University of the Witwatersrand

Ph.D. Thesis

Non-perturbative String Theory Dynamics from the Gauge / Gravity Duality

David Gossman[1]

supervised by
Prof. Robert de Mello Koch[2]

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[1]dmgossman@gmail.com
[2]robert.demellokoch@wits.ac.za

Declaration

I, David Gossman, declare that except where acknowledged in the customary manner, the material presented in this dissertation is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

Date: 21 April 2020 **Signature:**

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Table of Contents

Declaration **1**

Acknowledgments **2**

List of Figures **5**

Abstract **6**

Chapter 1. Introduction **7**

Chapter 2. Background **10**
2.1. Matrix Model Quantum Mechanics 10
2.1.1. Free theory 10
2.1.2. Interacting theory 12
2.1.3. Normalisation 13
2.1.4. t' Hooft expansion 14
2.1.5. Number of Fields Comparable to N 19
2.1.6. Schur polynomials and group theory 20
2.1.7. Schur Polynomials 26
2.1.8. Two Matrices 30
2.2. AdS/CFT correspondence 39
2.2.1. Anti-de Sitter space 39
2.2.2. Conformal symmetry 40
2.2.3. $\mathcal{N} = 4$ Super Yang-Mills 41
2.2.4. IIB strings on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM 41
2.3. Chaos 42
2.3.1. Classical Chaos 42
2.3.2. Quantum chaos 42
2.3.3. Operator growth and chaos 43
2.3.4. Qubit system 44
2.4. SYK Model 46
2.5. Tensor Models 50
2.6. Collective Field Theory 53
2.7. Collective Field Theory examples for a Tensor Model 55
2.7.1. Changing variables 56
2.7.2. Re-scaling and Jacobian 58
2.7.3. New variables 61

Chapter 3. Gauge Invariants, Correlators and Holography **66**
3.1. Construction of Gauge Invariant Operators 66
3.1.1. Counting and construction for bosonic tensors 68
3.1.2. Counting and construction for fermionic tensors 70
3.2. Correlators of Gauge Invariant Operators 72

TABLE OF CONTENTS—*Continued*

3.2.1. Bosonic Correlators . . . 72
3.2.2. Fermionic Correlators . . . 73
3.3. Algebra of the Gauge Invariant Operators . . . 74
3.4. Collective Field Theory . . . 77

CHAPTER 4. HOLOGRAPHY FOR TENSOR MODELS . . . **83**
4.1. Invariant Variables . . . 83
4.2. A "planar limit" for the tensor model . . . 86
4.3. The Collective Field Theory of Melons . . . 90
4.4. Finite N Contributions . . . 93

CHAPTER 5. CONCLUSION . . . **95**

APPENDIX A. . . . **98**
A.1. First correction to two point function . . . 98
A.2. Collective Field Theory Toy Model . . . 99
A.2.1. Investigating Hermiticity . . . 99
A.2.2. Calculating the Jacobian . . . 102
A.2.3. Testing H_{eff} . . . 105

APPENDIX B. . . . **108**
B.1. Check of counting formulas . . . 108
B.2. Examples of Operators and Correlators . . . 110
B.3. Identities needed to derive the collective field theory Hamiltonian . . . 112

APPENDIX C. . . . **113**
C.1. Notation for Equivalence Classes . . . 113
C.2. Perturbative Collective Field Theory . . . 113
C.3. Relating Matrix Model and Tensor Model Correlators . . . 115
C.4. Some Matrix Model Correlators . . . 116

REFERENCES . . . **118**

List of Figures

FIGURE 2.1. 4-point vertex . . . 13
FIGURE 2.2. Figure of eight diagram . . . 15
FIGURE 2.3. Triangulation of a sphere . . . 16
FIGURE 2.4. Section of a triangulated surface . . . 16
FIGURE 2.5. Section of a triangulated surface with two holes . . . 17
FIGURE 2.6. Perturbation expansion in terms of topologies of worldsheets in string theory . . . 17
FIGURE 2.7. Sizes of the terms in $Z_1(t)$ with $n = 4,\ g = -1.05$ & $h = 0.5$ . . . 45
FIGURE 2.8. $C_j(t)$'s with $n = 4$, $g = -1.05$ & $h = 0.5$ . . . 46
FIGURE 2.9. First order correction to two point function . . . 48
FIGURE 2.10. Some second order corrections to two point function . . . 49
FIGURE 2.11. Tensor propagator with strands explicit (*left*) and compressed (*right*) . . . 52
FIGURE 2.12. Tensor vertex with External lines suppressed . . . 52
FIGURE 2.13. The Feynman diagrams contributing to $\langle \phi^{abc}\bar{\phi}_{abc}\rangle$. The dashed lines indicate which operators are Wick contracted. . . . 52

FIGURE 3.1. The above figures label gauge invariant operators in the tensor model gauge theory. Black dots correspond to $\bar{\phi}^{ijk}$'s and white dots to ϕ_{ijk}'s. A line labeled by i is a gauge index for $U(N_i)$. The operator on the left corresponds to $\bar{\phi}^{i_1j_1k_1}\phi_{i_2j_1k_1}\bar{\phi}^{i_2j_2k_2}\phi_{i_3j_2k_2}\bar{\phi}^{i_3j_3k_3}\phi_{i_4j_3k_3}\bar{\phi}^{i_4j_4k_4}\phi_{i_1j_4k_4}$ and the operator on the right corresponds to $\bar{\phi}^{ijk}\phi_{ijk}$. . . . 78

FIGURE 4.1. Examples of melonic graphs. . . . 90
FIGURE 4.2. The tree labeling a melonic invariant. . . . 91
FIGURE 4.3. An example of tree joining. This figure shows the possible values for $C + C'$ for the given C and C'. The joining is performed by contracting a solid vertex in C with a hollow vertex in C'. . . . 91

FIGURE 5.1. The Feynman diagrams that contribute to $\langle \phi^{abc}\bar{\phi}_{abd}\phi^{efd}\bar{\phi}_{efc}\rangle$. . . . 97

FIGURE C.1. The Feynman diagrams contributing to $\langle \phi^{abc}\bar{\phi}_{abc}\rangle$. The dashed lines indicate which operators are Wick contracted. . . . 115
FIGURE C.2. $\mathrm{Tr}\,(Z\bar{Z}Z\bar{Z})$ is shown on the left and $\phi_{jka}\bar{\phi}^{jkb}\phi_{lmb}\bar{\phi}^{lma}$ on the right. . . . 116
FIGURE C.3. A Feynman diagram contributing to $\langle \mathrm{Tr}\,(Z\bar{Z}Z\bar{Z})\rangle$ is shown on the left and to $\langle \phi_{jka}\bar{\phi}^{jkb}\phi_{lmb}\bar{\phi}^{lma}\rangle$ on the right. . . . 116

Abstract

Motivated by the close connection of tensor models to the SYK model, and the success of the SYK model within holography, tensor models are studied. A brief introduction to relevant topics surrounding holography and tensor models is given. Representation theory is used to construct the complete set of gauge invariant observables for bosonic and fermionic tensor models. Correlation functions of the gauge invariant operators in the free theory are computed exactly. The gauge invariant operators close a ring. The structure constants of the ring are described explicitly. The holographic duals of tensor models are studied using collective field theory. A description of the gauge invariant variables of the tensor model, is developed. This is used to develop a collective field theory description of the dynamics. A matrix like subsector that develops an extra holographic dimension is considered. In particular, the collective field theory for the matrix like sector of an interacting tensor model is developed. The correctness of the large N collective field is checked by showing that it reproduces the perturbative expansion of large N expectation values. In contrast to this, it is argued that melonic large N limits do not develop an extra dimension. This conclusion follows from the large N value for the melonic collective field, which has delta function support. The finite N physics of the model is also developed and non-perturbative effects in the $1 = N$ expansion are exhibited.

Chapter 1

Introduction

By the end of the 19th century physics was widely thought to be able to explain all fundamental aspects of nature and that all the laws of physics had been discovered, with Classical Mechanics explaining the motion of objects, Thermodynamics explaining temperature and heat transfer and Maxwell's equations explaining optics and electromagnetism. But at the turn of the 20th century this had to be questioned with the emergence of problems that physics at that time could not answer, such as the ultraviolet catastrophe, the photoelectric effect and the seemingly bizarre prediction of Maxwell's equations: that the speed of light is constant regardless of observer. These questions saw the emergence of modern physics and what are thought of as the pillars of modern physics: Quantum Mechanics (QM): which explains the interactions of fundamental particles as well as the ultraviolet catastrophe and the photoelectric effect, and General Relativity (GR): which is the logical generalisation of Special Relativity which was birthed out of the fact that the speed of light is a constant for any observer. QM explains three of the four forces in nature (the Coulomb force, the weak nuclear force and the strong nuclear force) as the interactions between fundamental particles. GR explains the fourth force: gravity, as the warping of space-time by mass and energy. These two theories explain beautifully the physics in their domain; QM for very small objects and GR for very massive objects. But what happens when one needs to study objects that fall in both domains like an object that is very massive but also very small, a black hole or the very early universe for instance? For these objects a unified theory which accounts for both quantum and general relativistic physics is needed—a theory of Quantum Gravity. If, however, we try and write down a quantum field theory for gravity we run into difficulties because gravity is an irrelevant (in the sense of the renormalisation group) interaction. What this means is that at low energies the gravitational interaction will be weak, as we see in nature, but it will grow with energy scale so that at large energies the interaction will diverge and calculations are impossible. This is analogous to the Fermi interaction which was an interaction proposed by Enrico Fermi to explain Beta decay: the process in which a neutron decays into a proton, electron and neutrino. Fermi proposed a four point interaction to describe this process but the coupling for this process also turned out to be irrelevant so it worked for low energies but diverged at large energies. These kinds of divergences tell us that there is something, some physics, not being accounted for. In the case of Beta decay what was missing was the W and Z bosons proposed by Sheldon Glashow, Abdus Salam, and Steven Weinberg – eventually winning them the Nobel Prize in 1979. The missing element in quantum gravity is yet to be discovered. Just like the Fermi interaction and the problems which were seen at the turn of the 20th century this problem indicates that there is physics yet to be discovered.

The theory with the most promise in answering these questions seems to be string theory. In it the fundamental particles are defined as one dimensional strings. What this means is that one can no longer probe distances smaller than the size of a string. One would expect the size of the string to be smaller than the smallest distances we have been able to probe thus far. And so one can make a quantitative argument for the size of the string. The length of the string $l_s$would be a fundamental constant just like the speed of light c, which is a fundamental speed, and $\hbar$, which is a fundamental uncertainty. It is suspected that the string length will be related to the other fundamental constants. In the theory of gravity we also have

the gravitational constant G. Out of these constants one can define a length as follows

$$l_P = \sqrt{\frac{G\hbar}{c^3}} \tag{1.1}$$

This is known as the Plank length and it is on the order of 10^{-36}m. This is very small and we expect l_s will be proportional to this. In string theory all fundamental particles are the same object: the string, and their properties are determined by the vibrational mode of the string thus this theory unifies theories of fermions and bosons into a theory of strings.

A significant gap in our current understanding of string theory is that at present we do not know a non-perturbative definition of the theory. However, in the last 20 years genuinely new and fascinating insights into the large N expansion have been achieved. The ribbon graph expansion proposed by 't Hooft suggested a deep connection between non-Abelian gauge theories and string theories more than 40 years ago. The expansion identifies the surface triangulated by the ribbon graph with a string worldsheet[1]. This initial proposal has found beautiful confirmation in the duality between the large N expansion of $\mathcal{N} = 4$ super Yang-Mills theory (a conformal field theory (CFT)) and the loop expansion of IIB string theory on asymptotically $AdS_5 \times S^5$ spacetimes[2, 3, 4] (where AdS stands for anti-de Sitter space). This connection goes by the name of holography, the AdS/CFT correspondence or the gauge theory/gravity duality and for any theory with adjoint valued variables, one expects a duality with a string theory.

When studying the large N expansions of matrix models one usually studies the planar limit where classical operator dimensions are held fixed as one takes $N \to \infty$, there are non-planar large N limits of the theory [5] defined by considering operators with a bare dimension that is allowed to scale with N as one takes $N \to \infty$. These limits are also relevant for the AdS/CFT correspondence. Indeed, operators with a dimension that scales as N include operators relevant for the description of giant graviton branes [6, 7, 8] while operators with a dimension of order N^2 include operators that correspond to new geometries in supergravity [9, 10, 11, 12]. These convincing motivations have motivated sustained study of large N field theory. However, carrying out the large N expansion is mostly still beyond our current capabilities.

The mechanism by which gravitational physics is manifested from a strongly coupled gauge theory within holography remains elusive. The original CFT description has the field theory coupling as the loop expansion parameter. On the other hand, the gravitational description that emerges at strong coupling, must have $1/N$ as the loop counting parameter. This is a highly non-trivial hint into the structure of the holographic reorganization of the CFT. The collective field theory of Jevicki and Sakita [13, 14] achieves exactly this: by formulating the theory in terms of gauge invariant variables, the resulting field theory explicitly has $1/N$ as the loop expansion parameter. The reorganization of the dynamics is highly non-trivial, with non-linear collective dynamics being induced by the Jacobian of the change of variables[13, 14].

It would be very attractive to apply the collective field theory method to CFTs and explore the resulting field theory. In the case of a single matrix, this leads to a string field theory for the $c = 1$ string[15]. This is a beautiful example of how a quantum mechanical system can develop an extra dimension. In the much more interesting example of $\mathcal{N} = 4$ super Yang-Mills theory[2] (or even quantum chromodynamics (QCD)) the construction of collective field theory is frustrated by the fact that the space of gauge invariants (loop space) is enormous and an explicit construction of the dynamics of gauge invariants looks hopeless. It turns out that representation theory can provide a systematic approach towards the structure of loop space. Indeed, the use of representation theory in the half-BPS sector[10, 11] leads to a clear connection

to free fermions[10, 11, 12] and ultimately to a rather complete understanding of the mapping between the CFT operators and supergravity geometries[9]. This has been extended to more general bosonic sectors[16] -[17] and even for fermions and gauge fields[18, 19]. These bases allow the computations of anomalous dimensions of heavy operators in $\mathcal{N} = 4$ super Yang-Mills[20, 21, 22, 23] (in a large N but non planar limit[5]) that are dual to excited giant gravitons[24]-[25]. Up to now however, even with this improved understanding, it is not obvious how to build the collective field theory of these invariant variables.

The large N limits of vector models are relatively simple in comparison to matrix models. There are fewer gauge invariants that one can form and the large N limit of these models is explicitly solvable. The space of invariants is spanned by a bilocal field and one can explicitly build the collective dynamics[26]. In [27] the idea that the bilocal fields provide a reconstruction of the bulk fields of the dual higher spin gravity[28] was put forward. Using essentially kinematics [29]-[30] developed a map between the space of bilocals and the dual gravity. The bilocal description has also proved to be very useful for the SYK model[31, 32, 33], as well as for descriptions of supersymmetric versions of SYK[34, 35, 36, 37], in which case an elegant bilocal superspace formulation has been developed in [38].

There are suggestions that tensor models are richer than vector models, but perhaps simpler than matrix models. One might hope that the case of tensor models is, in a sense, intermediate between the vector and matrix models. It is then possible that the space of gauge invariants is richer than that of vectors, but still not as complex as that of matrices. If this is the case, this may provide a useful lesson towards managing the loop space of multi-matrix models. In light of this, it is natural to work out the large N expansion and holography for tensor models, which is the main goal of this Ph.D.

This thesis is structured as follows: an introduction to some of the important concepts and techniques used are given in Chapter 2. Then the novel work carried out on tensor models is presented in Chapter 3 and 4, these works are published in [39] and [40] respectively. Then in Chapter 5 some conclusions are given.

Chapter 2

Background

The aim of this chapter is to briefly introduce the concepts and techniques that will aid in the understanding of the work done in Chapters 3 and 4.

2.1 Matrix Model Quantum Mechanics

Matrix models are theories involving rank 2 tensors in which the $\frac{1}{N}$ expansion has a well known relation to strings. The large N limit exhibits factorisation and the $\frac{1}{N}$ expansion can be interpreted as a genus expansion. This provides a compelling link between matrix models and string theory. Since matrix models exhibit so much success in this regard there is a hope that a similar thing can be found for higher rank tensor models. The goal of this section is to introduce matrix models, motivate their duality to strings and show some of the success that has been achieved in calculating the large N expansion for Matrix Models.

2.1.1 Free theory

Start with a $N \times N$ Hermitian matrix M and define a free action:

$$S = -\frac{w}{2}\mathrm{Tr}(M^2) \tag{2.1}$$

then expectation values are given by:

$$\langle M_{ij}...M_{nm}\rangle_0 = \mathcal{N}\int [dM] e^{-\frac{w}{2}\mathrm{Tr}(M^2)} M_{ij}...M_{nm} \tag{2.2}$$

where the subscript 0 stands for free and the integral over $[dM]$ is understood as a functional integral over each element of the matrix M, making this N^2 integrals in total. The normalisation $\mathcal{N}$ is chosen such that:

$$\langle 1\rangle_0 = \mathcal{N}\int [dM] e^{-\frac{w}{2}\mathrm{Tr}(M^2)} = 1 \tag{2.3}$$

This implies for a general N:

$$\mathcal{N} = \left[\frac{1}{2}\left(\frac{\omega}{\pi}\right)^N\right]^{\frac{N}{2}} \tag{2.4}$$

By defining a generating function:

$$Z_0[J] = \mathcal{N}\int [dM] e^{-\frac{w}{2}\mathrm{Tr}(M^2)+\mathrm{Tr}(JM)} = e^{\frac{1}{2\omega}\mathrm{Tr}(J^2)} \tag{2.5}$$

correlators can be represented as follows:

$$\langle M_{ij}...M_{nm}\rangle_0 = \frac{d}{dJ_{ji}}...\frac{d}{dJ_{mn}} Z_0[J]\Big|_{J=0} \tag{2.6}$$

Gauge invariant operators are traces of powers of matrices so one should study correlators of traces of powers of M. For example:

$$\langle \mathrm{Tr}(M^2)\rangle_0 = \langle M_{ij}M_{ji}\rangle = \frac{N^2}{\omega} \tag{2.7}$$

$$\langle \mathrm{Tr}(M^4)\rangle_0 = \langle M_{ij}M_{jk}M_{kl}M_{li}\rangle = \frac{1}{\omega^2}\left[2N^3+N\right] \tag{2.8}$$

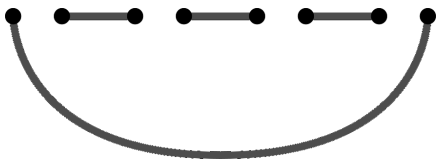
and

$$\langle \mathrm{Tr}(M^2)\mathrm{Tr}(M^2)\rangle_0 = \langle M_{ij}M_{ji}M_{kl}M_{lk}\rangle = \frac{1}{\omega^2}\left[N^4+2N^2\right] \tag{2.9}$$

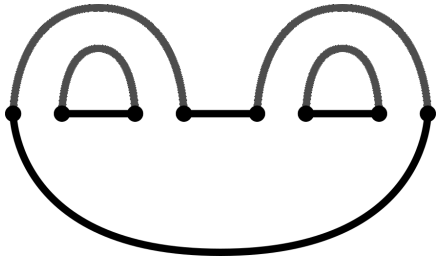
The Feynman rules for these correlators are as follows (illustrated with $\langle \mathrm{Tr}(M^4)\rangle$):

1. Draw two dots for each pair of indices on the operators

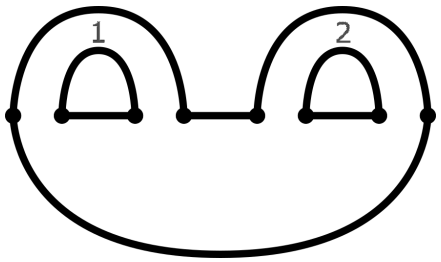
2. Connect the contracted indices below



3. Connect the pairs of indices above in ribbons, each different way of connecting is a different diagram

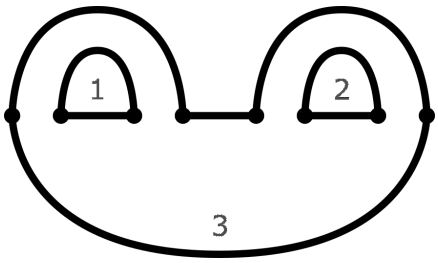


4. Each ribbon comes with a factor of $1/\omega$



$\equiv \frac{1}{\omega^2}\ldots$

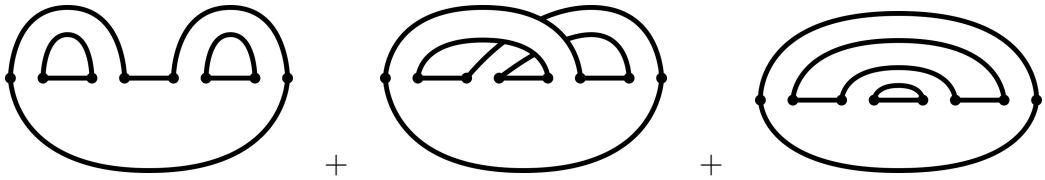
5. Each closed loop comes with a factor of N



$$\equiv \frac{N^3}{\omega^2}$$

Computing $\langle \mathrm{Tr}(M^4)\rangle_0$ with ribbon graphs gives:

$$\langle \mathrm{Tr}(M^4)\rangle_0 =$$



$$= \frac{N^3}{\omega^2} + \frac{N}{\omega^2} + \frac{N^3}{\omega^2} = \frac{1}{\omega^2}\left[2N^3 + N\right]$$

2.1.2 Interacting theory

Now the aim is to add interactions to the theory and this will be done by putting an interaction term into the action, quartic in M. The generating function becomes

$$\begin{aligned} Z[J] =& \mathcal{N}\int [dM] e^{-w/2\mathrm{Tr}(M^2) - g\mathrm{Tr}(M^4) + \mathrm{Tr}(JM)} && (2.10)\\ =& \mathcal{N}\sum_{n=0}^{\infty}\frac{1}{n!}(-g)^n \int [dM] e^{-w/2\mathrm{Tr}(M^2)+\mathrm{Tr}(JM)}\mathrm{Tr}(M^4)^n \\ =& \sum_{n=0}^{\infty}\frac{1}{n!}\left[-g\frac{d}{dJ_{ji}}\frac{d}{dJ_{kj}}\frac{d}{dJ_{lk}}\frac{d}{dJ_{il}}\right]^n Z_0[J] \\ =& \sum_{n=0}^{\infty}\frac{1}{n!}\left[-g\frac{d}{dJ_{ji}}\frac{d}{dJ_{kj}}\frac{d}{dJ_{lk}}\frac{d}{dJ_{il}}\right]^n e^{\frac{1}{2\omega}\mathrm{Tr}(J^2)} && (2.11)\\ \Rightarrow Z[J=0] =& \sum_{n=0}^{\infty}\frac{1}{n!}(-g)^n \left\langle \left(\mathrm{Tr}(M^4)\right)^n\right\rangle_0 && (2.12) \end{aligned}$$

The correlators in the interacting theory are given by:

$$\langle M_{ij}...M_{nm}\rangle = \frac{d}{dJ_{ji}}...\frac{d}{dJ_{mn}}Z[J]\Big|_{J=0} \qquad (2.13)$$

The interaction is put into the diagrams by a 4-point vertex of the following form:

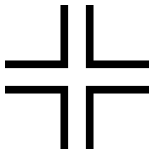


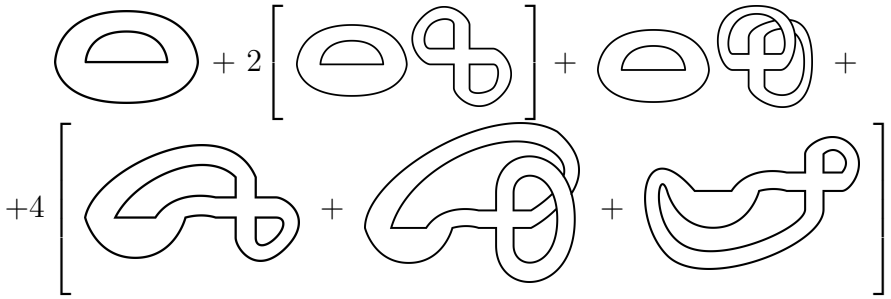
Figure 2.1: 4-point vertex

Each vertex comes with a factor of $(-g)$. Correlators in the interacting theory can now be calculated perturbatively in g. Consider $\langle \mathrm{Tr}(M^2)\rangle$ to first order in g.

$$\begin{aligned}
\langle \mathrm{Tr}(M^2)\rangle =& \frac{d}{dJ_{nm}}\frac{d}{dJ_{mn}}\sum_{n=0}^{1}\frac{1}{n!}\left[-g\frac{d}{dJ_{ji}}\frac{d}{dJ_{kj}}\frac{d}{dJ_{lk}}\frac{d}{dJ_{il}}\right]^n e^{\frac{1}{2\omega}\mathrm{Tr}(J^2)}\Big|_{J=0}\\
=&\langle \mathrm{Tr}(M^2)\rangle_0 - g\,\frac{d}{dJ_{nm}}\frac{d}{dJ_{mn}}\frac{d}{dJ_{ji}}\frac{d}{dJ_{kj}}\frac{d}{dJ_{lk}}\frac{d}{dJ_{il}}e^{\frac{1}{2\omega}\mathrm{Tr}(J^2)}\Big|_{J=0}\\
=&\frac{N^2}{\omega} - g\langle \mathrm{Tr}(M^2)\mathrm{Tr}(M^4)\rangle_0\\
=&\frac{N^2}{\omega} - \frac{g}{\omega^3}\left[2N^5+9N^3+4N\right]
\end{aligned} \tag{2.14}$$

This can also be done diagrammatically:

$\langle \mathrm{Tr}(M^2)\rangle =$



$$\begin{aligned}
=&\frac{N^2}{\omega} - \frac{g}{\omega^3}\left[2N^5+N^3+4\left(N^3+N+N^3\right)\right]\\
=&\frac{N^2}{\omega} - \frac{g}{\omega^3}\left[2N^5+9N^3+4N\right]
\end{aligned}$$

2.1.3 Normalisation

Now it is convenient and more physically insightful to normalise such that the vacuum contributions are removed. This is achieved by normalising as follows:

$$\begin{aligned}
Z_{nn}[J] =& \frac{Z[J]}{Z[J=0]}\\
=&\frac{Z[J]}{\sum_{n=0}^{\infty}\frac{1}{n!}(-g)^n\left\langle\left(\mathrm{Tr}(M^4)\right)^n\right\rangle_0}\\
=&\frac{Z[J]}{1-\frac{g}{\omega^2}(2N^2+N)+\frac{g^2}{2\omega^4}(4N^6+40N^4+61N^2)+\mathcal{O}(g^3)},
\end{aligned} \tag{2.15}$$

where the subscript nn stands for new normalisation. To illustrate this consider $\langle \mathrm{Tr}(M^2)\rangle_{nn}$.

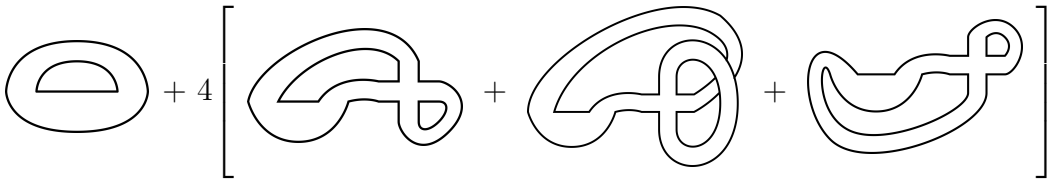
$$\begin{aligned}\langle \mathrm{Tr}(M^2)\rangle_{nn} =& \frac{\langle \mathrm{Tr}(M^2)\rangle}{1-\frac{g}{\omega^2}(2N^2+N)+\frac{g^2}{2\omega^4}(4N^6+40N^4+61N^2)+\mathcal{O}(g^3)} \\ =& \frac{\frac{N^2}{\omega}-\frac{g}{\omega^3}\left[2N^5+9N^3+4N\right]+\mathcal{O}(g^2)}{1-\frac{g}{\omega^2}(2N^2+N)+\frac{g^2}{2\omega^4}(4N^6+40N^4+61N^2)+\mathcal{O}(g^3)}\end{aligned} \tag{2.16}$$

Assuming g is small and expanding to first order in g gives:

$$\begin{aligned}\langle \mathrm{Tr}(M^2)\rangle_{nn} \cong& \left[\frac{N^2}{\omega}-\frac{g}{\omega^3}\left[2N^5+9N^3+4N\right]+\mathcal{O}(g^2)\right]\left[1+\frac{g}{\omega^2}(2N^2+N)+\mathcal{O}(g^2)\right] \\ =& \frac{N^2}{\omega}-\frac{g}{\omega^3}\left[8N^3+4N\right]\end{aligned} \tag{2.17}$$

Comparing this to adding up the diagrams excluding the diagrams with vacuum graphs.

$\langle \mathrm{Tr}(M^2)\rangle_{nn} =$



$$\begin{aligned}&= \frac{N^2}{\omega}-\frac{4g}{\omega^3}\left[N^3+N+N^3\right] \\ &= \frac{N^2}{\omega}-\frac{g}{\omega^3}\left[8N^3+4N\right]\end{aligned} \tag{2.18}$$

This is the same result and henceforth, this new normalisation will be used.

2.1.4 t' Hooft expansion

Consider the double scaling limit $N \to \infty$ and $g \to 0$ such that $\lambda = gN$ is a fixed small number. $\langle \mathrm{Tr}(M^2)\rangle_{nn}$ up to second order in g is given by

$$\begin{aligned}\langle \mathrm{Tr}(M^2)\rangle_{nn} =& \frac{N^2}{\omega}-\frac{g}{\omega^3}\left[8N^3+4N\right]+\frac{g^2}{\omega^5}\left[144N^4+224N^2\right] \\ =& \frac{N^2}{\omega}-\frac{N^2\lambda}{\omega^3}\left[8+\frac{4}{N^2}\right]+\frac{N^2\lambda^2}{\omega^5}\left[144+\frac{224}{N^2}\right]+\mathcal{O}(\lambda^3)\end{aligned} \tag{2.19}$$

This is an expansion in λ and $\frac{1}{N^2}$ which are both small quantities in this limit. These small quantities are analogous to $\hbar$ in a QFT and so imply that this theory has two sources of intrinsic uncertainty. Because the aim is to show the equivalence to a string theory one can interpret the extra source of uncertainty as

coming from the fact that a string is 1 dimensional with a finite size and so cannot probe position with the precision that a point particle could. In other words on top of the fuzzy uncertainty that comes from the quantum nature of the theory, the finite size of the string will give rise to another uncertainty in how precisely position can be measured. This is because the probes in this theory are strings and their finite nature ensures that positions can never be known to a precision smaller than the order of the length of the string. Intuitively this follows since it is uncertain which part of the string has interacted.

It is clear that the terms with a N to the power of 2 are the main contributors in the $N \to \infty$ limit so one would like to restructure the tools of the theory so that one can easily write down correlators as a perturbative expansion in $\frac{1}{N^2}$ and λ instead of just g. In doing this one will also find a clear link between this theory and a string theory. Puttting λ in and change variables to:

$$M = \sqrt{N}M' \tag{2.20}$$

$$\Rightarrow -\frac{\omega}{2}\mathrm{Tr}(M^2) - g\mathrm{Tr}(M^4) = -\frac{\omega N}{2}\mathrm{Tr}(M'^2) - N\lambda\mathrm{Tr}(M'^4) \tag{2.21}$$

Since this is just a scaling factor and the measure goes over all M's the measure will not change.

$$\Rightarrow Z[J'] = \mathcal{N}\int [dM']e^{-\frac{\omega N}{2}\mathrm{Tr}(M'^2)-N\lambda\mathrm{Tr}(M'^4)+\mathrm{Tr}(J'M')} \tag{2.22}$$

This changes the Feynman rules so that each ribbon comes with a $\frac{1}{\omega N}$ and each vertex comes with a $-\lambda N$. Now one would like to be able to understand the N dependencies of each diagram. Let the number of ribbons be E, the number of closed loops be F and the number of vertices be V. Then the N dependance of a diagram will be N^{F-E+V}. Shifting perspective, imagine that the ribbon diagrams triangulate surfaces. In this view the ribbons correspond to edges and the closed loops correspond to faces, and the quantity $F - E + V = \chi$ is a topological invariant known as the Euler characteristic of the surface. It is also given by $\chi = 2 - 2h - b$ where h is the number of handles and b is the number of boundaries. This is interesting because it implies that the diagrams with a specific N dependence correspond to the triangulation of a surface of specific topology. Consider a diagram with a N^2 dependance and what surface it triangulates. Consider the following diagram

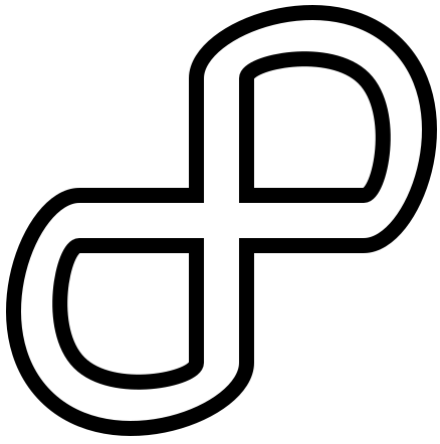


Figure 2.2: Figure of eight diagram

This diagram has two ribbons or edges, three closed loops or faces and one vertex $\Rightarrow \chi = F - E + V = 2$. As it turns out this value corresponds to a sphere as can be illustrated by this drawing:

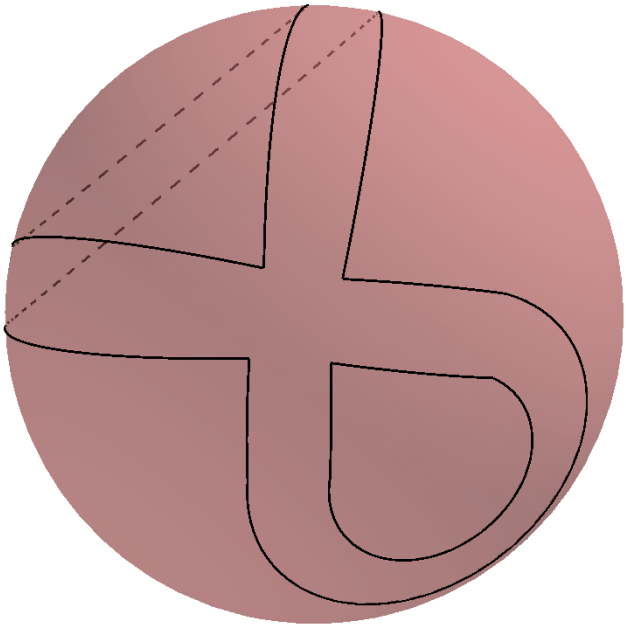


Figure 2.3: Triangulation of a sphere

This implies that all diagrams which triangulate spheres come with a N^2 dependance. Since it was seen that N^2 is the highest power of N a diagram can have one can conclude that the dominant diagrams will always triangulate spheres. These are also referred to as the planar diagrams. Now what about the lower power diagrams? It was seen that the N dependance drops by two powers in our expansion (Equation 2.19) but how does this fit into the surfaces picture? Consider the situation where the surface being triangulated by some ribbon diagram with a certain $\chi = F - E + V$ value. With a section of the surface looking as follows:

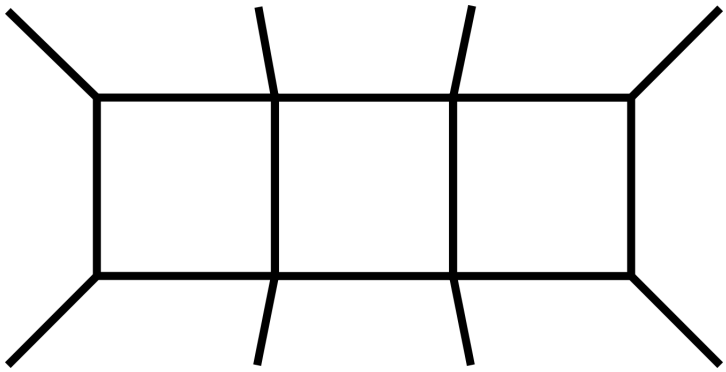


Figure 2.4: Section of a triangulated surface

Where each line corresponds to a ribbon. Now imagine cutting two holes (Figure 2.5) and gluing the edges of the holes together so that a new surface is formed.

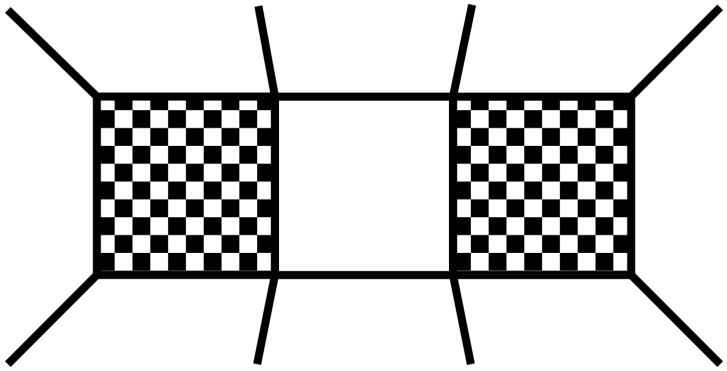


Figure 2.5: Section of a triangulated surface with two holes

This process reduces the number of edges and vertices by 4 and the number of faces by 2.

$$\begin{aligned} E' &= E - 4 \\ F' &= F - 2 \\ V' &= V - 4 \end{aligned} \tag{2.23}$$

$$\Rightarrow \chi' = F' - E' + V' = F - E + V - 2 = \chi - 2 \tag{2.24}$$

It corresponds to adding a handle to the surface and does indeed correspond to the desired reduction in the power of N that we were looking for. Topologically adding a handle is equivalent to increasing the genus of a surface. Adding a handle to a sphere (genus 0) creates a torus (genus 1), adding a handle to a torus makes a pretzel (genus 2) and so on. So that the genus of a surface is equivalent to the number of handles. This implies that the expansion can be understood as an expansion in the genuses of surfaces. This is very interesting because in a string theory the classical contribution to a string being created at x_1 and being detected at x_2 would map out a surface of genus 0 and the first quantum correction corresponding to one loop would map out a genus 1 surface, two loops a genus 2 surface and so on.

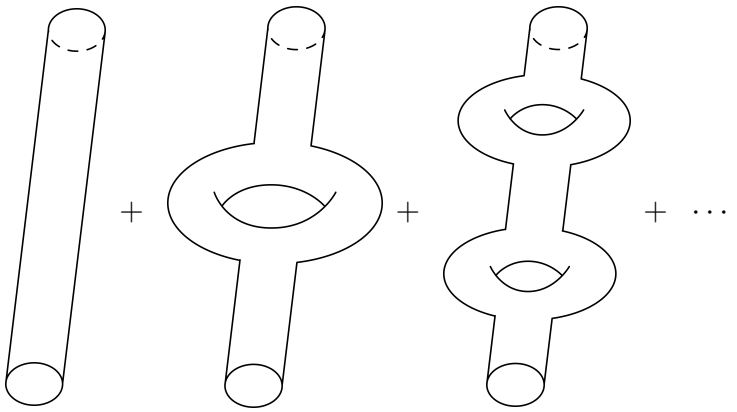


Figure 2.6: Perturbation expansion in terms of topologies of worldsheets in string theory

This has a significant implication: if one ignores the higher order corrections and only considers the planar contributions to correlators one should find the classical limit of the theory. In the classical limit one finds

that expectation values of products of observables are equivalent to the product of the expectation values of the individual observables. It is said that the theory exhibits factorization in this limit. This can be illustrated by some arbitrary system in state i with the probability of being in state i given by μ_i.

$$\Rightarrow \sum_i \mu_i = 1 \tag{2.25}$$

$$\mu_i \geq 0, \forall \mu_i \tag{2.26}$$

In this system there are a set of observables O_I whose value in state i is $O_I(i)$.

$$\Rightarrow \langle O_I \rangle = \sum_i \mu_i O_I(i) \tag{2.27}$$

Now this theory exhibits factorization in the classical limit because the expectation values of the observables in this limit behave in the following manner:

$$\langle O_{I_1} O_{I_2} ... O_{I_n} \rangle = \langle O_{I_1} \rangle \langle O_{I_2} \rangle ... \langle O_{I_n} \rangle \tag{2.28}$$

$$\Rightarrow \sum_i \mu_i O_{I_1}(i) O_{I_2}(i) ... O_{I_n}(i) = \sum_{i_1} \mu_{i_1} O_{I_1}(i_1) \sum_{i_2} \mu_{i_2} O_{I_2}(i_2) ... \sum_{i_n} \mu_{i_n} O_{I_n}(i_n) \tag{2.29}$$

This is only possible if there is only one accessible state i^*,

$$\Rightarrow \mu_i = \begin{cases} 1 & i = i^* \\ 0 & i \neq i^* \end{cases} \tag{2.30}$$

implying that this is indeed the classical limit of the theory.

Now consider the following correlators in the matrix model in the large N limit.

$$\langle \mathrm{Tr}(M^2) \rangle_{nn} = \frac{N^2}{\omega} - \frac{N^2 \lambda}{\omega^3} \left[8 + \frac{4}{N^2} \right] + \mathcal{O}(\lambda^2) \tag{2.31}$$

$$\cong \frac{N^2}{\omega} - \frac{8 N^2 \lambda}{\omega^3} + \mathcal{O}(\lambda^2) \tag{2.32}$$

and

$$\langle \mathrm{Tr}(M^2) \mathrm{Tr}(M^2) \rangle_{nn} = \frac{N^4}{\omega^2} \left[1 + \frac{2}{N^2} \right] - \frac{\lambda N^4}{\omega^4} \left[8 + \frac{52}{N^2} + \frac{24}{N^4} \right] + \mathcal{O}(\lambda^2) \tag{2.33}$$

$$\cong \frac{N^4}{\omega^2} - \frac{8 \lambda N^4}{\omega^4} + \mathcal{O}(\lambda^2) \tag{2.34}$$

$$= \langle \mathrm{Tr}(M^2) \rangle_{nn} \langle \mathrm{Tr}(M^2) \rangle_{nn} \tag{2.35}$$

This shows that factorization is exhibited in the large N limit and the theory reduces to a classical one. This discussion, together with our previous discussion of ribbon graphs, is substance for the claim of the equivalence between matrix models and a string theory.

What has been done so far corresponds to the singlet sector of matrix quantum mechanics of a single hermitian matrix[41]. One can also consider a complex matrix model containing a complex matrix Z as long as one is restricted to potentials that are analytic in Z (summed with the dagger of this which needs to be added to get a real potential) and observables constructed out of traces of a product of Zs or out of a product of $Z^\dagger$s [42]. In these situations one can reduce the problem to eigenvalue dynamics.

2.1.5 Number of Fields Comparable to N

A matrix model with one matrix field and observables built using a number of fields that does not grow parametrically with N has been studied. Here it is motivated why one cannot use the techniques developed for studying these observables at large N for observables whose number of fields grows parametrically with N (the non-planar large N limits of the theory [5]) and then outline the tools needed for studying these non-planar operators.

No mixing of trace structure When studying a complex matrix model with one complex field Z one can define physical operators $\mathcal{O}_i$ according to the trace structure of the operator. Let i correspond to the number of traces such that:

$$\begin{array}{ccc} \mathcal{O}_1(J) & \text{proportional to} & \mathrm{Tr}(Z^J) \\ \mathcal{O}_2(J_1,J_2) & \text{proportional to} & \mathrm{Tr}(Z^{J_1})\mathrm{Tr}(Z^{J_2}) \\ \mathcal{O}_3(J_1,J_2,J_3) & \text{proportional to} & \mathrm{Tr}(Z^{J_1})\mathrm{Tr}(Z^{J_2})\mathrm{Tr}(Z^{J_3}) \\ & \vdots & \end{array} \tag{2.36}$$

If one defines the operators so that the $\mathcal{O}_{A_1}$ is normalised to have a unit two point function in the large N limit, they have the following form:

$$\begin{array}{c} \mathcal{O}_1(n) = \frac{1}{\sqrt{nN^n}}\mathrm{Tr}(Z^n) \\ \mathcal{O}_2(n,m) = \mathcal{O}_1(n)\mathcal{O}_1(m) = \frac{1}{\sqrt{nmN^{n+m}}}\mathrm{Tr}(Z^n)\mathrm{Tr}(Z^m) \\ \vdots \end{array} \tag{2.37}$$

This implies that in the large N limit one has:

$$\left\langle \mathcal{O}_1(n)\mathcal{O}_1^\dagger(m)\right\rangle = \delta_{nm}\left(1+O\left(\frac{1}{N^2}\right)\right), \tag{2.38}$$

and

$$\left\langle \mathcal{O}_1(n)\mathcal{O}_1(m)\mathcal{O}_2^\dagger(n,m)\right\rangle = \frac{\sqrt{nm(m+n)}}{N}\left(1+O\left(\frac{1}{N^2}\right)\right). \tag{2.39}$$

From this one can clearly see that the right hand side of the latter result vanishes in the large N limit if the number of fields (n and m) are fixed as $N \longrightarrow \infty$. This is in fact a general result that different types of trace structures do not mix in the large N limit. This means that the set of observable quantities in the theory reduces to correlators of the following form:

$$\left\langle \mathcal{O}_i\mathcal{O}_i^\dagger\right\rangle \tag{2.40}$$

This however, would obviously not hold when n and m grow comparably to N and so all trace structures must be considered together in the large N limit.

Now consider correlators of this form:

$$\left\langle \mathrm{Tr}(Z^J)\mathrm{Tr}(Z^{\dagger J})\right\rangle \tag{2.41}$$

The planar contribution is:

$$\left\langle \mathrm{Tr}(Z^J)\mathrm{Tr}(Z^{\dagger J})\right\rangle = JN^J + \text{higher genus contributions}, \tag{2.42}$$

and it can be shown that the genus h contribution is given by:

$$\frac{1\cdot 3\cdot 5\cdots(4h-1)}{2h+1}J\frac{J(J-1)(J-2)\cdots(J-4h+1)}{(4h)!}N^{J-2h} \tag{2.43}$$

$$\approx\frac{1\cdot 3\cdot 5\cdots(4h-1)}{2h+1}\frac{J^{1+4h}N^{J-2h}}{(4h)!}. \tag{2.44}$$

From this it is clear that for J's that grow more slowly than $\sqrt{N}$ only planar (genus 0) diagrams will contribute in the large N limit. This drastically reduces the amount of work that needs to be done to calculate correlators because one only needs to sum planar ribbon graphs. This is not necessarily trivial in some cases but it is doable. If however, the number of fields grow parametrically with N such that $J = N^p$ the genus 0 contribution will grow like:

$$N^{p+J}, \tag{2.45}$$

and the genus h contributions will grow like:

$$N^{p+4ph+J-2h}. \tag{2.46}$$

So one can see that for $p < \frac{1}{2}$ the genus 0 contributions will dominate but if $p \geq \frac{1}{2}$ then the higher genus contributions will be the same as the genus 0 contributions at $p = \frac{1}{2}$ and dominate when $p > \frac{1}{2}$. This implies that in order to get any sort of realistic prediction one has to sum all diagrams and because there are N fields and $N \longrightarrow \infty$ there will also be an infinite number of diagrams. Thus, the drawing of ribbon graphs can not be used to calculate correlators anymore and one needs a new technique. Luckily there is a technique that will aid us in this endeavor. It is the Schur polynomials which have their roots in group theory.

2.1.6 Schur polynomials and group theory

In this section the goal is to show how the Schur polynomials can be used to sum all possible ribbon graphs, not just planar ones and that any multi-trace operator can be represented as a linear combination of Schur polynomials. Thus, using the Schur polynomials any correlator can be calculated with finite or large N. To this end it is necessary to recall some group theory results [43] and define some notation.

Group theory results and notation The symmetric group S_n is the group of all permutations of n distinct objects. Cycle notation is used to represent the group elements. For example for the S_3 permutation of the group element (a,b,c) to (c,a,b) is written as (1 2 3) in cycle notation and is understood as the element in position 1 goes to position 2, the element in position 2 goes to position 3 and the element in position 3 goes to position 1. The rearrangement (c,a,b) is achieved by the group element (1 2 3).

Representations: A representation ($\Gamma(g)$) of a group G is a set of matrices, one for each element of the group, such that

$$\Gamma(g_1)\cdot\Gamma(g_2) = \Gamma(g_1\cdot g_2) \quad \forall g_1, g_2 \in G \tag{2.47}$$

The space on which these matrices act is called the carrier space of the representation. There are an

infinite number of representations for a group but most are equivalent or reducible representations. An equivalent representation ($\Gamma_2(g)$) of $\Gamma_1(g)$ is related to $\Gamma_1(g)$ as follows:

$$\Gamma_2(g) = M\Gamma_1(g)M^{-1} \tag{2.48}$$

where M is some invertible matrix. A quick way to check whether two representations are equivalent is to check if their characters are the same for all elements of the group. The character of the representation r ($\chi_r(g)$) is defined as

$$\chi_r(g) = \mathrm{Tr}(\Gamma_r(g)) \tag{2.49}$$

So the check of equivalence between two group representations ($\Gamma_r(g)$ and $\Gamma_s(g)$) of group G becomes

$$\chi_r(g) = \chi_s(g) \quad \forall g \in G \tag{2.50}$$

A representation is said to be reducible if it is equivalent to a block diagonal representation. An equivalent statement is: a representation is irreducible if it has no invariant subspaces. The list of inequivalent, irreducible representation of a group are called the irreps of a group. The list of irreps of a group is finite for a finite group and infinite for an infinite group.

Labeling the irreps of a group: The irreps of the symmetric group S_n are labeled by Young diagrams that are n boxes arranged in a specific order. The rules for the boxes are: all boxes must be arranged in rows starting from a common vertical line and there can not be more boxes in subsequent rows, than previous rows. An example of a valid diagram for S_6 is:

An example of an invalid diagram for S_6 is:

Similarly labels for the irreps of the unitary group ($U(N)$) can be defined. The diagrams look the same and have the same rules except that the diagrams can have any total number of boxes (instead of only n boxes as with S_n) but may only have a maximum of N rows.

These Young diagrams can be used to determine the dimensions of the representations and the states of the carrier space of the representations. One gets the dimension of reps as follows:
For a representation of S_n one notates the dimension of representation R as d_R. It is found by dividing $n!$ by the product of the hook lengths (hooks). The hooks are defined as follows:

1. Draw a line starting from below the Young diagram into a box then make a perpendicular elbow from that line out of the diagram to the right.

2. The hook length of the box with the elbow is the number of boxes the line goes through in its path through the diagram.

The following shows the hook lengths of a diagram and the corresponding dimension of the representation.

5	3	1
3	1	
1		

$$d_{\ydiagram{3,1}} = \frac{6!}{5 \cdot 3 \cdot 3 \cdot 1 \cdot 1 \cdot 1} = 16 \tag{2.51}$$

Now when working with the unitary group the dimension of a representation is notated as $\mathrm{Dim}\, R$ and it is calculated by dividing the product of the factors (f_R) by the product of the hooks. The factors are defined as follows:

> Start at the top, leftmost box. Its factor is N. Then every consecutive box to the right has a factor 1 larger than the box to its left and every consecutive box down has a factor of 1 less then the box above it.

The factors of a representation of $U(N)$ are shown here with the corresponding dimension.

N	$N+1$	$N+2$
$N-1$	N	
$N-2$		

$$\mathrm{Dim}_R = \frac{f_R}{\prod \text{hooks}} = \frac{N \cdot N \cdot (N-2)(N-1)(N+1)(N+2)}{45} \tag{2.52}$$

These diagrams can now be used to find the actual irreps of S_n themselves. This is done with Young-Yamanouchi (YY) patterns. YY patterns are constructed by writing all the possible ordered ways of dropping boxes from a Young diagram that leave a valid Young diagram after each drop. For instance consider the irrep of S_4 labeled by $\ydiagram{3,1}$ ($\Gamma_{\ydiagram{3,1}}(\sigma)$), the YY patterns will be

$$\begin{array}{|c|c|c|}\hline 4&2&1\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 4&3&1\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 4&3&2\\\hline 1\\\cline{1-1}\end{array} \tag{2.53}$$

where the numbers represent the order of dropping boxes. These YY patterns are considered states that when acted on by $\Gamma_{\ydiagram{3,1}}(\sigma)$, where $\sigma \in$ adjacent 2-cycles of S_4, behave in a specific way. Let a specific state of the irrep R of S_n be $|R\rangle$ and $|R_{(i,j)}\rangle$ be the state were box i and j have been swaped. Then

$$\Gamma_R((i,i+1))|R\rangle = \frac{1}{c_i - c_{i+1}}|R\rangle + \sqrt{1 - \frac{1}{(c_i - c_{i+1})^2}}|R_{(i,i+1)}\rangle \tag{2.54}$$

where c_i is the content of box i. The content of a diagram is defined in much the same way as the factors of the irreps of $U(N)$ are defined except that one starts with 0 in the top left box instead of N. For example the content of $\ydiagram{3,1}$ is

0	1	2
-1	0	
-2		

(2.55)

Doing this for all the states of the irrep R gives equations that can be solved to find the matrix elements of the adjacent 2-cycles of $\Gamma_R(\sigma)$. Then all the other elements can be written as a product of the adjacent 2-cycles of the group.

Notation: Just like in special relativity where the position of an index tells us how a quantity transforms under the action of the Lorentz group we want to use the position of the indices here to tell us how

quantities transform under the action of the unitary group. Quantities with an upper index transform with U while quantities with lower indices transform with $U^\dagger$. So column vectors (notated as kets $|v\rangle$) will transform with U and row vectors (notated as bras $\langle v|$) will transform with $U^\dagger$. So the action of the unitary group will look as follows:

$$\Rightarrow U|v\rangle \equiv U^i{}_j v^j \tag{2.56}$$

$$\langle v|U^\dagger \equiv U^*{}_i{}^j v_j \tag{2.57}$$

Now Z is an $N \times N$ complex matrix and we declare that the action of the unitary group on Z will be

$$Z \longrightarrow UZU^\dagger \tag{2.58}$$

which implies that Z must be written with upper and lower indices (Z^i_j) so that

$$\Rightarrow U^i_j Z^j_k (U^\dagger)^k_l \longrightarrow Z^i_l \tag{2.59}$$

Z is an endomorphism of the N dimensional vector space V_N. Tensoring n Z's produces an operator $Z^{\otimes n} = Z \otimes Z \otimes \cdots \otimes Z$ that acts on $V_N^{\otimes n}$. The following index notation is used:

$$(Z^{\otimes n})^I_J = Z^{i_1}_{j_1} Z^{i_2}_{j_2} \cdots Z^{i_n}_{j_n} \tag{2.60}$$

The symmetric group S_n has a natural action on $V_N^{\otimes n}$given by:

$$(\sigma)^I_J = \delta^{i_1}_{j_{\sigma(1)}} \delta^{i_1}_{j_{\sigma(2)}} \cdots \delta^{i_n}_{j_{\sigma(n)}} \quad \sigma \in S_n \tag{2.61}$$

Using this notation:

$$\mathrm{Tr}(\sigma Z^{\otimes n}) = (\sigma)^I_J (Z^{\otimes n})^J_I = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \cdots Z^{i_n}_{i_{\sigma(n)}} \tag{2.62}$$

Consider $n = 4$ with $\sigma = (1)(2)(34)$:

$$\mathrm{Tr}((1)(2)(34) Z^{\otimes 4}) = Z^{i_1}_{i_1} Z^{i_2}_{i_2} Z^{i_3}_{i_4} Z^{i_4}_{i_3} = \mathrm{Tr}(Z)^2 \mathrm{Tr}(Z^2) \tag{2.63}$$

This is thus a unified notation for representing multi-trace operators of n fields.

The power of this notation can be illustrated with free field correlators. Consider the following correlator:

$$\left\langle Z^{i_1}_{j_1} Z^{i_2}_{j_2} \cdots Z^{i_n}_{j_n} Z^{\dagger k_1}_{l_1} \cdots Z^{\dagger k_n}_{l_n} \right\rangle = \left\langle (Z^{\otimes n})^I_J (Z^{\dagger \otimes n})^K_L \right\rangle \tag{2.64}$$

There are a total of $n!$ Wick contractions that need to be summed. The sum of Wick contractions can however be represented as a sum of actions by S_n. To illustrate this consider the following Wick contraction with $n = 3$

$$\begin{aligned} &\langle \overbrace{Z^{i_1}_{j_1} \overbrace{Z^{i_2}_{j_2} \underbrace{Z^{i_3}_{j_3} (Z^\dagger)^{k_1}_{l_1}}} \underbrace{(Z^\dagger)^{k_2}_{l_2}}} (Z^\dagger)^{k_3}_{l_3} \rangle \\ &= \delta^{i_1}_{l_3} \delta^{i_2}_{l_1} \delta^{i_3}_{l_2} \delta^{k_3}_{j_1} \delta^{k_1}_{j_2} \delta^{k_2}_{j_3} \\ &= ((132))^I_L \left((132)^{-1}\right)^K_J \end{aligned} \tag{2.65}$$

Thus, one can deduce that

$$\left\langle (Z^{\otimes n})^I_J (Z^{\dagger \otimes n})^K_L \right\rangle = \sum_{\sigma \in S_n} (\sigma)^I_L (\sigma^{-1})^K_J \tag{2.66}$$

Projection operators Now that the results and notation necessary have been defined the work towards summing all possible ribbon graphs can begin. To that end define an operator

$$(\hat{P}_R)^I_J = \frac{d_R}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma)(\sigma)^I_J \tag{2.67}$$

where R labels the irrep and d_R is the dimension of the irrep; put there for normalisation. This operator acts on $V_N^{\otimes n}$ and is in fact a projection operator as can be seen from the following.

Working in S_2 with irrep $R = \square\square$. Consider the operator above acting on $(v \otimes w)$ where $v, w \in V_N$

$$\begin{aligned}
(\hat{P}_{\square\square})^I_J (v \otimes w)^J =& (\hat{P}_{\square\square})^{i_1 i_2}_{j_1 j_2} v^{j_1} w^{j_2} \\
=& \frac{1}{2} \left[\chi_{\square\square}(\mathbb{I})(\mathbb{I})^{i_1 i_2}_{j_1 j_2} + \chi_{\square\square}((12))((12))^{i_1 i_2}_{j_2 j_1} \right] v^{j_1} w^{j_2} \\
=& \frac{1}{2} \left[1 \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} + 1 \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} \right] v^{j_1} w^{j_2} \\
=& \frac{1}{2} \left[v^{i_1} w^{i_2} + v^{i_2} w^{i_1} \right]
\end{aligned} \tag{2.68}$$

This has clearly projected onto the symmetric part of $(v \otimes w)$. This is in fact always the case when the irrep of the projector is a row of boxes. If on the other hand the irrep is a column of boxes it projects onto the antisymmetric part as can be seen if we choose $R = \begin{smallmatrix}\square\\\square\end{smallmatrix}$.

$$\begin{aligned}
(\hat{P}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}})^I_J (v \otimes w)^J =& (\hat{P}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}})^{i_1 i_2}_{j_1 j_2} v^{j_1} w^{j_2} \\
=& \frac{1}{2} \left[\chi_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(\mathbb{I})(\mathbb{I})^{i_1 i_2}_{j_1 j_2} + \chi_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}((12))((12))^{i_1 i_2}_{j_2 j_1} \right] v^{j_1} w^{j_2} \\
=& \frac{1}{2} \left[1 \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} + (-1) \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} \right] v^{j_1} w^{j_2} \\
=& \frac{1}{2} \left[v^{i_1} w^{i_2} - v^{i_2} w^{i_1} \right]
\end{aligned} \tag{2.69}$$

Now if one acts with the same projector twice it should be the same as acting once.

$$\begin{aligned}
(\hat{P}_{\square\square})^I_J (\hat{P}_{\square\square})^J_K =& \frac{1}{4} \left[\delta^{i_1}_{j_1} \delta^{i_2}_{j_2} + \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} \right] \left[\delta^{j_1}_{k_1} \delta^{j_2}_{k_2} + \delta^{j_1}_{k_2} \delta^{j_2}_{k_1} \right] \\
=& \frac{1}{4} \left[\delta^{i_1}_{k_1} \delta^{i_2}_{k_2} + \delta^{i_1}_{k_2} \delta^{i_2}_{k_1} + \delta^{i_1}_{k_2} \delta^{i_2}_{k_1} + \delta^{i_1}_{k_1} \delta^{i_2}_{k_2} \right] \\
=& \frac{1}{2} \left[\delta^{i_1}_{k_1} \delta^{i_2}_{k_2} + \delta^{i_1}_{k_2} \delta^{i_2}_{k_1} \right] \\
=& (\hat{P}_{\square\square})^I_K
\end{aligned} \tag{2.70}$$

This is a defining property of a projector. It holds for all irreps. Projectors of different irreps are also orthogonal. Consider acting with two projectors one of irrep R and the other S.

$$\begin{aligned}(\hat{P}_R)^I_J(\hat{P}_S)^J_K =&\frac{d_R}{n!}\sum_{\sigma_1\in S_n}\chi_R(\sigma_1)(\sigma_1)^I_J\frac{d_S}{n!}\sum_{\sigma_2\in S_n}\chi_S(\sigma_2)(\sigma_2)^J_K\\ =&\frac{d_R d_S}{n!\,n!}\sum_{\sigma_1\in S_n}\sum_{\sigma_2\in S_n}\chi_R(\sigma_1)\chi_S(\sigma_2)(\sigma_1)^I_J(\sigma_2)^J_K\\ =&\frac{d_R d_S}{n!\,n!}\sum_{\sigma_1\in S_n}\sum_{\sigma_2\in S_n}\chi_R(\sigma_1)\chi_S(\sigma_2)(\sigma_2\sigma_1)^I_K\end{aligned}\tag{2.71}$$

change variables:

$$\sigma_1,\sigma_2\longrightarrow\sigma_1,\psi=\sigma_2\sigma_1\tag{2.72}$$

$$\Rightarrow(\psi)^I_K=(\sigma_2\sigma_1)^I_K=(\sigma_1)^I_J(\sigma_2)^J_K\tag{2.73}$$

$$\Rightarrow(\sigma_1^{-1})^L_I(\psi)^I_K=(\sigma_1^{-1})^L_I(\sigma_1)^I_J(\sigma_2)^J_K=(\sigma_2)^L_K\tag{2.74}$$

$$\Rightarrow(\sigma_2)^L_K=(\psi\sigma_1^{-1})^L_K,\tag{2.75}$$

because S_n is a finite group the sum over ψ and $\psi\sigma_1^{-1}$ both sum every element in the group. So this can be written as

$$\begin{aligned}\Rightarrow(\hat{P}_R)^I_J(\hat{P}_S)^J_K =&\frac{d_R d_S}{n!\,n!}\sum_{\sigma_1\in S_n}\sum_{\psi\in S_n}\chi_R(\sigma_1)\chi_S(\psi\sigma_1^{-1})(\psi)^I_K\\ =&\frac{d_R d_S}{n!\,n!}\sum_{\sigma_1\in S_n}\sum_{\psi\in S_n}(\Gamma_R(\sigma_1))^A_A(\Gamma_S(\psi\sigma_1^{-1}))^B_B(\psi)^I_K\\ =&\frac{d_R d_S}{n!\,n!}\sum_{\sigma_1\in S_n}\sum_{\psi\in S_n}(\Gamma_R(\sigma_1))^A_A(\Gamma_S(\psi))^B_C(\Gamma_S(\sigma_1^{-1}))^C_B(\psi)^I_K\\ =&\frac{d_R d_S}{n!\,n!}\sum_{\sigma_1\in S_n}(\Gamma_R(\sigma_1))^A_A(\Gamma_S(\sigma_1^{-1}))^C_B\sum_{\psi\in S_n}(\Gamma_S(\psi))^B_C(\psi)^I_K\end{aligned}\tag{2.76}$$

The fundamental orthogonality relation states that

$$\sum_{g\in G}\Gamma_R(g^{-1})^A_B\Gamma_S(g)^C_D=\frac{|G|}{d_R}\delta_{RS}(\delta)^A_D(\delta)^C_B\tag{2.77}$$

Applying this to (2.76) where $|S_n|=n!$ gives

$$\begin{aligned}(\hat{P}_R)^I_J(\hat{P}_S)^J_K =&\frac{d_R d_S}{n!\,n!}\frac{n!}{d_R}\delta_{RS}(\delta)^A_B(\delta)^C_A\sum_{\psi\in S_n}(\Gamma_S(\psi))^B_C(\psi)^I_K\\ =&\delta_{RS}\frac{d_S}{n!}\sum_{\psi\in S_n}(\Gamma_S(\psi))^B_B(\psi)^I_K\\ =&\delta_{RS}(\hat{P}_S)^I_K\\ =&\delta_{RS}(\hat{P}_R)^I_K\end{aligned}\tag{2.78}$$

It will be useful to know if $\hat{P}_R$ commutes with the elements of the symmetric group ($\psi \in S_n$). So consider the following

$$(\hat{P}_R)^I_J(\psi)^J_K = \frac{d_R}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)(\sigma)^I_J(\psi)^J_K \tag{2.79}$$

Change σ to its conjugate τ related as follows

$$(\sigma)^I_J = \psi^I_L(\tau)^L_K(\psi^{-1})^K_J = (\psi^{-1}\tau\psi)^I_J \tag{2.80}$$

$$\begin{aligned}\Rightarrow (\hat{P}_R)^I_J(\psi)^J_K =& \frac{d_R}{n!}\sum_{\tau\in S_n}\chi_R(\psi^{-1}\tau\psi)(\psi)^I_L(\tau)^L_B(\psi^{-1})^B_J(\psi)^J_K \\ =& \frac{d_R}{n!}\sum_{\tau\in S_n}\chi_R(\psi^{-1}\tau\psi)(\psi)^I_L(\tau)^L_K \end{aligned}\tag{2.81}$$

The characters of the elements of a conjugacy class are the same since the trace has a cyclic symmetry. Consider the conjugacy class defined by $h = \sigma g\sigma^{-1}$. Then the characters of h and g are the same.

$$\begin{aligned}\chi_R(h) =& \mathrm{Tr}(\Gamma_R(\sigma g\sigma^{-1})) \\ =& \mathrm{Tr}(\Gamma_R(\sigma^{-1})\Gamma_R(g)\Gamma_R(\sigma)) \\ =& \mathrm{Tr}(\Gamma_R(\sigma)\Gamma_R(\sigma^{-1})\Gamma_R(g)) \\ =& \mathrm{Tr}(\Gamma_R(g)) \qquad (2.82)\\ =& \chi_R(g) \qquad (2.83)\end{aligned}$$

$$\Rightarrow (\hat{P}_R)^I_J(\psi)^J_K = (\psi)^I_L\frac{d_R}{n!}\sum_{\tau\in S_n}\chi_R(\tau)(\tau)^L_K = (\psi)^I_L(\hat{P}_R)^L_K \tag{2.84}$$

$$\Rightarrow \hat{P}_R\psi = \psi\hat{P}_R \tag{2.85}$$

So $\hat{P}_R$ commutes with the elements of the symmetric group.

2.1.7 Schur Polynomials

We are now in a position where we can define the Schur polynomials. The Schur polynomials for irrep R ($\chi_R(Z)$) are defined as:

$$\chi_{_R}(Z) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi_{_R}(\sigma)\mathrm{Tr}(\sigma Z^{\otimes n}) = \mathrm{Tr}(P_R Z^{\otimes n}) \tag{2.86}$$

where $P_R = \frac{\hat{P}_R}{d_R}$ in the un-normalized projection operator. Examine this for $R = \square\square\square$ for S_3.

$$\chi_{\square\square\square}(Z) = \frac{1}{6}\sum_{\sigma\in S_n}\chi_{\square\square\square}(\sigma)\mathrm{Tr}(\sigma Z^{\otimes 3}) \tag{2.87}$$

$$= \frac{1}{6}\left[Z^{i_1}_{i_1}Z^{i_2}_{i_2}Z^{i_3}_{i_3} + Z^{i_1}_{i_2}Z^{i_2}_{i_1}Z^{i_3}_{i_3} + Z^{i_1}_{i_1}Z^{i_2}_{i_3}Z^{i_3}_{i_2} + Z^{i_1}_{i_3}Z^{i_2}_{i_2}Z^{i_3}_{i_1} + Z^{i_1}_{i_2}Z^{i_2}_{i_3}Z^{i_3}_{i_1} + Z^{i_1}_{i_3}Z^{i_2}_{i_1}Z^{i_3}_{i_2}\right] \tag{2.88}$$

$$= \frac{1}{6}\left[\mathrm{Tr}(Z)^3 + \mathrm{Tr}(Z)\mathrm{Tr}(Z^2) + \mathrm{Tr}(Z)\mathrm{Tr}(Z^2) + \mathrm{Tr}(Z)\mathrm{Tr}(Z^2) + \mathrm{Tr}(Z^3) + \mathrm{Tr}(Z^2)\right] \tag{2.89}$$

$$= \frac{1}{6}\left[\mathrm{Tr}(Z)^3 + 3\mathrm{Tr}(Z)\mathrm{Tr}(Z^2) + 2\mathrm{Tr}(Z^3)\right] \tag{2.90}$$

The Schur polynomials are linear combinations of multi-trace operators and it should be noted that these multi-trace operators are related to the conjugacy classes of S_3. The conjugacy classes of the symmetric group are determined by the cycle structure. For instance the cycle structure of (1)(23), (3)(12) and (2)(13) are all $1^1 2^1$ in other words 1 times 1-cycle and 1 times 2-cycle. The conjugacy classes of S_3 are 1^3, $1^1 2^1$ and 3^1 and there is one element in 1^3, three elements in $1^1 2^1$ and 2 elements in 3^1. Each conjugacy class corresponds to an operator in which a 1-cycle corresponds to the trace of a single Z, a 2-cycle corresponds to the trace of a product of two Z's and so on.

The two point correlation function can be calculated as follows [10, 11]

$$\begin{aligned}\left\langle \chi_{_R}(Z)\chi^\dagger_{_S}(Z)\right\rangle =& \frac{1}{d_R d_S}(\hat{P}_R)^I_J(\hat{P}_S)^K_L \left\langle (Z^{\otimes n})^J_I (Z^{\dagger\otimes n})^L_K \right\rangle \\ =& \frac{1}{d_R d_S}\sum_{\sigma\in S_n}(\hat{P}_R)^I_J(\hat{P}_S)^K_L(\sigma)^L_I(\sigma^{-1})^J_K \\ =& \frac{1}{d_R d_S}\sum_{\sigma\in S_n}\mathrm{Tr}(\hat{P}\sigma^{-1}\hat{P}_S\sigma)\end{aligned} \tag{2.91}$$

As shown above $\hat{P}_S$ and σ commute so the σ's cancel and one is left with

$$\begin{aligned}\left\langle \chi_{_R}(Z)\chi^\dagger_{_S}(Z)\right\rangle =& \frac{1}{d_R d_S}\sum_{\sigma\in S_n}\mathrm{Tr}(\hat{P}_R\sigma^{-1}\hat{P}_S\sigma) \\ =& \frac{1}{d_R d_S}\mathrm{Tr}(\hat{P}_R\hat{P}_S)\sum_{\sigma\in S_n} \\ =& \frac{n!}{d_R d_S}\delta_{RS}\mathrm{Tr}(\hat{P}_R)\end{aligned} \tag{2.92}$$

Now $\mathrm{Tr}(\hat{P}_R)$ is just the dimension of the subspace that is projected to which in this case is $d_R\mathrm{Dim}_R$. So one has

$$\begin{aligned}\left\langle \chi_{_R}(Z)\chi^\dagger_{_S}(Z)\right\rangle =& \frac{n!}{d_R d_S}\delta_{RS}d_R\mathrm{Dim}_R = \delta_{RS}\frac{n!}{d_R}\mathrm{Dim}_R \\ =& \delta_{RS}f_R\end{aligned} \tag{2.93}$$

This is diagonal and so forms an orthogonal basis. This was first achieved in [10, 11].

For example the result for $R = S = \square\square\square$ will simply be

$$\left\langle \chi_{\square\square\square}(Z)\chi^\dagger_{\square\square\square}(Z)\right\rangle = N(N+1)(N+2) \tag{2.94}$$

which is in exact agreement with what is obtained by summing all ribbon graphs of

$$\left\langle \chi_{\square\square\square}(Z)\chi^\dagger_{\square\square\square}(Z)\right\rangle = \frac{1}{36}\left\langle \left[\mathrm{Tr}(Z)^3 + 3\mathrm{Tr}(Z)\mathrm{Tr}(Z^2) + 2\mathrm{Tr}(Z^3)\right]\left[\mathrm{Tr}(Z^\dagger)^3 + 3\mathrm{Tr}(Z^\dagger)\mathrm{Tr}(Z^{\dagger 2}) + 2\mathrm{Tr}(Z^{\dagger 3})\right]\right\rangle \tag{2.95}$$

Group of translations and the Fourier transform: Consider the group of translations $T(x)$ that translate by a distance x. They can be represented by plane waves since

$$T(a)T(b) = T(a+b) \tag{2.96}$$

and

$$e^{ika}e^{ikb} = e^{ik(a+b)} \tag{2.97}$$

These representations are labeled by k and each different k is a different irreducible, inequivalent representation as can be seen by the following. Consider a representation of $T(x)$ labeled by k and an invertible matrix M then

$$Me^{ikx}M^{-1} = e^{ikx}MM^{-1} = e^{ikx} \tag{2.98}$$

which implies that a representation can only be equivalent to itself. Also, they are irreducible since they are 1 dimensional representations. Now consider three group elements of $T(\cdot)$: $T(a)$, $T(b)$ and $T(c)$. If $T(a)$ and $T(b)$ are related as follows

$$T(a) = T(c)T(b)T^{-1}(c) = T(c)T(b)T(-c) = T(b)T(c)T(-c) = T(b) \tag{2.99}$$

This shows that the group of translations is abelian and that each element is its own conjugacy class. This is the case for every abelian group. Now because these representations are 1 dimensional they are also their own character.

$$\chi_k(x) = e^{ikx} \tag{2.100}$$

One can show that the characters of different representations are orthogonal as was shown before by summing the characters of two different irreps over all the group elements. In this case the space of group elements is continuous and so the sum is an integral.

$$\int_{-\infty}^{\infty} dx \chi_k(x)\chi_{k'}(x) = \int_{-\infty}^{\infty} dx e^{ikx}e^{ik'x} = 2\pi\delta(k-k') \tag{2.101}$$

This is a well known identity for plane waves and when viewed from the perspective of group theory is quite insightful. Another well known identity for plane waves is

$$\int_{-\infty}^{\infty} dk \chi_k(x)\chi_k(x') = \int_{-\infty}^{\infty} dk e^{ikx}e^{ikx'} = 2\pi\delta(x-x') \tag{2.102}$$

What this identity shows is that plane waves span the space. Another well known mathematical tool involving plane waves is plane wave decomposition or the Fourier transform. It shows that any arbitrary function ($F(k)$) can be represented as a "linear combination" of plane waves

$$F(k) = \int_{-\infty}^{\infty} dx e^{ikx} f(x) \tag{2.103}$$

where $f(x)$ is an arbitrary function of x (which is the conjugacy class) and can be thought of as the coefficients of the "linear combination." The inverse of this is also true

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx} F(k) \tag{2.104}$$

This shows that functions that take their values on the conjugacy classes (called class functions) can be represented as a sum of characters.

One would like to try and apply these ideas for the characters of the symmetric group. In other words one would like to develop the notion of a Fourier transform and inverse Fourier transform for the symmetric group characters. Consider the Schur polynomials

$$\chi_{_R}(Z) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi_{_R}(\sigma)\mathrm{Tr}(\sigma Z^{\otimes n}) \tag{2.105}$$

Here $\chi_{_R}(Z)$ is in fact an arbitrary function of the irrep by its dependance on R and as seen, $\mathrm{Tr}(\sigma Z^{\otimes n})$ is a function on the conjugacy class. So the Schur polynomials are in fact the notion of a Fourier transform for the symmetric group. So can one find the inverse? In other words does this hold

$$\mathrm{Tr}(\sigma Z^{\otimes n}) = A\sum_{R\vdash n}\chi_{_R}(\sigma^{-1})\chi_{_R}(Z) \tag{2.106}$$

and for what A does it hold? Putting $\mathrm{Tr}(\sigma Z^{\otimes n})$back into (2.105).

$$\begin{aligned}\chi_{_R}(Z) =&\frac{1}{n!}\sum_{\sigma\in S_n}\chi_{_R}(\sigma)A\sum_{R'\vdash n}\chi_{_{R'}}(\sigma^{-1})\chi_{_{R'}}(Z)\\ =&\frac{A}{n!}\sum_{R'\vdash n}\chi_{_{R'}}(Z)\sum_{\sigma\in S_n}\chi_{_R}(\sigma)\chi_{_{R'}}(\sigma^{-1})\end{aligned} \tag{2.107}$$

Taking the traces of the fundamental orthogonality relation gives

$$\sum_{\sigma\in S_n}\chi_{_{R'}}(\sigma^{-1})\chi_{_R}(\sigma) = \frac{|S_n|}{d_R}\delta_{RR'}(\delta)^A_B(\delta)^B_A = \frac{n!}{d_R}\delta_{RR'}d_R = n!\,\delta_{RR'} \tag{2.108}$$

$$\begin{aligned}\Rightarrow \chi_{_R}(Z) =&\frac{A}{n!}\sum_{R'\vdash n}\chi_{_{R'}}(Z)n!\,\delta_{RR'}\\ =&A\sum_{R'\vdash n}\chi_{_{R'}}(Z)\delta_{RR'}\\ =&A\chi_{_R}(Z)\end{aligned} \tag{2.109}$$

$$\Rightarrow A = 1 \tag{2.110}$$

So one has

$$\mathrm{Tr}(\sigma Z^{\otimes n}) = \sum_{R\vdash n}\chi_{_R}(\sigma^{-1})\chi_{_R}(Z) \tag{2.111}$$

and since it is known from above that the left hand side can be any multi-trace operator this equation proves that any multi-trace operator can be represented as a linear combination of Schur polynomials.

2.1.8 Two Matrices

Here the dynamics of two general complex matrices is considered. Expectation values in this model are computed as follows

$$\langle \cdots \rangle = \int [dZ][dZ^\dagger][dY][dY^\dagger] e^{-\mathrm{Tr}(ZZ^\dagger)-\mathrm{Tr}(YY^\dagger)} \cdots \tag{2.112}$$

So one will have

$$\left\langle (Z^{\otimes n})^I_J (Z^{\dagger\otimes n})^K_L \right\rangle = \sum_{\sigma\in S_n} (\sigma)^I_L (\sigma^{-1})^K_J = \left\langle (Y^{\otimes n})^I_J (Y^{\dagger\otimes n})^K_L \right\rangle \tag{2.113}$$

$$\left\langle (Z^{\otimes n})^I_J (Z^{\otimes n})^K_L \right\rangle = \left\langle (Y^{\otimes n})^I_J (Y^{\otimes n})^K_L \right\rangle = \left\langle (Z^{\otimes n})^I_J (Y^{\otimes n})^K_L \right\rangle = 0 \tag{2.114}$$

$$\left\langle (Z^{\otimes n})^I_J (Y^{\dagger\otimes n})^K_L \right\rangle = \left\langle (Z^{\dagger\otimes n})^I_J (Y^{\otimes n})^K_L \right\rangle = 0 \tag{2.115}$$

and

$$\left\langle (Z^{\otimes n})^I_J (Y^{\otimes m})^K_L (Z^{\dagger\otimes n})^M_N (Y^{\dagger\otimes m})^O_P \right\rangle = \sum_{\sigma\in S_n} (\sigma)^I_N (\sigma^{-1})^M_J \sum_{\psi\in S_m} (\psi)^K_P (\psi^{-1})^O_L \tag{2.116}$$

The following notation is used

$$\left(Z^{\otimes n}\otimes Y^{\otimes m}\right)^I_J = Z^{i_1}_{j_1} Z^{i_2}_{j_2} \cdots Z^{i_n}_{j_n} Y^{i_{n+1}}_{j_{n+1}} \cdots Y^{i_{n+m}}_{j_{n+m}} \tag{2.117}$$

where this is understood as acting on $V_N^{\otimes(n+m)}$ and

$$\mathrm{Tr}(\sigma Z^{\otimes n}\otimes Y^{\otimes m}) = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \cdots Z^{i_n}_{i_{\sigma(n)}} Y^{i_{n+1}}_{i_{\sigma(n+1)}} \cdots Y^{i_{n+m}}_{i_{\sigma(n+m)}} \tag{2.118}$$

As before this produces multi-trace operators but one needs to try and figure out what operators are produced by different permutations. Before only permutations from different conjugacy classes produced distinct observables but what permutations produce distinct observables in this case? In order to answer this let us distill why only permutations from different conjugacy classes produced distinct observables when we had one matrix. Consider operators A, B, C, D acting on V_N so that

$$(A\otimes B\otimes C\otimes D)^I_J = A^{i_1}_{j_1} B^{i_2}_{j_2} C^{i_3}_{j_3} D^{i_4}_{j_4} \tag{2.119}$$

acts on $V_N^{\otimes 4}$. Now consider the following permutations

$$(\sigma)^I_J = ((1432))^I_J = \delta^{i_1}_{j_4}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}\delta^{i_4}_{j_3} \tag{2.120}$$

$$\Rightarrow (\sigma^{-1})^I_J = ((1234))^I_J = \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_4}\delta^{i_4}_{j_1} \tag{2.121}$$

and the following

$$\begin{aligned}
(\sigma^{-1})^I_J (A\otimes B\otimes C\otimes D)^J_K (\sigma)^K_L =& (\sigma\cdot A\otimes B\otimes C\otimes D\cdot\sigma^{-1})^I_L \\
=& \delta^{i_1}_{j_2}\delta^{i_2}_{j_3}\delta^{i_3}_{j_4}\delta^{i_4}_{j_1} A^{j_1}_{k_1} B^{j_2}_{k_2} C^{j_3}_{k_3} D^{j_4}_{k_4} \delta^{k_1}_{l_4}\delta^{k_2}_{l_1}\delta^{k_3}_{l_2}\delta^{k_4}_{l_3} \\
=& A^{i_4}_{l_4} B^{i_1}_{l_1} C^{i_2}_{l_2} D^{i_3}_{l_3} \\
=& B^{i_1}_{l_1} C^{i_2}_{l_2} D^{i_3}_{l_3} A^{i_4}_{l_4} \\
=& (B\otimes C\otimes D\otimes A)^I_L
\end{aligned} \tag{2.122}$$

The operators $ABCD$ are rearranged according to the permutation σ. When one does this to an operator $Z^{\otimes n}$ or $Y^{\otimes n}$

$$(\sigma \cdot Z^{\otimes n} \cdot \sigma^{-1})^I_L = (Z^{\otimes n})^I_L \tag{2.123}$$

and

$$(\sigma \cdot Y^{\otimes n} \cdot \sigma^{-1})^I_L = (Y^{\otimes n})^I_L \tag{2.124}$$

because the Z's are indistinguishable as are the Y's. This is because these fields are bosonic. This and the cyclic property of the trace ensures that

$$\mathrm{Tr}(\rho Z^{\otimes n}) = \mathrm{Tr}(\rho \cdot \sigma \cdot Z^{\otimes n} \cdot \sigma^{-1}) = \mathrm{Tr}(\sigma^{-1} \cdot \rho \cdot \sigma \cdot Z^{\otimes n}) \tag{2.125}$$

for $\forall \rho, \sigma \in S_n$. This is why, with one matrix, all permutations from the same conjugacy class produced the same observable. But now what happens when there are Y's and Z's such as

$$\mathrm{Tr}(\sigma Z^{\otimes n} \otimes Y^{\otimes m}) = Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \cdots Z^{i_n}_{i_{\sigma(n)}} Y^{i_{n+1}}_{i_{\sigma(n+1)}} \cdots Y^{i_{n+m}}_{i_{\sigma(n+m)}} \tag{2.126}$$

Clearly, one cannot swap a Z and a Y and produce the same observable. In other words

$$\begin{aligned} \mathrm{Tr}(\rho Z^{\otimes n} \otimes Y^{\otimes m}) =& \mathrm{Tr}(\rho \cdot \sigma \cdot Z^{\otimes n} \otimes Y^{\otimes m} \cdot \sigma^{-1}) \\ =& \mathrm{Tr}(\sigma^{-1} \cdot \rho \cdot \sigma \cdot Z^{\otimes n} \otimes Y^{\otimes m}) \end{aligned} \tag{2.127}$$

only holds if the permutation σ only permutes Z's with Z's and Y's with Y's but not Z's with Y's. This implies that for (2.127) to hold one can only have permutations where $\sigma \in S_n \times S_m$ and the classes of permutations are defined by

$$g = \sigma h \sigma^{-1} \tag{2.128}$$

where $g, h \in S_{(n+m)}$ and $\sigma \in S_n \times S_m$, are called restricted conjugacy classes [24]. Only permutations from different restricted conjugacy classes produce distinct observables. Equation (2.116) can be rewritten as follows

$$\left\langle (Z^{\otimes n} Y^{\otimes m})^I_J (Z^{\dagger \otimes n} Y^{\dagger \otimes m})^K_L \right\rangle = \sum_{\sigma \in S_n \times S_m} (\sigma)^I_L (\sigma^{-1})^K_J \tag{2.129}$$

The goal now is to work towards defining restricted Schur polynomials that will do for us what the normal Schur polynomials did with one matrix. A trace on this restricted space $S_n \times S_m \subset S_{n+m}$ needs to be defined, so that a suitable generalization of the characters $\chi(\sigma) \quad \forall \sigma \in S_n \times S_m$ can be defined. At this point it is necessary to digress and talk about branching rules.

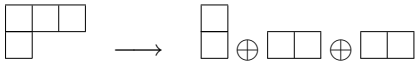
Branching rules: When restricting to a subgroup an irrep of the group decomposes into a number of irreps of the subgroup and more than one of each may appear. Consider the irrep of S_4, $T = \begin{array}{|c|c|c|}\hline & & \\ \hline \\ \cline{1-1} \end{array}$. There are three states in this irrep

$$\begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline 4 & 3 & 2 \\ \hline 1 \\ \cline{1-1} \end{array} \tag{2.130}$$

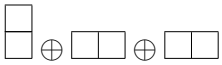
If one wants to restrict this to the subgroup $S_2 \subset S_4$ of permutations that leave 1 and 2 fixed. The subgroup will be

$$S_2 = \{1, (34)\} \tag{2.131}$$

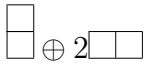
This implies that none of the states will mix with each other and after the restriction:

 (2.132)

After a restriction of S_p to S_{p-q} the irrep $T \vdash p$ will subduce every representation that can be obtained by dropping q boxes. One can write

 (2.133)

as

 (2.134)

Two representations of the same type ($\square\square$) that do not mix have been obtained. These two copies have come from the YY patterns

(2.135)

In general when multiple copies of an irrep appear they need to be labeled so we introduce a multiplicity index which can label them. In this case the partially labeled YY patterns have been used as multiplicity indices.

Restricted trace and Schur polynomials: In restricting from S_{n+m} to $S_n \times S_m$, an irrep for S_{n+m}, $R \vdash n+m$, an irrep for S_n, $r \vdash n$, and an irrep for S_m, $s \vdash m$ is needed. A multiplicity index α to tell us which copy of the irrep (r,s) of $S_n \times S_m$ one has and a state label (I) to tell us which state of $(r,s)\alpha$ one has is also needed. Thus, the states are denoted by $|R,(r,s)\alpha;I\rangle$. Now define the restricted character as follows [24]

$$\chi_{R,(r,s)\alpha\beta}(\sigma) = \mathrm{Tr}_{R,(r,s)\alpha\beta}((\sigma)) \tag{2.136}$$

$$= \sum_I \langle R,(r,s)\alpha;I|\Gamma_R(\sigma)|R,(r,s)\beta;I\rangle \tag{2.137}$$

One can choose a basis of the carrier space of R such that the different matrix elements for different copies of $\psi \in S_n \times S_m$ are equal:

$$\Gamma_{(r,s)\alpha}(\psi) = \Gamma_{(r,s)\beta}(\psi) \qquad \psi \in S_n \times S_m \quad \& \quad \forall \alpha,\beta \tag{2.138}$$

This basis will have the following form

$$|R,(r,s)\beta;I\rangle = \begin{pmatrix} |0\rangle \\ \vdots \\ |(r,s)\beta;I\rangle \\ \vdots \\ |0\rangle \end{pmatrix} \tag{2.139}$$

where every entry is a zero column vector except for the βth entry and

$$|(r,s)\alpha;I\rangle = |(r,s)\beta;I\rangle \qquad \forall \alpha,\beta \tag{2.140}$$

The states $|(r,s)\alpha;I\rangle$ represent a basis for the carrier space of $\Gamma_{(r,s)\alpha}(\psi)$ and since $|(r,s)\alpha;I\rangle = |(r,s)\beta;I\rangle$ one can choose $\Gamma_{(r,s)\alpha}(\psi) = \Gamma_{(r,s)\beta}(\psi)$ so that one can drop the multiplicity label:

$$\Gamma_{(r,s)\alpha}(\psi) = \Gamma_{(r,s)\beta}(\psi) = \Gamma_{(r,s)}(\psi) \tag{2.141}$$

This implies that for $\psi \in S_n \times S_m$ one has

$$\Gamma_R(\psi) = \begin{pmatrix} \Gamma_{(r,s)1}(\psi) & 0 & \cdots & 0 \\ 0 & \Gamma_{(r,s)2}(\psi) & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \Gamma_{(r,s)g(r,s,R)}(\psi) \end{pmatrix} \tag{2.142}$$

$$= \begin{pmatrix} \Gamma_{(r,s)}(\psi) & 0 & \cdots & 0 \\ 0 & \Gamma_{(r,s)}(\psi) & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \Gamma_{(r,s)}(\psi) \end{pmatrix} \tag{2.143}$$

where $g(r,s,R)$ is a Littlewood-Richardson number. Thus

$$\Gamma_R(\psi)|R,(r,s)\alpha;I\rangle = \Gamma_{(r,s)}(\psi)|R,(r,s)\alpha;I\rangle \qquad \psi \in S_n \times S_m \tag{2.144}$$

One can show that the restricted characters are functions on the restricted conjugacy class. To show this it must be shown that

$$\chi_{R,(r,s)\alpha\beta}(\sigma) = \chi_{R,(r,s)\alpha\beta}(\rho) \tag{2.145}$$

for

$$\rho = \psi^{-1}\sigma\psi \qquad \psi \in S_n \times S_m. \tag{2.146}$$

Consider

$$\begin{aligned} \chi_{R,(r,s)\alpha\beta}(\rho) &= \sum_I \langle R,(r,s)\alpha;I|\Gamma_R(\rho)|R,(r,s)\beta;I\rangle \\ &= \sum_I \langle R,(r,s)\alpha;I|\Gamma_R(\psi^{-1}\sigma\psi)|R,(r,s)\beta;I\rangle \\ &= \sum_I \langle R,(r,s)\alpha;I|\Gamma_R(\psi^{-1})\Gamma_R(\sigma)\Gamma_R(\psi)|R,(r,s)\beta;I\rangle \\ &= \sum_I \langle R,(r,s)\alpha;I|\Gamma_{(r,s)}(\psi)^{-1}\Gamma_R(\sigma)\Gamma_{(r,s)}(\psi)|R,(r,s)\beta;I\rangle \end{aligned} \tag{2.147}$$

Using the linear algebra identity

$$\mathrm{Tr}(|i\rangle\langle j|A) = \langle j|A|i\rangle \tag{2.148}$$

where A is a matrix that acts on the set of states $|i\rangle$, to rewrite (2.147) as follows.

$$\begin{aligned} \chi_{R,(r,s)\alpha\beta}(\rho) &= \sum_I \langle R,(r,s)\alpha;I|\Gamma_{(r,s)}(\psi)^{-1}\Gamma_R(\sigma)\Gamma_{(r,s)}(\psi)|R,(r,s)\beta;I\rangle \\ &= \sum_I \mathrm{Tr}(|R,(r,s)\beta;I\rangle\langle R,(r,s)\alpha;I|\Gamma_{(r,s)}(\psi)^{-1}\Gamma_R(\sigma)\Gamma_{(r,s)}(\psi)) \\ &= \mathrm{Tr}\left(\Gamma_{(r,s)}(\psi)\left[\sum_I |R,(r,s)\beta;I\rangle\langle R,(r,s)\alpha;I|\right]\Gamma_{(r,s)}(\psi)^{-1}\Gamma_R(\sigma)\right) \end{aligned} \tag{2.149}$$

In the basis chosen

$$\sum_I |R,(r,s)\beta;I\rangle\langle R,(r,s)\alpha;I| = \sum_I \begin{pmatrix} |0\rangle \\ \vdots \\ |(r,s)\beta;I\rangle \\ \vdots \\ |0\rangle \end{pmatrix} \begin{pmatrix} \langle 0| & \cdots & \langle (r,s)\alpha;I| & \cdots & \langle 0| \end{pmatrix} \tag{2.150}$$

Since $|(r,s)\beta;I\rangle = |(r,s)\alpha;I\rangle$, then

$$\sum_I |(r,s)\beta;I\rangle\langle (r,s)\alpha;I| = \mathbb{I} \tag{2.151}$$

where $\mathbb{I}$ is the identity matrix.

$$\Rightarrow \sum_I |R,(r,s)\beta;I\rangle\langle R,(r,s)\alpha;I| = \mathbb{I}\delta_{\beta\alpha} \tag{2.152}$$

where $\delta_{\beta\alpha}$ here is a matrix where all entries are zero except for the entry at row position β and column position α at which the entry is one. This tells us that

$$\left[\sum_I |R,(r,s)\beta;I\rangle\langle R,(r,s)\alpha;I|\right] \tag{2.153}$$

will change the multiplicity label of something that acts on it from β to α. Then

$$\begin{aligned} &\Gamma_{(r,s)\beta}(\psi)\left[\sum_I |R,(r,s)\beta;I\rangle\langle R,(r,s)\alpha;I|\right] \\ &= \left[\sum_I |R,(r,s)\beta;I\rangle\langle R,(r,s)\alpha;I|\right]\Gamma_{(r,s)\alpha}(\psi) \end{aligned} \tag{2.154}$$

Since the multiplicity label has been dropped

$$\begin{aligned} \chi_{R,(r,s)\alpha\beta}(\rho) &= \mathrm{Tr}\left(\left[\sum_I |R,(r,s)\beta;I\rangle\langle R,(r,s)\alpha;I|\right]\Gamma_{(r,s)}(\psi)\Gamma_{(r,s)}(\psi)^{-1}\Gamma_R(\sigma)\right) \\ &= \mathrm{Tr}\left(\left[\sum_I |R,(r,s)\beta;I\rangle\langle R,(r,s)\alpha;I|\right]\Gamma_R(\sigma)\right) \\ &= \sum_I \mathrm{Tr}\left(|R,(r,s)\beta;I\rangle\langle R,(r,s)\alpha;I|\Gamma_R(\sigma)\right) \\ &= \sum_I \langle R,(r,s)\alpha;I|\Gamma_R(\sigma)|R,(r,s)\beta;I\rangle \\ &= \chi_{R,(r,s)\alpha\beta}(\sigma) \end{aligned} \tag{2.155}$$

This proves that the restricted characters are functions on the restricted conjugacy classes. The restricted characters are orthogonal.

$$\sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\sigma)\chi_{T,(t,u)\gamma\delta}(\sigma^{-1})$$

$$= \sum_{\sigma\in S_{n+m}} \sum_I \langle R,(r,s)\alpha;I|\Gamma_R(\sigma)|R,(r,s)\beta;I \sum_J \langle T,(t,u)\gamma;J|\Gamma_T(\sigma^{-1})|T,(t,u)\delta;J\rangle$$

$$= \sum_{\sigma\in S_{n+m}} \sum_I (\langle R,(r,s)\alpha;I|)_N(\Gamma_R(\sigma))^N_K(|R,(r,s)\beta;I\rangle)^K \times$$
$$\times \sum_J (\langle T,(t,u)\gamma;J|)_L(\Gamma_T(\sigma^{-1}))^L_M(|T,(t,u)\delta;J\rangle)^M$$

$$= \sum_{\sigma\in S_{n+m}} (\Gamma_R(\sigma))^N_K(\Gamma_T(\sigma^{-1}))^L_M \sum_I (\langle R,(r,s)\alpha;I|)_N(|R,(r,s)\beta;I\rangle)^K \times$$
$$\times \sum_J (\langle T,(t,u)\gamma;J|)_L(|T,(t,u)\delta;J\rangle)^M \quad (2.156)$$

Using the fundamental orthogonality relation

$$\sum_{g\in G} \Gamma_R(g^{-1})^A_B \Gamma_S(g)^C_D = \frac{|G|}{d_R}\delta_{RS}(\delta)^A_D(\delta)^C_B \quad (2.157)$$

to get

$$\sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\sigma)\chi_{T,(t,u)\gamma\delta}(\sigma^{-1}) =$$
$$\frac{(n+m)!}{d_R}\delta_{RT}(\delta)^N_M(\delta)^L_K \sum_I (\langle R,(r,s)\alpha;I|)_N(|R,(r,s)\beta;I\rangle)^K \sum_J (\langle T,(t,u)\gamma;J|)_L(|T,(t,u)\delta;J\rangle)^M$$

$$= \frac{(n+m)!}{d_R}\delta_{RT}(\delta)^N_M(\delta)^L_K \sum_I (\langle R,(r,s)\alpha;I|)_N(|R,(r,s)\beta;I\rangle)^K \sum_J (\langle T,(t,u)\gamma;J|)_L(|T,(t,u)\delta;J\rangle)^M$$
$$= \frac{(n+m)!}{d_R}\delta_{RT} \sum_I \sum_J \langle R,(r,s)\alpha;I|T,(t,u)\delta;J\rangle\langle T,(t,u)\gamma;J|R,(r,s)\beta;I\rangle$$
$$= \frac{(n+m)!}{d_R}\delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}\delta_{\beta\gamma} \sum_I \sum_J \delta_{IJ} \quad (2.158)$$

$$\Rightarrow \sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\sigma)\chi_{T,(t,u)\gamma\delta}(\sigma^{-1}) = \frac{(n+m)!}{d_R}\delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}\delta_{\beta\gamma}d_r d_s \quad (2.159)$$

Now define an operator $P_{R,(r,s)\alpha\beta}$ such that

$$P_{R,(r,s)\alpha\beta} = \frac{1}{n!\,m!} \sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\sigma)\sigma \quad (2.160)$$

These will have the same role as the projection operators in the one matrix case but these are not in general projection operators. One can now check what $P_{R,(r,s)\alpha\beta}\cdot\psi$ is for $\psi \in S_n \times S_m$

$$(P_{R,(r,s)\alpha\beta})^I_J(\psi)^J_K = \frac{1}{n!\,m!}\sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\sigma)(\sigma)^I_J(\psi)^J_K \tag{2.161}$$

Change σ to its conjugate τ related as follows

$$(\sigma)^I_J = \psi^I_L(\tau)^L_K(\psi^{-1})^K_J = (\psi^{-1}\tau\psi)^I_J \tag{2.162}$$

This implies

$$\begin{aligned}(P_{R,(r,s)\alpha\beta})^I_J(\psi)^J_K =& \frac{1}{n!\,m!}\sum_{\tau\in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\psi^{-1}\tau\psi)(\psi)^I_L(\tau)^L_B(\psi^{-1})^B_J(\psi)^J_K \\ =& \frac{1}{n!\,m!}\sum_{\tau\in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\psi^{-1}\tau\psi)(\psi)^I_L(\tau)^L_K \end{aligned}\tag{2.163}$$

Now since the restricted characters are functions on the restricted conjugacy class one has

$$\begin{aligned}(P_{R,(r,s)\alpha\beta})^I_J(\psi)^J_K =& (\psi)^I_L\frac{1}{n!\,m!}\sum_{\tau\in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\tau)(\tau)^L_K \\ =& (\psi)^I_L(P_{R,(r,s)\alpha\beta})^L_K \end{aligned}\tag{2.164}$$

$$\Rightarrow P_{R,(r,s)\alpha\beta}\cdot\psi = \psi\cdot P_{R,(r,s)\alpha\beta} \tag{2.165}$$

The restricted Schur polynomials [24] ($\chi_{R,(r,s)\alpha\beta}(Z,Y)$) or "Fourier transform [44]" is defined as

$$\chi_{R,(r,s)\alpha\beta}(Z,Y) \equiv \mathrm{Tr}(P_{R,(r,s)\alpha\beta}Z^{\otimes n}\otimes Y^{\otimes m}) \tag{2.166}$$

$$\begin{aligned} =& \frac{1}{n!\,m!}\sum_{\sigma\in S_{n+m}} \mathrm{Tr}(\chi_{R,(r,s)\alpha\beta}(\sigma)\sigma\cdot\ Z^{\otimes n}\otimes Y^{\otimes m}) \\ =& \frac{1}{n!\,m!}\sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\sigma)\mathrm{Tr}(\sigma\cdot\ Z^{\otimes n}\otimes Y^{\otimes m}) \end{aligned}\tag{2.167}$$

The inverse Fourier transform can again be found by solving for A in the following

$$\mathrm{Tr}(\sigma\cdot\ Z^{\otimes n}\otimes Y^{\otimes m}) = A\sum_{R,(r,s)\alpha\beta} \chi_{R,(r,s)\alpha\beta}(\sigma^{-1})\chi_{R,(r,s)\alpha\beta}(Z,Y) \tag{2.168}$$

Putting this back into (2.167) one gets

$$\begin{aligned}\chi_{R,(r,s)\alpha\beta}(Z,Y) =& \frac{A}{n!\,m!}\sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\sigma)\sum_{T,(t,u)\gamma\delta}\chi_{T,(t,u)\gamma\delta}(\sigma^{-1})\chi_{T,(t,u)\gamma\delta}(Z,Y) \\ =& \frac{A}{n!\,m!}\sum_{T,(t,u)\gamma\delta}\chi_{T,(t,u)\gamma\delta}(Z,Y)\sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\sigma)\chi_{T,(t,u)\gamma\delta}(\sigma^{-1}) \end{aligned}\tag{2.169}$$

Given

$$\sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\sigma)\chi_{T,(t,u)\gamma\delta}(\sigma^{-1}) = \frac{(n+m)!}{d_R}\delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}\delta_{\beta\gamma}d_r d_s \tag{2.170}$$

$$\Rightarrow \chi_{R,(r,s)\alpha\beta}(Z,Y) = \frac{A(n+m)!\, d_r d_s}{d_R n!\, m!} \sum_{T,(t,u)\gamma\delta} \chi_{T,(t,u)\gamma\delta}(Z,Y)\delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}\delta_{\beta\gamma}$$
$$= \frac{A(n+m)!\, d_r d_s}{d_R n!\, m!} \chi_{R,(r,s)\alpha\beta}(Z,Y) \tag{2.171}$$

This implies

$$A = \frac{d_R n!\, m!}{d_r d_s (n+m)!} \tag{2.172}$$

and so the inverse Fourier transform [44] is given by

$$\mathrm{Tr}(\sigma \cdot\ Z^{\otimes n} \otimes Y^{\otimes m}) = \sum_{R,(r,s)\alpha\beta} \frac{d_R n!\, m!}{d_r d_s (n+m)!} \chi_{R,(r,s)\alpha\beta}(\sigma^{-1}) \chi_{R,(r,s)\alpha\beta}(Z,Y) \tag{2.173}$$

This again tells us that any multi-trace operator can be represented as a linear combination of restricted Schur polynomials. The 2 point function can be calculated. This was first done in [45].

$$\begin{aligned}
&\langle \chi_{R,(r,s)\alpha\beta}(Z,Y) \chi_{T,(t,u)\gamma\delta}(Z^\dagger, Y^\dagger) \rangle \\
=&\langle \mathrm{Tr}(P_{R,(r,s)\alpha\beta} Z^{\otimes n} \otimes Y^{\otimes m}) \mathrm{Tr}(P_{T,(t,u)\gamma\delta} Z^{\dagger\otimes n} \otimes Y^{\dagger\otimes m}) \rangle \\
=&(P_{R,(r,s)\alpha\beta})^I_J (P_{T,(t,u)\gamma\delta})^K_L \langle (Z^{\otimes n} \otimes Y^{\otimes m})^J_I (Z^{\dagger\otimes n} \otimes Y^{\dagger\otimes m})^L_K \rangle \\
=& \sum_{\sigma \in S_n \otimes S_m} (P_{R,(r,s)\alpha\beta})^I_J (P_{T,(t,u)\gamma\delta})^K_L (\sigma)^J_K (\sigma^{-1})^L_I \\
=& \sum_{\sigma \in S_n \otimes S_m} \mathrm{Tr}(P_{R,(r,s)\alpha\beta} \sigma P_{T,(t,u)\gamma\delta} \sigma^{-1}) \\
=& \sum_{\sigma \in S_n \otimes S_m} \mathrm{Tr}(P_{R,(r,s)\alpha\beta} P_{T,(t,u)\gamma\delta}) \\
=& n!\, m!\, \mathrm{Tr}(P_{R,(r,s)\alpha\beta} P_{T,(t,u)\gamma\delta})
\end{aligned} \tag{2.174}$$

One now needs to evaluate $\mathrm{Tr}(P_{R,(r,s)\alpha\beta} P_{T,(t,u)\gamma\delta})$. Consider

$$\begin{aligned}
&P_{R,(r,s)\alpha\beta} P_{T,(t,u)\gamma\delta} \\
=& \frac{1}{(n!\, m!\,)^2} \sum_{\sigma \in S_{n+m}} \sum_{\psi \in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\sigma) \chi_{T,(t,u)\gamma\delta}(\psi) (\sigma)^I_J (\psi)^J_K \\
=& \frac{1}{(n!\, m!\,)^2} \sum_{\sigma \in S_{n+m}} \sum_{\psi \in S_{n+m}} \chi_{R,(r,s)\alpha\beta}(\sigma) \chi_{T,(t,u)\gamma\delta}(\psi) (\psi\sigma)^I_K
\end{aligned} \tag{2.175}$$

Change variables

$$\sigma, \psi \longrightarrow \sigma, \tau = \psi\sigma \tag{2.176}$$
$$\Rightarrow (\tau)^I_K = (\psi\sigma)^I_K = (\sigma)^I_J (\psi)^J_K \tag{2.177}$$
$$\Rightarrow (\sigma^{-1})^L_I (\tau)^I_K = (\psi)^L_K \tag{2.178}$$
$$\Rightarrow (\psi)^L_K = (\tau\sigma^{-1})^L_K \tag{2.179}$$

So that

$$
\begin{aligned}
&P_{R,(r,s)\alpha\beta}P_{T,(t,u)\gamma\delta} \\
&= \frac{1}{(n!\,m!\,)^2}\sum_{\sigma\in S_{n+m}}\sum_{\tau\in S_{n+m}}\chi_{R,(r,s)\alpha\beta}(\sigma)\chi_{T,(t,u)\gamma\delta}(\tau\sigma^{-1})(\tau)^I_K \qquad (2.180)\\
&= \frac{1}{(n!\,m!\,)^2}\sum_{\tau\in S_{n+m}}\sum_{\sigma\in S_{n+m}}(\Gamma_R(\sigma))^N_K(\Gamma_T(\tau\sigma^{-1}))^L_M\sum_I(\langle R,(r,s)\alpha;I|)_N(|R,(r,s)\beta;I\rangle)^K\times \\
&\quad\times\sum_J(\langle T,(t,u)\gamma;J|)_L(|T,(t,u)\delta;J\rangle)^M(\tau)^I_K \\
&= \frac{1}{(n!\,m!\,)^2}\sum_{\tau\in S_{n+m}}(\Gamma_T(\tau))^L_O\sum_{\sigma\in S_{n+m}}(\Gamma_R(\sigma))^N_K(\Gamma_T(\sigma^{-1}))^O_M\times \\
&\quad\times\sum_I(\langle R,(r,s)\alpha;I|)_N(|R,(r,s)\beta;I\rangle)^K\sum_J(\langle T,(t,u)\gamma;J|)_L(|T,(t,u)\delta;J\rangle)^M(\tau)^I_K \\
&= \frac{1}{(n!\,m!\,)^2}\frac{(n+m)!}{d_R}\sum_{\tau\in S_{n+m}}(\Gamma_T(\tau))^L_O\delta_{RT}(\delta)^N_M(\delta)^O_K\sum_I(\langle R,(r,s)\alpha;I|)_N(|R,(r,s)\beta;I\rangle)^K\times \\
&\quad\times\sum_J(\langle T,(t,u)\gamma;J|)_L(|T,(t,u)\delta;J\rangle)^M(\tau)^I_K \\
&= \frac{1}{(n!\,m!\,)^2}\frac{(n+m)!}{d_R}\sum_{\tau\in S_{n+m}}\delta_{RT}\sum_I\sum_J\langle R,(r,s)\alpha;I|T,(t,u)\delta;J\rangle\langle T,(t,u)\gamma;J|\Gamma_T(\tau)|R,(r,s)\beta;I\rangle(\tau)^I_K \\
&= \frac{1}{(n!\,m!\,)^2}\frac{(n+m)!}{d_R}\sum_{\tau\in S_{n+m}}\delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}\sum_I\sum_J\delta_{IJ}\langle R,(r,s)\gamma;J|\Gamma_T(\tau)|R,(r,s)\beta;I\rangle(\tau)^I_K \\
&= \frac{(n+m)!}{(n!\,m!\,)^2}\frac{1}{d_R}\delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}\sum_{\tau\in S_{n+m}}\chi_{T,(t,u)\gamma\beta}(\tau)(\tau)^I_K \\
&= \frac{(n+m)!}{n!\,m!}\frac{1}{d_R}\delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}P_{T,(t,u)\gamma\beta} \qquad (2.181)
\end{aligned}
$$

It can further be shown that [45, 24]

$$\mathrm{Tr}(P_{T,(t,u)\gamma\beta}) = \frac{1}{n!\,m!}d_t d_u f_T\delta_{\gamma\beta} \qquad (2.182)$$

$$
\begin{aligned}
\Rightarrow \mathrm{Tr}(P_{R,(r,s)\alpha\beta}P_{T,(t,u)\gamma\delta}) =& \frac{(n+m)!}{n!\,m!}\frac{1}{d_R}\delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}\mathrm{Tr}(P_{T,(t,u)\gamma\beta}) \\
=& \frac{(n+m)!}{(n!\,m!\,)^2}\frac{d_r d_s}{d_R}f_R\delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}\delta_{\gamma\beta} \qquad (2.183)
\end{aligned}
$$

This implies that the two point function is given by

$$
\begin{aligned}
\langle\chi_{R,(r,s)\alpha\beta}(Z,Y)\chi_{T,(t,u)\gamma\delta}(Z^\dagger,Y^\dagger)\rangle =& \frac{(n+m)!}{n!\,m!}\frac{d_r d_s}{d_R}f_R\delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}\delta_{\gamma\beta} \\
=& \frac{\mathrm{hooks}_R}{\mathrm{hooks}_r\mathrm{hooks}_s}f_R\delta_{RT}\delta_{rt}\delta_{su}\delta_{\alpha\delta}\delta_{\gamma\beta} \qquad (2.184)
\end{aligned}
$$

which is diagonal and allows the easily summation of all ribbon graphs.

2.2 AdS/CFT correspondence

The AdS/CFT or gauge-gravity correspondence is a proposed equivalence between theories of quantum gravity in AdS spaces and conformally symmetric quantum field theories on the boundary of that space. Thus, it is a duality between a higher dimensional theory (quantum gravity in AdS) and a lower dimensional theory that is on the boundary. This concept of a theory living on the boundary of a space being able to encapsulate the physics of everything inside the space (often referred to as the bulk) is known as holography. The holographic principle was first proposed by Gerard 't Hooft in 1993 [46] and promoted by Leonard Susskind in 1995 [47]. These ideas culminated in Maldecena's landmark paper postulating the first explicit example of holography in 1997 [2]. This specific example related type IIB string theory on the product space $AdS_5 \times S^5$ to $\mathcal{N} = 4$ SYM theory on the four-dimensional boundary. Since then the AdS/CFT correspondence has been generalised to the statement that quantum gravity theories in $(d+1)$ dimensions can be entirely and completely equivalent to non-gravitational quantum field theories on the d dimensional boundary.

One of the very first ideas which lead up to this was the Kaluze-Klein theory. Theodor Kaluza tried to unify general relativity and electromagnetism by introducing an extra dimension above the usual 4. By applying the five-dimensional Einstein equations the four-dimensional Einstein field equations and the Maxwell equations for the electromagnetic field manifested. Later Oskar Klein proposed that the 5th dimension was curled up and microscopic which would explain why it is unobserved. Although this is not an example of holography it was the first time a lower dimensional theory was shown to proceed form a higher dimensional theory. An interesting insight from this is that the modes in the smaller compactified dimension exhibit themselves as particles in the lower dimensional theory. This is a key concept in string theory which explains why Kaluze-Klein theory is considered an important precursor to string theory.

2.2.1 Anti-de Sitter space

AdS stands for Anti-de Sitter. AdS spacetime is a spacetime with Lorentzian signature $(-++++\cdots+)$ and constant negative curvature (de Sitter space has constant positive curvature). In $d+1$ dimensions AdS is defined by the embedding of

$$-t_1^2 + x_1^2 + \cdots + x_{d-1}^2 - t_2^2 = -R^2 \tag{2.185}$$

in a flat Minkowski space with two time like coordinates

$$ds^2 = -dt_1^2 + dx_1^2 + \cdots + dx_{d-1}^2 - dt_2^2 \tag{2.186}$$

where R has dimensions of length and is referred to as the radius of curvature. R defines the length scale of the space even though AdS is not compact. It is a hyperboloid. From (2.185) AdS space in d dimensions is explicitly invariant under $SO(2, d-1)$. A global coordinate system can be defined as

$$t_1 = R\cosh\rho\cos\tau \tag{2.187}$$
$$x_i = R\sinh\rho\Omega_i \tag{2.188}$$
$$t_2 = R\cosh\rho\sin\tau \tag{2.189}$$

where Ω_i parameterises a $d-2$ dimensional unit sphere ($\Omega_i\Omega_i = 1$). In these coordinates the AdS_d metric is

$$ds^2 = R^2(-\cosh^2(\rho)d\tau^2 + d\rho^2 + \sinh^2(\rho)d\Omega_{d-2}^2)$$

where $d\Omega_{d-2}^2$ is the metric of a $d-2$ dimensional sphere, $\rho \in \mathbb{R}^+$ and $\tau \in [0, 2\pi]$. Now $\tau \in [0, 2\pi]$ allows closed time like curves which violates causality. This is solved by requiring $\tau \in \mathbb{R}$. These coordinates are known as the universal cover of AdS space.

2.2.2 Conformal symmetry

Conformal invariance is the local generalization of scale invariance. A conformal transformation is one which preserves angles. A conformably invariant theory is one which is invariant under a conformal transformation which is defined as any transformation $x^\mu \to x^{'\mu}$ which results in

$$g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu} \tag{2.190}$$
$$\Rightarrow ds^2 \to \Omega^2(x) ds^2 \tag{2.191}$$

Considering an infinitesimal conformal transformation in a d dimensional space

$$x^\mu \to x^{'\mu} = x^\mu + \xi^\mu(x) \tag{2.192}$$
$$\Rightarrow \Omega(x) = 1 + \omega(x) \tag{2.193}$$

one obtains the conformal Killing vector equation in d dimensions

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 2\eta_{\mu\nu} \frac{\partial \cdot \xi}{d} \tag{2.194}$$

Solving this equation gives

$$\xi^\mu = a^\mu + w^{\mu\nu} x_\nu + \lambda x^\mu + x^2 b^\mu - 2b \cdot x x^\mu \tag{2.195}$$

The first three terms are identified with translations, Lorentz transformations and scalings (dilatations) respectively. The final two terms are identified with the special conformal transformation and can be likened to an inversion followed by a translation and another inversion. The generators for these transformations are

Transformation:	**Generator:**
Translations	$P_\mu = -i\partial_\mu$
Lorentz	$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$
Scalings	$D = -ix \cdot \partial$
Special conformal	$K_\mu = 2ix_\mu x \cdot \partial - ix^2 \partial_\mu$

The algebra for these generators is as follows

$$[P_\alpha, M_{\mu\nu}] = i\eta_{\alpha\mu} P_\nu - i\eta_{\alpha\nu} P_\mu \tag{2.196}$$
$$[K_\alpha, M_{\mu\nu}] = i\eta_{\alpha\mu} K_\nu - i\eta_{\alpha\nu} K_\mu \tag{2.197}$$
$$[M_{\alpha\beta}, M_{\mu\nu}] = i\eta_{\mu\alpha} M_{\nu\beta} - i\eta_{\nu\alpha} M_{\mu\beta} + i\eta_{\mu\beta} M_{\alpha\nu} - i\eta_{\nu\beta} M_{\alpha\mu} \tag{2.198}$$
$$[D, P_\mu] = iP_\mu \tag{2.199}$$
$$[D, K_\mu] = -iK_\mu \tag{2.200}$$
$$[D, M_{\mu\nu}] = 0 \tag{2.201}$$
$$[P_\mu, K_\nu] = 2i(\eta_{\mu\nu} D + M_{\mu\nu}) \tag{2.202}$$

with all other combinations commuting. Let us now define new $(d+2)$ dimensional generators L_{MN} where $M, N = -1, 0, 1, ..., d-1, d$ and $L_{MN} = -L_{NM}$ as follows

$$\begin{aligned} L_{\mu\nu} &= M_{\mu\nu} & L_{\mu -1} &= \tfrac{1}{2}(P_\mu + K_\mu) \\ L_{-1d} &= D & L_{\mu d} &= \tfrac{1}{2}(P_\mu - K_\mu) \end{aligned} \tag{2.203}$$

where $\mu, \nu = 0, 1, \ldots, d-1$. By examining the Lie algebra of L_{MN} one finds the metric to be $\eta_{MN} = diag(-1, -1, 1, 1, ..., 1)$ and as such the symmetry group is $SO(2, d)$. With this metric the algebra is given by

$$[L_{AB}, L_{MN}] = i\eta_{MA}L_{NB} - i\eta_{NA}L_{MB} + i\eta_{MB}L_{AN} - i\eta_{NB}L_{AM} \tag{2.204}$$

2.2.3 $\mathcal{N} = 4$ Super Yang-Mills

$\mathcal{N} = 4$ Super Yang-Mills is a conformally symmetric field theory with an additional internal symmetry related to rotating the supercharges. The supercharges Q^i_α generate a symmetry between fermions and bosons, known as supersymmetry. There is an internal symmetry that rotates the supercharges which translates into a global symmetry of the theory that rotates the fields of the theory. This global symmetry is known as R-symmetry. In $\mathcal{N} = 4$ Super Yang-Mills there is one gauge field A_μ, four fermions ψ^i and six scalar fields ϕ_n. There are 4 supercharges which transform under $SU(4)$. This translates into a global R-symmetry $SU(4) = SO(6)$. The scalars fields transform in the fundamental representation of $SO(6)$ and the fermions transform in the the fundamental representation of $SU(4)$.

2.2.4 IIB strings on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM

The first hint at a duality is the fact that the symmetries match. The preceding sub-sections provide a basis to see this. As was discussed the isometry group of AdS_{d+1} is $SO(2, d)$ which is the same as the conformal symmetry of a CFT in d dimensions. Furthermore, when considering specifically $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM in 4 dimensions the isometry group of AdS_5 ($SO(2, 4)$) matches the conformal group of $\mathcal{N} = 4$ SYM in 4D and the isometrtry group of S^5 is $SO(6)$ which matches the R-symmetry of $\mathcal{N} = 4$ SYM. Thus, there is a matching of the symmetries of the theories. This fact is essential for there to be a duality but it is by no means a proof. There is no argument to date that constitutes a proof of the conjecture but there are some very compelling arguments and matches between the two sides of the duality. t' Hooft's argument outlined in section (2.1), his work on dimensional reduction [46] and Susskind's work in holography [47] provide general evidence that there should be a holographic duality between quantum gravity theories in the bulk and CFTs on the boundary. Maldacena [2] gave the first explicit example of this holographic duality. He examines the low energy limit of D-brains (objects on which open strings end) and shows that in certain limits there are two interpretations [48] one is $\mathcal{N} = 4$ SYM in 4D Minkowski space with gauge group $SU(N)$ and the other, in the near horizon limit, is type IIB string theory in $AdS_5 \times S^5$.

Within the duality there is what is known as a dictionary that describes what certain quantities on one side of the duality correspond to on the other side of the duality. One key fact is that one must study gauge invarient objects in the CFT because there is no gauge group in the gravity side [49]. Some of the note worthy parts of this dictionary are first of all that operators in the CFT correspond to fields in the bulk. These operators will have a conformal dimension of Δ which is related to the mass or energy of the dual fields. The stress-energy tensor $T_{\mu\nu}$ is related to the graviton $g_{\mu\nu}$ and as discussed earlier the string coupling (g_s) goes like $\frac{1}{N}$ where N is the order of the gauge group in the CFT.

The AdS/CFT correspondence links strongly coupled theories to weakly coupled ones. Consider a $U(N)$ gauge theory with coupling g_{YM} following the argument in Section (2.1) the effective coupling or t' Hooft coupling is $\lambda = g_{YM}^2 N$ and the string coupling g_s goes like $\frac{1}{N}$. Thus, the large N limit of the gauge theory corresponds to a strong coupling in the gauge theory and a weak coupling in the string theory. The general rule is that the correspondence is a strong/weak one in that if a theory is strongly coupled on one side of the duel the other side will be weakly coupled. This is of particular importance because strongly coupled theories are in general difficult to calculate things in because one cannot use the usual techniques of perturbation theory but having a duality to a weakly coupled theory can allow one to learn things about the strongly coupled theory from the weakly coupled one.

2.3 Chaos

Quantum chaos is the extension of classical chaos to quantum systems and is of particular interest in the domain of the AdS/CFT conjecture because chaotic quantum systems and black holes show similar behaviour [50, 51]. In particular it is proposed in [51] and further studied in [52] that the size of an operator in a chaotic quantum system is dual to the radial momentum of a particle falling into a black hole and the growth of the size of an operator is dual to the acceleration.

2.3.1 Classical Chaos

There is not a strict mathematical definition for a chaotic system but a simple and widely used definition is that a system is chaotic if small changes in the initial conditions produce profound changes to the system at later times where the time scale is the Lyapunov time. An implication of this is that even though a chaotic system is deterministic, the approximation of the system at later times using approximate initial conditions will become increasingly bad as time goes on and quickly not resemble the exact system at all. A classical example of chaos is the Bunimovich Stadium in which hard balls bounce around an oval stadium. In this example the trajectories of two different balls with slightly different initial conditions diverge at an exponential rate until the distance between the balls is of the order of the stadium. The rate of divergence is characterised by the Lyapunov exponent (λ_L) so that the difference in position (δx) grows like $\delta x \propto e^{\lambda_L t}$. This is a characteristic of chaotic systems: observables with slightly different initial conditions will diverge at a rate proportional to $e^{\lambda_L t}$ until the initial conditions are indeterminable. This is also known as the butterfly effect, the term came from the question of whether a small effect like the flapping of a butterflies wings in China could grow to be a large effect like a hurricane in the Caribbean. The answer for a chaotic systems is that one can not know because the extreme sensitivity to initial conditions means that the path between cause and effect becomes increasingly blurred as the system evolves. Consequently at late times one can not determine the initial conditions.

2.3.2 Quantum chaos

When extending this to a quantum system there is an immediate problem. In quantum mechanics it is impossible to know the initial conditions with absolute accuracy. It is however possible to see a semiclassical version of a chaotic system. The idea is that we relate the Poisson bracket $\{\cdot,\cdot\}$ with the commutator $[\cdot,\cdot]$ in the standard way

$$[A, B] = i\hbar\{a, b\} \tag{2.205}$$

where $\{a,b\} = \frac{\partial a}{\partial x(0)}\frac{\partial b}{\partial p(0)} - \frac{\partial a}{\partial p(0)}\frac{\partial b}{\partial x(0)}$ and study things like

$$[X(t), X(0)] = i\hbar\{x(t), x(0)\} = -i\hbar\frac{\partial x(t)}{\partial p(0)} \tag{2.206}$$

which is a quantity that tells us how sensitive $x(t)$ is to changes in $p(0)$. Thus, this quantity for a chaotic system should go like $e^{\lambda_L t}$ in the semiclassical limit.

$$[X(t), X(0)] \sim -i\hbar e^{\lambda_L t} \tag{2.207}$$

One would want to study the average of this quantity but it can be positive or negative so one studies the average of the square

$$\langle -[X(t), X(0)]^2\rangle_\beta \sim \hbar^2 e^{2\lambda_L t} \tag{2.208}$$

where $\langle\cdots\rangle_\beta = \frac{\mathrm{Tr}\,(e^{-\beta H}\cdots)}{\mathrm{Tr}\,(e^{-\beta H})}$. This should be true in the semiclassical limit but whether it is true in the full quantum dynamics or not is unclear. One would need to study this quantity and see if it does exhibit exponential growth or not.

For a general quantum system with Hermitian operators W and V each having order 1 energy and being localised at x and y one can study

$$\begin{aligned} -\langle [W_x(t), V_y(0)]^2\rangle_\beta &= \langle (W_x(t)V_y(0) - V_y(0)W_x(t))(W_x(t)V_y(0) - V_y(0)W_x(t))^\dagger\rangle_\beta \\ &= \langle W_x(t)V_y(0)V_y(0)W_x(t)\rangle_\beta - \langle W_x(t)V_y(0)W_x(t)V_y(0)\rangle_\beta \\ &\quad - \langle V_y(0)W_x(t)V_y(0)W_x(t)\rangle_\beta + \langle V_y(0)W_x(t)W_x(t)V_y(0)\rangle_\beta \end{aligned} \tag{2.209}$$

Identify $|\phi\rangle \equiv W_x(T)V_y(0)|\beta\rangle$ and $|\psi\rangle \equiv V_y(0)W_x(T)|\beta\rangle$ so that

$$-\langle [W_x(t), V_y(0)]^2\rangle_\beta = \langle\psi|\psi\rangle + \langle\phi|\phi\rangle - \langle\psi|\phi\rangle - \langle\phi|\psi\rangle \tag{2.210}$$

The first two terms are state norms and so should be of order 1. Thus, if this quantity were to exhibit exponential growth it would have to come from the last two terms. Schematically, for $-\langle [W_x(t), V_y(0)]^2\rangle_\beta \sim e^{ct}$ one must have that $\langle\psi|\phi\rangle \sim \langle\phi|\psi\rangle \sim (1 - e^{ct})$ so that

$$1 + 1 - (1 - e^{ct}) - (1 - e^{ct}) \sim e^{ct} \tag{2.211}$$

Thus, diagnosing chaos is done by studying the out of time ordered correlator (OTOC) $\langle V(0)W(t)V(0)W(t)\rangle_\beta$ which goes like $(1 - e^{ct})$ for a chaotic system.

2.3.3 Operator growth and chaos

It was suggested in [53] that AdS space could be understood as a type of tensor network representing the state of the dual gauge theory and asserted in [51] that in this picture the interior of a black hole should be proportional to the size of the minimal tensor network that can represent the state. This, as well as [51, 52] motivate that it is of interest to study how operators for such systems grow. The idea is to study localised operators of the form

$$W_x(t) = e^{-iHt}W_x e^{iHt} \tag{2.212}$$

where W_x is an operator localised at x and characterise their growth.

2.3.4 Qubit system

A good toy model to illustrate how this is done is a qubit system defined by the Ising Hamiltonian

$$H = \sum_i (-Z_i Z_{i+1} + gX_i + hY_i) \tag{2.213}$$

the operator to study is

$$Z_1(t) = e^{-iHt} Z_1 e^{iHt} \tag{2.214}$$

where $X_i,\ Y_i$ and Z_i are the Pauli operators on the ith site. At $t = 0$ it is localised at site 1 but at later times it will grow and eventually fill the space. Consider

$$C_j(t) = \langle [Z_1(t), A_j]^2 \rangle_\beta \tag{2.215}$$

where $A_j = X_j,\ Y_j$ or Z_j. One can identify the size of $Z_1(t)$ by the site j for which $C_j(t)$ transitions between its approximate maximum and minimum value. There are two ways in which one can calculate $C_j(t)$. The first is to expand $Z_1(t)$ using the Baker-Campbell-Hausdorff formula

$$Z_1(t) = Z_1 - it[H, Z_1] - \frac{t^2}{2!}[H, [H, Z_1]] + \frac{it^3}{3!}[H, [H, [H, Z_1]]] + \cdots \tag{2.216}$$

and then calculate $C_j(t)$ to powers of t. Consider the ith term; $\frac{t^i}{i!}[H, \cdots [H, Z_1] \cdots]$. The total contribution for this term will be of order $\frac{t^i}{i!}$ which means that the main contribution to $Z_1(t)$ comes from the terms where $t = i$. This can be seen from the following argument. The maximum of $\frac{t^i}{i!}$ occurs when $\frac{\partial}{\partial i}\frac{t^i}{i!} = 0$.

$$\frac{\partial}{\partial i}\frac{t^i}{i!} \cong \frac{\partial}{\partial i} e^{i\log(t) - i\log(i) + i} = (\log(t) - \log(i)) e^{i\log(t) - i\log(i) + i} \tag{2.217}$$

which is maximized when $t = i$. This implies that at late times the main contribution will come from terms which have deeply nested commutators. The amount of computational effort needed to calculate each consecutive correlator grows fast. This seems to be a dire situation but there is some hope in that as one takes successively higher commutators with H the types of terms we get eventually saturate. What is meant by this is that after taking many commutators of Z_1 with H one finds terms like $A_1 I_2 A_3 A_4 I_5 \cdots A_n$ where n is the size of the spin chain, I is the identity operator and $A_j = X_j,\ Y_j$ or Z_j as before. Eventually all the possible terms that can be made will be made and taking consecutive commutators with H will just change the number coefficients of each term. The total number of terms one can get for n sites is 4^n which grows quite fast with n but if the commutator can be calculated up to this number of terms then it can be calculated to high orders.

Another measure of the size of $Z_1(t)$ is to measure the number of terms where the highest site occupied by a non identity operator is j and to see how this changes for different j's with time. Following [54] one can do this by summing the squares of the coefficients of each term in $Z_1(t)$ with an operator at highest site j for different j's and plotting them. Using the Baker-Campbell-Hausdorff expansion for $n = 4,\ g = -1.05$ & $h = 0.5$ one finds the following

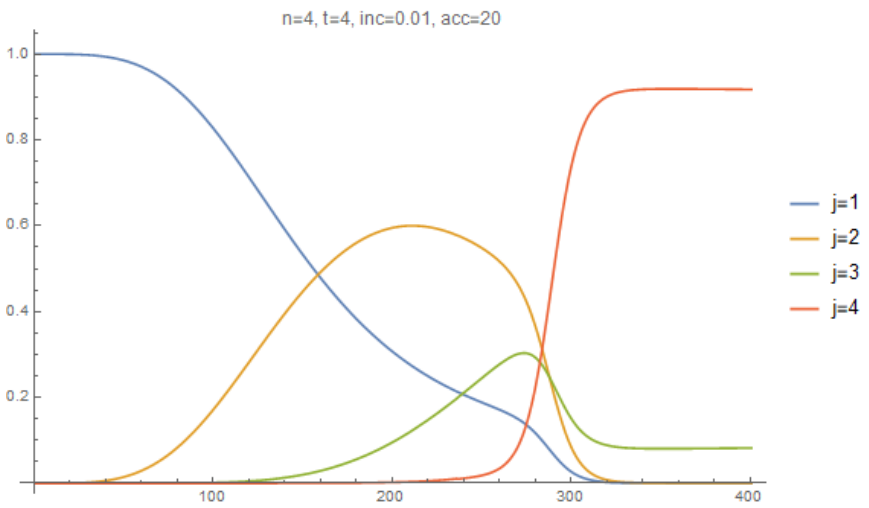


Figure 2.7: Sizes of the terms in $Z_1(t)$ with $n = 4,\ g = -1.05$ & $h = 0.5$

One can see that as time progresses the sizes of the terms in $Z_1(t)$ grow until it is dominated by terms that fill the space.

One can also calculate $C_j(t)$ exactly by getting an orthogonal basis from the eigenvectors of the Hamiltonian and taking the trace by summing over the inner product of these states.

$$\begin{aligned}
C_j(t) &= \langle [Z_1(t), Z_j]^2 \rangle_\beta \\
&\propto \mathrm{Tr}\left(e^{-\beta H}[Z_1(t), Z_j]^2\right) \\
&= \mathrm{Tr}\left(e^{-\beta H}(e^{-iHt}Z_1e^{iHt}Z_j - Z_je^{-iHt}Z_1e^{iHt})(e^{-iHt}Z_1e^{iHt}Z_j - Z_je^{-iHt}Z_1e^{iHt})^\dagger\right) \\
&= \mathrm{Tr}\left(e^{-\beta H}(e^{-iHt}Z_1e^{iHt}Z_jZ_je^{-iHt}Z_1e^{iHt} - e^{-iHt}Z_1e^{iHt}Z_je^{-iHt}Z_1e^{iHt}Z_j\right. \\
&\qquad \left. - Z_je^{-iHt}Z_1e^{iHt}Z_je^{-iHt}Z_1e^{iHt} + Z_je^{-iHt}Z_1e^{iHt}e^{-iHt}Z_1e^{iHt}Z_j)\right) \\
&= \mathrm{Tr}\left(e^{-\beta H}\left(\frac{\mathbb{I}}{8} - e^{-iHt}Z_1e^{iHt}Z_je^{-iHt}Z_1e^{iHt}Z_j\right.\right. \\
&\qquad \left.\left. - Z_je^{-iHt}Z_1e^{iHt}Z_je^{-iHt}Z_1e^{iHt}\right)\right)
\end{aligned} \tag{2.218}$$

Converting the trace to a sum over eigenstates ($|\psi_i\rangle$) of the Hamiltonian with energy E_i one finds

$$
\begin{aligned}
&\mathrm{Tr}\,(e^{-\beta H}[Z_1(t), Z_j]^2) \\
&= \sum_i \langle\psi_i|e^{-\beta H}(\frac{1}{8} - e^{-iHt}Z_1e^{iHt}Z_je^{-iHt}Z_1e^{iHt}Z_j \\
&\quad - Z_je^{-iHt}Z_1e^{iHt}Z_je^{-iHt}Z_1e^{iHt})|\psi_i\rangle \\
&= -\sum_{i,m,k,l} e^{-\beta E_i}e^{-iE_it}\langle\psi_i|Z_1e^{iHt}|\psi_m\rangle\langle\psi_m|Z_je^{-iHt}|\psi_k\rangle\langle\psi_k|Z_1e^{iHt}|\psi_l\rangle\langle\psi_l|Z_j|\psi_i\rangle \\
&\quad - \sum_{i,m,k,l} e^{-\beta E_i}e^{iE_it}\langle\psi_i|Z_je^{-iHt}|\psi_m\rangle\langle\psi_m|Z_1e^{iHt}|\psi_k\rangle\langle\psi_k|Z_je^{-iHt}|\psi_l\rangle\langle\psi_l|Z_1|\psi_i\rangle \\
&\quad + \frac{1}{8}\sum_i e^{-\beta E_i} \\
&= -\sum_{;i,m,k,l} je^{-\beta E_i}e^{-i(E_i-E_m+E_k-E_l)t}\langle\psi_i|Z_1|\psi_m\rangle\langle\psi_m|Z_j|\psi_k\rangle\langle\psi_k|Z_1|\psi_l\rangle\langle\psi_l|Z_j|\psi_i\rangle \\
&\quad - \sum_{i,m,k,l} e^{-\beta E_i}e^{i(E_i-E_m+E_k-E_l)t}\langle\psi_i|Z_j|\psi_m\rangle\langle\psi_m|Z_1|\psi_k\rangle\langle\psi_k|Z_j|\psi_l\rangle\langle\psi_l|Z_1|\psi_i\rangle \\
&\quad + \frac{1}{8}\sum_i e^{-\beta E_i}
\end{aligned}
\tag{2.219}
$$

Calculating this for $n = 4$, $g = -1.05$ & $h = 0.5$ one finds

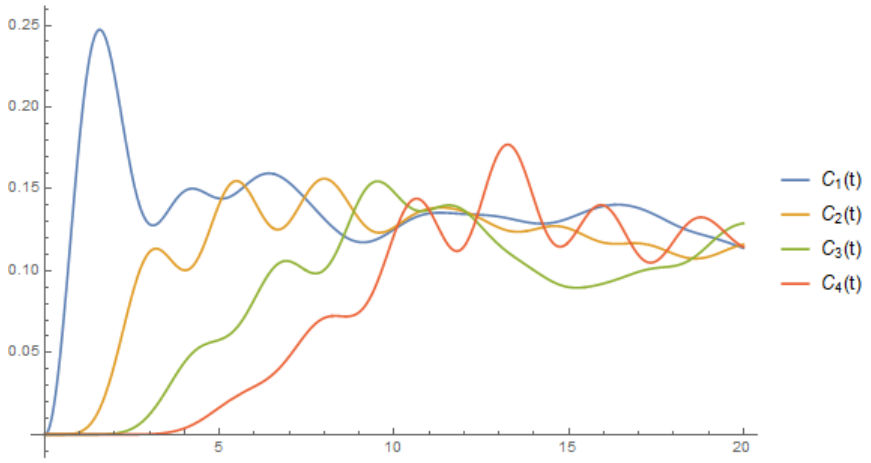


Figure 2.8: $C_j(t)$'s with $n = 4$, $g = -1.05$ & $h = 0.5$

From this it is clear that at late times the contributions for different values of j in $C_j(t)$ are roughly equal. This is a characteristic of chaos, that at late times the system is completely mixed and discorded.

2.4 SYK Model

SYK stands for Sachdev, Ye and Kitaev. It is a quantum many-body system of Majorana fermions, interacting with a random coupling. It was first devised by Subir Sachdev and Jinwu Ye [55], and later refined

by Alexei Kitaev [56]. It is of interest because it is an exactly solvable model at large N with an emergent conformal symmetry in the infrared (IR) with strong coupling indicating that it should have a holographic dual. It is maximally chaotic, saturating a chaotic bound [57] suggesting that it's holographic dual exhibits black holes. This makes the model a very interesting model to study because it could potentially give insights into holography and black holes.

As mentioned the SYK Model is a model of Majorana fermions (χ_i) living at N sites with a random coupling J_{ijkl}. The Majorana fermions (χ_i) obey the usual Clifford algebra

$$\{\chi_i(t), \chi_j(t)\} = \delta_{ij} \tag{2.220}$$

The interacting Hamiltonian is

$$H = \frac{J_{ijkl}}{4!}\chi_i\chi_j\chi_k\chi_l \tag{2.221}$$

with repeated indices summed. The coupling is completely anti-symmetric in its indices, it is a random normally distributed variable with zero mean ($\overline{J_{ijkl}} = 0$) and variance given by

$$\overline{J_{ijkl}J_{mnpq}} = \begin{cases} 0 & \text{if } mnpq \text{ not a permutation of } ijkl \\ \frac{3!J^2}{N^3} & \text{if } mnpq \text{ is an even permutation of } ijkl \\ -\frac{3!J^2}{N^3} & \text{if } mnpq \text{ an odd permutation of } ijkl \end{cases} \tag{2.222}$$

The idea is to examine the large N and large J limit of the theory. Given the Lagrangian for the theory

$$L = L_0 - H = -\frac{1}{2}\chi_i\frac{\partial}{\partial t}\chi_i - \frac{J_{ijkl}}{4!}\chi_i\chi_j\chi_k\chi_l \tag{2.223}$$

The free two point function $G_0(t)$ is given by

$$\langle T(\chi_i(t)\chi_j(0))\rangle = \Theta(t)\langle 0|\chi_i(t)\chi_j(0)|0\rangle - \Theta(-t)\langle 0|\chi_j(0)\chi_i(t)|0\rangle \tag{2.224}$$

$$= \int D\chi e^{S_0}\chi_i(t)\chi_j(0) \tag{2.225}$$

$$= G_0(t)\delta_{ij} \tag{2.226}$$

were $S_0 = \int dt L_0$. With the solution

$$G_0(t) = \frac{1}{2}\text{sgn}(t) \tag{2.227}$$

or

$$G_0(\omega) = -\frac{1}{i\omega} \tag{2.228}$$

in frequency space. The free m point function is

$$\int D\chi e^{S_0}(\chi_{i_1}(t_1)\chi_{i_2}(t_2)\cdots\chi_{i_m}(t_m)) \tag{2.229}$$

If m is odd its zero and if m is even it can be calculated by grouping the fields in every possible way (Wick's theorem) and inserting (2.226) for each pair.

The idea now is to determine the general two point function. When working with the interacting theory correlators must be disorder averaged so as to exploit the statistical properties of the random coupling J_{ijkl} (2.222). The corrections to the free two point function can be found perturbatively. Expanding the interacting part of the action gives

$$\langle T(\chi_a(t_1)\chi_b(t_2))\rangle \tag{2.230}$$

$$= \int D\chi e^{S}\chi_a(t_1)\chi_b(t_2) \tag{2.231}$$

$$= \int D\chi e^{S_0}\chi_a(t_1)\chi_b(t_2)\left[1 + \int H dt + \frac{1}{2}\left(\int H dt\right)^2 + \cdots\right] \tag{2.232}$$

$$= G_0(t)\delta_{ab} - \int D\chi e^{S_0}\int dt \frac{\overline{J_{ijkl}}}{4!}\chi_a(t_1)\chi_b(t_2)\chi_i(t)\chi_j(t)\chi_k(t)\chi_l(t)$$

$$+ \int D\chi e^{S_0}\int dt dt' \frac{\overline{J_{ijkl}J_{mnpq}}}{4!\,4!}\chi_a(t_1)\chi_b(t_2)\chi_i(t)\chi_j(t)\chi_k(t)\chi_l(t)\chi_m(t')\chi_n(t')\chi_p(t')\chi_q(t') + \cdots \tag{2.233}$$

The first term after the free contribution is zero because of the disorder averaging (as is every odd term) and the second term can be calculated diagrammatically. Assuming the vacuum contributions have already been normalised out the only diagram looks like

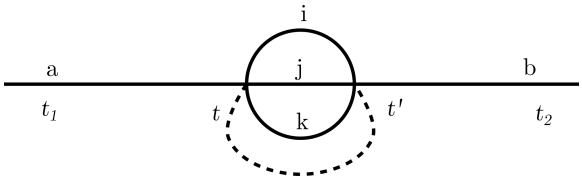


Figure 2.9: First order correction to two point function

where the dashed line represents the disorder averaging (in this diagram the disorder averaging must go between these vertices because they are the only ones but in higher order diagrams that will not be the case). This diagram is known as a melon. The contribution from this diagram at large N is (see appendix for calculation)

$$\int dt \int dt' \delta_{ab}\delta_{ii}\delta_{jj}\delta_{kk}G_0(t_1 - t)G_0(t' - t_2)(G_0(t - t'))^3\frac{J^2}{N^3} \tag{2.234}$$

The disorder averaging gives delta functions for the indices on either side. The next non-zero contribution has 4 vertices. Consider the following diagrams

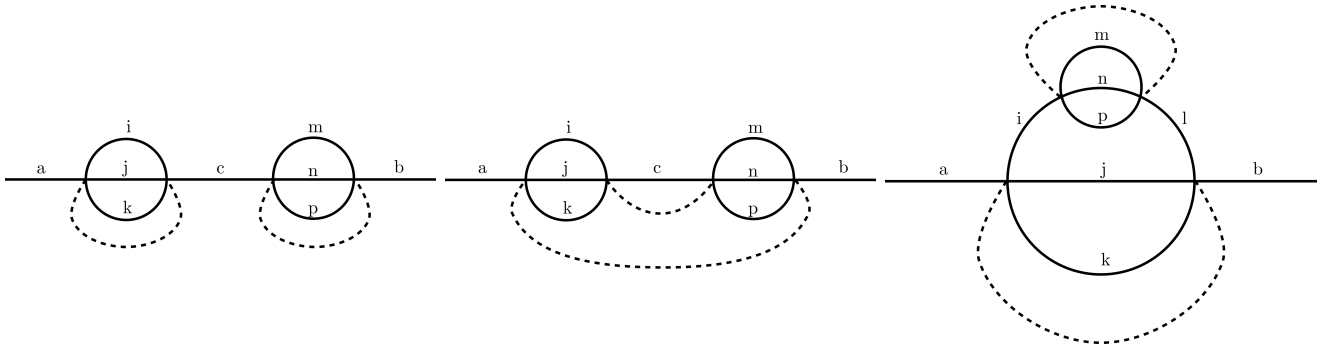


Figure 2.10: Some second order corrections to two point function

The first and the last are melonic and the middle one is not. Melonic diagrams contribute higher powers of N than non-melonic diagrams and so since the large N limit is of interest the non-melonic diagrams are dropped and one only needs to sum melonic diagrams. This point can be illustrated by examining the first and second diagrams in figure above. In the first diagram the disorder averaging will give a contribution of

$$\delta_{ac}\delta_{ii}\delta_{jj}\delta_{kk}\delta_{cb}\delta_{mm}\delta_{nn}\delta_{pp}\left(\frac{J^2}{N^3}\right)^2 = \delta_{ab}N^6\left(\frac{J^2}{N^3}\right)^2 = \delta_{ab}J^4 \tag{2.235}$$

and the second will give

$$\delta_{ab}\delta_{im}\delta_{jn}\delta_{kp}\delta_{cc}\delta_{mi}\delta_{nj}\delta_{pk}\left(\frac{J^2}{N^3}\right)^2 = \delta_{ab}N^4\left(\frac{J^2}{N^3}\right)^2 = \delta_{ab}\frac{J^4}{N^2} \tag{2.236}$$

Thus, in the large N limit one only needs to sum melonic diagrams and so the two point function can be written as

$$\langle T(\chi_a(t_1)\chi_b(t_2))\rangle = \delta_{ab}G(t_1-t_2) \tag{2.237}$$

where $G(t_1-t_2)$ is the sum of all the melonic diagrams. Since only melonic diagrams are summed one can expand $G(t_1-t_2)$ perturbitively by starting with the free propagator and systematically adding melons to propagators. The following illustrates this procedure

$$G(t_1-t_2) = G_0(t_1-t_2) + \mathcal{O}(J^2) \tag{2.238}$$

$$= G_0(t_1-t_2) + \int dt \int dt' G_0(t_1-t)G_0(t'-t_2)(G_0(t-t'))^3J^2 + \mathcal{O}(J^4) \tag{2.239}$$

$$\begin{aligned} &= G_0(t_1-t_2)+ \\ &+ \int dt \int dt' G_0(t_1-t)\left[\int dt'' \int dt''' G_0(t'-t'')G_0(t'''-t_2)(G_0(t''-t'''))^3J^2\right]\times \\ &\times (G_0(t-t'))^3J^2 + \mathcal{O}(J^6) \end{aligned} \tag{2.240}$$

This process can be repeated until all the propagators are the full propagators. This results in the following expression

$$G(t_1-t_2) = G_0(t_1-t_2) + \int dt \int dt' G_0(t_1-t)G(t'-t_2)(G(t-t'))^3J^2 \tag{2.241}$$

The first propagator in the second term stays the free propagator because no matter how many melons are added to that line there is always a free propagator in front. It is convenient to group the one particle irreducible part ($(G(t-t'))^3 J^2 = \Sigma(t-t')$) so that

$$G(t_1 - t_2) = G_0(t_1 - t_2) + \int dt \int dt' G_0(t_1 - t) G(t' - t_2) \Sigma(t - t') \tag{2.242}$$

The aim is to solve this in the IR. In the IR distances and times are large and $G(t)$ must be decreasing in t. Thus, in the IR (2.242) becomes

$$0 = G_0(t_1 - t_2) + \int dt G_0(t_1 - t) \int dt' G(t' - t_2) \Sigma(t - t') \tag{2.243}$$

Which means that

$$\int dt' G(t' - t_2) \Sigma(t - t') = -\delta(t - t_2) \tag{2.244}$$

This exhibits emergent conformal symmetry which allows the two point function to be determined up to a constant. In one dimension conformal symmetry is re-parameterisation symmetry. The conformal dimension Δ of $G(t)$ is $\Delta = \frac{1}{4}$. Re-parameterising with $t \to f(t) = \alpha t$

$$\Rightarrow G(t - t') \to \left[\frac{df(t)}{dt} \frac{df(t')}{dt'} \right]^{\frac{1}{4}} G(f(t) - f(t')) = \sqrt{\alpha} G(\alpha(t - t')) \tag{2.245}$$

$$\Rightarrow G(t - t') = \frac{b}{\sqrt{t - t'}} \text{sgn}(t - t') \tag{2.246}$$

(2.244) is holds for $b^4 = \frac{1}{4J^2\pi}$. The dominance of melonic diagrams in the large N limit of the SYK model means that it can be solved exactly in the IR. The four point function is calculated by summing what are known as ladder diagrams [58]. Every lader with a larger number of rungs can be obtained by adding what is know as the Kernel to a lader with one less rung and so the four point can be calculated exactly. Furthermore it was shown in [59] that all $2p$-point correlation functions can be calculated by the adding together of four and two point functions and using the operator product expansion (OPE). It is also argued in [59] that this result can be extended to any theory in which higher-point correlators are built out of four-point functions. This is a characteristic property of melonic theories of which the Tensor model is an example.

The SYK model may provide a simple solvable example of holography, realized as an AdS/CFT duality - see [58, 31, 60, 32]. This expectation is motivated by the fact that the model develops an approximate conformal symmetry in the infrared. Exact conformal symmetry is spontaneously and explicitly broken, leading to a pseudo-Goldstone mode. This mode is responsible for the exponential growth of out of time ordered correlators, which saturates the chaos bound[57]. Saturating the bound is a strong hint that the model is dual to something close to Einstein gravity. Much of the progress to date is possible because the large N limit is dominated by melonic diagrams. It is because these diagrams can be summed that the model is solvable, even at strong coupling.

2.5 Tensor Models

Tensor models have recently gained popularity, as close cousins of the SYK model[55, 56]. The SYK model describes fermions interacting with all-to-all random interactions. However, the large N physics of

the SYK model is identical to a tensor model, that has a conventional large N limit[61]. Thus, tensor models share many of the notable features of SYK models with one significant advantage: they have no disorder[61]. The large N limit of the tensor models is also dominated by melonic graphs[62, 63, 64, 65], which can be summed. For interesting related work on holographic tensor models see [66]-[67]. Earlier work on tensor models includes [68]-[69].

Tensor models are rather interesting in their own right. Tensor models are asymptotically free[70, 71, 72]. The large N limit of tensor models involves specific discretizations of the d-sphere, known as melonic [73], which provide an analytic description of the universality class of the branched polymer phase of Euclidean dynamical triangulations[74]. Further, the multi-critical behaviors [75] can be interpreted as critical, non-unitary matter [76, 77], just like in 1-matrix models [78].

A tensor model field theory is a theory in which the basic fields are tensors. They can be fermionic or bosonic and they can be of arbitrary rank. For this overview consider the rank 3 bosonic fields ϕ^{abc} and its conjugate $\bar{\phi}_{abc}$ which transform in the fundamental and anti-fundamental of $G = U(N_1)\times U(N_2)\times U(N_3)$ respectively. The free Hamiltonian is given by

$$H = -\frac{\partial}{\partial\bar{\phi}_{abc}}\frac{\partial}{\partial\phi^{abc}} + \frac{1}{4}\phi^{abc}\bar{\phi}_{abc} \tag{2.247}$$

Gauge invariants are made by contracting all the up indices with down indices so that there are no free indices. A convienient notation to use for collections of fields of size n is the following

$$\Phi^{IJK}(n) = \phi^{i_1j_1k_1}\phi^{i_2j_2k_2}\cdots\phi^{i_nj_nk_n} \tag{2.248}$$

on which the symmetric group has two natural actions

$$\sigma_1\circ\sigma_2\circ\sigma_3\cdot\Phi^{IJK} \to \Phi^{\sigma_1(I)\sigma_2(J)\sigma_3(K)} = \phi^{i_{\sigma_1(1)}j_{\sigma_2(1)}k_{\sigma_3(1)}}\phi^{i_{\sigma_1(2)}j_{\sigma_2(2)}k_{\sigma_3(2)}}\cdots\phi^{i_{\sigma_1(n)}j_{\sigma_2(n)}k_{\sigma_3(n)}} \tag{2.249}$$

where $\sigma_1\circ\sigma_2\circ\sigma_3 \in S_n\times S_n\times S_n$ and the diagonal action

$$\sigma\circ\sigma\circ\sigma\cdot\Phi_{IJK} = \sigma\cdot\Phi^{IJK} \to \Phi^{\sigma(I)\sigma(J)\sigma(K)} \tag{2.250}$$

Thus, the general observables are

$$f(n,\sigma_1,\sigma_2,\sigma_3) = \Phi^{IJK}\cdot\sigma_1\circ\sigma_2\circ\sigma_3\cdot\bar{\Phi}_{IJK} \tag{2.251}$$

where they are labeled by the size of the collection n and a number of permutations equal to the rank of the basic fields of the theory which in this case is σ_1, σ_2 and σ_3 for rank 3 a tensor. These observables are counted for both bosonic and fermionic tensors in Chapter 3. They can also be visualised diagrammatically with bubble/melon diagrams or trees. This is illustrated in Section 4.3.

The free two point function is given by

$$\langle\phi^{i_1j_1k_1}\bar{\phi}_{i_2j_2k_2}\rangle = \delta^{i_1}_{i_2}\delta^{j_1}_{j_2}\delta^{k_1}_{k_2} \tag{2.252}$$

and Wick's theorem can be applied to give

$$\begin{aligned}\langle\Phi^{IJK}\bar{\Phi}_{LMN}\rangle &= \sum_{\sigma\in S_n}\prod_{a=1}^{n}\delta^{i_a}_{l_{\sigma(a)}}\delta^{j_a}_{m_{\sigma(a)}}\delta^{k_a}_{n_{\sigma(a)}}\\ &= \sum_{\sigma\in S_n}(\sigma)^I_L(\sigma)^J_M(\sigma)^K_N\end{aligned} \tag{2.253}$$

There is only one possible melonic quartic vertex in the complex model which is

$$V = g\phi^{abc}\bar{\phi}_{abd}\phi^{efd}\bar{\phi}_{efc} \tag{2.254}$$

With this the corrections to the two point function are again obtained by summing melons. The propagator diagram can be either three coloured strands each representing a distinct index or the strands can be combined into one line in cases when there will be no ambiguity.

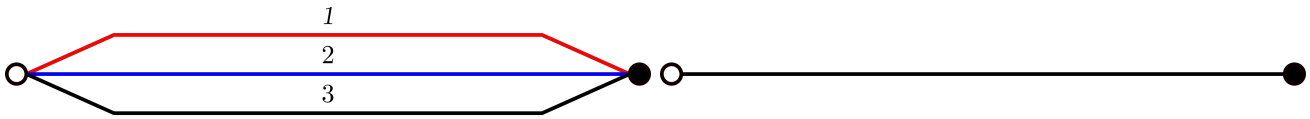


Figure 2.11: Tensor propagator with strands explicit (*left*) and compressed (*right*)

Each white point corresponds to a field ϕ^{abc} and each black point to a field $\bar{\phi}_{abc}$. Lines must go between white and black points. The vertex is known as the pillow (see Fig. 2.12).

Figure 2.12: Tensor vertex with External lines suppressed

With this the first corrections to the two point can be calculated. In 0+1 dimensions summing the diagrams in Figure 2.13 gives (for more details see Appendix C.2)

$$\begin{aligned}\langle\phi^{abc}\bar{\phi}_{abc}\rangle &= N_1N_2N_3G(t,t) - i2g(N_1^2N_2^2N_3 + N_1N_2N_3^2)\int_{-\infty}^{\infty} dt' G(t,t')^2 G(t',t') + O(g^2) \\ &= N_3^2 - 8gN_3^3 + O(g^2)\end{aligned} \tag{2.255}$$

using the following Green's function

$$G(t_1,t_2) = \int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{i}{E^2 - \frac{1}{4} + i\epsilon}e^{iE(t_1-t_2)} \tag{2.256}$$

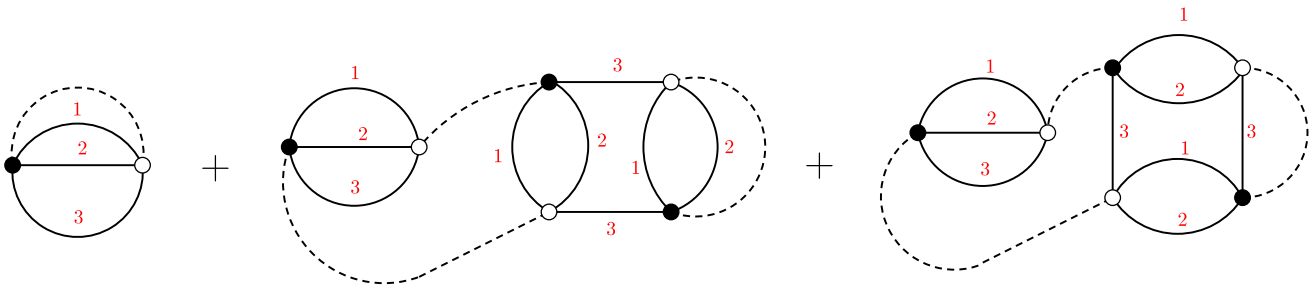


Figure 2.13: The Feynman diagrams contributing to $\langle\phi^{abc}\bar{\phi}_{abc}\rangle$. The dashed lines indicate which operators are Wick contracted.

2.6 Collective Field Theory

Collective field theory provides a systematic construction of the dynamics of invariant observables of the theory [13]. This is typically accompanied by the emergence of extra holographic dimensions as well as a reorganization of the dynamics such that $1/N$ is the new loop counting parameter [14]. It is therefore the ideal framework with which to pursue the holography of tensor models and to provide insight into the structure of the large N expansion for these models.

Collective field theory achieves a direct change of variables to the invariant observables, taking careful account of the Jacobian associated to this change of variables. The Jacobian is a highly non-trivial functional of the fields and it produces a non-linear (collective) Hamiltonian which provides a complete description of the theory.

This section aims to give a concise summary of collective field theory (a slightly more in depth investigation is presented in Appendix A.2 for a simple system). Consider a system with Hamiltonian

$$H = -\frac{1}{2}\sum_{i=1}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i} + V(x_i) \tag{2.257}$$

Here x_i are the original variables of the system. Denote the invariant field variables by ϕ_C where C is an index for the invariants. In the large N limit, the change of variables from the original variables to invariants is a reduction in the number of degrees of freedom. These variables are treated as independent in the large N limit. There are constraints that arise at finite N related to the stringy exclusion principle[79, 6], but these will play no role here.

Consider changing variables from the original variables to a set of gauge invariant variables. The kinetic term becomes

$$T = -\frac{1}{2}\sum_{i=1}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i} = -\frac{1}{2}\sum_{C,C'}\Omega(C,C')\frac{\partial}{\partial\phi_C}\frac{\partial}{\partial\phi_{C'}} + \frac{1}{2}\sum_{C}\omega(C)\frac{\partial}{\partial\phi_C} \tag{2.258}$$

where

$$\Omega(C,C') = \sum_{i=1}\frac{\partial\phi_C}{\partial x_i}\frac{\partial\phi_{C'}}{\partial x_i} \qquad\qquad \omega(C) = -\sum_{i=1}\frac{\partial^2\phi_C}{\partial x_i\partial x_i} \tag{2.259}$$

$\Omega(C,C')$ "joins" invariant variables so that we write schematically

$$\Omega(C,C') = \sum \phi_{C+C'} \tag{2.260}$$

where the sum runs over all possible ways of "joining" ϕ_C and $\phi_{C'}$ to produce $\phi_{C+C'}$. Similarly, ω "splits" loops so that

$$\omega(C) = \sum \phi_{C'}\phi_{C''} \tag{2.261}$$

where the sum runs over all possible ways of splitting the word C into C' and C''. This change of variables is accompanied by a nontrivial Jacobian. This Jacobian will be the source of the non-linear interactions of collective field theory. It ensures the unitarity of the new description. After the similarity transformation

$$\frac{\partial}{\partial\phi_C} \to J^{1/2}\frac{\partial}{\partial\phi_C}J^{-1/2} = \frac{\partial}{\partial\phi_C} - \frac{1}{2}\frac{\partial\ln J}{\partial\phi_C} \tag{2.262}$$

one requires that the Hamiltonian is explicitly hermitian, which implies a differential equation for the Jacobian, which determines the Hamiltonian in terms of the invariant variables. The differential equation can be obtained as follows. The effective Hamiltonian (H_{eff}) after the change of variables is given by

$$H_{\text{eff}} = J^{1/2} H J^{-1/2} \tag{2.263}$$

It is required that the effective Hamiltonian is hermitian which equates to requiring

$$J^{1/2} H J^{-1/2} - (J^{1/2} H J^{-1/2})^\dagger = J^{1/2} T J^{-1/2} - (J^{1/2} T J^{-1/2})^\dagger = 0 \tag{2.264}$$

$$\Rightarrow \frac{1}{2}\sum_C \left[2\omega\partial_C + (\partial_C \omega)\right] - \frac{1}{2}\sum_{C,C'}\Bigg[\Omega(\partial_C \ln J)(\partial_{C'} \ln J) - \frac{\Omega}{2}(\partial_C \partial_{C'} J) + $$

$$-2\Omega(\partial_{C'} \ln J)\partial_C - (\partial_C \partial_{C'}\Omega) - (\partial_C \Omega)(\partial_{C'} \ln J) - 2(\partial_{C'}\Omega)\partial_C\Bigg] = 0 \tag{2.265}$$

This equation must be true no mater what arbitrary function of ϕ_C and $\phi_{C'}$ it acts on. Thus, it must be zero at every level of the derivatives. This implies two differential equations

$$\frac{1}{2}\sum_C \left[\partial_C \omega - \sum_{C'}\left(\Omega(\partial_C \ln J)(\partial_{C'} \ln J) - \frac{\Omega}{2}(\partial_C \partial_{C'} J) - (\partial_C \partial_{C'}\Omega) - (\partial_C \Omega)(\partial_{C'} \ln J)\right)\right] = 0 \tag{2.266}$$

and

$$\omega(C) + \sum_{C'}\left[\Omega(C,C')(\partial_{C'} \ln J) + (\partial_{C'}\Omega(C,C'))\right] = 0 \tag{2.267}$$

The latter of which is a useful differential equation for J. One can alternatively separate H_{eff} into manifestly Hermitian and non-Hermitian parts and require the manifestly non-Hermitian part be zero to acquire the same equation. Using the notation $\Omega_{CC'}(\partial_{C'} \ln J) = \sum_{C'} \Omega(C,C')(\partial_{C'} \ln J)$

$$\Rightarrow \omega_C + \Omega_{CC'}\partial_{C'} \ln J + \partial_{C'}\Omega_{CC'} = 0$$

$$\Rightarrow \Omega^{-1}_{BC}\Omega_{CC'}\partial_{C'} \ln J = -\Omega^{-1}_{BC}(\omega_C + \partial_{C'}\Omega_{CC'})$$

$$\Rightarrow \delta_{BC'}\partial_{C'} \ln J = -\Omega^{-1}_{BC}(\omega_C + \partial_{C'}\Omega_{CC'})$$

$$\Rightarrow \partial_{C'} \ln J = -\Omega^{-1}_{C'C}(\omega_C + \partial_D \Omega_{CD})$$

$$\Rightarrow -(\partial_{C'} \ln J)\Omega(C',C) = \omega(C) + \partial_{C'}\Omega(C',C) \tag{2.268}$$

The collective field Hamiltonian is obtained by inserting the above to give

$$\Rightarrow H_{\text{eff}} = \Pi_C \Omega_{CC'}\Pi_{C'} + \frac{1}{4}(\omega_C + \partial_{C'}\Omega_{CC'})\Omega^{-1}_{CD}(\omega_D + \partial_E \Omega_{DE}) - \frac{1}{2}\partial_C \omega_C - \frac{1}{2}\partial_C \partial_{C'}\Omega_{CC'} + V \tag{2.269}$$

$$\Rightarrow H_{\text{eff}} = \sum_{C,C'} \Pi(C)\Omega(C,C')\Pi(C') + \frac{1}{4}\sum_{C,C'} \omega(C)\Omega^{-1}(C,C')\omega(C') + V + \Delta H \tag{2.270}$$

where

$$\Pi(C) = -i\frac{\partial}{\partial\phi(C)} \tag{2.271}$$

and ΔH contains counter terms (terms of order $\hbar$) whose form is determined by the hermiticity requirement

$$\begin{aligned}\Delta H = -\frac{1}{4}\sum_{C}\frac{\partial\omega(C)}{\partial\phi_C} + \frac{1}{4}\sum_{C,C',C''}\frac{\partial\Omega(C'',C')}{\partial\phi_{C''}}\,\Omega^{-1}(C',C)\omega(C) + \\ +\frac{1}{4}\sum_{C,C',C'',C'''}\frac{\partial\Omega(C'',C)}{\partial\phi_{C''}}\,\Omega^{-1}(C,C')\,\frac{\partial\Omega(C',C''')}{\partial\phi_{C'''}}\end{aligned} \tag{2.272}$$

The key obstacle in applying collective field theory, lies in handling the space of invariant variables. The set of invariants for both multi matrix models and tensor models is large and, consequently, difficult to handle. If one starts with a truncated set of invariants C and their conjugates C', through the process of "joining" (contained in $\Omega(C,C')$) one typically generates new loops not in the original set. This makes it hard to find simpler sub sectors whose dynamics can be studied. For relevant related work on this issue, that is used in this work, see [80, 81, 82].

2.7 Collective Field Theory examples for a Tensor Model

The aim of this section is to give a few examples of attempts to construct a collective field theory for a basic tensor model. It should illustrate some of the obstacles one encounters in constructing a collective field theory.

The basic fields of the model are:

$$\phi^{abc} \qquad \text{and} \qquad \bar{\phi}_{abc}$$

The gauge group of the theory is $U(N_1)\times U(N_2)\times U(N_3)$ and the fields transform under this gauge group like

$$\phi^{abc} \to (U_1)^a_d (U_2)^b_e (U_3)^c_f \phi^{def}$$

and

$$\bar{\phi}_{abc} \to (U_1)^d_a (U_2)^e_b (U_3)^f_c \bar{\phi}_{def}$$

The Hamiltonian of this theory is

$$H = -\frac{\partial}{\partial\bar{\phi}_{abc}}\frac{\partial}{\partial\phi^{abc}} + \frac{1}{4}\phi^{abc}\bar{\phi}_{abc} \tag{2.273}$$

2.7.1 Changing variables

The idea is to develop a large N expansion by systematically contracting indices of particular gauge groups so as to make their N_i dependance explicit. Thus, consider constructing a collective field theory by changing variables to

$$T(2n)^{i_1 i_3 \dots i_{2n-1}}_{i_2 i_4 \dots i_{2n}} = \phi^{a_1 b_1 i_1} \bar{\phi}_{a_1 b_2 i_2} \phi^{a_2 b_2 i_3} \bar{\phi}_{a_2 b_3 i_4} \cdots \phi^{a_n b_n i_{2n-1}} \bar{\phi}_{a_n b_1 i_{2n}} \tag{2.274}$$

By doing this the N_1 and N_2 dependance should become explicit in the Hamiltonian. Looking at the kinetic term

$$\begin{aligned}
&- \frac{\partial}{\partial \bar{\phi}_{abc}} \frac{\partial}{\partial \phi^{abc}} \\
&= i \sum_n - \frac{\partial^2 T(2n)^{i_1 i_3 \dots i_{2n-1}}_{i_2 i_4 \dots i_{2n}}}{\partial \bar{\phi}_{abc} \partial \phi^{abc}} \left(- i \frac{\partial}{\partial T(2n)^{i_1 i_3 \dots i_{2n-1}}_{i_2 i_4 \dots i_{2n}}} \right) \\
&+ \sum_{n,m} \frac{\partial T(2n)^{i_1 i_3 \dots i_{2n-1}}_{i_2 i_4 \dots i_{2n}}}{\partial \phi^{abc}} \frac{\partial T(2m)^{j_1 j_3 \dots j_{2m-1}}_{j_2 j_4 \dots j_{2m}}}{\partial \bar{\phi}_{abc}} \left(- i \frac{\partial}{\partial T(2n)^{i_1 i_3 \dots i_{2n-1}}_{i_2 i_4 \dots i_{2n}}} \right) \left(- i \frac{\partial}{\partial T(2m)^{j_1 j_3 \dots j_{2m-1}}_{j_2 j_4 \dots j_{2m}}} \right) \\
&= \sum_{n,m} \Omega(n,m)^{i_1 i_3 \dots i_{2n-1}, j_1 j_3 \dots j_{2m-1}}_{i_2 i_4 \dots i_{2n}, j_2 j_4 \dots j_{2m}} \Pi(2n)^{i_2 i_4 \dots i_{2n}}_{i_1 i_3 \dots i_{2n-1}} \Pi(2m)^{j_2 j_4 \dots j_{2m}}_{j_1 j_3 \dots j_{2m-1}} + i \sum_n \omega(n)^{i_1 i_3 \dots i_{2n-1}}_{i_2 i_4 \dots i_{2n}} \Pi(2n)^{i_2 i_4 \dots i_{2n}}_{i_1 i_3 \dots i_{2n-1}}
\end{aligned} \tag{2.275}$$

where

$$\Pi(2n)^{i_2 i_4 \dots i_{2n}}_{i_1 i_3 \dots i_{2n-1}} = \left(- i \frac{\partial}{\partial T(2n)^{i_1 i_3 \dots i_{2n-1}}_{i_2 i_4 \dots i_{2n}}} \right) \tag{2.276}$$

$$\begin{aligned}
\Omega(n,m)^{i_1 i_3 \dots i_{2n-1}, j_1 j_3 \dots j_{2m-1}}_{i_2 i_4 \dots i_{2n}, j_2 j_4 \dots j_{2m}} &= \frac{\partial T(2n)^{i_1 i_3 \dots i_{2n-1}}_{i_2 i_4 \dots i_{2n}}}{\partial \phi^{abc}} \frac{\partial T(2m)^{j_1 j_3 \dots j_{2m-1}}_{j_2 j_4 \dots j_{2m}}}{\partial \bar{\phi}_{abc}} \\
&= \sum_{k=1}^{n} \left(\phi^{a_1 b_1 i_1} \bar{\phi}_{a_1 b_2 i_2} \cdots \bar{\phi}_{a_{k-1} b i_{2k-2}} \delta^{i_{2k-1}}_{c} \bar{\phi}_{a b_{k+1} i_{2k}} \cdots \phi^{a_n b_n i_{2n-1}} \bar{\phi}_{a_n b_1 i_{2n}} \right) \\
&\times \sum_{l=1}^{m} \left(\phi^{a_1 b_1 j_1} \bar{\phi}_{a_1 b_2 j_2} \cdots \phi^{a b_l j_{2l-1}} \delta^{c}_{j_{2l}} \phi^{a_{l+1} b j_{2l+1}} \cdots \phi^{a_m b_m j_{2m-1}} \bar{\phi}_{a_m b_1 j_{2m}} \right) \\
&= \sum_{k,l=1}^{n,m} \delta^{i_{2k-1}}_{j_{2l}} \phi^{a_1 b_1 i_1} \bar{\phi}_{a_1 b_2 i_2} \cdots \bar{\phi}_{a_{k-1} b i_{2k-2}} \phi^{a_{l+1} b j_{2l+1}} \cdots \phi^{a_m b_m j_{2m-1}} \bar{\phi}_{a_m b_1 j_{2m}} \\
&\qquad \times \phi^{a_1 b_1 j_1} \bar{\phi}_{a_1 b_2 j_2} \cdots \phi^{a b_l j_{2l-1}} \phi_{a b_{k+1} i_{2k}} \cdots \phi^{a_n b_n i_{2n-1}} \bar{\phi}_{a_n b_1 i_{2n}} \\
&= \sum_{k,l=1}^{n,m} \delta^{i_{2k-1}}_{j_{2l}} T(2(n+m-1))^{i_1 i_3 \cdots i_{2k-3} j_{2l+1} j_{2l+3} \cdots j_{2m-1} j_1 j_3 \cdots j_{2l-3} j_{2l-1} i_{2k+1} \cdots i_{2n-1}}_{i_2 i_4 \cdots i_{2k-2} j_{2l+2} j_{2l+4} \cdots j_{2m} \quad j_2 j_4 \cdots j_{2l-2} i_{2k} \quad i_{2k+2} \cdots i_{2n}}
\end{aligned} \tag{2.277}$$

and

$$
\begin{aligned}
\omega(n)^{i_1 i_3 \ldots i_{2n-1}}_{i_2 i_4 \ldots i_{2n}} &= -\frac{\partial^2 T(2n)^{i_1 i_3 \ldots i_{2n-1}}_{i_2 i_4 \ldots i_{2n}}}{\partial\bar{\phi}_{abc}\partial\phi^{abc}} \\
&= -\frac{\partial}{\partial\bar{\phi}_{abc}}\sum_{k=1}^{n}\left(\phi^{a_1 b_1 i_1}\bar{\phi}_{a_1 b_2 i_2}\cdots\bar{\phi}_{a_{k-1} b i_{2k-2}}\delta_c^{i_{2k-1}}\bar{\phi}_{a b_{k+1} i_{2k}}\cdots\phi^{a_n b_n i_{2n-1}}\bar{\phi}_{a_n b_1 i_{2n}}\right) \\
&= -\sum_{\substack{k,l=1\\ l<k-1}}^{n}\delta^{i_{2k-1}}_{i_{2l}}\Bigg(\phi^{a_1 b_1 i_1}\bar{\phi}_{a_1 b_2 i_2}\cdots\bar{\phi}_{a_{l-1} b_l i_{2l-2}}\phi^{a b_l i_{2l-1}}\bar{\phi}_{a b_{k+1} i_{2k}}\phi^{a_{k+1} b_{k+1} i_{2k+1}}\cdots\phi^{a_n b_n i_{2n-1}}\bar{\phi}_{a_n b_1 i_{2n}} \\
&\qquad \times\phi^{a_{l+1} b i_{2l+1}}\bar{\phi}_{a_{l+1} b_{l+2} i_{2l+2}}\cdots\phi^{a_{k-1} b_{k-1} i_{2k-3}}\bar{\phi}_{a_{k-1} b i_{2k-2}}\Bigg)+ \\
&\quad -\sum_{\substack{k,l=1\\ l>k}}^{n}\delta^{i_{2k-1}}_{i_{2l}}\Bigg(\phi^{a_1 b_1 i_1}\bar{\phi}_{a_1 b_2 i_2}\cdots\bar{\phi}_{a_{k-1} b i_{2k-2}}\phi^{a_{l+1} b i_{2l+1}}\cdots\phi^{a_n b_n i_{2n-1}}\bar{\phi}_{a_n b_1 i_{2n}} \\
&\qquad \times\phi^{a b_l i_{2l-1}}\bar{\phi}_{a b_{k+1} i_{2k}}\cdots\bar{\phi}_{a_{l-1} b_l i_{2l-2}}\Bigg)+ \\
&\quad - N_2\sum_{k=1}^{n}\delta^{i_{2k-1}}_{i_{2k-2}}\left(\phi^{a_1 b_1 i_1}\bar{\phi}_{a_1 b_2 i_2}\cdots\phi^{a b_{k-1} i_{2k-3}}\bar{\phi}_{a b_{k+1} i_{2k}}\cdots\phi^{a_n b_n i_{2n-1}}\bar{\phi}_{a_n b_1 i_{2n}}\right)+ \\
&\quad - N_1\sum_{k=1}^{n}\delta^{i_{2k-1}}_{i_{2k}}\left(\phi^{a_1 b_1 i_1}\bar{\phi}_{a_1 b_2 i_2}\cdots\bar{\phi}_{a_{k-1} b i_{2k-2}}\phi^{a_{k+1} b i_{2k+1}}\cdots\phi^{a_n b_n i_{2n-1}}\bar{\phi}_{a_n b_1 i_{2n}}\right) \\
&= -\sum_{\substack{k,l=1\\ l<k-1}}^{n}\delta^{i_{2k-1}}_{i_{2l}}T(2(l+n-k))^{i_1 i_3\cdots i_{2l-3} i_{2l-1} i_{2k+1}\cdots i_{2n-1}}_{i_2 i_4\cdots i_{2l-2} i_{2k}\quad i_{2k+2}\cdots i_{2n}}T(2(k-l-1))^{i_{2l+1}\cdots i_{2k-3}}_{i_{2l+2}\cdots i_{2k-2}}+ \\
&\quad -\sum_{\substack{k,l=1\\ l>k}}^{n}\delta^{i_{2k-1}}_{i_{2l}}T(2(n+k-l-1))^{i_1 i_3\cdots i_{2k-3} i_{2l+1}\cdots i_{2n-1}}_{i_2 i_4\cdots i_{2k-2} i_{2l+2}\cdots i_{2n}}T(2(l-k))^{i_{2l-1}\cdots i_{2l-3}}_{i_{2k}\quad\cdots i_{2l-2}}+ \\
&\quad - N_1\sum_{k=1}^{n}\delta^{i_{2k-1}}_{i_{2k}}T(2n-2)^{i_1\cdots i_{2k-3} i_{2k+1}\cdots i_{2n-1}}_{i_2\cdots i_{2k-2} i_{2k+2}\cdots i_{2n}} - N_2\sum_{k=1}^{n}\delta^{i_{2k-1}}_{i_{2k-2}}T(2n-2)^{i_1\cdots i_{2k-3} i_{2k+1}\cdots i_{2n-1}}_{i_2\cdots i_{2k}\quad i_{2k+2}\cdots i_{2n}}
\end{aligned}
\tag{2.278}
$$

The index notation can be written more compactly. The convention that will be used from here is as follows:

- Pairs of capital letters, raised and lowered, will indicate a sequence of indices.
- The subscript of the capital index with the lowest value will indicate where the sequence starts.
- The subscript of the capital index with the highest value will indicate where the sequence ends.

For example

$$
T(2(n+m-1))^{I_1 J_{2l+1} J_1 j_{2l-1}}_{I_{2k-2} J_{2m} J_{2l-2} i_{2k} I_{2n}} = T(2(n+m-1))^{i_1 i_3\cdots i_{2k-3} j_{2l+1} j_{2l+3}\cdots j_{2m-1} j_1 j_3\cdots j_{2l-3} j_{2l-1} i_{2k+1}\cdots i_{2n-1}}_{i_2 i_4\cdots i_{2k-2} j_{2l+2} j_{2l+4}\cdots j_{2m}\quad j_2 j_4\cdots j_{2l-2} i_{2k}\quad i_{2k+2}\cdots i_{2n}}
$$

and

$$\Pi(2n)^{i_2 i_4 \ldots i_{2n}}_{i_1 i_3 \ldots i_{2n-1}} = \Pi(2n)^{I_{2n}}_{I_1}$$

$$\begin{aligned}
\implies \omega(n)^{i_1 i_3 \ldots i_{2n-1}}_{i_2 i_4 \ldots i_{2n}} &= \omega(n)^{I_1}_{I_{2n}} \\
&= -2 \sum_{\substack{k,l=1 \\ l<k-1}}^{n} \delta^{i_{2k-1}}_{i_{2l}} T(2(l+n-k))^{I_1 i_{2l-1} I_{2k+1}}_{I_{2l-2} i_{2k} I_{2n}} T(2(k-l)-2)^{I_{2l+1}}_{I_{2k-2}} + \\
&\quad - N_1 \sum_{k=1}^{n} \delta^{i_{2k-1}}_{i_{2k}} T(2n-2)^{I_1 I_{2k+1}}_{I_{2k-2} I_{2n}} - N_2 \sum_{k=1}^{n} \delta^{i_{2k-1}}_{i_{2k-2}} T(2n-2)^{I_1 i_{2k-3} I_{2k+1}}_{I_{2k-4} i_{2k} I_{2n}}
\end{aligned}$$

$$\implies \Omega(n,m)^{I_1, J_1}_{I_{2n}, J_{2m}} = \sum_{k,l=1}^{n,m} \delta^{i_{2k-1}}_{j_{2l}} T(2(n+m-1))^{I_1 J_{2l+1} J_1 j_{2l-1} I_{2k+1}}_{I_{2k-2} J_{2m} J_{2l-2} i_{2k} I_{2n}}$$

$$\implies -\frac{\partial}{\partial \bar{\phi}_{abc}} \frac{\partial}{\partial \phi^{abc}} = \sum_{n,m} \Omega(n,m)^{I_1, J_1}_{I_{2n}, J_{2m}} \Pi(2n)^{I_{2n}}_{I_1} \Pi(2m)^{J_{2m}}_{J_1} + i \sum_{n} \omega(n)^{I_1}_{I_{2n}} \Pi(2n)^{I_{2n}}_{I_1}$$

So the Hamiltonian will be

$$H = i \sum_{n} \omega(n)^{I_1}_{I_{2n}} \Pi(2n)^{I_{2n}}_{I_1} + \sum_{n,m} \Omega(n,m)^{I_1, J_1}_{I_{2n}, J_{2m}} \Pi(2n)^{I_{2n}}_{I_1} \Pi(2m)^{J_{2m}}_{J_1} + \frac{1}{4} T(2) \tag{2.279}$$

2.7.2 Re-scaling and Jacobian

The next task is to re-scale the theory and find $H_{\text{eff}} = J^{\frac{1}{2}} H J^{-\frac{1}{2}}$

$$\begin{aligned}
J^{\frac{1}{2}} H J^{-\frac{1}{2}} &= i \sum_{n} \omega(n)^{I_1}_{I_{2n}} J^{\frac{1}{2}} \Pi(2n)^{I_{2n}}_{I_1} J^{-\frac{1}{2}} + \sum_{n,m} \Omega(n,m)^{I_1, J_1}_{I_{2n}, J_{2m}} J^{\frac{1}{2}} \Pi(2n)^{I_{2n}}_{I_1} \Pi(2m)^{J_{2m}}_{J_1} J^{-\frac{1}{2}} + \frac{1}{4} T(2) \\
&= i \sum_{n} \omega(n)^{I_1}_{I_{2n}} \left(\frac{i}{2} \frac{\partial \ln J}{\partial T(2n)^{I_1}_{I_{2n}}} + \Pi(2n)^{I_{2n}}_{I_1} \right) + \\
&\quad + \sum_{n,m} \Omega(n,m)^{I_1, J_1}_{I_{2n}, J_{2m}} \left(\frac{i}{2} \frac{\partial \ln J}{\partial T(2n)^{I_1}_{I_{2n}}} + \Pi(2n)^{I_{2n}}_{I_1} \right) \left(\frac{i}{2} \frac{\partial \ln J}{\partial T(2m)^{J_1}_{J_{2m}}} + \Pi(2m)^{J_{2m}}_{J_1} \right) + \frac{1}{4} T(2) \\
&= i \sum_{n} \omega(n)^{I_1}_{I_{2n}} \left(\frac{i}{2} \frac{\partial \ln J}{\partial T(2n)^{I_1}_{I_{2n}}} + \Pi(2n)^{I_{2n}}_{I_1} \right) + \\
&\quad + \sum_{n,m} \Omega(n,m)^{I_1, J_1}_{I_{2n}, J_{2m}} \Bigg(- \frac{1}{4} \frac{\partial \ln J}{\partial T(2n)^{I_1}_{I_{2n}}} \frac{\partial \ln J}{\partial T(2m)^{J_1}_{J_{2m}}} + \frac{i}{2} \frac{\partial \ln J}{\partial T(2n)^{I_1}_{I_{2n}}} \Pi(2m)^{J_{2m}}_{J_1} + \\
&\quad + \frac{1}{2} \frac{\partial^2 \ln J}{\partial T(2n)^{I_1}_{I_{2n}} \partial T(2m)^{J_1}_{J_{2m}}} + \frac{i}{2} \frac{\partial \ln J}{\partial T(2m)^{J_1}_{J_{2m}}} \Pi(2n)^{I_{2n}}_{I_1} + \Pi(2n)^{I_{2n}}_{I_1} \Pi(2m)^{J_{2m}}_{J_1} \Bigg) + \frac{1}{4} T(2)
\end{aligned} \tag{2.280}$$

Using

$$\begin{aligned}
&\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}\Pi(2n)^{I_{2n}}_{I_1}\Pi(2m)^{J_{2m}}_{J_1}\\
&=\Pi(2n)^{I_{2n}}_{I_1}\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}\Pi(2m)^{J_{2m}}_{J_1}+\Big[\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}},\Pi(2n)^{I_{2n}}_{I_1}\Big]\Pi(2m)^{J_{2m}}_{J_1}\\
&=\Pi(2n)^{I_{2n}}_{I_1}\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}\Pi(2m)^{J_{2m}}_{J_1}+i\frac{\partial\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}}{\partial T(2n)^{I_1}_{I_{2n}}}\Pi(2m)^{J_{2m}}_{J_1}
\end{aligned}\tag{2.281}$$

gives

$$\begin{aligned}
H_{\text{eff}}=\sum_n\Bigg[&-\frac{1}{2}\omega(n)^{I_1}_{I_{2n}}\frac{\partial\ln J}{\partial T(2n)^{I_1}_{I_{2n}}}+i\omega(n)^{I_1}_{I_{2n}}\Pi(2n)^{I_{2n}}_{I_1}+\\
&-\sum_m\Bigg(\frac{1}{4}\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}\frac{\partial\ln J}{\partial T(2n)^{I_1}_{I_{2n}}}\frac{\partial\ln J}{\partial T(2m)^{J_1}_{J_{2m}}}-\frac{i}{2}\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}\frac{\partial\ln J}{\partial T(2n)^{I_1}_{I_{2n}}}\Pi(2m)^{J_{2m}}_{J_1}+\\
&-\frac{1}{2}\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}\frac{\partial^2\ln J}{\partial T(2n)^{I_1}_{I_{2n}}\partial T(2m)^{J_1}_{J_{2m}}}-\frac{i}{2}\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}\frac{\partial\ln J}{\partial T(2m)^{J_1}_{J_{2m}}}\Pi(2n)^{I_{2n}}_{I_1}+\\
&-\Pi(2n)^{I_{2n}}_{I_1}\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}\Pi(2m)^{J_{2m}}_{J_1}-i\frac{\partial\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}}{\partial T(2n)^{I_1}_{I_{2n}}}\Pi(2m)^{J_{2m}}\Bigg)\Bigg]+\frac{1}{4}T(2)\\
=-\sum_n\Bigg\{&\frac{1}{2}\omega(n)^{I_1}_{I_{2n}}\frac{\partial\ln J}{\partial T(2n)^{I_1}_{I_{2n}}}+\sum_m\Bigg(\frac{1}{4}\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}\frac{\partial\ln J}{\partial T(2n)^{I_1}_{I_{2n}}}\frac{\partial\ln J}{\partial T(2m)^{J_1}_{J_{2m}}}+\\
&-\frac{1}{2}\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}\frac{\partial^2\ln J}{\partial T(2n)^{I_1}_{I_{2n}}\partial T(2m)^{J_1}_{J_{2m}}}-\Pi(2n)^{I_{2n}}_{I_1}\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}\Pi(2m)^{J_{2m}}_{J_1}\Bigg)+\\
&-i\Bigg[\omega(n)^{I_1}_{I_{2n}}\Pi(2n)^{I_{2n}}_{I_1}+\sum_m\Bigg(\frac{1}{2}\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}\frac{\partial\ln J}{\partial T(2n)^{I_1}_{I_{2n}}}\Pi(2m)^{J_{2m}}_{J_1}+\\
&+\frac{1}{2}\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}\frac{\partial\ln J}{\partial T(2m)^{J_1}_{J_{2m}}}\Pi(2n)^{I_{2n}}_{I_1}+\\
&+\frac{\partial\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}}{\partial T(2n)^{I_1}_{I_{2n}}}\Pi(2m)^{J_{2m}}\Bigg)\Bigg]\Bigg\}+\frac{1}{4}T(2)
\end{aligned}\tag{2.282}$$

Consider the term

$$\sum_{n,m}\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}\frac{\partial^2\ln J}{\partial T(2n)^{I_1}_{I_{2n}}\partial T(2m)^{J_1}_{J_{2m}}}$$

Renaming indices $n\leftrightarrow m$ and $I\leftrightarrow J$ gives

$$\sum_{n,m}\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}\frac{\partial^2\ln J}{\partial T(2n)^{I_1}_{I_{2n}}\partial T(2m)^{J_1}_{J_{2m}}}=\sum_{n,m}\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}\frac{\partial^2\ln J}{\partial T(2m)^{J_1}_{J_{2m}}\partial T(2n)^{I_1}_{I_{2n}}}$$

Since partial derivatives commute this implies that

$$\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}=\Omega(m,n)^{I_1,J_1}_{I_{2n},J_{2m}}$$

Using this fact and renaming some summation variables

$$H_{\text{eff}} = -\sum_n \left\{ \frac{1}{2}\omega(n)^{I_1}_{I_{2n}} \frac{\partial \ln J}{\partial T(2n)^{I_1}_{I_{2n}}} + \sum_m \left(\frac{1}{4}\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}} \frac{\partial \ln J}{\partial T(2n)^{I_1}_{I_{2n}}} \frac{\partial \ln J}{\partial T(2m)^{J_1}_{J_{2m}}} + \right.\right.$$
$$- \frac{1}{2}\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}} \frac{\partial^2 \ln J}{\partial T(2n)^{I_1}_{I_{2n}} \partial T(2m)^{J_1}_{J_{2m}}} - \Pi(2n)^{I_{2n}}_{I_1} \Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}} \Pi(2m)^{J_{2m}}_{J_1} \Big) +$$
$$\left. - i\left[\omega(n)^{I_1}_{I_{2n}} + \sum_m \left(\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}} \frac{\partial \ln J}{\partial T(2m)^{J_1}_{J_{2m}}} + \frac{\partial \Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}}{\partial T(2m)^{J_1}_{J_{2m}}} \right) \right] \Pi(2n)^{I_{2n}}_{I_1} \right\} + \frac{1}{4}\text{Tr}\left[T(2)\right] \tag{2.283}$$

Now it is clear that the only part of H_{eff} which is not manifestly Hermitian are the terms in square brackets. So that

$$H_{\text{eff}} - H^\dagger_{\text{eff}} = i\sum_n \left\{ \left[\omega(n)^{I_1}_{I_{2n}} + \sum_m \left(\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}} \frac{\partial \ln J}{\partial T(2m)^{J_1}_{J_{2m}}} + \frac{\partial \Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}}{\partial T(2m)^{J_1}_{J_{2m}}} \right) \right] \Pi(2n)^{I_{2n}}_{I_1} + \right.$$
$$\left. + i\Pi(2n)^{I_{2n}}_{I_1} \left[\omega(n)^{I_1}_{I_{2n}} + \sum_m \left(\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}} \frac{\partial \ln J}{\partial T(2m)^{J_1}_{J_{2m}}} + \frac{\partial \Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}}{\partial T(2m)^{J_1}_{J_{2m}}} \right) \right] \right\}$$

Thus, for H_{eff} to be Hermitian the following must hold

$$\omega(n)^{I_1}_{I_{2n}} + \sum_m \left(\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}} \frac{\partial \ln J}{\partial T(2m)^{J_1}_{J_{2m}}} + \frac{\partial \Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}}{\partial T(2m)^{J_1}_{J_{2m}}} \right) = 0 \tag{2.284}$$

This is the differential equation for J. Consider the following term

$$\frac{\partial \Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}}{\partial T(2m)^{J_1}_{J_{2m}}} = \frac{\partial}{\partial T(2m)^{J_1}_{J_{2m}}} \sum_{k,l=1}^{n,m} \delta^{i_{2k-1}}_{j_{2l}} T(2(n+m-1))^{I_1 J_{2l+1} J_1 j_{2l-1} I_{2k+1}}_{I_{2k-2} J_{2m} J_{2l-2} i_{2k} I_{2n}}$$

The general derivative is

$$\frac{\partial T(2n)^{I_1}_{I_{2n}}}{\partial T(2m)^{J_1}_{J_{2m}}} = \delta_{mn} \prod_{k=1}^{n} \delta^{i_{2k-1}}_{j_{2k-1}} \delta^{j_{2k}}_{i_{2k}}$$

which implies that the derivative of $\Omega(n,m)$ is in general zero unless $n = 1$. If $n \neq 1$

$$\omega(n)^{I_1}_{I_{2n}} + \sum_m \left(\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}} \frac{\partial \ln J}{\partial T(2m)^{J_1}_{J_{2m}}} \right) = 0$$

The trouble with solving this is that it requires calculating the inverse of $\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}$ which is very difficult if not impossible. Attempting to use the analog of (2.266) yields the following

$$\frac{\partial \Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}}{\partial T(2n)^{I_1}_{I_{2n}}} \frac{\partial \ln J}{\partial T(2m)^{J_1}_{J_{2m}}} + \Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}} \frac{\partial^2 \ln J}{\partial T(2n)^{I_1}_{I_{2n}} \partial T(2m)^{J_1}_{J_{2m}}} + \frac{\partial^2 \Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}}{\partial T(2n)^{I_1}_{I_{2n}} \partial T(2m)^{J_1}_{J_{2m}}} = 0$$

For $n \neq 1$ this implies

$$\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}} \frac{\partial^2 \ln J}{\partial T(2n)^{I_1}_{I_{2n}} \partial T(2m)^{J_1}_{J_{2m}}} = 0$$

The trouble is that in order to simplify this we have to multiply both sides by the inverse of $\Omega(\,n,m)^{I_1,J_1}_{I_{2n},J_{2m}}$ which is likely not to exist. If however, it is assumed to exist.

$$\begin{aligned}
&\Rightarrow \frac{\partial^2 \ln J}{\partial T(2n)^{I_1}_{I_{2n}} \partial T(2m)^{J_1}_{J_{2m}}} = 0 \\
&\Rightarrow \frac{\partial \ln J}{\partial T(2m)^{J_1}_{J_{2m}}} = B \\
&\Rightarrow \ln J = BT(2m)^{J_1}_{J_{2m}} + C
\end{aligned}$$

Since m and the J_i's are arbitrary the general solution would be

$$\ln J = \sum_n \sum_{i_1,i_2,\ldots,i_n} B(n)^{I_1}_{I_{2n}} T(2n)^{I_1}_{I_{2n}} + C$$

$$J = Ae^{\sum_n \sum_{i_1,i_2,\ldots,i_n} B(n)^{I_1}_{I_{2n}} T(2n)^{I_1}_{I_{2n}}} \tag{2.285}$$

With the aim of getting the N dependance of the $B(n)$'s this expression for J can be put into (2.284) giving

$$\begin{aligned}
&\omega(n)^{I_1}_{I_{2n}} + \sum_m \left(\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}} \frac{\partial \ln J}{\partial T(2m)^{J_1}_{J_{2m}}} \right) = 0 \\
\Rightarrow &\omega(n)^{I_1}_{I_{2n}} + \sum_m \left(\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}} \frac{\partial}{\partial T(2m)^{J_1}_{J_{2m}}} \left(\sum_k \sum_{p_1,p_2,\ldots,p_k} B(k)^{P_1}_{P_{2k}} T(2k)^{P_1}_{P_{2k}} + C \right) \right) = 0 \\
\Rightarrow &\omega(n)^{I_1}_{I_{2n}} + \sum_m \left(\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}} \left(\sum_k \sum_{p_1,p_2,\ldots,p_k} B(k)^{P_1}_{P_{2k}} \delta_{mk} \prod_{l=1}^{m} \delta^{p_{2l-1}}_{j_{2l-1}} \delta^{j_{2l}}_{p_{2l}} \right) \right) = 0 \\
\Rightarrow &\omega(n)^{I_1}_{I_{2n}} + \sum_m \left(\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}} B(m)^{J_{2m}}_{J_1} \right) = 0
\end{aligned}$$

However, in order to solve for the $B(n)$'s one would again need the inverse of $\Omega(n,m)^{I_1,J_1}_{I_{2n},J_{2m}}$.

2.7.3 New variables

If new variables are considered which contract the $U(N_3)$ indices there is a guarantee of being able to invert Ω. There is the question of how to contract the indices since there are many possible ways and none of them stand out. So consider variables of the form

$$\chi(\sigma,n) \equiv T(2n)^{i_1 \quad i_3 \quad \ldots \quad i_n}_{i_{\sigma(1)} i_{\sigma(2)} \ldots i_{\sigma(n)}} \tag{2.286}$$

These represent the most general way of contracting the $U(N_3)$ indices where each different contraction is defined by an element of the symmetric group ($\sigma \in S_n$). For simplicity start by looking at the case where $\sigma = \mathbb{I}$ (these are the variables used in [39] and Chapter 3)

$$\chi(\mathbb{I}, n) \equiv T(2n)^{i_1 i_3 \dots i_n}_{i_1 i_2 \dots i_n}$$

the kinetic term of the Hamiltonian becomes

$$\begin{aligned} -\frac{\partial}{\partial\bar{\phi}_{abc}}\frac{\partial}{\partial\phi^{abc}} &= i\sum_{n=1}^{2} -\frac{\partial^2\chi(\mathbb{I},n)}{\partial\bar{\phi}_{abc}\partial\phi^{abc}}\Big(-i\frac{\partial}{\chi(\mathbb{I},n)}\Big) \\ &\quad + \sum_{n,m=1}^{2}\frac{\partial\chi(\mathbb{I},n)}{\partial\phi^{abc}}\frac{\partial\chi(\mathbb{I},m)}{\partial\bar{\phi}_{abc}}\Big(-i\frac{\partial}{\partial\chi(\mathbb{I},n)}\Big)\Big(-i\frac{\partial}{\partial\chi(\mathbb{I},m)}\Big) \\ &= i\sum_{n=1}^{2}\omega(n)\Pi(n) + \sum_{n,m=1}^{2}\Omega(n,m)\Pi(n)\Pi(m) \end{aligned} \tag{2.287}$$

where

$$\Pi(n) = -i\frac{\partial}{\partial\chi(\mathbb{I},n)}$$

$$\begin{aligned} \Omega(n,m) &= \frac{\partial\chi(\mathbb{I},n)}{\partial\phi^{abc}}\frac{\partial\chi(\mathbb{I},m)}{\partial\bar{\phi}_{abc}} \\ &= nm\chi(\mathbb{I},n+m-1) \end{aligned} \tag{2.288}$$

$$\begin{aligned} \omega(n) &= -\frac{\partial^2\chi(\mathbb{I},n)}{\partial\bar{\phi}_{abc}\partial\phi^{abc}} \\ &= -\sum_{k=1}^{n}\Bigg[\sum_{l=1}^{k-2}\chi(\mathbb{I},n+l-k)\chi(\mathbb{I},k-l-1) - \sum_{l=k+1}^{n}\chi(\mathbb{I},n-l+k-1)\chi(\mathbb{I},l-k)\Bigg]+ \\ &\qquad - (N_1N_3+N_2)n\chi(\mathbb{I},n-1) \end{aligned} \tag{2.289}$$

$$= -\sum_{k=1}^{n-1}(2(n-k)-1)\chi(\mathbb{I},n-k-1)\chi(\mathbb{I},k) - (N_1N_3+N_2)n\chi(\mathbb{I},n-1) \tag{2.290}$$

giving

$$\begin{aligned} H_{\text{eff}} = \sum_n\Bigg\{ &-\frac{1}{2}\omega(n)\frac{\partial\ln J}{\partial\chi(\mathbb{I},n)} + \sum_m\Bigg(\Pi(2n)\Omega(n,m)\Pi(2m) - \frac{1}{4}\Omega(n,m)\frac{\partial\ln J}{\partial\chi(\mathbb{I},n)}\frac{\partial\ln J}{\partial\chi(\mathbb{I},m)} \\ &+ \frac{1}{2}\Omega(n,m)\frac{\partial^2\ln J}{\partial\chi(\mathbb{I},n)\partial\chi(\mathbb{I},m)}\Bigg) + i\Bigg[\omega(n) + \sum_m\Bigg(\Omega(n,m)\frac{\partial\ln J}{\partial\chi(\mathbb{I},m)} + \frac{\partial\Omega(n,m)}{\partial\chi(\mathbb{I},m)}\Bigg)\Bigg]\Pi(2n)\Bigg\} + \frac{1}{4}\chi(\mathbb{I},2) \end{aligned}$$

$$\Rightarrow \omega(n) + \sum_m\Bigg(\Omega(n,m)\frac{\partial\ln J}{\partial\chi(\mathbb{I},m)} + \frac{\partial\Omega(n,m)}{\partial\chi(\mathbb{I},m)}\Bigg) = 0$$

Using the notation $\omega_n\partial_n \ln J = \sum_n \omega(n)\frac{\partial \ln J}{\partial\chi(\mathbb{I},n)}$

$$\Rightarrow \omega_n + \Omega_{nm}\partial_m \ln J + \partial_m\Omega_{mn} = 0$$

$$\Rightarrow \Omega^{-1}_{pn}\Omega_{nm}\partial_m \ln J = -\Omega^{-1}_{pn}(\omega_n + \partial_m\Omega_{mn})$$

$$\Rightarrow \delta_{pm}\partial_m \ln J = -\Omega^{-1}_{pn}(\omega_n + \partial_m\Omega_{mn})$$

$$\Rightarrow \partial_n \ln J = -\Omega^{-1}_{np}(\omega_p + \partial_q\Omega_{qp})$$

$$\begin{aligned}
\Rightarrow H_{\text{eff}} &= \frac{1}{2}\omega_n\Omega^{-1}_{np}(\omega_p + \partial_q\Omega_{qp}) - \frac{1}{4}\Omega_{nm}\Omega^{-1}_{np}(\omega_p + \partial_q\Omega_{qp})\Omega^{-1}_{mi}(\omega_i + \partial_j\Omega_{ji}) + \\
&\quad + \frac{1}{2}\Omega_{nm}\partial_n\partial_m \ln J + \Pi_n\Omega_{nm}\Pi_m + \frac{1}{4}\chi(\mathbb{I},1) \\
&= \frac{1}{2}\omega_n\Omega^{-1}_{np}(\omega_p + \partial_q\Omega_{qp}) - \frac{1}{4}(\omega_m + \partial_q\Omega_{qm})\Omega^{-1}_{mi}(\omega_i + \partial_j\Omega_{ji}) + \\
&\quad - \frac{1}{2}\Omega_{nm}[\Omega^{-1}_{mp}(\partial_n\omega_p + \partial_n\partial_q\Omega_{qp}) - \Omega^{-1}_{mk}\partial_n\Omega_{ki}\Omega^{-1}_{ip}(\omega_p + \partial_q\Omega_{qp})] + \Pi_n\Omega_{nm}\Pi_m + \frac{1}{4}\chi(\mathbb{I},1) \\
&= \frac{1}{2}\omega_n\Omega^{-1}_{np}(\omega_p + \partial_q\Omega_{qp}) - \frac{1}{4}\omega_m\Omega^{-1}_{mi}(\omega_i + \partial_j\Omega_{ji}) - \frac{1}{4}\partial_q\Omega_{qm}\Omega^{-1}_{mi}(\omega_i + \partial_j\Omega_{ji}) + \\
&\quad - \frac{1}{2}(\partial_n\omega_n + \partial_n\partial_q\Omega_{qn}) + \frac{1}{2}\partial_n\Omega_{ni}\Omega^{-1}_{ip}(\omega_p + \partial_q\Omega_{qp}) + \Pi_n\Omega_{nm}\Pi_m + \frac{1}{4}\chi(\mathbb{I},1) \\
&= \frac{1}{4}\omega_m\Omega^{-1}_{mi}(\omega_i + \partial_j\Omega_{ji}) + \frac{1}{4}\partial_n\Omega_{nm}\Omega^{-1}_{mi}(\omega_i + \partial_j\Omega_{ji}) + \\
&\quad - \frac{1}{2}(\partial_n\omega_n + \partial_n\partial_q\Omega_{qn}) + \Pi_n\Omega_{nm}\Pi_m + \frac{1}{4}\chi(\mathbb{I},1) \\
&= \Pi_n\Omega_{nm}\Pi_m + \frac{1}{4}(\omega_n + \partial_m\Omega_{mn})\Omega^{-1}_{ni}(\omega_i + \partial_j\Omega_{ji}) + \frac{1}{4}\chi(\mathbb{I},1) - \frac{1}{2}\partial_n\omega_n - \frac{1}{2}\partial_n\partial_q\Omega_{qn} \qquad (2.291)
\end{aligned}$$

Dropping the indices

$$\Rightarrow H_{\text{eff}} = \Pi\Omega\Pi + \frac{1}{4}(\omega + \partial\Omega)\Omega^{-1}(\omega + \partial\Omega) + \frac{1}{4}\chi(\mathbb{I},2) - \frac{1}{2}\partial\omega - \frac{1}{2}\partial(\partial\Omega) \qquad (2.292)$$

This seems to be the only case in which a collective field theory hamiltonian can be obtained. An attempt to construct a collective field theory with $\chi(\sigma_n, n)$ for general σ and the difficulties which result is illustrated below.

$$
\begin{aligned}
\Omega(n,m) &= \frac{\partial\chi(\sigma_n,n)}{\partial\phi^{abc}}\frac{\partial\chi(\sigma_m,m)}{\partial\bar{\phi}_{abc}} \\
&= \sum_{k=1}^{n}\left(\phi^{a_1b_1i_1}\bar{\phi}_{a_1b_2i_{\sigma_n(1)}}\cdots\bar{\phi}_{a_{k-1}bi_{\sigma_n(k-1)}}\delta^{i_k}_{c}\bar{\phi}_{ab_{k+1}i_{\sigma_n(k)}}\cdots\phi^{a_nb_ni_n}\bar{\phi}_{a_nb_1i_{\sigma_n(n)}}\right) \\
&\times\sum_{l=1}^{m}\left(\phi^{a_1b_1j_1}\bar{\phi}_{a_1b_2j_{\sigma_m(1)}}\cdots\phi^{ab_lj_l}\delta^{c}_{j_{\sigma_m(l)}}\phi^{a_{l+1}bj_{l+1}}\cdots\phi^{a_mb_mj_m}\bar{\phi}_{a_mb_1j_{\sigma_m(m)}}\right) \\
&= \sum_{k,l=1}^{n,m}\delta^{i_k}_{j_{\sigma_m(l)}}\phi^{a_1b_1i_1}\bar{\phi}_{a_1b_2i_{\sigma_n(1)}}\cdots\bar{\phi}_{a_{k-1}bi_{\sigma_n(k-1)}}\phi^{a_{l+1}bj_{l+1}}\cdots\phi^{a_mb_mj_m}\bar{\phi}_{a_mb_1j_{\sigma_m(m)}} \\
&\qquad\times\phi^{a_1b_1j_1}\bar{\phi}_{a_1b_2j_{\sigma_m(1)}}\cdots\phi^{ab_lj_l}\bar{\phi}_{ab_{k+1}i_{\sigma_n(k)}}\cdots\phi^{a_nb_ni_n}\bar{\phi}_{a_nb_1i_{\sigma_n(n)}} \\
&= \sum_{k,l=1}^{n,m}\delta^{i_k}_{j_{\sigma_m(l)}}\chi(\sigma_p,n+m-1)
\end{aligned}
\tag{2.293}
$$

where σ_p is defined as $\sigma_p(1\text{ to }k-1) = \sigma_n(1\text{ to }k-1)$, $\sigma_p(k\text{ to }k+m-l-1) = \sigma_m(l+1\text{ to }m)$, $\sigma_p(k+m-l\text{ to }k+m-2) = \sigma_m(1\text{ to }l-1)$ and $\sigma_p(k+m-1\text{ to }n+m-1) = \sigma_m(1\text{ to }l-1)$. The difficulty with this is that it seems to be impossible to find any set of σ_n and σ_m's other than the set of identities for which this can be satisfied.

ω is given by

$$
\begin{aligned}
\omega(n) &= -\frac{\partial^2 \chi(\sigma_n, n)}{\partial \bar{\phi}_{abc} \partial \phi^{abc}} \\
&= -\frac{\partial}{\partial \bar{\phi}_{abc}} \sum_{k=1}^{n} \left(\phi^{a_1 b_1 i_1} \bar{\phi}_{a_1 b_2 i_{\sigma_n(1)}} \cdots \bar{\phi}_{a_{k-1} b i_{\sigma_n(k-1)}} \delta_c^{i_k} \bar{\phi}_{a b_{k+1} i_{\sigma_n(k)}} \cdots \phi^{a_n b_n i_n} \bar{\phi}_{a_n b_1 i_{\sigma_n(n)}} \right) \\
&= -\sum_{\substack{k,l=1 \\ l<k-1}}^{n} \delta^{i_k}_{i_{\sigma_n(l)}} \Big(\phi^{a_1 b_1 i_1} \bar{\phi}_{a_1 b_2 i_{\sigma_n(1)}} \cdots \bar{\phi}_{a_{l-1} b_l i_{\sigma_n(l-1)}} \phi^{a b_l i_l} \bar{\phi}_{a b_{k+1} i_{\sigma_n(k)}} \phi^{a_{k+1} b_{k+1} i_{k+1}} \cdots \phi^{a_n b_n i_n} \bar{\phi}_{a_n b_1 i_{\sigma_n(n)}} \\
&\qquad \times \phi^{a_{l+1} b i_l} \bar{\phi}_{a_{l+1} b_{l+2} i_{\sigma_n(l+1)}} \cdots \phi^{a_{k-1} b_{k-1} i_{k-1}} \bar{\phi}_{a_{k-1} b i_{\sigma_n(k-1)}} \Big) + \\
&\quad - \sum_{\substack{k,l=1 \\ l>k}}^{n} \delta^{i_k}_{i_{\sigma_n(l)}} \Big(\phi^{a_1 b_1 i_1} \bar{\phi}_{a_1 b_2 i_{\sigma_n(1)}} \cdots \bar{\phi}_{a_{k-1} b i_{\sigma_n(k-1)}} \phi^{a_{l+1} b i_{l+1}} \cdots \phi^{a_n b_n i_n} \bar{\phi}_{a_n b_1 i_{\sigma_n(n)}} \\
&\qquad \times \phi^{a b_l i_l} \bar{\phi}_{a b_{k+1} i_{\sigma_n(k)}} \cdots \bar{\phi}_{a_{l-1} b_l i_{\sigma_n(l-1)}} \Big) + \\
&- N_2 \sum_{k=1}^{n} \delta^{i_k}_{i_{\sigma_n(k-1)}} \Big(\phi^{a_1 b_1 i_1} \bar{\phi}_{a_1 b_2 i_{\sigma_n(1)}} \cdots \phi^{a b_{k-1} i_{k-1}} \bar{\phi}_{a b_{k+1} i_{\sigma_n(k)}} \cdots \phi^{a_n b_n i_n} \bar{\phi}_{a_n b_1 i_{\sigma_n(n)}} \Big) + \\
&- N_1 \sum_{k=1}^{n} \delta^{i_k}_{i_{\sigma_n(k)}} \Big(\phi^{a_1 b_1 i_1} \bar{\phi}_{a_1 b_2 i_{\sigma_n(1)}} \cdots \bar{\phi}_{a_{k-1} b i_{\sigma_n(k-1)}} \phi^{a_{k+1} b i_{k+1}} \cdots \phi^{a_n b_n i_n} \bar{\phi}_{a_n b_1 i_{\sigma_n(n)}} \Big) \\
&= -\sum_{\substack{k,l=1 \\ l<k-1}}^{n} \delta^{i_k}_{i_{\sigma_n(l)}} \chi(\sigma_p, l+n-k) \chi(\sigma_q, k-l-1) - \sum_{\substack{k,l=1 \\ l>k}}^{n} \delta^{i_k}_{i_{\sigma_n(l)}} \chi(\sigma_r, n+k-l-1) \chi(\sigma_s, l-k) + \\
&\quad - N_1 \sum_{k=1}^{n} \delta^{i_k}_{i_{\sigma_n(k-1)}} \chi(\sigma_t, n-1) - N_2 \sum_{k=1}^{n} \delta^{i_k}_{i_{\sigma_n(k)}} \chi(\sigma_u, n-1)
\end{aligned}
\tag{2.294}
$$

which puts more constraints on the specific sets of permutations σ_n and σ_m needed for Ω and ω to be expressed entirely in terms of these variables. Thus, it seems to not be possible to construct a collective field theory Hamiltonian for these variables.

Chapter 3

Gauge Invariants, Correlators and Holography

The work presented in this chapter is based on the author's published research [39]. The basic goal of this work is to construct the gauge invariants of both bosonic and fermionic tensor models. For bosonic tensor (colored as well as non-colored) models, the paper [83] counted the gauge invariants uncovering a relationship with counting problems of branched covers of the 2-sphere. The rank d of the tensor is related to a number of branch points. Further, formulas for correlators of the tensor model invariants in a permutation basis were obtained. Correlators in the permutation basis have been related to the (Hurwitz) character calculus in [84] (see also [85]). A dual representation theory basis was developed in [86].

Our starting point (section 3.1) reconsiders the representation theory basis for the bosonic tensor models, in a way that naturally allows an extension to fermionic tensor models. The basic ideas are explained in the next section, where we obtain counting formulas for the number of gauge invariant operators in bosonic and fermionic tensor models. The counting results for the bosons agree with results presented in [83, 86, 87]. The counting formulas for the fermions are new. In section 3.2 we consider both the computations of the vacuum expectation values of our gauge invariant operators, as well as two point functions of normal ordered gauge invariant operators. These computations are performed exactly (i.e. to all orders in $1/N$), in the free theory. In section 3.3 we describe the algebraic structure of the gauge invariants: they form a ring. In section 3.4, we construct a collective field theory in terms of a subset of the gauge invariant variables in analogy to the construction for a single matrix model. We exhibit an emergent dimension and show that the Hamiltonian is local in this new dimension. We reproduce large N correlators of the tensor model quantum mechanics from the classical collective field theory.

3.1 Construction of Gauge Invariant Operators

In this section we simply want to count the number of gauge invariant operators that can be constructed, for both bosonic and fermionic tensor models. Once we have understood how to count the number of gauge invariants, a natural construction formula will be evident.

The fields that we consider are tensors, of rank r. We will denote the bosonic tensors by $\phi_{b_1 b_2 \cdots b_r}$ and the fermionic tensors by $\psi_{f_1 f_2 \cdots f_r}$. These fields transform in the fundamental of $G = U(N_1) \times U(N_2) \times \cdots \times U(N_r)$.

Let V_k denote the vector space carrying a copy of the fundamental representation of $U(N_k)$. Fields transforming in the fundamental of $G = U(N_1) \times U(N_2) \times \cdots \times U(N_r)$ belong to $\mathcal{V} \equiv V_1 \times V_2 \times \cdots \times V_r$. To build gauge invariants we will also need fields that transform in the anti-fundamental, denoted $\bar{\phi}^{b_1 b_2 \cdots b_r}$ and $\bar{\psi}^{f_1 f_2 \cdots f_r}$. Gauge invariants are then given by contracting corresponding upper and lower indices. The valid gauge invariants, built using two fields, are given by

$$\bar{\phi}^{b_1 b_2 \cdots b_r} \phi_{b_1 b_2 \cdots b_r} \qquad \bar{\psi}^{f_1 f_2 \cdots f_r} \psi_{f_1 f_2 \cdots f_r} \tag{3.1}$$

A valid gauge invariant operator is only obtained if we contract corresponding indices of the tensors, since the position of an index signifies which gauge group it belongs to. To simplify the arguments that

follow, we now specialize to rank 3 tensors ϕ_{ijk} or ψ_{ijk} with $i = 1, ..., N_1$, $j = 1, ..., N_2$ and $k = 1, ..., N_3$. The generalization to higher rank tensors is completely clear. A comment is in order: the symmetry $U(N) \times U(N) \times U(N)$ can not be realized in any interacting theory, whose large N expansion is dominated by melonic diagrams. The maximal symmetry in this case is only $U(N) \times U(N) \times O(N)$.

We will want to consider products of tensors to build the general gauge invariant operator. Here is an example

$$\phi_{i_1 j_1 k_1} \phi_{i_2 j_2 k_2} \cdots \phi_{i_n j_n k_n} \tag{3.2}$$

This notation will quickly get out of hand, as the number of indices rapidly proliferates. To avoid this, we will now use the notation first introduced in [10, 11]. The sleek notation uses a capital Roman letter to collect all of the little Roman letter indices, for example I stands for $i_1, i_2, \cdots, i_n$. We will also use a capital Greek letter to collect the tensors. Thus, for example, we write

$$\Phi_{IJK} = \phi_{i_1 j_1 k_1} \phi_{i_2 j_2 k_2} \cdots \phi_{i_n j_n k_n} \tag{3.3}$$

Similarly

$$\Psi_{IJK} = \psi_{i_1 j_1 k_1} \psi_{i_2 j_2 k_2} \cdots \psi_{i_n j_n k_n} \tag{3.4}$$

These fields belong to $\mathcal{V}^{\otimes n}$. There is a natural action of S_n on $\mathcal{V}^{\otimes n}$ defined as follows: For any $\sigma \in S_n$ we have

$$\sigma \cdot \Phi_{IJK} \to \Phi_{\sigma(I)\sigma(J)\sigma(K)} = \phi_{i_{\sigma(1)} j_{\sigma(1)} k_{\sigma(1)}} \phi_{i_{\sigma(2)} j_{\sigma(2)} k_{\sigma(2)}} \cdots \phi_{i_{\sigma(n)} j_{\sigma(n)} k_{\sigma(n)}} \tag{3.5}$$

We will sometimes call this the diagonal action of S_n since each type of index, i, j or k is permuted in exactly the same way. We could also define an action of $S_n \times S_n \times S_n$ that acts independently on these three indices. The notation distinguishing these two actions is

$$\sigma \cdot \Phi_{IJK} \to \Phi_{\sigma(I)\sigma(J)\sigma(K)} \qquad \sigma \in S_n \tag{3.6}$$

versus

$$\sigma_1 \circ \sigma_2 \circ \sigma_3 \cdot \Phi_{IJK} \to \Phi_{\sigma_1(I)\sigma_2(J)\sigma_3(K)} \qquad \sigma_1 \circ \sigma_2 \circ \sigma_3 \in S_n \times S_n \times S_n \tag{3.7}$$

Since the diagonal action swaps the tensors we have

$$\sigma \cdot \Phi_{IJK} = \Phi_{\sigma(I)\sigma(J)\sigma(K)} = \Phi_{IJK} \tag{3.8}$$

$$\sigma \cdot \Psi_{IJK} = \Psi_{\sigma(I)\sigma(J)\sigma(K)} = \mathrm{sgn}(\sigma) \Psi_{IJK} \tag{3.9}$$

We know that swapping fermions costs a sign which is what the above equation captures. In the last formula above $\mathrm{sgn}(\sigma)$ denotes the signature of the permutation σ. For example, if $n = 2$ and $\sigma = (12)$ we have

$$\begin{aligned} (12)\psi_{i_1 j_1 k_1} \psi_{i_2 j_2 k_2} &= \psi_{i_2 j_2 k_2} \psi_{i_1 j_1 k_1} \\ &= -\psi_{i_1 j_1 k_1} \psi_{i_2 j_2 k_2} \\ &= \mathrm{sgn}\,(\,(12)\,)\, \psi_{i_1 j_1 k_1} \psi_{i_2 j_2 k_2} \end{aligned} \tag{3.10}$$

since the fermions are described using Grassman numbers. The equations (3.8),(3.9) will be important in the next section.

3.1.1 Counting and construction for bosonic tensors

Our goal in this section is to count the number of gauge invariant operators that can be constructed from the bosonic tensors introduced above. To construct gauge invariants we need to completely contract the indices of Φ_{IJK} with the indices of $\bar{\Phi}^{IJK}$. In general, this is accomplished using three permutations $\sigma_1, \sigma_2, \sigma_3 \in S_n$ (or equivalently, one permutation $\sigma_1 \circ \sigma_2 \circ \sigma_3 \in S_n \times S_n \times S_n$) as follows

$$\bar{\Phi} \cdot \sigma_1 \circ \sigma_2 \circ \sigma_3 \cdot \Phi = \bar{\Phi}^{IJK} \Phi_{\sigma_1(I)\sigma_2(J)\sigma_3(K)} \tag{3.11}$$

The invariants given in equation (3.11) are over complete: the ϕ's and $\bar{\phi}$'s are bosons, so we have the symmetry given in (3.8) which must be accounted for. Let $\beta_1 \in S_n$ be an arbitrary permutation of the ϕ's and let $\beta_2 \in S_n$ be an arbitrary permutation of the $\bar{\phi}$'s. Then (we act to the right if we act on lower indices and to the left if we act on upper indices)

$$\bar{\Phi} \cdot \sigma_1 \circ \sigma_2 \circ \sigma_3 \cdot \Phi = (\bar{\Phi} \cdot \beta_2) \cdot \sigma_1 \circ \sigma_2 \circ \sigma_3 \cdot (\beta_1 \cdot \Phi) \tag{3.12}$$

Manipulating this a little, we have[1]

$$\begin{aligned} \Phi_{\sigma_1(I)\sigma_2(J)\sigma_3(K)} \bar{\Phi}^{IJK} &= \Phi_{\sigma_1(\beta_1(I))\sigma_2(\beta_1(J))\sigma_3(\beta_1(K))} \bar{\Phi}^{\beta_2(I)\beta_2(J)\beta_2(K)} \\ &= \Phi_{\beta_2^{-1}(\sigma_1(\beta_1(I)))\beta_2^{-1}(\sigma_2(\beta_1(J)))\beta_2^{-1}(\sigma_3(\beta_1(K)))} \bar{\Phi}^{IJK} \end{aligned} \tag{3.13}$$

Thus, $(\sigma_1, \sigma_2, \sigma_3)$ and $(\beta_1\sigma_1\beta_2, \beta_1\sigma_2\beta_2, \beta_1\sigma_3\beta_2)$ define the same gauge invariant operator. This implies that we have one gauge invariant operator for each element in the double coset

$$S_n \setminus S_n \times S_n \times S_n \,/\, S_n \tag{3.14}$$

This understanding of the structure of the space of gauge invariant observables was first achieved in [83]. The generalization to other ranks is obvious. For example, rank 5 tensors would be elements of the coset

$$S_n \setminus S_n \times S_n \times S_n \times S_n \times S_n \,/\, S_n \tag{3.15}$$

The number of elements in a double coset $|H_1 \setminus G / H_2|$ is given, by Burnside's Lemma, as

$$|H_1 \setminus G / H_2| = \frac{1}{|H_1||H_2|} \sum_{h_1 \in H_1} \sum_{h_2 \in H_2} \sum_{g \in G} \delta(h_1 g h_2 g^{-1}) \tag{3.16}$$

Thus, for example, the number $\mathcal{N}_3$ of rank 3 tensors built using n fields is given by

$$\mathcal{N}_3 = \frac{1}{(n!)^2} \sum_{\sigma_1, \sigma_2, \sigma_3 \in S_n} \sum_{\beta_1, \beta_2 \in S_n} \delta(\beta_1\sigma_1\beta_2\sigma_1^{-1}) \delta(\beta_1\sigma_2\beta_2\sigma_2^{-1}) \delta(\beta_1\sigma_3\beta_2\sigma_3^{-1}) \tag{3.17}$$

To make sure the generalization is clear, we simply quote the count for the number of rank q tensors built using n fields

$$\mathcal{N}_q = \frac{1}{(n!)^2} \sum_{\sigma_1, \cdots, \sigma_q \in S_n} \sum_{\beta_1, \beta_2 \in S_n} \prod_{i=1}^{q} \delta(\beta_1\sigma_i\beta_2\sigma_i^{-1}) \tag{3.18}$$

[1] A useful identity to keep in mind is the following: $\bar{\Phi}^{\gamma^{-1}(K)} \Phi_K = \bar{\Phi}^K \Phi_{\gamma(K)}$. This follows very simply by using the explicit representation $(\sigma)^I_J = \delta^{i_1}_{j_{\sigma(1)}} \cdots \delta^{i_n}_{j_{\sigma(n)}}$.

The arguments we have just outlined are not the most natural when we generalize to fermionic tensors. To perform the counting in a way that will generalize nicely to the fermionic case, we will change basis. The operators

$$\mathcal{O}(\sigma_1, \sigma_2, \sigma_3) = \bar{\Phi} \cdot \sigma_1 \circ \sigma_2 \circ \sigma_3 \cdot \Phi \tag{3.19}$$

define the "permutation basis". We will Fourier transform to the representation theory basis as follows

$$\begin{aligned}(\mathcal{O}_{r_1,r_2,r_3})_{\alpha_1\alpha_2\alpha_3,\beta_1\beta_2\beta_3} = \\ \sum_{\sigma_1,\sigma_2,\sigma_2} \mathcal{O}(\sigma_1,\sigma_2,\sigma_3)\Gamma^{r_1}{}_{\alpha_1\beta_1}(\sigma_1)\Gamma^{r_2}{}_{\alpha_2\beta_2}(\sigma_2)\Gamma^{r_3}{}_{\alpha_3\beta_3}(\sigma_3)\end{aligned} \tag{3.20}$$

All of the representations above are irreducible representations of S_n, i.e. $r_i \vdash n$, $i = 1, 2, 3$. We again have to deal with the symmetry present as a consequence of (3.8). The simplest way to do this is to couple the row indices to the trivial irreducible representation and to couple the column indices to the trivial irreducible representation of the diagonal S_n. The tensor product of the irreducible representations involved is

$$V_{r_1} \otimes V_{r_2} \otimes V_{r_3} = \bigoplus_r g_{r_1\, r_2\, r_3\, r} V_r \tag{3.21}$$

The Kronecker coefficients $g_{r_1\, r_2\, r_3\, r}$ are non-negative integers that count how many times irreducible representation r appears in the tensor product $r_1 \otimes r_2 \otimes r_3$. To perform the projection to the trivial, introduce the branching coefficients $B^{\gamma}_{\alpha_1\alpha_2\alpha_3}$ defined by

$$\frac{1}{n!}\sum_{\sigma\in S_n} \Gamma^{r_1}{}_{\alpha_1\beta_1}(\sigma)\Gamma^{r_2}{}_{\alpha_2\beta_2}(\sigma)\Gamma^{r_3}{}_{\alpha_3\beta_3}(\sigma) = \sum_{\gamma} B^{\gamma}_{\alpha_1\alpha_2\alpha_3}B^{\gamma}_{\beta_1\beta_2\beta_3} \tag{3.22}$$

The branching coefficients provide an orthonormal basis for the subspace of $r_1 \otimes r_2 \otimes r_3$ that carries the trivial representation, i.e.

$$B^{\gamma_1}_{\alpha_1\alpha_2\alpha_3}B^{\gamma_2}_{\alpha_1\alpha_2\alpha_3} = \delta^{\gamma_1\gamma_2} \tag{3.23}$$

and where we employ the usual convention that repeated indices are summed. The gauge invariant operators are now given by

$$\mathcal{O}^{\gamma_1\gamma_2}_{r_1,r_2,r_3} = B^{\gamma_1}_{\alpha_1\alpha_2\alpha_3}(\mathcal{O}_{r_1,r_2,r_3})_{\alpha_1\alpha_2\alpha_3,\beta_1\beta_2\beta_3}B^{\gamma_2}_{\beta_1\beta_2\beta_3} \tag{3.24}$$

We will also write this as

$$\mathcal{O}^{\gamma_1\gamma_2}_{r_1,r_2,r_3} = \sum_{\sigma_1\in S_n}\sum_{\sigma_2\in S_n}\sum_{\sigma_3\in S_n} C^{\gamma_1\gamma_2}_{r_1,r_2,r_3}(\sigma_1,\sigma_2,\sigma_3)\mathcal{O}(\sigma_1,\sigma_2,\sigma_3) \tag{3.25}$$

where

$$C^{\gamma_1\gamma_2}_{r_1,r_2,r_3}(\sigma_1,\sigma_2,\sigma_3) = B^{\gamma_1}_{\alpha_1\alpha_2\alpha_3}\Gamma^{r_1}{}_{\alpha_1\beta_1}(\sigma_1)\Gamma^{r_2}{}_{\alpha_2\beta_2}(\sigma_2)\Gamma^{r_3}{}_{\alpha_3\beta_3}(\sigma_3)B^{\gamma_2}_{\beta_1\beta_2\beta_3} \tag{3.26}$$

is in fact a restricted character, in the language introduced in [24, 88]. Thus, (3.25) provides the restricted Schur polynomial basis for the gauge invariant operators of the bosonic tensor model.

Since each multiplicity runs from 1 to $g_{r_1\, r_2\, r_3\, 1}$ and each operator is labeled by a pair of multiplicity labels, this second construction shows that the number of gauge invariant operators, constructed using n ϕ's and n $\bar{\phi}$'s, is given by

$$\sum_{r_i \vdash n \; l(r_i) \leq N_i} g^2_{r_1\, r_2\, r_3\, 1} \tag{3.27}$$

where we have used 1 to denote the rep labeled by a Young diagram with a single row of n boxes. This is in complete agreement with [87], as already pointed out in [86]. A standard result which follows from the orthogonality of characters is

$$g_{r_1\, r_2\, r_3\, 1} = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_{r_1}(\sigma)\chi_{r_2}(\sigma)\chi_{r_3}(\sigma) = g_{r_1\, r_2\, r_3} \tag{3.28}$$

Some checks of the counting formula (3.27) are given in Appendix B.1.

3.1.2 Counting and construction for fermionic tensors

Our goal in this section is to count the number of gauge invariant operators that can be constructed from the fermionic tensors introduced above. To construct gauge invariants we need to completely contract the indices of Ψ_{IJK} with the indices of $\bar{\Psi}^{IJK}$. In general, this is again accomplished using three permutations $\sigma_1, \sigma_2, \sigma_3 \in S_n$ (or equivalently $\sigma_1 \circ \sigma_2 \circ \sigma_3 \in S_n \times S_n \times S_n$) as follows

$$\bar{\Psi} \cdot \sigma_1 \circ \sigma_2 \circ \sigma_3 \cdot \Psi = \bar{\Psi}^{IJK} \Psi_{\sigma_1(I)\sigma_2(J)\sigma_3(K)} \tag{3.29}$$

The invariants given in equation (3.29) are again over complete: the ψ's and $\bar{\psi}$'s are fermions, so we have the symmetry given in (3.9) which must be accounted for. Following our discussion for the bosons, let $\beta_1 \in S_n$ be an arbitrary permutation of the ψ's and let $\beta_2 \in S_n$ be an arbitrary permutation of the $\bar{\psi}$'s. Then (exactly as for bosonic tensors, we act to the right if we act on lower indices and to the left, if we act on upper indices)

$$\bar{\Psi} \cdot \sigma_1 \circ \sigma_2 \circ \sigma_3 \cdot \Psi = \mathrm{sgn}(\beta_1)\mathrm{sgn}(\beta_2)(\bar{\Psi} \cdot \beta_2) \cdot \sigma_1 \circ \sigma_2 \circ \sigma_3 \cdot (\beta_1 \cdot \Psi) \tag{3.30}$$

Manipulating this a little, we have

$$\begin{aligned} \Psi_{\sigma_1(I)\sigma_2(J)\sigma_3(K)} \bar{\Psi}^{IJK} &= \mathrm{sgn}(\beta_1)\mathrm{sgn}(\beta_2) \Psi_{\sigma_1(\beta_1(I))\sigma_2(\beta_1(J))\sigma_3(\beta_1(K))} \bar{\Psi}^{\beta_2(I)\beta_2(J)\beta_2(K)} \\ &= \mathrm{sgn}(\beta_1)\mathrm{sgn}(\beta_2) \Psi_{\beta_2^{-1}(\sigma_1(\beta_1(I)))\beta_2^{-1}(\sigma_2(\beta_1(J)))\beta_2^{-1}(\sigma_3(\beta_1(K)))} \bar{\Psi}^{IJK} \end{aligned} \tag{3.31}$$

We will still have to account for this symmetry. To do this, it again proves useful to change basis. The operators

$$\mathcal{P}(\sigma_1, \sigma_2, \sigma_3) = \bar{\Psi} \cdot \sigma_1 \circ \sigma_2 \circ \sigma_3 \cdot \Psi \tag{3.32}$$

define the "permutation basis". Again, Fourier transform to the representation theory basis as follows

$$(\mathcal{P}_{r_1, r_2, r_3})_{\alpha_1\alpha_2\alpha_3, \beta_1\beta_2\beta_3} =$$

$$\sum_{\sigma_1,\sigma_2,\sigma_2} \mathcal{P}(\sigma_1,\sigma_2,\sigma_3)\Gamma^{r_1}{}_{\alpha_1\beta_1}(\sigma_1)\Gamma^{r_2}{}_{\alpha_2\beta_2}(\sigma_2)\Gamma^{r_3}{}_{\alpha_3\beta_3}(\sigma_3) \tag{3.33}$$

We now have to deal with the symmetry present as a consequence of (3.9). The simplest way to do this is to couple the row indices to the antisymmetric irreducible representation and to couple the column indices to the antisymmetric irreducible representation of the diagonal S_n. By the antisymmetric irreducible representation, (denoted (1^n)) we mean the irreducible representation labeled by a Young diagram that has a single column of n boxes. This is a one dimensional representation defined by

$$\Gamma^{(1^n)}(\sigma) = \mathrm{sgn}(\sigma) \tag{3.34}$$

To perform the projection to the antisymmetric irreducible representation, we again introduce branching coefficients

$$\frac{1}{n!}\sum_{\sigma\in S_n} \Gamma^{r_1}{}_{\alpha_1\beta_1}(\sigma)\Gamma^{r_2}{}_{\alpha_2\beta_2}(\sigma)\Gamma^{r_3}{}_{\alpha_3\beta_3}(\sigma)\mathrm{sgn}(\sigma) = \sum_{\gamma} \tilde{B}^{\gamma}_{\alpha_1\alpha_2\alpha_3}\tilde{B}^{\gamma}_{\beta_1\beta_2\beta_3} \tag{3.35}$$

We are using a tilde to distinguish the branching coefficients defined using the antisymmetric irreducible representation, from those relevant for the bosons which are defined using the symmetric representation. The branching coefficients again define an orthonormal basis

$$\tilde{B}^{\gamma_1}_{\alpha_1\alpha_2\alpha_3}\tilde{B}^{\gamma_2}_{\alpha_1\alpha_2\alpha_3} = \delta^{\gamma_1\gamma_2} \tag{3.36}$$

The gauge invariant operators are now given by

$$\mathcal{P}^{\gamma_1\gamma_2}_{r_1,r_2,r_3} = \tilde{B}^{\gamma_1}_{\alpha_1\alpha_2\alpha_3}(\mathcal{P}_{r_1,r_2,r_3})_{\alpha_1\alpha_2\alpha_3,\beta_1\beta_2\beta_3}\tilde{B}^{\gamma_2}_{\beta_1\beta_2\beta_3} \tag{3.37}$$

Once again, we can write this as

$$\mathcal{P}^{\gamma_1\gamma_2}_{r_1,r_2,r_3} = \sum_{\sigma_1\in S_n}\sum_{\sigma_2\in S_n}\sum_{\sigma_3\in S_n} \tilde{C}^{\gamma_1\gamma_2}_{r_1,r_2,r_3}(\sigma_1,\sigma_2,\sigma_3)\mathcal{P}(\sigma_1,\sigma_2,\sigma_3) \tag{3.38}$$

where

$$\tilde{C}^{\gamma_1\gamma_2}_{r_1,r_2,r_3}(\sigma_1,\sigma_2,\sigma_3) = \tilde{B}^{\gamma_1}_{\alpha_1\alpha_2\alpha_3}\Gamma^{r_1}{}_{\alpha_1\beta_1}(\sigma_1)\Gamma^{r_2}{}_{\alpha_2\beta_2}(\sigma_2)\Gamma^{r_3}{}_{\alpha_3\beta_3}(\sigma_3)\tilde{B}^{\gamma_2}_{\beta_1\beta_2\beta_3} \tag{3.39}$$

is again a restricted character. Thus, (3.38) provides the restricted Schur polynomial basis for the gauge invariant operators of the fermionic tensor model.

This construction shows that the number of gauge invariant operators is given by

$$\sum_{r_i\vdash n\ l(r_i)\le N_i} g^2_{r_1\, r_2\, r_3\, (1^n)} \tag{3.40}$$

A standard result which follows from the orthogonality of characters is

$$g_{r_1\, r_2\, r_3\, (1^n)} = \frac{1}{n!}\sum_{\sigma\in S_n} \chi_{r_1}(\sigma)\chi_{r_2}(\sigma)\chi_{r_3}(\sigma)\mathrm{sgn}(\sigma) \tag{3.41}$$

Some checks of this counting formula are given in Appendix B.1.

3.2 Correlators of Gauge Invariant Operators

In this section we will compute the correlation functions of the operators defined in the previous section. Since these operators are neutral under the gauge symmetry they can develop a nonzero vacuum expectation value. It is interesting to compute these values as their large N limit must be reproduced by the classical equations of motion of collective field theory. We also compute the two point functions of normal ordered gauge invariant operators. The large N limit of these two point functions must be reproduced by considering quadratic fluctuations about the classical collective configuration.

3.2.1 Bosonic Correlators

The free field two point function is

$$\langle \bar{\phi}^{ijk} \phi_{lmn} \rangle = \delta^i_l \delta^j_m \delta^k_n \tag{3.42}$$

This is valid both as a formula in a zero dimensional random tensor model, or as an equal time two point function in the tensor model quantum mechanics. Wick's theorem can be written as

$$\begin{aligned} \langle \bar{\Phi}^{IJK} \Phi_{LMN} \rangle &= \sum_{\sigma \in S_n} \prod_{a=1}^{n} \delta^{i_a}_{l_{\sigma(a)}} \delta^{j_a}_{m_{\sigma(a)}} \delta^{k_a}_{n_{\sigma(a)}} \\ &= \sum_{\sigma \in S_n} (\sigma)^I_L (\sigma)^J_M (\sigma)^K_N \end{aligned} \tag{3.43}$$

There are two interesting correlators to consider: first we could consider the one point functions $\langle \mathcal{O}^{\gamma_1 \gamma_2}_{r_1, r_2, r_3} \rangle$; second we could consider the two point function of normal ordered operators $\langle : \mathcal{O}^{\gamma_1 \gamma_2}_{r_1 r_2 r_3} :: \mathcal{O}^{\gamma_3 \gamma_4}_{s_1 s_2 s_3} : \rangle$.

One point functions: We will use the fact that

$$\mathrm{Tr}_{V_j}(\sigma) = \delta^{i_1}_{i_{\sigma(1)}} \cdots \delta^{i_n}_{i_{\sigma(n)}} = N_j^{C(\sigma)} \tag{3.44}$$

where $C(\sigma)$ denotes the number of cycles in the permutation σ. In addition, we will use the orthogonality relation

$$\frac{d_r}{n!} \sum_{\sigma \in S_n} \Gamma_r(\sigma)_{ab} \Gamma_s(\sigma^{-1})_{cd} = \delta_{rs} \delta_{bc} \delta_{ad} \tag{3.45}$$

to obtain

$$\frac{d_r}{n!} \sum_{s \vdash n} \sum_{\sigma_1 \in S_n} \Gamma_r(\sigma_1)_{cd} \chi_s(\sigma \sigma_1^{-1}) = \Gamma_r(\sigma)_{cd} \tag{3.46}$$

Finally, we will use the relation, valid for Schur polynomials

$$\mathrm{Tr}(\sigma Z) = \sum_R \chi_R(\sigma) \chi_R(Z) \tag{3.47}$$

evaluated at $Z = 1$ to find

$$\mathrm{Tr}(\sigma) = \frac{1}{n!} \sum_{R \vdash n} d_R \chi_R(\sigma) f_R(N) \tag{3.48}$$

$f_R(N)$ is the product of the factors of Young diagram R understood as a representation of $U(N)$. Recall that the factor of a box in row i and column j is $N-i+j$. We use $\chi_R(\sigma)$ to denote a character of the symmetric group and $\chi_R(Z)$ to denote a Schur polynomial. The two are distinguished only by their argument, which is either an element of the symmetric group $\sigma \in S_n$ or an $N \times N$ matrix Z. We are now ready to compute the one point function

$$\begin{aligned}
\langle \mathcal{O}^{\gamma_1\gamma_2}_{r_1r_2r_3} \rangle &= \sum_{\sigma_i \in S_n} \langle \bar{\Phi} \cdot \sigma_1 \circ \sigma_2 \circ \sigma_3 \cdot \Phi \rangle B^{\gamma_1} \Gamma_{r_1}(\sigma_1)\Gamma_{r_2}(\sigma_2)\Gamma_{r_3}(\sigma_3) B^{\gamma_2} \\
&= \sum_{\sigma,\sigma_i \in S_n} N_1^{C(\sigma\sigma_1)} N_2^{C(\sigma\sigma_2)} N_3^{C(\sigma\sigma_3)} B^{\gamma_1} \Gamma_{r_1}(\sigma_1)\Gamma_{r_2}(\sigma_2)\Gamma_{r_3}(\sigma_3) B^{\gamma_2} \\
&= \sum_{\sigma,\sigma_i \in S_n} \sum_{s_i \vdash n} \left(\frac{1}{n!}\right)^3 d_{s_1} \chi_{s_1}(\sigma\sigma_1^{-1}) f_{s_1}(N_1) d_{s_2} \chi_{s_2}(\sigma\sigma_2^{-1}) f_{s_2}(N_2) \\
&\quad d_{s_3} \chi_{s_3}(\sigma\sigma_3^{-1}) f_{s_3}(N_3) B^{\gamma_1} \Gamma_{r_1}(\sigma_1)\Gamma_{r_2}(\sigma_2)\Gamma_{r_3}(\sigma_3) B^{\gamma_2} \\
&= \sum_{\sigma \in S_n} f_{r_1}(N_1) f_{r_2}(N_2) f_{r_3}(N_3) B^{\gamma_1} \Gamma_{r_1}(\sigma)\Gamma_{r_2}(\sigma)\Gamma_{r_3}(\sigma) B^{\gamma_2} \\
&= n!\, f_{r_1}(N_1) f_{r_2}(N_2) f_{r_3}(N_3) \delta^{\gamma_1\gamma_2}
\end{aligned} \tag{3.49}$$

See Appendix B.2 for some checks of this formula.

Two point functions of normal ordered operators: Using the identities given above, it is straightforward to compute

$$\begin{aligned}
&\langle : \mathcal{O}^{\gamma_1\gamma_2}_{r_1r_2r_3} :: \mathcal{O}^{\gamma_3\gamma_4}_{s_1s_2s_3} : \rangle = \sum_{\sigma \in S_n} \sum_{\rho \in S_n} \sum_{\sigma_i \in S_n} \sum_{\tau_i \in S_n} \mathrm{Tr}(\sigma_1\sigma\tau_1\rho)\mathrm{Tr}(\sigma_2\sigma\tau_2\rho)\mathrm{Tr}(\sigma_3\sigma\tau_3\rho) \\
&\times B^{\gamma_1} \Gamma_{r_1}(\sigma_1)\Gamma_{r_2}(\sigma_2)\Gamma_{r_3}(\sigma_3) B^{\gamma_2}\ B^{\gamma_3} \Gamma_{s_1}(\tau_1)\Gamma_{s_2}(\tau_2)\Gamma_{s_3}(\tau_3) B^{\gamma_4} \\
&= \sum_{\sigma\rho\sigma_i\tau_i} N^{C(\sigma_1\sigma\tau_1\rho)} N^{C(\sigma_2\sigma\tau_2\rho)} N^{(\sigma_3\sigma\tau_3\rho)} \\
&\times B^{\gamma_1} \Gamma_{r_1}(\sigma_1)\Gamma_{r_2}(\sigma_2)\Gamma_{r_3}(\sigma_3) B^{\gamma_2}\ B^{\gamma_3} \Gamma_{s_1}(\tau_1)\Gamma_{s_2}(\tau_2)\Gamma_{s_3}(\tau_3) B^{\gamma_4} \\
&= \sum_{\sigma\rho\sigma_i\tau_i} \sum_{t_i \vdash n} \left(\frac{1}{n!}\right)^3 d_{t_1} \chi_{t_1}(\sigma_1\sigma\tau_1\rho) f_{t_1}(N_1) d_{t_2} \chi_{t_2}(\sigma_2\sigma\tau_2\rho) f_{t_2}(N_2) d_{t_3} \chi_{t_3}(\sigma_3\sigma\tau_3\rho) f_{t_3}(N_3) \\
&\times B^{\gamma_1} \Gamma_{r_1}(\sigma_1)\Gamma_{r_2}(\sigma_2)\Gamma_{r_3}(\sigma_3) B^{\gamma_2}\ B^{\gamma_3} \Gamma_{s_1}(\tau_1)\Gamma_{s_2}(\tau_2)\Gamma_{s_3}(\tau_3) B^{\gamma_4} \\
&= (n!)^2 \delta_{r_1s_1}\delta_{r_2s_2}\delta_{r_3s_3} f_{r_1}(N_1) f_{r_2}(N_2) f_{r_3}(N_3) \frac{n!}{d_{r_1}} \frac{n!}{d_{r_2}} \frac{n!}{d_{r_3}} \delta^{\gamma_1\gamma_4}\delta^{\gamma_2\gamma_3}
\end{aligned} \tag{3.50}$$

Some checks of this formula are given in Appendix B.2.

3.2.2 Fermionic Correlators

The relevant two point function for the fermionic tensor model is

$$\langle \bar{\psi}^{ijk} \psi_{lmn} \rangle = \delta^i_l \delta^j_m \delta^k_n \tag{3.51}$$

This is valid, as for the bosons, both as a formula in a zero dimensional random tensor model, or as an equal time two point function in the tensor model quantum mechanics. Since the fermionic fields anticommute, it is important to spell out the ordering of the fields. Order the fields in the following way

$$\bar{\Psi}^{IJK}\Psi_{LMN} = \bar{\psi}^{i_1 j_1 k_1}\bar{\psi}^{i_2 j_2 k_2}\cdots\bar{\psi}^{i_n j_n k_n}\psi_{l_n m_n n_n}\cdots\psi_{l_2 m_2 n_2}\psi_{l_1 m_1 n_1} \tag{3.52}$$

With this ordering spelled out, a simple application of Wick's theorem now gives

$$\langle\bar{\psi}^{IJK}\psi_{LMN}\rangle = \sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)\sigma_L^I\sigma_M^J\sigma_N^K \tag{3.53}$$

One point functions: A simple computation shows that

$$\begin{aligned}
\langle\mathcal{P}^{\gamma_1\gamma_2}_{r_1r_2r_3}\rangle &= \sum_{\sigma_i\in S_n}\langle\bar{\Psi}\cdot\sigma_1\circ\sigma_2\circ\sigma_3\cdot\Psi\rangle\tilde{B}^{\gamma_1}\Gamma_{r_1}(\sigma_1)\Gamma_{r_2}(\sigma_2)\Gamma_{r_3}(\sigma_3)\tilde{B}^{\gamma_2}\\
&= \sum_{\sigma,\sigma_i\in S_n}\mathrm{sgn}(\sigma)N_1^{C(\sigma\sigma_1)}N_2^{C(\sigma\sigma_2)}N_3^{C(\sigma\sigma_3)}\tilde{B}^{\gamma_1}\Gamma_{r_1}(\sigma_1)\Gamma_{r_2}(\sigma_2)\Gamma_{r_3}(\sigma_3)\tilde{B}^{\gamma_2}\\
&= \sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)f_{r_1}(N_1)f_{r_2}(N_2)f_{r_3}(N_3)\tilde{B}^{\gamma_1}\Gamma_{r_1}(\sigma)\Gamma_{r_2}(\sigma)\Gamma_{r_3}(\sigma)\tilde{B}^{\gamma_2}\\
&= n!\,f_{r_1}(N_1)f_{r_2}(N_2)f_{r_3}(N_3)\delta^{\gamma_1\gamma_2}
\end{aligned} \tag{3.54}$$

See Appendix B.2 for examples and checks of this formula.

Two point functions of normal ordered operators: Using the identities given above

$$\begin{aligned}
&\langle:\mathcal{P}^{\gamma_1\gamma_2}_{r_1r_2r_3}::\mathcal{P}^{\gamma_3\gamma_4}_{s_1s_2s_3}:\rangle = \sum_{\sigma\in S_n}\sum_{\rho\in S_n}\sum_{\sigma_i\in S_n}\sum_{\tau_i\in S_n}\mathrm{sgn}(\sigma)\mathrm{sgn}(\rho)\mathrm{Tr}(\sigma_1\sigma\tau_1\rho)\mathrm{Tr}(\sigma_2\sigma\tau_2\rho)\mathrm{Tr}(\sigma_3\sigma\tau_3\rho)\\
&\times\tilde{B}^{\gamma_1}\Gamma_{r_1}(\sigma_1)\Gamma_{r_2}(\sigma_2)\Gamma_{r_3}(\sigma_3)\tilde{B}^{\gamma_2}\ \tilde{B}^{\gamma_3}\Gamma_{s_1}(\tau_1)\Gamma_{s_2}(\tau_2)\Gamma_{s_3}(\tau_3)\tilde{B}^{\gamma_4}\\
&= \sum_{\sigma\rho\sigma_i\tau_i}\mathrm{sgn}(\sigma)\mathrm{sgn}(\rho)N^{C(\sigma_1\sigma\tau_1\rho)}N^{C(\sigma_2\sigma\tau_2\rho)}N^{(\sigma_3\sigma\tau_3\rho)}\\
&\times\tilde{B}^{\gamma_1}\Gamma_{r_1}(\sigma_1)\Gamma_{r_2}(\sigma_2)\Gamma_{r_3}(\sigma_3)\tilde{B}^{\gamma_2}\ \tilde{B}^{\gamma_3}\Gamma_{s_1}(\tau_1)\Gamma_{s_2}(\tau_2)\Gamma_{s_3}(\tau_3)\tilde{B}^{\gamma_4}\\
&= (n!)^2\delta_{r_1s_1}\delta_{r_2s_2}\delta_{r_3s_3}f_{r_1}(N_1)f_{r_2}(N_2)f_{r_3}(N_3)\frac{n!}{d_{r_1}}\frac{n!}{d_{r_2}}\frac{n!}{d_{r_3}}\delta^{\gamma_1\gamma_4}\delta^{\gamma_2\gamma_3}
\end{aligned} \tag{3.55}$$

Appendix B.2 illustrates and checks this formula in some simple cases.

3.3 Algebra of the Gauge Invariant Operators

The gauge invariant operators that we have introduced above close an interesting algebra: we will argue that the gauge invariant operators have a ring structure. Algebras of gauge invariant operators have also been considered in [89]. To develop the algebra for our tensor model, we will need to develop some properties of the restricted character. We can always assume that we work in an orthogonal representation of the symmetric group. In this case the restricted characters obey

$$C^{\gamma_1\gamma_2}_{r_1r_2r_3}(\sigma_1,\sigma_2,\sigma_3) = C^{\gamma_2\gamma_1}_{r_1r_2r_3}(\sigma_1^{-1},\sigma_2^{-1},\sigma_3^{-1})$$

$$\tilde{C}^{\gamma_1\gamma_2}_{r_1r_2r_3}(\sigma_1,\sigma_2,\sigma_3) = \tilde{C}^{\gamma_2\gamma_1}_{r_1r_2r_3}(\sigma_1^{-1},\sigma_2^{-1},\sigma_3^{-1}) \tag{3.56}$$

They also enjoy a "completeness identity" given by

$$\sum_{\sigma_1\in S_n}\sum_{\sigma_2\in S_n}\sum_{\sigma_3\in S_n} C^{\gamma_1\gamma_2}_{r_1r_2r_3}(\sigma_1,\sigma_2,\sigma_3)C^{\gamma_3\gamma_4}_{s_1s_2s_3}(\sigma_1,\sigma_2,\sigma_3) = \frac{n!}{d_{r_1}}\frac{n!}{d_{r_2}}\frac{n!}{d_{r_3}}\delta_{r_1s_1}\delta_{r_2s_2}\delta_{r_3s_3}\delta^{\gamma_1\gamma_3}\delta^{\gamma_2\gamma_4}$$
$$\sum_{\sigma_1\in S_n}\sum_{\sigma_2\in S_n}\sum_{\sigma_3\in S_n} \tilde{C}^{\gamma_1\gamma_2}_{r_1r_2r_3}(\sigma_1,\sigma_2,\sigma_3)\tilde{C}^{\gamma_3\gamma_4}_{s_1s_2s_3}(\sigma_1,\sigma_2,\sigma_3) = \frac{n!}{d_{r_1}}\frac{n!}{d_{r_2}}\frac{n!}{d_{r_3}}\delta_{r_1s_1}\delta_{r_2s_2}\delta_{r_3s_3}\delta^{\gamma_1\gamma_3}\delta^{\gamma_2\gamma_4} \tag{3.57}$$

Using these formulas, we find the following interesting Fourier transform pairs

$$\mathcal{O}^{\gamma_1\gamma_2}_{r_1r_2r_3} = \sum_{\sigma_1\in S_n}\sum_{\sigma_2\in S_n}\sum_{\sigma_3\in S_n} C^{\gamma_1\gamma_2}_{r_1r_2r_3}(\sigma_1,\sigma_2,\sigma_3)\mathcal{O}(\sigma_1,\sigma_2,\sigma_3) \tag{3.58}$$

$$\mathcal{O}(\sigma_1,\sigma_2,\sigma_3) = \sum_{s_1\vdash n}\sum_{s_2\vdash n}\sum_{s_3\vdash n}\sum_{\gamma_1,\gamma_2}\frac{d_{s_1}}{n!}\frac{d_{s_2}}{n!}\frac{d_{s_3}}{n!}C^{\gamma_1\gamma_2}_{s_1s_2s_3}(\sigma_1,\sigma_2,\sigma_3)\mathcal{O}^{\gamma_1\gamma_2}_{s_1s_2s_3} \tag{3.59}$$

$$\mathcal{P}^{\gamma_1\gamma_2}_{r_1r_2r_3} = \sum_{\sigma_1\in S_n}\sum_{\sigma_2\in S_n}\sum_{\sigma_3\in S_n} \tilde{C}^{\gamma_1\gamma_2}_{r_1r_2r_3}(\sigma_1,\sigma_2,\sigma_3)\mathcal{P}(\sigma_1,\sigma_2,\sigma_3) \tag{3.60}$$

$$\mathcal{P}(\sigma_1,\sigma_2,\sigma_3) = \sum_{s_1\vdash n}\sum_{s_2\vdash n}\sum_{s_3\vdash n}\sum_{\gamma_1,\gamma_2}\frac{d_{s_1}}{n!}\frac{d_{s_2}}{n!}\frac{d_{s_3}}{n!}\tilde{C}^{\gamma_1\gamma_2}_{s_1s_2s_3}(\sigma_1,\sigma_2,\sigma_3)\mathcal{P}^{\gamma_1\gamma_2}_{s_1s_2s_3} \tag{3.61}$$

These formulas provide the clearest way to understand the relation between the permutation and representation theory bases.

Now, in the permutation basis the gauge invariant operators close the following algebra

$$\mathcal{O}(\sigma_1,\sigma_2,\sigma_3)\mathcal{O}(\rho_1,\rho_2,\rho_3) = \mathcal{O}(\sigma_1\circ\rho_1,\sigma_2\circ\rho_2,\sigma_3\circ\rho_3) \tag{3.62}$$

$$\mathcal{P}(\sigma_1,\sigma_2,\sigma_3)\mathcal{P}(\rho_1,\rho_2,\rho_3) = \mathcal{P}(\sigma_1\circ\rho_1,\sigma_2\circ\rho_2,\sigma_3\circ\rho_3) \tag{3.63}$$

where $\sigma_i \in S_n$ and $\rho_i \in S_m$ for $i = 1, 2, 3$. Note that thanks to the way that we have ordered the fermions there are no -1 factors in this second equation.

We can now work out the details of this algebra in the representation basis. A straightforward computation shows

$$\mathcal{O}^{\gamma_1\gamma_2}_{r_1r_2r_3}\mathcal{O}^{\gamma_3\gamma_4}_{s_1s_2s_3} = \sum_{t_1\vdash n+m}\sum_{t_2\vdash n+m}\sum_{t_3\vdash n+m}\sum_{\gamma_5\gamma_6} f^{t_1t_2t_3;\gamma_1\gamma_2\gamma_3\gamma_4}_{r_1r_2r_3s_1s_2s_3;\gamma_5\gamma_6}\mathcal{O}^{\gamma_5\gamma_6}_{t_1t_2t_3} \tag{3.64}$$

where $r_i \vdash n$ and $s_i \vdash m$ for $i = 1, 2, 3$. The structure constants for this algebra are given by

$$f^{t_1t_2t_3;\gamma_1\gamma_2\gamma_3\gamma_4}_{r_1r_2r_3s_1s_2s_3;\gamma_5\gamma_6} = \frac{d_{t_1}}{(n+m)!}\frac{d_{t_2}}{(n+m)!}\frac{d_{t_3}}{(n+m)!}\sum_{\sigma_1\in S_n}\sum_{\sigma_2\in S_n}\sum_{\sigma_3\in S_n}\sum_{\rho_1\in S_m}\sum_{\rho_2\in S_m}\sum_{\rho_3\in S_m}$$

$$C^{\gamma_1\gamma_2}_{r_1r_2r_3}(\sigma_1,\sigma_2,\sigma_3)C^{\gamma_3\gamma_4}_{s_1s_2s_3}(\rho_1,\rho_2,\rho_3)C^{\gamma_5\gamma_6}_{t_1t_2t_3}(\sigma_1\circ\rho_1,\sigma_2\circ\rho_2,\sigma_3\circ\rho_2)$$
$$= \frac{d_{t_1}n!\,m!}{(n+m)!\,d_{r_1}d_{s_1}}\frac{d_{t_2}n!\,m!}{(n+m)!\,d_{r_2}d_{s_2}}\frac{d_{t_3}n!\,m!}{(n+m)!\,d_{r_3}d_{s_3}}B^{\gamma_1}_a\circ B^{\gamma_3}_b B^{\gamma_5}_{ab}\;\; B^{\gamma_6}_{cd}B^{\gamma_2}_c\circ B^{\gamma_4}_d \tag{3.65}$$

To get to the last line above, we have simply performed the sum over the σ_i and the ρ_i using the orthogonality relation (3.45). Remarkably, the structure constants are simply related to overlaps between branching coefficients! Computing these overlaps is a well defined problem in the representation theory of the symmetric group. Notice also that the structure constant, up to an overall factor, factorizes into a product of two overlaps of branching coefficients.

There is a similar algebra for the fermionic operators

$$\mathcal{P}^{\gamma_1\gamma_2}_{r_1r_2r_3}\mathcal{P}^{\gamma_3\gamma_4}_{s_1s_2s_3} = \sum_{t_1\vdash n+m}\sum_{t_2\vdash n+m}\sum_{t_3\vdash n+m}\sum_{\gamma_5\gamma_6} g^{t_1t_2t_3;\gamma_1\gamma_2\gamma_3\gamma_4}_{r_1r_2r_3s_1s_2s_3;\gamma_5\gamma_6}\mathcal{P}^{\gamma_5\gamma_6}_{t_1t_2t_3} \tag{3.66}$$

where the structure constants for this algebra are given by

$$g^{t_1t_2t_3;\gamma_1\gamma_2\gamma_3\gamma_4}_{r_1r_2r_3s_1s_2s_3;\gamma_5\gamma_6} = \frac{d_{t_1}}{(n+m)!}\frac{d_{t_2}}{(n+m)!}\frac{d_{t_3}}{(n+m)!}\sum_{\sigma_1\in S_n}\sum_{\sigma_2\in S_n}\sum_{\sigma_3\in S_n}\sum_{\rho_1\in S_m}\sum_{\rho_2\in S_m}\sum_{\rho_3\in S_m}$$
$$\tilde{C}^{\gamma_1\gamma_2}_{r_1r_2r_3}(\sigma_1,\sigma_2,\sigma_3)\tilde{C}^{\gamma_3\gamma_4}_{s_1s_2s_3}(\rho_1,\rho_2,\rho_3)\tilde{C}^{\gamma_5\gamma_6}_{t_1t_2t_3}(\sigma_1\circ\rho_1,\sigma_2\circ\rho_2,\sigma_3\circ\rho_2)$$
$$= \frac{d_{t_1}n!\,m!}{(n+m)!\,d_{r_1}d_{s_1}}\frac{d_{t_2}n!\,m!}{(n+m)!\,d_{r_2}d_{s_2}}\frac{d_{t_3}n!\,m!}{(n+m)!\,d_{r_3}d_{s_3}}\tilde{B}^{\gamma_1}_a\circ\tilde{B}^{\gamma_3}_b\tilde{B}^{\gamma_5}_{ab}\;\;\tilde{B}^{\gamma_6}_{cd}\tilde{B}^{\gamma_2}_c\circ\tilde{B}^{\gamma_4}_d \tag{3.67}$$

Again, the structure constants are simply related to overlaps between branching coefficients.

The existence of an algebraic structure for the gauge invariant operators has a remarkable consequence: we are able to solve the free theory exactly. To make this point, write the algebraic structure in a condensed notation as follows

$$\mathcal{O}^A\mathcal{O}^B = f^{AB}_C\mathcal{O}^C \tag{3.68}$$

with repeated indices summed. At the risk of being pedantic, A stands for two multiplicity labels (γ_1,γ_2 say) and three Young diagrams (r_1,r_2,r_3 say). Using this product repeatedly we find

$$\langle\mathcal{O}^{A_1}\mathcal{O}^{A_2}\mathcal{O}^{A_3}\cdots\mathcal{O}^{A_n}\rangle = f^{A_1A_2}_{C_1}f^{C_1A_3}_{C_2}\cdots f^{C_{n-2}A_n}_{C_{n-1}}\langle\mathcal{O}^{C_{n-1}}\rangle \tag{3.69}$$

so that the computation of an n-point correlation function is reduced to the computation of a one point function. We have already computed the most general one point function in the previous section. Of course, the structure constants of the algebra need to be evaluated and this is non-trivial. However, it does mean that the problem of solving the free tensor model has been reduced entirely to a problem in S_n representation theory. This simplification is highly non-trivial. There is a completely parallel argument for the fermionic tensor model.

3.4 Collective Field Theory

We have now constructed the complete set of gauge invariant variables and an algebra that these gauge invariants close. In this section we would like to construct a (collective) field theory governing the dynamics of these variables. Our discussion is guided by the dynamics of a single hermitian matrix $X = X^\dagger$ and we will review some relevant background before we consider the collective field theory relevant for the tensor model.

A complete set of gauge invariant variables for the one matrix model is provided by the Schur polynomials [90].

$$\chi_R(X) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\mathrm{Tr}(\sigma X) \tag{3.70}$$

These variables again close an interesting algebra, given by

$$\chi_R(X)\chi_S(X) = \sum_T g_{RST}\chi_T(X) \tag{3.71}$$

where g_{RST} are the Littlewood-Richardson coefficients. If we tried to quantize the Schur polynomial variables, it would be a mistake to treat them as independent, as the above algebra proves. In the case of a single matrix it is clear how we should proceed: one can select a smaller set of variables that are independent

$$\phi_n = \mathrm{Tr}(X^n) \tag{3.72}$$

where we should restrict $n \leq N$. The complete set of gauge invariant variables, the Schur polynomials, are polynomials in the ϕ_n. This is an important point: the ϕ_n are the set of variables that are independent and by considering polynomials in these variables, we recover the complete set of gauge invariant operators. In the large N limit, it is sensible to simply ignore the constraint $n \leq N$[13, 14]. We can then consider the field

$$\phi_k = \mathrm{Tr}(e^{ikX}) \tag{3.73}$$

or its Fourier transform, $\phi(x)$. The dynamics of this field, which is local in the emergent dimension x, is captured in the Das-Jevicki-Sakita Hamiltonian[13, 14, 15].

We now want to explore the possibility that there is a similar description possible for tensor models. Our first task is to identify the smaller set of independent variables which are independent and which we will quantize. Further, by considering polynomials in these variables, we should reconstruct the complete set of gauge invariant operators.

It proves convenient to work in the permutation basis

$$\bar{\Phi}\cdot\sigma_1\circ\sigma_2\circ\sigma_3\cdot\Phi = \bar{\Phi}^{IJK}\Phi_{\sigma_1(I)\sigma_2(J)\sigma_3(K)} \tag{3.74}$$

These invariants were first counted by Geloun and Ramgoolam in [83]. They have identified the number of invariants with the series A110143 on the OEIS website. This sequence counts the number of orbits

obtained when S_n acts on $S_n \times S_n$ via conjugacy, i.e. for $g \in S_n$ and $(x, y) \in S_n \times S_n$ we have $g(x, y) = (gxg^{-1}, gyg^{-1})$. The number of invariants grows extremely rapidly

$$1, 4, 11, 43, 161, 901, 5579, 43206, 378360, 3742738, ... \tag{3.75}$$

A useful way to label the invariants, following [83], is by bipartite cubic graphs with edges labeled by the gauge group the corresponding index belongs to.

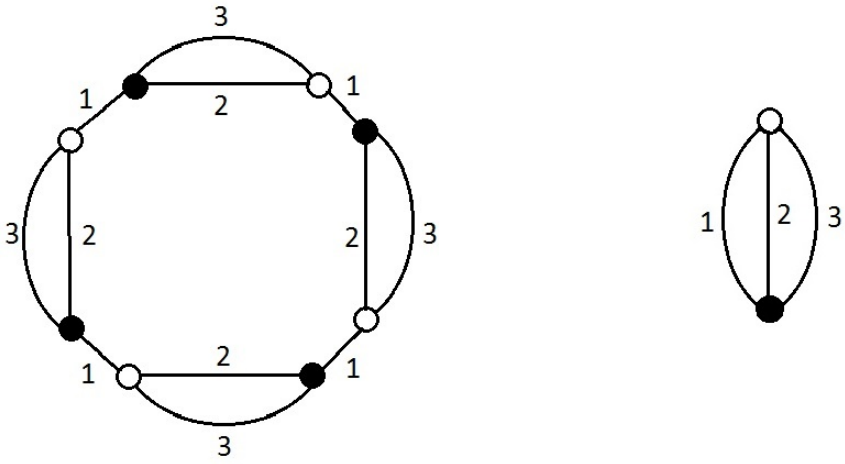


Figure 3.1: The above figures label gauge invariant operators in the tensor model gauge theory. Black dots correspond to $\bar{\phi}^{ijk}$'s and white dots to ϕ_{ijk}'s. A line labeled by i is a gauge index for $U(N_i)$. The operator on the left corresponds to $\bar{\phi}^{i_1 j_1 k_1}\phi_{i_2 j_1 k_1}\bar{\phi}^{i_2 j_2 k_2}\phi_{i_3 j_2 k_2}\bar{\phi}^{i_3 j_3 k_3}\phi_{i_4 j_3 k_3}\bar{\phi}^{i_4 j_4 k_4}\phi_{i_1 j_4 k_4}$ and the operator on the right corresponds to $\bar{\phi}^{ijk}\phi_{ijk}$.

In the language of graphs it is easy to identify the smaller set of independent variables: they are the variables that correspond to connected graphs. The number of connected graphs can be counted using the plethystic logarithm and is identified with the series A057005 on the OEIS website[83]. The number of independent variables still grows extremely rapidly

$$1, 3, 7, 26, 97, 624, 4163, 34470, 314493, 3202839, 35704007, 433460014, ... \tag{3.76}$$

This growth seems to be too rapid to manage. We will now argue that we can restrict the dynamics to an even smaller set of variables. To describe the smaller set of variables, it is useful to consider

$$T^{i_1}{}_{i_2} = \bar{\phi}^{i_1 jk}\phi_{i_2 jk} \tag{3.77}$$

T is a matrix on the vector space that carries the fundamental of $U(N_1)$. It thus makes sense to take powers of T

$$(T^n)^{i_1}{}_{i_2} = T^{i_1}{}_j T^j{}_k \cdots T^l{}_{i_2} \tag{3.78}$$

The smaller set of gauge invariants that we consider is given by

$$\phi_n = \mathrm{Tr}(T^n) \tag{3.79}$$

The Hamiltonian of the tensor model quantum mechanics we consider is given by

$$H = -\frac{\partial}{\partial\bar{\phi}^{ijk}}\frac{\partial}{\partial\phi_{ijk}} + \frac{1}{4}\bar{\phi}^{ijk}\phi_{ijk} \tag{3.80}$$

The coefficient of the second term has been chosen to ensure that the equal time two point function is given by (3.42). The kinetic terms of the Hamiltonian, when rewritten in terms of the new (collective) variables are

$$-\frac{\partial}{\partial\bar{\phi}^{ijk}}\frac{\partial}{\partial\phi_{ijk}} = -\sum_{n,m}\Omega(n,m)\frac{\partial}{\partial\phi_n}\frac{\partial}{\partial\phi_m} + \sum_n \omega(n)\frac{\partial}{\partial\phi_n} \tag{3.81}$$

where[2]

$$\Omega(n,m) = \frac{\partial\phi_n}{\partial\bar{\phi}^{ijk}}\frac{\partial\phi_m}{\partial\phi_{ijk}} = nm\phi_{n+m-1} \tag{3.82}$$

$$\omega(n) = -\frac{\partial}{\partial\bar{\phi}^{ijk}}\left(\frac{\partial\phi_n}{\partial\phi_{ijk}}\right) = -\sum_{r=0}^{n-2}\phi_r\phi_{n-r-1} - N_2N_3m\phi_{m-1} \tag{3.83}$$

It is nontrivial that $\Omega(n,m)$ and $\omega(n)$ can be expressed in terms of the ϕ_n. This implies that the Hamiltonian itself can be expressed in terms of this smaller set of variables, and hence that it is consistent with the dynamics to restrict to this smaller set of variables. When written in terms of the new variables, the Hamiltonian

$$H = -\sum_{n,m}\Omega(n,m)\frac{\partial}{\partial\phi_n}\frac{\partial}{\partial\phi_m} + \sum_n\omega(n)\frac{\partial}{\partial\phi_n} + \frac{1}{4}\phi_1 \tag{3.84}$$

is not hermitian. This simply reflects the fact that in the new variables the inner product is accompanied by a non-trivial Jacobian $J[\phi]$. Performing a similarity transformation to trivialize the measure, we arrive at a manifestly hermitian Hamiltonian[13, 14, 15]

$$H = \Pi\Omega\Pi + \frac{1}{4}\left(\omega + \frac{\partial\Omega}{\partial\phi}\right)\Omega^{-1}\left(\omega + \frac{\partial\Omega}{\partial\phi}\right) + \phi_1 - \frac{1}{2}\frac{\partial\omega}{\partial\phi} - \frac{1}{2}\frac{\partial^2\Omega}{\partial\phi\partial\phi} \tag{3.85}$$

where we have used an obvious matrix notation and have introduced the momentum $\Pi(n)$ conjugate to ϕ_n

$$\Pi_n = -i\frac{\partial}{\partial\phi_n} \tag{3.86}$$

As we have commented above, the variables ϕ_n that we have employed in the description so far are a natural generalization of the variables (3.72) used in the matrix model. The variables (3.72) are essentially the eigenvalues of the matrix model, so that it is natural to interpret the gauge invariant variables constructed out of $T^i{}_j$ as providing an eigenvalue like description of the tensor model. Just as in the matrix case, the quantum mechanical system develops an extra dimension. To see how this happens, we can explore the range of the eigenvalues of T. In the case of a single matrix, the change to eigenvalues induces a Van der Monde determinant which produces a repulsion between the eigenvalues ensuring they spread out to produce a macroscopic emergent geometry at large N. To get some insight into what is happening in the case of the tensor model, we compute the one point functions

$$\langle \mathrm{Tr}(T^k)\rangle = \langle\bar{\Phi}\cdot(k)\circ\mathbf{1}\circ\mathbf{1}\cdot\Phi\rangle = \sum_{\sigma\in S_k}N_1^{C((k)\sigma)}N_2^{C(\sigma)}N_3^{C(\sigma)} \tag{3.87}$$

[2]Note that these expressions are almost identical to the answers for the Hermitian one matrix model which are $\Omega(n,m) = nm\phi_{n+m-2}$ and $\omega(n) = -\sum_{r=0}^{n-2}\phi_r\phi_{n-r-2}$.

In the above (k) is a k-cycle which, for concreteness, we take to be $(123\cdots k)$. Lets study the limit that $N_i \to \infty$ holding $\frac{N_2}{N_3}$ fixed and taking $\alpha = \frac{N_2 N_3}{N_1}$ fixed. In this large N_i limit the above sum is then dominated by a nontrivial class of diagrams. For example

$$\begin{aligned}
\langle \mathrm{Tr}(T)\rangle &= N_1 N_2 N_3 = \alpha N_1^2 \\
\langle \mathrm{Tr}(T^2)\rangle &= N_1^2 N_2 N_3 + N_1 N_2^2 N_3^2 = (\alpha+\alpha^2) N_1^3 \\
\langle \mathrm{Tr}(T^3)\rangle &= N_1^3 N_2 N_3 + 3 N_1^2 N_2^2 N_3^2 + N_1 N_2^3 N_3^3 + N_1 N_2 N_3 \\
&= (\alpha + 3\alpha^2 + \alpha^3 + \frac{\alpha}{N_1^2}) N_1^4 \\
\langle \mathrm{Tr}(T^4)\rangle &= N_1^4 N_2 N_3 + 6 N_1^3 N_2^2 N_3^2 + 6 N_1^2 N_2^3 N_3^3 + N_1 N_2^4 N_3^4 + 5 N_1^2 N_2 N_3 + 5 N_1 N_2^2 N_3^2 \\
&= N_1^5 \left(\alpha + 6\alpha^2 + 6\alpha^3 + \alpha^4 + \frac{5\alpha}{N^2} + \frac{5\alpha^2}{N^2} \right) \\
\langle \mathrm{Tr}(T^5)\rangle &= N_1^5 N_2 N_3 + 10 N_1^4 N_2^2 N_3^2 + 20 N_1^3 N_2^3 N_3^3 + 10 N_1^2 N_2^4 N_3^4 + N_1 N_2^5 N_3^5 \\
&\quad + 15 N_1^3 N_2 N_3 + 40 N_1^2 N_2^2 N_3^2 + 15 N_1 N_2^3 N_3^3 + 8 N_1 N_2 N_3 \\
&= N_1^6 \Big(\alpha + 10\alpha^2 + 20\alpha^3 + 10\alpha^4 + \alpha^5 + \frac{15\alpha + 40\alpha^2 + 15\alpha^3}{N_1^2} + 8\frac{\alpha}{N_1^4} \Big) \\
\langle \mathrm{Tr}(T^k)\rangle &\sim N_1^{k+1}
\end{aligned} \tag{3.88}$$

The growth with k as N_1^{k+1} is a clear indication that the eigenvalues of T are spreading out and are potentially able to generate a new dimension. To construct the field theory in this extra dimension, it is useful to introduce the field

$$\phi(x) = \int \frac{dk}{2\pi}\, e^{-ikx}\, \phi_k \qquad \phi_k = \mathrm{Tr}(e^{ikT}) \tag{3.89}$$

Notice that $\phi(x)$ is nothing but the density of eigenvalues of the T matrix and consequently

$$\mathrm{Tr}(T^n) = \int dx \phi(x) x^n \tag{3.90}$$

The momentum dual to $\phi(x)$ is $\pi(x) = \frac{1}{i}\frac{\delta}{\delta\phi(x)}$ and similarly $\pi_k = \frac{1}{i}\frac{\delta}{\delta\phi_k}$. To perform the change of variables, note that the kinetic terms in the tensor model Hamiltonian can be written as

$$\begin{aligned}
-\frac{\partial}{\partial \bar\phi^{ijk}}\frac{\partial}{\partial \phi_{ijk}} &= -T^i{}_l \frac{\partial}{\partial T^j{}_l}\frac{\partial}{\partial T^i{}_j} - N_2 N_3 \frac{\partial}{\partial T^i{}_i} \\
&= -\int dk \int dk'\ \Omega_{k,k'}\, \pi_k\, \pi_{k'} + \int dk\ \omega_k\, \pi_k
\end{aligned} \tag{3.91}$$

where

$$\begin{aligned}
\Omega_{k,k'} &= T^i{}_l \frac{\partial \phi_k}{\partial T^j{}_l}\frac{\partial \phi_{k'}}{\partial T^i{}_j} = -kk' \mathrm{Tr}(T\, e^{i(k+k')T}) \\
&= ikk' \frac{\partial}{\partial k}\phi_{k+k'}
\end{aligned} \tag{3.92}$$

and

$$\begin{aligned}
\omega_k &= -T^i{}_l \frac{\partial}{\partial T^j{}_l}\left(\frac{\partial \phi_k}{\partial T^i{}_j}\right) - N_2 N_3 \frac{\partial \phi_k}{\partial T^i{}_i} \\
&= k\int_0^1 d\tau\ \phi_{\tau k} i\frac{\partial}{\partial\tau}\phi_{(1-\tau)k} - ikN_2N_3\phi_k
\end{aligned} \tag{3.93}$$

To obtain these results, we have used the formula

$$\frac{\partial}{\partial M^i{}_j}(e^{-ikM})^k_l = (-ik)\int_0^1 d\tau (e^{-i\tau kM})^k{}_i (e^{-i(1-\tau)kM})^j{}_l \tag{3.94}$$

In position space we obtain

$$\Omega(x,x') = \frac{\partial}{\partial x}\frac{\partial}{\partial x'}\left(x\phi(x)\delta(x-x')\right) \tag{3.95}$$

and

$$\omega(x) = 2\frac{\partial}{\partial x} dy\ \phi(x)\phi(y)\frac{x}{x-y} + (N_2N_3 - N_1)\frac{\partial\phi(x)}{\partial x} \tag{3.96}$$

It is interesting to note that the formula for $\Omega(x,x')$ is identical to the formula obtained from the radial sector of multi matrix models, and that the formula for $\omega(x)$ is very similar - see [91, 92, 93, 94]. This easily leads to the following Hamiltonian (we have dropped constant terms)

$$\begin{aligned} H &= \int dx \left[\frac{\partial\pi}{\partial x}\, x\phi(x)\, \frac{\partial\pi}{\partial x} + \frac{\phi(x)}{4x}\left(dy\ \frac{2x\phi(y)}{x-y}\right)^2 + \frac{(N_2N_3-N_1)^2}{4x}\phi(x) + \frac{x}{4}\phi(x)\right] \\ &\quad -\mu\int dx\,\phi(x) \end{aligned} \tag{3.97}$$

where the last term above enforces the constraint $\int dx\phi(x) = N_1$. To get this result, we used

$$\int dxdy\,\phi(x)\phi(y)\frac{x+y}{x-y} = 0 \tag{3.98}$$

As we explain in Appendix B.3, the Hamiltonian can be written as

$$H = \int dx \frac{\partial\pi}{\partial x} x\phi(x)\frac{\partial\pi}{\partial x} + V_{\text{eff}} \tag{3.99}$$

where the effective potential is

$$V_{\text{eff}} = \int dx \left[\frac{\pi^2 x}{3}\phi^3 + \frac{(\alpha-1)^2N_1^2}{4x}\phi(x) + \frac{x}{4}\phi(x) - \mu\phi(x)\right] \tag{3.100}$$

The classical field should minimize the effective potential, which leads to the following classical collective equation of motion

$$\begin{aligned} 0 = \frac{\delta V_{\text{eff}}}{\delta\phi(x)} &= \pi^2 x\phi^2 + \frac{(1-\alpha)^2N_1^2}{4x} + \frac{x}{4} - \mu \\ \Rightarrow \phi(x) &= \frac{1}{\pi}\sqrt{\frac{\mu}{x} - \frac{1}{4} - \frac{(1-\alpha)^2N_1^2}{4x^2}} \end{aligned} \tag{3.101}$$

The chemical potential μ should be fixed by requiring that

$$\int_{x_-}^{x_+} dx\phi(x) = N_1 \tag{3.102}$$

where the limits of integration are

$$x_\pm = 2\mu \pm \sqrt{4\mu^2 - N_1^2(1-\alpha)^2} \tag{3.103}$$

As a test of this classical solution, we would like to show that it reproduces the correct large N_1 correlators. To simplify the analysis that follows, we will set $\alpha = 1$. In this case, after solving for μ we have the density

$$\phi(x) = \frac{1}{\pi}\sqrt{\frac{N_1}{x} - \frac{1}{4}} \qquad x_+ = 4N_1 \quad x_- = 0 \tag{3.104}$$

A simple computation now gives

$$\begin{gathered}\int_0^{\frac{N_1}{4}} x\phi(x)dx = N_1^2 \qquad \int_0^{\frac{N_1}{4}} x^2\phi(x)dx = 2N_1^3 \\ \int_0^{\frac{N_1}{4}} x^3\phi(x)dx = 5N_1^4 \qquad \int_0^{\frac{N_1}{4}} x^4\phi(x)dx = 14N_1^5 \quad \int_0^{\frac{N_1}{4}} x^5\phi(x)dx = 42N_1^6\end{gathered} \tag{3.105}$$

in complete agreement with (3.88). This provides a nice test of classical collective solution.

Chapter 4

Holography for Tensor models

The work presented in this chapter is based on the author's original research [40]. The main focus of this work is on the holography of tensor models. For related studies, see [39, 95, 96], as well as [66, 63, 97, 98, 99, 100, 101, 102, 103]. One possible approach to the holography of tensor models, is to express the dynamics of the model in terms of gauge invariant variables. A systematic method to approach this problem is the collective field theory[13, 14].

Since there are suggestions that tensor models are richer than vector models, but perhaps simpler than matrix models, it is worth examining their collective description. This is our main goal in this work. If an example can be found, with a space of invariant variables that is richer than vector models, it may shed important insights into how to tackle the space of invariant variables for multi-matrix models.

Section 4.1 discusses one approach towards enumerating and then constructing the gauge invariants of tensor models. We focus on rank 3 tensors which transform in the fundamental of $U(N_1) \times U(N_2) \times U(N_3)$. This space of invariants is enormous and hard to handle. We start by constructing invariants under $U(N_1) \times U(N_2)$, which can be managed. The result is a collection of $U(N_3)$ tensors of arbitrary even rank, which is a definite intermediate result on the path towards characterizing the complete space of invariants. Section 4.2 looks for and identifies sectors that are dynamically decoupled. These correspond to stringy states and indicate that there are strings in the Hilbert space of the tensor model and further, that this is a useful language for classes of questions that can be pursued. We explicitly demonstrate how collective field theory reproduces correlators in a model with interactions. This physics is however the well known physics of matrix models. In an attempt to find some truly tensor model physics, Section 4.3 considers the construction of the collective field theory of melonic variables. Using certain known results for the melonic variables, we are able to construct the leading large N descriptions of the melonic invariant variables. We find that there is no holographic dimension emergent at large N. In Section 4.4 we use a complete basis of the tensor model invariants provided by the restricted Schur polynomials, to investigate non-perturbative contributions to the large N expansion.

4.1 Invariant Variables

In matrix models the basic gauge invariant observables are built from a matrix Z^a_b which transforms as $Z \to U^\dagger Z U$ under a gauge transformation. The gauge invariant observables are given by traces and products of traces. For theories with more than one matrix, traces are populated with arbitrary products of the different matrices. Although this space of invariants is complicated, many results have been established. We know how to count gauge invariant operators both at infinite and at finite N for any number of matrices[104, 105, 106, 107, 108, 19, 88]. There is a systematic $1/N$ expansion for correlation functions, the expansion is phrased in terms of ribbon graphs and corrections to the planar limit come in powers of $\frac{1}{N^2}$. The expansion has a graphical meaning as a sum over surfaces (string worldsheets)[1]. The goal of this work is to generalize as much of this description as possible to tensor models. The basic method employed in this work is the collective field theory. Applying collective field theory to matrix quantum mechanics leads to a highly non-trivial string field theory that beautifully captures the dynamics of the

$c = 1$ string[15]. Our primary motivation in this work is to initiate the construction of the analogous theory of the invariants of the tensor model quantum mechanics.

In the tensor model we consider, the basic field is ϕ^{abc} and its conjugate $\bar{\phi}_{abc}$. The gauge group is $U(N_1) \times U(N_2) \times U(N_3)$. Under a gauge transformation the fields transform as

$$\phi^{abc} \to (U_1)^a_d (U_2)^b_e (U_3)^c_f \phi^{def} \qquad\qquad \bar{\phi}_{abc} \to (U_1)^d_a (U_2)^e_b (U_3)^f_c \phi_{def} \tag{4.1}$$

where $U_i \in U(N_i)$. The complete set of gauge invariant operators is enormous and making sense of this space of invariants is the basic obstacle that we must overcome. Our approach is to deal with the problem in two steps: first construct the gauge invariants under $U(N_1) \times U(N_2)$, and then deal with the last gauge symmetry as the second step. The advantage of breaking the problem up in this way is that the first step can be tackled: there is a simple solution that can be written down explicitly. The set of $U(N_1) \times U(N_2)$ invariants is spanned by the $U(N_3)$ tensors given by

$$T(2n)^{i_1 i_3 \cdots i_{2n+1}}_{i_2 i_4 \cdots i_{2n}} = \phi^{a_1 b_1 i_1} \bar{\phi}_{a_1 b_2 i_2} \phi^{a_2 b_2 i_3} \bar{\phi}_{a_2 b_3 i_4} \cdots \bar{\phi}_{a_n b_1 i_{2n}} \tag{4.2}$$

To prove that these span the $U(N_1) \times U(N_2)$ invariants, we will count the number of $U(N_3)$ invariants and then compare to an independent count.

Our first task is to provide the independent count of the number of $U(N_1) \times U(N_2) \times U(N_3)$ invariants. This problem has been solved in [83, 39]. Invariants constructed using n ϕ^{abc}s and n $\bar{\phi}_{abc}$s are in one-to-one correspondence with elements of the double coset

$$S_n \setminus S_n \times S_n \times S_n \,/\, S_n \tag{4.3}$$

The number $\mathcal{N}_3$ of invariants, which is equal to the order of this double coset is given by

$$\mathcal{N}_3 = \frac{1}{(n!)^2} \sum_{\sigma_1, \sigma_2, \sigma_3 \in S_n} \sum_{\beta_1, \beta_2 \in S_n} \delta(\beta_1 \sigma_1 \beta_2 \sigma_1^{-1}) \delta(\beta_1 \sigma_2 \beta_2 \sigma_2^{-1}) \delta(\beta_1 \sigma_3 \beta_2 \sigma_3^{-1}) \tag{4.4}$$

The first few values of $\mathcal{N}_3$ are shown in Table 4.1 below.

Table 4.1: $\mathcal{N}_3$ counts the number of gauge invariant operators for rank 3 tensor fields constructed using n fields. We take $N_i = \infty$ for $i = 1, 2, 3$, i.e. we have not accounted for any relations between invariants that appear at finite N_i.

	n=1	n=2	n=3	n=4	n=5	n=6
$\mathcal{N}_3$	1	4	11	43	161	901

To argue that (4.2) are a complete set of $U(N_1) \times U(N_2)$ invariants, we want to reproduce the counting above, using (4.2) as the dynamical variables. The counting for $n = 1, 2$ is simple. For $n = 1$ we have a single invariant given by $T(2)^i_i$ in agreement with Table 4.1 above. For $n = 2$ we have $T(2)^i_i T(2)^j_j$, $T(2)^i_j T(2)^j_i$, $T(4)^{ij}_{ij}$ or $T(4)^{ij}_{ji}$. This gives a total of four gauge invariants, in agreement with Table 4.1. For higher values of n things are less trivial as there are non-trivial relations between different invariant variables. To see the origin of these relations, note that $T(2n)$ enjoys a Z_n generated by the n cycle $(12 \cdots n)$. For example, $T(6)$ enjoys a Z_3 generated by the three cycle (123). The symmetry acts as

$$T(6)^{i_1 i_3 i_5}_{i_2 i_4 i_6} = T(6)^{i_5 i_1 i_3}_{i_6 i_2 i_4} = T(6)^{i_3 i_5 i_1}_{i_4 i_6 i_2} \tag{4.5}$$

A nice example in which these symmetries play a role is for $n = 4$. For $n = 4$ there are 5 possible combinations of fields that can be used to construct the gauge invariant, namely

$$T(2)^{i_1}_{j_1}T(2)^{i_2}_{j_2}T(2)^{i_3}_{j_3}T(2)^{i_4}_{j_4} \quad \text{or} \quad T(4)^{i_1 i_2}_{j_1 j_2}T(2)^{i_3}_{j_3}T(2)^{i_4}_{j_4} \quad \text{or} \quad T(6)^{i_1 i_2 i_3}_{j_1 j_2 j_3}T(2)^{i_4}_{j_4}$$

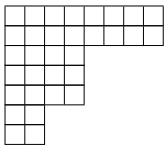
$$\text{or} \quad T(4)^{i_1 i_2}_{j_1 j_2}T(4)^{i_3 i_4}_{j_3 j_4} \quad \text{or} \quad T(8)^{i_1 i_2 i_3 i_4}_{j_1 j_2 j_3 j_4} \tag{4.6}$$

For each of these the number of gauge invariant variables is given by computing a sum of the form [1]

$$|S_4//G| = \frac{1}{|G|}\sum_{h_1 \in G}\sum_{g \in S_4}\delta(h_1 g h_1^{-1} g^{-1}) \tag{4.7}$$

The notation used for the LHS of this equation is explained in Appendix C.1. Using an element $g \in S_4$, the invariants constructed from a product of four $T(2)$s are given by $T(2)^{i_1}_{i_{g(1)}}T(2)^{i_2}_{i_{g(2)}}T(2)^{i_3}_{i_{g(3)}}T(2)^{i_4}_{i_{g(4)}}$. There is a symmetry under permuting the fields so that $G = S_4$ and we find 5 gauge invariant operators. G swaps the $T(2)$s. For $T(4)^{i_1 i_2}_{i_{g(1)} i_{g(2)}}T(2)^{i_3}_{i_{g(3)}}T(2)^{i_4}_{i_{g(4)}}$ we find that $G = S_2 \times Z_2$ and we find 10 gauge invariant operators. G swaps the two $T(2)$s and does cyclic permutations of the upper and lower indices of $T(4)$. For $T(6)^{i_1 i_2 i_3}_{i_{g(1)} i_{g(2)} i_{g(3)}}T(2)^{i_4}_{i_{g(4)}}$ we have $G = Z_3$ and we find 10 gauge invariant operators. G performs cyclic swaps of the upper and lower indices of $T(6)$. For $T(4)^{i_1 i_2}_{i_{g(1)} i_{g(2)}}T(4)^{i_3 i_4}_{i_{g(3)} i_{g(4)}}$ we have $G = S_2 \times Z_2 \times Z_2$ and we find 8 gauge invariant operators. G swaps the two $T(4)$s and performs cyclic permutations of the upper and lower indices. For $T(8)^{i_1 i_2 i_3 i_4}_{i_{g(1)} i_{g(2)} i_{g(3)} i_{g(4)}}$ we have $G = Z_4$ and we find 10 gauge invariant operators. G performs cyclic permutation of the upper and lower indices of $T(8)$. This gives a total of 43 gauge invariant operators which reproduces the $n = 4$ entry in Table 4.1.

For $n = 6$ we have also verified that there is a total of 901 operators. The pattern follows that of the computations above, but this example is complicated enough that it is worth the effort to set things up in a more general way. To illustrate the comments that follow, consider the number of gauge invariant operators constructed for $n = 32$ i.e. we use a total of 64 fields. Each possible collection of operators that can be used to construct an invariant can be labeled with a Young diagram that has n boxes. Further, we will see that the group G can be read straight from the Young diagram. Each row containing k boxes translates into a tensor $T(2k)^{i_1 \cdots}_{j_1 \cdots}$. As an example, the Young diagram



(4.8)

labels the gauge invariant operators constructed using $T(16)T(16)T(8)T(8)T(8)T(4)T(4)$. To read G from the Young diagram, first note that for each row we get a cyclic group Z_k, where k is the number of boxes in the row. For the above Young diagram we'd get

$$Z_8 \times Z_8 \times Z_4 \times Z_4 \times Z_4 \times Z_2 \times Z_2 \tag{4.9}$$

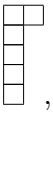
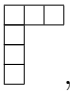
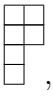
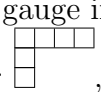
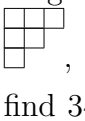
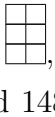
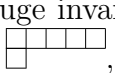
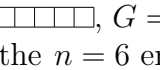
For p rows of equal length we get a factor of S_p. For the above Young diagram we get

$$S_2 \times S_3 \times S_2 \tag{4.10}$$

[1] This is an application of the orbit counting lemma. See [109] for relevant background.

G is a product of these cyclic and symmetric groups. For the above Young diagram we'd have

$$G = Z_8 \times Z_8 \times Z_4 \times Z_4 \times Z_4 \times Z_2 \times Z_2 \times S_2 \times S_3 \times S_2 \tag{4.11}$$

To illustrate these rules, we count the number of gauge invariant operators for $n = 6$. For , $G = S_6$ and we find 11 gauge invariant operators. For  , $G = Z_2 \times S_4$ and we find 34 gauge invariant operators. For  , $G = Z_3 \times S_3$ and we find 58 gauge invariant operators. For  , $G = Z_2 \times Z_2 \times S_2 \times S_2$ and we find 70 gauge invariant operators. For  , $G = Z_4 \times S_2$ and we find 108 gauge invariant operators. For  , $G = Z_3 \times Z_2$ and we find 136 gauge invariant operators. For  , $G = Z_2 \times Z_2 \times Z_2 \times S_3$ and we find 34 gauge invariant operators. For  , $G = Z_5$ and we find 148 gauge invariant operators. For , $G = Z_4 \times Z_2$ and we find 108 gauge invariant operators. For , $G = Z_3 \times Z_3 \times S_2$ and we find 58 gauge invariant operators. For  , $G = Z_6$ and we find 136 gauge invariant operators. This gives 901 operators in total, reproducing the $n = 6$ entry in Table 4.1.

The counting arguments we have given above provide convincing evidence that the $U(N_1) \times U(N_2)$ invariants given in (4.2) indeed provide a complete list. The simple model, of rank-3 tensors considered above, is equivalent to a model of interacting $U(N_3)$ tensors of every even rank. We can attempt a similar construction for higher rank tensors again constructing the invariants of two of the groups appearing. In this way tensor models of higher rank tensors are described by a collection of tensors that transform in all but 2 gauge groups. Dealing with this remaining gauge symmetry becomes more and more non-trivial as the rank is increased beyond 3 and for this reason we will restrict attention to rank 3 tensors. In the sections that follow we will consider constructing collective field theory based on a subset of these invariant variables.

4.2 A "planar limit" for the tensor model

In the previous section we have described the structure of the space of invariants for tensor models. The complete space is difficult to manage. Consequently, we seek subsets of invariant variables that are dynamically closed. In this section we will find a matrix like subsector, which has non-trivial dynamics but is still simple enough to manage.

Consider a model with two basic flavors of tensors, ϕ^{abc}, $\bar{\phi}_{abc}$ and ψ^{abc}, $\bar{\psi}_{abc}$. There are also momenta conjugate to these fields and we have the equal time commutation relations

$$\begin{aligned} [\Pi_{abc}, \phi^{def}] &= -i\delta^d_a\delta^e_b\delta^f_c \qquad [\bar{\Pi}^{def}, \bar{\phi}_{abc}] = -i\delta^d_a\delta^e_b\delta^f_c \\ [P_{abc}, \psi^{def}] &= -i\delta^d_a\delta^e_b\delta^f_c \qquad [\bar{P}^{def}, \bar{\psi}_{abc}] = -i\delta^d_a\delta^e_b\delta^f_c \end{aligned} \tag{4.12}$$

All commutators between barred and unbarred fields, as well as all others commutators not shown, vanish. The Hamiltonian for the free theory is

$$H = \Pi_{abc}\bar{\Pi}^{abc} + \omega^2\phi^{abc}\bar{\phi}_{abc} + P_{abc}\bar{P}^{abc} + \omega^2\psi^{abc}\bar{\psi}_{abc}$$

$$= -\frac{\partial}{\partial\phi^{abc}}\frac{\partial}{\partial\bar{\phi}_{abc}} + \omega^2\phi^{abc}\bar{\phi}_{abc} - \frac{\partial}{\partial\psi^{abc}}\frac{\partial}{\partial\bar{\psi}_{abc}} + \omega^2\psi^{abc}\bar{\psi}_{abc} \tag{4.13}$$

A simple set of invariants that are closed under both the splitting and joining operations of collective field theory, are the traces of products of $T_\phi(2)^i_j$ and $T_\psi(2)^i_j$, where

$$T_\phi(2)^i_j = \phi^{abi}\bar{\phi}_{abj} \qquad T_\psi(2)^i_j = \psi^{abi}\bar{\psi}_{abj} \tag{4.14}$$

Both transform in the adjoint of $U(N_3)$. Following [39] we can construct an exact collective description for $T_\phi(2)$. We will extend the discussion of [39] by considering a model with a quartic potential. Choosing a potential which is a sum $V(T_\phi(2)) + V(T_\psi(2))$ preserves the fact that the dynamics of $T_\phi(2)$ defines a dynamically closed subsector. Motivated by this remark, we consider the interacting dynamics of the Hamiltonian

$$H = \Pi_{abc}\bar{\Pi}^{abc} + \omega^2\phi^{abc}\bar{\phi}_{abc} + g\phi^{abc}\bar{\phi}_{abd}\phi^{efd}\bar{\phi}_{efc} \tag{4.15}$$

where we dropped the ψ terms since they play no role in this discussion. The collective field

$$\phi(x) = \int \frac{dk}{2\pi}\, e^{-ikx}\,\phi_k \qquad \phi_k = \mathrm{Tr}(e^{ikT_\phi(2)}) \tag{4.16}$$

is the density of eigenvalues of $T_\phi(2)$. A straight forward computation gives

$$\begin{aligned}\Omega_{k,k'} &= \frac{\partial\phi_k}{\partial\phi^{abc}}\frac{\partial\phi_{k'}}{\partial\bar{\phi}_{abc}} = ikk'\frac{\partial}{\partial k}\phi_{k+k'} \\ \omega_k &= -\frac{\partial}{\partial\phi^{abc}}\left(\frac{\partial\phi_k}{\partial\bar{\phi}_{abc}}\right) k\int_0^1 d\tau\ \phi_{\tau k} i\frac{\partial}{\partial\tau}\phi_{(1-\tau)k} - ikN_1N_2\phi_k\end{aligned} \tag{4.17}$$

In position space we obtain[2]

$$\begin{aligned}\Omega(x,x') &= \frac{\partial}{\partial x}\frac{\partial}{\partial x'}\left(x\phi(x)\delta(x-x')\right) \\ \omega(x) &= 2\frac{\partial}{\partial x}dy\ \phi(x)\phi(y)\frac{x}{x-y} + (N_1N_2 - N_3)\frac{\partial\phi(x)}{\partial x}\end{aligned} \tag{4.18}$$

This easily leads to the Hamiltonian (after discarding constant terms)

$$H = \int dx\frac{\partial\pi}{\partial x}x\phi(x)\frac{\partial\pi}{\partial x} + V_{\text{eff}} \tag{4.19}$$

where the effective potential is

$$V_{\text{eff}} = \int dx\left[\frac{\pi^2 x}{3}\phi^3 + \frac{N_3^2 - N_1^2N_2^2}{4x}\phi(x) + gx^2\phi(x) + \omega^2 x\phi(x) - \mu\phi(x)\right] \tag{4.20}$$

The last term above is a Lagrange multiplier enforcing the constraint $\int dx\phi(x) = N_3$. The large N classical field minimizes the effective potential, which implies the following equation of motion

$$0 = \frac{\delta V_{\text{eff}}}{\delta\phi(x)} = \pi^2 x\phi^2 + \frac{N_3^2 - N_1^2N_2^2}{4x} + \omega^2 x - \mu + gx^2$$

[2]The formula for $\Omega(x,x')$ coincides with the radial sector of multi matrix models and the formula for $\omega(x)$ is very similar - see [91, 92, 93, 94].

$$\Rightarrow \phi(x) = \frac{1}{\pi}\sqrt{\frac{\mu}{x} - \omega^2 - \frac{N_3^2 - N_1^2 N_2^2}{4x^2} - gx} \tag{4.21}$$

where μ is fixed by requiring that

$$\int_{x_-}^{x_+} dx\phi(x) = N_3 \qquad \phi(x_\pm) = 0 \tag{4.22}$$

The above dynamics simplifies if we set $N_1 N_2 = N_3$, something that we do from now on. In the Appendix C.2 we demonstrate that this collective field reproduces the correct large N expectation values in perturbation theory.

The discussion of this section shows that there is a matrix-like subsector in the Hilbert space of the tensor model. In particular, there will be string states. This has all been extracted from a single matrix subsector. To see stringy states and their excitations, we will construct the analog of the BMN loops[110]. We carry out the discussion in the context of the free theory. The fields have the following modes expansions

$$\begin{aligned} \phi^{abc}(t) &= \frac{1}{2\omega}\left(e^{-i\omega t}a^{abc} + e^{i\omega t}b^{abc\,\dagger}\right) \qquad \bar{\phi}_{abc}(t) = \frac{1}{2\omega}\left(e^{-i\omega t}b_{abc} + e^{i\omega t}a^\dagger_{abc}\right) \\ \psi^{abc}(t) &= \frac{1}{2\omega}\left(e^{-i\omega t}c^{abc} + e^{i\omega t}d^{abc\,\dagger}\right) \qquad \bar{\psi}_{abc}(t) = \frac{1}{2\omega}\left(e^{-i\omega t}d_{abc} + e^{i\omega t}c^\dagger_{abc}\right) \end{aligned} \tag{4.23}$$

and the oscillators obey the following commutation relations

$$[a^{abc}, a^\dagger_{def}] = [c^{abc}, c^\dagger_{def}] = 2\omega\delta^a_d\delta^b_e\delta^c_f = [b_{def}, b^{abc\,\dagger}] = [d_{def}, d^{abc\,\dagger}] \tag{4.24}$$

We can also write mode expansions for the conjugate momenta as follows

$$\begin{aligned} \bar{\Pi}^{abc}(t) &= -\frac{i}{2}\left(e^{-i\omega t}a^{abc} - e^{i\omega t}b^{abc\,\dagger}\right) \qquad \Pi_{abc}(t) = -\frac{i}{2}\left(e^{-i\omega t}b_{abc} - e^{i\omega t}a^\dagger_{abc}\right) \\ \bar{P}^{abc}(t) &= -\frac{i}{2}\left(e^{-i\omega t}c^{abc} - e^{i\omega t}d^{abc\,\dagger}\right) \qquad P_{abc}(t) = -\frac{i}{2}\left(e^{-i\omega t}d_{abc} - e^{i\omega t}c^\dagger_{abc}\right) \end{aligned} \tag{4.25}$$

Expressing the Hamiltonian in terms of oscillators, we have

$$H = \frac{1}{4}\left(a^\dagger_{abc}a^{abc} + b^{abc\,\dagger}b_{abc} + c^\dagger_{abc}c^{abc} + d^{abc\,\dagger}d_{abc}\right) + 2\omega \tag{4.26}$$

Introduce the pair of fields

$$Z^i_j \equiv \frac{a^\dagger_{klj}b^{kli\,\dagger}}{2\omega\sqrt{N_1N_2}} \qquad Y^i_j \equiv \frac{c^\dagger_{klj}d^{\dagger\,kli}}{2\omega\sqrt{N_1N_2}} \tag{4.27}$$

Using these we can construct the following gauge invariant observables

$$\Phi(n_1, n_2, \cdots) = \mathrm{Tr}(Z^{n_1}YZ^{n_2-n_1}YZ^{n_3-n_2}\cdots) \tag{4.28}$$

These loop variables have a large N dynamics that matches the dynamics of BMN loops. We want to compute correlators of the form

$$\langle 0|\Phi(n_1', n_2', \cdots)^\dagger\Phi(n_1, n_2, \cdots)|0\rangle \tag{4.29}$$

Lets start by making a few observations. Is is straight forward to argue that

$$\langle 0|(Z^\dagger)^i_j Z^k_l|0\rangle = \delta^k_j \delta^i_l = \langle 0|(Y^\dagger)^i_j Y^k_l|0\rangle \tag{4.30}$$

and

$$\begin{aligned}&\langle 0|(Z^\dagger)^{i_1}_{j_1}(Z^\dagger)^{i_2}_{j_2} Z^{k_1}_{l_1} Z^{k_2}_{l_2}|0\rangle = \langle 0|(Y^\dagger)^{i_1}_{j_1}(Y^\dagger)^{i_2}_{j_2} Y^{k_1}_{l_1} Y^{k_2}_{l_2}|0\rangle \\ &= \delta^{i_1}_{l_1}\delta^{k_1}_{j_1}\delta^{i_2}_{l_2}\delta^{k_2}_{j_2} + \delta^{i_1}_{l_2}\delta^{k_1}_{j_2}\delta^{i_2}_{l_1}\delta^{k_2}_{j_1} + \frac{1}{N_1 N_2}\left(\delta^{i_1}_{l_1}\delta^{k_1}_{j_2}\delta^{i_2}_{l_2}\delta^{k_2}_{j_1} + \delta^{i_1}_{l_2}\delta^{k_1}_{j_1}\delta^{i_2}_{l_1}\delta^{k_2}_{j_2}\right)\end{aligned} \tag{4.31}$$

The generalization of this formula to n fields is

$$\begin{aligned}\langle 0|(Z^\dagger)^{i_1}_{j_1}\cdots(Z^\dagger)^{i_n}_{j_n} Z^{k_1}_{l_1}\cdots Z^{k_n}_{l_n}|0\rangle &= \langle 0|(Y^\dagger)^{i_1}_{j_1}\cdots(Y^\dagger)^{i_n}_{j_n} Y^{k_1}_{l_1}\cdots Y^{k_n}_{l_n}|0\rangle \\ &= \sum_{\sigma\in S_n}\sigma^I_L(\sigma^{-1})^K_J\left(1+O\left(\frac{1}{N_1N_2}\right)\right)\end{aligned} \tag{4.32}$$

This is a nice result because it is what we would get from the matrix model so that we have a detailed and specific grasp on how the single matrix dynamics emerges from the tensor model. Note that the above observations continue to work if we use the fields

$$Z^i_j \equiv \frac{a^\dagger_{klj} b^{kli\,\dagger}}{2\omega\sqrt{N_1N_2}} \qquad\qquad Y^i_j \equiv \frac{a^\dagger_{klj} c^{\dagger\,kli}}{2\omega\sqrt{N_1N_2}} \tag{4.33}$$

which employ one less oscillator. The subleading corrections to the two descriptions (i.e. (4.27) versus (4.33)) do not agree. Again, we stress that these statements hold at the leading order in large N_1N_2.

The BMN spectrum of excited states is obtained by adding an interaction

$$\propto \mathrm{Tr}\left([Z^\dagger, Y^\dagger][Z,Y]\right) \tag{4.34}$$

to the free Z, Y matrix model. It is simple to check that the spectrum of the tensor model quantum mechanics, with loops (4.33) and with the interaction

$$\propto (b^{\dagger mnh} a^\dagger_{qrh} c^{\dagger qri} - c^{\dagger mnh} a^\dagger_{qrh} b^{\dagger qri})(b_{kli} a^{klj} c_{mnj} - c_{kli} a^{klj} b_{mnj}) \tag{4.35}$$

agrees with the spectrum of excited string states, at leading order at large N_1N_2.

The observation (4.32) above has a number of immediate implications. The subset of tensor model correlators we consider follow by using the usual rules for Wick contracting matrices. Consequently, we will have many of the matrix model results in this tensor model setting. This includes the fact that different trace structures don't mix, which implies a Fock space type description of this limit with the number of traces identified with the number of particles. At large N_1N_2 gauge invariant observables constructed using Z and Y will have a genus expansion, exactly like the usual 't Hooft expansion in matrix models. We will have ribbon graphs, that triangulate surfaces and the genus of these surfaces determines the power of N_3 of the graph. This is a nice geometrical interpretation for the $1/N_3$ expansion and it provides convincing evidence for emergent strings[3]. We will also have the usual simplifications of large N for matrix models including factorization in the planar limit. It is also possible to consider the finite N_3 physics of the model, using Schur polynomials [10, 11], restricted Schur polynomials [45] and other methods [112, 113, 114] that have been developed for matrix models.

[3]For another geometrical interpretation of the N dependence see Section 6 of [111]. This interpretation seems to be distinct to the interpretation we obtain here from triangulated worldsheets.

4.3 The Collective Field Theory of Melons

The space of all possible invariants of the tensor model appears to be too complicated to manage without further input. In the previous section we have studied some closed subsectors which are matrix like. The dynamics of these subsectors are certainly manageable, but one might worry that they don't exhibit behavior which is characteristic of the tensor model. For that reason, we turn in this section to an approximation scheme in which we take advantage of known results for the large N limit of tensor models[4]. The novelty of the large N limit of tensor models is that it is dominated by melonic diagrams and it is precisely this feature that it has in common with SYK. The large N limit we consider in this section is obtained by setting $N_1 = N_2 = N_3$. While reading this section, the reader will find it useful to consult [115, 116, 117, 118, 119, 120]. The first result we will make use of is the fact that the large N limit is dominated by melonic diagrams[73, 75]. For some examples of melonic diagrams see Figure 4.1 below. By closing the loop in these melonic graphs, we obtain a graphical representation of a gauge invariant operator. Each white vertex corresponds to a field ϕ^{abc} and each black vertex to a field $\bar{\phi}_{abc}$. The lines in the graph are an instruction for how to contract indices to obtain the invariant. The lines are labeled by an index i telling us that the corresponding index transforms under $U(N_i)$. We refer to the melon shown in (a) as an elementary melon. The melons shown in (b) and (c) are dressed. It is clear that each melon, after removing all dressing, is defined by a pair of vertices. There is a single Feynman diagram that contributes to the large N limit, given by contracting the pairs of vertices that define a given melon[5]. Later we will see that this gives us important insights into how the collective field theory of melons simplifies.

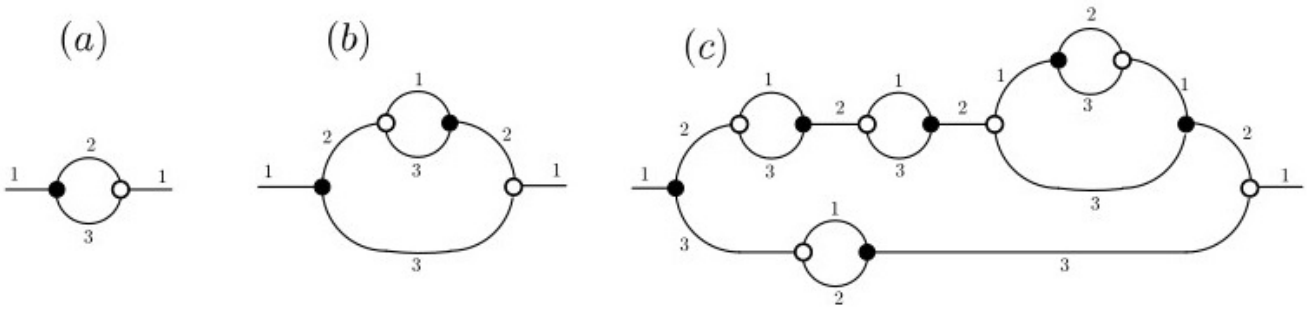


Figure 4.1: Examples of melonic graphs.

In what follows, we focus our attention on the invariants formed by closing the loop in the melon graphs. This is an enormous set of invariant variables. However, we can write equations for the resulting dynamics thanks largely to the fact that there is a very nice way to label these invariants: they can be labeled by D-ary trees, as explained in [73]. An example of this labeling is given in the Figure 4.2.

[4]Some of the equations we write only hold at the leading order in the large N expansion. With this approximation, the collective field theory does not take us outside the space of melonic gauge invariants.

[5]This follows immediately after computing a few examples. Alternatively, consult Section III A of [118] for a proof that uses the Schwinger-Dyson equations.

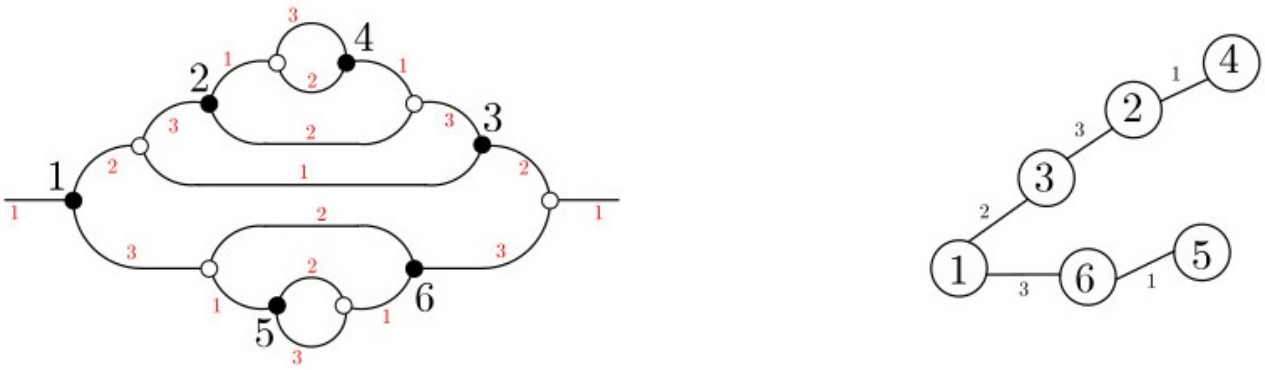


Figure 4.2: The tree labeling a melonic invariant.

The fact that we can label melonic invariants with trees has immediate implications for collective field theory. The operation of joining given by Ω implies that joining melonic invariants keeps us within the space of melonic invariants. Denote the loop for a melonic invariant labeled by a tree with p melons T_p by $\phi_{\mathcal{M}(T_p)}$. Joining a tree of p-melons and a tree of p'-melons gives a tree of $p + p' - 1$ melons

$$\Omega(\phi_{\mathcal{M}(T_p)}, \phi_{\mathcal{M}(T_{p'})}) = \sum_{T_{p+p'-1}} \phi_{\mathcal{M}(T_{p+p'-1})} \tag{4.36}$$

The sum on the right hand side above includes pp' terms.

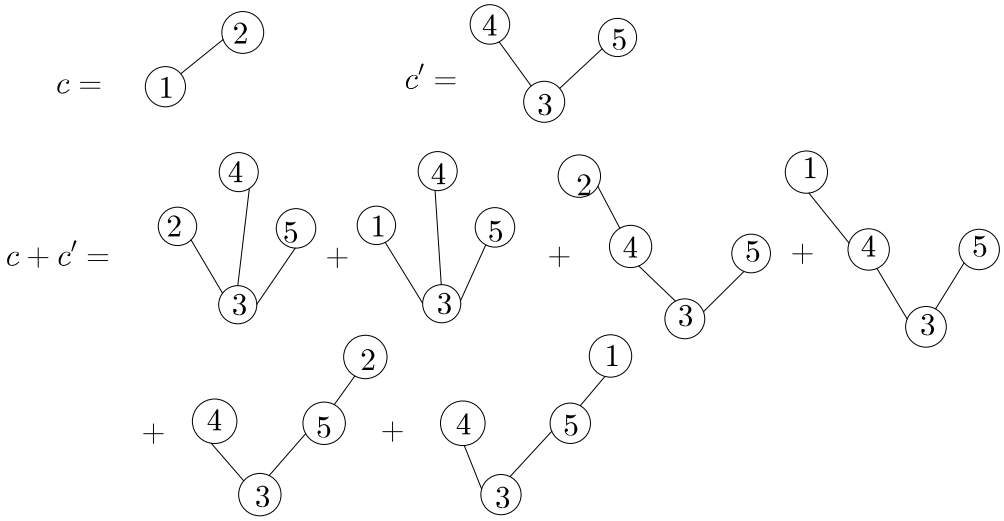


Figure 4.3: An example of tree joining. This figure shows the possible values for $C + C'$ for the given C and C'. The joining is performed by contracting a solid vertex in C with a hollow vertex in C'.

Splitting does not keep us inside the space of melons. Splitting a single melonic invariant can give us back a melonic invariant (this occurs when we split on vertices defining an elementary melon), a product of disconnected melonic invariants (this occurs when we split on vertices defining a melon which is not elementary) or an invariant which is not melonic (this occurs when we split on vertices that belong to

different melons). This means that the melonic invariants are not dynamically decoupled and there is mixing between melonic and non-melonic invariants. This is a significant complication. Our approach towards this issue is to treat the problem systematically in the $1/N$ expansion. At the leading order we can simply omit all terms involving non-melonic invariants, since these are subleading. Further, the expectation values of disconnected melonic invariants have the same large N behavior as the original melonic invariant that produced them (because the relevant splitting is a Wick contraction contributing to the unique Feynman diagram giving the complete large N expectation value of the melonic observable). The main obstacle to completing the construction of the collective Hamiltonian, is now determining the inverse of $\Omega(\phi_{\mathcal{M}(T_p)}, \phi_{\mathcal{M}(T_{p'})})$. This is a formidable task that is under current investigation.

Although we have not yet managed to determine the collective Hamiltonian, even at large N, there is a little more we can say. To illustrate the argument, return to the free Hermitian single matrix model. The collective field is given by

$$\phi(x) = \int \frac{dk}{2\pi} e^{-ikx} \phi_k \qquad \phi_k = \mathrm{Tr}\,(e^{ikM}) \tag{4.37}$$

Summing the planar diagrams we find

$$\langle \phi_k \rangle = \sum_{l=0}^{\infty} \frac{(ik)^{2l}}{(2l)!} \langle \mathrm{Tr}\,(M^{2l}) \rangle = \sum_{l=0}^{\infty} \frac{(ik)^{2l}}{(2l)!} \frac{N^{l+1}}{\omega^l} \frac{(2l-1)!\,!}{(l+1)!} = \frac{\sqrt{2N\omega}}{p} J_1\left(\sqrt{\frac{2N}{\omega}} p\right) \tag{4.38}$$

where $J_1(\cdot)$ is a Bessel function of the first kind. After a Fourier transform we obtain the usual Wigner distribution

$$\langle \phi(x) \rangle = \frac{\omega}{\pi} \sqrt{\frac{2N}{\omega} - x^2} \tag{4.39}$$

describing the large N eigenvalue density. The coordinate x here is the extra holographic coordinate that emerges from collective field theory. We will now carry out the corresponding computation in the free tensor model. Denote $N_1 = N_2 = N_3 = N$. Denote the melonic variable (before closing the loop as in Figure 4.1) by $\mathcal{M}(T_p)$. As a consequence of the fact that a single Feynman diagram contributes, we find

$$\langle \mathrm{Tr}\,(\mathcal{M}(T_p)^n) = \frac{N^{1+2pn}}{(2\omega)^{pn}} \tag{4.40}$$

Defining the collective field

$$\phi(x) = \int \frac{dk}{2\pi} e^{-ikx} \phi_k \qquad \phi_k = \mathrm{Tr}\,(e^{ik\mathcal{M}(T_p)}) \tag{4.41}$$

and working as we did above, we find

$$\langle \phi(x) \rangle = N\delta\left(x - \frac{N^{2p}}{(2\omega)^p}\right) \tag{4.42}$$

The density has support at a single point so that no holographic dimension emerges. This is a consequence of the simplicity of the melonic limit, namely that a single Feynman diagram contributes.

4.4 Finite N Contributions

An interesting limit of matrix model dynamics, quite distinct from the planar limit, is the limit of finite N. This limit of the matrix model is accessed by studying operators built from a parametrically (in N) large number of fields [5, 121, 122]. While the planar limit is dual to the fundamental string and its excitations, finite N sectors of the theory are dual to giant graviton branes and new space time geometries, including black holes [6, 10, 11, 9]. They also provide an interesting window into the structure of the large N expansion: in this limit one finds non-perturbative contributions that obstruct the perturbative $1/N$ expansion [123]. Another important point that should be stressed is that these non-perturbative effects are already visible in the free matrix model since they are concerned with the expansion in $1/N$ and not with the 't Hooft coupling expansion. Given this motivation, it is clearly interesting to probe the analogous limits of free tensor models, something that is within reach of existing methods.

One approach towards the finite N physics of the tensor model is to use representation theory techniques, as developed in [83, 39]. The basic idea is to construct gauge invariant operators as a product of the basic fields multiplied by a projection operator. Correlation functions reduce to traces of products of projectors. This implies that they can be computed exactly in the free theory. The projectors are labeled by Young diagrams and finite N effects are encoded as a cut off on the number of rows in the Young diagram labeling the operator. In what follows we will simply state and use the results we need. The reader requiring more details is encouraged to consult [83, 39], as well as [86, 124, 125, 84].

Similarly to before we consider products of tensors to build the general gauge invariant operator. As an example, we could use

$$\phi^{i_1 j_1 k_1}\phi^{i_2 j_2 k_2}\cdots\phi^{i_n j_n k_n} \tag{4.43}$$

A better notation [10, 11] uses a capital Roman letter to collect the little letter indices, so that for example I stands for $i_1, i_2, \cdots, i_n$. Thus, we write

$$\Phi^{IJK} = \phi^{i_1 j_1 k_1}\phi^{i_2 j_2 k_2}\cdots\phi^{i_n j_n k_n} \qquad\qquad \bar{\Phi}_{IJK} = \bar{\phi}_{i_1 j_1 k_1}\bar{\phi}_{i_2 j_2 k_2}\cdots\bar{\phi}_{i_n j_n k_n} \tag{4.44}$$

The basis we study is as follows

$$\mathcal{O}^{\gamma_1\gamma_2}_{r_1,r_2,r_3} = \sum_{\sigma_1\in S_n}\sum_{\sigma_2\in S_n}\sum_{\sigma_3\in S_n} C^{\gamma_1\gamma_2}_{r_1,r_2,r_3}(\sigma_1,\sigma_2,\sigma_3)\mathcal{O}(\sigma_1,\sigma_2,\sigma_3) \tag{4.45}$$

where

$$\mathcal{O}(\sigma_1,\sigma_2,\sigma_3) = \bar{\Phi}_{IJK}\Phi^{\sigma_1(I)\sigma_2(J)\sigma_3(K)} \tag{4.46}$$

$$C^{\gamma_1\gamma_2}_{r_1,r_2,r_3}(\sigma_1,\sigma_2,\sigma_3) = B^{\gamma_1}_{\alpha_1\alpha_2\alpha_3}\Gamma^{r_1}{}_{\alpha_1\beta_1}(\sigma_1)\Gamma^{r_2}{}_{\alpha_2\beta_2}(\sigma_2)\Gamma^{r_3}{}_{\alpha_3\beta_3}(\sigma_3)B^{\gamma_2}_{\beta_1\beta_2\beta_3} \tag{4.47}$$

is in fact a restricted character, in the language introduced in [126, 88]. Thus, (4.45) provides the restricted Schur polynomial basis for the gauge invariant operators of the bosonic tensor model. In this formula, $\Gamma^r{}_{\alpha\beta}(\sigma)$ denotes the matrix (with row label α and column label β) representing $\sigma\in S_n$ in irreducible representation r, and we have made use of the branching coefficients $B^{\gamma}_{\alpha_1\alpha_2\alpha_3}$ defined by

$$\frac{1}{n!}\sum_{\sigma\in S_n}\Gamma^{r_1}{}_{\alpha_1\beta_1}(\sigma)\Gamma^{r_2}{}_{\alpha_2\beta_2}(\sigma)\Gamma^{r_3}{}_{\alpha_3\beta_3}(\sigma) = \sum_{\gamma} B^{\gamma}_{\alpha_1\alpha_2\alpha_3}B^{\gamma}_{\beta_1\beta_2\beta_3} \tag{4.48}$$

The branching coefficients provide an orthonormal basis for the subspace of $r_1 \otimes r_2 \otimes r_3$ that carries the trivial representation, i.e.

$$B^{\gamma_1}_{\alpha_1\alpha_2\alpha_3} B^{\gamma_2}_{\alpha_1\alpha_2\alpha_3} = \delta^{\gamma_1\gamma_2} \tag{4.49}$$

and where we employ the usual convention that repeated indices are summed. The advantage of the restricted Schur polynomial basis follows because we are able, in the free theory, to compute correlators exactly. The results we will use are [39]

$$\langle \mathcal{O}^{\gamma_1\gamma_2}_{r_1r_2r_3} \rangle \;\; = \;\; n!\, f_{r_1}(N_1) f_{r_2}(N_2) f_{r_3}(N_3) \delta^{\gamma_1\gamma_2} \tag{4.50}$$

$$\langle : \mathcal{O}^{\gamma_1\gamma_2}_{r_1r_2r_3} :: \mathcal{O}^{\gamma_3\gamma_4}_{s_1s_2s_3} : \rangle = (n!)^2 \delta_{r_1s_1}\delta_{r_2s_2}\delta_{r_3s_3} f_{r_1}(N_1) f_{r_2}(N_2) f_{r_3}(N_3) \frac{n!}{d_{r_1}} \frac{n!}{d_{r_2}} \frac{n!}{d_{r_3}} \delta^{\gamma_1\gamma_4}\delta^{\gamma_2\gamma_3} \tag{4.51}$$

If we normalize the operators to have unit one point function as $N \to \infty$

$$O^{\gamma_1\gamma_2}_{r_1r_2r_3} = \frac{\mathcal{O}^{\gamma_1\gamma_2}_{r_1r_2r_3}}{n!\, f_{r_1}(N_1) f_{r_2}(N_2) f_{r_3}(N_3)} \tag{4.52}$$

the two point function becomes

$$\langle : O^{\gamma_1\gamma_2}_{r_1r_2r_3} :: O^{\gamma_3\gamma_4}_{s_1s_2s_3} : \rangle = \delta_{r_1s_1}\delta_{r_2s_2}\delta_{r_3s_3} \frac{1}{f_{r_1}(N_1) f_{r_2}(N_2) f_{r_3}(N_3)} \frac{n!}{d_{r_1}} \frac{n!}{d_{r_2}} \frac{n!}{d_{r_3}} \delta^{\gamma_1\gamma_4}\delta^{\gamma_2\gamma_3} \tag{4.53}$$

Now, for simplicity, set $N_1 = N_2 = N_3 = N$ and consider the case that $r_1 = r_2 = r_3 = 1^N$, that is, r is a single column with N boxes. This is the melonic large N limit used in Section 4.3 and not the matrix-like large N limit of Section 4.2. In the matrix model case this choise of representations corresponds to a giant graviton and we would expect this computation to exhibit non-perturbative effects, even for a real matrix where the one point function doesn't vanish. The giant graviton description nicely explains the stringy exclusion principle as a bound that exists because the giant graviton brane stretches to its maximum size in a compact space - see [6] for a detailed discussion. We have explicitly verified that the corresponding matrix model correlators receive non-perturbative corrections in Appendix C.4. The tensor model two point function becomes

$$\langle : O^{\gamma_1\gamma_2}_{1^N1^N1^N} :: O^{\gamma_3\gamma_4}_{1^N1^N1^N} : \rangle = \delta_{r_1s_1}\delta_{r_2s_2}\delta_{r_3s_3}\delta^{\gamma_1\gamma_4}\delta^{\gamma_2\gamma_3} \tag{4.54}$$

This is an exact result, with no sign of any non-perturbative effects, and no hint into the mechanism behind the finite N cut off. This is a significant difference as compared to matrix model physics.

Next consider the case that we set $N_1 = N_2 = N_3$ (i.e. melonic large N) and consider the case that $r_1 = r_2 = r_3 = N$, that is, r is a single row with N boxes. This choice of r_1, r_2, r_3 would correspond, in the matrix model description, to dual giant gravitons. In this case

$$\langle : O^{\gamma_1\gamma_2}_{1^N1^N1^N} :: O^{\gamma_3\gamma_4}_{1^N1^N1^N} : \rangle = \delta_{r_1s_1}\delta_{r_2s_2}\delta_{r_3s_3}\sqrt{\pi^3 N^3} e^{6N\log(2N)-6N} \delta^{\gamma_1\gamma_4}\delta^{\gamma_2\gamma_3} \tag{4.55}$$

This is only the leading order result. It is clearly signaling non-perturbative corrections to the large N expansion, which is similar to what we find for the matrix model physics.

Chapter 5

Conclusion

Motivated by the close connection between tensor models and the SYK model and the success the SYK model has had within holography, the aim of this Ph.D. was to study tensor models within holography. In order to do this a vast amount of background needed to be studied. It became clear that a good approach would be to study the space of gauge invariants with a specific focus on the collective field theories defined by gauge invariant operators.

Concerning the problem of counting and then constructing the gauge invariant operators of tensor models bosonic tensor models have already been considered in the literature, and the results obtained here are consistent with what is already known. The results for fermionic vector models are novel. Using the operators that have been constructed, an interesting algebra was exhibited, underlying the gauge invariant operators of the tensor model: the gauge invariant operators define a ring. Closed formulas for the structure constants of this ring were written down. This algebraic structure expresses arbitrary correlation functions as one point functions, which have been computed explicitly. Consequently, once the structure constants of the algebra are known, the free theory has been solved exactly. These structure constants are expressed as overlaps of branching coefficients so that their computation is now a well defined problem in the representation theory of the symmetric group.

To study the large N dynamics of tensor model quantum mechanics a smaller set of gauge invariant operators was identified that has lead to an eigenvalue like description. The system admits a collective field theory description which is similar but not identical to the collective field theory of a single hermitian matrix. This collective description shares all the good features of previous collective descriptions. Two such features are

1. The collective description manifests the fact that the tensor model quantum mechanics has emergent dimensions. Further, it is very attractive and highly non-trivial that the collective dynamics in this emergent dimension is local.

2. The loop expansion parameter of the collective field theory is not $\hbar$ of the quantum mechanics, but rather it is $\frac{1}{N_1}$ with N_1 set by the tensor model gauge group. Consequently the classical equations of motion of the collective field theory yield the answer obtained by summing the complete set of Feynman diagrams that contribute at large N_1. For the tensor model example considered this was explicitly demonstrated.

The above two features are highly suggestive of holography, which claims that a local (at large N_1) higher dimensional classical system is dual to the large N limit of the gauge theory.

An approach towards describing the space of gauge invariant variables associated to tensor models was developed for a rank 3 tensor model with gauge group $U(N_1) \times U(N_2) \times U(N_3)$. It was shown that the gauge invariant variables for the $U(N_1) \times U(N_2)$ gauge symmetries are tensors of the $U(N_3)$ gauge group. There is a single tensor with k indices transforming in the fundamental and k indices transforming in the anti fundamental for $k = 1, 2, \cdots$. There is an enormous space of gauge invariants that can be constructed

from this set of $U(N_3)$ tensors. Through simple counting arguments non-trivial evidence that the resulting set of gauge invariants is complete was given.

The dynamics of the tensor model were expressed in terms of the invariant variables. The initial strategy was to identify sets of invariant variables that are dynamically closed. Closed subsets that mimic the dynamics of a matrix model have been identified in Section 4.2, and they define a non-linear large N theory that is very similar to the large N dynamics of a matrix model. The interaction of the effective matrix degrees of freedom is attractive causing eigenvalues to clump. By moving from the matrix to gauge invariant degrees of freedom a Van der Monde like repulsion is effectively generated. The net result of the competition of these two effects is that there is a non-trivial function describing the density of eigenvalues and this leads to an extra holographic dimension. This is precisely the same mechanism at work in the $c = 1$ string [15], which is the only example in which a holographic duality has been proved. It is worth stressing that this limit can be obtained by holding $\frac{N_1 N_2}{N_3}$ fixed and taking $N_3 \to \infty$, which is quite different from the large N limit that is dominated by melons.

In a distinct effort to develop dynamics characteristic of the tensor model, the large N melonic limits of the theory were explored. This limit fixes $\frac{N_1}{N_2}$ and $\frac{N_1}{N_3}$ and takes $N_3 \to \infty$. The resulting collective field theory has a number of promising features: the joining operation leaves one within the space of melonic invariants. Although splitting does not leave us in the space of invariants, excursions away from melonic gauge invariants is suppressed at large N. Motivated by these initial encouraging signs, the collective field theory of melons was considered. Although, it was not possible to write a closed form for the resulting collective field theory Hamiltonian at this point, a direct evaluation of the large N collective field was carried out. It was found that the density of eigenvalues has delta function support and that an extra holographic dimension does not emerge. This is a consequence of the fact that a single Feynman diagram determines the large N value of melonic gauge invariants, so that the Schwinger-Dyson equations become linear, as commented in [118]. The non-linearity of the Van der Monde term is suppressed in the melonic limit.

The fact that there is an extra holographic dimension emerging in the matrix like limit that was considered but not in the melonic limit, is a consequence of how the large N limit was taken in these different settings. In Figure 5.1 the Feynman diagrams that contribute to $\langle \phi^{abc} \bar{\phi}_{abd} \phi^{efd} \bar{\phi}_{efc} \rangle$ were shown. In the matrix model limit $N_1 N_2$ was set equal to N_3 so that both diagrams survive, matching the fact that two planar diagrams contribute to the large N limit of $\langle \mathrm{Tr}(Z\bar{Z}Z\bar{Z}) \rangle$. On the other hand, in the melonic large N limit, where $N_1 = N_2 = N_3$, diagram (a) of Figure 5.1 is suppressed and the large N value is completely given by diagram (b).

One of the claims often made in the literature, is that tensor models and tensor field theories admit a $1/N$ expansion and a melonic large N limit which is simpler than the planar limit of random matrices and richer than the large N limit of vector models. This gives hope that tensor models may provide an interesting toy model that sheds light on the large N limit of matrix models. The results obtained here imply that the melonic limit does not lead to a loop space that has enough structure to provide a useful insight into how to organize the loop space of the matrix model. In contrast to the matrix model, there is no holographic dimension emerging from the color combinatorics. To see the emergence of an extra holographic dimension, in the melonic large N limit, it seems to be necessary to look at a bilocal description, along the lines of similar discussions for SYK[31, 32, 33]. In this collective field theory description, one

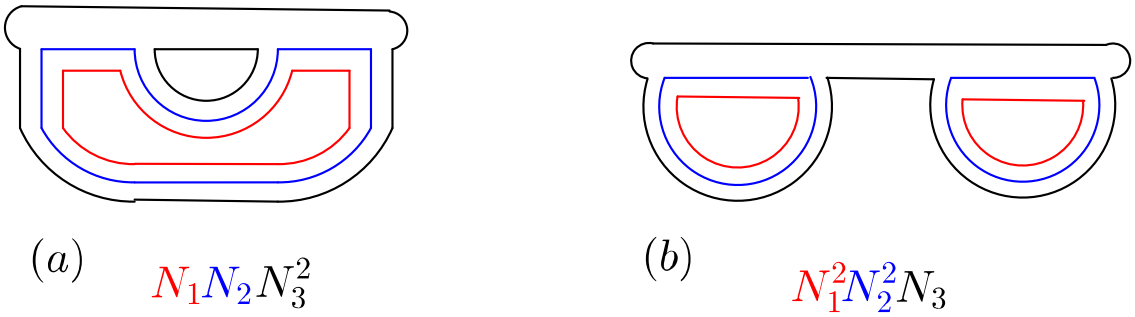


Figure 5.1: The Feynman diagrams that contribute to $\langle \phi^{abc}\bar{\phi}_{abd}\phi^{efd}\bar{\phi}_{efc}\rangle$.

changes to bilocal (two time) variables which is accompanied by a non-trivial Jacobian [26] which generates non-linear interactions. This is a key message of this study: for holography in the melonic limit, one must use a bilocal description since an extra holographic dimension does not emerge from the combinatorics of the gauge indices.

Non-perturbative contributions to the $1/N$ expansion were considered, in the large N melonic limit[1]and non-perturbative contributions were indeed found. It would be interesting to find the physical interpretation of these terms.

A study of the collective field theory of the melonic limit was initiated. Completing this description would provide a detailed understanding of corrections to the large N melonic limit. It is evident from the description sketched that this leads to a theory of interacting random trees which appears to make contact with the ideas put forward in [127].

[1]For the matrix like large N limit we have verified the answer matches the matrix model dynamics - the analysis contains no surprises.

Appendix A

A.1 First correction to two point function

Consider the first correction to the SYK two point function. It is given by the following expression

$$\frac{1}{2(4!)^2}\int dt\int dt'\overline{J_{ijkl}J_{mnpq}}\langle\chi_a(t_1)\chi_b(t_2)\chi_i(t)\chi_j(t)\chi_k(t)\chi_l(t)\chi_m(t')\chi_n(t')\chi_p(t')\chi_q(t')\rangle \tag{A.1}$$

The Wick contractions are restricted by the couplings because any Wick contraction that sets any of the $ijkl$ or any of the $nmpq$ indices equal will result in zero. If $\chi_a(t_1)\chi_b(t_2)$ are contracted the only diagrams which can be produced are vacuum graphs which we assume have been normalized out. Thus, the contractions of interest are the ones in which a contracts with one of $ijkl$ while b contracts with one of $nmpq$ or vise versa. There are 8×4 choices for contracting $\chi_a(t_1)\chi_b(t_2)$ but the ones where a contracts with one of $nmpq$ and b contracts with one of $ijkl$ will come with a minus sign since it amounts to swapping $\chi_a(t_1)\chi_b(t_2)$ around. The minuses that come from the different contractions of the remaining fields cancel with the swaps of the indices of the couplings so that one gets

$$\begin{aligned}
&= -\frac{1}{2(4!)^2}\int dt\int dt'\overline{J_{ijkl}J_{mnpq}}\delta_{ai}\delta_{bm}G_0(t_1-t)G_0(t_2-t')\langle\chi_j(t)\chi_k(t)\chi_l(t)\chi_n(t')\chi_p(t')\chi_q(t')\rangle+\\
&+\frac{1}{2(4!)^2}\int dt\int dt'\overline{J_{ijkl}J_{mnpq}}\delta_{bi}\delta_{am}G_0(t_2-t)G_0(t_1-t')\langle\chi_j(t)\chi_k(t)\chi_l(t)\chi_n(t')\chi_p(t')\chi_q(t')\rangle \qquad \text{(A.2)}\\
&= -\frac{4^2}{2(4!)^2}\int dt\int dt'\overline{J_{ajkl}J_{bnpq}}G_0(t_1-t)G_0(t_2-t')3!\,\delta_{jn}\delta_{kp}\delta_{lq}(G_0(t-t'))^3+\\
&+\frac{4^2}{2(4!)^2}\int dt\int dt'\overline{J_{bjkl}J_{anpq}}G_0(t_2-t)G_0(t_1-t')3!\,\delta_{jn}\delta_{kp}\delta_{lq}(G_0(t-t'))^3 \qquad \text{(A.3)}
\end{aligned}$$

In the first term change $G_0(t_2-t')\to -G_0(t'-t_2)$ and in the second do the swap $t\leftrightarrow t'$ and then $(G_0(t'-t))^3\to -(G_0(t-t'))^3$ and $G_0(t_2-t')\to -G_0(t'-t_2)$. This makes the terms the same and they can be added to give

$$= \frac{3!\,4^2}{2(4!)^2}\int dt\int dt'(\overline{J_{ajkl}J_{bjkl}}+\overline{J_{bjkl}J_{ajkl}})G_0(t_2-t)G_0(t_1-t')(G_0(t-t'))^3 \tag{A.4}$$

Now a and b must be equal in the coupling terms for this to ne non-zero so using a delta function to set them equal

$$= \frac{3!\,4^2}{(4!)^2}\int dt\int dt'\overline{J_{ajkl}J_{bjkl}}\delta_{ab}G_0(t_2-t)G_0(t_1-t')(G_0(t-t'))^3 \tag{A.5}$$

Recall that repeated indices are summed so the term $\overline{J_{ajkl}J_{bjkl}}\delta_{ab}=\delta_{ab}(N-1)(N-2)(N-3)\frac{3!J^2}{N^3}=\delta_{ab}\frac{3!J^2N!}{N^4(N-4)!}$. In the large N limit this becomes

$$\delta_{ab}\frac{3!\,J^2N!}{N^4(N-4)!}=\delta_{ab}\frac{3!\,J^2(N-1)(N-2)(N-3)}{N^3}=\delta_{ab}3!\,J^2 \tag{A.6}$$

and so the first correction to the SYK two point function in the large N limit is

$$= \frac{3!\,4^2}{(4!)^2} 3!\, J^2 \int dt \int dt' G_0(t_2 - t) G_0(t_1 - t') (G_0(t - t'))^3 \delta_{ab} \tag{A.7}$$

$$= J^2 \int dt \int dt' G_0(t_2 - t) G_0(t_1 - t') (G_0(t - t'))^3 \delta_{ab} \tag{A.8}$$

This is the factor that comes with a melon. It is convenient to call the one particle irreducible part $\Sigma_0(t - t') = J^2 (G_0(t - t'))^3$

A.2 Collective Field Theory Toy Model

A.2.1 Investigating Hermiticity

Consider a particle in N dimensions. Denote the coordinate of this particle by x^a with $a = 1, 2, \cdots, N$. Imagine that rotations are now considered to be a gauge symmetry. Things that are physical will be things that are invariant under rotations - and that means things with angular momentum $= 0$ or equivalently, things that don't depend on the angles - things that depend only on $x^a x^a = \rho$. It is clear that ρ is the only (independent) gauge invariant that can be made using the x^a. The simplest possible dynamics is given by taking

$$H = P_a P_a = \Big(-i \frac{\partial}{\partial x^a} \Big) \Big(-i \frac{\partial}{\partial x^a} \Big) \tag{A.9}$$

This is Hermitian because P^a is hermitian. As can be seen by the following

$$\langle f(x) | P^a | g(x) \rangle = \int d^d x f^*(x) (-i \frac{\partial}{\partial x^a}) g(x) \tag{A.10}$$

$$= \int d^d x (if(x))^* \frac{\partial}{\partial x^a} g(x) \tag{A.11}$$

$$= - \int d^d x \frac{\partial}{\partial x^a} (if(x))^* g(x) + \int d^d x \frac{\partial}{\partial x^a} [(if(x))^* g(x)] \tag{A.12}$$

$$= \int d^d x (-i \frac{\partial}{\partial x^a} f(x))^* g(x) \tag{A.13}$$

$$= \langle (P^a)^\dagger f(x) | g(x) \rangle \tag{A.14}$$

$$\Rightarrow P^a = (P^a)^\dagger$$

Now consider changing to gauge invarient variables ρ

$$H = \Big(-i \frac{\partial}{\partial x^a} \Big) \Big(-i \frac{\partial \rho}{\partial x^a} \frac{\partial}{\partial \rho} \Big) \tag{A.15}$$

$$= \Big(-i \frac{\partial^2 \rho}{\partial x^a \partial x^a} \Big) \Big(-i \frac{\partial}{\partial \rho} \Big) + \frac{\partial \rho}{\partial x^a} \frac{\partial \rho}{\partial x^a} \Big(-i \frac{\partial}{\partial \rho} \Big) \Big(-i \frac{\partial}{\partial \rho} \Big) \tag{A.16}$$

$$= i\omega \Pi + \Omega \Pi \Pi \tag{A.17}$$

where

$$\omega = \frac{\partial^2 \rho}{\partial x^a \partial x^a} \qquad \Omega = \frac{\partial \rho}{\partial x^a} \frac{\partial \rho}{\partial x^a} \qquad \Pi = -i \frac{\partial}{\partial \rho} \tag{A.18}$$

The aim now is to investigate (A.17). It is the same operator as (A.9) and so must also be Hermitian. We start by changing variables and restricting to functions which only depend on ρ.

$$\begin{aligned}
\langle f(\rho)|H|g(\rho)\rangle &= \int d^N x f^*(\rho) P^a P^a g(\rho) \\
&= \frac{1}{2}\int d\rho d\phi_1 d\phi_2 \cdots d\phi_{N-2} d\theta \sin^{N-2}(\phi_1)\sin^{N-3}(\phi_2)\cdots\sin(\phi_{N-2})\rho^{\frac{N-2}{2}} \\
&\quad \times f^*(\rho)\bigg[\bigg(-\frac{\partial^2\rho}{\partial x^a \partial x^a}\frac{\partial}{\partial\rho}\bigg)+\frac{\partial\rho}{\partial x^a}\frac{\partial\rho}{\partial x^a}\bigg(-i\frac{\partial}{\partial\rho}\bigg)\bigg(-i\frac{\partial}{\partial\rho}\bigg)\bigg]g(\rho) \\
&= \frac{1}{2}\int d\rho \rho^{\frac{N-2}{2}} f^*(\rho)\bigg[-2N\bigg(\frac{\partial}{\partial\rho}\bigg)-4\rho\bigg(\frac{\partial}{\partial\rho}\bigg)\bigg(\frac{\partial}{\partial\rho}\bigg)\bigg]g(\rho) \\
&\quad \times \int d\phi_1 d\phi_2 \cdots d\phi_{N-2} d\theta \sin^{N-2}(\phi_1)\sin^{N-3}(\phi_2)\cdots\sin(\phi_{N-2}) \\
&= \frac{1}{2}\int d\rho\bigg[2N\frac{\partial}{\partial\rho}\bigg(\rho^{\frac{N-2}{2}}f^*(\rho)\bigg)-4\frac{\partial}{\partial\rho}\frac{\partial}{\partial\rho}\bigg(\rho^{\frac{N}{2}}f^*(\rho)\bigg)\bigg]g(\rho)S_N \\
&= \frac{1}{2}\int d\rho\bigg[2N\bigg(\frac{N-2}{2}\rho^{\frac{N-4}{2}}f^*(\rho)+\rho^{\frac{N-2}{2}}\frac{\partial}{\partial\rho}f^*(\rho)\bigg) \\
&\quad -4\bigg(\frac{N}{2}\frac{N-2}{2}\rho^{\frac{N-4}{2}}f^*(\rho)+N\rho^{\frac{N-2}{2}}\frac{\partial}{\partial\rho}f^*(\rho)+\rho^{\frac{N}{2}}\frac{\partial}{\partial\rho}\frac{\partial}{\partial\rho}f^*(\rho)\bigg)\bigg]g(\rho)S_N \\
&= \frac{1}{2}\int d\rho\bigg[-2N\rho^{\frac{N-2}{2}}\frac{\partial}{\partial\rho}f^*(\rho)-4\rho^{\frac{N}{2}}\frac{\partial}{\partial\rho}\frac{\partial}{\partial\rho}f^*(\rho)\bigg]g(\rho)S_N \\
&= \frac{1}{2}\int d\rho\rho^{\frac{N-2}{2}}\bigg[\bigg(-2N\frac{\partial}{\partial\rho}f^*(\rho)-4\rho\frac{\partial}{\partial\rho}\frac{\partial}{\partial\rho}\bigg)f^*(\rho)\bigg]g(\rho)S_N \\
&= \frac{1}{2}\int d\rho\rho^{\frac{N-2}{2}}g(\rho)S_N\bigg[\bigg(-\frac{\partial^2\rho}{\partial x^a\partial x^a}\frac{\partial}{\partial\rho}\bigg)+\frac{\partial\rho}{\partial x^a}\frac{\partial\rho}{\partial x^a}\bigg(-i\frac{\partial}{\partial\rho}\bigg)\bigg(-i\frac{\partial}{\partial\rho}\bigg)\bigg]f^*(\rho) \\
&= \langle g(\rho)|H|f(\rho)\rangle^*
\end{aligned}$$

Thus, this is still Hermitian, as it should be, as long as the non-trivial measure is put in. In this case the important factor is $\rho^{\frac{N-2}{2}}$. When changing variables from gauge variant coordinates to gauge invariant coordinates collective field theory does not do it as it was done above. Above a simple coordinate transformation from rectilinear coordinates to spherical coordinates was done, the gauge invariant parts (stuff depending on ρ) and the gauge variant parts (stuff depending on angles) were kept. What is done in collective field theory is the gauge variant part is ignored and just the gauge invariant parts are considered. This may seem like parts of the theory are ignored but it is okay because the only part of a theory one cares about is in fact the gauge invariant part. If one wants to simply change the measure to only go over the gauge invariant quantities then one has to calculate the gauge invariant Jacobian. In general this Jacobian is a highly non-trivial functional of the fields. It is determined by requiring that the Hamiltonian is explicitly hermitian, which implies a differential equation for the Jacobian. The Jacobian can be thought of as a change to the inner product

$$\langle\phi|\psi\rangle \to \langle\phi|\psi\rangle_J = \langle\phi|\hat{J}|\psi\rangle$$

where

$$\langle x|\hat{J}|x'\rangle = J(x)\delta(x-x')$$

so that

$$\langle\phi|\psi\rangle_J = \int dx J(x)\phi^*(x)\psi(x)$$

$P = -i\frac{d}{dx}$ is hermitian with the standard inner product but when the inner product is changed by a non-trivial measure it is no longer hermitian as can be seen by

$$\begin{aligned}
\langle\phi|P|\psi\rangle_J &= \int dx J(x)\phi^*(x)\bigg(-i\frac{d}{dx}\psi(x)\bigg) \\
&= \int dx J(x)(i\phi(x))^*\bigg(\frac{d}{dx}\psi(x)\bigg) \\
&= \int dx\psi(x)\bigg(-i\frac{d}{dx}(J(x)\phi(x))\bigg)^* \\
&= \int dx J(x)\psi(x)\bigg(-i\frac{d}{dx}\phi(x)\bigg)^* + \int dx\psi(x)\phi(x)^*\bigg(i\frac{d}{dx}J(x)\bigg) \\
&= \langle\psi|P|\phi\rangle_J^* + \int dx\psi(x)\phi(x)^*\bigg(i\frac{d}{dx}J(x)\bigg)
\end{aligned}$$

Thus, when there is a non-trivial measure any operator that involves derivatives and naively looks hermitian will in fact not be. The Jacobian amounts to rescaling states and operators as follows

$$\begin{aligned}
|\psi\rangle &\to |\tilde{\psi}\rangle = \sqrt{\hat{J}}|\psi\rangle \\
\langle\psi| &\to \langle\tilde{\psi}| = \langle\psi|\sqrt{\hat{J}} \\
O &\to \tilde{O} = \sqrt{\hat{J}}O\frac{1}{\sqrt{\hat{J}}}
\end{aligned} \tag{A.19}$$

So that

$$\begin{aligned}
\langle\tilde{\psi}|\tilde{O}_1\tilde{O}_2\cdots\tilde{O}_n|\tilde{\phi}\rangle &= \langle\psi|\sqrt{\hat{J}}\sqrt{\hat{J}}O_1\frac{1}{\sqrt{\hat{J}}}\sqrt{\hat{J}}O_2\frac{1}{\sqrt{\hat{J}}}\cdots\sqrt{\hat{J}}O_n\frac{1}{\sqrt{\hat{J}}}\sqrt{\hat{J}}|\phi\rangle \\
&= \langle\psi|\hat{J}O_1O_2\cdots O_n|\phi\rangle \\
&= \int dx\psi^*(x)J(x)O_1O_2\cdots O_n\phi(x)
\end{aligned}$$

Under this transformation in position space we have

$$X \to \tilde{X} = \sqrt{\hat{J}}X\frac{1}{\sqrt{\hat{J}}} = X\sqrt{\hat{J}}\frac{1}{\sqrt{\hat{J}}} = X$$

and

$$\begin{aligned}
P \to \tilde{P} = \sqrt{\hat{J}}P\frac{1}{\sqrt{\hat{J}}} &= \sqrt{\hat{J}}\bigg(-i\frac{d}{dx}\bigg)\frac{1}{\sqrt{\hat{J}}} \\
&= \frac{i}{2}\sqrt{\hat{J}}\hat{J}^{-\frac{3}{2}}\bigg(\frac{d\hat{J}}{dx}\bigg) + \bigg(-i\frac{d}{dx}\bigg) \\
&= \frac{i}{2}\frac{1}{\hat{J}}\bigg(\frac{d\hat{J}}{dx}\bigg) + P
\end{aligned}$$

After this resealing gauge invarient operators that were Hermitian before the change of variables will remain Hermitian.

A.2.2 Calculating the Jacobian

The requirement that the Hamiltonian is Hermitian amounts to the requirement that

$$J^{\frac{1}{2}}HJ^{-\frac{1}{2}} - (J^{\frac{1}{2}}HJ^{-\frac{1}{2}})^{\dagger} = 0 \tag{A.20}$$

where

$$\begin{aligned} H &= \Big(- \frac{\partial^2 \rho}{\partial x^a \partial x^a}\frac{\partial}{\partial \rho}\Big) + \frac{\partial \rho}{\partial x^a}\frac{\partial \rho}{\partial x^a}\Big(- i\frac{\partial}{\partial \rho}\Big)\Big(- i\frac{\partial}{\partial \rho}\Big) \\ &= i\omega\Big(- i\frac{\partial}{\partial \rho}\Big) + \Omega(\rho)\Big(- i\frac{\partial}{\partial \rho}\Big)\Big(- i\frac{\partial}{\partial \rho}\Big) \end{aligned}$$

$$\begin{aligned} \Rightarrow J^{\frac{1}{2}}HJ^{-\frac{1}{2}} &= J^{\frac{1}{2}}i\omega\Big(- i\frac{\partial}{\partial \rho}\Big)J^{-\frac{1}{2}} + J^{\frac{1}{2}}\Omega(\rho)\Big(- i\frac{\partial}{\partial \rho}\Big)\Big(- i\frac{\partial}{\partial \rho}\Big)J^{-\frac{1}{2}} \\ &= -\frac{1}{2}\omega\frac{\partial}{\partial \rho}\ln J - \frac{3}{4}\Omega\Big(\frac{\partial}{\partial \rho}\ln J\Big)^2 + \frac{\Omega}{2}\frac{1}{J}\frac{\partial^2 J}{\partial \rho^2} + \Omega\Big(\frac{\partial}{\partial \rho}\ln J\Big)\frac{\partial}{\partial \rho} + \omega\frac{\partial}{\partial \rho} - \Omega\frac{\partial^2}{\partial \rho^2} \end{aligned}$$

To find $(J^{\frac{1}{2}}HJ^{-\frac{1}{2}})^{\dagger}$ one should use the inner product

$$\begin{aligned} &\langle\phi|J^{\frac{1}{2}}J^{\frac{1}{2}}HJ^{-\frac{1}{2}}J^{\frac{1}{2}}|\psi\rangle^* \\ &= \int d\rho\phi(\rho)JH\psi^*(\rho) \\ &= \int d\rho\phi(\rho)J\Big(i\omega\Big(- i\frac{\partial}{\partial \rho}\Big) + \Omega(\rho)\Big(- i\frac{\partial}{\partial \rho}\Big)\Big(- i\frac{\partial}{\partial \rho}\Big)\Big)\psi^*(\rho) \\ &= \int d\rho\psi^*(\rho)\Big(- \omega\Big(\frac{\partial}{\partial \rho}\Big)J\phi(\rho) - \Big(\frac{\partial}{\partial \rho}\Big)\Big(\frac{\partial}{\partial \rho}\Big)\Omega(\rho)J\phi(\rho)\Big) \\ &= \int d\rho\psi^*(\rho)\Big(- \omega\Big(\frac{\partial J}{\partial \rho}\Big) - \omega J\Big(\frac{\partial}{\partial \rho}\Big) - J\frac{\partial^2\Omega}{\partial \rho^2} - 2\Big(\frac{\partial \Omega}{\partial \rho}\Big)\Big(\frac{\partial J}{\partial \rho}\Big) - \Omega\Big(\frac{\partial^2 J}{\partial \rho^2}\Big) - 2J\Big(\frac{\partial \Omega}{\partial \rho}\Big)\Big(\frac{\partial}{\partial \rho}\Big) \\ &\quad - 2\Omega\Big(\frac{\partial J}{\partial \rho}\Big)\Big(\frac{\partial}{\partial \rho}\Big) - \Omega J\frac{\partial^2}{\partial \rho^2}\Big)\phi(\rho) \\ &= \langle\psi|J^{\frac{1}{2}}J^{-\frac{1}{2}}\Big(- \omega\Big(\frac{\partial J}{\partial \rho}\Big) - \omega J\Big(\frac{\partial}{\partial \rho}\Big) - J\frac{\partial^2\Omega}{\partial \rho^2} - 2\Big(\frac{\partial \Omega}{\partial \rho}\Big)\Big(\frac{\partial J}{\partial \rho}\Big) - \Omega\Big(\frac{\partial^2 J}{\partial \rho^2}\Big) - 2J\Big(\frac{\partial \Omega}{\partial \rho}\Big)\Big(\frac{\partial}{\partial \rho}\Big) \\ &\quad - 2\Omega\Big(\frac{\partial J}{\partial \rho}\Big)\Big(\frac{\partial}{\partial \rho}\Big) - \Omega J\frac{\partial^2}{\partial \rho^2}\Big)J^{-\frac{1}{2}}J^{\frac{1}{2}}|\phi\rangle \end{aligned}$$

$$\begin{aligned} \Rightarrow &(J^{\frac{1}{2}}HJ^{-\frac{1}{2}})^{\dagger} \\ &= J^{-\frac{1}{2}}\Big(- \omega\Big(\frac{\partial J}{\partial \rho}\Big) - \omega J\Big(\frac{\partial}{\partial \rho}\Big) - J\frac{\partial^2\Omega}{\partial \rho^2} - 2\Big(\frac{\partial \Omega}{\partial \rho}\Big)\Big(\frac{\partial J}{\partial \rho}\Big) - \Omega\Big(\frac{\partial^2 J}{\partial \rho^2}\Big) - 2J\Big(\frac{\partial \Omega}{\partial \rho}\Big)\Big(\frac{\partial}{\partial \rho}\Big) \\ &\quad - 2\Omega\Big(\frac{\partial J}{\partial \rho}\Big)\Big(\frac{\partial}{\partial \rho}\Big) - \Omega J\frac{\partial^2}{\partial \rho^2}\Big)J^{-\frac{1}{2}} \\ &= -\frac{\omega}{2}\frac{\partial}{\partial \rho}\ln J - \frac{\partial \Omega}{\partial \rho}\frac{\partial}{\partial \rho}\ln J - 2\frac{\partial \Omega}{\partial \rho}\frac{\partial}{\partial \rho} + \frac{\Omega}{4}\Big(\frac{\partial}{\partial \rho}\ln J\Big)^2 - \Omega\Big(\frac{\partial}{\partial \rho}\ln J\Big)\frac{\partial}{\partial \rho} - \frac{\partial^2\Omega}{\partial \rho^2} - \frac{1}{2}\frac{\Omega}{J}\frac{\partial^2 J}{\partial \rho^2} + \\ &\qquad - \omega\frac{\partial}{\partial \rho} - \Omega\frac{\partial^2}{\partial \rho^2} \end{aligned}$$

$$
\begin{aligned}
&\Rightarrow J^{\frac{1}{2}} H J^{-\frac{1}{2}} - (J^{\frac{1}{2}} H J^{-\frac{1}{2}})^\dagger \\
&\quad = -\Omega\left(\frac{\partial}{\partial\rho}\ln J\right)^2 + \Omega\frac{1}{J}\frac{\partial^2 J}{\partial\rho^2} + \frac{\partial\Omega}{\partial\rho}\frac{\partial}{\partial\rho}\ln J + \frac{\partial^2\Omega}{\partial\rho^2} + 2\frac{\partial\Omega}{\partial\rho}\frac{\partial}{\partial\rho} + 2\Omega\left(\frac{\partial}{\partial\rho}\ln J\right)\frac{\partial}{\partial\rho} + 2\omega\frac{\partial}{\partial\rho} \\
&\quad = 0
\end{aligned}
$$

Using

$$
\frac{1}{J}\frac{\partial^2 J}{\partial\rho^2} = \left(\frac{\partial \ln J}{\partial\rho}\right)^2 + \frac{\partial^2 \ln J}{\partial\rho^2}
$$

$$
\implies J^{\frac{1}{2}} H J^{-\frac{1}{2}} - (J^{\frac{1}{2}} H J^{-\frac{1}{2}})^\dagger = \Omega\frac{\partial^2 \ln J}{\partial\rho^2} + \frac{\partial\Omega}{\partial\rho}\frac{\partial}{\partial\rho}\ln J + \frac{\partial^2\Omega}{\partial\rho^2} + 2\frac{\partial\Omega}{\partial\rho}\frac{\partial}{\partial\rho} + 2\Omega\left(\frac{\partial}{\partial\rho}\ln J\right)\frac{\partial}{\partial\rho} + 2\omega\frac{\partial}{\partial\rho} = 0 \tag{A.21}
$$

This is an operator equation and so must hold for the action on any arbitrary function of ρ. Thus,

$$
[J^{\frac{1}{2}} H J^{-\frac{1}{2}} - (J^{\frac{1}{2}} H J^{-\frac{1}{2}})^\dagger] f(\rho) = 0
$$

for any f. If f is chosen to be a constant one must have

$$
-\Omega\left(\frac{\partial \ln J}{\partial\rho}\right)^2 f + \Omega\frac{1}{J}\frac{\partial^2 J}{\partial\rho^2} f + \frac{\partial\Omega}{\partial\rho}\frac{\partial \ln J}{\partial\rho} f = 0
$$

$$
\Rightarrow -\Omega\left(\frac{\partial \ln J}{\partial\rho}\right)^2 + \Omega\frac{1}{J}\frac{\partial^2 J}{\partial\rho^2} + \frac{\partial\Omega}{\partial\rho}\frac{\partial \ln J}{\partial\rho} = 0 \tag{A.22}
$$

Plugging this back into (A.21) gives

$$
2\frac{\partial\Omega}{\partial\rho}\frac{\partial}{\partial\rho} + 2\Omega\left(\frac{\partial}{\partial\rho}\ln J\right)\frac{\partial}{\partial\rho} + 2\omega\frac{\partial}{\partial\rho} = 0
$$

$$
\Rightarrow \frac{\partial\Omega}{\partial\rho} + \Omega\frac{\partial \ln J}{\partial\rho} + \omega = 0 \tag{A.23}
$$

This differential equation can be used to find J. For $\omega = -2N$ and $\Omega = 4\rho$ (A.23) implies $J = \rho^{\frac{N-2}{2}}$ as it should be.

Alternative method One can also come to this same equation for J by separating $H_{\text{eff}} = J^{\frac{1}{2}} H J^{-\frac{1}{2}}$ into manifestly Hermitian and not manifestly Hermitian parts as follows

$$
\begin{aligned}
H_{\text{eff}} &= i\omega J^{\frac{1}{2}}\left(-i\frac{\partial}{\partial\rho}\right)J^{-\frac{1}{2}} + \Omega(\rho) J^{\frac{1}{2}}\left(-i\frac{\partial}{\partial\rho}\right)\left(-i\frac{\partial}{\partial\rho}\right)J^{-\frac{1}{2}} \\
&= i\omega\left(\frac{i}{2}\frac{\partial \ln J}{\partial\rho} + \Pi\right) + \Omega(\rho)\left(\frac{i}{2}\frac{\partial \ln J}{\partial\rho} + \Pi\right)\left(\frac{i}{2}\frac{\partial \ln J}{\partial\rho} + \Pi\right) \\
&= -\omega\frac{1}{2}\frac{\partial \ln J}{\partial\rho} + i\omega\Pi - \Omega\frac{1}{4}\left(\frac{\partial \ln J}{\partial\rho}\right)^2 + \frac{i}{2}\Omega\frac{\partial \ln J}{\partial\rho}\Pi + \frac{i}{2}\Omega\Pi\frac{\partial \ln J}{\partial\rho} + \Omega\Pi\Pi \\
&= -\omega\frac{1}{2}\frac{\partial \ln J}{\partial\rho} + i\omega\Pi - \Omega\frac{1}{4}\left(\frac{\partial \ln J}{\partial\rho}\right)^2 + i\Omega\frac{\partial \ln J}{\partial\rho}\Pi + \frac{1}{2}\Omega\frac{\partial^2 \ln J}{\partial\rho^2} + \Omega\Pi\Pi
\end{aligned}
$$

Now

$$\begin{aligned}\Omega\Pi\Pi &= \Pi\Omega\Pi + [\Omega,\Pi]\Pi \\ &= \Pi\Omega\Pi + i\frac{\partial\Omega}{\partial\rho}\Pi\end{aligned}$$

$$\begin{aligned}\implies H_{\text{eff}} &= -\omega\frac{1}{2}\frac{\partial\ln J}{\partial\rho} + i\omega\Pi - \Omega\frac{1}{4}\left(\frac{\partial\ln J}{\partial\rho}\right)^2 + i\Omega\frac{\partial\ln J}{\partial\rho}\Pi + \frac{1}{2}\Omega\frac{\partial^2\ln J}{\partial\rho^2} + \Pi\Omega\Pi + i\frac{\partial\Omega}{\partial\rho}\Pi \\ &= -\omega\frac{1}{2}\frac{\partial\ln J}{\partial\rho} - \Omega\frac{1}{4}\left(\frac{\partial\ln J}{\partial\rho}\right)^2 + \frac{1}{2}\Omega\frac{\partial^2\ln J}{\partial\rho^2} + \Pi\Omega\Pi + i\left(\omega + \Omega\frac{\partial\ln J}{\partial\rho} + \frac{\partial\Omega}{\partial\rho}\right)\Pi\end{aligned}$$

The only part of this that is not manifestly Hermitian is the last term. That means that

$$H_{\text{eff}} - H_{\text{eff}}^\dagger = i\left(\omega + \Omega\frac{\partial\ln J}{\partial\rho} + \frac{\partial\Omega}{\partial\rho}\right)\Pi + i\Pi\left(\omega + \Omega\frac{\partial\ln J}{\partial\rho} + \frac{\partial\Omega}{\partial\rho}\right)$$

Thus, by setting

$$\omega + \Omega\frac{\partial\ln J}{\partial\rho} + \frac{\partial\Omega}{\partial\rho} = 0$$

the Hamiltonian is ensured to be Hermitian. The effective Hamiltonian is then given by

$$H_{\text{eff}} = -\omega\frac{1}{2}\frac{\partial\ln J}{\partial\rho} - \Omega\frac{1}{4}\left(\frac{\partial\ln J}{\partial\rho}\right)^2 + \frac{1}{2}\Omega\frac{\partial^2\ln J}{\partial\rho^2} + \Pi\Omega\Pi$$

Solving for $\frac{\partial\ln J}{\partial\rho}$ and eliminating it from the Hamiltonian

$$\frac{\partial\ln J}{\partial\rho} = -\Omega^{-1}\left(\omega + \frac{\partial\Omega}{\partial\rho}\right)$$

$$\begin{aligned}\Rightarrow H_{\text{eff}} &= \omega\frac{1}{2}\Omega^{-1}\left(\omega + \frac{\partial\Omega}{\partial\rho}\right) - \Omega\frac{1}{4}\left(\Omega^{-1}\left(\omega + \frac{\partial\Omega}{\partial\rho}\right)\right)^2 - \frac{1}{2}\Omega\frac{\partial}{\partial\rho}\left(\Omega^{-1}\left(\omega + \frac{\partial\Omega}{\partial\rho}\right)\right) + \Pi\Omega\Pi \\ &= \frac{\omega^2}{2\Omega} + \frac{\omega}{2}\frac{\partial\ln\Omega}{\partial\rho} - \frac{1}{4}\Omega^{-1}\left(\omega + \frac{\partial\Omega}{\partial\rho}\right)^2 - \frac{1}{2}\Omega\frac{\partial}{\partial\rho}\left(\Omega^{-1}\left(\omega + \frac{\partial\Omega}{\partial\rho}\right)\right) + \Pi\Omega\Pi \\ &= \frac{\omega^2}{4\Omega} + \frac{\omega}{2}\frac{\partial\ln\Omega}{\partial\rho} + \frac{1}{4\Omega}\left(\frac{\partial\Omega}{\partial\rho}\right)^2 - \frac{1}{2}\frac{\partial^2\Omega}{\partial\rho^2} + \Pi\Omega\Pi\end{aligned}$$

Plugging in $\omega = -2N$ and $\Omega = 4\rho$ gives

$$\begin{aligned}H_{\text{eff}} &= \frac{N^2}{4\rho} - \frac{N}{\rho} + \frac{1}{\rho} - 4i\Pi + 4\rho\Pi\Pi \\ &= \frac{(N-2)^2}{4\rho} - 4i\Pi + 4\rho\Pi\Pi\end{aligned}$$

A.2.3 Testing H_{eff}

This section aims to test the above Hamiltonian. In order to do that the potential $x^a x^a$ is added to $H = P^a P^a$ so that the theory now represents a system of harmonic oscillators. Thus, the usual wave functions should be eigenfunctions of the Hamiltonian

$$H' = P^a P^a + x^a x^a$$

After the change variables the corresponding effective Hamiltonian will be

$$H'_{\text{eff}} = \frac{(N-2)^2}{4\rho} - 4i\Pi + 4\rho\Pi\Pi + \rho$$

The N dimensional harmonic oscillator (HO) eigenfuntions for $\omega = m = 1$ are

$$\begin{aligned}\Psi_{\vec{n}}(x^a) &= A\prod_{i=1}^{N}\frac{1}{\sqrt{2^{n_i}n_i!}}e^{-\frac{(x^i)^2}{2}}H_{n_i}(x^i)\\ &= Ae^{-\frac{x^a x^a}{2}}\prod_{i=1}^{N}\frac{1}{\sqrt{2^{n_i}n_i!}}H_{n_i}(x^i)\end{aligned}$$

where A is a normalisation constant, $H_n(x)$ are the Hermite polynomials, n_i represents the excitation level in dimension i and $\vec{n} = (n_1, n_2, \cdots, n_N)$. The corresponding HO Hamiltonian is

$$H_{HO} = \frac{P^a P^a}{2} + \frac{x^a x^a}{2}$$

with the spectrum

$$E_{\vec{n}} = \frac{N}{2} + \sum_{i=1}^{N} n_i$$

Since the goal is to test H'_{eff} which comes from H' which differs from H_{HO} by a factor of half it is expected that the eigenvalues also differ by a factor of half. The states must also be re-scaled. The states will be

$$\begin{aligned}\tilde{\Psi}_{\vec{n}}(x^a) &= \sqrt{J}Ae^{-\frac{\rho}{2}}\prod_{i=1}^{N}\frac{1}{\sqrt{2^{n_i}n_i!}}H_{n_i}(x^i)\\ &= A\rho^{\frac{N-2}{4}}e^{-\frac{\rho}{2}}\prod_{i=1}^{N}\frac{1}{\sqrt{2^{n_i}n_i!}}H_{n_i}(x^i)\end{aligned} \tag{A.24}$$

Testing the ground state

$$\begin{aligned}
H'_{\text{eff}}\tilde{\Psi}_0(x^a) &= \left[\frac{(N-2)^2}{4\rho} - 4i\Pi + 4\rho\Pi\Pi + \rho\right] A\rho^{\frac{N-2}{4}} e^{-\frac{\rho}{2}} \\
&= \left[\frac{(N-2)^2}{4\rho} - 4\frac{\partial}{\partial\rho} - 4\rho\frac{\partial^2}{\partial\rho^2} + \rho\right] A\rho^{\frac{N-2}{4}} e^{-\frac{\rho}{2}} \\
&= \left[\frac{(N-2)^2}{4\rho} - 4\left(\frac{N-2}{4\rho} - \frac{1}{2}\right) - 4\rho\left(\frac{(N-2)(N-6)}{16\rho^2} - \frac{N-2}{4\rho} + \frac{1}{4}\right) + \rho\right]\tilde{\Psi}_0(x^a) \\
&= \left[\frac{(N-2)^2}{4\rho} - \frac{N-2}{\rho} + 2 - \frac{(N-2)(N-6)}{4\rho} + N - 2 - \rho + \rho\right]\tilde{\Psi}_0(x^a) \\
&= \left[\frac{(N-2)^2}{4\rho} - \frac{N^2-4N+4}{4\rho} + N\right]\tilde{\Psi}_0(x^a) \\
&= N\tilde{\Psi}_0(x^a)
\end{aligned}$$

Thus, $\tilde{\Psi}_0(x^a)$ is an eigenvector of H'_{eff} with eigenvalue N. Which agrees with what was expected. In order to test higher order states one has to choose states which only depend on ρ and not on angles. The problem is that the product of Hermite polynomials in (A.24) is not independent of angles. Thus one needs to find the eigenstates of H'_{eff} and see how they are related to the $\tilde{\Psi}_n(x^a)$. To do this start by finding the eigenstates of $H' = i\omega\Pi + \Omega\Pi\Pi + \rho$ and then scale those states with $\sqrt{J}$ to get the eigenstates of H'_{eff}. Solving the Schrodinger equation

$$H'\Psi_n(\rho) = E_n\Psi_n(\rho)$$

one finds that $\Psi_n(\rho)$ takes the form

$$\Psi_n(\rho) = Ae^{-\frac{\rho}{2}}h_n(\rho)$$

where

$$h_n(\rho) = \sum_{j=0}^{n} a_j\rho^j \qquad\qquad n = 0, 1, 2, \cdots$$

and the coefficients obey the following recursion relation

$$a_{j+1} = \frac{2(j-n)}{(2j+N)(j+1)}a_j \qquad n = 0, 1, 2, \cdots$$

The spectrum of the states is

$$E = N + 4n \qquad\qquad n = 0, 1, 2, \cdots$$

Comparing this to the eigenstates of $H_{HO} = P^aP^a + x^ax^a$, $\Psi_{HO\vec{n}}(x^a) = Ae^{-\frac{x^ax^a}{2}}\prod_{i=1}^{N} h_{HOn_i}(x^i)$ where

$$h_{HOn_i}(x^i) = \sum_{j=0}^{n_i} a_j(x^i)^j \qquad\qquad n = 0, 1, 2, \cdots$$

and

$$a_{j+2} = \frac{2(j-n)}{(j+1)(j+2)} a_j$$

with the spectrum

$$E_{HOn} = N + 2\sum_{i=1}^{N} n_i$$

One sees that each $\Psi_n(\rho)$ can be built out of a linear superposition of $\Psi_{HO\vec{m}}(x^a)$ states where $\sum_i m_i = 2n$ and each m_i is even. For example

$$\begin{aligned}
\Psi_2(\rho) &= \frac{1}{N(N+2)} \sum_{\substack{n_1,n_2,\cdots,n_N \\ n_1+n_2+\cdots+n_N=4}} c_{\vec{n}} \Psi_{HO\vec{n}}(x^a) \\
&= \frac{1}{N(N+2)} A e^{-\frac{\rho}{2}} \left[3\sum_{i=1}^{N} h_{HO4}(x^i) + \sum_{\substack{i,j \\ j\neq i}}^{N} h_{HO2}(x^i) h_{HO2}(x^j) \right] \\
&= \frac{1}{N(N+2)} A e^{-\frac{\rho}{2}} [2N + N^2 - 4(N+2)\rho + 4\rho^2] \\
&= A e^{-\frac{\rho}{2}} \left[1 - \frac{4\rho}{N} + \frac{4\rho^2}{N(N+2)} \right] \\
&= A e^{-\frac{\rho}{2}} h_2(\rho)
\end{aligned}$$

Similarly, for $\Psi_3(\rho)$

$$\begin{aligned}
\Psi_3(\rho) &= \frac{1}{N(N+2)(N+4)} \sum_{\substack{n_1,n_2,\cdots,n_N \\ n_1+n_2+\cdots+n_N=6}} c_{\vec{n}} \Psi_{HO\vec{n}}(x^a) \\
&= \frac{1}{N(N+2)(N+4)} A e^{-\frac{\rho}{2}} \Bigg[15\sum_{i=1}^{N} h_{HO6}(x^i) + 9\sum_{\substack{i,j \\ j\neq i}}^{N} h_{HO4}(x^i) h_{HO2}(x^j) + \\
&\qquad + \sum_{\substack{i,j,k \\ k\neq j\neq i}}^{N} h_{HO2}(x^i) h_{HO2}(x^j) h_{HO2}(x^k) \Bigg] \\
&= \frac{1}{N(N+2)(N+4)} A e^{-\frac{\rho}{2}} [8N + 6N^2 + N^3 - 48\rho - 36N\rho - 6N^2\rho + 48\rho^2 + 12N\rho^2 - 8\rho^3] \\
&= A e^{-\frac{\rho}{2}} \left[1 - \frac{6\rho}{N} + \frac{12\rho^2}{N(N+2)} - \frac{8\rho^3}{N(N+2)(N+4)} \right] \\
&= A e^{-\frac{\rho}{2}} h_3(\rho)
\end{aligned}$$

This is to be expected since the harmonic oscillator states form a basis and so any state can be built from a linear superposition of the HO states granted the state being built is in the same Hilbert space. We have not changed Hilbert spaces by changing variables. Furthermore, the eigenstates of H'_{eff} with energy E are built out of a symmetric superposition of the HO eigenstates with the same energy as would be expected.

Appendix B

B.1 Check of counting formulas

In this section we will explore the counting formulas obtained in Section 3.1. First consider the counting at infinite N. It is rather easy to use characters of the symmetric group to compute the Kronecker coefficients and then sum the squares of the coefficients, to compute the number of bosonic gauge invariant operators (N_b) and the number of fermionic gauge invariants operators (N_f). The results are shown in Table 1.

Table B.1: The number of bosonic N_b or fermionic N_f gauge invariant tensors constructed using n fields. This counting is for gauge group ranks $N_1 = N_2 = N_3 = \infty$.

n	1	2	3	4	5	6
N_b	1	4	11	43	161	901
N_f	1	4	11	43	161	901

Note that the number of fermionic gauge invariant operators is equal to the number of bosonic gauge invariant operators. This fact is easily explained: every time we have a non-zero bosonic Kronecker coefficient, there is a corresponding non-zero fermionic Kronecker coefficient. This is easily proved using the well known property of characters of the symmetric group

$$\chi_{R^T}(\sigma) = \mathrm{sgn}(\sigma)\chi_R(\sigma) \tag{B.1}$$

where R^T is the transposed Young diagram, i.e. the Young diagram obtained from R by swapping rows and columns. For example

$$R = \ydiagram{3,2} \qquad \Rightarrow \qquad R^T = \ydiagram{2,2,1} \tag{B.2}$$

Recall that 1^n represents the Young diagram with a single column of n boxes. Use n to denote the Young diagram that has a single row of n boxes. The proof is as follows

$$\begin{aligned}
g_{r_1,r_2,r_3,n} &= \frac{1}{n!}\sum_{\sigma\in S_n}\chi_{r_1}(\sigma)\chi_{r_2}(\sigma)\chi_{r_3}(\sigma)\chi_n(\sigma) \\
&= \frac{1}{n!}\sum_{\sigma\in S_n}\chi_{r_1}(\sigma)\chi_{r_2}(\sigma)\chi_{r_3}(\sigma) \\
&= \frac{1}{n!}\sum_{\sigma\in S_n}\chi_{r_1^T}(\sigma)\chi_{r_2^T}(\sigma)\chi_{r_3^T}(\sigma)(\mathrm{sgn}(\sigma))^3 \\
&= \frac{1}{n!}\sum_{\sigma\in S_n}\chi_{r_1^T}(\sigma)\chi_{r_2^T}(\sigma)\chi_{r_3^T}(\sigma)\mathrm{sgn}(\sigma)
\end{aligned}$$

$$= \frac{1}{n!}\sum_{\sigma\in S_n}\chi_{r_1^T}(\sigma)\chi_{r_2^T}(\sigma)\chi_{r_3^T}(\sigma)\chi_{1^n}(\sigma)$$
$$= g_{r_1^T,r_2^T,r_3^T,1^n} \tag{B.3}$$

Since the set of non-zero Kronecker coefficients are the same for the bosonic and the fermionic tensor models, and the number of gauge invariant operators is equal to the sum of the squares of these coefficients, this proves that the number of gauge invariant operators one can construct in bosonic tensor models equals the number of gauge invariant operators one can construct in fermionic tensor models.

The argument above has been for rank $d = 3$ tensors. It is clear that the above proof goes through for rank d tensors with d odd, since in this case $(\mathrm{sgn}(\sigma))^d = \mathrm{sgn}(\sigma)$. For even d however, the above proof does not go through: in this case $(\mathrm{sgn}(\sigma))^d = 1$. However, a simple variant of the proof does work: for rank four for example, it is simple to prove that

$$g_{r_1,r_2,r_3,r_4,n} = g_{r_1^T,r_2^T,r_3^T,r_4,1^n} \tag{B.4}$$

We have verified this equality explicitly for $n \leq 6$ and ranks $d \leq 8$, which is a nice check of the above arguments.

At finite N the number of fermionic and bosonic gauge invariant operators no longer matches. Recall that for a general rank d tensor model the gauge group is $U(N_1) \times U(N_2) \times \cdots \times U(N_d)$. As soon as n exceeds any of the N_i, it is possible to have Young diagrams $r \vdash n$ whose number of rows is greater than at least one of the N_i. In this case, the corresponding operator vanishes and it is for this reason that we must put a cut off on the number of rows. For example, for the bosons we have

$$N_b = \sum_{r_i\vdash n\ l(r_i)\leq N_i} g^2_{r_1\, r_2\, r_3\, 1} \tag{B.5}$$

The proof breaks because we can have, for example, $l(r_1) < N_1$ and $l(r_1^T) > N_1$. In Table 2 we have given the finite N counting for rank 3 tensors with $N_1 = N_2 = N_3 = 5$.

Table B.2: The number of bosonic N_b or fermionic N_f gauge invariant tensors constructed using n fields. Here $N_1 = N_2 = N_3 = 5$.

n	1	2	3	4	5	6
N_b	1	4	11	43	92	70
N_f	1	4	11	43	87	20

Notice that there are more bosonic gauge invariant operators than there are fermionic gauge invariant operators. This is in fact rather general: the Kronecker coefficients relevant for the bosonic gauge invariants are mostly short and wide Young diagrams. On the other hand, the Kronecker coefficients relevant for the fermionic gauge invariants are mostly tall and thin Young diagrams. In fact, for the fermionic tensor model, there is some value of n beyond which there are no new gauge invariants. In Table 3 we have shown the finite N counting for rank 3 tensors with $N_1 = N_2 = N_3 = 3$. For $n \geq 6$ there are no gauge invariant operators.

For the first few values of n, it is possible to explicitly construct the gauge invariant operators. For $n = 1$ there is a single bosonic and a single fermionic gauge invariant operator

$$\bar{\phi}^{ijk}\phi_{ijk} \qquad\qquad \bar{\psi}^{ijk}\psi_{ijk} \tag{B.6}$$

Table B.3: The number of bosonic N_b or fermionic N_f gauge invariant tensors constructed using n fields. Here $N_1 = N_2 = N_3 = 3$.

n	1	2	3	4	5	6
N_b	1	4	11	12	151	18
N_f	1	4	11	8	41	0

For $n = 2$ we have the following bosonic operators

$$\begin{array}{ll} \bar{\phi}^{i_1 j_1 k_1}\bar{\phi}^{i_2 j_2 k_2}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2} & \bar{\phi}^{i_2 j_1 k_1}\bar{\phi}^{i_1 j_2 k_2}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2} \\ \bar{\phi}^{i_1 j_2 k_1}\bar{\phi}^{i_2 j_1 k_2}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2} & \bar{\phi}^{i_1 j_1 k_2}\bar{\phi}^{i_2 j_2 k_1}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2} \end{array} \tag{B.7}$$

which nicely matches the counting given above. There is an identical set of operators for the fermions. For $n = 3$ we have the following bosonic operators

$$\begin{array}{ll} \bar{\phi}^{i_1 j_1 k_1}\bar{\phi}^{i_2 j_2 k_2}\bar{\phi}^{i_3 j_3 k_3}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2}\phi_{i_3 j_3 k_3} & \bar{\phi}^{i_1 j_1 k_1}\bar{\phi}^{i_2 j_2 k_3}\bar{\phi}^{i_3 j_3 k_2}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2}\phi_{i_3 j_3 k_3} \\ \bar{\phi}^{i_1 j_1 k_1}\bar{\phi}^{i_2 j_3 k_2}\bar{\phi}^{i_3 j_3 k_2}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2}\phi_{i_3 j_3 k_3} & \bar{\phi}^{i_1 j_1 k_1}\bar{\phi}^{i_2 j_3 k_3}\bar{\phi}^{i_3 j_2 k_2}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2}\phi_{i_3 j_3 k_3} \\ \bar{\phi}^{i_1 j_1 k_2}\bar{\phi}^{i_2 j_2 k_3}\bar{\phi}^{i_3 j_3 k_1}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2}\phi_{i_3 j_3 k_3} & \bar{\phi}^{i_1 j_1 k_2}\bar{\phi}^{i_2 j_3 k_1}\bar{\phi}^{i_3 j_2 k_3}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2}\phi_{i_3 j_3 k_3} \\ \bar{\phi}^{i_1 j_2 k_1}\bar{\phi}^{i_2 j_3 k_3}\bar{\phi}^{i_3 j_1 k_2}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2}\phi_{i_3 j_3 k_3} & \bar{\phi}^{i_1 j_2 k_1}\bar{\phi}^{i_2 j_3 k_2}\bar{\phi}^{i_3 j_1 k_3}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2}\phi_{i_3 j_3 k_3} \\ \bar{\phi}^{i_1 j_2 k_3}\bar{\phi}^{i_2 j_3 k_1}\bar{\phi}^{i_3 j_1 k_2}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2}\phi_{i_3 j_3 k_3} & \bar{\phi}^{i_1 j_2 k_2}\bar{\phi}^{i_2 j_3 k_3}\bar{\phi}^{i_3 j_1 k_1}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2}\phi_{i_3 j_3 k_3} \end{array} \tag{B.8}$$

which matches the counting given above. The set of fermionic operators is again the same.

B.2 Examples of Operators and Correlators

In the previous Appendix some of the gauge invariant operators in the permutation basis were written down. In this Appendix we will write down some operators in the representation theory basis. We will then explore correlators of gauge invariant operators, in both bases.

For $n = 2$ fields, there are no multiplicities, so these labels are dropped. There is a total of four gauge invariant operators that can be defined. We will give the complete set of gauge invariant operators, since this will allow us to test that they are indeed orthogonal and have the correct two point function. The operators are given by

$$\begin{aligned} \mathcal{O}_{\square\square,\square\square,\square\square} &= 2\bar{\phi}^{i_1 j_1 k_1}\bar{\phi}^{i_2 j_2 k_2}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2} + 2\bar{\phi}^{i_2 j_1 k_1}\bar{\phi}^{i_1 j_2 k_2}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2} \\ &\quad +2\bar{\phi}^{i_1 j_2 k_1}\bar{\phi}^{i_2 j_1 k_2}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2} + 2\bar{\phi}^{i_1 j_1 k_2}\bar{\phi}^{i_2 j_2 k_1}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2} \end{aligned} \tag{B.9}$$

$$\begin{aligned} \mathcal{O}_{\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix},\square\square} &= 2\bar{\phi}^{i_1 j_1 k_1}\bar{\phi}^{i_2 j_2 k_2}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2} - 2\bar{\phi}^{i_2 j_1 k_1}\bar{\phi}^{i_1 j_2 k_2}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2} \\ &\quad -2\bar{\phi}^{i_1 j_2 k_1}\bar{\phi}^{i_2 j_1 k_2}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2} + 2\bar{\phi}^{i_1 j_1 k_2}\bar{\phi}^{i_2 j_2 k_1}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2} \end{aligned} \tag{B.10}$$

$$\begin{aligned} \mathcal{O}_{\begin{smallmatrix}\square\\\square\end{smallmatrix},\square\square,\begin{smallmatrix}\square\\\square\end{smallmatrix}} &= 2\bar{\phi}^{i_1 j_1 k_1}\bar{\phi}^{i_2 j_2 k_2}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2} - 2\bar{\phi}^{i_2 j_1 k_1}\bar{\phi}^{i_1 j_2 k_2}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2} \\ &\quad +2\bar{\phi}^{i_1 j_2 k_1}\bar{\phi}^{i_2 j_1 k_2}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2} - 2\bar{\phi}^{i_1 j_1 k_2}\bar{\phi}^{i_2 j_2 k_1}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2} \end{aligned} \tag{B.11}$$

$$\mathcal{O}_{\square\square,\begin{smallmatrix}\square\\\square\end{smallmatrix},\begin{smallmatrix}\square\\\square\end{smallmatrix}} = 2\bar{\phi}^{i_1 j_1 k_1}\bar{\phi}^{i_2 j_2 k_2}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2} + 2\bar{\phi}^{i_2 j_1 k_1}\bar{\phi}^{i_1 j_2 k_2}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2}$$

$$-2\bar{\phi}^{i_1 j_2 k_1}\bar{\phi}^{i_2 j_1 k_2}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2} - 2\bar{\phi}^{i_1 j_1 k_2}\bar{\phi}^{i_2 j_2 k_1}\phi_{i_1 j_1 k_1}\phi_{i_2 j_2 k_2} \tag{B.12}$$

A simple but tedious computation confirms (4.50) and (4.51). Some sample computations are

$$\langle \mathcal{O}_{\square\square,\square\square,\square\square}\rangle = 2N_1(N_1+1)N_2(N_2+1)N_3(N_3+1) \tag{B.13}$$

$$\langle \mathcal{O}_{\square\square,\square\square,\square\square}\mathcal{O}_{\square\square,\square\square,\square\square}\rangle = 32N_1(N_1+1)N_2(N_2+1)N_3(N_3+1) \tag{B.14}$$

$$\langle \mathcal{O}_{\square\square,\square\square,\square\square}\mathcal{O}_{\square\square,\square\square,\square}\rangle = 0 \tag{B.15}$$

For $n = 2$, the complete set of fermionic operators in the representation basis is given by

$$\begin{aligned}\mathcal{P}_{\substack{\square\,\square\,\square\\ \square\,\square\,\square}} &= 2\bar{\psi}^{i_1 j_1 k_1}\bar{\psi}^{i_2 j_2 k_2}\psi_{i_2 j_2 k_2}\psi_{i_1 j_1 k_1} - 2\bar{\psi}^{i_1 j_2 k_1}\bar{\psi}^{i_2 j_1 k_2}\psi_{i_2 j_2 k_2}\psi_{i_1 j_1 k_1}\\ &\quad -2\bar{\psi}^{i_1 j_1 k_2}\bar{\psi}^{i_2 j_2 k_1}\psi_{i_2 j_2 k_2}\psi_{i_1 j_1 k_1} - 2\bar{\psi}^{i_2 j_1 k_1}\bar{\psi}^{i_1 j_2 k_2}\psi_{i_2 j_2 k_2}\psi_{i_1 j_1 k_1}\end{aligned} \tag{B.16}$$

$$\begin{aligned}\mathcal{P}_{\substack{\square\\ \square,\square\square,\square\square}} &= 2\bar{\psi}^{i_1 j_1 k_1}\bar{\psi}^{i_2 j_2 k_2}\psi_{i_2 j_2 k_2}\psi_{i_1 j_1 k_1} - 2\bar{\psi}^{i_2 j_1 k_1}\bar{\psi}^{i_1 j_2 k_2}\psi_{i_2 j_2 k_2}\psi_{i_1 j_1 k_1}\\ &\quad +2\bar{\psi}^{i_1 j_2 k_1}\bar{\psi}^{i_2 j_1 k_2}\psi_{i_2 j_2 k_2}\psi_{i_1 j_1 k_1} + 2\bar{\psi}^{i_1 j_1 k_2}\bar{\psi}^{i_2 j_2 k_1}\psi_{i_2 j_2 k_2}\psi_{i_1 j_1 k_1}\end{aligned} \tag{B.17}$$

$$\begin{aligned}\mathcal{P}_{\substack{\square\\ \square\square,\square,\square\square}} &= 2\bar{\psi}^{i_1 j_1 k_1}\bar{\psi}^{i_2 j_2 k_2}\psi_{i_2 j_2 k_2}\psi_{i_1 j_1 k_1} + 2\bar{\psi}^{i_2 j_1 k_1}\bar{\psi}^{i_1 j_2 k_2}\psi_{i_2 j_2 k_2}\psi_{i_1 j_1 k_1}\\ &\quad -2\bar{\psi}^{i_1 j_2 k_1}\bar{\psi}^{i_2 j_1 k_2}\psi_{i_2 j_2 k_2}\psi_{i_1 j_1 k_1} + 2\bar{\psi}^{i_1 j_1 k_2}\bar{\psi}^{i_2 j_2 k_1}\psi_{i_2 j_2 k_2}\psi_{i_1 j_1 k_1}\end{aligned} \tag{B.18}$$

$$\begin{aligned}\mathcal{P}_{\substack{\square\\ \square\square,\square\square,\square}} &= 2\bar{\psi}^{i_1 j_1 k_1}\bar{\psi}^{i_2 j_2 k_2}\psi_{i_2 j_2 k_2}\psi_{i_1 j_1 k_1} + 2\bar{\psi}^{i_2 j_1 k_1}\bar{\psi}^{i_1 j_2 k_2}\psi_{i_2 j_2 k_2}\psi_{i_1 j_1 k_1}\\ &\quad +2\bar{\psi}^{i_1 j_2 k_1}\bar{\psi}^{i_2 j_1 k_2}\psi_{i_2 j_2 k_2}\psi_{i_1 j_1 k_1} - 2\bar{\psi}^{i_1 j_1 k_2}\bar{\psi}^{i_2 j_2 k_1}\psi_{i_2 j_2 k_2}\psi_{i_1 j_1 k_1}\end{aligned} \tag{B.19}$$

Some sample computations confirming (3.54) and (3.55) are

$$\langle \mathcal{P}_{\substack{\square\,\square\,\square\\ \square\,\square\,\square}}\rangle = 2N_1(N_1-1)N_2(N_2-1)N_3(N_3-1) \tag{B.20}$$

$$\langle \mathcal{P}_{\substack{\square\,\square\,\square\\ \square\,\square\,\square}}\mathcal{P}_{\substack{\square\,\square\,\square\\ \square\,\square\,\square}}\rangle = 32N_1(N_1-1)N_2(N_2-1)N_3(N_3-1) \tag{B.21}$$

$$\langle \mathcal{P}_{\substack{\square\,\square\,\square\\ \square\,\square\,\square}}\mathcal{P}_{\substack{\square\,\square\\ \square\,\square\,\square\square}}\rangle = 0 \tag{B.22}$$

B.3 Identities needed to derive the collective field theory Hamiltonian

Using the identity

$$dy\frac{e^{-iky}}{x-y}=\epsilon(k)\pi ie^{-ikx} \tag{B.23}$$

we find

$$dy\ \frac{2x\,e^{-iky}}{y-x}=2x\epsilon(k)\pi ie^{-ikx} \tag{B.24}$$

Our main goal in this Appendix is to explain how to rewrite the term

$$T_1=\int dx\frac{\phi(x)}{x}dy_1\frac{2x\phi(y_1)}{y_1-x}dy_2\frac{2x\phi(y_2)}{y_2-x} \tag{B.25}$$

in a manifestly local form. This is the only term in the Hamiltonian that is not manifestly local. Use the Fourier transform

$$\phi(x)=\int\frac{dk}{2\pi}e^{-ikx}\phi_k \tag{B.26}$$

to write (this is the only non-local term in the Hamiltonian)

$$\begin{aligned}T_1 &= \int\frac{dk_1}{2\pi}\int\frac{dk_2}{2\pi}\int\frac{dk_3}{2\pi}\int dx\phi_{k_1}\phi_{k_2}\phi_{k_3}4xe^{-ik_1x}dy_1\frac{e^{-ik_2y_1}}{y_1-x}dy_2\frac{e^{-ik_3y_2}}{y_2-x}\\ &= \int\frac{dk_1}{2\pi}\int\frac{dk_2}{2\pi}\int\frac{dk_3}{2\pi}\int dx\phi_{k_1}\phi_{k_2}\phi_{k_3}4xe^{-ik_1x}\left[\pi i\epsilon(k_2)e^{-ik_2x}\right]\left[\pi i\epsilon(k_3)e^{-ik_3x}\right]\\ &= -4\pi^2\int\frac{dk_1}{2\pi}\int\frac{dk_2}{2\pi}\int\frac{dk_3}{2\pi}\int dx\phi_{k_1}\phi_{k_2}\phi_{k_3}xe^{-i(k_1+k_2+k_3)x}\epsilon(k_2)\epsilon(k_3)\end{aligned} \tag{B.27}$$

The expression on the last line can be manipulated, by renaming variables into

$$\begin{aligned}&-\frac{4\pi^2}{3}\int\frac{dk_1}{2\pi}\int\frac{dk_2}{2\pi}\int\frac{dk_3}{2\pi}\int dx\phi_{k_1}\phi_{k_2}\phi_{k_3}xe^{-i(k_1+k_2+k_3)x}(\epsilon(k_1)\epsilon(k_2)+\epsilon(k_1)\epsilon(k_3)+\epsilon(k_2)\epsilon(k_3))\\ &=\frac{4\pi^2}{3}\int\frac{dk_1}{2\pi}\int\frac{dk_2}{2\pi}\int\frac{dk_3}{2\pi}\phi_{k_1}\phi_{k_2}\phi_{k_3}(i\partial_{k_1}\delta(k_1+k_2+k_3))(\epsilon(k_1)\epsilon(k_2)+\epsilon(k_1)\epsilon(k_3)+\epsilon(k_2)\epsilon(k_3))\end{aligned} \tag{B.28}$$

Because of the delta function, one or two of the k_i's must be positive so that

$$\epsilon(k_1)\epsilon(k_2)+\epsilon(k_1)\epsilon(k_3)+\epsilon(k_2)\epsilon(k_3)=-1 \tag{B.29}$$

and we now find

$$T_1=\frac{4\pi^2}{3}\int\frac{dk_1}{2\pi}\int\frac{dk_2}{2\pi}\int\frac{dk_3}{2\pi}\phi_{k_1}\phi_{k_2}\phi_{k_3}(i\partial_{k_1}\delta(k_1+k_2+k_3))=\frac{4\pi^2}{3}\int dxx\phi^3(x) \tag{B.30}$$

so that, remarkably, this term is local and it gives rise to a cubic interaction!

Appendix C

C.1 Notation for Equivalence Classes

In Section 4.1, we have counted the number of invariants, by taking into account certain symmetries of the variables involved. This amounts to formulating an equivalence relation which implements the symmetry and then counting the resulting number of equivalence classes. Equivalence classes associated to the equivalence $g_1 \sim g_2$ if $g_2 = hg_1$ for $g_1, g_2 \in G$ and $h \in H$ lead to the usual notion of a coset. The equivalence just described is the right coset $H \setminus G$. The equivalence $g_2 \sim g_1$ if $g_2 = g_1 h$ defines elements of the left coset G/H.

The equivalences classes used in Section 4.1 are of a different type. They can be related to our usual notion of conjugacy classes. The equivalence relation used to define conjugacy classes says $g_1 \sim g_2$ if $g_1 = gg_2g^{-1}$ for some $g \in G$. Although the conjugacy classes are not related to cosets, they are related to double cosets. The conjugacy classes are given by the double coset

$$\mathrm{Diag}(G) \setminus (G \times G)/\mathrm{Diag}(G) \tag{C.1}$$

Here Diag denotes the diagonal action of G obtained by allowing G to act in the same way on both factors in $G \times G$. The equivalence is

$$(g_1, g_2) \sim (g_3 g_1 g_4, g_3 g_2 g_4) \tag{C.2}$$

Now, choose $g_3 = g_1^{-1}$ to map (g_1, g_2) to $(1, g_1^{-1} g_2)$. Then the g_4 action leads to

$$(1, g_4^{-1}(g_1^{-1} g_2) g_4) \tag{C.3}$$

We will use the notation $G//H$ when we want to talk about the equivalence classes of the relation $g_1 \sim g_2$ if $g_1 = hg_2h^{-1}$ for $g_1, g_2 \in G$ and for some $h \in H$.

C.2 Perturbative Collective Field Theory

In this section we set $N_1 N_2 = N_3$, which simplifies a number of the formulas that follow. We also set the frequency $\omega = \frac{1}{2}$. The collective field is given by

$$\phi(x) = \frac{1}{\pi}\sqrt{\frac{\mu}{x} - \frac{1}{4} - gx} \tag{C.4}$$

where the chemical potential μ is fixed by the requirement

$$\int_0^{x_0} \phi(x) dx = N_3 \tag{C.5}$$

where x_0 is the positive zero of the equation

$$x - 4\mu + 4gx^2 = 0 \tag{C.6}$$

At weak coupling g, we find

$$x_0 = 4\mu - 64\mu^2 g + O(g^2) \tag{C.7}$$

It is now straight forward to find $\mu = N_3 + 6gN_3^2 + O(g^2)$. Using this collective field we can compute correlators, at large N, perturbatively. As an example summing the diagrams in Figure C.1 gives

$$\langle \phi^{abc}\bar{\phi}_{abc}\rangle = N_1N_2N_3G(t,t) - i2g(N_1^2N_2^2N_3 + N_1N_2N_3^2)\int_{-\infty}^{\infty} dt' G(t,t')^2 G(t',t') + O(g^2) \tag{C.8}$$

using the following Green's function

$$G(t_1,t_2) = \int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{i}{E^2 - \frac{1}{4} + i\epsilon}e^{iE(t_1-t_2)} \tag{C.9}$$

one has

$$\begin{aligned} G(t,t) &= \int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{i}{E^2 - \frac{1}{4} + i\epsilon} \\ &= \frac{i}{2\pi}\int_{-\infty}^{\infty} dE\left[\frac{1}{E - \frac{1}{2} + i\epsilon} - \frac{1}{E + \frac{1}{2} - i\epsilon}\right] \\ &= \frac{i}{2\pi}(2\pi i)(-1) \qquad \text{(C.10)} \\ &= 1 \qquad \text{(C.11)} \end{aligned}$$

where the residue theorem was used in the second last line and the contour was closed in the upper half plane. Similarly,

$$\begin{aligned} \int_{-\infty}^{\infty} dt' G(t,t')^2 G(t',t') &= \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty}\frac{dE}{2\pi}\int_{-\infty}^{\infty}\frac{dE'}{2\pi}\frac{i}{E^2 - \frac{1}{4} + i\epsilon}\frac{i}{E'^2 - \frac{1}{4} + i\epsilon}e^{i(E+E')(t-t')} \\ &\quad \times \int_{-\infty}^{\infty}\frac{dE''}{2\pi}\frac{i}{E''^2 - \frac{1}{4} + i\epsilon} \qquad \text{(C.12)} \\ &= \int_{-\infty}^{\infty}\frac{dE}{2\pi}\int_{-\infty}^{\infty}\frac{dE'}{2\pi}\frac{i}{E^2 - \frac{1}{4} + i\epsilon}\frac{i}{E'^2 - \frac{1}{4} + i\epsilon}2\pi\delta(E+E')\times 1 \qquad \text{(C.13)} \\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} dE\frac{1}{(E^2 - \frac{1}{4} + i\epsilon)^2} \qquad \text{(C.14)} \\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} dE\left[\frac{2}{E + \frac{1}{2} - i\epsilon} + \frac{1}{(E + \frac{1}{2} - i\epsilon)^2} - \frac{2}{E - \frac{1}{2} + i\epsilon} + \frac{1}{(E - \frac{1}{2} + i\epsilon)^2}\right] \qquad \text{(C.15)} \\ &= -\frac{1}{2\pi}(2\pi i)2 \qquad \text{(C.16)} \\ &= -2i \qquad \text{(C.17)} \end{aligned}$$

where the second last step follows from the fact that only first order poles contribute to the residue and the contour was again closed in the upper half plane. Putting these results into (C.8) gives

$$\begin{aligned} \langle \phi^{abc}\bar{\phi}_{abc}\rangle &= N_1N_2N_3 - 4g(N_1^2N_2^2N_3 + N_1N_2N_3^2) + O(g^2) \qquad \text{(C.18)} \\ &= N_3^2 - 8gN_3^3 + O(g^2) \qquad \text{(C.19)} \end{aligned}$$

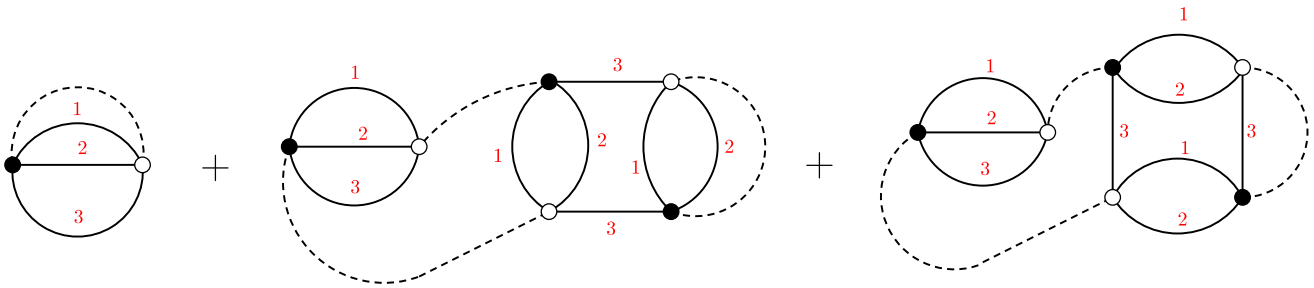


Figure C.1: The Feynman diagrams contributing to $\langle \phi^{abc} \bar{\phi}_{abc} \rangle$. The dashed lines indicate which operators are Wick contracted.

The collective field computation reproduces this result

$$\int_0^{x_0} \phi(x) x dx \;=\; N_3^2 - 8gN_3^3 + O\left(g^2\right) \tag{C.20}$$

C.3 Relating Matrix Model and Tensor Model Correlators

In Section 4.2 we have argued that there is a subsector of the tensor model that is closely related to a matrix model. In this Appendix we exhibit a connection between the correlators of the two. Consider first correlators in the tensor model. Assuming the basic contraction

$$\langle \phi_{ajk} \bar{\phi}^{blm} \rangle = \delta_a^b \delta_j^l \delta_k^m \tag{C.21}$$

we easily find

$$\begin{aligned}
\langle \phi_{jka} \bar{\phi}^{jkb} \phi_{lmb} \bar{\phi}^{lma} \rangle &= N_1^2 N_2^2 N_3 + N_1 N_2 N_3^2 \\
\langle \phi_{jka} \bar{\phi}^{jkb} \phi_{lmb} \bar{\phi}^{lmc} \phi_{pqc} \bar{\phi}^{pqa} \rangle &= 3N_1^2 N_2^2 N_3^2 + N_1 N_2 N_3^3 + N_1^3 N_2^3 N_3 + N_1 N_2 N_3 \\
\langle \phi_{jka} \bar{\phi}^{jkb} \phi_{lmb} \bar{\phi}^{lmc} \phi_{pqc} \bar{\phi}^{pqd} \phi_{rsd} \bar{\phi}^{rsa} \rangle &= N_1^4 N_2^4 N_3 + 5N_1^2 N_2^2 N_3 + 5N_1 N_2 N_3^2 \\
&\quad + 6N_1^3 N_2^3 N_3^2 + 6N_1^2 N_2^2 N_3^3 + N_1 N_2 N_3^4
\end{aligned} \tag{C.22}$$

Now consider correlators in the matrix model. Assuming the basic contraction

$$\langle Z_j^i \bar{Z}_l^k \rangle = \delta_l^i \delta_j^k \tag{C.23}$$

we easily find

$$\begin{aligned}
\langle \mathrm{Tr}(Z\bar{Z}Z\bar{Z}) \rangle &= 2N^3 \\
\langle \mathrm{Tr}(Z\bar{Z}Z\bar{Z}Z\bar{Z}) \rangle &= 5N^4 + N^2 \\
\langle \mathrm{Tr}(Z\bar{Z}Z\bar{Z}Z\bar{Z}Z\bar{Z}) \rangle &= 10N^3 + 14N^5
\end{aligned} \tag{C.24}$$

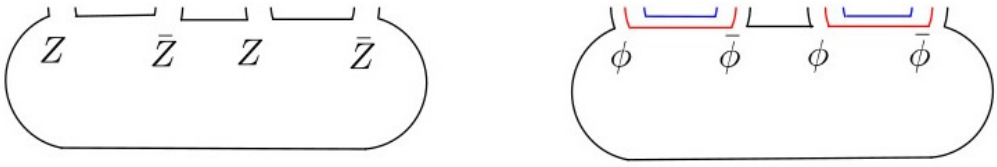


Figure C.2: $\mathrm{Tr}\,(Z\bar{Z}Z\bar{Z})$ is shown on the left and $\phi_{jka}\bar{\phi}^{jkb}\phi_{lmb}\bar{\phi}^{lma}$ on the right.

Notice that if we set $N_1N_2 = N$ and $N_3 = N$, the tensor model correlators map into the matrix model correlators. Its simple to understand this rule diagrammatically: the correspondence holds diagram by diagram. Consider the operators shown in Figure C.2

The black line in the tensor model operator is associated to $U(N_3)$, the red line to $U(N_1)$ and the blue line to $U(N_2)$. One Feynman diagram contributing to the correlator is shown in Figure C.3. Notice that in the tensor model graph, the red and blue lines follow each other, and if they are collapsed into a single black loop, we recover the matrix model diagram. This is the explanation for the rule we have found.

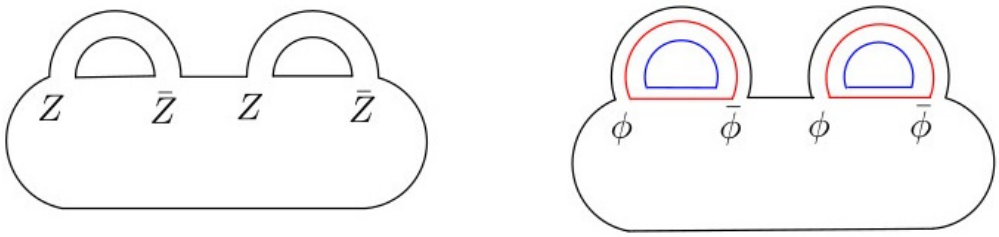


Figure C.3: A Feynman diagram contributing to $\langle \mathrm{Tr}\,(Z\bar{Z}Z\bar{Z})\rangle$ is shown on the left and to $\langle \phi_{jka}\bar{\phi}^{jkb}\phi_{lmb}\bar{\phi}^{lma}\rangle$ on the right.

There are a number of interesting results implied by this correspondence between the correlators. For example, factorization in the matrix model immediately implies that correlators of gauge invariant observables in the tensor model factorize at the leading order at large N. The fact that the correspondence holds diagram by diagram immediately implies that the Feynman diagrams contributing to tensor model observables can be identified as triangulations of string world sheets, with the N dependence of the graph fixed by the topology of the worldsheet.

C.4 Some Matrix Model Correlators

In this section we will consider correlators of Schur polynomials of a free Hermittian matrix model. As far as we are aware, these results are new. They are obtained using ideas and formulas developed in [85]. The Schur polynomial is given by

$$\langle \chi_R(M)\rangle = \frac{1}{(2n)!}\sum_{\sigma\in S_{2n}} \chi_R(\sigma)\mathrm{Tr}(\sigma M^{\otimes 2n}) \tag{C.25}$$

where $R \vdash 2n$, i.e. R is a Young diagram with $2n$ boxes. Wick's theorem for the hermittian matrix can be expressed as

$$\langle M^{i_1}_{j_1} M^{i_2}_{j_2} \cdots M^{i_{2n}}_{j_{2n}} \rangle = \sum_{\sigma \in [2^n]} \sigma^I_J \tag{C.26}$$

Using this result, we find that

$$\begin{aligned} \langle \chi_R(M) \rangle &= \frac{1}{(2n)!} \sum_{\sigma \in S_{2n}} \chi_R(\sigma) \langle \mathrm{Tr}(\sigma M^{\otimes 2n}) \rangle \\ &= \frac{1}{(2n)!} \sum_{\sigma \in S_{2n}} \sum_{\psi \in [2^n]} \chi_R(\sigma) \mathrm{Tr}(\sigma \psi) \\ &= \frac{1}{(2n)!} \sum_{\sigma \in S_{2n}} \sum_{\psi \in [2^n]} \sum_{T \vdash 2n} \chi_R(\sigma) \chi_T(\sigma \psi) \mathrm{Dim}_T \\ &= \frac{1}{(2n)!} \sum_{\sigma \in S_{2n}} \chi_R(\sigma) f_R \end{aligned} \tag{C.27}$$

For the case that R is a single column with $2n = N$ this becomes

$$\langle \chi_R(M) \rangle = \langle \chi_{(1^N)}(M) \rangle = (-1)^n (2n-1)!! \tag{C.28}$$

For the case that R is a single row with $2n = N$ this becomes

$$\langle \chi_R(M) \rangle = \langle \chi_{(N)}(M) = (2n-1)!! \, \frac{(4n-1)!}{(2n)! \, (2n-1)!} \tag{C.29}$$

The two point function of normal ordered operators is

$$\langle : \chi_R(M) :: \chi_S(M) : \rangle = \delta_{RS} f_R \tag{C.30}$$

We now introduce the normalized operators

$$O_{(1^N)} \equiv \frac{\chi_{(1^N)}}{(N-1)!!} \qquad O_{(N)} \equiv \frac{N! \, (N-1)!}{(N-1)!! \, (2N-1)!} \chi_{(N)} \tag{C.31}$$

The two point functions of these correlators are

$$\begin{aligned} \langle : O_{(1^N)} :: O_{(1^N)} : \rangle &= \sqrt{\frac{N\pi}{2}} \\ \langle : O_{(N)} :: O_{(N)} : \rangle &= \sqrt{2} N \pi e^{-N \log 4} \end{aligned} \tag{C.32}$$

Notice that both of these results are non-perturbative in N. As explained in Section 4.4, the analog of these correlators in the tensor model have a different behavior.

References

[1] G. t' Hooft, "A planar diagram theory for strong interactions," *Nuclear Physics, Section B*, vol. 72, no. 3, pp. 461–473, 1974.

[2] J. M. Maldacena, "The Large N Limit of Superconformal Field Theories and Supergravity," *Adv. Theor. Math. Phys.*, vol. 2, p. 20, nov 1997.

[3] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge Theory Correlators from Non-Critical String Theory," *Physical Letters B*, vol. 428, no. May, p. 15, 1998.

[4] E. Witten, "Anti-de Sitter space and holography," *Adv. Theor. Math. Phys.*, vol. 2, pp. 253–291, 1998.

[5] V. Balasubramanian, M. Berkooz, A. Naqvi, and M. J. Strassler, "Giant Gravitons in Conformal Field Theory," *JHEP*, vol. 0204, 2002.

[6] J. McGreevy, L. Susskind, and N. Toumbas, "Invasion of the Giant Gravitons from Anti de Sitter Space," *JHEP*, vol. 0006, pp. 0–14, 2000.

[7] A. Hashimoto, S. Hirano, and N. Itzhaki, "Large branes in AdS and their field theory dual," *Journal of High Energy Physics*, vol. 2000, pp. 051–051, aug 2000.

[8] M. T. Grisaru, R. C. Myers, and O. Tafjord, "SUSY and Goliath," *Journal of High Energy Physics*, vol. 040, no. 08, pp. 040–040, 2000.

[9] H. Lin, O. Lunin, and J. Maldacena, "Bubbling AdS space and 1/2 BPS geometries," *Journal of High Energy Physics*, vol. 2004, pp. 025–025, oct 2004.

[10] S. Corley, A. Jevicki, and S. Ramgoolam, "Exact correlators of giant gravitons from dual N=4 SYM theory," *Adv. Theor. Math. Phys.*, vol. 5, pp. 809–839, 2002.

[11] S. Corley and S. Ramgoolam, "Finite factorization equations and sum rules for BPS correlators in N=4 SYM theory," *Nuclear Physics B*, vol. 641, pp. 131–187, oct 2002.

[12] D. Berenstein, "A toy model for the AdS/CFT correspondence," *Journal of High Energy Physics*, vol. 018, no. 0407, pp. 018–018, 2004.

[13] A. Jevicki and B. Sakita, "The quantum collective field method and its application to the planar limit," *Nuclear Physics, Section B*, vol. 511, no. 165, pp. 511–527, 1980.

[14] A. Jevicki and B. Sakita, "Collective field approach to the large-N limit: Euclidean field theories," *Nuclear Physics, Section B*, vol. 89, no. 89, pp. 89–100, 1981.

[15] S. R. Das and A. Jevicki, "String Field Theory and Physical Interpretation of $D = 1$ Strings," *Mod. Phys. Lett.*, vol. A5, pp. 1639–1650, 1990.

[16] V. Balasubramanian, D. Berenstein, B. Feng, and M. x. Huang, "D-branes in Yang-Mills theory and emergent gauge symmetry," *JHEP*, vol. 006, no. 0503, 2005.

[17] Y. Kimura, "Correlation functions and representation bases in free N=4 Super Yang-Mills," *Nucl. Phys.*, vol. B865, pp. 568–594, 2012.

[18] R. d. M. Koch, P. Diaz, and H. Soltanpanahi, "Non-planar Anomalous Dimensions in the sl(2) Sector," *Physics Letters B*, vol. 713, pp. 509–513, nov 2011.

[19] R. de Mello Koch, P. Diaz, and N. Nokwara, "Restricted Schur polynomials for fermions and integrability in the su(2—3) sector," *Journal of High Energy Physics*, vol. 2013, p. 173, mar 2013.

[20] R. d. M. Koch, M. Dessein, D. Giataganas, and C. Mathwin, "Giant Graviton Oscillators," *Journal of High Energy Physics*, vol. 2011, p. 9, aug 2011.

[21] W. Carlson, R. de Mello Koch, and H. Lin, "Nonplanar Integrability," *JHEP*, vol. 03, p. 105, 2011.

[22] R. de Mello Koch, G. Kemp, and S. Smith, "From Large N Nonplanar Anomalous Dimensions to Open Spring Theory," *Phys. Lett.*, vol. B711, pp. 398–403, 2012.

[23] R. d. M. Koch and S. Ramgoolam, "A double coset ansatz for integrability in AdS/CFT," *Journal of High Energy Physics*, vol. 2012, p. 83, apr 2012.

[24] R. d. M. Koch, J. Smolic, and M. Smolic, "Giant gravitons with strings attached (I)," *Journal of High Energy Physics*, vol. 2007, pp. 074–074, jun 2007.

[25] R. de Mello Koch, N. H. Tahiridimbisoa, and C. Mathwin, "Anomalous Dimensions of Heavy Operators from Magnon Energies," *JHEP*, vol. 03, p. 156, 2016.

[26] R. de Mello Koch and J. P. Rodrigues, "Systematic 1/N corrections for bosonic and fermionic vector models without auxiliary fields," *Phys. Rev.*, vol. D54, pp. 7794–7814, 1996.

[27] S. R. Das and A. Jevicki, "Large N collective fields and holography," *Phys. Rev.*, vol. D68, p. 044011, 2003.

[28] I. R. Klebanov and A. M. Polyakov, "AdS dual of the critical O(N) vector model," *Phys. Lett.*, vol. B550, pp. 213–219, 2002.

[29] R. de Mello Koch, A. Jevicki, K. Jin, and J. P. Rodrigues, "AdS_4/CFT_3 Construction from Collective Fields," *Phys. Rev.*, vol. D83, p. 025006, 2011.

[30] R. de Mello Koch, A. Jevicki, J. P. Rodrigues, and J. Yoon, "Canonical Formulation of $O(N)$ Vector/Higher Spin Correspondence," *J. Phys.*, vol. A48, no. 10, p. 105403, 2015.

[31] K. S. A. Jevicki and J. Yoon, "Bi-local holography in the syk model," *JHEP* **1607**, *007*, 2016.

[32] A. Jevicki and K. Suzuki, "Bi-local holography in the syk model: perturbations," *Journal of High Energy Physics*, vol. 2016, p. 46, Nov 2016.

[33] S. R. Das, A. Jevicki, and K. Suzuki, "Three Dimensional View of the SYK/AdS Duality," *JHEP*, vol. 09, p. 017, 2017.

[34] W. Fu, D. Gaiotto, J. Maldacena, and S. Sachdev, "Supersymmetric Sachdev-Ye-Kitaev models," *Phys. Rev.*, vol. D95, no. 2, p. 026009, 2017. [Addendum: Phys. Rev.D95,no.6,069904(2017)].

[35] C. Peng, M. Spradlin, and A. Volovich, "A Supersymmetric SYK-like Tensor Model," *JHEP*, vol. 05, p. 062, 2017.

[36] C. Peng, M. Spradlin, and A. Volovich, "Correlators in the $\mathcal{N} = 2$ Supersymmetric SYK Model," *JHEP*, vol. 10, p. 202, 2017.

[37] J. Murugan, D. Stanford, and E. Witten, "More on Supersymmetric and 2d Analogs of the SYK Model," *JHEP*, vol. 08, p. 146, 2017.

[38] J. Yoon, "Supersymmetric SYK Model: Bi-local Collective Superfield/Supermatrix Formulation," *JHEP*, vol. 10, p. 172, 2017.

[39] R. de Mello Koch, D. Gossman, and L. Tribelhorn, "Gauge invariants, correlators and holography in bosonic and fermionic tensor models," *Journal of High Energy Physics*, vol. 2017, 07 2017.

[40] R. de Mello Koch, D. Gossman, N. H. Tahiridimbisoa, and A. L. Mahu, "Holography for tensor models," 2019.

[41] E. Brézin, C. Itzykson, G. Parisi, and J. B. Zuber, "Planar diagrams," *J. Math. Phys.*, vol. 35, no. 59, pp. 35–51, 1978.

[42] J. Ginibre, "Statistical Ensembles of Complex, Quaternion, and Real Matrices," *Journal of Mathematical Physics*, vol. 6, no. 440, 1965.

[43] M. Hammermesh, "Group Theory and Its Application to Physical Problems," 1963.

[44] R. Bhattacharyya, R. d. M. Koch, and M. Stephanou, "Exact multi-restricted Schur polynomial correlators," *Journal of High Energy Physics*, vol. 2008, pp. 101–101, jun 2008.

[45] R. Bhattacharyya, S. Collins, and R. d. M. Koch, "Exact multi-matrix correlators," *Journal of High Energy Physics*, vol. 2008, pp. 044–044, mar 2008.

[46] G. 't Hooft, "Dimensional reduction in quantum gravity," *Conf. Proc.*, vol. C930308, pp. 284–296, 1993.

[47] L. Susskind, "The world as a hologram," *Journal of Mathematical Physics*, vol. 36, no. 11, pp. 6377–6396, 1995.

[48] E. D'Hoker and D. Z. Freedman, "Supersymmetric gauge theories and the ads/cft correspondence," 2002.

[49] H. Nastase, *Introduction to the AdS/CFT Correspondence*. Cambridge University Press, 2015.

[50] A. R. Brown and L. Susskind, "The Second Law of Quantum Complexity," 2017.

[51] L. Susskind, "Why do Things Fall?," 2018.

[52] A. Brown, H. Gharibyan, A. Streicher, L. Susskind, and L. Thorlacius, "Falling Toward Charged Black Holes," 2018.

[53] J. Maldacena, "Fuzz or fire talk," KITP "Fuzz or Fire" conference, (http://online.kitp.ucsb.edu/online/fuzzorfire-m13/maldacena/), 2013.

[54] D. A. Roberts, D. Stanford, and L. Susskind, "Localized shocks," *Journal of High Energy Physics*, vol. 2015, no. 3, pp. 1–27, 2015.

[55] S. Sachdev and J. Ye, "Gapless spin-fluid ground state in a random quantum heisenberg magnet," *Phys. Rev. Lett.*, vol. 70, pp. 3339–3342, May 1993.

[56] A. Kitaev, "A simple model of quantum holography," KITP strings seminar and Entanglement 2015 program, (http://online.kitp.ucsb.edu/online/entangled15/), 2015.

[57] J. Maldacena, S. H. Shenker, and D. Stanford, "A bound on chaos," *Journal of High Energy Physics*, vol. 2016, p. 106, Aug 2016.

[58] J. Polchinski and V. Rosenhaus, "The spectrum in the sachdev-ye-kitaev model," *Journal of High Energy Physics*, vol. 2016, p. 1, Apr 2016.

[59] D. J. Gross and V. Rosenhaus, "All point correlation functions in syk," *Journal of High Energy Physics*, vol. 2017, p. 148, Dec 2017.

[60] J. Maldacena and D. Stanford, "Remarks on the sachdev-ye-kitaev model," *Phys. Rev. D*, vol. 94, p. 106002, Nov 2016.

[61] E. Witten, "An SYK-like model without disorder," *Journal of Physics A: Mathematical and Theoretical*, vol. 52, p. 474002, oct 2019.

[62] R. Gurau, "The complete 1/n expansion of a syk–like tensor model," *Nuclear Physics B*, vol. 916, pp. 386 – 401, 2017.

[63] I. R. Klebanov and G. Tarnopolsky, "Uncolored random tensors, melon diagrams, and the sachdev-ye-kitaev models," *Phys. Rev. D*, vol. 95, p. 046004, Feb 2017.

[64] F. Ferrari, "The Large D Limit of Planar Diagrams," 2017.

[65] H. Itoyama, A. Mironov, and A. Morozov, "Rainbow tensor model with enhanced symmetry and extreme melonic dominance," *Phys. Lett.*, vol. B771, pp. 180–188, 2017.

[66] S. Carrozza and A. Tanasa, "$O(N)$ Random Tensor Models," *Lett. Math. Phys.*, vol. 106, no. 11, pp. 1531–1559, 2016.

[67] J. Yoon, "SYK Models and SYK-like Tensor Models with Global Symmetry," *JHEP*, vol. 10, p. 183, 2017.

[68] R. Gurau, "Invitation to Random Tensors," *SIGMA*, vol. 12, p. 094, 2016.

[69] R. Gurau, "The 1/N expansion of colored tensor models," *Annales Henri Poincare*, vol. 12, pp. 829–847, 2011.

[70] J. Ben Geloun and D. O. Samary, "3D Tensor Field Theory: Renormalization and One-loop β-functions," *Annales Henri Poincare*, vol. 14, pp. 1599–1642, 2013.

[71] J. Ben Geloun, "Two and four-loop β-functions of rank 4 renormalizable tensor field theories," *Class. Quant. Grav.*, vol. 29, p. 235011, 2012.

[72] V. Rivasseau, "Why are tensor field theories asymptotically free?," *EPL*, vol. 111, no. 6, p. 60011, 2015.

[73] V. Bonzom, R. Gurau, A. Riello, and V. Rivasseau, "Critical behavior of colored tensor models in the large N limit," *Nucl. Phys.*, vol. B853, pp. 174–195, 2011.

[74] J. Ambjorn, B. Durhuus, and T. Jonsson, "Summing over all genera for $d > 1$: a toy model," *Physics Letters B*, vol. 244, no. 3, pp. 403 – 412, 1990.

[75] V. Bonzom, R. Gurau, and V. Rivasseau, "Random tensor models in the large N limit: Uncoloring the colored tensor models," *Phys. Rev.*, vol. D85, p. 084037, 2012.

[76] V. Bonzom and H. Erbin, "Coupling of hard dimers to dynamical lattices via random tensors," *J. Stat. Mech.*, vol. 1209, p. P09009, 2012.

[77] V. Bonzom, "Multicritical tensor models and hard dimers on spherical random lattices," *Phys. Lett.*, vol. A377, pp. 501–506, 2013.

[78] V. A. Kazakov, "The Appearance of Matter Fields from Quantum Fluctuations of 2D Gravity," *Mod. Phys. Lett.*, vol. A4, p. 2125, 1989. [,2125(1989)].

[79] A. Jevicki and S. Ramgoolam, "Noncommutative gravity from the AdS / CFT correspondence," *JHEP*, vol. 04, p. 032, 1999.

[80] R. de Mello Koch, A. Jevicki, and J. P. Rodrigues, "Collective string field theory of matrix models in the BMN limit," *Int. J. Mod. Phys.*, vol. A19, pp. 1747–1770, 2004.

[81] R. de Mello Koch, A. Donos, A. Jevicki, and J. P. Rodrigues, "Derivation of string field theory from the large N BMN limit," *Phys. Rev.*, vol. D68, p. 065012, 2003.

[82] A. Donos, A. Jevicki, and J. P. Rodrigues, "Matrix model maps in AdS/CFT correspondence," *Physical Review D*, vol. 72, no. 12, 2005.

[83] J. Ben Geloun and S. Ramgoolam, "Counting Tensor Model Observables and Branched Covers of the 2-Sphere," 2013.

[84] A. Mironov and A. Morozov, "Correlators in tensor models from character calculus," *Phys. Lett.*, vol. B774, pp. 210–216, 2017.

[85] R. de Mello Koch and S. Ramgoolam, "From Matrix Models and Quantum Fields to Hurwitz Space and the absolute Galois Group," 2010.

[86] P. Diaz and S.-J. Rey, "Orthogonal Bases of Invariants in Tensor Models," *JHEP*, vol. 02, p. 089, 2018.

[87] M. W. Hero and J. F. Willenbring, "Stable hilbert series as related to the measurement of quantum entanglement," *Discrete Mathematics*, vol. 309, no. 23, pp. 6508 – 6514, 2009.

[88] J. Pasukonis and S. Ramgoolam, "Quivers as Calculators: Counting, Correlators and Riemann Surfaces," *JHEP*, vol. 04, p. 094, 2013.

[89] P. Mattioli and S. Ramgoolam, "Permutation Centralizer Algebras and Multi-Matrix Invariants," *Phys. Rev.*, vol. D93, no. 6, p. 065040, 2016.

[90] A. Jevicki, "Nonperturbative collective field theory," *Nucl. Phys.*, vol. B376, pp. 75–98, 1992.

[91] M. Masuku and J. P. Rodrigues, "Laplacians in polar matrix coordinates and radial fermionization in higher dimensions," *Journal of Mathematical Physics*, vol. 52, p. 032302, mar 2011.

[92] M. Masuku and J. P. Rodrigues, "How universal is the Wigner distribution?," *Journal of Physics A: Mathematical and Theoretical*, vol. 45, p. 085201, mar 2012.

[93] M. Masuku, M. Mulokwe, and J. P. Rodrigues, "Large N matrix hyperspheres and the gauge-gravity correspondence," *Journal of High Energy Physics*, vol. 2015, p. 35, dec 2015.

[94] M. Masuku and J. P. Rodrigues, "De Alfaro, Fubini and Furlan from multi matrix systems," *Journal of High Energy Physics*, vol. 2015, p. 175, dec 2015.

[95] M. A. Vasiliev, "From Coxeter Higher-Spin Theories to Strings and Tensor Models," *JHEP*, vol. 08, p. 051, 2018.

[96] N. Delporte and V. Rivasseau, "The Tensor Track V: Holographic Tensors," 2018.

[97] I. R. Klebanov and G. Tarnopolsky, "On Large N Limit of Symmetric Traceless Tensor Models," *JHEP*, vol. 10, p. 037, 2017.

[98] K. Bulycheva, I. R. Klebanov, A. Milekhin, and G. Tarnopolsky, "Spectra of Operators in Large N Tensor Models," *Phys. Rev.*, vol. D97, no. 2, p. 026016, 2018.

[99] S. Giombi, I. R. Klebanov, F. Popov, S. Prakash, and G. Tarnopolsky, "Prismatic Large N Models for Bosonic Tensors," *Phys. Rev.*, vol. D98, no. 10, p. 105005, 2018.

[100] I. R. Klebanov, F. Popov, and G. Tarnopolsky, "TASI Lectures on Large N Tensor Models," *PoS*, vol. TASI2017, p. 004, 2018.

[101] F. Ferrari, V. Rivasseau, and G. Valette, "A New Large N Expansion for General Matrix–Tensor Models," *Commun. Math. Phys.*, vol. 370, no. 2, pp. 403–448, 2019.

[102] R. Pascalie, C. I. Pérez-Sánchez, A. Tanasa, and R. Wulkenhaar, "On the large N limit of Schwinger-Dyson equations of a rank-3 tensor field theory," *J. Math. Phys.*, vol. 60, no. 7, p. 073502, 2019.

[103] D. Benedetti, R. Gurau, and S. Harribey, "Line of fixed points in a bosonic tensor model," *JHEP*, vol. 06, p. 053, 2019.

[104] B. Sundborg, "The Hagedorn transition, deconfinement and N=4 SYM theory," *Nucl. Phys.*, vol. B573, pp. 349–363, 2000.

[105] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas, and M. Van Raamsdonk, "The Hagedorn - deconfinement phase transition in weakly coupled large N gauge theories," *Adv. Theor. Math. Phys.*, vol. 8, pp. 603–696, 2004. [,161(2003)].

[106] D. Gaiotto, L. Rastelli, and S. S. Razamat, "Bootstrapping the superconformal index with surface defects," *JHEP*, vol. 01, p. 022, 2013.

[107] H.-C. Kim, S.-S. Kim, and K. Lee, "5-dim Superconformal Index with Enhanced En Global Symmetry," *JHEP*, vol. 10, p. 142, 2012.

[108] F. A. Dolan, "Counting BPS operators in N=4 SYM," *Nucl. Phys.*, vol. B790, pp. 432–464, 2008.

[109] R. de Mello Koch and S. Ramgoolam, "Strings from Feynman Graph counting : without large N," *Phys. Rev.*, vol. D85, p. 026007, 2012.

[110] D. E. Berenstein, J. M. Maldacena, and H. S. Nastase, "Strings in flat space and pp waves from N=4 superYang-Mills," *JHEP*, vol. 04, p. 013, 2002.

[111] J. B. Geloun and S. Ramgoolam, "Tensor models, kronecker coefficients and permutation centralizer algebras," *Journal of High Energy Physics*, vol. 2017, p. 92, Nov 2017.

[112] Y. Kimura and S. Ramgoolam, "Branes, anti-branes and brauer algebras in gauge-gravity duality," *JHEP*, vol. 11, p. 078, 2007.

[113] T. W. Brown, P. J. Heslop, and S. Ramgoolam, "Diagonal multi-matrix correlators and BPS operators in N=4 SYM," *JHEP*, vol. 02, p. 030, 2008.

[114] T. W. Brown, P. J. Heslop, and S. Ramgoolam, "Diagonal free field matrix correlators, global symmetries and giant gravitons," *JHEP*, vol. 04, p. 089, 2009.

[115] A. Jevicki and J. P. Rodrigues, "Loop space hamiltonians and field theory of non-critical strings," *Nuclear Physics B*, vol. 421, no. 2, pp. 278 – 292, 1994.

[116] R. Gurau, "A generalization of the virasoro algebra to arbitrary dimensions," *Nuclear Physics B - NUCL PHYS B*, vol. 852, 05 2011.

[117] R. Gurau, "The Schwinger Dyson equations and the algebra of constraints of random tensor models at all orders," *Nucl. Phys.*, vol. B865, pp. 133–147, 2012.

[118] V. Bonzom, "Revisiting random tensor models at large N via the Schwinger-Dyson equations," *JHEP*, vol. 03, p. 160, 2013.

[119] H. Itoyama, A. Mironov, and A. Morozov, "Cut and join operator ring in tensor models," *Nucl. Phys.*, vol. B932, pp. 52–118, 2018.

[120] H. Itoyama, A. Mironov, and A. Morozov, "From Kronecker to tableau pseudo-characters in tensor models," *Phys. Lett.*, vol. B788, pp. 76–81, 2019.

[121] O. Aharony, Y. E. Antebi, M. Berkooz, and R. Fishman, "'Holey sheets': Pfaffians and subdeterminants as D-brane operators in large N gauge theories," *JHEP*, vol. 12, p. 069, 2002.

[122] D. Berenstein, "Shape and holography: Studies of dual operators to giant gravitons," *Nucl. Phys.*, vol. B675, pp. 179–204, 2003.

[123] R. de Mello Koch, E. Gandote, and J.-H. Huang, "Non-Perturbative String Theory from AdS/CFT," *JHEP*, vol. 02, p. 169, 2019.

[124] P. Diaz and S.-J. Rey, "Invariant Operators, Orthogonal Bases and Correlators in General Tensor Models," *Nucl. Phys.*, vol. B932, pp. 254–277, 2018.

[125] P. Diaz, "Tensor and matrix models: a one-night stand or a lifetime romance?," *Journal of High Energy Physics*, vol. 2018, p. 140, Jun 2018.

[126] R. de Mello Koch, J. Smolic, and M. Smolic, "Giant Gravitons - with Strings Attached (II)," *JHEP*, vol. 09, p. 049, 2007.

[127] N. Delporte and V. Rivasseau, "Perturbative Quantum Field Theory on Random Trees," 2019.