On the Application of the Double Reduction Theory to (n+1)-Dimensional Scalar PDEs and Systems of PDEs

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DECLARATION

I, Molahlehi Charles Kakuli, hereby declare that this thesis is my own, original, and unaided work. It is submitted in fulfillment of the requirements for the Degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. This work has not been submitted previously for any degree or examination at any other university.

M.C. Kakuli

This 27^{th} Day of January 2025.

Declaration of Publications

Details of the research work published or submitted for publication as part of this thesis are outlined below.

Chapter 2

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Chapter 3

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Dedication

To my princess, Mphoentle.

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Abstract

The generalization of the double reduction theory to higher-dimensional partial differential equations (PDEs) presents significant challenges, as highlighted by Sjöberg in prior studies. Despite these challenges, the application of the double reduction method to systems of PDEs and higher dimensional PDEs has been explored in the literature, albeit to a limited extent. Seminal contributions in this area include the work of Bokhari et al., who demonstrated that if a nontrivial conserved form exists with at least one associated symmetry in each reduction, a nonlinear system of qth order PDEs with n independent and m dependent variables can be systematically reduced to a nonlinear system of $(q-1)$th order ordinary differential equations (ODEs). This approach underscores the potential of the double reduction method in simplifying complex systems and advancing the analysis of higher-dimensional PDEs.

In this thesis, we contribute to the existing body of literature on the double reduction method by applying it to a variety of $(n+1)$-dimensional PDEs and $(1+1)$-dimensional systems of PDEs. We also introduce a variation of the double reduction method, demonstrating that the explicit use of associated conservation laws is not always necessary. Instead, canonical variables derived from symmetry alone can suffice for the reduction process.

We begin by illustrating the versatility of the double reduction method by applying it to specific $(1+1)$-dimensional PDEs, such as the Gibbons-Tsarev (GT) equation and the Hunter-Saxton (HS) equation. We construct conservation laws using the multiplier method, compute Lie point symmetries, and identify symmetries associated with conservation laws. These steps lead to invariant solutions through the double reduction process. Furthermore, the introduced variation of the double reduction method is also applied to several $(1+1)$-dimensional scalar PDEs, highlighting its effectiveness in simplifying the reduction process.

Additionally, we apply the double reduction method to $(2+1)$-dimensional scalar PDEs, in-

cluding the Zakharov-Kuznetsov (ZK) equation. By identifying and utilizing inherited symmetries, we achieve a second symmetry reduction, which significantly simplifies the equations and leads to invariant solutions.

Finally, we apply the generalized double reduction method to two $(1+1)$-dimensional systems of PDEs— the Broer-Kaup (BK) system and the Kaup-Boussinesq (K-B) system. In these cases, we identify symmetries, construct conservation laws using the multiplier method, and perform reductions, yielding exact solutions and demonstrating the practical applicability of the method.

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Introduction

Lie Symmetry Analysis (LSA) is a robust mathematical framework developed in the late 19th century by Norwegian mathematician Sophus Lie. Sophus Lie introduced the concept of Lie groups in his seminal works between 1842 and 1899, establishing a systematic approach to understanding symmetries in differential equations. His methods provided a unified framework for various integration techniques applicable to specific classes of differential equations, thus enhancing the analytical capabilities of mathematicians and physicists alike [1, 2]. LSA focuses on the study of continuous transformation groups, known as Lie groups, which leave differential equations invariant. This invariance has significant implications for the simplification and solution of ordinary and partial differential equations (ODEs and PDEs) [3]. Inspired by earlier contributions from Galois and Abel in the classification of algebraic equations, Lie extended the concept of symmetry to differential equations, providing a systematic and algorithmic approach to their analysis [4, 5].

Sophus Lie's pioneering work introduced the fundamental concepts of Lie groups and Lie algebras, establishing a foundation for the Lie group method. This method facilitates the discovery of exact solutions through techniques such as similarity solutions and order reductions. These solutions, referred to as group-invariant solutions, are derived by exploiting the invariance properties of differential equations under specific symmetry transformations. Lie's work laid the groundwork for a systematic method of solving differential equations, reducing the reliance on guesswork and heuristic approaches [3, 6–9].

The utility of LSA extends beyond the theoretical realm. It provides powerful routines for categorizing and simplifying PDEs based on their symmetries, making it an essential tool in applied mathematics, physics, and engineering. For example, LSA has been successfully applied to problems in fluid dynamics, reaction-diffusion systems, and other nonlinear phenomena, where analytical solutions are often highly desirable [10–17].

While the classical application of Lie symmetries relied heavily on manual computations, these calculations were often cumbersome and error-prone. The advent of computer algebra systems, such as Mathematica [18] and Maple [19], has revolutionized the field by automating the computation of symmetries and similarity solutions. This integration of computational techniques has significantly enhanced the applicability of LSA to complex systems of PDEs that were previously intractable [3, 6, 8, 20].

Building on Lie's principles, LSA has evolved to incorporate advanced methods such as the double reduction method, introduced by Sjöberg [21, 22]. This method extends classical symmetry analysis by incorporating conservation laws into the reduction process. For PDEs with two independent variables, the double reduction method reduces a q-th order PDE to a $(q-1)$-th order ODE, making it a powerful tool for simplifying equations and deriving exact solutions [21–25]. The applicability of this method has recently been extended to multi-variable PDEs and systems of PDEs, addressing challenges in higher-dimensional cases [17, 26–29].

The work of Bokhari et al. [30, 31] further advanced the double reduction method by applying it to nonlinear systems of PDEs. They demonstrated that a system of q-th order PDEs with n independent and m dependent variables could be systematically reduced to a system of $(q-1)$-th order ODEs using symmetries associated with nontrivial conservation laws.

Recent contributions by Anco and Gandarias [32] have generalized the double reduction method, introducing explicit algorithms to identify symmetry-invariant conservation laws for PDEs with multiple independent variables. These conservation laws simplify the corresponding ODEs, facilitating the construction of symmetry-invariant solutions. Applications of this generalized approach have been demonstrated by researchers such as Naz et al. [33], Bokhari et al. [31], and Sait et al. [34].

The theoretical framework of LSA also intersects with Noether's theorem [35], which links symmetries and conservation laws in the context of Euler-Lagrange PDEs. However, Ibragimov's nonlinear self-adjointness method extends Noether's theorem to arbitrary PDEs, providing a broader foundation for constructing conservation laws and enhancing the double reduction method's potential applications [36].

In summary, Lie Symmetry Analysis has evolved from Sophus Lie's foundational contributions into a sophisticated framework for analyzing and solving differential equations. Modern advancements, including the double reduction theory and its generalizations, underscore its ongoing relevance and applicability to higher-dimensional scalar PDEs and systems of PDEs. These developments form the foundation for this thesis, which explores the application of generalized double reduction theory to $(n+1)$-dimensional scalar PDEs and systems of PDEs.

An intriguing avenue for further extension lies in exploring nonclassical symmetries, which involve additional constraints on the infinitesimals, known as the invariant surface condition [37, 38]. Nonclassical symmetries can unveil special solutions beyond classical symmetry methods [37, 38], with applications in integrable systems and nonlinear dynamics. The prospect of integrating nonclassical symmetries into the double reduction method opens new horizons for achieving double reduction through a more diverse range of symmetries.

This thesis systematically explores and extends the application of double reduction method. The journey begins with an in-depth investigation of Lie's symmetry analysis, followed by a comprehensive exploration of the double reduction theory and its extension. Through this exploration, the thesis seeks to address the challenges of higher-dimensional PDEs and systems of PDEs and advance the understanding of the double reduction method in diverse mathematical landscapes.

The outline of the thesis is as follows:

In the first chapter, we present standard definitions and theorems relevant to the double reduction algorithm and the main elements of double reductions.

In Chapter Two, we perform the standard double reduction on two nonlinear (1+1)-dimensional scalar PDEs, the Gibbons-Tsarev (GT) equation [39]), and the Hunter-Saxton (HS) equation [40]. In both cases, we determine the admitted symmetries and use the multiplier method to compute the conservation laws. For the GT equation, we used the approach parallel to that of Sjöberg [21, 22], and for the HS equation, we followed the extended approach similar to Bokhari et al. [30, 31] and many other articles in the literature. We concluded this chapter by showing an alternative approach to perform double reduction using any symmetry of a PDE known to have an associated nontrivial conservation law without explicitly using the conservation law in the derivations. We also provided five examples.

In Chapter Three, we focus on applying the double reduction theorem to two (2+1)-dimensional PDEs. We begin this chapter by doing the extension of the seminal paper by Bokhari et al. [31] on the generalization of the double reduction theory where the theory was applied to the nonlinear $(2+1)$-dimensional wave equation. We have presented an extended account of the application and included a second multi-reduction of the conservation law of the equation by finding and using inherited symmetries that were not determined. We have also included, for illustrative purposes, solutions of the wave equation for particular specifications of the arbitrary functions. Next, we apply the generalization of the double reduction theory on the $(2+1)$-dimensional Zakharov-Kuznetsov (ZK) equation [13, 41–43]. We begin by computing the Lie point symmetries of the ZK equation and use the multiplier method to compute the conservation laws for the ZK equation.

In Chapter Four, we apply the generalized double reduction method to the Kaup-Boussinesq (K-B) and Broer-Kaup systems, both significant in nonlinear wave propagation studies. For each system, we identify Lie point symmetries and compute conservation laws using the multiplier method. Specifically, we construct four conservation laws for the K-B system and six for the Broer-Kaup system, with each symmetry associated with at least two conservation laws. These symmetries and conservation laws are then utilized to perform double reductions, simplifying the original PDE systems to first-order ODEs or algebraic equations in specific scenarios. In both cases, we report a solution obtained from a particular reduction, demonstrating the practical applicability of the method to nonlinear systems.

It is noteworthy that, for PDEs and systems of PDEs reported in this thesis, the Lie point symmetries are easily computed using MathLie [44], the symmetry-finding package for Mathematica [45] developed by G. Baumann [46]. For conservation laws, except for the Gibbons-Tsarev (GT) equation where we used Mathematica [45], they are computed using GeM [47] for Maple [19], the symbolic software package for computation of symmetries and conservation laws of nonlinear and linear differential equations developed by A. Shevyakov [48, 49].

Chapter 1

Preliminaries

1.1 Introduction

In this chapter we present basic operators, definitions, notations, and theorems that are well-known in the literature as preliminary information. These preliminaries will serve as the bedrock upon which we will build our applications. Each term and concept, carefully chosen and drawn from the literature [3, 6, 7, 20–22, 25, 30, 50–52], serves as a foundation, guiding us through the complex mathematical landscape and equipping us with the essential tools to address the forthcoming challenges.

1.2 Lie Point Symmetry

Consider a kth order system of partial differential equations of n independent variables $x = \left(x^1, x^2, \dots, x^n\right)$ and m dependent variables $u = \left(u^1, u^2, \dots, u^m\right)$:

$$\Delta^{\alpha}\left(x, u, u_{(1)}, u_{(2)}, \dots, u_{(k)}\right) = 0, \quad \alpha = 1, \dots, m. \tag{1.2.1}$$

The variables $u_{(1)}, u_{(2)}, \ldots, u_{(k)}$ denote the collections of all first, second, ..., kth-order partial derivatives, respectively, that is

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j D_i(u^\alpha), \ldots, \tag{1.2.2}$$

with the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} +, \cdots, \quad i = 1, \ldots, n. \tag{1.2.3}$$

A Lie-Bäcklund or generalized operator is defined by [50]

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \sum_{s \geq 1} \zeta_{i_1 \cdots i_s}^\alpha \frac{\partial}{\partial u_{i_1 \cdots i_s}^\alpha}, \quad \xi^i, \eta^\alpha \in \mathscr{F} \tag{1.2.4}$$

where $\mathscr{F}$ is the universal space of differential functions and the additional coefficients are determined uniquely by the following formulas:

$$\begin{aligned} \zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i\left(\xi^j\right), \\ \zeta_{i_1 \cdots i_s}^\alpha &= D_{i_s}\left(\zeta_{i_1 \cdots i_{s-1}}^\alpha\right) - u_{ji_1 \cdots i_{s-1}}^\alpha D_{i_s}\left(\xi^j\right), \quad s > 1. \end{aligned} \tag{1.2.5}$$

The Lie point symmetry of equation (1.2.1) is a generator X of the form (1.2.4) that satisfies

$$X^{[k]} \Delta^\alpha \Big|_{(1.2.1)} = 0, \tag{1.2.6}$$

where $X^{[k]}$ is the k^{th} prolongation of X defined by

$$X^{[k]} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta^\alpha \frac{\partial}{\partial u_i^\alpha} + \cdots + \zeta_{i_1 \ldots i_k}^\alpha \frac{\partial}{\partial u_{i_1 \ldots i_k}^\alpha}. \tag{1.2.7}$$

This means that equation (1.2.1) is invariant under the action of the generator X.

1.3 Conservation Laws

A conserved vector of (1.2.1) is n-tuple $T = \left(T^1, T^2, \ldots, T^n\right)$ satisfying the relation

$$D_i T^i\big|_{(1.2.1)} = 0, \tag{1.3.1}$$

where $T^i = T^i\left(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(q)}\right) \in \mathscr{F}, \quad i = 1, 2, \ldots, n.$ A conservation law can be expressed in characteristic form [6] as

$$D_i T^i = \Lambda_\alpha \Delta^\alpha, \quad \alpha = 1, 2, \ldots, m, \tag{1.3.2}$$

where $\Lambda_\alpha = \Lambda_\alpha\left(x, u, u_{(1)}, \ldots, u_{(q)}\right)$ are the characteristics or multipliers for the PDE system (1.2.1). The determining equations for multipliers are obtained by taking the variational derivative

$$E_u\left[\Lambda_\alpha\left(\Delta^\alpha\right)\right] = 0, \tag{1.3.3}$$

where the Euler operator E_u is defined by

$$E_u = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 \cdots i_s}}, \quad \alpha = 1, \ldots, m. \tag{1.3.4}$$

1.4 Fundamentals of the Double Reduction Theorem

A Lie-Bäcklund operator X given in (1.2.4) is said to be associated with the conserved vector T of (1.2.1) if it satisfies the following relation [53]:

$$\left[T^i, X\right] = X\left(T^i\right) + T^i D_j\left(\xi^j\right) - T^j D_j\left(\xi^i\right) = 0, \quad i = 1, 2, \ldots, n. \tag{1.4.1}$$

The equation (1.4.1) is also known as the symmetry conservation law relationship. Stemming from the relation (1.4.1), Sjöberg [21] suggested the double reduction theory for a scalar PDE with two independent variables. If the PDE admits a symmetry with operator X that is associated with a conservation law

$$D_tT^t + D_xT^x = 0, \tag{1.4.2}$$

canonical variables r, s and w such that

$$\begin{aligned} r &= r(t,x,u), \\ s &= s(t,x,u), \\ w &= w(t,x,u), \end{aligned} \tag{1.4.3}$$

are determined as the starting point . These variables transform the generator X to its canonical form $\widetilde{X} = \frac{\partial}{\partial s}$. This means that they must satisfy the equations:

$$\begin{aligned} &Xr = Xw = 0, \\ &Xs = 1. \end{aligned} \tag{1.4.4}$$

Clearly r and w are invariants of X and are therefore easily obtained from the characteristic equations:

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}. \tag{1.4.5}$$

The variable s can be determined by inspection. More systematically, it can be obtained from an invariant $J = v - s(x,y)$ of the extended operator $X + \partial_v$, where v is an auxiliary variable [6]. In this way, (1.4.2) can be written in canonical variables as [21, 22]

$$D_rT^r + D_sT^s = 0, \tag{1.4.6}$$

with

$$T^r = \frac{T^t D_t(r) + T^x D_x(r)}{D_t(r)D_x(s) - D_x(r)D_t(s)} \tag{1.4.7}$$

and

$$T^s = \frac{T^t D_t(s) + T^x D_x(s)}{D_t(r)D_x(s) - D_x(r)D_t(s)}. \tag{1.4.8}$$

The components T^t and T^x depend on $(t, x, u, u_{(1)}, u_{(2)}, \ldots, u_{(q-1)})$, which means that T^r and T^s depend on $(r, s, w, w_r, w_{rr}, \ldots, w_{r^{q-1}})$ for solutions invariant with respect to X. Therefore, the conservation law in canonical variables (1.4.6) becomes:

$$\frac{\partial T^s}{\partial s} + D_r T^r = 0. \tag{1.4.9}$$

From the association of X with T, it follows that in canonical variables

$$\widetilde{X} T^r \equiv \frac{\partial T^r}{\partial s} = 0 \quad \text{and} \quad \widetilde{X} T^s \equiv \frac{\partial T^s}{\partial s} = 0, \tag{1.4.10}$$

which leads to further reduction of the conservation law (1.4.9) to

$$D_r T^r = 0, \tag{1.4.11}$$

or, equivalently, the reduced ODE of order $q-1$, namely

$$T^r = k, \tag{1.4.12}$$

where k is an arbitrary constant.

Theorem 1.4.1 *(see [25]). Suppose that X is any Lie-Bäcklund symmetry generator of the system (1.2.1) and $T^i, i = 1, \ldots, n$ are the components of conserved vector of (1.2.1). Then*

$$T^{*i} = \left[T^i, X\right] = X\left(T^i\right) + T^i D_j\left(\xi^j\right) - T^j D_j\left(\xi^i\right), \quad i = 1, \ldots, n \tag{1.4.13}$$

also constitute the components of a conserved vector for (1.2.1); that is,

$$D_i T^{*i}\big|_{(1.2.1)} = 0. \tag{1.4.14}$$

Theorem 1.4.2 *(see [30]). Suppose that $D_i T^i = 0$ is a conservation law of the system (1.2.1). Then, under a contact transformation, there exists function $\widetilde{T}^i$ such that $J D_i T^i = \widetilde{D}_i \widetilde{T}^i$, where $\widetilde{T}^i$ is given by the following formula:*

$$\begin{pmatrix} \tilde{T}^1 \\ \widetilde{T}^2 \\ \vdots \\ \tilde{T}^n \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix}, \qquad J \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix} = A^T \begin{pmatrix} \tilde{T}^1 \\ \tilde{T}^2 \\ \vdots \\ \tilde{T}^n \end{pmatrix}, \tag{1.4.15}$$

in which

$$A = \begin{pmatrix} \widetilde{D}_1 x_1 & \widetilde{D}_1 x_2 & \cdots & \widetilde{D}_1 x_n \\ \widetilde{D}_2 x_1 & \widetilde{D}_2 x_2 & \cdots & \widetilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{D}_n x_1 & \widetilde{D}_n x_2 & \cdots & \widetilde{D}_n x_n \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} D_1 \tilde{x}_1 & D_1 \tilde{x}_2 & \cdots & D_1 \tilde{x}_n \\ D_2 \tilde{x}_1 & D_2 \tilde{x}_2 & \cdots & D_2 \tilde{x}_n \\ \vdots & \vdots & \vdots & \vdots \\ D_n \tilde{x}_1 & D_n \tilde{x}_2 & \cdots & D_n \tilde{x}_n \end{pmatrix}, \tag{1.4.16}$$

and $J = \det(A)$.

Theorem 1.4.3 *(fundamental theorem on generalized double reduction [30]). Suppose that $D_i T^i = 0$ is a conservation law of the system (1.2.1). Then, under a similarity transformation of a symmetry X of the system (1.2.1), there exist functions $\widetilde{T}^i$ such that X is still a symmetry* *for the PDE* $\widetilde{D}_i\widetilde{T}^i = 0$ *and*

$$\begin{pmatrix} X\widetilde{T}^1 \\ X\widetilde{T}^2 \\ \vdots \\ X\widetilde{T}^n \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} \left[T^1,X\right] \\ \left[T^2,X\right] \\ \vdots \\ \left[T^n,X\right] \end{pmatrix} \tag{1.4.17}$$

in which

$$A = \begin{pmatrix} \widetilde{D}_1 x_1 & \widetilde{D}_1 x_2 & \cdots & \widetilde{D}_1 x_n \\ \widetilde{D}_2 x_1 & \widetilde{D}_2 x_2 & \cdots & \widetilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{D}_n x_1 & \widetilde{D}_n x_2 & \cdots & \widetilde{D}_n x_n \end{pmatrix}, \qquad A^{-1} = \begin{pmatrix} D_1 \tilde{x}_1 & D_1 \tilde{x}_2 & \cdots & D_1 \tilde{x}_n \\ D_2 \tilde{x}_1 & D_2 \tilde{x}_2 & \cdots & D_2 \tilde{x}_n \\ \vdots & \vdots & \vdots & \vdots \\ D_n \tilde{x}_1 & D_n \tilde{x}_2 & \cdots & D_n \tilde{x}_n \end{pmatrix}, \tag{1.4.18}$$

and $J = \det(A)$.

Corollary 1.4.1 *(the necessary and sufficient condition for reduced conserved form [30]). The conserved form* $D_iT^i = 0$ *of the PDE system (1.2.1) can be reduced under a similarity transformation of a symmetry X to a reduced conserved form* $\widetilde{D}_i\widetilde{T}^i = 0$ *if and only if X is associated with the conservation law T, that is,* $[T,X]|_{(1.2.1)} = 0$.

Corollary 1.4.2 *(see [30]). A nonlinear system of qth-order PDEs with n independent and m dependent variables which admits a nontrivial conserved form that has at least one associated symmetry in every reduction from the n reductions (the first step of double reduction) can be reduced to a* $(q-1)$ *th-order nonlinear system of ODEs.*

Corollary 1.4.3 *(The Inherited Symmetries [30]). Any symmetry Y for the conserved form* $D_iT^i = 0$ *of PDE system (1.2.1) can be transformed under the similarity transformation of a symmetry X for the PDE to the symmetry* $\tilde{Y}$ *for the PDE* $\tilde{D}_i\widetilde{T}^i = 0$.

Chapter 2

Comparative Analysis: Double Reduction of $(1+1)$-Dimensional Scalar PDEs: With and Without Explicit Use of Conservation Laws

2.1 Introduction

This chapter has three main sections. In the first two sections, we perform the standard double reduction on two nonlinear (1+1)-dimensional scalar PDEs, the Gibbons-Tsarev (GT) equation [39], and the Hunter-Saxton (HS) equation [40]. In both cases, we determined the admitted symmetries using Mathematica [45] and used the multiplier method [47, 48, 54] to compute the conservation laws. For the GT equation, we used the approach parallel to that of Sjöberg [21, 22] in the reduction process, and for the HS equation, we followed the extended approach similar to Bokhari et al. [30, 31] and many other articles in the literature. In the last section, we show an alternative approach to performing double reduction using any symmetry of a PDE known to have an associated nontrivial conservation law without explicitly using the conservation law in the derivations and provide five examples.

2.2 Conservation Laws and Symmetry Reductions of the Gibbons-Tsarev (GT) Equation via the Double Reduction Method

2.2.1 Introduction

In this section, we apply the double reduction to a second-order nonlinear PDE:

$$u_{tt} - u_x u_{tx} + u_t u_{xx} - 1 = 0, \tag{2.2.1}$$

known as the GT equation, and arises in the theory of dispersionless integrable systems (see, for example, [39]). The equation has attracted interest from many researchers. For instance, Kaptsov et al. [55, 56] applied the method of differential constraints to find solutions of the GT equation that are expressible in terms of solutions of Painlevé equations. Lelito and Morozov [57], on the other hand, used methods of group analysis of differential equations to find solutions of the GT equation that arise as invariant solutions (in the "classical" way [58–63]) from Lie point symmetries admitted by the equation. Using a known Lax pair, Baran et al. [64] constructed an infinite series of conservation laws and the algebra of nonlocal symmetries associated with these conservation laws for the GT equation.

In this section, we employ the double reduction method and find invariant solutions to the GT equation. The multiplier method is utilized to construct low-order multipliers of the equation of a particular form, namely, polynomial functions of degree at most three in the variables t,

x, u, and the first-order derivatives of u. This leads to four non-trivial conservation laws of (2.2.1), two of which have associated Lie point symmetries. According to the double reduction algorithm, solutions are then constructed systematically from the two conservation laws and associated Lie point symmetries.

2.2.2 Conservation Laws and Symmetries of (2.2.1)

We seek conservation laws of (2.2.1) via the multiplier approach [6, 54, 65]. Let us consider first-order multipliers of the form

$$\Lambda = \Lambda(t, x, u, u_x, u_t), \tag{2.2.2}$$

on the equation (2.2.1). Determining equations for the multipliers can be obtained by taking the variational derivative defined in (1.3.3),

$$\frac{\delta}{\delta u}\left[\Lambda\left(u_{tt} - u_x u_{tx} + u_t u_{xx} - 1\right)\right] = 0. \tag{2.2.3}$$

The determining equations that arise from (2.2.3) are intractable for an arbitrary Λ. For the purpose of double reduction, however, a particular form of Λ would suffice provided that it leads to conservation laws that have symmetries of the GT equation associated with them. Considering this, we assume that Λ is a sum of monomials of degree up to three involving the variables t, x, u, u_t, u_x,

$$\begin{aligned}\Lambda(t,x,u,u_x,u_t) = {} & \delta_0 + \delta_1 x + \delta_2 t + \delta_3 u + \delta_4 u_x + \delta_5 u_t + \delta_6 u^2 + \delta_7 u u_x \\ & + \delta_8 u_x^2 + \delta_9 u u_t + \delta_{10} u_x u_t + \delta_{11} u_t^2 + \delta_{12} x u + \delta_{13} x u_x \\ & + \delta_{14} x u_t + \delta_{15} x^2 + \delta_{16} t u + \delta_{17} t u_x + \delta_{18} t u_t + \delta_{19} t x \\ & + \delta_{20} t^2 + \delta_{21} u^3 + \delta_{22} u^2 u_x + \delta_{23} u u_x^2 + \delta_{24} u_x^3 + \delta_{25} u^2 u_t\end{aligned}$$

$$+\delta_{26}uu_xu_t+\delta_{27}u_x^2u_t+\delta_{28}uu_t^2+\delta_{29}u_xu_t^2+\delta_{30}u_t^3$$
$$+\delta_{31}xu^2+\delta_{32}xuu_x+\delta_{33}xu_x^2+\delta_{34}xuu_t+\delta_{35}xu_xu_t$$
$$+\delta_{36}xu_t^2+\delta_{37}x^2u+\delta_{38}x^2u_x+\delta_{39}x^2u_t+\delta_{40}x^3+\delta_{41}tu^2$$
$$+\delta_{42}tuu_x+\delta_{43}tu_x^2+\delta_{44}tuu_t+\delta_{45}tu_xu_t+\delta_{46}tu_t^2$$
$$+\delta_{47}txu+\delta_{48}txu_x+\delta_{49}txu_t+\delta_{50}tx^2+\delta_{51}t^2u$$
$$+\delta_{52}t^2u_x+\delta_{53}t^2u_t+\delta_{54}t^2x+\delta_{55}t^3, \tag{2.2.4}$$

where δ_i are arbitrary constants. This assumption on the form of admitted multipliers allows the determining equations from (2.2.3) to be solved easily [45], and we obtain

$$\Lambda=\delta_0+\delta_4u_x+\delta_5\left(\frac{2u_t-3t-3u_x^2}{2}\right)+\delta_{24}\left(\frac{4tu_x+2u_x^3-3u_xu_t-x}{2}\right). \tag{2.2.5}$$

The multipliers Λ for (2.2.1) satisfy

$$\Lambda\left(u_{tt}-u_xu_{tx}+u_tu_{xx}-1\right)=D_tT^t+D_xT^x, \tag{2.2.6}$$

for all functions $u(t,x)$. From (2.2.5) and (2.2.6) we obtain

$$\begin{aligned}T^t=\delta_{24}&\left(tu_x^3-2tu_xu_t+\frac{u_x^5}{4}-u_x^3u_t-\frac{xu_x^2}{2}+\frac{3u_xu_t^2}{4}+\frac{xu_t}{2}\right)\\&+\delta_0\left(u_x^2-u_t\right)+\delta_4\left(\frac{u_x^3}{2}-u_xu_t\right)+\delta_5\left(\frac{3}{2}u_t\left(t+u_x^2\right)\right.\\&\left.-\frac{3tu_x^2}{2}-\frac{u}{2}-\frac{u_x^4}{2}-\frac{u_t^2}{2}\right)-\lambda u_x+\psi(x),\end{aligned} \tag{2.2.7}$$

and

$$T^x=\delta_{24}\left(2tu-tu_x^2u_t+tu_t^2-\frac{u_x^4u_t}{4}+\frac{3u_x^2u_t^2}{4}+\frac{xu_xu_t}{2}-\frac{u_t^3}{4}-\frac{x^2}{4}\right)$$

$$+\delta_5\left(\frac{1}{2}u_xu_t(3t-2u_t)-\frac{3tx}{2}+\frac{u_x^3u_t}{2}\right)+\delta_0\left(x-u_xu_t\right)$$

$$+\delta_4\left(u-\frac{u_x^2u_t}{2}+\frac{u_t^2}{2}\right)-\lambda u_t+\phi(t), \tag{2.2.8}$$

for arbitrary functions ψ and ϕ and a constant λ. If $u(t,x)$ is a solution of (2.2.1), the left hand side of (2.2.6) vanishes. Therefore, the following non-trivial conserved vectors of (2.2.1) arise from multipliers of the form (2.2.4):

$$T_1 \quad : \quad \begin{cases} T_1^t = u_x^2 - u_t, \\ T_1^x = x - u_xu_t, \end{cases} \tag{2.2.9}$$

$$T_2 \quad : \quad \begin{cases} T_2^t = \dfrac{u_x^3}{2} - u_xu_t, \\ T_2^x = u - \dfrac{u_x^2u_t}{2} + \dfrac{u_t^2}{2}, \end{cases} \tag{2.2.10}$$

$$T_3 \quad : \quad \begin{cases} T_3^t = \dfrac{3}{2}u_t\left(t+u_x^2\right) - \dfrac{3tu_x^2}{2} - \dfrac{u}{2} - \dfrac{u_x^4}{2} - \dfrac{u_t^2}{2}, \\ T_3^x = \dfrac{1}{2}u_xu_t(3t-2u_t) - \dfrac{3tx}{2} + \dfrac{u_x^3u_t}{2}, \end{cases} \tag{2.2.11}$$

$$T_4 \quad : \quad \begin{cases} T_4^t = tu_x^3 - 2tu_xu_t + \dfrac{u_x^5}{4} - u_x^3u_t - \dfrac{xu_x^2}{2} + \dfrac{3u_xu_t^2}{4} + \dfrac{xu_t}{2}, \\ T_4^x = 2tu - tu_x^2u_t + tu_t^2 - \dfrac{u_x^4u_t}{4} + \dfrac{3u_x^2u_t^2}{4} + \dfrac{xu_xu_t}{2} - \dfrac{u_t^3}{4} - \dfrac{x^2}{4}. \end{cases} \tag{2.2.12}$$

It is to be noted that a different form of Λ in (2.2.4) might lead to other conservation laws of (2.2.1). Next is to establish which of the constructed conservation laws have associated Lie point symmetries of (2.2.1). As did Lu and Zhang [63], we determined that (2.2.1) admits the following Lie point symmetries [44]:

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial x},\\
X_2 &= \frac{\partial}{\partial t},\\
X_3 &= \frac{3}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u},\\
X_4 &= \frac{\partial}{\partial u},\\
X_5 &= -\frac{1}{2}t\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}.
\end{aligned}
\tag{2.2.13}
$$

A linear combination of these symmetries is

$$
X = \Sigma_{i=1}^{5}\kappa_i X_i \equiv \xi^t\partial_t + \xi^x\partial_x + \eta\partial_u, \tag{2.2.14}
$$

where κ_i are arbitrary constants. Extending X once gives

$$
\begin{aligned}
X^{(1)} = {}& (\kappa_2 + \kappa_3 t)\,\partial_t + \left(\kappa_1 + \frac{3\kappa_3 x}{2} - \frac{\kappa_5 t}{2}\right)\partial_x + (2\kappa_3 u + \kappa_4 + \kappa_5 x)\,\partial_u\\
&+ \left(\kappa_3 u_t + \frac{\kappa_5 u_x}{2}\right)\partial_{u_t} + \left(\frac{\kappa_3 u_x}{2} + \kappa_5\right)\partial_{u_x}.
\end{aligned}
\tag{2.2.15}
$$

For each of the constructed conservation vectors (T_i^t, T_i^x), $i = 1,2,\ldots,4$, parameters can be determined in (2.2.14) for which (1.4.1) is satisfied. That is

$$
\begin{aligned}
X^{(1)}T^t + T^t D_x\xi^x - T^x D_x\xi^t &= 0,\\
X^{(1)}T^x + T^x D_t\xi^t - T^t D_t\xi^x &= 0.
\end{aligned}
\tag{2.2.16}
$$

It turns out that two of the conserved vectors (2.2.9) and (2.2.10) have associated Lie point symmetries of the form

$$
\Phi_1 = \kappa_1\partial_x + \kappa_2\partial_t = \kappa_1 X_1 + \kappa_2 X_2 \tag{2.2.17}
$$

and

$$\Phi_2 = \kappa_2 \partial_t + \kappa_4 \partial_u = \kappa_2 X_2 + \kappa_4 X_4, \tag{2.2.18}$$

respectively.

2.2.3 Double Reduction of (2.2.1) by $\langle \Phi_1 \rangle$

The generator, Φ_1, has a canonical form $\widetilde{\Phi_1} = \partial/\partial s$ when the characteristic equations

$$\frac{dt}{\kappa_2} = \frac{dx}{\kappa_1} = \frac{du}{0} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0}, \tag{2.2.19}$$

hold and thus the canonical variables are:

$$\begin{aligned} r &= \frac{\kappa_2 x - \kappa_1 t}{\kappa_2}, \qquad && \kappa_2 \neq 0, \\ s &= \frac{t}{\kappa_2}, && \kappa_2 \neq 0, \\ w &= u. \end{aligned} \tag{2.2.20}$$

where $w = w(r)$. The inverse canonical coordinates follow from (2.2.20) and are given by:

$$\begin{aligned} t &= \kappa_2 s, \\ x &= \kappa_1 s + r, \\ u &= w. \end{aligned} \tag{2.2.21}$$

From (2.2.21) partial derivatives of u in terms of the canonical variables are derived following the routine outlined in [8]. We obtain:

$$\begin{aligned}
u_x &= w_r, \\
u_t &= -\frac{\kappa_1}{\kappa_2} w_r, \qquad \kappa_2 \neq 0 \\
u_{xx} &= w_{rr}, \\
u_{tx} &= -\frac{\kappa_1}{\kappa_2} w_{rr}, \qquad \kappa_2 \neq 0 \\
u_{tt} &= \frac{\kappa_1^2}{\kappa_2^2} w_{rr}, \qquad \kappa_2 \neq 0.
\end{aligned} \tag{2.2.22}$$

By substituting (2.2.21) and (2.2.22) into T_1^t and T_1^x, and using (1.4.7) and (1.4.8), we obtain the following components of the reduced vector:

$$\begin{aligned}
T_1^r &= \kappa_2 w - \frac{\kappa_1^2 w_r^2}{2\kappa_2}, \qquad \kappa_2 \neq 0 \\
T_1^s &= -\frac{\kappa_1 w_r^2}{\kappa_2} - \frac{w_r^3}{2}, \qquad \kappa_2 \neq 0.
\end{aligned} \tag{2.2.23}$$

The first component in (2.2.23), in accordance with (1.4.12), results in a known first-order nonlinear ODE called Chrystal's equation (see [66] and the references therein)

$$2\kappa_2^2 w - \kappa_1^2 (w')^2 = k, \tag{2.2.24}$$

where k is an arbitrary constant. Equation (2.2.24) admits the translation symmetry ∂_r, and is thus tractable via Lie symmetry methods for ODEs (see, for example, [3, 6, 8, 66, 67]). The solution is

$$w = \frac{\kappa_2^4 (c_1 \kappa_1 + r)^2 + k\kappa_1^2}{2\kappa_1^2 \kappa_2^2}, \tag{2.2.25}$$

where c_1 is an arbitrary constant. Finally, from (2.2.21) and (2.2.25), a solution of the GT equation (2.2.1) obtained via the double reduction method and using Φ_1 is

$$u(t,x) = J + \frac{c_1\kappa_2^2 x}{\kappa_1} - c_1\kappa_2 t + \frac{\kappa_2^2 x^2}{2\kappa_1^2} - \frac{\kappa_2 t x}{\kappa_1} + \frac{t^2}{2}. \tag{2.2.26}$$

where $J = \frac{c_1^2\kappa_2^2}{2} + \frac{k}{2\kappa_2^2}$.

2.2.4 Double Reduction of (2.2.1) by $\langle\Phi_2\rangle$

The canonical coordinates in this case are:

$$\begin{aligned} r &= x, \\ s &= \frac{t}{\kappa_2}, \qquad \kappa_2 \neq 0 \\ w &= \frac{\kappa_2 u - \kappa_4 t}{\kappa_2}, \qquad \kappa_2 \neq 0. \end{aligned} \tag{2.2.27}$$

where $w = w(r)$. From (2.2.27) the inverse canonical coordinates are given by:

$$\begin{aligned} t &= \kappa_2 s, \\ x &= r, \\ u &= \kappa_4 s + w. \end{aligned} \tag{2.2.28}$$

It follows from (2.2.28) that the partial derivatives of u in terms of the canonical variables are given by:

$$\begin{aligned} u_x &= w_r, \\ u_t &= \frac{\kappa_4}{\kappa_2}, \qquad\qquad \kappa_2 \neq 0 \\ u_{xx} &= w_{rr}, \\ u_{tx} &= u_{tt} = 0. \end{aligned} \tag{2.2.29}$$

By substituting (2.2.28) and (2.2.29) into T_2^t and T_2^x, and using (1.4.7) and (1.4.8), we obtain components of the reduced vector:

$$\begin{aligned} T^r &= \kappa_2 r - \kappa_4 w_r, \\ T^s &= \frac{\kappa_4}{\kappa_2} - w_r^2. \qquad\qquad \kappa_2 \neq 0 \end{aligned} \tag{2.2.30}$$

From the first component in (2.2.30), we deduce, in accordance with (1.4.12), that the reduced conservation law in this case satisfies the simple ODE

$$\kappa_2 r - \kappa_4 w_r = k, \tag{2.2.31}$$

where k is an arbitrary constant. The solution of (2.2.31) is

$$w(r) = \frac{\frac{\kappa_2 r^2}{2} - kr}{\kappa_4} + c_1, \tag{2.2.32}$$

where c_1 is an arbitrary constant. The solution of the GT equation (2.2.1) obtained via the double reduction method using Φ_2 follows from (2.2.28) and (2.2.32):

$$u(t,x) = c_1 - \frac{kx}{\kappa_4} + \frac{\kappa_4 t}{\kappa_2} + \frac{\kappa_2 x^2}{2\kappa_4}. \tag{2.2.33}$$

2.2.5 Concluding Remarks

In this section, we used the double reduction method to find exact solutions to the GT equation, which arises from the dispersionless integrable systems theory. The multiplier method was employed to construct the conservation laws of the equation. Notably, we did not insist on determining all the admitted multipliers for the equation, in which case the determining equations are intractable. Instead, only multipliers (and hence conservation laws) corresponding to a particular form of the multiplier were determined. We determined multipliers for the GT equation that are polynomial functions in the variables t, x, u and the first derivatives of u, and with the degree of at most three. Four non-trivial conservation laws were determined, two of which were found to have associated Lie point symmetries. The double reduction method was then invoked in the two cases, leading to first-order ODEs that were easily solved and ultimately yielded two general solutions of the GT equation.

2.3 Conservation Laws and Symmetry Reductions of the Hunter–Saxton (HS) Equation via the Double Reduction Method

2.3.1 Introduction

In this section, we focus on the HS equation, a mathematical model described by the partial differential equation (PDE),

$$(u_t + uu_x)_x = \tfrac{1}{2}u_x^2, \tag{2.3.1}$$

which arises as an Euler-Lagrange equation of a variational principle in the study of a nonlinear wave equation for the director field of a nematic liquid crystal [40]. Equation (2.3.1) has attracted significant attention from researchers, prompting numerous studies on it and its derivatives. These investigations have often employed Lie symmetry analysis to explore various equation properties and, in certain instances, to uncover solutions.

Nadjafikhah and Ahangari [68] determined the Lie point symmetries of the equation and used the symmetries to find conservation laws and conduct symmetry reductions of the equation. An optimal system of one-dimensional subalgebras of the symmetry algebra of the HS equation was also constructed. San et al [69] investigated a modified version of the HS equation, a third-order nonlinear PDE. Their work featured the utilization of Ibragimov's nonlocal conservation method to derive conservation laws for the equation. Liu and Zhao [70] undertook the study of a generalized two-component HS system of equations. They determined similarity variables and executed symmetry reductions for this new generalized system, leading to the discovery of some exact solutions for the system. Yao et al [71] tackled the periodic Hunter–Saxton equation, introducing a variable coefficient into the generalized equation. They succeeded in finding exact solutions for specific selections of the variable coefficient by employing the classical approach to finding invariant solutions. Johnpillai and Khalique [72] also used Lie symmetry analysis to find exact solutions for yet another generalized version of the HS equation.

For this thesis, our study is dedicated to examining the symmetry reductions of the HS equation, utilizing the double reduction method. Our objectives encompass the identification of Lie point symmetries, the determination of conservation laws through the multiplier method, and the application of the double reduction method to achieve symmetry reductions. This research serves as a valuable addition to the existing body of work on the HS equation while also contributing insights into the double reduction method in the search for solutions for PDEs. It must be noted that the double reduction routine we have adopted in this article is based on the generalized approach proposed by Bokhari et al [30], which can be used to study PDEs such as those studied in [73–75], of higher dimensions.

2.3.2 Symmetries and Conservation Laws of (2.3.1)

The HS equation (2.3.1) is a $(1+1)$-dimensional scalar PDE with two independent variables $x=(x^1,x^2)=(t,x)$ and one dependent variable $u=u(t,x)$. It admits the following four symmetries [44]:

$$\begin{aligned}
X_1 &= x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u},\\
X_2 &= \frac{\partial}{\partial t},\\
X_3 &= t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x},\\
X_4 &= t^2\frac{\partial}{\partial t}+2tx\frac{\partial}{\partial x}+2x\frac{\partial}{\partial u}.
\end{aligned} \tag{2.3.2}$$

We will use the multiplier approach to derive conservation laws for the HS equation (2.3.1). We seek first-order multipliers

$$\Lambda=\Lambda(x,t,u,u_x,u_t) \tag{2.3.3}$$

of (2.3.1), for which the determining equation according to (1.3.3) is

$$\frac{\delta}{\delta u}\left[\Lambda\left((u_t+uu_x)_x-\frac{1}{2}ux^2\right)\right]=0, \tag{2.3.4}$$

where the standard Euler operator $\delta/\delta u$, as defined in (1.3.4), is

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}-D_t\frac{\partial}{\partial u_t}-D_x\frac{\partial}{\partial u_x}+D_t^2\frac{\partial}{\partial u_{tt}}+D_x^2\frac{\partial}{\partial u_{xx}}+D_xD_t\frac{\partial}{\partial u_{tx}}-\cdots, \tag{2.3.5}$$

and total derivative operators D_t and D_x using (1.2.3) are

$$D_t=\frac{\partial}{\partial t}+u_t\frac{\partial}{\partial u}+u_{tt}\frac{\partial}{\partial u_t}+u_{tx}\frac{\partial}{\partial u_x}+\cdots,$$
$$D_x=\frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+u_{xx}\frac{\partial}{\partial u_x}+u_{tx}\frac{\partial}{\partial u_t}+\cdots.$$

The determining equation for the multiplier Λ after expansion takes the following form:

$$\Omega_0+u_{tt}\Omega_1+u_{tx}\Omega_2+(u_{tx})^2\,\Omega_3+u_{xx}\Omega_4+u_{xx}u_{tt}\Omega_5=0, \tag{2.3.6}$$

where

$$\begin{aligned}
\Omega_0 &= u_x\Lambda_{tu}-\frac{1}{2}u_x^2\Lambda_{tu_t}+\Lambda_{tx}-\frac{1}{2}u_x^3\Lambda_{uu_x}-\frac{1}{2}u_x^2u_t\Lambda_{uu_t}+uu_x^2\Lambda_{uu}+u_xu_t\Lambda_{uu}\\
&\quad+2uu_x\Lambda_{xu}+u_t\Lambda_{xu}+u\Lambda_{xx}-\frac{1}{2}u_x^2\Lambda_{xu_x}+\frac{3u_x^2\Lambda_u}{2}+u_x\Lambda_x,\\
\Omega_1 &= u_x\Lambda_{uu_t}-\frac{1}{2}u_x^2\Lambda_{u_tu_t}+\Lambda_{xu_t},\\
\Omega_2 &= 2uu_x\Lambda_{uu_t}+2u\Lambda_{xu_t}-u_x^2\Lambda_{u_tu_x}+2\Lambda_u,\\
\Omega_3 &= u\Lambda_{u_tu_t}-\Lambda_{u_tu_x},\\
\Omega_4 &= \Lambda_{tu_x}-u\Lambda_{tu_t}+u_t\Lambda_{uu_x}+u\Lambda_{xu_x}+uu_x\Lambda_{uu_x}-uu_t\Lambda_{uu_t}-\frac{1}{2}u_x^2\Lambda_{u_xu_x}\\
&\quad+2u\Lambda_u-u_x\Lambda_{u_x}-u_t\Lambda_{u_t}+\Lambda,
\end{aligned}$$

$$\Omega_5 \;=\; \Lambda_{u_t u_x} - u\Lambda_{u_t u_t}.$$

The multiplier determining equation (2.3.6) splits with respect to different combinations of the derivatives u_{xx}, u_{tx} and u_{tt} yielding an overdetermined linear system of equations for the multiplier. The system of equations was solved using Mathematica [45] to obtain

$$\Lambda = u_t\left(\delta_2 + \delta_3 t - \frac{\delta_1 t^2}{2}\right) + u_x x(\delta_3 - \delta_1 t) + \delta_1 x + \delta_4 u_x + \frac{\delta_5}{u_x^2}, \tag{2.3.7}$$

where $\delta_i,\ i = 1,2,\ldots,5$, are arbitrary constants. From (1.3.2) and (2.3.7), we have

$$\begin{aligned}&\left[(u_t + uu_x)_x - \frac{1}{2}ux^2\right]\left[u_t\left(\delta_2 + \delta_3 t - \frac{\delta_1 t^2}{2}\right) + u_x x(\delta_3 - \delta_1 t)\right.\\&\left.+\,\delta_1 x + \delta_4 u_x + \frac{\delta_5}{u_x^2}\right] = D_t T^t + D_x T^x,\end{aligned} \tag{2.3.8}$$

where

$$\begin{aligned}T^t &= u_x^2\left(u\left(\frac{\delta_1 t^2}{4} - \frac{\delta_2}{2} - \frac{\delta_3 t}{2}\right) + x\left(\frac{\delta_3}{2} - \frac{\delta_1 t}{2}\right) + \frac{\delta_4}{2}\right) - \frac{\delta_5}{u_x} + \phi_2(x)\\&\quad + u_x(\delta_1 x - \delta_1 t u + \phi_1(u)),\\T^x &= u_t^2\left(\frac{\delta_2}{2} + \frac{\delta_3 t}{2} - \frac{\delta_1 t^2}{4}\right) + uu_x^2\left(x\left(\frac{\delta_3}{2} - \frac{\delta_1 t}{2}\right) + \frac{\delta_4}{2}\right) - \frac{\delta_5 u}{u_x} + \frac{3\delta_5 x}{2}\\&\quad + u_x\left(uu_t\left(\delta_2 + \delta_3 t - \frac{\delta_1 t^2}{2}\right) + \delta_1 ux\right) + u_t(\delta_1 tu - \phi_1(u)) + \phi_3(t),\end{aligned}$$

for arbitrary functions $u(t,x)$. When $u(t,x)$ is a solution of Eq. (2.3.1), the left hand side of (2.3.8) vanishes and we obtain conservation laws of HS equation (2.3.1) for which the conserved vectors $\left(T_i^1, T_i^2\right)$, $i = 1,2,\ldots,5$, are given by:

$$T_1 \ : \ \begin{cases} T_1^t = u_x\left(ux - \dfrac{1}{2}t^2uu_t\right) - \dfrac{t^2u_t^2}{4} - \dfrac{1}{2}tuu_x^2x + u_t(tu - \phi_1(u)) + \phi_3(t), \\ T_1^x = u_x^2\left(\dfrac{t^2u}{4} - \dfrac{tx}{2}\right) + u_x(x - tu + \phi_1(u)) + \phi_2(x), \end{cases}$$

$$T_2 \ : \ \begin{cases} T_2^t = \phi_3(t) + uu_xu_t - u_t\phi_1(u) + \dfrac{u_t^2}{2}, \\ T_2^x = u_x\phi_1(u) + \phi_2(x) - \dfrac{uu_x^2}{2}, \end{cases}$$

$$T_3 \ : \ \begin{cases} T_3^t = tuu_xu_t + \dfrac{tu_t^2}{2} + \phi_3(t) + \dfrac{1}{2}uu_x^2x - u_t\phi_1(u), \\ T_3^x = u_x^2\left(\dfrac{x}{2} - \dfrac{tu}{2}\right) + u_x\phi_1(u) + \phi_2(x), \end{cases}$$

$$T_4 \ : \ \begin{cases} T_4^t = \phi_3(t) - u_t\phi_1(u) + \dfrac{uu_x^2}{2}, \\ T_4^x = \phi_2(x) + u_x\phi_1(u) + \dfrac{u_x^2}{2}, \end{cases}$$

$$T_5 \ : \ \begin{cases} T_5^t = \phi_3(t) - u_t\phi_1(u) - \dfrac{u}{u_x} + \dfrac{3x}{2}, \\ T_5^x = \phi_2(x) + u_x\phi_1(u) - \dfrac{1}{u_x}. \end{cases}$$

According to (1.4.1), a symmetry X is associated with a conservation law $D_tT^t + D_xT^x = 0$ if the following formula is satisfied:

$$X\begin{pmatrix} T^t \\ T^x \end{pmatrix} - \begin{pmatrix} D_t\xi^t & D_x\xi^t \\ D_t\xi^x & D_x\xi^x \end{pmatrix}\begin{pmatrix} T^t \\ T^x \end{pmatrix} + \left(D_t\xi^t + D_x\xi^x\right)\begin{pmatrix} T^t \\ T^x \end{pmatrix} = 0. \qquad (2.3.9)$$

It turns out that the association of symmetries and conservation laws of (2.3.1) is obtained in the following cases:

$$\kappa_1(X_1+2X_3)+\kappa_2X_2 \to \begin{cases} T_2^t = \frac{u_t^2}{2} - \frac{\delta_1 u_t}{u} + uu_xu_t, \\ T_2^x = \frac{\delta_1 u_x}{u} + \frac{\delta_3}{x} - \frac{uu_x^2}{2}, \end{cases}$$

$$\kappa_1(X_1+X_3)+\kappa_2X_2 \to \begin{cases} T_4^t = \frac{uu_x^2}{2} - \frac{\delta_1 u_t}{u}, \\ T_4^x = \frac{\delta_1 u_x}{u} + \frac{\delta_3}{x} + \frac{u_x^2}{2}, \end{cases}$$

$$\kappa_1\left(X_1-\frac{X_3}{2}\right)+\kappa_2X_2 \to \begin{cases} T_5^t = \frac{\delta_2}{2\kappa_2-\kappa_1 t} - \frac{\delta_1 u_t}{u} - \frac{u}{u_x} + \frac{3x}{2}, \\ T_5^x = \frac{\delta_1 u_x}{u} + \frac{\delta_3}{x} - \frac{1}{u_x}, \end{cases}$$

$$X_3 \to \begin{cases} T_3^t = \frac{\delta_1}{t} + tuu_xu_t + \frac{tu_t^2}{2} + \frac{1}{2}uu_x^2x - u_t\phi_1(u), \\ T_3^x = \frac{\delta_2}{x} + u_x^2\left(\frac{x}{2} - \frac{tu}{2}\right) + u_x\phi_1(u). \end{cases}$$

It is important to observe that among the five computed conservation laws, we have identified associated Lie point symmetries for only four of them. Notably, the conservation law T_1 lacks any associated Lie point symmetry of the HS equation.

2.3.3 Double Reduction of (2.3.1) by $\langle \kappa_1(X_1+2X_3)+\kappa_2X_2 \rangle$

We transform the generator

$$Z = \kappa_1(X_1+2X_3)+\kappa_2X_2$$

to its canonical form

$$Y = 0\frac{\partial}{\partial r} + \frac{\partial}{\partial s} + 0\frac{\partial}{\partial w}.$$

Therefore canonical coordinates $r = r(t,x)$, $s = s(t,x)$ and
$w = w(t,x,u)$ must be found such that $Z(r) = 0, Z(s) = 1$ and $Z(w) = 0$. While the coordinates r and w are obtained from invariants of Z, the coordinate s may be determined by

inspection. More systematically, it can be obtained from an invariant $J = v - s(x,y)$ of the extended operator $Z + \partial_v$, where v is an auxiliary variable [6]. We obtain

$$
\begin{aligned}
r &= \frac{x}{(2\kappa_1 t + \kappa_2)^{3/2}}, \quad && \kappa_1 \neq 0 \\
s &= \frac{\ln x}{3\kappa_1}, && \kappa_1 \neq 0 \\
w &= \frac{u}{\sqrt{2\kappa_1 t + \kappa_2}}, && \kappa_1 \neq 0.
\end{aligned}
\tag{2.3.10}
$$

where $w = w(r)$. Inverse canonical coordinates follow from (2.3.10) and are given by

$$
\begin{aligned}
t &= \frac{e^{2\kappa_1 s} - \kappa_2 r^{2/3}}{2\kappa_1 r^{2/3}}, \quad \kappa_1 \neq 0 \\
x &= e^{3\kappa_1 s}, \\
u &= \frac{w e^{\kappa_1 s}}{r^{1/3}}.
\end{aligned}
\tag{2.3.11}
$$

Computing A and $\left(A^{-1}\right)^T$, we obtain

$$
A = \begin{pmatrix} D_r t & D_r x \\ D_s t & D_s x \end{pmatrix} = \begin{pmatrix} -\frac{e^{2\kappa_1 s}}{3\kappa_1 r^{5/3}} & 0 \\ \frac{e^{2\kappa_1 s}}{r^{2/3}} & 3e^{3\kappa_1 s}\kappa_1 \end{pmatrix}
$$

and

$$
\left(A^{-1}\right)^T = \begin{pmatrix} D_t r & D_x r \\ D_t s & D_x s \end{pmatrix} = \begin{pmatrix} -3e^{-2\kappa_1 s}\kappa_1 r^{5/3} & e^{-3\kappa_1 s} r \\ 0 & \frac{e^{-3\kappa_1 s}}{3\kappa_1} \end{pmatrix}.
$$

The partial derivatives of u from (2.3.11) are given by:

$$\begin{aligned}
u_t &= \kappa_1 \sqrt[3]{r} e^{-\kappa_1 s}(w - 3rw_r),\\
u_x &= r^{2/3} w_r e^{-2\kappa_1 s},\\
u_{tx} &= -\kappa_1 r^{4/3} e^{-4\kappa_1 s}(3rw_{rr} + 2w_r),\\
u_{xx} &= r^{5/3} w_{rr} e^{-5\kappa_1 s}.
\end{aligned} \tag{2.3.12}$$

The reduced conserved form is given by:

$$\begin{pmatrix} T_2^r \\ T_2^s \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T_2^t \\ T_2^x \end{pmatrix}, \tag{2.3.13}$$

where

$$J = \det(A) = -\frac{e^{5\kappa_1 s}}{r^{5/3}}.$$

By substituting (2.3.11) and (2.3.12) into (2.3.13), we obtain:

$$\begin{aligned}
T_2^r &= \delta_1\kappa_1 + 3\delta_3\kappa_1 + 3\kappa_1^2 rww_r - \frac{9}{2}\kappa_1^2 r^2 w_r^2 - \frac{\kappa_1^2 w^2}{2} + \frac{3}{2}\kappa_1 rww_r^2 - \kappa_1 w^2 w_r,\\
T_2^s &= w_r\left(\kappa_1 w - \frac{\delta_1}{w} - \frac{w^2}{3r}\right) + \frac{\delta_1}{3r} - \frac{\kappa_1 w^2}{6r} + w_r^2\left(w - \frac{3\kappa_1 r}{2}\right),
\end{aligned} \tag{2.3.14}$$

where the reduced conserved form satisfies

$$D_r T_2^r = 0. \tag{2.3.15}$$

From (2.3.14) and (2.3.15), we have

$$3\kappa_1^2 rww_r - \frac{9}{2}\kappa_1^2 r^2 w_r^2 - \frac{\kappa_1^2 w^2}{2} + \frac{3}{2}\kappa_1 rww_r^2 - \kappa_1 w^2 w_r = k,$$

where k is an arbitrary constant.

2.3.4 Double Reduction of (2.3.1) by $\langle \kappa_1(X_1+X_3)+\kappa_2 X_2\rangle$

Canonical coordinates determined from $\langle \kappa_1(X_1+X_3)+\kappa_2 X_2\rangle$ are:

$$\begin{aligned} r &= \frac{x}{(\kappa_1 t+\kappa_2)^2}, \quad \kappa_1 \neq 0 \\ s &= \frac{\ln x}{2\kappa_1}, \qquad\qquad \kappa_1 \neq 0 \\ w &= \frac{u}{\sqrt{x}}, \end{aligned} \tag{2.3.16}$$

where $w = w(r)$, and the inverse canonical coordinates are given by:

$$\begin{aligned} t &= -\frac{\kappa_2\sqrt{r}-e^{\kappa_1 s}}{\kappa_1\sqrt{r}}, \quad \kappa_1 \neq 0 \\ x &= e^{2\kappa_1 s}, \\ u &= we^{\kappa_1 s}. \end{aligned} \tag{2.3.17}$$

The partial derivatives of u from (2.3.17) are given by:

$$\begin{aligned} u_t &= -2\kappa_1 r^{3/2} w_r, \\ u_x &= \frac{1}{2}e^{-\kappa_1 s}(2rw_r+w), \\ u_{tx} &= -e^{-2\kappa_1 s}\kappa_1 r^{3/2}(2rw_{rr}+3w_r, \\ u_{xx} &= -\frac{1}{4}e^{-3\kappa_1 s}(w-4r(rw_{rr}+w_r)). \end{aligned} \tag{2.3.18}$$

As for A and $\left(A^{-1}\right)^T$, we obtain

$$A = \begin{pmatrix} D_r t & D_r x \\ D_s t & D_s x \end{pmatrix} = \begin{pmatrix} -\frac{e^{\kappa_1 s}}{2\kappa_1 r^{3/2}} & 0 \\ \frac{e^{\kappa_1 s}}{\sqrt{r}} & 2e^{2\kappa_1 s}\kappa_1 \end{pmatrix},$$

and

$$\left(A^{-1}\right)^T = \begin{pmatrix} D_t r & D_x r \\ D_t s & D_x s \end{pmatrix} = \begin{pmatrix} -2e^{-\kappa_1 s}\kappa_1 r^{3/2} & e^{-2\kappa_1 s} r \\ 0 & \frac{e^{-2\kappa_1 s}}{2\kappa_1} \end{pmatrix}.$$

Therefore, from

$$\begin{pmatrix} T_4^r \\ T_4^s \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T_4^t \\ T_4^x \end{pmatrix}, \tag{2.3.19}$$

where

$$J = \det(A) = -\frac{e^{3\kappa_1 s}}{r^{3/2}},$$

we obtain:

$$\begin{aligned} T_4^r &= \delta_1\kappa_1 + 2\delta_3\kappa_1 + \kappa_1 r^2 w_r^2 + \kappa_1 r w w_r + \frac{\kappa_1 w^2}{4} - \frac{1}{2}r^{3/2} w w_r^2 - \frac{w^3}{8\sqrt{r}} - \frac{1}{2}\sqrt{r} w^2 w_r, \\ T_4^s &= -\frac{\delta_1 w_r}{w} - \frac{w^3}{16\kappa_1 r^{3/2}} - \frac{w^2 w_r}{4\kappa_1\sqrt{r}} - \frac{\sqrt{r} w w_r^2}{4\kappa_1}. \end{aligned} \tag{2.3.20}$$

From the reduced conservation law $D_r T_4^r = 0$, we obtain

$$\kappa_1 r^2 w_r^2 + \kappa_1 r w w_r + \frac{\kappa_1 w^2}{4} - \frac{1}{2}r^{3/2} w w_r^2 - \frac{w^3}{8\sqrt{r}} - \frac{1}{2}\sqrt{r} w^2 w_r = k,$$

where k is an arbitrary constant.

2.3.5 Double Reduction of (2.3.1) by $\left\langle \kappa_1 \left(X_1 - \frac{X_3}{2} \right) + \kappa_2 X_2 \right\rangle$

Canonical coordinated determined from $\left\langle \kappa_1 \left(X_1 - \frac{X_3}{2} \right) + \kappa_2 X_2 \right\rangle$ are:

$$
\begin{aligned}
r &= x(2\kappa_2 - \kappa_1 t), \\
s &= \frac{2\ln x}{\kappa_1}, \quad \kappa_1 \neq 0 \\
w &= \frac{u}{x^2},
\end{aligned}
\tag{2.3.21}
$$

where $w = w(r)$, and the inverse canonical coordinates are given by:

$$
\begin{aligned}
t &= \frac{2\kappa_2 - re^{-\frac{1}{2}\kappa_1 s}}{\kappa_1}, \\
x &= e^{\frac{\kappa_1 s}{2}}, \\
u &= we^{\kappa_1 s}.
\end{aligned}
\tag{2.3.22}
$$

The partial derivatives of u from (2.3.22) are given by:

$$
\begin{aligned}
u_t &= -\kappa_1 w_r e^{\frac{3\kappa_1 s}{2}}, \\
u_x &= e^{\frac{\kappa_1 s}{2}}(rw_r + 2w), \\
u_{tx} &= -\kappa_1 e^{\kappa_1 s}(rw_{rr} + 3w_r), \\
u_{xx} &= r(rw_{rr} + 4w_r) + 2w.
\end{aligned}
\tag{2.3.23}
$$

For A and $\left(A^{-1}\right)^T$, we obtain

$$
A = \begin{pmatrix} D_r t & D_r x \\ D_s t & D_s x \end{pmatrix} = \begin{pmatrix} -\frac{e^{-\frac{1}{2}\kappa_1 s}}{\kappa_1} & 0 \\ \frac{1}{2}e^{-\frac{1}{2}\kappa_1 s} r & \frac{1}{2}e^{\frac{\kappa_1 s}{2}}\kappa_1 \end{pmatrix}
$$

and

$$\left(A^{-1}\right)^T = \begin{pmatrix} D_t r & D_x r \\ D_t s & D_x s \end{pmatrix} = \begin{pmatrix} -e^{\frac{\kappa_1 s}{2}}\kappa_1 & e^{-\frac{1}{2}\kappa_1 s} r \\ 0 & \frac{2e^{-\frac{1}{2}\kappa_1 s}}{\kappa_1} \end{pmatrix}.$$

Therefore, from

$$\begin{pmatrix} T_5^r \\ T_5^s \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T_5^t \\ T_5^x \end{pmatrix}, \tag{2.3.24}$$

where

$$J = \det(A) = -\frac{1}{2},$$

we obtain:

$$\begin{aligned} T_5^r &= \frac{2\kappa_1(2\delta_1 rw_r + 4\delta_1 w + \delta_3 rw_r + 2\delta_3 w - 1) - 2\delta_2(rw_r + 2w) - r(3rw_r + 4w)}{4rw_r + 8w}, \\ T_5^s &= -\frac{2\delta_1\kappa_1 r^2 w_r^2 + 4\delta_1\kappa_1 rww_r + 2\delta_2 rww_r + 4\delta_2 w^2 + 3r^2 ww_r + 4rw^2}{2\kappa_1 r^2 ww_r + 4\kappa_1 rw^2}. \end{aligned} \tag{2.3.25}$$

From the reduced conservation law $D_r T_5^r = 0$, we obtain

$$\frac{2\kappa_1(2\delta_1 rw_r + 4\delta_1 w + \delta_3 rw_r + 2\delta_3 w - 1) - 2\delta_2(rw_r + 2w) - r(3rw_r + 4w)}{4rw_r + 8w} = k,$$

where k is an arbitrary constant.

2.3.6 Double Reduction of (2.3.1) by $\langle X_3 \rangle$

Canonical coordinated determined from X_3 are:

$$\begin{aligned} r &= \frac{x}{t}, \\ s &= \ln x, \\ w &= u, \end{aligned} \tag{2.3.26}$$

where $w = w(r)$, and the inverse canonical coordinates are given by:

$$\begin{aligned} t &= \frac{e^s}{r}, \\ x &= e^s, \\ u &= w. \end{aligned} \tag{2.3.27}$$

The partial derivatives of u from (2.3.27) are given by:

$$\begin{aligned} u_t &= -r^2 e^{-s} w_r, \\ u_x &= r e^{-s} w_r, \\ u_{tx} &= -r^2 e^{-2s}(r w_{rr} + w_r), \\ u_{xx} &= r^2 e^{-2s} w_{rr}. \end{aligned} \tag{2.3.28}$$

For A and $\left(A^{-1}\right)^T$, we obtain

$$A = \begin{pmatrix} D_r t & D_r x \\ D_s t & D_s x \end{pmatrix} = \begin{pmatrix} -\frac{e^s}{r^2} & 0 \\ \frac{e^s}{r} & e^s \end{pmatrix},$$

and

$$\left(A^{-1}\right)^T = \begin{pmatrix} D_t r & D_x r \\ D_t s & D_x s \end{pmatrix} = \begin{pmatrix} -e^{-s} r^2 & e^{-s} r \\ 0 & e^{-s} \end{pmatrix}.$$

Therefore, from

$$\begin{pmatrix} T_3^r \\ T_3^s \end{pmatrix} = J \left(A^{-1}\right)^T \begin{pmatrix} T_3^t \\ T_3^x \end{pmatrix}, \tag{2.3.29}$$

where

$$J = \det(A) = -\frac{e^{2s}}{r^2},$$

we obtain:

$$
\begin{aligned}
T_3^r &= \delta_2 - \delta_1, \\
T_3^s &= \frac{1}{2}w_r^2(w-r) - \frac{\delta_1}{r} - w_r\phi_1(w).
\end{aligned}
\tag{2.3.30}
$$

It is remarkable that in this case, because T_3^r in (2.3.30) is simply a constant, the reduced conservation law $D_rT_3^r = 0$ does not result in an ODE that can be solved for w. Therefore, no invariant solution will arise via the double reduction method from the association of X_3 and the conservation law T_3.

2.3.7 Concluding Remarks

In this section, a study of the HS equation using Lie symmetry analysis was presented. Symmetry reductions of the equation were carried out by employing the generalised approach to double reduction theory proposed by Bokhari et al [30]. By utilising the multiplier method, nontrivial conservation laws for the HS equation were derived. These conservation laws, along with the Lie point symmetries of the equation, were employed to perform symmetry reductions via the double reduction method.

Through the analysis, a set of first-order ODEs was obtained, whose solutions represent invariant solutions for the HS equation. Out of the five nontrivial conservation laws constructed, it was observed that only four had associated Lie point symmetries according to the definition provided by Kara and Mahomed [24]. The conservation law T_1 did not have any linear combination of symmetries associated with it. Additionally, it is noteworthy that despite the conservation law T_3 having an associated Lie point symmetry, X_3, the application of the double reduction method in this case did not yield a symmetry reduction of the HS equation. This outcome could be attributed to the "collapse" of the first integral, which was expected to represent a reduced ODE for the PDE but instead resulted in a constant value.

2.4 Symmetry-Based Double Reduction of a $(1+1)$-dimensional scalar PDEs Without Direct Use of Conservation Laws: An Alternative Approach

2.4.1 Introduction

From the "classical" double reduction routine [21, 22], it follows that canonical variables, while reducing the conservation law (1.4.2)

$$D_t T^t + D_x T^x = 0,$$

into the ODE (1.4.11)

$$D_r T^r = 0,$$

must necessarily transform the a scalar PDE with two independent variables into an ODE equivalent to (1.4.11), albeit of the same order as the original PDE. The reduction of order by one of this "equivalent" ODE is subsequently achieved through routine calculations. The steps for implementing the alternative double reduction routine are presented below.

1. Identify a symmetry

   $$X = \xi^i(t,x,u)\partial_{x^i} + \eta(t,x,u)\partial_u$$

   of a scalar partial differential equation (PDE) with two independent variables, t and x, and one dependent variable, $u = u(t,x)$:

   $$\Theta\left(t,x,u,u_{(1)},u_{(2)},\ldots,u_{(q)}\right) = 0, \qquad (2.4.1)$$

   where $u_{(q)}$ denotes the collection $\{u_q\}$ of all q-th order partial derivatives. The goal is to identify such a symmetry X for which there exists an associated nontrivial conservation law.

2. Find similarity variables r, s and w, i.e., variables under which the symmetry X is transformed to its canonical form $Y = \frac{\partial}{\partial s}$. The similarity variables can be determined from the conditions

$$X(r) = 0, \quad X(s) = 1, \quad X(w) = 0.$$

The similarity variables constitute an invertible mapping

$$\begin{aligned} r =& R(t,x), \\ s =& S(t,x), \\ w =& W(t,x,u), \quad \frac{\partial w}{\partial u} \neq 0 \end{aligned}$$

from the space $(t,x,u,u_{(1)},\ldots,u_{(q)})$ to the space $(r,s,w,w_r,\ldots,w_{r^q})$.

3. Express all partial derivatives of u in the PDE (2.4.1) in terms of the canonical variables r,s,w, and the partial derivatives of w. This transformation reduces the PDE (2.4.1) to the ODE

$$\omega(r,w,w_r,w_{rr},\ldots,w_{r^q}) = 0, \tag{2.4.2}$$

for some function ω, where $w = w(r)$.

4. Through some tedious process of routine computations, write the ODE (2.4.2) in the form

$$D_r\psi(r,w,w_r,w_{rr},\ldots,w_{r^{q-1}}) = 0, \tag{2.4.3}$$

for some function ψ, where $w = w(r)$. It follows from (2.4.3) that

$$\psi(r,w,w_r,w_{rr},\ldots,w_{r^{q-1}}) = k, \tag{2.4.4}$$

where k is an arbitrary constant. Equation (2.4.4) is the desired ODE of order $q-1$.

In the illustrative examples that follow, we perform double reduction of five PDEs following the algorithm outlined above. Although we have provided associated nontrivial conservation laws for each of the symmetries used in the examples, the conservation laws are not used in the double reduction procedure.

2.4.2 Example 1. The generalized Boussinesq equation with $f(u) = u^3$ and $g(x) = \frac{1}{x^5}$

We consider the generalized Boussinesq equation with $f(u) = u^3$ and $g(x) = \frac{1}{x^5}$, given by

$$u_{tt} - u_{txx} - 3u^2 u_{xx} + u_{xxxx} - 6uu_x^2 - \frac{1}{x^5} = 0. \tag{2.4.5}$$

This equation was studied by Gandarias and Rosa in [76] and admits a Lie point symmetry with infinitesimal generator

$$X = 2t\partial_t + x\partial_x - u\partial_u. \tag{2.4.6}$$

The symmetry (2.4.6) is associated with the nontrivial conservation law

$$D_t(\phi^t) + D_x(\phi^x) = 0,$$

where

$$\begin{aligned} \phi^t &= (u_t - u_{xx})t - \frac{t^2}{2x^2}, \\ \phi^x &= tu_{xxx} - 3tu^2 u_x + u_x. \end{aligned} \tag{2.4.7}$$

To find canonical variables, we solve characteristic equations

$$\frac{dx}{x} = \frac{dt}{2t} = \frac{du}{-u} \tag{2.4.8}$$

corresponding to (2.4.6). We obtain canonical variables

$$\begin{aligned} r &= \frac{x}{\sqrt{t}}, \\ s &= \frac{\ln t}{2}, \\ w &= u\sqrt{t} \quad \text{or} \quad w = xu. \end{aligned} \tag{2.4.9}$$

where $w = w(r)$. In (2.4.9) we essentially have two sets of canonical variables, one corresponding to $w = u\sqrt{t}$ and the other to $w = xu$.

Case 1.1: Reduction using canonical variables with $w = u\sqrt{t}$

The inverse canonical variables of (2.4.9), in the case $w = u\sqrt{t}$, are given by

$$\begin{aligned} t &= e^{2s}, \\ x &= re^{s}, \\ u &= e^{-s}w, \end{aligned} \tag{2.4.10}$$

and the partial derivatives that appear in (2.4.5), expressed in terms of the canonical variables using the routine outlined in [8], are:

$$\begin{aligned} u_x &= e^{-2s}w_r, \\ u_{xx} &= e^{-3s}w_{rr}, \\ u_{tt} &= \frac{1}{4}e^{-5s}\left(r^2 w_{rr} + 5rw_r + 3w\right), \\ u_{xxt} &= -\frac{1}{2}e^{-5s}(rw_{rrr} + 3w_{rr}), \\ u_{xxxx} &= e^{-5s}w_{rrrr}. \end{aligned} \tag{2.4.11}$$

Substituting (2.4.10) and (2.4.11) into (2.4.5), we obtain an ODE of the same order as (2.4.5), namely

$$
\begin{aligned}
&r^7 w_{rr} + 5r^6 w_r + 2r^6 w_{rrr} - 12r^5 w^2 w_{rr} - 24r^5 w w_r^2 \\
&+ 3r^5 w + 6r^5 w_{rr} + 4r^5 w_{rrrr} - 4 = 0.
\end{aligned} \tag{2.4.12}
$$

To obtain a lower order ODE, we find functions h and ψ so that (2.4.12) can be written in the form

$$
h(r,w) D_r \psi(r,w,w_r,w_{rr},w_{rrr}) = 0. \tag{2.4.13}
$$

Equation (2.4.13) leads to a system of determining equations which we solve to obtain $h = r^5$, and

$$
\psi = \frac{1}{r^4} - 12w^2 w_r + 3rw + r^2 w_r + 4w_r + 2rw_{rr} + 4w_{rrr}. \tag{2.4.14}
$$

Case 1.2: Reduction using canonical variables with $w = xu$

The inverse canonical variables of (2.4.9), in the case $w = xu$, are given by

$$
\begin{aligned}
t &= e^{2s}, \\
x &= re^{s}, \\
u &= \frac{e^{-s} w}{r},
\end{aligned} \tag{2.4.15}
$$

and partial derivatives that appear in (2.4.5), expressed in terms of the canonical variables are:

$$
\begin{aligned}
u_x &= e^{-2s}\left(\frac{w_r}{r}-\frac{w}{r^2}\right),\\
u_{xx} &= e^{-3s}\left(\frac{2w}{r^3}-\frac{2w_r}{r^2}+\frac{w_{rr}}{r}\right),\\
u_{tt} &= \frac{1}{4}e^{-5s}\left(rw_{rr}+3w_r\right),\\
u_{xxt} &= -\frac{1}{2}e^{-5s}w_{rrr},\\
u_{xxxx} &= e^{-5s}\left(\frac{24w}{r^5}-\frac{24w_r}{r^4}+\frac{12w_{rr}}{r^3}-\frac{4w_{rrr}}{r^2}+\frac{w_{rrrr}}{r}\right).
\end{aligned}
\tag{2.4.16}
$$

Substituting (2.4.15) and (2.4.16) into (2.4.5), we obtain

$$
\begin{aligned}
&4r^4w_{rrrr}+2r^3\left(r^2-8\right)w_{rrr}+\left(r^6-12r^2w^2+48r^2\right)w_{rr}-24r^2ww_r^2\\
&+3r\left(r^4+24w^2-32\right)w_r-48w^3+96w-4=2r^5D_r\psi=0,
\end{aligned}
\tag{2.4.17}
$$

where

$$
\psi=\frac{1}{2r^4}+\frac{6w^3}{r^4}+\left(1-\frac{12}{r^4}\right)w+\left(\frac{r}{2}+\frac{12}{r^3}-\frac{6w^2}{r^3}\right)w_r+\left(1-\frac{6}{r^2}\right)w_{rr}+\frac{2w_{rrr}}{r}. \tag{2.4.18}
$$

2.4.3 Example 2. The generalized Boussinesq equation with $f(u)=e^{nu}$ and $g(x)=\frac{1}{x^4}$

We also consider the generalized Boussinesq equation with $f(u)=e^{nu}$ and $g(x)=\frac{1}{x^4}$, given by

$$
u_{xxxx}-u_{txx}-ne^{nu}u_{xx}+u_{tt}-n^2e^{nu}u_x^2-\frac{1}{x^4}=0. \tag{2.4.19}
$$

This equation is also discussed by Gandarias and Rosa in [76] and admits a Lie point symmetry with infinitesimal generator

$$X = t\partial_t + \frac{x}{2}\partial_x - \frac{1}{n}\partial_u. \tag{2.4.20}$$

The symmetry (2.4.20) has an associated nontrivial conservation law

$$D_t(\phi^t) + D_x(\phi^x) = 0,$$

where

$$\begin{aligned} \phi^t &= x(u_t - u_{xx}), \\ \phi^x &= -nxe^{nu}u_x + e^{nu} - u_{xx} + xu_{xxx} + \frac{1}{2x^2}. \end{aligned} \tag{2.4.21}$$

Canonical variables arising from (2.4.20) are:

$$\begin{aligned} r &= \frac{x}{\sqrt{t}}, \\ s &= \ln t, \\ w &= \frac{\ln t}{n} + u \quad \text{or} \quad w = \frac{2\ln x}{n} + u, \end{aligned} \tag{2.4.22}$$

where $w = w(r)$. Again, In (2.4.22) we have two sets of canonical variables, one corresponding to $w = \frac{\ln t}{n} + u$ and $w = \frac{2\ln x}{n} + u$.

Case 2.1: Reduction using canonical variables with $w = n^{-1}\ln t + u$

The inverse canonical variables of (2.4.22), in the case $w = n^{-1}\ln t + u$, are given by:

$$\begin{aligned} t &= e^s, \\ x &= re^{s/2}, \\ u &= \frac{nw - s}{n}, \end{aligned} \tag{2.4.23}$$

and the following partial derivatives of u in terms of the canonical variables are obtained:

$$
\begin{aligned}
u_x &= e^{-\frac{s}{2}} w_r, \\
u_{xx} &= e^{-s} w_{rr}, \\
u_{tt} &= \frac{e^{-2s}}{4n}\left(nr^2 w_{rr} + 3nrw_r + 4\right), \\
u_{xxt} &= -\frac{1}{2}e^{-2s}(rw_{rrr} + 2w_{rr}), \\
u_{xxxx} &= e^{-2s} w_{rrrr}.
\end{aligned}
\tag{2.4.24}
$$

Substituting (2.4.23) and (2.4.24) into (2.4.19), we obtain:

$$
\begin{aligned}
& w_{rrrr} + \frac{rw_{rrr}}{2} + \frac{r^2 w_{rr}}{4} + w_{rr} - ne^{nw} w_{rr} \\
& + \frac{3rw_r}{4} - n^2 e^{nw} w_r^2 - \frac{1}{r^4} + \frac{1}{n} = r^{-1} D_r \psi = 0,
\end{aligned}
\tag{2.4.25}
$$

where

$$
\psi = rw_{rrr} + \frac{r^2 w_{rr}}{2} - w_{rr} + \frac{r^3 w_r}{4} - nre^{nw} w_r + e^{nw} + \frac{r^2}{2n} + \frac{1}{2r^2}. \tag{2.4.26}
$$

Case 2.2: Reduction using canonical variables with $w = 2n^{-1}\ln x + u$

The inverse canonical variables of (2.4.22), in the case $w = 2n^{-1}\ln x + u$, are given by:

$$
\begin{aligned}
t &= e^s, \\
x &= re^{s/2}, \\
u &= w - \frac{2\ln r + s}{n}.
\end{aligned}
\tag{2.4.27}
$$

Therefore, the partial derivatives of u expressed in terms of the canonical variables are:

$$\begin{aligned}
u_x &= e^{-\frac{s}{2}}\left(w_r - \frac{2}{nr}\right),\\
u_{xx} &= e^{-s}\left(\frac{2}{nr^2} + w_{rr}\right),\\
u_{tt} &= \frac{1}{4}re^{-2s}\left(rw_{rr} + 3w_r\right),\\
u_{xxt} &= -\frac{1}{2}e^{-2s}\left(rw_{rrr} + 2w_{rr}\right),\\
u_{xxxx} &= e^{-2s}\left(\frac{12}{nr^4} + w_{rrrr}\right).
\end{aligned} \tag{2.4.28}$$

Upon substituting (2.4.27) and (2.4.28) into (2.4.19), we obtain:

$$\begin{aligned}
&4nr^4w_{rrrr} + 2nr^5w_{rrr} - nr^2\left(4ne^{nw} - r^4 - 4r^2\right)w_{rr} - 4n^3r^2w_r^2e^{nw}\\
&+ nr\left(16ne^{nw} + 3r^4\right)w_r - 24ne^{nw} - 4n + 48 = 4nr^3D_r\psi = 0,
\end{aligned} \tag{2.4.29}$$

where

$$\psi = rw_{rrr} + \left(\frac{r^2}{2} - 1\right)w_{rr} + \left(\frac{r^3}{4} - \frac{ne^{nw}}{r}\right)w_r + \frac{3e^{nw}}{r^2} + -\frac{6}{nr^2} + \frac{1}{2r^2} + \frac{1}{n}. \tag{2.4.30}$$

2.4.4 Example 3. The Benjamin–Bona–Mahony (BBM) equation

The BBM equation

$$u_{txx} - u_t + uu_x = 0 \tag{2.4.31}$$

was discussed by Sjöberg in [21] and admits Lie point symmetries with infinitesimal generators

$$X_1 = \partial_x \ \text{ and } \ X_2 = \partial_t. \tag{2.4.32}$$

These symmetries have an associated nontrivial conservation law

$$D_t(\phi^t) + D_x(\phi^x) = 0,$$

where

$$\begin{aligned} \phi^t &= \frac{u^3}{3}, \\ \phi^x &= u_t^2 - u_{tx}^2 - u^2 u_{tx} - \frac{u^4}{4}. \end{aligned} \tag{2.4.33}$$

Let $X = \alpha X_1 + X_2$, where α is an arbitrary constant. Canonical variables derived from X are:

$$\begin{aligned} r &= x - \alpha t, \\ s &= t, \\ w &= u. \end{aligned} \tag{2.4.34}$$

where $w = w(r)$. From (2.4.34), we obtain the inverse canonical variables:

$$\begin{aligned} t &= s, \\ x &= \alpha s + r, \\ u &= w. \end{aligned} \tag{2.4.35}$$

and the following partial derivatives of u expressed in terms of the canonical variables:

$$\begin{aligned} u_x &= w_r, \\ u_t &= -\alpha w_r, \\ u_{xxt} &= -\alpha w_{rrr}. \end{aligned} \tag{2.4.36}$$

Substituting (2.4.35) and (2.4.36) into (2.4.31), we obtain

$$\alpha w_r - \alpha w_{rrr} + w w_r = D_r \psi = 0, \tag{2.4.37}$$

where

$$\psi = \alpha w - \alpha w_{rr} + \frac{w^2}{2}. \tag{2.4.38}$$

2.4.5 Example 4. The Gardner equation

The Gardner equation

$$u_{xxx} + u_t + u^2 u_x + u u_x = 0 \tag{2.4.39}$$

is studied by Rosa et al. in [77] and admits Lie point symmetries with infinitesimal generators

$$X_1 = \partial_x \ \text{ and } \ X_2 = \partial_t. \tag{2.4.40}$$

Equation (2.4.39) has a nontrivial conservation law of

$$D_t(\phi^t) + D_x(\phi^x) = 0,$$

where

$$\begin{aligned} \phi^t &= \frac{u^2}{2} + \frac{u}{2} + \frac{1}{2}, \\ \phi^x &= \frac{u^4}{4} + \frac{u^3}{2} + \frac{u^2}{4} - \frac{u_x^2}{2} + \left(u - \frac{1}{2}\right) u_{xx}. \end{aligned} \tag{2.4.41}$$

This conservation law is associated with the symmetries in (2.4.40). Let $X = \alpha X_1 + X_2$, where α is an arbitrary constant. Canonical variables derived from X are:

$$\begin{aligned} r &= x - \alpha t, \\ s &= t, \\ w &= u. \end{aligned} \tag{2.4.42}$$

where $w = w(r)$. From (2.4.42), we obtain inverse canonical variables:

$$\begin{aligned} t &= s, \\ x &= \alpha s + r, \\ u &= w. \end{aligned} \tag{2.4.43}$$

and the partial derivatives:

$$\begin{aligned} u_x &= w_r, \\ u_t &= -\alpha w_r, \\ u_{xxx} &= w_{rrr}. \end{aligned} \tag{2.4.44}$$

Substituting (2.4.43) and (2.4.44) into (2.4.39), we obtain

$$w_{rrr} + w^2 w_r + w w_r - \alpha w_r = D_r \psi = 0, \tag{2.4.45}$$

where

$$\psi = w_{rr} + \frac{w^3}{3} + \frac{w^2}{2} - \alpha w. \tag{2.4.46}$$

2.4.6 Example 5. The Korteweg–De Vries (KdV) equation

In [78] (see also [79]), Morris and Kara discussed the scalar KdV equation

$$u_t - u u_x - u_{xxx} = 0, \tag{2.4.47}$$

and admits a Lie point symmetry with infinitesimal generator

$$X = x\partial_x + 3t\partial_t - 2u\partial_u. \tag{2.4.48}$$

Furthermore, the nontrivial conservation law of (2.4.47)

$$D_t(\phi^t) + D_x(\phi^x) = 0,$$

where

$$\begin{aligned} \phi^t &= \frac{tu^2}{2} + ux, \\ \phi^x &= -\frac{tu^3}{3} - tuu_{xx} + \frac{tu_x^2}{2} - \frac{xu^2}{2} + u_x - xu_{xx} \end{aligned} \tag{2.4.49}$$

is associated with the symmetry (2.4.48). Canonical variables derived from (2.4.48) are:

$$\begin{aligned} r &= \frac{x^3}{t}, \\ s &= \frac{\ln t}{3}, \\ w &= t^{2/3}u \quad \text{or} \quad w = x^2 u. \end{aligned} \tag{2.4.50}$$

where $w = w(r)$. Again, from (2.4.50) we have two sets of canonical variables, one corresponding to $w = t^{2/3}u$ and $w = x^2 u$.

Case 5.1: Reduction using canonical variables with $w = t^{2/3}u$

From (2.4.50), the inverse canonical variables are given by:

$$\begin{aligned} t &= e^{3s}, \\ x &= r^{1/3}e^s, \\ u &= e^{-2s}w. \end{aligned} \tag{2.4.51}$$

Therefore, the following partial derivatives of u expressed in terms of the canonical variables are obtained from (2.4.51):

$$\begin{aligned} u_x &= 3r^{2/3}e^{-3s}w_r, \\ u_r &= -\frac{1}{3}e^{-5s}(3rw_r+2w), \\ u_{xxx} &= e^{-5s}\left(27r^2w_{rrr}+54rw_{rr}+6w_r\right). \end{aligned} \tag{2.4.52}$$

Substituting (2.4.51) and (2.4.52) into (2.4.47), we obtain:

$$81r^2w_{rrr}+162rw_{rr}+3\left(3r^{2/3}w+r+6\right)w_r+2w=\frac{3r^{2/3}}{r^{1/3}+w}D_r\psi=0, \tag{2.4.53}$$

where

$$\psi=r^{2/3}w+2r^{1/3}w^2+w^3+9\left(r^{2/3}+2r^{1/3}w\right)w_r+27\left[\left(r^{4/3}w+r^{5/3}\right)w_{rr}-\frac{1}{2}r^{4/3}w_r^2\right]. \tag{2.4.54}$$

Case 5.2: Reduction using canonical variables with $w = x^2u$

We obtain from (2.4.50), in this case, inverse canonical variables:

$$\begin{aligned} t &= e^{3s}, \\ x &= r^{1/3}e^{s}, \\ u &= \frac{e^{-2s}w}{r^{2/3}}. \end{aligned} \tag{2.4.55}$$

and partial derivatives of u expressed in terms of the canonical variables:

$$\begin{aligned} u_x &= \frac{e^{-3s}(3rw_r-2w)}{r}, \\ u_t &= -r^{1/3}e^{-5s}w_r, \\ u_{xxx} &= e^{-5s}\left(27r^{4/3}w_{rrr}+\frac{24w_r}{r^{2/3}}-\frac{24w}{r^{5/3}}\right). \end{aligned} \tag{2.4.56}$$

Substituting (2.4.55) and (2.4.56) into (2.4.47), we obtain

$$6w^2 + w(72 - 9rw_r) - 3(r+24)rw_r - 81r^3 w_{rrr} = \frac{3r^3}{r+w} D_r \psi = 0, \tag{2.4.57}$$

where

$$\psi = -\frac{w^3}{r^2} - \frac{2(r+6)w^2}{r^2} - \frac{(r+24)w}{r} + \frac{27w_r^2}{2} + 27w_r - 27(r+w)w_{rr}. \tag{2.4.58}$$

2.4.7 Concluding remarks

In the standard double reduction method as proposed by Sjöberg [21,22], both a Lie symmetry of a PDE and an associated nontrivial conservation law are required in order to find invariant solutions of the PDE through the double reduction method. The algorithm involves writing a conservation law in terms of canonical variables derived from the associated Lie symmetry. The association of the conservation law with the symmetry ensures that, in canonical variables, the conservation law is reduced to an ODE of order one less than that of the PDE. In this section, we have demonstrated that double reduction can be realised by simply transforming the PDE using the constructed canonical variables. This means that any symmetry of a PDE known to have an associated nontrivial conservation law may be used to perform double reduction of the PDE, without explicitly using the conservation law in the derivations. We have provided examples involving five $(1+1)$-dimensional scalar PDEs.

For each of the PDEs in the illustrative examples, we started by identifying an admitted Lie point symmetry that has an associated nontrivial conservation law. We then found canonical variables r, s and w, i.e., variables under which the Lie symmetry is transformed into $X = \partial/\partial s$. The canonical variables constitute an invertible mapping from the $(t, x, u, u_{(1)}, \ldots, u_{(q)})$-space to the $(r, s, w, w_r, \ldots, w_{r^q})$-space. In the latter space, the PDE is reduced to an ODE of the same order, but one which could be written in the form $D_r(\cdot) = 0$ to complete the reduction.

Chapter 3

Conservation Laws and Symmetry Multi-Reductions of two (2+1)-dimensional equations, the Zakharov-Kuznetsov (ZK) equation and a nonlinear wave equation

3.1 Introduction

In this chapter we focus on the application of generalized double reduction theory to two (2+1)-dimensional PDEs. We begin this chapter by doing the extension of the seminal paper by Bokhari et al [31] on the generalisation of the double reduction theory where the theory was applied on the nonlinear $(2+1)$-dimensional wave equation. We have presented an extended account of the application and included a second multi-reduction of the conservation law of the equation by finding and using inherited symmetries that were not determined. We have also included for illustrative purposes solutions of the wave equation for particular specifications of the arbitrary functions.

Next, we apply the generalisation of the double reduction theory on the $(2+1)$-dimensional ZK equation [13,41–43]. We begin by computing the Lie point symmetries of the ZK equation using Mathematica [45] and used the multiplier method [47, 48, 54] to compute the conservation laws for the ZK equation.

The main goal of our present work is to obtain reductions of the ZK equation by exploiting the generalised double reduction theory [30]. We obtain four non-trivial conservation laws of the ZK equation (when $\mu = 0$ and $\nu = 1$) by the multiplier method. The generalised double reduction theorem is then applied leading to second-order ODEs in each of the cases where multi-reduction is possible. Our application of the generalised double reduction method is instructive in that we demonstrate the use of "nonassociated" symmetries in the reduction routine. We show that an associated symmetry with a reduced conserved form may be inherited from a nonassociated symmetry with the original conserved form.

3.2 A nonlinear wave equation

3.2.1 Introduction

This section is essentially the extension of the seminal paper by Bokhari et al [31] on the generalisation of the double reduction theory. In [31] the theory was applied on the nonlinear $(2+1)$-dimensional wave equation

$$u_{tt} - (f(u)u_x)_x - (g(u)u_y)_y = 0 \tag{3.2.1}$$

involving two arbitrary functions $f(u)$ and $g(u)$. We have presented an extended account of the application and included a second multi-reduction of the conservation law of the equation by finding and using inherited symmetries that were not determined. We have also included for illustrative purposes solutions of the wave equation for particular specifications of the arbitrary functions.

3.2.2 Application of the generalised double reduction theory to the nonlinear wave equation (3.2.1)

It was established in [30, 31] that for arbitrary functions $f(u)$ and $g(u)$, the nonlinear $(2+1)$-dimensional wave equation (3.2.1) has the conserved vector

$$\left(T^t, T^x, T^y\right) = \left(-u_t, f(u)u_x, g(u)u_y\right), \tag{3.2.2}$$

and admits the Lie point symmetries:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= \frac{\partial}{\partial y}, \\ X_4 &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \end{aligned} \tag{3.2.3}$$

Furthermore, the symmetries X_1, X_2 and X_3 are associated with the conserved vector (3.2.2), i.e., if

$$X = \kappa_1 X_1 + \kappa_2 X_2 + \kappa_3 X_3,$$

where κ_i's are arbitrary constants, then

$$X\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} - \begin{pmatrix} D_t\xi^t & D_x\xi^t & D_y\xi^t \\ D_t\xi^x & D_x\xi^x & D_y\xi^x \\ D_t\xi^y & D_x\xi^y & D_y\xi^y \end{pmatrix}\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} + \left(D_t\xi^t + D_x\xi^x + D_y\xi^y\right)\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} = 0. \quad (3.2.4)$$

So we can get a reduced conserved form by the combination of them

$$X = \frac{\partial}{\partial t} + \kappa_2\frac{\partial}{\partial x} + \kappa_3\frac{\partial}{\partial y},$$

where the generator X has a canonical form $\widetilde{X} = \frac{\partial}{\partial q}$. From the characteristic equations

$$\frac{dt}{1} = \frac{dx}{\kappa_2} = \frac{dy}{\kappa_3} = \frac{du}{0} = \frac{dr}{0} = \frac{ds}{0} = \frac{dq}{1} = \frac{dw}{0}, \quad (3.2.5)$$

we obtain canonical coordinates:

$$\begin{aligned} r &= y - \kappa_3 t, \\ s &= x - \kappa_2 t, \\ q &= t, \\ w(r,s) &= u. \end{aligned} \quad (3.2.6)$$

The inverse canonical coordinates are:

$$\begin{aligned} t &= q, \\ x &= \kappa_2 q + s, \\ y &= \kappa_3 q + r, \\ u &= w. \end{aligned} \quad (3.2.7)$$

It follows therefore that the partial derivatives u_t, u_x and u_y expressed in terms of the canonical variables (3.2.6) are:

$$u_t = -\kappa_2 w_s - \kappa_3 w_r,$$

$$u_x = w_s, \tag{3.2.8}$$

$$u_y = w_r.$$

According to Theorem 1.4.2, we obtain the reduced conserved form

$$\begin{pmatrix} T^r \\ T^s \\ T^q \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix}, \tag{3.2.9}$$

where

$$A = \begin{pmatrix} D_r t & D_r x & D_r y \\ D_s t & D_s x & D_s y \\ D_q t & D_q x & D_q y \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & \kappa_2 & \kappa_3 \end{pmatrix}, \tag{3.2.10}$$

$$A^{-1} = \begin{pmatrix} D_t r & D_t s & D_t q \\ D_x r & D_x s & D_x q \\ D_y r & D_y s & D_y q \end{pmatrix} = \begin{pmatrix} -\kappa_3 & -\kappa_2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{3.2.11}$$

with

$$J = \det(A) = -1.$$

Substituting for the partial derivatives in (3.2.9) using (3.2.8), we obtain:

$$T^r = \kappa_3^2 w_r + \kappa_2 \kappa_3 w_s - g(w) w_r,$$

$$T^s = \kappa_2 \kappa_3 w_r + \kappa_2^2 w_s - f(w) w_s, \tag{3.2.12}$$

$$T^q = -\kappa_3 w_r - \kappa_2 w_s.$$

Therefore, the reduced conserved form is

$$D_r T^r + D_s T^s = 0. \tag{3.2.13}$$

After writing the symmetries (3.2.3) in the canonical variables (3.2.6), we obtain:

$$\begin{aligned}
\widetilde{X}_1 &= \kappa_3 \frac{\partial}{\partial r} + \kappa_2 \frac{\partial}{\partial s}, \\
\widetilde{X}_2 &= \frac{\partial}{\partial s}, \\
\widetilde{X}_3 &= \frac{\partial}{\partial r}, \\
\widetilde{X}_4 &= r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s},
\end{aligned} \tag{3.2.14}$$

and it turns out that they are all inherited by (3.2.13). Furthermore, all the inherited symmetries (3.2.14) are associated with the conservation law (3.2.13), i.e., if

$$\widetilde{X} = \delta_1 \widetilde{X}_1 + \delta_2 \widetilde{X}_2 + \delta_3 \widetilde{X}_3 + \delta_4 \widetilde{X}_4,$$

then

$$\widetilde{X}\begin{pmatrix} T^r \\ T^s \end{pmatrix} - \begin{pmatrix} D_r\xi^r & D_s\xi^r \\ D_r\xi^s & D_s\xi^s \end{pmatrix}\begin{pmatrix} T^r \\ T^s \end{pmatrix} + (D_r\xi^r + D_s\xi^s)\begin{pmatrix} T^r \\ T^s \end{pmatrix} = 0. \tag{3.2.15}$$

We consider reduction of the conservation law (3.2.13) under two cases: under the inherited symmetry

$$Y_1 = r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s},$$

and under the inherited symmetry

$$Y_2 = \frac{\partial}{\partial r} + \gamma\frac{\partial}{\partial s},$$

where γ is an arbitrary constant.

3.2.3 Reduction of (T^r, T^s) under $\langle Y_1 \rangle$

The generator

$$Y_1 = r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s}$$

has a canonical form $\widetilde{Y}_1 = \frac{\partial}{\partial m}$ when

$$\frac{dr}{r} = \frac{ds}{s} = \frac{dw}{0} = \frac{dn}{0} = \frac{dm}{1} = \frac{dv}{0}, \tag{3.2.16}$$

or

$$\begin{aligned} n &= \frac{s}{r}, \\ m &= \ln r, \\ v(n) &= w. \end{aligned} \tag{3.2.17}$$

The inverse canonical coordinates are given by:

$$\begin{aligned} r &= e^m, \\ s &= e^m n, \\ w &= v, \end{aligned} \tag{3.2.18}$$

and so the partial derivatives in the conserved vector (T^r, T^s) in terms of the canonical coordinates (3.2.17) are given by:

$$\begin{aligned} w_r &= -e^{-m} n v_n, \\ w_s &= -e^{-m} n v_n. \end{aligned} \tag{3.2.19}$$

By using the formula (1.4.15) we get the reduced conserved form

$$\begin{pmatrix} T^n \\ T^m \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T^r \\ T^s \end{pmatrix}, \tag{3.2.20}$$

where

$$A = \begin{pmatrix} D_n r & D_n s \\ D_m r & D_m s \end{pmatrix} = \begin{pmatrix} 0 & e^m \\ e^m & e^m n \end{pmatrix},$$

$$A^{-1} = \begin{pmatrix} D_r n & D_r m \\ D_s n & D_s m \end{pmatrix} = \begin{pmatrix} -\frac{s}{r^2} & \frac{1}{r} \\ \frac{1}{r} & 0 \end{pmatrix},$$

and $J = \det(A) = -e^{2m}$.

From (3.2.19) and (3.2.20) we obtain:

$$\begin{aligned} T^n &= v_n\left(2\kappa_2\kappa_3 n - \kappa_3^2 n^2 + n^2 g(v) - \kappa_2^2 + f(v)\right), \\ T^m &= v_n\left(\kappa_3^2 n - \kappa_2\kappa_3 - n g(v)\right). \end{aligned} \tag{3.2.21}$$

Therefore, the reduced conservation law is

$$D_n T^n = 0, \tag{3.2.22}$$

or

$$v_n\left(2\kappa_2\kappa_3 n - \kappa_3^2 n^2 + n^2 g(v) - \kappa_2^2 + f(v)\right) = k, \tag{3.2.23}$$

where k is a constant. The solution of the nonlinear wave equation (3.2.1) follows from the solution of (3.2.23) via (3.2.17) and (3.2.6).

For illustrative purposes, let us consider particular choices of the arbitrary functions in equation (3.2.1). If we let $f(u) = 1$, $g(u) = u$ and set $\kappa_2 = 1$ and $\kappa_3 = 0$, then the ODE (3.2.23) reduces to

$$n^2 v(n) v'(n) = k, \tag{3.2.24}$$

the solution of which is

$$\frac{2k}{n} + v^2 = \lambda, \tag{3.2.25}$$

where λ is an arbitrary constant. In light of the change of variables (3.2.17) and (3.2.6), with $\kappa_2 = 1$, $\kappa_3 = 0$, $n = \frac{x-t}{y}$ and $v = u$, the solution (3.2.25) translates into the following solution of the nonlinear wave equation (3.2.1):

$$\frac{2ky}{x-t} + u^2 = \lambda. \tag{3.2.26}$$

3.2.4 Reduction of (T^r, T^s) under $\langle Y_2 \rangle$

From the generator Y_2 the canonical coordinates are:

$$\begin{aligned} n &= s - \gamma r, \\ m &= r, \\ v(n) &= w. \end{aligned} \tag{3.2.27}$$

and the inverse canonical coordinates are given by:

$$\begin{aligned} r &= m, \\ s &= \gamma m + n, \\ w &= v. \end{aligned} \tag{3.2.28}$$

Therefore, the partial derivatives in (T^r, T^s) in terms of the canonical coordinates (3.2.27) are given by:

$$w_r = -\gamma v_n,$$
$$w_s = v_n. \tag{3.2.29}$$

By using the formula (1.4.15) we get the reduced conserved form

$$\begin{pmatrix} T^n \\ T^m \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T^r \\ T^s \end{pmatrix}, \tag{3.2.30}$$

where

$$A = \begin{pmatrix} D_n r & D_n s \\ D_m r & D_m s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & \gamma \end{pmatrix},$$

$$A^{-1} = \begin{pmatrix} D_r n & D_r m \\ D_s n & D_s m \end{pmatrix} = \begin{pmatrix} -\gamma & 1 \\ 1 & 0 \end{pmatrix},$$

and

$$J = \det(A) = -1.$$

From (3.2.29) and (3.2.30) we obtain:

$$T^n = v_n\left(f(v) + \gamma^2 g(v) - (\kappa_2 - \gamma\kappa_3)^2\right),$$
$$T^m = v_n(\kappa_3(\gamma\kappa_3 - \kappa_2) - \gamma g(v)). \tag{3.2.31}$$

Therefore, the reduced conservation law is

$$D_n T^n = 0, \tag{3.2.32}$$

or

$$v_n\left(f(v) + \gamma^2 g(v) - (\kappa_2 - \gamma\kappa_3)^2\right) = k, \tag{3.2.33}$$

where k is a constant. The solution of the nonlinear wave equation (3.2.1) follows from the solution of (3.2.33) via (3.2.27) and (3.2.6).

If for illustrative purposes we let $f(u) = g(u) = u$ in the nonlinear wave equation (3.2.1), and set $\gamma = \frac{\kappa_3}{\kappa_2}$, the ODE (3.2.33) reduces to

$$(Mv(n) + v(n) - L)v'(n) = k, \tag{3.2.34}$$

where

$$M = \frac{\kappa_3^2}{\kappa_2^2}, \quad L = \left(\kappa_2 - \frac{\kappa_3^2}{\kappa_2}\right)^2.$$

The solution of (3.2.34) is

$$\lambda - 2kn - 2Lw + (M+1)w^2 = 0, \tag{3.2.35}$$

where λ is an arbitrary constant. In light of (3.2.6) and (3.2.27), and if we set $\gamma = \frac{\kappa_3}{\kappa_2}$, we obtain

$$n = \frac{\kappa_3^2 t}{\kappa_2} - \frac{\kappa_3 y}{\kappa_2} - \kappa_2 t + x \quad \text{and} \quad w = u. \tag{3.2.36}$$

Therefore (3.2.35) becomes

$$k\left(\sqrt{L}t + \frac{\kappa_3}{\kappa_2}y - x\right) + \frac{\lambda}{2} - Lu + \frac{1}{2}(M+1)u^2 = 0, \tag{3.2.37}$$

which is the solution in implicit form of the non-linear wave equation (3.2.1).

3.3 The Zakharov-Kuznetsov (ZK) equation

3.3.1 Introduction

In this section we consider another equation, the $(2+1)$-dimensional ZK equation [13,41–43]

$$u_t + \mu u_x + \nu u u_x + \alpha u_{xxx} + \beta u_{xyy} = 0, \tag{3.3.1}$$

where μ, ν, α, and β are arbitrary constants, are found by employing the generalised double reduction theory. The ZK equation originated from the study of weakly nonlinear ion-acoustic waves in a strongly magnetised plasma consisting of cold ions and hot isothermal electrons. It was first introduced by Vladimir Zakharov and Boris Kuznetsov in 1974 [80] and serves as a two-dimensional generalisation of the well-known Korteweg-de Vries (KdV) equation, alongside the Kadomtsev-Petviashvili (KP) equation [81, 82].

Over time, researchers have investigated various aspects of the ZK equation, including its local, global, and scattering properties, as well as seeking novel exact solutions through techniques such as the $\left(\frac{G'}{G}\right)^2$-expand method, Lie symmetry analysis, and the homotopy perturbation method [14, 81–84]. Research to find analytical solutions of the ZK equation and its variants has continued aimed at contributing to the understanding of complex physical phenomena modelled by the ZK equation that arise in diverse fields [14, 81–83].

3.3.2 Symmetries and Conservation Laws of (3.3.1)

Symmetries of (3.3.1)

First, we will derive the Lie point symmetries of (3.3.1), in the case $\mu = 0, \nu = 1$, $\alpha\beta \neq 0$, i.e.,

$$u_t + uu_x + \alpha u_{xxx} + \beta u_{xyy} = 0. \tag{3.3.2}$$

If the operator

$$X = \xi^1(t,x,y,u)\frac{\partial}{\partial t} + \xi^2(t,x,y,u)\frac{\partial}{\partial x} + \xi^3(t,x,y,u)\frac{\partial}{\partial y} + \eta(t,x,y,u)\frac{\partial}{\partial u}$$

is a generator of a Lie point symmetry of (3.3.2), then it must satisfy the invariance condition

$$X^{[3]}\left[u_t + uu_x + \alpha u_{xxx} + \beta u_{xyy}\right]\Big|_{(3.3.2)} = 0, \tag{3.3.3}$$

where $X^{(3)}$ is the third prolongation of X and can be computed according to (1.2.7). Equation (3.3.3), after expansion and separation, yields an overdetermined system of linear first-order partial differential equations for the unknown coefficients ξ^1, ξ^2, ξ^3, and η. The solution of the system leads to the following Lie point symmetries of (3.3.2) [44]:

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t},\\
X_2 &= \frac{\partial}{\partial x},\\
X_3 &= \frac{\partial}{\partial y},\\
X_4 &= t\frac{\partial}{\partial x} + \frac{\partial}{\partial u},\\
X_5 &= 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u}.
\end{aligned} \tag{3.3.4}$$

Conservation Laws of (3.3.1)

The conservation laws for (3.3.2) are constructed by the multiplier approach. We consider multipliers of the form $\Lambda(x,t,u,u_x,u_y,u_t,u_{xx},u_{xy},u_{tx},u_{yy},u_{tt},u_{ty})$ for (3.3.2). The determining equation for the multipliers is

$$\frac{\delta}{\delta u}\left[\Lambda\left(u_t+uu_x+\alpha u_{xxx}+\beta u_{xyy}\right)\right]=0, \tag{3.3.5}$$

where $\frac{\delta}{\delta u}$ is the standard Euler operator (1.3.4). Expanding and then separating (3.3.5) with respect to different combinations of partial derivatives of u results in the following overdetermined system for the multipliers:

$$\begin{aligned}
\Lambda_{tt}&=0, & \Lambda_{tu}-\frac{\Lambda_t}{u}&=0, & \Lambda_{tu_{yy}}&=0,\\
\Lambda_{uu}-\frac{\Lambda_{u_{yy}}}{\beta}&=0, & \Lambda_{u,u_{yy}}&=0, & \Lambda_{u_{yy}u_{yy}}&=0,\\
\Lambda_x+\frac{\Lambda_t}{u}&=0, & \Lambda_{ux}&=0, & \Lambda_{uy}&=0,\\
\Lambda_{ut}&=0, & \Lambda_{u_{xx}}-\frac{\alpha\Lambda_{u_{yy}}}{\beta}&=0, & \Lambda_{u_{xy}}&=0,\\
\Lambda_{u_{tx}}&=0, & \Lambda_{u_{tt}}&=0, & \Lambda_{u_{ty}}&=0,
\end{aligned} \tag{3.3.6}$$

provided that $\alpha\beta\neq 0$. The solution of system (3.3.6) is

$$\Lambda=c_1tu-c_1x+c_2u_{yy}+\frac{\alpha c_2}{\beta}u_{xx}+\frac{c_2}{2\beta}u^2+c_3u+c_4, \tag{3.3.7}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants. From (3.3.7), we obtain single parameter multipliers:

$$
\begin{aligned}
\Lambda_1 &= tu - x, \\
\Lambda_2 &= u_{yy} + \frac{1}{2\beta}u^2 + \frac{\alpha}{\beta}u_{xx}, \\
\Lambda_3 &= u, \\
\Lambda_4 &= 1.
\end{aligned}
\tag{3.3.8}
$$

According to (1.3.2), the multipliers in (3.3.8) satisfy

$$
\Lambda\left(u_t + uu_x + \alpha u_{xxx} + \beta u_{xyy}\right) = D_t T^t + D_x T^x + D_y T^y \tag{3.3.9}
$$

for arbitrary functions $u(t,x,y)$. We obtain four nontrivial conserved vectors:

$$
T_1 = \left(T_1^t, T_1^x, T_1^y\right), \quad T_2 = \left(T_2^t, T_2^x, T_2^y\right), \quad T_3 = \left(T_3^t, T_3^x, T_3^y\right), \quad T_4 = \left(T_4^t, T_4^x, T_4^y\right), \tag{3.3.10}
$$

where $T_i^t, T_i^x, T_i^y,\ i = 1, \ldots, 4$, are given by:

$$
T_1 \quad : \quad \begin{cases}
T_1^t = \dfrac{tu^2}{2} - ux, \\
T_1^x = \dfrac{tu^3}{3} - \dfrac{u^2 x}{2} + \alpha u_x - \dfrac{\alpha}{2}tu_x^2 + \alpha tuu_{xx} - \alpha xu_{xx} + \dfrac{\beta}{2}tuu_{yy} - \beta xu_{yy}, \\
T_1^y = \dfrac{\beta tuu_{xy}}{2} + \beta u_y - \dfrac{\beta tu_x u_y}{2},
\end{cases}
$$

$$
T_2 \quad : \quad \begin{cases}
T_2^t = \dfrac{uu_{yy}}{2} + \dfrac{u^3}{6\beta} + \dfrac{\alpha uu_{xx}}{2\beta}, \\
T_2^x = \alpha u_{xx}u_{yy} + \dfrac{u^2 u_{yy}}{2} + \dfrac{u^4}{8\beta} + \dfrac{\alpha u^2 u_{xx}}{2\beta} - \dfrac{\alpha uu_{tx}}{2\beta} + \dfrac{\alpha u_x u_t}{2\beta} + \dfrac{\alpha^2 u_{xx}^2}{2\beta} + \dfrac{\beta u_{yy}^2}{2}, \\
T_2^y = \dfrac{u_t u_y}{2} - \dfrac{uu_{ty}}{2},
\end{cases}
$$

$$
T_3 \quad : \quad \begin{cases}
T_3^t = \dfrac{u^2}{2}, \\
T_3^x = \dfrac{u^3}{3} - \dfrac{\alpha u_x^2}{2} + \alpha uu_{xx} + \dfrac{\beta uu_{yy}}{2}, \\
T_3^y = \dfrac{\beta uu_{xy}}{2} - \dfrac{\beta u_x u_y}{2},
\end{cases}
$$

$$T_4 \quad : \quad \begin{cases} T_4^t = u, \\ T_4^x = \dfrac{u^2}{2} + \alpha u_{xx} + \beta u_{yy}, \\ T_4^y = 0. \end{cases}$$

3.3.3 Application of the generalised Double Reduction Theory to (3.3.1)

To determine symmetries associated with the conserved vectors (3.3.10), we set a linear combination of the symmetries (3.3.4), i.e., $X = \sum_{i=1}^{5} \kappa_i X_i$, where κ_i's are arbitrary constants, and then apply the association condition

$$X \begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} - \begin{pmatrix} D_t\xi^1 & D_x\xi^1 & D_y\xi^1 \\ D_t\xi^2 & D_x\xi^2 & D_y\xi^2 \\ D_t\xi^3 & D_x\xi^3 & D_y\xi^3 \end{pmatrix} \begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} + \left(D_t\xi^1 + D_x\xi^2 + D_y\xi^3\right) \begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} = 0 \quad (3.3.11)$$

for each of conserved vector T_i in (3.3.10). We obtain that T_1 is associated with only X_3, T_2 and T_3 are associated with the linear combination $\kappa_1 X_1 + \kappa_2 X_2 + \kappa_4 X_4$, and T_4 is associated with the linear combination $\kappa_1 X_1 + \kappa_2 X_2 + \kappa_3 X_3 + \kappa_5 X_5$. For the multi-reductions that follow we use these associations:

$$\begin{aligned} X_3 &\longrightarrow T_1, \\ X_1 + \kappa_2 X_2 + \kappa_3 X_3 &\longrightarrow T_2, \\ X_1 + \kappa_2 X_2 + \kappa_3 X_3 &\longrightarrow T_3, \\ X_1 + \kappa_2 X_2 + \kappa_3 X_3 &\longrightarrow T_4, \\ X_5 &\longrightarrow T_4. \end{aligned} \quad (3.3.12)$$

3.3.4 Multi-Reduction of (3.3.1) by T_3

We obtain the first reduction of the conserved vector T_3 by writing the vector in canonical variables determined from the associated symmetry

$$X = X_1 + \kappa_2 X_2 + \kappa_3 X_3. \tag{3.3.13}$$

Writing (3.3.13) in the canonical form $\widetilde{X} = \frac{\partial}{\partial q}$, we obtain from the corresponding characteristic equations

$$\frac{dt}{1} = \frac{dx}{\kappa_2} = \frac{dy}{\kappa_3} = \frac{du}{0} = \frac{dr}{0} = \frac{ds}{0} = \frac{dq}{1} = \frac{dw}{0}, \tag{3.3.14}$$

the canonical coordinates:

$$\begin{aligned} r &= y - \kappa_3 t, \\ s &= x - \kappa_2 t, \\ q &= t, \\ w &= u, \end{aligned} \tag{3.3.15}$$

where $w = w(r,s)$. Inverse canonical coordinates are given by:

$$\begin{aligned} t &= q, \\ x &= \kappa_2 q + s, \\ y &= \kappa_3 q + r, \\ u &= w. \end{aligned} \tag{3.3.16}$$

From Theorem 1.4.2, we obtain the reduced conserved form

$$\begin{pmatrix} T_3^r \\ T_3^s \\ T_3^q \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T_3^t \\ T_3^x \\ T_3^y \end{pmatrix}, \tag{3.3.17}$$

where

$$A = \begin{pmatrix} D_r t & D_r x & D_r y \\ D_s t & D_s x & D_s y \\ D_q t & D_q x & D_q y \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & \kappa_2 & \kappa_3 \end{pmatrix}, \tag{3.3.18}$$

$$A^{-1} = \begin{pmatrix} D_t r & D_t s & D_t q \\ D_x r & D_x s & D_x q \\ D_y r & D_y s & D_y q \end{pmatrix} = \begin{pmatrix} -\kappa_3 & -\kappa_2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{3.3.19}$$

and

$$J = \det(A) = -1. \tag{3.3.20}$$

Expressing the first and second partial derivatives u_t, u_x, u_{tt}, u_{xy} and u_{xx} in terms of the canonical coordinates (3.3.15), we obtain:

$$\begin{aligned} u_t &= -\kappa_2 w_s - \kappa_3 w_r, \\ u_x &= w_s, \\ u_y &= w_r, \\ u_{xx} &= w_{ss}, \\ u_{tx} &= -\kappa_2 w_{ss} - \kappa_3 w_{rs}, \\ u_{yy} &= w_{rr}. \end{aligned} \tag{3.3.21}$$

Substituting for the partial derivatives u_{xx} and u_{yy} in (3.3.17) using (3.3.21), we obtain:

$$
\begin{aligned}
T_3^r &= \frac{\kappa_3 w^2}{2} - \frac{\beta w w_{rs}}{2} + \frac{\beta w_r w_s}{2}, \\
T_3^s &= \frac{\kappa_2 w^2}{2} - \frac{w^3}{3} - \frac{\beta w w_{rr}}{2} - \alpha w w_{ss} + \frac{\alpha w_s^2}{2}, \\
T_3^q &= -\frac{w^2}{2},
\end{aligned}
\tag{3.3.22}
$$

leading to the reduced conservation law

$$
D_r T_3^r + D_s T_3^s = 0. \tag{3.3.23}
$$

Equation (3.3.23) inherits the symmetries X_1, X_2 and X_3 from (3.3.4), which when written in terms of the canonical variables (3.3.15), are:

$$
\begin{aligned}
\widetilde{X}_1 &= \kappa_3 \frac{\partial}{\partial r} + \kappa_2 \frac{\partial}{\partial s}, \\
\widetilde{X}_2 &= \frac{\partial}{\partial s}, \\
\widetilde{X}_3 &= \frac{\partial}{\partial r}.
\end{aligned}
\tag{3.3.24}
$$

It turns out that all the symmetries in (3.3.24) are associated with the conservation law (3.3.23), i.e.

$$
\widetilde{X} \begin{pmatrix} T_3^r \\ T_3^s \end{pmatrix} - \begin{pmatrix} D_r \xi^r & D_s \xi^r \\ D_r \xi^s & D_s \xi^s \end{pmatrix} \begin{pmatrix} T_3^r \\ T_3^s \end{pmatrix} + (D_r \xi^r + D_s \xi^s) \begin{pmatrix} T_3^r \\ T_3^s \end{pmatrix} = 0, \tag{3.3.25}
$$

if $\widetilde{X} = \delta_1 \widetilde{X}_1 + \delta_2 \widetilde{X}_2 + \delta_3 \widetilde{X}_3$, where $\delta_i' s$ are arbitrary constants.

So, we can get a further reduction of the conserved vector $\left(T_3^r, T_3^s\right)$ by the linear combination

$$
Y = \frac{\partial}{\partial r} + \gamma \frac{\partial}{\partial s}, \tag{3.3.26}
$$

where γ is an arbitrary constant. The generator Y has a canonical form $\widetilde{Y} = \dfrac{\partial}{\partial m}$ when

$$\frac{dr}{r} = \frac{ds}{s} = \frac{dw}{0} = \frac{dn}{0} = \frac{dm}{1} = \frac{dv}{0}, \tag{3.3.27}$$

which results in canonical coordinates:

$$\begin{aligned} n &= s - \gamma r, \\ m &= r, \\ v &= w, \end{aligned} \tag{3.3.28}$$

where $v = v(n)$. The inverse canonical coordinates are given by:

$$\begin{aligned} r &= m, \\ s &= \gamma m + n, \\ w &= v. \end{aligned} \tag{3.3.29}$$

Therefore, the partial derivatives in the components (3.3.22) in terms of the canonical coordinates (3.3.28) are given by:

$$\begin{aligned} w_r &= -\gamma v_n, \\ w_s &= v_n, \\ w_{rr} &= \gamma^2 v_{nn}, \\ w_{rs} &= -\gamma v_{nn}, \\ w_{ss} &= v_{nn}. \end{aligned} \tag{3.3.30}$$

According to Theorem 1.4.2, we have that

$$\begin{pmatrix} T_3^n \\ T_3^m \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T_3^r \\ T_3^s \end{pmatrix}, \tag{3.3.31}$$

where

$$A = \begin{pmatrix} D_n r & D_n s \\ D_m r & D_m s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & \gamma \end{pmatrix}, \tag{3.3.32}$$

$$A^{-1} = \begin{pmatrix} D_r n & D_r m \\ D_s n & D_s m \end{pmatrix} = \begin{pmatrix} -\gamma & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.3.33}$$

and

$$J = \det(A) = -1. \tag{3.3.34}$$

Substituting the partial derivatives in (3.3.30) into (3.3.31) results in a reduced conserved vector with the following components:

$$\begin{aligned} T_3^n &= (\alpha+\beta\gamma^2)\left(vv_{nn} - \frac{v_n^2}{2}\right) + \frac{1}{2}v^2(\gamma\kappa_3 - \kappa_2) + \frac{v^3}{3}, \\ T_3^m &= \frac{1}{2}\beta\gamma v_n^2 - \frac{\kappa_3 v^2}{2} - \frac{1}{2}\beta\gamma v v_{nn}. \end{aligned} \tag{3.3.35}$$

This leads to the reduced conservation law $D_r T_3^r = 0$, from which we obtain the second-order ODE

$$(\alpha+\beta\gamma^2)\left(vv_{nn} - \frac{v_n^2}{2}\right) + \frac{1}{2}v^2(\gamma\kappa_3 - \kappa_2) + \frac{v^3}{3} = k, \tag{3.3.36}$$

where k is an arbitrary constant.

3.3.5 Multi-Reduction of (3.3.1) by T_4

We see from (3.3.11) that the symmetries associated with T_4 are X_1, X_2, X_3 and X_5. We perform multi-reduction by T_4 under two cases, namely reduction from using the linear combination

$$X = X_1 + \kappa_2 X_2 + \kappa_3 X_3$$

and reduction from using

$$X = X_5.$$

Case I: Reduction via $\langle X_1 + \kappa_2 X_2 + \kappa_3 X_3 \rangle$

The canonical variables (3.3.15) obtained earlier under T_3 apply here. Therefore, the reduced conserved form resulting from T_4 is

$$\begin{pmatrix} T_4^r \\ T_4^s \\ T_4^q \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T_4^t \\ T_4^x \\ T_4^y \end{pmatrix}, \tag{3.3.37}$$

where A, A^{-1}, J and the partial derivatives are the same as the ones computed earlier in (3.3.18)– (3.3.21). Therefore, we obtain:

$$\begin{aligned} T_4^r &= \kappa_3 w, \\ T_4^s &= \kappa_2 w - \frac{w^2}{2} - \beta w_{rr} - \alpha w_{ss} \\ T_4^q &= -w, \end{aligned} \tag{3.3.38}$$

and the reduced conservation law

$$D_r T_4^r + D_s T_4^s = 0. \tag{3.3.39}$$

Like in the T_3 case, the symmetries X_1, X_2, and X_3, in (3.3.4) written in terms of the canonical variables (3.3.15), become:

$$\begin{aligned}\widetilde{X}_1 &= \kappa_3 \frac{\partial}{\partial r} + \kappa_2 \frac{\partial}{\partial s},\\ \widetilde{X}_2 &= \frac{\partial}{\partial s},\\ \widetilde{X}_3 &= \frac{\partial}{\partial r},\end{aligned} \tag{3.3.40}$$

and are all inherited by (3.3.39). Also, they are all associated with the conservation law (3.3.39). Using the linear combination $Y = \frac{\partial}{\partial r} + \gamma \frac{\partial}{\partial s}$, where γ is an arbitrary constant, as in the T_3 case, we find canonical coordinates:

$$\begin{aligned}n &= s - \gamma r,\\ m &= r,\\ v &= w,\end{aligned} \tag{3.3.41}$$

where $v = v(n)$. Taking advantage of the calculations done in the T_3 case, in which the same canonical variables were used, the reduced conserved vector is given by

$$\begin{pmatrix} T_4^n \\ T_4^m \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T_4^r \\ T_4^s \end{pmatrix}, \tag{3.3.42}$$

where A, A^{-1}, and J are given by (3.3.32), (3.3.33) and (3.3.34), respectively.

Substituting the partial derivatives in (3.3.30) into (3.3.42) results in the following components of the reduced conserved vector:

$$\begin{aligned}T_4^n &= -\kappa_2 v + \gamma \kappa_3 v + \frac{v^2}{2} + v_{nn}\left(\alpha + \beta \gamma^2\right),\\ T_4^m &= -\kappa_3 v.\end{aligned} \tag{3.3.43}$$

This leads to the reduced conservation law $D_r T_4^r = 0$, from which it follows that

$$v(\gamma\kappa_3 - \kappa_2) + \frac{v^2}{2} + v_{nn}\left(\alpha + \beta\gamma^2\right) = k, \tag{3.3.44}$$

where k is an arbitrary constant.

Case II: Reduction via $\langle X_5 \rangle$

We obtain the first reduction of the conservation law T_4 by writing it in canonical variables determined from writing the associated symmetry X_5 in the canonical form $\widetilde{X}_5 = \frac{\partial}{\partial q}$. From the associated characteristic equations

$$\frac{dt}{3t} = \frac{dx}{x} = \frac{dy}{y} = \frac{du}{-2u} = \frac{dr}{0} = \frac{ds}{0} = \frac{dq}{1} = \frac{dw}{0}, \tag{3.3.45}$$

we obtain the canonical coordinates:

$$\begin{aligned} r &= \frac{y}{\sqrt[3]{t}}, \\ s &= \frac{x}{\sqrt[3]{t}}, \\ q &= \frac{\ln t}{3}, \\ w &= t^{2/3}u, \end{aligned} \tag{3.3.46}$$

where $w = w(r,s)$. Inverse canonical coordinates are given by:

$$\begin{aligned} t &= e^{3q}, \\ x &= e^{q}s, \\ y &= e^{q}r, \\ u &= e^{-2q}w. \end{aligned} \tag{3.3.47}$$

The partial derivatives in the conserved vector T_4, in terms of the canonical coordinates, are:

$$
\begin{aligned}
u_{xx} &= e^{-4q} w_{ss}, \\
u_{yy} &= e^{-4q} w_{rr}.
\end{aligned}
\tag{3.3.48}
$$

Thus, the reduced conserved vector is therefore

$$
\begin{pmatrix} T_4^r \\ T_4^s \\ T_4^q \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T_4^t \\ T_4^x \\ T_4^y \end{pmatrix}, \tag{3.3.49}
$$

where

$$
A = \begin{pmatrix} D_r t & D_r x & D_r y \\ D_s t & D_s x & D_s y \\ D_q t & D_q x & D_q y \end{pmatrix} = \begin{pmatrix} 0 & 0 & e^q \\ 0 & e^q & 0 \\ 3e^{3q} & e^q s & e^q r \end{pmatrix} \tag{3.3.50}
$$

$$
A^{-1} = \begin{pmatrix} D_t r & D_t s & D_t q \\ D_x r & D_x s & D_x q \\ D_y r & D_y s & D_y q \end{pmatrix} = \begin{pmatrix} -\frac{y}{3t^{4/3}} & -\frac{x}{3t^{4/3}} & \frac{1}{3t} \\ 0 & \frac{1}{\sqrt[3]{t}} & 0 \\ \frac{1}{\sqrt[3]{t}} & 0 & 0 \end{pmatrix}, \tag{3.3.51}
$$

and

$$
J = \det(A) = -3e^{5q}. \tag{3.3.52}
$$

Substituting the partial derivatives (3.3.48) into (3.3.49), we obtain:

$$
\begin{aligned}
T_4^r &= rw, \\
T_4^s &= sw - \frac{3w^2}{2} - 3\beta w_{rr} - 3\alpha w_{ss} \\
T_4^q &= -w.
\end{aligned}
\tag{3.3.53}
$$

Then the reduced conservation law is

$$D_r T_4^r + D_s T_4^s = 0. \tag{3.3.54}$$

None of the symmetries from (3.3.4) are inherited by (3.3.54), and therefore no further reduction of (3.3.54) is performed in this case.

3.3.6 Multi-Reduction of (3.3.1) by T_2

We see from (3.3.11) that the symmetry associated with T_2 is

$$X = \frac{\partial}{\partial t} + \kappa_2 \frac{\partial}{\partial x} + \kappa_3 \frac{\partial}{\partial y},$$

the same symmetry used in the multi-reduction by T_3. Therefore, the first reduction can be achieved through the canonical coordinates:

$$\begin{aligned} r &= y - \kappa_3 t, \\ s &= x - \kappa_2 t, \\ q &= t, \\ w &= u, \end{aligned} \tag{3.3.55}$$

where $w = w(r,s)$. The resulting reduced conserved form is

$$\begin{pmatrix} T_2^r \\ T_2^s \\ T_2^q \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T_2^t \\ T_2^x \\ T_2^y \end{pmatrix}, \tag{3.3.56}$$

where A, A^{-1}, and J are given by (3.3.18), (3.3.19) and (3.3.20), respectively. Substituting for the partial derivatives in the conserved vector T_2 using (3.3.21), we obtain:

$$\begin{aligned}
T_2^r &= -\frac{\kappa_2 w w_{rs}}{2} + \frac{\kappa_2 w_r w_s}{2} + \frac{\kappa_3 w^3}{6\beta} + \frac{\alpha\kappa_3 w w_{ss}}{2\beta} + \frac{\kappa_3 w_r^2}{2},\\
T_2^s &= \frac{\kappa_2 w^3}{6\beta} + \frac{\kappa_2 w w_{rr}}{2} + \frac{\alpha\kappa_2 w_s^2}{2\beta} - \frac{\alpha\kappa_3 w w_{rs}}{2\beta} - \frac{w^4}{8\beta} - \frac{w^2 w_{rr}}{2}\\
&\quad + \frac{\alpha\kappa_3 w_r w_s}{2\beta} - \frac{\alpha w^2 w_{ss}}{2\beta} - \frac{\beta w_{rr}^2}{2} - \alpha w_{rr} w_{ss} - \frac{\alpha^2 w_{ss}^2}{2\beta},\\
T_2^q &= -\frac{w^3}{6\beta} - \frac{w w_{rr}}{2} - \frac{\alpha w w_{ss}}{2\beta}.
\end{aligned} \tag{3.3.57}$$

Then the reduced conservation law is

$$D_r T_2^r + D_s T_2^s = 0. \tag{3.3.58}$$

Like in the T_3 case, the symmetries X_1, X_2, and X_3, in (3.3.4) written in terms of the canonical variables (3.3.15), become:

$$\begin{aligned}
\widetilde{X}_1 &= \kappa_3\frac{\partial}{\partial r} + \kappa_2\frac{\partial}{\partial s},\\
\widetilde{X}_2 &= \frac{\partial}{\partial s},\\
\widetilde{X}_3 &= \frac{\partial}{\partial r},
\end{aligned} \tag{3.3.59}$$

and are inherited by (3.3.58). Also, they are all associated with the conservation law (3.3.58). Using $Y = \frac{\partial}{\partial r} + \gamma\frac{\partial}{\partial s}$, where γ is an arbitrary constant, we find canonical coordinates:

$$\begin{aligned}
n &= s - \gamma r,\\
m &= r,\\
v &= w,
\end{aligned} \tag{3.3.60}$$

where $v = v(n)$. Taking advantage of the calculations done in the T_3 case, in which the same canonical variables were used, the reduced conserved vector is given by

$$\begin{pmatrix} T_2^n \\ T_2^m \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T_2^r \\ T_2^s \end{pmatrix}, \tag{3.3.61}$$

where A, A^{-1}, and J are given by (3.3.32), (3.3.33) and (3.3.34), respectively.

Substituting the partial derivatives in (3.3.30) into (3.3.61) results in the following components of the reduced conserved vector:

$$\begin{aligned} T_2^n &= \frac{Q^2 v_{nn}^2}{2\beta} - \frac{(\kappa_2 - \gamma\kappa_3)\left(Qv_n^2 + v^3/3\right)}{2\beta} + \frac{Qv^2 v_{nn}}{2\beta} + \frac{v^4}{8\beta}, \\ T_2^m &= \frac{1}{2}\gamma\kappa_2 v_n^2 - \frac{1}{2}\gamma\kappa_2 v v_{nn} - \frac{\kappa_3 v^3}{6\beta} - \frac{\alpha\kappa_3 v v_{nn}}{2\beta} - \frac{1}{2}\gamma^2\kappa_3 v_n^2, \end{aligned} \tag{3.3.62}$$

where $Q = \alpha + \beta\gamma^2$. This leads to the reduced conservation law $D_n T_2^n = 0$, from which it follows that

$$\frac{Q^2 v_{nn}^2}{2\beta} - \frac{(\kappa_2 - \gamma\kappa_3)\left(Qv_n^2 + v^3/3\right)}{2\beta} + \frac{Qv^2 v_{nn}}{2\beta} + \frac{v^4}{8\beta} = k, \tag{3.3.63}$$

where k is an arbitrary constant.

3.3.7 Multi-Reduction of (3.3.1) by T_1

The first reduction of the conserved vector T_1 is obtained from writing the vector in canonical variables determined from writing the associated symmetry $X = X_3$ in the form $\widetilde{X} = \frac{\partial}{\partial q}$. From the corresponding characteristic equations

$$\frac{dt}{0} = \frac{dx}{0} = \frac{dy}{1} = \frac{du}{0} = \frac{dr}{0} = \frac{ds}{0} = \frac{dq}{1} = \frac{dw}{0}, \tag{3.3.64}$$

we obtain the canonical coordinates:

$$\begin{aligned}
r &= t, \\
s &= x, \\
q &= y, \\
w &= u,
\end{aligned} \tag{3.3.65}$$

where $w = w(r,s)$. Inverse canonical coordinates are given by:

$$\begin{aligned}
t &= r, \\
x &= s, \\
y &= q, \\
u &= w.
\end{aligned} \tag{3.3.66}$$

Therefore, the partial derivatives in the conserved vector T_1 in terms of the canonical coordinates (3.3.65), are:

$$\begin{aligned}
u_x &= w_s, \\
u_y &= 0, \\
u_{xx} &= w_{ss}, \\
u_{xy} &= 0, \\
u_{yy} &= 0.
\end{aligned} \tag{3.3.67}$$

The reduced conserved vector is

$$\begin{pmatrix} T_1^r \\ T_1^s \\ T_1^q \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T_1^t \\ T_1^x \\ T_1^y \end{pmatrix}, \tag{3.3.68}$$

where

$$A = \begin{pmatrix} D_r t & D_r x & D_r y \\ D_s t & D_s x & D_s y \\ D_q t & D_q x & D_q y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.3.69}$$

$$A^{-1} = \begin{pmatrix} D_t r & D_t s & D_t q \\ D_x r & D_x s & D_x q \\ D_y r & D_y s & D_y q \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.3.70}$$

and

$$J = \det(A) = 1. \tag{3.3.71}$$

Substituting for the partial derivatives in (3.3.68) using (3.3.67), we obtain the components:

$$\begin{aligned} T_1^r &= \frac{rw^2}{2} - sw, \\ T_1^s &= \frac{rw^3}{3} + \alpha rww_{ss} - \frac{1}{2}\alpha rw_s^2 - \frac{sw^2}{2} - \alpha sw_{ss} + \alpha w_s, \\ T_1^q &= 0. \end{aligned} \tag{3.3.72}$$

The corresponding reduced conservation law

$$D_r T_1^r + D_s T_1^s = 0, \tag{3.3.73}$$

inherits the symmetries X_1, X_2, X_4, and X_5 from (3.3.4), or in terms of the canonical variables (3.3.65) we have:

$$
\begin{aligned}
\widetilde{X}_1 &= \frac{\partial}{\partial r},\\
\widetilde{X}_2 &= \frac{\partial}{\partial s},\\
\widetilde{X}_4 &= r\frac{\partial}{\partial s} + \frac{\partial}{\partial w},\\
\widetilde{X}_5 &= -\frac{3}{2}r\frac{\partial}{\partial r} - \frac{1}{2}s\frac{\partial}{\partial s} + w\frac{\partial}{\partial w}.
\end{aligned}
\tag{3.3.74}
$$

It turns out that none of the symmetries (3.3.74) are associated with the conservation law (3.3.73). Therefore, no further reduction of (3.3.73) is performed.

3.4 Concluding remarks

The work presented in this study significantly enhances the existing body of literature concerning the application of the generalised double reduction theory. The theory presents a powerful tool for obtaining invariant solutions of systems of PDEs from the association of symmetries of the system with its nontrivial conservation laws. Given a system of qth-order PDEs with n independent and m dependent variables, the theory provides for the reduction of the system to a system of $(q-1)$th-order ODEs.

We have applied the generalised double reduction theory to two $(2+1)$-dimensional partial differential equations, a nonlinear wave equation and the ZK equation. For the nonlinear wave equation, we extended the application presented in the seminal paper by Bokhari et al [31] on the generalised double reduction theory. We presented a more detailed account of the application and also determined and used additional inherited symmetries. We furthermore presented exact solutions of the equation for specific prescriptions of the arbitrary functions in the equation. As for the ZK equation, five Lie point symmetries of the equation were determined along with four nontrivial conservation laws. Upon establishing association of the conservation laws with the symmetries, multi-reductions were performed for each of the conservation laws.

Overall, our study has provided a comprehensive guide to using the generalised double reduction method. We have presented clear illustrative examples and included each key step of the method.

Chapter 4

Double Reduction of two (1+1)-Dimensional PDE Systems: the Kaup-Boussinesq (K-B) System and the Broer-Kaup (BK) system

4.1 Introduction

In this chapter we apply double reduction theorem [30] on two (1+1)-dimensional PDE systems, the Kaup-Boussinesq System [85–87] and the Broer-Kaup system [88, 89]. We focus on obtaining solutions to both the Kaup-Boussinesq system and the Broer-Kaup system by employing the generalised double reduction method [30, 31]. We find four local conservation laws of the Kaup-Boussinesq system and six local conservation laws of the Broer-Kaup system that arise form low-order multipliers and determine that each of the conservation laws have associated Lie point symmetries. This chapter consists of two main sections. In Section 4.2 we report the Kaup-Boussinesq System and in Section 4.5 we report the Broer-Kaup system.

4.2 The Kaup-Boussinesq (K-B) System

4.2.1 Introduction

Wave phenomena are pervasive in both nature and applied sciences, manifesting in various forms such as sound waves, light waves, and water waves [75, 90–93]. At the heart of understanding these phenomena lies the Boussinesq equation and its variants. One important variant is the Kaup-Boussinesq (K-B) system [85, 87, 94], described by the coupled partial differential equations

$$v_t - u_x - 2vv_x = 0, \quad u_t - v_{xxx} - 2vu_x - 2uv_x = 0, \tag{4.2.1}$$

representing a well-known model for the propagation of nonlinear waves in a variety of physical contexts, including surface waves in shallow water [95–97]. The system captures the intricate interactions between wave modes and nonlinear effects, making it particularly relevant for studying complex wave phenomena. Originally proposed as an integrable system [98, 99], the K-B equations have attracted considerable attention in the field of mathematical physics due to their rich structure and applicability to real-world problems [100–102].

The K-B system is classified as a completely integrable system [103, 104], meaning it admits an infinite number of conservation laws and exact solutions. Such systems are valuable because they provide insight into nonlinear wave dynamics, including soliton behavior, energy transfer mechanisms, and stability of wave patterns. As a result, researchers have extensively studied the K-B system in the context of soliton theory, integrable systems, and wave propagation in dispersive media [85–87, 103–105]. Previous research has explored various aspects of the K-B system. Babajanov et al. [104] have extended the class of initial functions of the

Cauchy problem for the K-B system and presented an efficient method to obtain the time evolution of scattering data, which allows applying the ITS method to solve the Cauchy problem for the K-B system in the class of rapidly decreasing functions. Again, Babajanov et al. [103] have also shown that the K-B system with an additional term is also an important theoretical model, since it is a completely integrable system. They found the time evolution of scattering data for a quadratic pencil of Sturm–Liouville operators associated with the solution of the K-B system with time-dependent coefficients. Zhou et al. [85] applied bifurcation theory to analyze traveling-wave solutions of a dual equation related to the K-B system and derived analytic expressions for solitary-wave solutions. Hosseini et al. [86] employed the first integral method to obtain exact solutions for the K-B system analytically. In addition, Motsepa et al. [87] used direct integration techniques to find traveling wave solutions and reported six conservation laws of the K-B system using the multiplier method with second-order multipliers.

In this section, we build on these previous efforts by applying the generalized double reduction method to the K-B system [13,26,30,76]. This study focuses on identifying Lie point symmetries of the K-B system, constructing four non-trivial conservation laws through the multiplier method, and determining the associated Lie point symmetries. Additionally, we present a specific solution for one case of the reduction. This work contributes to a deeper understanding of the generalized double reduction method and its application to nonlinear systems like the K-B system, enhancing both theoretical insights and practical solution techniques.

4.2.2 Symmetries and Conservation laws of (4.2.1)

The K-B system admits the following four Lie point symmetries [44]:

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial x}, \\
X_2 &= \frac{\partial}{\partial t}, \\
X_3 &= \frac{\partial}{\partial v} - 2t\frac{\partial}{\partial x}, \\
X_4 &= -2t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} - x\frac{\partial}{\partial x}.
\end{aligned} \tag{4.2.2}
$$

To construct conservation laws for (4.2.1) we employ the multiplier method [47, 48, 54]. We look for first-order multipliers of the form:

$$
\begin{aligned}
\Lambda_1 &= \Lambda_1\left(x, t, u, v, u_x, v_x\right), \\
\Lambda_2 &= \Lambda_2\left(x, t, u, v, u_x, v_x\right).
\end{aligned}
$$

The determining equations for multipliers Λ_1 and Λ_2 become

$$
\begin{aligned}
&\frac{\delta}{\delta u}\left(\Lambda_1\left(v_t - u_x - 2vv_x\right) + \Lambda_2\left(u_t - v_{xxx} - 2vu_x - 2uv_x\right)\right) \equiv 0, \\
&\frac{\delta}{\delta v}\left(\Lambda_1\left(v_t - u_x - 2vv_x\right) + \Lambda_2\left(u_t - v_{xxx} - 2vu_x - 2uv_x\right)\right) \equiv 0,
\end{aligned} \tag{4.2.3}
$$

where the Euler operators $\frac{\delta}{\delta u}$ and $\frac{\delta}{\delta v}$ are given by (1.3.4)

$$
\begin{aligned}
\frac{\delta}{\delta u} &= \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_x^2\frac{\partial}{\partial u_{xx}}\ldots, \\
\frac{\delta}{\delta v} &= \frac{\partial}{\partial v} - D_t\frac{\partial}{\partial v_t} - D_x\frac{\partial}{\partial v_x} + D_x^2\frac{\partial}{\partial v_{xx}} + \ldots,
\end{aligned} \tag{4.2.4}
$$

and total derivative operators D_t and D_x using (1.2.3) are

$$
\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + v_t\frac{\partial}{\partial v} + u_{tt}\frac{\partial}{\partial u_t} + v_{tt}\frac{\partial}{\partial v_t} + u_{tx}\frac{\partial}{\partial u_x} + v_{tx}\frac{\partial}{\partial v_x} + \cdots, \\
D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + v_x\frac{\partial}{\partial v} + u_{xx}\frac{\partial}{\partial u_x} + v_{xx}\frac{\partial}{\partial v_x} + u_{tx}\frac{\partial}{\partial u_t} + v_{tx}\frac{\partial}{\partial v_t} + \cdots.
\end{aligned} \tag{4.2.5}
$$

The system (4.2.3) after expansion and splitting with respect to the derivatives of u and v, gives the determining equations:

$$\begin{aligned} \Lambda_{2_{xx}} &= 0, & \Lambda_{2_{vx}} &= 0, & \Lambda_{2_{vv}} &= 0, \\ \Lambda_{1_t} - 2u\Lambda_{2_x} &= 0, & \Lambda_{2_t} - 2v\Lambda_{2_x} &= 0, & \Lambda_{1_x} &= 0, \\ \Lambda_{1_u} - \Lambda_{2_v} &= 0, & \Lambda_{2_u} &= 0, & \Lambda_{1_v} &= 0, \\ \Lambda_{1_{ux}} &= 0, & \Lambda_{2_{ux}} &= 0, & \Lambda_{1_{vx}} &= 0, \\ \Lambda_{2_{v_x}} &= 0. \end{aligned} \tag{4.2.6}$$

The system of determining equations (4.2.6) gives the solutions given by:

$$\begin{aligned} \Lambda_1 &= u(2C_1 t + C_2) + C_4, \\ \Lambda_2 &= C_1(2tv + x) + C_2 v + C_3, \end{aligned} \tag{4.2.7}$$

where $C_i,\ i = 1, 2, \ldots, 4$ are arbitrary constants. The multipliers of the K-B system satisfy

$$\Lambda_1(v_t - u_x - 2vv_x) + \Lambda_2(u_t - v_{xxx} - 2vu_x - 2uv_x) = D_t T^t + D_x T^x, \tag{4.2.8}$$

for all functions $u(t,x)$ and $v(t,x)$. From (4.2.7) and (4.2.8) using the direct method [54], we obtain four conserved vectors for (4.2.1):

$$T_1 \ : \ \begin{cases} T_1^t = 2tuv + ux, \\ T_1^x = -tu^2 - 4tuv^2 - 2tvv_{xx} + tv_x^2 - 2uvx + v_x - xv_{xx}, \end{cases} \tag{4.2.9}$$

$$T_2 \ : \ \begin{cases} T_2^t = uv, \\ T_2^x = -\dfrac{u^2}{2} - 2uv^2 - vv_{xx} + \dfrac{v_x^2}{2}, \end{cases} \tag{4.2.10}$$

$$T_3 \quad : \quad \begin{cases} T_3^t = u, \\ \\ T_3^x = -2uv - v_{xx}, \end{cases} \tag{4.2.11}$$

$$T_4 \quad : \quad \begin{cases} T_4^t = v, \\ \\ T_4^x = -u - v^2. \end{cases} \tag{4.2.12}$$

4.3 Double Reduction of K-B System (4.2.1)

We are now applying the double reduction theorem based on Lie symmetries and conservation laws of (4.2.1) to find the reductions and exact solutions. For two independent variables t and x, the formula (1.4.1) yields

$$X \begin{pmatrix} T^t \\ \\ T^x \end{pmatrix} - \begin{pmatrix} D_t\xi^t & D_x\xi^t \\ \\ D_t\xi^x & D_x\xi^x \end{pmatrix} \begin{pmatrix} T^t \\ \\ T^x \end{pmatrix} + \left(D_t\xi^t + D_x\xi^x\right) \begin{pmatrix} T^t \\ \\ T^x \end{pmatrix} = 0. \tag{4.3.1}$$

Using (4.3.1), we establish association of the symmetries (4.2.2) and the conserved vectors (4.2.9) - (4.2.12). The results are presented in the Table 4.3.1.

$[T_i, X_j]$	T_1	T_2	T_3	T_4
X_1		✓	✓	✓
X_2		✓	✓	✓
X_3	✓		✓	
X_4	✓			✓

Table 4.3.1: Association of Symmetries and Conservation Laws

4.3.1 Reduction of (4.2.1) using $\langle X_3 \rangle$

The generator X_3 takes canonical form $X = \partial/\partial s$ when

$$\frac{dt}{0} = \frac{dx}{-2t} = \frac{du}{0} = \frac{dv}{1} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0} = \frac{dq}{0}, \tag{4.3.2}$$

and from (4.3.2) we get the the canonical coordinates

$$\begin{aligned} r &= t, \\ s &= -\frac{x}{2t}, \\ w &= u, \\ q &= \frac{2tv+x}{2t}. \end{aligned} \tag{4.3.3}$$

where $w = w(r)$ and $q = q(r)$. From (4.3.3), the inverse canonical coordinates are given by

$$\begin{aligned} t &= r\gamma, \\ x &= -2rs, \\ u &= w, \\ v &= q+s. \end{aligned} \tag{4.3.4}$$

The partial derivatives of v from (4.3.4) are

$$\begin{aligned} v_x &= -\frac{1}{2r}, \\ v_{xx} &= 0. \end{aligned} \tag{4.3.5}$$

For two independent variables t and x, the formula (1.4.15) reduces to the conserved form

$$\begin{pmatrix} T^r \\ T^s \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T^t \\ T^x \end{pmatrix}, \tag{4.3.6}$$

where

$$A = \begin{pmatrix} D_r t & D_r x \\ D_s t & D_s x \end{pmatrix} = \begin{pmatrix} 1 & -2s \\ 0 & -2r \end{pmatrix},$$

$$
\left(A^{-1}\right)^T = \begin{pmatrix} D_t r & D_x r \\ D_t s & D_x s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -s/r & -1/2r \end{pmatrix},
$$

and $J = \det(A) = 2r$. Thus, the conserved vectors T_1 and T_3 reduce to

$$
\widetilde{T}_1 \quad : \quad \begin{cases} T_1^r = -4qr^2 w, \\ T_1^s = -rw\left(4q^2 + w\right) - \dfrac{1}{4r}, \end{cases} \tag{4.3.7}
$$

$$
\widetilde{T}_3 \quad : \quad \begin{cases} T_3^r = -2rw, \\ T_3^s = -2qw, \end{cases} \tag{4.3.8}
$$

and the conserved vectors $\widetilde{T}_1$ and $\widetilde{T}_3$ satisfy the reduced conserved form

$$
D_r T_i^r = 0, \quad i = 1, 3.
$$

Consequently, we obtain

$$
qr^2 w = \kappa_1, \tag{4.3.9}
$$

$$
rw = \kappa_2, \tag{4.3.10}
$$

where κ_1 and κ_2 are arbitrary constants. Solving (4.3.9) and (4.3.10) simultaneously for w and p and using (4.3.3) leads to a solution

$$
u(t,x) = \frac{\kappa_2}{t}, \tag{4.3.11}
$$

$$
v(t,x) = \frac{2\kappa_1 - \kappa_2 x}{2\kappa_2 t}, \tag{4.3.12}
$$

for the system (4.2.1).

4.3.2 Reduction of (4.2.1) using $\langle X_1 + \gamma X_2 \rangle$

According to Table 4.3.1, the symmetries X_1 and X_2 are both associated with the conserved vectors T_2, T_3, and T_4. For the reduction process involving X_1 and X_2, we will use the linear combination $X_1 + \kappa X_2$, where κ is a parameter. The generator $X_1 + \kappa X_2$ has the canonical form $X = \partial/\partial s$ when

$$\frac{dt}{\gamma} = \frac{dx}{1} = \frac{du}{0} = \frac{dv}{0} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0} = \frac{dq}{0}, \tag{4.3.13}$$

which results in canonical coordinates

$$\begin{aligned} r &= \frac{\gamma x - t}{\gamma}, \qquad \gamma \neq 0 \\ s &= \frac{t}{\gamma}, \qquad \gamma \neq 0 \\ w &= u, \\ q &= v, \end{aligned} \tag{4.3.14}$$

where $w = w(r)$ and $q = q(r)$. From (4.3.14), the inverse canonical coordinates are given by

$$\begin{aligned} t &= s\gamma, \\ x &= r + s, \\ u &= w, \\ v &= q. \end{aligned} \tag{4.3.15}$$

The partial derivatives of v from (4.3.15) are

$$\begin{aligned} v_x &= q_r, \\ v_{xx} &= q_{rr}. \end{aligned} \tag{4.3.16}$$

Again, using formula (1.4.15) the reduced conserved form is given by

$$\begin{pmatrix} T^r \\ T^s \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T^t \\ T^x \end{pmatrix}, \tag{4.3.17}$$

where

$$A = \begin{pmatrix} D_r t & D_r x \\ D_s t & D_s x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \gamma & 1 \end{pmatrix},$$

$$\left(A^{-1}\right)^T = \begin{pmatrix} D_t r & D_x r \\ D_t s & D_x s \end{pmatrix} = \begin{pmatrix} -1/\gamma & 1 \\ 1/\gamma & 0 \end{pmatrix},$$

and $J = \det(A) = -\gamma$.

This reduces the conserved vectors T_2, T_3, and T_4 to

$$\widetilde{T}_2 \quad : \quad \begin{cases} T_2^r = 2\gamma q^2 w + q(\gamma q_{rr} + w) + \dfrac{1}{2}\gamma\left(w^2 - q_r^2\right), \\ T_2^s = -qw, \end{cases} \tag{4.3.18}$$

$$\widetilde{T}_3 \quad : \quad \begin{cases} T_3^r = 2\gamma q w + \gamma q_{rr} + w, \\ T_3^s = -w, \end{cases} \tag{4.3.19}$$

$$\widetilde{T}_4 \quad : \quad \begin{cases} T_4^r = \gamma q^2 + q + \gamma w, \\ T_4^s = -q, \end{cases} \tag{4.3.20}$$

where the conserved vectors $\widetilde{T}_2$, $\widetilde{T}_3$, and $\widetilde{T}_4$ satisfy the reduced conserved form

$$D_r T_i^r = 0, \quad i = 2, 3, 4.$$

Therefore, this leads to the system of equations

$$
\begin{aligned}
2\gamma q^2 w + q(\gamma q_{rr} + w) + \frac{1}{2}\gamma\left(w^2 - q_r^2\right) =& \kappa_1, \\
2\gamma q w + \gamma q_{rr} + w =& \kappa_2, \\
\gamma q^2 + q + \gamma w =& \kappa_3,
\end{aligned}
\tag{4.3.21}
$$

where κ_1, κ_2, and κ_3 are arbitrary constants.

Any pair of equations (4.3.21) can be solved to obtain a solutions of the K-B system (4.2.1). However, directly solving the system (4.3.21) can pose significant difficulties. Nevertheless, a helpful simplification arises by setting $\kappa_3 = 0$ and solving for w in terms of q and r in the third equation of (4.3.21) to get

$$
w(r) = \frac{-\gamma q^2 - q}{\gamma}. \tag{4.3.22}
$$

Substituting (4.3.22) in the first equation of (4.3.21), we get the second-order ODE

$$
2\gamma q q_{rr} - \gamma q_r^2 - 3\gamma q^4 - \frac{q^2}{\gamma} - 4q^3 = \kappa_1. \tag{4.3.23}
$$

Similarly, substituting (4.3.22) into the second equation of (4.3.21) results in the second-order ODE

$$
\gamma q_{rr} - 2\gamma q^3 - \frac{q}{\gamma} - 3q^2 = \kappa_2. \tag{4.3.24}
$$

Now, expressing (4.3.24) as

$$
q_{rr} = \frac{\gamma\kappa_2 + 2\gamma^2 q^3 + 3\gamma q^2 + q}{\gamma^2} \tag{4.3.25}
$$

and replace q_{rr} in (4.3.23) by the right-hand side of (4.3.25) we obtain the first-order ODE

$$
\kappa_1 - \gamma q_r^2 + \gamma q^4 + \frac{q^2}{\gamma} + 2\kappa_2 q + 2q^3 = 0. \tag{4.3.26}
$$

By observing that (4.3.26) admits the translational symmetry $\Gamma = \partial/\partial r$, we apply the method of canonical variables [7] to solve (4.3.26) in the case $\kappa_1 = 0 = \kappa_2$. This yields the solution

$$q(r) = \frac{e^{r/\gamma}}{M - \gamma e^{r/\gamma}}, \tag{4.3.27}$$

where $M = e^{J/\gamma}$ and J is the constant of integration.

Finally, the equations (4.3.27) and (4.3.22), using (4.3.14) lead to the following solution for the K-B system (4.2.1)

$$u(t,x) = -\frac{M\, e^{(t+\gamma x)/\gamma^2}}{\gamma\left(M\, e^{t/\gamma^2} - \gamma e^{x/\gamma}\right)^2}, \tag{4.3.28}$$

$$v(t,x) = \frac{e^{x/\gamma}}{M\, e^{t/\gamma^2} - \gamma e^{x/\gamma}}, \tag{4.3.29}$$

arising from $X_1 + \gamma X_2$ via T_2, T_3, and T_4.

4.3.3 Reduction of (4.2.1) using $\langle X_4 \rangle$

The generator X_4 takes canonical form $X = \partial/\partial s$ when

$$\frac{dt}{-2t} = \frac{dx}{-x} = \frac{du}{2u} = \frac{dv}{v} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0} = \frac{dq}{0}, \tag{4.3.30}$$

and from (4.3.30) we get the the canonical coordinates

$$\begin{aligned} r &= \frac{x}{\sqrt{t}},\\ s &= -\frac{\log(t)}{2},\\ w &= tu,\\ q &= \sqrt{t}v. \end{aligned} \tag{4.3.31}$$

where $w = w(r)$ and $q = q(r)$. From (4.3.31), the inverse canonical coordinates are given by

$$\begin{aligned} t &= e^{-2s}, \\ x &= e^{-s}, \\ u &= e^{2s}w, \\ v &= -e^{s}q. \end{aligned} \tag{4.3.32}$$

The partial derivatives of v from (4.3.32) are

$$\begin{aligned} v_x &= -e^{2s}q_r, \\ v_{xx} &= -e^{3s}q_{rr}. \end{aligned} \tag{4.3.33}$$

Again, the formula (1.4.15) reduces to the conserved form

$$\begin{pmatrix} T^r \\ T^s \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T^t \\ T^x \end{pmatrix}, \tag{4.3.34}$$

where

$$A = \begin{pmatrix} D_r t & D_r x \\ D_s t & D_s x \end{pmatrix} = \begin{pmatrix} 0 & e^{-s} \\ -2e^{-2s} & -e^{-s}r \end{pmatrix},$$

$$\left(A^{-1}\right)^T = \begin{pmatrix} D_t r & D_x r \\ D_t s & D_x s \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}e^{2s}r & e^{s} \\ -\frac{e^{2s}}{2} & 0 \end{pmatrix},$$

and $J = \det(A) = 2e^{-3s}$.

The conserved vectors T_1 and T_4 so reduce to

$$\widetilde{T}_1 \quad : \quad \begin{cases} T_1^r = -8q^2w - 4qq_{rr} + 6qrw + 2q_r^2 - 2q_r + 2rq_{rr} - r^2w - 2w^2, \\ T_1^s = w(2q - r), \end{cases} \tag{4.3.35}$$

$$
\widetilde{T}_4 \quad : \quad \begin{cases} T_4^r = -2q^2 + qr - 2w, \\ T_4^s = q, \end{cases} \tag{4.3.36}
$$

where the conserved vectors $\widetilde{T}_1$ and $\widetilde{T}_4$ satisfy the reduced conserved form

$$
D_r T_i^r = 0, \quad i = 1, 4.
$$

Thus, we obtain the system of equations

$$
\begin{aligned}
-8q^2w - 4qq_{rr} + 6qrw + 2q_r^2 - 2q_r + 2rq_{rr} - r^2w - 2w^2 = \kappa_1, \\
-2q^2 + qr - 2w = \kappa_2,
\end{aligned} \tag{4.3.37}
$$

where κ_1 and κ_2 are arbitrary constants. Solving the system (4.3.37) directly can be quite challenging. However, a useful simplification is to solve for w in terms of q and r from the second equation of (4.3.37), and setting $\kappa_2 = 0$, to obtain

$$
w(r) = \frac{1}{2}qr - q^2. \tag{4.3.38}
$$

Substituting the expression for w from (4.3.38) into the first equation of (4.3.37) yields a second-order ODE

$$
4rq_{rr} - 8qq_{rr} + 4q_r^2 - 4q_r - r^3q + 7r^2q^2 + 12q^4 - 16rq^3 = \kappa_1. \tag{4.3.39}
$$

Thus, the solution of the K-B system arising from X_4 via T_1 and T_4 is given by

$$
\begin{aligned}
u(t,x) &= \frac{1}{t}w(r), \\
v(t,x) &= \frac{1}{\sqrt{t}}q(r),
\end{aligned} \tag{4.3.40}
$$

where q is the solution of the ODE (4.3.39), w is given by (4.3.38), and $r = x/\sqrt{t}$.

4.4 Concluding remarks

This paper illustrates the application of the generalized double reduction method by analyzing the (1+1)-dimensional Kaup-Boussinesq (K-B) system, which serves as a model for nonlinear wave propagation. This method takes advantage of the association of Lie point symmetries and conservation laws, and has proven to be an effective tool for simplifying and finding solutions to nonlinear systems. In this study, we began by computing Lie point symmetries for the K-B system and constructed four non-trivial conservation laws using the multiplier method. Using the generalized double reduction method, we successfully reduced the K-B system to second-order ordinary differential equations and algebraic equations. This reduction yielded two distinct exact solutions: one derived from the algebraic equations and another from the second-order ODEs, where further exploitation of Lie point symmetries allowed us to utilize the method of canonical variables for the solution process. The generalized double reduction method is versatile and can be applied to other nonlinear systems that are rich in symmetries and conservation laws, in which case exact solutions may be discovered. Future research will explore further applications of this methods to different PDE models, potentially contributing to advancements in the study of nonlinear wave equations and other related phenomena.

4.5 The Broer-Kaup (BK) system

4.5.1 Introduction

The phenomenon of shallow water waves occurs in various contexts, notably in the analysis of tsunami wave behavior. These waves are typically modeled using nonlinear partial differential equations. Several mathematical models have been developed to describe shallow water waves, including the Korteweg-de Vries (KdV) equation [106], the Boussinesq equation [107], the Degasperis-Procesi equation [108], the Benjamin-Bona-Mahony (BBM) equation [109], the Kadomtsev-Petviashvili (K-P) equation [110]. The mathematical model we study in the article is the Broer-Kaup (BK) system, an important physical model which arises in the modelling of the propagation of long waves in shallow water [111]. The BK system is a $(1+1)$-dimensional system of two nonlinear PDEs [88, 89],

$$\begin{aligned} &u_t + \frac{1}{2}u_{xx} - v_x - uu_x = 0, \\ &v_t - \frac{1}{2}v_{xx} - (uv)_x = 0. \end{aligned} \tag{4.5.1}$$

In the literature, variants of the Broer-Kaup system (BKs) have been analysed in several studies using various approaches. For example in [112] exact solutions of (2+1)-dimensional Broer–Kaup equations are investigated by using the method of direct integral based on a known transform. In [113] travelling wave solutions of Broer–Kaup equations are obtained by using the $\left(\frac{G'}{G}\right)$-expansion method. Then in [114] the double Wronskian solutions of the Whitham–Broer–Kaup (WBK) system, a (1+ 1)-dimensional coupled system, are investigated. Conditions are given for two double Wronskians to generate the non-singular, non-trivial and irreducible soliton solutions. In [115] the classical Whitham-Broer-Kaup equations are studied using the generalized projective Riccati-equation method, resulting in 20 sets of solutions, some of which are novel.

In this section we apply the generalised double reduction theory to perform symmetry reductions on the BK system [30, 31]. We find six local conservation laws of the BKs system that arise form low-order multipliers and determine that each of the conservation laws has associated Lie point symmetries.

4.5.2 Lie point symmetries and conservation laws of (4.5.1)

The BK system (4.5.1) admits the following symmetries [44]:

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial t},\\
X_2 &= 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v},\\
X_3 &= \frac{\partial}{\partial x},\\
X_4 &= t\frac{\partial}{\partial x} - \frac{\partial}{\partial u}.
\end{aligned}
\tag{4.5.2}
$$

We investigate the conservation laws of the system (4.5.1) arising from first-order multipliers

$$
\begin{aligned}
\Lambda_1 &= \Lambda_1\left(t, x, u, v, u_t, u_x, v_t, v_x\right),\\
\Lambda_2 &= \Lambda_2\left(t, x, u, v, u_t, u_x, v_t, v_x\right).
\end{aligned}
\tag{4.5.3}
$$

For the two independent variables (t, x) and two dependent variables (u, v) in the BK system (4.5.1), equation (1.3.2) takes the form

$$
\Lambda_1(u_t + \frac{1}{2}u_{xx} - v_x - uu_x) + \Lambda_2(v_t - \frac{1}{2}v_{xx} - (uv)_x) = D_tT^t + D_xT^x \tag{4.5.4}
$$

where D_t and D_x are total derivative operators given in (4.2.5). The determining equations for multipliers Λ_1 and Λ_2 become:

$$
\begin{aligned}
\frac{\delta}{\delta u}\left[\Lambda_1\left(u_t + \frac{1}{2}u_{xx} - v_x - uu_x\right) + \Lambda_2\left(v_t - \frac{1}{2}v_{xx} - (uv)_x\right)\right] &= 0,\\
\frac{\delta}{\delta v}\left[\Lambda_1\left(u_t + \frac{1}{2}u_{xx} - v_x - uu_x\right) + \Lambda_2\left(v_t - \frac{1}{2}v_{xx} - (uv)_x\right)\right] &= 0,
\end{aligned}
\tag{4.5.5}
$$

where the Euler operators $\dfrac{\delta}{\delta u}$ and $\dfrac{\delta}{\delta v}$ are given by (4.2.4). Solving determining equations arising from (4.5.5), we obtain:

$$\begin{aligned}
\Lambda_1 &= \frac{1}{2}c_1u^2v + \frac{c_1uv_x}{6} - \frac{c_1u_xv}{3} + \frac{c_1v^2}{2} + \frac{c_1v_t}{3} + c_2uv + \frac{c_2v_x}{2} + c_3tv + c_4v + c_6, \\
\Lambda_2 &= \frac{c_1u^3}{6} - \frac{c_1uu_x}{6} + c_1uv - \frac{c_1u_t}{3} + \frac{c_1v_x}{3} + \frac{c_2u^2}{2} - \frac{c_2u_x}{2} + c_2v + c_3tu + c_3x \\
&\quad + c_4u + c_5,
\end{aligned} \tag{4.5.6}$$

where $c_i,\ i = 1, \ldots, 6$ are arbitrary constants. The solution set (4.5.6) of the determining equations (4.5.5) yields six sets of multipliers:

$$\Lambda^1 \quad : \quad \begin{cases} \Lambda^1_1 = \dfrac{u^2v}{2} + \dfrac{uv_x}{6} - \dfrac{u_xv}{3} + \dfrac{v^2}{2} + \dfrac{v_t}{3}, \\ \Lambda^1_2 = \dfrac{u^3}{6} - \dfrac{uu_x}{6} + uv - \dfrac{u_t}{3} + \dfrac{v_x}{3}, \end{cases} \tag{4.5.7}$$

$$\Lambda^2 \quad : \quad \begin{cases} \Lambda^2_1 = uv + \dfrac{v_x}{2}, \\ \Lambda^2_2 = \dfrac{u^2}{2} - \dfrac{u_x}{2} + v, \end{cases} \tag{4.5.8}$$

$$\Lambda^3 \quad : \quad \begin{cases} \Lambda^3_1 = tv, \\ \Lambda^3_2 = tu + x, \end{cases} \tag{4.5.9}$$

$$\Lambda^4 \quad : \quad \begin{cases} \Lambda^4_1 = v, \\ \Lambda^4_2 = u, \end{cases} \tag{4.5.10}$$

$$\Lambda^5 \quad : \quad \begin{cases} \Lambda^5_1 = 0, \\ \Lambda^5_2 = 1, \end{cases} \tag{4.5.11}$$

$$\Lambda^6 \quad : \quad \begin{cases} \Lambda^6_1 = 1, \\ \Lambda^6_2 = 0. \end{cases} \tag{4.5.12}$$

The conserved vectors of (4.5.1) corresponding to the multipliers (4.5.6) are:

$$T_1 \quad : \quad \begin{cases} T_1^t = -\frac{1}{2}tv^2v_x + \frac{u^3v}{6} + \frac{u^2v_x}{4} + \frac{uv^2}{2} - \frac{u_xv_x}{6}, \\ T_1^x = \frac{1}{2}tv^2v_t - \frac{u^4v}{6} - \frac{u^3v_x}{12} + \frac{1}{4}u^2u_xv - \frac{3u^2v^2}{4} - \frac{u^2v_t}{4} + \frac{uu_xv_x}{12} \\ \qquad - \frac{uvv_x}{2} + \frac{u_tv_x}{6} - \frac{{u_x}^2v}{12} + \frac{u_xv^2}{4} + \frac{u_xv_t}{6} - \frac{{v_x}^2}{12}, \end{cases} \tag{4.5.13}$$

$$T_2 \quad : \quad \begin{cases} T_2^t = \frac{u^2v}{2} + \frac{uv_x}{2} + \frac{v^2}{2}, \\ T_2^x = -\frac{u^3v}{2} - \frac{u^2v_x}{4} + \frac{uu_xv}{2} - uv^2 - \frac{uv_t}{2} + \frac{u_xv_x}{4} - \frac{vv_x}{2}, \end{cases} \tag{4.5.14}$$

$$T_3 \quad : \quad \begin{cases} T_3^t = -\frac{1}{2}t^2vv_x + tuv + \frac{tv_x}{2} + vx, \\ T_3^x = \frac{1}{2}t^2vv_t - tu^2v - \frac{tuv_x}{2} + \frac{tu_xv}{2} - \frac{tv_t}{2} - uvx - \frac{v_xx}{2}, \end{cases} \tag{4.5.15}$$

$$T_4 \quad : \quad \begin{cases} T_4^t = uv - tvv_x, \\ T_4^x = tvv_t - u^2v - \frac{uv_x}{2} + \frac{u_xv}{2}, \end{cases} \tag{4.5.16}$$

$$T_5 \quad : \quad \begin{cases} T_5^t = v, \\ T_5^x = -uv - \frac{v_x}{2}, \end{cases} \tag{4.5.17}$$

$$T_6 \quad : \quad \begin{cases} T_6^t = u - tv_x, \\ T_6^x = tv_t - \frac{u^2}{2} + \frac{u_x}{2}. \end{cases} \tag{4.5.18}$$

4.5.3 Double Reduction of the system (4.5.1)

The associations between the symmetries (4.5.2) and conservation laws of the BK system are established using (1.4.1) and presented in Table 4.5.1.

$[T_i, X_j]$	T_1	T_2	T_3	T_4	T_5	T_6
X_1		✓			✓	
X_2			✓			✓
X_3	✓	✓		✓	✓	✓
X_4			✓		✓	

Table 4.5.1: Association of Symmetries and Conservation Laws

Case 1: Reduction of (4.5.1) using $\langle X_1 \rangle$

The symmetry X_1 is associated with the conservation laws T_2 and T_5 (see Table 4.3.1). The generator X_1 has a canonical form $\widetilde{X}_1 = \partial/\partial s$ when

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{0} = \frac{dv}{0} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0} = \frac{dq}{0}. \tag{4.5.19}$$

The characteristic equation (4.5.19) yields the canonical variables:

$$\begin{aligned} r &= x, \\ s &= t, \\ w(r) &= u, \\ q(r) &= v. \end{aligned} \tag{4.5.20}$$

The inverse canonical coordinates are given by:

$$\begin{aligned} t &= s, \\ x &= r, \\ u &= w \\ v &= q. \end{aligned} \tag{4.5.21}$$

The first-order partial derivatives u_x, v_x, and v_t in terms of the canonical coordinates (4.5.20) are given by:

$$\begin{aligned} u_x &= w_r, \\ v_t &= 0, \\ v_x &= q_r. \end{aligned} \tag{4.5.22}$$

According to Theorem 1.4.2, the conservation laws T_2 and T_5 are transformed under the canonical variable (4.5.20) into

$$\begin{pmatrix} T^r \\ T^s \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T^t \\ T^x \end{pmatrix}, \tag{4.5.23}$$

where

$$A = \begin{pmatrix} D_r t & D_r x \\ D_s t & D_s x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\left(A^{-1}\right)^T = \begin{pmatrix} D_t r & D_x r \\ D_t s & D_x s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and

$$J = \det(A) = -1.$$

Therefore, the reduced conserved vectors T_2 and T_5 are:

$$\widetilde{T_2} \quad : \quad \begin{cases} T_2^r = \dfrac{1}{4}(2qw + q_r)\left(2q + w^2 - w_r\right), \\ T_2^s = -\dfrac{1}{2}\left(q^2 + qw^2 + wq_r\right), \end{cases} \tag{4.5.24}$$

$$\widetilde{T_5} \quad : \quad \begin{cases} T_5^r = qw + \dfrac{q_r}{2}, \\ T_5^s = -q, \end{cases} \tag{4.5.25}$$

where the reduced conserved vectors $\widetilde{T}_2$ and $\widetilde{T}_5$ also satisfies

$$D_r T^r = 0.$$

The second step of double reduction is given by the following r-components of the reduced conserved vectors:

$$\begin{aligned} \frac{1}{4}(2qw+q_r)\left(2q+w^2-w_r\right) &= d_1, \\ qw+\frac{q_r}{2} &= d_2, \end{aligned} \tag{4.5.26}$$

where d_1 and d_2 are arbitrary constants.

If we set $d_1 = 0$ in equation (4.5.26) and solve for q in terms w, we obtain

$$q = \frac{1}{2}\left(w_r - w^2\right). \tag{4.5.27}$$

Substituting (4.5.27) in the second equation of (4.5.26), we obtain the second-order ordinary differential equation (ODE)

$$w_{rr} = 2\left(d_2 + w^3\right). \tag{4.5.28}$$

If we set $d_2 = 0$, equation (4.5.28) is reduced to the equation

$$w_{rr} = 2w^3, \tag{4.5.29}$$

which admits the two symmetries

$$\begin{aligned} Z_1 &= \partial_r \\ Z_2 &= r\partial_r - w\partial_w, \end{aligned} \tag{4.5.30}$$

for which the Lie bracket is

$$[Z_1, Z_2] = Z_1. \tag{4.5.31}$$

Therefore, we can use the method of differential invariants [7, 116] to integrate (4.5.29).

Taking the symmetry Z_1 for which first extension is

$$Z^{(1)} = 1 \cdot \partial_r + 0 \cdot \partial_w + 0 \cdot \partial_{w_r} \tag{4.5.32}$$

one easily solves the associated characteristic system

$$\frac{dr}{1} = \frac{dw}{0} = \frac{dw_r}{0} \tag{4.5.33}$$

to find the invariants

$$p = w, \quad q = w_r. \tag{4.5.34}$$

We now take p and q as the new independent variable and dependent variable, respectively. We now have

$$\frac{dq}{dp} = \frac{D_r q}{D_r p} = \frac{w_{rr}}{w_r}, \tag{4.5.35}$$

where D_r is the total differentiation operator defined by

$$D_r = \frac{\partial}{\partial r} + w_r \frac{\partial}{\partial w} + w_{rr} \frac{\partial}{\partial w_r} + w_{rrr} \frac{\partial}{\partial w_{rr}} + \cdots \tag{4.5.36}$$

The equation (4.5.35) reduces to the first order ODE

$$\frac{dq}{dp} = \frac{2p^2}{q} \tag{4.5.37}$$

when (4.5.29) is taken into account. The ODE (4.5.37) admits Z_2 transormed into the variables p and q via (4.5.34). We integrate the variables-separable ODE (4.5.37) and obtain

$$q^2 = p^4 + J, \tag{4.5.38}$$

where J is an arbitrary constant. In terms of r and w, as defined in (4.5.34), and setting $J=0$, equation (4.5.38) becomes

$$w_r = w^2, \tag{4.5.39}$$

the solution of which is

$$w^2 = \frac{1}{(\kappa+r)^2}, \tag{4.5.40}$$

or

$$w = \pm\frac{1}{\kappa+r}, \tag{4.5.41}$$

where κ is an arbitrary constant. The following solution of the BK system (4.5.1) follows from (4.5.41), (4.5.27) and (4.5.20):

$$\begin{aligned} u(t,x) &= \pm\frac{1}{\kappa+x}, &(4.5.42)\\ v(t,x) &= \frac{1}{2}\left(\frac{\partial u(t,x)}{\partial x} - u(t,x)^2\right). &(4.5.43)\end{aligned}$$

Case 2: Reduction of (4.5.1) using $\langle X_2\rangle$

In this case, the symmetry X_2 is associated with the conservation laws T_3 and T_6 (see Table 4.3.1). The generator X_2 provides a distinct set of canonical variables, defined as:

$$\begin{aligned} r &= \frac{x}{\sqrt{t}},\\ s &= \frac{\ln t}{2},\\ w(r) &= u\sqrt{t},\\ q(r) &= tv. \end{aligned} \tag{4.5.44}$$

The inverse canonical coordinates are given by:

$$
\begin{aligned}
t &= e^{2s},\\
x &= re^{s},\\
u &= -e^{-s}w,\\
v &= qe^{-2s}.
\end{aligned}
\tag{4.5.45}
$$

The expressions for the first-order partial derivatives u_x, v_x, and v_t in terms of these canonical variables are:

$$
\begin{aligned}
u_x &= -e^{-2s}w_r,\\
v_t &= -\frac{1}{2}e^{-4s}(2q+q_r r),\\
v_x &= q_r e^{-3s}.
\end{aligned}
\tag{4.5.46}
$$

Utilizing Theorem 1.4.2, the conserved vectors T_3 and T_6 transform under the canonical variables (4.5.44) as follows:

$$
\widetilde{T}_3 \quad : \quad \begin{cases} T_3^r = q^2+qr^2-3qrw+2qw^2+qw_r-q+q_r r-q_r w,\\ T_3^s = \dfrac{qq_r}{2}-qr+qw-\dfrac{q_r}{2}. \end{cases} \tag{4.5.47}
$$

$$
\widetilde{T}_6 \quad : \quad \begin{cases} T_6^r = 2q-rw+w^2+w_r,\\ T_6^s = q_r+w. \end{cases} \tag{4.5.48}
$$

Therefore, from the reduced conservation laws $D_r T_3^r = 0$ and $D_r T_6^r = 0$, we obtain the following equations for w and q :

$$
\begin{aligned}
qq_r-2qr+2qw-q_r &= d_1,\\
2q-rw+w^2+w_r &= d_2,
\end{aligned}
\tag{4.5.49}
$$

where d_1 and d_2 are arbitrary constants. It follows from (4.5.49) that w is a solution of the second-order ODE

$$\begin{aligned}
&d_2w-d_2r-\frac{1}{4}d_2w_{rr}-\frac{1}{2}d_2ww_r+\frac{1}{4}d_2rw_r+\frac{1}{4}w^2w_{rr}-\frac{1}{4}rww_{rr}+\frac{w_{rr}}{2}\\
&+\frac{1}{4}r^2ww_r+\frac{1}{2}w^3w_r-\frac{3}{4}rw^2w_r+\frac{1}{2}ww_r^2-\frac{1}{4}rw_r^2+\frac{1}{2}rw_r\\
&+\frac{1}{4}w_rw_{rr}-r^2w-w^3+2rw^2=d_1,
\end{aligned} \tag{4.5.50}$$

and q is given by

$$q(r)=\frac{1}{2}\left(d_2+rw-w^2-w_r\right). \tag{4.5.51}$$

Therefore, a solution of the BK system (4.5.1) arising from X_2 via T_3 and T_6 is

$$u(t,x) = \frac{1}{\sqrt{t}}w\left(x/\sqrt{t}\right), \tag{4.5.52}$$

$$v(t,x) = \frac{1}{t}q\left(x/\sqrt{t}\right), \tag{4.5.53}$$

where w is a solution of (4.5.50) and q is qiven by (4.5.51).

Case 3: Reduction of (4.5.1) using $\langle X_3\rangle$

The symmetry X_3 is associated with all the conservation laws of the BK system (4.5.1), except for T_3 (as shown in Table 4.3.1). The generator X_3 yields the following canonical variables:

$$\begin{aligned}
r&=t,\\
s&=x,\\
w(r)&=u,\\
q(r)&=v.
\end{aligned} \tag{4.5.54}$$

The inverse canonical coordinates are given by:

$$
\begin{aligned}
t &= r, \\
x &= s, \\
u &= w, \\
v &= q.
\end{aligned}
\tag{4.5.55}
$$

The expressions for the first-order partial derivatives u_x, u_t, v_x, and v_t in terms of these canonical variables are:

$$
\begin{aligned}
u_t &= w_r, \\
u_x &= 0, \\
v_t &= q_r, \\
v_x &= 0.
\end{aligned}
\tag{4.5.56}
$$

Applying Theorem 1.4.2 with these canonical variables (4.5.54), the conserved vectors T_1, T_2, T_4, T_5, and T_6 reduce as follows:

$$
\widetilde{T_1} \quad : \quad \begin{cases} T_1^r = \dfrac{q^2 w}{2} + \dfrac{q w^3}{6}, \\ T_1^s = \dfrac{1}{2} q^2 q_r r - \dfrac{3 q^2 w^2}{4} - \dfrac{q w^4}{6} - \dfrac{q_r w^2}{4}. \end{cases} \tag{4.5.57}
$$

$$
\widetilde{T_2} \quad : \quad \begin{cases} T_2^r = \dfrac{q^2}{2} + \dfrac{q w^2}{2}, \\ T_2^s = -q^2 w - \dfrac{q w^3}{2} - \dfrac{q_r w}{2}, \end{cases} \tag{4.5.58}
$$

$$
\widetilde{T_4} \quad : \quad \begin{cases} T_4^r = q w, \\ T_4^s = q q_r r - q w^2, \end{cases} \tag{4.5.59}
$$

$$
\widetilde{T_5} \quad : \quad \begin{cases} T_5^r = q, \\ T_5^s = -q w, \end{cases} \tag{4.5.60}
$$

$$
\widetilde{T_6} \quad : \quad \begin{cases} T_6^r = w, \\ T_6^s = q_r r - \dfrac{w^2}{2}, \end{cases} \tag{4.5.61}
$$

From the reduced conservation laws $D_rT_i^r = 0$, $i = 1,2,4,5,6$, we obtain the following equations for w and q :

$$\begin{aligned}
3q^2w + qw^3 &= d_1,\\
q^2 + qw^2 &= d_2,\\
qw &= d_4,\\
q &= d_5,\\
w &= d_6.
\end{aligned} \tag{4.5.62}$$

where d_1, d_2, d_4, d_5, and d_6 are arbitrary constants. The only solution of the BK system (4.5.1) in this case is the trivial solution in which both u and v are constant.

Case 4: Reduction of (4.5.1) using $\langle X_4 \rangle$

The symmetry X_4 is associated with the conservation laws T_3 and T_5 (as shown in Table 4.3.1). The generator X_4 leads to the following canonical variables:

$$\begin{aligned}
r &= t,\\
s &= \frac{x}{t},\\
w(r) &= u + \frac{x}{t},\\
q(r) &= v.
\end{aligned} \tag{4.5.63}$$

The inverse transformation to revert to the original coordinates is given by:

$$\begin{aligned}
t &= r,\\
x &= rs,\\
u &= w - s,\\
v &= q.
\end{aligned} \tag{4.5.64}$$

The first-order partial derivatives u_x, v_x, and v_t in terms of these canonical variables are:

$$\begin{aligned} u_x &= -\frac{1}{r}, \\ v_t &= q_r, \\ v_x &= 0. \end{aligned} \tag{4.5.65}$$

Applying Theorem 1.4.2 with these canonical variables (4.5.63), the conserved vectors T_3 and T_5 reduce as follows:

$$\widetilde{T}_3 \quad : \quad \begin{cases} T_3^r = qr^2 w, \\ T_3^s = \frac{1}{2}qq_r r^2 - qrw^2 - \frac{q}{2} - \frac{q_r r}{2}, \end{cases} \tag{4.5.66}$$

$$\widetilde{T}_5 \quad : \quad \begin{cases} T_5^r = 2qr, \\ T_5^s = -qw. \end{cases} \tag{4.5.67}$$

From the reduced conservation laws $D_r T_3^r = 0$ and $D_r T_5^r = 0$, we obtain the following equations for w and q :

$$qr^2 w = d_1, \tag{4.5.68}$$

$$qr = d_2, \tag{4.5.69}$$

where d_1 and d_2 are arbitrary constants.

Solving the system of equations (4.5.68) and (4.5.69) simultaneously for w and q and using (4.5.63) yields the solution

$$u(t,x) = \frac{d_1}{d_2 t} - \frac{x}{t}, \tag{4.5.70}$$

$$v(t,x) = \frac{d_2}{t}, \tag{4.5.71}$$

for the BK system (4.5.1).

4.5.4 Concluding remarks

In this section, we have applied the generalized double reduction theory to the $(1+1)$-dimensional Broer-Kaup system, a PDE model that arises in the study of long wave propagation in shallow water. We computed four Lie point symmetries of the Broer-Kaup system, and six local conservation laws, derived from low-order multipliers. Notably, each Lie point symmetry was found to be associated with at least two conservation laws. Utilizing these symmetries and the associated conservation laws, we were able to perform double reductions of the system, reducing the original PDE system to systems of first-order ordinary differential equations or algebraic equations in certain scenarios. These reductions facilitate the derivation of explicit solutions to the Broer-Kaup system demonstrating the practical applicability of the generalized double reduction method to nonlinear PDEs, particularly in the context of physical models like the Broer-Kaup system. Overall, our findings contribute to the broader understanding and application of the generalized double reduction method.

Chapter 5

Summary, Conclusions and Future Work

This thesis systematically applied the double reduction method, in both its standard and generalized forms, to a diverse set of partial differential equations (PDEs), offering significant contributions to the theoretical and practical understanding of these techniques. The results presented in this study highlight the power and versatility of symmetry-based methods in deriving exact solutions for complex nonlinear systems. The research began with the application of the double reduction method to the GT equation, a key equation in the theory of dispersionless integrable systems. By employing the multiplier method, conservation laws were constructed without the need for determining all possible multipliers, focusing instead on those with polynomial structures. The reduction process led to first-order ordinary differential equations (ODEs), ultimately yielding two general solutions of the GT equation. The method was further applied to the HS equation, where symmetry reductions using conservation laws derived through the multiplier method resulted in invariant solutions. Insights were gained from examining the collapse of first integrals in specific cases, enhancing our understanding of the method's applicability. Additionally, this research advanced the standard double reduction method by demonstrating that reductions could be achieved using only the canonical variables of the symmetries associated with conservation laws. This alternative approach, applied to five (1+1)-dimensional scalar PDEs, simplified the reduction process and expanded the method's applicability.vThis work also extended the generalized double reduction method to higher-dimensional systems, such as the nonlinear wave equation and the ZK equation, where symmetry multi-reductions were performed. In these cases, inherited symmetries were established and used, highlighting the method's adaptability to more complex PDEs. The generalized double reduction method was also successfully applied to the Kaup-Boussinesq and Broer-Kaup systems, both of which are significant in nonlinear wave propagation studies. The identification of Lie point symmetries and conservation laws facilitated the reduction of these systems to simpler ODEs or algebraic equations, in some cases leading to explicit solutions. This demonstrated the practical applicability of the method to real-world physical models.

In conclusion, this thesis has substantially advanced the application and understanding of the double reduction method. It has provided clear and illustrative examples across a variety of PDEs, significantly contributing to the broader literature by offering new methodologies for deriving exact solutions. The findings underscore the method's flexibility and utility in reducing nonlinear systems and highlight its potential for more complex scenarios. While the focus of this research was primarily on (n+1)-dimensional scalar PDEs and systems of PDEs, the results reinforce the value of symmetry-based reduction methods in theoretical and applied mathematics. The study has also demonstrated the potential of the alternative double reduction approach, which does not rely explicitly on conservation laws, broadening the scope of the method's applicability. This approach, along with the generalized double reduction theory, provides new tools for addressing higher-dimensional and more complex systems.

Looking ahead, there are several promising directions for future research. While this thesis has laid the groundwork for applying the double reduction method to (n+1)-dimensional scalar PDEs and systems, several avenues remain unexplored. One potential direction is the application of the method to systems of PDEs consisting of a PDE and its adjoint. The exploration of nonclassical symmetries, which could enable further double reductions in nonlinear scalar

PDEs, also presents an exciting opportunity for future work. Additionally, a comprehensive comparison between the alternative double reduction approach and the standard approach, particularly with regard to computational efficiency and broader applicability to various types of PDEs, is an area for further investigation.Moreover, there is a need for a deeper exploration of the physical interpretations of the symmetries and conservation laws derived in this study, particularly in the context of nonlinear wave propagation and other physical phenomena. Such a discussion would strengthen the connection between the mathematical findings and real-world applications. Extending the research to include higher-dimensional systems and examples from diverse physical contexts, such as fluid dynamics, plasma physics, and elasticity theory, would further demonstrate the generality and power of the double reduction method.

Finally, open problems such as extending the method to more complex systems of PDEs, exploring its application in higher-dimensional settings, and addressing the unsolved challenges in applying the method to certain nonlinear equations, offer a rich terrain for further research. This thesis, while contributing significantly to the field, lays the foundation for future investigations aimed at expanding the applicability of the double reduction method and deepening its theoretical underpinnings.

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