$$

Inter-branch correspondence functions provide control over the structure of the map being constructed. These functions may, for example, be selected as to introduce symmetry between the map's branches. This is illustrated in the following example.

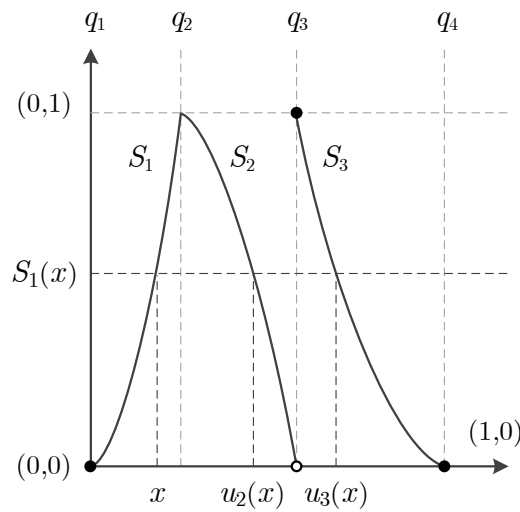


Figure 3.4.6: A piecewise monotonic map S with three branches and inter-branch correspondence functions u_2 and u_3 . Reproduced with permission from [1].

Example 3.4.2 (Direct solution [1]). Consider the two-interval partition $\mathcal{Q} = \{Q_1, Q_2\}$, where $Q_1 = (0, \beta)$, $Q_2 = (\beta, 1)$ and $0 < \beta < 1$. Two sets of complete maps S with branch functions $S_1 = S|_{Q_1}$ and $S_2 = S|_{Q_2}$ are constructed, such that each map in the first set is symmetric around $x = \beta$, and each map in the second set is translationally symmetric around $x = \beta$. Branch S_1 of each map is required to be monotonically increasing. Using piecewise linear maps, the expected branch correspondence of these symmetric and translationally symmetric maps is illustrated in Figs. 3.4.7(c, f), respectively. To preserve symmetry, it is required that the branch functions satisfy

$$S_2(x) = S_1\left(\frac{\beta(x-1)}{\beta-1}\right) \quad (\text{Map set 1}) \quad (3.51)$$

and

$$S_2(x) = S_1\left(\frac{\beta(x-\beta)}{1-\beta}\right) \quad (\text{Map set 2}). \quad (3.52)$$

The symmetric maps of set 1 are considered first. From (3.51), the function $u_2(x) = S_2^{-1} \circ S_1(x)$ is derived as

$$u_2(x) = \frac{\beta-1}{\beta}x + 1, \quad 0 < x < \beta. \quad (3.53)$$

Since S_1 and S_2 are monotonically increasing and decreasing, respectively, it follows that $d_1 = 0$ and $d_2 = 1$. As $F_P(0) = 0$ and $F_P(1) = 1$, (3.49) implies that

$$S_1(x) = F_P^{-1}\left(F_P(x) + 1 - F_P\left(\frac{\beta-1}{\beta}x + 1\right)\right), \quad 0 < x < \beta \quad (3.54)$$

and (3.50) implies that

$$S_2(x) = F_P^{-1}\left(F_P\left(-\frac{\beta(x-1)}{1-\beta}\right) + 1 - F_P(x)\right), \quad \beta < x < 1. \quad (3.55)$$

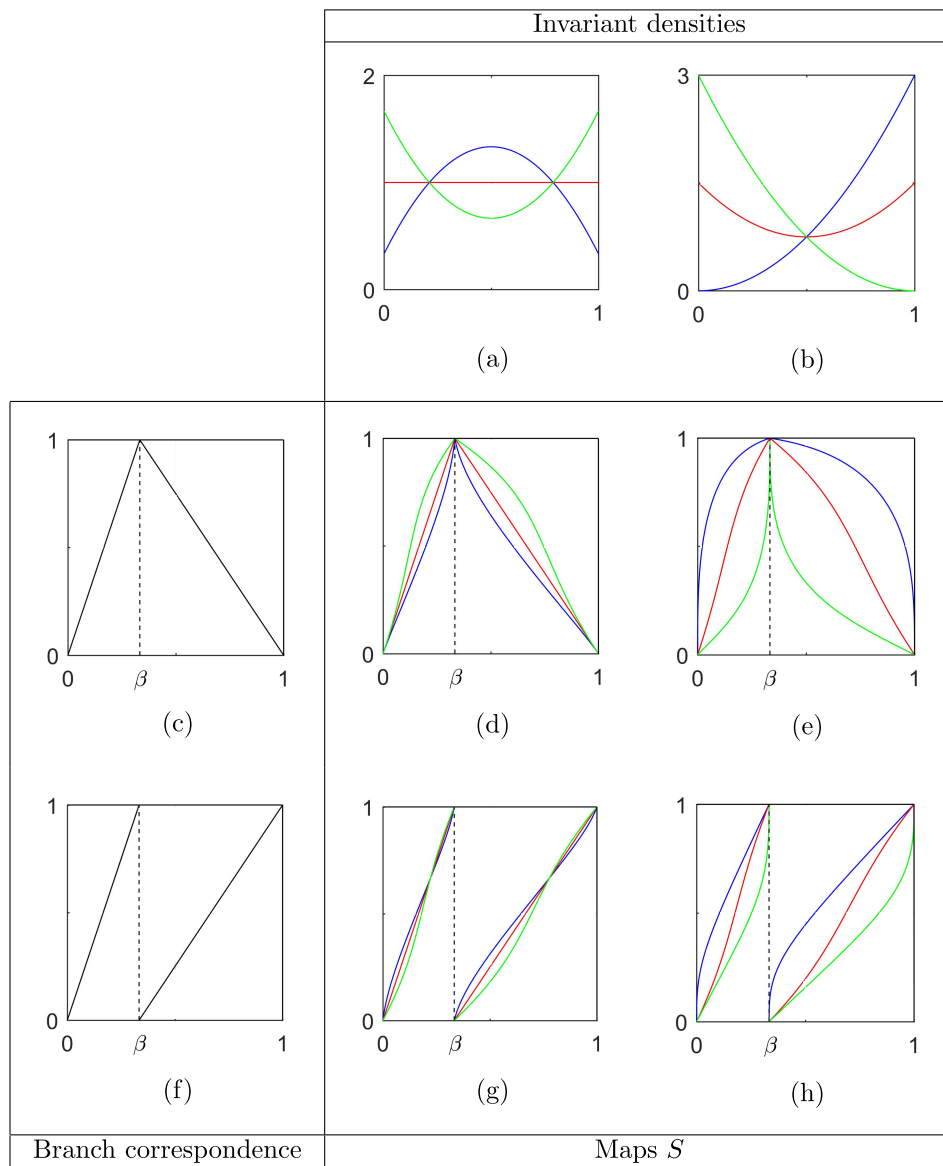


Figure 3.4.7: Symmetric and translationally symmetric maps S constructed by direct solution of IFPP-I. Reproduced with permission from [1].

Substituting the distribution function of (3.12) into these expressions yields the maps of Figs. 3.4.7(d, e). These maps possess the invariant densities of Figs. 3.4.7(a, b), which correspond to the parameter sets indicated in Figs. 3.4.1 and 3.4.2.

Considering the translationally symmetric maps of set 2, $u_2(x) = S_2^{-1} \circ S_1(x)$ is derived from (3.52) as

$$u_2(x) = \frac{1-\beta}{\beta}x + \beta, \quad 0 < x < \beta. \quad (3.56)$$

Since both branches of each map are monotonically increasing, it follows that $d_1 = 0$

and $d_2 = 0$. Hence,

$$S_1(x) = F_P^{-1} \left(F_P(x) + F_P \left(\frac{1-\beta}{\beta} x + \beta \right) - F_P(\beta) \right), \quad 0 < x < \beta \quad (3.57)$$

and

$$S_2(x) = F_P^{-1} \left(F_P \left(\frac{\beta(x-\beta)}{1-\beta} \right) + F_P(x) - F_P(\beta) \right), \quad \beta < x < 1. \quad (3.58)$$

Substitution of (3.12) into these expressions yields the maps plotted in Figs. 3.4.7(g, h), where the corresponding invariant densities are plotted in Figs. 3.4.7(a, b).

Remark 3.4.3. Example 3.4.2 illustrates that direct solution of IFPP-I, unlike the topological conjugation strategy, provides *direct* opportunity to specify not only the domain of each branch function through the choice of partition $\mathcal{Q}$, but also the functional relationship between the branches through the choice of inter-branch correspondence functions [1]. This affords an additional degree of freedom whereby the map may be constructed to possess specified structural characteristics such as symmetry. Whereas the direct solution strategy does not grant the freedom to directly specify a subset of the map's branch functions, it does allow for the shape of the branches to be manipulated.

The direct solution strategy presented thus far solves (3.40) for the unknown map S . An alternative approach is the solution of (3.39), which characterises the evolution of density functions under iteration of S [1]. In this approach, the partition $\mathcal{Q}$ and inter-branch correspondence functions u_m of the map are again specified. Consider the case where the first branch S_1 of the map is selected to be monotonically increasing. An expression for S_1 may then be obtained by solving the differential equation

$$S_1'(x) = \sum_{m=1}^M \frac{(-1)^{d_m} u_m'(x) f_P \circ u_m(x)}{f_P \circ S_1(x)}, \quad S_1(0) = 0, \quad q_1 < x < q_2, \quad (3.59)$$

where $u_m' \triangleq du_m/dx$ and f_P is the prescribed invariant density function (as before, $d_1 = 0$). The remaining branches of the map may then be derived using (3.50). Although this approach yields the same map as obtained by solving (3.40), the differential equation of (3.59) may be solved numerically if expressions for the distribution function and its inverse are unavailable.

The topological conjugation solution strategy may be interpreted as a special case of the direct solution strategy [1]. Let $\tilde{S} = h \circ S \circ h^{-1}$ denote a topological conjugate of a map S , and assume that $h : \Omega \rightarrow \Omega$ and its inverse are C^r functions, where $r \geq 1$. Suppose that $S \in \Gamma(\mathcal{Q})$ is a complete map. Then $\tilde{S} \in \Gamma(\tilde{\mathcal{Q}})$ is also a complete

map, where $\tilde{\mathcal{Q}} \triangleq \{\tilde{Q}_1, \tilde{Q}_2, \dots, \tilde{Q}_M\}$ and $\tilde{Q}_m \triangleq (h(q_m), h(q_{m+1}))$. The inter-branch correspondence functions $\tilde{u}_m : \tilde{Q}_1 \rightarrow \tilde{Q}_m$ of this map are defined as

$$\tilde{u}_m(x) \triangleq \begin{cases} x, & \text{if } m = 1, \\ \tilde{S}_m^{-1} \circ \tilde{S}_1(x), & \text{if } m = 2, 3, \dots, M, \end{cases} \quad (3.60)$$

It can then be shown that

$$\begin{aligned} \tilde{u}_m(x) &= h \circ S_m^{-1} \circ S_1 \circ h^{-1}(x) \\ &= h \circ u_m \circ h^{-1}(x), \quad h(q_1) < x < h(q_2), \quad m = 1, 2, \dots, M. \end{aligned} \quad (3.61)$$

Hence, the inter-branch correspondence functions of $\tilde{S}$ are topological conjugates of the inter-branch correspondence functions of the underlying map S . The topological conjugation solution strategy is thus equivalent to the direct solution strategy with the selection of a particular set of inter-branch correspondence functions.

This section is concluded by observing that maps constructed using the direct solution strategy are not necessarily guaranteed to be ergodic, unlike maps constructed via topological conjugation [1]. However, if the map $S \in \Gamma(\mathcal{Q})$ is complete, it is necessarily $\mathcal{Q}$ -Markov. If S then satisfies the remaining conditions of the Folklore theorem, it is ergodic, and the prescribed invariant density is unique. Hence, Theorem 2.6.3 may be used to verify whether the map is ergodic.

Literature

Ershov and Malinetskii [19, 55] solved IFPP-I directly to construct unimodal maps with an arbitrary invariant density $f_P > 0$ [1]. An expression for S that contains the complementary distribution function corresponding to the prescribed density was derived. The map constructed in the stated manner was proved to be the unique unimodal map with the prescribed inter-branch correspondence function and invariant density. However, conditions under which the map is ergodic were not stated.

Koga [56] solved IFPP-I directly to construct two-branch maps that are either symmetric or translationally symmetric (i.e., with inter-branch correspondence functions $u_2(x) = 1 - x$ and $u_2(x) = x + 1/2$, respectively) [1]. The symmetric and translationally symmetric maps were constructed by solving

$$S'_1(x) = \frac{f_P(x) + f_P(1 - x)}{f_P \circ S_1(x)}, \quad 0 < x < \frac{1}{2} \quad (3.62)$$

and

$$S'_1(x) = \frac{f_P(x) + f_P(x + \frac{1}{2})}{f_P \circ S_1(x)}, \quad 0 < x < \frac{1}{2}, \quad (3.63)$$

respectively. Expressions were derived for maps with sinusoidal and polynomial invariant densities, and maps with an invariant density that corresponds to the truncated Gaussian distribution were obtained numerically. It was verified that all these maps are ergodic, which implies that each invariant density is unique.

Diakonou and Schmelcher [57] solved IFPP-I directly to construct symmetric unimodal maps with invariant density equal to the symmetric beta density function (this is commonly referred to as the *doubly symmetric case* in the literature) [1]. An implicit defining equation was derived that is identical to the expression obtained using the method of Ershov and Malinetskii [19, 55]. Hence, the maps constructed in this manner are the unique unimodal and symmetric maps with the symmetric beta invariant density. The analytical properties of each map in the region around its maximum, as well as the bifurcation routes of the maps, were investigated.

By defining an explicit inter-branch correspondence function, Pingel et al. [53] generalised the method of [57] in order to construct asymmetric unimodal maps [1]. These maps are equivalent to those obtained using the method proposed in [19, 55], and are therefore the unique unimodal maps with the prescribed invariant density and branch correspondence. The maps were constructed to possess invariant density functions corresponding to the beta distribution with arbitrary parameter values. The analytical and dynamical properties of the maps were characterised, and their Lyapunov exponents and autocorrelation functions were investigated.

In a series of papers, Huang used the direct solution strategy to construct unimodal maps with branches that have a reciprocal relationship [58], unimodal maps with closed functional forms [59] and complete maps with more than two branches [60, 61] (see [1]). More recently, Venegeroles [62] used the direct solution strategy to construct maps with branches that are related by explicit inter-branch correspondence functions.

Boyarsky et al. [63] considered approximate solutions to IFPP-I for the case where the map S is restricted to a specific region of the codomain, as defined by *lower* and *upper* maps $S_L, S_U : \Omega \rightarrow \Omega$ such that $S_L \leq S \leq S_U$ [1]. An algorithm was proposed for directly solving an expression that contains the explicit form of the FPO corresponding to a sequence of piecewise defined maps, thereby approximating a prescribed invariant density. These maps are appropriate for situations where the trajectories of the system should not evolve in a free and unrestricted manner over time, such as the modelling of gross domestic product growth.

3.4.3 Matrix solution strategy

The novel solutions for the more general formulations of the IFPP, as developed in the subsequent chapters, use elements of the matrix solution strategy for IFPP-I. To reveal implicit design choices and assumptions of existing methods that use this strategy, and to unify the theory and results scattered throughout the literature, the matrix solution strategy is derived from first principles in this section.

The matrix strategy for solving IFPP-I permits construction of PWL Markov and semi-Markov maps, as defined and characterised in section 2.6. These maps possess branch functions that map intervals between partitions defined over the unit interval. Ergodic semi-Markov maps constructed using matrix solutions to IFPP-I possess a unique invariant density that is PWC over a prescribed Markov partition [21]. Although these solutions only permit the approximate realisation of prescribed invariant densities that are smooth, they afford greater freedom to select the branch function domains and co-domains, and permit maps that need not be complete [1].

It was proved in Remark 2.6.1 that the $\mathcal{Q}$ -Markov maps are a subset of the $\mathcal{R}(\mathcal{Q})$ -semi-Markov maps. Hence, in this work, attention is restricted to the matrix solution strategy for semi-Markov maps only. This strategy is outlined below.

Outline

Góra and Boyarsky [21] proposed a matrix solution strategy to IFPP-I that constructs a PWL and ergodic $\mathcal{R}(\mathcal{Q})$ -semi-Markov map S with a prescribed PWC invariant density. This strategy follows three steps, referred to as the *matrix solution process* [1]:

1. Prescribe a Markov partition $\mathcal{Q}$ and a density $\hat{f}_P \in \mathcal{D}_{\mathcal{Q}}^{(0)}(\Omega, \mathcal{B}, \mu)$ that is PWC over $\mathcal{Q}$ and strictly greater than zero almost everywhere. The ergodic map S will be constructed such that its invariant density is equal to $\hat{f}_P$. Although the selection of $\hat{f}_P$ to approximate an arbitrary prescribed density f_P was not considered in [21], this problem is addressed by the work presented here.
2. Construct a matrix $\mathbf{P}$ with characteristics that are sufficient to qualify it as the order-zero FPM of an ergodic semi-Markov map, in addition to the matrix possessing the required left and right Perron eigenvectors that yield the required Markov partition and invariant density (to be defined below).

3. Construct an ergodic map S such that its order-zero FPM $\mathbf{P}_S^{(0)}$ is equal to $\mathbf{P}$.

Suppose that S is an $\mathcal{R}(\mathcal{Q})$ -semi-Markov map that satisfies the conditions of the Folklore theorem (i.e., Theorem 2.6.3), such that S is ergodic. Proposition 2.6.4 states that the left Perron eigenvector of the order-zero FPM of S has elements that coincide with the lengths of the intervals in partition $\mathcal{Q}$. Its counterpart, Proposition 2.6.9, states that the right Perron eigenvector of the order-zero FPM is equal to $\mathbf{f}_S^*$, the vector representation of the map's invariant density f_S^* . Hence, by designing a semi-Markov map S using the matrix solution process to satisfy the conditions of the Folklore theorem and to possess an order-zero FPM with Perron eigenvectors selected to be consistent with the vector representation $\hat{\mathbf{f}}_P$ of the density $\hat{f}_P$ and the partition over which it is defined, its invariant density is necessarily equal to $\hat{f}_P$.

The following remark proves that any map constructed using the process outlined above is not unique. Stated differently, the process may be used to construct distinct maps S defined over the same Markov partition and with the same invariant density.

Remark 3.4.4. Multiple distinct maps may be constructed to possess the same prescribed order-zero FPM $\mathbf{P}_S^{(0)} = \mathbf{P}$ in step 3 of the above process. To prove this statement, let S denote any $\mathcal{R}(\mathcal{Q})$ -semi-Markov map that possesses an order-zero FPM equal to $\mathbf{P} = [p_{k,m}]$. This implies that the slopes $v_{j,m}$ of the map's branch functions $S_{j,m} = S|_{R_j^{(m)}}$ satisfy (2.122).

Now, define a second map $\tilde{S}$ to be identical to S , but redefine any of its branch functions $\tilde{S}_{j,m}$ such that its slope $\tilde{v}_{j,m}$ changes sign (i.e., $\tilde{v}_{j,m} \triangleq -v_{j,m}$) and that the domain and image of the branch remain unchanged. Equation (2.122) implies that the elements of the order-zero FPM of a semi-Markov map are only dependent on the *absolute values* of its branch functions' slopes. Hence, the order-zero FPM of $\tilde{S}$ remains unchanged, and is identical to that of S , despite the maps being distinct.

Boyarsky and Góra [55] recognised that changing any branch of a Markov map from monotonically increasing to decreasing (or vice versa) in the manner outlined above does not change the map's order-zero FPM. In particular, [55, Remark 9.2.2] states that there exists 2^K possible $\mathcal{R}$ -Markov maps that possess the same order-zero FPM, where K denotes the number of intervals in the partition $\mathcal{R}$.

The three steps of the matrix solution process are developed below. Two methods for deriving an approximation $\hat{f}_P$ of the prescribed invariant density f_P are defined. *These methods are not designed to nor intended to optimally estimate an arbitrary*

*prescribed invariant density from a specified class.*⁵ Sufficient characteristics that qualify matrix $\mathbf{P}$ as the order-zero FPM of an ergodic semi-Markov map with the prescribed Markov partition and invariant density are then identified, and the problem of generating matrices with these properties is considered. Thereafter, an algorithm is derived for constructing a semi-Markov map S such that its order-zero FPM is equal to $\mathbf{P}$. Sufficient conditions under which S is ergodic and possesses the required invariant density are then stated.

Step 1: Prescription of the Markov partition and invariant density The invariant density of an ergodic semi-Markov map is necessarily PWC over the intervals of its Markov partition, and can (in general) not be selected to equal an arbitrary invariant density f_P that may be prescribed. It is thus necessary to select the PWC invariant density to approximate the prescribed invariant density. Two methods are presented for deriving the approximation $\hat{f}_P \in \mathcal{D}_Q^{(0)}$ of the prescribed invariant density f_P . These methods, their approach to approximation, and the requirements imposed on the design of the order-zero FPM, are summarised in Table 3.4.1.

Both methods were developed by assuming that the number of intervals M in the Markov partition $\mathcal{Q}$ is predetermined and remains fixed. The free parameters available to any method then comprise the lengths of the intervals Q_m in partition $\mathcal{Q}$ (or equivalently, its interval endpoints $q_2, q_3, \dots, q_M$), and the level of the density $\hat{f}_P$ over each interval Q_m , subject to the area under the density being equal to unity.

Method 1 In this method, the partition $\mathcal{Q}$ is selected to be uniform. The level of the PWC density $\hat{f}_P$ over each interval Q_m is selected such that the areas under the corresponding densities, over each partition interval Q_m , are equal; i.e.,

$$\|\hat{f}_P|_{Q_m}\|_1 = \|f_P|_{Q_m}\|_1, \quad m = 1, 2, \dots, M. \quad (3.64)$$

The elements of the vector representation $\hat{\mathbf{f}}_P = [\hat{f}_1, \hat{f}_2, \dots, \hat{f}_M]^T$ of $\hat{f}_P$ are selected as

$$\hat{f}_m = \frac{1}{\mu(Q_m)} \int_{Q_m} f_P(x) dx. \quad (3.65)$$

The above selection guarantees $\|\hat{f}_P\|_1 = 1$. The distribution functions $\hat{F}_P$ and F_P corresponding to $\hat{f}_P$ and f_P then satisfy

$$\hat{F}_P(q_m) = F_P(q_m), \quad m = 1, 2, \dots, M + 1. \quad (3.66)$$

⁵This was not pursued, as the novel methods of the subsequent chapters that draw upon the theory presented here are constrained with respect to the class of PWC invariant densities realisable.

Table 3.4.1: Summary of methods 1 and 2 for approximating an arbitrary density over the intervals of a partition.

Method	Equations	Approach to approximation	Requirements imposed on the design of the order-zero FPM
Method 1	(3.64) to (3.66)	The partition is selected to possess intervals of uniform length. The approximating density is selected such that the area under it, when restricted to each partition interval, is equal to the area under the approximated density when restricted to the same interval.	Prescription of arbitrary right Perron eigenvector.
Method 2	(3.67) to (3.70)	The partition is selected such that the areas under the approximated density, when restricted to the partition intervals, are all equal. The approximating density is then selected such that the area under it, when restricted to each partition interval, is equal to the area under the approximated density when restricted to the same interval.	Prescription of right Perron eigenvector with uniform elements.

Method 2 In this method, the partition $\mathcal{Q}$ is selected such that the areas under $f_P|_{Q_1}, f_P|_{Q_2}, \dots, f_P|_{Q_M}$ are all equal; i.e.,

$$\int_{Q_m} f_P(x) dx = \frac{1}{M}, \quad m = 1, 2, \dots, M. \quad (3.67)$$

The distribution function $\hat{F}_P$ corresponding to $\hat{f}_P$ is then selected such that

$$\hat{F}_P(q_m) = F_P(q_m) = \frac{m-1}{M}, \quad m = 1, 2, \dots, M+1, \quad (3.68)$$

where F_P is the distribution function corresponding to f_P . The interval endpoints q_m are then given by

$$q_m = F_P^{-1}\left(\frac{m-1}{M}\right), \quad m = 1, 2, \dots, M+1. \quad (3.69)$$

The elements of $\hat{\mathbf{f}}_P = [\hat{f}_1, \hat{f}_2, \dots, \hat{f}_M]^T$ are selected as

$$\hat{f}_m = \frac{1}{M\mu(Q_m)}. \quad (3.70)$$

Accuracy of the methods Methods 1 and 2 described above realise a PWL approximation $\hat{F}_P$ of the prescribed distribution function F_P . The following propositions are proved.

Proposition 3.4.1. Let $F_P \in C^2$. The L^∞ error between $\hat{F}_P$ and F_P is then bounded as

$$\left\| \hat{F}_P - F_P \right\|_\infty \leq \max_{m=1,2,\dots,M} \frac{\mu(Q_m)^2}{8} \|f'_{P,m}\|_\infty, \quad (3.71)$$

where

$$f'_{P,m} = \left. \frac{df_P}{dx} \right|_{Q_m}. \quad (3.72)$$

Proof. The proposition is proved in section 3.A.1. □

Proposition 3.4.2. Suppose that $F_P \in C^2$ and let $\hat{f}_P$ be derived using method 1. If $|f'_P|$ is bounded, the L^∞ error between the distribution functions $\hat{F}_P$ and F_P can be made arbitrarily small by choosing a partition $\mathcal{Q}$ with a sufficient number of intervals.

Proof. Method 1 chooses the partition $\mathcal{Q}$ to be uniform, such that $\mu(Q_m) = 1/M$ for all $m = 1, 2, \dots, M$. It follows that

$$\lim_{M \rightarrow \infty} \left\| \hat{F}_P - F_P \right\|_\infty \leq \lim_{M \rightarrow \infty} \frac{1}{8M^2} \|f'_P\|_\infty = 0. \quad (3.73)$$

□

Remark 3.4.5. Although a similar result is conjectured to hold for method 2, it is not proved here.

Remark 3.4.6. In the IFPP literature, methods that use the matrix solution generally do not consider how the PWC density $\hat{f}_P$ is selected to approximate an arbitrary smooth density f_P prescribed for the map. Although methods 1 and 2 presented above are elementary and known in other branches of the literature, these methods impose different design requirements on the matrix $\mathbf{P}$ constructed in step 2, as investigated in the next subsection. These design requirements become relevant in chapter 4, where a novel method is developed that imposes additional design requirements on the matrix. This justifies the formal definition of the two methods.

Step 2: Construction of the matrix $\mathbf{P}$ Proposition 2.6.5 states that the order-zero FPM of *any* $\mathcal{R}(\mathcal{Q})$ -semi-Markov map S is non-negative and similar to a column stochastic matrix. In particular, the order-zero FPM of S may be expressed as

$$\mathbf{P}_S^{(0)} = \mathbf{D}\tilde{\mathbf{P}}\mathbf{D}^{-1}, \quad (3.74)$$

where $\tilde{\mathbf{P}}$ is column stochastic and $\mathbf{D}$ is the diagonal matrix

$$\mathbf{D} = \text{diag}\{\mu(Q_1)^{-1}, \mu(Q_2)^{-1}, \dots, \mu(Q_M)^{-1}\}. \quad (3.75)$$

In this expression, $\mu(Q_m) > 0$ for all $m = 1, 2, \dots, M$. This implies that only the construction of matrices with the form of (3.74) need be considered in step 2 of the matrix solution process. Furthermore, these equations imply that a candidate order-zero FPM $\mathbf{P}$ that corresponds to a semi-Markov map over an arbitrary partition $\mathcal{Q}$ may be constructed by first constructing a column stochastic matrix $\tilde{\mathbf{P}}$ and then computing the similarity transform [1]

$$\mathbf{P} = \mathbf{D}\tilde{\mathbf{P}}\mathbf{D}^{-1}. \quad (3.76)$$

In the context of step 2, attention may thus be restricted to methods for constructing column stochastic matrices.

Propositions 2.6.4 and 2.6.9 imply that the matrix $\mathbf{P}$ serving as the candidate order-zero FPM of a semi-Markov map that satisfies the conditions of the Folklore theorem (and is therefore ergodic) must possess a left Perron eigenvector that coincides with the lengths of the intervals in the partition $\mathcal{Q}$ and a right Perron eigenvector that coincides with $\hat{\mathbf{f}}_P$. The similarity transform defined by (3.76) bestows⁶ the required

⁶Recall that similar matrices share the same characteristic equation, and therefore share the same eigenspectrum.

left Perron eigenvector on the matrix $\mathbf{P}$. The same equation implies that the right Perron eigenvector $\mathbf{w}_1$ of $\mathbf{P}$ satisfies [1]

$$\mathbf{w}_1 = \mathbf{D}\tilde{\mathbf{w}}_1, \quad (3.77)$$

where $\tilde{\mathbf{w}}_1$ is the right Perron eigenvector of $\tilde{\mathbf{P}}$. Hence, in order to ensure that $\mathbf{w}_1 = \hat{\mathbf{f}}_P$, the matrix $\tilde{\mathbf{P}}$ must be designed to possess the right Perron eigenvector [1]

$$\tilde{\mathbf{w}}_1 = \mathbf{D}^{-1}\hat{\mathbf{f}}_P. \quad (3.78)$$

Now, consider the problem of constructing a column stochastic matrix $\tilde{\mathbf{P}}$ with the required right eigenvector. Suppose that $\tilde{\mathbf{P}}$ is primitive, such that the Perron-Frobenius theorem [64] holds. This implies that $\mathbf{P}$ has a unique unity Perron eigenvalue. If method 1 of step 1 is used to derive the approximating density $\hat{f}_P$, which selects the Markov partition $\mathcal{Q}$ to be uniform, no similarity transform is required. This is evident from (3.75), which implies that $\mathbf{D} = M\mathbf{I}$.

Although the similarity transform is unnecessary, the matrix $\tilde{\mathbf{P}}$ is required to possess a right Perron eigenvector equal to $\hat{\mathbf{f}}_P$. This eigenvector depends on the (arbitrarily) prescribed invariant density f_P . This implies that the method for constructing the column stochastic matrix $\tilde{\mathbf{P}}$ must also permit selection of the right Perron eigenvector of the matrix. Two such methods [21, 30] have been successfully applied to solve IFPP-I; the method in [21] is demonstrated in Example 3.4.3.

Remark 3.4.7. A novel method for solving IFPP-II that uses the matrix strategy is developed in chapter 4 (see also [3]). This method requires that the eigenvalues of the matrix $\tilde{\mathbf{P}}$ be prescribable. If method 1 for density approximation was used in this context, the stochastic matrix $\tilde{\mathbf{P}}$ must then be designed with a particular right Perron eigenvector and eigenspectrum, which is a complex problem. It is shown below that method 2 imposes less stringent design requirements on the matrix $\tilde{\mathbf{P}}$.

Suppose that method 2 of step 1 is instead used to construct the approximating density $\hat{f}_P$. Substitution of (3.70) and (3.75) into (3.78) yields

$$\tilde{\mathbf{w}}_1 = [1/M, 1/M, \dots, 1/M]^T. \quad (3.79)$$

This implies that the matrix $\tilde{\mathbf{P}}$ is required to be doubly stochastic. However, the expression also implies that the method for generating $\tilde{\mathbf{P}}$ need not permit the selection of an *arbitrary* right eigenvector associated with the unity eigenvalue of the matrix. Hence, it imposes less stringent design requirements on the stochastic matrix.

Remark 3.4.8. The problem of generating column stochastic matrices with the additional requirement of specifying the eigenspectrum of the matrix is referred to

in the literature as the *stochastic inverse eigenvalue problem* [65]. Several solutions to this problem have appeared (an overview of these solutions, as well as a novel analytical solution by the author of this thesis [14], are presented in section 4.6.2 of chapter 4). Any such solution that permits the construction of a doubly stochastic matrix may be successfully applied to solve IFPP-I using method 2 of step 1.⁷

Step 3: Construction of the map S An algorithm for constructing an $\mathcal{R}(\mathcal{Q})$ -semi-Markov map S with its order-zero FPM equal to a non-negative matrix $\mathbf{P}$, where $\mathbf{P}$ is similar to a stochastic matrix, is developed below. This algorithm was first proposed in [21]. Proposition 3.4.4, which follows thereafter, states conditions for S to be ergodic and possesses the invariant density $f_S^* = \hat{f}_P$.

The algorithm is developed by iteratively selecting certain features of S and determining the implications of each selection. Let $\mathcal{Q}$ denote a partition of the unit interval and let $\mathbf{P}$ be similar to a column stochastic matrix. Let $\mathbf{P}$ possess a left eigenvector $\mathbf{v}_1^T$ associated with the unity eigenvalue that has strictly positive elements given by

$$\mathbf{v}_1^T = [\mu(Q_1), \mu(Q_2), \dots, \mu(Q_M)]. \quad (3.80)$$

A partition $\mathcal{R}^{(m)} = \{R_1^{(m)}, R_2^{(m)}, \dots, R_{c(m)}^{(m)}\}$ of the closure of each interval Q_m is constructed, where $m = 1, 2, \dots, M$. To select a valid partition, the conditions imposed by (2.122) need to be considered. Equation (2.122) implies that any non-zero element $p_{k,m}$ in column m of $\mathbf{P}$ requires *at least one* interval in $\mathcal{R}^{(m)}$ to be mapped to the corresponding interval Q_k . The same equation implies that any zero element $p_{k,m}$ in column m of $\mathbf{P}$ requires that *no* interval in $\mathcal{R}^{(m)}$ be mapped to Q_k . As $\mathbf{P}$ is stochastic, it must have at least one non-zero element in each column.

Suppose that the partition $\mathcal{R}^{(m)}$ is selected such that (i) each of its intervals is mapped to a *unique* interval Q_k (i.e., no other interval from $\mathcal{R}^{(m)}$ is also mapped to Q_k), and (ii) its intervals are mapped in the same order as the corresponding non-zero elements appear in column m of the matrix $\mathbf{P}$. Let

$$z_{1,m} < z_{2,m} < \dots < z_{w(m),m}, \quad m = 1, 2, \dots, M, \quad (3.81)$$

denote the indices of the $w(m)$ non-zero entries in column m of matrix $\mathbf{P}$. Requirement (i) implies that

$$c(m) = w(m), \quad m = 1, 2, \dots, M. \quad (3.82)$$

Let $\mathcal{R}^{(m)} \triangleq \{R_1^{(m)}, R_2^{(m)}, \dots, R_{c(m)}^{(m)}\}$. Requirement (ii) then implies that

$$R_j^{(m)} \xrightarrow{S} Q_{U(j,m)}, \quad j = 1, 2, \dots, c(m), \quad m = 1, 2, \dots, M, \quad (3.83)$$

⁷The extra degree of freedom of selecting the eigenspectrum is applied in a later chapter to specify the non-repeating poles of the map's rational PSD during its design.

where $U(j, m) \triangleq z_{j,m}$. In turn, this implies that the function T , defined by

$$T(k, m) \triangleq \{j : S(R_j^{(m)}) = Q_k\}, \quad k, m = 1, 2, \dots, M, \quad (3.84)$$

must be selected as

$$T(k, m) \triangleq \begin{cases} \{j\}, & \text{if } \exists j : U(j, m) = k, \\ \emptyset, & \text{otherwise,} \end{cases} \quad (3.85)$$

for all $k, m = 1, 2, \dots, M$.

Let the branch functions $S_{j,m}(x) \triangleq S|_{R_j^{(m)}}$ be defined as

$$S_{j,m}(x) = v_{j,m}x + w_{j,m}, \quad x \in R_j^{(m)}. \quad (3.86)$$

Equation (2.122) then implies that the branch slopes must satisfy

$$|v_{j,m}|^{-1} = p_{U(j,m),m}, \quad j = 1, 2, \dots, c(m), \quad m = 1, 2, \dots, M. \quad (3.87)$$

The branch slopes are selected to be positive; i.e.,

$$v_{j,m} = p_{U(j,m),m}^{-1}, \quad j = 1, 2, \dots, c(m), \quad m = 1, 2, \dots, M. \quad (3.88)$$

The intervals of partition $\mathcal{R} = \{R_j^{(m)}\}$, where $R_j^{(m)} \triangleq (r_j^{(m)}, r_{j+1}^{(m)})$, then have endpoints that satisfy

$$r_j^{(m)} = \begin{cases} q_m + \sum_{k=1}^{U(j,m)-1} \mu(Q_k) |v_{k,m}|^{-1}, & \text{if } j = 1, 2, \dots, c(m), \\ q_{m+1}, & \text{if } j = c(m) + 1, \end{cases} \quad (3.89)$$

where $Q_m = (q_m, q_{m+1})$ and $m = 1, 2, \dots, M$. As each branch function $S_{j,m}$ has a positive slope, it passes through the point $(r_j^{(m)}, q_{U(j,m)})$. The y -intercept $w_{j,m}$ of each branch may then be determined from (3.86) as

$$w_{j,m} = q_{U(j,m)} - v_{j,m} r_j^{(m)}, \quad j = 1, 2, \dots, c(m), \quad m = 1, 2, \dots, M. \quad (3.90)$$

It follows that (3.88), (3.89) and (3.90) define the branches of map S in (3.86) as well as the partition $\mathcal{R}$. Hence, the map S of (2.109) is completely specified.

The above algorithm for constructing the semi-Markov map S is referred to as Algorithm 3.4.1 (canonical map construction), and is formally specified on page 128; the term *canonical* is to be formally defined in what follows.

Remark 3.4.9. An example of a map constructed using Algorithm 1 (canonical map construction) is plotted in Fig. 3.4.8. Observe that the algorithm arbitrarily

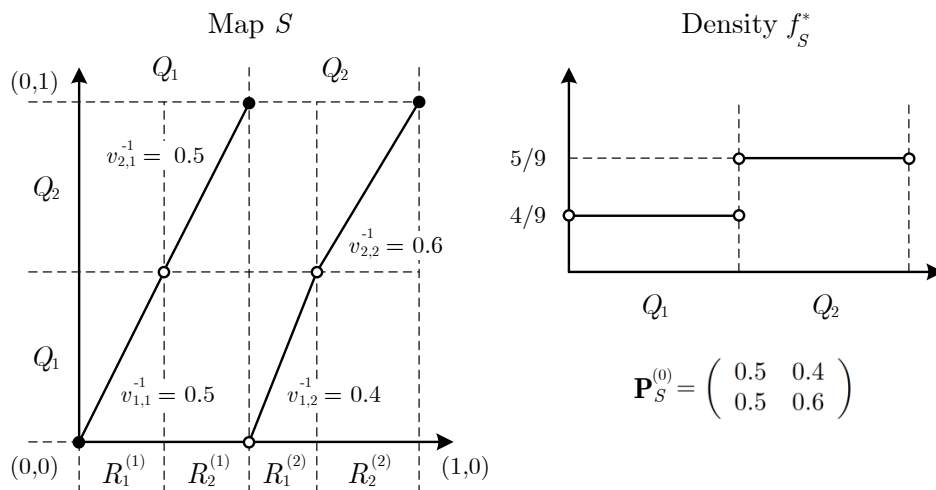


Figure 3.4.8: Example of a semi-Markov map constructed using Algorithm 3.4.1 (canonical map construction).

Algorithm 3.4.1 (canonical map construction (see [1])): Construction of a canonical semi-Markov map S such that its order-zero FPM equals a matrix $\mathbf{P}$, where $\mathbf{P}$ is non-negative and similar to a column stochastic matrix.

- 1: Input: Matrix $\mathbf{P} = [p_{k,m}]_{k,m=1}^M$ and Markov partition $\mathcal{Q} = \{Q_1, Q_2, \dots, Q_M\}$.
 - 2: Find the indices $z_{j,m}$ of the non-zero elements in row m of $\mathbf{P}$ (refer to (3.81)).
 - 3: Define the number of intervals $c(m)$ in partition $\mathcal{R}^{(m)}$ using (3.82).
 - 4: Define the branch slopes $v_{j,m}$ using (3.88).
 - 5: Compute the intervals of $\mathcal{R} = \{R_j^{(m)}\}$, where $R_j^{(m)} \triangleq (r_j^{(m)}, r_{j+1}^{(m)})$, using (3.89).
 - 6: Compute the y -intercepts $w_{j,m}$ of the branches using (3.90).
 - 7: Return the map S defined by (2.109).
-

selects the slope of each branch function to be positive. This choice is arbitrary, and any of the branches may be redefined such that the slope is negative, without changing the order-zero FPM of the map (provided the domain and image of each branch function remains the same).

Algorithm 1 (canonical map construction) assigns a strictly ascending order, over the unit interval, on the images of the intervals in each of the partitions $\mathcal{R}^{(m)} = \{R_1^{(m)}, R_2^{(m)}, \dots, R_{c(m)}^{(m)}\}$. PWL semi-Markov maps that possess this property are referred to as canonical [2]; this map class is formally defined below.

Definition 3.4.5 (Canonical PWL semi-Markov map [2]). Let $S : \Omega \rightarrow \Omega$ denote a PWL $\mathcal{R}(\mathcal{Q})$ -semi-Markov map and let

$$R_j^{(m)} \xrightarrow{S} Q_{U(j,m)}, \quad (3.91)$$

where $U \in \{1, 2, \dots, M\}$. Map S is canonical iff

$$j_1 < j_2 \implies U(j_1, m) < U(j_2, m) \quad (3.92)$$

for all $m = 1, 2, \dots, M$.

Figure 3.4.9(a) illustrates the structure of a canonical map by representing the region spanned by each branch function's domain and image with a shaded rectangle [2]. The condition of (3.92) imposes the strictly ascending order on the images of the intervals. Non-canonical maps do not possess this property; examples of such maps are provided in Fig. 3.4.9(b) and Fig. 3.4.9(c).

The following propositions are proved.

Proposition 3.4.3 (Maps constructed using Algorithm 3.4.1 (canonical map construction) are semi-Markov). Let $\mathcal{Q}$ denote a partition of the unit interval, and let $\mathbf{P}$ denote a non-negative $M \times M$ matrix. Let $\mathbf{P}$ possess a left eigenvector $\mathbf{v}_1^T$ associated with the unity eigenvalue that has strictly positive elements given by (3.80). The map S constructed using Algorithm 3.4.1 (canonical map construction) is then an $\mathcal{R}(\mathcal{Q})$ -semi-Markov map.

Proof. The proposition is proved in section 3.A.2. □

Observe that Proposition 3.4.3 states sufficient conditions for a matrix $\mathbf{P}$ to qualify as the order-zero FPM of a semi-Markov map. The following proposition states sufficient conditions for the ergodicity of the semi-Markov map constructed by Algorithm 3.4.1 (canonical map construction) to possess this matrix as its order-zero FPM.

Proposition 3.4.4 (Ergodicity of maps constructed using Algorithm 3.4.1 (canonical map construction)). Let $\mathcal{Q}$ denote a partition of the unit interval, and let $\mathbf{P}$ denote a non-negative $M \times M$ matrix that possesses a unity eigenvalue. Let the left eigenvector $\mathbf{v}_1^T$ associated with the unity eigenvalue be given by (3.80), and let its corresponding right eigenvector $\mathbf{w}_1$ coincide with the vector representation of a density $\hat{f}_P \in \mathcal{D}_{\mathcal{Q}}^{(0)}$ that is greater than zero almost everywhere. Furthermore, let S denote the map constructed from $\mathcal{Q}$ and $\mathbf{P}$ using Algorithm 3.4.1 (canonical map construction).

Define

$$\tilde{\mathbf{P}} = \mathbf{H}\mathbf{P}\mathbf{H}^{-1}, \quad (3.93)$$

where

$$\mathbf{H} = \text{diag}\{\mu(Q_1), \mu(Q_2), \dots, \mu(Q_M)\}. \quad (3.94)$$

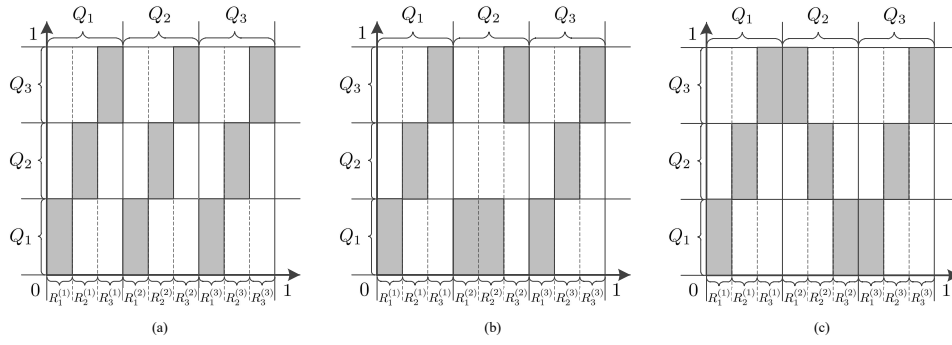


Figure 3.4.9: Branch structure of $\mathcal{R}(\mathcal{Q})$ -semi-Markov maps that are (a) canonical and (b, c) not canonical, where the region spanned by each branch domain and image is represented by a shaded rectangle. Reproduced with permission from [2] ((© [2023] IEEE)).

If $\tilde{\mathbf{P}} = [\tilde{p}_{k,m}]$ is primitive and possesses at least one element $\tilde{p}_{k,m}$ in each of its columns that satisfies

$$0 < \tilde{p}_{k,m} < 1, \quad m = 1, 2, \dots, M, \quad (3.95)$$

then S is an ergodic $\mathcal{R}(\mathcal{Q})$ -semi-Markov map with invariant density $f_S^* = \hat{f}_P$.

Proof. The proposition is proved in section 3.A.3. $\square$

Remark 3.4.10. Equations (3.94) and (3.75) imply that $\mathbf{H} = \mathbf{D}^{-1}$. Hence, (3.93) and (3.76) are equivalent. Thus, in order for the map S constructed using Algorithm 1 (canonical map construction) to be ergodic, the column stochastic matrix $\tilde{\mathbf{P}}$ constructed during step 2 of the matrix solution process need only be primitive and possess at least one element in each of its columns that is strictly larger than zero and smaller than unity.

In the following example, the matrix solution of IFPP-I is illustrated by constructing ergodic and PWL semi-Markov maps with invariant densities selected to approximate the quadratic densities defined in (3.10) and plotted in Figs. 3.4.1 and 3.4.2.

Example 3.4.3 (Matrix solution [1]). Step 1 of the matrix solution process requires the specification of the Markov partition $\mathcal{Q}$ and the construction of the PWC density approximation $\hat{f}_P \approx f_P$. The Markov partition is selected to be uniform and possess M intervals. Method 1 is used to construct the approximation $\hat{f}_P$ in each case. The elements of the vector representation $\hat{\mathbf{f}}_P = [\hat{f}_1, \hat{f}_2, \dots, \hat{f}_M]^T$ of each approximation $\hat{f}_P$ are then defined by (3.65), and computed as

$$\hat{f}_m = \frac{a}{3} \left[\frac{m^3 - (m-1)^3}{M^2} \right] + \frac{b}{2} \left[\frac{m^2 - (m-1)^2}{M} \right] + c, \quad m = 1, 2, \dots, M, \quad (3.96)$$

where the coefficients a , b and c are defined in (3.11).

A column stochastic matrix $\tilde{\mathbf{P}}$ is constructed in step 2 of the matrix solution process. By using method 1 to construct the approximation $\hat{f}_P$ in step 1, the method used to construct $\tilde{\mathbf{P}}$ in step 2 must permit the prescription of the right Perron eigenvector of the matrix. The method proposed by Góra and Boyarsky [21] is used to construct each column stochastic matrix $\tilde{\mathbf{P}}$ with right Perron eigenvector equal to $\hat{\mathbf{f}}_P$.

The method of [21] constructs a *three-band* stochastic matrix, which is defined as a stochastic matrix having non-zero elements only on the main diagonal and the diagonals immediately above and below the main diagonal. Using this method, the vector $\mathbf{g} = [g_1, g_2, \dots, g_M]^T$ is defined as

$$\mathbf{g} = \frac{1}{2 \max\{\hat{f}_m\}} \hat{\mathbf{f}}_P. \quad (3.97)$$

The non-zero elements in the first and last columns of $\tilde{\mathbf{P}} = [\tilde{p}_{k,m}]_{k,m=1}^M$ are selected as

$$\begin{aligned} \tilde{p}_{1,1} &= 1 - g_2, \\ \tilde{p}_{2,1} &= g_2, \\ \tilde{p}_{M-1,M} &= g_{M-1}, \\ \tilde{p}_{M,M} &= 1 - g_{M-1}. \end{aligned} \quad (3.98)$$

The remaining non-zero elements of $\tilde{\mathbf{P}}$ are selected as

$$\begin{aligned} \tilde{p}_{m-1,m} &= g_{m-1}, \\ \tilde{p}_{m,m} &= 1 - g_{m-1} - g_{m+1}, \\ \tilde{p}_{m+1,m} &= g_{m+1}, \end{aligned} \quad (3.99)$$

where $m = 2, 3, \dots, M-1$. By using method 1 to construct the approximation $\hat{f}_P$ in step 1, the similarity transformation of matrix $\tilde{\mathbf{P}}$, as defined in (3.76), is unnecessary. Hence, matrix $\mathbf{P}$ is selected equal to $\tilde{\mathbf{P}}$.

In the final step of the matrix solution process, Algorithm 3.4.1 (canonical map construction) is used to construct an $\mathcal{R}(\mathcal{Q})$ -semi-Markov map S such that its FPM $\mathbf{P}_S$ is equal to each matrix $\mathbf{P}$ derived in the second step. Figure 3.4.10 contains plots of the semi-Markov maps that were constructed to possess invariant densities that approximate the quadratic density f_P of (3.10) with parameters $y_0 = y_1 = 1/3$ (this density is plotted in blue). Three maps were constructed over uniform Markov partitions of 4, 8 and 16 intervals. These maps are plotted in Figs. 3.4.10(a, c, e), respectively. The PWC invariant density function of each map is plotted in black in Figs. 3.4.10(b, d, f). The three-band property of the FPM is reflected by the

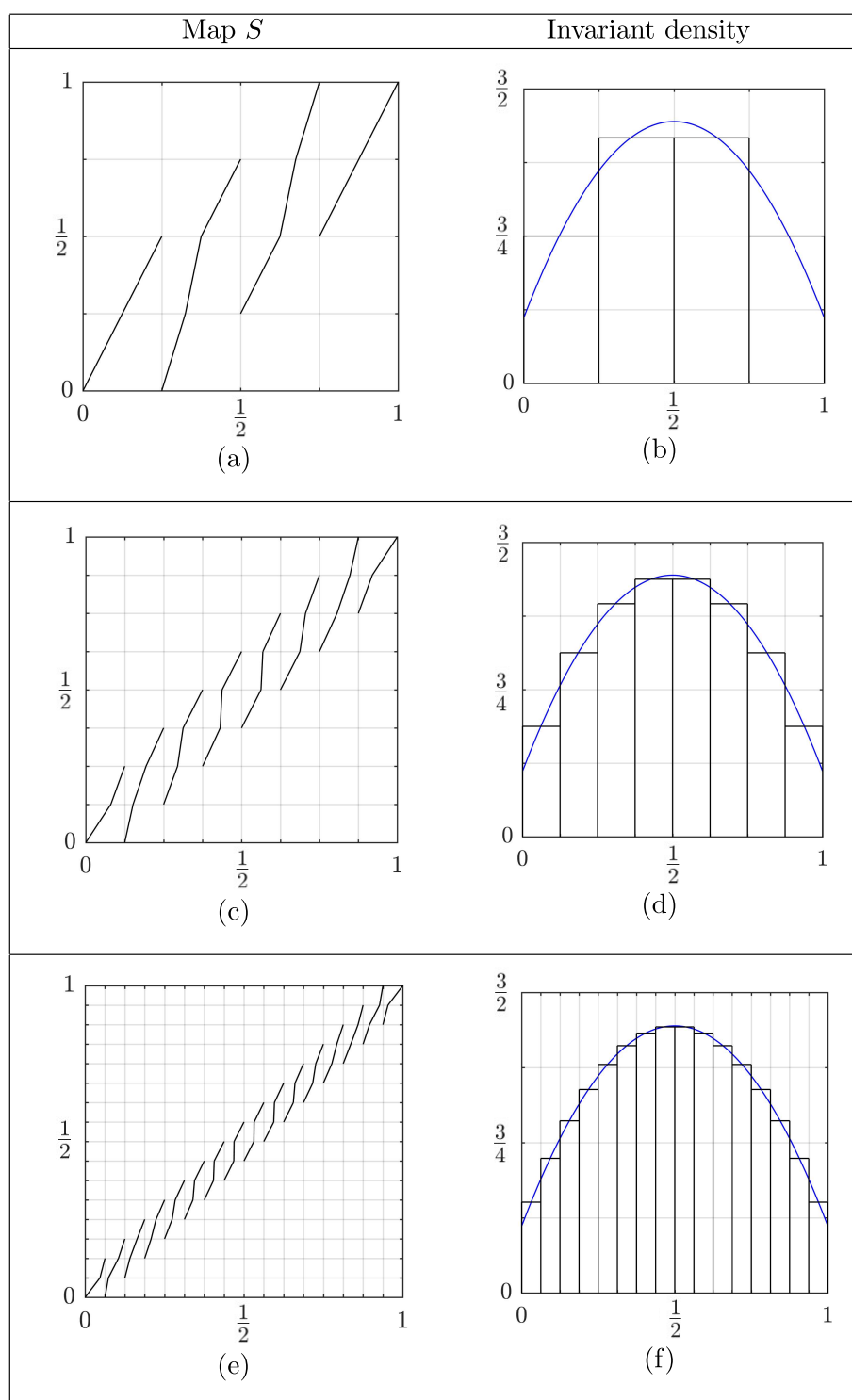


Figure 3.4.10: Semi-Markov maps with invariant density approximating the quadratic density f_P defined in (3.10) (with parameters $y_0 = y_1 = 1/3$). Reproduced with permission from [1].

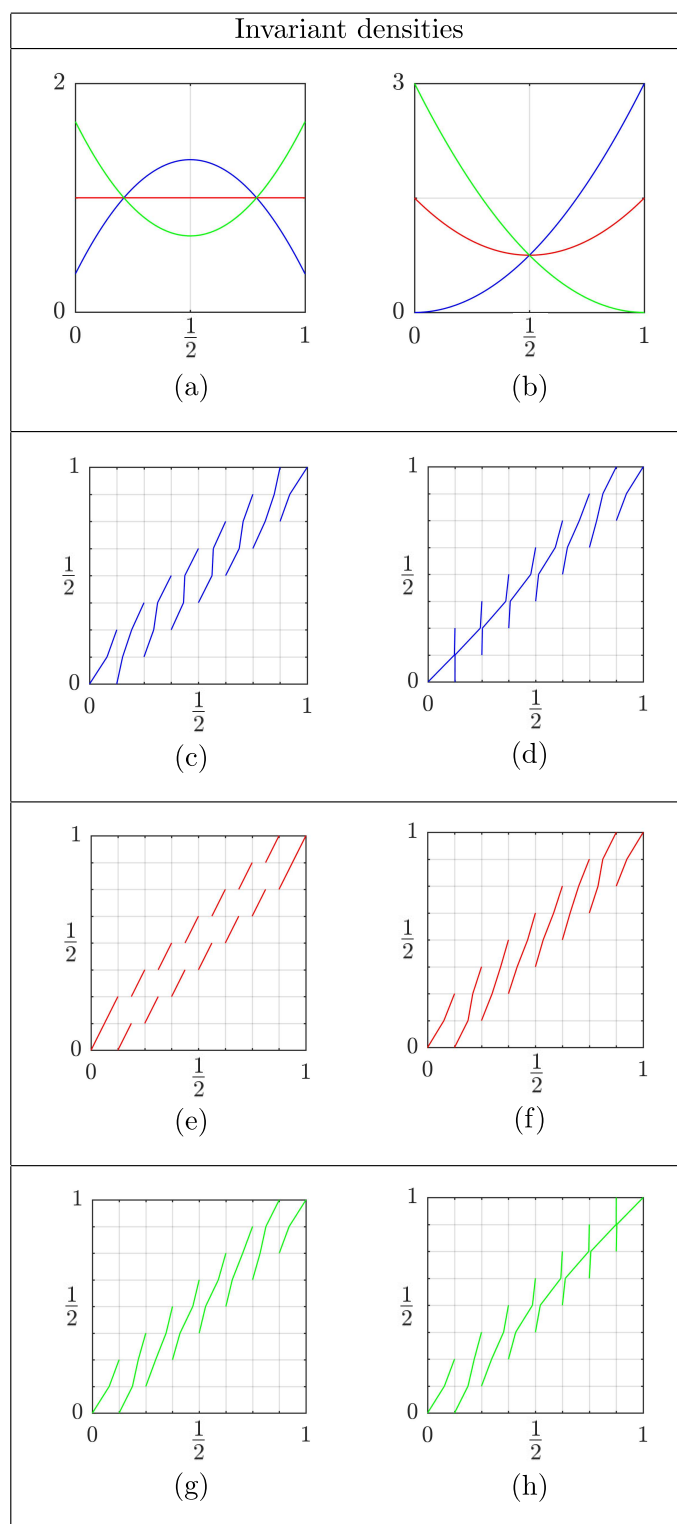


Figure 3.4.11: Semi-Markov maps with invariant densities approximating the quadratic densities f_P of Figs. 3.4.1 and 3.4.2. Reproduced with permission from [1].

structure of each map, where branches are concentrated around the line $y = x$. This observation reveals that the approach towards constructing matrix $\mathbf{P}$ has a direct impact on the structure of the resulting map. The plots also reveal that a finer partition produces a more accurate fit to the prescribed invariant density function.

To illustrate how the choice of invariant density affects the structure of the map, the matrix solution strategy was used to construct maps with invariant densities that approximate the quadratic invariant density functions f_P of Figs. 3.4.1 and 3.4.2. The resulting maps are plotted in Figures 3.4.11(c) to 3.4.11(h). These maps were constructed over a uniform Markov partition of eight intervals, and possess the invariant densities plotted in Figures 3.4.11(a, b).

Remark 3.4.11. Proposition 3.4.4 implies that Algorithm 3.4.1 (canonical map construction) can realise an $\mathcal{R}(\mathcal{Q})$ -semi-Markov map with an arbitrary PWC invariant density defined over an arbitrary unit-interval partition $\mathcal{Q}$, provided that a qualifying matrix $\mathbf{P}$ can be realised such that the left and right eigenvectors associated with its unity eigenvalue coincide with the lengths of the intervals in the partition $\mathcal{Q}$ and the vector representation of the required invariant density. The method used to construct a three-band column stochastic matrix, as proposed in [21] and demonstrated in example 3.4.3, together with an appropriate similarity transform of the column stochastic matrix, can realise a qualifying matrix $\mathbf{P}$ with eigenvectors that coincide with *any* PWC invariant density that may be specified over an arbitrary unit-interval partition $\mathcal{Q}$. This leads directly to a constructive proof that there exists at least one PWL and expanding semi-Markov map with an arbitrary PWC invariant density defined over any compact real interval [21, Theorem 5].

Literature

Friedman and Boyarsky [20] proposed a matrix technique for constructing ergodic maps with invariant densities from a class of PWC density functions with relative minima equal to zero [1]. The technique uses methods from the domain of graph theory. However, due to the restrictiveness of the class of achievable invariant densities, this technique was found to be of limited practicality [21].

Góra and Boyarsky [21] proved that there exists at least one PWL and expanding semi-Markov map with any PWC invariant density defined over a partition of a compact real interval, as stated in Remark 3.4.11 [1]. The proof is constructive, as demonstrated in example 3.4.3, and thus qualifies as a solution to IFPP-I. Recently, this solution was generalised by introducing K -band matrices, where $K > 3$ [30].

Rogers et al. [22] proposed a matrix solution to IFPP-I that constructs PWL and ergodic semi-Markov maps with invariant densities that are PWC over a uniform partition [1]. The proposed solution uses an $M \times M$ order-zero FPM with a predefined structure that is fully specified by a set of $2M$ real valued parameters. This structure leads to a simplified expression for the right Perron eigenvector of the order-zero FPM, which coincides with the invariant density of the map. Rogers [23] demonstrated how a suitable choice of parameters allows for the selection of a right Perron eigenvector that coincides with an arbitrary PWC invariant density over the uniform partition, and generalised the solution to non-uniform partitions.

The correlation properties of the maps constructed using Rogers' method were investigated numerically in [24] (see [1]). The solution was also extended to randomly switched chaotic maps and chaotic maps of higher dimensions. Convergence properties of Rogers' solution to IFPP-I [22], together with further examples of ergodic maps that were constructed using this solution, were provided in [25]. This solution to IFPP-I is considered in detail in section 3.5.3, where its extension to IFPP-II is explored.

3.5 Review of Literature: IFPP-II

IFPP-II requires the design of an ergodic map that simultaneously satisfies a prescribed invariant density and ACF⁸ as specifications [1, 3]. Existing techniques aimed at solving IFPP-II follows the approach of generalising the three strategies for solving IFPP-I to permit the prescription of the map's ACF. In the following sections, the methods developed under each solution strategy are described.

3.5.1 Topological conjugation solution strategy

The topological conjugation solution strategy, as applied to solve IFPP-I in section 3.4.1, first selects an underlying ergodic map S that possesses a known invariant density. A topological conjugate $\tilde{S}$ of the underlying map is then designed to possess the prescribed invariant density. The expression for a conjugating transformation that realises the prescribed invariant density for the conjugate map is given by (3.24).

⁸In the remainder of this section, reference will generally be made to prescribing the ACF, where it is understood that the PSD may equivalently be prescribed instead of the ACF.

Recall that the topological conjugation solution strategy affords freedom to choose the underlying map. In particular, Remark 3.4.2 states that the prescribed invariant density is realised for the conjugate map *irrespective of* the invariant density of the underlying map, provided the appropriate conjugating transformation is used and the latter density is greater than zero almost everywhere. Recall also that the appropriate conjugating transformation is a function of the underlying map's invariant density.

Now, consider the problem of generalising this strategy to permit the prescription of the conjugate map's ACF. The most straightforward and natural approach to generalising this strategy utilises the freedom that topological conjugation affords to ideally select the underlying map in such a manner that the conjugate map possesses the prescribed ACF. This is interpreted by some authors as a problem of selecting the ACF of the underlying map to realise the prescribed ACF for the conjugate map. However, the ACF of the underlying map and its topological conjugate are not equal in general; this is due to the required conjugating transformation being a non-linear function. This complicates the problem of selecting an appropriate underlying map.

The remainder of this section is dedicated to the extensive article of Baranovsky and Daems [33] on the above approach to generalising topological conjugation and its potential use as a method for solving IFPP-II. The theory is presented using surjective maps⁹ with domain and codomain equal to $\Omega' \triangleq [-1, 1]$. In [33], maps from two classes Γ_1 and Γ_2 were investigated as possible candidates for the underlying map. These are classes of piecewise monotonic and linear maps that are complete (Γ_1) and (possibly) non-Markov (Γ_2), respectively.

Although it was shown that the maps from the second class have greater spectral diversity than the maps from the first class, their more sophisticated structure and the resulting difficulty of their design ultimately disqualified them from consideration as the underlying map in [33]. Below, class Γ_1 is defined and characterised, and the topological conjugation strategy is defined using underlying maps from this class. Map class Γ_2 is summarised in section 3.B.

Map class Γ_1 The class $\Gamma_1(\mathcal{Q}')$ comprises the complete piecewise monotonic and linear maps defined on the partition $\mathcal{Q}' \triangleq \{Q_1, Q_2, \dots, Q_M\}$ of Ω' . This class is defined below. Thereafter, conditions are stated for the ergodicity of the maps from this class and expressions for the maps' ACF and PSD are provided.

⁹The interval Ω' is momentarily considered instead of the unit interval Ω , as it leads to simplified expressions for the ACF and PSD of the maps considered next.

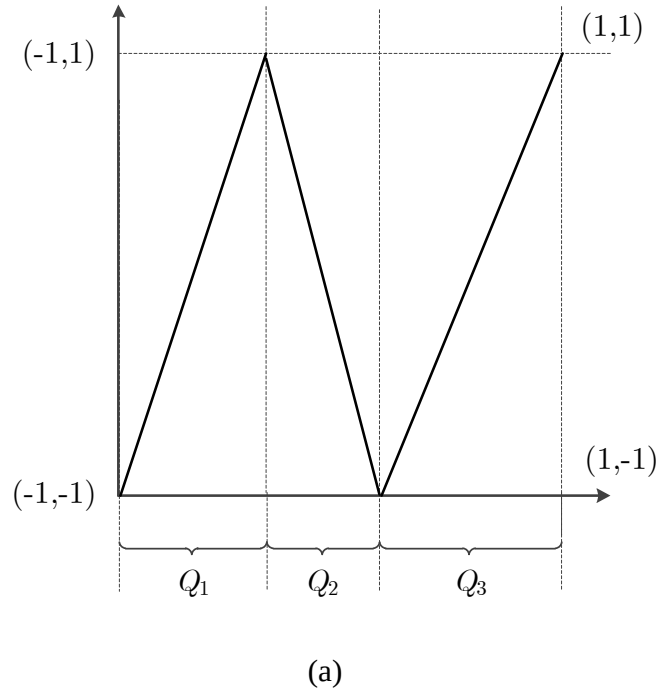


Figure 3.5.1: Example of a map belonging to the class $\Gamma_1(\mathcal{Q}')$.

Definition 3.5.1 (Map class Γ_1 [33]). A piecewise monotonic and linear map $S \in \Gamma(\mathcal{Q}')$ is also a member of $\Gamma_1(\mathcal{Q}')$ if it is complete (refer to Definition 3.4.3). Completeness requires that the image of each of the map's branches is equal to the map's codomain. More specifically, completeness requires that

$$\text{cl}(S(Q_m)) = \Omega', \quad \forall m = 1, 2, \dots, M. \quad (3.100)$$

Let $Q_m \triangleq (q_m, q_{m+1})$. The preceding definition implies that each branch function of $S \in \Gamma_1(\mathcal{Q}')$ may be expressed as

$$S_m(x) \triangleq S|_{Q_m}(x) = v_m x + w_m, \quad m = 1, 2, \dots, M, \quad (3.101)$$

where

$$v_m = \pm \frac{2}{q_{m+1} - q_m} \quad (3.102)$$

and

$$w_m = \pm \frac{q_m + q_{m+1}}{q_m - q_{m+1}}. \quad (3.103)$$

An example of a map $S \in \Gamma_1(\mathcal{Q}')$ is plotted in Fig. 3.5.1.

The following proposition is stated in [33].

Proposition 3.5.1. Let $\mathcal{Q}'$ denote a partition of Ω' with M intervals. If $M > 1$, then any map $S \in \Gamma_1(\mathcal{Q}')$ is ergodic and possesses the unique invariant density

$$f_S^*(x) = \frac{1}{2}, \quad x \in \Omega'. \quad (3.104)$$

Proof. The proposition is proved in [33, Section 2]. $\square$

The *normalised* ACF $\mathcal{R}_S^*(\tau)$ associated with an ergodic map $S \in \Gamma_1(\mathcal{Q}')$ is defined as

$$\mathcal{R}_S^*(\tau) \triangleq \lim_{n \rightarrow \infty} \frac{1}{n\bar{\sigma}_S^2} \sum_{k=0}^{n-1} (S^k(x_1) - \bar{\mathcal{M}}_S)(S^{|\tau|+k}(x_1) - \bar{\mathcal{M}}_S), \quad \tau \in 0, \pm 1, \pm 2, \dots \quad (3.105)$$

for almost every $x_1 \in \Omega'$. In this expression, $\bar{\mathcal{M}}_S \equiv \bar{\mathcal{M}}_S(x_1)$ is the mean value of the trajectory starting from almost every x_1 , as defined in (2.23), and $\bar{\sigma}_S^2$ is the corresponding variance, defined by

$$\bar{\sigma}_S^2 \equiv \bar{\sigma}_S^2(x_1) \triangleq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} (S^k(x_1) - \bar{\mathcal{M}}_S)^2 \quad (3.106)$$

for almost every $x_1 \in \Omega'$. In the preceding expressions, the extension to the Birkhoff individual ergodic theorem (Theorem 2.4.3), which implies that the mean value and variance of almost every trajectory is equal to the corresponding average computed over the state space, was invoked. Using the same theorem, and observing that $\bar{\mathcal{M}}_S = 0$ due to the uniformity of the invariant density f_S^* , it was shown in [33] that

$$\mathcal{R}_S^*(\tau) = \lambda^{|\tau|}, \quad \tau = 0, \pm 1, \pm 2, \dots, \quad (3.107)$$

where

$$\lambda \triangleq \sum_{m=1}^M \text{sgn}(v_m) \mu(Q_m)^2. \quad (3.108)$$

In the latter expression, $\text{sgn}(\cdot)$ denotes the sign of its argument, as defined in (2.121), and μ denotes the Borel measure.

Remark 3.5.1. Any complete piecewise monotonic and linear map $S \in \Gamma_1(\mathcal{Q}')$ is by definition also a semi-Markov map with the single-interval Markov partition $(-1, 1)$. Hence, the expression for the normalised ACF in (3.107) may be derived by adapting the theory presented for semi-Markov maps in section 2.6.4.

Remark 3.5.2. Three types of normalised ACFs may be realised by designing the map such that λ possesses a value from a particular range. The terminology of [33] refers to these ACFs as *exponential* (if $0 < \lambda < 1$), *delta-type* (if $\lambda = 0$) and *periodically modulated exponential* (if $-1 < \lambda < 0$).

The normalised PSD $\mathcal{S}_S^*$ associated with map S is expressed as a rational function. An expression for this PSD was derived in [33]:

$$\mathcal{S}_S^*(\omega) = \frac{1}{2\pi} \frac{1 - \lambda^2}{1 - 2\lambda \cos(\omega) + \lambda^2}. \quad (3.109)$$

Remark 3.5.3. Equation (3.109) may be rewritten as

$$\mathcal{S}_S^*(\omega) = \frac{1}{2\pi} \frac{1 - \lambda^2}{(e^{i\omega} - \lambda)(e^{-i\omega} - \lambda)}. \quad (3.110)$$

This expression reveals that, if $\lambda \neq 0$, the PSD has a single pole equal to λ ; otherwise, the PSD is uniform over frequency.

Remark 3.5.4. Equation (3.107) reveals that any map $S \in \Gamma_1(\mathcal{Q}')$ is restricted to possessing a single spectral mode that corresponds to λ . As $\lambda \in \mathbb{R}$, this spectral mode necessarily has a centre frequency $\omega \in \{0, \pi\}$. Hence, these maps are severely restricted with respect to the PSDs realisable. In general, semi-Markov maps may possess multiple spectral modes that correspond to the eigenvalues of the map's order-one FPM, such that the centre frequency of each mode coincides with the argument of the eigenvalue (which may be complex-valued). *It is observed that the restrictiveness of these maps' PSDs is a consequence of the maps being complete.*

Topological conjugation of maps in class Γ_1 Baranovsky and Daems [33] investigated the construction of topological conjugates $\tilde{S} = h \circ S \circ h^{-1}$ of the maps $S \in \Gamma_1(\mathcal{Q})$ with the aim of solving IFPP-II. In what follows, it is assumed that the partition $\mathcal{Q}$ of the unit interval has $M > 1$ intervals, such that the map S is ergodic and possesses a uniform invariant density. Equation (3.24) then implies that the required conjugating transformation is given by $h = F_P^{-1}$, where F_P is the distribution function associated with the prescribed invariant density f_P . The conjugate map $\tilde{S}$ is therefore given by

$$\tilde{S} = F_P^{-1} \circ S \circ F_P. \quad (3.111)$$

If arbitrary prescribed invariant densities $f_P > 0$ are considered, the distribution function F_P is in general non-linear. Topological conjugation with a non-linear conjugating transformation distorts the ACF of the underlying map S . To utilise topological conjugation to solve IFPP-II analytically, it then becomes necessary to derive expressions that relate the underlying map S (or its ACF) to the ACF of its conjugate map $\tilde{S}$. It was shown in [33] that the normalised ACF $\mathcal{R}_{\tilde{S}}^*$ of the conjugate map is given by

$$\mathcal{R}_{\tilde{S}}^*(\tau) = \frac{1}{\sigma_{\tilde{S}}^2} \int_0^1 F_P^{-1}(z) F_P^{-1}(S^{|\tau|}(z)) dz - \mathcal{M}_{\tilde{S}}^2, \quad \tau = 0, \pm 1, \pm 2, \dots \quad (3.112)$$

The authors of [33] observed that this general expression is analytically unsolvable for the unknown map S .

Remark 3.5.5. Equation (3.112) relates the ACF of the map S to the map itself. Due to the integral and the non-linearity of F_P , this expression is difficult to solve in an exact fashion for an unknown map S that generates a prescribed ACF.

It is instructive to compare (3.112) with the equivalent expression (2.144) for a PWL semi-Markov map. The latter expression exactly relates the ACF to a finite collection of the map's branch parameters through its order-one FPM in an unsophisticated manner. This admits viable methods for analytically designing the map.

It was shown in [33] that the restriction of the conjugating transformation to belong to the class of PWL functions leads to the simplification of (3.112). Suppose that an estimate $\hat{F}_P^{-1} \approx F_P^{-1}$ is constructed, such that $\hat{F}_P^{-1}$ is PWL on the intervals in partition $\mathcal{Q}$ (i.e., on the same intervals over which the branches of S are defined). The approximation may be constructed such that it equals F_P^{-1} when evaluated at the partition points $q_1, q_2, \dots, q_M$. This approximation is given by

$$\hat{F}_P^{-1}(z) = \sum_{m=1}^M \alpha_m x + \beta_m, \quad (3.113)$$

where

$$\alpha_m = \frac{F_P^{-1}(q_{m+1}) - F_P^{-1}(q_m)}{q_{m+1} - q_m}, \quad m = 1, 2, \dots, M \quad (3.114)$$

and

$$\beta_m = -q_m \alpha_m + F_P^{-1}(q_{m+1}), \quad m = 1, 2, \dots, M. \quad (3.115)$$

Now, let the topological conjugate $\tilde{S}$ be constructed using $h = \hat{F}_P^{-1}$ as conjugating transformation; i.e.,

$$\tilde{S} = \hat{F}_P^{-1} \circ S \circ \hat{F}_P. \quad (3.116)$$

If it were assumed that $\bar{\mathcal{M}}_{\tilde{S}} = 0$, the normalised ACF $\mathcal{R}_{\tilde{S}}^*$ of the map $\tilde{S}$, as given in (3.112), may be simplified to obtain [33, Eq. 91]:

$$\mathcal{R}_{\tilde{S}}^*(\tau) = [\mathcal{R}_S^*(1)]^{|\tau|}, \quad \tau = 0, \pm 1, \pm 2, \dots \quad (3.117)$$

This expression implies that the selection of a conjugating transformation that is PWL on the partition $\mathcal{Q}$ permits the normalised ACF of the conjugate map $\tilde{S}$ to be characterised by the normalised ACF of the underlying map S . However, as the conjugating transformation is only a PWL approximation of the transformation required to realise the exact prescribed invariant density f_P , the realised invariant

density f_S^* of the conjugate map $\tilde{S}$ does not equal the prescribed invariant density f_P . That is, $f_S^* \neq f_P$, unless F_P^{-1} itself is PWL on the intervals of the partition $\mathcal{Q}$.

Based on the analysis above, a method was proposed in [33] to construct a topological conjugate $\tilde{S}$ of an underlying map $S \in \Gamma_1(\mathcal{Q})$, such that $\tilde{S}$ has a prescribed invariant density f_P and a normalised ACF of the form

$$\mathcal{R}_S^*(\tau) \approx [\mathcal{R}_S^*(1)]^{|\tau|} = \lambda^{|\tau|}, \quad \tau = 0, \pm 1, \pm 2, \dots, \quad (3.118)$$

where λ is prescribed during the design of the underlying map. The method thus utilises the result that the normalised ACF of the conjugate map is characterised by its counterpart of the underlying map. However, instead of using a PWL approximation $\hat{F}_P^{-1}$ as a conjugating transformation, the method instead uses the transformation F_P^{-1} required to realise the exact prescribed invariant density (i.e., it constructs the conjugate map using (3.111)). The method then relies on the assumption that the approximation in (3.118) will be accurate if F_P^{-1} can be accurately approximated using a PWL function defined over the same intervals on which the map S is defined.

More specifically, the method proposed in [33] first requires the selection of a partition $\mathcal{Q} = \{Q_1, Q_2, \dots, Q_M\}$ of the unit interval that satisfies

$$\|F_P^{-1} - \hat{F}_P^{-1}\|_p \triangleq \left[\int_0^1 |F_P^{-1}(z) - \hat{F}_P^{-1}(z)|^p dz \right]^{1/p} < \epsilon, \quad (3.119)$$

for a sufficiently small value $\epsilon > 0$ and where $\hat{F}_P^{-1}$ is the PWL approximation of F_P^{-1} as defined in (3.113). A map $S \in \Gamma_1(\mathcal{Q})$ is then constructed with its parameters selected to satisfy¹⁰ (3.108); i.e.,

$$\sum_{m=1}^M \text{sgn}(v_m) \mu(Q_m)^2 = \lambda. \quad (3.120)$$

The map $\tilde{S} = F_P^{-1} \circ S \circ F_P$ then simultaneously possesses the required invariant density $f_S^* = f_P$ and a normalised ACF that approximates the prescribed normalised ACF (as characterised by the parameter λ) according to (3.118).

Remark 3.5.6. Equation (3.108) was derived under the assumption that the map S is defined over a partition $\mathcal{Q}'$ of the interval $[-1, 1]$. Equation (3.120) then implicitly assumes the normalised ACF of map S defined over a partition $\mathcal{Q}$ of the unit interval shares the same expression for the normalised ACF. As the two maps may be related using a linear conjugating transformation, the conjecture that both share the same normalised ACF appears reasonable. However, this was not formally proved in [33].

¹⁰Equation (3.120) depends on the lengths of the intervals in partition $\mathcal{Q}$. This implies that the intervals of $\mathcal{Q}$ must be selected to simultaneously ensure an accurate approximation $\hat{F}_P^{-1}$ and to satisfy (3.120).

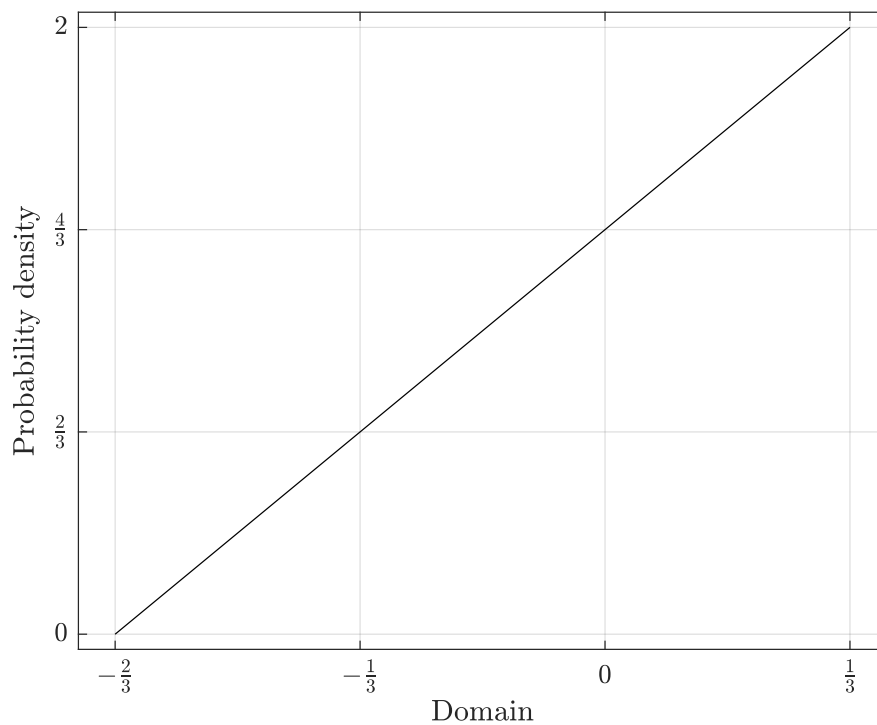


Figure 3.5.2: Prescribed invariant density function f_P of (3.121).

Example 3.5.1. The numerical example presented here was reproduced by this author from the details given in [33, Ex. 5.1]. Suppose that an ergodic map $\tilde{S} : [-\frac{2}{3}, \frac{1}{3}] \rightarrow [-\frac{2}{3}, \frac{1}{3}]$ is to be designed with the invariant density

$$f_{\tilde{S}}^*(x) = 2\left(x + \frac{2}{3}\right), \quad x \in \left[-\frac{2}{3}, \frac{1}{3}\right], \quad (3.121)$$

and the normalised ACF

$$\mathcal{R}_{\tilde{S}}^*(\tau) = \left(-\frac{1}{r}\right)^{|\tau|}, \quad \tau = 0, \pm 1, \pm 2, \dots \quad (3.122)$$

In the preceding equation, r denotes the golden mean, given by

$$r = \frac{1 + \sqrt{5}}{2} \approx 1.618. \quad (3.123)$$

The prescribed invariant density function and normalised ACF are plotted in Figs. 3.5.2 and 3.5.3. The figure reveals that the prescribed ACF comprises a single period-2 mode with a decaying envelope.

The first step in the design of $\tilde{S}$ is to select a partition $\mathcal{Q}$. In this example, a partition $\mathcal{Q}$ of four intervals $Q_m = (q_m, q_{m+1})$, $m = 1, \dots, 4$, is selected. The map $S \in \Gamma_1(\mathcal{Q})$ is constructed such that all its branches have negative slopes. Selecting $\lambda = -1/r$ in

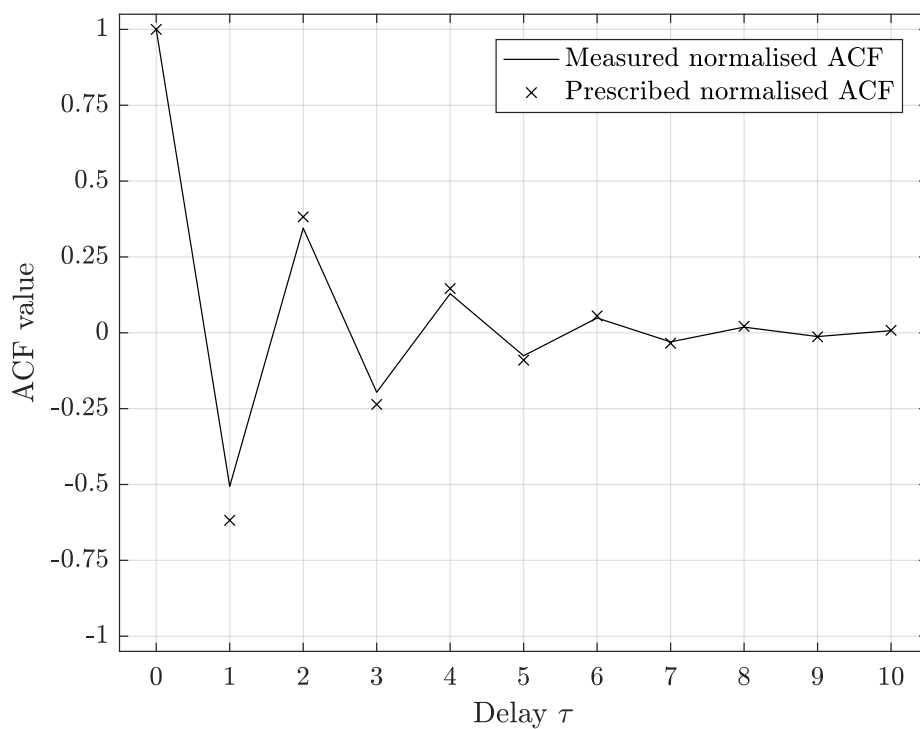


Figure 3.5.3: Prescribed and measured normalised ACFs of the conjugate map $\tilde{S}$. The results of [33] were reproduced via numerical experimentation.

(3.120) then yields the expression

$$\sum_{m=1}^4 (q_{m+1} - q_m)^2 = \frac{1}{r}, \quad (3.124)$$

which must be satisfied to realise the prescribed normalised ACF. In [33, Ex. 5.1], these interval endpoints are selected as

$$\begin{aligned} q_1 &= 0, \\ q_2 &= \frac{1}{4} - \frac{1}{2} \sqrt{-\frac{1}{3} \left(\frac{1}{4} - \frac{1}{r} \right)}, \\ q_3 &= 2q_2, \\ q_4 &= 3q_2, \\ q_5 &= 1, \end{aligned} \quad (3.125)$$

thereby satisfying (3.124). The underlying map $S : \Omega \rightarrow \Omega$ is then defined by the expression

$$S(x) = \sum_{m=1}^4 \left[-\frac{x - q_m}{q_{m+1} - q_m} + 1 \right] \chi_{Q_m}(x). \quad (3.126)$$

This map is plotted in Fig. 3.5.4.

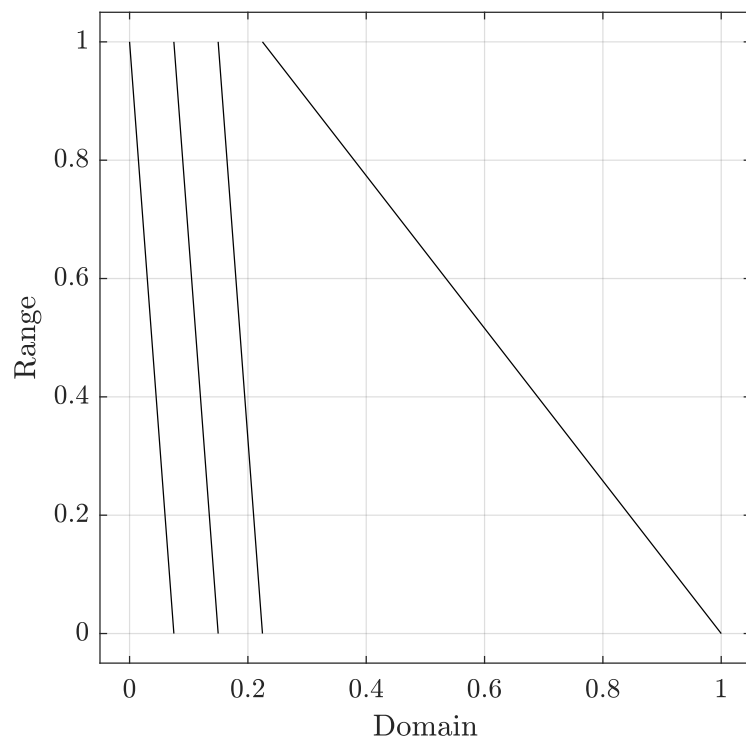


Figure 3.5.4: Piecewise monotonic and complete map $S \in \Gamma_1(\mathcal{Q})$.

The distribution function F_P associated with the prescribed invariant density f_P is given by

$$F_P(x) = \left(x + \frac{2}{3}\right)^2, \quad x \in \left[-\frac{2}{3}, \frac{1}{3}\right]. \quad (3.127)$$

The function F_P is plotted in Fig. 3.5.5. The required conjugating transformation is derived as

$$h(x) = F_P^{-1}(x) = \sqrt{x} - \frac{2}{3}. \quad (3.128)$$

The conjugate map $\tilde{S} = h \circ S \circ h^{-1}$ is then derived as

$$\tilde{S}(x) = \sum_{m=1}^4 F_P^{-1} \left(-\frac{F_P(x) - q_m}{q_{m+1} - q_m} + 1 \right) \chi_{\tilde{Q}_m}(x), \quad (3.129)$$

where $\tilde{Q}_m = (\tilde{q}_m, \tilde{q}_{m+1})$ and $\tilde{q}_m = F_P^{-1}(q_m)$.

The conjugate map $\tilde{S}$, which is the solution to problem stated at the beginning of the example, is plotted in Fig. 3.5.6. The invariant density of this map is equal to the prescribed invariant density, which is plotted in Fig. 3.5.2. The normalised ACF associated with $\tilde{S}$ is plotted in Fig. 3.5.3. The figure reveals that the map's ACF is approximately equal to the prescribed ACF, but the envelope of the realised mode decays more rapidly than the prescribed mode.

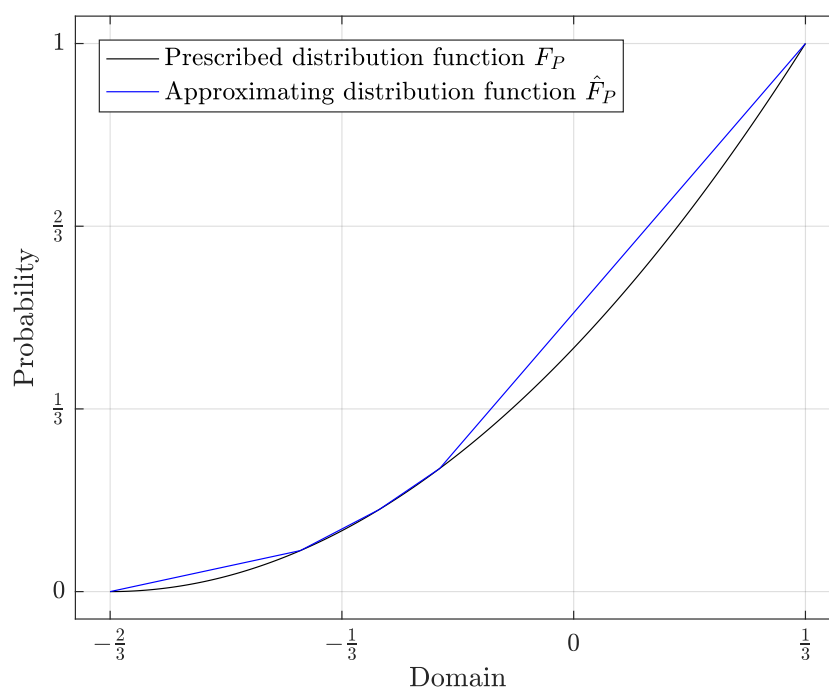


Figure 3.5.5: Distribution function F_P associated with the prescribed invariant density function f_P , and its PWL approximation $\hat{F}_P$.

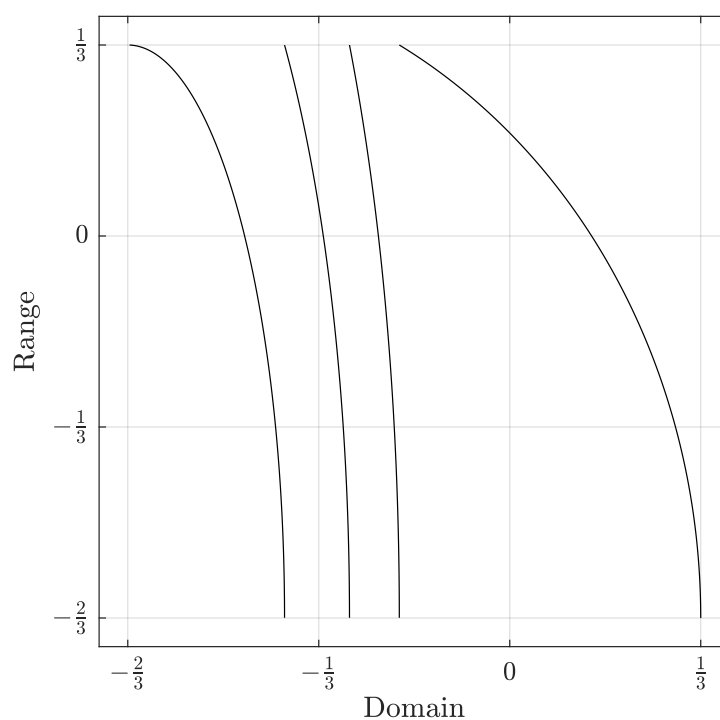


Figure 3.5.6: Conjugate map $\tilde{S}$ obtained by transforming the map S of Fig. 3.5.4. The results of [33] were reproduced via numerical experimentation.

Recall that the derivation of the expression for the conjugate map's ACF depends on a PWL function $\hat{F}_P^{-1}$, defined over the intervals of $\mathcal{Q}$, that is an accurate approximation of the inverse distribution function F_P^{-1} associated with the prescribed invariant density. If this condition is violated, the possibility exists that the ACF of the constructed map $\tilde{S}$ will deviate from the prescribed ACF. The PWL function $\hat{F}_P$ for the example is plotted in Fig. 3.5.5. Although the figure suggests that $\hat{F}_P$ is a reasonable approximation of F_P , it is not perfect. The implication of this non-zero error agrees with the observed deviation of the map's ACF from the prescribed ACF.

Remark 3.5.7. To satisfy the requirement of $\hat{F}_P$ being a close approximation of the inverse distribution function associated with the prescribed invariant density, one may consider increasing the number of intervals $\mathcal{Q}$ in the partition over which S is defined. However, as the number of intervals grows, the range of values that may be accommodated for the parameter λ , which determines the characteristics of the realised spectral mode, contracts. This is evident from (3.120), which reveals that the largest absolute value of λ that may be accommodated is equal to the sum of the *squared* lengths of the intervals that constitute $\mathcal{Q}$.

Remark 3.5.8. The method presented above relies on the conjugate map inheriting the ACF of the underlying map. Hence, Remark 3.5.4 regarding the restricted nature of the PSDs realisable using maps $S \in \Gamma_1$ also apply to maps constructed using the proposed method of topological conjugation. In particular, by selecting any underlying map $S \in \Gamma_1$, the PSD realisable for the conjugate map is restricted to a single spectral mode [3]. Whereas one may consider deriving topological conjugates to underlying maps with richer ACFs, depending on the underlying map considered, the derivation of an expression for the ACF of the conjugate map becomes unfeasible. This is evident from [33], in which a second class Γ_2 of maps with richer spectral characteristics were investigated, but ultimately not considered as candidates for the underlying map due to the complexity of the expression for the ACF (these maps are characterised in section 3.B, and the corresponding ACF is provided in (3.220)).

Remark 3.5.9. It is observed that the restriction to maps from the class Γ_1 yielded progress towards solving (3.112) for the unknown map. In particular, these maps permitted the approximation of F_P with a PWL function in a manner that causes the ACF of the underlying map and the topological conjugate to be approximately equal. The simplified branch structure of the maps also yielded the unsophisticated expression of (3.118) for the underlying map's ACF, as a function of the branch parameters. However, these maps only yield power spectra with single modes.

3.5.2 Direct solution strategy

Diakonou et al. [32] proposed a ground-breaking numerical method for solving a relaxed formulation of IFPP-II. This method uses the direct solution strategy to construct an expression for all unimodal maps that share the prescribed invariant density. The inter-branch correspondence function in this expression is parameterised, and a numerical search is performed over the parameter space for a map which possesses an ACF that approximates the prescribed ACF.

The method proposed in [32] is outlined below. The relaxed formulation of IFPP-II addressed by the method is defined. A general expression is then derived for all unimodal maps that possess the prescribed invariant density. Thereafter, two models of the parameterised inter-branch correspondence function are presented, and the numerical search algorithm for the map with the required ACF is described. Examples from [32] that illustrate the method are then presented.

Relaxed formulation of IFPP-II

The relaxed formulation of IFPP-II requires the construction of an ergodic map S with a prescribed invariant density f_S^* and an ACF that matches the first $K + 1$ values of the prescribed ACF. More formally, if $\mathcal{R}_P$ denotes the prescribed ACF, it is required that the ACF $\mathcal{R}_S$ of the map S satisfy

$$\mathcal{R}_S(\tau) = \begin{cases} \mathcal{R}_P(\tau) & \text{if } \tau = 0, 1, \dots, K, \\ \mathcal{R}_P(-\tau) & \text{if } \tau = -1, -2, \dots, -K. \end{cases} \quad (3.130)$$

Derivation of the unimodal map

Let $S : \Omega \rightarrow \Omega$ denote a unimodal map over the two-interval partition $\mathcal{Q} = \{Q_1, Q_2\}$, where $Q_1 = (q_1, q_2)$ and $Q_2 = (q_2, q_3)$. The branch functions $S_1 = S|_{Q_1}$ and $S_2 = S|_{Q_2}$ are defined to be monotonically increasing and decreasing, respectively. To construct a unimodal map with a prescribed invariant density using the direct solution strategy, the inter-branch correspondence function $u_2 : Q_1 \rightarrow Q_2$, as defined in (3.47), must satisfy

$$\frac{du_2}{dx} < 0 \quad (3.131)$$

almost everywhere, and its values at the interval endpoints must be selected as

$$\begin{aligned} u_2(q_1) &= 1, \\ u_2(q_2) &= q_2. \end{aligned} \tag{3.132}$$

The branches of the unimodal map are then given by (3.49) and (3.50) as

$$S_1(x) = F_P^{-1}(F_P(x) + 1 - F_P \circ u_2(x)), \quad x \in Q_1 \tag{3.133}$$

and

$$S_2(x) = S_1 \circ u_2^{-1}(x), \quad x \in Q_2. \tag{3.134}$$

Substitution of (3.133) into (3.134) yields

$$S_2(x) = F_P^{-1}(F_P \circ u_2^{-1}(x) + 1 - F_P(x)), \quad x \in Q_2. \tag{3.135}$$

Equations (3.133) and (3.135) may be combined into a single expression for the unimodal map, given by [32]

$$S(x) = F_P^{-1}(1 - |F_P(x) - F_P \circ U_2(x)|), \tag{3.136}$$

where

$$U_2(x) = \begin{cases} u_2(x) & \text{if } x \in Q_1, \\ u_2^{-1}(x) & \text{if } x \in Q_2. \end{cases} \tag{3.137}$$

Parameterisation of the inter-branch correspondence function

Two parametric models of the inter-branch correspondence function u_2 were proposed in [32]. These models are presented in what follows.

A partition $\mathcal{R}^{(1)} = \{R_1, R_2, \dots, R_{c(1)}\}$ of the closure of interval Q_1 is defined.¹¹ This partition has the intervals defined by

$$R_j = [r_j, r_{j+1}), \quad j = 1, 2, \dots, c(1), \tag{3.138}$$

where $r_1 \triangleq 0$ and $r_{c(1)+1} \triangleq q_2$. The inter-branch correspondence function of each model is defined to be piecewise polynomial over the intervals of partition $\mathcal{R}^{(1)}$ and to be a continuous function.

¹¹Here, $c(m)$ denotes the number of intervals in partition $\mathcal{R}^{(m)}$.

Model I In the first model, the order of the inter-branch correspondence function at the left endpoint r_j of each interval of the partition $\mathcal{R}^{(1)}$ is directly specifiable through parameter selection, thereby permitting the derivative of the map at $x = 0$ to be fitted. The disadvantage of this model is that the order of the maximum of the map at the right endpoint $x = q_2$ of the final interval cannot be fitted independently.

In model I, the branch functions $u_{2,j} \triangleq u_2|_{R_j}$ of the inter-branch correspondence function u_2 are defined by

$$u_{2,j}(x) = (y_{j+1} - y_j) \left(\frac{x - r_j}{r_{j+1} - r_j} \right)^{\alpha_j} + y_{j+1}, \quad r_j \leq x \leq r_{j+1}, \quad j = 1, 2, \dots, c(1), \quad (3.139)$$

where $y_j \triangleq S(x_j)$.

The parameters of the inter-branch correspondence function u_2 in model I are the endpoints $(r_j)_{j=2}^{c(1)+1}$ of the intervals that comprise partition $\mathcal{R}^{(1)}$, the interval endpoints $(y_j)_{j=2}^{c(1)}$ of the images of each interval R_j under u_2 , and the orders $(\alpha_j)_{j=1}^{c(1)}$ of the polynomials defined over each interval of $\mathcal{R}^{(1)}$.

Model II In model II, the expression for each branch $u_{2,j} \triangleq u_2|_{R_j}$ involves expansions around each of the endpoints r_j and r_{j+1} of the interval R_j , which coincide at a point $\bar{r}_j \in [r_j, r_{j+1})$ inside the interval. This necessitates the introduction of the additional variables $c_{j,1}$ and $c_{j,2}$ for each branch $u_{2,j}$. The branch functions of u_2 in model II are defined as

$$u_{2,j}(x) = \begin{cases} y_j - c_{j,1}(x - r_j)^{\alpha_{j,1}} & \text{if } r_j \leq x < \bar{r}_j, \\ y_{j+1} + c_{j,2}(r_{j+1} - x)^{\alpha_{j,2}} & \text{if } \bar{r}_j \leq x \leq r_{j+1}. \end{cases} \quad (3.140)$$

The parameters $c_{j,1}$ and $c_{j,2}$ are then selected to ensure that the branch and its first derivative are continuous at $\bar{r}_j$, thereby yielding

$$u_{2,j}(x) = \begin{cases} y_j - d_j \alpha_{j,2} (\bar{r}_j - r_j)^{1-\alpha_{j,1}} (x - r_j)^{\alpha_{j,1}} & \text{if } r_j \leq x < \bar{r}_j, \\ y_j + d_j \alpha_{j,1} (\bar{r}_{j+1} - \bar{r}_j)^{1-\alpha_{j,2}} (r_{j+1} - x)^{\alpha_{j,2}} & \text{if } \bar{r}_j \leq x \leq \bar{r}_{j+1}, \end{cases} \quad (3.141)$$

where

$$d_j \triangleq \frac{r_j - r_{j+1}}{\alpha_2(\bar{r}_j - r_j) + \alpha_1(r_{j+1} - \bar{r}_j)}. \quad (3.142)$$

The parameters of the inter-branch correspondence function in model II are the endpoints $(r_j)_{j=2}^{c(1)+1}$ of the intervals that comprise partition $\mathcal{R}^{(1)}$, the interval endpoints $(y_j)_{j=2}^{c(1)}$ of the images of each interval R_j under u_2 , the exponents $(\alpha_{j,1})_{j=1}^{c(1)}$ and $(\alpha_{j,2})_{j=1}^{c(1)}$ associated with the branch defined over each interval of $\mathcal{R}^{(1)}$, and the points $(\bar{r}_j)_{j=1}^{c(1)}$.

Numerical search algorithm

Consider any candidate unimodal map S given by (3.136). A cost function ε that quantifies the deviation of this map's ACF from the first $K+1$ values of the prescribed ACF is defined in [32]. This cost function is given by

$$\varepsilon(S) = \sqrt{\sum_{\tau=1}^K \left(\frac{\mathcal{R}_S(\tau) - \mathcal{R}_P(\tau)}{\mathcal{R}_P(\tau)} \right)^2}. \quad (3.143)$$

Observe that this cost function does not account for the value of the ACF at $\tau = 0$. This is motivated by observing that $\mathcal{R}_S(0) = \bar{\mathcal{M}}^2 + \bar{\sigma}_S^2$, where $\bar{\mathcal{M}}$ and $\bar{\sigma}_S^2$ denote the mean and variance of almost every trajectory generated by the map, and that these statistics are fully determined by the map's invariant density (this is a consequence of the extension of the Birkhoff individual ergodic theorem). As the invariant density of each map S is equal to the prescribed invariant density, it follows that the ACF of each map S at $\tau = 0$ will always equal the value prescribed.

A numerical search is performed over the parameter space of the inter-branch correspondence functions from either of the two models presented [32]. Below, the numerical search is outlined using the inter-branch correspondence function of model I; the search is similar for the function of model II. First, a partition $\mathcal{R}^{(1)}$ with a single interval $R_1 = [0, r_2)$ is defined, where r_2 and the polynomial order α_1 are parameters to be determined. A first round of Monte-Carlo minimisation is used to search for parameters r_2 and α_1 that minimise the cost function of (3.143). Let $r_2^{(1)}$ and $\alpha_1^{(1)}$ denote the parameter values obtained after the first round of minimisation.

Once the Monte-Carlo minimisation algorithm has converged and $r_2^{(1)}$ and $\alpha_1^{(1)}$ have been determined, a new partition $\mathcal{R}^{(1)}$ with two intervals $R_1 = [r_1, r_2)$ and $R_2 = [r_2, r_3)$ is defined, where $r_1 = 0$ [32]. The parameters to be determined for the inter-branch correspondence function defined over this partition are the partition points r_2 and r_3 and the polynomial orders α_1 and α_2 . Values are selected for parameters r_2 and α_1 using a Gaussian pseudorandom number generator. Specifically, r_2 is selected as a Gaussian random variate with mean value selected equal to $r_2^{(1)}/2$, and α_1 as a Gaussian random variate with mean value equal to $\alpha_1^{(1)}$. The variance of these random variates are selected to be sufficiently small as to allow relatively weak fluctuation around the mean value (the exact variance used is not reported in [32], but a figure of 10% fluctuation is reported). A second round of Monte-Carlo minimisation is then performed to search for the new parameters r_3 and α_2 that minimise the cost function of (3.143)

Once the second round of minimisation has converged, a partition with three intervals

is defined and the process repeats [32].¹² In this manner, the size of the partition over which the inter-branch correspondence function is defined increases by one interval over each iteration. It was proposed that a sufficient number of iterations are performed until the value of the cost function becomes acceptably small.

Remark 3.5.10. In the numerical search outlined above, each evaluation of the cost function requires that the ACF of the candidate map be estimated. In [32], the authors attempted to generate trajectories of the candidate map and estimate the ACF as a time average, but it was observed that sensitive dependence on the initial state led to inaccurate results.

Instead, the use of numeric integration to estimate the values of the ACF as statistical averages according to the expression

$$\mathcal{R}_S(\tau) = \int_0^1 x S^\tau(x) f_S^*(x) dx, \quad \tau = 1, 2, \dots, K, \quad (3.144)$$

was proposed [32]. However, this approach is computationally very intensive. The number of intervals over which the τ^{th} iterate of the map S is piecewise monotonic equals 2^τ . In [32], the authors report the use of a routine to extract these intervals and to separately evaluate the integral of (3.144) over each interval. Although this improves the accuracy of the computed ACF, the exponential growth of the number of intervals greatly increases the computational complexity. Observe that (3.144) has to be evaluated K times (i.e., for each value of $\tau = 1, 2, \dots, K$) for each candidate map during the Monte-Carlo minimisation.

Remark 3.5.11. The authors of [32] did not simplify (3.144) analytically to relate the map's ACF to the parameters of its branch functions. It is conjectured that the complexity of models I and II prohibited the simplification of this expression.

The Metropolis algorithm [66] is used in each round of Monte-Carlo minimisation in [32]. In iteration $n = 1, 2, \dots, N$, the parameters r_{n+1} and α_n are assumed to follow uniform distributions over $(0, 1)$ and $(0, c)$ respectively, where $c \gg 1$ is selected as a finite value. Annealing is introduced through an exponential probability distribution in an attempt to avoid convergence to a local minimum of the cost function.

Example 3.5.2. The method described above was applied in [32] to numerically construct examples of eight maps with prescribed invariant density and partially prescribed ACF. The experiment was not reproduced by the author of this thesis, but the results were adapted with permission from [32] and are repeated below.

¹²It is not explicitly stated in [32] how the leftmost partition points are selected in the third and later iterations from the previous iteration's endpoints.

Table 3.5.1: The maps constructed in [32] using the numerical solution to IFPP-II.

Map	Model	Inv. density	Figure	ACF	Figure
1	Model I	Uniform	3.5.7(a)	Exponentially decaying	3.5.7(b)
2	Model II	Uniform	3.5.8(a)	Exponentially decaying	3.5.8(b)
3	Model I	Linear	3.5.9(a)	Exponentially decaying	3.5.9(b)
4	Model II	Linear	3.5.10(a)	Exponentially decaying	3.5.10(b)
5	Model I	Power-law	3.5.11(a)	Oscillating and decaying	3.5.11(b)
6	Model II	Power-law	3.5.12(a)	Oscillating and decaying	3.5.12(b)
7	Model I	Uniform	3.5.13(a)	Power-law decaying	3.5.13(b)
8	Model II	Uniform	3.5.14(a)	Power-law decaying	3.5.14(b)

Table 3.5.1 lists the maps that were constructed, the model of the inter-branch correspondence function used as well as the type of invariant density and ACF prescribed. Each row of the table also lists figures that contain plots of the map constructed as well as the prescribed ACF.

In each case, only the first six values of the ACF was prescribed for the map to be constructed; i.e., $K = 5$ was selected in (3.130). The inter-branch correspondence functions of models I and II were constructed over partitions of three and one intervals, respectively. Several tens of thousands of steps were performed during each round of Monte-Carlo minimisation.

Maps 1 and 2 were required to possess the uniform invariant density and the same exponentially decaying ACF, as plotted in Figs. 3.5.7(b) and 3.5.8(b). These statistical metrics were selected such that they belong to a known member from the class of skew tent maps. The maps constructed using the inter-branch correspondence functions of models I and II, as plotted in Figs. 3.5.7(a) and 3.5.8(a), both approximate this tent map. The mean relative deviation between the realised ACF $\mathcal{R}_S$ and the prescribed ACF $\mathcal{R}_P$ over the first $K + 1 = 6$ values, defined by

$$\bar{\varepsilon}(S) = \frac{\varepsilon(S)}{\sqrt{K}}, \quad (3.145)$$

is approximately 2.2% for the maps constructed.

Maps 3 and 4 were constructed to possess a linear invariant density and an exponentially decaying ACF, as plotted in 3.5.9(b) and 3.5.10(b). This ACF is similar to the ACF prescribed for maps 1 and 2. It is observed that the maps constructed using branch functions from models I and II, as plotted in Figs. 3.5.9(a) and 3.5.10(a), are distinct. Several structural features of the maps are identified in [32]. It is observed

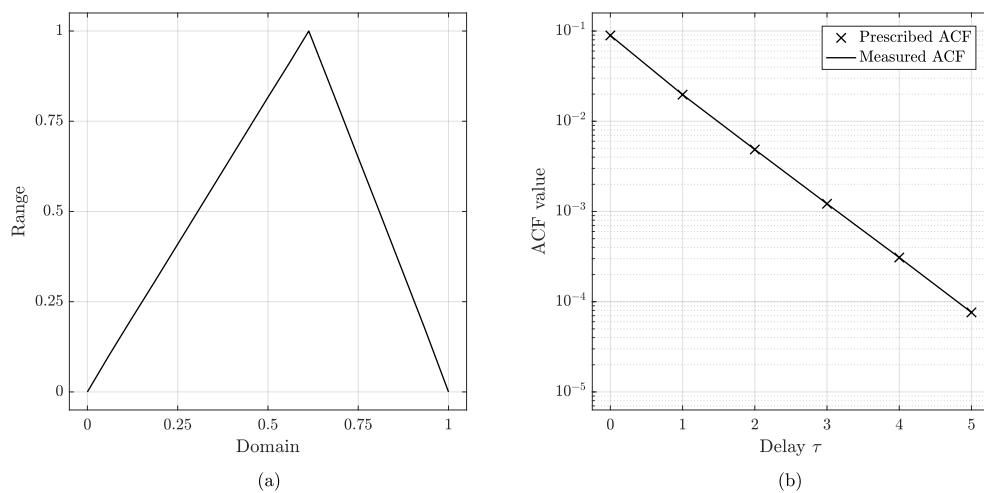


Figure 3.5.7: Map 1 (left), designed using model I with a prescribed uniform invariant density and exponentially decaying ACF (right). These figures were reproduced with permission from the figures presented in [32].

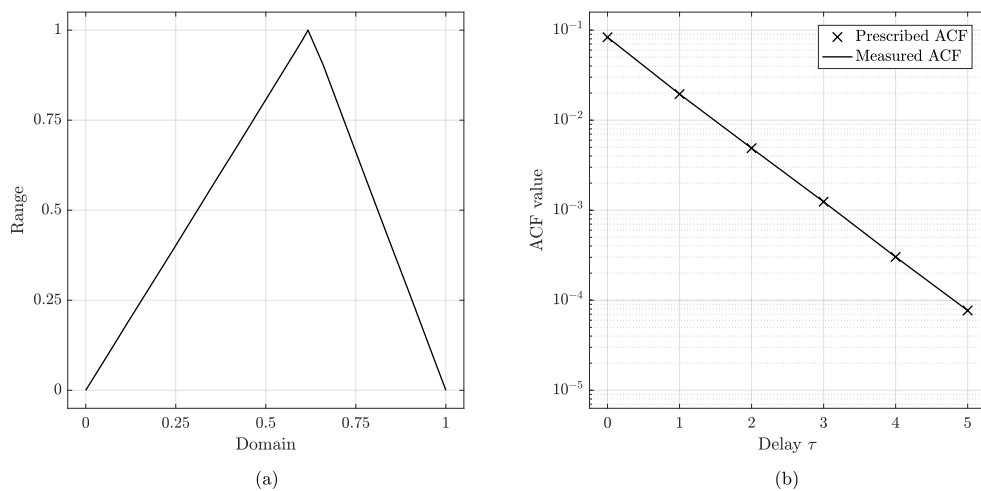


Figure 3.5.8: Map 2 (left), designed using model II with a prescribed uniform invariant density and exponentially decaying ACF (right). These figures were reproduced with permission from the figures presented in [32].

that the ACF of map 3 deviates significantly from that of the prescribed ACF at a delay of $\tau = 1$, leading to a mean relative deviation $\bar{\varepsilon}(S)$ of 129%. In contrast, the ACF of map 4 is a closer approximation of the prescribed ACF, yielding a mean relative deviation of only 5.4%. Map 4 is observed to be smoother than map 3, which is attributed to the additional flexibility of the inter-branch correspondence function of model II (more specifically, this model permits independent adjustment of the left and right derivatives in each interval).

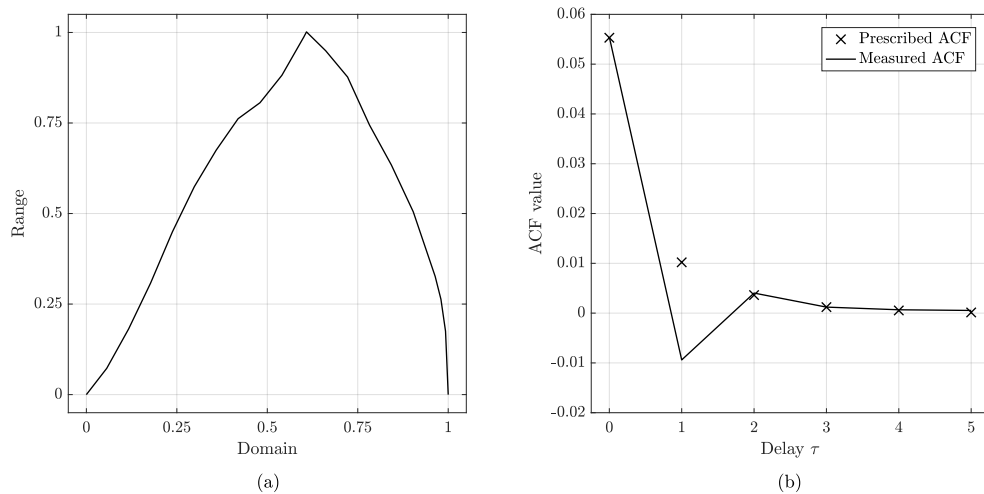


Figure 3.5.9: Map 3 (left), designed using model I with a prescribed linear invariant density and exponentially decaying ACF (right). These figures were reproduced with permission from the figures presented in [32].

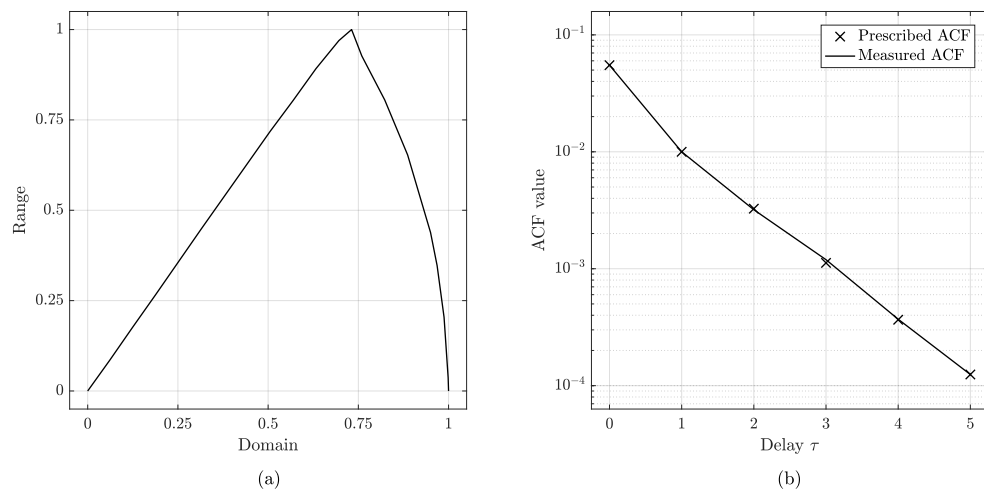


Figure 3.5.10: Map 4 (left), designed using model II with a prescribed linear invariant density and exponentially decaying ACF (right). These figures were reproduced with permission from the figures presented in [32].

Maps 5 and 6 were prescribed an invariant density that obeys a power law with exponent $\beta = -0.5$ and a decaying ACF that oscillates, as plotted in Figs. 3.5.11(b) and 3.5.12(b). These maps are plotted in Figs. 3.5.11(a) and 3.5.12(a). Although map 6 (constructed using model II) is smoother than map 5, its ACF has a mean relative deviation $\bar{\varepsilon}(S)$ of 45% from the prescribed ACF, as compared to the 22% mean relative deviation of the ACF belonging to map 5 (constructed using model I). Hence, the more sophisticated model II is not necessarily the better choice in all

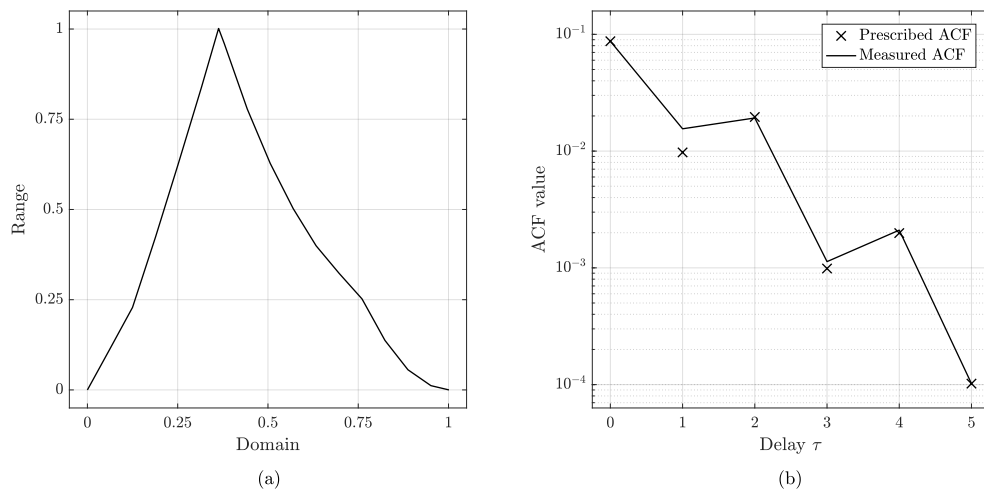


Figure 3.5.11: Map 5 (left), designed using model I with a prescribed power-law invariant density and an oscillating and decaying ACF (right). These figures were reproduced with permission from the figures presented in [32].

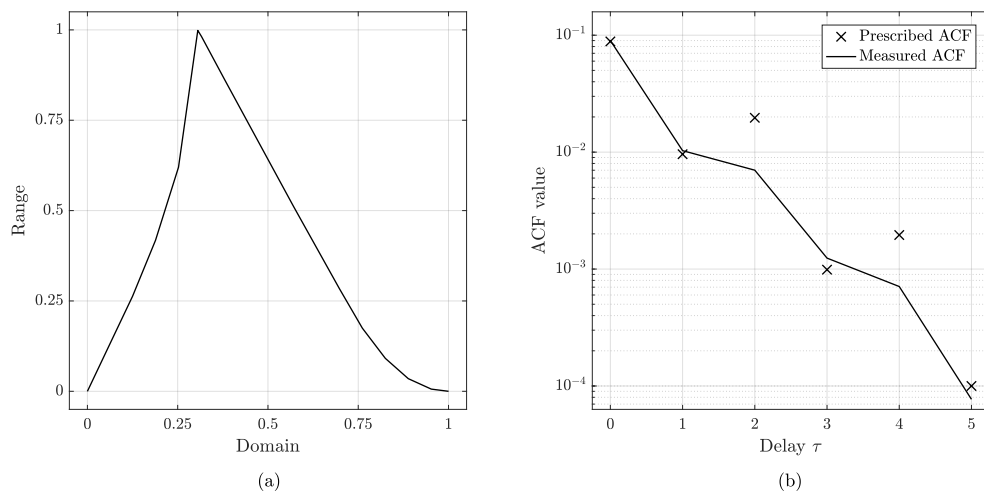


Figure 3.5.12: Map 6 (left), designed using model II with a prescribed power-law invariant density and an oscillating and decaying ACF (right). These figures were reproduced with permission from the figures presented in [32].

scenarios; this may partly be attributed to fact that the numerical search is performed over a larger parameter space, and that the solution found after convergence might coincide with a local minimum of the cost function.

Maps 7 and 8 were prescribed the uniform invariant density and an ACF that decays algebraically with exponent -2.5 , as plotted in Figs. 3.5.13(b) and 3.5.14(b). The maps constructed using the direct solution are plotted in Figs. 3.5.13(a) and 3.5.14(a).

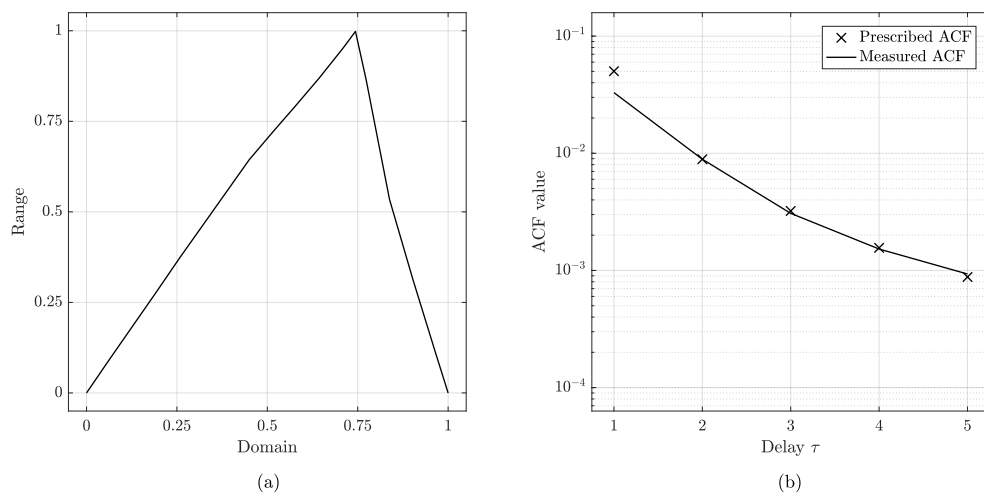


Figure 3.5.13: Map 7 (left), designed using model I with a prescribed uniform invariant density and a power-law decaying ACF (right). These figures were reproduced with permission from the figures presented in [32].

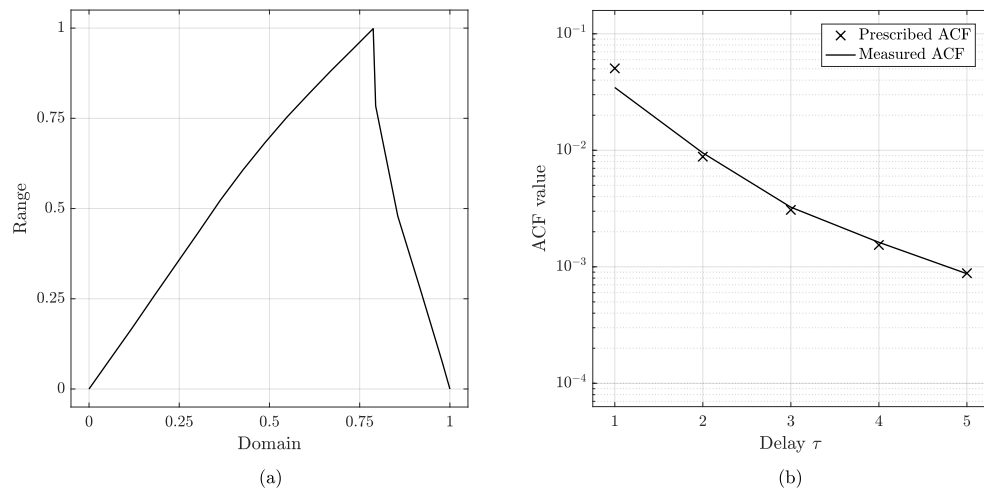


Figure 3.5.14: Map 8 (left), designed using model II with a prescribed uniform invariant density and a power-law decaying ACF (right). These figures were reproduced with permission from the figures presented in [32].

It is found that the ACF of both maps, constructed using inter-branch correspondence functions from models I and II, approximate the prescribed ACF reasonably well, with a mean relative deviation $\bar{\varepsilon}(S)$ of only 13%.

Several remarks are presented regarding the numerical method that uses the direct strategy for solving IFPP-II.

Remark 3.5.12. The numerical search for parameters values that minimise the cost function is suboptimal, as the Monte-Carlo minimisation is not performed simultaneously over the entire parameter space. Hence, it is unlikely to yield the optimal solution. The reason for only performing the search over the subspace of newly defined parameters in each iteration is necessary due to the unmanageably high dimensionality of the parameter space as a whole. In particular, the parameter space of models I and II have $3c(1) + 1$ and $5c(1) + 1$ dimensions, respectively, where $c(1)$ denotes the number of intervals over which the inter-branch correspondence function is defined.

A single numerical search over the entire parameter space is unlikely to converge with reasonable computational resources and in reasonable time [3]. Even by only performing the numerical search over the reduced parameter space, the authors of [32] report three months of computation time to construct the eight examples presented above using a powerful workstation, where the branch functions are constructed over a modest number of intervals (three intervals and one interval for models I and II, respectively).

Remark 3.5.13. The numerical search for a map with the prescribed ACF, as described above, is not guaranteed to converge to a solution [3]. Furthermore, should the algorithm converge, it is not guaranteed that the ACF of the map exhibits the same characteristics as the prescribed ACF. This applies to map 3, which appears to possess an ACF with one or more modes that oscillate, whereas the prescribed ACF is required not to possess such a mode; refer to Fig. 3.5.9(b). Even should the characteristics of the ACFs coincide, it is also not guaranteed that the ACF of the prescribed map is a sufficiently accurate approximation of the prescribed ACF. This is the case with map 6, which has a mean relative deviation of 45% when compared to the prescribed ACF; refer to Fig. 3.5.12(b).

Remark 3.5.14. Although the maps constructed in the example possess ACFs with distinct and diverse characteristics, the class of ACFs achievable using the inter-branch correspondence functions of models I and II are not known [3]. More specifically, explicit expressions for these ACFs are not available, due to the structural complexity of the inter-branch correspondence functions of the models. Hence, when using the proposed solution to construct a map with an arbitrarily specified ACF, it is not known beforehand whether a map with the desired ACF can be constructed regardless of whether the numerical search successfully converges to the global minimum of the cost function. Stated differently, it is not known beforehand whether a map with the required ACF is member of the solution space, or if it can be approximated arbitrarily closely using members of the solution space.

Remark 3.5.15. The method proposed above constructs a map with an ACF designed to match the first $K + 1$ values of the prescribed ACF. It is not guaranteed that the remaining values of the ACF will match or even approximate those value of the prescribed ACF. Although the value of K may be increased, Remark 3.5.10 states that the computational complexity associated with the estimation of a candidate map's ACF grows exponentially as a function of K .

The examples presented in [32] motivate the claim that the proposed method is a useful and valuable tool for studying the relationship between the structural characteristics of an ergodic map and its invariant density and ACF. However, due to its computational complexity, the possibility of non-convergence and possible inaccuracy, and the unknown nature of the class of ACFs achievable using the method, it cannot be considered a *practical solution* of IFPP-II for arbitrarily specified ACFs.

3.5.3 Matrix solution strategy

The literature review revealed a gap for a matrix solution to IFPP-II that constructs members from the class of PWL semi-Markov maps. In this section, observations are made regarding the design of these maps and their capacity to approximate arbitrary statistical metrics. Thereafter, a recent matrix solution for IFPP-I is investigated, and its extension to solve IFPP-II is briefly considered.

PWL semi-Markov maps

The work presented in chapter 2 revealed that the expressions for the statistical metrics of ergodic members from the class of PWL semi-Markov maps are functions of the map's order-one FPM. The elements of this matrix are simple functions of a finite collection of the map's branch parameters (i.e., the branch slopes and y -intercepts). Hence, the statistical metrics of these maps are related to their branch parameters in a relatively unsophisticated manner.¹³ This increases the potential for an analytical method that designs these maps to satisfy prescribed metrics.

¹³Classes which permit maps with infinite dimensional FPOs may yield sophisticated expressions for the statistical metrics that cannot be analytically utilised for design. An example is the class Γ_2 considered in [33] and described in section 3.B, which yields the sophisticated expression of (3.220) for the map's ACF. The complexity of these expressions forces reliance on numerical methods for designing the map, thereby leading to issues such as non-convergence and high computational complexity.

Although the PWL semi-Markov maps have the potential to yield analytical methods, there exists a trade-off in that the invariant density and PSD of these maps can only be designed to approximate arbitrary continuous functions that may be prescribed. In particular, chapter 2 revealed that an ergodic map from this class necessarily possesses a PWC invariant density function and a rational power spectrum.

In section 3.4.3, two methods were presented for approximating an arbitrary smooth function that may be prescribed for the invariant density; refer to Table 3.4.1. The same section presented an algorithm for constructing a PWL semi-Markov map to realise the approximation. It was proved that the approximation error of the first method approaches zero as the number of intervals in the map's Markov partition grows. Hence, the semi-Markov maps may be designed to approximate smooth invariant densities as closely as desired.

The capacity of a PWL semi-Markov map to approximate an arbitrary continuous function prescribed for the map's PSD is uncertain. Stoica [67] observed that a fully specifiable rational power spectrum can be chosen to approximate continuous power spectra as closely as desired. This observation was motivated by the Weierstrass approximation theorem [68], which implies that the rational power spectra form a dense set over all continuous power spectra.

Not all rational power spectra may necessarily be realisable using PWL Markov maps [69]. The same statement holds for the class of PWL semi-Markov maps. This follows from the observation that a subset of the poles of a semi-Markov map's power spectrum coincide with the eigenvalues of its order-zero FPM, which is similar to a stochastic matrix. A stochastic matrix need not exist for an arbitrary collection of eigenvalues that may be specified [70]. This implies that a semi-Markov map with an arbitrary rational power spectrum may not exist.

Although it may not be possible to design a PWL semi-Markov map to approximate an arbitrary continuous power spectrum, the relationship between the poles of the power spectrum and the eigenvalues of the order-one FPM appears to grant some freedom to select characteristics for the power spectrum. In particular, by prescribing the eigenvalues of this matrix, it is conjectured that multiple peaks may be realised in the power spectrum, such that their centre frequencies and bandwidths are adjustable. This conjecture is confirmed in the numerical examples of chapter 4, where maps are constructed using the novel method for approximately solving IFPP-II (see also [3]).

Existing solution for IFPP-I and its extension

In a series of articles [22–25], Rogers and his co-workers developed a matrix solution for IFPP-I. This method follows the matrix solution process outlined in section 3.4.3 and designs a semi-Markov map with an invariant density that is PWC over a partition of the unit interval. Although the authors investigated the ACFs of maps designed using the method, the solution was not extended to solve IFPP-II in the literature. However, it is observed here that the method has certain properties that afford a degree of freedom to specify the power spectrum modes of the map.

The method proposed in [22–25] designs the semi-Markov map to possess an $M \times M$ order-zero FPM with a predefined structure that is fully specified by a collection of $2M$ real-valued parameters. This structure leads to a simplified expression for the right Perron eigenvector of the order-zero FPM. This eigenvector coincides with the map's PWC invariant density and is directly specifiable by selecting values for the matrix parameters. Hence, the map is designed directly through parameter selection.

Recall that the eigenvalues of a semi-Markov map's order-zero FPM coincide with a subset of the map's power spectrum modes (specifically, a subset of the map's power spectrum poles); refer to Remarks 2.6.8 and 2.6.10. The eigenvalues of the matrix prescribed as the order-zero FPM in [22–25] satisfy a particular collection of inequalities, as shown in [71]. These inequalities permit the specification of upper and lower bounds between which the real eigenvalues of the matrix must fall by appropriate selection of its free parameters. Thus, the method provides the means to manipulate the spectral characteristics of the map.

In the remainder of this section, the parameterised matrix prescribed as the order-zero FPM is defined and its properties are described. The method proposed in [22–25] is then outlined. The findings reported in [24] on the ACFs of maps designed using the proposed method are subsequently presented. Thereafter, the method's extension to define bounds for a subset of the poles belonging to the map's PSD is explored.

Parameterised column stochastic matrix Let $(\alpha_m)_{m=1}^M$ and $(\beta_m)_{m=1}^M$ denote real-valued parameters that satisfy

$$\alpha_m \geq 0, \quad m = 1, 2, \dots, M \quad (3.146)$$

and

$$0 < \beta_m < 1, \quad m = 1, 2, \dots, M. \quad (3.147)$$

An $M \times M$ matrix $\mathbf{P}$ is constructed from these parameters according to [22]

$$\mathbf{P} \triangleq \text{diag}\{[\beta_1, \beta_2, \dots, \beta_M]\} + \frac{1}{K} \mathbf{a} \mathbf{b}^T, \quad (3.148)$$

where

$$\begin{aligned} \mathbf{a} &\triangleq [\alpha_1, \alpha_2, \dots, \alpha_M]^T, \\ \mathbf{b} &\triangleq [1 - \beta_1, 1 - \beta_2, \dots, 1 - \beta_M]^T, \end{aligned} \quad (3.149)$$

and

$$K \triangleq \sum_{m=1}^M \alpha_m. \quad (3.150)$$

It is readily verified that $\mathbf{P}$ is column stochastic and possesses strictly positive elements. The Perron-Frobenius theorem [64] thus applies to matrix $\mathbf{P}$. This theorem implies that $\mathbf{P}$ has a spectral radius of unity and a unique maximal (Perron) eigenvalue

$$\lambda_1 = 1. \quad (3.151)$$

All the remaining eigenvalues of $\mathbf{P}$ satisfy

$$|\lambda_m| < \lambda_1, \quad m = 2, 3, \dots, M. \quad (3.152)$$

Now, without loss of generality, suppose that the parameters β_m satisfy

$$0 < \beta_M \leq \beta_{M-1} \leq \dots \leq \beta_1 < 1. \quad (3.153)$$

It is proved in [71] that all the eigenvalues of $\mathbf{P}$ are real-valued and strictly positive, and that these eigenvalues satisfy the inequalities

$$\beta_m \leq \lambda_m \leq \beta_{m-1}, \quad m = 2, 3, \dots, M. \quad (3.154)$$

Furthermore, the right Perron eigenvector $\mathbf{w}_1$ of $\mathbf{P}$ has positive elements given by [71]

$$\mathbf{w}_1 = \left[\frac{\alpha_1}{1 - \beta_1}, \frac{\alpha_2}{1 - \beta_2}, \dots, \frac{\alpha_M}{1 - \beta_M} \right]^T. \quad (3.155)$$

Method for constructing an ergodic semi-Markov map Proposition 3.4.3 implies that the matrix $\mathbf{P}$ defined in (3.148) qualifies as a candidate order-zero FPM of a semi-Markov map S with a uniform Markov partition. A canonical map S possessing an order-zero FPM $\mathbf{P}_S^{(0)} = \mathbf{P}$ is constructed in [22–25] using an algorithm that is identical to Algorithm 3.4.1 (canonical map construction) introduced in section 3.4.3. As $\mathbf{P}$ is strictly positive, it is primitive, and therefore satisfies the conditions of Proposition 3.4.4. This implies that S is ergodic with the invariant density f_S^* that coincides with the right Perron eigenvector $\mathbf{w}_1 = \mathbf{f}_S^*$ of its order-zero FPM.

Now, observe that the elements of the right Perron eigenvector of $\mathbf{P}_S^{(0)} = \mathbf{P}$, as given by (3.155), may be specified by choosing appropriate values for the parameters (α_m) and (β_m) . As the right Perron eigenvector of $\mathbf{P}$ and the PWC invariant density of the map S coincide, the latter density may thus be specified through the selection of the aforementioned parameters. The method proposed in [22–25] follows this approach.

Let $\hat{f}_P = [\hat{f}_1, \hat{f}_2, \dots, \hat{f}_M]^T$ denote the vector representation of the prescribed invariant density, which is defined to be PWC over the uniform partition $\mathcal{Q}$. The method proposed in [24] selects the parameters (α_m) and (β_m) as follows. First, each of the parameters $(\beta_m)_{m=1}^M$ is selected arbitrarily to satisfy $0 < \beta_m < 1$. To realise the prescribed PWC invariant density, the corresponding value of each parameter α_m is then selected as

$$\alpha_m = \hat{f}_m(1 - \beta_m), \quad m = 1, 2, \dots, M. \quad (3.156)$$

The freedom that this method affords to select the parameters (β_m) provides the opportunity to manipulate the power spectrum characteristics of the map S during its design; this is investigated towards the end of this section.

Investigation of the ACF The ACFs of the semi-Markov maps designed using the above method were investigated in [24]. However, an exact expression was not provided for the normalised ACF $\mathcal{R}_S^*$ of the relevant maps. Instead, an expression was provided for an approximation $\hat{\mathcal{R}}_S^*$ that is claimed to converge to the true normalised ACF $\mathcal{R}_S^*$ in the limit as the number of intervals M in the map's Markov partition $\mathcal{Q}$ increases.

This approximation of $\mathcal{R}_S^*$ provided in [24] was derived by representing each interval of the Markov partition by its midpoint. This expression is given by

$$\hat{\mathcal{R}}_S^*(\tau) = \frac{1}{\bar{\sigma}_S^2} \sum_{j=1}^M \sum_{k=1}^M \left(\frac{j}{M} - \frac{1}{2M} \right) \left(\frac{k}{M} - \frac{1}{2M} \right) \frac{\alpha_j}{1 - \beta_j} p_{j,k}^{(|\tau|)} - \bar{\mathcal{M}}_S^2, \quad \tau = 0, \pm 1, \dots \quad (3.157)$$

In this expression, $[\mathbf{P}_S^{(0)}]^{|\tau|} \triangleq [p_{j,k}^{(|\tau|)}]_{j,k=1}^M$,

$$\bar{\mathcal{M}}_S^2 = \sum_{m=1}^M \frac{\alpha_m(2m-1)}{2M(1-\beta_m)}, \quad (3.158)$$

and

$$\bar{\sigma}_S^2 \equiv \bar{\sigma}_S^2(x_1) \triangleq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} (S^k(x_1) - \bar{\mathcal{M}}_S)^2. \quad (3.159)$$

Maps that were designed using the proposed method to possess linear and uniform invariant densities, as well as an invariant density defined to be zero except over

two intervals at the edges of the unit interval, were investigated in [24]. During the design of these maps, all parameters β_m were selected as

$$\beta_m = \beta, \quad m = 1, 2, \dots, M, \quad (3.160)$$

where β was varied over the interval $(0, 1)$. It was found that the value of the ACF at delay $\tau = 1$ appears to grow linearly in proportion to the value of β selected, for each of the maps constructed. However, this value of the ACF grows at different rates depending on the invariant density selected. The authors of [24] suggest that further investigation of the impact that parameter β has on the ACF may be a topic for future work.

Remark 3.5.16. Expressions that relate the eigenvalues of a semi-Markov map's order-zero FPM to its ACF and PSD were not developed in [24]. These expressions were derived in chapter 2 and revealed that the eigenvalues coincide with a subset of the map's power spectrum modes. The results obtained in [24] may be interpreted using this insight.

The continuous component $\mathcal{R}_{S,C}$ of the ACF belonging to map S is given by (2.141):

$$\mathcal{R}_{S,C}(\tau) = \sum_{m=2}^M c_m \lambda_m^{|\tau|} + \sum_{m=M+1}^{2M} c_m \lambda_m^{|\tau|}, \quad (3.161)$$

where $(\lambda_m)_{m=2}^M$ denotes the eigenvalues of the map's order-zero FPM, and $(\lambda_m)_{m=M+1}^{2M}$ denotes the eigenvalues of submatrix $\mathbf{Q}_S$ of the map's order-one FPM, as defined in (2.117). Now, recall that the maps of the examples in [24] were designed by selecting parameters $\beta_m = \beta$ for all $m = 1, 2, \dots, M$. Equation (3.154) then implies that all of the eigenvalues of each map's order-zero FPM are equal to β . The ACFs of these maps are therefore given by

$$\mathcal{R}_{S,C}(\tau) = C\beta^{|\tau|} + \sum_{m=M+1}^{2M} c_m \lambda_m^{|\tau|}, \quad (3.162)$$

where

$$C = \sum_{m=2}^M c_m. \quad (3.163)$$

This implies that the choice of parameters in (3.160) leads to $M - 1$ of the spectral modes of the map combining into a single spectral mode.

Remark 3.5.17. Equation (3.157) approximates the semi-Markov map using a Markov chain with outputs selected to coincide with the interval midpoints. This reveals a connection between these systems. This connection is further illustrated in chapter 4, where a Markov chain is designed with a transition matrix that coincides

with the order-zero FPM of the map (refer to [3, 14]). Expressions for the power spectra of signals generated by Markov chains are well known in the literature; see, for example, [72–74].

Possible extension of the method to solve IFPP-II The method described above to construct an ergodic semi-Markov map only satisfies the specification of IFPP-I for PWC invariant densities. It is not a solution of IFPP-II, which prescribes the map’s ACF or PSD in addition to its invariant density. The extension of the method to solve IFPP-II was not considered in [22–25]. In the remainder of this section, the method’s potential as a solution for IFPP-II is investigated and evaluated. The extension of this method is briefly described in [3].

Equation (3.154) implies that the parameters β_m may be selected to impose a lower and upper bound on each of the eigenvalues of the map’s order-zero FPM, albeit in a interdependent fashion (specifically, the upper bound of any eigenvalue is the lower bound of the next largest eigenvalue). Equation (3.161) implies that the specification of these bounds provide some control over the properties of the spectral modes corresponding to these eigenvalues. In particular, the rate of decay of the mode in the ACF is inversely proportional to the modulus of the corresponding eigenvalue.

To investigate the potential of the above approach, suppose that a 4×4 order-zero FPM is to be designed with the eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3/4$, $\lambda_3 = 1/2$ and $\lambda_4 = 1/4$. Due to the interdependency of the lower and upper bounds that are specified, (3.154) cannot be used to guarantee arbitrarily tight bounds on each of these eigenvalues. Suppose that tight bounds are desired for λ_2 and λ_4 . The parameters $(\beta_m)_{m=1}^4$ of the order-zero FPM may then be selected as

$$\begin{aligned}\beta_{1,2} &= 3/4 \pm \delta, \\ \beta_{3,4} &= 1/4 \pm \delta,\end{aligned}\tag{3.164}$$

where δ is a small positive value, such that

$$\begin{aligned}\lambda_2 &\in [3/4 - \delta, 3/4 + \delta], \\ \lambda_3 &\in [1/4 + \delta, 3/4 - \delta], \\ \lambda_4 &\in [1/4 - \delta, 1/4 + \delta].\end{aligned}\tag{3.165}$$

Suppose that the map is to be constructed with a uniform invariant density. Substituting $\hat{f}_m = 1$ in (3.156) then yields the parameters

$$\begin{aligned}\alpha_{1,2} &= 1/4 \mp \delta, \\ \alpha_{3,4} &= 3/4 \mp \delta.\end{aligned}\tag{3.166}$$

Substituting the constant $\delta = 10^{-3}$ into the parameter expressions of (3.164) and (3.166), and then substituting the resulting parameters into (3.148), yields the order-zero FPM

$$\mathbf{P}_S^{(0)} \approx \begin{pmatrix} 0.7820 & 0.0312 & 0.0933 & 0.0935 \\ 0.0312 & 0.7805 & 0.0940 & 0.0943 \\ 0.0933 & 0.0940 & 0.5315 & 0.2812 \\ 0.0935 & 0.0943 & 0.2812 & 0.5310 \end{pmatrix}. \quad (3.167)$$

This matrix has the eigenvalues

$$\begin{aligned} \tilde{\lambda}_1 &= 1, \\ \tilde{\lambda}_2 &= 3/4 + \varepsilon_1, \\ \tilde{\lambda}_3 &\approx 0.6250, \\ \tilde{\lambda}_4 &= 1/4 + \varepsilon_2, \end{aligned} \quad (3.168)$$

where $|\varepsilon_1| < 10^{-4}$ and $|\varepsilon_2| < 10^{-4}$. Hence, a tight bound is obtained for eigenvalues $\tilde{\lambda}_2$ and $\tilde{\lambda}_4$, but eigenvalue $\tilde{\lambda}_2$ deviates from the prescribed eigenvalue λ_2 by

$$|\lambda_2 - \tilde{\lambda}_2| \approx 0.125. \quad (3.169)$$

The above example leads to several remarks on the potential of extending the matrix method of [22–25] to the solution of IFPP-II.

Remark 3.5.18. Selecting upper and lower bounds for the eigenvalues of the order-zero FPM using (3.154) *cannot* guarantee a tight bound for every eigenvalue of the matrix [3]. This is due to the fact that the lower and upper bounds of the eigenvalues are interdependent. This was demonstrated using the preceding example.

Remark 3.5.19. The eigenvalues of the matrix given by (3.148) are restricted to be real-valued and positive [3]. Hence, it is not possible to select delta-type modes (eigenvalues equal to zero) and oscillating modes (negative eigenvalues) for the component of the ACF that corresponds to the eigenvalues of any order-zero FPM given by (3.148). Stated differently, the corresponding poles of the map's PSD are all required to lie on the positive real axis in the complex plane. In general, a stochastic matrix may possess complex-valued eigenspectra, which correspond to spectral modes with centre frequencies (in the PSD) equal to the argument of the eigenvalue. It is thus concluded that the freedom afforded by the proposed extension of the method to select the spectral characteristics of the map is limited.

Remark 3.5.20. Equation (3.161) reveals that the ACF of the map has a second term that coincides with the eigenvalues $\lambda_{M+1}, \lambda_{M+2}, \dots, \lambda_{2M}$ of submatrix $\mathbf{Q}_S$ of the map's order-one FPM. The elements of the order-zero FPM and $\mathbf{Q}_S$ are closely related, as is evident from (2.118) and (2.122). As the $2M$ parameters of

the matrix in (3.148) are chosen to select the required invariant density and bound its eigenvalues, it follows that there are insufficient free parameters in the model to select the eigenvalues of $\mathbf{Q}_S$ independently from those of the order-zero FPM. This implies that a subset of the spectral characteristics of the map constructed using the extended method will not be prescribable.

The shortcomings identified in Remarks 3.5.18 and 3.5.19 are inherent to the particular parameterisation of the order-zero FPM, as defined in (3.148). Hence, in order to extend the work of [22–25] to solve IFPP-II, a more general parameterisation of the order-zero FPM would need to be developed. It is shown in chapter 4 that the shortcoming identified in Remark 3.5.20 can be overcome by utilising the additional degrees of freedom afforded in the final design step of constructing a map that possesses a prescribed order-zero FPM (see also [3]). In particular, it is shown that by modifying Algorithm 3.4.1 (canonical map construction) to construct a novel map (a so-called *hat map*) with the same order-zero FPM,¹⁴ the eigenvalues of $\mathbf{Q}$ can be manipulated in a meaningful way.

3.6 Review of Literature: IFPP-III

Nie and Coca [35, 36] proposed the first solution to IFPP-III, referred to in this thesis as the *original solution*. This solution was designed to reconstruct an unknown PWL and ergodic semi-Markov map from sequences of state density functions generated under iteration of the map. The original solution was then extended to construct approximations to a class of continuous maps [36]. Subsequent work further extended the original solution to address more general formulations of IFPP-III [37–39].

The original solution and its extension to construct approximations to continuous maps are presented in sections 3.6.1 and 3.6.2, respectively. A summary of work that addresses the more general formulations of IFPP-III is provided in section 3.6.3.

3.6.1 The original solution

An outline of the original solution to IFPP-III is presented in this section. In order to create context for the novel work that addresses IFPP-III in chapter 5, several propositions regarding the original solution’s characteristics are stated. Two examples

¹⁴It was shown in Remark 3.4.4 that multiple distinct maps may be constructed to possess the same prescribed order-zero FPM.

are provided to demonstrate the original solution and illustrate its properties.¹⁵ The material presented in this section appears in [2].

Outline

The original solution proposed in [35, 36] assumes that the unknown ergodic map S to be reconstructed from density function sequences is PWL and $\mathcal{R}(\mathcal{Q})$ -semi-Markov, where partitions $\mathcal{Q}$ and $\mathcal{R}$ are both unknown. This solution reconstructs the map in three steps, thereby generating a map $\hat{S}$ that is (ideally) a close approximation of S .

The Markov partition of the unknown map is estimated in the first step of the solution [35, 36]. This step depends on the unknown map being ergodic and possessing an invariant density function that is PWC over the true Markov partition. The estimate $\hat{\mathcal{Q}}$ of the Markov partition is used in both of the remaining solution steps.

Several *probing experiments* are carried out in the second solution step [35, 36]. In each experiment, a chosen ensemble of system states is allowed to evolve under iteration of the unknown map. The initial ensemble of each probing experiment comprises states that are drawn randomly and independently from a prescribed *probing density function*, which is selected to be PWC over the estimated Markov partition.

A sequence of density functions is then constructed from the measurements collected in each probing experiment [35, 36]. Each density in the sequence is an estimate of the evolved probing density at a particular time instant (i.e., after iterating the map a specified number of times), and is constructed from the ensemble measured at that time instant. An estimate of the unknown map's order-zero FPM is then constructed from these density sequences, thereby concluding the second solution step.

The third solution step constructs a PWL $\hat{\mathcal{R}}(\hat{\mathcal{Q}})$ -semi-Markov $\hat{S}$ from the outputs of the previous steps [35, 36]. More specifically, the map $\hat{S}$ is constructed under the condition that its order-zero FPM is equal to the estimate of the unknown map's order-zero FPM obtained in the second solution step. This implies that the Markov partition of $\hat{S}$ coincides with the estimate $\hat{\mathcal{Q}}$ of the unknown map's Markov partition, as obtained in the first solution step. The third solution step is thus identical to the third step of the matrix solution process for IFPP-I presented in section 3.4.3.

¹⁵These examples are included in chapter 5, as they permit comparison with the novel method presented in that chapter. Refer to Examples 5.6.1 and 5.6.2.

Each of the original solution's steps is detailed below. In what follows, the use of the hat symbol implies that the partition, function or matrix is an estimate.

Step 1: Estimation of the Markov partition [2] Let $\{x_k^*\}_{k=1}^{K_0}$ denote a set of independent state observations drawn from the invariant state density f_S^* of S . In practice, this set is constructed by allowing an initial ensemble of states drawn from the uniform distribution over Ω to evolve over a sufficient number of time instants under iteration of S .

Since f_S^* is PWC over $\mathcal{Q}$, the problem of estimating the Markov partition $\mathcal{Q}$ is formulated as a search for a partition $\hat{\mathcal{Q}} = \{\hat{Q}_1, \hat{Q}_2, \dots, \hat{Q}_{\hat{M}}\}$ that satisfies

$$f_S^*(x) \approx \sum_{m=1}^{\hat{M}} h_m \chi_{\hat{Q}_m}(x), \quad (3.170)$$

where h_m is given by

$$h_m = \frac{1}{K_0 \mu(\hat{Q}_m)} \sum_{k=1}^{K_0} \chi_{\hat{Q}_m}(x_k^*). \quad (3.171)$$

The search for $\hat{\mathcal{Q}}$ proceeds as follows (the interested reader is referred to [36] for specifics). A uniform partition $\mathcal{U}$ of the unit interval Ω is constructed, where the number of partition intervals is selected to minimise a penalised log-likelihood function. Next, a histogram of the observations $\{x_k^*\}_{k=1}^{K_0}$ is derived, such that the histogram bins coincide with the intervals of $\mathcal{U}$. Different candidates for $\hat{\mathcal{Q}}$ are constructed by merging neighbouring bins of the histogram with absolute height differences that fall below a threshold, and by varying the threshold value. The candidate partition associated with the minimum integrated squared error between the densities coinciding with its histogram and the histogram over the uniform partition is then selected as the partition $\hat{\mathcal{Q}}$.

Step 2: Estimation of the order-zero FPM [2] An estimate $\hat{\mathbf{P}}_S^{(0)}$ of the order-zero FPM $\mathbf{P}_S^{(0)}$ belonging to S is obtained from state density function estimates constructed after conducting N probing experiments. In each probing experiment $n = 1, 2, \dots, N$, an ensemble $\{x_{1,k}^n\}_{k=1}^K$ of K states drawn independently from a probing density f_1^n evolves under iteration of S . The ensemble is measured at time instants $t = 2, \dots, T$. Let these measurements be denoted by $\{x_{t,k}^n\}_{k=1}^K$.

Whereas an arbitrary number of probing experiments may be performed, in the method presented here, a total of $N = \hat{M}$ probing experiments $n = 1, 2, \dots, \hat{M}$ are conducted, where $\hat{M}$ denotes the number of intervals in the estimated Markov

partition $\hat{\mathcal{Q}}$. The probing densities $f_1^1, f_1^2, \dots, f_1^{\hat{M}}$ are selected to be PWC over $\hat{\mathcal{Q}}$ according to the expression

$$f_1^n(x) = \sum_{m=1}^{\hat{M}} f_{1,0,m}^n \chi_{\hat{\mathcal{Q}}_m}(x), \quad (3.172)$$

where

$$f_{1,0,m}^n = \frac{1}{\mu(\hat{\mathcal{Q}}_m)} \delta[n - m] \quad (3.173)$$

and $\delta[\cdot]$ denotes the Kronecker delta function. These probing densities are plotted in Fig. 3.6.1. The motivation for the selection of these densities is to be provided below.

If it was assumed that $\hat{\mathcal{Q}} = \mathcal{Q}$, the evolved density functions f_t^n , where $t = 2, 3, \dots, T$, would also be PWC over $\hat{\mathcal{Q}}$. This observation motivates the construction of density function estimates $\hat{f}_t^n \approx f_t^n$ that are PWC over $\hat{\mathcal{Q}}$. Specifically, the estimates

$$\hat{f}_t^n(x) = \begin{cases} f_1^n(x) & \text{if } t = 1, \\ \sum_{m=1}^{\hat{M}} \hat{f}_{t,0,m} \chi_{\hat{\mathcal{Q}}_m}(x) & \text{if } t = 2, 3, \dots, T, \end{cases} \quad (3.174)$$

where

$$\hat{f}_{t,0,m} = \frac{1}{K \mu(\hat{\mathcal{Q}}_m)} \sum_{k=1}^K \chi_{\hat{\mathcal{Q}}_m}(x_{t,k}^n), \quad (3.175)$$

are constructed.

Let $\hat{\mathbf{f}}_t^n \triangleq [\hat{f}_{t,0,1}, \hat{f}_{t,0,2}, \dots, \hat{f}_{t,0,\hat{M}}]^T$ denote the order-zero vector representation of the density $\hat{f}_t^n$. Furthermore, let

$$\hat{\mathbf{A}} \triangleq [\hat{\mathbf{F}}_1, \hat{\mathbf{F}}_2, \dots, \hat{\mathbf{F}}_{T-1}] \quad (3.176)$$

and

$$\hat{\mathbf{B}} \triangleq [\hat{\mathbf{F}}_2, \hat{\mathbf{F}}_3, \dots, \hat{\mathbf{F}}_T], \quad (3.177)$$

where

$$\hat{\mathbf{F}}_t \triangleq [\hat{\mathbf{f}}_t^1, \hat{\mathbf{f}}_t^2, \dots, \hat{\mathbf{f}}_t^N]. \quad (3.178)$$

If it were assumed that $\hat{\mathcal{Q}} = \mathcal{Q}$ and each density estimate satisfies $\hat{f}_t^n = f_t^n$, (2.114) implies that

$$\hat{\mathbf{B}} = \mathbf{P}_S^{(0)} \hat{\mathbf{A}}. \quad (3.179)$$

Now, Proposition 3.4.3 states sufficient conditions for a matrix to qualify as the order-zero FPM of a semi-Markov map. More specifically, the matrix qualifies if it is non-negative and possesses a left eigenvector, associated with a unity eigenvalue,

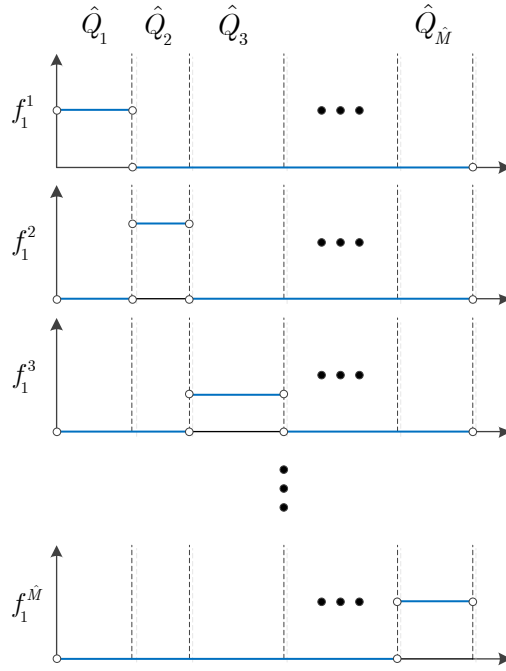


Figure 3.6.1: Probing densities used in the original solution of IFPP-III. Reproduced with permission from [2] (© [2023] IEEE).

that coincides with the lengths of the intervals in the map's Markov partition. An estimate $\hat{\mathbf{P}}_S^{(0)}$ of the order-zero FPM $\mathbf{P}_S^{(0)}$ is thus constructed under the constraints that the elements of $\hat{\mathbf{P}}_S^{(0)}$ are non-negative and possess the left eigenvector

$$\hat{\mathbf{v}}_1^T = [\mu(\hat{Q}_1), \mu(\hat{Q}_2), \dots, \mu(\hat{Q}_M)] \quad (3.180)$$

associated with a unity eigenvalue.¹⁶ Hence, the estimate $\hat{\mathbf{P}}_S^{(0)}$ is ideally selected as a solution to the constrained optimisation problem

$$\mathbf{P}_S^* = \arg \min_{\mathbf{P}} : \|\hat{\mathbf{B}} - \mathbf{P}\hat{\mathbf{A}}\|_F, \quad (3.181)$$

subject to

$$\hat{\mathbf{v}}_1^T \mathbf{P} = \hat{\mathbf{v}}_1^T \quad (3.182)$$

and to the elements of $\mathbf{P} \triangleq [p_{k,m}]_{k,m=1}^{\hat{M}}$ satisfying

$$p_{k,m} \geq 0, \quad k, m = 1, 2, \dots, \hat{M}, \quad (3.183)$$

where $\|\cdot\|_F$ denotes the Frobenius norm.

The following proposition is proved.

¹⁶These constraints are sufficient for $\hat{\mathbf{P}}_S^{(0)}$ to be similar to a column stochastic matrix under the similarity transform of (3.74).

Proposition 3.6.1 (Convexity of the optimisation problem of (3.181) [75]). The constrained optimisation problem defined by (3.181) is convex.

Proof. The Proposition is proved in section 5.C.2, where $\mathbf{q} = \hat{\mathbf{v}}_1$. $\square$

As the problem defined in (3.181) is convex, standard methods for solving convex problems may be used to approximate its optimal solution. However, the estimate

$$\bar{\mathbf{P}}_S^{(0)} = \hat{\mathbf{B}}\hat{\mathbf{A}}^\dagger, \quad (3.184)$$

is considered in this work, where $\hat{\mathbf{A}}^\dagger$ denotes the Moore-Penrose pseudoinverse of $\hat{\mathbf{A}}$, as defined by

$$\hat{\mathbf{A}}^\dagger \triangleq \hat{\mathbf{A}}^T(\hat{\mathbf{A}}\hat{\mathbf{A}}^T)^{-1}. \quad (3.185)$$

This estimate is motivated by the property that it minimises the Frobenius norm of (3.181) [76]. Furthermore, observe that by selecting the probing densities defined by (3.172) and (3.173), the matrix $\hat{\mathbf{F}}_1$ defined in (3.178) becomes a diagonal matrix with strictly positive entries along its diagonal. This implies that matrix $\hat{\mathbf{A}}$ has a rank of $N = \hat{M}$. Now, suppose that all density functions are estimated with perfect accuracy, such that $\hat{\mathbf{B}} = \mathbf{P}_S^{(0)}\hat{\mathbf{A}}$. It then follows from [76, Th. 1] that

$$\bar{\mathbf{P}}_S^{(0)} = \mathbf{P}_S^{(0)}\hat{\mathbf{A}}\hat{\mathbf{A}}^\dagger = \mathbf{P}_S^{(0)}. \quad (3.186)$$

Hence, with perfect density estimation, $\hat{\mathbf{P}}_S^{(0)} \triangleq \bar{\mathbf{P}}_S^{(0)}$, where $\bar{\mathbf{P}}_S^{(0)}$ is defined in (3.184), is the optimal solution to the constrained optimisation problem of (3.181).

In the general case where density estimation is not perfect, the estimate $\bar{\mathbf{P}}_S^{(0)}$ defined in (3.184) does not necessarily satisfy the constraints of (3.182) and (3.183). In this work, the matrix $\bar{\mathbf{P}}_S^{(0)}$ is manipulated to satisfy the constraints, thereby yielding the estimate $\hat{\mathbf{P}}_S^{(0)}$. All negative elements of $\bar{\mathbf{P}}_S$ are first replaced with zeros. Let the columns of the resulting matrix be denoted by $\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_2, \dots, \bar{\mathbf{p}}_{\hat{M}}$. The approximate solution

$$\hat{\mathbf{P}}_S^{(0)} \triangleq [\hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2, \dots, \hat{\mathbf{p}}_{\hat{M}}] \quad (3.187)$$

is obtained by normalising the columns of the resulting matrix as

$$\hat{\mathbf{p}}_m = \frac{\mu(\hat{Q}_m)}{\hat{\mathbf{v}}_1^T \bar{\mathbf{p}}_m} \bar{\mathbf{p}}_m, \quad m = 1, 2, \dots, \hat{M}. \quad (3.188)$$

Remark 3.6.1. The estimate defined by (3.187) and (3.188) is a suboptimal solution to the constrained optimisation problem of (3.181) in the general case where density estimation is imperfect [2]. An imperfect estimate of $\mathbf{P}_S^{(0)}$ introduces errors into the reconstructed map. In particular, it is shown in Example 5.6.1, as presented in chapter 5, that the domains and slopes of the reconstructed branches deviate from those of the unknown map.

Step 3: Construction of a map with matching order-zero FPM [2] The final solution step is the construction of a PWL $\hat{\mathcal{R}}(\hat{\mathcal{Q}})$ -semi-Markov map $\hat{S}$ with order-zero FPM $\mathbf{P}_{\hat{S}}^{(0)}$ that is equal to the estimate $\hat{\mathbf{P}}_S^{(0)}$ of the unknown map's order-zero FPM. As presented in [35, 36], the original solution uses Algorithm 3.4.1 (canonical map construction), which was introduced in section 3.4.3, to construct the map $\tilde{S}$. Since the matrix $\hat{\mathbf{P}}_S^{(0)}$ was designed to possess a left eigenvector $\hat{\mathbf{v}}_1^T$ associated with the unity eigenvalue (this corresponds to constraint (3.182) of the constrained optimisation problem of (3.181)), Proposition 3.4.3 then implies that $\tilde{S}$ is $\hat{\mathcal{R}}(\hat{\mathcal{Q}})$ -semi-Markov for some sub-partition $\hat{\mathcal{R}}$ of the estimated Markov partition $\hat{\mathcal{Q}}$.

Two remarks regarding the map $\tilde{S}$ constructed in the final step of the original solution are stated below.

Remark 3.6.2. Although it cannot be guaranteed that each map $\tilde{S}$ constructed in step 3 of the original solution is ergodic, Proposition 3.4.4 may be used to verify whether any particular map $\tilde{S}$ constructed using the solution is ergodic. In particular, it may be verified whether the order-zero FPM $\mathbf{P}_{\tilde{S}}^{(0)}$ of the map $\tilde{S}$ satisfies the conditions stated in the proposition.

Remark 3.6.3. As Algorithm 3.4.1 (canonical map construction) was used to construct the map in step 3 of the original solution, the map is necessarily canonical [2]. It is important to note that this represents only one of multiple possible maps $\tilde{S}$ that may be constructed to possess the matrix $\hat{\mathbf{P}}_S^{(0)}$ as its order-zero FPM. This observation is exploited in chapter 5 to develop an improved solution to IFPP-III.

The following propositions regarding the original solution to IFPP-III are proved.

Proposition 3.6.2. Assume S is canonical and possesses only increasing branches. If the Markov partition and the order-zero FPM of S are estimated with perfect accuracy (i.e., $\hat{\mathcal{Q}} = \mathcal{Q}$ and $\hat{\mathbf{P}}_S^{(0)} = \mathbf{P}_S^{(0)}$), the original solution reconstructs S perfectly (i.e., $\hat{S} = S$) [2]. Conversely, if S is not canonical or possesses any decreasing branches, S cannot be reconstructed successfully (i.e., $\hat{S} \neq S$), regardless of estimation accuracy.

Proof. The proposition is proved in section 3.A.4. □

Proposition 3.6.3. If the original solution estimates the Markov partition and the order-zero FPM of S with perfect accuracy (i.e., $\hat{\mathcal{Q}} = \mathcal{Q}$ and $\hat{\mathbf{P}}_S^{(0)} = \mathbf{P}_S^{(0)}$), the invariant density $f_{\hat{S}}^*$ of $\hat{S}$ equals the invariant density f_S^* of S , regardless of whether S is canonical or the monotonicity of its branches [2].

Proof. The proposition is proved in section 3.A.5. □

Proposition 3.6.4. If S possesses any decreasing branches, a subset of the map's power spectrum mode characteristics (i.e., the centre frequencies and bandwidths of the $2M$ modes that comprise the map's power spectrum) are in general unequal to that of S , regardless of estimation accuracy [2].

Proof. The proposition is proved in section 3.A.6. □

Remark 3.6.4. Propositions 3.6.2 and 3.6.4 reveal two shortcomings of the original solution. In particular, if the unknown map possesses any decreasing branch functions, the method is unable to consistently reconstruct the map. Furthermore, the map constructed using the solution possesses a power spectrum with mode characteristics (i.e., mode centre frequencies and bandwidths) that are inconsistent with those of the unknown map in general. This limits the applicability of the original solution.

3.6.2 Extension of the original solution

Nie and Coca [36] considered the problem of extending the original solution of IFPP-III presented in section 3.6.1 to permit the approximation of more general maps that do not necessarily possess a finite-dimensional FPO. The use of the original solution as a starting point was motivated by Ulam's conjecture that the FPO of a one-dimensional map can be approximated arbitrarily well by the finite-dimensional FPO of a Markov map defined over a uniform partition of the interval [48]. Li proved Ulam's conjecture and presented an algorithm for constructing the finite dimensional representation of the FPO if the map is known [77]. In the context of the original solution to IFPP-III, the order-zero FPM of the map coincides with the finite dimensional representation of the FPO.

The maps $S : \Omega \rightarrow \Omega$ to be approximated using the extension of the original solution presented in [36] are required to be continuous. Furthermore, the FPO $\mathcal{P}_S$ associated with S is required to possess a unique stationary density f_S^* , and the sequence $(\mathcal{P}_S^n)$ is required to be *asymptotically stable*, as defined below.

Definition 3.6.1 (Asymptotic stability [49]). Let $(\Omega, \mathcal{B}, \nu)$ denote a measure space and let $S : \Omega \rightarrow \Omega$ be non-singular with respect to ν . Let $\mathcal{P}_S$ denote the FPO corresponding to S . The sequence $(\mathcal{P}_S^n)$ is said to be asymptotically stable if $\mathcal{P}_S$ has a stationary density $f_S^* \in \mathcal{D}(\Omega, \mathcal{B}, \nu)$ and

$$\lim_{n \rightarrow \infty} \|\mathcal{P}_S^n f - f_S^*\| = 0, \quad \text{for every } f \in \mathcal{D}(\Omega, \mathcal{B}, \nu). \quad (3.189)$$

The following theorem regarding the asymptotic stability of $(\mathcal{P}_S^n)$ for piecewise monotonic maps $S \in \Gamma(\mathcal{Q})$ was proved.

Theorem 3.6.1 (Asymptotic stability and piecewise monotonic maps [49]). *Let $S \in \Gamma(\mathcal{Q})$ be a complete map such that $S_m \triangleq S|_{Q_m}$ is a C^2 function. Suppose that*

$$|S'(x)| > 1, \quad \forall x \neq q_m, \quad m = 1, 2, \dots, M, \quad (3.190)$$

where $\overline{P}(\mathcal{Q}) = \{q_1, q_2, \dots, q_{M+1}\}$ are the partition points of $\mathcal{Q}$. If there exists some finite constant c such that

$$-\frac{S''(x)}{[S'(x)]^2} \leq c, \quad x \neq q_m, \quad m = 1, 2, \dots, M, \quad (3.191)$$

then $\{\mathcal{P}_S^n\}$ is asymptotically stable.

Proof. The interested reader is referred to [49, Theorem 6.2.1] for a proof. □

Theorem 3.6.1 imposes restrictive conditions on the derivatives of the map S . However, it is in some cases possible to extend the applicability of the above theorem to more general maps, such as the logistic map, by using a change of variables; refer to [49, Section 6.5].

Let $\hat{S}$ denote a semi-Markov map constructed to approximate an unknown map S using the extended solution to IFPP-III [36]. The extended solution follows the same three steps as the original solution outlined in section 3.6.1. In the first step, a Markov partition is identified for the map to be constructed. The Markov partition is constructed from an estimate of the map's invariant density f_S^* using the same method as the original solution, despite the fact that the invariant density may not be PWC over any partition of the unit interval. In what follows, $\hat{\mathcal{Q}}$ is used to denote the Markov partition that was identified, even though it may not be an estimate of some underlying Markov partition $\mathcal{Q}$.

An order-zero matrix representation $\hat{\mathbf{P}}_S^{(0)}$ of the unknown map's FPO is constructed in the second step [36]. This matrix is constructed in the same manner that the estimate of the unknown semi-Markov map's order-zero FPM is constructed in the original solution. It is shown in [36] that $\hat{\mathbf{P}}_S^{(0)}$ qualifies as the order-zero FPM of an $\hat{\mathcal{R}}(\hat{\mathcal{Q}})$ -semi-Markov map that approximates the underlying map S . However, it is observed in [36] that the continuity of S bestows the additional property whereby the non-zero entries in each column of $\hat{\mathbf{P}}_S^{(0)}$ are contiguous.

In the final step of the extended solution, a semi-Markov map $\hat{S}$ is constructed to possess the Markov partition $\hat{\mathcal{Q}}$ and matrix $\hat{\mathbf{P}}_S^{(0)}$ as its order-zero FPM [36].

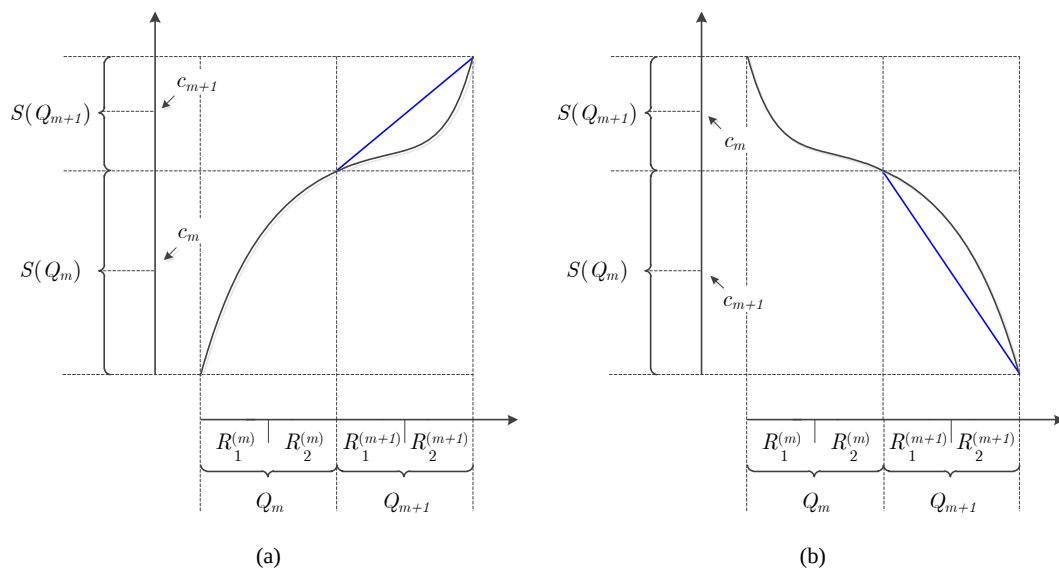


Figure 3.6.2: Identification of monotonically increasing and decreasing branches of a continuous map.

Algorithm 3.4.1 (canonical map construction) is used for this purpose. However, the matrix representation of the unknown map's FPO does not reveal whether each of the unknown map's branches are increasing or decreasing. Nie and Coca [36] devised a method for estimating the monotonicity of each branch that exploits the fact that the unknown map is a continuous function.

Assume that S restricted to any interval of the partition $\mathcal{Q}$ is either monotonically increasing or decreasing. To define the method for estimating branch monotonicity, consider Fig. 3.6.2(a), which is a partial plot of a continuous map S . As the non-zero elements in each column of $\hat{\mathbf{P}}_S^{(0)}$ is contiguous, the closure of the image of Q_m under S is an interval. Let c_m denote the midpoint of this interval. As the map is continuous and increasing over the intervals Q_m and Q_{m+1} , it follows that $c_m < c_{m+1}$.

Alternatively, if the map is continuous and decreasing over the intervals Q_m and Q_{m+1} , as is the case in Fig. 3.6.2(b), it follows that $c_m > c_{m+1}$. Using an ensemble of states drawn from each of the probing densities of Fig. 3.6.1, the interval $S(Q_m)$ and its midpoint may be estimated for each $m = 1, 2, \dots, M$. The signs $s_{j,m+1}$ of the slopes belonging to all branch functions $S_{j,m+1}$ defined over Q_{m+1} may then be

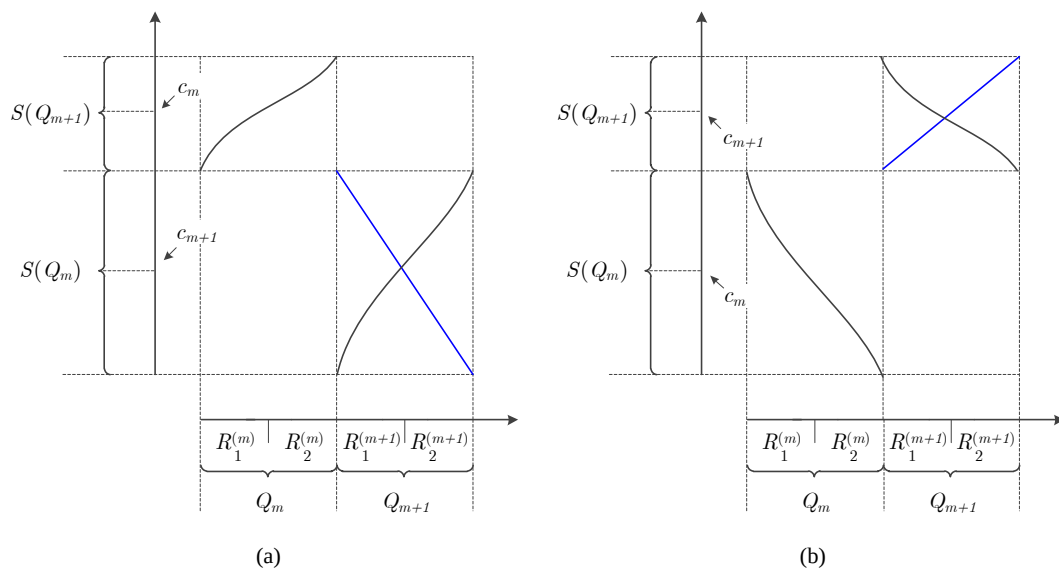


Figure 3.6.3: Attempted identification of monotonically increasing and decreasing branches of a map with discontinuities.

selected as [36]

$$s_{j,m+1} = \begin{cases} 1, & \text{if } c_m < c_{m+1}, \\ -1, & \text{if } c_m > c_{m+1}, \\ s_{1,m}, & \text{if } c_m = c_{m+1}, \end{cases} \quad (3.192)$$

for all $j = 1, 2, \dots, c(m)$. The branch functions obtained using this expression are plotted in blue in Figs. 3.6.2(a, b).

Remark 3.6.5. The continuity of S is a requirement for the method defined by (3.192) to successfully identify whether each map branch is increasing or decreasing. If the map S has any discontinuities, the proposed method may fail. This is illustrated with Figs. 3.6.3(a, b), which are partial plots of maps that are not continuous. Applying the proposed method to these maps yields the branch functions plotted in blue. It is apparent that the slope of these branch functions is incorrectly estimated.

Nie and Coca [36] presented several examples involving the logistic map that demonstrate the ability of the proposed method to reconstruct an unknown continuous map from density sequences. It was also shown that the approximation accuracy remains high if artificially generated noise was added to the measurements from which the map was constructed.

3.6.3 Solutions to generalisations of IFPP-III

Nie and Coca [37] generalised the technique proposed in [36] to approximate the evolution rule of an unknown stochastically perturbed dynamical system, which is of practical relevance when constructing models from measurements that are corrupted by noise [1]. Under the assumption that the unknown map is semi-Markov, a matrix representation of the system's transfer operator was formulated in terms of the map's order-zero FPM. An algorithm was proposed for estimating this FPM from sequences of causal density functions generated by the stochastically perturbed system, where it is assumed that the density function of the perturbation is known. The unknown map is reconstructed from the estimate of the FPM as in [36]. The technique was also extended to approximate continuous non-linear maps. Examples were presented that demonstrate the improved accuracy of the technique when used to approximate an unknown semi-Markov map and the logistic map in higher noise levels, as compared to the original technique presented in [36].

Nie et al. [38] considered the problem of constructing a semi-Markov map that approximates the unknown map of a one-dimensional dynamical system that is driven by an external control input and subjected to an additive stochastic perturbation [1]. Following a similar approach as in [37], a matrix representation of the system's transfer operator was formulated in terms of the order-zero FPM corresponding to the semi-Markov map. An algorithm was proposed for estimating the latter matrix from sequences of density functions generated by the system (here it is assumed that the density functions of the noise and control inputs are both known). The map is constructed from the FPM as in [36]. It is noteworthy that this solution to the IFPP presents a new approach to solving the control problem for stochastically perturbed autonomous systems, where the goal is to determine the optimal input density function such that the dynamical system has the desired equilibrium distribution.

Nie et al. [39] proposed a technique for inferring the evolution rule of an unknown one-dimensional dynamical system driven by an external stochastic input from stationary densities generated by the system [1]. In this scenario the density function of the input is varied, and the stationary density of the system is observed in each case. Conditions under which each input density leads to a unique stationary density were stated. The proposed technique constructs a matrix approximation of the unknown map's FPO and generates a semi-Markov map with order-zero FPM equal to this matrix. An example was presented wherein a semi-Markov approximation of the logistic map was obtained from stationary density functions generated by the map.

3.7 Summary and Conclusions

This section presents a summary of existing methods that have been developed to solve the three formulations of the IFPP, and identifies gaps in existing work. Conclusions regarding the methods' characteristics and practicality are presented.

3.7.1 IFPP-I

IFPP-I requires the design of an ergodic map that possesses an invariant density function prescribed as a specification. The topological conjugation, direct and matrix strategies have emerged for solving this problem. These strategies have matured over the years and methods that make use of them are well understood. Hence, these methods may be formally considered solutions to IFPP-I.

The review presented in this chapter reveals that the prescription of the invariant density is a relatively weak constraint, and that considerable freedom remains to specify the structural features of the map during its design. The three solution strategies afford different degrees of freedom to select these structural features.

Although the topological conjugation solution strategy realises the prescribed invariant density exactly and guarantees the ergodicity of the map, it requires an existing ergodic map with a known invariant density as a starting point and does not permit direct selection of the domain, image or shape of the map's branch functions. However, this strategy permits multiple conjugating transformations that yield the same invariant density. Different classes of such conjugating transformations may potentially be designed to yield distinct conjugate maps that share the same invariant density. This has been identified as a gap in the literature.

The direct solution strategy realises an ergodic map with the exact invariant density prescribed. This strategy requires the specification of inter-branch correspondence functions, which permit selection of the domains and images of the map's branch functions and grants freedom to specify characteristics such as symmetry. This strategy yields expressions for *all* maps that share the prescribed invariant density and satisfy the correspondence functions, but ergodicity is not guaranteed.

In the literature, the direct solution strategy has only been applied to design *complete* maps to solve IFPP-I. The specification of inter-branch correspondence functions for *incomplete* maps may yield new map classes that share the same invariant density.

This has been identified as a gap in existing work.

The matrix solution strategy permits construction of PWL semi-Markov maps. The FPO of these maps admits an exact finite-dimensional matrix representation (the order-zero FPM). The solution strategy designs a candidate order-zero FPM to realise a prescribed PWC invariant density, and then designs the map from this matrix.

The design requirements for the order-zero FPM and an algorithm for constructing the map from this matrix are developed from first principles. It is shown that multiple distinct maps may be constructed to possess the same order-zero FPM and the same invariant density. Hence, the matrix solution grants the freedom to specify structural characteristics such as the number of map branches and their monotonicity.

Two methods are developed for selecting the PWC invariant density to approximate a smooth density function prescribed for the metric. The design requirements that each method imposes on the FPM are determined (these requirements are relevant to the work presented in chapter 4). A bound on the approximation error of the methods is derived, and it is shown that the approximation error of the first method converges to zero as the number of intervals in the map's Markov partition grows.

Existing matrix solutions to IFPP-I construct a canonical map using Algorithm 3.4.1 (canonical map construction). The canonical maps are only one subclass of the semi-Markov maps. The design of maps from distinct subclasses of the semi-Markov maps that yield the same invariant density has been identified as a gap in the literature. This gap is addressed with the design of maps from two novel classes in chapters 4 and 5; these are the classes of *hat maps* and *generalised hat maps*.

3.7.2 IFPP-II

IFPP-II requires the design of an ergodic map to simultaneously possess an invariant density function and an ACF or PSD that are prescribed as specifications. This problem is more complicated and the methods developed have shortcomings that prohibit their practical use. These methods aim to solve IFPP-II by generalising one of the three strategies proposed for solving IFPP-I.

The chapter investigates the feasibility of topological conjugation as a solution strategy for IFPP-II. Although this strategy admits a conceptually simple and elegant design approach, the development of an analytical method must overcome multiple issues. An analytical method relies on an expression for the ACF of the conjugate map as a

function of the underlying map's ACF and its branch parameters. However, these expressions are often exceedingly sophisticated and troublesome to derive, especially if the underlying maps are chosen to realise ACFs with multiple modes. To simplify these expressions and attain feasibility, the analytical method investigated in this chapter relies on an underlying map with only a single power spectrum mode [33].

The above issues appear to have halted further work aimed at developing analytical methods that use topological conjugation to solve IFPP-II. However, the numerical design of conjugate maps with richer statistical metrics has been identified as a gap in existing work. The map class Γ_2 investigated in section 3.A.6 appears to be a suitable candidate class for the design of the underlying map. These maps are defined over a low dimensional parameter space and admit power spectra with multiple modes. A numerical search for a map with the prescribed ACF over this parameter space is less likely to suffer from non-convergence than the method proposed in [32].

The feasibility of direct solutions to IFPP-II is also investigated in this chapter. This strategy derives an expression for all maps that possess the prescribed invariant density and satisfy a collection of parameterised inter-branch correspondence functions. The aim is then to find parameter values that yield a map with the prescribed ACF.

To permit maps with greater statistical diversity, multiple parameters are specified for each inter-branch correspondence function. This increases the complexity of the implicit expressions that relate the ACF to the parameters, thereby forcing reliance on a numerical search for parameters values that yield the desired map. However, this search has to be carried out over a parameter space that is of high dimensionality, which leads to convergence problems and excessive computational requirements.

The numerical method of [32] successfully used the direct strategy to design maps with prescribed invariant densities and multiple power spectrum modes. However, the method's convergence is not guaranteed, and its computational complexity is impractical. The latter drawback is a consequence of having to numerically estimate the ACF of each candidate map during the search. A further drawback of this method is the lack of theory regarding the realisable class of ACFs or PSDs.

Future work on using the direct strategy to solve IFPP-II involves the definition of simplified models for the map's inter-branch correspondence functions. If these functions are defined to be piecewise and to possess the Markov property, expressions for the map's ACF may possibly be derived using a matrix approach. It is anticipated that these expressions will relate the ACF to the parameters in a less sophisticated manner, thereby admitting an analytical method for map design.

A gap was identified in existing work for a matrix solution to IFPP-II that constructs members from the class of PWL semi-Markov maps. These matrix representation of these maps' FPO admits unsophisticated expressions for the statistical metrics as functions of a finite collection of branch parameters. This increases the potential for an analytical method that designs these maps to satisfy prescribed metrics.

The PWC invariant density of the PWL semi-Markov maps may be selected to approximate smooth functions prescribed for this metric arbitrarily closely. However, the capacity of these maps to approximate an arbitrary continuous functions prescribed for the power spectrum is uncertain. Whereas these maps possess rational power spectra, the structure of the FPO's matrix representation restricts the poles that may be directly prescribed under the current design approach. This implies that a semi-Markov map with an arbitrary rational power spectrum may not exist.

Although the approximation of arbitrary power spectra may not be possible, there exists design freedom to select the characteristics of a PWL semi-Markov map's power spectrum. It is conjectured that multiple peaks may be realised in the power spectrum, and that their centre frequencies and bandwidths may be adjusted. This conjecture is confirmed in the numerical examples of chapter 4, where maps are constructed using the novel method for approximately solving IFPP-II.

The extension of a more recent matrix solution of IFPP-I [22–25] to address IFPP-II is investigated. The structure of the FPM used by this method is found to severely restrict the characteristics of the power spectrum modes that may be realised. This is a consequence of the matrix only admitting real eigenspectra. This prompted the development of a novel analytical method¹⁷ for designing the FPM to possess a prescribed complex-valued eigenspectrum, thereby yielding more diverse power spectrum mode characteristics.

3.7.3 IFPP-III

IFPP-III requires the reconstruction of an unknown ergodic map from sequences of density function estimates, which are derived from state ensembles that evolve under iteration of the map. The chapter focusses on the formulation that requires the reconstruction of a PWL semi-Markov map. An existing matrix solution to this problem [35, 36], referred to here as the original solution, is presented in detail.

¹⁷This method is presented in chapter 4.

Results from the literature demonstrate the original solution’s successful reconstruction of unknown canonical semi-Markov maps with increasing branch functions, thereby yielding maps with consistent dynamical and statistical characteristics. Other features of this method are its ability to identify the Markov partition of the unknown map and to accurately reconstruct the map under impairments such as measurement noise. The chapter also describes several extensions of the original solution [37–39].

The chapter identifies two shortcomings of the original solution. It was proved that the solution fails to identify the monotonicity of the unknown map’s branch functions, and is thus unable to consistently reconstruct maps that possess one or more decreasing branch functions. In this case, the reconstructed map possesses inconsistent power spectrum mode characteristics.

It is shown that the above shortcomings are due to unknown map’s reconstruction from an estimate of its order-zero FPM, which does not depend on the monotonicity of the map’s branches. Chapter 5 develops a novel method that instead reconstructs the map from an estimate of its order-one FPM, which is dependent on branch function monotonicity. This novel method is shown to resolve the shortcomings of the original solution.

3.7.4 Gaps in the literature

The gaps that were identified in the literature are enumerated in Table 3.7.1.

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Table 3.7.1: Gaps identified in the literature on the IFPP.

Problem	Solution strategy	Gap
IFPP-I	Topological conjugation	The design of conjugating transformation classes to yield distinct conjugate maps that share the same invariant density.
IFPP-I	Direct	The definition of inter-branch correspondence functions for <i>incomplete</i> maps to yield new map classes that share the same invariant density.
IFPP-I	Matrix	The design of non-canonical maps in the final solution step to yield distinct semi-Markov maps that share the same invariant density.
IFPP-II	Topological conjugation	The <i>numerical</i> design of conjugate maps with the prescribed invariant density and richer ACFs from underlying maps with <i>multimodal</i> power spectra (such as class Γ_2).
IFPP-II	Direct	The <i>analytical</i> design of a map with the prescribed invariant density and ACF by deriving expressions for the latter function under simplified models for the inter-branch correspondence function, where possible.
IFPP-II	Matrix	<i>The absence of a matrix solution strategy for IFPP-II.</i>
IFPP-III	Topological conjugation	The absence of a method that uses the topological conjugation solution strategy.
IFPP-III	Direct	The absence of a method that uses the direct solution strategy.
IFPP-III	Matrix	The absence of method that yields canonical maps with consistent branch function monotonicity and power spectrum mode characteristics.

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Appendix

3.A Proofs of Propositions

3.A.1 Proof of Proposition 3.4.1

This proof was adapted from [78]. The estimate $\hat{F}_P$ is a PWL approximation of the distribution function F_P that satisfies

$$\hat{F}_P(q_m) = F_P(q_m), \quad m = 1, 2, \dots, M + 1. \quad (3.193)$$

The function $\hat{F}_{P,m} \triangleq \hat{F}_P|_{Q_m}$ is given by

$$\hat{F}_{P,m}(x) = \frac{F_P(q_{m+1}) - F_P(q_m)}{q_{m+1} - q_m}(x - q_m) + F_P(q_m). \quad (3.194)$$

Define the functions

$$\omega_m(x) = (x - q_m)(x - q_{m+1}), \quad (3.195)$$

$$\varepsilon_m(x) = F_{P,m}(x) - \hat{F}_{P,m}(x) \quad (3.196)$$

and

$$G_m(x) = \varepsilon_m(x) - \frac{\omega_m(x)}{\omega_m(a)}\varepsilon_m(a), \quad (3.197)$$

where $a \in (q_m, q_{m+1})$.

From the definition of the functions, $G_m(a) = 0$ and $\omega_m(q_m) = \omega_m(q_{m+1}) = 0$. It follows that

$$G_m(q_m) = \varepsilon_m(q_m) = 0 \quad (3.198)$$

and

$$G_m(q_{m+1}) = \varepsilon_m(q_{m+1}) = 0. \quad (3.199)$$

Rolle's theorem [79] then implies the existence of the points $b \in (q_m, a)$ and $c \in (a, q_{m+1})$ such that $G'_m(b) = 0$ and $G'_m(c) = 0$. Invoking Rolle's theorem a second time, a point $d \in (b, c)$ exists such that $G''_m(d) = 0$. This implies that

$$G''_m(d) = F''_{P,m}(d) - \frac{2}{\omega_m(a)}\varepsilon_m(a) = 0. \quad (3.200)$$

This implies that

$$\varepsilon_m(a) = \frac{1}{2} \omega_m(a) F''_{P,m}(d). \quad (3.201)$$

It follows that

$$\begin{aligned} |\varepsilon_m(x)| &\leq \frac{1}{2} \left[\max_{[q_m, q_{m+1}]} |\omega_m(x)| \right] \cdot \max_{[q_m, q_{m+1}]} |F''_{P,m}| \\ &\leq \frac{1}{8} (q_{m+1} - q_m)^2 \max_{[q_m, q_{m+1}]} |F''_{P,m}|. \end{aligned}$$

This expression implies that

$$\left\| \hat{F}_{P,m} - F_{P,m} \right\|_{\infty} \leq \frac{\mu(Q_m)^2}{8} \|f'_{P,m}\|_{\infty}, \quad m = 1, 2, \dots, M. \quad (3.202)$$

Furthermore,

$$\begin{aligned} \left\| \hat{F}_P - F_P \right\|_{\infty} &\leq \max_{m=1,2,\dots,M} \left\| \hat{F}_{P,m} - F_{P,m} \right\|_{\infty} \\ &\leq \max_{m=1,2,\dots,M} \frac{\mu(Q_m)^2}{8} \|f'_{P,m}\|_{\infty}. \end{aligned} \quad (3.203)$$

3.A.2 Proof of Proposition 3.4.3

The length of the domain of branch function $S_{j,m}$ is derived from (3.89) as

$$\mu(R_j^{(m)}) = \mu(Q_{U(j,m)}) |v_{j,m}|^{-1}, \quad j = 1, 2, \dots, c(m), \quad m = 1, 2, \dots, M, \quad (3.204)$$

where $v_{j,m}$ is the branch function's slope. This expression implies that the length of the image of $R_j^{(m)}$ under S is given by

$$\mu(S(R_j^{(m)})) = |v_{j,m}| \mu(R_j^{(m)}) = \mu(Q_{U(j,m)}). \quad (3.205)$$

Substitution of (3.90) and $x = r_j^{(m)}$ into (3.86) yields

$$\inf(S(R_j^{(m)})) = q_{U(j,m)}. \quad (3.206)$$

As $S_{j,m}$ is linear, it then follows that

$$S(R_j^{(m)}) = Q_{U(j,m)} \in \mathcal{Q}, \quad j = 1, 2, \dots, c(m), \quad m = 1, 2, \dots, M. \quad (3.207)$$

It remains to prove that each set of intervals $\mathcal{R}^{(m)} = \{R_1^{(m)}, R_2^{(m)}, \dots, R_{c(m)}^{(m)}\}$ is a valid partition of Q_m . Substitution of (3.88) into (3.204) yields

$$\mu(R_j^{(m)}) = \mu(Q_{U(j,m)}) p_{U(j,m),m}. \quad (3.208)$$

Since $\mathbf{v}_1^T$ defined in (3.80) is a left eigenvector of $\mathbf{P}$ associated with the unity eigenvalue, it follows that

$$\begin{aligned} \sum_{j=1}^{c(m)} \mu(R_j^{(m)}) &= \sum_{j=1}^{c(m)} \mu(Q_{U(j,m)}) p_{U(j,m),m} \\ &= \sum_{j=1}^M \mu(Q_j) p_{j,m} \\ &= \mu(Q_m), \quad m = 1, 2, \dots, M. \end{aligned} \tag{3.209}$$

As the right endpoint of each interval $R_j^{(m)}$ is defined as the left endpoint of interval $R_{j+1}^{(m)}$, it then follows that each set of intervals $\mathcal{R}_j^{(m)}$ is a valid partition of Q_m . Hence, $\mathcal{R}$ is a valid sub-partition of $\mathcal{Q}$.

It was proved that each interval R in the valid sub-partition $\mathcal{R}$ of partition $\mathcal{Q}$ is mapped by S to an interval in $\mathcal{Q}$. Hence, S is $\mathcal{R}(\mathcal{Q})$ -semi-Markov.

3.A.3 Proof of Proposition 3.4.4

Suppose that the conditions of the proposition are satisfied. Proposition 3.4.3 then states that S is an $\mathcal{R}(\mathcal{Q})$ -semi-Markov map with order-zero FPM $\mathbf{P}_S^{(0)} = \mathbf{P}$. Let $h : \Omega \rightarrow \Omega$ denote the strictly increasing homeomorphism

$$h(x) = \sum_{m=1}^M h_m \chi_{Q_m}(x), \tag{3.210}$$

where

$$h_m = \frac{1}{M\mu(Q_m)}(x - q_m) + \frac{m-1}{M}, \quad m = 1, 2, \dots, M. \tag{3.211}$$

Let $\tilde{\mathcal{Q}} = \{\tilde{Q}_1, \tilde{Q}_2, \dots, \tilde{Q}_M\}$ denote the uniform partition of the unit interval, such that

$$\tilde{Q}_m = \left(\frac{m-1}{M}, \frac{m}{M} \right), \quad m = 1, 2, \dots, M. \tag{3.212}$$

Each function h_m maps the interval Q_m to the interval $\tilde{Q}_m$. It is shown in [21, Theorem 5] that the topological conjugate $\tilde{S}$, defined by

$$\tilde{S} = h \circ S \circ h^{-1} \tag{3.213}$$

is an $\tilde{\mathcal{R}}(\tilde{\mathcal{Q}})$ -semi-Markov map with order-zero FPM $\tilde{\mathbf{P}}_S^{(0)}$ given by

$$\tilde{\mathbf{P}}_S^{(0)} = \mathbf{H} \mathbf{P}_S^{(0)} \mathbf{H}^{-1}, \tag{3.214}$$

where

$$\mathbf{H} = \text{diag}\{[\mu(Q_1), \mu(Q_2), \dots, \mu(Q_M)]\}. \tag{3.215}$$

As the conditions of Proposition 3.4.4 are satisfied, it follows that $\tilde{\mathbf{P}}_S^{(0)} = [\tilde{p}_{k,m}]$ is primitive. Now, since $\tilde{\mathbf{P}}_S^{(0)}$ is column stochastic (Proposition 2.6.5), its elements are non-negative and its columns sum to unity. As there exists at least one entry in each column of $\tilde{\mathbf{P}}_S^{(0)}$ that satisfies $0 < \tilde{p}_{k,m} < 1$, it follows that all non-zero entries of $\tilde{\mathbf{P}}_S^{(0)}$ must be strictly smaller than unity. Equation (3.88) then implies that

$$\tilde{v}_{j,m} = \tilde{p}_{U(j,m),m}^{-1} > 1, \quad j = 1, 2, \dots, c(m), \quad m = 1, 2, \dots, M, \quad (3.216)$$

where $\tilde{v}_{j,m}$ is the slope of the branch function $\tilde{S}_{j,m}$ of map $\tilde{S}$.

The primitivity of $\tilde{\mathbf{P}}_S^{(0)}$ implies that $\tilde{S}$ is aperiodic (Proposition 2.6.7). This property of S , together with (3.216), satisfy the conditions of the Folklore theorem (refer to Theorem 2.6.3). Hence, $\tilde{S}$ is ergodic and possesses a PWC invariant density larger than zero almost everywhere. As S is a topological conjugate of $\tilde{S}$, Theorem 3.4.1 then implies that S is ergodic. Since the right eigenvector of $\mathbf{P}_S^{(0)} = \mathbf{P}$ associated with the unity eigenvalue is equal to $\hat{\mathbf{f}}_P$, the map S must then have the unique PWC invariant density $f_S^* = \hat{f}_P$.

3.A.4 Proof of Proposition 3.6.2

If S is canonical, the number of branches of S over each $\mathcal{R}^{(m)}$ is equal to the number of non-zero entries in column m of its order-zero FPM [2]. The order of the branches over each $\mathcal{R}^{(m)}$ is also uniquely determined (i.e., equal to the order of the non-zero entries of the column). Now, suppose that S is indeed canonical, and that the Markov partition and order-zero FPM estimates are perfect. It follows that the original solution correctly discerns the number of branches of S over each $\mathcal{R}^{(m)}$, their domains and codomains. By further assuming that all branches are increasing, it is able to construct the map uniquely from the order-zero FPM estimate. However, by assuming canonicity and increasing branches, the original solution cannot successfully reconstruct S if any of these assumptions are invalid, regardless of estimation accuracy.

3.A.5 Proof of Proposition 3.6.3

The invariant density of S coincides with the right Perron eigenvector of the map's order-zero FPM, and is PWC over the Markov partition [75]. Thus, if the assumptions of Proposition 3.6.3 hold, the invariant density functions are also equal.

3.A.6 Proof of Proposition 3.6.4

Equations (2.134) and (2.146) imply that each spectral mode's centre frequency and bandwidth coincide with the argument and modulus, respectively, of a particular eigenvalue of the map's order-one FPM (refer to Remark 2.6.8) [75]. The eigenvalues of this matrix are comprised of the eigenvalues of its top-left submatrix $\mathbf{Q}_S$ and the eigenvalues of the order-zero FPM (Proposition 2.6.10), and each non-zero element of $\mathbf{Q}_S$ is a function of the monotonicity of one or more branches of S , as is evident from (2.118). Due to the original solution's assumption of S possessing no decreasing branches, submatrix $\mathbf{Q}_{\hat{S}}$ of the order-one FPM of $\hat{S}$ generally does not match its counterpart in the order-one FPM of S if the latter map does possess any decreasing branches. Thus, the eigenvalues of these matrices are generally not equal.

3.B Map Class Γ_2

In this section, the class Γ_2 of piecewise monotonic and linear maps is characterised. These maps differ from those in class Γ_1 in that they do not necessarily possess the Markov property. These maps were investigated in [33].

The maps of class $\Gamma_2(S)$ are generated by transforming the map $S \in \Gamma_1(\mathcal{Q}')$, where the class Γ_1 was defined in section 3.5.1 (refer to Definition 3.5.1) [33]. Here, $\mathcal{Q}'$ denotes a partition of the interval $\Omega' = [-1, 1]$. Let d denote a *shift parameter* that satisfies $0 < d < 1 - q_M$, and let $\mathcal{W}' = \{W_1, W_2, \dots, W_{M+1}\}$ denote a partition of Ω' , where $W_m = (w_m, w_{m+1})$ and

$$w_m = \begin{cases} -1, & \text{if } m = 1, \\ q_{m-1} + d, & \text{if } m = 2, 3, \dots, M + 1, \\ 1 & \text{if } m = M + 2. \end{cases} \quad (3.217)$$

Let $S \in \Gamma_1(\mathcal{Q}')$ and define $\rho_d : S \mapsto \hat{S}$ such that $\hat{S} \in \Gamma(\mathcal{W}')$ is defined with branch functions $\hat{S}_m \triangleq \hat{S}|_{W_m}$ given by

$$\hat{S}_m(x) = \begin{cases} S_M(x - d + 2), & \text{if } m = 1, \\ S_{m-1}(x - d), & \text{if } m = 2, 3, \dots, M + 1. \end{cases} \quad (3.218)$$

The map $\hat{S}$ obtained by transforming map S in Fig. 3.B.1(a) using the above expressions is plotted in Fig. 3.B.1(b). The figure reveals that $\hat{S}$ is constructed from S by ‘shifting’ each branch function a distance of d to the right, but where the part of the final branch that exceeds $x = 1$ is ‘reinjected’ from the left at $x = -1$.

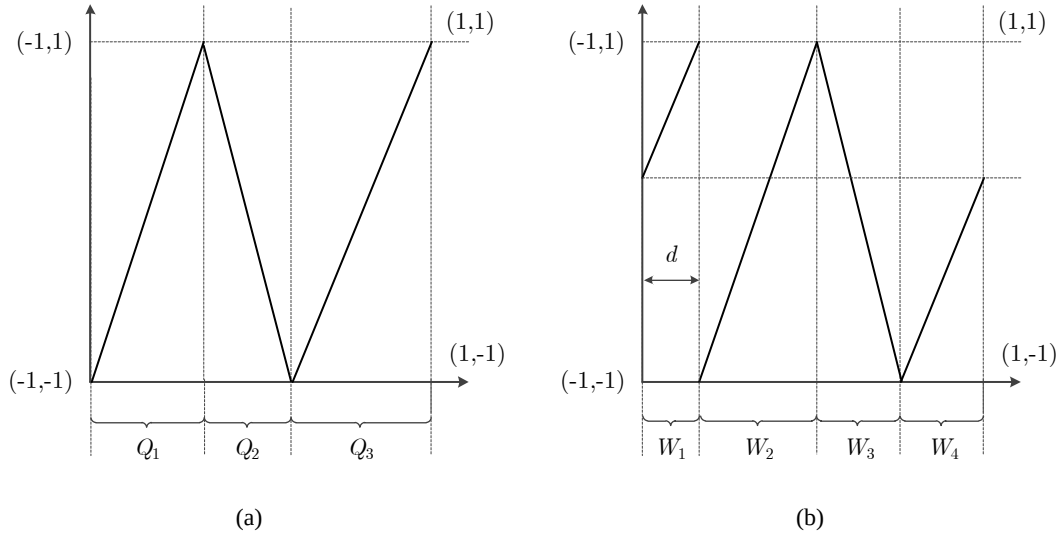


Figure 3.B.1: Piecewise monotonic maps (a) S from class $\Gamma_1(Q')$ and (b) $\dot{S}$ from class $\Gamma_2(S)$.

Definition 3.B.1 (Map class $\Gamma_2(S)$). Let $S \in \Gamma_1(Q')$. The map class $\Gamma_2(S)$ is defined as

$$\Gamma_2(S) \triangleq \{\dot{S} : \dot{S} = \rho_d(S), d \in (0, q_m)\}. \quad (3.219)$$

In general, maps from the class $\Gamma_2(S)$ do not satisfy the Markov property. However, it was proved in [33] that any map $\dot{S} \in \Gamma_2(S)$ is ergodic if $M > 1$, except if S is the tent map or the skew tent map. It was also proved that the invariant density of $\dot{S} \in \Gamma_2(S)$ is equal to that of S [33].

The normalised ACF $\mathcal{R}_{\dot{S}}^*$ of an ergodic map $\dot{S} \in \Gamma_2(S)$ is given by [33],

$$\mathcal{R}_{\dot{S}}^*(\tau) = \begin{cases} 1 & \text{if } \tau = 0, \\ \lambda \mathcal{R}_{\dot{S}}^*(\tau - 1) + \frac{1}{\bar{\sigma}_{\dot{S}}^2 v_M} \mathcal{I}_{\tau-1}(\beta), & \text{if } \tau \geq 1, \\ \mathcal{R}_{\dot{S}}^*(-\tau) & \text{if } \tau \leq -1, \end{cases} \quad (3.220)$$

where v_m denotes the slope of the branch function $S_m \triangleq S|_{Q_m}$ and $\beta \triangleq \dot{S}_1(-1)$. In this expression, $\mathcal{I}_\tau$ is defined by

$$\mathcal{I}_{\tau-1}(\beta) \triangleq -\frac{1}{M} \mathcal{I}_{\tau-2}(\beta) + \begin{cases} \frac{1}{v_M} \mathcal{I}_{\tau-2}(\dot{S}(\beta)), & \text{if } \beta \in W_1, \\ \frac{1}{v_m} \mathcal{I}_{\tau-2}(\dot{S}(\beta)), & \text{if } \beta \in W_{m+1}, m = 1, 2, \dots, M, \end{cases} \quad (3.221)$$

where

$$\mathcal{I}_0(\beta) = \frac{1}{2}(\beta^2 - 1). \quad (3.222)$$

Maps from the class Γ_2 may possess more sophisticated ACFs than those from class Γ_1 . To motivate this claim, an example of a map from class Γ_2 was presented in [33] that possesses an ACF with a period-2 oscillating spectral mode that is modulated by a slowly decaying and oscillating envelope. It was also demonstrated that the maps from class Γ_2 may possess multiple competing spectral modes, in contrast to the single spectral mode of maps from class Γ_1 .

Remark 3.B.1. Although the class Γ_2 has greater diversity with respect to their PSDs than the maps from Γ_1 , it is observed that (3.220) for the ACF of the map is highly sophisticated. An analytical approach to selecting map parameters to realise a prescribed ACF using (3.220) appears unfeasible, and is not pursued in [33]. However, the complexity of (3.220) does not disqualify numerical solutions wherein a search is performed over the parameter space for a prescribed ACF.

Chapter 4

A NOVEL SOLUTION FOR A RELAXED FORMULATION OF IFPP-II

4.1 Introduction

IFPP-II requires the design of an ergodic map with a prescribed invariant density and a prescribed ACF or PSD [1–4]. Motivated by the promise of highly efficient yet versatile random signal generators to be designed using solutions to this problem, the work presented in this chapter develops a novel solution to a relaxed formulation of IFPP-II. This method addresses the shortcomings of existing methods.

Existing methods that have been developed to solve IFPP-II generalise the solution strategies for IFPP-I to permit prescription of the map’s ACF. Chapter 3 reveals that the generalisation of the topological conjugation and direct solution strategies yielded methods with shortcomings that render them impractical [2, 3]. These shortcomings are attributed to the sophistication of the expressions that relate the statistical metrics to the map’s structural parameters. This sophistication excludes the analytical and practical numerical solution of these expressions for the purpose of map design.

The PWL semi-Markov maps admit an exact matrix representation for the FPO. The statistical metrics of these maps are related to the parameters of the maps’ branch functions via this matrix. These expressions are comparatively unsophisticated, which increases the potential for analytical methods that design these maps to satisfy prescribed metrics. However, ergodic members from this class necessarily possess a

PWC invariant density and a rational PSD. These statistical metrics can only be selected to approximate arbitrary continuous functions that may be prescribed.

The statistical characteristics of Markov maps are well-known [5, 6] and several methods that design semi-Markov maps to solve IFPP-I have been proposed in more recent years [7–10]. However, to the best of this author’s knowledge, no method has been developed to solve IFPP-II by constructing these maps. The work presented in chapter 4 addresses this gap in the literature by developing such a method.

IFPP-II is a complex problem that imposes strict specifications on the map. This motivates the adoption of an incremental approach to solving IFPP-II in the work presented here. In this approach, the problem formulation is initially relaxed. More specifically, the problem is reformulated to require the design of an ergodic map to possess (i) an invariant density that approximates a prescribed invariant density, and (ii) a rational PSD with prescribed poles. This new problem formulation is labelled IFPP-IIa. By developing a method that solves IFPP-IIa, the opportunity is created for its future generalisation to address the more sophisticated problem of IFPP-II.

This chapter develops a novel analytical method for solving IFPP-IIa by designing a PWL semi-Markov map. Two solution strategies for designing these maps to possess the required statistical metrics are proposed, and several design issues related to the implementation of these strategies are identified. The most feasible solution strategy, which involves the generalisation of the existing matrix solution strategy for IFPP-I, is pursued. Novel contributions that resolve the associated design issues are presented. The novel method for solving IFPP-IIa implements the solution strategy and uses the abovementioned contributions to successfully resolve the design issues.

Unlike the existing numerical method for solving IFPP-II [2], the novel method has negligible computational complexity and is purely analytical, which bestows immunity to non-convergence. The proposed method also improves upon the existing analytical method [3] by permitting the prescription of an *arbitrary number* of non-repeating power spectrum poles that may be selected as *complex-valued* numbers. However, the poles that are successfully prescribable using the novel method become increasingly restricted as the number of poles in the power spectrum grows. Furthermore, unlike methods that utilise topological conjugation, the prescribed invariant density cannot be realised in an exact fashion.

The new method for solving IFPP-IIa is demonstrated by constructing several examples of maps with prescribed invariant densities and power spectrum poles. These examples include both elementary maps with the uniform invariant density

and a maximum of two prescribed poles, as well as more sophisticated maps with a non-uniform invariant density and up to three prescribed poles. The maps and their statistical metrics are plotted and characterised, and a relationship between the structural features of the maps and their spectral characteristics is identified. Numerical simulations are performed to verify that the statistical metrics prescribed for the maps are successfully realised.

The examples demonstrate that the novel method permits the design of maps with power spectra that possess multiple peaks, such that the centre frequencies and bandwidths of the peaks are adjustable.¹ As such, it is anticipated that this method will find application in DRFM-based random signal generation [11–15], which trades accuracy of the statistical metrics for greater efficiency (it is anticipated that the PWL nature of the maps will permit their efficient implementation). However, as the method does not permit the realisation of a fully prescribed PSD, it cannot be considered a solution to IFPP-II.

The remainder of this chapter is set out as follows. The novel contributions presented in the chapter are enumerated in section 4.2. The relaxed formulation of IFPP-II is defined in section 4.3. The novel strategies for solving this problem, and their associated design issues, are outlined in section 4.4. Section 4.5 presents an overview of preliminary work carried out by the author of this thesis towards solving IFPP-IIa.

Section 4.6 presents the novel contributions that resolve the three design issues identified for the most viable solution strategy. The novel method for solving IFPP-IIa is presented in section 4.7, and the numerical examples are presented in section 4.8. A summary of the chapter is presented in section 4.9. Several conclusions are drawn in section 4.10.

The material presented in this chapter appears in [4, 16].

4.2 Novel Contributions of this Chapter

The primary novel contributions of this chapter are outlined in Table 4.2.1, and the contributions published as conference articles are outlined in Table 4.2.2.

¹However, it is shown in this chapter that the novel method is restricted with respect to the values that may be prescribed for the poles. This restricts the number of sharp peaks in the power spectrum and the frequencies around which these peaks may be realised.

Table 4.2.1: Primary novel contributions presented in chapter 4.

Reference	Publications by author	Description
Sec. 4.4.3	[4]	The formulation of a novel strategy for solving IFPP-IIa that successfully generalises the existing matrix solution strategy for IFPP-I.
Sec. 4.4.3	[4]	The identification of three design issues that must be overcome by a method that seeks to implement the steps of the novel solution strategy.
Sec. 4.6.1, Alg. 4.6.1	[4]	A novel method for approximating an arbitrary invariant density under the constraint of using a stochastic matrix with a non-prescribable right Perron eigenvector as basis for the order-zero FPM (resolution of design issue 1).
Sec. 4.6.2, Alg. 4.6.2	[4, 16]	A novel analytical solution to the StIEP that permits construction of stochastic matrices with an arbitrary number of prescribable complex eigenvalues (resolution of design issue 2).
Sec. 4.6.3 Alg. 4.6.3	[4, 17]	The design of a novel class of maps, referred to as the PWL hat maps. The structural features of these maps are selected to prevent unwanted poles from appearing in the power spectrum of the map (resolution of design issue 3).
Sec. 4.7, Alg. 4.7.1	[4]	A novel analytical method for solving IFPP-IIa that implements the novel solution strategy. The novel method permits the approximation of a prescribed invariant density and the prescription of an arbitrary number of non-repeating complex-valued poles for the map's power spectrum.
Sec. 4.8, Ex. 4.8.1, 4.8.2, 4.8.3	[4]	The characterisation of maps constructed as examples using the novel method, including the identification of the structural features of the maps that give rise to the observed power spectrum peaks.

Table 4.2.2: Contributions published as peer-reviewed conference articles and presented as talks.

Reference	Publications by author	Description
Sec. 4.5	[18]	A novel analytical method for designing a canonical semi-Markov map to approximate a prescribed invariant density and to possess a power spectrum with an arbitrary number of real-valued poles. The method permits the prescription of a subset of the power spectrum poles.
Sec. 4.5	[19]	A novel analytical method for designing a canonical semi-Markov map to possess a uniform invariant density and to possess a power spectrum with up to 5 complex-valued poles. The method permits the prescription of two power spectrum poles.
Sec. 4.5	[17, 20, 21]	A novel numerical method for designing a semi-Markov map with non-linear branches to possess an (exact) prescribed invariant density and to possess a power spectrum that approximates a rational power spectrum with a number of prescribed complex-valued poles. The successful prescription of three poles was demonstrated.
Sec. 4.5	[22]	Experimental work on developing a novel random signal generator that implements an ergodic map.

4.3 The Relaxed Problem Formulation: IFPP-IIa

IFPP-II is reformulated in this section to relax the requirement of having to realise the prescribed invariant density in an exact fashion. This relaxed problem also requires the map to possess a rational PSD with prescribed poles, instead of prescribing an arbitrary PSD. The relaxed problem formulation is labelled IFPP-IIa.

More formally, IFPP-IIa requires the construction of a map $S : \Omega \rightarrow \Omega$ that is non-singular with respect to the Borel measure μ . The map is required to be ergodic and possess an invariant density $f_S^* = \hat{f}_P$. Here, $\hat{f}_P$ is an arbitrary and non-prescribed PWC approximation of a prescribed invariant density function $f_P \in \mathcal{D}(\Omega, \mathcal{B}, \mu)$, where $f_P \in C^2$. Simultaneously, the PSD $\mathcal{S}_S$ of the map is required to be rational and possess the complex-valued numbers $\kappa_1, \kappa_2, \dots, \kappa_M$ as its poles, where each pole satisfies

$$0 < |\kappa_m| < 1, \quad m = 1, 2, \dots, M. \quad (4.1)$$

Furthermore, it is required that any strictly complex (i.e., non-real) poles appear in complex conjugate pairs.

Remark 4.3.1. The motivation for conditions regarding the poles is as follows. Any rational PSD has a z -domain representation of the form

$$\Psi_S(z) = K_S H_S(z) H_S(z^{-1}) \quad (4.2)$$

with an associated region of convergence, where K_S is a real constant and H_S is a rational function [5]. The prescription of a pole $\kappa_m = 0$ yields a multiplicative term z in the function $H_S(z)$, which will necessarily be cancelled by a corresponding term z^{-1} in $H_S(z^{-1})$. Hence, the prescription of zero-valued poles are not considered.

The prescription of a pole $|\kappa_m| = 1$ corresponds with an impulsive mode in the map's PSD (i.e., the PSD possesses a spectral line at the coinciding frequency). This work does not consider the design of the impulsive component of the map's PSD, and thus excludes the prescription of such power spectrum modes.

The requirement for the strictly complex poles to be prescribed in complex conjugate pairs is necessary for the PSD $\mathcal{S}_S(\omega) = \Psi_S(z)|_{z=e^{i\omega}}$ to be a real function.

Remark 4.3.2. The formulation of IFPP-IIa does not prescribe the accuracy of the density $\hat{f}_P$, which is required to approximate the prescribed invariant density f_P . The approach of this work is not to require a specified level of accuracy, but to rather derive a bound on the error of any method developed to solve the problem.

IFPP-IIa does not permit the prescription of the power spectrum zeros. Although this restricts the freedom to specify the characteristics of the power spectrum, it still permits the realisation of power spectrum modes that coincide with the poles. It is shown in the examples of section 4.8 that the centre frequencies and bandwidths of these modes may be independently adjusted by selecting the poles, and that modes may appear as peaks in the power spectrum.

4.4 Novel Strategies for Solving IFPP-IIa

In this section, two novel strategies for solving IFPP-IIa by designing PWL semi-Markov maps are presented. To create context, the section first describes the relevant characteristics of the matrix representations for the maps' FPO, as well as the relationship between its eigenvalues and eigenvectors and the statistical metrics. The strategies are then outlined, and the design issues implied by each are identified.

4.4.1 Relevant characteristics

In section 2.6.3, expressions were derived for the order-zero and order-one FPM of a semi-Markov map, which characterise the evolution of PWC and PWL density functions under iteration of the map, respectively. These expressions revealed that the order-zero FPM is a submatrix of the order-one FPM. It was shown in Propositions 2.6.4 and 2.6.9 that the left and right Perron eigenvectors of an ergodic semi-Markov map's order-zero FPM coincide with the lengths of the intervals in its Markov partition and the vector representation of its invariant density, respectively.

Section 2.6.5 derived expressions for the rational PSD of an ergodic semi-Markov map with a diagonalisable order-one FPM; refer to (2.152), (2.154) and (2.155). These expressions revealed that the poles of the map's PSD coincide with the unique non-zero and non-unity (in absolute value) eigenvalues of the map's order-one FPM

It was proved in Proposition 2.6.10 that the eigenvalues of the order-one FPM comprise the eigenvalues of the order-zero FPM and submatrix $\mathbf{Q}_S$ of the order-one FPM. The preceding statements then imply that the poles of the map's PSD coincide with the eigenvalues of the latter two matrices.

4.4.2 Strategy 1: Design of a candidate order-one FPM

The characteristics described in section 4.4.1 lead to the conjecture that the simultaneous prescription of a semi-Markov map's PWC invariant density and power spectrum poles may be achieved by (i) designing a candidate order-one FPM with the required eigenvalues and eigenvectors, and (ii) designing an ergodic semi-Markov map to possess this candidate matrix as its order-one FPM. This approach is referred to as *solution strategy 1* in this chapter.

A method that implements solution strategy 1 faces two design issues.

- The design of a matrix that qualifies as a valid order-one FPM of an ergodic semi-Markov map is complicated by the property that a subset of the elements of any order-one FPM² depend on the y -intercepts of the map's branches. As the number of map branches is arbitrary and the branch slopes are not fixed in advance, it is difficult to derive conditions that a candidate matrix must satisfy to qualify as a valid order-one FPM.
- The design of a map to possess a valid candidate order-one FPM is complicated by the same property mentioned above.

These design issues have led to the pursuit of a second strategy for solving IFPP-IIa, as outlined in the next section.

4.4.3 Strategy 2: Generalisation of the matrix solution strategy for IFPP-I

Solution strategy 2 generalises the existing matrix solution strategy for IFPP-I, as presented in section 3.4.3, to permit prescription of the power spectrum poles. This is achieved by imposing additional constraints on the order-zero FPM and by selecting the map's structural features to prevent unwanted poles from appearing in the PSD.

The existing matrix solution strategy for IFPP-I follows three steps to design an $\mathcal{R}(\mathcal{Q})$ -semi-Markov map S with a prescribed PWC invariant density over a Markov partition $\mathcal{Q}$. An in-depth description of these steps was provided in section 3.4.3. This description is not repeated here; instead, the three steps are briefly outlined.

²Specifically, the elements of submatrix $\mathbf{R}_S$; refer to (2.119).

1. Specify a Markov partition $\mathcal{Q}$ and a density $\hat{f}_P > 0$ that is PWC over the intervals of $\mathcal{Q}$, such that $\hat{f}_P$ approximates the prescribed invariant density f_P .
2. Construct a non-negative matrix $\mathbf{P}$ with a unity Perron eigenvalue. The elements of the left Perron eigenvector must coincide with the lengths of the intervals in $\mathcal{Q}$, and the elements of the right Perron eigenvector must coincide with the vector representation of $\hat{f}_P$.
3. Construct an ergodic $\mathcal{R}(\mathcal{Q})$ -semi-Markov map S to possess the order-zero FPM equal to $\mathbf{P}$.

By following the above steps, the semi-Markov map S is designed to possess an invariant density f_S^* equal to the PWC density $\hat{f}_P \approx f_P$.

The above solution strategy is generalised in this work by requiring that the candidate order-zero FPM be designed to possess a *prescribed eigenspectrum* and simultaneously permit a map with an invariant density that approximates the prescribed invariant density. As the order-zero FPM eigenvalues coincide with a subset of the map's power spectrum poles, this approach affords some freedom to prescribe these poles.

Section 4.4.1 implies that the above approach will yield *unwanted* power spectrum poles that coincide with the eigenvalues of submatrix $\mathbf{Q}_S$ of the order-one FPM. These poles are unwanted, as step 2 of the matrix solution strategy does not provide any opportunity to select the eigenvalues of this submatrix. However, the third step of the matrix solution strategy grants some freedom to manipulate the entries of this submatrix.³ Strategy 2 requires the design of a semi-Markov map with structural features that prevent the remaining eigenvalues of submatrix $\mathbf{Q}_S$ from manifesting as unwanted poles, thereby permitting prescription of *all* the power spectrum poles during solution step 2.

The steps of solution strategy 2 are outlined below.

1. Design a candidate order-zero FPM $\mathbf{P}$ with eigenvalues and eigenvectors that simultaneously realise a PWC invariant density (that approximates the prescribed invariant density) and a collection of prescribed *non-repeating*⁴ power spectrum poles.

³In particular, it was shown in Proposition 3.4.4 that multiple maps may be designed to possess the same order-zero FPM.

⁴Only non-repeating poles can be accommodated, as the poles of the maps under consideration coincide with the *unique* eigenvalues of the order-one FPM.

2. Design a semi-Markov map that possesses $\mathbf{P}$ as its order-zero FPM. The branch structure of the map is selected to prevent unwanted poles from appearing in the power spectrum.

In section 4.7, a novel analytical method for solving IFPP-IIa that implements solution strategy 2 is developed. However, any method that implements this strategy faces several design issues that need to be resolved. These issues are described below. The novel contributions that led to their resolution are described in section 4.6.

Design issue 1: Simultaneous prescription of eigenvalues and eigenvalues for the order-zero FPM

To design an ergodic semi-Markov map that realises a prescribed PWC invariant density and prescribed power spectrum poles using strategy 2, the candidate order-zero FPM of the map must be designed to simultaneously possess

- a left Perron eigenvector that coincides with the lengths of the intervals in the chosen Markov partition $\mathcal{Q}$,
- a right Perron eigenvector that coincides with the vector representation of the PWC approximating density $\hat{f}_P$ that was selected, and
- an eigenspectrum chosen to coincide with the prescribed non-repeating power spectrum poles.⁵

The problem of constructing a non-negative matrix to simultaneously possess a prescribed eigenspectrum as well as prescribed left and right Perron eigenvectors is exceedingly complicated. In the literature, there has been progress in solving the relaxed problem of constructing a matrix with a prescribed eigenspectrum, labelled the *inverse eigenvalue problem* (IEP); refer to [23] and the more recent publication [16] by the author of this thesis. However, solutions to the IEP are not sufficient to address the problem under consideration.

In solution strategy 2, the difficulty of designing the candidate order-zero FPM is resolved by relaxing the requirement of having to realise an *arbitrary* Markov partition and approximating density $\hat{f}_P$ that may be prescribed by the designer.

⁵Recall from Remark 2.6.10 that the poles of the map's power spectrum comprise the unique non-zero and non-unity (in absolute value) eigenvalues of its order-zero FPM and submatrix $\mathbf{Q}_S$ of its order-one FPM.

Stated differently, the freedom afforded to the designer to select the approximating invariant density in step one of the matrix solution strategy for IFPP-I is sacrificed to relax the matrix design problem.

The above approach was inspired by method 2 for density approximation (refer to section 3.4.3 and Table 3.4.1). This method proved that any smooth density may be approximated while only requiring that the right Perron eigenvector of the order-zero FPM has uniform elements (i.e., the elements need not be arbitrarily selectable). The novel method for solving IFPP-IIa generalises this approach by approximating a smooth density *regardless of* the right Perron eigenvector of the order-zero FPM that is available for design. This is accomplished by implicitly selecting an appropriate Markov partition, as a function of the right Perron eigenvector, that permits approximation.

By using the above approach, the matrix design problem simplifies to the construction of a stochastic matrix with a prescribed eigenspectrum. An appropriate similarity transform of this matrix then yields an order-zero FPM that realises a particular PWC invariant density. The latter density is an approximation of the prescribed invariant density, but over a partition that is not arbitrarily selectable.

Design issue 2: Analytical construction of an order-zero FPM with the required eigenspectrum

By sacrificing the freedom to select the Markov partition, the problem of designing a candidate order-zero FPM simplifies to the construction of a stochastic matrix with a prescribed eigenspectrum. This problem is referred to as the stochastic IEP, or StIEP, in the literature [23]. Although methods that depend on numerical convergence to a matrix with the required eigenspectrum via minimisation of a cost function have been proposed [24–29], it is well known that their successful and rapid convergence depends strongly on the eigenvalues selected [25]. Thus, analytical methods for solving the StIEP that are not subject to possible non-convergence are required.

Existing analytical methods for solving the StIEP are only able to realise matrices of restricted dimensionality [30, 31] or only permit prescription of real-valued eigenspectra [32–34]. Hence, these methods are of limited applicability to the matrix construction problem considered in this work, which is required to yield an arbitrary number of complex eigenvalues (to permit prescription of arbitrary poles).

In section 4.6.2, an existing analytical solution to the StIEP that permits prescription

of an arbitrary number of real eigenvalues is generalised, thereby realising a novel method for constructing stochastic matrices that permits the prescription of complex eigenvalues. This analytical method overcomes the shortcomings of previous methods and is directly applied to solve the current matrix design problem.

Design issue 3: Designing the map to prevent unwanted poles

The original matrix solution strategy for IFPP-I designs a semi-Markov map to possess a specified order-zero FPM. Hence, it provides no opportunity to select or manipulate the poles contributed by submatrix $\mathbf{Q}_S$ of the order-one FPM during the matrix design step. Instead, the elements of submatrix $\mathbf{Q}_S$, and thereby the eigenvalues that it contributes to the PSD, are implicitly determined by the branch structure of the map that is constructed to possess the candidate order-zero FPM in the final design step. In the context of solution strategy 2, these eigenvalues yield unwanted power spectrum poles.

The final step of solution strategy 2, which constructs a semi-Markov map to possess the candidate order-zero FPM, grants freedom to select the map's structural features. This freedom is exploited in solution strategy 2 by designing a map with a branch structure that prevents unwanted poles in its PSD.

Design issue 3 refers to the fact that the structural features required by the map to prevent unwanted poles in its power spectrum is neither known nor immediately apparent. In section 4.6.3, a novel semi-Markov map with a carefully selected branch structure that achieves the goal of preventing unwanted poles is presented. Maps that possess this branch structure are referred to as *hat maps* due to the visual appearance of consecutive pairs of their branch functions.

This branch structure of hat maps yields an order-one FPM with a submatrix $\mathbf{Q}_S$ containing only elements equal to zero. All eigenvalues of this matrix are equal to zero; hence, it contributes no poles to the PSD of the hat map. Thus, all poles of a hat map's PSD may be prescribed by selecting only the eigenspectrum of the order-zero FPM, as permitted by the novel analytical method for solving the StIEP.

4.5 Preliminary Work

Preliminary work on evaluating different approaches for resolving the design issues identified in section 4.4.3 was carried out. This work involved the implementation of solution strategy 2 and yielded several methods that partially solve IFPP-IIa. Whereas these methods do not permit full prescription of the power spectrum poles, and thus cannot be considered full solutions to IFPP-IIa, they serve to demonstrate the viability of the mechanisms and algorithms that were ultimately applied to resolve the design issues. These methods yielded a number of conference publications [17–19, 22] and presentations [20, 21]. In this section, a brief overview of these methods is provided.

The method proposed in [18] designs a canonical semi-Markov map to possess an invariant density that approximates a smooth prescribed density and a PSD with an arbitrary number of poles, where non-repeating real values may be prescribed for *a subset of* the poles [4]. The method uses recursive Markov state disaggregation [32] to construct a doubly stochastic matrix of arbitrary dimensionality and with a real eigenspectrum selected to equal the prescribed poles. The canonical semi-Markov map S is then constructed using Algorithm 3.4.1 (canonical map construction) to possess this matrix as its order-zero FPM.

To realise a non-uniform invariant density, the method of [18] constructs a topological conjugate $\tilde{S}$ of the map S in the final step, such that $\tilde{S}$ is also a canonical semi-Markov map [4]. The PWC invariant density of the conjugate map is selected to approximate the prescribed invariant density. It is shown that the eigenspectra of the original and conjugate maps' order-zero FPMs are equal. Hence, a subset of the poles belonging to the conjugate map's PSD (i.e., those that correspond to the order-zero FPM eigenspectrum) may be prescribed. However, the method permits prescription of real-valued poles only, and neither the zeros nor the remaining poles of the PSD (corresponding to the order-one FPM's remaining eigenvalues) can be prescribed. Experimental work was carried out to develop a random signal generator using this method [22].

The results presented in [18] confirmed the relationship between the eigenvalues of the order-zero FPM and the poles of the map's PSD. The work also proved that topological conjugation is a viable option for realising a non-uniform invariant density, and that Markov state disaggregation can realise stochastic matrices with arbitrary dimensionality and prescribable eigenspectra. However, the method that was developed is limited to realising matrices with real eigenspectra only. In particular,

the results proved that spectral modes with arbitrary centre frequencies were not realisable using the existing technique for Markov state disaggregation.

The success of Markov state disaggregation to yield stochastic matrices of arbitrary dimensionality and with prescribable eigenspectra spurred interest in its generalisation to permit complex eigenspectra. The generalisation of Markov state disaggregation first required the design of 3×3 stochastic matrices with a prescribable pair of complex conjugate eigenvalues. The technique proposed in [35] was successfully used to construct such matrices. Work was then carried out in [19] to determine whether these matrices yield spectral modes with prescribable centre frequencies.

Algorithm 3.4.1 (canonical map construction) was used to design a canonical semi-Markov map to possess an order-zero FPM equal to each of these matrices [4]. These maps possess the uniform invariant density and permit the prescription of the same subset of PSD poles as the method of [18]. Although the method used to design these maps permits the prescription of complex-valued poles, the low dimensionality of the order-zero FPM restricts the maps to a maximum of five poles, where two poles are prescribable.

The results presented in [19] proved that spectral modes with arbitrary centre frequencies may be prescribed by selecting complex eigenvalues for the order-zero FPM. Furthermore, the 3×3 matrices used in the design of these maps yielded a generalisation of the original technique for Markov state disaggregation that led to a method for constructing stochastic matrices of arbitrary dimensionality and with prescribable *complex-valued* eigenspectra. This method is successfully used to solve IFPP-IIa in the later sections of this chapter, and is presented in section 4.6.2.

The results presented in [18, 19] revealed unexpected modes in the maps' power spectra. These modes are associated with unwanted power spectrum poles that coincide with the eigenvalues of submatrix $\mathbf{Q}_S$ of the map's order-one FPM. This led to specification of the class of PWL hat maps, which was formally introduced in [17, 20]. The structural features of these maps are selected to ensure that the elements of submatrix $\mathbf{Q}_S$ are all zero-valued. This excludes the eigenvalues of this matrix as poles of the power spectrum, and permits full prescription of all the poles by selecting the eigenvalues of the order-zero FPM.

The method proposed in [17, 20, 21] constructs a semi-Markov map with non-linear branches and prescribed invariant density such that its PWL approximation, which is itself a PWL hat map, possesses an order-zero FPM equal to a prescribed stochastic matrix [4]. A numerical algorithm is used to construct a stochastic matrix with a

specified eigenspectrum, thereby prescribing all the poles of the PWL hat map's PSD. Whereas numerical examples suggest that the PSDs of the map with non-linear branches and the PWL hat map are approximately equal, the relationship is not established analytically. In certain cases, it was also observed that the method fails to converge to a map.

The results presented in [17, 20, 21] confirmed that PWL hat maps prevent unwanted poles in the power spectrum. This led to the selection of these maps as design candidates for solving IFPP-IIa in this chapter. These maps are formally introduced in section 4.6.3.

4.6 Resolution of the Design Issues

This section details three contributions that resolve the design issues identified for solution strategy 2 in section 4.4.3. These contributions are utilised in section 4.7 to realise the novel analytical method for solving IFPP-IIa.

4.6.1 Resolution of design issue 1: Relaxation of the order-zero FPM specifications

Consider the first step of solution strategy 2 presented in section 4.4.3. To implement this step in a manner that permits the simultaneous realisation of a prescribed approximating PWC invariant density and a subset of the map's power spectrum poles, it is necessary to design a non-negative matrix $\mathbf{P}$ to possess a prescribed eigenspectrum with a unity Perron eigenvalue as well as prescribed left and right Perron eigenvectors with positive elements. This matrix is the candidate order-zero FPM for the map that is to be constructed. The characteristics required for its eigenspectrum and eigenvectors are listed on page 206.

Suppose that the above approach is to be followed. Now, consider any non-negative matrix $\mathbf{P}$ with a left Perron eigenvector, associated with the unity Perron eigenvalue, that coincides with the (strictly positive) lengths of the intervals in a prescribed partition $\mathcal{Q}$ of the unit interval. This matrix is necessarily similar to a stochastic matrix $\tilde{\mathbf{P}}$. Specifically, [4]

$$\mathbf{P} = \mathbf{D}\tilde{\mathbf{P}}\mathbf{D}^{-1}, \quad (4.3)$$

where $\mathbf{D}$ is the diagonal matrix with elements

$$\mathbf{D} = \text{diag}\{\mu(Q_1)^{-1}, \mu(Q_2)^{-1}, \dots, \mu(Q_M)^{-1}\}. \quad (4.4)$$

This implies that the problem of designing a candidate order-zero FPM $\mathbf{P}$ with the desired eigenspectrum and left and right Perron eigenvectors simplifies to the design of an appropriate stochastic matrix $\tilde{\mathbf{P}}$ with the required eigenspectrum⁶ and right Perron eigenvector, and its subsequent transformation according to (4.3) and (4.4). Equation (4.3) implies that the stochastic matrix $\tilde{\mathbf{P}}$ must be designed to possess the right Perron eigenvector $\tilde{\mathbf{w}}_1$ given by [4]

$$\tilde{\mathbf{w}}_1 = \mathbf{D}^{-1} \hat{\mathbf{f}}_P \quad (4.5)$$

to ensure that $\mathbf{P}$ has the required right Perron eigenvector. Here, $\hat{\mathbf{f}}_P = [\hat{f}_1, \hat{f}_2, \dots, \hat{f}_M]^T$ denotes the vector representation of the PWC invariant density $\hat{f}_P$ that the map must realise and which approximates the prescribed invariant density f_P .

Since $\hat{f}_P$ is chosen to be PWC over the intervals of $\mathcal{Q}$, its elements satisfy

$$\hat{f}_m = \frac{\alpha_m}{\mu(Q_m)}, \quad (4.6)$$

where

$$\alpha_m \triangleq \int_{Q_m} \hat{f}_P(x) dx. \quad (4.7)$$

Substitution of (4.4) and (4.6) into (4.5) yields

$$\tilde{\mathbf{w}}_1 = [\alpha_1, \alpha_2, \dots, \alpha_M]^T. \quad (4.8)$$

This expression implies that the elements of the right Perron eigenvector belonging to the stochastic matrix $\tilde{\mathbf{P}}$ must coincide with the areas α_m under the approximating density $\hat{f}_P$ when restricted to each of the intervals Q_m in partition $\mathcal{Q}$.

Now, suppose instead that the map is not required to realise an *arbitrarily prescribed* Markov partition and approximating invariant density $\hat{f}_P$. Assume that a particular positive⁷ stochastic matrix $\tilde{\mathbf{P}}$ with a right Perron eigenvector⁸ $\tilde{\mathbf{w}}_1 = [\tilde{w}_1, \tilde{w}_2, \dots, \tilde{w}_M]^T$ that is not prescribable is used in the first step of solution strategy 2. The set of approximating densities $\hat{f}_P$ that may be realised by choosing different

⁶Recall that the eigenspectrum of a matrix is preserved under a similarity transformation [36].

⁷By assuming the matrix is positive, the Perron-Frobenius theorem implies that it has a unique maximal (Perron) eigenvalue equal to unity, which is a requirement for the ergodicity of the map being designed. Equivalently, it may be assumed that the matrix is non-negative and primitive.

⁸It is assumed that the elements of $\tilde{\mathbf{w}}_1$, which are necessarily positive, are normalised and sum to unity.

Markov partitions $\mathcal{Q}$ and computing the candidate order-zero FPM $\mathbf{P}$ according to (4.3) is considered. Using (4.6) to (4.8), this class is derived as

$$\hat{f}_P \in \left\{ f \in \mathcal{D}_{\mathcal{Q}}^{(0)} : \int_{Q_m} f(x) dx = \tilde{w}_m, \quad m = 1, 2, \dots, M \right\}. \quad (4.9)$$

Stated differently, the matrix $\tilde{\mathbf{P}}$ can yield an approximating density $\hat{f}_P$ that is PWC over the intervals of *any* partition $\mathcal{Q}$, provided that the area under the density when restricted to each interval Q_m in the partition is equal to element $\tilde{w}_m$ of its right Perron eigenvector. The partition $\mathcal{Q}$ may thus be interpreted as an index into the class of densities realisable using the stochastic matrix $\tilde{\mathbf{P}}$.

Consider the problem of approximating an arbitrary prescribed invariant density f_P using the class of densities defined by (4.9). The partition $\mathcal{Q}$ may be selected such that the area under both the approximating density $\hat{f}_P$ and the prescribed invariant density f_P are equal when both functions are restricted to each of the intervals in the partition. As the area under the approximating density, when restricted to each interval Q_m , is fixed and determined by the corresponding vector element $\tilde{w}_m$, it follows that the intervals of the partition $\mathcal{Q}$ must be then selected to satisfy

$$\int_{Q_m} f_P(x) dx = \tilde{w}_m, \quad m = 1, 2, \dots, M. \quad (4.10)$$

This yields the PWC approximating density

$$\hat{f}_P(x) = \sum_{m=1}^M \frac{\tilde{w}_m}{\mu(Q_m)} \chi_{Q_m}(x). \quad (4.11)$$

This choice for the partition $\mathcal{Q}$ yields the distribution function $\hat{F}_P$, associated with the density $\hat{f}_P$, that satisfies [4]

$$\hat{F}_P(q_m) = F_P(q_m) = \sum_{k=1}^{m-1} \tilde{w}_k \quad m = 1, 2, \dots, M+1, \quad (4.12)$$

where

$$F_P(x) \triangleq \int_0^x f_P(z) dz. \quad (4.13)$$

If it is assumed that F_P is continuous, it is invertible, and the partition points of the partition $\mathcal{Q}$, as required to construct the approximation $\hat{f}_P$, are given by

$$q_m = F_P^{-1} \left(\sum_{k=1}^{m-1} \tilde{w}_k \right), \quad m = 1, 2, \dots, M+1. \quad (4.14)$$

This above expression explicitly relates the elements of the right Perron eigenvector of the stochastic matrix $\tilde{\mathbf{P}}$ to the intervals of the required partition $\mathcal{Q}$.

The above approach is implemented in Algorithm 4.6.1 (determination of approximating density). This algorithm determines an approximating PWC invariant density

$\hat{f}_P \approx f_P$ to be realised by a semi-Markov map, under the constraint of using a particular positive stochastic matrix $\tilde{\mathbf{P}}$ with a right Perron eigenvector that is not selectable in the first step of the map's design. Note that the similarity transform that relates $\tilde{\mathbf{P}}$ and the candidate order-zero FPM $\mathbf{P}$ depends on the Markov partition selected using Algorithm 4.6.1 (determination of approximating density).

Algorithm 4.6.1 (determination of approximating density): The determination of an approximating invariant density to be realised by a semi-Markov map, under the constraint of using a positive stochastic matrix with a right Perron eigenvector that is not selectable.

- 1: Inputs: Prescribed invariant density f_P and strictly positive stochastic matrix $\tilde{\mathbf{P}} = [\tilde{p}_{k,m}]_{k,m=1}^M$ with right Perron eigenvector $\tilde{\mathbf{w}}_1 = [\tilde{w}_1, \tilde{w}_2, \dots, \tilde{w}_M]^T$.
 - 2: Outputs: Markov partition $\mathcal{Q}$ and approximating density $\hat{f}_P$.
 - 3: Derive the distribution function F_P of (4.13).
 - 4: Return the Markov partition $\mathcal{Q} = \{Q_1, Q_2, \dots, Q_M\}$, where $Q_m = (q_m, q_{m+1})$ and q_m is given by (4.14), as well as the approximating density $\hat{f}_P$ of (4.11).
-

Remark 4.6.1. Equations (4.11) and (4.12) imply that the PWC approximating density constructed using Algorithm 4.6.1 (determination of approximating density) yields a distribution function that is a PWL approximation of the distribution function associated with the prescribed invariant density. The accuracy of this PWL approximation depends on the right Perron eigenvector of the particular stochastic matrix that was used. As such, results regarding the approximation's convergence in accuracy, with a growing number of intervals permitted for the partition $\mathcal{Q}$, cannot be developed without further assumptions regarding the stochastic matrix.

However, a bound may be derived on the accuracy of the distribution function yielded by a particular stochastic matrix $\tilde{\mathbf{P}}$ and Algorithm 4.6.1 (determination of approximating density). Let $\mathcal{Q}$ denote the Markov partition determined by the algorithm and suppose that $F_P \in C^2$. Using an approach identical to the proof of Proposition 3.4.1, it can be shown that

$$\left\| \hat{F}_P - F_P \right\|_{\infty} \leq \max_{m=1,2,\dots,M} \left\{ \frac{\mu(Q_m)^2}{8} \|f'_{P,m}\|_{\infty} \right\}, \quad (4.15)$$

where

$$f'_{P,m} = \left. \frac{df_P}{dx} \right|_{Q_m}. \quad (4.16)$$

This error bound is identical to the error bound shared by methods 1 and 2 of section 3.4.3 (refer to Table 3.4.1).

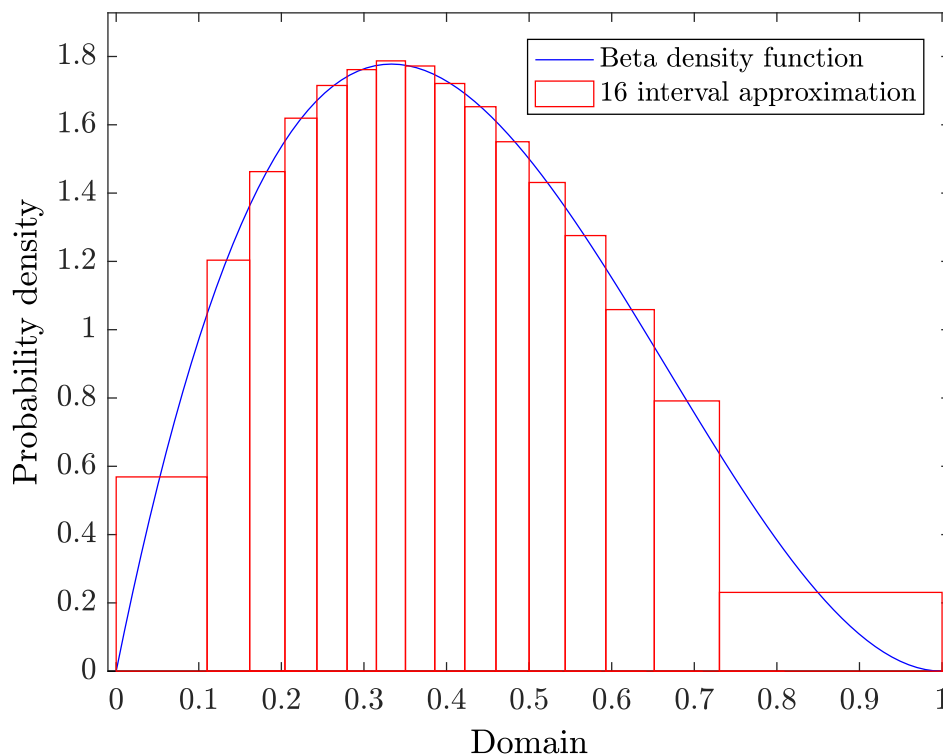


Figure 4.6.1: Approximation of the beta density function over a 16 interval partition (derived from [18]).

Example 4.6.1. Consider the beta density function defined by

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}, \quad x \in (0,1), \quad (4.17)$$

with parameters $a = 2$ and $b = 3$. In this expression,

$$B(a,b) \triangleq \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad (4.18)$$

and

$$\Gamma(a) \triangleq \int_0^\infty x^{a-1}e^{-x}dx. \quad (4.19)$$

This density function is plotted in Figs. 4.6.1 and 4.6.2 (blue curves).

Algorithm 4.6.1 (determination of approximating density) was used to construct two PWC density functions $\hat{f}_P$ that approximate the beta density function defined above (see [18]). These densities were constructed to coincide with two doubly stochastic matrices $\mathbf{P}$ of dimensionality 16×16 and 32×32 elements. The approximating density functions $\hat{f}_P$ are plotted in red in Figs. 4.6.1 and 4.6.2, respectively. The figures suggest that an increase in the matrix dimensionality (which yields a greater number of intervals in the Markov partition) results in a closer approximation of the beta density function. This is confirmed by Table 4.6.1, which lists the L^2 error

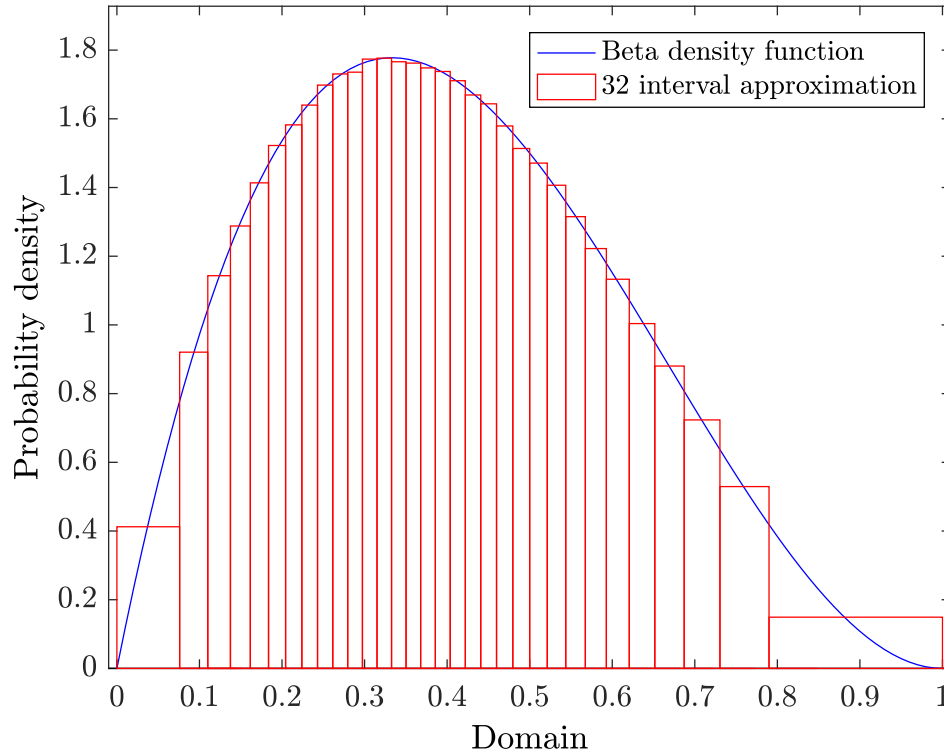


Figure 4.6.2: Approximation of the beta density function over a 32 interval partition (derived from [18]).

between the beta density function and the approximating densities (in the table, the errors over the entire domain, the mode of the density and the domain endpoints are listed separately).

Remark 4.6.2. The novel analytical method for solving IFPP-IIa, as presented in section 4.7, uses the approach outlined in this section to implement the first step of solution strategy 2. More specifically, it relaxes the order-zero FPM specifications by not permitting the prescription of an arbitrary Markov partition $\mathcal{Q}$ and approximating invariant density $\hat{f}_P$. Instead, the novel method designs a stochastic matrix without prescribing its right Perron eigenvector, and then derives the appropriate candidate order-zero FPM to realise the approximating density determined by Algorithm 4.6.1 (determination of approximating density). Thus, by sacrificing the freedom to arbitrarily choose the Markov partition $\mathcal{Q}$ and approximating density $\hat{f}_P$, and by developing Algorithm 4.6.1 (determination of approximating density), design issue 1 identified in section 4.4.3 is resolved.

In the novel method, the stochastic matrix is designed to possess a prescribed eigenspectrum. A similarity transform of the stochastic matrix then yields the candidate order-zero FPM of the map. As a similarity transform preserves the

Table 4.6.1: L^2 errors of the approximating invariant densities of Figs. 4.6.1 and 4.6.2 (derived from [18]).

Region	16-interval approximation	32-interval approximation
Entire domain	0.149	0.0903
Mode	0.0302	0.0168
Domain endpoints	0.189	0.115

eigenspectrum of the matrix, the poles of the map may be selected by prescribing the eigenvalues of the stochastic matrix. A novel method for solving the StIEP, which is used to construct the stochastic matrix, is presented in the next section.

4.6.2 Resolution of design issue 2: A novel analytical solution to the StIEP

This section presents a novel analytical solution to the StIEP that permits prescription of an arbitrary number of complex eigenvalues for the matrix. This method addresses design issue 2 identified in section 4.4.3. The material presented in this section appears in [4, 16].

A brief overview of existing methods for solving the StIEP is presented. Next, an existing analytical method [32] that uses *Markov state disaggregation* to solve the StIEP and realise matrices with real-valued eigenspectra is outlined. The section concludes with the presentation of the novel analytical method for solving the StIEP and two examples. This novel method generalises the existing method proposed in [32] to permit realisation of matrices with *complex-valued* eigenspectra.

Overview of existing methods

The StIEP is closely related to the IEP that requires the construction of non-negative matrices; this problem is referred to as the *non-negative IEP*, or *NIEP* [16]. More specifically, any non-negative matrix with positive maximal eigenvalue and eigenvector can be mapped to a stochastic matrix using a scalar-weighted similarity transformation⁹ [37]. Since similar matrices share the same eigenspectrum, a generic

⁹This fact was used throughout chapters 2 and 3 to uncover relevant properties of the order-zero FPM of a semi-Markov map, and to derive sufficient conditions for the ergodicity of these maps.

solution to the NIEP then qualifies as a solution to the StIEP [23]. Hence, the overview presented here considers solutions to both the StIEP and NIEP.

The StIEP and NIEP are associated with work aimed at deriving necessary and sufficient conditions for a specified list of numbers to be realisable as the eigenvalues of a stochastic or non-negative matrix [16]. Theorems regarding necessary and sufficient conditions have appeared in the literature [30, 31], but the corresponding proofs are generally not constructive in nature [38]. Hence, these theorems cannot be used to solve the IEP in a direct manner. When used to solve the IEP, those theorems that were proved in a constructive manner are generally limited to producing matrices of small dimensions. The work of Suleimanova [34] and Perfect [33] are exceptions; however, the theory presented in these papers is restricted to real-valued eigenspectra.

An alternative approach to solving the StIEP involves the use of numerical methods to converge to a matrix with the required eigenspectrum [16]. One class of methods aims to minimise a predefined cost function by using optimization methods to perform a descent on the underlying matrix manifolds (refer to [25]). The first method of this class was developed by Chu and Driessel [28] to solve the NIEP for symmetric matrices. This method was later generalised by Chu and Guo [29] to solve the NIEP for matrices that are not necessarily symmetric. Although the authors reported promising results, problems related to the methods' convergence were observed.

More recently, Zhao et al. [27] proposed a numerical method from the same class as [28] and [29] (see [16]). This method solves the StIEP using a conjugate gradient descent algorithm, where minimisation is carried out over the product manifold of the orthogonal matrices and matrices with rows on the unit sphere. Whereas numerical results presented by the authors demonstrate the method's effectiveness, the solution depends on an existence theorem for isospectral stochastic matrices that has not been proved thus far (as reported in [25]). Steidl and Winkler [25] extended the method proposed in [27], thereby addressing this shortcoming. At the same time, the shortcoming of the method presented in [27] was remedied in [24]. Additional methods for solving the NIEP and StIEP and that belong to the same class appeared in [39] and [26], respectively.

Remark 4.6.3. Although considerable progress has been made regarding numerical methods for solving the StIEP, it is observed that the performance of methods that rely on the minimisation of a cost function and numerical convergence depend strongly on the distribution of the eigenvalues prescribed for the matrix [16, 25]. Motivated by this observation, only analytical solutions to the StIEP are considered for use in the novel method that implements solution strategy 2. A particular analytical

solution with properties that permit its generalisation and application in this method is presented next.

Existing solution using Markov state disaggregation

Ciampolini et al. [32] proposed an analytical solution to the StIEP that directly constructs a stochastic matrix with prescribed real-valued eigenspectrum. Unlike numerical methods that minimise a cost function, the matrix is constructed after a finite and predetermined number of iterations. This solution to the StIEP uses *Markov state disaggregation* to construct the matrix.

The process whereby one or more states of a discrete-time and homogenous Markov chain are disaggregated (or divided) into multiple new states is referred to as *Markov state disaggregation* [16, 32]. A stochastic matrix may be constructed by disaggregating a Markov chain's states over several rounds, starting with the elementary single-state Markov chain. During each round, a particular *disaggregation technique* is applied to divide one of the Markov chain's states and to select the transition probabilities to and from the new states. After completing the required number of rounds, the Markov chain's transition matrix (a stochastic matrix) is produced as output.

Within the context of the StIEP, consideration is given to the family of disaggregation techniques that cause the transition matrix of the Markov chain obtained after one round of disaggregation to (i) inherit the eigenvalues of the transition matrix belonging to the Markov chain at the start of the round, and (ii) gain one or more additional eigenvalues that may be prescribed by appropriately selecting the transition probabilities associated with the new states [16]. These disaggregation techniques may be mathematically formulated as mappings between the transition matrices at the start and end of each round. These mappings have parameters that are selected to specify the eigenvalue(s) that are inserted into the eigenspectrum of the transition matrix.

Ciampolini et al. [32] proposed a disaggregation technique that permits the insertion of a single real eigenvalue into the eigenspectrum of the transition matrix during each round [16]. This technique is referred to here as the *original disaggregation technique*, and its application as *original disaggregation*. The original disaggregation technique is detailed in the following subsection.

As a point of departure for defining the disaggregation techniques, let the sequence

$Z_1, Z_2, \dots$ of random variables denote a discrete-time and homogenous Markov chain, referred to as the *original* Markov chain [16]. Let this Markov chain have states $s_1, s_2, \dots, s_N$ and transition probabilities defined by

$$p_{i,j} \triangleq \Pr(Z_{t+1} = s_i | Z_t = s_j). \quad (4.20)$$

The transition matrix of the original Markov chain is defined as $\mathbf{P} \triangleq [p_{i,j}]_{i,j=1}^N$; this matrix may be expressed as

$$\mathbf{P} = \left(\begin{array}{c|c|c} \mathbf{P}_{1,1} & \mathbf{v}_{1,k} & \mathbf{P}_{1,2} \\ \hline \mathbf{w}_{k,1}^T & p_{k,k} & \mathbf{w}_{k,2}^T \\ \hline \mathbf{P}_{2,1} & \mathbf{v}_{2,k} & \mathbf{P}_{2,2} \end{array} \right), \quad (4.21)$$

where $1 \leq k \leq N$ is an arbitrary integer. Since the transition matrix has at least one eigenvalue of unity, there exists at least one *stationary probability vector* $\boldsymbol{\pi} = [\pi_1, \pi_2, \dots, \pi_N]^T$ satisfying $\mathbf{P}\boldsymbol{\pi} = \boldsymbol{\pi}$. The elements of any such vector are referred to as *stationary state probabilities*.

Original disaggregation technique The following description of the original disaggregation technique appears in [16]. Ciampolini et al. [32] proposed the original disaggregation technique, which divides a particular state of a Markov chain into two states. Suppose that original disaggregation of state s_k in the original Markov chain is to be performed, with the requirement of inserting a real eigenvalue λ into the eigenspectrum of the transition matrix. This produces a longer Markov chain with states $s'_1, s'_2, \dots, s'_{N+1}$, where s'_k and s'_{k+1} denote the new states obtained by dividing state s_k , and

$$s'_i = \begin{cases} s_i, & \text{if } i < k, \\ s_{i-1}, & \text{if } i > k+1. \end{cases} \quad (4.22)$$

Original disaggregation is performed by replacing the element $p_{k,k} > 0$ of the original Markov chain's transition matrix in (4.21) by a 2×2 matrix $\mathbf{T}$, to be specified below, and selecting the remaining transition matrix elements in a systematic manner, thus obtaining a larger transition matrix $\mathbf{P}'$. Let $\mathbb{P}_N$ denote the set of $N \times N$ stochastic matrices. The larger transition matrix is obtained by applying the map $\mathcal{T} : \mathbb{P}_N \rightarrow \mathbb{P}_{N+1}$ to the original transition matrix, such that

$$\mathbf{P}' = \mathcal{T}(\mathbf{P}, \lambda, \alpha, k) \quad (4.23)$$

and

$$\mathbf{P}' = \left(\begin{array}{c|cc|c} \mathbf{P}_{1,1} & \mathbf{v}_{1,k} & \mathbf{v}_{1,k} & \mathbf{P}_{1,2} \\ \hline \alpha \mathbf{w}_{k,1}^T & & & \alpha \mathbf{w}_{k,2}^T \\ (1-\alpha) \mathbf{w}_{k,1}^T & \mathbf{T} & & (1-\alpha) \mathbf{w}_{k,2}^T \\ \hline \mathbf{P}_{2,1} & \mathbf{v}_{2,k} & \mathbf{v}_{2,k} & \mathbf{P}_{2,2} \end{array} \right). \quad (4.24)$$

Here, $\mathbf{T} \triangleq T(p_{k,k}, \lambda, \alpha)$ is defined according to three real-valued scalar parameters. Parameter λ denotes the eigenvalue to be inserted into the eigenspectrum of the transition matrix and α is selected in order to choose the stationary probabilities of the new states obtained by disaggregating state s_k . Specifically, the operator $T(p_{k,k}, \lambda, \alpha)$ is defined as

$$T(p_{k,k}, \lambda, \alpha) \triangleq \frac{p_{k,k}}{h_{\max}} \begin{pmatrix} \alpha & \alpha - h_{\min} \\ h_{\max} - \alpha & 1 - \alpha \end{pmatrix}, \quad (4.25)$$

where

$$h_{\max} \triangleq \frac{p_{k,k}}{p_{k,k} + \lambda} \quad (4.26)$$

and

$$h_{\min} \triangleq \frac{\lambda}{p_{k,k} + \lambda}. \quad (4.27)$$

To maintain a valid (i.e., stochastic) transition matrix, the selected parameters λ and α must satisfy

$$|\lambda| < p_{k,k} \quad (4.28)$$

and

$$\max\{h_{\min}, 0\} \leq \alpha \leq \min\{h_{\max}, 1\}. \quad (4.29)$$

The original disaggregation technique has the following properties, as proved¹⁰ in [32]. It is assumed in each case that the original disaggregation of state s_k is carried out in the original Markov chain, thereby producing the transition matrix $\mathbf{P}'$. It is also assumed that the disaggregation parameters are selected to satisfy (4.28) and (4.29) in each case, thereby ensuring a valid Markov chain after disaggregation (i.e., such that the transition probabilities satisfy $p_{i,j} \in [0, 1]$).

Theorem 4.6.1. *The transition matrix $\mathbf{P}'$ inherits the eigenvalues of $\mathbf{P}$, and gains the additional real eigenvalue λ .*

Proof. Refer to [32, Th. 3.1] for a proof of this theorem. □

Theorem 4.6.1 implies that the original disaggregation successfully inserts the selected real eigenvalue λ into the eigenspectrum of the Markov chain's transition matrix during each round of disaggregation, while leaving the existing eigenvalues of the matrix unaltered.

¹⁰Reference [32] follows the row-stochastic convention. By transposing the relevant matrices in each proof of [32], the corresponding proof is obtained for the column-stochastic convention followed in this work.

Theorem 4.6.2. *The Markov chain obtained after one round of original disaggregation possesses a stationary probability vector $\boldsymbol{\pi}' = [\pi'_1, \pi'_2, \dots, \pi'_{N+1}]^T$ that is associated with a corresponding stationary probability vector $\boldsymbol{\pi} = [\pi_1, \pi_2, \dots, \pi_N]^T$ of the original Markov chain. The elements of these vectors are related according to*

$$\pi'_i = \begin{cases} \pi_i, & \text{if } i < k, \\ \beta\pi_k, & \text{if } i = k, \\ (1 - \beta)\pi_k, & \text{if } i = k + 1, \\ \pi_{i-1}, & \text{if } i > k + 1, \end{cases} \quad (4.30)$$

where

$$\beta \triangleq \frac{\alpha(1 + \lambda) - \lambda}{1 - \lambda}. \quad (4.31)$$

Proof. Refer to [32, Th. 2.6] for a proof of this theorem. $\square$

Theorem 4.6.2 is particularly useful when attention is restricted to irreducible and aperiodic Markov chains, which possess a unique stationary probability vector [40]. In this case, there exists some freedom to select the stationary probabilities of the new states obtained by disaggregating state s_k . This is achieved by selecting an appropriate value for α , thereby choosing β according to (4.31).

The original disaggregation technique only permits insertion of a *real* eigenvalue into the eigenspectrum of the Markov chain's transition matrix during each round. Hence, the practical applicability of the StIEP solution proposed in [32], which relies solely on this disaggregation technique, is limited. In particular, it is of limited use when applied in matrix solutions to IFPP-IIa, which ideally require that the order-zero FPM of the semi-Markov map be designed to possess a complex-valued eigenspectrum.

A novel StIEP solution that generalises the analytical solution described thus far is presented next.

Novel solution using Markov state disaggregation

The following description of the novel solution for the StIEP appears in [4, 16]. The novel solution for the StIEP proposes a complementary disaggregation technique, referred to here as the *novel disaggregation technique*, that permits insertion of *strictly complex* eigenvalues (i.e., eigenvalues from the set $\mathbb{C} \setminus \mathbb{R}$) into the transition matrix.

By applying the original and novel disaggregation techniques in tandem, the novel solution constructs stochastic matrices with complex-valued eigenspectra.

Below, the novel disaggregation technique is detailed, and a novel algorithm that applies the original and novel techniques in tandem to solve the StIEP is described. Two examples that demonstrate the solution's ability to construct stochastic matrices with prescribed complex-valued eigenspectra are provided.

Novel disaggregation technique The novel disaggregation technique divides a particular state of a Markov chain into three states. The application of this technique is referred to as *novel disaggregation*. Suppose that novel disaggregation of state s_k in the original Markov chain is to be performed, with the requirement of inserting a pair of strictly complex (i.e., not real-valued) eigenvalues λ and λ^* into the eigenspectrum of the transition matrix. This produces a longer Markov chain with states $s'_1, s'_2, \dots, s'_{N+2}$, where s'_k, s'_{k+1} and s'_{k+2} denote the new states obtained by dividing state s_k , and

$$s'_i = \begin{cases} s_i, & \text{if } i < k, \\ s_{i-2}, & \text{if } i > k+2. \end{cases} \quad (4.32)$$

Novel disaggregation is performed by replacing the element $p_{k,k} > 0$ of the original Markov chain's transition matrix in (4.21) by a 3×3 matrix $\mathbf{U}$, to be specified below, and selecting the remaining transition matrix elements in a systematic manner, thus obtaining a larger transition matrix $\mathbf{P}'$. The larger transition matrix is obtained by applying the map $\mathcal{U} : \mathbb{P}_N \rightarrow \mathbb{P}_{N+2}$ to the original transition matrix, such that

$$\mathbf{P}' = \mathcal{U}(\mathbf{P}, r, \theta, k) \quad (4.33)$$

and

$$\mathbf{P}' = \left(\begin{array}{c|ccc|c} \mathbf{P}_{1,1} & \mathbf{v}_{1,k} & \mathbf{v}_{1,k} & \mathbf{v}_{1,k} & \mathbf{P}_{1,2} \\ \hline \mathbf{w}_{k,1}^T/3 & & & & \mathbf{w}_{k,2}^T/3 \\ \mathbf{w}_{k,1}^T/3 & & \mathbf{U} & & \mathbf{w}_{k,2}^T/3 \\ \mathbf{w}_{k,1}^T/3 & & & & \mathbf{w}_{k,2}^T/3 \\ \hline \mathbf{P}_{2,1} & \mathbf{v}_{2,k} & \mathbf{v}_{2,k} & \mathbf{v}_{2,k} & \mathbf{P}_{2,2} \end{array} \right). \quad (4.34)$$

Here, $\mathbf{U} \triangleq U(p_{k,k}, r, \theta)$ is defined according to three real-valued scalar parameters, where $\lambda \triangleq re^{i\theta}$ denotes one of the eigenvalues of the complex conjugate pair to be inserted into the eigenspectrum of the transition matrix. Specifically, the operator $U(p_{k,k}, r, \theta)$ is defined as

$$U(p_{k,k}, r, \theta) \triangleq \begin{pmatrix} \nu_1 & \nu_2 & \nu_3 \\ \nu_3 & \nu_1 & \nu_2 \\ \nu_2 & \nu_3 & \nu_1 \end{pmatrix}, \quad (4.35)$$

where

$$\nu_1 \triangleq \nu_1(p_{k,k}, r, \theta) = \frac{p_{k,k}}{3} + \frac{2r}{3} \cos \theta, \quad (4.36)$$

$$\nu_2 \triangleq \nu_2(p_{k,k}, r, \theta) = \frac{p_{k,k}}{3} - \frac{r}{3}(\cos \theta - \sqrt{3} \sin \theta), \quad (4.37)$$

$$\nu_3 \triangleq \nu_3(p_{k,k}, r, \theta) = \frac{p_{k,k}}{3} - \frac{r}{3}(\cos \theta + \sqrt{3} \sin \theta). \quad (4.38)$$

The matrix on the right-hand side of (4.35) was first applied towards the design of semi-Markov maps in the article of [19] by the author of this thesis.

The region of permissible values for λ is determined by first observing that (4.34) is required to be a valid transition matrix (i.e., stochastic). Since $\nu_1 + \nu_2 + \nu_3 = p_{k,k}$ by definition, each column of $\mathbf{P}'$ sums to unity. Thus, to ensure a stochastic matrix $\mathbf{P}'$, the elements of $U(p_{k,k}, r, \theta)$ are required to be non-negative. Necessary and sufficient conditions for $\mathbf{P}'$ to be stochastic are

$$\operatorname{Re}(\lambda) \geq -\frac{p_{k,k}}{2} \quad (4.39)$$

and

$$\frac{1}{\sqrt{3}}(\operatorname{Re}(\lambda) - p_{k,k}) \leq \operatorname{Im}(\lambda) \leq -\frac{1}{\sqrt{3}}(\operatorname{Re}(\lambda) - p_{k,k}). \quad (4.40)$$

Second, it is required that λ must be strictly complex; this implies that

$$\theta \neq k\pi, \quad k \in \mathbb{Z}. \quad (4.41)$$

The preceding expressions define the region of permissible values for $\lambda = re^{i\theta}$ when applying the novel disaggregation technique. This region coincides with the shaded triangle in the complex plane of Fig. 4.6.3, with the real line excluded.

Remark 4.6.4. The matrix in (4.35) was derived by this author from the work presented in [35]. In the case where $p_{k,k} = 1$, this matrix realises *all* possible strictly complex eigenvalues of a stochastic 3×3 matrix [25, Corollary 3.4], [41]. Hence, this matrix affords the freedom to choose *any* pair of strictly complex eigenvalues that may be realised during the first step of disaggregation.

The following theorems describe two properties of the novel disaggregation technique. It is assumed in each case that the novel disaggregation of state s_k is carried out in the original Markov chain, thereby producing the transition matrix $\mathbf{P}'$. It is also assumed that the disaggregation parameters are selected to satisfy (4.39) to (4.41) in each case, thereby ensuring a valid Markov chain after disaggregation (i.e., such that the transition probabilities satisfy $p_{i,j} \in [0, 1]$).

Theorem 4.6.3. *The transition matrix $\mathbf{P}'$ inherits the eigenvalues of $\mathbf{P}$, and gains the complex conjugate pair of eigenvalues $\lambda \triangleq re^{i\theta}$ and $\lambda^* \triangleq re^{-i\theta}$.*

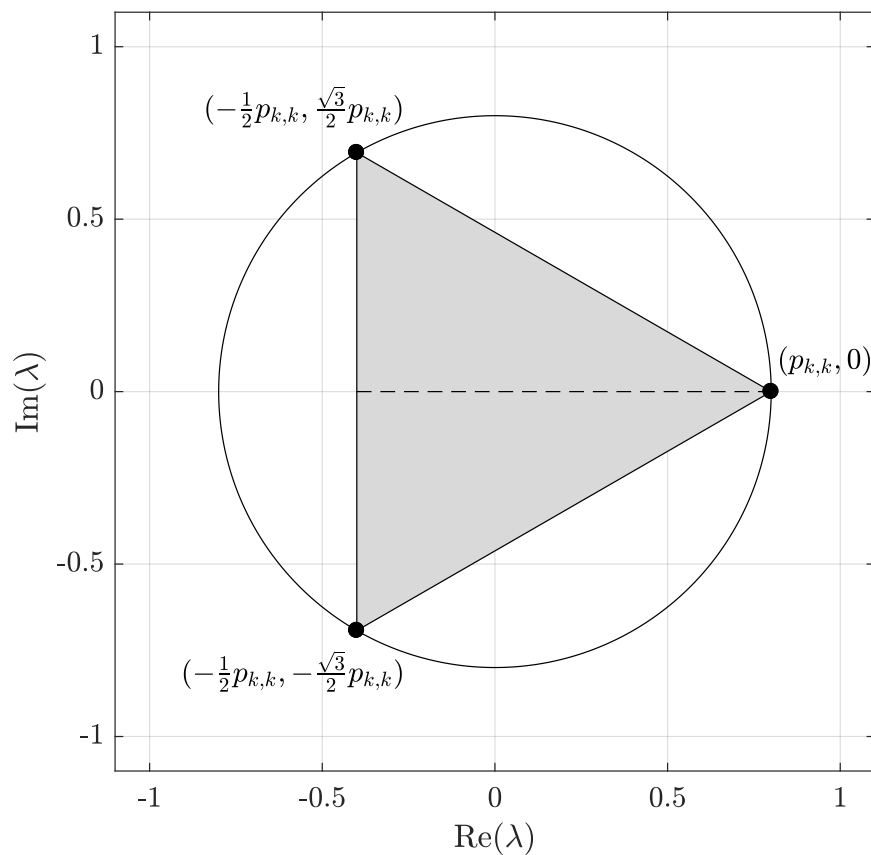


Figure 4.6.3: Complex plane with shaded region representing the permissible values for eigenvalue $\lambda = re^{i\theta}$ when applying the novel disaggregation technique. The parameter $p_{k,k} = 0.8$ was used in the figure. Reproduced with permission from [16].

Proof. The theorem is proved in section 4.A.1. $\square$

Theorem 4.6.3 implies that the novel disaggregation successfully inserts a selected complex conjugate pair into the eigenspectrum of the Markov chain's transition matrix during each round of disaggregation, while leaving the existing eigenvalues of the matrix unaltered. In particular, by selecting parameters r and θ , both $\lambda = re^{i\theta}$ and its complex conjugate $\lambda^* = re^{-i\theta}$ are inserted as eigenvalues, provided λ is within the region of permissible values illustrated in Fig. 4.6.3.

Theorem 4.6.4. *The Markov chain obtained after one round of novel disaggregation possesses a stationary probability vector $\boldsymbol{\pi}' = [\pi'_1, \pi'_2, \dots, \pi'_{N+2}]^T$ that is associated with a corresponding stationary probability vector $\boldsymbol{\pi} = [\pi_1, \pi_2, \dots, \pi_N]^T$ of the original*

Markov chain. The elements of these vectors are related according to

$$\pi'_i = \begin{cases} \pi_i, & \text{if } i < k, \\ \pi_k/3, & \text{if } i = k, k+1, k+2, \\ \pi_{i-2}, & \text{if } i > k+2. \end{cases} \quad (4.42)$$

Proof. The theorem is proved in section 4.A.2. □

Although the original disaggregation technique affords a measure of freedom to select the stationary probabilities of the new states obtained by disaggregating state s_k , Theorem 4.6.4 implies that the novel disaggregation technique affords no such freedom. The new states are always equiprobable.

Solution algorithm The original and novel disaggregation techniques may be applied in tandem to solve the StIEP. Starting with the elementary single-state Markov chain¹¹, the techniques are used to insert the prescribed eigenvalues into the eigenspectrum of the Markov chain's transition matrix over several rounds. An algorithm for solving the StIEP in this fashion is developed next.

There exist multiple degrees of freedom when applying the two disaggregation techniques to solve the StIEP. These are (i) the order in which the prescribed eigenvalues are inserted into the eigenspectrum of the transition matrix, (ii) the state selected for disaggregation during each round, and (iii) during those rounds where original disaggregation is performed, a measure of freedom to select the stationary probabilities of the two new states obtained through disaggregation. Simultaneously, there exist multiple constraints with regards to the parameter values that may be selected during each round of disaggregation. The parameters of the original disaggregation technique are constrained according to (4.28) and (4.29), and the parameters of the novel disaggregation technique are constrained according to (4.39), (4.40) and (4.41) (the region of permissible values is plotted in Fig. 4.6.3).

Equations (4.28), (4.39) and (4.40) imply that the disaggregation of a state with a larger self-transition probability during any round permits the insertion of a real eigenvalue with larger modulus or a pair of complex conjugate eigenvalues with larger modulus into the eigenspectrum of the transition matrix. This property, together

¹¹This Markov chain has a transition matrix $\mathbf{P} = [1]$ with single eigenvalue equal to unity. Any stochastic matrix has at least one unity eigenvalue for being stochastic. Hence, starting with the elementary single-state Markov chain does not prohibit the realisation of any valid eigenspectrum in future rounds.

with the observation that the self-transition probabilities of the Markov chain's states become smaller as disaggregation proceeds, motivates the following guideline: *Eigenvalues are to be inserted into the eigenspectrum of the Markov chain's transition matrix in non-increasing order of their modulus when solving the StIEP.* The intention behind this guideline is to reduce the likelihood of a scenario where the prescribed eigenspectrum cannot be realised due to incompatible values selected for parameter α during earlier rounds where original disaggregation is performed, because this parameter determines, in part, the new states' self-transition probabilities.

The guideline presented above does not determine which state to select for disaggregation during each round. Whereas one may select any state with self-transition probability large enough to accommodate the eigenvalue(s) to be inserted during each round, in the proposed algorithm the state with the *largest* self-transition probability is arbitrarily selected for disaggregation. In addition, the guideline does not determine how the stationary probabilities of the new states obtained during each round of original disaggregation are to be selected (i.e., the selection of a value for parameter α). Selecting a value of $\alpha = 1/2$ is permissible in any round, regardless of the eigenvalue λ selected for insertion. However, the values chosen for α during earlier rounds may affect whether certain of the prescribed eigenvalues can be successfully inserted into the eigenspectrum of the transition matrix during future rounds. If required, the parameter α may be selected to preserve states with larger self-transition probabilities, thereby accommodating the insertion of eigenvalues with large modulus during subsequent rounds. Alternatively, α may be considered a free parameter and remains unspecified. By following this approach, an expression may be derived for a set of stochastic matrices that share the same eigenspectrum, provided that (4.29) is satisfied during each round. This approach is illustrated in Example 4.6.2 in the next section.

Algorithm 4.6.2 (StIEP solution) for solving the StIEP is developed according to the above observations. Let $\lambda_1, \lambda_2, \dots, \lambda_M$ denote the eigenspectrum prescribed for the stochastic matrix; it is assumed that one of the eigenvalues equals unity, and that the prescription of a strictly complex eigenvalue implies the prescription of its complex conjugate.¹² To simplify the presentation of the algorithm, let $\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_{\tilde{M}}$ denote the eigenvalues that remain after removing one of the unity eigenvalues from the prescribed eigenspectrum, and removing one of the strictly complex eigenvalues from each of the complex conjugate eigenvalue pairs prescribed.

¹²This statement follows from the observation that the eigenvalues of any matrix with real elements are the roots of a characteristic polynomial in one variable and with *real* coefficients; thus, the strictly complex eigenvalues of these matrices appear in complex conjugate pairs.

These remaining eigenvalues are indexed in non-increasing order of their modulus; i.e., $i < j \Rightarrow |\tilde{\lambda}_i| \geq |\tilde{\lambda}_j|$. In addition, the notation $\mathbf{P}_{(m)}$ is used to denote the Markov chain's transition matrix at the end of round m , where $m \geq 0$.

Algorithm 4.6.2 (StIEP solution): Solution of the StIEP using Markov state disaggregation. Reproduced with permission from [16].

```

1: Input: Remaining eigenvalues  $\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_{\tilde{M}}$  ▷ Refer to section 4.6.2.
2:  $\mathbf{P}_{(0)} \leftarrow [1]$ ; ▷ Transition matrix of single-state chain.
3: for  $m \leftarrow 1, 2, \dots, \tilde{M}$  do ▷ Perform  $\tilde{M}$  disaggregation rounds.
4:    $k \leftarrow \arg \max(\text{diag}(\mathbf{P}_{(m-1)}))$ ; ▷ Index of state to disaggregate.
5:   if  $\tilde{\lambda}_m \in \mathbb{R}$  then ▷ Original disaggregation to be performed?
6:     if  $|\tilde{\lambda}_m| > p_{k,k}$  then
7:       return; ▷ Eigenspectrum cannot be realised by this algorithm.
8:     end if
9:     Select value for  $\alpha$  satisfying (4.29). ▷ Refer to section 4.6.2.
10:     $\mathbf{P}_{(m)} \leftarrow \mathcal{T}(\mathbf{P}_{(m-1)}, \tilde{\lambda}_m, \alpha, k)$ ; ▷ Apply original disaggregation.
11:  end if
12:  if  $\tilde{\lambda}_m \in \mathbb{C} \setminus \mathbb{R}$  then ▷ Novel disaggregation to be performed?
13:    if  $\lambda \triangleq \tilde{\lambda}_m$  does not satisfy (4.39) and (4.40) then
14:      return; ▷ Eigenspectrum cannot be realised by this algorithm.
15:    end if
16:     $r \leftarrow |\tilde{\lambda}_m|$ ;
17:     $\theta \leftarrow \arg(\tilde{\lambda}_m)$ ;
18:     $\mathbf{P}_{(m)} \leftarrow \mathcal{U}(\mathbf{P}_{(m-1)}, r, \theta, k)$ ; ▷ Apply novel disaggregation.
19:  end if
20: end for
21: return  $\mathbf{P}_{(\tilde{M})}$  ▷ Column stochastic matrix with prescribed eigenspectrum.

```

Examples Algorithm 4.6.2 (StIEP solution) is applied to solve two examples of the StIEP. Both examples prescribe complex-valued eigenspectra for the stochastic matrix. These examples appear in [16].

Example 4.6.2. The StIEP of deriving an expression for a set of 4×4 stochastic matrices with eigenspectrum $\lambda_1 = 1$, $\lambda_2 = 0.6$, $\lambda_3 = 0.5e^{i2\pi/3}$ and $\lambda_4 = 0.5e^{-i2\pi/3}$ is solved. After removing the unity eigenvalue λ_1 and eigenvalue λ_4 from the prescribed complex conjugate pair (λ_3, λ_4) , the eigenvalues $\tilde{\lambda}_1 = 0.6$ and $\tilde{\lambda}_2 = 0.5e^{i2\pi/3}$ remain.

The elementary single-state Markov chain, which possesses the transition matrix

$\mathbf{P}_{(0)} = [1]$ with unity eigenvalue, is used as the starting point. Original disaggregation of state s_1 is performed during the first round to insert the real eigenvalue $\tilde{\lambda}_1 = 0.6$ into the eigenspectrum of the transition matrix. The parameter α is not specified during this round. The transition matrix $\mathbf{P}_{(1)} = \mathcal{T}(\mathbf{P}_{(0)}, \tilde{\lambda}_1, \alpha, k)$ is produced at the end of the first round, where $k = 1$,

$$\mathbf{P}_{(1)} = \frac{1}{10} \begin{pmatrix} 16\alpha & 16\alpha - 6 \\ 10 - 16\alpha & 16 - 16\alpha \end{pmatrix} \quad (4.43)$$

and

$$\frac{6}{16} \leq \alpha \leq \frac{10}{16}. \quad (4.44)$$

During the second round, the eigenvalue $\tilde{\lambda}_2 = 0.5e^{i2\pi/3}$ and its complex conjugate are inserted into the eigenspectrum of the transition matrix using the novel disaggregation technique. State s'_2 is selected¹³ for disaggregation with parameters $r = 0.5$ and $\theta = 2\pi/3$. The transition matrix $\mathbf{P}_{(2)} = \mathcal{U}(\mathbf{P}_{(1)}, r, \theta, k)$ is produced at the end of the second round, where $k = 2$ and

$$\mathbf{P}_{(2)} = \frac{1}{30} \begin{pmatrix} 48\alpha & 48\alpha - 18 & 48\alpha - 18 & 48\alpha - 18 \\ 10 - 16\alpha & 11 - 16\alpha & 26 - 16\alpha & 11 - 16\alpha \\ 10 - 16\alpha & 11 - 16\alpha & 11 - 16\alpha & 26 - 16\alpha \\ 10 - 16\alpha & 26 - 16\alpha & 11 - 16\alpha & 11 - 16\alpha \end{pmatrix}. \quad (4.45)$$

Equations (4.44) and (4.45) define a set of stochastic matrices with the prescribed eigenspectrum.

Example 4.6.3. The StIEP of constructing a 5×5 stochastic matrix with eigenspectrum $\lambda_1 = 1$, $\lambda_2 = 0.8$, $\lambda_3 = 0.3i$, $\lambda_4 = -0.3i$ and $\lambda_5 = 0.1$ is solved. After removing the unity eigenvalue λ_1 and eigenvalue λ_4 from the prescribed complex conjugate pair (λ_3, λ_4) , the eigenvalues $\tilde{\lambda}_1 = 0.8$, $\tilde{\lambda}_2 = 0.3i$ and $\tilde{\lambda}_3 = 0.1$ remain.

The elementary single-state Markov chain with transition matrix $\mathbf{P}_{(0)} = [1]$ is again used as the starting point. Original disaggregation of state s_1 is performed during the first round to insert the real eigenvalue $\tilde{\lambda}_1 = 0.8$ into the eigenspectrum of the transition matrix. The parameter $\alpha = 0.5$ is selected during this round. The transition matrix $\mathbf{P}_{(1)} = \mathcal{T}(\mathbf{P}_{(0)}, \tilde{\lambda}_1, \alpha, k)$ is produced at the end of the first round, where $k = 1$ and

$$\mathbf{P}_{(1)} = \frac{1}{10} \begin{pmatrix} 9 & 1 \\ 1 & 9 \end{pmatrix}. \quad (4.46)$$

¹³Strict application of Algorithm 4.6.2 will result in state s'_2 being selected for disaggregation only if $6/16 \leq \alpha < 8/16$; however, the final solution remains valid over the original range of parameter values specified in (4.44).

During the second round, the eigenvalue $\tilde{\lambda}_2 = 0.3i$ and its complex conjugate are inserted into the eigenspectrum of the transition matrix using the novel disaggregation technique. State s'_2 is selected for disaggregation with parameters $r = 0.3$ and $\theta = \pi/2$. The transition matrix $\mathbf{P}_{(2)} = \mathcal{U}(\mathbf{P}_{(1)}, r, \theta, k)$ is produced at the end of the second round, where $k = 2$ and

$$\mathbf{P}_{(2)} = \frac{1}{30} \begin{pmatrix} 27 & 3 & 3 & 3 \\ 1 & 9 & 9 + \sqrt{27} & 9 - \sqrt{27} \\ 1 & 9 - \sqrt{27} & 9 & 9 + \sqrt{27} \\ 1 & 9 + \sqrt{27} & 9 - \sqrt{27} & 9 \end{pmatrix}. \quad (4.47)$$

During the third round, the eigenvalue $\tilde{\lambda}_3 = 0.1$ is inserted into the eigenspectrum of the transition matrix using the original disaggregation technique. State s''_1 is selected for disaggregation with parameter $\alpha = 0.6$. The transition matrix $\mathbf{P}_{(3)} = \mathcal{T}(\mathbf{P}_{(2)}, \tilde{\lambda}_3, \alpha, k)$ is produced at the end of the third round, where $k = 1$ and

$$\mathbf{P}_{(3)} = \frac{1}{300} \begin{pmatrix} 180 & 150 & 18 & 18 & 18 \\ 90 & 120 & 12 & 12 & 12 \\ 10 & 10 & 90 & 90 + 10\sqrt{27} & 90 - 10\sqrt{27} \\ 10 & 10 & 90 - 10\sqrt{27} & 90 & 90 + 10\sqrt{27} \\ 10 & 10 & 90 + 10\sqrt{27} & 90 - 10\sqrt{27} & 90 \end{pmatrix}. \quad (4.48)$$

Matrix $\mathbf{P}_{(3)}$ is stochastic and possesses the prescribed eigenspectrum.

4.6.3 Resolution of design issue 3: Design of PWL hat maps

This section introduces a novel class of semi-Markov maps, referred to as the *PWL hat maps*. The structural features of these maps prevent the eigenvalues of submatrix $\mathbf{Q}_S$ of each map's order-one FPM from appearing as unwanted poles in its PSD. These maps resolve design issue 3 identified in section 4.4.3, which concerns the fact that the structural features of a semi-Markov map that prevent unwanted poles in its power spectrum are unknown.

Below, the class of PWL hat maps is defined, and algorithms are presented for constructing these maps to possess a prescribed order-zero FPM. Relevant properties of these maps are then presented. Examples of hat maps constructed using the algorithms are not presented in this section; instead, the algorithms are used to construct several maps in Examples 4.8.1, 4.8.2 and 4.8.3 of section 4.8, which demonstrate the novel method for solving IFPP-IIa.

The material presented in this section appears in [4, 17].

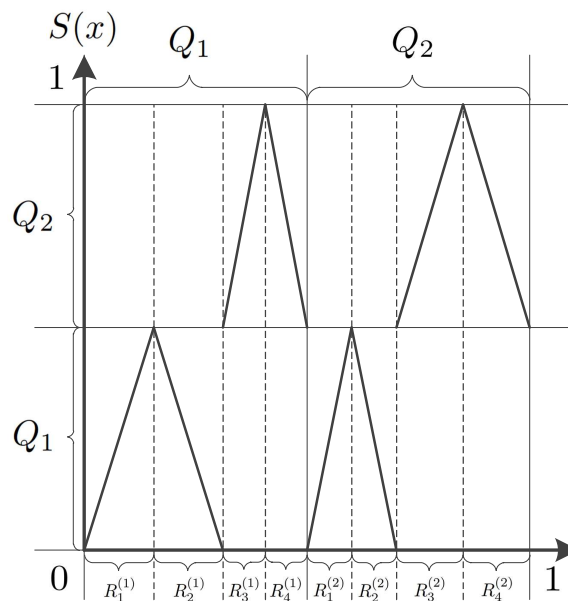


Figure 4.6.4: Example of a PWL hat map. Reproduced with permission from [4].

Definition

A PWL hat map S is a semi-Markov map where each consecutive pair of neighbouring branches is shaped like a hat; an example is plotted in Fig. 4.6.4. Each pair's left and right branches possess a positive and negative slope of equal magnitude and share the same image. This symmetry between the branches that constitute each hat prevents unwanted poles from appearing in the map's PSD.

The class of PWL hat maps is defined formally in a constructive fashion. This involves the design of a PWL hat map to be $\mathcal{R}(\mathcal{Q})$ -semi-Markov and to possess an order-zero FPM equal to a prescribed candidate matrix $\mathbf{P}$. The map is designed by (i) specifying its Markov partition $\mathcal{Q}$, (ii) specifying the subpartition $\mathcal{R}$ of partition $\mathcal{Q}$ and selecting the image of each subpartition interval under the map, and (iii) defining each of its branch functions.

Let $\mathbf{P}$ denote a positive $M \times M$ matrix that possesses a positive left Perron eigenvector associated with the unity Perron eigenvalue; this is sufficient to qualify the matrix as the order-zero FPM of an ergodic semi-Markov map (refer to Proposition 3.4.4). It is assumed that $\mathbf{v}_1$ is normalised such that its elements sum to unity. The steps for designing a hat map to possess $\mathbf{P}$ as its order-zero FPM are outlined below.

Step 1: Select the Markov partition The PWL hat map S is designed to be an $\mathcal{R}(\mathcal{Q})$ -semi-Markov map. This requires that the lengths of the intervals in its Markov partition $\mathcal{Q}$ coincide with the elements of $\mathbf{v}_1^T$ (refer to Proposition 2.6.4). The Markov partition $\mathcal{Q} = \{Q_1, \dots, Q_M\}$ is therefore selected as

$$Q_m = \left(\sum_{\ell=1}^{m-1} v_\ell, \sum_{\ell=1}^m v_\ell \right). \quad (4.49)$$

Step 2: Specify the subpartition $\mathcal{R}$ and subinterval images The subpartition $\mathcal{R}$ of partition $\mathcal{Q}$ is specified and the images of its intervals under S are selected. The subpartition $\mathcal{R}$ is constructed by selecting the partition $\mathcal{R}^{(m)}$ of each interval Q_m . As implied by (2.122), if element $p_{k,m}$ of any order-zero FPM associated with an $\mathcal{R}(\mathcal{Q})$ -semi-Markov map S is greater than zero, then S must possess at least one branch with a domain that coincides with an interval of $\mathcal{R}^{(m)}$ and an image under S that is equal to Q_k . All elements of $\mathbf{P}$ are greater than zero. Hence, each interval Q_k of $\mathcal{Q}$ must have a pre-image under S that overlaps with at least one interval in $\mathcal{R}^{(m)}$. Two intervals in $\mathcal{R}^{(m)}$ are reserved for each element in column m of $\mathbf{P}$, which implies that $\mathcal{R}^{(m)} = \{R_1^{(m)}, R_2^{(m)}, \dots, R_{2M}^{(m)}\}$. It is specified that S maps the intervals of $\mathcal{R}^{(m)}$ to the intervals of $\mathcal{Q}$ according to the expression

$$R_{2k-1}^{(m)}, R_{2k}^{(m)} \xrightarrow{S} Q_k, \quad k = 1, 2, \dots, M. \quad (4.50)$$

Equation (2.122) requires that the partition intervals must satisfy

$$\mu(R_{2k-1}^{(m)}) + \mu(R_{2k}^{(m)}) = p_{k,m}\mu(Q_k), \quad k, m = 1, 2, \dots, M, \quad (4.51)$$

if the order-zero FPM of S is to be equal to $\mathbf{P}$. It is specified that the intervals $R_{2k-1}^{(m)}$ and $R_{2k}^{(m)}$ are to be of equal length; i.e.,

$$\mu(R_{2k-1}^{(m)}) = \mu(R_{2k}^{(m)}) = p_{k,m}\mu(Q_k)/2. \quad (4.52)$$

This implies that the intervals of $\mathcal{R}$ must be selected as

$$R_j^{(m)} = \left(q_m + \sum_{\ell=1}^{j-1} \mu(R_\ell^{(m)}), q_m + \sum_{\ell=1}^j \mu(R_\ell^{(m)}) \right), \quad m = 1, 2, \dots, M, \quad j = 1, 2, \dots, 2M. \quad (4.53)$$

Step 3: Define the map's branch functions The branches defined over $R_{2k-1}^{(m)}$ and $R_{2k}^{(m)}$ are specified to be linear and to possess a positive and a negative slope, respectively. The branch functions of S are thus defined as

$$S_{2k-1,m}(x) \triangleq S|_{R_{2k-1}^{(m)}}(x) = \frac{2}{p_{k,m}}(x - r_{2k-1}^{(m)}) + q_k, \quad k, m = 1, 2, \dots, M, \quad (4.54)$$

and

$$S_{2k,m}(x) \triangleq S|_{R_{2k}^{(m)}}(x) = -\frac{2}{p_{k,m}}(x - r_{2k+1}^{(m)}) + q_k, \quad k, m = 1, 2, \dots, M, \quad (4.55)$$

where

$$\begin{aligned} r_{2k-1}^{(m)} &\triangleq \inf(R_{2k-1}^{(m)}), \\ r_{2k+1}^{(m)} &\triangleq \sup(R_{2k}^{(m)}). \end{aligned} \quad (4.56)$$

These expressions uniquely define a PWL hat map S that is $\mathcal{R}(\mathcal{Q})$ -semi-Markov and possesses a prescribed order-zero FPM $\mathbf{P}_S^{(0)} = \mathbf{P}$. This map is given by

$$S(x) = \sum_{m=1}^M \sum_{k=1}^M \left[S_{2k-1,m}(x) \chi_{R_{2k-1}^{(m)}}(x) + S_{2k,m}(x) \chi_{R_{2k}^{(m)}}(x) \right]. \quad (4.57)$$

The above steps are implemented as Algorithm 4.6.3 (hat map construction I) for constructing a PWL hat map to possess a candidate matrix $\mathbf{P}$ as its order-zero FPM. This algorithm assumes that the matrix is positive and possesses a unity Perron eigenvalue, which qualifies it as the order-zero FPM of an ergodic semi-Markov map.

Algorithm 4.6.3 (hat map construction I): Construction of a PWL hat map from a qualifying order-zero FPM (refer to [4]).

- 1: Input: Candidate order-zero FPM $\mathbf{P} = [p_{k,m}]_{k,m=1}^M$ that is positive and possesses a unity Perron eigenvalue.
 - 2: Output: PWL hat map S that is $\mathcal{R}(\mathcal{Q})$ -semi-Markov and possesses $\mathbf{P}$ as its order-zero FPM.
 - 3: Step 1: Construct the Markov partition $\mathcal{Q} = \{Q_1, Q_2, \dots, Q_M\}$, where $Q_m = (q_m, q_{m+1})$ is given by (4.49) and $\mathbf{v}_1^T = [v_1, v_2, \dots, v_M]$ is the left Perron eigenvector of $\mathbf{P}$, normalised such that its elements sum to unity.
 - 4: Step 2: Construct the subpartition $\mathcal{R} = \{R_j^{(m)}\}$ using (4.52) and (4.53), and define the image of each subinterval $R_j^{(m)}$ under S according to (4.50).
 - 5: Step 3: Define the branch functions of map S according to (4.54) and (4.55).
 - 6: Return the map S defined by (4.57).
-

Remark 4.6.5. Figure 4.6.5 contains plots of an example canonical semi-Markov map (grey dashed curve) and example PWL hat map (black curve) designed to possess the same order-zero FPM using Algorithm 3.4.1 (canonical map construction) and Algorithm 4.6.3 (hat map construction I), respectively. This figure reveals a relationship between the branches of these maps, which yields a second approach to constructing a hat map. In this second approach, a canonical semi-Markov map $\bar{S}$ is first constructed using Algorithm 3.4.1 (canonical map construction) and then modified to yield a corresponding hat map S that possesses the same order-zero FPM

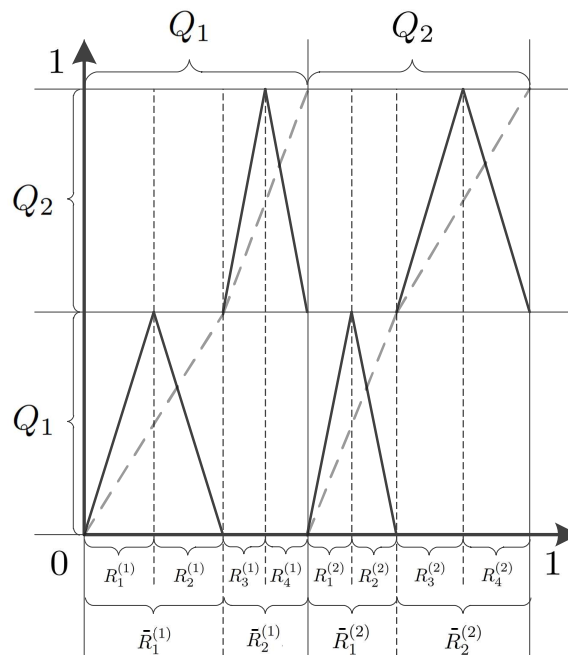


Figure 4.6.5: Canonical semi-Markov map (grey, dashed) and PWL hat map (black) designed to possess the same candidate matrix as their order-zero FPM.

as the canonical map. This two-step approach to construction yields the same hat map as that obtained by using Algorithm 4.6.3 (hat map construction I) to construct the map directly.

The modification of the canonical map $\bar{S}$ to yield a corresponding hat map S is outlined in what follows (refer to Fig. 4.6.5 as a guide). As both $\bar{S}$ and S possess the same order-zero FPM, they necessarily share the same Markov partition $\mathcal{Q}$. Hence, only the subpartition $\bar{\mathcal{R}}$ of the map $\bar{S}$ has to be modified. The subpartition $\bar{\mathcal{R}}$ is modified by uniformly dividing each of its intervals $\bar{R} \in \bar{\mathcal{R}}$, thereby yielding the intervals of the subpartition $\mathcal{R}$. The branches of the hat map S are defined over the shorter intervals of the subpartition $\mathcal{R}$. An increasing and decreasing branch function is defined for the hat map over each pair of intervals $R_{2j-1}^{(m)}$ and $R_{2j}^{(m)}$ obtained by dividing the interval $\bar{R}_j^{(m)} \in \bar{\mathcal{R}}$. These branches of the hat map are defined to have images that are identical to the images of the canonical map's branch defined over the original interval $\bar{R}_j^{(m)}$.

The alternative steps outlined in Remark 4.6.5 for constructing a hap map are implemented as Algorithm 4.6.4 (hat map construction II). In this algorithm, a canonical semi-Markov map is first designed to possess the specified order-zero FPM. Thereafter, the corresponding hat map is constructed from the canonical map.

Algorithm 4.6.4 (hat map construction II): Construction of a PWL hat map from a canonical semi-Markov map designed to possess a qualifying order-zero FPM.

- 1: Input: Candidate order-zero FPM $\mathbf{P} = [p_{k,m}]_{k,m=1}^M$ that is positive and possesses a unity Perron eigenvalue.
 - 2: Output: PWL hat map S that is $\mathcal{R}(\mathcal{Q})$ -semi-Markov and possesses $\mathbf{P}$ as its order-zero FPM.
 - 3: Construct the Markov partition $\mathcal{Q} = \{Q_1, Q_2, \dots, Q_M\}$, where $Q_m = (q_m, q_{m+1})$ is given by (4.49) and $\mathbf{v}_1^T = [v_1, v_2, \dots, v_M]$ is the left Perron eigenvector of $\mathbf{P}$.
 - 4: Use Algorithm 3.4.1 (canonical map construction) to construct an $\bar{\mathcal{R}}(\mathcal{Q})$ -semi-Markov map $\bar{S}$ to possess $\mathbf{P}$ as its order-zero FPM, and let $\bar{S}_{j,m}(x) = \bar{S}|_{\bar{R}_j^{(m)}}(x) = \bar{a}_{j,m}(x - \bar{r}_j^{(m)}) + \bar{b}_{j,m}$, where $\bar{R}_j^{(m)} = (\bar{r}_j^{(m)}, \bar{r}_{j+1}^{(m)})$.
 - 5: **for** $m \leftarrow 1, 2, \dots, M$ **do**
 - 6: **for** $j \leftarrow 1, 2, \dots, M$ **do**
 - 7: Define $R_{2j-1}^{(m)} = (\inf(\bar{R}_j^{(m)}), \inf(\bar{R}_j^{(m)}) + \mu(\bar{R}_j^{(m)})/2)$
 - 8: Define $S_{2j-1,m}(x) = S|_{R_{2j-1}^{(m)}}(x) = 2\bar{a}_{j,m}(x - \bar{r}_j^{(m)}) + \bar{b}_{j,m}$.
 - 9: Define $R_{2j}^{(m)} = (\inf(\bar{R}_j^{(m)}) + \mu(\bar{R}_j^{(m)})/2, \sup(\bar{R}_j^{(m)}))$.
 - 10: Define $S_{2j,m}(x) = S|_{R_{2j}^{(m)}}(x) = -2\bar{a}_{j,m}(x - \bar{r}_{j+1}^{(m)}) + \bar{b}_{j,m}$.
 - 11: **end for**
 - 12: **end for**
 - 13: Return the map S defined by (4.57).
-

In what follows, it is formally proved that the hat maps constructed using Algorithm 4.6.3 (hat map construction I) and Algorithm 4.6.4 (hat map construction II) are semi-Markov and possess the order-zero FPM prescribed during their design. The proof utilises the fact that the hat maps constructed using these algorithms are identical.

Proposition 4.6.1 (Hat maps constructed using Algorithm 4.6.3 (hat map construction I) and 4.6.4 (hat map construction II) are semi-Markov and possess the prescribed order-zero FPM). Let $\mathbf{P}$ denote a positive $M \times M$ matrix with Perron eigenvalue equal to unity, and let S denote the hat map constructed using either Algorithm 4.6.3 (hat map construction I) or Algorithm 4.6.4 (hat map construction II) to possess $\mathbf{P}$ as its order-zero FPM. Then S is an $\mathcal{R}(\mathcal{Q})$ -semi-Markov map that possesses $\mathbf{P}$ as its order-zero FPM, where the intervals of the Markov partition $\mathcal{Q}$ are defined by (4.49).

Proof. The proposition is proved in section 4.A.3. □

Properties

The following propositions outline two properties of PWL hat maps that are relevant to the solution of IFPP-IIa. In particular, these properties prove that the hat-like structure of these maps prevent unwanted poles in their PSDs.

Proposition 4.6.2. The top-left submatrix $\mathbf{Q}_S$ of the order-one FPM realised by any PWL hat map S is the zero matrix [4].

Proof. Equation (4.50) implies that any hat map associated with a positive order-zero FPM satisfies [4]

$$T(k, m) = \{j : S(R_j^{(m)}) = Q_k\} = \{2k - 1, 2k\}, \quad m = 1, 2, \dots, M. \quad (4.58)$$

Equation (2.118) then implies that each element $q_{k,m}$ of $\mathbf{Q}$ is given by

$$q_{k,m} = s_{2k-1,m}/v_{2k-1,m}^2 + s_{2k,m}/v_{2k,m}^2, \quad k, m = 1, 2, \dots, M, \quad (4.59)$$

where $v_{\cdot,m}$ is the slope of $S_{\cdot,m}$ and $s_{\cdot,m} \triangleq \text{sgn}(v_{\cdot,m})$. But (4.54) and (4.55) imply that

$$v_{2k-1,m}^2 = 4/p_{2k-1,m}^2 = v_{2k,m}^2, \quad k, m = 1, 2, \dots, M, \quad (4.60)$$

and that

$$s_{2k-1,m} = -s_{2k,m}, \quad k, m = 1, 2, \dots, M. \quad (4.61)$$

Substitution of (4.60) and (4.61) into (4.59) then implies that each element $q_{k,m}$ of submatrix $\mathbf{Q}_S$ equals zero. $\square$

Proposition 4.6.3. Consider any PWL hat map S that possesses a $2M \times 2M$ order-one FPM. The eigenspectrum of the map's order-one FPM comprises the eigenspectrum of its order-zero FPM and M zero eigenvalues [4].

Proof. Equation (2.117) reveals that the map's order-one FPM has a block lower triangular structure [4]. The eigenspectrum of a block triangular matrix equals the eigenspectra of its diagonal blocks [36]. Since the order-one FPM's bottom-right submatrix is the order-zero FPM, and its top-left submatrix $\mathbf{Q}_S$ is the zero matrix (Proposition 4.6.2), which possesses M zero eigenvalues, the proposition follows. $\square$

Remark 2.154 observes that a semi-Markov map possesses a rational PSD with poles that coincide with the unique non-zero and non-unity (in absolute value) eigenvalues of its order-one FPM. For a hat map, which is by definition a semi-Markov map, Proposition 4.6.3 then implies that the PSD poles coincide with the unique non-zero

and non-unity (in absolute value) eigenvalues of the system's *order-zero* FPM *only*. Stated differently, submatrix $\mathbf{Q}_S$ of a PWL hat map does not contribute any poles to the map's PSD.

The above propositions imply that design issue 3 identified in section 4.4 is resolved by constructing a PWL hat map with the required order-zero FPM. The novel method for solving IFPP-IIa, presented in the next section, constructs a hat map to implement the second step of solution strategy 2.

4.7 A Novel Method for Solving IFPP-IIa

In this section, a novel method for solving IFPP-IIa is presented. This method constructs a semi-Markov map with a PWC invariant density that approximates a prescribed invariant density, and which possesses a rational PSD with prescribed non-repeating poles. The method implements solution strategy 2 of section 4.4.3 and thus generalises the existing matrix solution strategy for IFPP-I. The novel method appears in [4].

The novel method overcomes the design issues identified for solution strategy 2 in section 4.4 by systematically applying the theory and algorithms developed in section 4.6. In what follows, the steps of the proposed method are outlined. Thereafter, an algorithm that implements the proposed method is presented. Relevant properties of the algorithm are subsequently stated.

4.7.1 Outline of steps

The novel method for solving IFPP-IIa implements each of the steps of solution strategy 2 described in section 4.4.3. Below, the specifics of each step are described.

Step 1: Design a candidate order-zero FPM with the required properties

The implementation of step 1 of solution strategy 2 requires the resolution of design issues 1 and 2, as identified in section 4.4.3. The resolution of these issues and the implementation of the design step are described below.

Design issue 1 concerns the lack of a method for designing a candidate order-zero FPM with a specified eigenspectrum and (simultaneously) specified left and right Perron eigenvectors. The novel method resolves this issue by sacrificing the ability to prescribe an arbitrary PWC invariant density and Markov partition for the map, and specifies that the invariant density and Markov partition instead be selected to accommodate the candidate order-zero FPM.

The above approach permits the relaxation of the design specifications for the candidate order-zero FPM; specifically, the matrix no longer required to possess a prescribed left and right Perron eigenvector, but is instead only required to possess a prescribed eigenspectrum. This is achieved while simultaneously permitting a PWC invariant density that approximates an arbitrary prescribed invariant density. This density is determined using the novel approach developed in section 4.6.1 and implemented in Algorithm 4.6.1 (determination of approximating density).

Design issue 2 concerns the unavailability of an analytical solution to the StIEP that permits construction of a stochastic matrix of arbitrary dimensionality and with complex-valued eigenspectra.¹⁴ This issue was resolved by introducing the novel analytical solution to the StIEP in section 4.6.2. Using Algorithm 4.6.2 (StIEP solution), a stochastic matrix of arbitrary dimensionality and with complex-valued eigenspectra is realised by the novel method.

The novel method first constructs a positive stochastic matrix $\tilde{\mathbf{P}}$ with a prescribed eigenspectrum that coincides with the prescribed poles using Algorithm 4.6.2 (StIEP solution). Thereafter, the Markov partition $\mathcal{Q}$ and the approximating invariant density $\hat{f}_P$ implied by the particular stochastic matrix $\tilde{\mathbf{P}}$ are determined using Algorithm 4.6.1 (determination of approximating density). Having derived the appropriate Markov partition $\mathcal{Q}$, the candidate order-zero FPM $\mathbf{P}$ is derived via the similarity transformation

$$\mathbf{P} = \mathbf{D}\tilde{\mathbf{P}}\mathbf{D}^{-1}, \quad (4.62)$$

where

$$\mathbf{D} = \text{diag}\{\mu(Q_1)^{-1}, \mu(Q_2)^{-1}, \dots, \mu(Q_M)^{-1}\}. \quad (4.63)$$

The map is designed to possess the matrix $\mathbf{P}$ as its order-zero FPM in the next step.

¹⁴The requirement of realising complex-valued eigenspectra is motivated by observing that the eigenvalue argument coincides with the centre frequency of the corresponding spectral mode; hence, complex eigenvalues permit realisation of spectral modes with arbitrary centre frequencies.

Step 2: Design a map with the prescribed order-zero FPM

The implementation of step 2 of solution strategy 2 requires the resolution of design issue 3, as identified in section 4.4.3. The resolution of this issue and the implementation of the design step are described below.

Design issue 3 concerns the problem of selecting structural features for a map that prevent unwanted poles in its PSD (specifically, the poles that are contributed by submatrix $\mathbf{Q}_S$ of the map's order-one FPM). This issue was resolved with the introduction of the family of hat maps in section 4.6.3. It was shown that these maps may be designed to possess a prescribed order-zero FPM and simultaneously yield a submatrix $\mathbf{Q}_S$ that only contains zero-valued elements. This implies that the eigenvalues of this matrix are zero. As only the unique non-zero and non-unity (in absolute values) eigenvalues are contributed as poles to the power spectrum, submatrix $\mathbf{Q}_S$ is prevented from contributing unwanted poles.

The novel method for solving IFPP-IIa uses Algorithm 4.6.3 (hat map construction I) to construct a hat map that possesses the order-zero FPM $\mathbf{P}$ designed in step one. In doing so, only the unique non-zero and non-unity (in magnitude) eigenvalues selected for matrix $\mathbf{P}$ in step 1 are contributed as poles to the map's power spectrum. Simultaneously, the approximating density $\hat{f}_P$ is realised as the invariant density of the map; this is due to the right Perron eigenvector of matrix $\mathbf{P}$ coinciding with the vector representation of $\hat{f}_P$.

4.7.2 Algorithm and properties

The novel method for solving IFPP-IIa is implemented as Algorithm 4.7.1 (novel method for solving IFPP-IIa). The input of the algorithm is the prescribed invariant density $f_P \in \mathcal{D}(\Omega, \mathcal{B}, \mu)$, which is approximately realised by the map as a PWC invariant density, and a collection of non-repeating poles $\kappa_1, \kappa_2, \dots, \kappa_M$ for the map's power spectrum, which are realised exclusively (i.e., the PSD possesses no other poles). The prescribed poles are required to satisfy $0 < |\kappa_m| < 1$, and the prescription of a strictly complex pole has to be accompanied by the prescription of its complex conjugate. The algorithm then constructs a hat map with these statistical metrics.

Remark 4.7.1. Equations (2.152), (2.154) and (2.155) for the PSD of a semi-Markov map were derived under the assumption that the order-one FPM of the semi-Markov map is diagonalisable. As the novel method for solving IFPP-IIa was developed from these expressions, it is therefore required that the order-one FPM of any hat

map constructed using Algorithm 4.7.1 (novel method for solving IFPP-IIa) be diagonalisable.

The condition that the map's order-one FPM be diagonalisable also restricts the power spectrum of the map to only possess non-repeating poles. Hence, Algorithm 4.7.1 (novel method for solving IFPP-IIa) does not permit prescription of repeating poles for the map's PSD. The generalisation of the method to permit prescription of repeating poles is a topic for future work.

Remark 4.7.2. Algorithm 4.7.1 (novel method for solving IFPP-IIa) requires that the stochastic matrix $\tilde{\mathbf{P}}$ constructed using the novel solution to the StIEP be positive. This condition may be weakened to a stochastic matrix that is non-negative, primitive and possesses at least one element in each of its columns that satisfies $0 < p_{k,m} < 1$. This observation follows from Proposition 3.4.4.

Algorithm 4.7.1 (novel method for solving IFPP-IIa): Construction of a PWL hat map to possesses an invariant density $f_S^* = \hat{f}_P$ that approximates a prescribed density f_P , and a rational PSD with prescribed non-repeating poles $\kappa_1, \kappa_2, \dots, \kappa_M$. Refer to [4].

- 1: Inputs: Prescribed invariant density $f_P \in \mathcal{D}(\Omega, \mathcal{B}, \mu)$ and non-repeating PSD poles $\mathcal{K} = \{\kappa_1, \kappa_2, \dots, \kappa_M\}$, where $0 < |\kappa_m| < 1$ and $\kappa_m \in \mathbb{C} \setminus \mathbb{R} \Rightarrow \kappa_m^* \in \mathcal{K}$.
 - 2: Output: PWL hat map S that is $\mathcal{R}(\mathcal{Q})$ -semi-Markov, possesses a PWC invariant density $\hat{f}_P \approx f_P$ and a rational PSD that possesses only the poles $\kappa \in \mathcal{K}$.
 - 3: Construct an $(M + 1) \times (M + 1)$ positive stochastic matrix $\tilde{\mathbf{P}}$ with prescribed eigenvalues $\lambda_1 = 1$ and $\lambda_m = \kappa_{m-1}$, $m = 2, 3, \dots, M + 1$, using Algorithm 4.6.2 (StIEP solution).
 - 4: **if** $\tilde{\mathbf{P}}$ is not strictly positive **then**
 - 5: return; ▷ Map cannot be realised by this algorithm.
 - 6: **end if**
 - 7: Derive the approximating density $\hat{f}_P$ and Markov partition $\mathcal{Q}$ using Algorithm 4.6.1 (determination of approximating density) using matrix $\tilde{\mathbf{P}}$ as input.
 - 8: Compute the candidate order-zero FPM $\mathbf{P}$ using (4.62) and (4.63).
 - 9: Construct a hat map S from $\mathbf{P}$ using Algorithm 4.6.3 (hat map construction I).
 - 10: Derive the order-one FPM $\mathbf{P}_S^{(1)}$ associated with the map S using (2.117).
 - 11: **if** $\mathbf{P}_S^{(1)}$ is not diagonalisable **then**
 - 12: return; ▷ Map cannot be realised by this algorithm.
 - 13: **end if**
 - 14: Return the map S .
-

The following proposition is proved.

Proposition 4.7.1. Suppose that the distribution function F_P associated with the prescribed invariant density f_P is continuous. If Algorithm 4.7.1 (novel method for solving IFPP-IIa) returns a map S , the map is necessarily ergodic.

Proof. The proposition is proved in section 4.A.4. □

4.8 Examples

This section presents several examples of hat maps that were constructed using the novel method. An invariant density and a collection of non-repeating power spectrum poles were prescribed for each map. The examples distinguish between *elementary maps* and *more sophisticated maps*, as described below.

- The elementary maps were prescribed the uniform invariant density and PSDs with a maximum of two poles.
- The more sophisticated maps were prescribed a non-uniform invariant density and PSDs with a maximum of three poles.

Each of the examples is set out as follows. The statistical metrics that were prescribed for the maps are presented. The steps followed to design the maps are outlined, and a subset of the maps is plotted. The dominant behaviour of trajectories generated by the maps are identified and related to the maps' structural features. Estimates of the maps' invariant density, as computed from numerically generated trajectories, are compared to the prescribed invariant density. The power spectra of the maps are characterised and compared against numerical estimates as verification.

The examples presented in this section appear in [4].

4.8.1 Elementary maps

Two sets of elementary maps were designed as examples. A single real-valued pole was prescribed for the maps in the first set, and a complex conjugate pair of non-real poles was prescribed for the maps in the second set.

Example 4.8.1 (First map set: Single real-valued pole).

Prescribed statistical metrics The single real-valued pole prescribed for the maps from the first set satisfies¹⁵

$$0 < |\kappa_1| < 1. \quad (4.64)$$

Results are presented that characterise the zero realised by prescribing the pole over this range. Individual maps were constructed by prescribing the pole κ_1 from the set

$$\kappa_1 \in \{\pm 0.1, \pm 0.3, \pm 0.5, \pm 0.7, \pm 0.9\}. \quad (4.65)$$

Map design Refer to section 4.B.1 for details on the steps that were followed to design the maps presented in this example.

Map structure and trajectories The hat maps designed by prescribing the poles $\kappa_1 = \pm 0.7$ are plotted in Figs. 4.8.1 and 4.8.2, respectively. Each map possesses four hats. Two pairs of hats are identified in each map, such that the hats from each pair have equally wide support. This narrative distinguishes between the two hat pairs of each map by referring to their hats as *wider* or *narrower*, based on the relative width of the support.

Additional plots of maps in the first set revealed that the wider hats in Fig. 4.8.1 become progressively wider as a *positive* pole κ_1 with increasing modulus is prescribed. In the limit as κ_1 approaches unity, the narrower hats disappear. Similarly, as a *negative* pole κ_1 with increasing modulus is prescribed, the wider hats in Fig. 4.8.2 widen further. The narrower hats in this figure disappear in the limit as κ_1 approaches negative unity. As the pole κ_1 approaches zero from either side, all the hats of Figs. 4.8.1 and 4.8.2 become equally wide. These observations are summarised by stating that the disparity in the width of the wider and narrower hats grows as $|\kappa_1|$ increases.

In the map designed by prescribing the pole $\kappa_1 = 0.7$ (Fig. 4.8.1), the wider hats map states from interval Q_1 or Q_2 back to the same interval between time instants. This is a characteristic of maps designed by prescribing positive poles κ_1 . However, in the map designed by prescribing the pole $\kappa_1 = -0.7$ (Fig. 4.8.2), the wider hats map states from interval Q_1 to Q_2 (and vice versa) between time instants. Maps associated with negative poles κ_1 possess this characteristic.

¹⁵Remark 4.3.1 motivates the exclusion of the poles $\kappa_1 = \pm 1$ and $\kappa_1 = 0$. Experimentation revealed that prescription of the pole $\kappa_1 = 0$ yielded a map with a non-diagonalisable order-one FPM; refer to Remark 4.7.1.

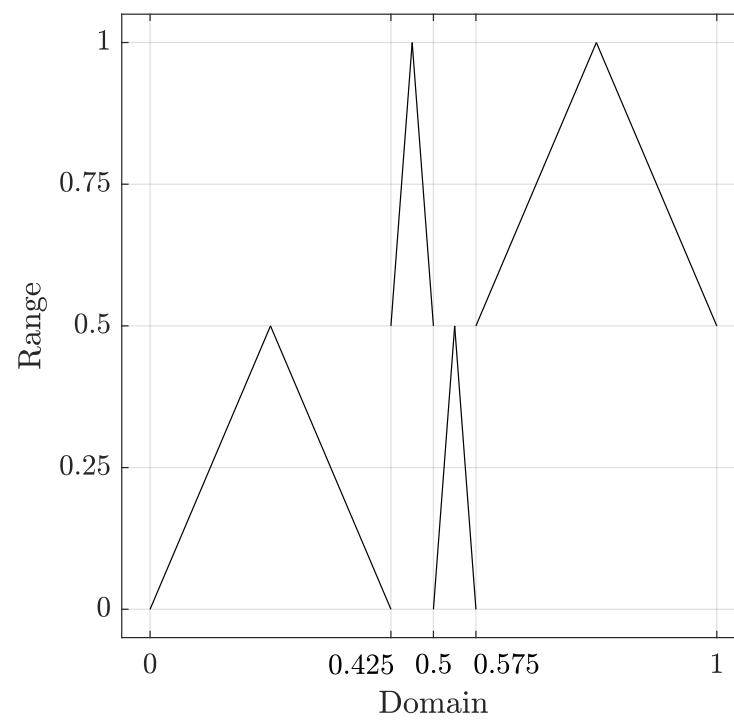


Figure 4.8.1: Plot of the hat map from the first set, as designed by prescribing the pole $\kappa_1 = 0.7$. Adapted from [4] with permission.

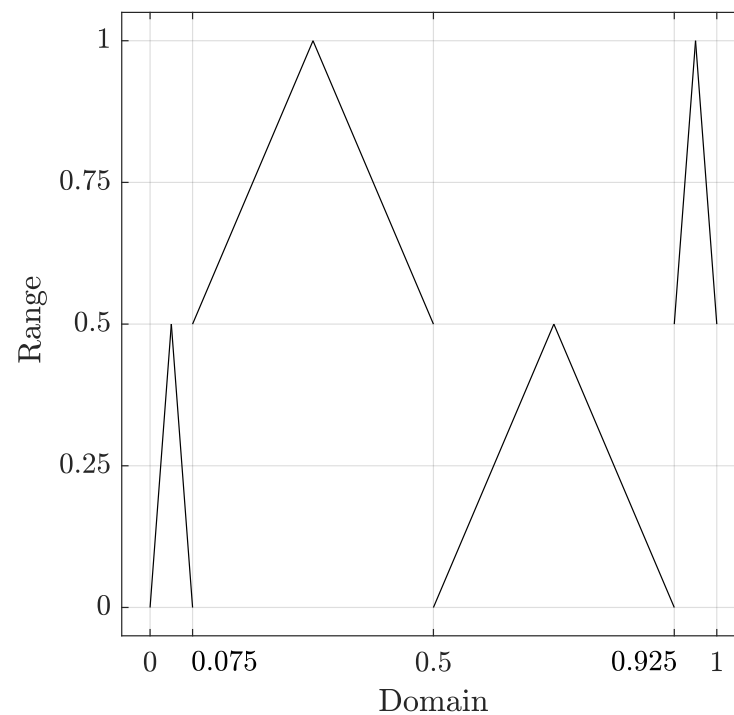


Figure 4.8.2: Plot of the hat map from the first set, as designed by prescribing the pole $\kappa_1 = -0.7$. Adapted from [4] with permission.

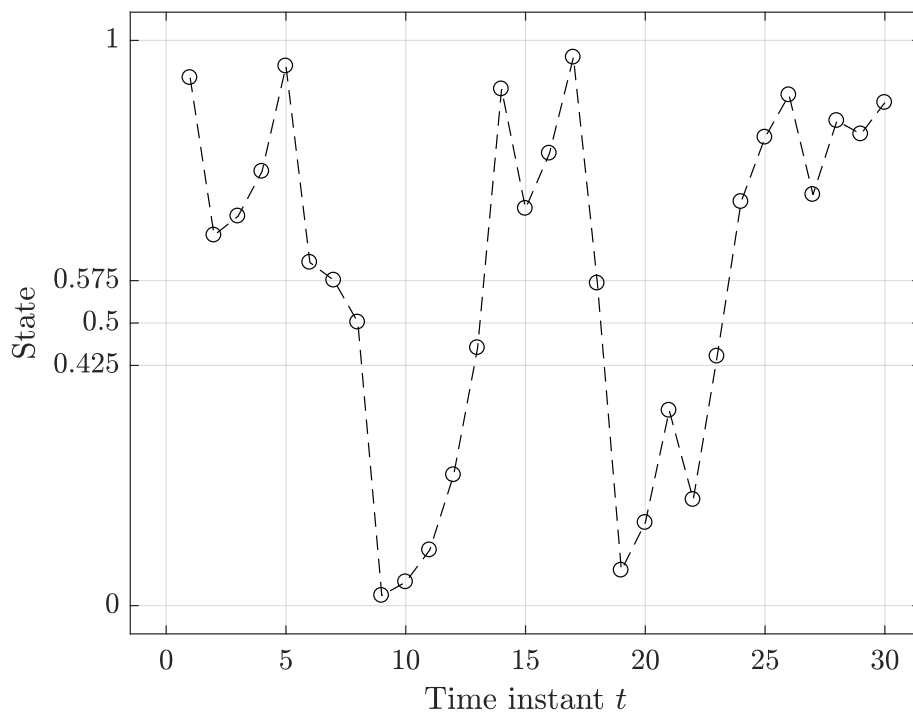


Figure 4.8.3: Plot of a trajectory generated by the map of Fig. 4.8.1 (as designed by prescribing the pole $\kappa_1 = 0.7$).

The above characteristics are discernible in trajectories generated by the maps. Figure 4.8.3 is a plot of a trajectory generated by the map in Fig. 4.8.1. The horizontal grid lines in this figure coincide with the domain endpoints of the hats in Fig. 4.8.1. The figure reveals that the trajectory, once entering an interval that coincides with the domain of a wider hat, typically remains in this interval for multiple time instants. Figure 4.8.4 is a plot of a trajectory generated by the map in Fig. 4.8.2; the horizontal grid lines in this figure also coincide with the domain endpoints of the corresponding hats. The figure reveals that the trajectory tends to alternate between the two intervals that coincide with the domains of the wider hats.

There exists a relationship between the disparity in hat width and the PSD characteristics of the hat maps that belong to the first set. Consider the evolution of the uniform (stationary) state density under iteration of the maps in Figs. 4.8.1 and 4.8.2. In this *stationary regime*, the Borel measure of those states mapped between time instants by the wider hats is larger than the Borel measure of those mapped by the narrower hats in proportion to the disparity in hat width. As the wider hats of Fig. 4.8.1 (where $\kappa_1 = 0.7$) map states from Q_1 or Q_2 back to the same interval, the dominant behaviour of the trajectories in this stationary regime is to remain in the same interval Q_1 or Q_2 from one time instant to the next (as observed in Fig. 4.8.3). These *slower* trajectories gives rise to a peak in the PSD at zero radians per sample

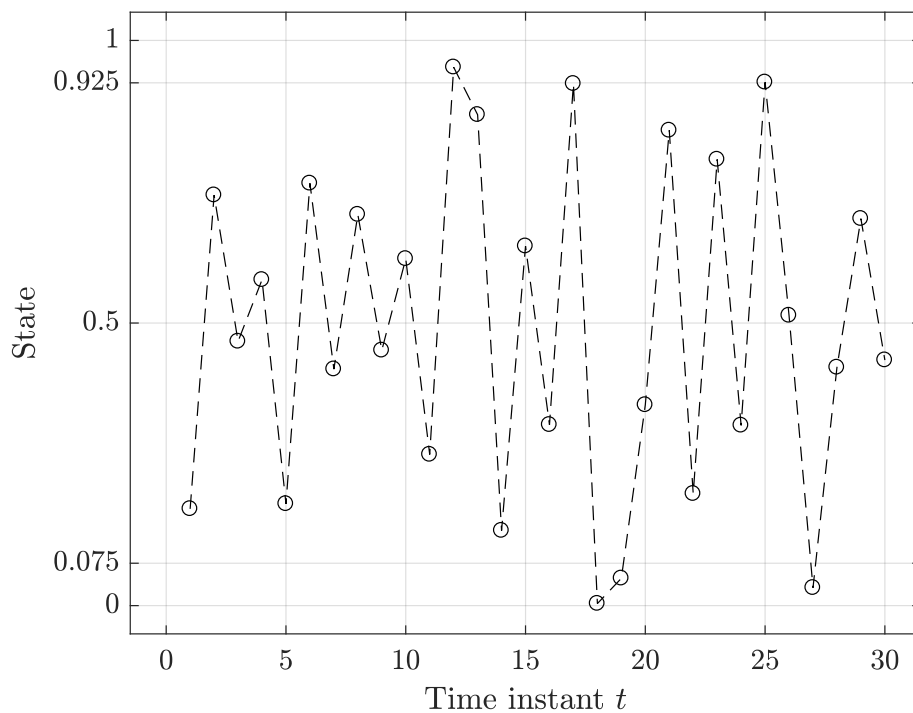


Figure 4.8.4: Plot of a trajectory generated by the map of Fig. 4.8.2 (as designed by prescribing the pole $\kappa_1 = -0.7$).

(the PSD of this map is plotted in Fig. 4.8.7 and characterised using a pole-zero plot towards the end of the example).

In contrast, the wider hats of Fig. 4.8.2 (where $\kappa_1 = -0.7$) map states from Q_1 to Q_2 and vice versa. The dominant behaviour of the trajectories in the stationary regime is to alternate between the partition intervals (as observed in Fig. 4.8.3). These *faster* trajectories frequently enter period-2 cycles and thus give rise to an observed PSD peak centred at π radians per sample (the PSD of this map is plotted in Fig. 4.8.8 and also characterised using a pole-zero plot towards the end of the example). As $|\kappa_1|$ increases, the disparity in hat width grows, the dominant behaviour of the trajectories intensifies, and a higher PSD peak is realised.

Invariant density The maps of the first set were designed to possess the uniform invariant density. The invariant density of each map in the set was verified by numerically computing a map trajectory of length 10^6 states and then constructing a histogram of the states. Each histogram was normalised to obtain an estimate $\hat{f}_S^*$, defined at the centres of the histogram bins, of the map's invariant density. These estimates were compared against the prescribed uniform invariant density.

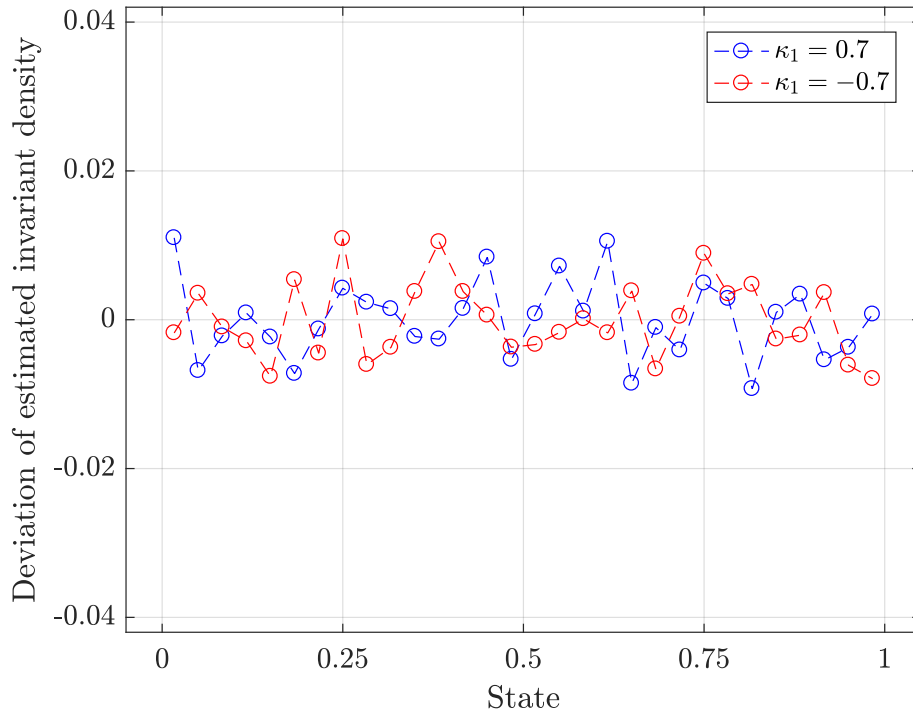


Figure 4.8.5: Deviation of the invariant density function estimates, as compared to the prescribed invariant density, of maps realised by prescribing the poles $\kappa_1 = \pm 0.7$. The estimates were constructed from map trajectories.

Figure 4.8.5 is a plot of the deviation $\hat{f}_S^* - f_P$ between the estimated invariant density and the prescribed uniform invariant density for the maps plotted in Figs. 4.8.1 and 4.8.2, where the circle markers in the figure coincide with the centres of the histogram bins. The figure reveals that the estimated invariant densities closely approximate the prescribed invariant density. The deviation observed in the figure is attributed to the finite number of states used to construct each density estimate.

Power spectral density Let $\Psi_S(z) = K_S H_S(z) H_S(z^{-1})$ denote the PSD of each map S from the first set, as expressed in the z -domain; refer to (2.154). Equation (2.155) and Proposition 4.6.3 imply that each PSD associated with a map from the first set has a single pole κ_1 and at most a single real-valued zero η_1 . Analysis of the power spectra revealed that

$$H_S(z) = \begin{cases} \frac{1}{z - \kappa_1}, & \text{if } \kappa_1 = \zeta, \\ \frac{z - \eta_1}{z - \kappa_1}, & \text{if } \kappa_1 \in (-1, 1) \setminus \{0, \zeta\}, \end{cases} \quad (4.66)$$

where

$$\zeta \triangleq -1/3 + \sqrt{10/9} \approx 0.721. \quad (4.67)$$

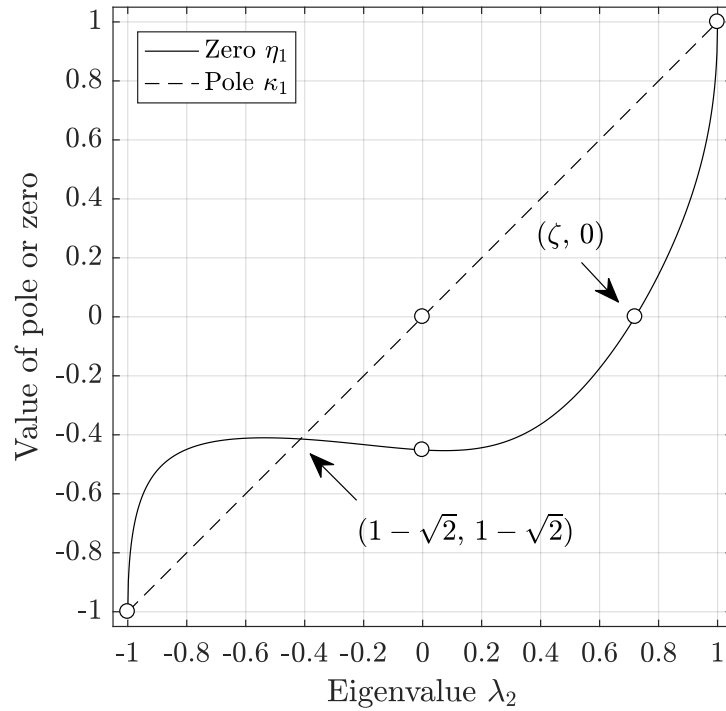


Figure 4.8.6: Plots of the PSD pole and zero of the maps belonging to the first set, as a function of eigenvalue λ_2 specified for the order-zero FPM. Adapted from [4] with permission.

The zero η_1 disappears if the pole $\kappa_1 = \zeta$ is prescribed. Furthermore, the pole and zero cancel if the pole

$$\kappa_1 = 1 - \sqrt{2} \approx -0.414 \quad (4.68)$$

is prescribed. Both the pole and zero are plotted, as functions of the eigenvalue λ_2 selected for the order-zero FPM, in Fig. 4.8.6.

The PSDs of the maps that were constructed with the pole κ_1 prescribed from the set of (4.65) are plotted in Figs. 4.8.7 and 4.8.8. In these figures, solid curves represent the PSD predicted by theory, as obtained by evaluating (2.152). The dots in the figures represent values computed numerically to verify the theoretical results. The numerical results were generated by computing a trajectory of length 5×10^6 states, and then computing the (discrete-time) short-time Fourier transform of the trajectory. In this approach, the trajectory is divided into shorter non-overlapping segments of 500 states each, and the discrete-time Fourier transform of each segment is computed. The squared magnitudes of the resulting frequency-domain representations of the segments are then averaged to obtain an estimate of the PSD associated with the trajectory.

Figs. 4.8.7 and 4.8.8 reveal that the PSDs computed numerically closely approximate

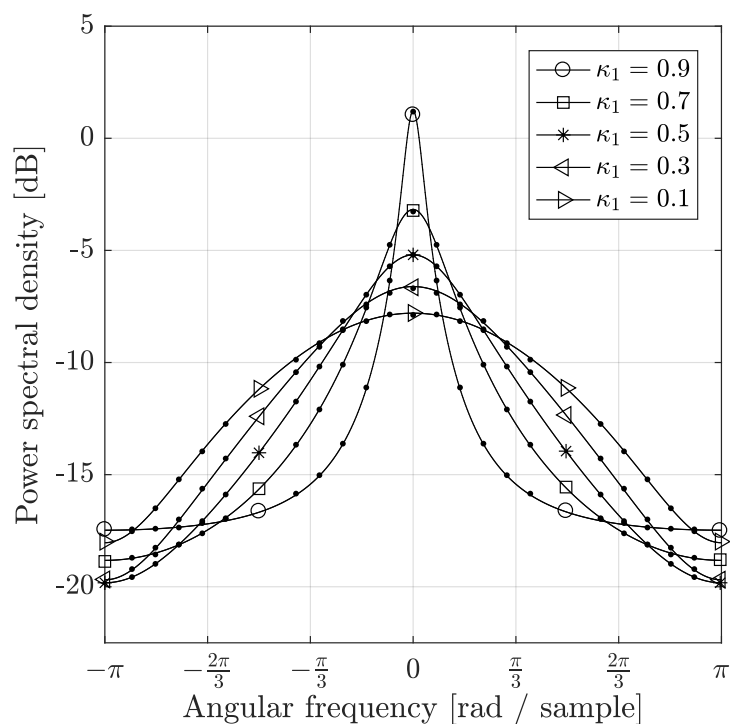


Figure 4.8.7: PSDs of the maps from the first map set and with positive pole κ_1 . Adapted from [4] with permission.

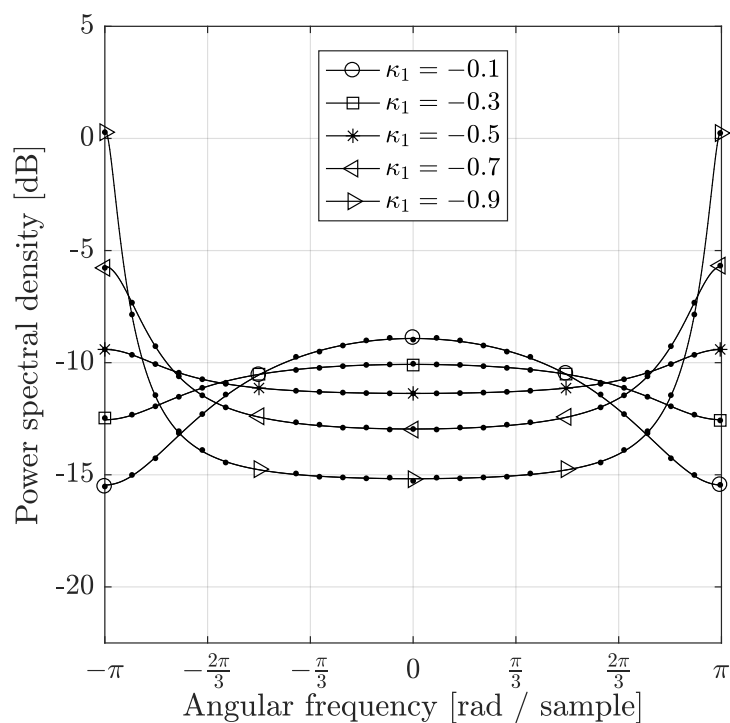


Figure 4.8.8: PSDs of the maps from the first map set and with negative pole κ_1 . Adapted from [4] with permission.

the PSDs predicted by theory. This verifies that the poles realised by the maps agree with those prescribed during the design of the map. These results also verify (2.152), which was derived from first principles in section 2.6.5.

Recall that the selection of a positive or negative pole κ_1 leads to the realisation of a spectral mode with centre frequency equal to zero or π radians per sample, respectively. The figures reveal that the spectral modes associated with poles $\kappa_1 > 1 - \sqrt{2} \approx -0.414$ appear as peaks at zero radians per sample in the PSDs. This coincides with the zero η_1 appearing to the left-hand side of the pole κ_1 , on the real line in the z -plane, for these values of κ_2 (as implied by Fig. 4.8.6). As progressively smaller poles $\kappa_1 > 1 - \sqrt{2}$ are selected, the peak is lowered and its bandwidth becomes wider.

Selecting $\kappa_1 = 1 - \sqrt{2} \approx -0.414$ causes the pole and zero to cancel each other, thereby leading to a uniform (flat) PSD. The maps realised by prescribing poles $\kappa_1 < 1 - \sqrt{2}$ possess PSDs with a peak centred at π radians per sample, which equals the centre frequency of the spectral mode associated with κ_1 . The appearance of this peak coincides with the zero η_1 now appearing to the right-hand side of the pole κ_1 on the real line in the z -plane. As $\kappa_1 < 1 - \sqrt{2}$ becomes progressively more negative, the peak is elevated and its bandwidth becomes narrower.

Example 4.8.2 (Second map set: Complex conjugate pair of non-real poles).

Prescribed statistical metrics Each map of the second set was designed to possess a complex conjugate pair of non-real poles κ_1 and κ_2 . The novel disaggregation technique of the StIEP solution was used in each case to construct a stochastic matrix with eigenvalues equal to the prescribed poles. Hence, the region of permissible poles coincided with the region of permissible eigenvalues of the disaggregation technique; refer to Fig. 4.8.9 and section 4.6.2. Specific pairs of poles were selected from this region of permissible values¹⁶ by defining two loci, which correspond to the concentric blue and red triangles in the figure.

The points on the loci denoted by blue and red markers (i.e., on the top half of each triangle) were prescribed as the poles κ_1 ; these poles are listed in Table 4.8.1. The markers also lie on radial lines of the unit circle that form angles of $k\pi/6$ radians with the real line, where $k = 1, 2, \dots, 5$. This implies that the poles κ_1 denoted by

¹⁶All pairs of strictly complex eigenvalues that may belong to a 3×3 doubly stochastic matrix, and where the eigenvalue modulus is strictly less than unity, coincide with this region [25, Corollary 3.4], [41]. Hence, the novel StIEP solution *does not* restrict the set of possible poles that may be prescribed in the first round of disaggregation.

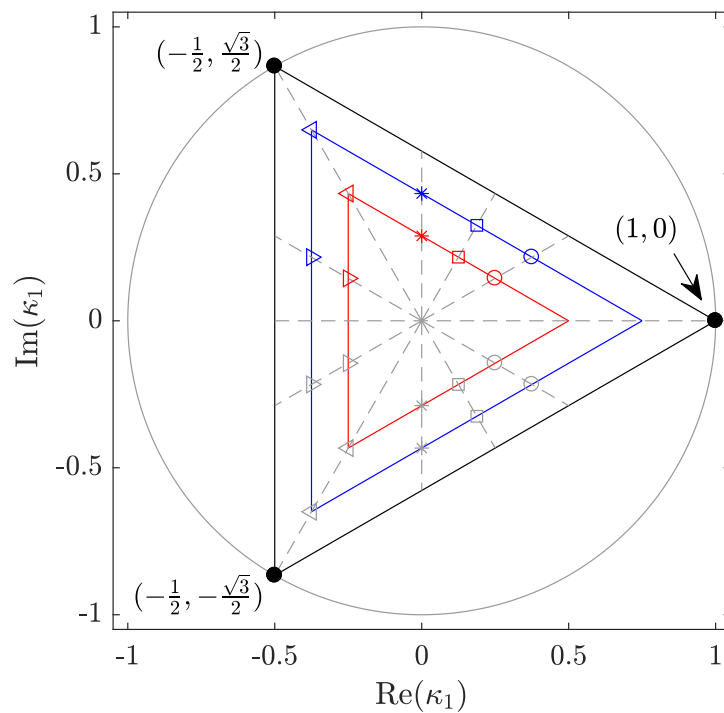


Figure 4.8.9: The region of permissible poles κ_1 that may be prescribed for the maps of the second set (i.e., the interior of the black triangle), where blue and red markers denote the poles prescribed in the example. Adapted from [4] with permission.

Table 4.8.1: Poles κ_1 denoted by the markers in Fig. 4.8.9. Reproduced from [4] with permission.

Marker					
$r = \kappa_1 $	0.433	0.375	0.433	0.750	0.433
$\theta = \arg(\kappa_1)$	$\pi/6$	$2\pi/6$	$3\pi/6$	$4\pi/6$	$5\pi/6$

Marker					
$r = \kappa_1 $	0.289	0.250	0.289	0.500	0.289
$\theta = \arg(\kappa_1)$	$\pi/6$	$2\pi/6$	$3\pi/6$	$4\pi/6$	$5\pi/6$

the markers satisfy $\arg(\kappa_1) = k\pi/6$. The corresponding poles κ_2 are denoted by grey markers. The set of all poles represented by the markers of Fig. 4.8.9, regardless of the marker's colour, is denoted by Λ .

Map design Refer to section 4.B.2 for details on the steps that were followed to design the maps presented in this example.

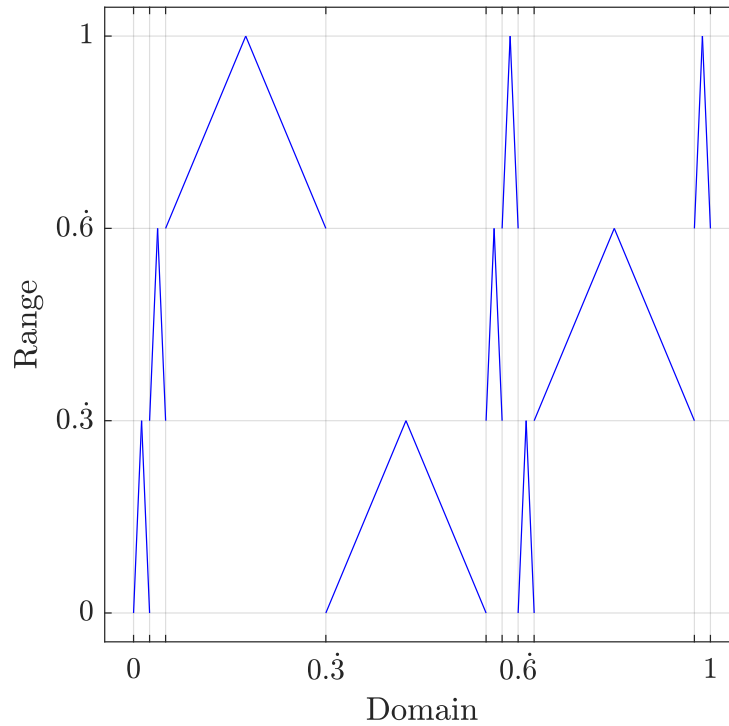


Figure 4.8.10: Plot of a hat map from the second set, as realised by prescribing the eigenvalue $\kappa_1 = 0.75 \exp(i4\pi/6)$. Adapted from [4] with permission.

Map structure and trajectories In this section, the structure and trajectories of two hat maps from the second set that possess more pronounced spectral characteristics (specifically, PSDs with sharper peaks) than the remaining maps are investigated. The hat maps designed by prescribing the poles $\kappa_1 = 0.75 \exp(i4\pi/6)$ and $\kappa_1 = 0.5 \exp(i4\pi/6)$, which are located on the blue and red triangles of Fig. 4.8.9, are plotted in Figs. 4.8.10 and 4.8.11, respectively. Each map possesses nine hats, where three of the hats have equal but significantly wider support than the remaining hats. The hats are distinguished by labelling them as *wider* or *narrower*.

The wider hats of each map plotted in Figs 4.8.10 and 4.8.11 maps states between intervals in the repeating pattern

$$Q_1 \rightarrow Q_3 \rightarrow Q_2 \rightarrow Q_1 \rightarrow \dots \quad (4.69)$$

from one time instant to the next. This characteristic is discernible in trajectories generated by these maps. Figures 4.8.12 and 4.8.13 are plots of trajectories generated by the maps plotted in Figs 4.8.10 and 4.8.11, respectively. The horizontal grid lines in Figs. 4.8.12 and 4.8.13 coincide with the domain endpoints of the hats belonging to the maps. The figures reveal that these trajectories frequently enter period-three cycles that are occasionally interrupted by a run of one or more irregular state transitions. These cycles coincide with the pattern of (4.69). The first irregular state

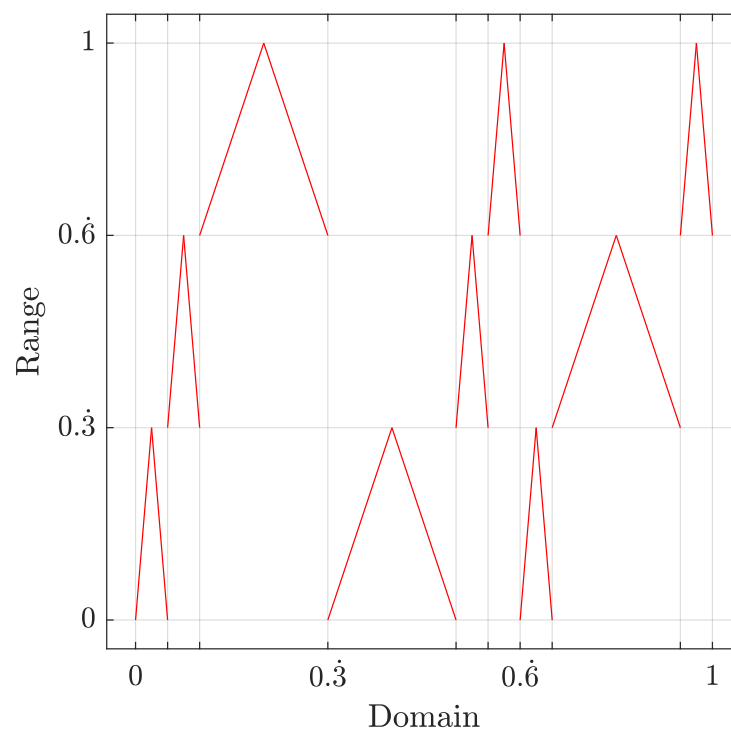


Figure 4.8.11: Plot of a hat map from the second set, as realised by prescribing the eigenvalue $\kappa_1 = 0.5 \exp(i4\pi/6)$.

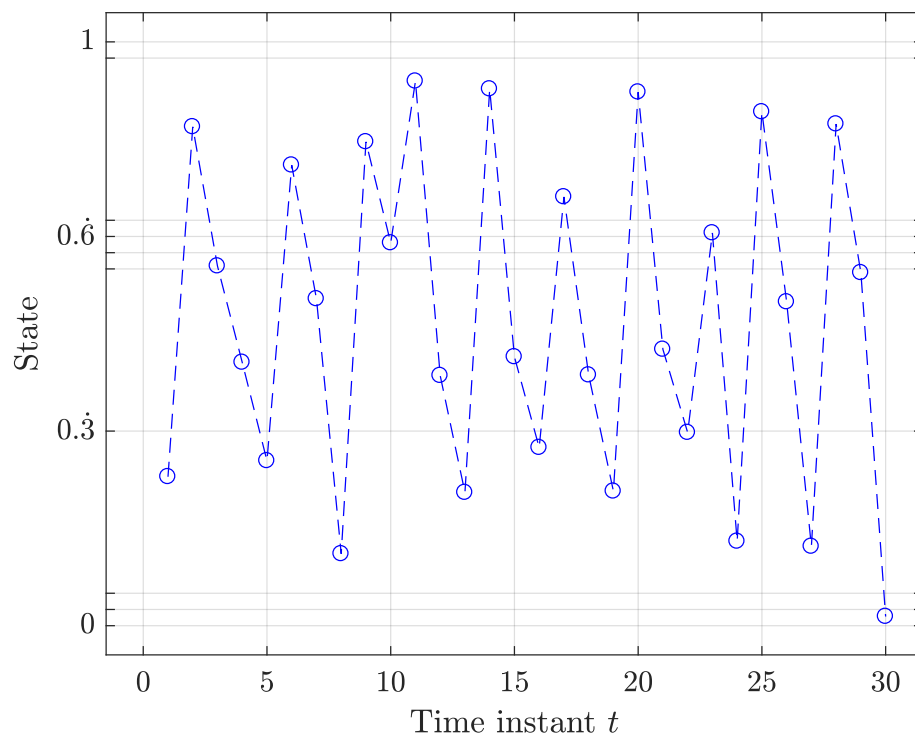


Figure 4.8.12: Plot of a trajectory generated by the map of Fig. 4.8.10 (as designed by prescribing the pole $\kappa_1 = 0.75 \exp(i4\pi/6)$).

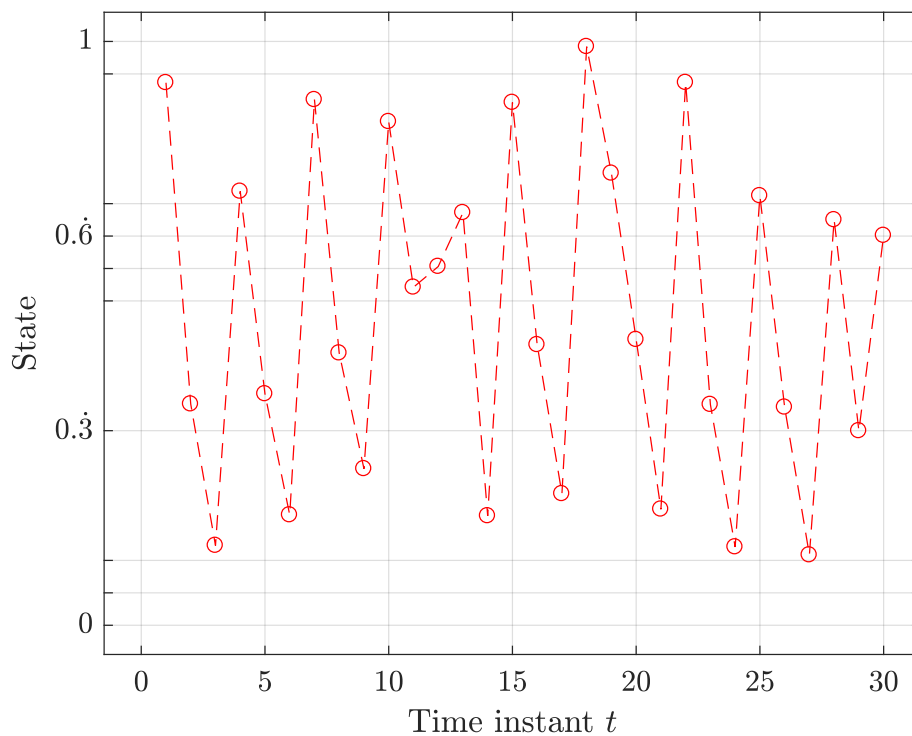


Figure 4.8.13: Plot of a trajectory generated by the map of Fig. 4.8.11 (as designed by prescribing the pole $\kappa_1 = 0.5 \exp(i4\pi/6)$).

transition in each run occurs immediately after the state is mapped to the domain of a narrower hat, thereby interrupting the pattern.

The above observations reveal a relationship between the map's structure and the characteristics of its PSD. Using the same argument as the preceding example, it is found that the dominant behaviour of the map's trajectories in the stationary regime is to move between the intervals in the repeating pattern of (4.69). This yields more frequent period-three cycles in these trajectories, which gives rise to observed peaks in the PSD with centre frequencies of $\omega = \pm 4\pi/6$ radians per sample (the PSDs of these maps are plotted in Figs. 4.8.15 and 4.8.16 and characterised using a pole-zero plots towards the end of the example). These frequencies coincide with the centre frequencies of the spectral modes associated with the poles κ_1 and κ_2 .

Invariant density The maps of the second set were designed to possess the uniform invariant density. As in the preceding example, the invariant density of each map in the set was verified by numerically computing a trajectory of length 10^6 states and then constructing a histogram of the states. Each histogram was normalised to obtain an estimate $\hat{f}_S^*$ of the map's invariant density. These estimates were compared against the prescribed uniform invariant density.

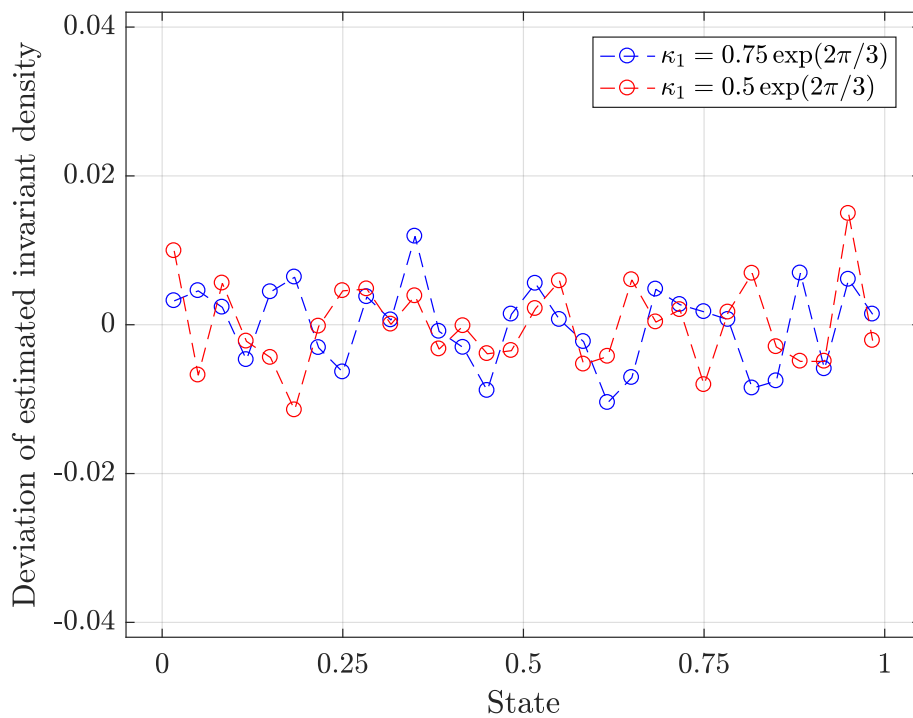


Figure 4.8.14: Deviation of the invariant density function estimates, as compared to the prescribed invariant density, of maps realised by prescribing the poles in the legend. The estimates were constructed from map trajectories.

Figure 4.8.14 is a plot of the deviation $\hat{f}_S^* - f_P$ between the estimated invariant density and the prescribed uniform invariant density for the maps plotted in Figs. 4.8.10 and 4.8.11, where the circle markers correspond to the centres of the histogram bins. Similar to the preceding example, the figure reveals that the estimated invariant densities closely approximate the prescribed density. The deviation observed in the figure is attributed to the finite number of states used to construct the histogram.

Power spectral density The PSDs of the maps in the second set are plotted in Figs. 4.8.15 and 4.8.16. These figures coincide with the maps designed by prescribing the pole κ_1 denoted by the blue and red markers of Fig. 4.8.9, respectively, together with its complex conjugate. In these figures, solid curves represent the PSD predicted by theory, as obtained by evaluating (2.152). The dots represent values computed numerically as verification. The PSDs computed to verify the theoretical expressions were again obtained by computing the (discrete-time) short-time Fourier transform of a numerically computed map trajectory, using the identical method as outlined in the previous example.

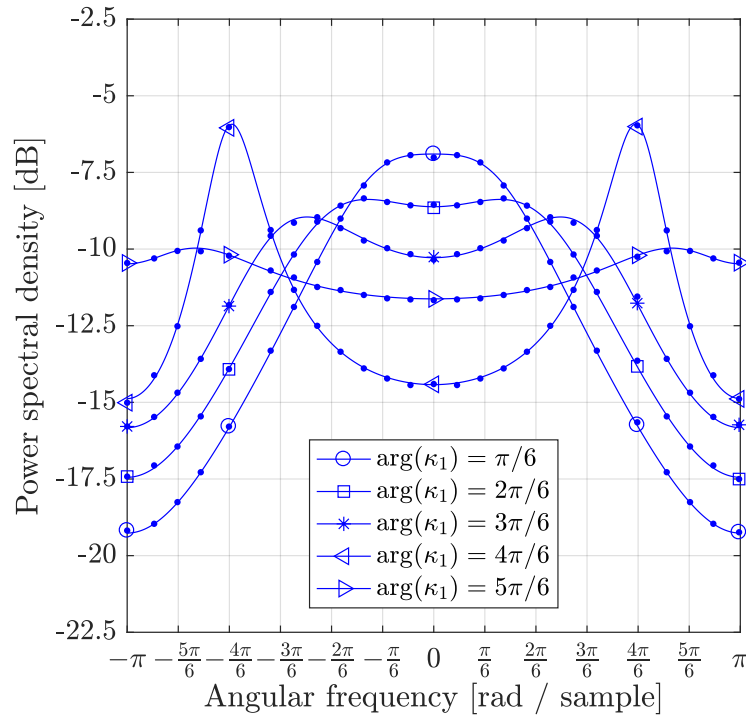


Figure 4.8.15: PSDs of the maps from the second set and with poles denoted by the blue markers in Fig. 4.8.9. Adapted from [4] with permission.

Figs. 4.8.15 and 4.8.16 reveal that the PSDs computed numerically closely approximate the PSDs predicted by theory. This verifies that the poles realised by the maps agree with those prescribed during the design of the map. The results again verify (2.152), which was derived from first principles in section 2.6.5.

The poles and zeros of the PSDs plotted in Figs. 4.8.15 and 4.8.16 are investigated to aid the interpretation of the power spectra. Let $\Psi_S(z) = K_S H_S(z) H_S(z^{-1})$ denote the PSD of a hat map S from the second map set, as expressed in the z -domain; refer to (2.154). Equation (2.155) and Proposition 4.6.3 imply that the PSD of each map in the second set has two poles and at most two zeros. An investigation revealed that the power spectrum of each map constructed in the example has two zeros. Hence,

$$H_S(z) = \frac{(z - \eta_1)(z - \eta_2)}{(z - \kappa_1)(z - \kappa_1^*)}, \quad \kappa_1 \in \Lambda. \quad (4.70)$$

The pole-zero plot associated with each PSD plotted in Figs. 4.8.15 and 4.8.16 are presented in Figs. 4.8.17(a) to 4.8.17(j).

Recall that each pole yields a spectral mode with centre frequency equal to the pole's argument. Since κ_2 is predetermined as the complex conjugate of κ_1 during Markov state disaggregation, two spectral modes with centre frequencies of $\pm \arg(\kappa_1)$ radians per sample are realised in each map's PSD.

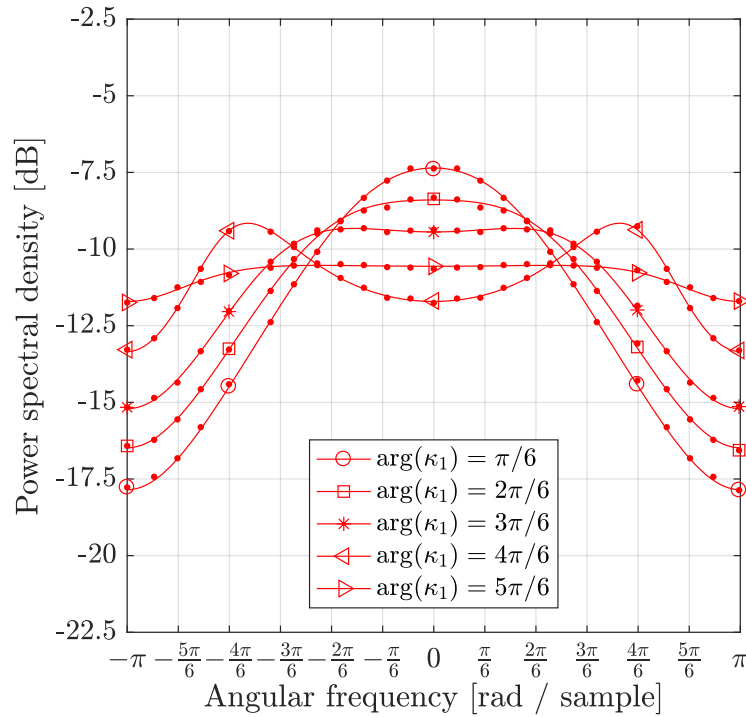


Figure 4.8.16: PSDs of the maps from the second set and with poles denoted by the red markers in Fig. 4.8.9. Adapted from [4] with permission.

The two spectral modes associated with the selection of $\arg(\kappa_1) = \pi/6$ appear as a single peak at zero radians per sample in the corresponding PSD of Fig. 4.8.15. This PSD is associated with the pole-zero plot of Fig. 4.8.17(a). As $\arg(\kappa_1)$ increases from $2\pi/6$ to $4\pi/6$, Figs. 4.8.17(b) to 4.8.17(d) reveal that the poles κ_1 and κ_2 move apart and closer to the unit circle. The spectral modes associated with κ_1 and κ_2 are thus distinguishable as separate peaks in the corresponding PSDs of Fig. 4.8.15. The poles reach their closest point to the unit circle when $\arg(\kappa_1) = 4\pi/6$, which corresponds with the PSD having two elevated and narrow peaks at frequencies of approximately $\pm 4\pi/6$ radians per sample. The PSD associated with $\arg(\kappa_1) = 5\pi/6$ and appearing in Fig. 4.8.15 has significantly lower and wider peaks when compared to the preceding PSDs. Fig. 4.8.17(e) reveals that, in this case, the poles and zeros are closer to one another in the z -plane, relative to their distance from the unit circle.

The PSDs of the maps realised by prescribing poles κ_1 denoted by the red markers of Fig. 4.8.9 are plotted in Fig. 4.8.16. The figure reveals that these PSDs have similar characteristics to those plotted in Fig. 4.8.15, except that the power ratio between the peak(s) and the valley of each PSD is significantly smaller and the bandwidths of the peaks appear to be wider. Figs. 4.8.17(f) to 4.8.17(j) imply that this is due to each PSD's poles and zeros being located closer to one another and further from the unit circle in the z -plane, as compared to the pole-zero plots of Figs. 4.8.17(a) to

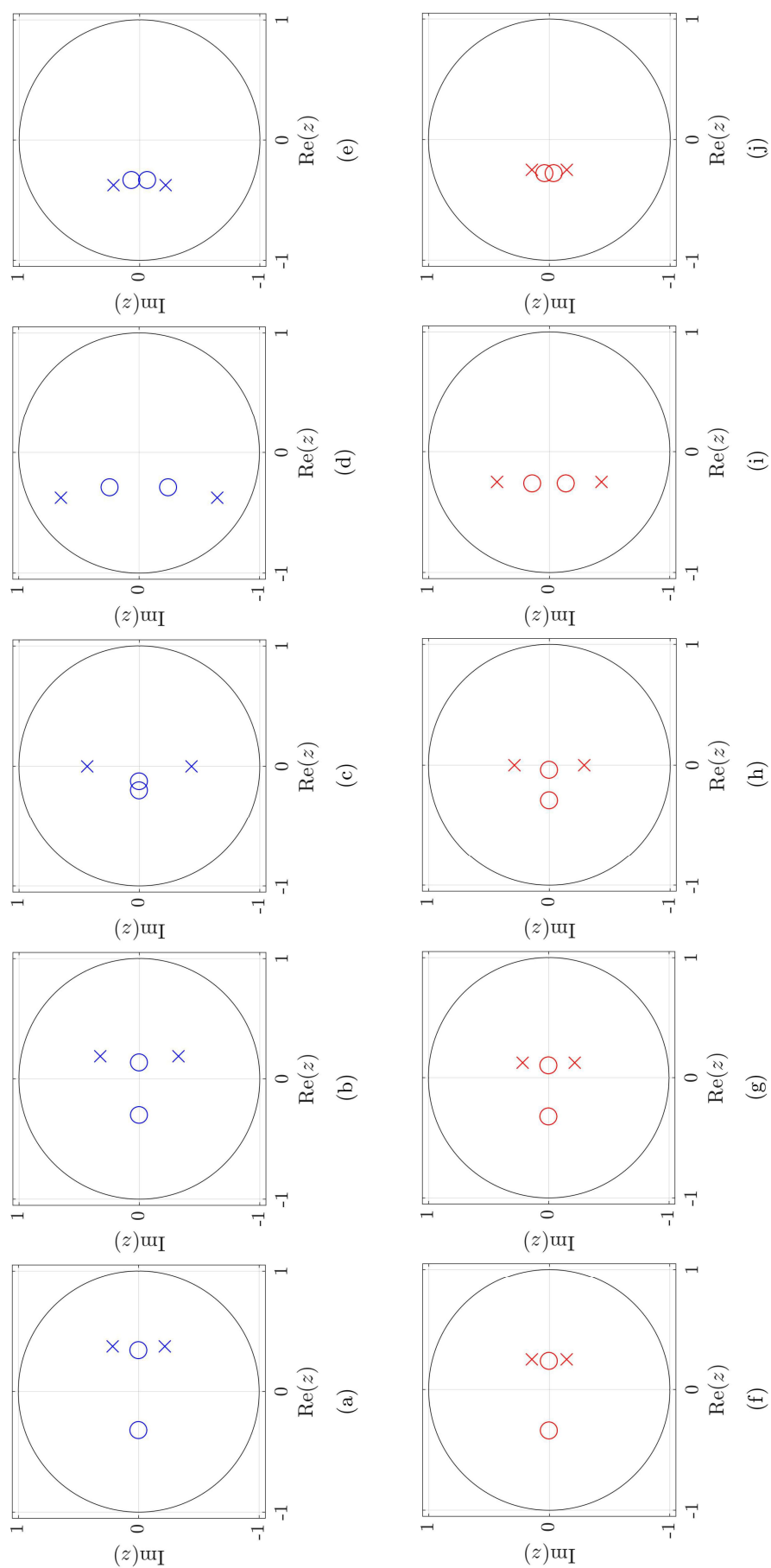


Figure 4.8.17: Pole-zero plots associated with the maps of the second set. Plots (a) to (e) and plots (f) to (j) are associated with the pole κ_1 denoted by the blue and red markers of Fig. 4.8.9, respectively. From left to right, the pole-zero plots of each row correspond to $\arg(\kappa_1) = \pi/6, 2\pi/6, \dots, 5\pi/6$. Reproduced from [4] with permission.

4.8.17(e).

Remark 4.8.1. Example 4.8.2 above demonstrates that the novel method permits the independent adjustment of the centre frequencies and bandwidths of the power spectrum modes. In particular, by independently selecting the argument and the modulus of the poles prescribed during the design of the map, the centre frequencies of the corresponding spectral modes may be selected and their bandwidths adjusted. As the bandwidths of these modes become narrower (i.e., as the poles approach the unit circle), they appear as peaks in the power spectrum at the mode's centre frequency.

Remark 4.8.2. Considering that power spectrum poles from anywhere inside the unit circle may be prescribed, the region of permissible poles for the novel method, as indicated in Fig. 4.8.9, is restrictive. Any pair of complex conjugate eigenvalues realisable by a 3×3 stochastic matrix falls within this region. This implies that the structure of the order-zero FPM restricts the poles prescribable using solution strategy 2, and thus the novel method. By extension, this restricts the novel method's ability to realise sharp peaks in the power spectrum at arbitrary frequencies.

4.8.2 More sophisticated maps

The example of this section presents results for more sophisticated maps that were designed using the novel method for solving IFPP-IIa. These maps were designed to possess a non-uniform invariant density and a rational PSD with three poles. The example is followed by an investigation of existing analytical methods aimed at solving IFPP-II and their potential to realise maps with the same statistical metrics as those constructed in the example.

Example 4.8.3 (Beta invariant density and three power spectrum poles).

Prescribed statistical metrics The beta PDF with parameters $a = 3/2$ and $b = 5/2$ was prescribed for the invariant density of each map. This density function is defined in (4.17) and plotted in Fig. 4.8.18 (black curve).

Three poles were prescribed for the power spectra of the maps:

$$\begin{aligned}\kappa_1 &= 3/4, \\ \kappa_2 &= \kappa_3^* = \tilde{r} \exp(i2\pi/3).\end{aligned}\tag{4.71}$$

The parameter values

$$\tilde{r} \in \{0.3, 0.5, 0.7, 0.9\}\tag{4.72}$$

were prescribed in order to realise distinct maps.

Map design Refer to section 4.B.3 for details on the steps that were followed to design the maps presented in this example.

Map structure Figure 4.8.19 is a plot of the hat map realised by prescribing the parameter value $\tilde{r} = 0.7$. The map possesses sixteen hats, where four of the hats have significantly wider support than the remaining hats. It is observed that the three wider hats with domains contained by the intervals Q_1 to Q_3 (i.e., the three leftmost wider hats in Fig. 4.8.19) map states between intervals in the repeating pattern

$$Q_1 \rightarrow Q_3 \rightarrow Q_2 \rightarrow Q_1 \dots \quad (4.73)$$

The period of this pattern is equal to three time instants. Using the same line of reasoning as presented in Examples 4.8.1 and 4.8.2, it is found that this pattern of state transitions gives rise to two spectral modes centred at $\omega = \pm 2\pi/3$ radians per sample. These frequencies coincide with the arguments of the prescribed poles κ_2 and κ_3 (the PSD of this map is plotted in Fig. 4.8.20).

In contrast to the leftmost wider hats in Fig. 4.8.19, the rightmost wider hat (which has a domain contained by interval Q_4) maps states back to the same interval Q_4 . This gives rise to a spectral mode centred at zero radians per sample.

Invariant density The invariant density shared by all the maps in this example is plotted in Fig. 4.8.18 (blue curve). The maps realise an invariant density that is a PWC approximation $\hat{f}_P$ of the prescribed invariant density f_P over the Markov partition. The approximating invariant density $\hat{f}_P$ is determined such that the areas under both densities $\hat{f}_P$ and f_P , when restricted to each interval of the partition, are equal. Whereas this approach does yield a valid approximation of the prescribed invariant density, the approximation in the example is not accurate due to the small number of intervals in the Markov partition (recall that the number of intervals coincide with the dimensionality of the stochastic matrix designed in the first step).

Similar to Examples 4.8.1 and 4.8.2, the invariant density of each map in the current example was verified by numerically computing a trajectory of length 10^6 states and then constructing a histogram of the states. The histogram bins were selected such that no bin contains a partition point of the map's Markov partition. Each histogram was then normalised to obtain an estimate $\hat{f}_S^*$ of the map's invariant density.

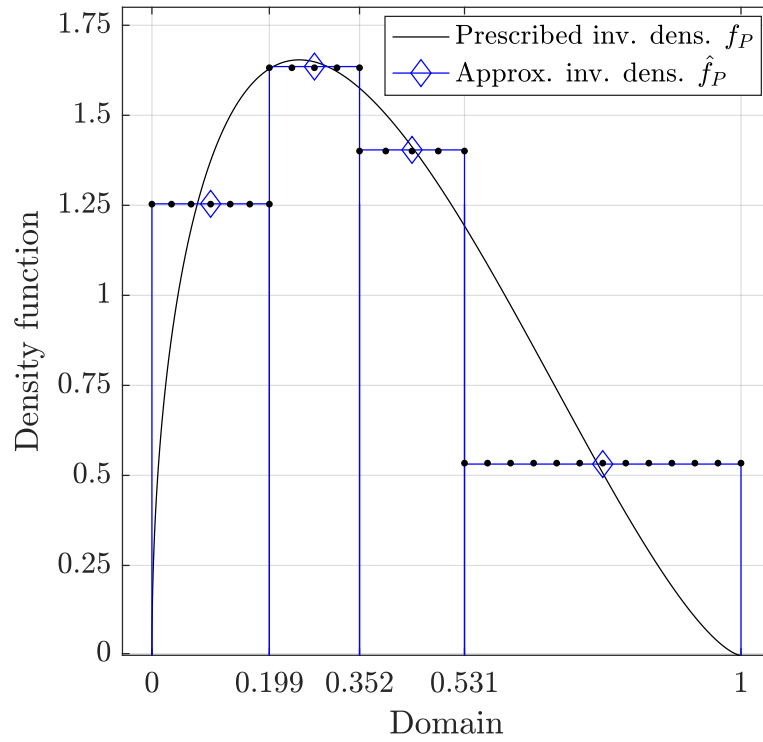


Figure 4.8.18: Plots of the prescribed invariant density f_P (black) and the approximating invariant density $\hat{f}_P$ realised by the more sophisticated maps (blue). Adapted from [4] with permission.

The density function estimate $\hat{f}_S^*$ obtained for the map that was realised by prescribing parameter $\tilde{r} \approx 0.7$ is superimposed on the densities of Fig. 4.8.18 as a sequence of dots. It is observed that the density function estimate closely approximates the density $\hat{f}_P$ that the map was designed to realise as its invariant density. This observation supports the claim that the method for designing the map correctly realises the required PWC invariant density $\hat{f}_P$.

Power spectral density The PSDs of the maps realised by prescribing the parameter values $\tilde{r}$ of (4.72) are plotted in Fig. 4.8.20. In this figure, solid curves represent the PSDs predicted by theory and as obtained by evaluating (2.152). The dots represent values computed numerically as verification. The PSDs computed to verify the theoretical expressions were again derived by computing the (discrete-time) short-time Fourier transform of a numerically generated map trajectory, using the identical method as outlined in Examples 4.8.1 and 4.8.2.

Similar to the previous examples, Fig. 4.8.20 reveals that the PSDs computed numerically closely approximate the PSDs predicted by theory. This verifies that the poles realised by the maps agree with those prescribed during the design of the map.

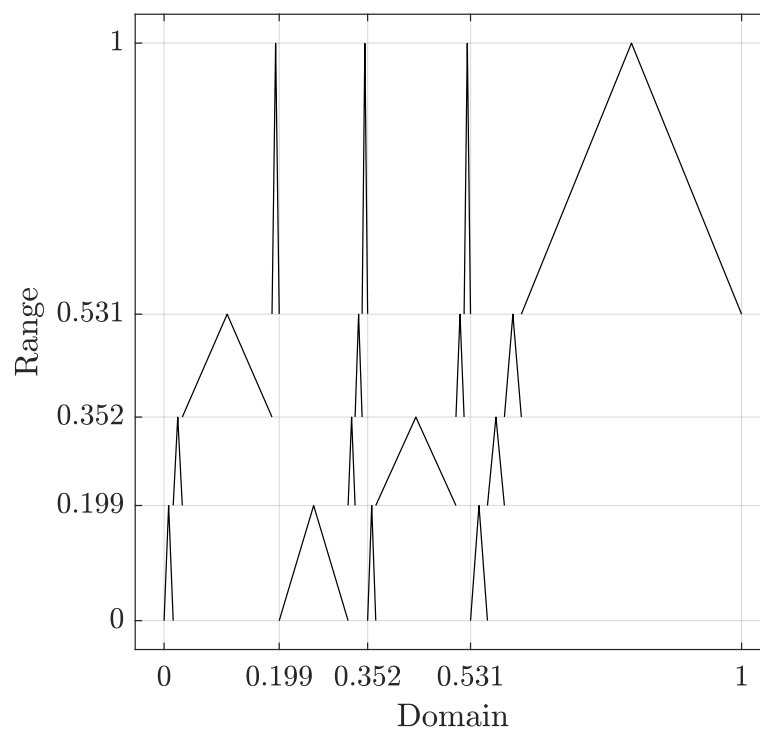


Figure 4.8.19: Plot of the hat map realised by prescribing the parameter $\tilde{r} = 0.7$. Reproduced from [4] with permission.

Figure 4.8.20 reveals that the prescribed poles yield three spectral modes with centre frequencies $\arg(\kappa_1) = 0$ and $\arg(\kappa_{2,3}) = \pm 2\pi/3$ radians per sample. As $\tilde{r}$ increases, these spectral modes begin to appear as peaks in the PSD at their centre frequencies. These peaks are elevated as progressively larger values of $\tilde{r}$ are selected, due to the poles moving closer to the unit circle in the z -plane.

Remark 4.8.3. The PWC density $\hat{f}_P$ that the maps in Example 4.8.3 were designed to realise as their invariant density is not a close approximation of the prescribed invariant density f_P , due to the Markov partition only possessing four intervals. Increasing the number of Markov partition intervals holds potential to improve the accuracy of the approximation. This statement is supported by the results reported in Example 4.6.1.

The dimensionality of the stochastic matrix constructed in the first solution step coincides with the number of intervals of the Markov partition. Although the novel solution to the StIEP permits the realisation (in theory) of a stochastic matrix with arbitrary dimensionality, this implies the insertion of further eigenvalues into the eigenspectrum of the matrix. To prevent these eigenvalues from manifesting as poles in the map's PSD, they may be selected to equal zero. This approach is to be investigated in future work.

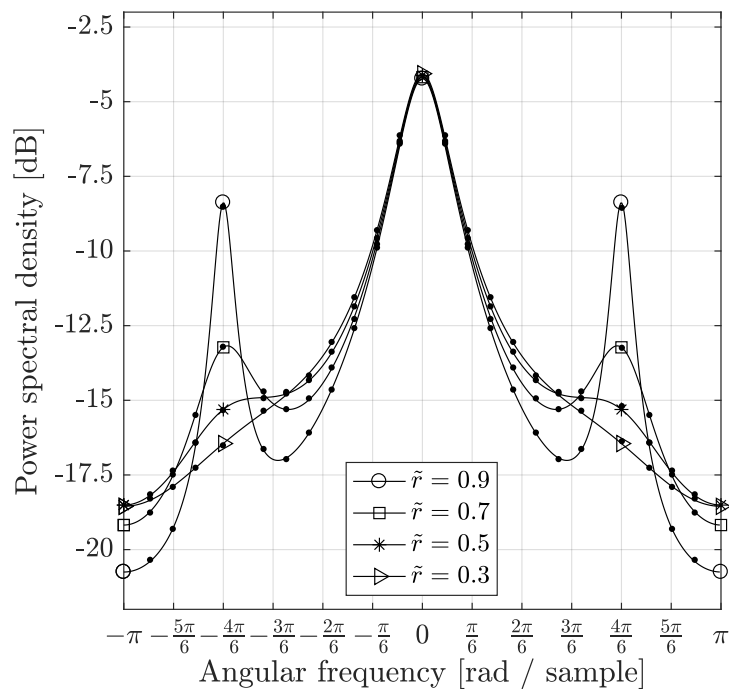


Figure 4.8.20: PSDs of the more sophisticated maps. Adapted from [4] with permission.

Remark 4.8.4. Example 4.8.3 demonstrated the novel method’s ability to realise a power spectrum with multiple peaks, and to adjust the bandwidths of the peaks. However, the poles selected for the example are aligned with the vertices of the triangle in the region of permissible poles (refer to Fig. 4.8.9), which implies that the poles may be selected to lie close to the unit circle and thereby realise *sharp* peaks at the corresponding frequencies. The novel method does not permit realisation of sharp peaks at arbitrary frequencies.

In the novel solution to the StIEP presented in section 4.6.2, it is observed that the region in the complex plane from which eigenvalues may be selected for insertion into the transition matrix progressively contracts around zero in each consecutive round of Markov state disaggregation. This implies that the permissible region for each consecutive pole also contracts, which restricts the number of sharp spectral peaks that may be realised in the power spectrum.

Existing analytical methods and their application

This section investigates the application of existing analytical methods¹⁷ to design maps that share the prescribed beta invariant density of (4.17) and possess rational PSDs with three poles. The poles are to be selected according to (4.71) for the maps. These maps are referred to as the *required maps* in what follows.

The method developed in [3] permits the construction of topological conjugates of complete and piecewise monotonic maps. This method grants the freedom to select the conjugate map's single spectral mode [4]. As these maps possess a power spectrum with at most a single pole, the method is disqualified from realising the required maps. However, in contrast to the novel method for solving IFPP-IIa, the method of [3] designs a map with an invariant density that is *equal* to the prescribed beta density of (4.17). The latter method is more versatile in this regard.

The method proposed in [18] by the author of this thesis may be used to construct maps that possess a rational PSD with at most $2M - 1$ poles, of which at most $M - 1$ are prescribable as real values (here, $M > 1$ is a parameter that may be selected) [4]. As the required maps possess PSDs with complex-valued poles, this method is also disqualified. However, similar to the novel method proposed in this paper, the method of [18] permits the PWC approximation of the prescribed invariant density.

The method of [19] constructs maps with rational power spectra that possess complex-valued poles [4]. However, it only permits construction of maps that possess at most five poles, of which at most two are prescribable. As the required maps possess three prescribed poles, the method is disqualified. Also, the method can only realise a uniform invariant density.

The extension of the matrix method of [7–10] for solving IFPP-I, as described in section 3.5.3, is considered [4]. It was shown in [9] that the eigenvalues of this matrix are restricted to being real-valued and strictly positive. Hence, any method that uses this matrix as a candidate order-zero FPM cannot construct the required maps, regardless of the particulars of the method.

¹⁷Two of the methods developed by the author of this thesis as preliminary work are included in this comparison [18, 19].

4.9 Summary

This chapter addresses IFPP-II, which requires the design of an ergodic map to simultaneously possess a prescribed invariant density and a prescribed ACF or PSD. An incremental approach to solving IFPP-II, which involves the definition of a relaxed problem formulation (IFPP-IIa), is adopted. This relaxed problem requires the design of an ergodic map to possess an invariant density that approximates a prescribed invariant density, as well as a rational PSD with prescribed poles.

A novel method that solves IFPP-IIa by designing PWL semi-Markov maps is developed. The choice of the semi-Markov maps is motivated by the comparatively unsophisticated relationships that exist between the statistical metrics and structural parameters of these maps. However, there exists a trade-off in that these maps can only permit approximation of arbitrary continuous functions that may be prescribed for the invariant density and PSD.

Two novel strategies for designing semi-Markov maps to solve IFPP-IIa are presented. Both these strategies design a candidate FPM that yields the prescribed statistical metrics, and then design the map to satisfy this matrix. However, the two strategies differ with respect to the order of the FPM prescribed for the map. Due to the complexity of qualifying a candidate matrix as a valid order-one FPM of a semi-Markov map, and the difficulty involved in designing the map to satisfy such a matrix, the alternative strategy of designing a candidate order-zero FPM is pursued. In this approach, additional design constraints are imposed on the matrix and the map.

Three design issues are identified for any method that seeks to implement the novel solution strategy for IFPP-IIa. The first and second design issues concern the difficulty of constructing an order-zero FPM to simultaneously possess the eigenspectrum and Perron eigenvectors required to yield the prescribed statistical metrics. The third issue concerns the selection of the map's structural features to prevent unwanted power spectrum poles. The latter step is necessitated by the property that the map's poles coincide with the eigenvalues of its *order-one* FPM, which is implicitly determined when a particular *order-zero* FPM and map structure are selected.

Three novel contributions that resolve the above design issues are presented. The first and second design issues are resolved by relaxing the specifications of the order-zero FPM in a manner that still permits the approximation of the prescribed invariant density, and by developing a novel analytical solution to the problem of constructing a stochastic matrix with a prescribed eigenspectrum (in the context of the solution, a

subset of the map's poles coincides with the eigenspectrum of this matrix). The third issue is resolved by introducing a novel subclass of semi-Markov maps with a hat-like branch structure, referred to as the hat maps. It is proved that the symmetry of these branch functions prevents unwanted poles from appearing in the map's power spectrum, thereby permitting full prescription of all the power spectrum poles. This is accomplished by specifying the eigenspectrum of the stochastic matrix.

A novel method for solving IFPP-IIa that implements the solution strategy and utilises the above contributions to resolve the design issues is proposed. This novel method realises an ergodic hat map with a PWC invariant density that approximates a prescribed invariant density, and permits the prescription of complex-valued and non-repeating power spectrum poles. Conditions are derived for the ergodicity of maps constructed using the novel method.

Several examples of hat maps designed using the novel method are presented. These maps illustrate the characteristics of the novel method and were designed to possess prescribed statistical metrics. Numerical experimentation verified that the statistical metrics of the maps are equal to those prescribed during the design step in each case.

Multiple elementary maps were designed to possess the uniform invariant density and power spectra with a maximum of two poles. These examples reveal the emergence of peaks in the power spectrum that coincide with the prescribed poles. It is shown that the centre frequencies of the peaks and their bandwidths may be adjusted. This is accomplished by independently prescribing the arguments and moduli of the poles. However, the restrictiveness of the region in the complex plane from which the poles may be prescribed limits the ability to realise sharp peaks at arbitrary frequencies.

A deeper investigation of the elementary maps constructed using the novel method reveals a relationship between the structural features of the maps and the characteristics of their power spectrum peaks. It is conjectured and then numerically verified that the structure of the hat maps give rise to cycles in their trajectories, which contributes to the realisation of peaks in the power spectrum. By identifying this relationship, the mechanism whereby a peak emerges at a particular centre frequency and with a particular bandwidth is revealed.

The novel method was also applied to construct more sophisticated maps. A non-uniform beta invariant density and three power spectrum poles were prescribed for these maps. It is shown that the novel method yielded maps with an invariant density that approximates the prescribed density over a four-interval partition. Although the approximation is not close, supplementary results suggest that the approximation may

be improved by increasing the dimensionality of the order-zero FPM during the design step, thereby yielding an approximation over a finer partition of intervals. A design technique that accomplishing this while simultaneously preventing additional poles in the power spectrum is proposed but not investigated further (this is considered a topic for future work).

The power spectra of the more sophisticated maps revealed the emergence of three peaks that coincide with power spectrum poles that are selected to approach the unit circle in the complex plane. This demonstrates the novel method's ability to realise power spectra with multiple adjustable peaks. However, the method's capacity to realise sharp peaks at arbitrary frequencies is limited. The capacity to realise sharp peaks also diminishes as the number of poles prescribed for the map grows.

The chapter concludes by comparing existing analytical methods [3, 7–10, 18, 19] to the novel method. The potential of these methods to construct maps with the same statistical metrics than those successfully constructed using the novel method is investigated. Although the topological conjugation method [3] permits exact realisation of the invariant density (and is thus more versatile than the novel method in this regard), none of the existing methods are able to realise a rational power spectrum with one real-valued pole and a complex-conjugate pair of non-real poles. Hence, none of these methods may be used to design maps with the same statistical metrics as the maps constructed using the novel method.

4.10 Conclusions

The following conclusions are drawn regarding the novel method for solving IFPP-IIa.

- The design issues that arise when generalising the existing matrix strategy for solving IFPP-I to permit the prescription of power spectrum poles may be overcome by using the novel contributions presented in this chapter.
- The novel method that implements the proposed solution strategy successfully realises the prescribed statistical metrics, as verified by numerical simulations.
- Unlike existing numerical methods, the proposed method has negligible computational complexity and is purely analytical, which bestows immunity to possible non-convergence. The proposed method also improves upon the capabilities of existing analytical methods by permitting the prescription of an

arbitrary number of power spectrum poles that may be selected as *complex-valued* numbers. However, unlike methods based on topological conjugation, it cannot realise the prescribed invariant density in an exact fashion.

- The restricted nature of the eigenvalues realisable for any stochastic matrix limits the power spectra that may be successfully realised using the novel method. However, the method permits the design of power spectra with multiple peaks, such that the centre frequencies and bandwidths of the peaks are adjustable. This suggests that the novel method will be relevant in applications such as DRFM-based random signal generation [11–15].
- Although the method permits the realisation of power spectra with multiple adjustable peaks, its capacity to realise *numerous* sharp peaks, and to realise sharp peaks at *arbitrary frequencies*, is limited.
- By not permitting the prescription of power spectrum zeros, the novel method does not permit the approximation of an arbitrary PSD and can therefore not be considered a solution to IFPP-II. However, it permits the approximation of a prescribed invariant density and the prescription of power spectrum poles, which qualifies it as a solution to the relaxed problem formulation (IFPP-IIa).
- The novel method for solving IFPP-IIa only permits the prescription of non-repeating power spectrum poles. Its generalisation to permit the prescription of repeating poles is a topic for future work.

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Appendix

4.A Proofs of Propositions and Theorems

4.A.1 Proof of Theorem 4.6.3

This proof appears in [16]. The theorem is first proved for novel disaggregation of the elementary single-state Markov chain with transition matrix $\mathbf{P} = [1]$. Thereafter, the proof is completed by addressing novel disaggregation of the original Markov chain possessing an $N \times N$ transition matrix, where $N > 1$.

Elementary single-state Markov chain

Let $\mathbf{P}'$ denote the transition matrix obtained after novel disaggregation of the single-state Markov chain with parameters r and θ satisfying (4.39) to (4.41). It is proved that $\mathbf{P}'$ inherits the unity eigenvalue λ_1 from the original transition matrix $\mathbf{P}$ and gains the additional eigenvalues $\lambda_2 = re^{i\theta}$ and $\lambda_3 = re^{-i\theta}$.

Suppose that disaggregation is carried out as stated above. Equations (4.33) to (4.35) imply that $\mathbf{P}' = U(1, r, \theta)$, where

$$\mathbf{P}' = \begin{pmatrix} \nu_1 & \nu_2 & \nu_3 \\ \nu_3 & \nu_1 & \nu_2 \\ \nu_2 & \nu_3 & \nu_1 \end{pmatrix} \quad (4.74)$$

and $\nu_q \triangleq \nu_q(1, r, \theta)$ is defined for $q \in \{1, 2, 3\}$ by (4.36) to (4.38). The eigenvalues of $\mathbf{P}'$ are the roots of its characteristic polynomial $\det(\mathbf{P}' - \mu\mathbf{I})$. Let the second and third rows of $\mathbf{P}' - \mu\mathbf{I}$ both be added to its first row, thereby yielding $\mathbf{W}_1 - \mu\mathbf{I}$.

Equations (4.36) to (4.38) imply that $\nu_1 + \nu_2 + \nu_3 = 1$; hence,

$$\mathbf{W}_1 = \begin{pmatrix} 1 & 1 - \mu & 1 - \mu \\ \nu_3 & \nu_1 & \nu_2 \\ \nu_2 & \nu_3 & \nu_1 \end{pmatrix}. \quad (4.75)$$

Next, let the first column of $\mathbf{W}_1 - \mu\mathbf{I}$ be subtracted from its second and third columns, thereby yielding $\mathbf{W}_2 - \mu\mathbf{I}$. It follows that

$$\mathbf{W}_2 = \begin{pmatrix} 1 & 0 & 0 \\ \nu_3 & \nu_1 - \nu_3 & \nu_2 - \nu_3 \\ \nu_2 & \nu_3 - \nu_2 & \nu_1 - \nu_2 \end{pmatrix}. \quad (4.76)$$

The determinant of $\mathbf{P}' - \mu\mathbf{I}$ remains unchanged if a scalar multiple of any matrix column (row) is added to any other column (row) [36]. This implies that

$$\begin{aligned} \det(\mathbf{P}' - \mu\mathbf{I}) &= \det(\mathbf{W}_2 - \mu\mathbf{I}) \\ &= (1 - \mu)((\nu_1 - \nu_3 - \mu)(\nu_1 - \nu_2 - \mu) + (\nu_2 - \nu_3)^2). \end{aligned} \quad (4.77)$$

Substitution of (4.36) to (4.38) into the latter equation yields

$$\begin{aligned} \det(\mathbf{P}' - \mu\mathbf{I}) &= (1 - \mu)(\mu^2 - 2r\mu \cos \theta + r^2) \\ &= -(\mu - \lambda_1)(\mu - \lambda_2)(\mu - \lambda_3), \end{aligned} \quad (4.78)$$

where $\lambda_1 = 1$ and $\lambda_2 = \lambda_3^* = re^{i\theta}$.

Markov chain with $N \times N$ transition matrix, where $N > 1$

Let $\mathbf{P} = [p_{i,j}]_{i,j=1}^N$ denote the transition matrix of the original Markov chain prior to disaggregation. Let state s_k of this Markov chain be disaggregated using the novel technique with parameters r and θ satisfying (4.39) to (4.41), where $1 \leq k \leq N$ is arbitrary. It is proved that the transition matrix $\mathbf{P}'$ obtained after disaggregation inherits the eigenspectrum $\lambda_1, \lambda_2, \dots, \lambda_N$ of $\mathbf{P}$, and gains the additional eigenvalues $\lambda_{N+1} = re^{i\theta}$ and $\lambda_{N+2} = re^{-i\theta}$.

Define $\mathbf{P}' = \mathcal{U}(\mathbf{P}, r, \theta, k)$ and consider the matrix

$$\tilde{\mathbf{P}}' = \left(\begin{array}{ccc|ccc} \tilde{p}_{1,1} & \cdots & \tilde{p}_{1,N-1} & \tilde{p}_{1,N} & \tilde{p}_{1,N} & \tilde{p}_{1,N} \\ \tilde{p}_{2,1} & \cdots & \tilde{p}_{2,N-1} & \tilde{p}_{2,N} & \tilde{p}_{2,N} & \tilde{p}_{2,N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \tilde{p}_{N-1,1} & \cdots & \tilde{p}_{N-1,N-1} & \tilde{p}_{N-1,N} & \tilde{p}_{N-1,N} & \tilde{p}_{N-1,N} \\ \hline \tilde{p}_{N,1}/3 & \cdots & \tilde{p}_{N,N-1}/3 & \nu_1 & \nu_2 & \nu_3 \\ \tilde{p}_{N,1}/3 & \cdots & \tilde{p}_{N,N-1}/3 & \nu_3 & \nu_1 & \nu_2 \\ \tilde{p}_{N,1}/3 & \cdots & \tilde{p}_{N,N-1}/3 & \nu_2 & \nu_3 & \nu_1 \end{array} \right) \quad (4.79)$$

that originates from $\mathbf{P}'$ via the similarity transformation $\tilde{\mathbf{P}}' = \mathbf{Q}\mathbf{P}'\mathbf{Q}^{-1}$, where $\mathbf{Q}$ is a permutation matrix. Element $\nu_q \triangleq \nu_q(\tilde{p}_{N,N}, r, \theta)$ appearing in the matrix of (4.79) is defined for $q \in \{1, 2, 3\}$ by (4.36) to (4.38). Matrix $\tilde{\mathbf{P}}'$ inherits the characteristic equation and eigenspectrum of $\mathbf{P}'$.

The eigenvalues of $\tilde{\mathbf{P}}'$ are the roots of its characteristic polynomial $\det(\tilde{\mathbf{P}}' - \mu\mathbf{I})$. Let rows $N+1$ and $N+2$ of $\tilde{\mathbf{P}}' - \mu\mathbf{I}$ both be added to row N of the same matrix, thereby yielding $\mathbf{W}_1 - \mu\mathbf{I}$. Equations (4.36) to (4.38) imply that $\nu_1 + \nu_2 + \nu_3 = \tilde{p}_{N,N}$; hence,

$$\mathbf{W}_1 = \left(\begin{array}{ccc|ccc} \tilde{p}_{1,1} & \cdots & \tilde{p}_{1,N-1} & \tilde{p}_{1,N} & \tilde{p}_{1,N} & \tilde{p}_{1,N} \\ \tilde{p}_{2,1} & \cdots & \tilde{p}_{2,N-1} & \tilde{p}_{2,N} & \tilde{p}_{2,N} & \tilde{p}_{2,N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \\ \tilde{p}_{N-1,1} & \cdots & \tilde{p}_{N-1,N-1} & \tilde{p}_{N-1,N} & \tilde{p}_{N-1,N} & \tilde{p}_{N-1,N} \\ \hline \tilde{p}_{N,1} & \cdots & \tilde{p}_{N,N-1} & \tilde{p}_{N,N} & \tilde{p}_{N,N} - \mu & \tilde{p}_{N,N} - \mu \\ \tilde{p}_{N,1}/3 & \cdots & \tilde{p}_{N,N-1}/3 & \nu_3 & \nu_1 & \nu_2 \\ \tilde{p}_{N,1}/3 & \cdots & \tilde{p}_{N,N-1}/3 & \nu_2 & \nu_3 & \nu_1 \end{array} \right). \quad (4.80)$$

Next, let column N of $\mathbf{W}_1 - \mu\mathbf{I}$ be subtracted from columns $N+1$ and $N+2$ of the same matrix, thereby yielding $\mathbf{W}_2 - \mu\mathbf{I}$. It follows that

$$\mathbf{W}_2 = \begin{pmatrix} \tilde{\mathbf{P}} & \mathbf{0} \\ \mathbf{M} & \mathbf{N} \end{pmatrix}, \quad (4.81)$$

$$\mathbf{M} = \begin{pmatrix} \tilde{p}_{N,1}/3 & \cdots & \tilde{p}_{N,N-1}/3 & \nu_3 \\ \tilde{p}_{N,1}/3 & \cdots & \tilde{p}_{N,N-1}/3 & \nu_2 \end{pmatrix} \quad (4.82)$$

and

$$\mathbf{N} = \begin{pmatrix} \nu_1 - \nu_3 & \nu_2 - \nu_3 \\ \nu_3 - \nu_2 & \nu_1 - \nu_2 \end{pmatrix}, \quad (4.83)$$

where $\tilde{\mathbf{P}}$ is related to $\mathbf{P}$ via the similarity transformation $\tilde{\mathbf{P}} = \mathbf{R}\mathbf{P}\mathbf{R}^{-1}$ and $\mathbf{R}$ is a permutation matrix. This implies that $\tilde{\mathbf{P}}$ and $\mathbf{P}$ share the same characteristic equation.

The determinant of $\tilde{\mathbf{P}}' - \mu\mathbf{I}$ remains unchanged if a scalar multiple of any matrix column (row) is added to any other column (row). Furthermore, since the determinant of the block triangular matrix $\mathbf{W}_2 - \mu\mathbf{I}$ is the product of the determinants of its diagonal blocks [36], it follows that

$$\det(\tilde{\mathbf{P}}' - \mu\mathbf{I}) = \det(\mathbf{W}_2 - \mu\mathbf{I}) = \det(\tilde{\mathbf{P}} - \mu\mathbf{I})\det(\mathbf{N} - \mu\mathbf{I}). \quad (4.84)$$

Since $\det(\tilde{\mathbf{P}}' - \mu\mathbf{I}) = \det(\mathbf{P}' - \mu\mathbf{I})$ and $\det(\tilde{\mathbf{P}} - \mu\mathbf{I}) = \det(\mathbf{P} - \mu\mathbf{I})$, it follows that

$$\det(\mathbf{P}' - \mu\mathbf{I}) = \det(\mathbf{P} - \mu\mathbf{I})\det(\mathbf{N} - \mu\mathbf{I}). \quad (4.85)$$

This expression implies that $\mathbf{P}'$ inherits the eigenspectrum $\lambda_1, \lambda_2, \dots, \lambda_N$ of $\mathbf{P}$ and gains additional eigenvalues equal to the roots of $\det(\mathbf{N} - \mu\mathbf{I})$. Substitution of (4.36) to (4.38) into (4.83) yields

$$\begin{aligned}\det(\mathbf{N} - \mu\mathbf{I}) &= \mu^2 - 2r\mu \cos \theta + r^2 \\ &= (\mu - \lambda_{N+1})(\mu - \lambda_{N+2}),\end{aligned}\tag{4.86}$$

where $\lambda_{N+1} = \lambda_{N+2}^* = re^{i\theta}$.

4.A.2 Proof of Theorem 4.6.4

This proof appears in [16]. The theorem is first proved for novel disaggregation of the elementary single-state Markov chain with transition matrix $\mathbf{P} = [1]$. Thereafter, the proof is completed by addressing novel disaggregation of the original Markov chain possessing an $N \times N$ transition matrix, where $N > 1$.

Elementary single-state Markov chain

The single-state Markov chain possesses the stationary probability vector $\boldsymbol{\pi} = [1]$. It is proved that novel disaggregation of this Markov chain, with parameters r and θ satisfying (4.39) to (4.41), produces a Markov chain with transition matrix $\mathbf{P}'$ and corresponding stationary probability vector $\boldsymbol{\pi}' = [1/3, 1/3, 1/3]^T$.

Suppose that disaggregation is carried out as stated above. Equations (4.33) and (4.34) imply that $\mathbf{P}' = U(1, r, \theta)$, where $\mathbf{P}'$ is given by (4.74) and $\nu_q \triangleq \nu_q(1, r, \theta)$ is defined for $q \in \{1, 2, 3\}$ by (4.36) to (4.38). By observing that $\nu_1 + \nu_2 + \nu_3 = 1$, it is directly verified that $\mathbf{P}'\boldsymbol{\pi}' = \boldsymbol{\pi}'$. Hence, $\boldsymbol{\pi}'$ is a stationary probability vector of the Markov chain obtained after disaggregation.

Markov chain with $N \times N$ transition matrix, where $N > 1$

Let $\mathbf{P} = [p_{i,j}]_{i,j=1}^N$ of (4.21) denote the transition matrix of the original Markov chain prior to disaggregation. Consider the matrix

$$\tilde{\mathbf{P}} = \left(\begin{array}{c|c} \tilde{\mathbf{P}}_{1,1} & \tilde{\mathbf{v}}_{1,N} \\ \hline \tilde{\mathbf{w}}_{N,1}^T & \tilde{p}_{N,N} \end{array} \right)\tag{4.87}$$

that originates from $\mathbf{P}$ via the similarity transform $\tilde{\mathbf{P}} = \mathbf{R}\mathbf{P}\mathbf{R}^{-1}$, where $\mathbf{R}$ is a permutation matrix. The existence of a stationary vector $\boldsymbol{\pi} = [\pi_1, \pi_2, \dots, \pi_N]^T$ of $\mathbf{P}$

implies that a stationary vector $\tilde{\pi} = [\tilde{\pi}_1, \tilde{\pi}_2, \dots, \tilde{\pi}_N]^T$ of $\tilde{\mathbf{P}}$ exists, where $\tilde{\pi} = \mathbf{R}\pi$ is a permutation of the elements of π .

Suppose that state s_k of the original Markov chain is disaggregated using the novel technique with parameters r and θ satisfying (4.39) to (4.41), where $1 \leq k \leq N$ is arbitrary. Let $\mathbf{P}' = \mathcal{U}(\mathbf{P}, r, \theta, k)$ denote the transition matrix of the Markov chain obtained after disaggregation. Consider the matrix

$$\tilde{\mathbf{P}}' = \left(\begin{array}{c|ccc} \tilde{\mathbf{P}}_{1,1} & \tilde{\mathbf{v}}_{1,N} & \tilde{\mathbf{v}}_{1,N} & \tilde{\mathbf{v}}_{1,N} \\ \hline \tilde{\mathbf{w}}_{N,1}^T/3 & & & \\ \tilde{\mathbf{w}}_{N,1}^T/3 & & \mathbf{U} & \\ \tilde{\mathbf{w}}_{N,1}^T/3 & & & \end{array} \right), \quad (4.88)$$

that originates from $\mathbf{P}'$ via the similarity transformation $\tilde{\mathbf{P}}' = \mathbf{Q}\mathbf{P}'\mathbf{Q}^{-1}$, where $\mathbf{Q}$ is a permutation matrix. Matrix $\mathbf{U} \triangleq U(\tilde{p}_{N,N}, r, \theta)$ in (4.88) is given by (4.35). The existence of a stationary vector $\pi' = [\pi'_1, \pi'_2, \dots, \pi'_{N+2}]^T$ of $\mathbf{P}'$ implies that a stationary vector $\tilde{\pi}' = [\tilde{\pi}'_1, \tilde{\pi}'_2, \dots, \tilde{\pi}'_{N+2}]^T$ of $\tilde{\mathbf{P}}'$ exists, where $\tilde{\pi}' = \mathbf{Q}\pi'$ is a permutation of the elements of π' .

The theorem is proved by showing that

$$\tilde{\pi}' \triangleq [\bar{\pi}^T, \tilde{\pi}_N/3, \tilde{\pi}_N/3, \tilde{\pi}_N/3]^T \quad (4.89)$$

is a stationary probability vector of $\tilde{\mathbf{P}}'$, where $\bar{\pi} \triangleq [\tilde{\pi}_1, \tilde{\pi}_2, \dots, \tilde{\pi}_{N-1}]^T$. Specifically, it is proved that $\tilde{\mathbf{P}}'\tilde{\pi}' = \tilde{\pi}'$ by showing that

$$[\tilde{\mathbf{P}}_{1,1}, \tilde{\mathbf{v}}_{1,N}, \tilde{\mathbf{v}}_{1,N}, \tilde{\mathbf{v}}_{1,N}]\tilde{\pi}' = \bar{\pi} \quad (4.90)$$

and

$$[\mathbf{A}, \mathbf{U}]\tilde{\pi}' = \frac{1}{3}[\tilde{\pi}_N, \tilde{\pi}_N, \tilde{\pi}_N]^T, \quad (4.91)$$

where $\mathbf{A} \triangleq [\tilde{\mathbf{w}}_{N,1}, \tilde{\mathbf{w}}_{N,1}, \tilde{\mathbf{w}}_{N,1}]^T/3$.

Since $\tilde{\mathbf{P}}\tilde{\pi} = \tilde{\pi}$, (4.87) implies that $[\tilde{\mathbf{P}}_{1,1}, \tilde{\mathbf{v}}_{1,N}]\tilde{\pi} = \tilde{\mathbf{P}}_{1,1}\bar{\pi} + \tilde{\pi}_N\tilde{\mathbf{v}}_{1,N} = \bar{\pi}$. It follows that

$$\begin{aligned} [\tilde{\mathbf{P}}_{1,1}, \tilde{\mathbf{v}}_{1,N}, \tilde{\mathbf{v}}_{1,N}, \tilde{\mathbf{v}}_{1,N}]\tilde{\pi}' &= \tilde{\mathbf{P}}_{1,1}\bar{\pi} + \tilde{\mathbf{v}}_{1,N}(\tilde{\pi}_N/3 + \tilde{\pi}_N/3 + \tilde{\pi}_N/3) \\ &= \bar{\pi}, \end{aligned} \quad (4.92)$$

thereby yielding (4.90).

Next, it is found that $\tilde{\mathbf{P}}\tilde{\pi} = \tilde{\pi}$ and (4.87) imply

$$\tilde{\mathbf{w}}_{N,1}^T\bar{\pi} + \tilde{p}_{N,N}\tilde{\pi}_N = \tilde{\pi}_N, \quad (4.93)$$

or $\tilde{\mathbf{w}}_{N,1}^T \bar{\boldsymbol{\pi}} = \tilde{\pi}_N(1 - \tilde{p}_{N,N})$. Furthermore, (4.36) to (4.38) imply that the rows of $\mathbf{U}$ sum to $\tilde{p}_{N,N}$. It follows that

$$\begin{aligned} [\mathbf{A}, \mathbf{U}] \tilde{\boldsymbol{\pi}}' &= \mathbf{A} \bar{\boldsymbol{\pi}} + \frac{1}{3} \mathbf{U} [\tilde{\pi}_N, \tilde{\pi}_N, \tilde{\pi}_N]^T \\ &= \frac{1 - \tilde{p}_{N,N}}{3} [\tilde{\pi}_N, \tilde{\pi}_N, \tilde{\pi}_N]^T + \frac{\tilde{p}_{N,N}}{3} [\tilde{\pi}_N, \tilde{\pi}_N, \tilde{\pi}_N]^T \\ &= \frac{1}{3} [\tilde{\pi}_N, \tilde{\pi}_N, \tilde{\pi}_N]^T, \end{aligned} \quad (4.94)$$

thereby yielding (4.91).

4.A.3 Proof of Proposition 4.6.1

Suppose that the hat map S is constructed using Algorithm 4.6.4 (hat map construction II) to possess the matrix $\mathbf{P} = [p_{k,m}]$ as its order-zero FPM, and let $\bar{S}$ denote the canonical map constructed during the course of this algorithm. Proposition 3.4.3 implies that $\bar{S}$ is $\bar{\mathcal{R}}(\mathcal{Q})$ -semi-Markov and possess the matrix $\mathbf{P}$ as its order-zero FPM. Due to the positivity of $\mathbf{P}$ and the canonicity of $\bar{S}$, the branch functions of $\bar{S}$ satisfy

$$p_{k,m} = |\bar{S}'_{k,m}|^{-1}, \quad (4.95)$$

where

$$\bar{S}'_{k,m} = \left. \frac{d\bar{S}}{dx} \right|_{\bar{R}_k^{(m)}}. \quad (4.96)$$

The construction of the hat map S replaces each branch function $\bar{S}_{k,m}$ with two branch functions defined over a partition of the closure of interval $\bar{R}_k^{(m)}$ and with the same images as the original branch function $\bar{S}_{k,m}$. The definition of a semi-Markov map (Definition 2.6.1) then implies that S is an $\mathcal{R}(\mathcal{Q})$ -semi-Markov map, where partition $\mathcal{R}$ comprises the intervals obtained by dividing each interval $\bar{R}_k^{(m)}$.

Now, suppose that the replacement of each branch function $\bar{S}_{k,m}$ changes the map's order-zero FPM. Let $\mathbf{P}^* = [p_{k,m}^*]$ denote the new order-zero FPM obtained after replacing all the branch functions. Element $p_{k,m}^*$ is then derived using (2.122) as

$$p_{k,m}^* = |S'_{2k-1,m}|^{-1} + |S'_{2k,m}|^{-1}. \quad (4.97)$$

Since the new branch functions have equal slopes of opposite sign, $|S'_{2k-1,m}| = |S'_{2k,m}| = 2|\bar{S}'_{k,m}|$. Hence,

$$p_{k,m}^* = |\bar{S}'_{k,m}|^{-1} = p_{k,m}. \quad (4.98)$$

This implies that the order-zero FPMs of S and $\bar{S}$ are identical.

As the hat maps obtained using Algorithm 4.6.3 (hat map construction I) and Algorithm 4.6.4 (hat map construction II) are identical, the above result applies to both algorithms.

4.A.4 Proof of Proposition 4.7.1

The algorithm requires that the stochastic matrix $\tilde{\mathbf{P}}$ constructed using the solution to the StIEP be positive. The Perron-Frobenius theorem [37] therefore applies to this matrix, and the elements of its right Perron eigenvector $\tilde{\mathbf{w}}_1 = [\tilde{w}_1, \tilde{w}_2, \dots, \tilde{w}_M]^T$ are positive. Now, consider Algorithm 4.6.1 (determination of approximating density). The partition points $q_1, q_2, \dots, q_{M+1}$ of the Markov partition constructed using this algorithm are given by (4.14). Since F_P is continuous, it follows that $q_m < q_{m+1}$ for each $m = 1, 2, \dots, M$. Thus, each interval $Q_m \in \mathcal{Q}$ has length strictly greater than zero; i.e., $\mu(Q_m) > 0$. This implies that the candidate order-zero FPM $\mathbf{P}$ constructed using (4.63) exists and possesses positive elements. Furthermore, as similarity transforms preserve eigenvalues, the matrix $\mathbf{P}$ has a left Perron eigenvector with positive elements that coincide with the lengths of the intervals in partition $\mathcal{Q}$.

Now, construct a hat map $\tilde{S}$ to possess the stochastic matrix $\tilde{\mathbf{P}}$ as its order-zero FPM. As $\tilde{\mathbf{P}}$ is positive, it follows that each of its elements is strictly smaller than unity. Each branch of the map therefore possesses a slope that is strictly larger than unity in magnitude. The map $\tilde{S}$ then satisfies the conditions of the Folklore theorem (Theorem 2.6.3), and is therefore ergodic. The proof then follows by recognising that the hat map S designed to possess the matrix $\mathbf{P}$ as order-zero FPM is a topological conjugate of the map $\tilde{S}$, and that topological conjugation preserves ergodicity (Theorem 3.4.1).

4.B Construction of the Maps in the Examples

4.B.1 Example 4.8.1

Algorithm 4.7.1 (novel method for solving IFPP-IIa) was applied to design each map. A positive 2×2 stochastic matrix $\tilde{\mathbf{P}}$ that possesses a Perron eigenvalue $\lambda_1 = 1$ and a second eigenvalue $\lambda_2 = \kappa_1$ was designed using Algorithm 4.6.2 (StIEP solution). The latter algorithm performed a single round of Markov state disaggregation by applying the original disaggregation technique to the initial matrix $\tilde{\mathbf{P}}_{(0)} = [1]$. Markov state disaggregation was applied in manner that did not require the selection of the pole κ_1 in advance (i.e., an expression that contains κ_1 as a variable was derived). Algorithm 4.6.2 (StIEP solution) then selected the disaggregation parameters

$$\lambda = \kappa_1, \quad k = 1, \quad \alpha = 1/2. \quad (4.99)$$

Disaggregation yielded the matrix $\tilde{\mathbf{P}}_{(1)} = \mathcal{T}(\tilde{\mathbf{P}}_{(0)}, \lambda, \alpha, k)$, where

$$\tilde{\mathbf{P}}_{(1)} = \frac{1}{2} \begin{pmatrix} 1 + \kappa_1 & 1 - \kappa_1 \\ 1 - \kappa_1 & 1 + \kappa_1 \end{pmatrix}. \quad (4.100)$$

Let $\tilde{\mathbf{P}} = \tilde{\mathbf{P}}_{(1)}$. The choice of $\alpha = 1/2$ yielded $\beta = 1/2$ in (4.31), which in turn yielded a right Perron eigenvector $\mathbf{w}_1$ for matrix $\tilde{\mathbf{P}}$ with equal elements (Theorem 4.6.2).¹⁸ This eigenvector was normalised such that its elements sum to unity,

$$\mathbf{w}_1 = [1/2, 1/2]^T. \quad (4.101)$$

The prescription of the uniform invariant density f_P coincided with the prescription of the distribution function $F_P(x) = x$, where $x \in \Omega$. Algorithm 4.6.1 (determination of approximating density) was then used to construct the PWC invariant density $\hat{f}_P$ that the map had to realise, as well as the Markov partition $\mathcal{Q} = \{Q_1, Q_2\}$. Equation (4.14) yielded the endpoints of the Markov partition,

$$q_m = (m - 1)/2, \quad m = 1, 2, 3. \quad (4.102)$$

This implied that $Q_1 = (0, 1/2)$ and $Q_2 = (1/2, 1)$. The approximating density $\hat{f}_P$ was then derived from (4.11),

$$\hat{f}_P(x) = \chi_{Q_1}(x) + \chi_{Q_2}(x). \quad (4.103)$$

The map was thus designed to possess the uniform invariant density $\hat{f}_P$, which is equal to the prescribed invariant density.

Algorithm 4.7.1 (novel method for solving IFPP-IIa) next required the design of the candidate order-zero FPM $\mathbf{P}$ using (4.62) and (4.63). Since $\mathbf{D} = \frac{1}{2}\mathbf{I}$, where $\mathbf{I}$ denotes the identity matrix, it follows that $\mathbf{P} = \tilde{\mathbf{P}}$. The algorithm concluded by designing the hat map S . This was accomplished by substituting κ_1 in (4.100) with a prescribed value from the set in (4.65), and then using Algorithm 4.6.3 (hat map construction I) to construct the map to possess the matrix $\mathbf{P}$ as its order-zero FPM.

4.B.2 Example 4.8.2

Algorithm 4.7.1 (novel method for solving IFPP-IIa) was applied to design each map. A positive 3×3 stochastic matrix $\mathbf{P}$ that possesses a Perron eigenvalue $\lambda_1 = 1$ and the complex conjugate pair of eigenvalues $\lambda_2 = \lambda_3^*$ was designed using Algorithm 4.6.2 (StIEP solution). The latter algorithm performed a single round of Markov

¹⁸This was confirmed by observing that $\tilde{\mathbf{P}}$ is doubly stochastic.

state disaggregation by applying the novel disaggregation technique to the initial matrix $\tilde{\mathbf{P}}_{(0)} = [1]$. Algorithm 4.6.2 (StIEP solution) then selected the disaggregation parameters

$$r = |\lambda_2|, \quad \theta = \arg(\lambda_2), \quad k = 1. \quad (4.104)$$

Disaggregation yielded the matrix $\tilde{\mathbf{P}}_{(1)} = \mathcal{U}(\tilde{\mathbf{P}}_{(0)}, r, \theta, k)$, where

$$\tilde{\mathbf{P}}_{(1)} \triangleq \begin{pmatrix} \nu_1 & \nu_2 & \nu_3 \\ \nu_3 & \nu_1 & \nu_2 \\ \nu_2 & \nu_3 & \nu_1 \end{pmatrix}. \quad (4.105)$$

The elements of this matrix are given by (4.36) to (4.38) with $p_{k,k} \triangleq 1$.

The elements of the right Perron eigenvector of $\tilde{\mathbf{P}}_{(1)}$ were equal (Theorem 4.6.4).¹⁹ The elements of this eigenvector were normalised such that they sum to unity, thereby yielding

$$\mathbf{w}_1 = [1/3, 1/3, 1/3]^T. \quad (4.106)$$

The prescription of a uniform invariant density f_P implied the distribution function $F_P(x) = x$, where $x \in \Omega$. Algorithm 4.6.1 (determination of approximating density) was then used to construct the PWC invariant density $\hat{f}_P$ for the map, as well as the Markov partition $\mathcal{Q}$. Equation (4.14) yielded

$$q_m = (m - 1)/3, \quad m = 1, \dots, 4. \quad (4.107)$$

This implied that $Q_1 = (0, 1/3)$, $Q_2 = (1/3, 2/3)$ and $Q_3 = (2/3, 1)$. The approximating density $\hat{f}_P$ was then obtained from (4.11) as

$$\hat{f}_P(x) = \chi_{Q_1}(x) + \chi_{Q_2}(x) + \chi_{Q_3}(x). \quad (4.108)$$

Hence, the approximating density $\hat{f}_P$ that the map was designed to possess is uniform and equal to the prescribed invariant density almost everywhere.

Algorithm 4.7.1 (novel method for solving IFPP-IIa) next required the design of the candidate order-zero FPM $\mathbf{P}$ using (4.62) and (4.63). Since $\mathbf{D} = \frac{1}{3}\mathbf{I}$, where $\mathbf{I}$ denotes the identity matrix, it follows that $\mathbf{P} = \tilde{\mathbf{P}}$. The algorithm then concluded by designing the hat map S . This was accomplished by using Algorithm 4.6.3 (hat map construction I) to construct the map to possess the matrix of (4.105) as its order-zero FPM.

¹⁹This was confirmed by observing that $\tilde{\mathbf{P}}_{(1)}$ is doubly stochastic.

4.B.3 Example 4.8.3

Algorithm 4.7.1 (novel method for solving IFPP-IIa) was applied to design each map. A positive 4×4 stochastic matrix $\tilde{\mathbf{P}}$ that possesses a Perron eigenvalue $\lambda_1 = 1$ and the remaining eigenvalues

$$\lambda_k = \kappa_{k-1}, \quad k = 2, 3, 4, \quad (4.109)$$

was designed using Algorithm 4.6.2 (StIEP solution). The latter algorithm first performed a single round of original disaggregation on $\tilde{\mathbf{P}}_{(0)} = [1]$ with the parameters

$$\lambda = 3/4, \quad k = 1, \quad \alpha = 15/28. \quad (4.110)$$

Here, the value of parameter α was chosen in order to realise a matrix $\tilde{\mathbf{P}}_{(1)}$ (after disaggregation) with the right Perron eigenvector $\tilde{\mathbf{w}}_1 = [3/4, 1/4]^T$; the parameter value was derived by solving (4.31) for α , where $\beta = 3/4$. Disaggregation then yielded the matrix $\tilde{\mathbf{P}}_{(1)} = \mathcal{T}(\tilde{\mathbf{P}}_{(0)}, \lambda, \alpha, k)$, where

$$\tilde{\mathbf{P}}_{(1)} = \frac{1}{16} \begin{pmatrix} 15 & 3 \\ 1 & 13 \end{pmatrix}. \quad (4.111)$$

Algorithm 4.6.2 (StIEP solution) then performed a single round of novel disaggregation on matrix $\tilde{\mathbf{P}}_{(1)}$ to insert the eigenvalues $\lambda_2 = \tilde{r} \exp(i2\pi/3)$ and $\lambda_3 = \lambda_2^*$ into its eigenspectrum; it was verified that these eigenvalues fall within the region of permissible eigenvalues as defined by (4.39) to (4.41). The algorithm selected the parameters

$$r = \tilde{r}, \quad \theta = 2\pi/3, \quad k = 1. \quad (4.112)$$

Disaggregation then yielded $\tilde{\mathbf{P}}_{(2)} = \mathcal{U}(\tilde{\mathbf{P}}_{(1)}, r, \theta, k)$, where

$$\tilde{\mathbf{P}}_{(2)} = \frac{1}{48} \begin{pmatrix} 15 - 16\tilde{r} & 15 + 32\tilde{r} & 15 - 16\tilde{r} & 3 \\ 15 - 16\tilde{r} & 15 - 16\tilde{r} & 15 + 32\tilde{r} & 3 \\ 15 + 32\tilde{r} & 15 - 16\tilde{r} & 15 - 16\tilde{r} & 3 \\ 3 & 3 & 3 & 39 \end{pmatrix}, \quad (4.113)$$

and

$$0 < \tilde{r} < \frac{15}{16}. \quad (4.114)$$

Theorem 4.6.4 implied that the elements of the right Perron eigenvector of $\tilde{\mathbf{P}}_{(2)}$ are equal.²⁰ The elements of this eigenvector were normalised such that they sum to unity, thereby yielding

$$\tilde{\mathbf{w}}_1 = [1/4, 1/4, 1/4, 1/4]^T. \quad (4.115)$$

²⁰This was confirmed by observing that $\tilde{\mathbf{P}}_{(1)}$ is doubly stochastic.

The prescription of an invariant density f_P equal to the beta PDF of (4.17) implied the distribution function

$$F_P(x) = \int_0^x \frac{z^{a-1}(1-z)^{b-1}}{B(a,b)} dz, \quad x \in \Omega. \quad (4.116)$$

Algorithm 4.6.1 (determination of approximating density) was then used to construct the PWC invariant density $\hat{f}_P$ for the map, as well as the Markov partition $\mathcal{Q} = \{Q_m\}_{m=1}^4$. Equation (4.14) yielded the partition endpoints

$$\tilde{q}_1 = 0, \quad \tilde{q}_2 \approx 0.199, \quad \tilde{q}_3 \approx 0.352, \quad \tilde{q}_4 \approx 0.531, \quad \tilde{q}_5 = 1. \quad (4.117)$$

The approximating density $\hat{f}_P$ was then derived using (4.11),

$$\hat{f}_P(x) = \sum_{m=1}^M \hat{f}_m \chi_{Q_m}(x), \quad (4.118)$$

where

$$\hat{f}_1 \approx 1.256, \quad \hat{f}_2 \approx 1.634, \quad \hat{f}_3 \approx 1.397, \quad \hat{f}_4 \approx 0.533. \quad (4.119)$$

This density was realised as the PWC invariant density of each map, and is plotted in Fig. 4.8.18 (blue curve).

Algorithm 4.7.1 (novel method for solving IFPP-IIa) next required the design of the candidate order-zero FPM $\mathbf{P}$ using (4.62) and (4.63). This matrix is given by

$$\mathbf{P} = \mathbf{D} \tilde{\mathbf{P}}_{(2)} \mathbf{D}^{-1}, \quad (4.120)$$

where

$$\mathbf{D} = \text{diag}\{\mu(Q_1)^{-1}, \mu(Q_2)^{-1}, \mu(Q_3)^{-1}, \mu(Q_4)^{-1}\} \quad (4.121)$$

and

$$\mu(Q_1) \approx 0.199, \quad \mu(Q_2) \approx 0.153, \quad \mu(Q_3) \approx 0.179, \quad \mu(Q_4) \approx 0.469. \quad (4.122)$$

The algorithm concluded by designing each hat map S . This was accomplished by first substituting each prescribed value for parameter $\tilde{r}$, as enumerated in the set of (4.72), into (4.113). Algorithm 4.6.3 (hat map construction I) was then used to construct each hat map to possess the matrix of (4.120) as its order-zero FPM.

Chapter 5

A NOVEL SOLUTION FOR IFPP-III

5.1 Introduction

IFPP-III requires the reconstruction of an unknown ergodic map from sequences of density function estimates, which are derived from measured state ensembles that evolve under iteration of the map [1–4]. Solutions to this problem are of practical importance in applications where it is impossible to isolate and track individual state trajectories from one time instant to the next. Examples of such applications include particle image velocimetry for flow visualisation [5], flow cytometry [6], the modelling of emulsion polymerisation processes [7], papermaking systems [8] and certain communication systems [9].

The original formulation of IFPP-III is addressed in this chapter. This problem requires the reconstruction of a PWL and ergodic semi-Markov map. An existing solution for this problem, referred to in this work as the *original solution*, is presented in [2, 3]. This solution and its characteristics are described in section 3.6.1.

The original solution derives a matrix estimate of the unknown map’s finite dimensional FPO from the density estimates, and then designs a PWL semi-Markov map with a canonical¹ structure to satisfy this matrix [2, 3]. This approach was shown to consistently reconstruct canonical semi-Markov maps with increasing branch functions, thereby yielding a map with the correct dynamical and statistical characteristics. This is achieved without tracking the time evolution of individual trajectories.

Although the original solution for IFPP-III is ground-breaking, it was proved in

¹The *canonical* semi-Markov maps were introduced in Definition 3.4.5.

section 3.6.1 that the solution has two shortcomings (refer to Propositions 3.6.2 and 3.6.4). More specifically, if the unknown map possesses any decreasing branch functions, the method is unable to consistently reconstruct the map, and the map constructed using the solution possesses power spectrum modes with characteristics (i.e., mode centre frequencies and bandwidths) that are inconsistent with those of the unknown map. This restricts the applicability of the original solution.

This chapter presents a novel method for solving IFPP-III that resolves the shortcomings of the original solution. It is shown that the cause of the original solution's shortcomings is its reconstruction of the unknown map from an estimate of its *order-zero* FPM, which does not depend on the monotonicity of the unknown map's branch functions. Hence, the original solution is not able to resolve branch function monotonicity using this matrix alone.

Two novel solution strategies are presented. These strategies require the reconstruction of the map to satisfy elements of the unknown map's *order-one* FPM, which is selected for its dependency on the monotonicity of the map's branch functions. The novel method implements the more feasible of the two strategies by first constructing density estimates from state measurements collected in two rounds of probing experiments. These density estimates are used to derive estimates of two submatrices from the unknown map's order-one FPM. A map from the newly proposed class of *generalised hat maps* is then constructed to satisfy these matrices.

Multiple propositions are proved regarding the characteristics of the novel method. It is proved that the novel method is able to discern the monotonicity of any canonical map's branch functions and consistently reconstruct such maps, provided the density estimates are sufficiently accurate. Furthermore, it is proved that the novel method constructs a map with consistent power spectrum mode characteristics, regardless of the unknown map's canonicity or the monotonicity of its branches. These propositions imply that the novel method resolves the shortcomings of the original solution.

Two numerical examples that illustrate the relevant characteristics of the original solution and the novel method are presented. It is shown that the numerical results support the propositions regarding the characteristics of both methods. The results also demonstrate that the reconstruction error of the novel method decreases as the estimation accuracy for the density functions improves, provided the relevant conditions for consistent reconstruction are met. By virtue of enlarging the classes of maps that are reconstructible to possess consistent characteristics, it is concluded that the novel method is more versatile than the original solution.

The remainder of this chapter is set out as follows. Section 5.2 presents an overview of the novel contributions of the chapter. Section 5.3 describes the shortcomings of the original solution for IFPP-III. The novel solution strategies are outlined in section 5.4, and the novel method for solving IFPP-III is presented in section 5.5. A numerical experiment that was conducted to reconstruct two maps as examples is described in section 5.6. A summary of the chapter is presented in section 5.7, and conclusions are drawn in section 5.8.

The material presented in this chapter appears in [4].

5.2 Novel Contributions of this Chapter

The novel contributions of this chapter are outlined in Tables 5.2.1 and 5.2.2.

5.3 Shortcomings of the Original Solution

The original solution to IFPP-III [2, 3] reconstructs an unknown PWL and ergodic semi-Markov map by first deriving an estimate of its order-zero FPM, and then designing a canonical semi-Markov map to possess this matrix as its order-zero FPM (refer to section 3.6.1 and Definition 3.4.5). Numerical results presented in [2, 3] demonstrate the original solution's consistent reconstruction of canonical semi-Markov maps with increasing branch functions. However, the original solution does not successfully reconstruct an arbitrary semi-Markov map, as proved in section 3.6.1.

Three propositions were proved in section 3.6.1 regarding the characteristics of any semi-Markov map constructed using the original solution presented in [2, 3]. These propositions are summarised below.

- **Proposition 3.6.2:** If the unknown map S is canonical and possesses only increasing branches, and both the Markov partition and the order-zero FPM of S are estimated with perfect accuracy, the original solution reconstructs S perfectly [4]. Conversely, if S is not canonical or possesses any decreasing branches, S cannot be consistently reconstructed, regardless of estimation accuracy.
- **Proposition 3.6.3:** If the Markov partition and the order-zero FPM of the unknown map S are estimated with perfect accuracy, the invariant density of

Table 5.2.1: Novel contributions presented in chapter 5 (part 1 of 2).

Reference	Publ.	Description
Sec. 5.3	[4]	The identification of the cause for the shortcomings of the original solution to IFPP-III as its exclusive reliance on the order-zero matrix representation of the map's FPO for the reconstruction of the map.
Sec. 5.4.1, 5.4.2	[4]	The development of two novel strategies for solving IFPP-III that successfully address the shortcomings of the original solution, and the identification of the design issues associated with each strategy.
Sec. 5.5, Alg. 5.5.2	[4]	The development of a novel method for solving IFPP-III that implements the more viable of the two novel solution strategies and resolves the relevant design issues.
Sec. 5.5.2, eq. (5.4), (5.11)	[4]	The design of the probing experiments to permit accurate estimation of the relevant matrix representations of the unknown map's FPO.
Sec. 5.5.2, 5.B, eq. (5.15) to (5.17)	[4]	The development of the method-of-moments estimate for piecewise linear density functions that is novel in the context of IFPP-III.
Sec. 5.5.2, Opt. Prob. 5.5.1 to 5.5.3, Prop. 5.5.1, 5.5.2	[4]	The formulation of the matrix estimation problem as joint and separate constrained optimisation problems, and the provision of formal proofs regarding the convexity and non-convexity of these problems.
Sec. 5.5.2, eq. (5.38), (5.39), Prop. 5.5.3	[4]	The specification of a matrix estimate that is novel in the context of IFPP-III, and the statement of conditions for its optimality.
Sec. 5.5.3, Prop. (5.5.4), (5.5.5), Alg. 5.5.1	[4]	The specification of a novel subclass of the semi-Markov maps (i.e., the generalised hat maps), the provision of formal proofs regarding the relevant properties of these maps, and the design of an algorithm for constructing maps from this subclass to satisfy matrix estimates of the unknown map's FPO.

Table 5.2.2: Novel contributions presented in chapter 5 (part 2 of 2).

Reference	Publ.	Description
Sec. 5.5.5, Prop. 5.5.7, 5.5.8	[4]	The provision of formal proofs on the characteristics of the novel method for solving IFPP-III that address the shortcomings of the original solution.
Sec. 5.6, Ex. 5.6.1, 5.6.2	[4]	The demonstration of the novel method, the identification, illustration and verification of its characteristics, and its quantitative comparison with the original solution using two examples. The identification of the design assumptions of both methods that give rise to characteristics such as incorrect branch monotonicity and unnecessary map branches.
Sec. 5.6.3, Fig. 5.6.13, Tab. 5.6.5	–	The empirical analysis of the mean reconstruction time of the novel method and the original solution and the provision of empirical evidence that suggests linear computational complexity.

the reconstructed map equals the invariant density of S , regardless of whether S is canonical or the monotonicity of its branches [4].

- **Proposition 3.6.4:** If the unknown map S possesses any decreasing branches, a subset of the power spectrum mode characteristics of the reconstructed map (i.e., the centre frequencies and bandwidths of the $2M$ modes that comprise the map's power spectrum) are in general unequal to that of S , regardless of estimation accuracy [4].

Propositions 3.6.2 and 3.6.4 reveal fundamental shortcomings of the original solution. More specifically, if the unknown map possesses any decreasing branches, then the reconstructed map and the characteristics of its power spectrum cannot be equal to the unknown map and its power spectrum characteristics, regardless of the accuracy of the estimates constructed during the solution steps. In particular, the map is reconstructed to possess branches that are only increasing.

The original solution's inability to consistently reconstruct map branches with decreasing monotonicity stems from its strategy for map reconstruction. In this strategy, the unknown map is reconstructed solely from estimates of its Markov partition and order-zero FPM. The elements of the order-zero FPM are functions of the *absolute values* of the branch slopes, and thus do not permit discernment of branch monotonicity [4]; refer to (2.122). Hence, the original solution is unable to identify the monotonicity of each branch, regardless of estimation accuracy. Although multiple maps may be designed to satisfy the same estimate of the unknown map's order-zero FPM, the original solution constructs a canonical map with increasing branches only.²

Although any semi-Markov map with one or more decreasing branches is inconsistently reconstructed by the original solution, Proposition 3.6.3 states that the map constructed by the solution possesses the same invariant density as the unknown map if perfect estimation accuracy is assumed [4]. This proposition follows from the observation that the invariant density of any ergodic semi-Markov map depends only on the map's order-zero FPM. More specifically, the invariant density coincides with the right Perron eigenvector of this matrix; refer to Proposition 2.6.9. Hence, by designing the map to satisfy an estimate of the unknown map's order-zero FPM, its invariant density will be consistent with that of the unknown map in the limit of perfect estimation accuracy.

²This implies that the unknown map is correctly reconstructed if it is canonical and possesses branches that only increase.

The observation regarding the invariant density of the reconstructed map does not extend to the modes of its power spectrum [4]. An expression for the ACF of a PWL and ergodic semi-Markov map³ was derived in Proposition 2.6.12:

$$\mathcal{R}_S(\tau) = \sum_{m=1}^{2M} c_m \lambda_m^{|\tau|}, \quad \tau = 0, \pm 1, \pm 2, \dots, \quad (5.1)$$

where λ_m denotes the eigenvalues of the map's order-one FPM, and c_m denotes weights that depend on the map's order-one FPM. It follows that the eigenvalues λ_m coincide with the power spectrum modes of the map; refer to Remark 2.6.8.

Now, the original solution constructs a canonical map with increasing branches to possess a particular order-zero FPM, which then *implies* a particular order-one FPM for the map. Although this order-one FPM possesses the order-zero FPM as one of its submatrices, its eigenvalues are not fully determined by the order-zero FPM. Hence, the power spectrum modes of the map constructed using the original solution (and, by implication, its PSD) will in general not be consistent with those of the unknown map if the latter possesses any decreasing branches. This result holds regardless of the accuracy of the estimates constructed during the solution steps.

5.4 Solution Strategies

The shortcomings identified in the previous section are consequences of the original solution's reconstruction of the unknown map from an estimate of its *order-zero* FPM. Instead, the reconstruction of the unknown map from an estimate of its *order-one* FPM is considered in this work. This approach is motivated by two observations.

- The elements of submatrix $\mathbf{Q}_S$ of a semi-Markov map's order-one FPM depend on the monotonicity of the map's branch functions; refer to (2.118). It is conjectured (and proved in this chapter) that an estimate of submatrix $\mathbf{Q}_S$ permits discernment of the monotonicity of a canonical map's branch functions.
- The eigenvalues of a semi-Markov map's order-one FPM coincide with the map's power spectrum modes. It is conjectured (and proved in this chapter) that by constructing the map to satisfy a subset of elements from an estimate of the unknown map's order-one FPM, the characteristics of its power spectrum modes coincide with those of the unknown map.

³It is implicitly assumed here that the order-one FPM of the map is diagonalisable.

Two novel strategies for solving IFPP-III by reconstructing an unknown semi-Markov map from estimates of its order-one FPM elements are presented below.

5.4.1 Strategy 1: Reconstruction from the full order-one FPM

The original solution to IFPP-III, as proposed in [2, 3] and outlined in section 3.6.1, uses the following strategy to reconstruct an unknown semi-Markov map.

- Estimate the unknown map's *order-zero* FPM from sequences of *PWC* density function estimates.
- Design a *canonical* semi-Markov map to possess this matrix estimate as its order-zero FPM.

In this strategy, the PWC density function estimates are derived from evolving state ensembles, as generated during experiments wherein the unknown map is probed. These probing experiments use PWC probing densities (refer to section 3.6.1).

A novel strategy, referred to as *solution strategy 1* in this chapter, is proposed. This strategy generalises the original strategy in a direct fashion, as outlined below.

- Estimate the unknown map's *order-one* FPM from sequences of *PWL* density function estimates.
- Design a semi-Markov map with a *more general structure* to possess this matrix as its order-one FPM.

As the order-one FPM characterises the evolution of PWL density functions under iteration of the map, the estimate for this matrix is derived from estimates of PWL densities. Hence, the probing experiments are required to use PWL probing densities, and PWL density estimates are derived from the evolving state ensembles.

A design issue in solution strategy 1 is identified. This issue concerns the final step of designing the map to satisfy the order-one FPM estimate. It is observed that each non-zero element of submatrix $\mathbf{R}_S$ belonging to a semi-Markov map's order-one FPM is a function of the y -intercepts of one or more of the map's branch functions; refer to (2.119). Designing a semi-Markov map to satisfy the elements of submatrix $\mathbf{R}_S$ is deemed unfeasible. Although the semi-Markov maps permit an arbitrary number of branch functions, the design of the map requires the impractical enumeration

of candidate maps with different branch configurations in an attempt to find the solution that best satisfies the order-one FPM estimate.

The above design issue led to the development and pursuit of an alternative solution strategy for IFPP-III. This design strategy is outlined below.

5.4.2 Strategy 2: Reconstruction from the partial order-one FPM

The unfeasibility of solution strategy 1 is a consequence of requiring that the map satisfy the elements of submatrix $\mathbf{R}_S$ from the order-one FPM estimate. To remedy this problem, a second solution strategy, labelled *solution strategy 2*, is developed and pursued. This strategy designs a semi-Markov map to satisfy all the non-zero⁴ submatrices of the order-one FPM estimate, except for submatrix $\mathbf{R}_S$. More specifically, the map is to satisfy the matrix estimates of the order-zero FPM $\mathbf{P}_S^{(0)}$ and submatrix $\mathbf{Q}_S$ of the order-one FPM; refer to (2.117).

The implied characteristics of solution strategy 2 are as follows. In each case, perfect estimation accuracy is assumed.

- By designing the map to satisfy the order-zero FPM estimate, it is conjectured (and proved in this chapter) that its invariant density will be consistent with the invariant density of the unknown map. This is consequence of the invariant density coinciding with the right Perron eigenvector of the order-zero FPM; refer to Proposition 2.6.9.
- By designing the map to satisfy the estimate of submatrix $\mathbf{Q}_S$, it is conjectured that the reconstructed map's branches may be chosen with monotonicity that is consistent with the branches of an unknown canonical map. This follows from the previous observation that the elements of submatrix $\mathbf{Q}_S$ depend on the monotonicity of the map's branch functions and (are conjectured to) permit discernment of branch monotonicity in canonical maps; refer to (2.118).
- By designing the map to simultaneously satisfy the order-zero FPM estimate and the estimate of submatrix $\mathbf{Q}_S$, it is conjectured⁵ (and proved in this chapter) that the characteristics of its power spectrum modes coincide with those of the

⁴The zero-valued submatrix of the order-one FPM is satisfied by default.

⁵An implicit assumption of this statement is that the order-one FPM is diagonalisable. This is a consequence of deriving the expressions for the ACF and PSD of a PWL semi-Markov map under this assumption in chapter 2.

unknown map. This follows from the observation that the eigenvalues of the order-one FPM coincide with its power spectrum modes; refer to Remark 2.6.8.

The above conjectures are proved formally in Propositions 5.5.7 and 5.5.8 of section 5.5.5, and confirmed by results presented in Examples 5.6.1 and 5.6.2 of section 5.6.

The conjectures presented above imply that solution strategy 2 permits the realisation of branch functions with consistent monotonicity in unknown canonical maps and the consistent realisation of an unknown semi-Markov map's invariant density and power spectrum mode characteristics. This is achievable despite only partially satisfying the order-one FPM and irrespective of the monotonicity of the unknown map's branches. However, the conjectures do not imply the realisation of a power spectrum that is fully consistent with the unknown map.

The formal steps of solution strategy 2 are enumerated below (refer to [4]).

1. Construct an estimate of the unknown map's Markov partition.
2. Estimate the unknown map's order-zero FPM and submatrix $\mathbf{Q}_S$ of its order-one FPM from density function estimates. The latter estimates are derived from evolving state ensembles generated during multiple probing experiments.
3. Design a semi-Markov map with *a sufficiently general structure* to possess the order-zero FPM and submatrix $\mathbf{Q}_S$ of its order-one FPM that are equal to their counterparts from the order-one FPM estimate.

The successful implementation of steps 2 and 3 requires the resolution of two design issues. Specifically, the probing densities to be selected in each probing experiment that yield accurate estimates of the relevant submatrices are unknown. Furthermore, the structure of the semi-Markov map in the final step must be sufficiently diverse as to permit the realisation of any valid submatrix estimates, but simultaneously constrained to yield a feasible design algorithm. In the next section, a novel method that resolves these design issues to successfully implement the strategy is presented.

5.5 A Novel Method for Solving IFPP-III

This section presents a novel method for solving IFPP-III that implements the steps of solution strategy 2. The material presented in this section appears in [4].

The novel method constructs a PWL and ergodic semi-Markov map $\hat{S}$ to approximate an unknown map S , which is also assumed to be a PWL and ergodic semi-Markov map. It resolves the shortcomings of the original solution to IFPP-III [2, 3] as described in the previous section.

The steps of the novel method are outlined in the flowchart of Fig. 5.5.1. These steps are first summarised and then described in full detail.

- The first step of the novel method constructs an estimate $\hat{\mathcal{Q}}$ of the unknown map's Markov partition $\mathcal{Q}$.
- In the second step, two rounds of probing experiments are performed. The two rounds use probing densities that are defined to be PWC and PWL over the estimated Markov partition, respectively (the motivation for this design choice is to be provided in section 5.5.2). The unknown map's order-zero FPM $\mathbf{P}_S^{(0)}$ and submatrix $\mathbf{Q}_S$ of its order-one FPM $\mathbf{P}_S^{(1)}$ are estimated using these density estimates, thereby yielding the matrix estimates $\hat{\mathbf{P}}_S^{(0)}$ and $\hat{\mathbf{Q}}_S$.
- In the final step, the map $\hat{S}$ is constructed such that its order-zero FPM $\mathbf{P}_{\hat{S}}^{(0)}$ and submatrix $\mathbf{Q}_{\hat{S}}$ of its order-one FPM $\mathbf{P}_{\hat{S}}^{(1)}$ satisfy

$$\mathbf{P}_{\hat{S}}^{(0)} = \hat{\mathbf{P}}_S^{(0)} \quad (5.2)$$

and

$$\mathbf{Q}_{\hat{S}} = \hat{\mathbf{Q}}_S. \quad (5.3)$$

The map S constructed in the final step is a member of a novel subclass of the semi-Markov maps, referred to as the *generalised hat maps*. These maps are to be introduced in section 5.5.3.

5.5.1 Step 1: Estimation of the Markov partition

The novel method constructs an estimate $\hat{\mathcal{Q}} = \{\hat{Q}_1, \hat{Q}_2, \dots, \hat{Q}_{\hat{M}}\}$ of the unknown map's Markov partition $\mathcal{Q} = \{Q_1, Q_2, \dots, Q_M\}$ in a manner that is identical to that of the original solution [2, 3] and as described in section 3.6.1 on page 168.

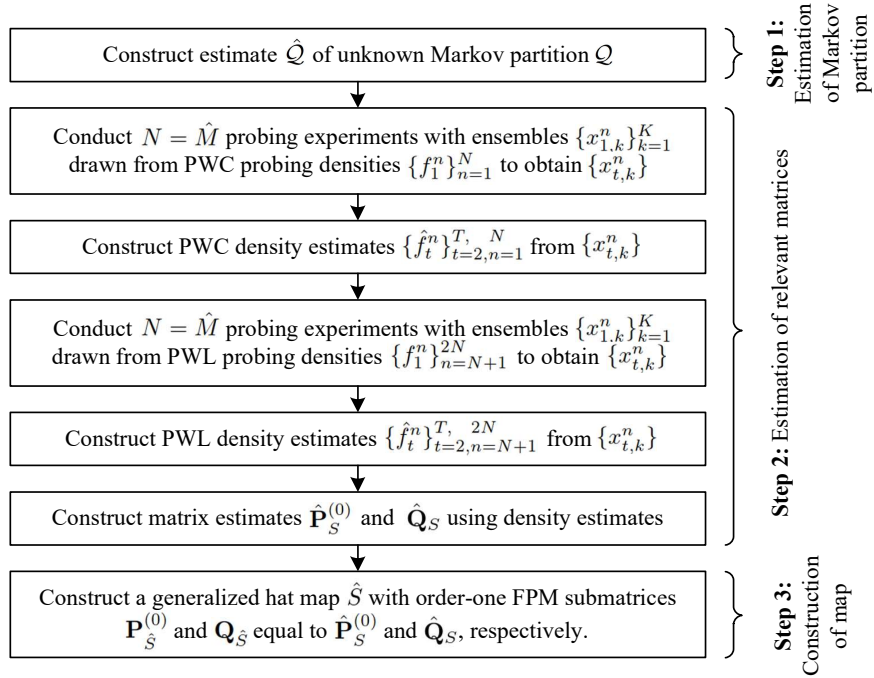


Figure 5.5.1: Flowchart of the novel method for solving IFPP-III. Reproduced with permission from [4] (© [2023] IEEE).

5.5.2 Step 2: Estimation of the relevant matrices

The approach to selecting probing densities and constructing density estimates in each round of probing experiments, such that accurate estimation of the relevant submatrices of the unknown map's order-one FPM is achieved, is outlined below.

- Submatrix $\mathbf{P}_S^{(0)}$ of the unknown map's order-zero FPM (i.e., the order-zero FPM) characterises the evolution of density functions that are PWC over the map's Markov partition. This leads to its estimation by specifying probing densities and constructing density estimates that are PWC over the estimated Markov partition.
- Submatrix $\mathbf{Q}_S$ of the order-one FPM describes the evolution of *the slopes* of densities that are PWL over the Markov partition; refer to (2.114) and (2.117). This leads to its estimation by specifying probing densities and constructing density estimates that are PWL over the estimated Markov partition, but selecting a non-zero value for the slope of each probing density.

The two rounds of probing experiments are described below.

First round of probing experiments

A first round of probing experiments $n = 1, 2, \dots, \hat{M}$ is conducted, where $\hat{M}$ denotes the number of intervals in the estimated Markov partition $\hat{\mathcal{Q}}$. The probing densities $f_1^1, f_1^2, \dots, f_1^{\hat{M}}$ are selected to be PWC over $\hat{\mathcal{Q}}$ according to the expression

$$f_1^n(x) = \sum_{m=1}^{\hat{M}} f_{1,0,m}^n \chi_{\hat{\mathcal{Q}}_m}(x), \quad (5.4)$$

where

$$f_{1,0,m}^n = \frac{1}{\mu(\hat{\mathcal{Q}}_m)} \delta[n - m] \quad (5.5)$$

and $\delta[\cdot]$ denotes the Kronecker delta function. These probing densities are plotted in Fig. 5.5.2(a). These probing densities are identical to the probing densities selected for the probing experiments of the original solution, as outlined in section 3.6.1.

The probing densities are selected to ensure that the matrix $\hat{\mathbf{A}}$, which is to be defined below, has full rank. This matrix contains as a subset of its columns the vector representations of the probing densities. The full rank of this matrix is a condition for the asymptotic optimality of the matrix estimate given by (5.38) for the unknown map's order-zero FPM (refer to Proposition 5.5.3).

In each probing experiment $n = 1, 2, \dots, \hat{M}$, an ensemble $\{x_{1,k}^n\}_{k=1}^K$ of K states is drawn independently from the probing density f_1^n and permitted to evolve under iteration of S . The ensemble is measured at time instants $t = 2, \dots, T$. Let these measurements be denoted by $\{x_{t,k}^n\}_{k=1}^K$.

After measuring the state ensembles, the density function estimates $\hat{f}_t^1, \hat{f}_t^2, \dots, \hat{f}_t^{\hat{M}}$, where $t = 1, 2, \dots, T$, are constructed to be PWC over $\hat{\mathcal{Q}}$ according to the expression

$$\hat{f}_t^n(x) = \begin{cases} f_1^n(x) & \text{if } t = 1, \\ \sum_{m=1}^{\hat{M}} \hat{f}_{t,0,m}^n \chi_{\hat{\mathcal{Q}}_m}(x) & \text{if } t = 2, 3, \dots, T, \end{cases} \quad (5.6)$$

where

$$\hat{f}_{t,0,m}^n = \frac{1}{K \mu(\hat{\mathcal{Q}}_m)} \sum_{k=1}^K \chi_{\hat{\mathcal{Q}}_m}(x_{t,k}^n). \quad (5.7)$$

These density estimates are identical to those constructed in the probing experiments of the original solution, as outlined in section 3.6.1.

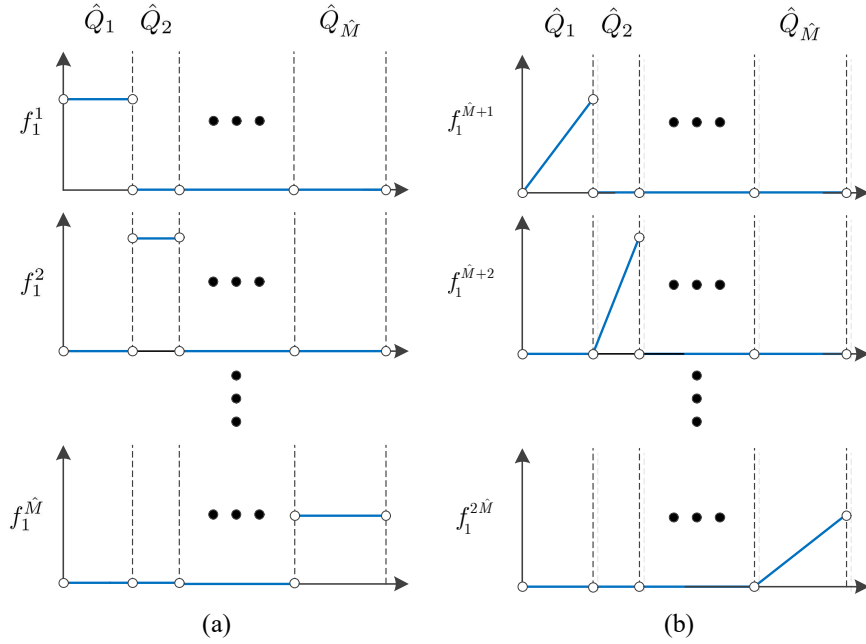


Figure 5.5.2: Probing densities used in the (a) first round and (b) second round of probing experiments of the novel method. Reproduced with permission from [4] (© [2023] IEEE).

The matrices $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$, which possess as columns the order-zero vector representations of the density function estimates, are then computed according to the expressions

$$\hat{\mathbf{A}} \triangleq [\hat{\mathbf{F}}_1, \hat{\mathbf{F}}_2, \dots, \hat{\mathbf{F}}_{T-1}] \quad (5.8)$$

and

$$\hat{\mathbf{B}} \triangleq [\hat{\mathbf{F}}_2, \hat{\mathbf{F}}_3, \dots, \hat{\mathbf{F}}_T], \quad (5.9)$$

where

$$\hat{\mathbf{F}}_t \triangleq [\hat{f}_t^1, \hat{f}_t^2, \dots, \hat{f}_t^{\hat{M}}] \quad (5.10)$$

and $\hat{f}_t^n$ denotes the vector representation of the density f_t^n over the partition $\hat{\mathcal{Q}}$.

Second round of probing experiments

A second round of probing experiments $n = \hat{M} + 1, \hat{M} + 2, \dots, 2\hat{M}$ is performed next. Let the partition points of the partition $\hat{\mathcal{Q}}$ be denoted by $\bar{P}(\hat{\mathcal{Q}}) = \{\hat{q}_1, \hat{q}_2, \dots, \hat{q}_{\hat{M}+1}\}$. The probing densities of these experiments, denoted by $f_1^{\hat{M}+1}, f_1^{\hat{M}+2}, \dots, f_1^{2\hat{M}}$, are selected to be PWL over $\hat{\mathcal{Q}}$ according to the expression⁶

$$f_1^n(x) = \sum_{m=1}^{\hat{M}} (f_{1,1,m}^n x + f_{1,0,m}^n) \chi_{\hat{Q}_m}(x), \quad (5.11)$$

⁶The expressions for these densities are derived in section 5.A.

where

$$f_{1,0,m}^n = -\frac{2\hat{q}_m}{\mu(\hat{Q}_m)^2}\delta[n-m] \quad (5.12)$$

and

$$f_{1,1,m}^n = \frac{2}{\mu(\hat{Q}_m)^2}\delta[n-m]. \quad (5.13)$$

These probing densities are plotted in Fig. 5.5.2(b).

The probing densities are selected to ensure that the matrix $\hat{\mathbf{C}}$, which is to be defined below, has full rank. This matrix contains as a subset of its columns the slopes of the probing densities. The full rank of this matrix is a condition for the asymptotic optimality of the matrix estimate given by (5.39) for submatrix $\mathbf{Q}_S$ of the unknown map's order-one FPM (refer to Proposition 5.5.3).

In each probing experiment $n = \hat{M} + 1, \hat{M} + 2, \dots, 2\hat{M}$, an ensemble $\{x_{1,k}^n\}_{k=1}^K$ of K states is drawn independently from the probing density f_1^n and permitted to evolve under iteration of S . The ensemble is measured at time instants $t = 2, \dots, T$. Let these measurements be denoted by $\{x_{t,k}^n\}_{k=1}^K$.

Densities that are PWL over $\mathcal{Q}$ remain PWL over the same partition under iteration of an $\mathcal{R}(\mathcal{Q})$ -semi-Markov map (refer to section 2.6.3). Thus, after conducting the probing experiments, the density function estimates $\hat{f}_t^{\hat{M}+1}, \hat{f}_t^{\hat{M}+2}, \dots, \hat{f}_t^{2\hat{M}}$, where $t = 1, 2, \dots, T$, are constructed to be PWL over the estimated partition $\hat{\mathcal{Q}}$ according to the expression

$$\hat{f}_t^n(x) = \begin{cases} f_1^n(x) & \text{if } t = 1, \\ \sum_{m=1}^{\hat{M}} (\hat{f}_{t,1,m}^n x + \hat{f}_{t,0,m}^n) \chi_{\hat{Q}_m}(x) & \text{if } t = 2, \dots, T. \end{cases} \quad (5.14)$$

The coefficients $\hat{f}_{t,0,m}^n$ and $\hat{f}_{t,1,m}^n$ are derived by applying a method-of-moments technique,⁷ which is developed in section 5.B, to estimate the density $f_t^n|_{\hat{Q}_m}$. The coefficients yielded by this technique are given by

$$\hat{f}_{t,0,m}^n = \frac{1}{\mu(\hat{Q}_m)} \left[\hat{c}_{t,0,m}^n - \frac{6(\hat{q}_{m+1} + \hat{q}_m)}{\mu(\hat{Q}_m)^2} \hat{c}_{t,1,m}^n \right] \quad (5.15)$$

and

$$\hat{f}_{t,1,m}^n = \frac{12\hat{c}_{t,1,m}^n}{\mu(\hat{Q}_m)^3}, \quad (5.16)$$

where

$$\hat{c}_{t,p,m}^n \triangleq \frac{1}{K} \sum_{k=1}^K \left(x_{t,k}^n - \frac{\hat{q}_m + \hat{q}_{m+1}}{2} \right)^p \chi_{\hat{Q}_m}(x_{t,k}^n) \quad (5.17)$$

⁷This technique is novel in the context of IFPP-III.

denotes the p^{th} empirical moment of $f_t^n|_{\hat{Q}_m}$ centred around the midpoint of $\hat{Q}_m$.

Let

$$\hat{\mathbf{F}}_{t,1} \triangleq [\hat{\mathbf{f}}_{t,1}^{\hat{M}}, \hat{\mathbf{f}}_{t,1}^{\hat{M}+1}, \dots, \hat{\mathbf{f}}_{t,1}^{2\hat{M}}], \quad (5.18)$$

where

$$\hat{\mathbf{f}}_{t,1}^n \triangleq [\hat{f}_{t,1,1}^n, \hat{f}_{t,1,2}^n, \dots, \hat{f}_{t,1,\hat{M}}^n]^T. \quad (5.19)$$

The matrices $\hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$, which possess as columns the slopes of the density function estimates, are then derived according to the expressions

$$\hat{\mathbf{C}} \triangleq [\hat{\mathbf{F}}_{1,1}, \hat{\mathbf{F}}_{2,1}, \dots, \hat{\mathbf{F}}_{T-1,1}] \quad (5.20)$$

and

$$\hat{\mathbf{D}} \triangleq [\hat{\mathbf{F}}_{2,1}, \hat{\mathbf{F}}_{3,1}, \dots, \hat{\mathbf{F}}_{T,1}]. \quad (5.21)$$

All density function estimates required to estimate the unknown map's order-zero FPM $\mathbf{P}_S^{(0)}$ and submatrix $\mathbf{Q}_S$ of its order-one FPM are represented by the matrices $\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$. The next step is to estimate $\mathbf{P}_S^{(0)}$ and $\mathbf{Q}_S$ from these matrices.

Estimation of matrices

In this section, conditions are first derived that estimates of the unknown map's order-zero FPM and submatrix $\mathbf{Q}_S$ of its order-one FPM must satisfy to qualify as valid matrices of any semi-Markov map. The problems of estimating the matrices are then formulated as optimisation problems with constraints that coincide with these conditions. The matrix estimates used by the novel method for solving IFPP-III are then presented, and their properties stated.

Conditions for the matrix estimates Candidate estimates $\hat{\mathbf{P}}_S^{(0)}$ and $\hat{\mathbf{Q}}_S$ need to satisfy certain conditions to qualify as valid submatrices of a PWL and ergodic semi-Markov map's order-one FPM. It is observed that every order-zero FPM $\mathbf{P}_S^{(0)}$ associated with a map from this class is necessarily non-negative and necessarily possesses a unity Perron eigenvalue with left Perron eigenvector that coincides with the lengths of the intervals in the relevant Markov partition; refer to (2.122) and Proposition 2.6.4. Furthermore, every piecewise expanding $\mathcal{R}(\mathcal{Q})$ -semi-Markov map S has branch functions $S_{j,m} \triangleq S|_{R_j^{(m)}}$ with slopes $|v_{j,m}| > 1$ that satisfy

$$\sum_{j \in T(k,m)} \frac{s_{j,m}}{v_{j,m}^2} \leq \sum_{j \in T(k,m)} \frac{1}{v_{j,m}^2} \leq \left[\sum_{j \in T(k,m)} \frac{1}{|v_{j,m}|} \right]^2, \quad (5.22)$$

where $s_{j,m} = \text{sgn}(v_{j,m})$ and $T(k, m)$ is defined in (2.120). Together with (2.118) and (2.122), this inequality implies that the elements of matrices $\mathbf{Q}_S = [q_{k,m}]_{k,m=1}^M$ and $\mathbf{P}_S^{(0)} = [p_{k,m}]_{k,m=1}^M$ must satisfy

$$|q_{k,m}| \leq p_{k,m}^2. \quad (5.23)$$

The matrix estimates are required to satisfy these three conditions.

Formulation of the constrained optimisation problems If it were assumed that $\hat{\mathcal{Q}} = \mathcal{Q}$ and all density estimates are perfect (i.e., $\hat{f}_t^n = f_t^n$), the matrices $\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$ satisfy

$$\hat{\mathbf{B}} = \mathbf{P}_S^{(0)} \hat{\mathbf{A}} \quad (5.24)$$

and

$$\hat{\mathbf{D}} = \mathbf{Q}_S \hat{\mathbf{C}}. \quad (5.25)$$

This motivates the approach of estimating $\mathbf{P}_S^{(0)}$ and $\mathbf{Q}_S$ by solving the following *joint* constrained optimisation problem.

Optimisation Problem 5.5.1 (Joint estimation of $\mathbf{P}_S^{(0)}$ and $\mathbf{Q}_S$ [4]).

Solve

$$\mathbf{P}_S^*, \mathbf{Q}_S^* = \arg \min_{\mathbf{P}, \mathbf{Q}} : \|\hat{\mathbf{W}} - \mathbf{D}_{\mathbf{P}, \mathbf{Q}} \hat{\mathbf{V}}\|_F \quad (5.26)$$

subject to the following constraints.

1. The left Perron eigenvector of $\mathbf{P}$ coincides with the lengths of the intervals in the estimated Markov partition,

$$\hat{\mathbf{q}}^T \mathbf{P} = \hat{\mathbf{q}}^T. \quad (5.27)$$

2. The matrix $\mathbf{P} \triangleq [p_{k,m}]_{k,m=1}^{\hat{M}}$ is non-negative.

3. The elements of $\mathbf{Q} \triangleq [q_{k,m}]_{k,m=1}^{\hat{M}}$ satisfy

$$|q_{k,m}| \leq p_{k,m}^2. \quad (5.28)$$

In these expressions, $\|\cdot\|_F$ denotes the Frobenius norm and

$$\hat{\mathbf{q}}^T = [\mu(\hat{Q}_1), \mu(\hat{Q}_2), \dots, \mu(\hat{Q}_{\hat{M}})]. \quad (5.29)$$

The matrices in (5.26) are defined as

$$\hat{\mathbf{V}} \triangleq [\hat{\mathbf{C}}; \hat{\mathbf{A}}] \quad (5.30)$$

and

$$\hat{\mathbf{W}} \triangleq [\hat{\mathbf{D}}; \hat{\mathbf{B}}], \quad (5.31)$$

where ‘;’ denotes vertical matrix concatenation, and

$$\mathbf{D}_{\mathbf{P}, \mathbf{Q}} \triangleq \begin{pmatrix} \mathbf{Q} & \mathbf{0}_{\hat{M} \times \hat{M}} \\ \mathbf{0}_{\hat{M} \times \hat{M}} & \mathbf{P} \end{pmatrix}. \quad (5.32)$$

The following proposition is proved.

Proposition 5.5.1. Optimisation Problem 5.5.1 is not convex [4].

Proof. The proposition is proved in section 5.C.1. $\square$

The non-convexity of Optimisation Problem 5.5.1 motivates the formulation and sequential solution of two *separate* optimisation problems. The first problem (and its solution in the novel method) is identical to the problem described in section 3.6.1 for estimating the order-zero FPM. This problem is formally defined below.

Optimisation Problem 5.5.2 (Estimation of $\mathbf{P}_S^{(0)}$ [4]).

Solve

$$\mathbf{P}_S^* = \arg \min_{\mathbf{P}} : \|\hat{\mathbf{B}} - \mathbf{P}\hat{\mathbf{A}}\|_F, \quad (5.33)$$

subject to the following constraints.

1. The left Perron eigenvector of $\mathbf{P}$ coincides with the lengths of the intervals in the estimated Markov partition,

$$\hat{\mathbf{q}}^T \mathbf{P} = \hat{\mathbf{q}}^T. \quad (5.34)$$

2. The matrix $\mathbf{P} \triangleq [p_{k,m}]_{k,m=1}^{\hat{M}}$ is non-negative; i.e.,

$$p_{k,m} \geq 0, \quad k, m = 1, 2, \dots, \hat{M}. \quad (5.35)$$

The second problem is formulated to permit estimation of submatrix $\mathbf{Q}_S$. This problem is formally defined below.

Optimisation Problem 5.5.3 (Estimation of $\mathbf{Q}_S$ [4]).

Solve

$$\mathbf{Q}_S^* = \arg \min_{\mathbf{Q}} : \|\hat{\mathbf{D}} - \mathbf{Q}\hat{\mathbf{C}}\|_F, \quad (5.36)$$

subject to the elements of $\mathbf{Q} \triangleq [q_{k,m}]_{k,m=1}^{\hat{M}}$ satisfying

$$|q_{k,m}| \leq \hat{p}_{k,m}^2, \quad (5.37)$$

where $\hat{\mathbf{P}}_S^{(0)} \triangleq [\hat{p}_{k,m}]_{k,m=1}^{\hat{M}}$ is the matrix estimate of $\mathbf{P}_S^{(0)}$ obtained by solving Optimisation Problem 5.5.2.

The following proposition follows.

Proposition 5.5.2. Optimisation Problems 5.5.2 and 5.5.3 are both convex [4].

Proof. The proposition is proved in sections 5.C.2 and 5.C.3 for Optimisation Problems 5.5.2 and 5.5.3, respectively. $\square$

Proposition 5.5.2 implies that standard methods for solving convex problems may be used to approximate the optimal solutions of Optimisation Problems 5.5.2 and 5.5.3. However, by considering separate problems, the resulting estimates are generally a suboptimal solution to the joint problem (i.e., Optimisation Problem 5.5.1), regardless of the method used.

Solution of the constrained optimisation problems Alternative solutions to Optimisation Problems 5.5.2 and 5.5.3 are developed. These solutions are used by the novel method for solving IFPP-III to derive estimates for the unknown map's order-zero FPM and submatrix $\mathbf{Q}_S$ of its order-one FPM. Note that the solution presented for Optimisation Problem 5.5.2 is identical to that presented in section 3.6.1 on page 171; it is repeated here for convenience.

The solutions to Optimisation Problems 5.5.2 and 5.5.3 use the estimates

$$\overline{\mathbf{P}}_S^{(0)} = \hat{\mathbf{B}}\hat{\mathbf{A}}^\dagger \quad (5.38)$$

and

$$\overline{\mathbf{Q}}_S = \hat{\mathbf{D}}\hat{\mathbf{C}}^\dagger, \quad (5.39)$$

where $\mathbf{Z}^\dagger$ denotes the Moore-Penrose pseudoinverse of $\mathbf{Z}$; i.e.,

$$\mathbf{Z}^\dagger \triangleq \mathbf{Z}^T(\mathbf{Z}\mathbf{Z}^T)^{-1}. \quad (5.40)$$

These estimates are motivated by the property that they minimise the Frobenius norms of (5.33) and (5.36) [10]. Furthermore, they possess the property outlined in the following proposition.

Proposition 5.5.3 (Optimality of estimates [4]). Suppose that all densities are estimated with perfect accuracy, such that

$$\hat{\mathbf{B}} = \mathbf{P}_S^{(0)} \hat{\mathbf{A}} \quad (5.41)$$

and

$$\hat{\mathbf{D}} = \mathbf{Q}_S \hat{\mathbf{C}}. \quad (5.42)$$

both hold. Then, the estimates $\hat{\mathbf{P}}_S^{(0)} = \overline{\mathbf{P}}_S^{(0)}$ and $\hat{\mathbf{Q}}_S = \overline{\mathbf{Q}}_S$, where $\overline{\mathbf{P}}_S^{(0)}$ and $\overline{\mathbf{Q}}_S$ are defined in (5.38) and (5.39), are the *optimal solutions* to the separate optimisation problems (i.e., Optimisation Problems 5.5.2 and 5.5.3), as well as the joint optimisation problem (i.e., Optimisation Problem 5.5.1).

Proof. A condensed version of this proof appears in [4]. By selecting the probing densities defined by (5.4) and (5.11), the matrices $\hat{\mathbf{F}}_1$ and $\hat{\mathbf{F}}_{1,1}$, as defined in (5.10) and (5.18), become diagonal matrices with strictly positive elements along the diagonal. This implies that matrices $\hat{\mathbf{A}}$ and $\hat{\mathbf{C}}$, as defined in (5.8) and (5.20), each have full rank.

Now, suppose that all density functions are estimated with perfect accuracy, such that (5.41) and (5.42) both hold. It then follows from [10, Th. 1] that

$$\overline{\mathbf{P}}_S^{(0)} = \mathbf{P}_S^{(0)} \hat{\mathbf{A}} \hat{\mathbf{A}}^\dagger = \mathbf{P}_S^{(0)}. \quad (5.43)$$

and

$$\overline{\mathbf{Q}}_S = \mathbf{Q}_S \hat{\mathbf{C}} \hat{\mathbf{C}}^\dagger = \mathbf{Q}_S. \quad (5.44)$$

This implies the optimality of the estimates for each of the problems. $\square$

In the more general case where density estimation is not perfect, the estimates $\overline{\mathbf{P}}_S^{(0)}$ and $\overline{\mathbf{Q}}_S$, as defined in (5.38) and (5.39), do not necessarily satisfy the constraints of (5.34), (5.35) and (5.37). In the novel method, the matrices $\overline{\mathbf{P}}_S^{(0)}$ and $\overline{\mathbf{Q}}_S$ are manipulated to satisfy these constraints, thereby yielding the estimates $\hat{\mathbf{P}}_S^{(0)}$ and $\hat{\mathbf{Q}}_S$. The manipulation of the matrices to yield each estimate is described below.

For the estimate $\hat{\mathbf{P}}_S$, all negative elements of $\overline{\mathbf{P}}_S^{(0)}$ are first replaced with zeros. Let the columns of the resulting matrix be denoted by $\overline{\mathbf{p}}_1, \overline{\mathbf{p}}_2, \dots, \overline{\mathbf{p}}_{\hat{M}}$. The approximate solution

$$\hat{\mathbf{P}}_S^{(0)} \triangleq [\hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2, \dots, \hat{\mathbf{p}}_{\hat{M}}] \quad (5.45)$$

is then constructed by normalising the columns of the resulting matrix as

$$\hat{\mathbf{p}}_m = \frac{\mu(\hat{Q}_m)}{\hat{\mathbf{q}}^T \overline{\mathbf{p}}_m} \overline{\mathbf{p}}_m, \quad m = 1, 2, \dots, \hat{M}. \quad (5.46)$$

The estimate $\hat{\mathbf{P}}_S^{(0)}$ then satisfies the constraints of (5.34) and (5.35).

The elements of the estimate $\hat{\mathbf{Q}}_S \triangleq [\hat{q}_{k,m}]_{k,m=1}^{\hat{M}}$ are selected from $\overline{\mathbf{Q}}_S = [\bar{q}_{k,m}]_{k,m=1}^{\hat{M}}$ and $\hat{\mathbf{P}}_S^{(0)} \triangleq [\hat{p}_{k,m}]_{k,m=1}^{\hat{M}}$ according to the expression

$$\hat{q}_{k,m} = \begin{cases} \bar{q}_{k,m} & \text{if } -\hat{p}_{k,m}^2 \leq \bar{q}_{k,m} \leq \hat{p}_{k,m}^2, \\ -\hat{p}_{k,m}^2 & \text{if } \bar{q}_{k,m} < -\hat{p}_{k,m}^2, \\ \hat{p}_{k,m}^2 & \text{if } \bar{q}_{k,m} > \hat{p}_{k,m}^2, \end{cases} \quad (5.47)$$

to satisfy the constraint of (5.37).

Remark 5.5.1. The matrix estimates defined by (5.46) and (5.47) are suboptimal solutions to Optimisation Problems 5.5.2 and 5.5.3 in the general case where density estimation is imperfect [4]. As stated in Remark 3.6.1, an imperfect estimate of $\mathbf{P}_S^{(0)}$ introduces errors into the reconstructed map – specifically, the domains and slopes of the reconstructed branches deviate from those of the unknown map. In addition, an imperfect estimate of $\mathbf{Q}_S$ introduces *unnecessary branches* into the reconstructed map. Both of these errors are observed in Example 5.6.1 of section 5.6. However, Proposition 5.5.3 leads to the conjecture that the overall reconstruction accuracy will improve with greater density estimation accuracy (i.e., by assigning more states per ensemble in the probing experiments). The numerical results presented in Example 5.6.1 support this conjecture.

5.5.3 Step 3: Construction of the map

In the final step, the novel solution to IFPP-III constructs a PWL semi-Markov map to satisfy the matrix estimates derived in the previous step. However, the structure of the maps belonging to the candidate class of solutions must be sufficiently diverse as to permit the realisation of any valid submatrix estimates, while simultaneously constrained to yield a feasible design algorithm.

To resolve the above design issue, a novel subclass of the semi-Markov maps, referred to as the *generalised hat maps*, is defined. These maps are used in the final step of the novel method. In what follows, these maps are defined, their relevant properties are outlined, and an algorithm is provided for constructing a generalised hat map to satisfy the matrix estimates.

Generalised hat maps

The class of *hat maps* is introduced in section 4.6.3. Maps from this class possess pairs of neighbouring branches with a hat-like structure; refer to Fig. 4.6.4 for an example. The left and right branches of each hat are defined to have a positive and negative slope of equal magnitude and are symmetric around the hat's midpoint. In this chapter, these maps are referred to as the *ordinary* hat maps.

The *generalised hat maps* introduced in this section generalises the ordinary hat maps by permitting for the asymmetry of the left and right branches of each hat. These branches are permitted slopes that are unequal in magnitude, but it is still required that the left and right branch are increasing and decreasing, respectively. It is permissible to define any hat of a generalised hat map such that the left or right branch disappears. An example of a generalised hat map is plotted in Fig. 5.5.3.

Conceptually, any generalised hat map may be constructed from a canonical semi-Markov map, similar to the manner in which any ordinary hat map may be constructed from a canonical semi-Markov map; refer to Algorithm 4.6.4 (hat map construction II). This approach is followed below to define the class of generalised hat maps.

Definition Let $\tilde{S}$ denote a PWL and canonical $\tilde{\mathcal{R}}(\tilde{\mathcal{Q}})$ -semi-Markov map with the Markov partition $\tilde{\mathcal{Q}} = \{\tilde{Q}_1, \tilde{Q}_2, \dots, \tilde{Q}_{\tilde{M}}\}$, where $\tilde{\mathcal{R}} = \{\tilde{\mathcal{R}}^{(1)}, \tilde{\mathcal{R}}^{(2)}, \dots, \tilde{\mathcal{R}}^{(\tilde{M})}\}$ denotes the relevant sub-partition of $\tilde{\mathcal{Q}}$ and $\tilde{\mathcal{R}}^{(m)} \triangleq \{\tilde{R}_1^{(m)}, \tilde{R}_2^{(m)}, \dots, \tilde{R}_{\tilde{c}(m)}^{(m)}\}$ denotes the relevant partition of the interval $\tilde{Q}_m$. Suppose that the intervals

$$\begin{aligned}\tilde{A}_j^{(m)} &\triangleq (\inf(\tilde{R}_j^{(m)}), \inf(\tilde{R}_j^{(m)}) + \tilde{\beta}_{j,m}\mu(\tilde{R}_j^{(m)})), \\ \tilde{B}_j^{(m)} &\triangleq (\inf(\tilde{R}_j^{(m)}) + \tilde{\beta}_{j,m}\mu(\tilde{R}_j^{(m)}), \sup(\tilde{R}_j^{(m)}))\end{aligned}\tag{5.48}$$

are defined from some predetermined parameter $\tilde{\beta}_{j,m} \in [0, 1]$. These intervals form a partition of $\tilde{R}_j^{(m)}$, as indicated in Fig. 5.5.4; in the case where $\tilde{\beta}_{j,m} = 0$ or $\tilde{\beta}_{j,m} = 1$, the interval $\tilde{A}_j^{(m)}$ or $\tilde{B}_j^{(m)}$ is defined to be the empty interval, respectively.

The generalised hat map $\hat{S}$ is derived from its counterpart $\tilde{S}$ and a collection of specified parameters $\tilde{\beta}_{j,m}$, where $m = 1, 2, \dots, \tilde{M}$ and $j = 1, 2, \dots, \tilde{c}(m)$. The map $\hat{S}$ is defined to be $\hat{\mathcal{R}}(\hat{\mathcal{Q}})$ -semi-Markov, such that it inherits its Markov partition $\hat{\mathcal{Q}} = \{\hat{Q}_1, \hat{Q}_2, \dots, \hat{Q}_{\hat{M}}\}$ from $\tilde{S}$, and where $\hat{\mathcal{R}} = \{\hat{\mathcal{R}}^{(1)}, \hat{\mathcal{R}}^{(2)}, \dots, \hat{\mathcal{R}}^{(\hat{M})}\}$ denotes a sub-partition of $\hat{\mathcal{Q}}$. The partition $\hat{\mathcal{R}}^{(m)}$ of the interval $\hat{Q}_m$ is selected as

$$\hat{\mathcal{R}}^{(m)} \triangleq \{\tilde{A}_j^{(m)}\}_{j=1}^{\tilde{c}(m)} \cup \{\tilde{B}_j^{(m)}\}_{j=1}^{\tilde{c}(m)}.\tag{5.49}$$

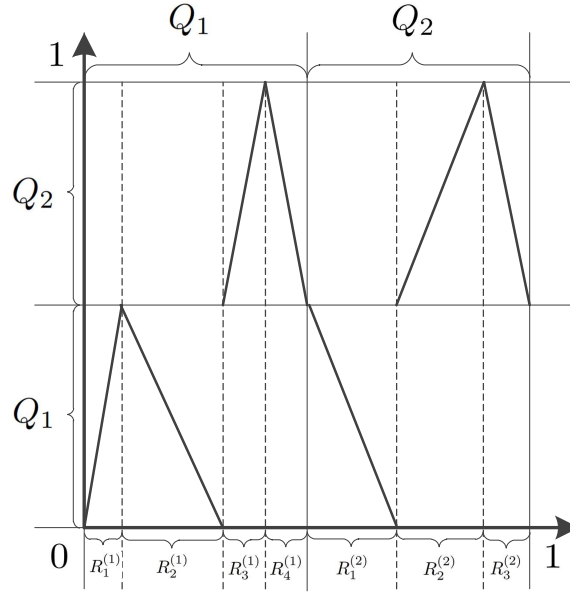


Figure 5.5.3: Example of a generalised hat map.

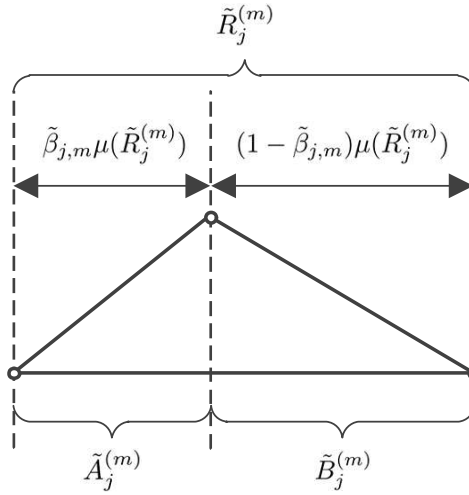


Figure 5.5.4: A hat belonging to a generalised hat map, illustrating the division of the interval $\tilde{R}_j^{(m)}$ and the definition of the branch functions. Reproduced with permission from [4] (© [2023] IEEE).

The map $\hat{S}$ is designed to map all non-empty intervals $\tilde{A}_j^{(m)}$ and $\tilde{B}_j^{(m)}$ to the same interval that $\tilde{R}_j^{(m)}$ is mapped to by $\tilde{S}$; i.e.,

$$\tilde{A}_j^{(m)} \xrightarrow{\hat{S}} \tilde{S}(\tilde{R}_j^{(m)}), \quad \text{if } \tilde{\beta}_{j,m} > 0, \quad (5.50)$$

$$\tilde{B}_j^{(m)} \xrightarrow{\hat{S}} \tilde{S}(\tilde{R}_j^{(m)}), \quad \text{if } \tilde{\beta}_{j,m} < 1. \quad (5.51)$$

It is specified that the branches of $\hat{S}$ defined over $\tilde{A}_j^{(m)}$ and $\tilde{B}_j^{(m)}$ are increasing and decreasing, respectively (refer to Fig. 5.5.4). This implies that the single branch of $\hat{S}$ defined over $\tilde{R}_j^{(m)}$ is decreasing or increasing if $\tilde{\beta}_{j,m} = 0$ or $\tilde{\beta}_{j,m} = 1$, respectively.

Any map $\hat{S}$ derived from a PWL and canonical $\tilde{\mathcal{R}}(\tilde{\mathcal{Q}})$ -semi-Markov map $\tilde{S}$ and a collection of parameters $\tilde{\beta}_{j,m}$ in the manner outlined above is referred to as a generalised hat map. By selecting appropriate values for parameters $\tilde{\beta}_{j,m} \in \{0, 1\}$ according to the monotonicity of the branches of any canonical map $\tilde{S}$, it is always possible to construct a generalised hat map that satisfies $\hat{S} = \tilde{S}$. Thus, the canonical maps are a subset of the generalised hat maps.

Remark 5.5.2. By defining the generalised hat maps via their construction from a canonical semi-Markov map, the hats of any generalised hat map necessarily appear in an ascending order over each interval $\hat{Q}_m$ of the Markov partition; this is illustrated in Fig. 5.5.3. It is possible to further generalise these maps by permitting an arbitrary order for the hats over each interval of the Markov partition.

Properties The following propositions describe two properties of generalised hat maps that are relevant to the final step of the novel solution for IFPP-III.

Proposition 5.5.4. The generalised hat map $\hat{S}$ inherits the order-zero FPM $\mathbf{P}_{\hat{S}}^{(0)} \triangleq [p_{k,m}^{(\hat{S})}]_{k,m=1}^{\tilde{M}}$ of its counterpart canonical map $\tilde{S}$ [4].

Proof. The proposition is proved in section 5.D.1 □

Proposition 5.5.5. Let $\mathbf{Q}_{\hat{S}} \triangleq [q_{k,m}^{(\hat{S})}]_{k,m=1}^{\tilde{M}}$ denote the top-left submatrix of the order-one FPM belonging to any generalised hat map $\hat{S}$. The zero-valued elements of this matrix and the zero-valued elements of the order-zero FPM $\mathbf{P}_{\tilde{S}}^{(0)}$ of its counterpart canonical map $\tilde{S}$ coincide, whereas their non-zero elements satisfy [4]

$$q_{k,m}^{(\hat{S})} = (2\tilde{\beta}_{\tilde{T}(k,m),m} - 1)(p_{k,m}^{(\hat{S})})^2, \quad (5.52)$$

where $\tilde{T}(k, m)$ denotes the index j satisfying $\tilde{S}(\tilde{R}_j^{(m)}) = \tilde{Q}_k$.

Proof. The proposition is proved in section 5.D.2 □

Construction The properties outlined in Propositions 5.5.4 and 5.5.5 are used to construct a generalised hat map $\hat{S}$, given by

$$\hat{S}(x) = \sum_{m=1}^M \sum_{j=1}^{c(m)} \left[\hat{S}_{2j-1,m}(x) \chi_{\hat{R}_{2j-1}^{(m)}}(x) + \hat{S}_{2j,m}(x) \chi_{\hat{R}_{2j}^{(m)}}(x) \right]. \quad (5.53)$$

The map $\hat{S}$ is designed such that its order-zero FPM and the top-left submatrix $\mathbf{Q}_{\hat{S}}$ of its order-one FPM are equal to the estimates $\hat{\mathbf{P}}_S^{(0)} \triangleq [\hat{p}_{k,m}]_{k,m=1}^{\tilde{M}}$ and $\hat{\mathbf{Q}}_S \triangleq [\hat{q}_{k,m}]_{k,m=1}^{\tilde{M}}$

derived in the preceding solution step. A canonical semi-Markov map $\tilde{S}$ is first designed to possess an order-zero FPM that is equal to the estimate $\hat{\mathbf{P}}_S^{(0)}$ using Algorithm 3.4.1 (canonical map construction). The generalised hat map $\hat{S}$ is then constructed from $\tilde{S}$, thereby ensuring that the order-zero FPM of $\hat{S}$ also equals $\hat{\mathbf{P}}_S^{(0)}$ (Proposition 5.5.4). The parameters $\tilde{\beta}_{j,m}$ used during the construction of $\hat{S}$ are selected to ensure that $\mathbf{Q}_{\hat{S}}$ matches the estimate $\hat{\mathbf{Q}}_S$. Proposition 5.5.5 and (5.52) imply that this may be achieved by selecting

$$\tilde{\beta}_{j,m} = \frac{1}{2} \left(\frac{\hat{q}_{\tilde{U}(j,m),m}}{\hat{p}_{\tilde{U}(j,m),m}^2} + 1 \right) \quad (5.54)$$

for each $m = 1, 2, \dots, \tilde{M}$ and $j = 1, 2, \dots, \tilde{c}(m)$, where $\tilde{c}(m)$ denotes the number of intervals in the partition $\tilde{\mathcal{R}}^{(m)}$ and $\tilde{U}(j, m)$ is the index k satisfying $\tilde{S}(\tilde{R}_j^{(m)}) = \tilde{Q}_k$. An algorithm that implements the above steps is provided on page 307.

Algorithm 5.5.1 (generalised hat map construction): Construction of a generalised hat map $\hat{S}$ to possess a candidate order-zero FPM $\hat{\mathbf{P}}_S^{(0)}$ and a candidate submatrix $\hat{\mathbf{Q}}_S$ for its order-one FPM. Refer to [4].

- 1: Inputs: Markov partition estimate $\hat{\mathcal{Q}}$, candidate order-zero FPM $\hat{\mathbf{P}}_S^{(0)}$ that satisfies (5.34) and (5.35), and candidate submatrix $\hat{\mathbf{Q}}_S$ for the order-one FPM that satisfies (5.37).
 - 2: Output: Generalised hat map $\hat{S}$ that possesses $\hat{\mathbf{P}}_S^{(0)}$ as its order-zero FPM and $\hat{\mathbf{Q}}_S$ as the corresponding submatrix of its order-one FPM.
 - 3: Use Algorithm 3.4.1 (canonical map construction) to construct an $\tilde{\mathcal{R}}(\hat{\mathcal{Q}})$ -semi-Markov map $\tilde{S}$ to possess $\hat{\mathbf{P}}_S^{(0)}$ as its order-zero FPM, and let $\tilde{S}_{j,m}(x) = \tilde{S}|_{\tilde{R}_j^{(m)}}(x) = \tilde{a}_{j,m}(x - \tilde{r}_j^{(m)}) + \tilde{b}_{j,m}$, where $\tilde{R}_j^{(m)} = (\tilde{r}_j^{(m)}, \tilde{r}_{j+1}^{(m)})$.
 - 4: **for** $m \leftarrow 1, 2, \dots, \hat{M}$ **do** $\triangleright \hat{M}$ denotes the number of intervals in $\hat{\mathcal{Q}}$.
 - 5: **for** $j \leftarrow 1, 2, \dots, \tilde{c}(m)$ **do** $\triangleright \tilde{c}(m)$ denotes the number of intervals in $\tilde{\mathcal{R}}^{(m)}$.
 - 6: Derive $\tilde{\beta}_{j,m}$ using (5.54).
 - 7: Define $\hat{R}_{2j-1}^{(m)} = (\inf(\tilde{R}_j^{(m)}), \inf(\tilde{R}_j^{(m)}) + \tilde{\beta}_{j,m}\mu(\tilde{R}_j^{(m)}))$
 - 8: Define $\hat{S}_{2j-1,m}(x) = \hat{S}|_{\hat{R}_{2j-1}^{(m)}}(x) = 2\tilde{a}_{j,m}(x - \tilde{r}_j^{(m)}) + \tilde{b}_{j,m}$. $\triangleright$ Left branch
 - 9: Define $\hat{R}_{2j}^{(m)} = (\inf(\tilde{R}_j^{(m)}) + \tilde{\beta}_{j,m}\mu(\tilde{R}_j^{(m)}), \sup(\tilde{R}_j^{(m)}))$.
 - 10: Define $\hat{S}_{2j,m}(x) = \hat{S}|_{\hat{R}_{2j}^{(m)}}(x) = -2\tilde{a}_{j,m}(x - \tilde{r}_{j+1}^{(m)}) + \tilde{b}_{j,m}$. $\triangleright$ Right branch
 - 11: **end for**
 - 12: **end for**
 - 13: Return the map $\hat{S}$ defined by (5.53).
-

The following proposition is proved.

Proposition 5.5.6. Let $\hat{S}$ denote a map constructed using Algorithm 5.5.1 (generalised hat map construction). The order-zero FPM of $\hat{S}$ can be selected to equal *any*

order-zero FPM estimate $\hat{\mathbf{P}}_S^{(0)}$ that satisfies (5.34) and (5.35) [4]. Simultaneously, the top-left submatrix $\mathbf{Q}_{\hat{S}}$ of the order-one FPM belonging to $\hat{S}$ can be selected to equal *any* estimate $\hat{\mathbf{Q}}_S$ that satisfies (5.37).

Proof. The proposition is proved in section 5.D.3 □

Remark 5.5.3. The above proposition implies that a generalised hat map may be designed to satisfy *any* valid estimates that may be derived of the unknown semi-Markov map's matrices.

5.5.4 Algorithm

An algorithm that implements the novel method for solving IFPP-III follows.

Algorithm 5.5.2 (novel method for solving IFPP-III): Implementation of the novel method for solving IFPP-III. Refer to [4].

- 1: Inputs: Unknown semi-Markov map S that is PWL and ergodic (black box).
 - 2: Output: Reconstructed map $\hat{S}$.
 - 3: Construct an estimate $\hat{\mathcal{Q}}$ of the unknown map's Markov partition $\mathcal{Q}$ using the existing method (refer to section 3.6.1 on page 168).
 - 4: Conduct the first round of $N = \hat{M}$ probing experiments using the probing densities of (5.4) to yield the ensembles $\{x_{t,k}^n\}$, $n = 1, 2, \dots, \hat{M}$.
 - 5: Construct the density estimates $\hat{f}_t^1, \hat{f}_t^2, \dots, \hat{f}_t^{\hat{M}}$, where $t = 1, 2, \dots, T$, from the above ensembles using (5.6).
 - 6: Conduct the second round of $N = \hat{M}$ probing experiments using the probing densities of (5.11) to yield the ensembles $\{x_{t,k}^n\}$, $n = \hat{M} + 1, \hat{M} + 2, \dots, 2\hat{M}$.
 - 7: Construct the density estimates $\hat{f}_t^{\hat{M}+1}, \hat{f}_t^{\hat{M}+2}, \dots, \hat{f}_t^{2\hat{M}}$, where $t = 1, 2, \dots, T$, from the above ensembles using (5.14).
 - 8: Construct the matrices $\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$ using (5.8), (5.9), (5.20) and (5.21).
 - 9: Construct the matrix estimates $\overline{\mathbf{P}}_S^{(0)}$ and $\overline{\mathbf{Q}}_S$ using (5.38) and (5.39).
 - 10: Construct the matrix estimates $\hat{\mathbf{P}}_S^{(0)}$ and $\hat{\mathbf{Q}}_S$ using (5.46) and (5.47).
 - 11: Construct the map $\hat{S}$ using Algorithm 5.5.1 (generalised hat map construction).
 - 12: Return the map $\hat{S}$.
-

5.5.5 Characteristics of the novel method

Two shortcomings of the original solution to IFPP-III [2, 3] were identified in section 3.6.1 and reiterated in section 5.3. In particular, Propositions 3.6.2 and 3.6.4 state that any map reconstructed using the original solution

- cannot be consistent with the unknown semi-Markov map, if the latter map is non-canonical or possesses any decreasing branches, and
- has power spectrum mode characteristics that are, in general, inconsistent with those of the unknown semi-Markov map, if the latter map possesses any decreasing branches.

The novel method for solving IFPP-III implements solution strategy 2, which is designed to address these shortcomings. In this section, two propositions are proved which state that any map reconstructed using the novel method

- is consistent with the unknown semi-Markov map, if the latter map is canonical (i.e., it may possess branches of *arbitrary monotonicity*), and
- has an invariant density and power spectrum mode characteristics that are consistent with those of the unknown semi-Markov map, regardless of the canonicity of the latter map or the monotonicity of its branches.

These propositions imply that the novel method addresses the shortcomings of the original solution. The propositions are proved below. In each case, it is assumed that the unknown map S is a PWL and ergodic semi-Markov map.

Proposition 5.5.7. Suppose that the estimates of the unknown map's Markov partition and its matrices are perfectly accurate – i.e.,

$$\begin{aligned}\hat{Q} &= Q, \\ \hat{\mathbf{P}}_S^{(0)} &= \mathbf{P}_S^{(0)} \\ \hat{Q}_S &= Q_S.\end{aligned}\tag{5.55}$$

If the unknown map S is canonical, it is perfectly reconstructed to yield the map $\hat{S} = S$ using the novel solution, *regardless of the monotonicity of its branch functions* [4].

Proof. The proposition is proved in section 5.D.4. □

Remark 5.5.4. It is possible to strengthen Proposition 5.5.7 to any unknown map S that is a member of the generalised hat maps. In particular, the canonical semi-Markov maps comprise a subset of the generalised hat maps. However, it is important to observe that the class of generalised hat maps were defined in a manner that restricts the hats in each interval of the Markov partition to appear in ascending order. Hence, a more general class of these maps cannot be successfully reconstructed using the novel method.

Proposition 5.5.8. Suppose that the estimates of the unknown map's Markov partition and its matrices are perfectly accurate, such that (5.55) is satisfied. Let $\hat{S}$ denote the map constructed from these estimates using the novel solution. The invariant density and power spectrum mode characteristics of $\hat{S}$ are then equal to those of the unknown map S , *regardless of the canonicity of S or the monotonicity of its branches* [4].

Proof. The proposition is proved in section 5.D.5. □

5.6 Numerical Experiment

This section describes a numerical experiment that was conducted to reconstruct two maps as examples. In each example, both the original solution [2, 3] and the novel method for solving IFPP-III were applied to reconstruct an unknown PWL and ergodic semi-Markov map. The examples illustrate relevant characteristics of the methods and permit a comparison of their reconstruction accuracy.

The experimental setup used to generate the numerical results, which includes the parameters selected for each method, is described below. Thereafter, each example and its results are presented. The section concludes by investigating the computation time required to reconstruct the maps.

The numerical experiment and examples presented in this section appear in [4].

5.6.1 Experimental setup

The numerical experiment was carried out using a laptop with an Intel i7-9850H processor and the MATLAB numeric computing environment. In each example, the original solution and the novel method were applied as described in sections 3.6.1

and 5.5, with the exception of the step for estimating the Markov partition. This step is identical for both the original solution and the novel method and its accuracy was evaluated in [3]. Hence, the estimated Markov partition is selected to be equal to that of the unknown map in each example. The original solution involved M probing experiments and the probing densities of Fig. 5.5.2(a), and the proposed method involved $2M$ probing experiments and the probing densities of Fig. 5.5.2(a) and 5.5.2(b); here, M denotes the number of intervals in the Markov partition of each unknown map. Each probing experiment yielded a sequence of $T = 2$ densities.

Each method was applied in multiple rounds for each example. A different value was selected for the number of states per ensemble K in each round, with K ranging between $K = 1.25 \times 10^3$ and $K = 80 \times 10^3$. To permit the estimation of the mean reconstruction error, each solution was repeated 10^3 times per round, with the initial ensemble of each probing experiment drawn randomly from the probing density at the start of each repetition.

The reconstruction error of the methods was quantified as the L^1 error

$$\varepsilon_S \triangleq \|S - \hat{S}\|_1 \quad (5.56)$$

between the constructed map $\hat{S}$ and the unknown map S , as well as the L^1 errors between their invariant densities and PSDs, as denoted by

$$\begin{aligned} \varepsilon_{f_S^*} &\triangleq \|f_S^* - f_{\hat{S}}^*\|_1, \\ \varepsilon_{\mathcal{J}_S} &\triangleq \|\mathcal{J}_S - \mathcal{J}_{\hat{S}}\|_1. \end{aligned} \quad (5.57)$$

The mean of each error was computed over the repetitions in each round.

5.6.2 Examples

The two examples are presented below.

Example 5.6.1. The reconstruction of an unknown canonical map with several decreasing branches was carried out. Any map of this type is *not* successfully reconstructible using the original solution [2, 3], but can be successfully reconstructed using the novel method of section 5.5. This example was selected to illustrate the novel method's resolution of the original solution's shortcomings.

The unknown map selected for the example is defined. The maps reconstructed using both methods, their invariant densities and their power spectra are then characterised and compared. The example concludes by investigating the mean reconstruction error of each method.

Unknown map

The canonical $\mathcal{R}(\mathcal{Q})$ -semi-Markov map⁸ plotted in Fig. 5.6.1(a) was selected as the unknown map S . The map and its branch functions are defined by

$$S(x) = \sum_{m=1}^M \sum_{j=1}^{c(m)} S_{j,m}(x) \chi_{R_j^{(m)}}(x), \quad (5.58)$$

where

$$S_{j,m}(x) \triangleq v_{j,m}x + w_{j,m}. \quad (5.59)$$

The map has a total of 16 branches, of which four branches are decreasing. Table 5.6.1 lists the domain, codomain and parameters of each branch function belonging to the map. The Markov partition of the map comprises the intervals

$$\begin{aligned} Q_1 &= (0, 0.40), \\ Q_2 &= (0.40, 0.50), \\ Q_3 &= (0.50, 0.80), \\ Q_4 &= (0.80, 1.00). \end{aligned} \quad (5.60)$$

Constructed maps

Figures 5.6.1(b) and 5.6.1(c) are plots of the maps constructed using the novel method and the original solution with $K = 10^4$ states per ensemble. Since S is canonical, Proposition 5.5.7 implies its successful reconstruction by the novel method under perfect estimation accuracy, despite the fact that it possesses decreasing branches. A comparison of Figs. 5.6.1(a) and 5.6.1(b) reveals that the map from the novel method closely approximates the unknown map, despite the matrix estimates not being equal to their counterparts of the unknown map.

Figure 5.6.1(b) indicates that the four decreasing branches of the unknown map were successfully reconstructed as decreasing branches by the novel method (these branches are coloured magenta in the figure), and that the remaining branches were correctly reconstructed with increasing monotonicity. This implies that the decreasing branches' monotonicity was successfully identified and distinguished from that of the increasing branches by the novel method; recall that this is made possible by estimating submatrix $\mathbf{Q}_S$ of the unknown map's order-one FPM, in addition to its order-zero FPM. In contrast, Fig. 5.6.1(c) reveals that the original solution

⁸A closely related map was presented as an example in [3].

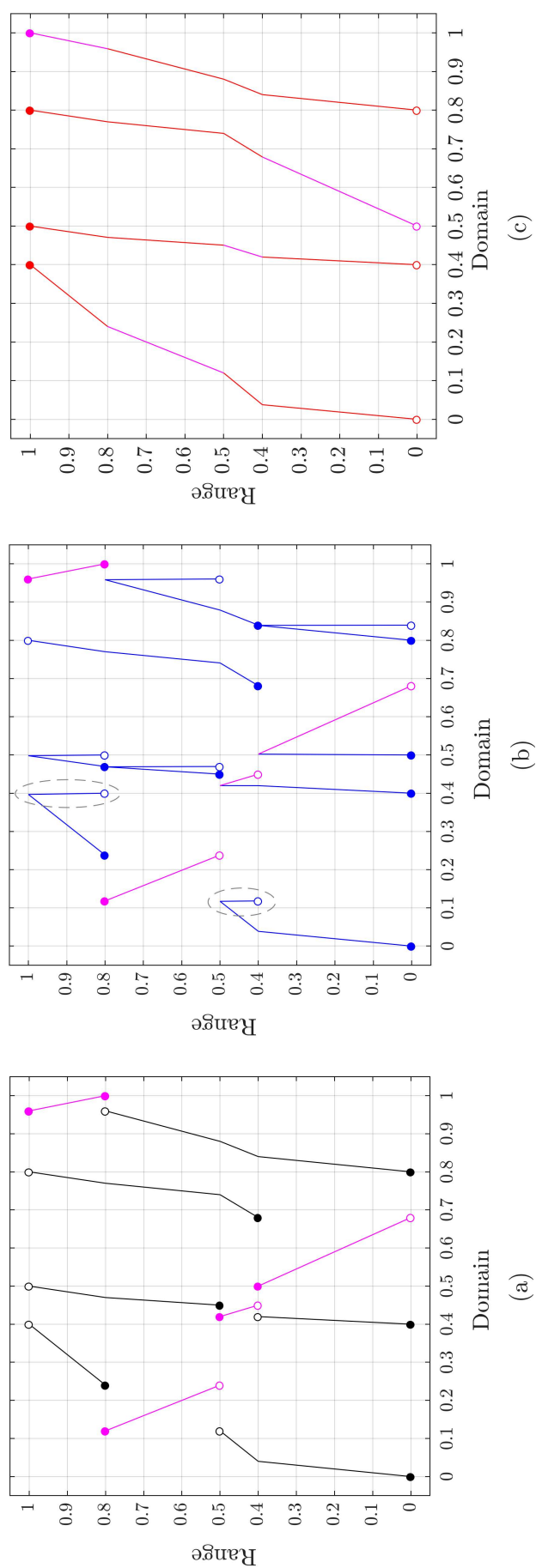


Figure 5.6.1: The maps associated with Example 5.6.1. The unknown map S is plotted in (a), and the maps $\hat{S}$ constructed using the novel method and the original solution (both with $K = 10^4$ states per ensemble) are plotted in (b) and (c), respectively. Reproduced with permission from [4] (© [2023] IEEE).

Table 5.6.1: Branch function parameters of the unknown map in Example 5.6.1. Reproduced with permission from [4] (© [2023] IEEE).

Branch	Domain	Codomain	$v_{j,m}$	$w_{j,m}$
$S_{1,1}$	$R_1^{(1)} = (0, 0.04)$	$Q_1 = (0, 0.40)$	10.00	0
$S_{2,1}$	$R_2^{(1)} = (0.04, 0.12)$	$Q_2 = (0.40, 0.50)$	1.25	0.35
$S_{3,1}$	$R_3^{(1)} = (0.12, 0.24)$	$Q_3 = (0.50, 0.80)$	-2.50	1.10
$S_{4,1}$	$R_4^{(1)} = (0.24, 0.40)$	$Q_4 = (0.80, 1.00)$	1.25	0.50
$S_{1,2}$	$R_1^{(2)} = (0.40, 0.42)$	$Q_1 = (0, 0.40)$	20.00	-8.00
$S_{2,2}$	$R_2^{(2)} = (0.42, 0.45)$	$Q_2 = (0.40, 0.50)$	-3.33	1.90
$S_{3,2}$	$R_3^{(2)} = (0.45, 0.47)$	$Q_3 = (0.50, 0.80)$	15.00	-6.25
$S_{4,2}$	$R_4^{(2)} = (0.47, 0.50)$	$Q_4 = (0.80, 1.00)$	6.67	-2.33
$S_{1,3}$	$R_1^{(3)} = (0.50, 0.68)$	$Q_1 = (0, 0.40)$	-2.22	1.51
$S_{2,3}$	$R_2^{(3)} = (0.68, 0.74)$	$Q_2 = (0.40, 0.50)$	1.67	-0.73
$S_{3,3}$	$R_3^{(3)} = (0.74, 0.77)$	$Q_3 = (0.50, 0.80)$	10.00	-6.90
$S_{4,3}$	$R_4^{(3)} = (0.77, 0.80)$	$Q_4 = (0.80, 1.00)$	6.67	-4.33
$S_{1,4}$	$R_1^{(4)} = (0.80, 0.84)$	$Q_1 = (0, 0.40)$	10.00	-8.00
$S_{2,4}$	$R_2^{(4)} = (0.84, 0.88)$	$Q_2 = (0.40, 0.50)$	2.50	-1.70
$S_{3,4}$	$R_3^{(4)} = (0.88, 0.96)$	$Q_3 = (0.50, 0.80)$	3.75	-2.80
$S_{4,4}$	$R_4^{(4)} = (0.96, 1.00)$	$Q_4 = (0.80, 1.00)$	-5.00	5.80

incorrectly reconstructed the decreasing branches of the unknown map as increasing branches, as it cannot discern branch monotonicity from the order-zero FPM estimate alone. This observation supports Proposition 3.6.2, which states increasing branches for the unknown map as a necessary condition for its consistent reconstruction.

Figure 5.6.1(b) reveals that the map from the novel method possesses unnecessary branches. Two of these branches are indicated using dashed ellipses in the figure. It is also observed that certain branch pairs of this map possess the characteristic hat-like shape. Both these imperfections are a consequence of the estimate $\hat{\mathbf{Q}}_S$ not being equal to $\mathbf{Q}_S$, which causes Algorithm 5.5.1 (generalised hat map construction) to unnecessarily divide certain intervals of the canonical map's sub-partition. However, the support of the unnecessary branches is significantly smaller than the support of the remaining branches.

In contrast to the novel method, the original solution does not produce unnecessary branches, as it does not use an estimate of the submatrix $\mathbf{Q}_S$. However, by incorrectly estimating the monotonicity of some branches, its reconstruction error is significantly larger. This error is quantified towards the end of the example.

Invariant density functions

The invariant density function of the unknown map S is plotted in Fig. 5.6.2. This density function is PWC over the intervals of the Markov partition $\mathcal{Q}$ and is given by

$$f_S^*(x) = 0.68\chi_{Q_1}(x) + 2.22\chi_{Q_2}(x) + 0.84\chi_{Q_3}(x) + 1.26\chi_{Q_4}(x). \quad (5.61)$$

The invariant densities of the maps from both the novel method and the original solution, as constructed with $K = 10^4$ states per ensemble, are plotted in the same figure. The latter plots were obtained using the right Perron eigenvector of each map's order-zero FPM. In each case, these invariant densities were verified by numerically computing a map trajectory of length 10^6 states and then constructing a normalised histogram of the states. The normalised histogram constructed for each map was found to closely approximate its invariant density as predicted by theory.

Propositions 3.6.3 and 5.5.8 state that the invariant densities of the maps from both the original solution and the novel method are equal to the invariant density of the unknown map in the example, under perfect estimation accuracy. More specifically, the map is not required to be canonical, nor is its branches required to be increasing. Figure 5.6.2 reveals that the invariant densities of the reconstructed maps closely approximate the invariant density of the unknown map, despite imperfect estimation accuracy. This is a consequence of both the novel method and the original solution designing the map to satisfy the estimate of the unknown map's order-zero FPM, thereby estimating its right Perron eigenvector.

Power spectral densities

The expression for the continuous component of the unknown map's PSD is given by (2.152) (in this expression, $z = e^{i\omega}$), and its parameters are listed in Table 5.6.2. The continuous component of the unknown map's PSD is plotted in Fig. 5.6.3.

The continuous components of the PSDs belonging to the maps from the novel method and the original solution, as constructed using $K = 10^4$ states per ensemble, are plotted in Fig. 5.6.3. These plots were obtained from (2.152). In each case, these theoretical results were verified numerically by computing a map trajectory of length 5×10^6 states, and then computing the (discrete-time) short-time Fourier transform of the trajectory. In this approach, the trajectory is divided into shorter non-overlapping segments of 500 states each, and the discrete-time Fourier transform

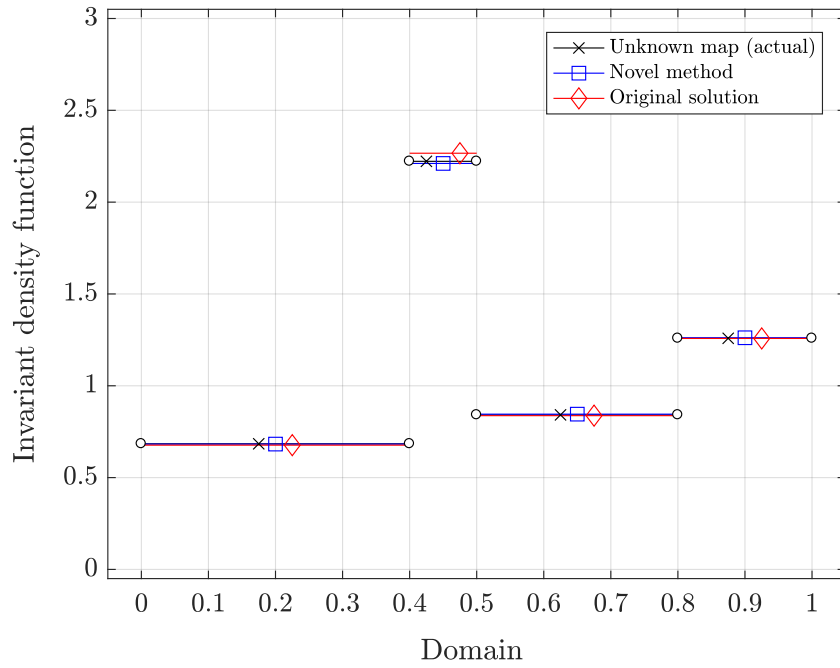


Figure 5.6.2: Plots of the invariant densities of the unknown map and the maps constructed using the novel method and the original solution in Example 5.6.1. Both methods used $K = 10^4$ states per ensemble. Reproduced with permission from [4] ((© [2023] IEEE)).

of each segment is computed. The squared magnitudes of the resulting frequency-domain representations of the segments are then averaged to obtain an estimate of the PSD.

As the unknown map of the example is canonical, Proposition 5.5.7 implies its consistency with the map constructed using the novel method, despite its decreasing branches. Under perfect matrix estimation, these maps are equal, which then implies that their PSDs are also equal. Figure 5.6.3 reveals that the continuous components of these PSDs remain approximately equal despite the imperfection of the matrix estimates. Whereas non-canonicity of the unknown map does not guarantee inconsistency between its PSD and that belonging to the map from the novel method, it generally yields inconsistent power spectra. In particular, non-canonicity generally yields a submatrix $\mathbf{R}_S$ for the order-one FPM of the map from the novel method that is inconsistent with that of the unknown map (recall that the PSD is a function of its map's order-one FPM).

The canonicity of the unknown map is not sufficient for its PSD to be consistent with that belonging to the map from the original solution. In particular, Proposition 3.6.4 states increasing monotonicity for all of the unknown map's branches as a further

Table 5.6.2: Power spectral density parameters for the unknown map of Example 5.6.1. Reproduced with permission from [4] (© [2023] IEEE).

Mode m	Coefficient c_m	Eigenvalue λ_m
1	2.97×10^{-1}	1.00
2	$(1.41 - 1.38i) \times 10^{-1}$	$(-2.00 + 1.73i) \times 10^{-1}$
3	$(1.41 + 1.38i) \times 10^{-1}$	$(-2.00 - 1.73i) \times 10^{-1}$
4	8.22×10^{-3}	1.00×10^{-1}
5	9.18×10^{-3}	-2.75×10^{-1}
6	2.61×10^{-4}	-1.23×10^{-1}
7	$(1.86 + 1.35i) \times 10^{-2}$	$(1.44 + 1.18i) \times 10^{-1}$
8	$(1.86 - 1.35i) \times 10^{-2}$	$(1.44 - 1.18i) \times 10^{-1}$

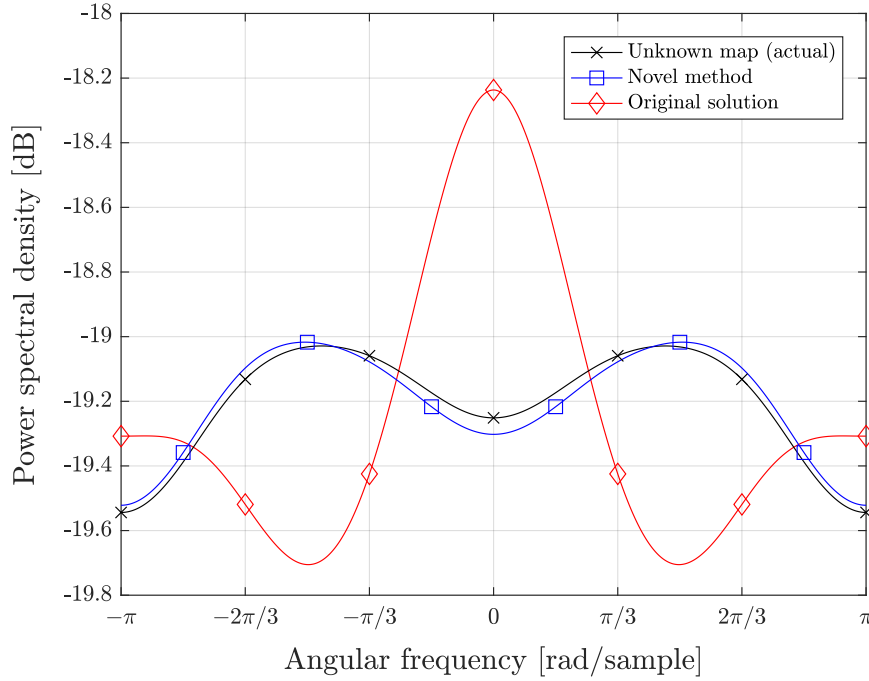


Figure 5.6.3: Plots of the continuous components of the PSDs belonging to the unknown map and the maps constructed using the novel method and the original solution in Example 5.6.1. Both methods used $K = 10^4$ states per ensemble. Reproduced with permission from [4] (© [2023] IEEE).

necessary condition for the PSDs to be consistent (in general). Figure 5.6.3 reveals that the continuous component of the reconstructed map's PSD deviates from that of the unknown map.

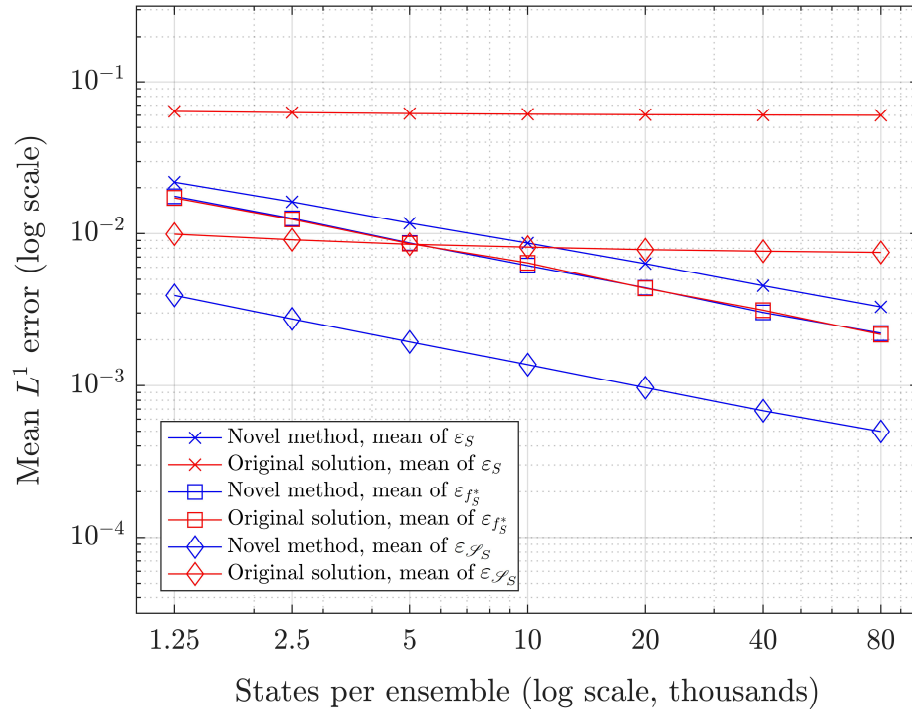


Figure 5.6.4: The mean L^1 errors of the maps, invariant densities and PSDs in Example 5.6.1, plotted as a function of the number of states per ensemble. Reproduced with permission from [4] (© [2023] IEEE).

Mean reconstruction error

The mean L^1 errors associated with the constructed maps, their invariant densities and PSDs are plotted as a function of the number of states per ensemble K in Fig. 5.6.4. The curves reveal that all errors associated with the novel method decrease as K grows and estimation accuracy improves, whereas the errors associated with the original solution (i.e., the errors for the reconstructed map and its PSD) remain constant regardless of the estimation accuracy. This supports the claim that maps constructed using the novel method are consistent with the unknown map, whereas the maps of the original solution are inconsistent.

The mean L^1 error ε_S associated with the maps from the novel method is non-zero due to imperfect estimation of the order-zero FPM, which leads to an imperfect match between the branch domains and slopes of the constructed maps and their counterparts of the unknown map. The appearance of unnecessary branches in the reconstructed maps also contribute towards this error. However, the figure suggests that this error decreases as the estimation accuracy improves.⁹ It is observed that

⁹This supports the conjecture of Remark 5.5.1 made in the context of justifying the matrix estimate used by the novel method.

the error ε_S associated with the maps from the original solution is significantly larger than that of the novel method.

The mean L^1 error $\varepsilon_{f_S^*}$ associated with the invariant densities of the novel method and the original solution, as plotted in Fig. 5.6.4, suggests the accuracy of the two methods are identical in this regard. The invariant densities coincide with the right Perron eigenvector of the order-zero FPM estimate; as both methods estimate this matrix in the same manner, the densities are expected to be equally accurate. As the estimate of the order-zero FPM becomes more accurate with larger values of K , the error for the invariant densities decreases.

Figure 5.6.4 reveals that the mean L^1 error $\varepsilon_{\mathcal{J}_S}$ associated with the PSDs of maps constructed using the novel method decreases with improved estimation accuracy, whereas the corresponding error for the original solution does not. This is due to the consistency between the maps constructed using the novel method and the unknown map. As the original solution does not design a map to possess an order-one FPM that is consistent with that of the unknown map, the error associated with its PSD does not decrease. The figure indicates the latter error is significantly larger than that of the novel method.

Example 5.6.2. The reconstruction of an unknown non-canonical map with increasing and decreasing branches was carried out. Due to its lack of canonicity, this map is *not* successfully reconstructible using the original solution [2, 3]. Non-canonical maps are, in general, also not successfully reconstructible using the novel method (unless they are members of the class of generalised hat maps). However, unlike the original solution, the map constructed using the novel method has power spectrum mode characteristics that are consistent with those of the unknown map. This example was selected to illustrate this characteristic.

The unknown map selected for the example is defined. The maps reconstructed using both methods, their invariant densities and their power spectra are then characterised and compared. The example concludes by investigating the reconstruction error and the accuracy of the power spectrum mode characteristics yielded by the methods.

Unknown map

The non-canonical $\mathcal{R}(\mathcal{Q})$ -semi-Markov map plotted in Fig. 5.6.5(a) was selected as the unknown map S . This map and its branch functions are defined by (5.58) and (5.59). The unknown map has a total of five branches, of which three branches are

decreasing. Table 5.6.3 lists the domain, codomain and parameters of each branch function belonging to the map. The Markov partition of the map comprises the intervals

$$\begin{aligned} Q_1 &= (0, 0.50), \\ Q_2 &= (0.50, 1.00). \end{aligned} \tag{5.62}$$

Constructed maps

Figures 5.6.5(b) and 5.6.5(c) are plots of the maps constructed using the novel method and the original solution with $K = 10^4$ states per ensemble. The figures reveal that neither map is a close approximation of the unknown map (more specifically, neither map is consistent with the unknown map). This observation is in agreement with Proposition 3.6.2, which disqualifies the consistency of the map from the original solution on the basis of the unknown map's non-canonicity. Furthermore, as S is not a generalised hat map, the proposed solution also fails to construct a consistent map.

Figures 5.6.5(a) and 5.6.5(b) reveal the underlying assumptions of the novel method in reconstructing the unknown map. Observe that the two branch functions of the unknown map defined over the interval $Q_1 = (0, 0.50)$ appear in descending order (i.e., the image of the first branch occupies the top half of the unit interval, and the image of the second branch the bottom half). As the generalised hat maps are constructed from canonical maps, for which the branches over each interval of the Markov partition appear in strictly ascending order, the generalised hat map of Fig. 5.6.1(b) is constructed under the incorrect assumption of ascending branches.

Figures 5.6.5(a) and 5.6.5(b) also reveal that the novel method fails to reconstruct the first two branches defined over the interval $Q_2 = (0.5, 1)$. This is due to the definition of the class of generalised hat maps in section 5.5.3, which does not permit the right branch function of a hat to possess an increasing slope. Hence, the first two branches defined over Q_2 cannot be reconstructed as a hat using the novel method.

Figures 5.6.5(a) and 5.6.5(c) reveal that the original solution constructs a canonical map with increasing branches only. This example clearly illustrates the original solution's assumption regarding the canonicity of the unknown map, and its reliance on the order-zero FPM, which does not permit the resolution of branch monotonicity.

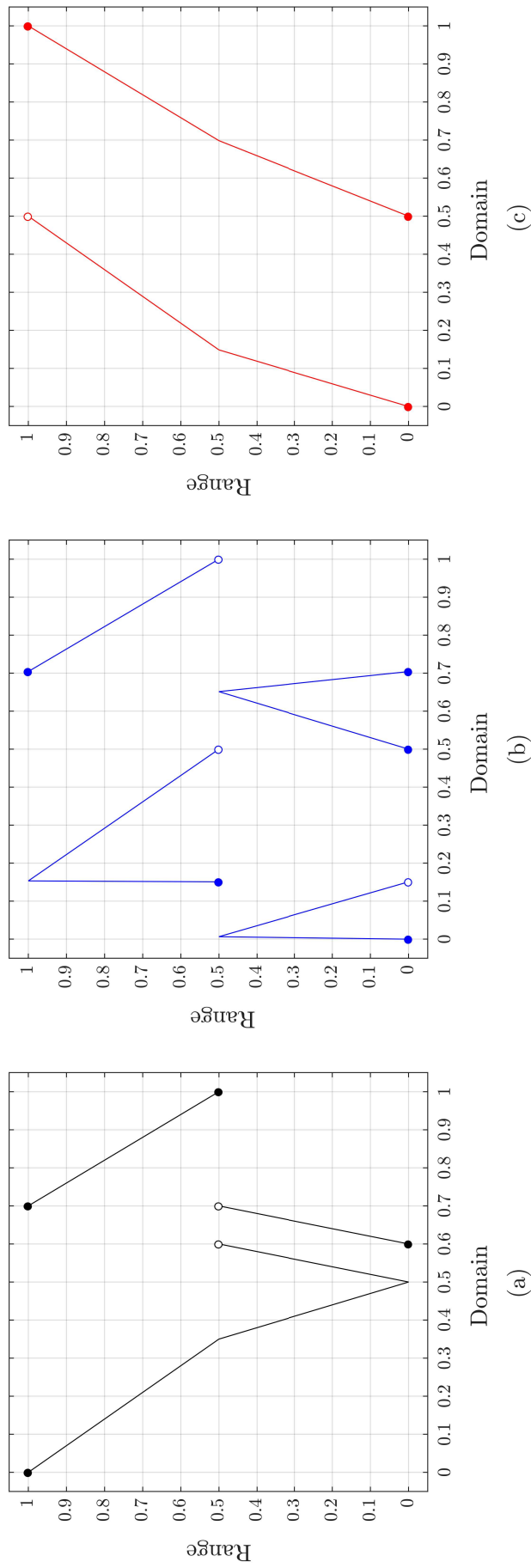


Figure 5.6.5: The maps associated with Example 5.6.2. The unknown map S is plotted in (a), and the maps $\hat{S}$ constructed using the novel method and the original solution (both with $K = 10^4$ states per ensemble) are plotted in (b) and (c), respectively. Reproduced with permission from [4] (© [2023] IEEE).

Table 5.6.3: Branch function parameters of the unknown map in Example 5.6.2. Reproduced with permission from [4] (© [2023] IEEE).

Branch	Domain	Codomain	$v_{j,m}$	$w_{j,m}$
$S_{1,1}$	$R_1^{(1)} = (0, 0.35)$	$Q_2 = (0.50, 1.00)$	-1.43	1.00
$S_{1,2}$	$R_2^{(1)} = (0.35, 0.5)$	$Q_1 = (0, 0.50)$	-3.33	1.67
$S_{1,2}$	$R_1^{(2)} = (0.50, 0.60)$	$Q_1 = (0, 0.50)$	5.00	-2.5
$S_{2,2}$	$R_2^{(2)} = (0.60, 0.70)$	$Q_1 = (0, 0.50)$	5.00	-3.00
$S_{3,2}$	$R_3^{(2)} = (0.70, 1.00)$	$Q_2 = (0.50, 1.00)$	-1.67	2.17

Invariant density functions

The invariant density function of the unknown map S is plotted in Fig. 5.6.6. This density function is PWC over the intervals of the Markov partition $\mathcal{Q}$ and is given by

$$f_S^*(x) = 0.73\chi_{Q_1}(x) + 1.27\chi_{Q_2}(x). \quad (5.63)$$

The invariant densities of the maps from both the novel method and the original solution, as constructed with $K = 10^4$ states per ensemble, are plotted in the same figure. The latter plots were obtained using the right Perron eigenvector of each map's order-zero FPM. In each case, these invariant densities were verified by numerically computing a map trajectory of length 10^6 states and then constructing a normalised histogram of the states. The normalised histogram constructed for each map was found to closely approximate its invariant density as predicted by theory.

Propositions 3.6.3 and 5.5.8 imply that the invariant densities of the unknown map and the maps from the novel method and the original solution are consistent, regardless of the canonicity of the unknown map or the monotonicity of its branches. Figure 5.6.6 reveals that the invariant densities of the reconstructed maps closely approximate that of the unknown map, despite imperfect estimation accuracy. This is a consequence of designing maps that satisfies the order-zero FPM estimate.

Power spectral densities

The expression for the continuous component of the unknown map's PSD is given by (2.152) (in this expression, $z = e^{i\omega}$), and its parameters are listed in Table 5.6.4. The continuous component of the unknown map's PSD is plotted in Fig. 5.6.7.

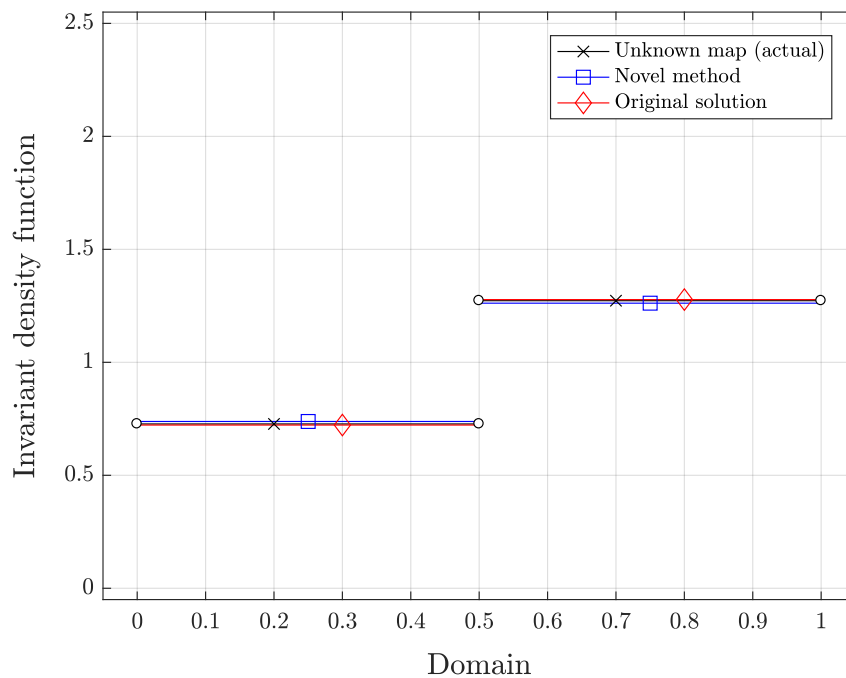


Figure 5.6.6: Plots of the invariant densities of the unknown map and the maps constructed using the novel method and the original solution in Example 5.6.2. Both methods used $K = 10^4$ states per ensemble. Reproduced with permission from [4] (© [2023] IEEE).

The continuous components of the PSDs belonging to the maps from the novel method and the original solution, as constructed using $K = 10^4$ states per ensemble, are plotted in Fig. 5.6.7. These plots were obtained from (2.152). In each case, these theoretical results were verified numerically by computing a map trajectory of length 5×10^6 states, and then computing the (discrete-time) short-time Fourier transform of the trajectory, as in the previous example.

Figure 5.6.7 reveals that PSDs of the maps from the novel method and the original solution are both inconsistent with the PSD of the unknown map S . The unknown map in this example is non-canonical. As observed in the previous example, non-canonicity of the unknown map, in general, yields maps from the novel method and the original solution with inconsistent PSDs. Whereas this relationship is not absolute, the unknown map considered here is one such example. However, canonicity is not required for the power spectrum mode characteristics of the maps to be consistent; this is demonstrated towards the end of the example.

Table 5.6.4: Power spectral density parameters for the unknown map of Example 5.6.2. Reproduced with permission from [4] (© [2023] IEEE).

Mode m	Coefficient c_m	Eigenvalue λ_m
1	3.23×10^{-1}	1.00
2	-1.98×10^{-1}	-1.00×10^{-1}
3	$(1.38 - 1.31i) \times 10^{-1}$	$(-2.25 + 1.45i) \times 10^{-1}$
4	$(1.38 + 1.31i) \times 10^{-1}$	$(-2.25 - 1.45i) \times 10^{-1}$

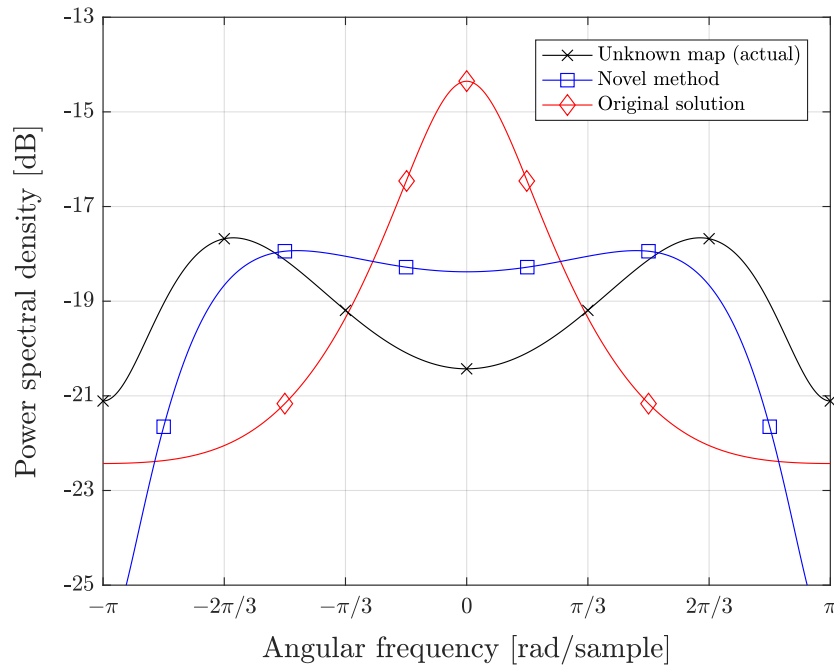


Figure 5.6.7: Plots of the continuous components of the PSDs belonging to the unknown map and the maps constructed using the novel method and the original solution in Example 5.6.2. Both methods used $K = 10^4$ states per ensemble. Reproduced with permission from [4] (© [2023] IEEE).

Mean reconstruction error

The mean L^1 errors associated with the constructed maps, their invariant densities and PSDs are plotted as a function of the number of states per ensemble K in Fig. 5.6.8. The curves confirm that the error $\varepsilon_{f_S^*}$ associated with the invariant density for both the novel method and the original solution decreases as K grows and the estimation accuracy improves. In contrast, the errors ε_S and $\varepsilon_{\mathcal{J}_S}$ associated with the map and its PSD remain constant for both methods. This implies that the inability to reconstruct the map does not stem from estimation inaccuracy, but from the

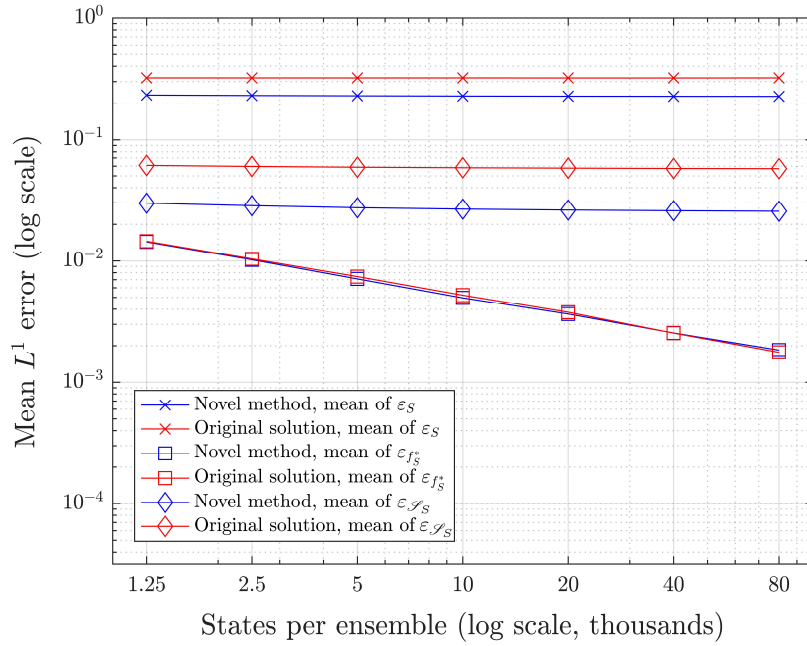


Figure 5.6.8: The mean L^1 errors of the maps, invariant densities and PSDs in Example 5.6.2, plotted as a function of the number of states per ensemble. Reproduced with permission from [4] (© [2023] IEEE).

underlying assumptions of the methods used to reconstruct the map.

Power spectrum mode characteristics

Proposition 5.5.8 implies that canonicity or increasing branch monotonicity are not conditions for the unknown map to be reconstructed with consistent power spectrum mode characteristics using the novel method. In particular, if perfect estimation accuracy is assumed, the novel method reconstructs the map such that the eigenvalues of its order-one FPM, which coincide with map's power spectrum modes, equal those of the unknown map. However, the original solution does not share this property.

The power spectrum mode characteristics of maps constructed using both methods are investigated with scatter plots of the order-one FPM eigenvalues.¹⁰ Figures 5.6.9 and 5.6.10 are scatter plots of the eigenvalues belonging to the order-one FPMs of 100 distinct maps $\hat{S}$ constructed using the novel method, for choices of $K = 2.5 \times 10^3$ and $K = 80 \times 10^3$ states per ensemble, respectively. The figures reveal that the eigenvalues associated with the constructed maps are clustered around the eigenvalues

¹⁰The exact relationship between the eigenvalues and the power spectrum modes are outlined in section 5.3; refer to (5.1) and the discussion that follows this equation, as well as Remark 2.6.8.

associated with the order-one FPM of the unknown map S (indicated using crosses), and that an increase in the number of states per ensemble causes the boundaries of these clusters to constrict. This implies that the PSD mode characteristics of maps constructed using the novel method approximate those of the unknown map more accurately as estimation accuracy improves.

Figures 5.6.11 and 5.6.12 are scatter plots of the eigenvalues belonging to the order-one FPMs of 100 distinct maps constructed using the original solution, for choices of $K = 2.5 \times 10^3$ and $K = 80 \times 10^3$ states per ensemble, respectively. These figures reveal that only a subset of the eigenvalues associated with the constructed maps cluster around the eigenvalues of S , regardless of estimation accuracy. Hence, the power spectrum mode characteristics of these maps deviate from the unknown map.

5.6.3 Mean reconstruction time

The results for the mean map reconstruction time, as presented in this subsection, appear in [11]. The mean time t_m required to reconstruct the unknown maps of Examples 5.6.1 and 5.6.2 using the novel method and the original solution is plotted in Fig. 5.6.13 as a function of the number of states per ensemble K . These mean reconstruction times were obtained by repeating the construction of each map 10^3 times and averaging the times that were measured for the repetitions.

Figure 5.6.13 reveals that the novel method requires more time to reconstruct the map than the original solution in both examples, with a maximum of $t_m = 33.5$ seconds required to reconstruct the most sophisticated map (i.e., the unknown map of Example 5.6.1 with 80×10^3 states per ensemble). Further investigation revealed that the probing experiments, which include the generation of the initial state ensemble, require 95.3% to 99.8% of the overall reconstruction time. The longer execution time of the novel method can thus be ascribed to it carrying out twice the number of probing experiments as compared to the original solution.

The power law curve was fitted to the measured reconstruction times of each solution and example, such that

$$t_m(K) \approx aK^b. \quad (5.64)$$

The fitted curves are plotted as solid lines in Fig. 5.6.13. The figure suggests a close fit between the measurements and the curves. The parameters a and b of each curve are listed in Table 5.6.5. The table reveals that the exponent b is approximately

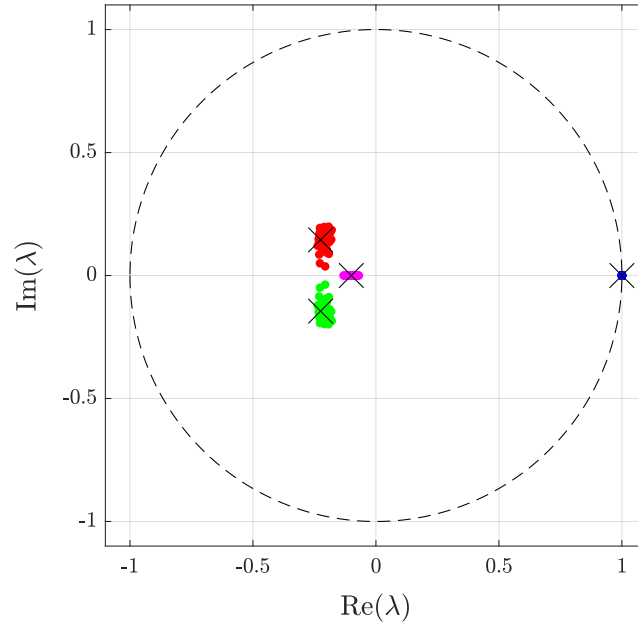


Figure 5.6.9: Scatter plot of the eigenvalues λ belonging to the order-one FPMs of 100 maps constructed using the novel method in Example 5.6.2, for $K = 2.5 \times 10^3$ states per ensemble. The eigenvalues of the unknown map are indicated using crosses. Reproduced with permission from [4] (© [2023] IEEE).

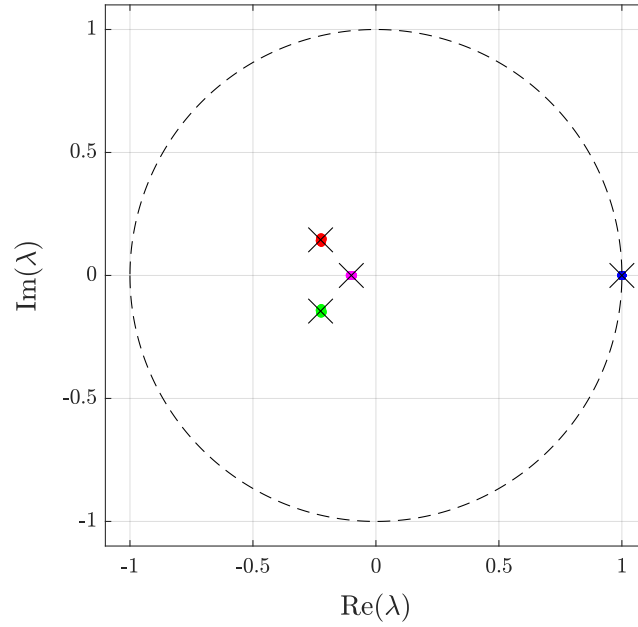


Figure 5.6.10: Scatter plot of the eigenvalues λ belonging to the order-one FPMs of 100 maps constructed using the novel method in Example 5.6.2, for $K = 80 \times 10^3$ states per ensemble. The eigenvalues of the unknown map are indicated using crosses. Reproduced with permission from [4] (© [2023] IEEE).

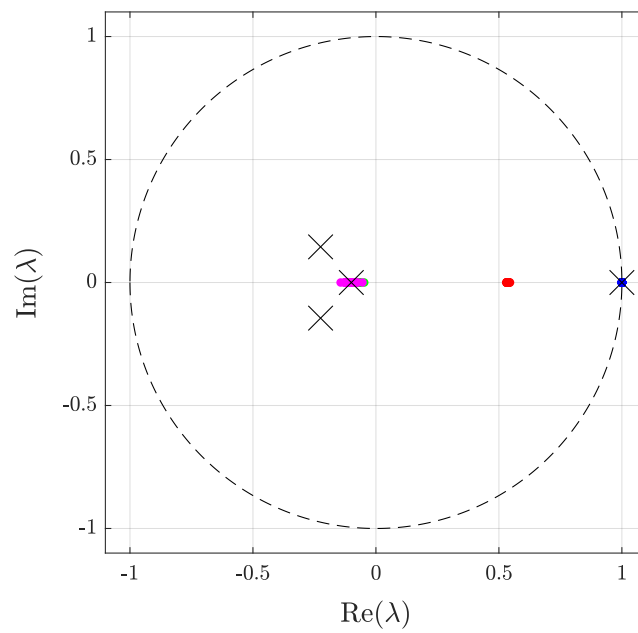


Figure 5.6.11: Scatter plot of the eigenvalues λ belonging to the order-one FPMs of 100 maps constructed using the original solution in Example 5.6.2, for $K = 2.5 \times 10^3$ states per ensemble. The eigenvalues of the unknown map are indicated using crosses. Reproduced with permission from [4] (© [2023] IEEE).

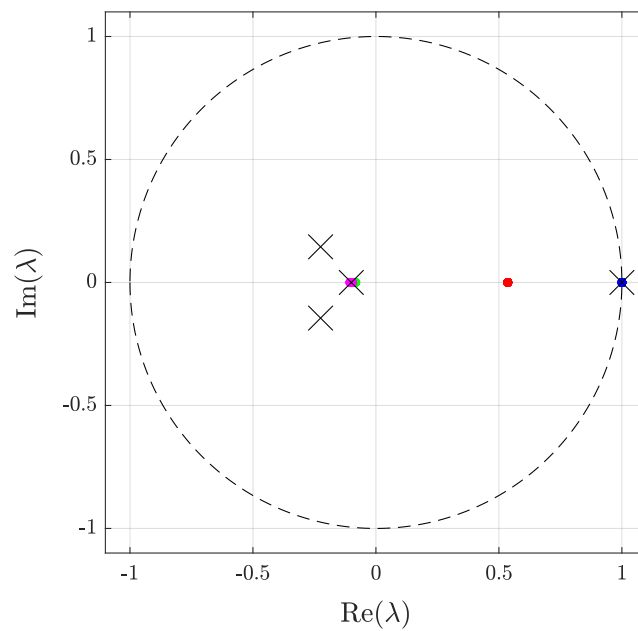


Figure 5.6.12: Scatter plot of the eigenvalues λ belonging to the order-one FPMs of 100 maps constructed using the original solution in Example 5.6.2, for $K = 80 \times 10^3$ states per ensemble. The eigenvalues of the unknown map are indicated using crosses. Reproduced with permission from [4] (© [2023] IEEE).

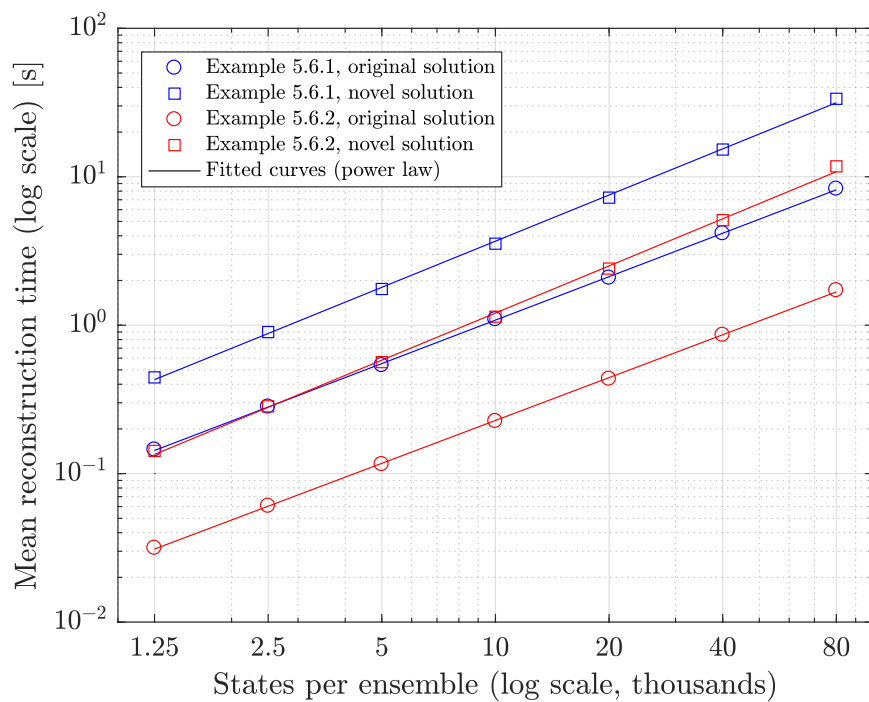


Figure 5.6.13: Mean time required to reconstruct the maps of Examples 5.6.1 and 5.6.2 using each solution. Reproduced with permission from [11].

Table 5.6.5: Parameters of the power law curve fitted to the measured reconstruction times. Reproduced with permission from [11].

Curve	Parameter a	Parameter b
Original solution, experiment 5.6.1	1.395×10^{-4}	9.724×10^{-1}
Proposed solution, experiment 5.6.1	2.721×10^{-4}	1.033
Original solution, experiment 5.6.2	3.321×10^{-5}	9.592×10^{-1}
Proposed solution, experiment 5.6.2	7.295×10^{-5}	1.055

equal to unity in each case, which implies a (near) linear relationship between the measured reconstruction time and the number of states per ensemble.

5.7 Summary

This chapter presents a novel solution to the original formulation of IFPP-III, which requires the reconstruction of an unknown PWL and ergodic semi-Markov map from density function estimates of its evolving state ensembles. Solutions to this problem are practically relevant in applications where it is impossible to isolate and track

individual state trajectories from one time instant to the next [5–9].

The original solution to IFPP-III [2, 3] constructs a canonical semi-Markov map to satisfy a matrix estimate of the unknown map’s finite-dimensional FPO. Two shortcomings of this solution are identified. It is proved that the original solution cannot consistently reconstruct any map with one or more decreasing branch functions. In general, this solution also constructs a map with inconsistent power spectrum mode characteristics if any of the unknown map’s branches are decreasing.

The original solution’s reconstruction of the unknown map from its *order-zero* FPM, which is not dependent on the monotonicity of the map’s branch functions, is identified as the cause of its shortcomings. Two novel solution strategies are proposed to address these shortcomings. These strategies reconstruct the unknown map from estimates of its *order-one* FPM. Certain elements from this matrix depend on the monotonicity of the map’s branch functions and may be utilised to consistently reconstruct any canonical map with correct branch function monotonicity.

The first strategy requires the construction of a semi-Markov map to satisfy *all* the elements of the unknown map’s order-one FPM. This strategy is deemed unfeasible, as it necessitates the enumeration of candidate solutions in a search for the map that best satisfies the matrix. The second strategy, which requires that the map only satisfy two submatrices of the order-one FPM, is instead pursued. This strategy permits identification of branch monotonicity in canonical maps and their consistent reconstruction. The strategy also yields a map with power spectrum mode characteristics (i.e., mode centre frequencies and bandwidths) that are consistent with those of any unknown PWL and ergodic semi-Markov map presented for reconstruction.

A novel method that implements the second strategy to solve IFPP-III is developed. The method conducts two rounds of probing experiments to generate the necessary state ensembles. The probing densities of these experiments are designed to permit accurate estimation of the relevant submatrices in the unknown map’s order-one FPM. A novel method-of-moments density estimation technique is derived.¹¹ The novel method applies this technique to derive density estimates from the evolved state ensembles, and then estimates the relevant submatrices from these densities.

The problem of estimating the relevant submatrices of the order-one FPM from the density estimates is formulated as a joint constrained optimisation problem. It

¹¹The novel method requires the estimation of PWL density functions, in contrast to the original solution’s estimation of PWC density functions. This necessitated the derivation of the method-of-moments estimator.

is proved that this problem is not convex, which leads to the formulation of two separate constrained optimisation problems for estimating the matrices. The separate problems are proved to be convex. These problems are solved by deriving matrix estimates based on the Moore-Penrose pseudoinverse, and then manipulating these estimates to satisfy the constraints imposed by the optimisation problems.

It is proved that the above estimates are the optimal solution for the joint and separate optimisation problems in the limit of perfect estimation accuracy for the density functions. This leads to the conjecture that the accuracy of the matrix estimates, and, by implication, the accuracy of the reconstructed map, improve as the densities are estimated from a larger collection of measured states. A numerical experiment yielded empirical evidence that supports this conjecture.

The novel solution designs a map from the newly proposed class of generalised hat maps to satisfy the estimates of the two submatrices from the unknown map's order-one FPM. The generalised hat maps define a more general structure for the hat maps introduced in chapter 4. It is proved that this generalisation permits an analytical and computationally feasible algorithm for designing the map to satisfy *any* submatrix estimates that adhere to the necessary constraints. Hence, the generalised hat maps do not restrict the matrix estimates that may be satisfied.

Two propositions on the characteristics of the novel method for solving IFPP-III are presented. These propositions prove the novel method's consistent reconstruction of any unknown map that is canonical, and the construction of a map with consistent power spectrum mode characteristics regardless of the unknown map's canonicity. Both these propositions hold regardless of the monotonicity of the unknown map's branches, which implies that the method resolves the original solution's shortcomings.

Two numerical examples are presented to illustrate the relevant characteristics of the original solution and the novel method, and to test the relevant propositions. The first example involved the reconstruction of a canonical map with both increasing and decreasing branches. It is shown that the original solution inconsistently reconstructs the map regardless of estimation accuracy, as predicted by the relevant proposition, but that the novel solution consistently reconstructs the map despite imperfect estimation accuracy. Although the novel solution is found to introduce unnecessary branches to the map because of the latter imperfection, the overall reconstruction error was empirically shown to decrease with growing estimation accuracy. These observations are in agreement with the proposition statements regarding the characteristics of the methods. In particular, the novel method enlarges the class of maps

Table 5.7.1: Classes of PWL and ergodic semi-Markov maps that may be reconstructed using the original solution and the novel method to possess consistent characteristics.

Characteristic	Class reconstructible by original solution	Class reconstructible by novel method
Consistent map	Canonical maps with increasing branches (necessary conditions)	Generalised hat maps (necessary condition)
Consistent invariant density	All PWL and ergodic semi-Markov maps	All PWL and ergodic semi-Markov maps
Consistent power spectrum mode characteristics	Canonical maps with increasing branches (sufficient conditions)	All PWL and ergodic semi-Markov maps with diagonalisable order-one FPM (sufficient condition)
Consistent power spectrum	Canonical maps with increasing branches (sufficient conditions)	Generalised hat maps (sufficient condition)

that are consistently reconstructible, as summarised in Table 5.7.1.¹²

The second example required the reconstruction of a non-canonical map that is not a member of the generalised hat maps. The numerical results indicate that neither the original solution nor the novel method consistently reconstructs the map, as predicted. However, it is confirmed empirically that the maps from the novel method possess power spectrum mode characteristics that are consistent with those of the unknown map, as predicted by the relevant proposition. This represents an improvement over the original solution, which is shown to yield a map with inconsistent power spectrum mode characteristics.

The chapter concludes with the presentation of measured data regarding the time required to reconstruct the maps using the original solution and the novel method. The novel solution is found to require more time on average to reconstruct each map than the original solution, due to its requirement of conducting two rounds of computationally intensive probing experiments. Further investigation reveals that the mean reconstruction time grows in a (nearly) linear fashion with respect to the number of states selected for each ensemble of the probing experiments.

¹²Recall that the generalised hat maps are a superset of the canonical maps.

5.8 Conclusions

The following conclusions are drawn regarding the novel method for solving IFPP-III.

- The novel solution strategy's approximation of the unknown map's order-one FPM permits successful identification of branch function monotonicity in canonical maps and the consistent modelling of the power spectrum mode characteristics of any PWL and ergodic semi-Markov map.
- The novel method is more versatile than the original solution, as it enlarges the classes of maps that may be reconstructed to possess consistent characteristics (refer to Table 5.7.1).
- The numerical results support the conjecture that the reconstruction error of the novel method decreases with increasing estimation accuracy, provided that the conditions for consistent map reconstruction are met.
- Unlike the original solution, the novel method introduces unnecessary branches in the reconstructed map, which is an additional source of reconstruction error. It is possible that examples of maps with increasing branches may be constructed, such that the reconstruction error of the original solution is lower than that of the novel method. This is a topic for future investigation.
- Due to its reconstruction of the map to *partially* satisfy the order-one FPM, the novel method does not consistently reconstruct the *full* power spectrum of an arbitrary PWL and ergodic semi-Markov map (in general).
- The mean time required to reconstruct a map using the novel method exceeds that of the original solution, as implied by numerical results. Exact figures in this regard depend on the structure of the unknown map.

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Appendix

5.A Expressions for the PWL Probing Density Functions

A probing density f_1^n that is linear over the non-empty interval $\hat{Q}_m = (\hat{q}_m, \hat{q}_{m+1})$, and zero elsewhere, is defined. The density is selected to satisfy $f_1^n(\hat{q}_m) = 0$ and thus possess a positive slope over the interval $\hat{Q}_m$.

Let $f_1^n(x) \triangleq \Delta_m(x - \hat{q}_m)$. It follows that

$$\int_{\hat{q}_m}^{\hat{q}_{m+1}} f_1^n(x) dx = \frac{\Delta_m \mu(\hat{Q}_m)^2}{2}. \quad (5.65)$$

Since f_1^n is a density defined to be non-zero only over interval $\hat{Q}_m$, (5.65) must equal unity. Setting (5.65) equal to unity and solving for Δ_m yields

$$\Delta_m = \frac{2}{\mu(\hat{Q}_m)^2}. \quad (5.66)$$

Hence, the expression for the probing density f_1^n is given by

$$f_1^n(x) = \frac{2}{\mu(\hat{Q}_m)^2} (x - \hat{q}_m) \chi_{\hat{Q}_m}(x). \quad (5.67)$$

5.B The Method-of-Moments Density Estimation Technique

Let f_t^n denote a density function that is PWL over the intervals of the partition $\hat{Q} = \{\hat{Q}_1, \hat{Q}_2, \dots, \hat{Q}_{\hat{M}}\}$, where $\hat{Q}_m = (\hat{q}_m, \hat{q}_{m+1})$. Let the restriction of the density over each interval be given by

$$f_t^n|_{\hat{Q}_m}(x) \triangleq \Delta_m(x - \hat{q}_m^*) + r_m^*, \quad m = 1, 2, \dots, \hat{M}, \quad (5.68)$$

where $\hat{q}_m^* = (\hat{q}_m + \hat{q}_{m+1})/2$. The p^{th} moment of the restriction of f_t^n to interval $\hat{Q}_m$, centred around the midpoint of the interval, is defined as

$$c_{t,p,m}^n \triangleq \int_{\hat{q}_m}^{\hat{q}_{m+1}} (x - \hat{q}_m^*)^p f_t^n(x) dx. \quad (5.69)$$

The zeroth moment ($p = 0$) is derived as

$$\begin{aligned} c_{t,0,m}^n &\triangleq \int_{\hat{q}_m}^{\hat{q}_{m+1}} [\Delta_m(x - \hat{q}_m^*) + r_m^*] dx \\ &= r_m^* \mu(\hat{Q}_m). \end{aligned} \quad (5.70)$$

It follows that

$$r_m^* = \frac{c_{t,0,m}^n}{\mu(\hat{Q}_m)}. \quad (5.71)$$

The first moment ($p = 1$) is derived as

$$\begin{aligned} c_{t,1,m}^n &\triangleq \int_{\hat{q}_m}^{\hat{q}_{m+1}} (x - \hat{q}_m^*) [\Delta_m(x - \hat{q}_m^*) + r_m^*] dx \\ &= \frac{\Delta_m \mu(\hat{Q}_m)^3}{12}. \end{aligned} \quad (5.72)$$

It follows that

$$\hat{\Delta}_m = \frac{12c_{t,1,m}^n}{\mu(\hat{Q}_m)^3}. \quad (5.73)$$

Substitution of (5.71) and (5.73) into (5.68) yields

$$f_t^n|_{\hat{Q}_m}(x) = \frac{12c_{t,1,m}^n}{\mu(\hat{Q}_m)^3} (x - \hat{q}_m^*) + \frac{c_{t,0,m}^n}{\mu(\hat{Q}_m)}. \quad (5.74)$$

It follows that $f_t^n|_{\hat{Q}_m}(x) \triangleq a_m x + b_m$, where

$$a_m = \frac{12c_{t,1,m}^n}{\mu(\hat{Q}_m)^3} \quad (5.75)$$

and

$$b_m = \frac{1}{\mu(\hat{Q}_m)} \left[-\frac{12c_{t,1,m}^n}{\mu(\hat{Q}_m)^2} \hat{q}_m^* + c_{t,0,m}^n \right]. \quad (5.76)$$

The p^{th} empirical moment of $f_t^n|_{\hat{Q}_m}$, centred around the midpoint of the interval $\hat{Q}_m$, is computed from the ensemble $\{x_{t,k}^n\}_{k=1}^K$ using the expression

$$\hat{c}_{t,1,m}^n \triangleq \frac{1}{K} \sum_{k=1}^K (x_{t,k}^n - \hat{q}_m^*)^p \chi_{\hat{Q}_m}(x_{t,k}^n). \quad (5.77)$$

Substituting the true moments in (5.75) and (5.76) with the empirical moments yields the estimates

$$a_m \approx \frac{12\hat{c}_{t,1,m}^n}{\mu(\hat{Q}_m)^3} \quad (5.78)$$

and

$$b_m \approx \frac{1}{\mu(\hat{Q}_m)} \left[\hat{c}_{t,0,m}^n - \frac{6(\hat{q}_{m+1} + \hat{q}_m)}{\mu(\hat{Q}_m)^2} \hat{c}_{t,1,m}^n \right]. \quad (5.79)$$

5.C Proofs Regarding Convexity of Optimisation Problems

5.C.1 Proof of the non-convexity of optimisation problem 5.5.1

This proof appears in [11]. It is proved that the feasible set associated with (5.28) is not convex. Assume that $(\mathbf{P}_1, \mathbf{Q}_1)$ and $(\mathbf{P}_2, \mathbf{Q}_2)$ are both elements of the feasible set. If the problem were convex, then

$$(\alpha \mathbf{P}_1 + (1 - \alpha) \mathbf{P}_2, \alpha \mathbf{Q}_1 + (1 - \alpha) \mathbf{Q}_2) \quad (5.80)$$

must be an element of the feasible set for all $\alpha \in [0, 1]$ and hence satisfy (5.28); i.e.,

$$|\alpha q_{k,m}^{(1)} + (1 - \alpha) q_{k,m}^{(2)}| \leq [\alpha p_{k,m}^{(1)} + (1 - \alpha) p_{k,m}^{(2)}]^2 \quad (5.81)$$

for all $k, m = 1, 2, \dots, \hat{M}$, where $\mathbf{P}_j \triangleq [p_{k,m}^{(j)}]_{k,m=1}^{\hat{M}}$ and $\mathbf{Q}_j \triangleq [q_{k,m}^{(j)}]_{k,m=1}^{\hat{M}}$.

As $(\mathbf{P}_1, \mathbf{Q}_1)$ and $(\mathbf{P}_2, \mathbf{Q}_2)$ are elements of the feasible set, they satisfy (5.28) and hence

$$-(p_{k,m}^{(1)})^2 \leq q_{k,m}^{(1)} \leq (p_{k,m}^{(1)})^2 \quad (5.82)$$

and

$$-(p_{k,m}^{(2)})^2 \leq q_{k,m}^{(2)} \leq (p_{k,m}^{(2)})^2. \quad (5.83)$$

Multiplying (5.82) with α and (5.83) with $1 - \alpha$ and adding these expressions yield

$$|\alpha q_{k,m}^{(1)} + (1 - \alpha) q_{k,m}^{(2)}| \leq \alpha (p_{k,m}^{(1)})^2 + (1 - \alpha) (p_{k,m}^{(2)})^2. \quad (5.84)$$

Thus, if there exists any $\alpha \in [0, 1]$ such that

$$\alpha (p_{k,m}^{(1)})^2 + (1 - \alpha) (p_{k,m}^{(2)})^2 > [\alpha p_{k,m}^{(1)} + (1 - \alpha) p_{k,m}^{(2)}]^2, \quad (5.85)$$

the feasible set associated with (5.28) is not convex. As $h(\cdot) = (\cdot)^2$ is a convex function, (5.85) holds for all $\alpha \in (0, 1)$. Hence, the joint optimization problem is not convex.

5.C.2 Proof of the convexity of optimisation problem 5.5.2

This proof appears in [11]. As norms are convex functions, the objective function is convex. It is proved that the feasible set is a convex set. Suppose that $\mathbf{P}_1$ and $\mathbf{P}_2$ are both elements of the feasible set. It is proved that $\alpha \mathbf{P}_1 + (1 - \alpha) \mathbf{P}_2$, for any $\alpha \in [0, 1]$, is also an element of the feasible set; i.e.,

$$\hat{\mathbf{q}}^T(\alpha \mathbf{P}_1 + (1 - \alpha) \mathbf{P}_2) = \hat{\mathbf{q}}^T \quad (5.86)$$

and

$$\alpha p_{k,m}^{(1)} + (1 - \alpha) p_{k,m}^{(2)} \geq 0, \quad k, m = 1, 2, \dots, \hat{M}, \quad (5.87)$$

where $\mathbf{P}_1 = [p_{k,m}^{(1)}]_{k,m=1}^{\hat{M}}$ and $\mathbf{P}_2 = [p_{k,m}^{(2)}]_{k,m=1}^{\hat{M}}$.

The left-hand side of (5.86) is manipulated as

$$\begin{aligned} \hat{\mathbf{q}}^T(\alpha \mathbf{P}_1 + (1 - \alpha) \mathbf{P}_2) &= \alpha \hat{\mathbf{q}}^T \mathbf{P}_1 + (1 - \alpha) \hat{\mathbf{q}}^T \mathbf{P}_2 \\ &= \alpha \hat{\mathbf{q}}^T + (1 - \alpha) \hat{\mathbf{q}}^T \\ &= \hat{\mathbf{q}}^T, \end{aligned} \quad (5.88)$$

where the observation that, as elements of the feasible set, both $\mathbf{P}_1$ and $\mathbf{P}_2$ satisfy (5.34) was used. Thus, the feasible set associated with (5.34) is convex.

The elements of both $\mathbf{P}_1$ and $\mathbf{P}_2$ are non-negative due to the matrices' membership of the feasible set; refer to (5.35). As $\alpha \in [0, 1]$, the left-hand side of (5.87) is non-negative. Thus, the feasible set associated with (5.35) is convex.

As the intersection of convex sets is convex, the problem's feasible set is also convex. Hence, the problem is convex.

5.C.3 Proof of the convexity of optimisation problem 5.5.3

This proof appears in [11]. As norms are convex functions, the objective function is convex. It is proved that the feasible set is a convex set. Suppose that $\mathbf{Q}_1 = [q_{k,m}^{(1)}]_{k,m=1}^{\hat{M}}$ and $\mathbf{Q}_2 = [q_{k,m}^{(2)}]_{k,m=1}^{\hat{M}}$ are both elements of the feasible set. It is proved that $\alpha \mathbf{Q}_1 + (1 - \alpha) \mathbf{Q}_2$, for any $\alpha \in [0, 1]$, is also an element of the feasible set; i.e.,

$$-\hat{p}_{k,m}^2 \leq \alpha q_{k,m}^{(1)} + (1 - \alpha) q_{k,m}^{(2)} \leq \hat{p}_{k,m}^2, \quad k, m = 1, 2, \dots, \hat{M}. \quad (5.89)$$

As $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are both members of the feasible set, (5.37) implies that their elements satisfy

$$-\hat{p}_{k,m}^2 \leq q_{k,m}^{(1)} \leq \hat{p}_{k,m}^2, \quad k, m = 1, 2, \dots, \hat{M}, \quad (5.90)$$

and

$$-\hat{p}_{k,m}^2 \leq q_{k,m}^{(2)} \leq \hat{p}_{k,m}^2, \quad k, m = 1, 2, \dots, \hat{M}. \quad (5.91)$$

Multiplying (5.90) by α and (5.91) by $1 - \alpha$ and adding these expressions yields (5.89). This implies that the problem's feasible set is convex, and that the problem itself is convex.

5.D Proofs of Propositions

5.D.1 Proof of Proposition 5.5.4

This proof appears in [11]. Let $\mathbf{P}_{\tilde{S}}^{(0)} \triangleq [p_{k,m}^{(\tilde{S})}]_{k,m=1}^{\tilde{M}}$ and $\mathbf{P}_{\hat{S}}^{(0)} \triangleq [p_{k,m}^{(\hat{S})}]_{k,m=1}^{\tilde{M}}$. When constructing $\hat{S}$, each branch $\tilde{S}_{j,m}$ is (i) left unchanged if $\tilde{\beta}_{j,m} = 1$, or (ii) modified to have a negative slope with the same codomain as $\tilde{S}_{j,m}$ if $\tilde{\beta}_{j,m} = 0$, or (iii) divided into two branches, each having the same codomain as $\tilde{S}_{j,m}$, if $0 < \tilde{\beta}_{j,m} < 1$. Therefore, if no branch of $\tilde{S}$ defined over an interval of the partition $\tilde{\mathcal{R}}^{(m)}$ has the codomain $\tilde{Q}_k$, the map $\hat{S}$ will have no branch defined over an interval of the partition $\hat{\mathcal{R}}^{(m)}$ with codomain $\tilde{Q}_k$. Thus,

$$p_{k,m}^{(\tilde{S})} = 0 \Rightarrow p_{k,m}^{(\hat{S})} = 0. \quad (5.92)$$

Now, suppose $p_{k,m}^{(\tilde{S})} \neq 0$. The canonicity of $\tilde{S}$ implies that it has one branch $\tilde{S}_{\tilde{T}(k,m),m}$ defined over an interval of the partition $\tilde{\mathcal{R}}^{(m)}$ with codomain equal to $\tilde{Q}_k$. If $\tilde{\beta}_{\tilde{T}(k,m),m} \in \{0, 1\}$, the map $\hat{S}$ will have a single branch defined over an interval of the partition $\hat{\mathcal{R}}^{(m)}$ with codomain $\tilde{Q}_k$. This branch has a slope equal to $\pm \tilde{S}'_{\tilde{T}(k,m),m}$. It follows that,

$$\begin{aligned} p_{k,m}^{(\hat{S})} &= |\pm \tilde{S}'_{\tilde{T}(k,m),m}|^{-1} \\ &= p_{k,m}^{(\tilde{S})}. \end{aligned} \quad (5.93)$$

If instead $0 < \tilde{\beta}_{\tilde{T}(k,m),m} < 1$, the map $\hat{S}$ will have two branches defined over intervals of the partition $\hat{\mathcal{R}}^{(m)}$ with the same codomain $\tilde{Q}_k$. These branches have slopes equal to

$$\hat{S}'|_{\tilde{A}_{\tilde{T}(k,m)}^{(m)}} = \frac{\mu(\tilde{Q}_k)}{\tilde{\beta}_{\tilde{T}(k,m),m} \mu(\tilde{R}_{\tilde{T}(k,m)}^{(m)})} \quad (5.94)$$

and

$$\hat{S}'|_{\tilde{B}_{\tilde{T}(k,m)}^{(m)}} = -\frac{\mu(\tilde{Q}_k)}{(1 - \tilde{\beta}_{\tilde{T}(k,m),m}) \mu(\tilde{R}_{\tilde{T}(k,m)}^{(m)})}. \quad (5.95)$$

It then follows that

$$\begin{aligned} p_{k,m}^{(\hat{S})} &= |\hat{S}'|_{\tilde{A}_{\tilde{T}(k,m)}^{(m)}}|^{-1} + |\hat{S}'|_{\tilde{B}_{\tilde{T}(k,m)}^{(m)}}|^{-1} \\ &= \frac{\mu(\tilde{R}_{\tilde{T}(k,m)}^{(m)})}{\mu(\tilde{Q}_k)} \\ &= p_{k,m}^{(\tilde{S})}. \end{aligned} \quad (5.96)$$

5.D.2 Proof of Proposition 5.5.5

This proof appears in [11]. Let $\mathbf{Q}_{\tilde{S}} \triangleq [q_{k,m}^{(\tilde{S})}]_{k,m=1}^{\tilde{M}}$ and $\mathbf{Q}_{\hat{S}} \triangleq [q_{k,m}^{(\hat{S})}]_{k,m=1}^{\tilde{M}}$. Using the same argument as in section 5.D.1, it is proved that

$$q_{k,m}^{(\tilde{S})} = 0 \Rightarrow q_{k,m}^{(\hat{S})} = 0. \quad (5.97)$$

As $p_{k,m}^{(\tilde{S})} = 0 \Rightarrow q_{k,m}^{(\tilde{S})} = 0$, it follows that

$$p_{k,m}^{(\tilde{S})} = 0 \Rightarrow q_{k,m}^{(\hat{S})} = 0. \quad (5.98)$$

Now, suppose that $q_{k,m}^{(\tilde{S})} \neq 0$. If $\tilde{\beta}_{\tilde{T}(k,m),m} \in \{0, 1\}$, the map $\hat{S}$ will have a single branch defined over an interval of the partition $\hat{\mathcal{R}}^{(m)}$ with codomain $\tilde{Q}_k$. This branch has a slope equal to $\pm \tilde{S}'_{\tilde{T}(k,m),m}$. Thus,

$$\begin{aligned} q_{k,m}^{(\hat{S})} &= \pm |\tilde{S}'_{\tilde{T}(k,m),m}|^{-2} \\ &= (2\tilde{\beta}_{\tilde{T}(k,m),m} - 1)(p_{k,m}^{(\tilde{S})})^2. \end{aligned} \quad (5.99)$$

If instead $0 < \tilde{\beta}_{\tilde{T}(k,m),m} < 1$, the map $\hat{S}$ will have two branches defined over intervals of the partition $\hat{\mathcal{R}}^{(m)}$ with the same codomain $\tilde{Q}_k$. These branches have slopes given by (5.94) and (5.95). Thus,

$$\begin{aligned} q_{k,m}^{(\hat{S})} &= |\hat{S}'|_{\tilde{A}^{(m)}_{\tilde{T}(k,m)}}|^{-2} - |\hat{S}'|_{\tilde{B}^{(m)}_{\tilde{T}(k,m)}}|^{-2} \\ &= (2\tilde{\beta}_{\tilde{T}(k,m),m} - 1)(p_{k,m}^{(\tilde{S})})^2. \end{aligned} \quad (5.100)$$

Equation (5.52) then follows.

5.D.3 Proof of Proposition 5.5.6

This proof appears in [4]. The statement regarding the order-zero FPM follows from $\hat{S}$ inheriting the order-zero FPM of $\tilde{S}$ (Proposition 5.5.4), and from the order-zero FPM of map $\tilde{S}$, as constructed using Algorithm 3.4.1 (canonical map construction), being equal to any estimate satisfying the constraints (Proposition 3.4.3). The statement regarding submatrix $\mathbf{Q}_{\hat{S}}$ follows from Proposition 5.5.5. More specifically, (5.52) implies that any estimate satisfying the constraint of (5.37) can be realised by selecting an appropriate value for each parameter $\tilde{\beta}_{\tilde{T}(k,m),m}$ from $[0, 1]$ using (5.54), and then constructing the generalised hat map $\hat{S}$, using these parameters, from $\tilde{S}$.

5.D.4 Proof of Proposition 5.5.7

This proof appears in [4]. Suppose that the conditions stated in the proposition hold. Using Algorithm 3.4.1 (canonical map construction), an $\tilde{\mathcal{R}}(\tilde{\mathcal{Q}})$ -semi-Markov map $\tilde{S}$ is constructed such that its order-zero FPM is equal to that of the unknown map (i.e., $\mathbf{P}_{\tilde{S}}^{(0)} = \mathbf{P}_S^{(0)}$). The equality of the order-zero FPMs and the canonicity of S imply that $\tilde{\mathcal{R}} = \mathcal{R}$ and that each $R_j^{(m)}$ has the same codomain under $\tilde{S}$ and S .

Since S is canonical, (2.118) and (2.122) imply that the non-zero elements of $\mathbf{P}_S^{(0)}$ and $\mathbf{Q}_S$ satisfy

$$q_{k,m} = s_{T(k,m),m} p_{k,m}^2, \quad (5.101)$$

where $T(k, m)$ denotes the index j such that $S(R_j^{(m)}) = Q_k$,

$$s_{j,m} = \begin{cases} -1 & \text{if } v_{j,m} < 0, \\ 1 & \text{if } v_{j,m} \geq 0, \end{cases} \quad (5.102)$$

and $v_{j,m}$ denotes the slope of branch function $S_{j,m} = S|_{R_j^{(m)}}$. The novel solution constructs a generalised hat map $\hat{S}$ from $\tilde{S}$ using Algorithm 5.5.1 (generalised hat map construction) with the parameters

$$\tilde{\beta}_{j,m} = \begin{cases} 0 & \text{if } s_{j,m} = -1, \\ 1 & \text{if } s_{j,m} = 1, \end{cases} \quad (5.103)$$

selected using (5.54). The map $\hat{S}$ is therefore $\mathcal{R}(\mathcal{Q})$ -semi-Markov, and it is designed such that its branch over $R_j^{(m)}$ is decreasing if $\tilde{\beta}_{j,m} = 0$ and increasing if $\tilde{\beta}_{j,m} = 1$. Thus, corresponding branches from S and $\hat{S}$ have the same monotonicity. As these branches also have the same domains and codomains, it follows that $\hat{S} = S$.

5.D.5 Proof of Proposition 5.5.8

This proof appears in [4]. Suppose that the conditions stated in the proposition hold. Proposition 5.5.6 then implies that the order-zero FPM and the top-left submatrix $\mathbf{Q}_{\hat{S}}$ of the order-one FPM belonging to $\hat{S}$ are equal to their counterparts of map S . Since the order-zero FPMs share the same (normalised) right Perron eigenvector, the order-zero vector representations of their invariant densities are identical. The equality of the maps' Markov partitions then imply that their invariant densities are equal. Furthermore, recall that the power spectrum mode characteristics of $\hat{S}$ coincide with the eigenvalues of its order-zero FPM and the eigenvalues of submatrix $\mathbf{Q}_{\hat{S}}$ of its order-one FPM (refer to Proposition 2.6.10). As these matrices equal

their counterparts of S , their eigenvalues, and thus the maps' power spectrum mode characteristics, are equal.

Chapter 6

SUMMARY AND CONCLUSIONS

This work presented in this thesis develops novel solutions to IFPP-II and IFPP-III. The following sections provide a summary of each chapter and present conclusions regarding the novel solutions and their characteristics. The chapter concludes with topics for future work.

6.1 Chapter 2: Mathematical Background

Chapter 2 lays the mathematical foundation for the development of novel solutions to the IFPP. The class of dynamical systems that are designed using the novel methods is introduced. These systems use an ergodic map in a recursive configuration to generate a zero-input time response with prescribed statistical metrics.

The PDSA framework for analysing the dynamical systems from a probabilistic perspective is defined. Using this framework, general expressions are derived for the statistical metrics of the system's response. It is shown that the expressions for the response PDF, ACF and PSD are functions of the map's FPO. This operator governs the evolution of state density functions under repeated iteration of the map.

The class of PWL semi-Markov maps is defined and characterised. The structure of these maps admits exact finite dimensional matrix representations of the FPO. This permits simplification of the general expressions for the system's statistical metrics by substituting the FPO with the relevant matrix.

It is revealed that the elements of the above matrices are simple functions of a finite collection of the map's structural parameters. This leads to expressions that directly relate the statistical metrics of the system to the map's branch functions. These

expressions have potential to yield analytical solutions for the IFPP (as proved by the novel analytical methods developed in the subsequent chapters).

It is shown that the class of PWL and ergodic semi-Markov maps realise a response PDF that is a PWC function, and a response PSDs expressed as a rational function. The poles of the response PSD coincide with the eigenvalues of the FPO's matrix representation.¹

6.2 Chapter 3: Literature Review

The IFPP requires the design of an ergodic map to satisfy one or more prescribed statistical metrics. Chapter 3 presents an in-depth review of existing strategies and methods that have been developed to solve this problem. Three distinct formulations of the IFPP and three strategies for solving these problems are identified. Existing methods are classified according to the problem addressed and their solution strategy, thereby permitting their comparison and the identification of gaps in existing work.

IFPP-I requires the design of an ergodic map with a prescribed invariant density. Existing methods that use each strategy to solve IFPP-I are reviewed and numerous examples are presented to illustrate their characteristics. Existing methods for solving IFPP-I are found to be mature and well understood.

The matrix solution strategy for IFPP-I constructs members from the class of PWL semi-Markov maps. Due to its relevance, the strategy is developed from first principles. The problem of selecting the PWC invariant density of the map to approximate a smooth function prescribed for the metric is addressed. Two methods are proposed and a bound on their approximation error is derived. The error of one method is proved to converge to zero with approximation over a progressively finer partition. The design constraints imposed by both methods are also determined.

IFPP-II requires the design of an ergodic map with a prescribed invariant density and a prescribed ACF or PSD. Existing methods were found to either generalise the topological conjugation [1] or direct solution strategies [2] for IFPP-I to permit prescription of the map's ACF. These methods have shortcomings that restrict their practicality.

¹In the interest of brevity, system-level attributes and characteristics are instead associated with the ergodic map in the remainder of the chapter.

The shortcomings of existing methods stem from the sophistication of the expressions that relate the ACF to the parameters of the candidate maps. The candidate maps for these methods do not admit finite dimensional representations for the FPO, which precludes simplification through the use of matrices. To make headway, existing methods either simplify the expressions by restricting the candidate class to maps with elementary power spectra [1], or rely on a computationally impractical search over a high dimensional space of solutions for a map that satisfies the prescribed metrics [2].

A gap is uncovered in the literature for a method that constructs PWL semi-Markov maps to solve IFPP-II. Due to the availability of unsophisticated expressions for their statistical metrics as functions of the branch parameters, these maps are conjectured (and confirmed in chapter 4) to admit analytical design methods that are practical. However, the statistical metrics of these maps can only approximate continuous functions that may be prescribed, and the maps' structure restricts the power spectra that are realisable. Motivated by the potential of realising multiple spectral modes with adjustable characteristics, a novel method for designing these maps to solve a relaxed formulation of IFPP-II is developed in chapter 4.

The original formulation of IFPP-III requires the reconstruction of an unknown PWL and ergodic semi-Markov map from sequences of density function estimates that are derived from evolving state ensembles. The original solution to this problem is detailed and two of its shortcomings are identified. These shortcomings revolve around the inconsistent reconstruction of a map with decreasing branch functions. A novel method that resolves these shortcomings is developed in chapter 5.

6.3 Chapter 4: A Novel Solution for a Relaxed Formulation of IFPP-II

Chapter 4 addresses IFPP-II, which requires the construction of an ergodic map with a prescribed invariant density and a prescribed ACF or PSD. Motivated by the complexity of this problem, an incremental approach to its solution is adopted. A relaxed problem formulation (IFPP-IIa), which requires the design of an ergodic map to possess an invariant density that approximates an arbitrary prescribed density and a rational power spectrum with prescribed poles, is defined.

A novel method for solving IFPP-IIa by constructing a PWL semi-Markov map is developed. A novel solution strategy is proposed for the method. This strategy (i)

derives a candidate order-zero FPM with eigenvalues and Perron eigenvectors chosen to yield the required statistical metrics, and (ii) designs a map to satisfy this matrix and simultaneously prevent unwanted poles from appearing in its power spectrum.

To implement the above strategy, any method must overcome the difficulty of deriving the order-zero FPM to possess the required characteristics, and resolve the uncertainty of how the map's structural features may be designed to prevent the manifestation of unwanted poles. Three novel contributions are presented to overcome these issues. These contributions are

- the relaxation of the design specifications for the order-zero FPM in a novel manner that still permits the approximation of the prescribed invariant density,
- the development of a novel solution to the StIEP, which permits the analytical construction of a candidate order-zero FPM of arbitrary dimensionality and with a prescribable complex-valued eigenspectrum, and
- the definition of a novel subclass of the semi-Markov maps, referred to as the hat maps. The hat maps possess structural features designed to prevent the manifestation of unnecessary poles in the map's power spectrum.

The novel method implements the above solution strategy and utilises the novel contributions to successfully resolve the design issues. This method simultaneously permits the approximation of a prescribed smooth invariant density and the selection of all the map's power spectrum poles.

Unlike the existing analytical method developed to solve IFPP-II [1], the novel method permits the realisation of a power spectrum with multiple modes. In contrast to the existing numerical method [2], the novel method has negligible computational complexity and is purely analytical, which bestows immunity to possible non-convergence. In addition, the novel method improves upon earlier analytical methods developed by the author of this thesis by permitting the prescription of an *arbitrary number* of power spectrum poles that may be selected as *complex-valued* numbers [3–8].

Several examples of hat maps with uniform and non-uniform invariant densities, and up to three prescribed power spectrum poles, were constructed using the novel method. Numerical simulations of the maps verify that their statistical metrics are equal to those prescribed during the design step. The examples demonstrate the method's ability to approximate an arbitrary invariant density and realise maps with

multiple power spectrum modes, which manifest as peaks in the power spectrum if appropriately selected. It is demonstrated that the centre frequencies and bandwidths of the power spectrum peaks may be adjusted. This implies that the novel method will be relevant in applications such as DRFM-based random signal generation [9–13].

Whereas the novel method grants freedom to adjust the peaks in the map's power spectrum, the structure of the PWL semi-Markov maps imposes restrictions on the poles that may be prescribed. This restricts the method's capacity to approximate arbitrary continuous power spectra and to realise sharp power spectrum peaks at arbitrary frequencies. The method's capacity to realise sharp peaks also diminishes as the number of poles prescribed for the map grows.

Motivated by the above results, the following conclusions are drawn.

- The matrix representation of a semi-Markov map's FPO admits simplified expressions for its statistical metrics as functions of the branch parameters, thereby yielding a practical analytical solution to the relaxed problem formulation of IFPP-IIa.
- The simplicity of the design equations come at the cost of only being able to approximate arbitrary smooth functions that may be prescribed for the invariant density and power spectrum. Hence, methods developed to design these maps cannot be considered *exact* solutions to IFPP-II.
- Although the structure of semi-Markov maps restricts the poles that may be prescribed and therefore the power spectra that may be successfully realised, the examples demonstrate that enough freedom remains to specify multiple power spectrum peaks and to adjust their centre frequencies and bandwidths.
- The numerical results prove that the novel method qualifies as a valid solution of IFPP-IIa.

6.4 Chapter 5: A Novel Solution for IFPP-III

Chapter 5 addresses the original formulation of IFPP-III, which requires the reconstruction of an unknown PWL and ergodic semi-Markov map from sequences of density function estimates that are derived from evolving state ensembles. The original solution to this problem [14, 15] has two shortcomings that stem from its inability to identify branch function monotonicity. In particular, the solution cannot

consistently reconstruct any map with one or more decreasing branch functions, and the map constructed in this case has inconsistent power spectrum mode characteristics (in general).

A novel method for solving IFPP-III that addresses the shortcomings of the original solution is developed. This method uses a novel strategy that requires the design of a semi-Markov map to satisfy two submatrices of the unknown map's order-one FPM. These matrices are chosen to permit the resolution of branch monotonicity in canonical maps and their consistent reconstruction. The strategy also yields a map with power spectrum mode characteristics (i.e., mode bandwidths and centre frequencies) that are consistent with any semi-Markov map presented for reconstruction.

Three contributions are made to overcome the design issues associated with the novel solution strategy. These contributions are

- a method-of-moments technique for estimating PWL density functions that is novel in the context of IFPP-III,
- estimates for the two submatrices of the unknown map's order-one FPM, which utilise the Moore-Penrose pseudoinverse and which are shown to be optimal in the limit of perfect estimation accuracy, and
- a novel subclass of the semi-Markov maps, labelled the generalised hat maps. The maps from this class are designed to simultaneously satisfy the two sub-matrix estimates.

The novel method uses the above contributions to overcome the design issues and successfully implement the novel strategy for solving IFPP-III.

Two propositions on the characteristics of the novel method for solving IFPP-III are presented. These propositions prove the novel method's consistent reconstruction of any unknown semi-Markov map that is canonical, and the construction of a map with consistent power spectrum mode characteristics regardless of the unknown map's canonicity. Both these propositions hold regardless of the monotonicity of the unknown map's branches, thereby implying that the method resolves the original solution's shortcomings.

Two numerical examples are presented to illustrate the relevant characteristics of the original solution and the novel method, and to test the relevant propositions. The numerical results are found to be in agreement with the proposition statements

regarding the characteristics of the methods. These results demonstrate that the novel method resolves the shortcomings of the original solution, thereby enlarging the classes of maps that may be reconstructed with consistent characteristics. However, unlike the original solution, the novel method introduces unnecessary branches in the reconstructed map.

Motivated by the above results, the following conclusions are drawn.

- The novel solution strategy's approximation of the unknown map's order-one FPM permits successful resolution of branch function monotonicity in canonical maps and the consistent modelling of the power spectrum mode characteristics of any semi-Markov map.
- The novel method is more versatile than the original solution, as it enlarges the classes of maps that may be reconstructed to possess consistent characteristics (refer to Table 5.7.1).
- The numerical results suggest that the reconstruction error of the novel method decreases with increasing estimation accuracy, provided that the conditions for consistent map reconstruction are met.
- The novel solution's introduction of unwanted branches in the reconstructed map is an additional source of reconstruction error. The original solution may be preferable from the perspective of reconstruction error if the unknown map is canonical and only possesses increasing branches.
- Due to its reconstruction of the map to *partially* satisfy the order-one FPM, the novel method does not consistently reconstruct the *full* power spectrum of an arbitrary PWL and ergodic semi-Markov map (in general).

6.5 Future Work

It is anticipated that the novel methods for solving IFPP-IIa and IFPP-III will find practical application. Future work in this context is the design and performance evaluation of random signal generators that utilise maps generated by the novel solution for IFPP-IIa, and their deployment in DRFM systems. The application of the novel method for solving IFPP-III towards the modelling of systems encountered in practice is another topic for future work.

On the theoretical front, future work includes the extension of the novel method for solving IFPP-IIa to permit the prescription of the power spectrum zeros, with the aim of yielding more diverse power spectra. Methods that solve IFPP-III by reconstructing the unknown map from the *full* order-one FPM (and possibly higher-order FPMs) may yield larger classes of maps that are reconstructible with consistent characteristics.

A further topic for future work is the application of the generalised hat maps towards probabilistic analysis and modelling problems outside the context of the IFPP. In particular, these maps may be used to construct approximations to singular maps that may permit their asymptotic analysis.

Particularly interesting future work, somewhat removed from possible extensions of the work reported herein, pertains to the Frobenius-Perron operator equation itself. Firstly, since infinite-dimensional iterative linear systems are able to exhibit chaos given appropriate conditions [16], the question arises as to whether iterates of the Frobenius-Perron equation can exhibit chaos. This could shed light on possible complications that might arise during numerical experiments. Secondly and somewhat esoteric, is notion of the Frobenius-Perron equation converted to a stochastic integral, e.g., Ito integral or even more general, its meaning and possible applications.

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