A non-perturbative theory of giant gravitons using AdS/CFT

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Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the Degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

(Signature of candidate)

6th of February, 2015

(Date)

Abstract

We explore the non-perturbative physics of giant gravitons in type IIB string theory on the $\mathrm{AdS}_5 \times S^5$ background in this thesis. The gauge theory dual is $\mathcal{N} = 4$ super Yang-Mills theory with a $U(N)$ gauge group. We diagonalise the one and two-loop dilatation operators acting on the restricted Schur polynomial basis. These operators are dual to a system of giant gravitons with strings attached. Hence, we present evidence for integrability in certain non-planar sectors of the gauge theory.

In the second half of the thesis, we turn our focus to $\mathcal{N} = 4$ super Yang-Mills theory with an $SO(N)$ gauge group. In this case, the geometry of the dual gravity theory is $\mathrm{AdS}_5 \times \mathbb{R}P^5$. The non-planar physics of the $SO(N)$ theory is distinct from that of the $U(N)$ theory. To pursue the goal of searching for non-planar integrability in the $SO(N)$ gauge theory, one might try to generalise the restricted Schur basis to the $SO(N)$ case. We propose such a basis and evaluate their two-point functions exactly in the free theory. Further, we develop techniques to compute correlation functions of multi-trace operators involving two scalar fields exactly. Lastly, we extend these results to the theory with an $Sp(N)$ gauge group.

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Contents

Preface

- Chapter 2 of this thesis is based on the work entitled ‘From large N nonplanar anomalous dimensions to open spring theory’ which was published in Physics Letters B, Volume 711: 398-403, 2012.

- Chapter 3 is based on the work entitled ‘Non-planar integrability at two loops’ and was published in the Journal of High Energy Physics (JHEP), 1210 (2012) 144.

- Chapter 4, ‘$SO(N)$ restricted Schur polynomials’, is based on work that has been accepted for publication in the Journal of Mathematical Physics (JMP). The ArXiv number for this work is 1405.7017.

- Finally, chapter 5, ‘Restricted Schurs and correlators for $SO(N)$ and $Sp(N)$’, was published in the Journal of High Energy Physics (JHEP), 1408 (2014), 137.

Chapter 1

Introduction: Non-perturbative quantum gravity

1.1 Setting the scene

This thesis is primarily interested in non-perturbative quantum gravity. Traditionally, quantum field theories are studied perturbatively using Feynman diagrams. The coupling of the theory is considered to be small and then expansions are performed in the coupling. Generally, the higher the power of the coupling, the more complicated the Feynman diagrams become. Furthermore, perturbative expansions for QED and QCD are asymptotic, as we will explain. So even perturbation theory breaks down after a while. Therefore, finding non-perturbative descriptions of quantum field theory is of paramount importance. Indeed, non-perturbative techniques seem to be necessary for theories like low energy QCD. In fact, one of the most challenging open problems in physics today is finding a low energy description of QCD. If, for example, a correlation function is non-perturbative we mean that it cannot be expanded as a power-series in the coupling. In this thesis, we will use group representation theory to evaluate certain correlation functions exactly. We will study a gauge theory with two couplings or expansion parameters. Its fields are $N \times N$ matrices, with $1/N$ being one of the parameters. We then succeed in calculating correlation functions exactly in $1/N$. The other coupling is the 't Hooft coupling, λ. For very supersymmetric settings, our results are exact in λ as well. This is because the large amount of supersymmetry prevents the correlation functions from being corrected with λ.

Being able to compute correlation functions exactly becomes especially interesting when the gauge theory has a gravity dual. According to AdS/CFT, every operator in the field theory must have an interpretation in the dual gravity theory. Thus, by studying the physics of the operators, one is also studying the physics of the dual objects in the gravity theory. Being able to compute things exactly is especially useful if these objects are inherently non-perturbative. Indeed, such is the nature of the objects we study in this thesis. We study particular D-brane states called giant gravitons living in the $\mathrm{AdS}_5 \times S^5$ spacetime. To see that D-branes are non-perturbative, note that the Born-Infeld action for D-branes is proportional to the brane tension. The tension itself is proportional to $1/g_s$, with g_s being the string coupling. Now, in perturbation theory g_s would be small, making the D-brane tension very large. Thus, the D-brane would be very massive, and consequently non-perturbative.

Further, when considering correlation functions involving giant gravitons, one simply cannot include quantum corrections perturbatively, as we will explain. In this thesis, we capture the full quantum theory of giant gravitons using group representation theory. One of the main motivations for our studies is to obtain a better understanding of the duality between excited giant gravitons and their proposed duals, the restricted Schur polynomials. Using representation theory to construct these operators, we have managed to capture the non-perturbative physics of these operators in the $1/N$ parameter.

The gauge theory we focus on has conformal symmetry. Having calculated the two-point functions of the restricted Schur polynomials exactly in the free theory, it is then natural to compute their anomalous dimensions. We then 'switch' on the other coupling, the 't Hooft coupling λ, and compute the spectrum of anomalous dimensions perturbatively in λ. By calculating anomalous dimensions of something in the gauge theory, one is calculating the excitation energy spectrum of the corresponding state in the dual theory. In this way, we probe the detailed relationship between restricted Schur polynomials and excited giant gravitons.

Calculating anomalous dimensions amounts to solving an eigenvalue problem. In the gauge theory, the anomalous dimensions are the eigenvalues of the dilatation operator, the generator of one of the conformal symmetries. We focus on restricted Schur polynomials describing a system of separated giants with strings attached. Acting on such operators, we have managed to diagonalise the dilation operator and solve for the anomalous dimensions. As we will explain in later chapters, the results we have obtained are startling. They display strong evidence for integrability. Before, it was widely thought that integrability was restricted to only a very simple regime of the gauge theory. Now, we claim that integrability persists in a much more general setting. This is what we will try show in this thesis. We will discuss the representation theory that made this all possible, we will show how we constructed our operators and how we solved for their anomalous dimensions.

AdS/CFT, which we formally introduce in the next section, forms the central theme in this thesis. It is the very non-trivial statement that a gauge theory without gravity in d dimensions is equivalent to a string theory, with gravity, in $d+1$ dimensions. There is a tremendous amount of literature supporting it. Before AdS/CFT, quantum field theories were considered to be only effective theories - valid up to some scale where string theory would take over as the most fundamental theory. AdS/CFT tells us that we can again consider quantum field theory as the most fundamental theory. Ideally, one would like to reap all the fruits AdS/CFT has to offer, not just in some special limits of the theory. The work presented in this thesis is thus an important contribution to the AdS/CFT picture and offers a new piece to this great puzzle.

1.2 The AdS/CFT correspondence

In this section, we briefly motivate that 3+1 dimensional $\mathcal{N} = 4$ SYM theory with $U(N)$ gauge group is equivalent to type IIB string theory on the $\mathrm{AdS}_5 \times S^5$ background. We begin by presenting some background information on AdS spacetime which is then followed by the motivation for the AdS/CFT correspondence.

1.2.1 AdS Space

Anti-de-Sitter spacetime is the maximally symmetric Lorentzian manifold with constant negative curvature. Maximally symmetric space means that the space admits the maximum number of isometries or Killing vector fields. For a d-dimensional space to be maximally symmetric, it must have $d(d+1)/2$ symmetries. This is indeed the case for AdS space. A $d+1$ dimensional AdS space has an $SO(2,d)$ isometry group consisting of $(d+1)(d+2)/2$ generators.

Consider a $d+1$ dimensional hyperboloid,

$$(X^0)^2 + (X^{d+1})^2 - \sum_{i=1}^{d}(X^i)^2 = R^2, \tag{1.1}$$

in a flat $(d+2)$-dimensional space with the following metric in the embedding coordinates

$$ds^2 = -dX^0 dX^0 - dX^{d+1} dX^{d+1} + \sum_{i=1}^{d} dX^i dX^i. \tag{1.2}$$

By setting

$$\begin{aligned} X^0 &= R\cosh(\rho)\cos(\tau), \qquad X^{d+1} = R\cosh(\rho)\sin(\tau), \\ X^i &= R\sinh(\rho)\omega_i, \end{aligned} \tag{1.3}$$

where

$$\begin{aligned} \omega_1 &= \cos(\theta_1), \\ \omega_2 &= \sin(\theta_1)\cos(\theta_2), \\ &\vdots \\ \omega_{d-1} &= \sin(\theta_1)\sin(\theta_2)\cdots\cos(\theta_{d-1}), \\ \omega_d &= \sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{d-1}), \end{aligned} \tag{1.4}$$

the metric can be rewritten as

$$ds^2 = R^2 d\rho^2 - R^2\cosh^2(\rho)d\tau^2 + R^2\sinh^2(\rho)(d\Omega_{d-1})^2. \tag{1.5}$$

This is AdS_{d+1} in *global* coordinates, ρ, τ and Ω_i. For $0 \leqslant \rho$ and time $\tau \in [0, 2\pi)$, these coordinates cover the hyperboloid exactly once. Close to $\rho = 0$, the metric looks like $S^1 \times \mathbb{R}^d$, with τ coordinate corresponding to the S^1. Thus, τ describes a family of S^1's around the hyperboloid. To get rid of these closed time-like curves, one can unwrap the circle by taking $-\infty < \tau < \infty$. Here is another useful reparameterization of the metric. Define

$$X^\mu = \frac{r x^\mu}{R}, \qquad X^d + X^{d+1} = r, \qquad X^d - X^{d+1} = v. \tag{1.6}$$

After some algebra, the metric in (1.2) becomes

$$ds^2 = \frac{r^2}{R^2} dx \cdot dx + \frac{R^2}{r^2} dr^2. \tag{1.7}$$

This is the metric of AdS in the so-called Poincarè patch. For $z = (R^2)/r$, (1.7) becomes

$$ds^2 = \frac{R^2}{z^2}(dz^2 + dx \cdot dx). \tag{1.8}$$

Finally, AdS space solves the Einstein equations in a vacuum for a negative cosmological constant Λ, given by

$$\Lambda_{AdS_{d+1}} = -\frac{d(d-1)}{r^2}. \tag{1.9}$$

1.2.2 Motivation for the correspondence

Begin with type IIB string theory in flat ten-dimensional space-time. Next, consider N D3-branes stacked on top of each other so that they are coincident. We choose D3-branes because we live in a 3+1 dimensional spacetime. An open string will have its end-points stuck onto any one of the N branes. Say we have a string with one end-point stuck to brane i and the other on brane j. In the limit of large string tension, the end-points want to be very close together. This situation is also thought to describe a field with colour indices i and j. This is how we generate fields in the adjoint of a non-abelian $U(N)$ gauge theory [1]. Consider a scattering of strings on this stack of branes. In the limit of large string tension, the interactions of the open strings on the branes reproduce the interactions of a non-abelian gauge theory with N colours. At low energies, say lower than the string scale $1/l_s$, only the massless string states can be excited. The effective (effective in the Wilsonian sense) Lagrangian for this theory is

$$\mathcal{L} = \mathcal{L}_{Brane} + \mathcal{L}_{Int} + \mathcal{L}_{Bulk}, \tag{1.10}$$

where all the massive modes have been integrated out. Here, $\mathcal{L}_{Brane}$ is the Lagrangian for $\mathcal{N} = 4$ $U(N)$ super Yang-Mills plus higher derivative corrections, proportional to powers of α'. $\mathcal{L}_{Bulk}$ is the Lagrangian for ten-dimensional supergravity plus higher derivative corrections also proportional to powers of α'. $\mathcal{L}_{Int}$ describes interactions between the branes and the bulk. Now take the low energy limit. More precisely, we mean the dimensionless quantity $\alpha'\vec{k}^2$ is much less than 1, where $\vec{k}$ is momentum. In practise, however, this amounts to sending $\alpha' \to 0$. The higher derivative corrections, as well as $\mathcal{L}_{Int}$ vanish, leaving two decoupled Lagrangians - one for the

3+1 dimensional gauge theory living on the branes and one for the 10-dimensional supergravity in the bulk.

Now consider the same system from a different point of view. The D3-brane system may also be seen as massive charged objects giving rise to a non-trivial background. What kind of background does this D-brane system produce? Symmetries determine the metric up to a coefficient for the $(-dt^2 + d\vec{x} \cdot d\vec{x})$ piece, corresponding to the brane, and a coefficient for the $dx^M dx_M$ piece, corresponding to the bulk. Symmetry again determines that these coefficients depend only on the radial coordinate $r = \sqrt{x^M x_M}$ from the brane. After computing the energy-momentum tensor, one may then solve the Einstein equations. Demanding the metric to reduce to flat space far from the brane gives

$$ds^2 = \frac{1}{f(r)}\eta_{\mu\nu}dx^\mu dx^\nu + f(r)dx^M dx_M, \qquad f(r) = \sqrt{1 + \frac{R^4}{r^4}}, \qquad R^4 \equiv g_s N 4\pi\alpha'^2, \tag{1.11}$$

where $\mu = 0, 1, 2, 3,\ M = 4, ..., 9$. Far away from the brane, $r \gg R$, the metric becomes that of flat 10-dimensional Minkowski space

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + dx^M dx_M. \tag{1.12}$$

Close to the brane, $r \ll R$, the metric, in spherical coordinates, becomes

$$ds^2 = \frac{r^2}{R^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{R^2}{r^2}dr^2 + R^2 d\Omega^2_{S^5}. \tag{1.13}$$

Thus, close to the branes, space-time is $\mathrm{AdS}_5 \times S^5$ (see equation (1.7)). We can now ask what are the lowest energy excitations an observer at infinity will see? The relation between the energy of an object as seen by an observer at some distance r from the branes and by an observer at infinity is

$$E_\infty = \frac{1}{1 + (R^2/r^2)} E_r. \tag{1.14}$$

Thus, any mode approaching the branes will appear to have lower and lower energy according to an observer at infinity. Thus, such an observer will see 10-dimensional supergravity in the bulk, and full type IIB string theory in the region that is asymptotically close to the branes, where the space-time is $\mathrm{AdS}_5 \times S^5$. Furthermore, these two are decoupled from each other.

Hence, in the low energy limit, we identify the 10D supergravity in both descriptions and make the conjecture that $\mathcal{N} = 4$ super Yang-Mills with $U(N)$ gauge group in 3+1-dimensions is equivalent to type IIB string theory on $\mathrm{AdS}_5 \times S^5$. This discussion was for the low energy limit. However, the duality is thought to hold in general. For any object one can define on the one side of the correspondence, there should be a dual object on the other side. One's job now is to work out the dictionary between the two theories. We turn to this topic next.

1.3 The AdS/CFT dictionary

The main purpose of this section is to discuss the conjecture that Schur polynomials with scaling dimensions $O(N)$ in the large N limit are dual to giant gravitons, and that restricted Schur polynomials are dual to excited giant gravitons. First, we discuss a key idea we will need for our goal - the operator-state correspondence in

AdS/CFT. We also classify the degrees of freedom of the string theory according to $\mathcal{R}$-charge of the dual gauge theory operators before our Schur polynomial discussion.

1.3.1 Operator-state correspondence in AdS/CFT

Begin with $\mathcal{N} = 4$ super Yang-Mills theory in four-dimensional Minkowski space. Performing a Wick rotation gives the field theory in four-dimensional Euclidean space $\mathbb{R}^4$. We can parameterize $\mathbb{R}^4$ using spherical coordinates, and hence write the metric as

$$ds^2 = dr^2 + r^2 d\Omega_3^2. \tag{1.15}$$

Now perform the following change of coordinates. Let $r = e^\tau$ so that the $\mathbb{R}^4$ metric becomes

$$ds^2 = e^{2\tau}(d\tau^2 + d\Omega_3^2). \tag{1.16}$$

This can be done for any field theory. For a conformal field theory, we can rescale the metric to obtain

$$ds^2 = d\tau^2 + d\Omega_3^2. \tag{1.17}$$

This is the metric for $\mathbb{R} \times S^3$. Space-time that was $\mathbb{R}^4$ is now a cylinder. Since $r = e^\tau$, the origin of $\mathbb{R}^4$, $r = 0$, maps to a circle on the cylinder at $\tau = -\infty$. Recall that operators in the quantum field theory are local. Locality, for example, means that a derivative of a field at some point x is related to the field at point x. In quantum field theories, locality also means requiring the commutator of fields evaluated at space-like separated points vanish. Since $\mathcal{O}(x)$ is local, it will act on some state, and the only place it can change the state is at x. If you want to do something at $r = 0$, you act with an operator $\mathcal{O}(0)$. This will map to a state $|\psi_i\rangle$ on the corresponding circle on the cylinder. One can then ask to which state will the operator $\mathcal{O}(0)$ map? To answer this, consider $\tau = 0$. This corresponds to an S^1 with $r = 1$ in $\mathbb{R}^4$. Now consider evaluating the path integral

$$\int [D\phi]\, e^{iS_{CFT}} \mathcal{O}(0). \tag{1.18}$$

Here we are integrating over the space of all fields ϕ which is determined by the boundary conditions. The boundary in this case is the unit circle. Thus, we integrate over the space of fields ϕ everywhere inside the unit disc with the following boundary conditions

$$\phi|_{S^1} = \phi_b. \tag{1.19}$$

The result of the path integral will then be some functional of the field ϕ_b

$$\int [D\phi]\, e^{iS_{CFT}} \mathcal{O}(0) = F[\phi_b], \tag{1.20}$$

where ϕ_b is defined at all points on the S^1. $F[\phi_b]$ may be seen as some state $|F\rangle$ in the basis $|\phi_b\rangle$.

On the cylinder, we start with the initial state $|\psi_i\rangle$ at $\tau = -\infty$, evolve this forwards in time, from $\tau = -\infty$ to $\tau = 0$, and require the result to be the same state

$|F\rangle$. Above, we obtained $|F\rangle$ in some basis. Here $|F\rangle$ is defined by

$$|F\rangle = T\Big(e^{-i\int_{-\infty}^{\tau} H(\tau')d\tau'}\Big)|\psi_i\rangle, \tag{1.21}$$

where T is the time ordering and the H is the Hamiltonian of the CFT. $|F\rangle$ in equation (1.21) may be rewritten as

$$|F\rangle = U(\tau)|\psi_i\rangle. \tag{1.22}$$

Solving for $|\psi_i\rangle$, we get

$$|\psi_i\rangle = U^{\dagger}(\tau)|F\rangle. \tag{1.23}$$

So, $|F\rangle$ may be obtained from the CFT on $\mathbb{R}^4$ using equation (1.20). Once we have $|F\rangle$, the state $|\psi_i\rangle$ may be obtained from (1.23). In this way, a given operator $\mathcal{O}$ in the CFT on $\mathbb{R}^4$ may be mapped to a state in the CFT on $\mathbb{R}\times S^3$. For a comprehensive discussion of the operator-state correspondence in a CFT the reader is referred to [2]. As an example, [2] shows that the unit operator maps to ground state on the cylinder.

The statement of the AdS/CFT correspondence is that quantum gravity on $\mathrm{AdS}_5 \times S^5$ is equivalent to $\mathcal{N} = 4$ SYM on $\mathbb{R}^4$ (or spaces conformal to $\mathbb{R}^4$). The boundary of $\mathrm{AdS}_5 \times S^5$ is $\mathbb{R} \times S^3$. To see this, take the metric for AdS_5 in global coordinates,

$$ds^2 = -\cosh^2(\rho)d\tau^2 + d\rho^2 + \sinh^2(\rho)d\Omega_3^2 \tag{1.24}$$

and set $\rho \gg 1$ and constant. This metric then reduces to that of $\mathbb{R} \times S^3$. The boundary of S^n vanishes. By studying how the $SO(2,4)$ isometry acts, we can identify the $\mathbb{R}$ of the $\mathrm{AdS}_5 \times S^5$ boundary with the $\mathbb{R}$ of the $\mathcal{N} = 4$ super Yang-Mills theory on $\mathbb{R} \times S^3$. This means that time on the $\mathrm{AdS}_5 \times S^5$ boundary is the same time in the field theory on $\mathbb{R} \times S^3$. For canonical quantization, one selects a time and for each time there is a Hilbert space of states. If the times on either side of the correspondence are the same we may identify the Hilbert space of the $\mathrm{AdS}_5 \times S^5$ boundary with the Hilbert space of the conformal field theory on $\mathbb{R} \times S^3$. Thus, AdS/CFT identifies a state in the quantum gravity on $\mathbb{R} \times S^3$ with a state in the CFT on $\mathbb{R} \times S^3$. Once we have this state in the CFT it can be mapped to an operator on $\mathbb{R}^4$.

1.3.2 Matching scaling dimensions with energies

$\mathcal{N} = 4$ SYM theory in four dimensions is a conformal field theory. The conformal symmetries are Poincare, scalings or dilatations and special conformal. The conformal group is $SO(2,4)$. By construction, AdS_5 has an $SO(2,4)$ isometry group. It follows that there is a one-to-one correspondence between the generators of the conformal symmetry of $\mathbb{M}^4$ and the generators of the isometry of AdS_5. For example, the dilatation operator in the CFT is associated to the Hamiltonian in AdS. To see this, consider the CFT on $\mathbb{R}^4$ again. Perform the following rescaling of r

$$r \to \lambda r. \tag{1.25}$$

With $r = e^\tau$, we can write

$$e^\tau \to \lambda e^\tau = e^{\tau + \ln \lambda}. \tag{1.26}$$

This can be viewed as a time translation on the cylinder

$$\tau \to \tau + \ln \lambda. \tag{1.27}$$

Thus, scalings in the CFT on $\mathbb{R}^4$ may be viewed as time translations in the CFT on $\mathbb{R} \times S^3$. One may therefore identify the generators of these two transformations - the dilatation operator and the Hamiltonian. Hence, one may identify their respective conserved charges - scaling dimensions and energies. Recall that AdS/CFT identified the Hilbert space of states of the CFT on $\mathbb{R} \times S^3$ with the Hilbert space of states of the string theory on the boundary of $\mathrm{AdS}_5 \times S^5$. For example, the energy of two identified states must be the same. AdS/CFT maps a state with some energy in the quantum gravity on the $\mathrm{AdS}_5 \times S^5$ boundary to some state in the CFT on $\mathbb{R} \times S^3$ with the same energy. This state may then be mapped to an operator in the CFT on $\mathbb{R}^4$ with some scaling dimension.

Furthermore, $\mathcal{N} = 4$ SYM theory contains six real scalar fields with an $SO(6) \simeq SU(4)$ symmetry. We refer to this symmetry as an $\mathcal{R}$-symmetry. It also rotates the $\mathcal{N} = 4$ supercharges amongst themselves. The conserved quantity for this symmetry is $\mathcal{R}$-charge. The $\mathcal{R}$-charge commutes with bosonic space-time symmetries but not with the fermionic ones. The $SO(6)$ symmetry must also be present in the string theory. A space with an $SO(6)$ isometry is an S^5. Denote the six scalar fields by $\Phi_i, i = 1, 2, ..., 6$. These fields are the embedding coordinates for the S^5. Combine the Φ's to form complex matrix fields

$$X = \Phi_1 + i\Phi_2, \tag{1.28}$$

$$Y = \Phi_3 + i\Phi_4, \tag{1.29}$$

$$Z = \Phi_5 + i\Phi_6. \tag{1.30}$$

Consider now acting with the $U(1)$ element $e^{-i\theta}$ on Z, say. This is a rotation in the complex $\Phi_5\Phi_6$-plane and defines a $U(1)$ conserved charge. This may also be viewed as rotation in the $\Phi_5\Phi_6$-plane of the S^5, the conserved charge of which is angular momentum.

1.3.3 Classifying the degrees of freedom

One observable that one may compute in any CFT is an operator's scaling dimension Δ. These quantities receive quantum corrections which we call *anomalous dimensions*. The scaling dimensions of 1/2-BPS operators are equal to their $\mathcal{R}$-charges with zero anomalous dimension. The supersymmetry forces these quantum corrections to vanish. Operators built out of only a single $N \times N$ scalar field are half-BPS. For example

$$\mathcal{O}_n = \mathrm{Tr}(Z^n) \tag{1.31}$$

is half-BPS. The CFT dual, type IIB string theory on $\mathrm{AdS}_5 \times S^5$, reduces to free type IIB supergravity in the ‘t Hooft and large λ limits[1]. Since $\Delta = \mathcal{R}$ for $\mathcal{O}_n$ in (1.31), its dual state in the string theory has its energy equal to its angular momentum. One

[1]See section 1.3

can think of the dual state as a massless mode propagating in the 10-dimensional supergravity theory, i.e., a graviton. In addition to gravitons the half-BPS sector of type IIB string theory on $\mathrm{AdS}_5 \times S^5$ consists of strings, branes, as well as new background geometries. If AdS/CFT is true, then all these objects must be captured in the quantum field theory. To see how this is possible, begin by noting that the graviton carries a D3 dipole charge. In the presence of a Ramond-Ramond five-form flux, this dipole charge couples to the flux thus polarizing the graviton. As the particle gains angular momentum J, the coupling of the dipole charge to the flux increases and hence the object expands in the space-time. The state expands to a radius of [3]

$$R = \sqrt{\frac{J}{N}} R_{AdS}, \qquad R^2_{\mathrm{AdS}} = \sqrt{g^2_{YM} N}\alpha'. \tag{1.32}$$

Thus, as the angular momentum of the state increases its geometric interpretation changes. When $J \sim O(1)$ we think of the state as a point-like graviton. For $J \sim O(\sqrt{N})$, the interpretation is a string and for $J \sim O(N)$, the interpretation is a membrane whose size is comparable to R_{AdS}. These objects were introduced as *giant gravitons* in [4]. There, they studied gravitons moving in the S^5 and showed that they blow up into spherical D3 branes whose $J_{max} = N$. In [5] and [6], it was shown that gravitons can also blow up into three-spheres in the AdS_5 space. In addition, it was shown that gravitons lose none of their supersymmetries when expanding into either the S^5 or the AdS_5.

How can these geometric objects be studied in the gauge theory? The conserved charges of a state in the string theory helps to identify the corresponding quantum field theory operator. $\mathrm{AdS}_5 \times S^5$ has an $SO(2,4) \times SO(6)$ isometry corresponding to the AdS_5 and S^5 spaces respectively. The $SO(2,4)$ and $SO(6)$ groups both have rank 3 and thus have three elements in their Cartan sub-algebra. This means that a string state can, in general, be labeled by $3+3$ charges: energy and two spins for the AdS, and three angular momenta for the S^5.

If we then consider a state in the quantum gravity with angular momentum J in a single S^5 plane corresponding to the Z field, the dual operator would be

$$\mathcal{O}_J = \mathrm{Tr}(Z^J) \tag{1.33}$$

where its $\mathcal{R}$-charge is J. The states in the quantum gravity with different values of angular momenta J (as described above) are identified with CFT states created by operators with matching $\mathcal{R}$-charges J. In this way, the half-BPS sector of the gauge theory can be used to study geometric objects in the half-BPS sector of the string theory. This is an important idea which will be used in this thesis. To study the energy spectrum of a system of giant gravitons, we construct their dual operators in the $\mathcal{N}=4$ SYM theory and compute their spectra of scaling dimensions. The issue of identifying gauge theory operators dual to giant gravitons is discussed next.

1.3.4 Gravitons and the free Fock space

Consider again the following BPS operator in four-dimensional $\mathcal{N}=4$ super Yang-Mills theory

$$\mathcal{O}_1 = \mathrm{Tr}(Z^n), \tag{1.34}$$

where n is fixed at $O(1)$. This is an example of a single-trace chiral primary operator transforming in the $(0,n,0)$ representation of $SU(4)$. A primary operator is the highest-weight state in a representation of the conformal group. It's therefore annihilated by the generator of special conformal transformations, K_μ. To specify an irrep of the conformal algebra, we need to specify an irrep of the Lorentz algebra as well as the scaling dimension of the highest weight state in that irrep of the conformal algebra. A chiral operator is annihilated by the supercovariant derivative D. This requirement is a way of extracting the irreps of the supersymmetry algebra. An operator that is both annihilated by K_μ and D is a chiral-primary operator. Recall that this operator corresponds to a CFT state which is then identified as a graviton in the dual quantum gravity. Next, consider

$$\mathcal{O}_2 = \mathrm{Tr}(Z^{n_1})\mathrm{Tr}(Z^{n_2}), \tag{1.35}$$

where $n = n_1+n_2$. Operators $\mathcal{O}_1$ and $\mathcal{O}_2$ have the same dimension but different trace structures - $\mathcal{O}_2$ is a product of two single-traces and may be thought of as creating a two-particle state in the $\mathrm{AdS}_5 \times S^5$. The two-point function results are [7]

$$\begin{aligned} \langle \mathcal{O}_1 \mathcal{O}_1 \rangle &\sim nN^n, &(1.36)\\ \text{and} \quad \langle \mathcal{O}_2 \mathcal{O}_2 \rangle &\sim n_1 n_2 N^n. &(1.37) \end{aligned}$$

The dependence of the two-point functions on space-time coordinates in a CFT is exactly determined by conformal symmetry. As a result, we have not included this dependence on the coordinates in the above results. When evaluating correlators the main task is to solve the combinatorial problem of taking all possible Wick contractions into account.

Furthermore, the two-point function $\langle\mathcal{O}_1\mathcal{O}_2\rangle$ can be shown to be

$$\frac{\langle\mathcal{O}_1\mathcal{O}_2\rangle}{\sqrt{\langle\mathcal{O}_1\mathcal{O}_1\rangle}\sqrt{\langle\mathcal{O}_2\mathcal{O}_2\rangle}} \sim \frac{\sqrt{n_1 n_2 n}}{N}. \tag{1.38}$$

Since n, n_1 and n_2 are fixed at $O(1)$, $\langle\mathcal{O}_1\mathcal{O}_2\rangle$ vanishes in the $N \to \infty$ limit. In general operators of dimension $O(1)$ with different trace structures have vanishing two-point functions in the large N limit. Two-point functions of operators may be thought of as overlaps between their dual states in the string theory. If we identify trace number in the CFT operator with particle number in the dual string theory state it follows that states with different particle number are orthogonal. In this way the notion of a free Fock space of gravitons emerges in the supergravity theory. For a more detailed discussion, see [7].

1.3.5 Giant Gravitons

Giants are not single traces

Consider again half-BPS operators built from a single $N \times N$ scalar field Z. Recall that their scaling dimensions are equal to their $\mathcal{R}$-charges which is also equal to the number of Z fields comprising the operator. We are interested in operators whose scaling dimension scales parametrically with N in the large N limit. In this situation, not only does the dimension of each Z matrix become large, the number of Z fields in $\mathcal{O}$ also becomes large.

In this thesis, we study giant graviton physics by computing scaling dimensions of their dual operators in the gauge theory. Exactly what are the operator duals of giant gravitons? In light of our discussion in the previous section, one might expect multi-trace operators to be natural candidates.

For giants large in the S^5 there is a bound on their size given by the S^5 radius, R_{AdS}. This leads to a bound on the giant's angular momentum. From (1.32), we see that $J_{max} = N$. Thus, we require operators whose $\mathcal{R}$-charges are also be bounded by N. The trace operators do satisfy this requirement since any higher dimensional operator can always be written in terms of lower-dimensional operators. For example, consider Z to be a 2×2 matrix. The single trace operator, $\mathrm{Tr}(Z^3)$ can be rewritten as

$$\mathrm{Tr}(Z^3) = \frac{1}{2}[3\mathrm{Tr}(Z)\mathrm{Tr}(Z^2) - \mathrm{Tr}(Z)^3]. \tag{1.39}$$

One might therefore suggest that a single giant graviton state should be identified with the state created by a single-trace operator whose $\mathcal{R}$-charge matches the giant's angular momentum.

However, giants can not be single traces. As argued in [7] we lose orthogonality of states with different particle number. This can easily be seen from (1.38); when n_1 and n are $O(N)$, and n_2 is $O(1)$, for example, the two-point function is $O(1)$. The situation is even worse when all three are $O(N)$. In such a case, the two-point function is $O(\sqrt{N})$. Therefore we can no longer identify trace number of large dimension operators with particle number of the string theory states. Clearly, to describe giant gravitons we need a different set of operators. Ideally, the new set must have orthogonal two-point functions for any scaling dimension Δ and satisfy the $\mathcal{R}$-charge bound required for sphere giants.

Schur Polynomials

The work of [8] was instrumental in developing the dictionary between giant gravitons and gauge theory operators. They studied a class of half-BPS multi-trace operators called Schur polynomials. [8] established a one-to-one correspondence between half-BPS representations of charge n and partitions of n. As we will see, the Schur polynomial corresponds to some partition of n, where n is the number of constituent matrix fields.

There is a duality between symmetric and unitary groups - a single Young diagram, with n boxes and no more than N rows, can label both an irrep of $U(N)$ and an irrep of S_n. If $U \in U(N)$, then the Schur polynomial is the character of U in some irreducible representation R of $U(N)$ and may be written as [8]

$$\chi_R(U) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\mathrm{Tr}(\sigma U^{\otimes n}), \tag{1.40}$$

where $\chi_R(\sigma)$ is the character of the permutation $\sigma \in S_n$ in the irrep R of S_n. The details of this definition are reserved for a later section. For now, our goal is simply to establish that Schur polynomials are the natural candidates for giant graviton duals. Consider Schur polynomials built from the complex matrix field $Z = \Phi_5 + i\Phi_6$, where Φ_5 and Φ_6 are two of the six scalar fields taking values in the adjoint representation of $u(N)$ in $\mathcal{N} = 4$ SYM theory. Schur polynomials satisfy two important properties:

1. They diagonalize their two-point functions in the free theory limit for any value of their $\mathcal{R}$-charge [8]

$$\langle\chi_R(Z)\chi_S(Z^\dagger)\rangle \propto \delta_{RS}. \tag{1.41}$$

2. Schur polynomials also display the required upper bound on its $\mathcal{R}$-charge. A Young diagram consisting of a single column labeling a $u(N)$ irrep can have a maximum number of N rows. Consider our 2×2 Z matrix example again. It is straightforward to verify that

$$\chi_{\tiny\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}}(Z) = 0. \tag{1.42}$$

We therefore identify a Schur polynomial labeled by a Young diagram of a single long column ($O(N)$ boxes) as the operator dual to a single giant graviton in the S^5. We identify a Schur polynomial with Young diagram of $O(1)$ columns, each with $O(N)$ boxes, as a bound state of sphere giants, where the number of columns equals the number of sphere giants.

There is no restriction on the size of an AdS giant. Thus, there should also be no restriction on the $\mathcal{R}$-charge of the dual operator. This is exactly the property displayed by Schur polynomials labeled by a single long row with $O(N)$ boxes - an irrep of $U(N)$ can have any number of boxes in its rows. This offers some support to the claim that a Schur polynomial having $O(1)$ rows, each with $O(N)$ boxes, is dual to a bound state of AdS giants, where the number of rows is identified with the number of AdS giants. Of course, one should investigate whether such a Schur polynomial can reproduce other properties of a system of AdS giants.

How does the operator-state correspondence work for Schur polynomials? We have already seen that two-point correlation functions of CFT operators map to overlaps between their corresponding states. Let's evaluate the two-point correlation function for Schur polynomials in the free theory

$$\langle \chi_R(Z)\chi_S^\dagger(Z)\rangle = \int [dZdZ^\dagger] e^{-\mathrm{Tr}(ZZ^\dagger)} \chi_R(Z)\chi_S^\dagger(Z). \tag{1.43}$$

The Z and $Z^\dagger$ can not be simultaneously diagonalized since they are not Hermitian. But they can be brought into the following form: Z will be upper triangular with its eigenvalues appearing along the diagonal, and $Z^\dagger$ will be lower triangular with its eigenvalues appearing along the diagonal. We can perform a change of coordinates to the eigenvalues of Z and $Z^\dagger$

$$\langle \chi_R(Z)\chi_S^\dagger(Z)\rangle = \int dz_i d\bar{z}_i \; \Delta(z_i)\Delta(\bar{z}_i)\, e^{-\sum_i z_i\bar{z}_i}\chi_R(z_i)\chi_S(\bar{z}_i) \tag{1.44}$$

where $\Delta(z_i)$ is the Van-der Monde determinant for changing coordinates from the matrix elements of Z to the eigenvalues of Z. Separating into two pieces

$$\langle \chi_R(Z)\chi_S^\dagger(Z)\rangle = \int dz_i d\bar{z}_i \Big(\Delta(z_i)\chi_R(z_i)e^{-\frac{1}{2}\sum_i z_i\bar{z}_i}\Big)\Big(\Delta(\bar{z}_i)\chi_S(\bar{z}_i)e^{-\frac{1}{2}\sum_i z_i\bar{z}_i}\Big). \tag{1.45}$$

The Van-der-Monde determinant multiplied by the Schur polynomial may be thought of as the Slater determinant of an N-particle fermionic wave-function. Thus, the quantities in round brackets may be thought of as a fermonic wave-function for N non-interacting fermions

$$\psi_R = \Delta(z_i)\chi_R(z_i)e^{-\frac{1}{2}\sum_i z_i\bar{z}_i}. \tag{1.46}$$

The Young diagram labeling the Schur polynomial is related to the energy of the free fermion system in the following way [8]

$$E = \sum_i r_i + i - 1, \tag{1.47}$$

where $\vec{r} = (r_1, r_2, ..., r_N)$ denote the row lengths of R. Thus, the two-point function between two Schur polynomials, labeled by R and S, may be interpreted as an inner product of two states describing a system of N free fermions of different energies

$$\langle \chi_R(Z) \chi_S^\dagger(Z) \rangle = \langle \psi_R | \psi_S \rangle. \tag{1.48}$$

For a further discussion, see [9], [8] and [3].

1.3.6 Excited giant gravitons

To further study the correspondence between giant gravitons and Schur polynomials, one may study a system of excited giant gravitons. Such a system may be described by attaching open strings between the giants. One may even hope to learn something about the giant graviton's geometry by studying a system of giant gravitons with strings attached [3], [10]. The end-points of these strings, which are stuck onto the D-brane world-volumes, are charged. At low energies the open strings should realise a gauge theory on the giant graviton world-volume itself [11]. The world-volume of the brane is not embedded in the space on which the dual Yang-Mills is defined. If AdS/CFT is valid, this gauge theory on the world-volume of the brane must somehow emerge in the Yang-Mills theory [11]. We thus speak of this gauge theory as an emergent gauge theory.

The total number of allowed excited giant graviton states is restricted in the following way. The two end-points of an open string are assigned opposite charges

- one positive, the other negative. Thus, we can associate a direction to each string whose end-points are attached to a giant graviton. Because the world-volume of the giant is compact, Gauss' law forces the net charge on the world-volume to vanish. In other words, the surface integral in the Gauss law,

$$\oint_{\partial S} \vec{E} \cdot d\vec{S} = \frac{Q_{tot}}{\epsilon_0}, \tag{1.49}$$

where ∂S is the world-volume of the giant and Q_{tot} is the total charge on the world-volume, vanishes. This then forces $Q_{tot} = 0$. Consequently, the only allowed configurations are the ones where the number of strings leaving a giant equals the number of strings arriving on the giant.

These D-brane states should correspond to some set of operators in the Yang-Mills theory. Further, the number of allowed excited D-brane states should match the number of dual operators one can define. Schur polynomials may also be built out of more than one type of scalar field. For example, one may construct a Schur polynomial out of Z and Y, or even Z, Y and X. In so doing, it is possible to build less supersymmetric operators. Such operators would not have their scaling dimensions equal to their $\mathcal{R}$-charges. Instead, their scaling dimensions are corrected by anomalous dimensions. We consider restricted Schur polynomials, built from n Zs and m Ys. The $n+m$ matrix indices are organised into an irrep R of S_{n+m}. The precise definition of restricted Schurs is reserved for a later discussion. The purpose of this section is to explore the possibility that excited giant gravitons are dual to restricted Schur polynomials. It is worth noting that these operators form an orthogonal basis for the $\frac{1}{4}$-BPS sector of the gauge theory at zero coupling [12]. Let $n \gg m$. Thus, these operators are composed mainly of Z's. Hence, the main

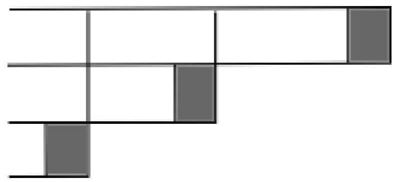


Figure 1.1: Young diagram R with 3 long rows. The last box in each row is shaded to indicate the box removed to subduce irreps of $S_n \times S_m$.

contribution to this object's angular momentum in the $\mathrm{AdS}_5 \times S^5$ is coming mainly from the Zs. This angular momentum again causes the object to expand into a giant graviton but with extra angular momentum contributions coming from the Y fields. It is natural to associate the Z's with the giant's worldvolume and the Y's with the open strings. For all m strings being identical, we decompose R into irreps of the $S_n \times S_m$ subgroup. The n Z indices are organised into the irrep r of S_n and the m Y indices are organised into the irrep s of S_m. Furthermore, R subduces irreps of $S_n \times S_m$ with some multiplicity. For a given $R, (r, s)$ labels, there are $g(r, s; R)^2$ different restricted Schurs one can define [13]. This number was also shown to match the number of states in the 1/4-BPS sector of the free theory with a $U(N)$ gauge group. Here is an example of how the restricted Schur counting and the open-string D-brane state counting agrees. For more examples, see [11].

Consider a Young diagram R with three long columns as shown in figure (1.1). We remove one box from each row in R and then assemble the $S_n \times S_3$ irreps. The shaded boxes in (1.1) indicate the boxes we remove and we denote by r the Young diagram left behind. The result is

$$r \times \yng(3) + 2\, r \times \yng(2,1) + r \times \yng(1,1,1). \tag{1.50}$$

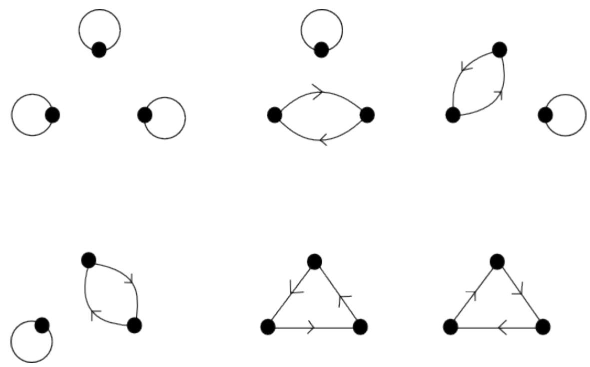


Figure 1.2: The six open-string configurations for a three giant graviton system with three strings stretching between them.

The counting of restricted Schurs goes according to $g(r, s; R)^2$, thus giving a total of 6 operators. The 6 open-string configurations are shown in figure (1.2). To get the 6th configuration, we reversed the orientation of the strings.

It was later proved in [14] that the counting of restricted Schur polynomials and open-string D-brane states obeying the Gauss law are the same. This is a highly non-trivial test of the proposal that restricted Schur polynomials are dual to excited giant gravitons. A restricted Schur polynomial with n Z's and m Y's ($n, m \sim O(N)$ and $n \gg m$), whose R has p long columns or rows each separated by $O(N)$ boxes, is now thought to be dual to a system of p giant gravitons in the S^5 or AdS$_5$ with m string attached. It is now natural to turn one's attention to their spectrum of anomalous dimensions. Using AdS/CFT, these anomalous dimensions map to the excitation energies of the giant graviton system. In this thesis, we compute the spectrum of one and two-loop anomalous dimensions of restricted Schurs. The spectrum will be shown to be equivalent to that of a set of decoupled harmonic oscillators.

1.4 Testing the correspondence

In this section, we discuss some of the initial tests of the AdS/CFT correspondence. We explain why testing the correspondence is difficult except in some highly symmetric settings and motivate why integrability can play an important role in this regard.

1.4.1 Weak-strong duality

The statement of AdS/CFT is that $\mathcal{N} = 4$ super Yang-Mills theory in flat four-dimensional Minkwoski space is equivalent to type IIB superstring theory on the $\mathrm{AdS}_5 \times S^5$ background. This is a powerful conjecture and, in the absence of a rigorous proof, we would like to think of ways in which it may be tested. One may study the correspondence in certain limits in which perturbation theory is applicable for either one side of the correspondence or the other, hence allowing us to perform calculations.

The ‘t Hooft limit is defined as $\lambda = g_{YM}^2 N$, with λ fixed and $N \to \infty$. The gauge theory has two parameters - the ‘t Hooft coupling λ and N. The string theory also has two parameters - the string length l_s and the string coupling, g_s. AdS/CFT relates these parameters in the following way

$$\frac{R_{AdS}^4}{l_s^4} = \lambda, \qquad g_{YM}^2 = g_s \Rightarrow g_s = \frac{\lambda}{N}. \tag{1.51}$$

Using the effective string tension $T = R_{AdS}^2/2\pi\alpha'$, and that $l_s = \sqrt{\alpha'}$, the first relation in (1.51) may be written as

$$\lambda = 4\pi^2 T^2. \tag{1.52}$$

A $1/N$ expansion in the Yang-Mills theory has been interpreted as corresponding to an expansion in the string theory in the string coupling g_s [15], [16] and [17]. With λ fixed and $N \to \infty$, type IIB string theory on $\mathrm{AdS}_5 \times S^5$ becomes *classical* (but still stringy) in the sense that there are no string loops, and any string interactions may be treated perturbatively. In this way, the theory simplifies significantly in the 't Hooft limit.

After taking the 't Hooft limit, one may take either the strong ($\lambda \gg 1$) or weak ($\lambda \ll 1$) coupling limits. For $\lambda \ll 1$, the Yang-Mills theory is *weakly coupled*. One might therefore be tempted to use perturbation theory. However, a perturbative expansion in $\mathcal{N} = 4$ SYM is typically an asymptotic expansion and thus has a zero radius of convergence. In other words, it's possible to expand something in terms of a small parameter and still have the series diverge. Asymptotic expansions are still useful in that they typically begin with decreasing error for the first few terms and as a result can still yield a good approximation of what is being calculated. Thus, the weakly coupled gauge theory may be studied using perturbation theory where only the first few terms of the expansion are studied[2]. However, from (1.52) it follows that $\lambda \ll 1$ corresponds to very small string tension and the shape of the string is highly unconstrained. The other complication with $\lambda \ll 1$ is that curvature corrections become important. This can be seen from (1.51); If $\lambda \ll 1$, it follows that

[2]QED is an example of an asymptotic expansion [18]. And yet, as of 2012, a tenth order calculation in the electric charge, or 5th order in the fine structure constant, has given an anomalous magnetic moment value of 1 159 652 182.78 (77) $\times 10^{-12}$ for the electron [19]. A grand total of 12672 Feynman diagrams were evaluated for this calculation. This is compared with the experimental measurement of 1 159 652 180.73 (0.28) $\times 10^{-12}$ [20].

$$\frac{R_{AdS}}{l_s} \ll 1. \tag{1.53}$$

In this case, the radius of curvature is small compared to the string length and curvature corrections cannot be neglected when studying this string.

One may also study the large λ limit, $\lambda \gg 1$. Equation (1.51) shows that R_{AdS} is large compared to l_s in this case. Curvature corrections can then safely be ignored. Also, from (1.52) it follows that the string tension becomes large. Therefore, at strong coupling, the string can effectively be treated as a rigid rod in flat space. The classical type IIB string theory on $\mathrm{AdS}_5 \times S^5$ reduces to classical type IIB supergravity on $\mathrm{AdS}_5 \times S^5$. Curvature corrections may now be included perturbatively where the relevant expansion parameter is l_s^2/R_{AdS}^2. Using AdS/CFT, $l_s^2/R_{AdS}^2 \propto 1/\sqrt{\lambda}$ and hence curvature corrections are proportional to powers of $1/\sqrt{\lambda}$.

We have now seen that the Yang-Mills theory in the 't Hooft limit may be described by classical type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$. The large λ limit leads to classical type IIB supergravity on $\mathrm{AdS}_5 \times S^5$. Effects due to the curvature of the background are proportional to powers of $1/\sqrt{\lambda}$ and quantum effects are proportional to powers of $1/N$. In this region of the parameter space, the string theory may be studied using perturbation theory. However, the gauge theory is now strongly coupled and perturbation theory breaks down. This is what makes AdS/CFT difficult to test - the regions in which the two theories may be treated perturbatively do not overlap.

1.4.2 Testing the correspondence in BPS sectors

One way to proceed is the following. The half-BPS sector of $\mathcal{N} = 4$ super Yang-Mills contains the most supersymmetry in the sense that half of the supercharges survive. The number of half-BPS states is the same in both the free theory and the interacting theory. On the gauge theory side, one may study the half-BPS sector in which quantities are protected by supersymmetry. Supersymmetry forces the quantum corrections to these quantities to vanish. These quantities may be computed in the free theory ($\lambda = 0$) and the results extrapolated to strong coupling. This is precisely the regime in which calculations in the string theory become tractable. The dual quantity in the string theory is then computed and the results are compared. However, in such a highly symmetric setting, one cannot be sure if any agreement is a consequence of the symmetry or of the actual dynamics of the two theories being the same. We would therefore like to test the duality when there is no supersymmetry. In other words, we want to study sectors of the theory in which objects are *not* protected by supersymmetry at finite values of the coupling λ.

1.4.3 Using integrability in non-BPS sectors

The number of half-BPS states in the free theory and the interacting theory are the same. This is not true however for 1/4-BPS. For example

$$\mathcal{O}_1 = \mathrm{Tr}(Z^n) \tag{1.54}$$

is 1/2-BPS at any value of the coupling, but

$$\mathcal{O}_2 = \mathrm{Tr}(YZ^nYZ^m) \tag{1.55}$$

is 1/4-BPS only at $g = 0$; It's not 1/4-BPS for $g \neq 0$. The one-loop dilatation operator acting on $\mathcal{O}_2$ is not zero

$$D\mathcal{O}_2 = \mathrm{Tr}\Big([Z,Y]\Big[\frac{\partial}{\partial Z},\frac{\partial}{\partial Y}\Big]\Big)\mathrm{Tr}(YZ^nYZ^m) \neq 0. \tag{1.56}$$

It is conjectured that the number of BPS states at infinitesimal coupling is equal to the number of BPS states at infinite coupling. So, the only discontinuity going from $g = 0$ to $g = \epsilon$ [21].

The discovery of integrability in the planar limit has allowed us to probe $\mathcal{N} = 4$ SYM theory and the classical type IIB string theory on $\mathrm{AdS}_5 \times S^5$ at finite values of the coupling λ. Integrability of the string world-sheet theory was discovered in [22]. Here, the Green-Schwarz superstring on $\mathrm{AdS}_5 \times S^5$ was studied for $\lambda \gg 1$. This theory was shown to possess an infinite number of non-local conserved charges which generate a non-Abelian algebra called the Yangian. The Yangian is an example where the conserved charges of a theory do not close a Lie algebra. The discovery of these extra conserved charges implied that the $1 + 1$ dimensional world-sheet theory is integrable. Meanwhile for the gauge theory side, the one-loop dilatation operator in the planar limit was argued to be the Hamiltonian of an integrable spin-chain [23]. The question of how the integrability in the string theory related to the integrability found in the gauge theory was answered in [24] and [25]. In [24], a conjecture was put forward of how the Yangian symmetry is realized in the spin-chain for $\lambda \to 0$. It was then shown that the conjectured form of the non-local charges commute with the one-loop dilatation operator. [25] then demonstrated that these charges indeed satisfy the standard Yangian algebra. This linked the integrability found on both sides of the correspondence.

To test the correspondence, one must calculate a quantity (for given values of the parameters) in both theories and check for agreement. We have already discussed the importance of testing the correspondence where there is no supersymmetry. Progress in this regard has been achieved in [26]. This work considered classical closed strings rotating in $\mathrm{AdS}_5 \times S^5$. Recall that strings in this background have 3+3 quantum numbers, $(E, S_1, S_2, J_1, J_2, J_3)$. [12] studied the string states (J_1, J_2) and (S, J), corresponding to the string rotating in two planes in the S^5, and to the string having in a single spin in AdS and rotating in one S^5 plane respectively. These solutions are not BPS. In the regime

$$J \gg 1, \qquad \frac{\lambda}{J^2} \ll 1, \qquad Q_i \sim J, \tag{1.57}$$

where $J = J_1 + J_2 + J_3$, the string energy is given by the following expansion

$$E(Q_i, \lambda) = S + J\Big(1 + \frac{\lambda}{J^2}\epsilon_1(Q_i/J) + \frac{\lambda^2}{J^4}\epsilon_2(Q_i/J) + \cdots\Big) + O(J^0) \tag{1.58}$$

where quantum corrections to E are $O(J^0)$. To test AdS/CFT, the anomalous dimension $\Delta(Q_i, \lambda)$, corresponding to these string states must be calculated. But diagonalizing the dilatation operator in regime (1.57) is very difficult. [26] then restricts λ to be small so they can make use of the integrability of the $\mathcal{N} = 4$ SYM at the one loop level. This allowed them to solve for the anomalous dimension for large J. They obtain a similar expansion for Δ

$$\Delta(Q_i, \lambda) = S + J\Big(1 + \frac{\lambda}{J^2}\delta_1(Q_i/J) + \frac{\lambda^2}{J^4}\delta_2(Q_i/J) + \cdots\Big). \tag{1.59}$$

Remarkably, the expansion for E is valid for large J and any λ, while Δ is valid for small λ and any J. Thus, there is a region of overlap in which perturbation theory may be applied to both theories - large J and small λ. Studying the (J_1, J_2) case, [26] demonstrates that the leading contribution to the one-loop anomalous dimension and the leading contribution to the classical energy are equivalent: $\epsilon_1 = \delta_1$. Furthermore, they were able to analytically continue these results, both in the gauge and string theories, to demonstrate $\epsilon_1 = \delta_1$ for the (S, J) case. Lastly, [26] presents numerical evidence that $\epsilon_2 = \delta_2$.

Including quantum corrections the energy expansion may be written as [27]

$$E = J\Big[1 + \frac{\lambda}{J^2}\Big(\epsilon_0^{(1)} + \frac{\epsilon_1^{(1)}}{J} + \frac{\epsilon_2^{(1)}}{J^2} + \cdots\Big) + \frac{\lambda^2}{J^4}\Big(\epsilon_0^{(2)} + \frac{\epsilon_1^{(2)}}{J} + \frac{\epsilon_2^{(2)}}{J^2} + \cdots\Big) + \cdots\Big] \quad (1.60)$$

where, in this expansion, $\epsilon_l^{(n)}$ refers to the nth term in the expansion for the lth loop correction to the string energy. For large J, the scaling dimension Δ in the gauge theory has a similar structure

$$\Delta = J\Big[1 + \frac{\lambda}{J^2}\Big(\delta_1^{(0)} + \frac{\delta_1^{(1)}}{J} + \frac{\delta_1^{(2)}}{J^2} + \cdots\Big) + \frac{\lambda^2}{J^4}\Big(\delta_2^{(0)} + \frac{\delta_2^{(1)}}{J} + \frac{\delta_2^{(2)}}{J^2} + \cdots\Big) + \cdots\Big], \quad (1.61)$$

where $\delta_l^{(n)}$ corresponds to the nth term in the lth loop anomalous dimension. Extending the agreement between E and Δ, the leading contribution to the one-loop string energy and the leading correction to the one-loop anomalous dimension were shown to match [27]

$$\epsilon_1^{(1)} = \delta_1^{(1)}. \quad (1.62)$$

Adding to this discussion, [28] considers the full space of local operators, including fermions and covariant derivatives, in the thermodynamic limit of planar $\mathcal{N} = 4$ SYM at one-loop. The thermodynamic limit simply means that they consider operators composed of a very large number of fields, which would be dual to large J states in the string theory. They use the integrability to construct an algebraic curve describing the solution for the dimensions of these operators. This spectral curve is then compared to what was found in [29], which derived the solution of the spectrum of classical type IIB string theory, including fermions, in terms of algebraic curves. In the limit of large spins the agreement between the two spectral curves is perfect.

Up till now, almost all checks of AdS/CFT have been performed in the planar limit of the gauge theory. One reason for this is that integrability is thought to break down in the non-planar limit [30]. One of the remarkable aspects in this thesis is that we demonstrate evidence for integrability in certain non-planar sectors. In our particular non-planar limit we argue that the dilatation operator is integrable up to two-loops. This could indeed pave the way for testing the gauge/gravity duality in non-planar limits.

1.4.4 What exactly do we mean by 'integrability'

What precisely do we mean by integrability here? $\mathcal{N} = 4$ SYM theory is a $3 + 1$ dimensional interacting quantum field theory. A theory being integrable implies that there are an infinite number of conserved charges. Now, the Coleman-Mandula theorem says that there can be no scalar conserved charges in a $3 + 1$ dimensional interacting quantum field theory. Thus, the Coleman-Mandula theorem implies that $\mathcal{N} = 4$ SYM theory can not be integrable. By integrability, we mean that the dilatation operator in the planar limit was shown to be identical to the Hamiltonian of an integrable spin-chain. As a result, the exact spectrum of anomalous dimensions of all operators in the planar limit can be calculated. By exact, we mean the anomalous dimensions may be determined as a function of λ.

That the gauge theory is exactly integrable in the planar limit means that we can study operators not protected by supersymmetry, and calculate their spectrum of scaling dimensions exactly. Integrability of the string theory allows one to calculate the energy spectrum of the dual states exactly. Thus, integrability allows us to calculate things exactly on both sides of the duality and thus perform non-trivial tests of AdS/CFT. Complete agreement has been observed in all considered cases [31], implying that the dynamics of the two theories are indeed equivalent.

1.5 Non-perturbative string theory

In this section, we explain why studying Schur polynomials, constructed in the gauge theory, allows us to study non-perturbative physics in the dual string theory, i.e., the physics of giant gravitons. We first discuss how the planar limit fails to capture the large N limit physics of the operators we wish to study. This is followed by a discussion of Schur polynomials and restricted Schur polynomials. Here we discuss why these operators are better suited to probing the dynamics of giant gravitons.

1.5.1 The simplification of the planar limit

To evaluate correlation functions of operators in quantum field theory, the usual approach is to sum Feynman diagrams order by order in the coupling, which is regarded to be small. $\mathcal{N} = 4$ super Yang-Mills theory is an example of a matrix model. Recall that this theory has two parameters: λ and N. When calculating a correlation function of operators, we can organise the terms into an expansion in $1/N^2$ and into an expansion in λ. All Feynman diagrams weighted by N^2 triangulate a sphere and are called *planar*. All diagrams triangulating surfaces with a more complicated topology (for example a torus or a pretzel and so on) are weighted by factors $1/N^2$. If we consider operators whose scaling dimension are fixed and $O(1)$, the leading contribution to correlators come from the planar diagrams. In the large N limit the planar diagrams then give the only contribution to correlation functions. In this case, the large N limit and the planar limit coincide. Recall that the $1/N^2$ expansion has been interpreted as corresponding to an expansion in the string theory in g_s. The planar limit in the gauge theory corresponds to summing tree-level processes in the string theory. We say that planar limit corresponds to a classical string theory. This is how the theory in the planar limit simplifies. The infinite number of non-planar diagrams are suppressed and may be neglected and what is left corresponds to a classical string theory. Furthermore, it was in the planar limit of the theory that integrability was discovered which allowed people to make some spectacular confirmations of AdS/CFT as discussed in section 1.4.

1.5.2 The failure of the planar limit

If we now study operators whose scaling dimension scales with N, the situation is rather different. In this case, the number of matrix fields inside the operator, as well as the size of each matrix, is going to infinity. When computing correlation functions with these operators, the number of non-planar diagrams becomes very large and cancels their $1/N^2$ suppression. In fact, the non-planar diagrams can even become the dominant contribution to the correlation function. In this case, the planar limit and the large N limit are distinct. The large N limit is correctly captured by summing all diagrams - an infinite number of them. Integrability was thought to be spoilt in the non-planar limit. And yet, for a thorough study of the AdS/CFT correspondence, we need to investigate all limits of the two theories. Studying non-planar limits therefore is unavoidable, and integrability would be a powerful tool in such a study. In this thesis, we describe how representation theory of the symmetric and unitary groups has led to substantial evidence of $\mathcal{N} = 4$ super Yang-Mills theory being integrable in a particular large N, non-planar limit. In the next section, we introduce one of our basic tools to achieve this goal - the Schur polynomial.

1.5.3 Schur polynomials

Recall that for operators whose scaling dimension was $O(N)$, there was a mixing between different trace structures when evaluating the two-point function of these operators. This was the main reason why giant gravitons could not be described by single trace operators. We have since argued that Schur polynomials are natural candidates for giant graviton duals. In this section, we discuss Schur polynomials in a bit more detail.

Consider an N-dimensional vector space V. Tensoring n copies of V, we get the space $V^{\otimes n}$. A basis vector in this space may be written as $|i_1, i_2, \cdots i_n\rangle$. Each index i_k can label any one of the N states inside the particular V space. Thus, $dim_{V^{\otimes n}} = N^n$. $V^{\otimes n}$ admits an action of the symmetric group S_n and of the unitary group $U(N)$. The action of S_n will simply permute the factors of V within $V^{\otimes n}$. So, $\sigma \in S_n$ is a map $\sigma : V^{\otimes n} \to V^{\otimes n}$ and its action on the $|i_1 \cdots i_n\rangle$ kets is

$$\sigma \, |i_1 i_2 \cdots i_n\rangle = |i_{\sigma(1)} i_{\sigma(2)} \cdots i_{\sigma(n)}\rangle \, . \tag{1.63}$$

The matrix elements of σ look like

$$\sigma^I_J = \langle i_1 \cdots i_n | \, \sigma \, | j_1 \cdots j_n \rangle = \delta^{i_1}_{j_{\sigma(1)}} \cdots \delta^{i_n}_{j_{\sigma(n)}} \tag{1.64}$$

where I is a shorthand for $i_1 \cdots i_n$. The action of $U(N)$ on V is to take $|i\rangle$ into some other $|i'\rangle$ in V. Further, the action of $U(N)$ is the same on every V in $V^{\otimes n}$

$$\begin{aligned} D(U) \, |i_1, \cdots i_n\rangle &= D(U) \, |i_1\rangle \otimes D(U) \, |i_1\rangle \otimes \cdots \otimes D(U) \, |i_n\rangle \\ &= |i'_1, i'_2, \cdots i'_n\rangle \, , \end{aligned} \tag{1.65}$$

where $D(U)$ is a matrix representation of $U(N)$ acting on the above basis. It is not difficult to see that the actions of S_n and $U(N)$ commute and may thus be diagonalised simultaneously. As a result, a single Young diagram labels both an irrep of S_n and of $U(N)$. The space $V^{\otimes n}$ may be decomposed in the following way

$$V^{\otimes n} = \bigoplus_R \left(V_R^{S_n} \bigotimes V_R^{U(N)} \right), \tag{1.66}$$

where the first R labels an irrep of S_n and the second labels a $U(N)$ irrep. Consider the operator

$$(\mathcal{P}_R)^I_J = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) (\sigma)^I_J \tag{1.67}$$

acting on $V^{\otimes n}$. In (1.67), R is a Young diagram with n boxes, $\chi_R(\sigma)$ is the character of the matrix representing σ in the irrep R. $(\mathcal{P}_R)^I_J$ is a projection operator satisfying

$$\begin{aligned} \mathcal{P}_R\mathcal{P}_S &\propto \delta_{RS}\mathcal{P}_R && (1.68) \\ \mathcal{P}_R\sigma &= \sigma\mathcal{P}_R \qquad \text{for all } \sigma \in S_n. && (1.69) \end{aligned}$$

These two equations tell us that the representation of $\sigma \in S_n$ acting on $V^{\otimes n}$ is reducible. $\mathcal{P}_R$ projects from the $V^{\otimes n}$ space onto the subspace that carries the irrep R. For example, consider

$$R = \square\!\square. \tag{1.70}$$

Then $(\mathcal{P}_R)^{k_1k_2}_{i_1i_2}$ acting on $|i_1, i_2\rangle$ in $V^{\otimes 2}$ gives

$$\begin{aligned} (\mathcal{P}_R)^{k_1k_2}_{i_1i_2}\,|i_1, i_2\rangle &= \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\delta^{k_1}_{i_{\sigma(1)}}\delta^{k_2}_{i_{\sigma(2)}}\,|i_1, i_2\rangle \\ &= \frac{1}{2}(\,|i_1, i_2\rangle + |i_2, i_1\rangle\,). \end{aligned} \tag{1.71}$$

Thus, the result is symmetric under swapping the upper indices, k_1 and k_2.

Now, let $V^{\otimes n}$ consist of taking n products of Z^i_j, where Z takes values in the adjoint of $u(N)$ in $\mathcal{N} = 4$ SYM theory. To generate a gauge invariant operator, we take the trace,

$$\chi_R(Z) = (\mathcal{P}_R)^I_J(Z^{\otimes n})^J_I = \mathrm{Tr}(\mathcal{P}_R Z^{\otimes n}) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\mathrm{Tr}(\sigma Z^{\otimes n}). \tag{1.72}$$

This is precisely the Schur polynomial in the field Z. If we replace Z with an element of $U(N)$, then $\chi_R(U)$ calculates the character of U in the irrep R of $U(N)$ [8].

Schur polynomials are an effective tool for describing multi-trace structures, as we now explain. $\mathrm{Tr}(\sigma Z^{\otimes n})$ is a single trace operator in the space $V^{\otimes n}$. However, specifying σ and contracting the indices produces a multi-trace operator. For example, let $n = 3$, and $\sigma = (12)$. Then

$$\mathrm{Tr}((12)Z^{\otimes 3}) = Z^{i_1}_{i_2} Z^{i_2}_{i_1} Z^{i_3}_{i_3} = \mathrm{Tr}(Z^2)\mathrm{Tr}(Z). \tag{1.73}$$

Indeed, any multi-trace operator involving n Z fields may be obtained from a single trace of an S_n permutation acting on $Z^{\otimes n}$ in $V^{\otimes n}$. Thus, a permutation of S_n corresponds to a unique multi-trace operator. A multi-trace operator does not however correspond to a unique permutation. All permutations with the same cycle structure give the same multi-trace operator, thus forming a conjugacy class. The number of unique cycle structures in S_n, or conjugacy classes, is equal to the number of partitions of n. Thus, there are as many different multi-trace operators of n Z fields as there are partitions of n.

Now, let's evaluate the two-point function.

$$\langle \chi_R(Z) \chi_T^\dagger(Z) \rangle \;\; = \;\; (\mathcal{P}_R)^J_I (\mathcal{P}_T)^L_K \langle (Z^{\otimes n})^I_J (Z^{\dagger \otimes n})^K_L \rangle. \tag{1.74}$$

The two point function between Z and $Z^\dagger$ in the $U(N)$ gauge theory is

$$\langle Z^i_j Z^{\dagger k}_l \rangle = \delta^i_l \delta^k_j. \tag{1.75}$$

This simple formula allows one to write the two-point function between n Z's and n $Z^\dagger$'s as a sum of permutations of S_n

$$\langle Z^{i_1}_{j_1} \cdots Z^{i_n}_{j_n} Z^{\dagger k_1}_{l_1} \cdots Z^{\dagger k_n}_{l_n} \rangle = \sum_{\sigma \in S_n} (\sigma)^I_L (\sigma^{-1})^K_J. \tag{1.76}$$

Thus

$$\begin{aligned} \langle \chi_R(Z) \chi^\dagger_T(Z) \rangle &= (\mathcal{P}_R)^J_I (\mathcal{P}_T)^L_K \sum_{\sigma \in S_n} (\sigma)^I_L (\sigma^{-1})^K_J \\ &= \sum_{\sigma \in S_n} \mathrm{Tr}(\mathcal{P}_R \sigma \mathcal{P}_T \sigma^{-1}). \end{aligned} \tag{1.77}$$

Since $\mathcal{P}_R$ and σ commute, we end up having the two projectors multiplying to produce δ_{RT}.

$$\begin{aligned} \langle \chi_R(Z) \chi^\dagger_T(Z) \rangle &= \sum_{\sigma \in S_n} \frac{\delta_{RT}}{d_R} \mathrm{Tr}(\mathcal{P}_R) \\ &= \frac{\delta_{RT} n!}{d_R} \mathrm{Tr}(\mathcal{P}_R), \end{aligned} \tag{1.78}$$

where d_R is the dimension of representation R of S_n. Since Schur polynomials compute the characters of the unitary group, we may replace Z by the identity so that $\chi_R(1)$ is simply the character of the identity. The character of the identity gives the dimension of the irrep, dim_R. Thus

$$\langle \chi_R(Z) \chi^\dagger_T(Z) \rangle = \delta_{RT} \frac{n!}{d_R} dim_R. \tag{1.79}$$

Instead of summing Feynman diagrams to evaluate the two-point function of Schur polynomials, we have used representation theory and evaluated it exactly. In this way, we have summed *all* the Feynman diagrams, not only the planar ones.

1.5.4 Restricted Schurs

Recall that $V^{\otimes n}$ was decomposed into irreps of the symmetric and unitary groups according to (1.66). Now, consider the space $V^{\otimes n+m}$. This space decomposes into a sum of subspaces $V_R^{S_{n+m}} \otimes V_R^{U_N}$. We want to decompose $V_R^{S_{n+m}}$ into subspaces carrying irreps of the subgroup $S_n \times S_m$. We start with an irrep R of S_{n+m}. R is a Young diagram with $n+m$ boxes. If one restricts to the $S_n \times S_m$ subgroup R is, in general, reducible. That is, R may be written as a direct sum of $S_n \times S_m$ irreps. Label the $S_n \times S_m$ irreps by (r, s), where r is a Young diagram with n boxes and s is a Young diagram with m boxes. In general, R can subduce a particular $S_n \times S_m$ irrep more than once. Thus, we need to introduce a multiplicity index α to label these copies. We may write this decomposition as

$$V_R^{S_{n+m}} = \bigoplus_{r,s} (V_r^{S_n} \otimes V_s^{S_m})^{\oplus g(r,s;R)} \tag{1.80}$$

where $g(r, s; R)$ is the multiplicity with which irrep (r, s) appears.

Next, consider the operator

$$\mathcal{P}_{R,(r,s)\alpha\beta} = \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\sigma) \sigma^{i_1 i_2 \cdots i_{n+m}}_{j_1 j_2 \cdots j_{n+m}} \tag{1.81}$$

acting on $(Z^{\otimes n} \otimes Y^{\otimes m})^J_K = Z^{j_1}_{k_1} \cdots Z^{j_n}_{k_n} Y^{j_{n+1}}_{k_{n+1}} \cdots Y^{j_{n+m}}_{k_{n+m}}$. The α and β are the multiplicity labels for the (r, s) irrep. Note that we need two multiplicity labels. The reason for this is as follows. The character of σ in some irrep R is defined to be

$$\sum_i \Gamma_R(\sigma)_{ii} \tag{1.82}$$

where the indices correspond to states in the carrier space of R. However, $\chi_{R,(r,s)\alpha\beta}(\sigma)$ is the character of the matrix representing σ in irrep R that is restricted to a particular subspace of R. That is, we decompose the carrier space of R into carrier spaces of $S_n \times S_m$ irreps and trace over one of these subspaces. The matrix indices therefore correspond to states in one of these subspaces. One such subspace can be the carrier space of $(r, s)\alpha$. Such a space is spanned by states with the labels $|R(r, s)i; \alpha\rangle$, where i labels a particular a combination of Young-Yamanouchi states for r and s. If $\gamma \in S_n \times S_m$, then $\Gamma_R(\gamma)$ takes on a block diagonal form in this basis, where the blocks are matrices representing γ in the irreducible representations of $S_n \times S_m$. Each index in $\Gamma_R(\sigma)_{ii}$ corresponds to a state in such a subspace. In general, the two indices could correspond to states in different copies of (r, s). The row index may be restricted to the $(r, s)\alpha$ subspace and the column index may be restricted to the $(r, s)\beta$ subspace. Thus

$$\chi_{(r,s)\alpha\beta}(\sigma) = \sum_i \Gamma_R(\sigma)_{[R(r,s)i;\alpha],[R(r,s)i;\beta]}. \tag{1.83}$$

We call $\chi_{(r,s)\alpha\beta}(\sigma)$ a *restricted character*. The restricted character may also be calculated by using a projection operator

$$\chi_{(r,s)\alpha\beta}(\sigma) = \mathrm{Tr}_{(r,s)\alpha\beta}(\Gamma_R(\sigma)) = \mathrm{Tr}_R(P_{R\to(r,s)\alpha\beta}\Gamma_R(\sigma)) \tag{1.84}$$

where we trace over the entire carrier space R in the last equation. When multiplicities are involved $P_{R,(r,s)\alpha\beta}$ is an intertwiner for irreps $(r,s)\alpha$ and $(r,s)\beta$. That is,

$$\begin{aligned} P_{R,(r,s)\alpha\beta} : |R(r,s),\beta;i\rangle &\rightarrow |R(r,s),\alpha;i\rangle\,, &(1.85)\\ P_{R,(r,s)\alpha\beta}\Gamma_{(r,s)\beta}(\gamma) &= \Gamma_{(r,s)\alpha}(\gamma)P_{R,(r,s)\alpha\beta}. &(1.86)\end{aligned}$$

Furthermore, $\mathcal{P}_{R,(r,s)\alpha\beta}$ satisfies the following properties

$$\begin{aligned} \mathcal{P}_{R,(r,s)\alpha\beta}\mathcal{P}_{T,(t,u)\rho\tau} &\propto \delta_{RT}\delta_{rt}\delta_{su}\delta_{\beta\rho}\mathcal{P}_{R,(r,s)\alpha\tau}, &(1.87)\\ \mathcal{P}_{R,(r,s)\alpha\beta}\gamma &= \gamma\mathcal{P}_{R,(r,s)\alpha\beta}, \qquad \gamma\in S_n\times S_m. &(1.88)\end{aligned}$$

$\mathcal{P}_{R(r,s)\alpha\beta}$ will act on $(Z^{\otimes n}\otimes Y^{\otimes m})^J_K$ and will organise the $n+m$ indices of this tensor into an irrep of S_{n+m}, the n indices of $Z^{\otimes n}$ into an irrep of S_n and the m indices of $Y^{\otimes m}$ into an irrep of S_m. To then produce a gauge invariant operator, we take the trace, i.e., contract the upper indices with the lower indices. The result is a restricted Schur polynomial

$$\chi_{R,(r,s)\alpha\beta}(Z,Y)=\frac{1}{n!m!}\sum_{\sigma\in S_{n+m}}\chi_{(r,s)\alpha\beta}(\sigma)\mathrm{Tr}(\sigma Z^{\otimes n}\otimes Y^{\otimes m}). \tag{1.89}$$

The trace structure of (1.89), as determined by σ, is invariant under permuting the Z and Y fields separately. If $\gamma\in S_n\times S_m$, and if

$$\tau = \gamma^{-1}\sigma\gamma \tag{1.90}$$

then the trace structures of τ and σ are the same. The restricted characters also have this symmetry. Indeed, if $\gamma \in S_n \times S_m$, the matrix $\Gamma_R(\gamma)$ commutes with the projector $P_{R\to(r,s)\alpha\beta}$ and hence

$$\mathrm{Tr}(P_{R\to(r,s)\alpha\beta}\Gamma_R(\tau)) = \mathrm{Tr}(P_{R\to(r,s)\alpha\beta}\Gamma_R(\sigma)). \tag{1.91}$$

We thus also have a conjugacy class for restricted Schurs. This conjugacy class is called a *restricted conjugacy class* [32]

The counting of restricted Schur polynomials have been shown to match the counting of states, as given by the partition function of free $\frac{1}{4}$-BPS sector of the SYM theory with $U(N)$ gauge group [13]. The partition function for the $U(N)$ gauge theory can be re-written in terms of a product of Schur polynomials. Using the product rule for Schur polynomials, which introduces Littlewood-Richardson coefficients $g(r, s; R)$, and an inner product defined for the $U(N)$ gauge theory, the partition function may be written as a sum of Littlewood-Richardson coefficients

$$\mathcal{Z}_{U(N)}(t_1, t_2) = \sum_R \sum_r \sum_s g(R; r, s)^2 t_1^n t_2^m. \tag{1.92}$$

The Littlewood-Richardson coefficients count how many times irrep (r, s) appears when we decompose the S_{n+m} irrep R in terms of its $S_n \times S_m$ irreps. Thus, for a given $R, (r, s)$, $g(R; r, s)^2$ is the total number of $\frac{1}{4}$-BPS states in the $U(N)$ gauge theory. By letting the multiplicity labels run over their allowed values, $\alpha, \beta = 1, 2, ...g(R; r, s)$, we get the required $g(R; r, s)^2$ restricted Schur polynomials. Furthermore, their two-point functions have been calculated in [12] and shown to be diagonal,

$$\langle \chi_{R,(r,s)\alpha\beta}(Z,Y)\chi^{\dagger}_{T,(t,u)\gamma\eta}(Z,Y)\rangle = \delta_{R,T}\delta_{r,t}\delta_{s,u}\delta_{\alpha,\eta}\delta_{\beta,\gamma}\frac{hooks_R}{hooks_r \times hooks_s}f_R, \qquad (1.93)$$

where $hooks_R$ is the product of hook lengths of Young diagram R and f_R is the product of weights.

1.5.5 Gauss Graph Operators

That the counting of these open-string D-brane states and restricted Schur polynomials agree was proved in [14]. Consider again excited giant graviton states with m identical strings attached. Let m_i be the number of strings leaving brane i, $i = 1, 2, .., p$. The Gauss law then forces the number of strings arriving on brane i to also be m_i. However, there is a symmetry in that each set i having m_i strings, leaving or arriving on the brane, may be permuted amongst themselves without changing the configuration. I.e., the first m_1 strings may be permuted amongst themselves, the next m_2 strings may be permuted amongst themselves and so on. The open string configurations consistent with the Gauss law, called Gauss graphs, are shown to be in one-to-one correspondence with elements of the double coset

$$H \setminus S_m/H, \qquad (1.94)$$

where H,

$$H = S_{m_1} \times S_{m_2} \times \cdots \times S_{m_p} \tag{1.95}$$

is the subgroup of S_m that leaves the open-string configuration invariant. The number of restricted Schurs one can define is then shown to be equal to the number of these double coset elements. A complete set of functions is then constructed on the double coset

$$O_{R,r}(\sigma) = \frac{|H|}{\sqrt{m!}} \sum_{j,k} \sum_{s \vdash m} \sum_{\mu_1 \mu_2} \sqrt{d_s} \Gamma^s_{jk}(\sigma) B^{s \to 1_H}_{j\mu_1} B^{s \to 1_H}_{k\mu_2} O_{R,(r,s)\mu_1\mu_2}, \tag{1.96}$$

where $B^{s \to 1_H}_{j\mu_1}$ is a branching coefficient for irrep s to the μ_1th copy of the trivial rep of the subgroup H. The index j labels a state in the carrier space of s. This is essentially a change of basis from the basis of restricted Schur polynomials $O_{R,(r,s)\mu_1\mu_2}$ to a basis on the double coset. These new operators $O_{R,r}(\sigma)$ are called Gauss graph operators. The action of the one-loop dilatation operator on the restricted Schurs factorises into two pieces - one piece acting on the Z's and one piece acting on the Y's. The Gauss graph operators were shown to diagonalise the Y piece of the action. The idea therefore is for a system of p giant gravitons, draw a Gauss graph, then construct the corresponding operator, and this operator diagonalises the Y piece of the dilation operator. Furthermore, the diagonal expression may simply be read off the Gauss graph. In chapters 2 and 3, we diagonalise the Z part of the dilatation operator action at the one and two-loop level.

1.6 Overview of progress made

We are concerned with the dilatation operator acting on restricted Schur polynomials defined in (1.89) and solving for the spectrum of anomalous dimensions. Progress in this regard first began in [33] and [34] where the action of the dilatation operator was evaluated numerically for small values of m and two row (and column) Young diagrams. Evidence was presented for the anomalous dimension spectrum to be equivalent to that of a system of decoupled harmonic oscillators. In [35], the dilatation operator was studied for arbitrary m, with n, m $O(N)$ and $n \gg m$. Young diagrams with two long rows and two long columns, with each row or column separated by order n boxes, were considered. In this large N limit the non-planar diagrams may no longer be neglected. In other words the large N limit of correlation functions of these operators involves summing all planar and non-planar diagrams. The dilatation operator acting on the restricted Schurs was diagonalised exactly and the spectrum was, again shown to be that of a system of decoupled harmonic oscillators. Thus, evidence was presented in this paper for integrability in this non-planar limit.

Progress was then extended to Young diagrams with $p \sim O(1)$ rows also each separated by $O(n)$ boxes [36]. For more than two rows, R may subduce multiple copies of the $S_n \times S_m$ irrep (r, s). Schur-Weyl duality was used to resolve the multiplicity of these irreps. The basic idea here is that s is both an irrep for S_m and the unitary group $U(p)$. The states in the $U(p)$ irrep s then were found to be in one-to-one correspondence with the copies of the $S_n \times S_m$ irrep (r, s). The action of the dilatation operator on restricted Schurs split into two pieces - one acting on the Z's and one acting on the Y's. For small values of m, ($m = 3$ for example) it was noted that the action on the Y's could be diagonalised in a very simple way. For a given m and p, one may draw a simple picture representing the p giant gravitons with an open string configuration of m strings as allowed by the Gauss law. This picture corresponds to an expression for the Y-piece of the dilatation operator after it has been diagonalized. Indeed, the expression may simply be read off the graph itself. These pictures were called Gauss graphs and the operators, corresponding to the pictures and diagonalising the Y piece were called Gauss graph operators. The remaining piece of the dilatation operator - the piece acting on the Z's - may then be diagonalised by treating the problem as a set of p coupled oscillators. The action of the dilatation operator again reduced to a system of decoupled harmonic oscillators. Operators corresponding to the Gauss graphs, called Gauss graph operators (1.96), were explicitly constructed in [14]. It was shown that the operators in (1.96) diagonalise the Y-piece of the dilatation operator for arbitrary m.

Progress was also made for restricted Schur polynomials built from other kinds of fields. [37] and [38] considered restricted Schurs built from three scalar fields. Leading and subleading terms to the one-loop dilatation operator were computed. For both cases, the double coset ansatz gave an anomalous dimension spectrum of decoupled harmonic oscillators. For the fermion fields, the one-loop dilatation operator was also diagonalised by the double-coset ansatz and the spectrum was shown to be equivalent to decoupled harmonic oscillators [39]. The same result was also obtained for restricted Schurs belonging to the $sl(2)$ sector of the theory [40]. Next, the two-loop dilatation operator in the $su(2)$ sector was diagonalised by the same Gauss graph operators with the spectrum related to decoupled harmonic oscillators [41]. Finally, using arguments of symmetry, the dilatation operator was computed to all loops in [42].

Thus, evidence has been presented in the above works for $\mathcal{N} = 4$ super Yang-Mills theory being integrable in this large N, non-planar limit.

Chapter 2

From Large N Nonplanar Anomalous Dimensions to Open Spring Theory

2.1 Outline of chapter

In this chapter we compute the non-planar one loop anomalous dimension of restricted Schur polynomials that have a bare dimension of $O(N)$. This is achieved by mapping the restricted Schur polynomials into states of a specific $U(p)$ irreducible representation. In this way the dilatation operator is mapped into a $u(p)$ valued operator and, as a result, can easily be diagonalized. The resulting spectrum is reproduced by a classical model of springs between masses. This work was reported in Physics Letters B, Volume 711: 398-403, 2012.

2.2 Introduction

According to the AdS/CFT correspondence [43], the conformal dimension of an operator in the $\mathcal{N} = 4$ super Yang-Mills theory maps into the energy of the corresponding state in IIB string theory on the $\mathrm{AdS}_5\times\mathrm{S}^5$ background. In this chapter we are interested in computing the energies of excited giant graviton systems in string theory by computing the anomalous dimensions of restricted Schur polynomials [11], [12], [32] in $\mathcal{N} = 4$ super Yang-Mills theory. These operators have a classical dimension which is order N and consequently summing the planar diagrams does not give an accurate large N approximation [7]. Fortunately, in the last few years, starting from [8], [44] methods to study the large N limit of such correlators have been developed [12], [32], [45], [46], [47], [48]. In particular, there are now powerful methods [33], [34], [35], [37], [36] to evaluate the action of the one loop dilatation operator in the $su(2)$ sector [49].

In this chapter we will consider the diagonalization of the one loop dilatation operator when acting on restricted Schur polynomials $\chi_{R,(r,s)}(Z,Y)$ built from n Z fields and m Y fields, with $m \ll n$ and m, n both order N, as in [36]. There is a concrete proposal for the AdS/CFT duals to these operators. A system of p sphere giant gravitons is dual to an operators labeled by a Young diagram R with p columns and $m + n$ boxes. In this case, r is a Young diagram with p columns and m boxes and s is a Young diagram with at most p columns. After diagonalizing on the s label, [36] finds that the resulting equations for the action of the dilatation operator can be labeled by what appears to be a configuration of open strings, as well as labels specifying Young diagram r, as defined in figure (2.1). It is a nontrivial fact that the only labels that appear correspond to open string configurations that are consistent with the Gauss Law. This strongly supports the dual interpretation of these gauge

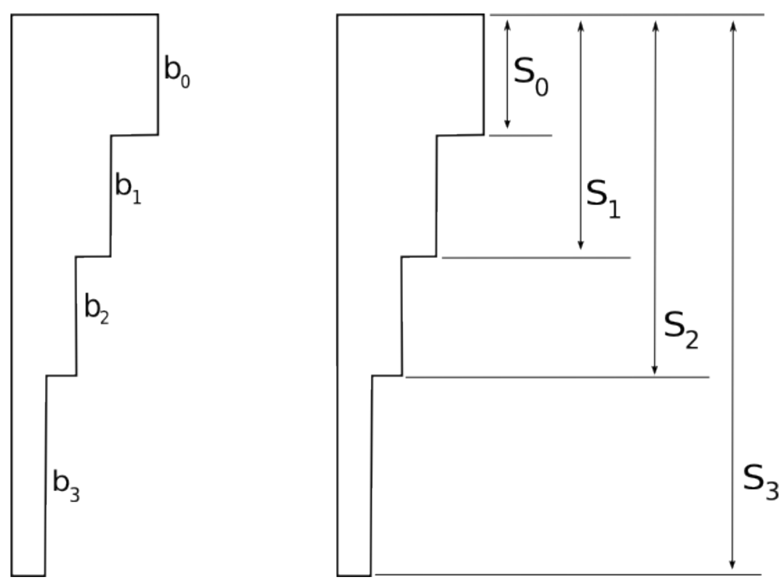


Figure 2.1: Definition of the b_is and s_is in terms of a Young diagram for $c = 4$ columns. The relation between the s_i and the b_i is easily read from the figure. For example, $s_2 = b_0 + b_1 + b_2$. Columns are ordered so that column length increases. They are then numbered starting from 0. For the Young diagram shown, the right most column is column 0 and the left most is column 3. The generalization to any c should be obvious.

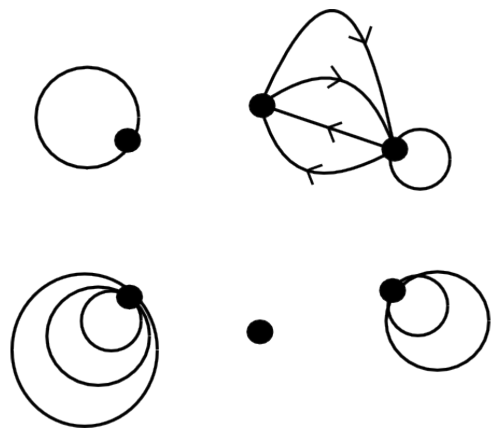


Figure 2.2: The label C for a system of 6 giants. $n_{12} = 4$ strings stretch between branes 1 and 2. There is one more string attached to brane 2. Two strings are attached to brane 3, 3 strings to brane 5 and a single string to brane 6. The dilatation operator action depends only on the strings stretching between different branes [36].

theory operators. The worldvolume of a giant graviton is compact so that the Gauss Law implies that, for any particular giant graviton, one must have the same number of strings starting on the giant as there are strings ending on the giant (see [11] for a detailed discussion). For an example of an allowed open string configuration, see figure (2.2). The operators $O_C(\{s_i\})$ below are specified by giving a Young diagram encoded in the $\{s_i\}$ (see fig. 2.1) and an open string configuration C, equivalent to the data provided for example in figure (2.2) or (2.3) below. For the configuration C with n_{ij} open strings stretching between branes i and j the one loop dilatation operator is given by

$$DO_C(\{s_i\}) = -g_{YM}^2 \sum_{\alpha\beta} n_{\alpha\beta}\Delta_{\alpha\beta}\, O_C(\{s_i\}), \tag{2.1}$$

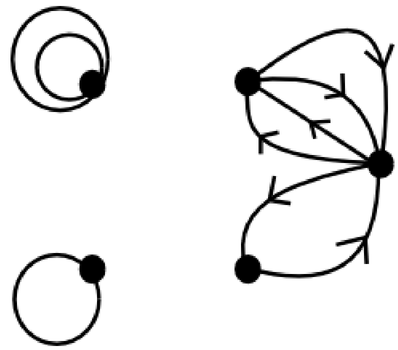


Figure 2.3: The label C for a system of 5 giants. $n_{12} = 4$ strings stretch between branes 1 and 2 and $n_{23} = 2$ strings stretching between branes 2 and 3. A string is attached to brane 4 and two strings are attached to brane 5.

where the operator Δ_{ij} acts as follows (Δ_{ij} only changes the values of s_i and s_j so that these are the only two variables that we display in the next equation)

$$\Delta_{ij} O_C(s_i, s_j) = -(c_i + c_j) O_C(s_i, s_j) + \sqrt{c_i c_j} (O_C(s_i + 1, s_j - 1) + O_C(s_i - 1, s_j + 1)) . \tag{2.2}$$

In this last equation c_a is the factor of the last box in column a. Recall that a box in row i and column j has a factor $N - i + j$. The primary goal of this chapter is to explain how to diagonalize (2.1). This is achieved by mapping the operators $O_C(s_i, s_j)$ into states in the carrier space of a specific $U(p)$ irreducible representation. The dilatation operator is mapped into a $u(p)$ valued operator and, as a result, can easily be diagonalized.

The article [50] proposed that for states with a large R-charge (which is the case we study here) one can carry out precise tests of the AdS/CFT correspondence (in a non-BPS sector) by comparing the expansion of the classical string energy with the corresponding quantum anomalous dimensions. As reviewed in [51], this was a very fruitful line of investigation. The large R-charge sector we consider has not been studied before. In an attempt to better understand our gauge theory results, we construct a system that reproduces some features of the anomalous dimensions we obtain. We will show that the dependence of the spectrum of anomalous dimensions on the parameters contained in the open string configuration C can be reproduced by the energies of normal modes of a system of masses coupled by springs.

2.3 Nonplanar dilatation operator

To start we will review a few elementary facts, familiar from angular momentum in quantum mechanics, that will play an important role later. The fundamental representation of $u(N)$ represents the elements of the Lie algebra as $N \times N$ matrices. The generators can be taken as

$$(E_{kl})_{ab} = \delta_{ak}\delta_{bl}, \qquad k, l, a, b = 1, 2, ..., N \, . \tag{2.3}$$

We will study the operators (the labeling is such that $i > j$ i.e. Q_{ij} is not defined if $i < j$)

$$Q_{ij} = \frac{E_{ii} - E_{jj}}{2}, \qquad Q_{ij}^{+} = E_{ij}, \qquad Q_{ij}^{-} = E_{ji} \, , \tag{2.4}$$

which obey the familiar algebra of angular momentum raising and lowering operators

$$[Q_{ij}, Q_{ij}^{+}] \;\; = \;\; Q_{ij}^{+}, \tag{2.5}$$

$$
\begin{aligned}
[Q_{ij}, Q^-_{ij}] &= -Q^-_{ij}, && (2.6)\\
[Q^+_{ij}, Q^-_{ij}] &= 2Q_{ij}\,. && (2.7)
\end{aligned}
$$

Although these commutators have been computed making use of the fundamental representation, we know that they would be the same if they had been computed in any representation and they define the representation independent Lie algebra.

General representations of these $su(2)$ subalgebras can be labeled with the eigenvalue of

$$
L^2_{ij} \equiv Q^-_{ij}Q^+_{ij} + Q^2_{ij} + Q_{ij} = Q^+_{ij}Q^-_{ij} + Q^2_{ij} - Q_{ij} \qquad (2.8)
$$

and states in the representation are labeled by the eigenvalue of Q_{ij}

$$
\begin{aligned}
Q_{ij}|\lambda, \Lambda\rangle &= \lambda|\lambda, \Lambda\rangle\,, && (2.9)\\
L^2_{ij}|\lambda, \Lambda\rangle &= (\Lambda^2 + \Lambda)|\lambda, \Lambda\rangle\,, \qquad -\Lambda \leq \lambda \leq \Lambda\,. && (2.10)
\end{aligned}
$$

Recall that

$$
Q^+_{ij}|\lambda, \Lambda\rangle = c_+|\lambda + 1, \Lambda\rangle\,, \qquad c_+ = \sqrt{(\Lambda + \lambda + 1)(\Lambda - \lambda)}\,, \qquad (2.11)
$$

and

$$
Q^-_{ij}|\lambda, \Lambda\rangle = c_-|\lambda - 1, \Lambda\rangle\,, \qquad c_- = \sqrt{(\Lambda + \lambda)(\Lambda - \lambda + 1)}\,. \qquad (2.12)
$$

The c operators E_{ii} commute so that we can always choose a basis in which they are simultaneously diagonal. Recall the definition of b_i $i = 0, 1, ..., c-1$ for a Young diagram with c columns, given in figure (2.1). The restricted Schur polynomials labeled by the Young diagram shown is identified with the state with $E_{ii} = 2(\,N - s_i\,)$. The advantage of identifying the restricted Schur polynomials with states of a $U(c)$ representation is that we can now write the dilatation operator as a $u(c)$ valued operator. In particular, the operators Δ_{ij} are

$$\Delta_{ij} = -\frac{1}{2}(E_{ii} + E_{jj}) + Q^{-}_{ij} + Q^{+}_{ij}\,. \tag{2.13}$$

For simplicity we will now focus on the case $c = 2$. In this case, identify

$$c_{-} = \sqrt{(N - b_0)(N - b_0 - b_1 + 1)}, \qquad c_{+} = \sqrt{(N - b_0 + 1)(N - b_0 - b_1)} \tag{2.14}$$

so that

$$\Lambda = \frac{1}{2}b_{1,\max}, \qquad \lambda = \frac{1}{2}b_1\,. \tag{2.15}$$

We will focus on $b_{1,\max}$ even so that Λ is integer. Not all states of the irreducible representation participate: because $b_1 \geq 0$ we have $\lambda \geq 0$. Thus, of the $2b_{1,\max} + 1$ states, only $b_{1,\max} + 1$ of them remain. Finally, we are interested in the limit $b_{1,\max} \sim \sqrt{N}$ with $N \to \infty$. It is only in this limit that (2.1) holds. Away from this limit (2.1) picks up corrections of order $1/b_{1,\max}$ [36]. There is an obvious extension of this discussion for $c > 2$.

2.4 Strings between 2 giants

Consider a system of p-giants with p arbitrary except that we fix it to be $O(1)$. The Young diagrams relevant for these states have p columns. Consider the situation for which we have $2n_{ij}$ strings stretching between giants i and j. See figure 2.2 for an example of the label C when $p = 6$ and $n_{12} = 4$. The results of this section are also directly applicable to the case that pairs of mutually distinct branes have strings stretching between them. In this case, the action of the dilatation operator is given by a sum of terms which commute and can each be diagonalized using the same method.

2.4.1 Construction of creation and annihilation operators:

In this case

$$D = -2n_{ij} g_{YM}^2 \Delta_{ij} \, . \tag{2.16}$$

For a creation operator we want

$$[D, A^\dagger] = \alpha A^\dagger \tag{2.17}$$

with $\alpha > 0$. Make the ansatz

$$A^\dagger = aE_{ii} + bE_{jj} + cE_{ij} + dE_{ji} \, . \tag{2.18}$$

It is straight forward to verify that (2.17) implies

$$A^\dagger = \frac{1}{2}(E_{ii} - E_{jj}) + \frac{1}{2}E_{ij} - \frac{1}{2}E_{ji} \tag{2.19}$$

and $\alpha = 4n_{ij}g_{YM}^2$. To implement the condition $b_1 > 0$ we need to require that the oscillator wave function has a node at the origin - thus only odd parity (i.e. odd under $b_1 \to -b_1$) states are kept. This implies that half the states are kept so that we land up with a frequency of $8n_{ij}g_{YM}^2$. For $n_{ij} = 1$ this is in complete agreement with spectrum computed in [33], [34]. Thus, the spectrum of the dilatation operator is

$$\lambda = (8n_{ij}g_{YM}^2)n \tag{2.20}$$

with n a not negative integer. This is in complete agreement with the spectrum computed in [35]. There is a simple algebra obeyed by the creation and annihilation operators of this oscillator

$$[A, A^\dagger] = \frac{1}{2}(E_{ii} + E_{jj}) + \Delta_{ij} = 2N - 2b_0 - b_1 - \frac{D}{2g_{YM}^2} = b_{1,\max} - \frac{D}{2g_{YM}^2}\,. \tag{2.21}$$

If we introduce the oscillators $A = \sqrt{b_{1,\max}}a$ we find, for any state of finite energy in the $b_{1,\max} \to \infty$ limit

$$[a, a^\dagger] = 1 - \frac{D}{2b_{1,\max}g_{YM}^2} = 1\,. \tag{2.22}$$

2.4.2 Connection to continuum limit:

We can ask how this compares to the frequencies computed after we have taken the continuum limit of the Δ_{ij}, described in appendix H of [36]. From that appendix, we find

$$D = -2g_{YM}^2 n_{ij} M_{ab} \left(\frac{\partial}{\partial x_a} \frac{\partial}{\partial x_b} - \frac{x_a x_b}{4} \right) \tag{2.23}$$

with

$$M_{11} = M_{22} = 1, \qquad M_{ij} = M_{ji} = -1. \tag{2.24}$$

The two frequencies are $4n_{ij}g_{YM}^2$ and 0. The zero frequency corresponds to the motion of the center of mass ($x_{cm} \propto x_i + x_j$). Fix this center of mass motion because the system of giants is fixed. The nonzero frequency reproduces what we found above, again after dropping half the states. Clearly then, the continuum limit catches the complete large $b_{1,\max}$ dynamics.

2.4.3 Open spring model:

The operators we study are nearly supersymmetric so that it is natural to expect that they correspond to fast moving strings on the D-brane. It is thus natural to associate them with null trajectories in $\mathrm{AdS}_5 \times \mathrm{S}^5$ that are contained in the D-brane worldvolume. This analysis has been performed in [52]. See [53, 54] for additional relevant and useful discussion. The resulting null trajectory leads to a pp-wave and the light cone Hamiltonian is related to the anomalous dimension

$$H_{\text{light cone}} = \frac{1}{P^+} H_\perp = \Delta - n_Z - n_Y = D, \tag{2.25}$$

where $H_\perp$ describes string oscillations in the perpendicular (to string motion) directions and $n_Z(n_Y)$ are the number of Zs (Ys) in the operator. See also [55] which is relevant to our discussion. What should we use for $H_\perp$? When we change the number of Z's in the giant we change the radius of the circle on which it is orbiting; this corresponds to the direction transverse to the giants direction of motion - i.e. the oscillator that we have diagonalized above is describing oscillations in the perpendicular (to string motion) directions. The Gauss Law picture of [36] suggests that the configuration we study consists of $2n_{12}$ strings stretching between the two giants. Each string is a single Y - so these are short strings that we will model as two endpoints. The spring constant for springs connected in parallel is the sum of the individual spring constants. Thus, the configuration we study will have $k \propto n_{12}$. The scale of the anomalous dimension is set by g_{YM}^2. Under AdS/CFT the anomalous dimension maps to an energy, so that g_{YM}^2 naturally sets the energy scale. To ensure that the scale of the potential energy is set by g_{YM}^2 we will choose the spring constant $k \propto g_{YM}^2$. Making a choice of a constant that will prove to be convenient below, we set $k = 4g_{YM}^2 n_{12}$. Notice that this matches the second term in (2.23). The usual rules for quantization identify ∂_x with ip so that the effective Hamiltonian is

$$H_\perp = 2g_{YM}^2 n_{ij} \Big((p_i - p_j)^2 + \frac{(x_i - x_j)^2}{4} \Big). \tag{2.26}$$

It would be very interesting to see if an honest derivation of this Hamiltonian can be given, starting from a system of excited giant gravitons. The energy of the normal modes of this system match the anomalous dimensions.

2.5 Strings between 3 giants

In this section we consider the situation for which we have n_{ij} strings stretching between giants i and j, n_{jk} strings stretching between giants j and k and n_{ik} strings stretching between giants i and k. See figure (2.3) for an example of the label C when $p = 5$ and $n_{12} = 4$, $n_{23} = 2$ and $n_{13} = 0$. The results of this section are also directly applicable to the case that any number of pairs and/or triples of mutually distinct branes have strings stretching between them. Just like in the last section, in this case the action of the dilatation operator is given by a sum of terms which commute and can each be diagonalized using the same method.

2.5.1 Construction of Creation and Annihilation Operators:

In this case, to be general, we should introduce the parameters n_{ij}, n_{ik} and n_{jk} (repeated indices are not summed)

$$D = -g_{YM}^2(n_{ij}\Delta_{ij} + n_{ik}\Delta_{ik} + n_{jk}\Delta_{jk})\,. \tag{2.27}$$

For any label C, the Gauss Law implies that $n_{ij} + n_{ik}$ is even, $n_{ij} + n_{jk}$ is even and $n_{ik} + n_{jk}$ is even. For a creation operator we again want (2.17). Make the ansatz

$$A^\dagger = aE_{ii} + bE_{ij} + cE_{ik} + dE_{ji} + eE_{jj} + fE_{jk} + gE_{ki} + hE_{kj} + iE_{kk}\,. \tag{2.28}$$

Then (2.17) gives 3 $A^\dagger$s. There is a nice analytic formula for the frequencies of these operators $\Omega_i = 2g_{YM}^2\omega_i$ where

$$\omega_1 = 2\gamma, \qquad \omega_2 = n_{ik} + n_{ij} + n_{jk} + \gamma, \qquad \omega_3 = n_{ik} + n_{ij} + n_{jk} - \gamma, \tag{2.29}$$

where

$$\gamma = \sqrt{n_{ij}^2 + n_{ik}^2 + n_{jk}^2 - n_{ij}n_{ik} - n_{jk}n_{ik} - n_{ij}n_{jk}}\,. \tag{2.30}$$

This proves that the spectrum of three giant system is indeed that of a set of oscillators. For the frequency ω_1 we find

$$\begin{aligned} A_1 \;=\; & \mathcal{N}_1\Big[(n_{ij} - n_{ik})(n_{ik} - n_{ij} - \gamma)E_{ii} \\ & + \;\left((n_{ik} - n_{ij})(n_{ik} - n_{ij} - \gamma) + (n_{ik} - n_{jk})(n_{ik} - n_{jk} - \gamma)\right) E_{ij} \\ & -(n_{ik} - \gamma - n_{jk})(n_{ik} - n_{jk})E_{ik} - (n_{ij} - n_{jk} + \gamma)(n_{ij} - n_{jk})E_{ji} \\ & +(n_{ij} - n_{ik} + \gamma)(n_{ij} - n_{jk})E_{jj} + (n_{ij} - n_{jk})(n_{ik} - n_{jk})E_{jk} \\ & + \left((n_{jk} - n_{ij})(n_{jk} - n_{ij} - \gamma) + (n_{ik} - n_{ij})(n_{ik} - n_{ij} - \gamma)\right) E_{ki} \\ & -((n_{ij} - n_{ik})(n_{ij} - n_{ik} + 2\gamma) + \gamma^2)E_{kj} \\ & -(n_{ij} - n_{ik} + \gamma)(n_{ik} - n_{jk})E_{kk}\Big], \end{aligned} \tag{2.31}$$

where

$$\begin{aligned} \mathcal{N}_1^{-2} \;=\; & (n_{ij} - n_{ik})^2(n_{ik} - n_{ij} - \gamma)^2 + ((n_{ik} - n_{ij})(n_{ik} - n_{ij} - \gamma) + (n_{ik} - n_{jk})(n_{ik} - n_{jk} - \gamma))^2 \\ & +(-n_{ik} + n_{jk} + \gamma)^2(n_{ik} - n_{jk})^2 + (-n_{ij} + n_{jk} - \gamma)^2(n_{ij} - n_{jk})^2 + (n_{ij} - n_{ik} + \gamma)^2(n_{ij} - n_{jk}) \end{aligned}$$

$$+(n_{ij}-n_{jk})^2(n_{ik}-n_{jk})^2+((n_{jk}-n_{ij})(-n_{ij}+n_{jk}-\gamma)+(n_{ik}-n_{ij})(n_{ik}-n_{ij}-\gamma))^2$$
$$+(-(n_{ij}-n_{ik})(n_{ij}-n_{ik}+2\gamma)-\gamma^2)^2+(n_{ik}-n_{ij}-\gamma)^2(n_{ik}-n_{jk})^2\,.$$

For the frequency ω_2 we find

$$\begin{aligned} A_2 &= \mathcal{N}_2\Big((n_{jk}-n_{ij}-\gamma)(E_{ii}+E_{ji}+E_{ki})+(n_{ij}-n_{ik}+\gamma)(E_{ij}+E_{jj}+E_{kj}) \\ &\quad +(n_{ik}-n_{jk})(E_{ik}+E_{jk}+E_{kk})\Big), \end{aligned}$$

where

$$\mathcal{N}_2^{-1}=\sqrt{6\gamma(2\gamma+2n_{ij}-n_{jk}-n_{ik})}\,. \tag{2.32}$$

For the frequency ω_3 we find

$$\begin{aligned} A_3 &= \mathcal{N}_3\Big((n_{jk}-n_{ij}+\gamma)(E_{ii}+E_{ji}+E_{ki}) \\ &\quad +(n_{ij}-n_{ik}-\gamma)(E_{ij}+E_{jj}+E_{kj})+(n_{ik}-n_{jk})(E_{ik}+E_{jk}+E_{kk})\Big), \end{aligned} \tag{2.33}$$

where

$$\mathcal{N}_3^{-1}=\sqrt{6\gamma(2\gamma-2n_{ij}+n_{jk}+n_{ik})}\,. \tag{2.34}$$

These oscillators close the following algebra

$$\begin{aligned}
[A_2, A_2^\dagger] &= \frac{4}{3}(3N - 3b_0 - 2b_1 - b_2) + \frac{1}{3}(\Delta_{ij} + \Delta_{ik} + \Delta_{jk}) - P_2, && (2.35)\\
[A_2, A_3^\dagger] &= -A_1, \qquad [A_2, A_1^\dagger] = A_3, \qquad [A_2, A_3] = 0 = [A_1, A_2], && (2.36)\\
[A_3, A_3^\dagger] &= \frac{4}{3}(3N - 3b_0 - 2b_1 - b_2) + \frac{1}{2}(\Delta_{ij} + \Delta_{ik} + \Delta_{jk}) - P_3, && (2.37)\\
[A_3, A_1] &= A_2, \qquad [A_3, A_1^\dagger] = 0, \qquad [A_1, A_1^\dagger] = P_3 - P_2\,, && (2.38)
\end{aligned}$$

where

$$\begin{aligned}
(4\gamma^2 - 2\gamma(n_{jk} + n_{ik} - 2n_{ij})P_2 &= (n_{ij} - n_{jk} + \gamma)^2 E_{ii} + (n_{ij} - n_{ik} + \gamma)^2 E_{jj}\\
&+ (n_{jk} - n_{ik})(n_{ij} - n_{jk} + \gamma)(E_{ik} + E_{ki})\\
&+ (n_{jk} - n_{ij} - \gamma)(n_{ij} - n_{ik} + \gamma)(E_{ji} + E_{ij})\\
&+ (n_{ij} - n_{ik} + \gamma)(n_{ik} - n_{jk})(E_{jk} + E_{kj})\\
&+ (n_{ik} - n_{jk})^2 E_{kk} && (2.39)
\end{aligned}$$

and

$$\begin{aligned}
(4\gamma^2 + 2\gamma(n_{jk} + n_{ik} - 2n_{ij})P_3 &= (n_{ij} - n_{jk} - \gamma)^2 E_{ii} + (n_{ij} - n_{ik} - \gamma)^2 E_{jj}\\
&+ (n_{ij} - n_{ik} - \gamma)(n_{jk} - n_{ij} + \gamma)(E_{ij} + E_{ji})\\
&+ (n_{jk} - n_{ik})(-n_{jk} + n_{ij} - \gamma)(E_{ik} + E_{ki})\\
&+ (n_{ik} - n_{jk})(n_{ij} - n_{ik} - \gamma)(E_{jk} + E_{kj})\\
&+ (n_{ik} - n_{jk})^2 E_{kk}\,. && (2.40)
\end{aligned}$$

Note also that

$$[A_2, A_2^\dagger] = [A_3, A_3^\dagger] + [A_1, A_1^\dagger], \tag{2.41}$$

$$[P_2, A_2^\dagger] = A_2^\dagger, \qquad [P_3, A_3^\dagger] = A_3^\dagger\,. \tag{2.42}$$

Thus, if we set

$$A_1 = \sqrt{3N - 3b_0 - 2b_1 - b_2}\sqrt{\frac{4}{3}}a_1, \tag{2.43}$$

$$A_2 = \sqrt{3N - 3b_0 - 2b_1 - b_2}\sqrt{\frac{4}{3}}a_2, \tag{2.44}$$

$$A_3 = \sqrt{3N - 3b_0 - 2b_1 - b_2}\sqrt{\frac{4}{3}}a_3 \tag{2.45}$$

and consider the limit in which $\sqrt{3N - 3b_0 - 2b_1 - b_2} \sim \sqrt{N} \to \infty$ we find

$$[a_1, a_1^\dagger] = 0, \qquad [a_2, a_2^\dagger] = 1, \qquad [a_3, a_3^\dagger] = 1 \tag{2.46}$$

and all other commutators vanish. Thus, we only have 2 oscillators. After keeping only the states that have a node at $b_1 = 0$, we find that these oscillators have a frequency $4g_{YM}^2\omega_2$ and $4g_{YM}^2\omega_3$.

2.5.2 Connection to Continuum Limit:

We can again ask how this compares to the frequencies computed after we have taken the continuum limit of the Δ_{ij}, described in appendix H of [36]. From that appendix, we find

$$D = -g_{YM}^2 M_{ab}\left(\frac{\partial}{\partial x_a}\frac{\partial}{\partial x_b} - \frac{x_a x_b}{4}\right) \tag{2.47}$$

with

$$M = \begin{bmatrix} n_{ij}+n_{ik} & -n_{ij} & -n_{ik} \\ -n_{ij} & n_{ij}+n_{jk} & -n_{jk} \\ -n_{ik} & -n_{jk} & n_{ik}+n_{jk} \end{bmatrix}. \tag{2.48}$$

The three frequencies are $\Lambda_i = 2g_{YM}^2\lambda_i$, where

$$\lambda_1 = 0, \qquad \lambda_2 = n_{ik}+n_{ij}+n_{jk}+\gamma, \qquad \lambda_3 = n_{ik}+n_{ij}+n_{jk}-\gamma, \tag{2.49}$$

and γ is defined as above. The zero frequency again corresponds to the center of mass, which we fix. Only the states with a node at $b_1 = 0$ will be retained, which doubles the above frequencies. Notice that the continuum limit has caught the full large b_1 spectrum.

2.5.3 Open string model:

Arguing exactly as we did in the last section leads to

$$\begin{aligned} H_\perp &= g_{YM}^2 n_{ij}\Big((p_i-p_j)^2 + \frac{(x_i-x_j)^2}{4}\Big) \\ &\quad + g_{YM}^2 n_{ik}\Big((p_i-p_k)^2 + \frac{(x_i-x_k)^2}{4}\Big) \end{aligned} \tag{2.50}$$

$$+g_{YM}^2 n_{jk}\Big((p_j - p_k)^2 + \frac{(x_j - x_k)^2}{4}\Big).$$

It is easy to solve for the two normal modes of this oscillator. The energy of these normal modes again match the anomalous dimensions.

2.6 Strings between 4 giants

The methods that we have outlined above work generally for any configuration C of open strings. However, not surprisingly, it becomes increasingly difficult to obtain simple analytic expressions. Obviously it is a simple matter to get explicit numerical results for any C. In this section we will simply write the equations one needs to solve in the case that strings stretch in an arbitrary way between four giant gravitons.

2.6.1 Construction of Creation and Annihilation Operators:

In this case, to be general, we should introduce the parameters n_{ij}, n_{ik}, n_{il}, n_{jk}, n_{jl} and n_{kl}

$$D = -g_{YM}^2(n_{ij}\Delta_{ij} + n_{ik}\Delta_{ik} + n_{il}\Delta_{il} + n_{jk}\Delta_{jk} + n_{jl}\Delta_{jl} + n_{kl}\Delta_{kl}). \tag{2.51}$$

For any C, $n_{ij} + n_{ik} + n_{il}$ is even, $n_{ij} + n_{jk} + n_{jl}$ is even, $n_{ik} + n_{jk} + n_{kl}$ is even and $n_{il} + n_{jl} + n_{kl}$ is even. For a creation operator we again want (2.17). This leads us to the eigenproblem of a 16×16 matrix. For general parameters we get 6 $A^{\dagger}$s. Only three of these survive in the large $b_{1,\max}$ limit. The frequencies of the oscillators which survive are roots of

$$
\begin{aligned}
&x^3 - 2(n_{ij} + n_{jl} + n_{ik} + n_{jk} + n_{kl} + n_{il})x^2 \\
&+(3n_{ik}n_{kl} + 4n_{ij}n_{kl} + 3n_{ij}n_{il} + 3n_{ik}n_{il} + 3n_{ik}n_{jk} + 3n_{il}n_{jl} + 3n_{kl}n_{il} + 4n_{ik}n_{jl} + 4n_{jk}n_{il} \\
&+3n_{jk}n_{jl} + 3n_{kl}n_{jl} + 3n_{ij}n_{jk} + 3n_{ij}n_{jl} + 3n_{ij}n_{ik} + 3n_{jk}n_{kl})x \\
&-4n_{ij}n_{kl}n_{jl} - 4n_{ij}n_{jk}n_{kl} - 4n_{ij}n_{ik}n_{kl} - 4n_{ij}n_{kl}n_{il} - 4n_{jk}n_{ik}n_{kl} \\
&-4n_{jk}n_{ik}n_{il} - 4n_{jk}n_{ik}n_{jl} - 4n_{jl}n_{ik}n_{kl} - 4n_{jl}n_{ik}n_{il} - 4n_{jk}n_{kl}n_{il} - 4n_{ij}n_{jk}n_{il} \\
&-4n_{ij}n_{jk}n_{jl} - 4n_{ij}n_{ik}n_{jl} - 4n_{ij}n_{ik}n_{il} - 4n_{jl}n_{jk}n_{il} - 4n_{jl}n_{kl}n_{il} = 0\,.
\end{aligned}
\tag{2.52}
$$

It is now straightforward to construct the algebra of the resulting oscillators as well as their large b_1 limit. We again find that this result is consistent with both the continuum limit of D (as outlined in appendix H of [36]) and the model of masses and springs. This computation (as well as the extension to situations in which strings interconnect more than 4 giants) is straight forward but a little tedious.

In summary, two things have been achieved in this chapter. The continuum limit of the dilatation operator was obtained in appendix H of [36]. What is the relation between the study of [36] and our result here? In [36] the large $b_{1,\max}$ limit was taken and the resulting eigenvalue problem was solved. Here we have first solved the eigenvalue problem and have then taken the large $b_{1,\max}$ limit. Our result is in perfect agreement with the continuum limit obtained in [36], and justifies the use of the simple Harmonic oscillator Hamiltonian obtained there. In particular, in the continuum limit the variables s_i become continuous coordinates and the operators of a good scaling dimension are obtained by summing restricted Schur polynomials with coefficients given by the harmonic oscillator wave functions. The second thing we have achieved is that the values of the anomalous dimensions have been reproduced by the normal mode frequencies of a coupled system of open strings. This provides non-trivial support for their interpretation in the dual theory as excited giant gravitons.

These results are in complete agreement with a recent paper [56]. In this work, it is shown that the continuum limit of the oscillator algebra can also be reached by an Inonu-Wigner contraction of the $u(2)$ algebra living inside the $u(p)$.

Chapter 3

Nonplanar integrability at two loops

3.1 Outline of chapter

In this chapter we compute the action of the two loop dilatation operator on restricted Schur polynomials that belong to the su(2) sector, in the displaced corners approximation. In this non-planar large N limit, operators that diagonalize the one loop dilatation operator are not corrected at two loops. The resulting spectrum of anomalous dimensions is related to a set of decoupled harmonic oscillators, indicating integrability in this sector of the theory at two loops. The anomalous dimensions are a non-trivial function of the 't Hooft coupling, with a spectrum that is continuous and starting at zero at large N, but discrete at finite N. This work was reported in JHEP 1210 (2012) 144.

3.2 Introduction and Questions

The discovery of a rich integrable structure [23], [49] underlying the planar limit of $\mathcal{N} = 4$ super Yang-Mills theory is fascinating. It has allowed tremendous progress in exploring planar $\mathcal{N} = 4$ super Yang-Mills theory and free IIB superstrings on the $AdS_5 \times S^5$ background, providing novel support for the AdS/CFT correspondence [43], [57], [58]. Indeed, there is reason to hope that the exact spectrum of anomalous dimensions can be found in the planar limit. We refer the reader to [31] for a comprehensive recent review. Even more recently, attempts to compute the three point functions of the theory using integrability have begun [59], [60], [61], [62], [63], [64]. An optimist might hope that planar $\mathcal{N} = 4$ super Yang-Mills theory can be solved exactly.

A natural question to ask now, is if integrability is present beyond the planar limit. In this chapter we will study the large N limit of the anomalous dimensions of a class of operators, restricted Schur polynomials [11], [12], that have classical dimension of order N. For these operators, summing the planar diagrams does not capture the large N limit [7].

There has by now been some progress in the study of these highly non-trivial large N limits. The basic new ingredient has been to use the representation theory of symmetric and unitary groups. In particular, the problem of diagonalizing the free field inner product for single and multi-matrix operators, to all orders in $1/N$, has been solved. In the half-BPS sector a complete set of operators is given by the Schur polynomials $\chi_R(Z)$ [8]. They are labeled with Young diagrams R. Operators with R having order one rows of length order N or order one columns of length order N are dual to giant gravitons [4], [6], [5]. If R has $O(N^2)$ boxes the corresponding operator is dual to an LLM geometry [65]. The problem of diagonalizing the free field inner product for multi-matrix operators, while preserving global symmetries, was solved in [46], [66]. A basis relevant for the description of brane-antibrane systems was given in [45], [67]. Finally, the restricted Schur basis was proved to diagonalize the inner product in [12] and to be complete in [32].

In this chapter our focus is on restricted Schur polynomials and on the su(2) sector of the theory. In the su(2) sector one considers a restricted Schur polynomial built mainly from one type of matrix field Z, doped with impurities Y. The restricted Schur polynomial is given by

$$\chi_{R,(r,s)\alpha\beta}(Z,Y) = \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \mathrm{Tr}_{(r,s)\alpha\beta} \left(\Gamma^{(R)}(\sigma) \right) \mathrm{Tr}(\sigma Z^{\otimes n} \otimes Y^{\otimes m}). \tag{3.1}$$

The polynomial above is built using n Zs and m Ys. $R \vdash m+n$ is an irreducible representation (irrep) of S_{n+m} and $\Gamma^{(R)}(\sigma)$ is a matrix representing σ in this irrep R. Spelling out the structure of the second trace

$$\mathrm{Tr}(\sigma Z^{\otimes n} \otimes Y^{\otimes m}) = Y^{i_1}_{i_{\sigma(1)}} \cdots Y^{i_n}_{i_{\sigma(n)}} Z^{i_{n+1}}_{i_{\sigma(n+1)}} \cdots Z^{i_{n+m}}_{i_{\sigma(n+m)}} \tag{3.2}$$

we see that σ acts on $n+m$ indices (we think of each index as a “slot” that can be populated with a matrix) with the first m slots associated to Ys and the next n slots associated to Zs. Consider the $S_n \times S_m$ subgroup that permutes the Z and Y slots separately. The irrep $R \vdash m+n$ will, in general, subduce many irreps of this subgroup. Denote these irreps by $(r,s)\alpha$. $r \vdash n$ is an irrep of S_n and $s \vdash m$ is an irrep of S_m so that (r,s) is an irrep of the $S_n \times S_m$ subgroup. In general (r,s) will be subduced more than once. The label α specifies which copy of the irrep we consider. There is a significant simplification when R has only two rows or columns: all irreps of $S_n \times S_m$ are subduced without multiplicity so that the α index can be dropped. To summarize, the restricted Schur polynomial is labeled by three Young diagrams, one for the Zs (which is a representation of S_n), one for the impurities (which is a representation of S_m) and one for the "composition" (which is any representation of S_{n+m} that subduces (r, s) when restricted to the $S_n \times S_m$ subgroup). A comment on notation is in order. We will sometimes need to refer to two representations of the same group in a single equation. In this case, the letters r, t will be used to denote representations of S_n, the letters s, u will be used to denote representations of S_m and the letters R, T will be used to denote representations of S_{n+m}.

Another ingredient appearing in (3.1) is the restricted trace $\mathrm{Tr}_{(r,s)\alpha\beta}\left(\Gamma^{(R)}(\sigma)\right)$. The restricted trace is a "trace" over the elements of a subspace whose column index belongs to $S_n \times S_m$ irrep $(r, s)\alpha$ and whose row index belongs to $S_n \times S_m$ irrep $(r, s)\beta$[1]. To concretely see how this works it is useful to review the method of [36], [68] which decomposes an S_{n+m} irrep into $S_n \times S_m$ irreps. Recall that standard tableaux are labelings of the Young diagram with integers 1 to $m + n$ that are strictly decreasing down the columns and along the rows. The standard tableaux provide a basis of states for irrep R of S_{n+m}. Following [36], [68], we perform the reduction from S_{n+m} to S_n by introducing partially labeled Young tableaux, which have m boxes labeled, and the remaining n boxes unlabeled. These partially labeled Young tableaux stand for a collection of states. The integers 1 to m are in fixed locations and the integers $m + 1, ..., m + n$ are then distributed in all possible locations thereby recovering collections of standard tableaux. Each such collection is a complete irrep of S_n. The unlabeled boxes determine the irrep r of S_n. Our task is now to combine the partially labeled Young tableaux with r fixed, into good irreps s of S_m. An irrep s can occur

[1] We put the word "trace" in inverted commas because if $\alpha \neq \beta$ one is not even summing over diagonal matrix elements!

with some multiplicity. The labels α, β run over this multiplicity. Concretely we can write the restricted trace as

$$\mathrm{Tr}_{(r,s)\alpha\beta}\left(\Gamma^{(R)}(\sigma)\right) = \mathrm{Tr}_R\left(P_{R,(r,s)\alpha\beta}\Gamma^{(R)}(\sigma)\right), \tag{3.3}$$

where

$$P_{R,(r,s)\alpha\beta} = \mathbf{1}_r \otimes \sum_{i=1}^{d_s} |s\,\alpha\,;\,i\rangle\langle s\,\beta\,;\,i|. \tag{3.4}$$

Here i is a state label for the irrep s, d_s is the dimension of irrep s and 1_r is the projector inside the carrier space of R onto irrep r. By definition, 1_r gives 1 when acting on a partially labeled Young tableaux whose unlabeled boxes have shape r and gives zero otherwise. To proceed further, we need to construct the factor

$$\sum_{i=1}^{d_s} |s\,\alpha\,;\,i\rangle\langle s\,\beta\,;\,i| \tag{3.5}$$

appearing in (3.4). The construction of this factor developed in [36], [68], applies when the corners of Young diagram R are well separated. Each box in the Young diagram can be assigned a factor; the box in row j and column i has a factor $N+i-j$. We consider only operators labeled by R that have all rows of different length. The right most box of each row therefore defines a corner on the right hand side of the Young diagram. These corners of the Young diagram are well separated when the difference between the factors of these right most boxes are large. In our case, these differences for all corners on the right hand side of the Young diagram, go to infinity as we take the large N limit. We will call this the *displaced corners approximation.* As soon as the length of any pair of rows becomes similar, the displaced corners approximation no longer applies. In this limit [36], [68] have shown that the factor (3.4) can be constructed using simple Unitary group representation theory.

Given this progress on diagonalizing the free field inner product it is natural to turn next to the spectrum of the dilatation operator in these non-planar large N limits. Initial numerical studies showed, remarkably, that the spectrum of the dilatation operator is that of a set of decoupled oscillators. Early studies computed the exact action of the dilatation operator and then took the large N limit as a final step. These computations are quite involved and it is not easy to obtain general results. Indeed, [33] focused on two impurities while [34] considered 3 or 4 impurities. By working in the displaced corners approximation, [35] was able to directly implement the simplifications of the large N limit allowing the computation of results for an arbitrary number of impurities but under the constraint that R, r, s have at most two columns or rows. This was then extended beyond the su(2) sector in [37] and to an arbitrary number of rows in [36]. This extension used a novel Schur-Weyl duality [36], [68] that emerges at large N in the displaced corners approximation. Using this novel Schur-Weyl duality, the states $|s\,\mu_1\,;\,i\rangle$ appearing in (3.4) are states of a $\mathrm{U}(p)$ representation where p is the number of rows or columns of the restricted Schur polynomial. This allows us to trade the pair $s\,\mu_1$ for a Gelfand-Tsetlin pattern if we wish. These results have a direct application to the sl(2) sector [40]. In the displaced corners approximation the action of the dilatation operator has an interesting structure. The eigenproblem of the anomalous dimensions factors into a product of two problems, one for the Zs involving Young diagram r and one for the Ys involving Young diagram s. In [36], based on numerical results, a conjecture for the solution to the eigenproblem involving the s label was given. This conjecture has now been proven in [14]. The starting point of [14] is a proof that the number of excited giant graviton states as constrained by the Gauss Law, matches the number of restricted

Schur polynomials in the gauge theory. The proof proceeds by associating excited giant graviton states to elements of a double coset involving permutation groups. Making heavy use of the ideas and methods of [46], [66]. Fourier transformation on the double coset suggests an ansatz for the operators of a good scaling dimension. The operators obtained in this way, denoted $O_{R,r}(\sigma)$, are labeled by an element of a double coset σ and by the Young diagrams R and r. In [14] it was proven that this ansatz indeed provides the conjectured diagonalization. Further, since the double coset structure is determined entirely by the Gauss Law which holds at all loops, these results suggest that the operators constructed in [14], may be relevant at higher loops. This is an issue we will manage to probe in this chapter. The eigenproblem on the r label has been considered in [69]. It is written in terms of a difference operator. The basic observation of [69] is to realize that this difference operator is an element of the Lie algebra of $U(p)$ when r has p rows or columns. Exploiting this insight [69] argued that the eigenproblem on the r label is related to a system of p particles in a line with 2-body harmonic oscillator interactions.

We can now give the set of questions that motivated this chapter. As just discussed, the dilatation operator of $\mathcal{N} = 4$ super Yang-Mills theory is integrable in the large N displaced corners approximation of the su(2) sector at one loop [35], [36], [14]. The first question we wish to address is

1 **Is the dilatation operator integrable in the large N displaced corners approximation at higher loops?**

Although we are not able to give a complete answer to this question, we will test integrability at two loops. We can sharpen the above question. As described above, the action of the dilatation operator factorizes into an action on the Young diagram associated with the Zs and an action on the Young diagram associated with the Ys.

The eigenproblem associated with the Ys appears (see [14]) to be determined by the Gauss Law constraint, which should hold at all higher loops. This motivates the question

2 **Do the $O_{R,r}(\sigma)$ of [14] continue to solve the Y eigenproblem at higher loops?**

The Z eigenproblem was solved in [69], and as described in chapter 2, by mapping it to a system of p particles in a line with 2-body harmonic oscillator interactions. The basic observation was to show that the operator to be diagonalized is an element of the Lie algebra of $U(p)$ when r has p rows or columns. Our third question is

3 **Can the two loop Z eigenproblem be mapped to a system of p particles, again using the Lie algebra of $U(p)$?**

The one loop spectrum of anomalous dimensions has some interesting features. One would have expected the eigenvalues of the one loop dilatation operator to be a function of the 't Hooft coupling. We find they are given by an integer times g_{YM}^2. It is not completely clear how this should be interpreted. By computing the two loop correction to the anomalous dimension and requiring that it is small compared to the leading term, we hope to gain insight into both the interpretation of our results and in the precise limit that should be taken to get a sensible perturbative expansion. This motivates our fourth question

4 **Does the two loop correction to the anomalous dimension determine the precise limit that should be taken to get a sensible perturbative expansion?**

These questions are all answered in the discussion section of this chapter. We will find that this limit of the theory continues to be integrable at two loops, that the one loop operators with a good scaling dimension are not modified at two loops and finally, that our perturbative expansion is sensible in the conventional 't Hooft limit.

This chapter is organized as follows: In the next section we will compute the action of the two loop dilatation operator in the large N limit. We do this under the assumption that the number of Zs (which we denote by n) is much larger than the number of Ys (which we denote by m). The condition $m \ll n$ is needed to ensure that the displaced corners approximation is justified. The result of this computation is given in (3.24). In section 3.4, we diagonalize the dilatation operator and obtain the spectrum of anomalous dimensions to two loops. Section 3.5 is used to discuss these results and their relevance for the AdS/CFT correspondence.

3.3 Two Loop Dilatation Operator

Our goal is to evaluate the action of the two loop dilatation operator [49]

$$\begin{aligned} D_4 = & - 2g^2 : \mathrm{Tr}\left(\left[[Y,Z], \frac{\partial}{\partial Z}\right]\left[\left[\frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}\right], Z\right]\right) : \\ & - 2g^2 : \mathrm{Tr}\left(\left[[Y,Z], \frac{\partial}{\partial Y}\right]\left[\left[\frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}\right], Y\right]\right) : \\ & - 2g^2 : \mathrm{Tr}\left(\left[[Y,Z], T^a\right]\left[\left[\frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}\right], T^a\right]\right) : \end{aligned} \tag{3.6}$$

$$g = \frac{g_{YM}^2}{16\pi^2} \tag{3.7}$$

on restricted Schur polynomials. The normal ordering symbols here indicate that derivatives within the normal ordering symbols do not act on fields inside the normal ordering symbols. For the operators we study, $n \gg m$ so that only the first term in D_4 will contribute. We have in mind a systematic expansion in two parameters: $\frac{1}{N}$ and $\frac{m}{n}$. In Appenidx B we show that keeping only the first term in D_4 corresponds to the computation of the leading term in this double expansion. The evaluation of the action of the one loop dilatation operator was carried out in [34]. The two loop computation uses many of the same techniques but there are a number of subtle points that must be treated correctly. The computation can be split into the evaluation of two types of terms, one having all derivatives adjacent to each other (for example $\mathrm{Tr}(ZYZ\partial_Z\partial_Y\partial_Z)$) and one in which only two of the derivatives are adjacent (for example $\mathrm{Tr}(YZ\partial_Z Z\partial_Y\partial_Z)$). We will deal with an example of each term paying special attention to points that must be treated with care.

First Term: Start by allowing the derivatives to act on the restricted Schur polynomial

$$\begin{aligned}\mathrm{Tr}(ZYZ\partial_Z\partial_Y\partial_Z)\,\chi_{R,(r,s)\alpha\beta}(Z,Y) = \frac{mn(n-1)}{n!m!}\sum_{\psi\in S_{n+m}} \mathrm{Tr}_{(r,s)\alpha\beta}(\Gamma^{(R)}((1,m+2)\psi(m+1,m+2)))\\ \times\delta^{i_1}_{i_{\psi(1)}} Y^{i_2}_{i_{\psi(2)}}\cdots Y^{i_m}_{i_{\psi(m)}}(ZYZ)^{i_{m+1}}_{i_{\psi(m+1)}}\delta^{i_{m+2}}_{i_{\psi(m+2)}} Z^{i_{m+3}}_{i_{\psi(m+3)}}\cdots Z^{i_{m+n}}_{i_{\psi(m+n)}}. \qquad (3.8)\end{aligned}$$

The two delta functions will reduce the sum over S_{n+m} to a sum over an S_{n+m-2} subgroup. This sum is most easily evaluated using the reduction rule of [70], [10]. The reduction rule rewrites the sum over S_{n+m} as a sum over S_{n+m-2} and its cosets. This is most easily done by making use of Jucys-Murphy elements whose action is easily evaluated. To employ the same strategy in the current computation, the action of the Jucys-Murphy element will only be the simple one if we swap the delta function from slot $m+2$ to slot 2. This gives

$$\frac{mn(n-1)}{n!m!}\sum_{\psi\in S_{n+m-2}}\mathrm{Tr}_{(r,s)\alpha\beta}(\Gamma^{(R)}((1,m+2)(2,m+2)\psi(2,m+2)\hat{C}(m+1,m+2)))$$
$$\times Y^{i_3}_{i_{\psi(3)}}\cdots Y^{i_m}_{i_{\psi(m)}}(ZYZ)^{i_{m+1}}_{i_{\psi(m+1)}}Y^{i_{m+2}}_{i_{\psi(m+2)}}Z^{i_{m+3}}_{i_{\psi(m+3)}}\cdots Z^{i_{m+n}}_{i_{\psi(m+n)}}, \tag{3.9}$$

where $\hat{C}=(N+J_2)(N+J_3)$ with J_i a Jucys-Murphy element

$$J_i=\sum_{k=i}^{n+m}(i-1,k). \tag{3.10}$$

Since we sum over the S_{n+m-2} subgroup, we can decompose $R\vdash m+n$ into a direct sum of terms which involve the irreps $R''\vdash m+n-2$ of the subgroup[2]. As usual [70], [10] for each term in the sum, $\hat{C}$ is equal to the product of the factors of the boxes that must be removed from R to obtain R''. To rewrite the result in terms of restricted Schur polynomials, note that

$$\begin{aligned}
Y^{i_3}_{i_{\psi(3)}}&\cdots Y^{i_m}_{i_{\psi(m)}}(ZYZ)^{i_{m+1}}_{i_{\psi(m+1)}}Y^{i_{m+2}}_{i_{\psi(m+2)}}Z^{i_{m+3}}_{i_{\psi(m+3)}}\cdots Z^{i_{m+n}}_{i_{\psi(m+n)}}\\
&=\mathrm{Tr}\left(\psi(2,m+1,1)Y\otimes Z\otimes Y^{\otimes m-2}\otimes Z\otimes Y\otimes Z^{\otimes n-2}\right)\\
&=\mathrm{Tr}\left((2,m+2)\psi(2,m+1,1)(2,m+2)Y^{\otimes m}\otimes Z^{\otimes n}\right)
\end{aligned} \tag{3.11}$$

and make use of the identity [32]

$$\mathrm{Tr}(\sigma Z^{\otimes n}\otimes Y^{\otimes m})=\sum_{T,(t,u)\beta\alpha}\frac{d_T n!m!}{d_t d_u(n+m)!}\mathrm{Tr}_{(t,u)\alpha\beta}(\Gamma^{(T)}(\sigma^{-1}))\chi_{T,(t,u)\beta\alpha}(Z,Y). \tag{3.12}$$

[2]In general if R denotes a Young diagram, then R' denotes a Young diagram that can be obtained from R by removing one box, R'' denotes a Young diagram that can be obtained from R by removing two boxes etc.

After this rewriting the sum over S_{n+m-2} can be carried out using the fundamental orthogonality relation. The result is

$$\sum_{T,(t,u)\gamma\delta}\sum_{R'',T''}\frac{d_T n(n-1)m}{d_t d_u d_{R''}(n+m)(n+m-1)}c_{RR'}c_{R'R''}\ \chi_{T,(t,u)\gamma\delta}(Z,Y)$$
$$\times\mathrm{Tr}(I_{T''R''}(2,m+2,m+1)P_{R,(r,s)\alpha\beta}(1,m+2,2)I_{R''T''}(2,m+2)P_{T,(t,u)\delta\gamma}(m+2,2,1,m+1)).$$

The intertwiner $I_{R''T''}$ is a map (see Appendix D of [36] for details on its properties) from irrep R'' to irrep T''. It is only non-zero if R'' and T'' have the same shape. Thus, to get a non-zero result R and T must differ at most, by the placement of two boxes. We make further comments relevant for this trace before equation (3.13) below.

Second Term: Evaluation of the second term is very similar. In this case however, taking the derivatives produces a single delta function, which will reduce the sum over S_{n+m} to a sum over S_{n+m-1}. The delta function should be in slot 1. The reader wanting to check an example may find it useful to verify that

$$:\mathrm{Tr}(YZ\partial_Z Z\partial_Y\partial_Z):\chi_{R,(r,s)\alpha\beta}(Z,Y)=\sum_{T,(t,u)\gamma\delta}\sum_{R',T'}\frac{d_T n(n-1)m}{d_t d_u d_{R'}(n+m)}c_{RR'}$$
$$\times\mathrm{Tr}(I_{T'R'}(1,m+2,m+1)P_{R,(r,s)\alpha\beta}I_{R'T'}(1,m+1)P_{T,(t,u)\delta\gamma})\chi_{T,(t,u)\gamma\delta}(Z,Y).$$

The intertwiner $I_{R'T'}$ is a map from irrep R' to irrep T'. It is only non-zero if R' and T' have the same shape. Thus, to get a non-zero result R and T must differ at most, by the placement of a single box. It is perhaps useful to spell out explicitely the meaning of the trace above. The above trace is taken over the reducible S_{n+m} representation $R\oplus T$. In addition, the projectors within the trace allow us to rewrite the permutations appearing in the trace as

$$\mathrm{Tr}\left(I_{T'R'}\Gamma^{(R)}\Big((1,m+2,m+1)\Big)P_{R,(r,s)\alpha\beta}I_{R'T'}\Gamma^{(T)}\Big((1,m+1)\Big)P_{T,(t,u)\delta\gamma}\right). \quad (3.13)$$

The final result for the action of the dilatation operator is (this includes only the first term in (3.6) since $n \gg m$)

$$\begin{aligned} D_4\chi_{R,(r,s)\alpha\beta}(Z,Y) &= -2g^2\sum_{T,(t,u)\gamma\delta}\sum_{R'\,T'}\frac{d_T n(n-1)mc_{RR'}}{d_t d_u d_{R'}(n+m)}M^{(b)}_{R,(r,s)\alpha\beta\ T,(t,u)\gamma\delta}\chi_{T,(t,u)\delta\gamma}(Z,Y) \\ &-2g^2\sum_{T,(t,u)\gamma\delta}\sum_{R''\,T''}\frac{d_T n(n-1)mc_{RR'}c_{R'R''}}{d_t d_u d_{R''}(n+m)(n+m-1)}M^{(a)}_{R,(r,s)\alpha\beta\ T,(t,u)\gamma\delta}\chi_{T,(t,u)\delta\gamma}(Z,Y), \end{aligned}$$

where

$$\begin{aligned} M^{(a)}_{R,(r,s)\alpha\beta\ T,(t,u)\gamma\delta} &= \mathrm{Tr}\left(I_{T''R''}(2,m+2)P_{R,(r,s)\alpha\beta}C_1 I_{R''T''}(2,m+2)P_{T,(t,u)\gamma\delta}C_1\right) \\ &+\mathrm{Tr}\left(I_{T''R''}C_2 P_{R,(r,s)\alpha\beta}(2,m+2)I_{R''T''}C_2 P_{T,(t,u)\gamma\delta}(2,m+2)\right), \end{aligned} \quad (3.14)$$

$$C_1 = [(m+2,2,1),(1,m+1)], \qquad C_2 = -C_1^T = [(m+2,1,2),(1,m+1)] \quad (3.15)$$

and

$$\begin{aligned} M^{(b)}_{R,(r,s)\alpha\beta\ T,(t,u)\gamma\delta} &= \mathrm{Tr}\left(I_{T'R'}C_3 I_{R'T'}[(1,m+1),P_{T,(t,u)\gamma\delta}]\right) \\ &+\mathrm{Tr}\left(I_{T'R'}C_4 I_{R'T'}[(1,m+1),P_{T,(t,u)\gamma\delta}]\right) \end{aligned} \quad (3.16)$$

$$C_3 = [(1, m+2, m+1), P_{R,(r,s)\alpha\beta}], \qquad C_4 = [(1, m+1, m+2), P_{R,(r,s)\alpha\beta}]. \quad (3.17)$$

This formula is correct to all orders in $1/N$. Denote the number of rows in the Young diagram R labeling the restricted Schur polynomial by p. This implies that, since R subduces $S_n \times S_m$ representation (r, s) and $n \gg m$ that r has p rows and s has at most p rows. Now we will make use of the displaced corners approximation. To see how this works, recall that to subduce $r \vdash n$ from $R \vdash m + n$ we remove m boxes from R. Each removed box is associated with a vector in a p dimensional vector space V_p. Thus, the m removed boxes associated with the Ys thus define a vector in $V_p^{\otimes m}$. In the displaced corners approximation, the trace over $R \oplus T$ factorizes into a trace over $r \oplus t$ and a trace over $V_p^{\otimes m}$. The structure of the projector (3.4) makes it clear that the bulk of the work is in evaluating the trace over $V_p^{\otimes m}$. This trace can be evaluated using the methods developed in [36]. Introduce a basis for the fundamental representation of the Lie algebra u(p) given by $(E_{ij})_{ab} = \delta_{ia}\delta_{jb}$. Recall the product rule

$$E_{ij}E_{kl} = \delta_{jk}E_{il} \quad (3.18)$$

which we use extensively below. If a box is removed from row i it is associated to a vector v_i which is an eigenstate of E_{ii} with eigenvalue 1. The intertwining maps can be written in terms of the E_{ij}. For example, if we remove two boxes from row i of R and two boxes from row j of T, assuming that R'' and T'' have the same shape, we have

$$I_{T''R''} = E^{(1)}_{ji} E^{(2)}_{ji} . \tag{3.19}$$

A big advantage of realizing the intertwiners in this way is that it is simple to evaluate the product of symmetric group elements with the intertwiners. For example, using the identification (for background, see for example [71]

$$(1, 2, m+1) = \mathrm{Tr}(E^{(1)} E^{(2)} E^{(m+1)}) \tag{3.20}$$

we easily find

$$(1, 2, m+1) I_{T''R''} \; = \; E^{(1)}_{kl} E^{(2)}_{lm} E^{(m+1)}_{mk} E^{(1)}_{ji} E^{(2)}_{ji} \; = \; E^{(1)}_{ki} E^{(2)}_{ji} E^{(m+1)}_{jk} . \tag{3.21}$$

This is now enough to evaluate the traces appearing in (3.14) and (3.16).

We will consider the action of the dilatation operator on normalized restricted Schur polynomials. The two point function for restricted Schur polynomials is [12]

$$\langle \chi_{R,(r,s)\alpha\beta}(Z, Y) \chi_{T,(t,u)\gamma\delta}(Z, Y)^\dagger \rangle = \delta_{R,(r,s)\, T,(t,u)} \delta_{\alpha\gamma} \delta_{\beta\delta}\, f_R \frac{\mathrm{hooks}_R}{\mathrm{hooks}_r\, \mathrm{hooks}_s} . \tag{3.22}$$

where f_R is the product of the factors in Young diagram R and hooks_R is the product of the hook lengths of Young diagram R. The normalized operators are thus given by

$$\chi_{R,(r,s)}(Z, Y) = \sqrt{\frac{f_R\, \mathrm{hooks}_R}{\mathrm{hooks}_r\, \mathrm{hooks}_s}} O_{R,(r,s)}(Z, Y) \,. \tag{3.23}$$

The components m_i of the vector $\vec{m}(R)$ record the number of boxes removed from row i of R to produce r. In the su(2) sector, both the one loop dilatation operator [36] and the two loop dilatation operator conserve $\vec{m}(R)$, recorded in the factor $\delta_{\vec{m}(R)\vec{m}(T)}$ in (3.24) below. In terms of these normalized operators the dilatation operator takes the form

$$D_4 O_{R,(r,s)\mu_1\mu_2} = -2g^2 \sum_{u\,\nu_1\,\nu_2} \delta_{\vec{m}(R)\vec{m}(T)} M^{(ij)}_{s\mu_1\mu_2\,;\,u\nu_1\nu_2} \left(\Delta^{(1)}_{ij} + \Delta^{(2)}_{ij}\right) O_{R,(r,u)\nu_1\nu_2}, \quad (3.24)$$

where

$$\begin{aligned} M^{(ij)}_{s\mu_1\mu_2\,;\,u\nu_1\nu_2} \quad &= \frac{m}{\sqrt{d_u d_s}} \Big(\Big\langle \vec{m}, s, \mu_2\,;\, a | E^{(1)}_{ii} | \vec{m}, u, \nu_2\,;\, b \Big\rangle \Big\langle \vec{m}, u, \nu_1\,;\, b | E^{(1)}_{jj} | \vec{m}, s, \mu_2\,;\, a \Big\rangle \\ &+ \Big\langle \vec{m}, s, \mu_2\,;\, a | E^{(1)}_{jj} | \vec{m}, u, \nu_2\,;\, b \Big\rangle \Big\langle \vec{m}, u, \nu_1\,;\, b | E^{(1)}_{ii} | \vec{m}, s, \mu_2\,;\, a \Big\rangle \Big). \quad (3.25) \end{aligned}$$

To spell out the action of the operators $\Delta^{(1)}_{ij}$ and $\Delta^{(2)}_{ij}$ we will need a little more notation. Denote the row lengths of r by r_i. The Young diagram r^+_{ij} is obtained by deleting a box from row j and adding it to row i. The Young diagram r^-_{ij} is obtained by deleting a box from row i and adding it to row j. In terms of these Young diagrams define

$$\Delta^0_{ij} O_{R,(r,s)\mu_1\mu_2} = -(2N + r_i + r_j) O_{R,(r,s)\mu_1\mu_2}, \quad (3.26)$$

$$\Delta^+_{ij} O_{R,(r,s)\mu_1\mu_2} = \sqrt{(N + r_i)(N + r_j)}\, O_{R^+_{ij},(r^+_{ij},s)\mu_1\mu_2}, \quad (3.27)$$

$$\Delta_{ij}^{-}O_{R,(r,s)\mu_1\mu_2} = \sqrt{(N+r_i)(N+r_j)}\, O_{R_{ij}^{-},(r_{ij}^{-},s)\mu_1\mu_2}. \tag{3.28}$$

We can now write

$$\Delta_{ij}^{(1)} = n(\Delta_{ij}^{+} + \Delta_{ij}^{0} + \Delta_{ij}^{-}), \tag{3.29}$$

$$\Delta_{ij}^{(2)} = (\Delta_{ij}^{+})^2 + \Delta_{ij}^{0}\Delta_{ij}^{+} + 2\Delta_{ij}^{+}\Delta_{ij}^{-} + \Delta_{ij}^{0}\Delta_{ij}^{-} + (\Delta_{ij}^{-})^2. \tag{3.30}$$

This completes the evaluation of the dilatation operator.

Our result for $\Delta_{ij}^{(2)}$ deserves a comment. The intertwiners $I_{T''R''}$ appearing in (3.14) only force the shapes of T and R to agree when two boxes have been removed from each. One might imagine removing a box from rows i,j of R to obtain R'' and from rows k,l of T to obtain T'', implying that in total four rows could participate. We see from $\Delta_{ij}^{(2)}$ that this is not the case - the mixing is much more constrained with only two rows participating. We discuss this point further in Appendix C.

3.4 Spectrum

An interesting feature of the result (3.24) is that the action of the dilatation operator has factored into the product of two actions: $\Delta_{ij}^{(1)} + \Delta_{ij}^{(2)}$ acts only on Young diagram r i.e. on the Zs, while $M^{(ij)}_{s\mu_1\mu_2\,;\,u\nu_1\nu_2}$ acts only on Young diagram s, i.e. on the Ys. This factored form, which also arises at one loop, implies that we can diagonalize on the $s\mu_1\mu_2;u\nu_1\nu_2$ and the $R,r;T,t$ labels separately. The diagonalization on the $s\mu_1\mu_2;u\nu_1\nu_2$ labels is identical to the diagonalization problem which arises at one loop. The solution was obtained analytically for two rows in [35] and then in general in [14]. Each possible open string configuration consistent with the Gauss Law constraint can be identified with an element of a double coset. A very natural basis of functions, constructed from representation theory, is suggested by Fourier transformation applied to this double coset. In this way [14] constructed an explicit formula for the wavefunction which solves the $s\mu_1\mu_2;u\nu_1\nu_2$ diagonalization. The resulting Gauss graph operators are labeled by elements of the double coset. The explicit solution obtained in [14] is

$$O_{R,r}(\sigma)=\frac{|H|}{\sqrt{m!}}\sum_{j,k}\sum_{s\vdash m}\sum_{\mu_1,\mu_2}\sqrt{d_s}\Gamma^{(s)}_{jk}(\sigma)B^{s\to 1_H}_{j\mu_1}B^{s\to 1_H}_{k\mu_2}O_{R,(r,s)\mu_1\mu_2}, \tag{3.31}$$

where the group $H=S_{m_1}\times S_{m_2}\times\cdots\times S_{m_p}$ and the branching coefficients $B^{s\to 1_H}_{j\mu_1}$ provide a resolution of the projector from irrep s of S_m onto the trivial representation of H

$$\frac{1}{|H|}\sum_{\sigma\in H}\Gamma^{(s)}_{ik}(\sigma)=\sum_{\mu}B^{s\to 1_H}_{i\mu}B^{s\to 1_H}_{k\mu}. \tag{3.32}$$

The action of the dilatation operator on the Gauss graph operator is

$$D_4O_{R,r}(\sigma)=-2g^2\sum_{i<j}n_{ij}(\sigma)\left(\Delta^{(1)}_{ij}+\Delta^{(2)}_{ij}\right)O_{R,r}(\sigma). \tag{3.33}$$

The numbers $n_{ij}(\sigma)$ can be read off of the element of the double coset σ. Each

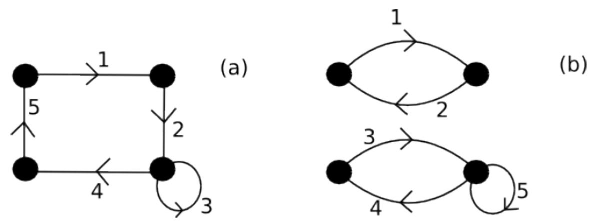


Figure 3.1: Two possible configurations for operators with $p = 4$ and $m = 5$.

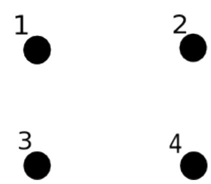


Figure 3.2: Labeling of the giant graviton branes.

possible Gauss operator is given by a set of m open strings stretched between p different giant graviton branes. As an example, consider $p = 4$ with $m = 5$. Two possible configurations are shown in figure (3.1). Label the open strings with integers from 1 to $m = 5$ for our example. The double coset element can then be read straight from the open string configuration by recording how the open strings are ordered as closed circuits in the graph are traversed. For the graphs shown, (a) corresponds to $\sigma = (1245)(3)$ and (b) corresponds to $\sigma = (12)(34)(5)$. The numbers $n_{ij}(\sigma)$ tell us how many strings stretch between branes i and j. The branes themselves are numbered with integers from 1 to p, as shown in figure (3.2) for our example. Thus, for (a) the non-zero n_{ij} are $n_{12} = 1$, $n_{23} = 1$, $n_{34} = 1$, and $n_{14} = 1$. Notice that we don't record strings that emanate and terminate on the same brane - string 3 in (a) or string 5 in (b), in this example. For (b) the non-zero n_{ij} are $n_{12} = 2$ and $n_{34} = 2$. For the details, see [14].

To obtain the anomalous dimensions, inspection of (3.33) shows that we now have to solve the eigenproblem of $\Delta_{ij}^{(1)}$ and $\Delta_{ij}^{(2)}$. The operator $\Delta_{ij}^{(1)}$ is simply a scaled version of the operator which plays a role in the one loop dilatation operator. The corresponding operator which participates at one loop was identified as an element of u(p) [69]. It is related to a system of p particles in a line with 2-body harmonic oscillator interactions [69]. The operator $\Delta_{ij}^{(2)}$ is new. Following [69], a useful approach is to study the continuum limit of $\Delta_{ij}^{(1)}$ and $\Delta_{ij}^{(2)}$. Towards this end, introduce the variables

$$y_j = \frac{r_{j+1} - r_1}{\sqrt{N + r_1}}, \qquad j = 1, 2, 3, ..., p-1 \tag{3.34}$$

which become continuous variables in the large N limit. We have numbered rows so that $r_1 < r_2 < \cdots < r_p$. In the continuum limit our Gauss graph operators become functions of y_i

$$O_{R,r}(\sigma) \equiv O^{\vec{m}(R)}(\sigma, r_1, r_2, \cdots, r_p) \to O^{\vec{m}(R)}(r_1, y_1, \cdots, y_{p-1}). \tag{3.35}$$

Using the expansions

$$\sqrt{(N + r_i)(N + r_j)} = N + r_1 + \frac{y_i + y_j}{2}\sqrt{N + r_1} - \frac{(y_i - y_j)^2}{8} + O\left(\frac{1}{\sqrt{N + r_1}}\right) \tag{3.36}$$

and

$$\begin{aligned} O^{\vec{m}(R)}\ &(r_1, y_1, \cdots, y_i + \frac{1}{\sqrt{N + r_1}}, \cdots, y_j - \frac{1}{\sqrt{N + r_1}}, \cdots, y_{p-1}) = O^{\vec{m}(R)}(r_1, y_1, \cdots, y_{p-1}) \\ &+ \frac{1}{\sqrt{N + r_1}} \frac{\partial}{\partial y_i} O^{\vec{m}(R)}(r_1, y_1, \cdots, y_{p-1}) - \frac{1}{\sqrt{N + r_1}} \frac{\partial}{\partial y_j} O^{\vec{m}(R)}(r_1, y_1, \cdots, y_{p-1}) \end{aligned}$$

$$+\frac{1}{2(N+r_1)}\left(\frac{\partial}{\partial y_i}-\frac{\partial}{\partial y_j}\right)^2 O^{\vec{m}(R)}(r_1,y_1,\cdots,y_{p-1}) \qquad (3.37)$$

we find that in the continuum limit

$$\Delta^{(1)}_{i+1\,j+1}O_{R,r}(\sigma)\to n\left[\left(\frac{\partial}{\partial y_i}-\frac{\partial}{\partial y_j}\right)^2-\frac{(y_i-y_j)^2}{4}\right]O^{\vec{m}(R)}(r_1,y_1,\cdots,y_{p-1}), \quad (3.38)$$

$$\Delta^{(1)}_{1\,i+1}O_{R,r}(\sigma)\to n\left[\left(2\frac{\partial}{\partial y_i}+\sum_{j\neq i}\frac{\partial}{\partial y_j}\right)^2-\frac{y_i^2}{4}\right]O^{\vec{m}(R)}(r_1,y_1,\cdots,y_{p-1}), \qquad (3.39)$$

and

$$\Delta^{(2)}_{i+1\,j+1}O_{R,r}(\sigma)\to 2(N+r_1)\left[\left(\frac{\partial}{\partial y_i}-\frac{\partial}{\partial y_j}\right)^2-\frac{(y_i-y_j)^2}{4}\right]O^{\vec{m}(R)}(r_1,y_1,\cdots,y_{p-1}), \quad (3.40)$$

$$\Delta^{(2)}_{1\,i+1}O_{R,r}(\sigma)\to 2(N+r_1)\left[\left(2\frac{\partial}{\partial y_i}+\sum_{j\neq i}\frac{\partial}{\partial y_j}\right)^2-\frac{y_i^2}{4}\right]O^{\vec{m}(R)}(r_1,y_1,\cdots,y_{p-1}). \qquad (3.41)$$

Remarkably, in the continuum limit both $\Delta^{(1)}_{ij}$ and $\Delta^{(2)}_{ij}$ have reduced to scaled versions of exactly the same operator that appears in the one loop problem. In the Appendix A we argue for the same conclusion without taking a continuum limit. This implies that the operators that have a good scaling dimension at one loop are uncorrected at two loops.

It is now straight forward to obtain the two loop anomalous dimension for any operator of interest. An instructive and simple example is provided by $p=2$ with[3] $n_{12}=n^+_{12}+n^-_{12}\neq 0$. In this case, the anomalous dimension $\gamma(g^2)$ which is the eigenvalue of

$$D=D_2+D_4 \qquad (3.42)$$

[3]The number n^+_{12} counts the number of open strings stretching from giant graviton 1 to giant graviton 2; the number n^-_{12} counts the number of open strings stretching from giant graviton 2 to giant graviton 1. The Gauss Law constraint forces $n^+_{12}=n^-_{12}$. See [14] for more details.

with[4]

$$D_2 = -2g : \mathrm{Tr}\left([Y,Z]\left[\frac{\partial}{\partial Y},\frac{\partial}{\partial Z}\right]\right) : \tag{3.43}$$

and D_4 given in (3.6), is

$$\gamma = 16qn_{12}^{+}(g + (2N + 2r_1 + n)g^2), \tag{3.44}$$

$$q = 0,1,2,..,M \quad n_{12} = 0,1,2,... \tag{3.45}$$

where the upper cut off M is itself a number of order N. Clearly, if the g^2 term is to be a small correction to the leading term, we must hold $\lambda_g \equiv gN$ fixed, which corresponds to the usual 't Hooft limit. The fact that the usual 't Hooft scaling leads to a sensible perturbative expansion in this sector of the theory was already understood in [72]. We then find

$$\gamma = \frac{16qn_{12}}{N}(\lambda_g + (2 + 2\frac{r_1}{N} + \frac{n}{N})\lambda_g^2). \tag{3.46}$$

For a given open string plus giant system (i.e. a given n_{12}), in the large N limit, $x = \frac{q}{N}$ varies continuously from 0 to $x = \frac{M}{N}$ implying that the spectrum of anomalous dimensions

$$\gamma = 16xn_{12}(\lambda_g + (2 + 2\frac{r_1}{N} + \frac{n}{N})\lambda_g^2) \tag{3.47}$$

is itself continuous. At finite N this spectrum is discrete. Notice that since both n and r_1 are of order N, all three terms multiplying λ_g^2 in (3.47) are of the same size. Note that the value for γ (3.47) will receive both $\frac{1}{N}$ corrections and $\frac{m}{n}$ corrections.

3.5 Answers and Discussion

We can now return to the questions we posed in the introduction:

[4]The normalization for both D_2 and D_4 follows [49]. This normalization for D_2 is a factor of 2 larger than the normalization used in [33], [34], [35], [40], [37], [36], [14], [69].

1 **Is the dilatation operator integrable in the large N displaced corners approximation at higher loops?**

We don't know. We have however been able to argue that the dilatation operator is integrable in the large N displaced corners approximation at two loops. This requires both sending $N \to \infty$ and keeping $m \ll n$ to ensure the validity of the displaced corners approximation. At large N with $m \sim n$ we do not know how to compute the action of the dilatation operator and hence integrability in this situation is an interesting open problem. It seems reasonable to hope that integrability will persist in the large N displaced corners approximation at higher loops.

2 **Do the $O_{R,r}(\sigma)$ of [14] continue to solve the Y eigenproblem at higher loops?**

Yes, the Gauss graph operators do indeed solve the Y eigenproblem at two loops. The Y eigenproblem at two loops is identical to the Y eigenproblem at one loop, so that even the eigenvalues (given by $n_{ij}(\sigma)$ in (3.33)) are unchanged. The fact that the Gauss operators continue to solve the Y eigenproblem does not depend sensitively on the coefficients of the individual terms in the two loop dilatation operator (see Appendix C).

3 **Can the two loop Z eigenproblem be mapped to a system of p particles, again using the Lie algebra of U(p)?**

We have indeed managed to map the Z eigenproblem to the dynamics of p particles (in the center of mass frame). The two loop problem again has a very natural phrasing in terms of the Lie algebra of U(p). The one loop and two loop problems are different: they share the same eigenstates but have different eigenvalues. The fact that the eigenstates are the same does depend sensitively on the coefficients of the individual terms in the two loop dilatation operator (see Appendix C).

4 **Does the two loop correction to the anomalous dimension determine the precise limit that should be taken to get a sensible perturbative expansion?**

Yes - requiring that the two loop correction in (3.46) is small compared to the one loop term clearly implies that we should be taking the standard 't Hooft limit. Our result then has an interesting consequence: at large N, $x = q/N$ becomes a continuous parameter and we recover a continuous energy spectrum. This is clearly related to [73]. At any finite N the spectrum is discrete.

Our discussion has been developed for operators with a label R that has p long rows, which are dual to giant gravitons wrapping an $\mathrm{S}^3 \subset \mathrm{AdS}_5$. Operators labeled by an R that has p long columns are dual to giant gravitons wrapping an $\mathrm{S}^3 \subset \mathrm{S}^5$. The anomalous dimensions for these operators are easily obtained from our results in this chapter (see section D.6 of [36] for a discussion of this connnection). The $\Delta_{ij}^{(1)}$ for this case is obtained by replacing the $r_i \to -r_i$ and $r_j \to -r_j$ in (3.38) and (3.39), while $\Delta_{ij}^{(2)}$ for this case is obtained by replacing the $r_i \to -r_i$ and $r_j \to -r_j$ in (3.40) and (3.41). The result (3.33) is unchanged when written in terms of the new $\Delta_{ij}^{(1)}$ and $\Delta_{ij}^{(2)}$.

Finally, the fact that our operators are not corrected at two loops is remarkable. It is natural now to conjecture that they are in fact exact and will not be corrected at any higher loop. This is somewhat reminiscent of the BMN operators [74]. In that case it is possible to determine the exact anomalous dimensions as a function of the 't Hooft coupling λ_g [75]. Can we use similar methods to achieve this for the operators discussed in this chapter?

Chapter 4

$SO(N)$ restricted Schur polynomials

4.1 Outline of chapter

We focus on the 1/4-BPS sector of free super Yang-Mills theory with an $SO(N)$ gauge group. This theory has an AdS/CFT dual in the form of type IIB string theory with $\mathrm{AdS}_5 \times \mathcal{R}\mathrm{P}^5$ geometry. With the aim of studying excited giant graviton dynamics, we construct an orthogonal basis for this sector of the gauge theory in this chapter. First, we demonstrate that the counting of states, as given by the partition function, and the counting of restricted Schur polynomials matches by restricting to a particular class of Young diagram labels. We then give an explicit construction of these gauge invariant operators and evaluate their two-point function exactly. This paves the way to studying the spectral problem of these operators and their D-brane duals.

4.2 Introduction

The most studied example of AdS/CFT is $\mathcal{N} = 4$ super Yang-Mills theory with $U(N)$ gauge group, and its dual, type IIB string theory on $\mathrm{AdS}_5 \times S^5$. Testing the AdS/CFT correspondence in non-planar limits of the gauge theory and its dual string theory is a very interesting problem. In this regard, integrability would be a powerful tool. Therefore, it is interesting to investigate if the integrability, that was found in the planar limit [31], is also present in non-planar limits. Furthermore, testing AdS/CFT in less supersymmetric settings is also an interesting problem. In particular we have the 1/4-BPS sector of the theory in mind. The study of restricted Schur polynomials has yielded progress in both these directions. A restricted Schur is a local operator in the gauge theory which can be built from a variety of fields. For instance, we can use the complex scalar Higgs fields, the fermion fields as well as the gauge fields to build these operators. We consider restricted Schurs built using two types of complex scalar fields, Z and Y say. In this case, the definition is

$$\chi_{R(r,s)\alpha\beta}(Z,Y) = \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \mathrm{Tr}(P_{R \to (r,s)\alpha\beta} \Gamma_R(\sigma)) \mathrm{Tr}(\sigma Z^{\otimes n} \otimes Y^{\otimes m}). \tag{4.1}$$

In (4.1), R is a Young diagram with $n + m$ boxes, corresponding to an irreducible representation (irrep) of S_{n+m} and (r, s) are a pair of Young diagrams with n and m boxes respectively corresponding to an irrep of $S_n \times S_m$. $P_{R \to (r,s)\alpha\beta}$ is a projector in the labels R and (r, s) and an intertwiner in the labels α and β. These labels are the multiplicity labels with which irrep (r, s) is subduced from R when restricting from S_{n+m} to $S_n \times S_m$. Further details of (4.1) can be found in [10], [36] for example. There are many good reasons to study the operators defined in (4.1) as we now explain.

The counting of states given by the partition function of the 1/4-BPS sector of the free $U(N)$ theory was shown to match the counting of restricted Schurs that can be defined [13]. Essentially, the partition function was expressed in terms of Littlewood Richardson coefficients which count the number of restricted Schurs for a given set of Young diagram labels, $R, (r, s)$. This result was generalised to an arbitrary product of $U(N)$ gauge groups in [76].

We are interested in operators whose bare dimension grows parametrically with N. For these operators, the large N limit of correlation functions is not captured by summing only planar diagrams [36]. The usual $1/N^2$ factor suppressing non-planar diagrams is over-powered by combinatorial factors resulting from evaluating all possible Wick contractions. Indeed, when the number of matrix fields in the operator scales with N, we are forced to sum an infinite number of non-planar diagrams in the large N limit. A solution to this problem was given in [8] for the 1/2-BPS case. In this important and influential work, it was observed that there is a one-to-one correspondence between the space of 1/2-BPS representations and Young diagrams, i.e., Schur polynomials. Using representation theory of the symmetric and unitary group, the two-point function of Schur polynomials was computed *exactly* in the free theory limit. [12] then achieved the remarkable result of computing the two-point function of restricted Schurs (4.1) exactly in the free theory. The operators diagonalised the two-point function and a very simple formula for the final result was given. Thus, the restricted Schurs provide an exactly orthogonal basis for the 1/4-BPS sector of the free $U(N)$ gauge theory.

Apart from the interesting properties of (4.1) as gauge theory objects, restricted Schurs also have an AdS/CFT dual. When the scaling dimension of operators grows with N, $\chi_{R(r,s)\alpha\beta}(Z, Y)$, with R having long columns or rows ($O(N)$ boxes), has been argued to be dual to a system of excited giant gravitons [4], [36]. Concretely, such a system is realised as a system of giant gravitons with strings attached [10]. Thus, restricted Schurs are a useful tool for studying non-perturbative physics of both the gauge theory and its dual string theory.

In a similar approach, [46] also studied 1/4 and 1/8-BPS operators at zero coupling, $g_{YM}^2 = 0$. Also by making heavy use of representation theory, the multi-matrix multi-trace operators constructed here were argued to form a basis and to diagonalise their two-point function. When the operators in [46] consist of two matrix fields Z and Y say, they were shown to be an alternative basis of restricted Schur polynomials [13]. Progress has also been made in the counting and constructing of 1/4 and 1/8-BPS operators at weak coupling [77].

Tremendous progress has already been achieved in the study of the anomalous dimension spectrum of restricted Schur polynomials; see [36], [35], [69], [41], [42] for examples. Anomalous dimensions of the restricted Schurs are dual to excitation energies of the excited giant graviton system. One of the main results of these works is that the spectrum of anomalous dimensions reduces to the set of normal mode frequencies of a system of decoupled harmonic oscillators. The fact that we get decoupled oscillators is evidence of integrability in the non-planar sectors studied in these works.

Is there a similar story for the theory with an $SO(N)$ gauge group? In this chapter, we take the first steps toward answering this question. $\mathcal{N} = 4$ super Yang-Mills theory with an $SO(N)$ gauge group is dual to type IIB string theory with $\mathrm{AdS}_5 \times \mathcal{R}\mathrm{P}^5$ geometry. At the non-planar level, the spectral problems of $U(N)$ and $SO(N)$ gauge theories are distinct [78]. Thus, there is a good chance that a study of anomalous dimensions of large dimension operators will reveal new aspects of D-brane physics. The 1/2-BPS case for free $SO(N)$ gauge theory was studied in [79], [80] and [81] . The results of [79] and [80] include defining Schur polynomials in the square of the eigenvalues and computing their two-point functions exactly in the free theory. Here too, the Schur polynomial was shown to diagonalise the two-point function, a property that was argued to be independent of the gauge group [81].

We show that restricted Schur polynomials provide an exactly orthogonal basis for the 1/4-BPS sector of free super Yang-Mills theory with $SO(N)$ gauge group. In this chapter, we take N to be even. First we show that the counting of states obtained from the partition function in this sector matches the number of restricted Schurs that can be defined. This is achieved by expressing the partition function in terms of the Littlewood Richardson coefficients. The counting comes out differently from the 1/4-BPS $U(N)$ case as we will see. $SO(N)$ has two invariant tensors. They are δ_{ij} and $\epsilon_{i_1 i_2 \cdots i_N}$. We focus on operators constructed using δ_{ij}. We construct restricted Schur polynomials and manage to compute the two-point function exactly in the free theory. The result is expressed in a relatively simple formula. In the next section, we present some background theory needed for our computations.

4.3 Projectors

In this section, we briefly discuss some of the projectors and states that appear frequently in our calculations. All projectors we discuss here project from the carrier space of irrep R of S_{2q}, $q = n+m$, onto the carrier space of an irrep of some subgroup.

4.3.1 From R onto (r, s)

Firstly we discuss the projectors $P_{R\to(r,s)\alpha\beta}$ constructed in [36]. They project from R onto irrep (r, s) of the subgroup $S_{2n} \times S_{2m}$. In the multiplicity labels, however, $P_{R\to(r,s)\alpha\beta}$ is an intertwiner [82]. We write these operators as

$$P_{R\to(r,s)\alpha\beta} = \sum_{l=1}^{d_r\times d_s} |R(r,s)\alpha;l\rangle\langle R(r,s)\beta;l| \,, \tag{4.2}$$

where $|R(r,s)\alpha;l\rangle$ is a state in the carrier space of the α-th copy of irrep (r,s) and $d_r \times d_s$ is the dimension of (r,s). In this basis, the matrix representation of $\gamma \in S_{2n} \times S_{2m}$ in irrep R is block diagonal with the $S_{2n} \times S_{2m}$ irreps as the diagonal blocks. For example

$$\Gamma_R(\gamma) = \begin{pmatrix} \Gamma_{(r',s')}(\gamma) & 0 & 0 & 0 \\ 0 & \Gamma_{(r'',s'')}(\gamma) & 0 & 0 \\ 0 & 0 & \Gamma_{(r,s)1}(\gamma) & 0 \\ 0 & 0 & 0 & \Gamma_{(r,s)2}(\gamma) \end{pmatrix}, \tag{4.3}$$

where we have shown two copies of irrep (r,s). There are four projectors that can be defined for (r,s). Acting on $\Gamma_R(\gamma)$ each give the following

$$P_{R\to(r,s)11}\Gamma_R(\gamma) \;\; = \;\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Gamma_{(r,s)1}(\gamma) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.4}$$

$$P_{R\to(r,s)22}\Gamma_R(\gamma) \;\; = \;\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Gamma_{(r,s)2}(\gamma) \end{pmatrix}, \tag{4.5}$$

$$P_{R\to(r,s)12}\Gamma_R(\gamma) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Gamma_{(r,s)2}(\gamma) \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.6}$$

$$P_{R\to(r,s)21}\Gamma_R(\gamma) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Gamma_{(r,s)1}(\gamma) & 0 \end{pmatrix}. \tag{4.7}$$

4.3.2 From R onto $[S]$

Let R have an even number of boxes in each row. Restricting S_{2q} to the subgroup $S_q[S_2]$, one can find a basis in which $\Gamma_R(\xi)$, $\xi \in S_q[S_2]$, is block diagonal with the $S_q[S_2]$ irreps appearing on the diagonal. For R having even rows, there exists a 1-dimensional irrep of $S_q[S_2]$ which we denote $[S]$. In this irrep, all matrices are represented by 1. Let $|[S]\rangle$ be the state spanning the carrier space of $[S]$. Then we have [83]

$$\Gamma_R(\xi)\,|[S]\rangle = |[S]\rangle\,, \qquad \xi \in S_q[S_2]. \tag{4.8}$$

We can also define a projector to go from R to $[S]$.

$$P_{[S]} = \frac{1}{q!2^q} \sum_{\xi\in S_q[S_2]} \Gamma_R(\xi). \tag{4.9}$$

The state $|[S]\rangle$ may be calculated as the eigenvector of $P_{[S]}$ with eigenvalue 1. Lastly, $[S]$ is subduced from R with multiplicity 1 [79]. If R does not have even rows, then no such irrep exists.

4.3.3 From $(r,s)\beta$ to $([A],[A])\beta$

Recall that $(r,s)\beta$ is the β-th copy of the irrep of $S_{2n}\times S_{2m}$. If r has an even number of boxes in each column, then, upon restricting to the $S_n[S_2]$ subgroup another type of 1 dimensional irrep may be subduced [83]. Call this irrep $[A]$. In this irrep,

$$\Gamma_r(\eta) = \mathrm{sgn}(\eta), \qquad \eta \in S_n[S_2]. \tag{4.10}$$

Denote by $|[A]\rangle$ the state spanning the 1-dimensional carrier space of $[A]$. Irrep $[A]$ is also subduced from r with no multiplicity [79]. If r and s both have even columns, then each may subduce the irrep $[A]$ of $S_n[S_2]$ and $S_m[S_2]$ respectively. Thus, $(r,s)\beta$ subduces the irrep $([A],[A])\beta$ of $S_n[S_2]\times S_m[S_2]$. Denote by $|[A],[A]\beta\rangle$ the state spanning this 1-dimensional irrep. Define

$$P_{R\to([A],[A])\beta} = P_{R\to(r,s)\beta\beta}P_{[A,A]}, \qquad P_{[A,A]} = \frac{1}{n!m!2^q}\sum_{\eta\in S_n[S_2]\times S_m[S_2]} \mathrm{sgn}(\eta)\Gamma_R(\eta). \tag{4.11}$$

The state $|[A],[A]\beta\rangle$ may be defined to be the eigenvector of $P_{R\to([A],[A])\beta}$ with eigenvalue 1. This state has the following properties

1.
$$P_{R\to(r,s)\alpha\beta}\,|[A],[A]\beta\rangle = |[A],[A]\alpha\rangle\,. \tag{4.12}$$

2.
$$\Gamma_R(\eta)\,|[A],[A]\beta\rangle = \mathrm{sgn}(\eta)\,|[A],[A]\beta\rangle\,, \qquad \eta\in S_n[S_2]\times S_m[S_2]. \tag{4.13}$$

3. The quantity $\langle[A],[A]\beta|R(r,s)\beta;l\rangle$ does not depend on the multiplicity index β. This will be important when we construct the restricted Schurs.

4.4 Counting

In this section we express the partition function for the 1/4-BPS sector of free $SO(2n)$ gauge theory in terms of the Littlewood-Richardson coefficients depending on three Young diagram labels. Two Young diagrams will have an even number of boxes in each *column*, and the third one (induced from the previous two) will have an even number of boxes in each *row*. This is first demonstrated for the $SO(4)$ case for simplicity. It is then easy to extend the argument to the general $SO(2n)$ case, which follows thereafter. The Littlewood Richardson number $g(\lambda,\mu,\xi)$ counts the number of times a Young diagram ξ is induced from two smaller Young diagrams μ and λ, say. Thus, $g(\lambda,\mu,\xi)$ counts the number of restricted Schurs that may be defined for labels $\{\lambda,\mu,\xi\}$.

The partition function for the 1/4-BPS sector of free $SO(2n)$ gauge theory is [79], [84]

$$\begin{aligned} G(t_1,t_2) \;&=\; \frac{1}{2^{n-1}n!}\int_{T_n}\prod_{j=1}^{n}\frac{dx_j}{2\pi i x_j}\Delta(x+x^{-1})^2\times \\ &\prod_{k=1}^{2}\prod_{1\leqslant i<j\leqslant n}\frac{1}{(1-t_k)^n}\frac{1}{(1-t_kx_ix_j)(1-t_kx_ix_j^{-1})(1-t_kx_i^{-1}x_j)(1-t_kx_i^{-1}x_j^{-1})}. \end{aligned} \tag{4.14}$$

Similar expressions for the partition function and the adjoint character (which we use in the partition function) exist for the $SO(2n+1)$ gauge group. Let's study the simple $SO(4)$ case first and generalise to $SO(2n)$ thereafter. For $SO(4)$, this becomes

$$\begin{aligned} G(t_1,t_2) &= \frac{1}{4}\int \frac{dx_1}{2\pi i x_1}\int \frac{dx_2}{2\pi i x_2}\Delta(x+x^{-1})^2 \times \\ &\prod_{k=1}^{2}\frac{1}{(1-t_k)^2}\frac{1}{(1-t_kx_1x_2)(1-t_kx_1x_2^{-1})(1-t_kx_1^{-1}x_2)(1-t_kx_1^{-1}x_2^{-1})}. \end{aligned} \tag{4.15}$$

The factor in the second line may be written as

$$\begin{aligned} &\frac{1}{(1-t_kx_1x_1^{-1})(1-t_kx_2x_2^{-1})(1-t_kx_1x_2)(1-t_kx_1x_2^{-1})(1-t_kx_1^{-1}x_2)(1-t_kx_1^{-1}x_2^{-1})} \\ &= \prod_{1\leqslant i<j\leqslant 4}\frac{1}{1-t_ky_iy_j} \end{aligned} \tag{4.16}$$

with $y_1=x_1, y_2=x_1^{-1}, y_3=x_2, y_4=x_2^{-1}$. Now we may expand this product in terms of Schur polynomials using the formula [85]

$$\prod_{1\leqslant i<j\leqslant L}\frac{1}{1-y_iy_j}=\sum_{\mu}s_{\mu^2}(y_1,...,y_L), \tag{4.17}$$

where μ^2 is defined as the partition with parts

$$(\mu^2)_i=\mu_{\lceil i/2\rceil}. \tag{4.18}$$

This means the following. Take, for example, $i = \{1, 2, 3, 4\}$. Then, $i/2 = \{1/2, 1, 3/2, 2\}$. The ceiling symbol then tells us to take: $\lceil i/2 \rceil = \{1, 1, 2, 2\}$. So, if ${\mu^2}_i$ denotes the ith part of μ^2, then

$$\mu^2 = ({\mu^2}_1, {\mu^2}_2, {\mu^2}_3, {\mu^2}_4) = (\mu_1, \mu_1, \mu_2, \mu_2). \tag{4.19}$$

Thus, in general for a partition μ^2 with n parts,

$$\mu^2 = ({\mu^2}_1, {\mu^2}_2, {\mu^2}_3, {\mu^2}_4, \cdots, {\mu^2}_{n-1}, {\mu^2}_n) = (\mu_1, \mu_1, \mu_2, \mu_2, \cdots, \mu_{n/2}, \mu_{n/2}). \tag{4.20}$$

Partitions μ^2 then correspond to Young diagrams with an even number of boxes in each column. Including the prefactor t_k in (4.17), we get

$$\prod_{1 \leqslant i < j \leqslant L} \frac{1}{1 - t_k y_i y_j} = \sum_{\mu} s_{\mu^2}(\sqrt{t_k} y_1, ..., \sqrt{t_k} y_L) = \sum_{\mu} (t_k)^{|\mu^2|/2} s_{\mu^2}(y_1, ..., y_L), \tag{4.21}$$

where we used the fact that Schur polynomial $s_\lambda(y_1, y_2, \cdots, y_L)$ may be written as

$$s_\lambda(ty_1, ty_2, \cdots, ty_L) = t^{|\lambda|} s_\lambda(y_1, y_2, \cdots, y_L). \tag{4.22}$$

In the above formulas, $|\,|$ stands for the weight of the partition. The partition function (4.15) becomes

$$\begin{aligned} G(t_1,t_2) &= \frac{1}{4}\int\frac{dx_1}{2\pi i x_1}\int\frac{dx_2}{2\pi i x_2}\Delta(x+x^{-1})^2\times \\ &\sum_\lambda\sum_\mu (t_1)^{|\lambda^2|/2}(t_2)^{|\mu^2|/2}s_{\lambda^2}(x_1,x_1^{-1},x_2,x_2^{-1})s_{\mu^2}(x_1,x_1^{-1},x_2,x_2^{-1}). \end{aligned} \tag{4.23}$$

Now, we use the product rule involving Schur polynomials. This means that we can write the product of two Schur polynomials with partitions λ^2 and μ^2 as the sum over a single Schur with partition ξ times by the Littlewood-Richardson coefficient $g(\lambda^2,\mu^2,\xi)$ [86]. The Littlewood-Richardson coefficient is how many times ξ subduces (λ^2,μ^2). Thus,

$$\begin{aligned} G(t_1,t_2) &= \sum_\xi\sum_\lambda\sum_\mu g(\lambda^2,\mu^2,\xi)(t_1)^{|\lambda^2|/2}(t_2)^{|\mu^2|/2}\times \\ &\frac{1}{4}\int\frac{dx_1}{2\pi i x_1}\int\frac{dx_2}{2\pi i x_2}\Delta(x+x^{-1})^2 s_\xi(x_1,x_1^{-1},x_2,x_2^{-1}). \end{aligned} \tag{4.24}$$

Next, we generalise this to $SO(2n)$. In the same way as in equation (4.16)

$$\frac{1}{(1-t_k)^n}\prod_{1\leqslant i<j\leqslant n}\frac{1}{(1-t_kx_ix_j)(1-t_kx_ix_j^{-1})(1-t_kx_i^{-1}x_j)(1-t_kx_i^{-1}x_j^{-1})}=\prod_{1\leqslant i<j\leqslant N}\frac{1}{1-t_ky_iy_j} \tag{4.25}$$

with $y_i = x_{\lceil i/2\rceil}$ and $y_{2i} = x_i^{-1}$ for $i = 1,2,\cdots,n$. Using equation (4.17) for each factor,

$$\prod_{k=1}^{2}\prod_{1\leqslant i<j\leqslant N}\frac{1}{1-t_ky_iy_j}=\sum_\lambda\sum_\mu(t_1)^{|\lambda^2|/2}(t_2)^{|\mu^2|/2}s_{\lambda^2}(y_1,..,y_N)s_{\mu^2}(y_1,..,y_N). \tag{4.26}$$

After using the product rule again, the $SO(2n)$ partition function in (4.14) becomes

$$\begin{aligned} G(t_1,t_2) \;&=\; \sum_{\xi}\sum_{\lambda}\sum_{\mu} g(\lambda^2,\mu^2,\xi)(t_1)^{|\lambda^2|/2}(t_2)^{|\mu^2|/2}\times \\ &\quad \frac{1}{2^{n-1}n!}\int_{T_n}\prod_{j=1}^{n}\frac{dx_j}{2\pi i x_j}\Delta(x+x^{-1})^2 s_\xi(x_1,x_1^{-1},...,x_n,x_n^{-1}). \end{aligned} \tag{4.27}$$

Recognising the Haar (or G-invariant) measure for $SO(2n)$, the integral in (4.27) may be written as

$$I_n = \int_{SO(2n)} [dO]\; s_\xi(\mathrm{x}). \tag{4.28}$$

This integral is equal to 1 for two cases. The first is if ξ has an even number of boxes in each *row*. The second is if ξ has a Young diagram with an even number of boxes in each row stuck onto a single column of $2n$ boxes[1]. Here are examples of each for $SO(4)$

$$\xi = \ydiagram{4,2,2}\;, \qquad \text{or} \qquad \xi = \ydiagram{3,3,1,1}\;. \tag{4.29}$$

The integral vanishes in all other cases. See [85] [87] and [88]. We have checked these results for many examples for $SO(4)$ and $SO(6)$. With this result, the $SO(2n)$ partition function becomes

[1]Recall that there are only two invariant tensors for $SO(2n)$. They are δ_{ij} and $\epsilon_{i_1 i_2 \cdots i_{2n}}$. This case is relevent for operators that are built using $\epsilon_{i_1 i_2 \cdots i_{2n}}$.

$$G(t_1, t_2) = \sum_{\xi} \sum_{\lambda} \sum_{\mu} g(\lambda^2, \mu^2, \xi)(t_1)^{|\lambda^2|/2}(t_2)^{|\mu^2|/2}, \tag{4.30}$$

where ξ is a Young diagram with even rows and λ^2 and μ^2 are Young diagrams with even columns. This result is truly different from the $U(N)$ 1/4-BPS case, studied in [13]. For $U(N)$ the counting went according to the square of the Littlewood-Richardson number. Here, the counting goes according to the Littlewood Richardson number itself. In other words, for a given $S_{2n} \times S_{2m}$ irrep, (r, s), each with even columns, and given S_{2q} irrep R with even rows, the number of restricted Schurs one can define is exactly $g(r, s, R)$. We present some counting examples in appendix D.

4.5 Defining the restricted Schurs

We now discuss the problem of defining the restricted Schurs for the 1/4-BPS free $SO(N)$ gauge theory. We focus on operators that can be constructed using δ rather than $\epsilon_{i_1 i_2 \cdots i_{2n}}$. We construct these operators out of two complex matrix fields Z and Y. The two-point function of these fields is given by [79]

$$\langle Z^{ij} \overline{Z}^{kl} \rangle = \langle Z^{ij} Z_{kl} \rangle = \delta^i_k \delta^j_l - \delta^i_l \delta^j_k \tag{4.31}$$

and similarly for Y. We define our $SO(N)$ restricted Schurs to be

$$O_{R(r,s)\alpha}(Z, Y) = \frac{1}{(2n)!(2m)!} \sum_{\sigma \in S_{2q}} \mathrm{Tr}(\mathcal{O}_{R(r,s)\alpha} \Gamma_R(\sigma)) C^{4\nu}_I \sigma^I_J (Z^{\otimes 2n} \otimes Y^{\otimes 2m})^J, \tag{4.32}$$

where

$$\mathcal{O}_{R(r,s)\alpha} = |[S]\rangle\langle[A],[A]\beta|\, P_{R\to(r,s)\beta\alpha}. \tag{4.33}$$

In (4.32), the tensor $(Z^{\otimes 2n} \otimes Y^{\otimes 2m})^J$ is defined as

$$(Z^{\otimes 2n} \otimes Y^{\otimes 2m})^J = Z^{j_1 j_2} \cdots Z^{j_{2n-1}j_{2n}} Y^{j_{2n+1}j_{2n+2}} \cdots Y^{j_{2q-1}j_{2q}}. \tag{4.34}$$

$C_I^{4\nu}$ is the contractor defined in [80]

$$C_I^{4\nu} = \delta_{k_1 k_2} \cdots \delta_{k_{2q-1}k_{2q}} (\sigma_{4\nu})_I^K. \tag{4.35}$$

$C_I^{4\nu}$ is responsible for contracting the free indices in such a way to make the operator gauge invariant. For $\sigma_{4\nu}$, we take

$$\sigma_{4\nu} = (1,2,3,4)(5,6,7,8)\cdots(2q-3,2q-2,2q-1,2q). \tag{4.36}$$

This permutation contracts indices 1&4, 2&3 $\cdots$, $2q-3$&$2q$ and $2q-2$&$2q-1$, i.e.,

$$C_I^{4\nu}(\sigma)_J^I = \sigma^{i_1 i_2 i_2 i_1 \cdots i_{q-1} i_q i_q i_{q-1}}_{j_1 j_2 j_3 j_4 \cdots j_{2q}}. \tag{4.37}$$

R is an irrep of S_{2q} and (r,s) is an irrep of the subgroup $S_{2n} \times S_{2m}$. Indices α and β are multiplicity labels for the irrep (r,s). State $|[A],[A]\beta\rangle$ is the state spanning the one-dimensional carrier space $([A],[A])\beta$ of $S_n[S_2] \times S_m[S_2]$ as defined in section 4.3. Concretely, it is the eigenvector of the operator $P_{R\to([A],[A])\beta}$ with eigenvalue 1. State

$|[S]\rangle$ is the state spanning the 1-dimensional carrier space $[S]$ of $S_q[S_2]$ as defined in section 2. The $S_q[S_2]$ here is chosen to stabilise the set $(1,4),(2,3),\cdots,(2q-3,2q),(2q-2,2q-1)$. Concretely, $|[S]\rangle$ is the eigenvector of $P_{[S]}$.

The coefficients in (4.32) satisfy the following property

$$\mathrm{Tr}(\mathcal{O}_{R(r,s)\alpha}\Gamma_R(\sigma))\mathrm{sgn}(\eta)=\mathrm{Tr}(\mathcal{O}_{R(r,s)\alpha}\Gamma_R(\eta\sigma\xi)),\qquad \xi\in S_q[S_2],\eta\in S_n[S_2]\times S_m[S_2]. \tag{4.38}$$

Lastly, the operators in (4.32) depend only on a single multiplicity label and thus, match the counting found in section 4.4.

4.6 Two-point function

In this section, we evaluate the two-point function of the operators defined in section 4.5. Recall the definition for our restricted Schurs,

$$O_{R(r,s)\alpha} = \frac{1}{(2n)!(2m)!}\sum_{\sigma\in S_{2q}}\mathrm{Tr}(\mathcal{O}_{R(r,s)\alpha}\Gamma_R)C^{4\nu}_I\sigma^I_J(Z^{\otimes 2n}\otimes Y^{\otimes 2m})^J \tag{4.39}$$

$$\overline{O}_{T(t,u)\beta} = \frac{1}{(2n)!(2m)!}\sum_{\sigma\in S_{2q}}\mathrm{Tr}(\mathcal{O}_{T(t,u)\beta}\Gamma_R)C^K_{4\nu}(\overline{\sigma})^L_K(Z^{\otimes 2n}\otimes Y^{\otimes 2m})_L \tag{4.40}$$

where

$$(\overline{\sigma})^J_I=\delta^{j_{\sigma(1)}}_{i_1}\delta^{j_{\sigma(2)}}_{i_2}\cdots\delta^{j_{\sigma(2q)}}_{i_{2q}}=\delta^{j_1}_{i_{\sigma^{-1}(1)}}\delta^{j_2}_{i_{\sigma^{-1}(2)}}\cdots\delta^{j_{2q}}_{i_{\sigma^{-1}(2q)}}=(\sigma^{-1})^J_I. \tag{4.41}$$

The first step in the calculation is evaluating the correlator

$$\langle (Z^{\otimes 2n} \otimes Y^{\otimes 2m})^J (Z^{\otimes 2n} \otimes Y^{\otimes 2m})_L \rangle. \tag{4.42}$$

In [79] the correlator $(\langle Z^{\otimes 2n})^J (Z^{\otimes 2n})_L \rangle$ was found to be a sum over permutations belonging to the wreath product $S_n[S_2]$, where each term in the sum was weighted by the sgn of the permutation. The correlator in (4.42) generalises to a sum over the subgroup $S_n[S_2] \times S_m[S_2]$

$$\langle (Z^{\otimes 2n} \otimes Y^{\otimes 2m})^J (Z^{\otimes 2n} \otimes Y^{\otimes 2m})_L = \sum_{\eta \in S_n[S_2] \times S_m[S_2]} \mathrm{sgn}(\eta) \eta^J_L. \tag{4.43}$$

This formula may be easily checked for the example of $n = m = 2$.

$$\begin{aligned}
& \langle Z^{i_1 i_2} Z^{i_3 i_4} Y^{i_5 i_6} Y^{i_7 i_8} \bar{Z}^{j_1 j_2} \bar{Z}^{j_3 j_4} \bar{Y}^{j_5 j_6} \bar{Y}^{j_7 j_8} \rangle = \\
& \langle Z^{i_1 i_2} \bar{Z}^{j_1 j_2} \rangle \langle Z^{i_3 i_4} \bar{Z}^{j_3 j_3} \rangle \Big[\langle Y^{i_5 i_6} \bar{Y}^{j_5 j_6} \rangle \langle Y^{i_7 i_8} \bar{Y}^{j_7 j_8} \rangle + \langle Y^{i_5 i_6} \bar{Y}^{j_7 j_8} \rangle \langle Y^{i_7 i_8} \bar{Y}^{j_5 j_6} \rangle \Big] + \\
& \langle Z^{i_1 i_2} \bar{Z}^{j_3 j_4} \rangle \langle Z^{i_3 i_4} \bar{Z}^{j_1 j_2} \rangle \Big[\langle Y^{i_5 i_6} \bar{Y}^{j_5 j_6} \rangle \langle Y^{i_7 i_8} \bar{Y}^{j_7 j_8} \rangle + \langle Y^{i_5 i_6} \bar{Y}^{j_7 j_8} \rangle \langle Y^{i_7 i_8} \bar{Y}^{j_5 j_6} \rangle \Big] \\
= \; & \langle Z^{i_1 i_2} \bar{Z}^{j_1 j_2} \rangle \langle Z^{i_3 i_4} \bar{Z}^{j_3 j_3} \rangle \langle Y^{i_5 i_6} Y^{i_7 i_8} \bar{Y}^{j_5 j_6} \bar{Y}^{j_7 j_8} \rangle + \langle Z^{i_1 i_2} \bar{Z}^{j_3 j_4} \rangle \langle Z^{i_3 i_4} \bar{Z}^{j_1 j_2} \rangle \langle Y^{i_5 i_6} Y^{i_7 i_8} \bar{Y}^{j_5 j_6} \bar{Y}^{j_7 j_8} \rangle \\
= \; & \langle Z^{i_1 i_2} Z^{i_3 i_4} \bar{Z}^{j_1 j_2} \bar{Z}^{j_3 j_4} \rangle \langle Y^{i_5 i_6} Y^{i_7 i_8} \bar{Y}^{j_5 j_6} \bar{Y}^{j_7 j_8} \rangle \\
= \; & \Big[\sum_{\rho \in S_2[S_2]} \mathrm{Sgn}(\rho) \rho \Big] \Big[\sum_{\sigma \in S_2[S_2]} \mathrm{Sgn}(\sigma) \sigma \Big] \\
= \; & \sum_{\psi \in S_2[S_2] \times S_2[S_2]} \mathrm{Sgn}(\psi) \psi^I_J.
\end{aligned} \tag{4.44}$$

where $\psi^I_J = \rho\sigma$ and $\mathrm{Sgn}(\psi) = \mathrm{Sgn}(\rho)\mathrm{Sgn}(\sigma)$. Now compute the two-point function

$$\langle O_{R(r,s)\alpha} \overline{O}_{T(t,u)\beta} \rangle = \frac{1}{((2n)!(2m)!)^2} \sum_{\sigma, \rho \in S_{2q}} \mathrm{Tr}(\mathcal{O}_{R(r,s)\alpha} \Gamma_R(\sigma)) \mathrm{Tr}(\mathcal{O}_{T(t,u)\beta} \Gamma_T(\rho)) C^{4\nu}_I C^L_{4\nu} \sigma^I_J (\rho^{-1})^K_L$$

$$\begin{aligned}
&\times\langle (ZY)^J (ZY)_K\rangle \\
= \;& \frac{1}{((2n)!(2m)!)^2}\sum_{\sigma,\rho\in S_{2q}} \mathrm{Tr}(\mathcal{O}_{R(r,s)\alpha}\Gamma_R(\sigma))\mathrm{Tr}(\mathcal{O}_{T(t,u)\beta}\Gamma_T(\rho)) C_I^{4\nu} C_{4\nu}^L \sigma_J^I (\rho^{-1})_L^K \\
& \sum_{\eta\in S_n[S_2]\times S_m[S_2]} \mathrm{sgn}(\eta)\eta_K^J \\
= \;& \frac{1}{((2n)!(2m)!)^2}\sum_{\sigma,\rho\in S_{2q}}\sum_{\eta\in S_n[S_2]\times S_m[S_2]} \mathrm{Tr}(\mathcal{O}_{R(r,s)\alpha}\Gamma_R(\sigma))\mathrm{Tr}(\mathcal{O}_{T(t,u)\beta}\Gamma_T(\rho))\mathrm{sgn}(\eta). \\
& \qquad \times C_I^{4\nu} C_{4\nu}^L (\rho^{-1}\eta\sigma)_L^I . \qquad (4.45)
\end{aligned}$$

Relabelling the sum over $\sigma = \eta^{-1}\psi$ where $\psi \in S_{2q}$, we obtain

$$\langle O_{R(r,s)\alpha}\overline{O}_{T(t,u)\beta}\rangle = \frac{n!m!2^q}{((2n)!(2m)!)^2}\sum_{\psi,\rho\in S_{2q}} \mathrm{Tr}(\mathcal{O}_{R(r,s)\alpha}\Gamma_R(\psi))\mathrm{Tr}(\mathcal{O}_{T(t,u)\beta}\Gamma_T(\rho)) C_I^{4\nu} C_{4\nu}^L (\rho^{-1}\psi)_L^I . \qquad (4.46)$$

We also used the fact that $\eta \in S_n[S_2]\times S_m[S_2]$, which is a subgroup of $S_{2n}\times S_{2m}$ and that

$$\langle [A],[A]\gamma|\,\Gamma_R(\eta^{-1}) = \langle [A],[A]\gamma|\,\mathrm{sgn}(\eta^{-1}) \qquad (4.47)$$

which then canceled with $\mathrm{sgn}(\eta)$ in (4.45). Next, relabel the sum over ψ by letting $\psi = \rho\tau$, with $\tau \in S_{2q}$. After using the orthogonality relation

$$\sum_{\rho\in S_{2q}} \Gamma_R(\rho)_{ij}\Gamma_T(\rho)_{kl} = \frac{(2q)!}{d_R}\delta_{RT}\delta_{ik}\delta_{jl} \qquad (4.48)$$

and using

$$\mathcal{O}_{R(r,s)\alpha}\mathcal{O}^T_{T(t,u)\beta} = |[S]\rangle\langle[A],[A]\gamma|\, P_{R\to(r,s)\gamma\zeta}\, |[A],[A]\zeta\rangle\langle[S]|\, \delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\beta}, \tag{4.49}$$

where T in the superscript is the transpose of $\mathcal{O}$, equation (4.46) becomes

$$\langle O_{R(r,s)\alpha}\overline{O}_{T(t,u)\beta}\rangle = \delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\beta}\frac{(2q)!n!m!2^q}{d_R((2n)!(2m)!)^2}\sum_{\tau\in S_{2q}}\mathrm{Tr}(\hat{P}_{[S]}\Gamma_R(\tau))C_I^{4\nu}C^L_{4\nu}(\tau)^I_L, \tag{4.50}$$

where we also used the fact that

$$\langle[A],[A]\gamma|\, P_{R\to(r,s)\gamma\zeta}\, |[A],[A],\zeta\rangle = 1 \tag{4.51}$$

to simplify the two-point function. In (4.50), the $S_q[S_2]$ in $\hat{P}_{[S]}$ is the stabiliser for $(1,4)(2,3)\cdots$. For reasons that will become clear shortly, we want to change the embedding of the $S_q[S_2]$. Let ρ be a permutation such that

$$C_I^{4\nu}(\rho)^I_K = C_K^{4\mu}, \tag{4.52}$$

where $\sigma_{4\mu} = (1,2)(3,4)\cdots(2q-1,2q)$. Defining

$$\tau = \rho^{-1}\psi\rho \tag{4.53}$$

equation (4.50) becomes

$$\langle O_{R(r,s)\alpha}\overline{O}_{T(t,u)\beta}\rangle = \delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\beta}\frac{(2q)!n!m!2^q}{d_R((2n)!(2m)!)^2}\sum_{\psi\in S_{2q}}\mathrm{Tr}(\hat{P}_{[S]}\Gamma_R(\rho^{-1}\psi\rho))C_K^{4\mu}C_{4\mu}^L(\psi)_L^K. \tag{4.54}$$

With this relabelling, we are now contracting indices 1&2, 3&4 $\cdots(2q-1\&2q)$,

$$C_K^{4\mu}C_{4\mu}^L(\psi)_L^K = (\psi)^{k_1k_1k_2k_2\cdots k_qk_q}_{l_1l_1l_2l_2\cdots l_ql_q}. \tag{4.55}$$

Inside the trace, $\Gamma_R(\rho)\hat{P}_{[S]}\Gamma_R(\rho^{-1})$ is now a sum over $S_q[S_2]$ which stabilises the set $(1,2),(3,4)\cdots(2q-1,2q)$. We now have the following for the two-point function

$$\langle O_{R(r,s)\alpha}\overline{O}_{T(t,u)\beta}\rangle = \delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\beta}\frac{(2q)!n!m!2^q}{d_R((2n)!(2m)!)^2}\sum_{\psi\in S_{2q}}\mathrm{Tr}(P_{[S]}\Gamma_R(\psi))C_K^{4\mu}C_{4\mu}^L(\psi)_L^K. \tag{4.56}$$

Now split the sum over S_{2q} up into a sum over the Brauer algrebra $\mathcal{B}_q$ and a sum over $S_q[S_2]$ as was done in [79]. $\mathcal{B}_q$ is isomorphic to the coset $S_{2q}/S_q[S_2]$.

$$\langle O_{R(r,s)\alpha}\overline{O}_{T(t,u)\beta}\rangle = \delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\beta}\frac{(2q)!n!m!2^q}{d_R((2n)!(2m)!)^2}\sum_{\psi_1\in\mathcal{B}_q}\sum_{\psi_2\in S_q[S_2]}\mathrm{Tr}(P_{[S]}\Gamma_R(\psi_1\psi_2))C_K^{4\mu}C_{4\mu}^L(\psi_1\psi_2)_L^K. \tag{4.57}$$

Now ψ_2 is an element of the stabiliser of $\sigma_{4\mu}$, which means that

$$C_K^{4\mu}(\psi_2)_I^K = C_I^{4\mu}. \tag{4.58}$$

Then

$$\langle O_{R(r,s)\alpha}\overline{O}_{T(t,u)\beta}\rangle = \delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\beta}\frac{(2q)!q!n!m!2^{2q}}{d_R((2n)!(2m)!)^2}\sum_{\psi_1\in\mathcal{B}_q}\mathrm{Tr}(\Gamma_R(\psi_1)P_{[S]})C_I^{4\mu}C_{4\mu}^L(\psi_1)_L^I. \tag{4.59}$$

Since $\mathcal{B}_q$ is the set of all coset representatives, and therefore not unique, we again choose the following elements for $\mathcal{B}_q$ [79][2]

$$\prod_{j=0}^{q-1}\prod_{i=1}^{2j+1}(i,2j+1). \tag{4.60}$$

It then follows that

$$\langle O_{R(r,s)\alpha}\overline{O}_{T(t,u)\beta}\rangle = \delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\beta}\frac{(2q)!q!n!m!2^{2q}}{d_R((2n)!(2m)!)^2}\langle[S]|\,(\prod_{j=0}^{q-1}(N+J_{2j+1})\,|[S]\rangle\,, \tag{4.61}$$

where we wrote $P_{[S]} = |[S]\rangle\langle[S]|$. J_{2j+1} is the Jucys-Murphy element in irrep R. J_{2j+1} acting on a state in R returns the content of the box labeled $2j+1$. Thus, $N+j_{2j+1}$ gives the weight of that box. Keep in mind that the $|[S]\rangle$ in (4.61) is symmetric in boxes $(1,2)$ and $(3,4)\cdots$, whereas the $|[S]\rangle$ in the definition of the restricted Schurs is symmetric in boxes $(1,4)$ and $(2,3)$ and so on.

The state $|[S]\rangle$ in (4.61) only consists of states in R that has boxes 1&2 next to each other, and 3&4 next to each other $\cdots$ and boxes $2q-1\&2q$ next to each other. As examples, we have

[2]Indeed, this is why we changed the embedding in the first place - so we could use the permutations in (4.60) as the set of coset representatives.

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2} 7 & 8 \\ \cline{1-2}\end{array} \qquad and \qquad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 \\ \cline{1-2} 7 & 8 \\ \cline{1-2}\end{array} \tag{4.62}$$

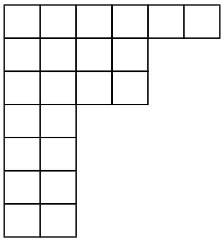
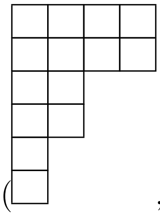
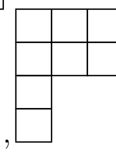
contributing to $|[S]\rangle$ for this particular R. In the end, it is only boxes in the odd columns whose weights contribute to the product in (4.61). By odd columns, we mean the weights of the all the boxes in the 1st column and the weights of all the boxes in the 3rd column and then the 5th column and so on. For example we take the weights of the starred boxes

$$\begin{array}{|c|c|}\hline * & \\ \hline * & \\ \hline\end{array}, \quad \begin{array}{|c|c|c|c|}\hline * & & * & \\ \hline * & & * & \\ \hline\end{array}, \quad \begin{array}{|c|c|}\hline * & \\ \hline * & \\ \hline * & \\ \hline * & \\ \hline\end{array}, \quad \begin{array}{|c|c|c|c|}\hline * & & * & \\ \hline * & \\ \cline{1-2} * & \\ \cline{1-2}\end{array} \tag{4.63}$$

We then arrive at the following formula for the two-point function

$$\langle O_{R(r,s)\alpha}\overline{O}_{T(t,u)\beta}\rangle = \delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\beta}\frac{(2q)!q!n!m!2^{2q}}{d_R((2n)!(2m)!)^2}\prod_{i\,\in\,odd\;columns\;in\;R} c_i. \tag{4.64}$$

We test our formula for a variety of S_4 and S_8 examples in appendix E of this thesis. Unfortunately, the first multiplicity in the $S_{2n} \times S_{2m}$ irreps we found occurred for the case of

$R =$  subducing 2 copies of $(r,s) = ($, $)$, (4.65)

which is beyond our computational capability.

4.7 Discussion

In this chapter, we have constructed an exactly orthogonal basis for the 1/4-BPS sector of free $\mathcal{N} = 4$ super Yang-Mills theory with $SO(2n)$ gauge group. We have shown that the counting of states in this sector exactly matches the number of restricted Schurs the can be defined for (r, s) having even columns and R having even rows. Furthermore, the counting went according to $g(r, s, R)$. This means that there are less operators than for the $U(N)$ case for which the counting went according to $g(r, s, R)^2$. We have also presented a variety of simple examples in support of our findings. We then constructed a basis of operators which matched the counting. The basis we constructed also has the nice symmetry property of being invariant under sending

$$\sigma \to \eta\sigma\xi \qquad \eta \in S_n[S_2] \times S_m[S_2] \text{ and } \xi \in S_q[S_2]. \tag{4.66}$$

Finally we achieved an analytic formula for the two-point function in the form of

(4.64). The restricted Schurs (4.32) are orthogonal and its two-point function is relatively simple to evaluate. We take the product of the weights of the boxes in the odd columns. We also computed a few simple, but non-trivial, examples to check this formula. A next natural step is to study the spectrum of anomalous dimensions of these operators. Using AdS/CFT, we can then study physics of the dual brane objects. We hope to make progress in this direction soon.

Chapter 5

Restricted Schurs and correlators for $SO(N)$ and $Sp(N)$

5.1 Outline of chapter

In the previous chapter, restricted Schur polynomials have been argued to form a complete orthogonal set of gauge invariant operators for the 1/4-BPS sector of free $\mathcal{N} = 4$ super Yang-Mills theory with an $SO(N)$ gauge group. In this chapter, we extend these results to the theory with an $Sp(N)$ gauge group. Using these operators, we develop techniques to compute correlation functions of any multi-trace operators with two scalar fields exactly in the free theory limit for both $SO(N)$ and $Sp(N)$.

5.2 Introduction

Restricted Schur polynomials have been argued to form a complete orthogonal set of gauge invariant operators for the 1/4-BPS sector of free $\mathcal{N} = 4$ super Yang-Mills theory with an $SO(N)$ gauge group [89]. For even N, [89] employed representation theory of the symmetric and hyper octahedral groups to construct a complete set of local operators depending on two scalar fields. Their two-point function was computed exactly and shown to be diagonal in the labels of these operators. More specifically, these operators were the generalisations of the so-called restricted Schur polynomials of the $U(N)$ gauge theory [10], [90] and [91].

Schur polynomials $\chi_R(Z)$ in the 1/2-BPS sector of the free $U(N)$ gauge theory were first studied in [8], in which these operators, labeled by irreducible representations (irreps) R of the symmetric and unitary groups, were shown to be an exactly orthogonal basis. Thus, these gauge invariant operators could be used as a basis to study the large N limit of operators whose dimension scales parametrically with N. The trace basis, for example, is no longer orthogonal in this case and computing non-planar corrections in correlation functions is a difficult task. When the Young diagram R labelling the Schur polynomial has long columns, or long rows, (order N boxes in each row/column) the operator is dual to a system of giant gravitons moving in the S^5 or AdS_5 [4], [5], [6], [8] and [3].

By adding a different type of field to $\chi_R(Z)$, one can build the restricted Schur polynomial. A restricted Schur may simply be thought of as a linear combination all possible multi-matrix, multi-trace operators, where the sum is over permutations of the symmetric group. We can add different types of complex scalar fields, fermion fields or even gauge fields [37], [39] and [40]. For two scalar fields, the counting of restricted Schurs was shown to match the counting of states of the free theory in [13]. Their two-point function was computed exactly in [12] and was shown to be diagonal in the operator labels. [32] succeeded in writing any multi-trace operator involving two scalar fields as a linear combination of restricted Schurs. They also derived a product rule for these operators. This allowed the product of any two restricted Schurs to be written in terms of restricted Schurs and restricted Littlewood-Richardson coefficients.

At the level of summing Feynman diagrams, computing correlation functions of multi-trace multi-matrix operators is a difficult task at finite N. This is because the non-planar corrections are no longer suppressed and must be taken into account. The results of [12] and [32] transformed the problem of computing correlation functions of the multi-matrix, multi-trace operators into the problem of computing correlation functions of restricted Schur polynomials - something we can do exactly for any N.

Other orthogonal bases for the 1/4 and 1/8-BPS sectors of the free gauge theory has been proposed. [45] constructed operators from Z and $Z^\dagger$, while [46] constructed operators depending on X, Y and Z. At weak coupling some progress towards understanding the anomalous dimensions of an orthogonal basis of the free theory, which is manifestly covariant under the global symmetry, was achieved in [77].

Restricted Schur polynomials with n Z's and m Y's are labeled by three Young diagram labels, R, r, s, corresponding to irreducible representations of S_{n+m} and $S_n \times S_m$ respectively. When the number of Z's and Y's is order N, with $n \gg m$, these operators again have a D-brane interpretation in the string theory. For R having long columns (or rows), each with order N boxes, the operator is dual to excited giant gravitons [10]. A system of excited giant gravitons can be thought of as a system of giants with strings attached. Amongst the other bases found for the 1/4-BPS sector it has been argued that restricted Schur polynomials is the most natural basis for studying open string dynamics of their dual D-brane states [32]. To this end, the one-loop dilation operator has been diagonalised and the spectrum of anomalous dimensions computed in [33], [34], [35], [36] [69], [14]. The two-loop case was studied in [41]. Remarkably, for the non-planar limit studied in these works the spectrum was shown to be that of a system of decouple harmonic oscillators. This is evidence of integrability in these non-planar sectors of the gauge theory.

A similar program has been initiated for the gauge theory with an orthogonal gauge group in [79], and [80]. In the 1/2-BPS sector, Schur polynomials were constructed from the basic building block $\yng(2,2)$ which diagonalised the two-point function in the free theory limit. Indeed, orthogonality of Schur polynomials built from a single scalar field Z is a gauge group-independent property [81]. The counting of states, as given by the partition function, for $SO(4)$ and $SO(6)$, was shown to match the number of Schur polynomials that could be defined. This basis was related to the trace basis and a product rule was derived for the Schurs. These results were then extended to the theory with symplectic gauge group, $Sp(N)$. $Sp(N)$ is related to $SO(N)$ by exchanging symmetrisations of irreps and replacing N by $-N$ [80], [78].

This study then progressed to the 1/4-BPS sector of the free $SO(N)$ gauge theory [89]. The counting of states, as given by the partition function, and the counting of restricted Schur polynomials was shown to agree by restricting to a particular class of Young diagram labels. R must have an even number of boxes in each row, while r and s are restricted to have an even number of boxes in each column. An explicit construction of these operators was given and their two-point function was evaluated exactly and shown to be diagonal.

Physics in the non-planar limit of $SO(N)$ and $Sp(N)$ gauge theory is different from that of $U(N)$. The $SO(N)$ and $Sp(N)$ gauge theories are matrix models with anti-symmetric and symplectic matrices respectively. When evaluating correlation functions, the leading non-planar correction comes from non-orientable Feynman diagrams with a single cross-cap - an effect not present in the $U(N)$ theory [80], [78], [92]. Furthermore, $\mathcal{N}=4$ super Yang-Mills theory with an $SO(N)$ or $Sp(N)$ gauge group is dual to type IIB string theory with $\mathrm{AdS}_5 \times \mathcal{R}P^5$ geometry. The Schur polynomials of [79], [80] and the restricted Schur polynomials of [89] may prove useful as a basis of operators to study non-planar limits of these gauge and dual string theories.

In this chapter, we extend our results found in [89], and described in chapter 4, for $SO(N)$ to $Sp(N)$. First, we express the free 1/4-BPS $Sp(N)$ partition function in terms the Littlewood-Richardson coefficients which count the number of $Sp(N)$ restricted Schurs. We then give a gauge invariant construction of these operators and evaluate their two-point function. As expected, the results are identical to those for $SO(N)$ except the Young diagrams are transposed and N is replaced by $-N$. We then relate the trace basis to the restricted Schur basis for both gauge groups. Lastly, we derive a product rule for our operators.

5.3 Recap of $SO(N)$

With a more convenient normalisation, the restricted Schurs defined in [89] were

$$O^{SO(N)}_{R(r,s)\alpha}(Z,Y) = \frac{1}{\sqrt{n!m!q!2^{2q}}} \sum_{\sigma \in S_{2q}} \chi^{SO(N)}_{R(r,s)\alpha}(\sigma) C^{4\nu}_{I} \sigma^{I}_{J} (Z^{\otimes 2n} \otimes Y^{\otimes 2m})^{J}, \tag{5.1}$$

where we defined the $SO(N)$ restricted character to be

$$\chi^{SO(N)}_{R(r,s)\alpha}(\sigma) = \mathrm{Tr}(\mathcal{O}^{SO(N)}_{R(r,s)\alpha} \Gamma_R(\sigma)), \qquad \text{and} \qquad \mathcal{O}^{SO(N)}_{R(r,s)\alpha} = |[S]\rangle\langle [A]_r, [A]_s, \beta| \, P_{R \to (r,s)\beta\alpha} \tag{5.2}$$

In (5.2), we have explicitly indicated which irreps subduce the two $[A]$'s. Irrep r of S_{2n} subduces irrep $[A]_r$ of $S_n[S_2]$, and irrep s of S_{2m} subduces irrep $[A]_s$ of $S_m[S_2]$. β still labels the particular copy of (r,s) subduced from irrep R of S_{2q}. Recall that the state $|[A]_r,[A]_s,\beta\rangle$ spans the 1-dimensional carrier space of the $S_n[S_2]\times S_m[S_2]$ irrep $([A]_r,[A]_s)\beta$, and is calculated as the eigenvector of $P_{R\to([A]_r,[A]_s)\beta}$ with eigenvalue 1. The intertwiner $P_{R\to(r,s)\alpha\beta}$ maps this state from one copy of (r,s) to another

$$P_{R\to(r,s)\alpha\beta}\,|[A]_r,[A]_s,\beta\rangle = |[A]_r,[A]_s,\alpha\rangle\,. \tag{5.3}$$

Thus, we can write

$$\mathcal{O}^{SO(N)}_{R(r,s)\alpha} = |[S]\rangle\langle[A]_r,[A]_s,\alpha|\,. \tag{5.4}$$

For the normalisation in (5.1), the two-point function is

$$\langle O^{SO(N)}_{R(r,s)\alpha}(Z,Y)\overline{O}^{SO(N)}_{T(t,u)\beta}(Z,Y)\rangle = \delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\beta}\frac{(2q)!}{d_R}\prod_{i\,\in\,\text{odd columns}} c_i\,. \tag{5.5}$$

The tensor

$$C^{4\nu}_I = \delta_{k_1k_2}\cdots\delta_{k_{2q-1}k_{2q}}(\sigma_{4\nu})^K_I = \delta_K(\sigma_{4\nu})^K_I \tag{5.6}$$

contracts the free indices in such a way as to produce a gauge invariant operator. In chapter 4 we chose $\sigma_{4\nu} = (1,2,3,4)\cdots(2q-3,2q-2,2q-1,2q)$. This contracted indices $1\&4, 2\&3\cdots 2q-3\&2q$ and $2q-2\&2q-1$. The permutation $\sigma_{4\nu} = (1,2)(3,4)\cdots(2q-1,2q)$, or simply the identity permutation, gives the same gauge invariant operator. Thus, an equivalent definition for $O_{R(r,s)\alpha}(Z,Y)$ is[1]

[1]The $|[S]\rangle$ in this operator is symmetric in boxes $1\&2$, $3\&4\cdots$, $2q-1\&2q$.

$$O^{SO(N)}_{R(r,s)\alpha}(Z,Y) = \frac{1}{\sqrt{n!m!q!2^{2q}}} \sum_{\sigma \in S_{2q}} \chi^{SO(N)}_{R(r,s)\alpha}(\sigma)\, \delta_I \sigma^I_J (Z^{\otimes 2n} \otimes Y^{\otimes 2m})^J. \tag{5.7}$$

For the permutations $\sigma \in S_{2q}$, $\delta_I \sigma^I_J (Z^{\otimes 2n} \otimes Y^{\otimes 2m})^J$ gives all the possible multi-trace operators involving the two scalar fields Z and Y. For example, consider $q = 4$ with $n = m = 2$. There are only 4 multi-trace operators we can define. Here they are for 4 examples of σ

$$\sigma = (2,4,6,8,3) \quad \text{gives} \quad \delta_I \sigma^I_J (Z^{\otimes 4} \otimes Y^{\otimes 4})^J = \mathrm{Tr}(Z^2 Y^2), \tag{5.8}$$

$$\sigma = (2,5)(3,4,5) \quad \text{gives} \quad \delta_I \sigma^I_J (Z^{\otimes 4} \otimes Y^{\otimes 4})^J = \mathrm{Tr}(ZY)^2, \tag{5.9}$$

$$\sigma = (1,3,2)(5,8,7) \quad \text{gives} \quad \delta_I \sigma^I_J (Z^{\otimes 4} \otimes Y^{\otimes 4})^J = \mathrm{Tr}(Z^2)\mathrm{Tr}(Y^2), \tag{5.10}$$

$$\sigma = (1,5,2)(3,4,8,6,7) \quad \text{gives} \quad \delta_I \sigma^I_J (Z^{\otimes 4} \otimes Y^{\otimes 4})^J = \mathrm{Tr}(ZYZY). \tag{5.11}$$

5.4 Counting for $Sp(N)$

The partition function for the 1/4-BPS sector of free $Sp(N)$ gauge theory is [86], [93]

$$G_{Sp(N)}(t_1, t_2) = \int_{O \in Sp(N)} [dO] e^{\sum_{m=1}^{\infty} \left(\frac{t_1^m + t_2^m}{m}\right) \chi_{\mathrm{adj}}(O^m)} \tag{5.12}$$

where $\chi_{\mathrm{adj}}(O)$ is the character of O in the adjoint representation of $Sp(N)$, and $[dO]$ is the $Sp(N)$-invariant measure. We take $N = 2n$. In terms of the eigenvalues of O, the adjoint character and the integration measure are [84]

$$\chi^{\mathrm{adj}}_{Sp(N)}(\mathrm{x}) = \sum_{1 \leqslant i < j \leqslant n} (x_i x_j + x_i^{-1} x_j + x_i x_j^{-1} + x_i^{-1} x_j^{-1}) + \sum_{i=1}^{n} (x_i^2 + x_i^{-2}) + n \tag{5.13}$$

and

$$\int_{O\in Sp(N)}[dO]f(\mathrm{x}) = \frac{(-1)^n}{2^n n!}\int_{T_n}\prod_{j=1}^{n}\frac{dx_j}{2\pi x_j}\prod_{j=1}^{n}(x_j-x_j^{-1})^2\Delta(\mathrm{x}+\mathrm{x}^{-1})^2 f(\mathrm{x}) \quad (5.14)$$

where $T_n = S^1\times S^1\cdots\times S^1$ and $f(\mathrm{x}), \mathrm{x}=(x_1,x_2\cdots x_n)$, is any symmetric function. Using (5.13) the exponential in (5.12), after some algebra, becomes

$$\begin{aligned} e^{\sum_{m=1}^{\infty}\left(\frac{t_1^m+t_2^m}{m}\right)\chi_{\mathrm{adj}}(O^m)} &= e^{\sum_{m=1}^{\infty}\left(\frac{t_1^m+t_2^m}{m}\right)\sum_{1\leqslant i<j\leqslant n}(x_i^m x_j^m+x_i^{-m}x_j^m+x_i x_j^{-m}+x_i^{-m}x_j^{-m})+\sum_{i=1}^{n}(x_i^{2m}+x_i^{-2m})+n} \\ &= \prod_{k=1}^{2}\prod_{1\leqslant i<j\leqslant n}\frac{1}{(1-t_k x_i x_j)(1-t_k x_i^{-1}x_j)(1-t_k x_i x_j^{-1})(1-t_k x_i^{-1}x_j^{-1})} \\ &\quad\times\frac{1}{(1-t_k)^n}\prod_{1\leqslant i\leqslant n}\frac{1}{(1-t_k x_i^2)(1-t_k x_i^{-2})}. \end{aligned} \quad (5.15)$$

Changing variables

$$y_i = x_{\lceil i/2\rceil} \quad \text{for odd } i \quad (5.16)$$

$$y_{2i} = x_i^{-1} \quad \text{for even } i \quad (5.17)$$

the exponential becomes

$$e^{\sum_{m=1}^{\infty}\left(\frac{t_1^m+t_2^m}{m}\right)\chi_{\mathrm{adj}}(O^m)} = \prod_{k=1}^{2}\prod_{1\leqslant i\leqslant j\leqslant N}\frac{1}{1-t_k y_i y_j} \quad (5.18)$$

$$= \prod_{k=1}^{2}\left(\sum_{\lambda}s_{2\lambda}(\sqrt{t_k}y_1,\sqrt{t_k}y_2\cdots\sqrt{t_k}y_N)\right) \quad (5.19)$$

$$= \sum_{\mu}\sum_{\lambda}(t_1)^{|2\lambda|/2}(t_2)^{|2\mu|/2}s_{2\lambda}(y_1,\cdots y_N)s_{2\mu}(y_1,\cdots,y_N). \tag{5.20}$$

The expansion in terms of Schur functions we used in (5.19) is an identity found in [88]. Using the product rule for Schur polynomials, (5.20) may be written in terms of the Littlewood-Richardson coefficients

$$e^{\sum_{m=1}^{\infty}\left(\frac{t_1^m+t_2^m}{m}\right)\chi_{\mathrm{adj}}(O^m)} = \sum_{\xi}\sum_{\mu}\sum_{\lambda}(t_1)^{|2\lambda|/2}(t_2)^{|2\mu|/2}g(2\lambda,2\mu,\xi)s_{\xi}(y_1,\cdots,y_N). \tag{5.21}$$

The partition function (5.12) becomes

$$\begin{aligned} G_{Sp(N)}(t_1,t_2) &= \sum_{\xi}\sum_{\mu}\sum_{\lambda}(t_1)^{|2\lambda|/2}(t_2)^{|2\mu|/2}g(\lambda,\mu,\xi) \\ &\times \frac{(-1)^n}{2^n n!}\int_{T_n}\prod_{j=1}^{n}\frac{dx_j}{2\pi x_j}\prod_{j=1}^{n}(x_j-x_j^{-1})^2\Delta(\mathrm{x}+\mathrm{x}^{-1})^2 s_{\xi}(x_1,x_1^{-1},\cdots,x_n,x_n^{-1}). \end{aligned} \tag{5.22}$$

The integral in (5.22) is 1 for partitions ξ that have even multiplicity, i.e., an even number of boxes in each column, and 0 otherwise [94], [87]. Therefore we write ξ^2 for this partition. As we did for $SO(N)$, we succeeded in writing the partition function for $Sp(N)$ gauge theory in terms of the Littlewood-Richardson coefficients

$$G_{Sp(N)}(t_1,t_2) = \sum_{\xi^2}\sum_{\mu}\sum_{\lambda}(t_1)^{|2\lambda|/2}(t_2)^{|2\mu|/2}g(2\lambda,2\mu,\xi^2). \tag{5.23}$$

Only partitions 2μ and 2λ, with an even number of boxes in each row, and ξ^2, with an even number of boxes in each column, contribute to $G_{Sp(N)}$. The Littlewood-Richardson coefficients $g(2\lambda, 2\mu, \xi^2)$ count the number of restricted Schur polynomials that can be defined for labels $(2\lambda, 2\mu, \xi^2)$. Thus, the counting of restricted Schur polynomials for this class of Young diagram labels matches the counting of states in the free $Sp(N)$ gauge theory.

5.5 $Sp(N)$ restricted Schurs

5.5.1 Constructing the operators

We now give an explicit construction of restricted Schurs for $Sp(N)$ gauge theory. We continue to consider only $N = 2n$. The group $Sp(N)$ is the set of $N \times N$ matrices, S, satisfying

$$S^T J S = J, \qquad J = \begin{pmatrix} 0 & \mathbb{I}_{N/2} \\ -\mathbb{I}_{N/2} & 0 \end{pmatrix}. \tag{5.24}$$

The matrix fields living in the adjoint representation of the $sp(N)$ algebra satisfy

$$Z^T J + JZ = 0, \qquad \text{or} \qquad Z^T = JZJ. \tag{5.25}$$

Our $Sp(N)$ restricted Schurs must match the counting found in (5.23). Since $(r,s)\alpha$ has an even number of boxes in each row, the irrep $([S]_r, [S]_s)\alpha$ of $S_n[S_2] \times S_m[S_2]$ may be subduced. To match the counting, we construct our operators using the state $|[S]_r, [S]_s, \alpha\rangle$. Furthermore, because R has an even number of boxes in each column, the irrep $[A]$ of $S_q[S_2]$ may be subduced. This leads us to a natural definition for the $Sp(N)$ restricted characters. Define

$$\mathcal{O}^{Sp(N)}_{R(r,s)\alpha} = |[A]\rangle\langle[S]_r, [S]_s, \alpha| = |[A]\rangle\langle[S]_r, [S]_s, \beta|\, P_{R\to(r,s)\beta\alpha}. \tag{5.26}$$

The $Sp(N)$ restricted character, $\chi^{Sp(N)}_{R(r,s)\alpha}(\sigma)$, clearly has the property

$$\chi^{Sp(N)}_{R(r,s)\alpha}(\sigma)\mathrm{sgn}(\xi) = \chi^{Sp(N)}_{R(r,s)\alpha}(\eta\sigma\xi), \qquad \xi \in S_q[S_2], \eta \in S_n[S_2] \times S_m[S_2]. \tag{5.27}$$

This function, and its $SO(N)$ counterpart (5.2), resembles the bi-spherical functions discussed in [83], [81], where one of the hyperoctahedral groups in (5.27) is now a subgroup of the other. We construct the restricted Schurs to be invariant under sending $\sigma \to \eta\sigma\xi$. Define

$$O^{Sp(N)}_{R(r,s)\alpha}(Z,Y) = \frac{1}{\sqrt{n!m!q!2^{2q}}} \sum_{\sigma\in S_{2q}} \chi^{Sp(N)}_{R(r,s)\alpha}(\sigma)\, J_I \sigma^I_J((JZ)^{\otimes 2n} \otimes (JY)^{\otimes 2m})^J, \tag{5.28}$$

where

$$J_I = J_{i_1 i_2} J_{i_3 i_4} \cdots J_{i_{2q-1} i_{2q}}. \tag{5.29}$$

The quantity $J_I(\sigma)^I_J((JZ)^{\otimes 2n} \otimes (JY)^{\otimes 2m})^J$ indeed gives all the possible multitrace operators for n Z's, m Y's and for the permutations of S_{2q}. For example, 2 Z's and 2 Y's, give only 4 possible multi-trace operators. They are

$$\mathrm{Tr}(Z^2Y^2), \quad \mathrm{Tr}(ZYZY), \quad \mathrm{Tr}(Z^2)\mathrm{Tr}(Y^2) \quad \text{and} \quad \mathrm{Tr}(ZY)^2. \tag{5.30}$$

Indeed, $J_I(\sigma)^I_J((JZ)^{\otimes 4} \otimes (JY)^{\otimes 4})^J$ generate all of these for $\sigma \in S_8$. Next, we note that $J_I(\sigma)^I_J((JZ)^{\otimes 2n} \otimes (JY)^{\otimes 2m})^J$ has the desired symmetry property when $\sigma \to \eta\sigma\xi$. Since the J's are anti-symmetric under transposition, ξ acting on J_I gives $\mathrm{sgn}(\xi)J_I$, and since (JZ) and (JY) are symmetric under transposition, η leaves the tensor invariant. The state $|[A]\rangle$ is calculated as the eigenvector of

$$P_{[A]} = \frac{1}{q!2^q} \sum_{\xi \in S_q[S_2]} \mathrm{sgn}(\xi)\Gamma_R(\xi) \tag{5.31}$$

with eigenvalue 1. The $|[S]_r, [S]_s, \alpha\rangle$ is calculated as the eigenvector of

$$P_{R\to([S]_r,[S]_s)\alpha} = P_{R\to(r,s)\alpha\alpha}P_{[S,S]}, \qquad P_{[S,S]} = \frac{1}{n!m!} \sum_{\eta \in S_n[S_2]\times S_m[S_2]} \Gamma_R(\eta) \tag{5.32}$$

with eigenvalue 1.

5.5.2 Two-point function

Now consider evaluating the two-point function of the operators in (5.28). First, it is straightforward to show that

$$\langle ((JZ)^{\otimes 2n} \otimes (JY)^{\otimes 2m})^J ((JZ)^{\otimes 2n} \otimes (JY)^{\otimes 2m})_L \rangle = \sum_{\eta \in S_n[S_2]\times S_m[S_2]} \eta^J_L. \tag{5.33}$$

Using (5.33), the evaluation of the two-point function is analogous to that of the $SO(N)$ case [89]. Following the same steps, we arrive at

$$\langle O^{Sp(N)}_{R(r,s)\alpha} \overline{O}^{Sp(N)}_{T(t,u)\beta} \rangle = \delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\beta}\frac{(2q)!}{d_R}\sum_{\psi\in\mathcal{B}_q} \mathrm{Tr}(P_{[A]}\Gamma_R(\psi))J_I J^M(\psi)^I_M \tag{5.34}$$

where $\mathcal{B}_q$ is again the set of representatives of the coset $S_{2q}/S_q[S_2]$, chosen to be the permutations

$$\prod_{j=0}^{q-1}\prod_{i=1}^{2j+1}(i, 2j+1). \tag{5.35}$$

For the permutations in (5.35), $J_I J^M(\psi)^I_M$ gives exactly the same result as $\delta_I \delta^J(\psi)^I_J$. The identity permutation, for example, gives

$$\begin{aligned} J_{i_1 i_2} J_{i_3 i_4} \cdots J_{i_{2q-1} i_{2q}} J^{i_1 i_2} J^{i_3 i_4} \cdots J^{i_{2q-1} i_{2q}} &= (J^T J)^{i_1}_{i_1} (J^T J)^{i_3}_{i_3} \cdots (J^T J)^{i_{2q-1}}_{i_{2q-1}} \\ &= \delta^{i_1}_{i_1}\delta^{i_3}_{i_3}\cdots\delta^{i_{2q-1}}_{i_{2q-1}} \\ &= N^q. \end{aligned} \tag{5.36}$$

A two-cycle $(i, 2j+1)$ returns N^{q-1}. For example, consider $q = 4$ and $(3, 5)$. This gives

$$(J^T J)^{i_1}_{i_1}(J^T J J^T J)^{i_3}_{i_3}(J^T J)^{i_7}_{i_7} = \delta^{i_1}_{i_1}\delta^{i_3}_{i_3}\delta^{i_7}_{i_7} = N^3. \tag{5.37}$$

For $(4, 7)$, we get

$$(J^T J)^{i_1}_{i_1}(J^2 J^2)^{i_3}_{i_3}(J^T J)^{i_5}_{i_5} = \delta^{i_1}_{i_1}\delta^{i_3}_{i_3}\delta^{i_7}_{i_7} = N^3. \tag{5.38}$$

For any $\psi \in \mathcal{B}_q$, $J_I J^M(\psi)^I_M$ always consists of products of $(J^T J)$ and an even number of J^2's. If p is the number of two-cycles in ψ, then $J_I J^M(\psi)^I_M$ gives N^{q-p}. After summing over $\mathcal{B}_q$, equation (5.34) becomes

$$\langle O^{Sp(N)}_{R(r,s)\alpha} \overline{O}^{Sp(N)}_{T(t,u)\beta} \rangle = \delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\beta}\frac{d_R}{(2q)!}\langle [A]| \prod_{j=0}^{q-1}(N + J_{2j+1})|[A]\rangle. \tag{5.39}$$

Instead of giving a product of weights coming from the odd columns in R, as it did for $SO(N)$, (5.39) now gives the product of weights coming from the odd *rows* of R. The two-point function for the $Sp(N)$ restricted Schurs is

$$\langle O^{Sp(N)}_{R(r,s)\alpha} \overline{O}^{Sp(N)}_{T(t,u)\beta} \rangle = \delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\beta}\frac{(2q)!}{d_R} \prod_{i\ \in \text{odd rows in R}} c_i. \tag{5.40}$$

This gives precisely the same result as the $SO(N)$ case, but with $N \to -N$, as expected. Thus, to calculate the two-point function of operators $O^{Sp(N)}_{R(r,s)\alpha}$, calculate the two-point function of $O^{SO(N)}_{R_c(r_c,s_c)\alpha}$ in $SO(N)$ gauge theory, and send $N \to -N$. This gives the result for $Sp(N)$. By R_c, we mean the conjugate (or transpose) of the Young diagram. As a check of these conclusions, we present two examples in appendix G.

5.6 Exact multi-trace correlators with two scalar fields

In this section, we express any multi-trace operator involving Z's and Y's as a linear combination of the $SO(N)$ and $Sp(N)$ restricted Schur polynomials, (5.1) and (5.28).

Thereafter, we derive a product rule for these operators. First, let's discuss $SO(N)$.

Define the 'dual restricted character'

$$\chi^{R(r,s)\alpha}_{SO(N)}(\sigma) \equiv \frac{d_R\sqrt{n!m!q!2^{2q}}}{(2q)!}\mathrm{Tr}(\mathcal{O}^T_{R(r,s)\alpha}\Gamma_R(\sigma^{-1})). \tag{5.41}$$

The aim is to use (5.41) to express any multi-trace operator in terms of the restricted Schurs in (5.1). The calculation is analogous to that in [32]. Calculate

$$\sum_{R,r,s,\alpha}\chi^{R(r,s)\alpha}_{SO(N)}(\sigma)\mathrm{Tr}(\mathcal{O}_{R(r,s)\alpha}\Gamma_R(\tau)) = \frac{\sqrt{n!m!q!2^{2q}}}{(2q)!}\sum_{R,r,s,\alpha}d_R\mathrm{Tr}(\mathcal{O}^T_{R(r,s)\alpha}\Gamma_R(\sigma^{-1}))\mathrm{Tr}(\mathcal{O}_{R(r,s)\alpha}\Gamma_R(\tau)). \tag{5.42}$$

The Γ's may be expanded in the R basis in the following way

$$\Gamma_R(\sigma) = \sum_{I,J}|R,I\rangle\langle R,J|\ \Gamma_R(\sigma)_{IJ}. \tag{5.43}$$

Using this, the trace $\mathrm{Tr}(\mathcal{O}_{R(r,s)\alpha}\Gamma_R(\tau))$ may be written as

$$\begin{aligned}\mathrm{Tr}(\mathcal{O}_{R(r,s)\alpha}\Gamma_R(\tau)) &= \sum_K \langle R,K|\,\mathcal{O}_{R(r,s)\alpha}\Gamma_R(\tau)\,|R,K\rangle \\ &= \sum_{K,L}\langle R,K|\,\mathcal{O}_{R(r,s)\alpha}\,|R,L\rangle\,\Gamma_R(\tau)_{LK}.\end{aligned} \tag{5.44}$$

Then

$$\sum_{r,s,\alpha}\mathrm{Tr}(\mathcal{O}^T_{R(r,s)\alpha}\Gamma_R(\sigma^{-1}))\mathrm{Tr}(\mathcal{O}_{R(r,s)\alpha}\Gamma_R(\tau)) =$$

$$
\begin{aligned}
&\sum_{r,s,\alpha}\Big(\sum_{I,J}\langle R,I|\,\mathcal{O}^T_{R(r,s)\alpha}\,|R,J\rangle\,\Gamma_R(\sigma^{-1})_{JI}\Big)\\
&\times\Big(\sum_{K,L}\langle R,K|\,\mathcal{O}_{R(r,s)\alpha}\,|R,L\rangle\,\Gamma_R(\tau)_{LK}\Big)\\
=&\sum_{r,s,\alpha}\sum_{I,J,K,L}\Gamma_R(\sigma^{-1})_{JI}\Gamma_R(\tau)_{LK}\\
&\times\langle R,I|[A]_r,[A]_s,\alpha\rangle\langle[S]|R,J\rangle\langle R,K|[S]\rangle\langle[A]_r,[A]_s,\alpha|R,L\rangle\\
=&\sum_{r,s,\alpha}\sum_{I,J,K,L}\Gamma_R(\sigma^{-1})_{JI}\Gamma_R(\tau)_{LK}\\
&\times\langle R,K|[S]\rangle\langle[S]|R,J\rangle\ \langle R,I|[A]_r,[A]_s,\alpha\rangle\langle[A]_r,[A]_s,\alpha|R,L\rangle. \qquad (5.45)
\end{aligned}
$$

Recognising

$$
\begin{aligned}
P_{[S]} &= \frac{1}{q!2^q}\sum_{\xi\in S_q[S_2]}\Gamma_R(\xi)=|[S]\rangle\langle[S]| &(5.46)\\
P_{[A,A]} &= \frac{1}{n!m!2^q}\sum_{\eta\in S_n[S_2]\times S_m[S_2]}\mathrm{sgn}(\eta)\Gamma_R(\eta)\\
&= \sum_{r,s,\alpha}|[A]_r,[A]_s,\alpha\rangle\langle[A]_r,[A]_s,\alpha|\,. &(5.47)
\end{aligned}
$$

In appendix F, we discuss going from (5.47) to (5.47) in more detail. Equation (5.45) then becomes

$$
\begin{aligned}
\sum_{r,s,\alpha}\mathrm{Tr}(\mathcal{O}^T_{R(r,s)\alpha}\Gamma_R(\sigma^{-1}))\mathrm{Tr}(\mathcal{O}_{R(r,s)\alpha}\Gamma_R(\tau)) &= \frac{1}{n!m!q!2^{2q}}\sum_{\xi\in S_q[S_2]}\sum_{\eta\in S_{n[S_2]\times S_{m[S_2]}}}\sum_{I,J,K,L}\\
&\quad\times\mathrm{sgn}(\eta)\Gamma_R(\sigma^{-1})_{JI}\Gamma_R(\tau)_{LK}\Gamma_R(\xi)_{KJ}\Gamma_R(\eta)_{IL}\\
&= \frac{1}{n!m!q!2^{2q}}\sum_{\xi\in S_q[S_2]}\sum_{\eta\in S_{n[S_2]\times S_{m[S_2]}}}\mathrm{sgn}(\eta)\chi_R(\sigma^{-1}\eta\tau\xi).
\end{aligned}
\qquad (5.48)
$$

Using (5.48), equation (5.42) becomes

$$\sum_{R,r,s,\alpha}\chi^{R(r,s)\alpha}_{SO(N)}(\sigma)\mathrm{Tr}(\mathcal{O}_{R(r,s)\alpha}\Gamma_R(\tau)) = \frac{1}{\sqrt{n!m!q!2^{2q}}}\sum_R\frac{d_R}{(2q)!}\sum_{\xi\in S_q[S_2]}\sum_{\eta\in S_{n[S_2]\times S_m[S_2]}}\mathrm{sgn}(\eta)\chi_R(\sigma^{-1}\eta\tau\xi). \tag{5.49}$$

In this expression, we are summing over all possible $R\vdash 2q$ and terms for which R does not have an even number of boxes in each row vanish[2].

$$\begin{aligned}\sum_{R,r,s,\alpha}\chi^{R(r,s)\alpha}_{SO(N)}(\sigma)\mathrm{Tr}(\mathcal{O}_{R(r,s)\alpha}\Gamma_R(\tau)) &= \frac{1}{\sqrt{n!m!q!2^{2q}}}\sum_{\xi\in S_q[S_2]}\sum_{\eta\in S_{n[S_2]\times S_m[S_2]}}\mathrm{sgn}(\eta)\sum_R\frac{d_R}{(2q)!}\chi_R(\sigma^{-1}\eta\tau\xi)\\ &= \frac{1}{\sqrt{n!m!q!2^{2q}}}\sum_{\xi\in S_q[S_2]}\sum_{\eta\in S_{n[S_2]\times S_m[S_2]}}\mathrm{sgn}(\eta)\delta(\sigma^{-1}\eta\tau\xi). \end{aligned}\tag{5.50}$$

We now use result (5.50) to write any multi-trace operator in terms of $O_{R(r,s)\alpha}(Z,Y)$. Consider

$$\begin{aligned}\sum_{R(r,s)\alpha}\chi^{R(r,s)\alpha}_{SO(N)}(\sigma)O_{R(r,s)\alpha}(Z,Y) &= \frac{1}{\sqrt{n!m!q!2^{2q}}}\sum_{\tau\in S_{2q}}\Big[\sum_{R(r,s)\alpha}\chi^{R(r,s)\alpha}_{SO(N)}(\sigma)\mathrm{Tr}(\mathcal{O}_{R(r,s)\alpha}\Gamma_R(\tau))\Big],\\ &\qquad\times C^{4\nu}_I\tau^I_J(Z^{\otimes 2n}\otimes Y^{\otimes 2m})^J\\ &= \frac{1}{n!m!q!2^{2q}}\sum_{\tau\in S_{2q}}\sum_{\xi\in S_q[S_2]}\sum_{\eta\in S_{n[S_2]\times S_m[S_2]}}\mathrm{sgn}(\eta)\delta(\sigma^{-1}\eta\tau\xi)\\ &\qquad\times C^{4\nu}_I\tau^I_J(Z^{\otimes 2n}\otimes Y^{\otimes 2m})^J. \end{aligned}\tag{5.51}$$

Summing over τ, $\delta(\sigma^{-1}\eta\tau\xi)$ sets $\tau=\eta^{-1}\sigma\xi^{-1}$. Thus

[2]This is a simple consequence of the fact that only R having even rows is capable of subducing the $S_q[S_2]$ irrep $[S]$.

$$\sum_{R(r,s)\alpha} \chi^{R(r,s)\alpha}_{SO(N)}(\sigma) O_{R(r,s)\alpha}(Z,Y) = \frac{1}{n!m!q!2^{2q}} \sum_{\xi\in S_q[S_2]} \sum_{\eta\in S_{n[S_2]\times S_m[S_2]}} \mathrm{sgn}(\eta) C^{4\nu}_I (\eta^{-1}\sigma\xi^{-1})^I_J (Z^{\otimes 2n}\otimes Y^{\otimes 2m})^J. \tag{5.52}$$

The tensor $C^{4\nu}_I$ acted on by the permutation $\xi \in S_q[S_2]$ is invariant, and the $\eta \in S_n[S_2] \times S_m[S_2]$ acting on the ZY tensor picks up a $\mathrm{sgn}(\eta)$. After summing over ξ and η, we obtain

$$\sum_{R(r,s)\alpha} \chi^{R(r,s)\alpha}_{SO(N)}(\sigma) O_{R(r,s)\alpha}(Z,Y) = C^{4\nu}_K \sigma^K_L (Z^{\otimes 2n}\otimes Y^{\otimes 2m})^L. \tag{5.53}$$

Each σ gives some multi-trace operator. Using the restricted Schurs in (5.1) for the S_8 irreps,

$$\yng(4,4), \left(\yng(2,2), \yng(2,2)\right) \quad \yng(2,2,2,2), \left(\yng(2,2), \yng(2,2)\right) \quad \yng(2,2,2,2), \left(\yng(1,1,1,1), \yng(1,1,1,1)\right) \quad \yng(4,2,2), \left(\yng(2,2), \yng(2,2)\right) \tag{5.54}$$

we have checked formula (5.53) for a large number of permutations σ. We present two examples in appendix H.

The calculation is exactly the same for $Sp(N)$. Defining the $Sp(N)$ 'dual restricted character'

$$\chi^{R(r,s)\alpha}_{Sp(N)}(\sigma) = \frac{d_R\sqrt{n!m!q!2^{2q}}}{(2q)!} \mathrm{Tr}(\mathcal{O}^{Sp(N)\,T}_{R(r,s)} \Gamma_R(\sigma^{-1})), \tag{5.55}$$

we find that any multi-trace operator may be written as

$$\sum_{R,r,s,\alpha} \chi^{R(r,s)\alpha}_{Sp(N)}(\sigma) O^{Sp(N)}_{R(r,s)\alpha}(Z,Y) = J_I(\sigma)^I_K((JZ)^{\otimes 2n} \otimes (JY)^{\otimes 2m})^K. \tag{5.56}$$

We are now able to express any multi-trace operator, involving two scalar fields, in the free $SO(N)$ and $Sp(N)$ theory in terms of our restricted Schur basis.

A product rule for operators $O^{SO(N)}_{R(r,s)\alpha}(Z,Y)$ in terms of restricted Littlewood-Richardson coefficients is easily derived. The basic idea is exactly the same as in [32]. We multiply two restricted Schurs, one having labels $R_1(r_1,s_1)\alpha_1$ and the other $R_2(r_2,s_2)\alpha_2$, to produce a linear combination of restricted Schurs with labels $R_1+R_2,(r_1+r_2,s_1+s_2)\beta$. R_1 and R_2 are irreps of S_{2q_1} and S_{2q_2} respectively. (r_1,s_1) and (r_2,s_2) are irreps of $S_{2n_1}\times S_{2m_1}$ and $S_{2n_2}\times S_{2m_2}$ respectively, where $q_i=n_i+m_i$. α_1 and α_2 are the multiplicity labels for the two respective subgroup irreps. Thus, $R_i(r_i,s_i)\alpha_i$ defines a restricted Schur having n_i Z's and m_i Y's. R_1+R_2 is an irrep of $S_{2q_1+2q_2}$, r_1+r_2 is an irrep of $S_{2n_1+2n_2}$ and s_1+s_2 is an irrep of $S_{2m_1+2m_2}$. β simply labels the (r_1+r_2,s_1+s_2) copy subduced from R_1+R_2. The set of labels $\{R_1+R_2,(r_1+r_2,s_1+s_2),\beta\}$ defines a restricted Schur having n_1+n_2 Z's and m_1+m_2 Y's. As in [32], it is convenient to streamline the notation. Thus, denote $\{i\}\equiv R_i(r_i,s_i)\alpha_i$ and $\{1+2\}\equiv(R_1+R_2)(r_1+r_2,s_1+s_2)\beta$. Also, write $n_{12}=n_1+n_2$, $m_{12}=m_1+m_2$ and $q_{12}=q_1+q_2$. Thus, for example, (5.7) may be written as

$$O^{SO(N)}_{\{1+2\}}(Z,Y) = \frac{1}{\sqrt{n_{12}!m_{12}!q_{12}!2^{2q_{12}}}} \sum_{\rho\in S_{2q_{12}}} \chi^{SO(N)}_{\{1+2\}}(\rho)\delta_I\rho^I_J(Z^{\otimes 2n_{12}}\otimes Y^{\otimes 2m_{12}})^J. \tag{5.57}$$

In the following derivation, we use the operators defined in (5.7) which are equivalent to the ones defined in (5.1). The factorisation we need in our product rule occurs naturally for the operators defined using the δ_I tensor, rather than the $C_I^{4\nu}$ tensor. The following derivation is for $SO(N)$, but the derivation for $Sp(N)$ is the same. Define the restricted Littlewood-Richardson coefficients to be

$$f_{\{1\},\{2\}}^{\{1+2\}} = \frac{1}{\sqrt{n_1!m_1!n_2!m_1!m_2!2^{2q_1+2q_2}}} \sum_{\sigma_1\in S_{2q_1}} \sum_{\sigma_2\in S_{2q_2}} \chi_{\{1\}}^{SO(N)}(\sigma_1)\chi_{\{2\}}^{SO(N)}(\sigma_2)\chi_{SO(N)}^{\{1+2\}}(\sigma_1\cdot\sigma_2) \tag{5.58}$$

where $\chi_{SO(N)}^{\{1+2\}}$ is the dual restricted character defined in (5.41). We want to evaluate

$$\sum_{\{1+2\}} f_{\{1\},\{2\}}^{\{1+2\}} O_{\{1+2\}}^{SO(N)}(Z,Y). \tag{5.59}$$

To evaluate (5.59), we use equation (5.50) to write

$$\begin{aligned}\sum_{\{1+2\}} \chi_{SO(N)}^{\{1+2\}}(\sigma_1\cdot\sigma_2)\chi_{\{1+2\}}^{SO(N)}(\rho) \quad = \quad & \frac{1}{\sqrt{n_{12}!m_{12}!q_{12}!2^{q_{12}!}}} \sum_{\xi\in S_{q_{12}}[S_2]} \sum_{\eta\in S_{n_{12}}[S_2]\times S_{m_{12}}[S_2]} \mathrm{sgn}(\eta) \\ & \times\, \delta(\sigma_1^{-1}\cdot\sigma_2^{-1}\eta\rho\xi). \end{aligned} \tag{5.60}$$

Summing over ρ sets $\rho = \eta^{-1}\sigma_1\sigma_2\xi^{-1}$. As before, ξ acting on δ_I is invariant and η acting on the ZY tensor gives back an extra $\mathrm{sgn}(\eta)$. Summing over ξ and η then cancels the normalisation factor in (5.57), and thus we write

$$\begin{aligned}\sum_{\{1+2\}} f_{\{1\},\{2\}}^{\{1+2\}} O_{\{1+2\}}^{SO(N)}(Z,Y) = & \frac{1}{\sqrt{n_1!m_1!n_2!m_1!m_2!2^{2q_1+2q_2}}} \sum_{\sigma_1\in S_{2q_1}} \sum_{\sigma_2\in S_{2q_2}} \chi_{\{1\}}^{SO(N)}(\sigma_1)\chi_{\{2\}}^{SO(N)}(\sigma_2) \\ & \delta_I(\sigma_1\cdot\sigma_2)^I_J(Z^{\otimes 2n_{12}}\otimes Y^{\otimes 2m_{12}})^J. \end{aligned} \tag{5.61}$$

By σ_1, we mean the permutation that acts on the first $2n_1$ Z indices and first $2m_1$ Y indices. By σ_2, we mean the permutation that acts on the second $2n_2$ Z indices and second $2m_2$ Y indices. The second line in (5.61) factories and we may write

$$\begin{aligned}\sum_{\{1+2\}} f^{\{1+2\}}_{\{1\},\{2\}} O^{SO(N)}_{\{1+2\}}(Z,Y) = & \frac{1}{\sqrt{n_1!m_1!q_1!2^{2q_1}}} \sum_{\sigma_1\in S_{2q_1}} \chi^{SO(N)}_{\{1\}}(\sigma_1)\delta_{I_1}(\sigma_1)^{I_1}_{J_1}(Z^{\otimes 2n_1}\otimes Y^{\otimes 2m_1})^{J_1} \\ & \times \frac{1}{\sqrt{n_2!m_2!q_2!2^{2q_2}}} \sum_{\sigma_2\in S_{2q_2}} \chi^{SO(N)}_{\{2\}}(\sigma_2)\delta_{I_2}(\sigma_2)^{I_2}_{J_2}(Z^{\otimes 2n_2}\otimes Y^{\otimes 2m_2})^{J_2}.\end{aligned} \tag{5.62}$$

Thus, we have achieved the desired result

$$\sum_{\{1+2\}} f^{\{1+2\}}_{\{1\},\{2\}} O^{SO(N)}_{\{1+2\}}(Z,Y) = O^{SO(N)}_{\{1\}}(Z,Y) O^{SO(N)}_{\{2\}}(Z,Y). \tag{5.63}$$

In appendix I, we check this product with a simple example.

5.7 Discussion

In this chapter, we have defined a basis for the 1/4-BPS sector of the free super Yang-Mills theory with a symplectic gauge group. These operators are very similar to those defined for the theory with orthogonal gauge group. The difference between the two cases is that the symmetrisations of the irreducible representations defining the operators have been exchanged, as expected. The two-point function for the symplectic gauge theory operators was related to its orthogonal gauge theory counterpart by replacing N by $-N$.

The results of this chapter make it possible to compute correlation functions of any kind of multi-matrix, multi-trace operators involving two scalar fields. In such a correlation function, each trace operator may be expressed in terms of restricted Schur polynomials. Using the product rule derived above, computing the correlation function of many restricted Schurs may be transformed into a simple two-point function computation, the formula for which is given in (5.5) and (5.40).

Studying the spectrum of anomalous dimensions of our restricted Schurs is an interesting problem, especially in the limit that these operators become dual to excited giant gravitons. Such a study may yield new insights into the non-perturbative physics of their D-brane duals. Pursuing this direction may also allow us to make some concrete statements about whether or not integrability is preserved in non-planar limits of the $SO(N)$ or $Sp(N)$ gauge theory.

Chapter 6

Concluding the thesis

In this thesis, we have computed the spectrum of the one and two-loop anomalous dimensions of restricted Schur polynomials in the limit of the gauge theory where they are thought to be dual to excited giant gravitons. At the two-loop level, the same set of operators, the Gauss graph operators, were found to diagonalise the dilatation operator. The one and two-loop spectra were found to be those a system of decoupled harmonic oscillators. This provides evidence for integrability in the particular non-planar limit of the theory we studied. Furthermore, from our one and two-loop calculations, we found that the conventional t'Hooft limit yields a sensible perturbative expansion for the anomalous dimension spectrum.

This was all for the gauge theory with a $U(N)$ gauge group. A similar program has been initiated for the gauge theory for an $SO(N)$ gauge group. We managed to write down a set of operators for the 1/4-BPS sector of the free $SO(N)$ Yang-Mills theory. We argued that these $SO(N)$ restricted Schur polynomials form an exact basis for this theory. By exact, we mean that their two-point functions are diagonal and have been calculated for any value of N. Furthermore, we were able to write any trace operator involving two scalar fields in terms of the restricted Schur basis. We were also able to write down a product rule for our operators, thus enabling us to evaluate correlators involving two-matrix trace operators exactly in the free theory limit. The next logical step is to begin computing the spectrum of anomalous dimensions of our new operators. Interesting questions such as

1. Will the spectrum, again, be that of a system of decoupled harmonic oscillators?
2. Will the Gauss graph operators, with some modification perhaps (since we are now working with an $SO(N)$ gauge group), continue to diagonalise the dilatation operator?
3. What will be the appropriate limit to take to obtain a sensible perturbative expansion for the anomalous dimensions

and more, are hoped to be answered in due course.

Appendix A

$\Delta_{ij}^{(2)}$ as an element of $\mathbf{u}(p)$

In this appendix we will argue that, at large N, the eigenstates of $\Delta_{ij}^{(1)}$ are also eigenstates of $\Delta_{ij}^{(2)}$. We focus on the case that $p = 2$. Towards this end we will review relevant background from [69]. Recall that in the fundamental representation of $u(N)$ the generators can be taken as

$$(E_{kl})_{ab} = \delta_{ak}\delta_{bl} \qquad k, l, a, b = 1, 2, ..., N. \tag{A.1}$$

Introduce the operators (the labeling is such that $i > j$ i.e. Q_{ij} is not defined if $i < j$)

$$Q_{ij} = \frac{E_{ii} - E_{jj}}{2}, \qquad Q_{ij}^{+} = E_{ij}, \qquad Q_{ij}^{-} = E_{ji}, \tag{A.2}$$

which obey the familiar algebra of angular momentum raising and lowering operators

$$[Q_{ij}, Q_{ij}^{+}] = Q_{ij}^{+}, \qquad [Q_{ij}, Q_{ij}^{-}] = -Q_{ij}^{-}, \qquad [Q_{ij}^{+}, Q_{ij}^{-}] = 2Q_{ij}. \tag{A.3}$$

Irreps of these su(2) subalgebras can be labeled with the eigenvalue of

$$L^2_{ij} \equiv Q^-_{ij} Q^+_{ij} + Q^2_{ij} + Q_{ij} = Q^+_{ij} Q^-_{ij} + Q^2_{ij} - Q_{ij} \tag{A.4}$$

and states in the representation are labeled by the eigenvalue of Q_{ij}

$$Q_{ij}|\lambda, \Lambda\rangle = \lambda|\lambda, \Lambda\rangle, \qquad L^2_{ij}|\lambda, \Lambda\rangle = (\Lambda^2 + \Lambda)|\lambda, \Lambda\rangle, \qquad -\Lambda \le \lambda \le \Lambda. \tag{A.5}$$

The restricted Schur polynomials can be identified with particular states in a definite irrep. The reader may consult [69] for the details. Identifying the restricted Schur polynomials with states of a $U(p)$ representation allows us to write $\Delta^{(1)}_{ij}$ as a $u(p)$ valued operator

$$\Delta^{(1)}_{ij} = n\left(-\frac{1}{2}(E_{ii} + E_{jj}) + Q^-_{ij} + Q^+_{ij}\right) \equiv n\Delta_{ij}. \tag{A.6}$$

Note that

$$\mathcal{C} = E_{ii} + E_{jj} \tag{A.7}$$

commutes with all elements (A.2) of the su(2) algebra and hence defines a Casimir of this algebra. It is simply a constant times the identity in a given u(p) irrep. It is not difficult to check [69] that $\Delta^{(1)}_{ij}$ defines a discrete oscillator with creation operator given by

$$A^\dagger = \frac{1}{2}(E_{ii} - E_{jj}) + \frac{1}{2}E_{ij} - \frac{1}{2}E_{ji} \qquad [\Delta_{ij}, A^\dagger] = -2A^\dagger. \tag{A.8}$$

As pointed out in [69], a correctly normalized creation operator is given by $a^\dagger$ with $A^\dagger = \sqrt{M} a^\dagger$, where M is introduced in (3.45). It is straight forward to verify that $\Delta_{ij}^{(2)}$ is given by

$$\Delta_{ij}^{(2)} = (Q^+)^2 - \frac{\mathcal{C}}{2}Q^+ + 2Q^+Q^- - \frac{\mathcal{C}}{2}Q^- + (Q^-)^2, \tag{A.9}$$

and hence that

$$[\Delta_{ij}^{(2)}, A^\dagger] = -4(\Delta_{ij} + \frac{\mathcal{C}}{4})A^\dagger - 4Q^+ - 4Q. \tag{A.10}$$

In terms of a correctly normalized operator at large N we have (the last two terms in (A.10) can be dropped in the limit)

$$[\Delta_{ij}^{(2)}, a^\dagger] = -4(\Delta_{ij} + \frac{\mathcal{C}}{4})a^\dagger. \tag{A.11}$$

There are two things worth noting at this point. First, when acting in the basis of energy eigenstates, it is clear that $a^\dagger$ is indeed a creation operator but, due to the appearance of Δ_{ij}, with a "state dependent frequency". Said differently, $a^\dagger$ continues to move us to higher eigenstates but the energies of these states are not equally spaced. Second, we can show that this result is in perfect agreement with section 3.4. To make a comparison with section 3.4 we need to restrict attention to states for which the eigenvalue of Δ_{ij} is finite, so that on this subspace we can replace $\Delta_{ij} + \frac{\mathcal{C}}{4} \to \frac{\mathcal{C}}{4}$. Using the value for $\mathcal{C}$ computed in [69], for any state of finite energy, we have

$$[\Delta^{(2)}_{ij}, a^{\dagger}] = -2\,(2N + 2r_1)\,a^{\dagger} \tag{A.12}$$

in perfect agreement with section 3.4.

Appendix B

Simplifications of the $m \ll n$ limit

In this Appendix we will explain why keeping the first term in (3.6) corresponds to computing the leading term in a systematic expansion of the anomalous dimension in a series expansion in $\frac{1}{N}$ and $\frac{m}{n}$. Notice that the first term in (3.6) contains two derivatives with respect to Z and one derivative with respect to Y, whilst the second term contains one derivative with respect to Z and two derivatives with respect to Y. Since the number of Zs (given by n) is much greater than the number of Ys (given by m) we should expect the leading contribution to come from the first term in (3.6). In this Appendix we will demonstrate that this is indeed the case.

It is simplest to consider the expression (3.24). The factor $M^{(ij)}_{s\mu_1\mu_2\,;\,u\nu_1\nu_2}$ includes

$$\begin{aligned}&\left\langle \vec{m}, s, \mu_2\,;\, a|E^{(1)}_{ii}|\vec{m}, u, \nu_2\,;\, b\right\rangle \left\langle \vec{m}, u, \nu_1\,;\, b|E^{(1)}_{jj}|\vec{m}, s, \mu_2\,;\, a\right\rangle \\ &+ \left\langle \vec{m}, s, \mu_2\,;\, a|E^{(1)}_{jj}|\vec{m}, u, \nu_2\,;\, b\right\rangle \left\langle \vec{m}, u, \nu_1\,;\, b|E^{(1)}_{ii}|\vec{m}, s, \mu_2\,;\, a\right\rangle \end{aligned} \tag{B.1}$$

which involves traces over interwiners acting in $V^{\otimes m}$. It has no dependence on the representation r of the Zs and hence, has no dependence on n. Thus, all n dependence comes from the coefficient multiplying the above term (B.1). We will therefore study the coefficient of this term. As a consequence of the fact that the first term in (3.6) contains two derivatives with respect to Z and one derivative with respect to Y, this term will have a coefficient which includes the factor

$$\frac{d_T n(n-1) m d_{r''}}{d_t d_u d_{R''}(n+m)(n+m-1)}. \tag{B.2}$$

Recall that r'' is obtained by removing two boxes from r. The factor of $d_{r''}$ is produced when we take two derivatives with respect to Z. In the limit that $m \ll n$ we now find

$$\frac{d_T n(n-1) m d_{r''}}{d_t d_u d_{R''}(n+m)(n+m-1)} = \frac{m}{d_u}\left[1 + O\left(\frac{m}{n}\right)\right]. \tag{B.3}$$

For the second term in (3.6), the corresponding factor is now

$$\frac{d_T m(m-1) n d_{r'}}{d_t d_u d_{R''}(n+m)(n+m-1)}. \tag{B.4}$$

The Young diagram r' is obtained by removing one box from r. The factor of $d_{r'}$ is produced when we take a single derivative with respect to Z. In the limit that $m \ll n$ we now find

$$\frac{d_T m(m-1) n d_{r'}}{d_t d_u d_{R''}(n+m)(n+m-1)} = \frac{m(m-1)}{n d_u}\left[1 + O\left(\frac{m}{n}\right)\right]. \tag{B.5}$$

Notice that (B.5) is smaller than (B.4) by a factor of $\frac{m}{n}$ as we expected. The second term in (3.6) will thus contribute at higher order in a systematic $\frac{m}{n}$ expansion.

Finally, performing the sum over the Lie algebra index in the third term in (3.6) gives a term that is identical to the one loop dilatation operator, except that it is supressed by a power of N. Thus, it does not contribute to the leading order in a large N expansion.

Thus, to summarize, keeping only the first term in D_4 in (3.6) corresponds to the computation of the leading term in the double expansion in the parameters $\frac{1}{N}$ and $\frac{m}{n}$.

Appendix C

On the action of the Dilatation Operator

In this Appendix we want to discuss how sensitively integrability depends on the coefficients of the individual terms appearing in D_4. We will start by making a few comments on the structure of $\Delta_{ij}^{(2)}$ that we obtained in (3.30).

Recall that we argued

$$\mathrm{Tr}(ZYZ\partial_Z\partial_Y\partial_Z)\chi_{R,(r,s)\alpha\beta}(Z,Y) = \sum_{T,(t,u)\gamma\delta}\sum_{R'',T''}\frac{d_T n(n-1)m}{d_t d_u d_{R''}(n+m)(n+m-1)}c_{RR'}c_{R'R''}\ \ \chi_{T,(t,u)\gamma\delta}(Z,Y)$$
$$\times\mathrm{Tr}(I_{T''R''}(2,m+2,m+1)P_{R,(r,s)\alpha\beta}(1,m+2,2)I_{R''T''}(2,m+2)P_{T,(t,u)\delta\gamma}(m+2,2,1,m+1))$$

in section 3.3. Focus on the trace appearing in the second line above. Assume that we obtain R' from R by dropping a box from row i and that we obtain R'' from R' by dropping a box from row j. Further, assume that we obtain T' from T by dropping a box from row k and that we obtain T'' from T' by dropping a box from row l.

Clearly then, we are allowing four rows of the Young diagram to participate when the dilatation operator acts. With these assumptions, we easily find (see (3.19), (3.20) and (3.21) as well as the discussion around these equations)

$$I_{R''T''} = E^{(1)}_{ik} E^{(2)}_{jl}, \qquad I_{T''R''} = E^{(1)}_{ki} E^{(2)}_{lj}, \tag{C.1}$$

and

$$(m+2,2,1,m+1) I_{T''R''}(2,m+2,m+1) = E^{(1)}_{li} E^{(m+1)}_{kj}, \tag{C.2}$$

$$(1,m+2,2) I_{R''T''}(2,m+2) = E^{(1)}_{jk} E^{(m+2)}_{il}. \tag{C.3}$$

In obtaining these results we have made heavy use of the simplifications in the action of the symmetric group that arise in the displaced corners approximation. It is now a simple matter to find

$$\mathrm{Tr}(I_{T''R''}(2,m+2,m+1) P_{R,(r,s)\alpha\beta}(1,m+2,2) I_{R''T''}(2,m+2) P_{T,(t,u)\delta\gamma}(m+2,2,1,m+1)), \tag{C.4}$$

$$= \mathrm{Tr}(E^{(1)}_{li} E^{(m+1)}_{kj} P_{R,(r,s)\alpha\beta} E^{(1)}_{jk} E^{(m+2)}_{il} P_{T,(t,u)\delta\gamma}). \tag{C.5}$$

Since the projectors $P_{R,(r,s)\alpha\beta}$ and $P_{T,(t,u)\delta\gamma}$ have a trivial action on slots $m+1$ and

$m+2$, the above result is only non-zero when $i=l$ and $k=j$ - so that only two rows participate.

This reduction from four possible rows participating to two rows participating is determined by (C.2) and (C.3). These equations are corrected when going beyond the displaced corners approximation and, in that case, all four rows do indeed enter. For all of the terms appearing in the first line of D_4, we find this reduction to two rows for each term separately. Further, we find that each trace is individualy proportional to $M^{(ij)}_{s\mu_1\mu_2\,;\,u\nu_1\nu_2}$ defined in (3.25). This implies that the answer to question 2 that we posed in chapter 3's introduction is completely insensitive to the precise coefficients of the terms appearing in D_4[1].

At this point it is natural to ask if the reduction of the dilatation operator to a set of decoupled oscillators (and thus the observed integrability) is likewise also insensitive to the detailed coefficients. We will see that this is not the case - the emergence of an oscillator does depend sensitively on the precise values of the coefficients of the terms appearing in D_4.

Consider equation (3.30). Individual terms appearing in (3.30) can be traced back to particular terms appearing in D_4. For example, the terms proportional to $(\Delta^+_{ij})^2$ and $(\Delta^-_{ij})^2$ come from the terms $\mathrm{Tr}(ZZY\partial_Z\partial_Z\partial_Y)$ and $\mathrm{Tr}(YZZ\partial_Y\partial_Z\partial_Z)$. Notice that these two terms are related by daggering. Similarly, the terms $\Delta^0_{ij}\Delta^+_{ij}$ and $\Delta^0_{ij}\Delta^-_{ij}$ come from the terms $\mathrm{Tr}(ZYZ\partial_Z\partial_Z\partial_Y)$ $\mathrm{Tr}(ZZY\partial_Z\partial_Y\partial_Z)$, $\mathrm{Tr}(ZYZ\partial_Y\partial_Z\partial_Z)$ and $\mathrm{Tr}(YZZ\partial_Z\partial_Y\partial_Z)$ which are again related by daggering. Changing the relative weights of terms appearing in D_4 will change the relative weight of terms appearing

[1]If one includes the remaining (subleading) terms in D_4 that we have discarded in the $m \ll n$ limit, the dilatation operator starts to mix different Gauss graph operators. This suggests that the integrability we study here is a property of the large N limit and of the displaced corners approximation (i.e. $m << n$) and may not survive when subleading corrections are included.

in (3.30).

To explore the effect of these changed coefficients on integrability, imagine we assign coefficient α to the terms $\mathrm{Tr}(ZZY\partial_Z\partial_Z\partial_Y)$ and $\mathrm{Tr}(YZZ\partial_Y\partial_Z\partial_Z)$ in D_4. We now find $\Delta_{ij}^{(2)}$ is replaced by

$$\Delta_{ij}^{\alpha(2)} = \alpha(\Delta_{ij}^{+})^2 + \Delta_{ij}^{0}\Delta_{ij}^{+} + 2\Delta_{ij}^{+}\Delta_{ij}^{-} + \Delta_{ij}^{0}\Delta_{ij}^{-} + \alpha(\Delta_{ij}^{-})^2. \tag{C.6}$$

It is straight forward to check, using the approach of [69] that this operator does not admit creation and annihilation operators and hence does not define an oscillator. A very instructive way to get some insight into what is going on, is to consider the continuum limit of section 3.4. We find

$$\Delta_{ij}^{\alpha(2)} O_{R,r}(\sigma) \to 2N^2(\alpha-1)O_{R,r}(\sigma) + 2(r_i+r_j)N(\alpha-1)O_{R,r}(\sigma) + O(N). \tag{C.7}$$

Compare this to (3.40) and (3.41). Even the scaling with N of the eigenvalues of $\Delta_{ij}^{\alpha(2)}$ and $\Delta_{ij}^{(2)}$ disagree. Indeed, with $\alpha = 1$ we have a delicate cancelation of the leading order terms - as we clearly see in (C.7). It is the subleading terms that combine to produce an oscillator. Note that all of the terms in (3.30) contribute at the leading order. Thus, the sensitive dependence we see on the coefficient of the terms $\mathrm{Tr}(ZZY\partial_Z\partial_Z\partial_Y)$ and $\mathrm{Tr}(YZZ\partial_Y\partial_Z\partial_Z)$ extends to the other terms in D_4 too.

This last point deserves explanation. The terms in $\Delta_{ij}^{(2)}$ can be collected into three groups which are each hermittian: $(\Delta_{ij}^{+})^2 + (\Delta_{ij}^{-})^2$, $\Delta_{ij}^{0}\Delta_{ij}^{+} + \Delta_{ij}^{0}\Delta_{ij}^{-}$ and finally $2\Delta_{ij}^{+}\Delta_{ij}^{-}$. The relative coefficients of the terms producing these pieces is fixed by hermitticity. For example $\mathrm{Tr}(ZZY\partial_Z\partial_Z\partial_Y) + \beta\mathrm{Tr}(YZZ\partial_Y\partial_Z\partial_Z)$ is only hermittian if $\beta = 1$ and in this case the terms sum to $(\Delta_{ij}^{+})^2 + (\Delta_{ij}^{-})^2$. The particular coefficients of the terms that appear in $\Delta_{ij}^{(2)}$ ensure that when we take the continuum limit (i) the terms proportional to N^2 cancel, (ii) the terms proportional to $(r_i + r_j)N$ cancel and (iii) the surviving terms sum to produce an operator that admits exactly the same creation and annihilation operators as the one loop dilatation operator does. The integrability we have studied here depends on a careful fine tuning of the terms appearing in D_4.

Appendix D

$SO(N)$ Counting examples

D.1 1 Z and 1 Y for $SO(4)$

First consider the case of having only 1 Z and 1 Y for $SO(4)$. The partition function gives a total of 2 operators. We now try to match this number with the number of restricted Schurs that can be defined. There is only 1 possible diagram for r,

$$\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array} \tag{D.1}$$

and one possible Young diagram for s,

$$\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}\,. \tag{D.2}$$

Multiplying r and s together, we find

$$\ydiagram{1,1} \otimes \begin{ytableau} 1 \\ 2 \end{ytableau} = \begin{ytableau} \ & 1 \\ \ & 2 \end{ytableau} \otimes \begin{ytableau} \ \\ \ \\ 1 \\ 2 \end{ytableau}. \tag{D.3}$$

Using δ_{ij} and ϵ_{ijkl}, we can construct only are two operators:

$$\mathcal{O}_1 = \mathrm{Tr}(ZY) \tag{D.4}$$

$$\mathcal{O}_2 = \epsilon_{ijkl} Z^{ij} Y^{kl}. \tag{D.5}$$

Thus, we have the following correspondence

$$\mathrm{Tr}(ZY) \leftrightarrow \ydiagram{2,2}, (\ydiagram{1,1}, \ydiagram{1,1}) \tag{D.6}$$

$$\epsilon_{ijkl} Z^{ij} Y^{kl} \leftrightarrow \ydiagram{1,1,1,1}, (\ydiagram{1,1}, \ydiagram{1,1}) \tag{D.7}$$

where the last operator is known as the Pfaffian [79].

D.2 2 Z's and 2 Y's for $SO(4)$

For 2 Z's and 2 Y's for $SO(4)$, the partition function gives a total of 7 operators. We thus expect 7 different restricted Schur labels. For r we can have

$$\ydiagram{2,2}\,, \qquad \ydiagram{1,1,1,1}\,. \tag{D.8}$$

For s we can have

$$\ydiagram{2,2}\,, \qquad \ydiagram{1,1,1,1}\,. \tag{D.9}$$

Multiplying r and s together and only taking the results with even rows, we get

$$\ydiagram{2,2} \otimes \begin{ytableau} 1 & 1 \\ 2 & 2 \end{ytableau} \quad \text{gives} \quad \begin{ytableau} {} & {} & 1 & 1 \\ {} & {} & 2 & 2 \end{ytableau}\,, \begin{ytableau} {} & {} & 1 & 1 \\ {} & {} \\ 2 & 2 \end{ytableau}\,, \begin{ytableau} {} & {} & 1 \\ {} & {} & 2 \\ 1 \\ 2 \end{ytableau}\,, \begin{ytableau} {} & {} \\ {} & {} \\ 1 & 1 \\ 2 & 2 \end{ytableau}\,. \tag{D.10}$$

Next

$$\ydiagram{2,2} \otimes \begin{ytableau} 1 \\ 2 \\ 3 \\ 4 \end{ytableau} \quad \text{gives} \quad \begin{ytableau} {} & {} & 1 \\ {} & {} & 2 \\ 3 \\ 4 \end{ytableau}\,. \tag{D.11}$$

Then

$$\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline 1&1\\ \hline 2&2\\ \hline\end{array} \quad \text{gives} \quad \begin{array}{|c|c|c|}\hline \ &1&1\\ \hline \ &2&2\\ \hline \multicolumn{1}{|c|}{\ } & \multicolumn{2}{c}{} \\ \cline{1-1} \multicolumn{1}{|c|}{\ } & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \; . \tag{D.12}$$

Finally

$$\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array} \otimes \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline 4\\ \hline\end{array} \quad \text{gives} \quad \begin{array}{|c|c|}\hline \ &1\\ \hline \ &2\\ \hline \ &3\\ \hline \ &4\\ \hline\end{array} . \tag{D.13}$$

Counting all the Young diagrams we get 7 with the correct labelling. We also check that we find 7 operators using δ_{ij} and ϵ_{ijkl}. They are listed below

$$\begin{aligned}
\mathcal{O}_1 &= \mathrm{Tr}(Z^2Y^2) \\
\mathcal{O}_2 &= \mathrm{Tr}(ZYZY) \\
\mathcal{O}_3 &= \mathrm{Tr}(Z^2)\mathrm{Tr}(Y^2) \\
\mathcal{O}_4 &= \mathrm{Tr}(ZY)^2 \\
\mathcal{O}_5 &= \epsilon_{ijkl}Z^{ij}Z^{kl}\mathrm{Tr}(Y^2) \\
\mathcal{O}_6 &= \epsilon_{ijkl}Z^{ij}Y^{kl}\mathrm{Tr}(ZY) \\
\mathcal{O}_7 &= \epsilon_{ijkl}Y^{ij}Y^{kl}\mathrm{Tr}(Z^2).
\end{aligned} \tag{D.14}$$

D.3 3 Z's and 2 Y's for $SO(6)$

Here we do not expect multitrace operators. This is because there is an odd number of fields; we will always end up with a $\mathrm{Tr}(Z)$ or $\mathrm{Tr}(Y)$ which vanishes. We can, however, obtain operators built using the ϵ_{ijklmn}. The partition function gives a total of 4 operators. We thus expect 4 restricted Schurs and each of the should have the Pfaffian as a factor. For r, the possible Young diagrams are

$$\yng(3,3)\,, \qquad \yng(2,2,1,1)\,, \qquad \yng(1,1,1,1,1,1) \tag{D.15}$$

and for s, the possible Young diagrams are

$$\yng(2,2)\,, \qquad \yng(1,1,1,1)\,. \tag{D.16}$$

Multiplying each r with each s, and taking only the diagrams which contribute, we find

$$\yng(3,3) \otimes \young(11,22) \quad \text{gives} \quad 0 \tag{D.17}$$

$$\ydiagram{2,2,1,1} \otimes \begin{ytableau} 1 & 1 \\ 2 & 2 \end{ytableau} \quad \text{gives} \quad \begin{ytableau} \none[] & & 1 \\ & & 2 \\ \\ \\ 1 \\ 2 \end{ytableau} \tag{D.18}$$

$$\ydiagram{1,1,1,1,1,1} \otimes \begin{ytableau} 1 & 1 \\ 2 & 2 \end{ytableau} \quad \text{gives} \quad \begin{ytableau} & 1 & 1 \\ & 2 & 2 \\ \\ \\ \\ \\ \end{ytableau} \tag{D.19}$$

$$\ydiagram{3,3} \otimes \begin{ytableau} 1 \\ 2 \\ 3 \\ 4 \end{ytableau} \quad \text{gives} \quad \begin{ytableau} & & \\ & & \\ 1 \\ 2 \\ 3 \\ 4 \end{ytableau} \tag{D.20}$$

$$\ydiagram{2,2,1,1} \otimes \begin{ytableau} 1 \\ 2 \\ 3 \\ 4 \end{ytableau} \quad \text{gives} \quad \begin{ytableau} & & 1 \\ & & 2 \\ \\ \\ 3 \\ 4 \end{ytableau} \tag{D.21}$$

$$\ydiagram{1,1,1,1,1,1} \otimes \begin{ytableau} 1 \\ 2 \\ 3 \\ 4 \end{ytableau} \quad \text{gives} \quad 0. \tag{D.22}$$

We indeed obtain 4 restricted Schurs as expected.

D.4 4 Z's and 2Y's for SO(8)

The partition function gives a total of 13 operators. The possible diagrams for r are

$$\ydiagram{4,4}\,,\quad \ydiagram{2,2,2,2}\,,\quad \ydiagram{3,3,1,1}\,,\quad \ydiagram{2,2,1,1,1,1}\,,\quad \ydiagram{1,1,1,1,1,1,1,1}\,, \tag{D.23}$$

and for s

$$\ydiagram{2,2}\,,\quad \ydiagram{1,1,1,1}\,. \tag{D.24}$$

Multiplying each r with each s and taking only the diagrams which contribute to the counting, we find

$$\ydiagram{4,4} \otimes \begin{ytableau} 1 & 1 \\ 2 & 2 \end{ytableau} \quad \text{gives} \quad \begin{ytableau} \ & \ & \ & \ & 1 & 1 \\ \ & \ & \ & \ & 2 & 2 \end{ytableau}\,,\ \begin{ytableau} \ & \ & \ & \ & 1 & 1 \\ \ & \ & \ & \ \\ 2 & 2 \end{ytableau}\,,\ \begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ & \ \\ 1 & 1 \\ 2 & 2 \end{ytableau}$$

$$\ydiagram{4,4} \otimes \begin{ytableau} 1 \\ 2 \\ 3 \\ 4 \end{ytableau} \quad \text{gives} \quad 0$$

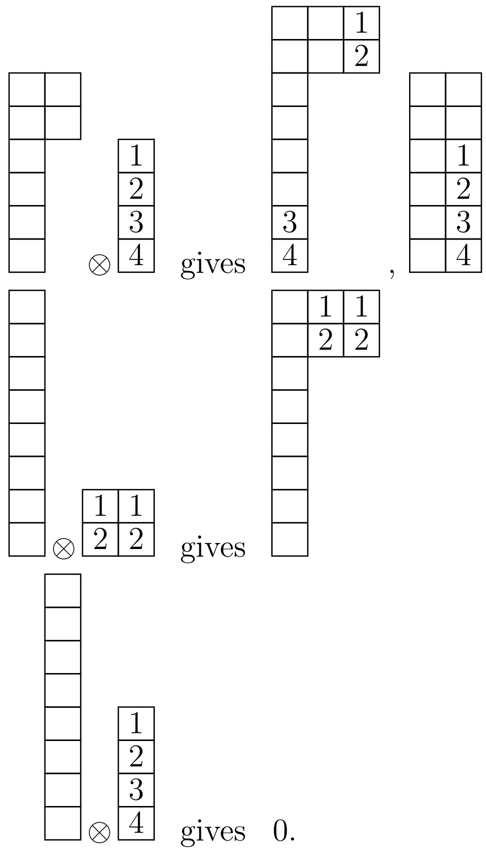
$$\ydiagram{2,2,2,2} \otimes \begin{ytableau} 1 & 1 \\ 2 & 2 \end{ytableau} \quad \text{gives} \quad \begin{ytableau} \, & \, & 1 & 1 \\ \, & \, & 2 & 2 \\ \, & \, \\ \, & \, \end{ytableau} \; , \; \begin{ytableau} \, & \, & 1 & 1 \\ \, & \, \\ \, & \, \\ \, & \, \\ 2 & 2 \end{ytableau} \; , \; \begin{ytableau} \, & \, \\ \, & \, \\ \, & \, \\ \, & \, \\ 1 & 1 \\ 2 & 2 \end{ytableau}$$

$$\ydiagram{2,2,2,2} \otimes \begin{ytableau} 1 \\ 2 \\ 3 \\ 4 \end{ytableau} \quad \text{gives} \quad 0 \tag{D.25}$$

$$\ydiagram{3,3,1,1} \otimes \begin{ytableau} 1 & 1 \\ 2 & 2 \end{ytableau} \quad \text{gives} \quad \begin{ytableau} \, & \, & \, & 1 \\ \, & \, & \, & 2 \\ \, & 1 \\ \, & 2 \end{ytableau}$$

$$\ydiagram{3,3,1,1} \otimes \begin{ytableau} 1 \\ 2 \\ 3 \\ 4 \end{ytableau} \quad \text{gives} \quad \begin{ytableau} \, & \, & \, & 1 \\ \, & \, & \, & 2 \\ \, & 3 \\ \, & 4 \end{ytableau} \; , \; \begin{ytableau} \, & \, & \, \\ \, & \, & \, \\ \, \\ \, \\ 1 \\ 2 \\ 3 \\ 4 \end{ytableau}$$

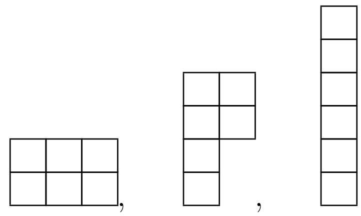
$$\ydiagram{2,2,1,1,1,1} \otimes \begin{ytableau} 1 & 1 \\ 2 & 2 \end{ytableau} \quad \text{gives} \quad \begin{ytableau} \, & \, & 1 \\ \, & \, & 2 \\ \, \\ \, \\ \, \\ \, \\ 1 \\ 2 \end{ytableau}$$



Adding up all the Young diagram labels, we indeed get 13 operators. The reader is invited to check that for $SO(4)$, there are 12 operators and for $SO(6)$, there are 9 operators.

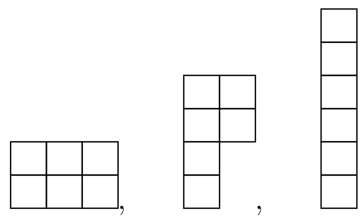
D.5 3 Z's and 3 Y's for $SO(8)$

As a final example, we consider 3 Z's and 3 Y's for $SO(8)$. The partition function gives a total of 14 operators. For r, we have



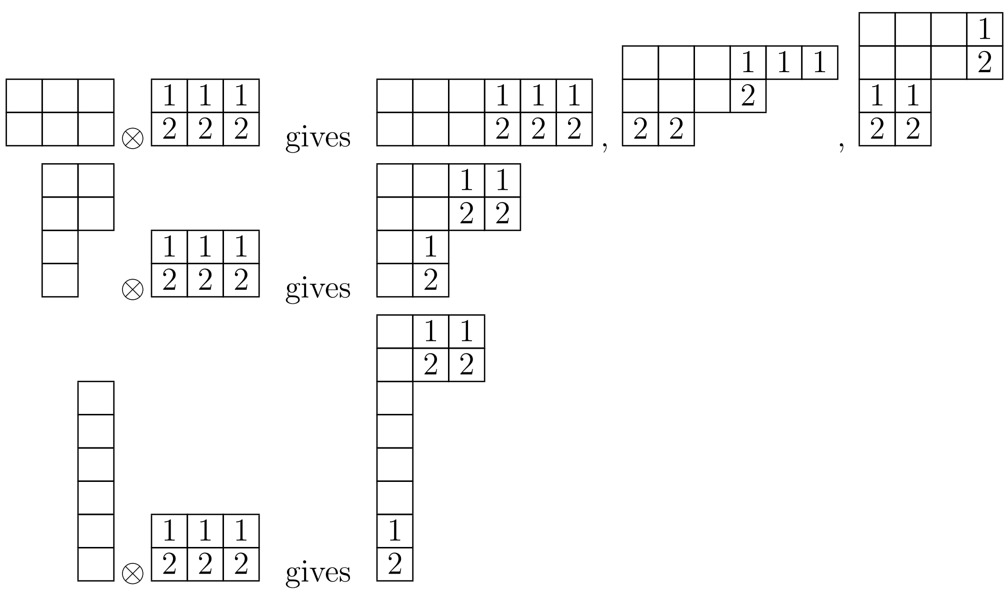
(D.26)

and for s



(D.27)

Multiplying, we find



$$\begin{ytableau} \ & \ & \ \\ \ & \ & \ \end{ytableau} \otimes \begin{ytableau} 1 & 1 \\ 2 & 2 \\ . \\ 4 \end{ytableau} \quad \text{gives} \quad \begin{ytableau} \ & \ & \ & 1 \\ \ & \ & \ & 2 \\ 1 & 3 \\ 2 & 4 \end{ytableau} \tag{D.28}$$

$$\begin{ytableau} \ & \ \\ \ & \ \\ \ \\ \ \end{ytableau} \otimes \begin{ytableau} 1 & 1 \\ 2 & 2 \\ 3 \\ 4 \end{ytableau} \quad \text{gives} \quad \begin{ytableau} \ & \ & 1 & 1 \\ \ & \ & 2 & 2 \\ \ & 3 \\ \ & 4 \end{ytableau}\ , \quad \begin{ytableau} \ & \ & 1 & 1 \\ \ & \ \\ \ & 2 \\ \ & 3 \\ 2 & 4 \end{ytableau}\ , \quad \begin{ytableau} \ & \ & 1 \\ \ & \ & 2 \\ \ \\ \ \\ 1 \\ 2 \\ 3 \\ 4 \end{ytableau}\ , \quad \begin{ytableau} \ & \ \\ \ & \ \\ \ & 1 \\ \ & 2 \\ 1 & 3 \\ 2 & 4 \end{ytableau}$$

$$\begin{ytableau} \ \\ \ \\ \ \\ \ \\ \ \\ \ \end{ytableau} \otimes \begin{ytableau} 1 & 1 \\ 2 & 2 \\ 3 \\ 4 \end{ytableau} \quad \text{gives} \quad \begin{ytableau} \ & 1 & 1 \\ \ & 2 & 2 \\ \ \\ \ \\ \ \\ \ \\ 3 \\ 4 \end{ytableau}$$

$$\begin{ytableau} \ & \ & \ \\ \ & \ & \ \end{ytableau} \otimes \begin{ytableau} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{ytableau} \quad \text{gives} \quad \begin{ytableau} \ & \ & \ \\ \ & \ & \ \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{ytableau}$$

$$\begin{ytableau} \ & \ \\ \ & \ \\ \ \\ \ \end{ytableau} \otimes \begin{ytableau} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{ytableau} \quad \text{gives} \quad \begin{ytableau} \ & \ & 1 \\ \ & \ & 2 \\ \ \\ \ \\ 3 \\ 4 \\ 5 \\ 6 \end{ytableau}$$

$$\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array} \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline 5 \\ \hline 6 \\ \hline \end{array} \quad \text{gives} \quad \begin{array}{|c|c|}\hline & 1 \\ \hline & 2 \\ \hline & 3 \\ \hline & 4 \\ \hline & 5 \\ \hline & 6 \\ \hline \end{array} .$$

Adding up all the Young diagram labels, we get our expected 14 restricted Schur polynomials. Again the reader is invited to check that for $SO(6)$ there are 10 restricted Schurs and for $SO(4)$, there are 12 restricted Schurs.

Appendix E

Two-point function examples

In the following examples, we denote the eigenvector of $\hat{P}_{[S]}$ by $|[\hat{S}]\rangle$.

E.1 $\quad q = 2$

For $q = 2$, we have

$$R = \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}, \qquad (r,s) = \left(\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}, \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}\right). \tag{E.1}$$

Firstly

$$|[\hat{S}]\rangle = \frac{1}{2}\begin{array}{|c|c|}\hline 4 & 3 \\ \hline 2 & 1 \\ \hline \end{array} + \frac{\sqrt{3}}{2}\begin{array}{|c|c|}\hline 4 & 2 \\ \hline 3 & 1 \\ \hline \end{array}, \qquad |[A],[A]\rangle = \begin{array}{|c|c|}\hline 4 & 2 \\ \hline 3 & 1 \\ \hline \end{array}. \tag{E.2}$$

The restricted Schur is

$$O_{R(r,s)} = 2\sqrt{3}\mathrm{Tr}(ZY). \tag{E.3}$$

Evaluating the Wick contractions, its two point function is

$$\langle O_{R(r,s)}\overline{O}_{T(t,u)}\rangle = 12\langle \mathrm{Tr}(ZY)\mathrm{Tr}(\overline{ZY})\rangle = 24N(N-1). \tag{E.4}$$

Using (4.61), with $q=2, n=m=1$ and $d_R=2$, we get

$$\langle O_{R(r,s)}\overline{O}_{T(t,u)}\rangle = 24N(N-1). \tag{E.5}$$

E.2 $q=4$

First consider

$$R = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \\ \hline \end{array}, \qquad (r,s) = \left(\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}, \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}\right). \tag{E.6}$$

We caluculate

$$|[\hat{S}]\rangle = \frac{\sqrt{5}}{4}\begin{array}{|c|c|c|c|}\hline 8&6&4&2\\ \hline 7&5&3&1\\ \hline \end{array} + \sqrt{\frac{5}{48}}\begin{array}{|c|c|c|c|}\hline 8&7&4&2\\ \hline 6&5&3&1\\ \hline \end{array} + \frac{\sqrt{5}}{12}\begin{array}{|c|c|c|c|}\hline 8&7&4&3\\ \hline 6&5&2&1\\ \hline \end{array} + \frac{2}{3}\begin{array}{|c|c|c|c|}\hline 8&7&6&5\\ \hline 4&3&2&1\\ \hline \end{array} + \sqrt{\frac{5}{48}}\begin{array}{|c|c|c|c|}\hline 8&6&4&3\\ \hline 7&5&2&1\\ \hline \end{array}, \tag{E.7}$$

and

$$|[A],[A]\rangle = \begin{array}{|c|c|c|c|}\hline 8 & 6 & 4 & 2 \\ \hline 7 & 5 & 3 & 1 \\ \hline \end{array}\,. \tag{E.8}$$

The operator is

$$O_{R(r,s)} = \frac{2\sqrt{5}}{3}\Big(2\mathrm{Tr}(ZY)^2 + 2\mathrm{Tr}(ZYZY) + \mathrm{Tr}(Z^2)\mathrm{Tr}(Y^2) + 4\mathrm{Tr}(Z^2Y^2)\Big) \tag{E.9}$$

and, after evaluating all the Wick contractions, its two-point function is

$$\langle O_{R(r,s)}\overline{O}_{R(r,s)}\rangle = \frac{640}{3}N(N-1)(N+1)(N+2). \tag{E.10}$$

Equation (4.61), for $q=4, n=m=2, d_R=14$ and taking only the weights of the odd columns, gives

$$\langle O_{R(r,s)}\overline{O}_{R(r,s)}\rangle = \frac{640}{3}N(N-1)(N+1)(N+2). \tag{E.11}$$

Next consider

$$R = \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}\,, \qquad (r,s) = \left(\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}\,,\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}\right). \tag{E.12}$$

The state $|[\hat{S}]\rangle$ was found to be

$$|[\hat{S}]\rangle = \frac{3}{4}\begin{array}{|c|c|}\hline 8&6\\\hline 7&5\\\hline 4&2\\\hline 3&1\\\hline\end{array} + \frac{\sqrt{3}}{4}\begin{array}{|c|c|}\hline 8&7\\\hline 6&5\\\hline 4&2\\\hline 3&1\\\hline\end{array} + \frac{1}{4}\begin{array}{|c|c|}\hline 8&7\\\hline 6&5\\\hline 4&3\\\hline 2&1\\\hline\end{array} + \frac{\sqrt{3}}{4}\begin{array}{|c|c|}\hline 8&6\\\hline 7&5\\\hline 4&3\\\hline 2&1\\\hline\end{array}, \tag{E.13}$$

and

$$|[A],[A]\rangle = \begin{array}{|c|c|}\hline 8&6\\\hline 7&5\\\hline 4&2\\\hline 3&1\\\hline\end{array}. \tag{E.14}$$

The operator is

$$O_{R(r,s)} = \frac{2}{3}\Big(2\mathrm{Tr}(ZY)^2 + 2\mathrm{Tr}(ZYZY) + 3\mathrm{Tr}(Z^2)\mathrm{Tr}(Y^2) - 12\mathrm{Tr}(Z^2Y^2)\Big), \tag{E.15}$$

and its two-point function is

$$\langle O_{R(r,s)}\overline{O}_{R(r,s)}\rangle = \frac{640}{3}N(N-1)(N-2)(N-3). \tag{E.16}$$

Equation (4.61), for $q = 4, n = m = 2, d_R = 14$ and taking the weights from the odd columns, gives

$$\langle O_{R(r,s)}\overline{O}_{R(r,s)}\rangle = \frac{640}{3}N(N-1)(N-2)(N-3), \tag{E.17}$$

Next consider

$$R = \ydiagram{2,2,2,2}\,, \qquad (r,s) = \left(\ydiagram{1,1,1,1}\,,\ydiagram{1,1,1,1}\right). \tag{E.18}$$

We find

$$|[\hat{S}]\rangle = \frac{3}{4}\,\ytableaushort{86,75,42,31} + \frac{\sqrt{3}}{4}\,\ytableaushort{87,65,42,31} + \frac{1}{4}\,\ytableaushort{87,65,43,21} + \frac{\sqrt{3}}{4}\,\ytableaushort{86,75,43,21}\,, \tag{E.19}$$

and

$$|[A],[A]\rangle = \ytableaushort{84,73,62,51}\,. \tag{E.20}$$

The operator is

$$O_{R(r,s)} = \frac{4\sqrt{5}}{3}\Big(\mathrm{Tr}(ZY)^2 - 2\mathrm{Tr}(ZYZY)\Big). \tag{E.21}$$

and its two-point function is

$$\langle O_{R(r,s)}\overline{O}_{R(r,s)}\rangle = \frac{640}{3}N(N-1)(N-2)(N-3). \tag{E.22}$$

Equation (4.61) gives

$$\langle O_{R(r,s)}\overline{O}_{R(r,s)}\rangle = \frac{640}{3}N(N-1)(N-2)(N-3). \tag{E.23}$$

For the final example for which $n = m = 2$, consider

$$R = \ydiagram{4,2,2}\,, \qquad (r,s) = \left(\ydiagram{2,2}\,,\ydiagram{2,2}\right). \tag{E.24}$$

We find the following for $|[\hat{S}]\rangle$ and $|[A],[A]\rangle$

$$\begin{aligned}
|[\hat{S}]\rangle \;=\; & \frac{1}{4\sqrt{6}}\ytableaushort{8721,65,43} + \frac{1}{4\sqrt{2}}\ytableaushort{8621,75,43} + \frac{1}{8\sqrt{3}}\ytableaushort{8731,65,42} + \frac{1}{8}\ytableaushort{8631,75,42} + \frac{\sqrt{5}}{8}\ytableaushort{8741,65,32} \\
& + \frac{\sqrt{15}}{8}\ytableaushort{8641,75,32} - \frac{1}{12\sqrt{2}}\ytableaushort{8743,65,21} - \frac{1}{4\sqrt{6}}\ytableaushort{8643,75,21} - \frac{1}{8\sqrt{3}}\ytableaushort{8742,65,31} - \frac{1}{8}\ytableaushort{8642,75,31} \\
& + \frac{\sqrt{5}}{8}\ytableaushort{8732,65,41} + \frac{\sqrt{15}}{8}\ytableaushort{8632,75,41} + \frac{1}{6}\sqrt{\frac{5}{2}}\ytableaushort{8765,43,21} + \frac{1}{2}\sqrt{\frac{5}{6}}\ytableaushort{8765,42,31}\,,
\end{aligned} \tag{E.25}$$

and

$$|[A],[A]\rangle = -\frac{3}{8}\ytableaushort{8631,75,42} + \frac{\sqrt{15}}{8}\ytableaushort{8641,75,32} - \frac{5}{8}\ytableaushort{8642,75,31} + \frac{\sqrt{15}}{8}\ytableaushort{8632,75,41}\,. \tag{E.26}$$

The operator is

$$O_{R(r,s)} = \frac{4}{3}\Big(- \mathrm{Tr}(ZY)^2 - \mathrm{Tr}(ZYZY) + \mathrm{Tr}(Z^2)\mathrm{Tr}(Y^2) + \mathrm{Tr}(Z^2Y^2)\Big) \tag{E.27}$$

and its two-point function, after evaluating all the Wick contractions, is

$$\langle O_{R(r,s)}\overline{O}_{R(r,s)}\rangle = \frac{160}{3}N(N-1)(N-2)(N+2). \tag{E.28}$$

Equation (4.61), for $q = 4, n = m = 2, d_R = 56$ and odd columns, gives

$$\langle O_{R(r,s)}\overline{O}_{R(r,s)}\rangle = \frac{160}{3}N(N-1)(N-2)(N+2). \tag{E.29}$$

Now let's try two examples where $n = 3, m = 1$, i.e., the number of Z's is not the equal to the number of Y's. Consider

$$R = \yng(4,4), \qquad (r,s) = \left(\yng(3,3), \yng(1,1)\right). \tag{E.30}$$

States $|[\hat{S}]\rangle$ and $|[A],[A]\rangle$ are still the same as in (E.7) and (E.8). The operator is

$$O_{R(r,s)} = \frac{4}{\sqrt{5}}\Big(\mathrm{Tr}(ZY)\mathrm{Tr}(Z^2) + 2\mathrm{Tr}(Z^3Y)\Big), \tag{E.31}$$

and its two-point function is

$$\langle O_{R(r,s)}\overline{O}_{R(r,s)}\rangle = \frac{256}{5}N(N-1)(N+1)(N+2). \tag{E.32}$$

Formula (4.61) reproduces exactly this result. As the final example, we consider

$$R = \yng(2,2,2,2)\,, \qquad (r,s) = \left(\yng(2,2,1,1)\,,\,\yng(1,1)\right). \tag{E.33}$$

The $|[\hat{S}]\rangle$ is the same as in (E.19). However

$$|[A],[A]\rangle = \frac{\sqrt{5}}{3}\begin{ytableau} 8 & 4 \\ 7 & 3 \\ 6 & 2 \\ 5 & 1 \end{ytableau} + \frac{2}{3}\begin{ytableau} 8 & 6 \\ 7 & 5 \\ 4 & 2 \\ 3 & 1 \end{ytableau}. \tag{E.34}$$

The operator is

$$O_{R(r,s)} = \frac{4}{\sqrt{5}}\Big(\mathrm{Tr}(ZY)\mathrm{Tr}(Z^2) - 2\mathrm{Tr}(Z^3Y)\Big). \tag{E.35}$$

The two-point function is

$$\langle O_{R(r,s)}\overline{O}_{R(r,s)}\rangle = \frac{256}{5}N(N-1)(N-2)(N-3). \tag{E.36}$$

which, again, is exactly reproduced by (4.61).

Appendix F

An expression for $P_{[A,A]}$

In this appendix, we prove that

$$P_{[A,A]} = \frac{1}{n!m!2^q} \sum_{\eta \in S_n[S_2] \times S_m[S_2]} \mathrm{sgn}(\eta)\Gamma_R(\eta) = \sum_{r,s,\nu} |[A]_r, [A]_s, \nu\rangle\langle [A]_r, [A]_s, \nu| \,. \quad \text{(F.1)}$$

Since $\eta_1 \cdot \eta_2 \in S_{2n} \times S_{2m}$, we can write [76]

$$\Gamma_R(\eta_1 \cdot \eta_2)_{IJ} = \sum_{r,s,\nu} \sum_{j_1 j_2 k_1 k_2} \langle R, I | R(r,s)\nu; j_1, j_2 \rangle \Gamma_r(\eta_1)_{j_1 k_1} \Gamma_s(\eta_2)_{j_2 k_2} \langle R(r,s)\nu; k_1, k_2 | R, J \rangle. \quad \text{(F.2)}$$

Writing out the IJ-th matrix element of $P_{[A,A]}$,

$$(P_{[A,A]})_{IJ} = \langle R, I| \left(\sum_{r,s,\nu} (P_{[A]_r})_{j_1 k_1} (P_{[A]_s})_{j_2 k_2} \, |R(r,s)\nu; j_1, j_2\rangle\langle R(r,s)\nu; k_1, k_2| \right) |R, J\rangle \,, \quad \text{(F.3)}$$

where we wrote

$$P_{[A]_r} = \frac{1}{n!2^n} \sum_{\eta_1 \in S_n[S_2]} \mathrm{sgn}(\eta_1)\Gamma_r(\eta_1) \tag{F.4}$$

with a similar expression for $P_{[A]_s}$. Using the fact the P's are projectors, i.e., $P_{[A]}^2 = P_{[A]}$, this matrix element becomes

$$\begin{aligned}(P_{[A,A]})_{IJ} &= \\ &\langle R,I|\Big(\sum_{r,s,\nu}(P_{[A]_r})_{j_1l_1}(P_{[A]_r})_{l_1k_1}(P_{[A]_s})_{j_2l_2}(P_{[A]_s})_{l_2k_2}\,|R(r,s)\nu;j_1,j_2\rangle\langle R(r,s)\nu;k_1,k_2|\Big)|R,J\rangle\,.\end{aligned} \tag{F.5}$$

We then have

$$\begin{aligned}(P_{[A,A]})_{IJ} &= \langle R,I|\Big(\sum_{r,s,\nu}\sum_{l_1l_2}|[A]_r,[A]_s,\nu;l_1l_2\rangle\langle[A]_r,[A]_s,\nu|\,;l_1l_2\Big)|R,J\rangle \\ &= \sum_{r,s,\nu}\langle R,I|[A]_r,[A]_s,\nu\rangle\langle[A]_r,[A]_s,\nu|R,J\rangle.\end{aligned} \tag{F.6}$$

To get the last line, we used the fact that $([A],[A])$ is a 1-dimensional irrep.

Appendix G

$Sp(N)$ two-function examples

First, consider for $Sp(N)$

$$R = \yng(2,2), (r,s) = (\yng(2), \yng(2)). \tag{G.1}$$

The restricted Schur (5.28) for this operator was calculated to be

$$O^{Sp(N)}_{\yng(2,2)(\yng(2),\yng(2))} = -\sqrt{6}\mathrm{Tr}(ZY). \tag{G.2}$$

It's two-point function is

$$\langle O^{Sp(N)}_{\yng(2,2)(\yng(2),\yng(2))} \overline{O}^{Sp(N)}_{\yng(2,2)(\yng(2),\yng(2))} \rangle = 12N(N+1). \tag{G.3}$$

Firstly, this agrees with (5.40) and secondly, we compare this to the $SO(N)$ two-point function for

$$R_c = \yng(2,2), (r_c, s_c) = \left(\yng(1,1), \yng(1,1)\right). \tag{G.4}$$

The restricted Schur for $SO(N)$ (5.1) is (see [89])

$$O^{SO(N)}_{\yng(2,2)(\yng(1,1),\yng(1,1))} = \sqrt{6}\mathrm{Tr}(ZY) \tag{G.5}$$

with two-point function

$$\langle O^{SO(N)}_{\yng(2,2)(\yng(1,1),\yng(1,1))} \overline{O}^{SO(N)}_{\yng(2,2)(\yng(1,1),\yng(1,1))} \rangle = 12N(N-1). \tag{G.6}$$

Clearly, sending N to $-N$ gives us the $Sp(N)$ result (G.3).

Next, consider for $Sp(N)$

$$R = \yng(4,4), (r, s) = (\yng(4), \yng(4)). \tag{G.7}$$

The restricted Schur was calculated to be

$$O^{Sp(N)}_{\yng(4,4)(\yng(4),\yng(4))} = 2\sqrt{30}\Big(\mathrm{Tr}(ZY)^2 + 2\mathrm{Tr}(ZYZY)\Big). \tag{G.8}$$

Its two-point function is

$$\langle O^{Sp(N)}_{\yng(4,4)(\yng(4),\yng(4))} \overline{O}^{Sp(N)}_{\yng(4,4)(\yng(4),\yng(4))} \rangle = 2880N(N+1)(N+2)(N+3), \tag{G.9}$$

agreeing with (5.40). We now compare this with the $SO(N)$ two-point function for

$$R_c = \yng(2,2,2,2), (r_c, s_c) = \left(\yng(1,1,1,1), \yng(1,1,1,1)\right). \tag{G.10}$$

The restricted Schur (5.1) is

$$O^{SO(N)}_{\yng(2,2,2,2)(\yng(1,1,1,1),\yng(1,1,1,1))} = 2\sqrt{30}\Big(\mathrm{Tr}(ZY)^2 - 2\mathrm{Tr}(ZYZY)\Big) \tag{G.11}$$

with two-point function

$$\langle O^{SO(N)}_{\yng(2,2,2,2)(\yng(1,1,1,1),\yng(1,1,1,1))} \overline{O}^{SO(N)}_{\yng(2,2,2,2)(\yng(1,1,1,1),\yng(1,1,1,1))} \rangle = 2880N(N-1)(N-2)(N-3). \tag{G.12}$$

Sending $N \to -N$, yields the $Sp(N)$ result.

Appendix H

Examples of multi-trace operators

In this appendix, we give two examples of our formula (5.53). Recall from [89], we calculated (with the normalisation of (5.1))

$$
\begin{aligned}
O_{\yng(4,4),(\yng(2,2),\yng(2,2))} &= \sqrt{30}\Big(2\mathrm{Tr}(ZY)^2 + 2\mathrm{Tr}(ZYZY) + \mathrm{Tr}(Z^2)\mathrm{Tr}(Y^2) + 4\mathrm{Tr}(Z^2Y^2)\Big), && \text{(H.1)} \\
O_{\yng(2,2,2,2),(\yng(2,2),\yng(2,2))} &= \sqrt{6}\Big(2\mathrm{Tr}(ZY)^2 + 2\mathrm{Tr}(ZYZY) + 3\mathrm{Tr}(Z^2)\mathrm{Tr}(Y^2) - 12\mathrm{Tr}(Z^2Y^2)\Big), && \text{(H.2)} \\
O_{\yng(2,2,2,2),(\yng(1,1,1,1),\yng(1,1,1,1))} &= 2\sqrt{30}\Big(\mathrm{Tr}(ZY)^2 - 2\mathrm{Tr}(ZYZY)\Big), && \text{(H.3)} \\
O_{\yng(4,2,2),(\yng(2,2),\yng(2,2))} &= 2\sqrt{6}\Big(- \mathrm{Tr}(ZY)^2 - \mathrm{Tr}(ZYZY) + \mathrm{Tr}(Z^2)\mathrm{Tr}(Y^2) + \mathrm{Tr}(Z^2Y^2)\Big). && \text{(H.4)}
\end{aligned}
$$

Let $\sigma = (1,3,5)(4,8,6,7)$. The dual restricted characters evaluated to

$$
\chi_{SO(N)}^{\yng(4,4)(\yng(2,2),\yng(2,2))}(\sigma) = \frac{1}{6\sqrt{30}}, \qquad \text{(H.5)}
$$

$$
\chi_{SO(N)}^{\yng(2,2,2,2)(\yng(2,2),\yng(2,2))}(\sigma) = \frac{1}{30\sqrt{6}}, \qquad \text{(H.6)}
$$

$$\chi_{SO(N)}^{\yng(2,2,2,2)(\yng(1,1,1,1),\yng(1,1,1,1))}(\sigma) = \frac{1}{6\sqrt{30}}, \tag{H.7}$$

$$\chi_{SO(N)}^{\yng(4,2,2)(\yng(2,2),\yng(2,2))}(\sigma) = -\frac{1}{15}\sqrt{\frac{2}{3}}. \tag{H.8}$$

Then adding the 4 Schurs with these coefficients, we found

$$\frac{1}{6\sqrt{30}}O^{SO(N)}_{\yng(4,4)(\yng(2,2),\yng(2,2))} + \frac{1}{30\sqrt{6}}O^{SO(N)}_{\yng(2,2,2,2)(\yng(2,2),\yng(2,2))} + \frac{1}{6\sqrt{30}}O^{SO(N)}_{\yng(2,2,2,2)(\yng(1,1,1,1),\yng(1,1,1,1))} - \frac{1}{15}\sqrt{\frac{2}{3}}O^{SO(N)}_{\yng(4,2,2)(\yng(2,2),\yng(2,2))} = \mathrm{Tr}(ZY)^2. \tag{H.9}$$

We then calculated the right-hand-side of (5.53) and found

$$C_I^{4\nu}(\sigma)^I_J(Z^{\otimes 4} \otimes Y^{\otimes 4}) = \mathrm{Tr}(ZY)^2. \tag{H.10}$$

For one more example, Let $\sigma = (3, 4, 7, 8)$. The dual restricted characters evaluated to

$$\chi_{SO(N)}^{\yng(4,4)(\yng(2,2),\yng(2,2))}(\sigma) = \frac{1}{12\sqrt{30}}, \tag{H.11}$$

$$\chi_{SO(N)}^{\yng(2,2,2,2)(\yng(2,2),\yng(2,2))}(\sigma) = \frac{-1}{20\sqrt{6}}, \tag{H.12}$$

$$\chi_{SO(N)}^{\yng(2,2,2,2)(\yng(1,1,1,1),\yng(1,1,1,1))}(\sigma) = 0, \tag{H.13}$$

$$\chi_{SO(N)}^{\yng(4,2,2)(\yng(2,2),\yng(2,2))}(\sigma) = \frac{1}{30\sqrt{6}}. \tag{H.14}$$

Adding the 4 Schurs with these coefficients, we found

$$\frac{1}{12\sqrt{30}}O^{SO(N)}_{\yng(4,4)(\yng(2,2),\yng(2,2))} - \frac{1}{20\sqrt{6}}O^{SO(N)}_{\yng(2,2,2,2)(\yng(2,2),\yng(2,2))} + \frac{1}{30\sqrt{6}}O^{SO(N)}_{\yng(4,2,2)(\yng(2,2),\yng(2,2))} = \mathrm{Tr}(Z^2Y^2). \tag{H.15}$$

We then calculated the right-hand-side of (5.53) and found

$$C_I^{4\nu}(\sigma)^I_J(Z^{\otimes 4} \otimes Y^{\otimes 4}) = \mathrm{Tr}(Z^2Y^2). \tag{H.16}$$

Appendix I

Example of the product rule

In this appendix, we give one simple example of our product rule in (5.63). We try evaluate the following product: $O_{\yng(2,2),(\yng(1,1),\yng(1,1))}(Z,Y)O_{\yng(2,2),(\yng(1,1),\yng(1,1))}(Z,Y)$. For the definition in (5.7),

$$O_{\yng(2,2),(\yng(1,1),\yng(1,1))}(Z,Y) = -\sqrt{6}\mathrm{Tr}(ZY). \tag{I.1}$$

This means that

$$O_{\yng(2,2),(\yng(1,1),\yng(1,1))}(Z,Y)O_{\yng(2,2),(\yng(1,1),\yng(1,1))}(Z,Y) = 6\mathrm{Tr}(ZY)^2. \tag{I.2}$$

According to (5.63), this product should also be given as a sum over restricted Schurs, each multiplied by restricted Littlewood-Richardson coefficients, corresponding to the labels in (5.54). The operators for these labels evaluated to

$$O_{\yng(4,4),(\yng(2,2),\yng(2,2))} = \sqrt{30}\Big(2\mathrm{Tr}(ZY)^2 + 2\mathrm{Tr}(ZYZY) + \mathrm{Tr}(Z^2)\mathrm{Tr}(Y^2) + 4\mathrm{Tr}(Z^2Y^2)\Big), \tag{I.3}$$

$$O_{\yng(2,2,2,2),(\yng(2,2),\yng(2,2))} = \sqrt{6}\Big(2\mathrm{Tr}(ZY)^2 + 2\mathrm{Tr}(ZYZY) + 3\mathrm{Tr}(Z^2)\mathrm{Tr}(Y^2) - 12\mathrm{Tr}(Z^2Y^2)\Big), \tag{I.4}$$

$$O_{\yng(2,2,2,2),(\yng(1,1,1,1),\yng(1,1,1,1))} = 2\sqrt{30}\Big(\mathrm{Tr}(ZY)^2 - 2\mathrm{Tr}(ZYZY)\Big), \tag{I.5}$$

$$O_{\yng(4,2,2),(\yng(2,2),\yng(2,2))} = -2\sqrt{6}\Big(- \mathrm{Tr}(ZY)^2 - \mathrm{Tr}(ZYZY) + \mathrm{Tr}(Z^2)\mathrm{Tr}(Y^2) + \mathrm{Tr}(Z^2Y^2)\Big). \tag{I.6}$$

Only the last operator differs from the operators defined in (5.1) and the difference is a minus sign. Here, $n_1 = m_1 = n_2 = m_2 = 1$, and $q_1 = q_2 = 2$. The restricted Littlewood-Richardson coefficients evaluated to

$$f^{\yng(4,4),(\yng(2,2),\yng(2,2))}_{\yng(2,2),(\yng(1,1),\yng(1,1))\,;\,\yng(2,2),(\yng(1,1),\yng(1,1))} = \frac{1}{\sqrt{30}}, \tag{I.7}$$

$$f^{\yng(2,2,2,2),(\yng(2,2),\yng(2,2))}_{\yng(2,2),(\yng(1,1),\yng(1,1))\,;\,\yng(2,2),(\yng(1,1),\yng(1,1))} = \frac{1}{5\sqrt{6}}, \tag{I.8}$$

$$f^{\yng(2,2,2,2),(\yng(1,1,1,1),\yng(1,1,1,1))}_{\yng(2,2),(\yng(1,1),\yng(1,1))\,;\,\yng(2,2),(\yng(1,1),\yng(1,1))} = \frac{1}{\sqrt{30}}, \tag{I.9}$$

$$f^{\yng(4,2,2),(\yng(2,2),\yng(2,2))}_{\yng(2,2),(\yng(1,1),\yng(1,1))\,;\,\yng(2,2),(\yng(1,1),\yng(1,1))} = \frac{2}{5}\sqrt{\frac{2}{3}}. \tag{I.10}$$

In the f's above, there were two sums over the permutation group S_4. The first sum was over permutations that permuted $1, 2, 5, 6$ amongst themselves, and the second sum was over permutations that permuted $3, 4, 7, 8$ amongst themselves. For the coefficients in (I.7) to (I.10) and operators (I.3) to (I.6), we found

$$\begin{aligned} & f^{\yng(4,4),(\yng(2,2),\yng(2,2))}_{\yng(2,2),(\yng(1,1),\yng(1,1))\,;\,\yng(2,2),(\yng(1,1),\yng(1,1))} O_{\yng(4,4),(\yng(2,2),\yng(2,2))}(Y,Z) + f^{\yng(2,2,2,2),(\yng(2,2),\yng(2,2))}_{\yng(2,2),(\yng(1,1),\yng(1,1))\,;\,\yng(2,2),(\yng(1,1),\yng(1,1))} O_{\yng(2,2,2,2),(\yng(2,2),\yng(2,2))} + f^{\yng(2,2,2,2),(\yng(1,1,1,1),\yng(1,1,1,1))}_{\yng(2,2),(\yng(1,1),\yng(1,1))\,;\,\yng(2,2),(\yng(1,1),\yng(1,1))} O_{\yng(2,2,2,2),(\yng(1,1,1,1),\yng(1,1,1,1))} \\ & + f^{\yng(4,2,2),(\yng(2,2),\yng(2,2))}_{\yng(2,2),(\yng(1,1),\yng(1,1))\,;\,\yng(2,2),(\yng(1,1),\yng(1,1))} O_{\yng(4,2,2),(\yng(2,2),\yng(2,2))} = 6\mathrm{Tr}(ZY)^2 \end{aligned} \tag{I.11}$$

precisely as expected.

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