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GROUP CLASSIFICATION OF COUPLED PARTIAL DIFFERENTIAL EQUATIONS WITH APPLICATIONS TO FLOW IN A COLLAPSIBLE CHANNEL AND DIFFUSION PROCESSES

MOTLATSI MOLATI

GROUP CLASSIFICATION OF COUPLED PARTIAL DIFFERENTIAL EQUATIONS WITH APPLICATIONS TO FLOW IN A COLLAPSIBLE CHANNEL AND DIFFUSION PROCESSES

MOTLATSI MOLATI

A thesis submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in fulfilment of the requirements for the degree of Doctor of Philosophy.

Johannesburg 2010

DECLARATION

I declare that the contents of this thesis are original except where due reference has been made. It is being submitted for the Degree of Doctor of Philosophy in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

_____

M. Molati

_____ day of _____ 20_____

ABSTRACT

The main purpose of this work is to perform the symmetry classification of the coupled systems of partial differential equations modelling flow in a collapsible tube and diffusion phenomenon. The idea of symmetry is exploited in the two models of paramount importance in applications: the one-dimensional model of flow through a collapsible tube and the coupled diffusion system. These models contain several unknown functions (tube law, friction, diffusion coefficients etc.) the forms of which are specified via the method of group classification.

The solution procedure to carry out the group classification involves the use of two approaches. In both of them the first part of the classification procedure deals with the execution of the usual approach of Lie to obtain the general form of the symmetry generator and the determining equations for the underlying models. The second part is concerned with the utilization of the structure of the low-dimensional Lie algebras and the Lie algebras of higher dimension to find the symmetry operators admitted by the given models. The procedure is continued until the functional forms of the unknown functions are completely specified. The latter part is different for each approach. In one approach no use of the equivalence transformations is required whereas for the other approach the equivalence group is a necessary component. In the former the classification with respect to the subalgebras of three- and four-dimensional Lie algebras is used. Thereafter the symmetry analysis of every case arising from the classification is performed in order to determine the maximal Lie symmetry algebra. On the other hand the equivalence group is used to obtain the canonical forms of the symmetry operators which satisfy the models under consideration. The canonical forms of the low-dimensional Lie algebras and the Lie algebras of higher dimension provide a means to specify the arbitrary functions of the models.

The secondary aim is to employ the symmetries to construct the invariant solutions wherever applicable. The solution of the optimal system problem allows the classification of all the invariant solutions, i.e. solutions that are left unchanged by subalgebras of the

symmetry Lie algebra. The invariant solutions of a given equation satisfy an equation with a reduced number of the independent variables.

DEDICATION

To my wife Vicky, my son Tlholo, our families and our well-wishers

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The administrative staff of the School of Computational and Applied Mathematics (CAM) should also be applauded for their hospitality and assistance in related matters. I also thank the entire academic and nonacademic staff of the Department of Mathematics and Computer Science, National University of Lesotho (NUL). Their support in many things is appreciated; one of which was to grant me permission to use the department's facilities on vacations during my study leave.

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INTRODUCTION

Several real-world models are formulated in terms of partial differential equations (PDEs). Quite often these models involve parameters, arbitrary elements or functions which may not be straightforwardly determined through experiment. In such situations the ability to predict the results becomes the sole test for the suitability of unknown parameters. The *symmetry principle* [10, 65, 67] which presumes that nature prefers maximally symmetric models has proven to be a powerful tool in determining these unknown parameters appearing in physical models. For instance *Newton's inverse square law* can be obtained solely from the symmetry principle. Once the unknown parameters of a model are determined, the next logical step is to find its solutions given a set of initial or boundary conditions. Depending upon the nature of the problem being investigated, one might be interested in obtaining numerical or analytical solutions and to compare them with experimental results.

Many branches of mathematics deal with techniques for obtaining solutions for PDEs. From the late 1950s Lie group analysis, also known as Lie group theory or Lie symmetry analysis of differential equations (DEs) and its extensions, has proven to be a powerful technique in finding the closed-form solutions of DEs. The Lie group theory combines analysis and algebra, and was initiated by the Norwegian mathematician Sophus Lie.

Marius Sophus Lie was born on December 17, 1842, in Nordfjordied, Norway. He was the penultimate child of the seven children of the protestant minister Johan Herman Lie and Mette Maren, née Stabell. While at university Lie attended lectures on the works of Abel and Galois on algebraic equations given by Ludvig Sylow. It became clear during the year 1868 which branch of mathematics that Lie would follow after studying papers on geometry by Julius Plüker and Jean Victor Poncelet. In 1869 during his visit to Berlin Lie met Felix Klein, a student of Plüker, and later collaborated with him. Their collaboration on transformation groups was fruitful and would later result in publication of several papers including Klein's *Erlangen Program* [38]. In 1870 Lie and Klein met again in Paris. It was during his temporary stay in Paris that Lie discovered contact

transformations. Their collaboration spanned about three years from 1869. From 1871 Lie began the study of PDEs using his new ideas on geometry and towards the end of 1874 Lie's theory of continuous transformation groups was born.

After having studied the theory of algebraic equations by Évariste Galois (a French mathematician who continued the work of Lie's countryman, Niels Henrik Abel), Lie began to explore PDEs. His dream was to develop a similar theory for DEs. In a letter dated 1874 to his friend Adolph Mayer Lie wrote [7]

> In the theory of algebraic equations before Galois only these questions were proposed: Is an equation solvable by radicals, and how is it to be solved? Since Galois, among other questions proposed is this: How is an equation to be solved by radicals in the *simplest* way possible? I believe the time is come to make a similar progress in differential equations.

The dream of Lie in developing a 'Galois theory' for differential equations could not come to pass. Émile Picard who was one of the French mathematicians who got interested in Lie's theory of continuous groups, together with his colleagues sent Arthur Tresse and Ernest Vessiot to study under Lie for a semester. It was years later after Lie's death that his ideas became a basis for Picard and Vessiot to somehow generalize Galois theory to DEs. Hawkins [26, 27] gives an in-depth study on how the theory of Lie groups originated.

Lie's seminal work on Lie groups (originally known as theory of continuous transformation groups) was not widely known and accepted in the mathematical community of his time. In September, 1884, Klein and Mayer sent their able student, Friedrich Engel, to help Lie in the edition of his work. This collaboration culminated in the publication of three volumes of Lie's work under the title *Theorie der Transformationsgruppen* [41] between 1888 and 1893. This was the fundamental work of Lie with the assistance of Engel. The reference [77] gives a complete bibliography of all Lie's published work.

Sophus Lie died on February 18, 1899 and his theories have become tools for understanding and solving problems in all areas of science and engineering today. This is indeed a confirmation of his words [77]:

> I am certain, absolutely certain that ... these theories will be recognized as fundamental at some point in the future.

The testimony below is a summary of his life and work [22]:

> My life is actually quite incomprehensible for me. As a young man, I had no idea that I was blessed with originality. Then, as a 26-year-old, I suddenly realized that I could create. I read a little and began to produce. In these years, 1869–1874, I had a lot of ideas which, in the course of time, I had developed only very imperfectly.
>
> In particular, it was group theory and its great importance for the differential equations which interested me. But publications in this area went woefully slow. I could not structure it properly, and I was always afraid of making mistakes. Not the small inessential mistakes ... No, it was the deep-rooted errors I feared. I am glad that my group theory in its present state doesn't contain any fundamental errors.

This is a brief historical background on Sophus Lie. More can be found in [7, 22, 87, 88, 89]. There are many great mathematicians not mentioned who contributed to his work and those who were (and still influenced) by his theories. A more detailed account on the life and work of Lie is given in Stubhaug [77] and the references therein.

The theory of Lie groups is a *local* theory. In simple terms it means that the group of transformations is only defined in the neighbourhood of the identity transformation. Lie discovered that the better way to understand Lie groups was to investigate the corresponding vector fields (also known as infinitesimal generators or symmetries) and he was able to relate the theory to its infinitesimal group, i.e. its Lie algebra as it is called today in his honour. Working simultaneously but independently of Lie, Wilhelm Killing also studied the relationship between Lie groups and Lie algebras.

The original Lie group theory was extended to a global theory in the 1930s by Élie Cartan and Herman Weyl. Cartan was a former classmate of Tresse in Paris' famous college, *l'École Normale Supérieure*, the institution where most renown French mathematicians

including Galois studied. Cartan improved the work of Killing and built on it after being encouraged to do so by Tresse and later, Weyl in turn built upon the work of Cartan. Emmy Noether [64] used some of these ideas in the early 1900s and succeeded in proving the correspondence between symmetries of a variational problem and the conservation laws of the Euler–Lagrange equations. Chevalley [14] and those that followed continued to work on Lie groups around the middle of the 20th century. From the late 1950s to the 1960s Morozov [56] and Mubarakzyanov [57, 58, 59, 60] worked on the classification of Lie algebras and others that have followed since such as Patera and Winternitz [69], Turkowski [80] and Mahomed [46] just a few to mention.

Around the mid-1980s Lie group theory was made popular in South Africa by P. G. L. Leach and his student, Fazal M. Mahomed[1][45]. Thereafter this area of study has been attracting many of researchers even from other disciplines in which differential equations play important role. Nowadays modern group analysis brings together many researchers in various fields of natural sciences and engineering. Lie symmetry theory plays a pivotal role in fluid or solid mechanics, modern physics, biological and physical systems, stochastic forecasting in finance and meteorology and across all areas in engineering technology.

Despite efforts to popularize Lie's work through publications, the application of Lie group theory to DEs remained dormant until Ovsiannikov [66] revived it in the late 1950s. Thereafter, the Lie group theory has been applied in many problems (described by linear or nonlinear equations) modelling some physical or abstract phenomena. The basic theory and developments in the Lie group analysis of differential equations can be obtained from [9, 10, 32, 33, 65, 67] and the literature that followed, some of which are unpublished lecture notes and theses [20, 43, 46, 66, 81].

The mathematical formulation of symmetries was already present in the theory of algebraic equations as developed by Galois. In fact he introduced and established the concept of *group* (see Stubhaug [77], p.114). Lie discovered that the concept of symmetry could be used to obtain solutions of DEs. He considered only the *local symmetries* (point and contact symmetries). Point symmetries depend only upon independent and dependent

[1]I was introduced to the Lie group theory of differential equations by Prof. F. M. Mahomed about a decade ago. Ever since then my main concentration has been in application of Lie group theory to solve differential equations modelling various phenomena [51, 52, 53, 54].

variables, whereas contact symmetries can also depend upon the first-order derivatives of the dependent variables. From around 1970s until today there has been an interest in exploring *nonlocal symmetries* (see [1, 9, 16, 61] just a few to mention). These are the symmetries that depend on integrals of the dependent variables.

Ever since the revival of the application of Lie group theory to DEs, Lie's work has been developed with many topics having being researched and continue to be researched. These include the following: Generation of new solutions from known ones, linearization of ordinary differential equations (ODEs) and PDEs, construction of equivalence group, solving group classification problems, reductions of PDEs (by invariant or similarity solutions), construction of generalized local symmetries and nonlocal symmetries, solving initial and boundary value problems, approximate symmetries, symmetries of stochastic DEs, symmetries of integro differential equations, symmetries of difference equations, symmetries of functional DEs, symmetries of geodesic equations, construction of conservation laws, construction of invariants of algebraic equations and DEs etc.

Roughly, a symmetry is a change or transformation that leaves an object apparently unchanged. For instance any rotation of a circle about its centre is a symmetry. Different objects can have different degrees of symmetries: intuitively a circle has more symmetries than a rectangle. Therefore symmetry can be employed as a classification criterion. In the case of DEs a symmetry is an invertible transformation of the dependent and independent variables that does not affect the form of the underlying equation or system of equations. Looking at a DE one can deduce symmetries like translations, scalings and rotations. Certain discrete transformations can also be deduced by inspection. In general finding all the symmetries is a difficult task and requires an algorithmic approach. If we consider symmetries that depend continuously upon a one-parameter and that constitute a group, we can use Lie's algorithm to compute them. In this algorithm the symmetry conditions (also known as the *determining equations*) upon expansion yield an over-determined system of linear homogeneous PDEs which are solved for symmetries.

Most problems even of no interest generating the determining equations turns out to be tedious. Fortunately Lie's method for calculating symmetries is algorithmic and can be implemented using packages for symbolic computation (Mathematica, Maple, Reduce,

MuPad, ...). In this work we use the program, `YaLie` [19], written in Mathematica for the generation and manipulation of the determining equations for symmetries.

When the original system contains arbitrary elements (parameters or functions), the consistency conditions of the determining equations provide a means to specify their forms. This is the essence of the group classification method. The importance of group classification stems from the fact that many models in application contain parameters which cannot be determined from any known physical law, but some of them can be found through experiment. However, by analyzing the consistency conditions of the determining equations we are able to identify maximally symmetric submodels. The problem of group classification of a DE is quite difficult compared with just calculating symmetries.

The approach by Ovsiannikov [66, 67] for solving a group classification problem is founded on the idea of an *equivalence transformation* connected to an identity, which is a Lie vector field acting in the extended space of independent variables, dependent variables, functions and their derivatives leaving unchanged the underlying equation. After obtaining the equivalence group, one constructs the *optimal system* [24, 65, 66, 67] of the subgroups of the (equivalence) group. The use of the optimal system reduces the task of searching for all the possible invariant solutions.

The basic or classical Lie technique is not effective when the underlying equation involves arbitrary functions which depend upon both independent and dependent variables, or even the first order derivatives of the dependent variables, that is, the technique would fail to classify the arbitrary functions completely. It is more desirable to perform the complete group classification, but in some problems of interest the analysis of the determining equations can be quite challenging. In such a case one can employ the method of preliminary group classification [21, 29, 30]. The results of the preliminary group classification could then be used to obtain solutions for the known models. However, when the determining equations are difficult to analyze, an approach based upon abstract Lie algebras of low dimension discussed in [5, 86] can be used. Also the classification with respect to low-dimensional Lie algebras was used for ODEs by Mahomed and Leach [47]. Thus depending upon the type of problem at hand solving the problem of group clas-

sification completely requires combining the standard Lie approach and the approach based on the structure of low-dimensional Lie algebras and the Lie algebras of higher dimension. Also the direct approach of Lie can be employed provided the analysis of the determining equations which include those of the arbitrary elements (compatibility conditions) can be handled without much difficulty [51, 82].

We anticipate that the determining equations and the classifying relations for the models investigated in this work may be complicated and difficult to analyze. Therefore we use both the classical Lie approach and the approach based upon the structure of Lie algebras for group classification.

It is well-known in most applications that from a computational standpoint ODEs are easier to solve compared to PDEs. Thus in many applications it is desirable to be able to reduce PDEs to ODEs or at least reduce the number of independent variables. The reduced system is easier to analyze both numerically and analytically than the original system. Therefore, whenever the need arises, the symmetries are used to construct the invariant solutions; the invariant solutions in turn are used to reduce the underlying system of PDEs to the systems of ODEs which could be solved or approximated numerically.

The outline of this work is given below.

The thesis comprises two parts. Part I is devoted to collapsible tube models and Part II deals with systems of coupled diffusion equations. In the preliminary chapter we provide the mathematical techniques that are used to analyze the models under consideration. In other words we present the basic theory to execute Lie's algorithm. This includes notations, definitions, theorems and examples.

Chapters 2 and 3 concentrate upon the group classification of the collapsible tube models. The former is concerned with a general model while the later involves the model which incorporates the tension and curvature of the tube wall. We use the classification with respect to the subalgebras of three- and four-dimensional Lie algebras. Even though this classification provides a way to specify the unknown functions of the model under study, the maximal Lie symmetry algebra is not guaranteed. In order to achieve the maximal symmetry Lie algebra, the group analysis of each of the cases obtained via the

classification is performed.

A general coupled diffusion system and its special case which has applications in soil science are treated in Chapters 4 and 5 respectively. The classification is performed up to an equivalence group, that is, the equivalence transformations are used to find the canonical forms of the symmetry generators (operators) which satisfy the underlying models. As a result the arbitrary functions which appear in the models are specified.

Finally we summarize our investigations and make suggestions for further work.

Chapter 1

Lie-Point Symmetries of Differential Equations

In this Chapter we introduce mathematical techniques that are used throughout this work based upon references [33, 48, 81], that is, we present the basic theory on Lie symmetry method for DEs. We adopt some examples, notations and most of the definitions and theorems from these references. Precisely we discuss Lie's algorithm for the calculation of *infinitesimal generators* or *symmetries*. A special emphasis is put upon the use of symmetries to find *invariant solutions*. Also a brief discussion on the application of the Lie symmetry theory in calculating the equivalence transformations is given. For more knowledge on the basic theory and some advanced concepts on symmetry analysis of DEs the texts [10, 33, 65, 67] can be consulted. The extensive collection of results on Lie group theory of DEs obtained by Lie himself and those that followed who embraced and developed his ideas is given in [32]. The new results and ideas in this field of research can be found in recent publications of which some are given in the Bibliography.

1.1 Differential equations

Definition 1.1.1. A kth-order ($k \geq 1$) system E of s differential equations is defined by

$$E^{\sigma}(x, u, u_{(1)}, \ldots, u_{(k)}) = 0, \quad \sigma = 1, \ldots, s, \tag{1.1}$$

where $u \equiv (u^1, u^2, \ldots, u^m)$ is the dependent vector, $x \equiv (x^1, x^2, \ldots, x^n)$ is the independent vector and $u_{(1)}, u_{(2)}, \ldots, u_{(k)}$ are respectively the collection of all first, second, up to kth-order partial derivatives. In expanded form

$$u_{(1)} = \{u^\alpha_i\}, \quad u_{(2)} = \{u^\alpha_{ij}\}, \ldots, u_{(k)} = \{u^\alpha_{i_1,\cdots,i_k}\},$$

where $\alpha = 1, \ \ldots, \ m; \ i, j, i_1 \cdots, i_k = 1, \ldots, n.$

The equations appearing in (1.1) are of maximal order k. In many applications $s = m$, i.e., the number of equations is equal to the number of unknowns.

Note that the system (1.1) becomes a system of ODEs if $n = 1$. Otherwise it is a system of PDEs.

The following is assumed throughout this thesis:

- The solutions and transformations of all system of DEs are locally analytic functions of their arguments.
- The system E^α is *locally solvable*, i.e, given any point $(x_0, u_0, u_{(1)0}, \ldots, u_{(k)0})$ on the Eq. (1.1) there must pass a solution $u = h(x)$ of E^α.
- The rank of the Jacobian matrix $\left(\dfrac{\partial E^\nu}{\partial x^i} \cdots \dfrac{\partial E^\nu}{\partial u^\alpha_{i_1 \cdots i_k}}\right)$ is s on (1.1).

Definition 1.1.2. A system E is of maximal rank s if it satisfies the *maximal rank condition*.

The maximal rank condition gives an assurance that a system of DEs can be written in *solved form*, meaning that the derivatives can be isolated on the left hand side.

Example 1.1.1. In order to illustrate the maximal rank condition, we calculate the Jacobian matrix for the heat equation above, i.e., if

$$E \equiv u_t - u_{xx} = 0,$$

then the Jacobian matrix with respect to all the variables is

$$\left(\frac{\partial E}{\partial t}, \frac{\partial E}{\partial x}, \frac{\partial E}{\partial u}, \frac{\partial E}{\partial u_t}, \frac{\partial E}{\partial u_x}, \frac{\partial E}{\partial u_{tx}}, \frac{\partial E}{\partial u_{tt}}, \frac{\partial E}{\partial u_{xx}}\right) = (0, 0, 0, 1, 0, 0, 0, -1)$$

which is of rank 1 on $E = 0$. Hence the maximal rank condition is satisfied.

However, the Jacobian matrix for

$$\widetilde{E} \equiv (u_t - u_{xx})^2 = 0$$

is $(0, 0, 0, 0, 0, 0, 0, 0)$ on $\widetilde{E} = 0$, meaning that the maximal rank condition is not satisfied. It is true that $(u_t - u_{xx})^2 = 0$ could be simplified to $u_t - u_{xx} = 0$ which satisfies the maximal rank condition. Thus the maximal rank condition is not a sufficient condition for E to be consistent.

From now throughout the chapter we use system E and system (1.1) interchangeably.

1.2 One-parameter group of transformations

Definition 1.2.1. A symmetry transformation of the system (1.1) is an invertible transformation of the variables x and u, namely

$$\bar{x}^i = f^i(x, u), \ \bar{u}^\alpha = \phi^\alpha(x, u), \quad i = 1, \ldots, n; \ \ \alpha = 1, \ldots, m, \tag{1.2}$$

that leaves (1.1) form-invariant in the new variables $\bar{x}$ and $\bar{u}$, i.e.,

$$E^\sigma(\bar{x}, \bar{u}, \bar{u}_{(1)}, \ldots, \bar{u}_{(k)}) = 0, \quad \sigma = 1, \ldots, s \tag{1.3}$$

whenever (1.1) is satisfied.

Example 1.2.1. The heat equation

$$u_t - u_{xx} = 0$$

is invariant under the transformations $\bar{t} = t$, $\bar{x} = x$, $\bar{u} = au$ for $a \in \mathbb{R}^+$.

The examples of transformations illustrated above are easy to deduce from the corresponding equations, but in general the calculation of symmetry transformations (1.2) admitted by the system (1.1) leads to nonlinear equations which are not easy to solve. However, if we consider symmetries that depend on a small parameter and that form

a *one-parameter group of transformations*, we can 'linearize' these equations and easily solve them: this is an important discovery made by Sophus Lie. The transformation groups can be either *local* or *global*, they can be of continuous, discontinuous and mixed type. This thesis concentrates on continuous one-parameter local groups.

Definition 1.2.2. A set G of transformations

$$T_a : \bar{x}^i = f^i(x, u, a),\ \bar{u}^\alpha = \phi^\alpha(x, u, a),\ i = 1, \ldots, n;\ \alpha = 1, \ldots, m, \tag{1.4}$$

where a is a real parameter which continuously takes values in a neighbourhood $\mathcal{D} \subset \mathbb{R}$ of $a = 0$ and f^i, ϕ^α are differentiable functions, is a *continuous one-parameter (local) Lie group of transformations* in $\mathbb{R}^{n+m}$ provided the group properties of closure, identity and inverses are satisfied, namely:

(i) **Closure**. If $T_a, T_b \in G$ and $a, b \in \mathcal{D}' \subset \mathcal{D}$, then

$$T_a T_b = T_c \in G,\ c = \varphi(a, b) \in \mathcal{D}.$$

(ii) **Identity**. There exists $T_0 \in G$ such that

$$T_0 T_a = T_a T_0 = T_a$$

for any $a \in \mathcal{D}' \subset \mathcal{D}$. T_0 is known as the *identity* of the group.

(iii) **Inverses**. There exists $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that

$$T_a^{-1} T_a = T_a T_a^{-1} = T_0\ ,$$

for any $T_a \in G,\ a \in \mathcal{D}' \subset \mathcal{D}$.

In the sequel we use the term 'a one-parameter group' to mean a continuous one-parameter (local) group.

Example 1.2.2. The set of transformations

$$T_a : \quad \bar{x} = (1 + a)x,\ \bar{u} = (1 + a)u, \quad a \in \mathbb{R}^+,$$

forms a one-parameter group.

The composition of the above transformations is

$$\bar{\bar{x}} = (1+b)\bar{x} = (1+b)(1+a)x = (1+a+b+ab)x.$$

Similarly

$$\bar{\bar{u}} = (1+a+b+ab)u.$$

Therefore

$$\bar{\bar{x}} = (1+c)x,\ \bar{\bar{u}} = (1+c)u,$$

where $c = \varphi(a,b) = a+b+ab$.

The closure property is established since $T_b T_a = T_c \in G,\ c = a+b+ab > -1$. Since $\bar{x} = x$, $\bar{u} = u$ when $a = 0$, T_0 is the identity. It can be shown that the group property of inverses hold. Hence, T_a is a one-parameter group known as the *scaling group*.

The choice of the parameter of a one-parameter group is not unique. Different choices of this parameter might produce different composition laws. It is desirable to have an additive composition law.

Definition 1.2.3. A group parameter a is *canonical* if the group composition law is additive, i.e. $\varphi(a,b) = a+b$.

The following theorem shows that a canonical parameter can always be constructed.

Theorem 1.2.1. For any composition law $\varphi(a,b)$, there exists a canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{da'}{\chi(a')}\ ,$$

where

$$\chi(a') = \frac{\partial \varphi(a',b)}{\partial b}\Big|_{b=0}\ .$$

Example 1.2.3. Considering the one-parameter group of transformations from Example 1.2.2,

$$T_a:\ \ \bar{x} = (a+1)x, \quad \bar{u} = (a+1)u, \quad a \in \mathbb{R}^+,$$

it has been shown that the composition law is $\varphi(a,b) = a+b+ab$. Hence the parameter a is not canonical. Using Theorem 1.2.1 above we have

$$\chi(a') = \frac{\partial(a'+b+a'b)}{\partial b}\Big|_{b=0} = 1+a'$$

and

$$\tilde{a} = \int_0^a \frac{da'}{1+a'} \ = \ln(1+a).$$

Therefore written in the new parameter the group consists of the transformations

$$T_{\tilde{a}} : \quad \bar{x} = e^{\tilde{a}} x, \quad \bar{u} = e^{\tilde{a}} u, \quad \tilde{a} \in \mathbb{R}\,.$$

It can be checked that the composition law is now

$$\tilde{\varphi}(\tilde{a}, \tilde{b}) = \tilde{a} + \tilde{b}\ .$$

Thus the new parameter is canonical.

From now we assume that group parameters are canonical.

Infinitesimal generator and Lie's equations

Lie's theory allows the construction of one-parameter group elements from their first-order approximations

$$\bar{x}^i \approx x^i + a\xi(x,u), \quad \bar{u}^\alpha \approx u^\alpha + a\eta^\alpha(x,u). \tag{1.5}$$

Equation (1.5) is the first-order Taylor expansion of T_a about $a = 0$ with the initial conditions

$$f^i|_{a=0} = x^i, \quad \phi^\alpha|_{a=0} = u^\alpha.$$

Hence we have

$$\xi^i(x,u) = \frac{\partial f^i(x,u,a)}{\partial a}\Big|_{a=0}, \qquad \eta^\alpha(x,u) = \frac{\partial \phi^\alpha(x,u,a)}{\partial a}\Big|_{a=0}. \tag{1.6}$$

The vector (ξ^i, η^α) is the tangent vector at (x,u) to the curve $(\bar{x}, \bar{u})$ parametrized by a.

The first-order approximations (1.5) can be written as

$$\bar{x}^i \approx (1 + aX)x^i, \quad \bar{u}^i \approx (1 + aX)u^\alpha,$$

where

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}\ . \tag{1.7}$$

The differential operator (1.7) is called the *infinitesimal generator* or *vector field* of the group G. It is also known as the symmetry generator (operator).

Theorem 1.2.2. For any given infinitesimal transformations (1.5), or an infinitesimal generator, X, the corresponding one-parameter group G is obtained by solution of the *Lie equations*

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x}, \bar{u}), \quad \frac{d\bar{u}^\alpha}{da} = \xi^\alpha(\bar{x}, \bar{u}) \tag{1.8}$$

subject to the initial conditions

$$\bar{x}^i|_{a=0} = x^i, \quad \bar{u}^\alpha|_{a=0} = u^\alpha \ .$$

The solution of Lie equations (1.8) involves exponentiating the generator X, i.e.,

$$\bar{x}^i = \exp(aX)x^i, \quad \bar{u}^\alpha = \exp(aX)u^\alpha, \tag{1.9}$$

where

$$\exp(aX) = 1 + aX + \frac{a^2}{2}X^2 + \cdots = \sum_{j=0}^{\infty} \frac{a^j}{j!}X^j \tag{1.10}$$

is known as the *Lie series operator*.

Example 1.2.4. Consider the following infinitesimal transformations

$$\bar{x} \approx x - au, \quad \bar{u} \approx u + ax$$

or equivalently

$$X = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u} \cdot$$

The corresponding Lie's equations are

$$\frac{d\bar{x}}{da} = -\bar{u} \quad \text{subject to} \quad \bar{x}_{|_{a=0}} = x.$$

$$\frac{d\bar{u}}{da} = \bar{x} \quad \text{subject to} \quad \bar{u}_{|_{a=0}} = u.$$

The solution of these equations leads to the one-parameter group (rotation group):

$$\bar{x} = x\cos a - u\sin a, \quad \bar{u} = x\sin a + u\cos a, \ -\pi < a < \pi.$$

Alternatively we can use the exponential map (1.9) in the form

$$\bar{x} = e^{aX}(x), \quad \bar{u} = e^{aX}(u),$$

where the generator X is as given above and e^{aX} is defined by (1.10).

Invariants

Definition 1.2.4. A point $(x, u) \in \mathbb{R}^{n+m}$ is an invariant point if it remains unchanged by every transformation of a group G, i.e.,

$$(\bar{x}, \bar{u}) = (x, u) \quad \forall\, a \in \mathcal{D}' \subset \mathcal{D}.$$

Theorem 1.2.3. A point $(x, u) \in \mathbb{R}^{n+m}$ is an invariant point of a group G with generator

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}$$

if and only if

$$\xi^i(x, u) = \eta^\alpha(x, u) = 0.$$

Definition 1.2.5. A function $F(x, u)$ is an invariant of a group G if and only if

$$F(\bar{x}, \bar{u}) = F(x, u) \quad \forall\, x, u, a \in \mathcal{D}' \subset \mathcal{D}.$$

Theorem 1.2.4. A function $F(x, u)$ is an invariant of a group G with the generator X if and only if

$$X(F) = 0. \tag{1.11}$$

The characteristic system for Eq. (1.11) is given by

$$\frac{dx^1}{\xi^1(x, u)} = \cdots = \frac{dx^n}{\xi^n(x, u)} = \frac{du^1}{\eta^1(x, u)} = \cdots = \frac{du^m}{\eta^m(x, u)}\,.$$

Thus an arbitrary invariant $F(x, u)$ of the group G is

$$F = \Lambda\left(I_1(x, u), \ldots, I_{m+n-1}(x, u)\right),$$

where $I_1(x, u), \ldots, I_{m+n-1}(x, u)$ is called a *basis of invariants* of G (i.e., group G has exactly $m + n - 1$ functionally independent invariants). The basis is not unique. One can take, as basic invariants, the left hand side of $m + n - 1$ first integrals

$$I_1(x, u), \ldots, I_{m+n-1}(x, u) = c_{m+n-1}.$$

Canonical variables

Definition 1.2.6. Two groups are *similar* if one can be transformed into the other by means of a suitable point change of variables.

Theorem 1.2.5. Every one-parameter group has canonical variables.

Proof. Using the chain rule we find that under the change of variables,

$$y^i = h^i(x,u), \quad v^\alpha = \theta^\alpha(x,u),$$

the generator X transforms as

$$\begin{aligned} X &= \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} \\ &= \xi^i\left(\frac{\partial h^k}{\partial x^i}\frac{\partial}{\partial y^k} + \frac{\partial \theta^\beta}{\partial x^i}\frac{\partial}{\partial v^\beta}\right) + \eta^\alpha\left(\frac{\partial h^k}{\partial u^i}\frac{\partial}{\partial y^k} + \frac{\partial \theta^\beta}{\partial u^i}\frac{\partial}{\partial v^\beta}\right) \\ &= \left(\xi^i\frac{\partial h^k}{\partial x^i} + \eta^\alpha\frac{\partial h^k}{\partial u^\alpha}\right)\frac{\partial}{\partial y^k} + \left(\xi^i\frac{\partial \theta^\beta}{\partial x^i} + \eta^\alpha\frac{\partial \theta^\beta}{\partial u^\alpha}\right)\frac{\partial}{\partial v^\beta} \\ &= X(h^k)\frac{\partial}{\partial y^k} + X(\theta^\beta)\frac{\partial}{\partial v^\beta}\,. \end{aligned}$$

In more suggestive form

$$X = X(y^k)\frac{\partial}{\partial y^k} + X(v^\beta)\frac{\partial}{\partial v^\beta}\,.$$

Then the coefficient operators $X(y^k)$ and $X(v^\beta)$ are written in terms of the new variables (y,v) which can be reduced to translation along one of the directions and are known as *canonical variables*.

Remark. One-parameter groups defined by transformations (1.4) can be transformed to a translation group

$$\bar{y}^{\,1} = y^1 + a,\ \bar{y}^{\,2} = y^2, \ldots, \bar{y}^{\,n} = y^n, \quad \bar{v}^{\,1} = v^1, \ldots, \bar{v}^{\,m} = v^m$$

with generator

$$X = \frac{\partial}{\partial y^1}\,.$$

Canonical variables (y,v) for translations along the y^1 axis are determined by solution of the system

$$X(y^1) = 1, \quad X(y^2) = 0, \ldots, X(y^n) = 0, \quad X(v^1) = 0, \ldots, X(v^m) = 0. \tag{1.12}$$

In order to solve system (1.12) we require a basis of invariants and a solution of $X(y^1) = 1$.

Example 1.2.5. Consider the rotation group with generator X in the (x, y) plane given by

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \cdot$$

Canonical variables for rotation along the v axis are obtained from the system

$$X(u) = 0, \quad X(v) = 1,$$

i.e.,

$$-y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0, \quad -y\frac{\partial v}{\partial x} + x\frac{\partial v}{\partial y} = 1.$$

Solving the first system, we write the corresponding characteristic system:

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{du}{0} \cdot$$

Therefore the basis invariant is $x^2 + y^2 = c_1$, the arbitrary invariant is $u = c_2$ and the general invariant is $u = f(x^2 + y^2)$.

Likewise the invariants of the second system which are in terms of v can be found.

1.3 Prolongation formulas

The transformations (1.4) form a symmetry group G of the system E if its invariant form (1.3) is satisfied whenever equation (1.1) holds. The transformed derivatives in (1.3) are obtained by employing the chain rule, $D_i = D_i(f^j)\bar{D}_j$, where

$$D_j = \frac{\partial}{\partial x^j} + u^\alpha_j \frac{\partial}{\partial u^\alpha} + u^\alpha_{jk}\frac{\partial}{\partial u^\alpha_k} + \cdots \ ; \ u^\alpha_j = D_j(u^\alpha) \ , \quad u^\alpha_{jk} = D_j(u^\alpha_k),$$

is the *total derivative operator* with respect to x^i, as is $\bar{D}_i$ for the transformed variables.

Applying $D_i = D_i(f^j)\bar{D}_j$ on $\bar{u}^\alpha$ and using the form of $\bar{u}^\alpha$ from (1.4) on the left hand side of the result, we arrive at

$$\begin{aligned} D_i(\phi^\alpha) &= D_i(f^j)\bar{D}_j(\bar{u}^\alpha) \ , \\ &= D_i(f^j)\bar{u}^\alpha_j \ . \end{aligned}$$

When one expands the last equation yields

$$\left(\frac{\partial f^j}{\partial x^i} + u_i^\beta \frac{\partial f^j}{\partial u^\beta}\right) \bar{u}_j^\alpha = \frac{\partial \phi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \phi^\alpha}{\partial u^\beta} \cdot \tag{1.13}$$

Now we solve equation (1.13) to get $\bar{u}_i^\alpha = \psi_i^\alpha(x, u, u_{(1)}, a)$. It can be verified that $\psi_i^\alpha|_{a=0} = u_i^\alpha$ and $\bar{u}_i^\alpha = \psi_i^\alpha(x, u, u_{(1)}, a)$ is locally solvable for $u_{(1)}$ given small a.

The transformations (1.2) together with the transformations $\bar{u}_{(1)} = \psi(x, u, u_{(1)}, a)$ form a one-parameter group, $G^{[1]}$, which is the *first prolonged group* acting on the space $(x, u, u_{(1)})$. In similar fashion higher-order prolonged (extended) groups $G^{[2]}$ up to $G^{[k]}$ can be obtained.

Let the infinitesimal transformations of the extended groups $G^{[1]}$ up to $G^{[k]}$ be given by (1.5) and

$$\begin{aligned}
\bar{u}_i^\alpha &\approx u_i^\alpha + a\zeta_i^\alpha(x, u, u_{(1)}), \\
\bar{u}_{ij}^\alpha &\approx u_{ij}^\alpha + a\zeta_{ij}^\alpha(x, u, u_{(1)}, u_{(2)}), \\
&\vdots \\
\bar{u}_{i_1\cdots i_k}^\alpha &\approx u_{i_1\cdots i_k}^\alpha + a\zeta_{i_1\cdots i_k}^\alpha(x, u, u_{(1)}, \ldots, u_{(k)}).
\end{aligned} \tag{1.14}$$

Then the functions ζs are given recursively by the prolongation formulae,

$$\begin{aligned}
\zeta_i^\alpha &= D_i(W^\alpha) + \xi^j u_{ji}^\alpha, \\
\zeta_{ij}^\alpha &= D_i D_j(W^\alpha) + \xi^k u_{kj}^\alpha, \\
&\vdots \\
\zeta_{i_1\cdots i_k}^\alpha &= D_{i_1}\cdots D_{i_k}(W^\alpha) + \xi^j u_{ji_1\cdots i_k}^\alpha,
\end{aligned} \tag{1.15}$$

where

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$$

is the so-called *Lie characteristic function*. Equivalently the formulae (1.15) can be written as

$$\begin{aligned}
\zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \\
\zeta_{ij}^\alpha &= D_j(\zeta_i^\alpha) - u_{il}^\alpha D_j(\xi^l), \\
&\vdots \\
\zeta_{i_1\cdots i_k}^\alpha &= D_{i_k}(\zeta_{i_1\cdots i_{k-1}}^\alpha) - u_{i_1\cdots i_{k-1}l}^\alpha D_{i_k}(\xi^l).
\end{aligned} \tag{1.16}$$

We now illustrate how to obtain one of the prolongation formulae (1.15), say, ζ_i^α. Consider $D_i(\phi^\alpha) = D_i(f^j)\bar{u}_j^\alpha$. Using the infinitesimal transformations (1.5) and $\bar{u}_i^\alpha$ from (1.14), we obtain

$$u_i^\alpha + a\zeta_i^\alpha + au_j^\alpha D_i(\xi^j) = u_i^\alpha + aD_i(\eta^\alpha).$$

Therefore we have

$$\zeta_j^\alpha = D_i(W^\alpha) + \xi^j u_{ji}^\alpha,$$

where the Einstein summation convention is adopted.

The corresponding prolonged generators of the prolonged groups $G^{[1]}$ up to $G^{[k]}$ are

$$X^{[1]} = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha(x,u,u_{(1)})\frac{\partial}{\partial u_i^\alpha}\ , \tag{1.17}$$

$$\vdots$$

$$X^{[k]} = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha(x,u,u_{(1)})\frac{\partial}{\partial u_i^\alpha} + \cdots + \zeta_{i_1\cdots i_k}^\alpha(x,u,\ldots,u_{(k)})\frac{\partial}{\partial u_{i_1\cdots i_k}^\alpha}\ , \tag{1.18}$$

where $X^{[1]}$ is the first prolongation of X. The converse is also true in that, given the prolonged operators (1.18) or their corresponding infinitesimal transformations (1.5), we can find the respective one-parameter prolonged groups. This is achieved by using Theorem 1.2.2 or the exponential map for the prolonged variables.

Definition 1.3.1. A differential function, $F(x,u,u_{(1)},\ldots,u_{(p)})$ for $p \geq 0$, is a pth-order differential invariant of a group G if

$$F(x,u,u_{(1)},\ldots,u_{(p)}) = F(\bar{x},\bar{u},\bar{u}_{(1)},\ldots,\bar{u}_{(p)}),$$

i.e., F is invariant under the prolonged group $G^{[p]}$, where for $p=0$, $u_{(0)} \equiv u$ and $G^{[0]} \equiv G$.

Theorem 1.3.1. A differential function, $F(x,u,u_{(1)},\ldots,u_{(p)})$ for $p \geq 0$, is a pth-order differential invariant of a group G if

$$X^{[p]}(F) = 0, \tag{1.19}$$

where $X^{[p]}$ is the pth prolongation of X and for $p=0$, $X^{[0]} \equiv X$.

The differential invariants can be obtained by solving the characteristic equations for Eq. (1.19).

Example 1.3.1. The prolongation of the operator

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$

for a second-order ODE with y depending on x is given by

$$X^{[2]} = X + \zeta_1\frac{\partial}{\partial y'} + \zeta_2\frac{\partial}{\partial y''}\cdot$$

The variables ζ_i, defined in (1.16) are

$$\begin{aligned}
\zeta_1 &= D_x(\eta) - y' D_x(\xi) \\
&= \eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y, \qquad (1.20)\\
\zeta_2 &= D_x(\zeta_1) - y'' D_x(\xi) \\
&= \eta_{xx} + y'(2\eta_{xy} - \xi_{xx}) + y'^2(\eta_{yy} - 2\xi_{xy}) - y'^3\xi_{yy} + y''(\eta_y - 2\xi_x - 3y'\xi_y). \qquad (1.21)
\end{aligned}$$

Example 1.3.2. For a second-order PDE having dependent variable, u, and independent variables, (t,x), we have

$$\begin{aligned}
X &= \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}\ , \\
X^{[2]} &= X + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{tt}\frac{\partial}{\partial u_{tt}} + \zeta_{tx}\frac{\partial}{\partial u_{tx}} + \zeta_{xx}\frac{\partial}{\partial u_{xx}}\ ,
\end{aligned}$$

where

$$\begin{aligned}
\zeta_t &= D_t(\eta) - u_t D_t(\xi^1) - u_x D_t(\xi^2) \\
&= \eta_t + u_t(\eta_u - \xi^1_t) - u_t^2\xi^1_u - u_x\xi^2_t - u_x u_t\xi^2_u\ , \qquad (1.22)\\
\zeta_x &= D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2) \\
&= \eta_x + u_x(\eta_u - \xi^2_x) - u_x^2\xi^2_u - u_t\xi^1_x - u_x u_t\xi^1_u, \qquad (1.23)\\
\zeta_{tt} &= D_t(\zeta_t) - u_{tt} D_t(\xi^1) - u_{tx} D_t(\xi^2) \\
&= \eta_{tt} + 2u_t\eta_{tu} + u_{tt}\eta_u + u_t^2\eta_{uu} - 2u_{xt}\xi^2_t - u_x\xi^2_{tt} - 2u_x u_t\xi^2_{tu} - \xi^2_u(u_x u_{tt} + 2u_t u_{xt}) \\
&\quad -u_x u_t^2\xi^2_{uu} - 2u_{tt}\xi^1_t - u_t\xi^1_{tt} - 2u_t^2\xi^1_{tu} - 3u_t u_{tt}\xi^1_u - u_t^3\xi^1_{uu}, \qquad (1.24)\\
\zeta_{tx} &= \zeta_{xt} = D_x(\zeta_t) - u_{tt} D_x(\xi^1) - u_{tx} D_x(\xi^2)
\end{aligned}$$

$$
\begin{aligned}
&= \eta_{tx} + 2u_t\eta_{xu} + u_{xt}\eta_u + u_tu_x\eta_{uu} - u_{tt}\xi^1_x - u_{tx}\xi^1_t - \xi^1_u(u_xu_{tt} + 2u_tu_{tx}) \\
&\quad -u_t(\xi^1_{xt} + u_t\xi^1_{xu} + u_x\xi^1_{tu} + u_tu_x\xi^1_{uu}) - \xi^2_u(u_tu_{xx} + 2u_xu_{xt}) \\
&\quad -u_x(\xi^2_{tx} + u_t\xi^2_{xu} + u_x\xi^2_{tu} + u_tu_x\xi^2_{uu}), \qquad (1.25)
\end{aligned}
$$

$$
\begin{aligned}
\zeta_{xx} &= D_x(\zeta_x) - u_{tx}D_x(\xi^1) - u_{xx}D_x(\xi^2) \\
&= \eta_{xx} + 2u_x\eta_{xu} + u_{xx}\eta_u + u_x^2\eta_{uu} - 2u_{xx}\xi^2_x - u_x\xi^2_{xx} - 2u_x^2\xi^2_{xu} - \xi^1_u(u_tu_{xx} + 2u_xu_{xt}) \\
&\quad -u_x^3\xi^2_{uu} - 2u_{xt}\xi^1_x - u_t\xi^1_{xx} - 2u_xu_t\xi^1_{xu} - u_x^2u_t\xi^1_{uu} - 3u_xu_{xx}\xi^2_u. \qquad (1.26)
\end{aligned}
$$

1.4 Determining equations for Lie-point symmetries

In this Section we introduce Lie's algorithm for calculating point symmetries of DEs and also give the illustrative examples.

Definition 1.4.1. An invertible transformation acting on the space (x, u) of E is a *point symmetry* of E provided every solution h of E is mapped onto another solution $\bar{h}$ of E.

Theorem 1.4.1. Let G be a group of transformations, (1.4), admitted by the system E. Performing the first-order Taylor expansion of (1.3) around $a = 0$, we arrive at the fact that

$$
X^{[k]}\Big(E^\sigma(x, u, u_{(1)}, \ldots, u_{(k)})\Big) = 0, \quad \sigma = 1, \ldots, s, \qquad (1.27)
$$

whenever (1.1) is satisfied for every group operator X of G. Then G consists of symmetries of the system E. It can be shown that the converse is also true.

The symmetry conditions (1.27) can be written compactly as

$$
X^{[k]}\Big(E^\sigma(x, u, u_{(1)}, \ldots, u_{(k)})\Big)\Big|_{(1.1)} = 0, \quad \sigma = 1, \ldots, s, \qquad (1.28)
$$

where $\Big|_{(1.1)}$ means evaluated on the surface (1.1).

Equations (1.28) are the so-called *determining equations*. In general the determining equations comprise an over-determined system of linear homogeneous PDEs for the unknown coordinates ξ^i and η^α of the symmetry generator X. The solutions of the determining system form a vector space, that is, any finite linear combination of symmetries is again a symmetry. This stems from the fact that the determining equations are linear.

Lie's algorithm

Below we give a layout of the steps involved in the execution of the procedure for calculating symmetries of E [10, 65, 67].

1. Write E such that all the terms are on the left hand side.

2. Write the generator of symmetry

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}\cdot$$

3. Prolong the symmetry generator X to the order which is the same as that of E, i.e.,

$$X^{[k]} = X + \zeta_i^\alpha(x,u,u_{(1)})\frac{\partial}{\partial u_i^\alpha} + \cdots + \zeta_{i_1\cdots i_k}^\alpha(x,u,\ldots,u_{(k)})\frac{\partial}{\partial u_{i_1\cdots i_k}^\alpha}\ ,$$

   where the variables ζ_i^α are given by (1.16).

4. Apply the prolonged generator $X^{[k]}$ on E evaluated on the surface (1.1) yielding the symmetry conditions

$$X^{[k]}\Big(E^\sigma(x,u,u_{(1)},\ldots,u_{(k)})\Big)\Big|_{(1.1)} = 0, \quad \sigma = 1,\ldots,s.$$

5. Substitute the ζ_i^α upon expansion of the symmetry conditions and replace the derivatives which are to be eliminated.

6. Separate the expanded expression with respect to the derivatives of the dependent variables and their powers resulting in an over-determined system of linear homogeneous PDEs in terms of ξ^i and η^α.

7. Solve the over-determined system for the infinitesimals ξ^i and η^α.

8. Construct one-parameter groups using Theorem 1.2.2.

Lie algebra of operators

The symmetries have an additional beautiful structure: they form a *Lie algebra*.

Definition 1.4.2. A *Lie algebra* is formed by a vector space L over a field, $\mathbb{F}$, say, together with a binary operation $[\ ,\ ]$ called a *Lie Bracket* commonly and incorrectly knows as *commutator*, defined on L such that the following properties hold:

(i) **Bilinearity**. If $X_1, X_2, X_3 \in L$, then

$$[aX_1 + bX_2, X_3] = a[X_1, X_3] + b[X_2, X_3] \quad \text{for scalars } a \text{ and } b \text{ in } \mathbb{F}.$$

(ii) **Skew-Symmetry**. If $X_1, X_2 \in L$, then

$$[X_1, X_2] = -[X_2, X_1].$$

(iii) **Jacobi Identity**. If $X_1, X_2, X_3 \in L$, then

$$[[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] = 0\ .$$

Definition 1.4.3. Consider a Lie algebra L. If the vector space L is finite-dimensional, its dimension is the dimension of the Lie algebra, that is, the finite-dimensional Lie algebra of dimension r is denoted by L_r.

In this thesis we take $\mathbb{F}$ to be the field of real numbers $\mathbb{R}$. We define the Lie Bracket $[\ ,\ ]$ on the set of vector fields $\mathcal{V}$ as

$$[X_1, X_2] = X_1X_2 - X_2X_1 \text{ for any } X_1, X_2 \in \mathcal{V}, \tag{1.29}$$

where

$$X_1 = \xi_1^i(x,u)\frac{\partial}{\partial x^i} + \eta_1^\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

and

$$X_2 = \xi_2^i(x,u)\frac{\partial}{\partial x^i} + \eta_2^\alpha(x,u)\frac{\partial}{\partial u^\alpha}\cdot$$

The binary operation (1.29) makes the space of vector field $\mathcal{V}$ a Lie algebra.

Theorem 1.4.2. The set of all symmetries forms a Lie algebra called a *symmetry Lie algebra*.

Definition 1.4.4. A subalgebra of a Lie algebra L_r is a subspace $\mathcal{S}$ of L_r such that $[X_1, X_2] \in \mathcal{S}$ for all $X_1, X_2 \in \mathcal{S}$.

Definition 1.4.5. An ideal of a Lie algebra L_r is a subspace $\mathcal{S}$ of L_r if $[\mathcal{S}, L_r] \subset \mathcal{S}$, i.e., $[\mathcal{S}, L_r] = \{[X_1, X_2] \text{ such that } X_1 \in \mathcal{S} \text{ and } X_2 \in L_r\} \subset \mathcal{S}$.

Definition 1.4.6. A k-derived algebra $L_r^{(k)}$ of a Lie algebra L_r is such that

$$L_r^{(1)} = [L_r, L_r], \ldots, L_r^{(k+1)} = [L_r^{(k)}, L_r^{(k)}], \quad k = 1, 2, \ldots .$$

The derived algebra $L_r^{(1)}$ is an ideal of the Lie algebra L_r.

Definition 1.4.7. A Lie algebra L_r is abelian if all of its elements commute, i.e., $L_r^{(1)} = 0$.

Definition 1.4.8. A Lie algebra L_r is solvable if there is a sequence of subalgebras

$$L_r \supset L_{r-1} \supset \cdots \supset L_1$$

of dimensions r, $r-1$, ..., 1 respectively such that each subalgebra L_{p-1} is an ideal of L_p for $p = 2, \ldots, r$.

Theorem 1.4.3. A Lie algebra L_r is solvable if and only if it has a vanishing kth derived algebra, i.e., $L_r^{(k)} = 0$ for a positive integer k.

Example 1.4.1. Suppose $[X_1, X_2] = 0$, $[X_1, X_3] = X_1$, $[X_2, X_3] = X_2$. Then the derived algebra is $L_3^{(1)} = \{X_1, X_2\}$. The algebra L_3 is solvable since by Theorem 1.4.3 its derived algebra vanishes, $L_3^{(2)} = [X_1, X_2] = \{0\}$.

Multi parameter groups

Lie algebras generate a multi parameter local group, i.e., G_r (where r denotes the dimension of a local group parameter). The definition of one-parameter local groups introduced earlier in Section 1.2 also applies to r-parameter local groups by defining the vector parameter $a = (a_1, a_2, \ldots, a_r)$.

Theorem 1.4.4. Let L_r be an r-dimensional vector space of operators

$$X_l = \xi_l^i(x, u)\frac{\partial}{\partial x^i} + \eta_l^\alpha(x, u)\frac{\partial}{\partial u^\alpha}, \quad l = 1, \ldots, r. \tag{1.30}$$

The product $T_a = T_{a_r} \cdots T_{a_1}$ of r one-parameter groups of transformations T_{a_l} generated by each operator X_l via the Lie equations

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x}, \bar{u}), \quad \frac{d\bar{u}^\alpha}{da} = \xi^\alpha(\bar{x}, \bar{u}),$$

subject to the initial conditions

$$\bar{x}^i|_{a=0} = x^i, \quad \bar{u}^\alpha_{|_{a=0}} = u^\alpha$$

is an r-parameter (local) group G_r if and only if L_r is a Lie algebra.

Example 1.4.2. Consider the operators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial u}, \quad X_3 = u\frac{\partial}{\partial x}\,\cdot$$

These span a Lie algebra L_3 since $[X_1, X_2] = 0$, $[X_1, X_3] = 0$ and $[X_2, X_3] = X_1$. The three one-parameter groups of transformations with the vector parameter $a = (a_1, a_2, a_3)$ obtained via Lie equations are

$$\begin{aligned} T_{a_1}: &\quad \bar{x} = x + a_1, \quad \bar{u} = u, \\ T_{a_2}: &\quad \bar{x} = x, \quad \bar{u} = u + a_2, \\ T_{a_3}: &\quad \bar{x} = x + a_3 u, \quad \bar{u} = u. \end{aligned}$$

Their composition $T_a = T_{a_3} T_{a_2} T_{a_1}$ is

$$T_a: \quad \bar{x} = x + a_3 u + a_2 a_3 + a_1, \quad \bar{u} = u + a_2, \quad a \in \mathbb{R}^3.$$

The composition of T_a and T_b, where $b = (b_1, b_2, b_3)$, is

$$T_b T_a: \quad \bar{\bar{x}} = x + (a_3 + b_3)u + a_2(a_3 + b_3) + b_2 b_3 + a_1 + b_1, \quad \bar{\bar{u}} = u + a_2 + b_2.$$

Now $T_b T_a = T_c$, i.e., the equations $\bar{\bar{x}} = x + c_3 u + c_2 c_3 + c_1, \quad \bar{\bar{u}} = u + c_2$

$$c_1 = a_1 + b_1 - a_3 b_2, \quad c_2 = a_2 + b_2, \quad c_3 = a_3 + b_3.$$

Thus the composition law is given by

$$\varphi_1(a, b) = a_1 + b_1 - a_3 b_2, \quad \varphi_2(a, b) = a_2 + b_2, \quad \varphi_3(a, b) = a_3 + b_3.$$

If $a = (0, 0, 0)$, then $T_{(0,0,0)}$ is the identity transformation. Now, $T_b T_a = T_0$ implies that $T_b = T_{a^{-1}}$ for $c = 0$, i.e., $a_1 + b_1 - a_3 b_2 = 0$, $a_2 + b_2 = 0$ and $a_3 + b_3 = 0$. Hence the inverse transformation $T_{a^{-1}} = T_{(-a_1 - a_2 a_3, -a_2, a_3)}$.

Therefore the transformations $T_a = T_{a_3} T_{a_2} T_{a_1}$ form a three-parameter group.

Theorem 1.4.5. A function $F(x,u)$ is an invariant of an r-parameter group G_r with the generators X_l given by (1.30) if and only if

$$X_l(F) = 0. \tag{1.31}$$

Theorem 1.4.6. Let $r^* = \text{rank}(\xi_l^i, \eta_l^\alpha)$, where (ξ_l^i, η_l^α) are taken at generic points (x,u). Then the group G_r has $m+n-r^*$ functionally independent invariants $I_1(x,u)$, $I_2(x,u)$, $\ldots, I_{m+n-r^*}(x,u)$ and any other invariant is a function of these. A set of $m+n-r^*$ functionally independent invariants is a basis of invariants for G_r .

Similar consideration applies to prolonged groups.

Example 1.4.3. Applying Theorem 1.4.6 to the previous example, we find that there are no invariants for the group, G_3, that is, $m+n-r^* = 1+1-2 = 0$ where

$$n=1, \quad m=1, \quad r^* = \text{rank}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ u & 0 \end{pmatrix} = 2.$$

Symmetry Calculations

The following examples illustrate Lie's algorithm for calculating symmetries of DEs. From now throughout the thesis we use the notation ∂_{x^i} and ∂_{u^α} instead of $\dfrac{\partial}{\partial x^i}$ and $\dfrac{\partial}{\partial u^\alpha}$ respectively.

Example 1.4.4. Consider the equation

$$y'' = \frac{\alpha}{y^3}, \quad \alpha \neq 0 \tag{1.32}$$

which is a special case of the Ermakov–Pinney equation.

According to Lie's algorithm the vector field

$$X = \xi(x,y)\partial_x + \eta(x,y)\partial_y \tag{1.33}$$

is a symmetry generator of (1.32) if and only if we have the symmetry condition

$$X^{[2]}\left(y'' - \frac{\alpha}{y^3}\right)\Big|_{(1.32)} = 0,$$

where

$$X^{[2]} = X + \zeta_1 \partial_{y'} + \zeta_2 \partial_{y''}.$$

The variables ζ_1 and ζ_2 are given by (1.20) and (1.21) respectively. However, the term involving ζ_1 does not contribute in calculations because there is no y' appearing in the equation under consideration.

Thus we have the determining equation

$$\left(\frac{3\alpha}{y^4}\eta + \zeta_2 \right)\Big|_{(1.32)} = 0. \tag{1.34}$$

Upon expansion of (1.34) we have

$$\frac{3\alpha}{y^4}\eta + \eta_{xx} + y'(2\eta_{xy} - \xi_{xx}) + (y')^2(\eta_{yy} - 2\xi_{xy}) - (y')^3\xi_{yy} + \frac{\alpha}{y^3}(\eta_y - 2\xi_x) - \frac{3\alpha}{y^3}y'\xi_y = 0.$$

The separation of the above expression with respect to powers of y' yields the equations

$$\begin{aligned}
(y')^3 &: \ \xi_{yy} = 0, \\
(y')^2 &: \ \eta_{yy} - 2\xi_{xy} = 0, \\
y' &: \ 2\eta_{xy} - \xi_{xx} - \frac{3\alpha}{y^3}\xi_y = 0, \\
(y')^0 &: \ \frac{3\alpha}{y^4}\eta + \eta_{xx} + \frac{\alpha}{y^3}(\eta_y - 2\xi_x) = 0.
\end{aligned}$$

The general solution of the above system is

$$\xi = c_0 x^2 + 2xc_1 + c_2, \quad \eta = (c_0 x + c_1)y.$$

Thus the symmetry Lie algebra of (1.32) is generated by the operators

$$X_1 = \partial_x, \quad X_2 = 2x\partial_x + y\partial_y, \quad X_3 = x^2\partial_x + xy\partial_y.$$

Hence the symmetry Lie algebra is three-dimensional. The Lie Bracket of the symmetry generators is

$$[X_1, X_2] = 2X_1, \quad [X_1, X_3] = X_2, \quad [X_2, X_3] = 2X_3.$$

Example 1.4.5. The heat equation

$$u_t - u_{xx} = 0 \tag{1.35}$$

has the symmetries:

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = u\partial_u, \quad X_4 = 2t\partial_t + x\partial_x, \quad X_5 = 2t\partial_x - xu\partial_u,$$
$$X_6 = 4t^2\partial_t + 4tx\partial_x - u(x^2 + 2t)\partial_u, \quad X_\beta = \beta(t,x)\partial_u,$$

where $\beta(t,x)$ is an arbitrary solution of the heat equation, i.e. $\beta_t = \beta_{xx}$. Note that the symmetry Lie algebra is infinite-dimensional. The commutator brackets are given in the following table:

Table 1.1: Table of commutators.

$\nearrow [X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6	X_β
X_1	0	0	0	$2X_1$	$2X_2$	$4X_4 - 2X_3$	X_{β_t}
X_2	0	0	0	X_2	$-X_3$	$2X_5$	X_{β_x}
X_3	0	0	0	0	0	0	$-X_\beta$
X_4	$-2X_1$	$-X_2$	0	0	X_5	$2X_6$	X_β
X_5	$-2X_2$	X_3	0	$-X_5$	0	0	$X_{\beta''}$
X_6	$2X_3 - 4X_4$	$-2X_5$	0	$-2X_6$	0	0	$X_{\beta'''}$
X_β	$-X_{\beta_t}$	$-X_{\beta_x}$	X_β	$-X_{\beta'}$	$-X_{\beta''}$	$-X_{\beta'''}$	0

In the table above we have adopted the following notations:

$$\begin{aligned} \beta' &= x\beta_x + 2t\beta_t, \\ \beta'' &= 2t\beta_x + x\beta, \\ \beta''' &= 4tx\beta_x + 4t^2\beta_t + (x^2 + 2t)\beta. \end{aligned}$$

Moreover β_t, β_x, β', β'' and β''' are solutions of the heat equation because $\beta(t,x)$ is any solution of the heat equation.

The one-parameter groups T_{a_i} for each X_i via the Lie equations are

$$T_{a_1}: \quad \bar{t} = t + a_1, \quad \bar{x} = x, \quad \bar{u} = u,$$
$$T_{a_2}: \quad \bar{t} = t, \quad \bar{x} = x + a_2, \quad \bar{u} = u,$$
$$T_{a_3}: \quad \bar{t} = t, \quad \bar{x} = x, \quad \bar{u} = a_3 u; \quad a_3 > 0,$$

$$
\begin{aligned}
T_{a_4}: &\quad \bar{t} = a_4^2 t, \quad \bar{x} = a_4 x, \quad \bar{u} = u; \quad a_4 > 0,\\
T_{a_5}: &\quad \bar{t} = t, \quad \bar{x} = 2a_5 t + x, \quad \bar{u} = u\exp\left[-a_5(a_5 t + x)\right]; \quad a_5 > 0,\\
T_{a_6}: &\quad \bar{t} = \frac{t}{1-4a_6 t}, \quad \bar{x} = \frac{x}{1-4a_6 t}, \quad \bar{u} = u\sqrt{1-4a_6 t}\,\exp\left[\frac{-a_6 x^2}{1-4a_6 t}\right]; \quad a_6 \neq 0,\\
T_a: &\quad \bar{t} = t, \quad \bar{x} = x, \quad \bar{u} = u + a\beta(t,x).
\end{aligned}
$$

1.5 The use of symmetries

Symmetries can be used to reduce the order of a DE. In fact, when the equation admits solvable Lie algebra, the ideals of the algebra can be used to perform the reduction. Knowing the symmetry of the reduced equation provides further reduction until in some cases the solution is obtained by quadratures.

Example 1.5.1. *Reduction of order.*

Consider the equation

$$y'' + by' + \frac{2b^2}{9}y + cy^3 = 0, \tag{1.36}$$

for arbitrary constants b and c. The symmetry Lie algebra admitted by the equation is two-dimensional and generated by the operators

$$X_1 = \partial_x, \quad X_2 = -\frac{3}{b}\exp\left(\frac{b}{3}x\right)\partial_x + y\exp\left(\frac{b}{3}x\right)\partial_y.$$

The Lie Bracket of the operators is

$$[X_1, X_2] = \frac{3}{b}X_2.$$

Rewriting L_2 in solvable form we use the skew-symmetry property that

$$[X_1, X_2] = -[X_2, X_1].$$

Thus we let $Y_1 = X_2$ and $Y_2 = X_1$ to obtain

$$[Y_1, Y_2] = -\frac{3}{b}Y_1.$$

The ideal is spanned by

$$Y_1 = -\frac{3}{b}\exp\left(\frac{b}{3}x\right)\partial_x + y\exp\left(\frac{b}{3}x\right)\partial_y.$$

Therefore we start the reduction with Y_1. For a second-order equation we require the invariants of the first prolongation of the operator to reduce the equation. In reducing a third-order equation the second prolongation of the operator is needed and so on.

The first prolongation of Y_1 is

$$Y_1^{[1]} = \exp\left(\frac{b}{3}x\right)\left[-\frac{3}{b}\partial_x + y\partial_y + (2y' + \frac{b}{3}y)\partial_{y'}\right].$$

The invariants of the group generated by $Y_1^{[1]}$ are

$$u = y\exp\left(\frac{b}{3}x\right), \quad v = \frac{y'}{y^2} + \frac{b}{3y}\,.$$

The second-order equation (1.36) becomes the first-order equation in the new variables, u and v, i.e.,

$$\frac{dv}{du} = -\frac{2v^2 + c}{uv}\,. \tag{1.37}$$

The reduced equation (1.37) is variables separable and there is no need for consecutive reduction using Y_2. However, Eq. (1.37) admits

$$\widetilde{Y}_2 = \left(\frac{b}{3}\right)u\partial_u$$

written in the new coordinates. The solution of Eq. (1.37) can be easily found and then written in original coordinates (x, y).

Example 1.5.2. *Generating solutions.*

The one-parameter group of transformations can be used to generate new solutions from the known ones. The new solutions can comprise nontrivial solutions compared to the known (trivial) solutions from which they were generated.

If $\bar{u} = \bar{h}(\bar{t}, \bar{x})$ is a solution of the heat equation, then so is

$$\phi(t, x, u, a) = h(f^1(t, x, u, a), f^2(t, x, u, a)),$$

where the f^i are differentiable functions (c.f. Definition 1.2.2).

Equivalently in solved form with respect to u: $u = H_a(t, x)$ is a one-parameter family of solutions. For instance, if $\bar{u} = \bar{h}(\bar{t}, \bar{x})$ is a solution of the heat equation corresponding to the transformations

$$T_{a_1}:\ \bar{t} = t + a_1, \quad \bar{x} = x, \quad \bar{u} = u,$$

then so is $u = h(t - a_1, x)$.

Also consider the transformations,

$$T_{a_6} : \bar{t} = \frac{t}{1 - 4a_6 t}, \quad \bar{x} = \frac{x}{1 - 4a_6 t}, \quad \bar{u} = u\sqrt{1 - 4a_6 t}\, \exp\left[\frac{-a_6 x^2}{1 - 4a_6 t}\right]; \quad a_6 \neq 0.$$

Given the constant solution of the heat equation $u = u_0 =$ constant and expressing $\bar{u}$ in terms of $\bar{t}$ and $\bar{x}$, a new solution (dropping the bars),

$$u = \frac{u_0}{\sqrt{1 + 4a_6 t}}\, \exp\left[\frac{-a_6 x^2}{1 + 4a_6 t}\right]$$

is generated.

In similar manner the generated solutions for the other one-parameter groups of transformations can be obtained.

Invariant Solutions

The main motivation for calculating the symmetries of DEs is to use them to discover the structure of the solution space. A symmetry has the interesting property of mapping a solution to not the same one. In the case of scalar ODEs symmetries are used for reduction of order [10, 31, 33, 65, 67]. When dealing with PDEs, finding the general solution is sometimes very difficult or impossible. In most of the cases one has to make do with particular solutions. Amongst these particular solutions, *invariant solutions* can be systematically determined when the symmetries of the underlying equation are known.

Definition 1.5.1. A solution $u^\alpha = F^\alpha(x^1, x^2, \ldots x^n)$ of E^α is invariant under the one-parameter group of transformations (1.4) if

$$\bar{u}^\alpha = F^\alpha(\bar{x}^1, \bar{x}^2, \ldots \bar{x}^n). \tag{1.38}$$

If we perform the first-order Taylor expansion of (1.38) around $a = 0$ and use the first-order approximations of one-parameter group (1.5), we obtain

$$u^\alpha + a\eta^\alpha + \cdots = F^\alpha + aXF^\alpha + \cdots$$

whenever $u = F$. This equation implies that

$$\eta^\alpha = XF^\alpha$$

whenever $u = F$. Therefore

$$u = F(x^1, \ldots, x^n), \tag{1.39}$$

is invariant under transformations (1.4) provided

$$X(u^\alpha - F^\alpha)_{|_{(1.39)}} = 0. \tag{1.40}$$

Conversely it can be shown that, if (1.40) be satisfied, then $u = F$ is invariant.

Equation (1.40) is a system of first-order quasilinear PDEs that can be solved to obtain the functional form of F. Once the functional form of the invariant solution is obtained, it is substituted back into the original system. This substitution leads to PDEs with fewer independent variables. If the symmetries of the reduced equations are known, we can perform further reductions. In favourable cases, we find closed-form solutions of the original system.

Example 1.5.3. *Travelling-wave solutions.*

Consider the symmetry $X = X_1 + cX_2$ of the heat equation, where c is a constant (representing wave speed). In order to find the form of the solution invariant under X, we have to solve the quasilinear PDE (1.40) which is in this case

$$\frac{\partial F}{\partial t} + c\frac{\partial F}{\partial x} = 0. \tag{1.41}$$

The method of characteristics for (1.41) gives

$$\frac{dt}{1} = \frac{dx}{c} = \frac{dF}{0}\,.$$

Thus $F = g(x - ct)$, where g is an arbitrary smooth function of its argument. Therefore any solution invariant under X assumes the form

$$u(t, x) = g(x - ct). \tag{1.42}$$

Substitution of (1.42) into the heat equation yields

$$g'' + cg' = 0, \tag{1.43}$$

where a prime denotes differentiation with respect to $\lambda = x - ct$. Solving (1.43), we get

$$g = K_1 \exp(-c\lambda) + K_2,$$

where K_1 and K_2 are arbitrary constants. Finally the most general solution of the heat equation invariant under X is

$$u(t, x) = K_1 \exp(c^2 t - cx) + K_2.$$

1.6 Equivalence transformations

Equivalence transformations play a vital role in solving the problem of group classification. They can be calculated directly [39, 42] or using the infinitesimal approach suggested by Ovsiannikov [66] and was later modified by Akhatov *et al* [1]. The infinitesimal approach is tedious but easy to follow. According to Ovsiannikov [67], an equivalence transformation is a generator of continuous equivalence group of transformations acting in the extended space of independent variables, dependent variables, functions and their derivatives which does not change the form of the equation under study.

Definition 1.6.1. An equivalence transformation for a system of DE is a change of both the independent and dependent variables into new independent and dependent variables taking the original system into a system of the same form. In general the arbitrary elements of the new system differ from those appearing in the original system.

A more technical definition can be given in conjunction with Definition 1.2.1 for symmetry transformation.

Definition 1.6.2. The set of all equivalence transformations forms an *equivalence group*, denoted by $\mathcal{E}$.

Definition 1.6.3. The continuous group of equivalence transformations $\mathcal{E}_s$ is a subgroup of $\mathcal{E}$.

Definition 1.6.4. The Lie algebra of the continuous equivalence group is called *the equivalence algebra* $L_{\mathcal{E}}$.

Example 1.6.1. We illustrate the infinitesimal approach by calculating the equivalence transformations for the nonlinear heat equation [66, 67]

$$u_t = (k(u)u_x)_x\ , \quad k(u) \neq \text{constant}. \tag{1.44}$$

We rewrite the equation as

$$u_t = k_u u_x^2 + k(u)u_{xx}.$$

We seek for the generator of equivalence group having the form

$$Y = \xi^1(t,x,u)\partial_t + \xi^2(t,x,u)\partial_x + \eta(t,x,u)\partial_u + \mu(t,x,u,k)\partial_k. \tag{1.45}$$

The operator (1.45) is the generator of the equivalence group provided it is admitted by the extended system

$$u_t - k_u u_x^2 - k(u)u_{xx} = 0, \tag{1.46}$$

$$k_t = 0, \quad k_x = 0. \tag{1.47}$$

The invariance conditions for system (1.46)–(1.47) require the following prolongation of the operator (1.45)

$$\widetilde{Y} = Y + \zeta_t\partial_{u_t} + \zeta_x\partial_{u_x} + \zeta_{xx}\partial_{u_{xx}} + \omega_t\partial_{k_t} + \omega_x\partial_{k_x} + \omega_u\partial_{k_u}, \tag{1.48}$$

where the ζ_i are as given by (1.22), (1.23) and (1.26). The variables ω_i are given by the prolongation formulae

$$\omega_t = \widetilde{D}_t(\mu) - k_t\widetilde{D}_t(\xi^1) - k_x\widetilde{D}_t(\xi^2) - k_u\widetilde{D}_t(\eta), \tag{1.49}$$

$$\omega_x = \widetilde{D}_x(\mu) - k_t\widetilde{D}_x(\xi^1) - k_x\widetilde{D}_x(\xi^2) - k_u\widetilde{D}_x(\eta), \tag{1.50}$$

$$\omega_u = \widetilde{D}_u(\mu) - k_t\widetilde{D}_u(\xi^1) - k_x\widetilde{D}_u(\xi^2) - k_u\widetilde{D}_u(\eta), \tag{1.51}$$

where

$$\widetilde{D}_t = \partial_t + k_t\partial_k, \quad \widetilde{D}_x = \partial_x + k_x\partial_k, \quad \widetilde{D}_u = \partial_u + k_u\partial_k$$

are the total derivative operators for the extended system (1.46)–(1.47). The invariance condition for Eqs. (1.46)–(1.47) yields the determining equations

$$\zeta_t - 2k_u u_x\zeta_x - k\,\zeta_{xx} - \mu u_{xx} - \omega_u u_x^2 = 0, \tag{1.52}$$

$$\omega_t = 0, \quad \omega_x = 0. \tag{1.53}$$

The general solution of the determining equations (1.52) and (1.53) is given by

$$\xi^1 = c_1 t + c_2, \quad \xi^2 = c_3 x + c_4, \quad \eta = c_5 u + c_6, \quad \mu = (2c_3 - c_1)k,$$

where $c_1,\ c_2, \ldots,\ c_6$ are the arbitrary constants. Thus the nonlinear heat equation (1.44) has a six-dimensional equivalence Lie algebra spanned by the operators

$$Y_1 = \partial_t,\ Y_2 = \partial_x,\ Y_3 = \partial_u,\ Y_4 = t\partial_t - k\partial_k,\ Y_5 = x\partial_x + 2k\partial_k,\ Y_6 = u\partial_u. \tag{1.54}$$

The one-parameter group of transformations T_{a_i} for each Y_i are

$$\begin{aligned}
T_{a_1}: &\quad \bar{t} = t + a_1, \quad \bar{x} = x, \quad \bar{u} = u, \quad \bar{k} = k,\\
T_{a_2}: &\quad \bar{t} = t, \quad \bar{x} = x + a_2, \quad \bar{u} = u, \quad \bar{k} = k,\\
T_{a_3}: &\quad \bar{t} = t, \quad \bar{x} = x, \quad \bar{u} = u + a_3, \quad \bar{k} = k,\\
T_{a_4}: &\quad \bar{t} = a_4 t, \quad \bar{x} = x, \quad \bar{u} = u, \quad \bar{k} = k/a_4\,; \quad a_4 > 0,\\
T_{a_5}: &\quad \bar{t} = t, \quad \bar{x} = a_5 x, \quad \bar{u} = u, \quad \bar{k} = a_5^2 k\,; \quad a_5 > 0,\\
T_{a_6}: &\quad \bar{t} = t, \quad \bar{x} = x, \quad \bar{u} = a_6 u, \quad \bar{k} = k\,; \quad a_6 > 0.
\end{aligned}$$

Taking the composition of the above transformations, we obtain the equivalence transformations for Eq. (1.44) written in finite form

$$\bar{t} = \alpha_1 t + \gamma_1, \quad \bar{x} = \alpha_2 x + \gamma_2, \quad \bar{u} = \alpha_3 u + \gamma_3, \quad \bar{k} = k\alpha_2^2/\alpha_1, \tag{1.55}$$

where

$$\alpha_1 = a_4 > 0,\ \alpha_2 = a_5 > 0,\ \alpha_3 = a_6 > 0,\ \gamma_1 = a_4 a_1,\ \gamma_2 = a_5 a_2,\ \gamma_3 = a_6 a_3.$$

Group classification is performed up to equivalence transformations (1.55); consult [66, 67] for details.

1.7 Conclusion

In this Chapter we have introduced basic techniques that are employed in the subsequent chapters. Particularly we have described Lie's algorithm for the calculation of symmetries. Examples have been given to illustrate the different steps in the calculation of

symmetries. Moreover we have shown how to use symmetries for reduction of order, generation of new solutions from known ones and the construction of invariant solutions. We have also illustrated by an example the computation of the equivalence transformations.

Part I

Collapsible Tube Model

It is well known in physiological, clinical or surgical applications that the flow of a fluid changes the geometry of the collapsible tube through which it is flowing. The phenomenon of fluid flow through collapsible tubes is complex due to the fluid flow interaction with the motion of the tube wall. Owing to this complexity much of the research (analytical and numerical investigations [23, 35, 44, 71] just a few to mention) have been based on one-dimensional and two-dimensional models. However, a complete understanding and comprehension of the complex dynamics involved in the flow through a collapsible tube requires a three-dimensional approach (experimental investigations and three-dimensional numerical simulations). Due to the extensive computational resources needed, currently there are limited three-dimensional numerical simulations for fluid flow through collapsible tubes [49].

For a given real-life situation the flow in a collapsible channel or tube taking into account the structure interactions can be modelled in various ways depending upon the type of flow. The flow can be steady or unsteady at low/high Reynolds numbers ignoring or incorporating elastic forces. Steady flow may breakdown under certain conditions, i.e., the flow ceases to exist for some given parameter values (this is known as *choking*). On the other hand for other conditions the flow does exist, but with some oscillatory instabilities. These instabilities are responsible for the "Korotkoff sounds" during blood-pressure measurements (sphygmomanometry) and snoring noises during forced expiration. The models of unsteady flow include wave propagation and other types of instabilities such as Tollmien-Schlichting waves and travelling-wave flutter.

There is a number of physiological, clinical and surgical applications of fluid flow through collapsible channels, for example: blood flow in arteries or veins during sphygmomanometry; urethral flow during micturition; ureteral flow during peristaltic pumping; airflow in the bronchial airways during forced expiration [8, 35] and in other circumstances which are less usual, such as life-sustaining and life-threatening situations. The review articles [8, 25] give applications of flow in collapsible channels and tubes, analytical and numerical models, experimental investigations and comments on some outstanding issues on the subject.

In an experimental setup the physics of fluid flow through a collapsible tube is designated

by the term "Starling Resistor". The Starling Resistor is a device introduced by Knowlton and Starling [40] and is commonly used by physiologists to represent the model of a deformable channel. The elastic tubing section collapses at any point along its length at which the transmural pressure (internal minus external pressure) is less than zero. Inside a pressure chamber a collapsible tube of finite-length is mounted between two rigid tubes and a fluid with a given volume flux is pumped into the setup.

Many features of the systems of PDEs describing compressible gas and shallow-water flows [3, 36, 76] also arise in collapsible tube flow. These include models for steady and unsteady flows. The next two chapters are devoted to the investigation of the models of flow in collapsible tube from the point of view of Lie group theory. Similar models (systems of equations) have been studied in [13, 50, 61, 67] and the references therein using this approach. Most of the analytical work done on the flow in collapsible channels involves approximation of solutions. However, the Lie symmetry method has proven to be a powerful tool in obtaining solutions naturally.

Chapter 2

Classification with respect to Three- and Four-Dimensional Lie Algebras

In this Chapter we perform a classification of a coupled system of PDEs modelling a flow in collapsible tubes. This system of PDEs contains unknown functions of the dependent variables the forms of which are specified through the classification with respect to subalgebras of real three- and four-dimensional Lie algebras.

2.1 The governing model

We consider one-dimensional flow through a collapsible tube ([25, 35]) governed by the equations:

• Conservation of mass

$$\alpha_t + (\alpha u)_x = 0, \tag{2.1}$$

where subscripts t and x denote time and spatial derivatives respectively.

• Conservation of momentum

$$\rho(u_t + uu_x) = -p_x - R(\alpha, u), \tag{2.2}$$

where ρ is the density of fluid assumed to be constant and $R(\alpha, u) > 0$ is a term representing distributed frictional losses.

• Pressure-area relation

$$p - p_e = \mathcal{P}(\alpha). \tag{2.3}$$

The tube law (2.3) can be accompanied by more terms to represent additional physical effects

$$p - p_e = \mathcal{P}(\alpha) - T\alpha_{xx} + D\alpha_t + M\alpha_{tt}, \tag{2.4}$$

in which $T \geq 0$ and α_{xx} approximate the effect of longitudinal tension and curvature of the tube wall respectively. The constant $D \geq 0$ represents the viscous damping in the wall and $M \geq 0$ represents wall inertia.

The dependent variables, $\alpha(t,x)$, $u(t,x)$ and $p(t,x)$, are the cross-sectional area of the tube, cross-sectional axial velocity and cross-sectional internal pressure respectively. The independent variable x measures the tube length and t the time. The nonlinear tube law function $\mathcal{P}(\alpha)$ can be computed using thin-shell theory, but a quite similar function is assumed for some models in applications [2, 11, 55]. The form of the term representing viscous effects, $R(\alpha, u)$, depends upon the type of flow considered. Equations (2.1)–(2.3) must be solved subject to appropriate initial and boundary conditions.

2.2 Equivalence generators

An equivalence transformation of system is a change of both dependent and independent variables into new dependent and independent variables taking the original system into a system of the same form [29, 30, 31, 32, 34], see also Chapter 1. Using the infinitesimal approach to calculate the generators of an equivalence group, we rewrite Eqs. (2.1)–(2.3) as the system

$$\begin{aligned} &\alpha_t + \alpha u_x + u\alpha_x = 0, \\ &\rho(u_t + uu_x) + \alpha_x \mathcal{P}_\alpha + R(\alpha, u) = 0. \end{aligned} \tag{2.5}$$

Since the independent variables t and x do not appear explicitly in the above system (2.5), the kernel of the maximal Lie invariance algebras of the system (i.e. the so-called

principal Lie algebra) is spanned by at least two operators

$$X_1 = \partial_t \quad \text{and} \quad X_2 = \partial_x. \tag{2.6}$$

We seek the generator of an equivalence group having the form

$$Y = \xi^1\partial_t + \xi^2\partial_x + \eta^1\partial_\alpha + \eta^2\partial_u + \mu^1\partial_{\mathcal{P}} + \mu^2\partial_R, \tag{2.7}$$

where

$$\xi^i = \xi^i(t,x,\alpha,u),\ \eta^i = \eta^i(t,x,\alpha,u),\ \mu^i = \mu^i(t,x,\alpha,u,\mathcal{P},R), \quad i = 1,2.$$

The operator (2.7) is the generator of the equivalence group provided it is admitted by the extended system

$$\alpha_t + \alpha u_x + u\alpha_x = 0, \quad \rho(u_t + uu_x) + \alpha_x\mathcal{P}_\alpha + R = 0, \tag{2.8}$$

$$\mathcal{P}_t = 0, \quad \mathcal{P}_x = 0, \quad \mathcal{P}_u = 0, \quad R_t = 0, \quad R_x = 0. \tag{2.9}$$

The invariance conditions for system (2.8)–(2.9) require the following prolongation of the operator (2.7)

$$\widetilde{Y} = Y + \zeta_1^1\partial_{\alpha_t} + \zeta_2^1\partial_{\alpha_x} + \zeta_1^2\partial_{u_t} + \zeta_1^2\partial_{u_x} + \omega_1^1\partial_{\mathcal{P}_t} + \omega_2^1\partial_{\mathcal{P}_x} + \omega_3^1\partial_{\mathcal{P}_\alpha} + \omega_4^1\partial_{\mathcal{P}_u} + \omega_1^2\partial_{R_t} + \omega_2^2\partial_{R_x}. \tag{2.10}$$

The variables ζ_j^i and ω_k^i are given by the prolongation formulae

$$\begin{aligned}
\zeta_1^1 &= D_t(\eta^1) - \alpha_t D_t(\xi^1) - \alpha_x D_t(\xi^2) \\
&\equiv \eta_t^1 + \alpha_t\eta_\alpha^1 + u_t\eta_u^1 - \alpha_t(\xi_t^1 + \alpha_t\xi_\alpha^1 + u_t\xi_u^1) - \alpha_x(\xi_t^2 + \alpha_t\xi_\alpha^2 + u_t\xi_u^2), && (2.11)\\
\zeta_2^1 &= D_x(\eta^1) - \alpha_t D_x(\xi^1) - \alpha_x D_x(\xi^2) \\
&\equiv \eta_x^1 + \alpha_x\eta_\alpha^1 + u_x\eta_u^1 - \alpha_t(\xi_x^1 + \alpha_x\xi_\alpha^1 + u_x\xi_u^1) - \alpha_x(\xi_x^2 + \alpha_x\xi_\alpha^2 + u_x\xi_u^2), && (2.12)\\
\zeta_1^2 &= D_t(\eta^2) - u_t D_t(\xi^1) - u_x D_t(\xi^2) \\
&\equiv \eta_t^2 + \alpha_t\eta_\alpha^2 + u_t\eta_u^2 - \alpha_t(\xi_t^1 + \alpha_t\xi_\alpha^1 + u_t\xi_u^1) - \alpha_x(\xi_t^2 + \alpha_t\xi_\alpha^2 + u_t\xi_u^2), && (2.13)\\
\zeta_2^2 &= D_x(\eta^2) - u_t D_x(\xi^1) - u_x D_x(\xi^2) \\
&\equiv \eta_x^2 + \alpha_x\eta_\alpha^2 + u_x\eta_u^2 - \alpha_t(\xi_x^1 + \alpha_x\xi_\alpha^1 + u_x\xi_u^1) - \alpha_x(\xi_x^2 + \alpha_x\xi_\alpha^2 + u_x\xi_u^2) && (2.14)
\end{aligned}$$

and according to Eqs (2.9)

$$\begin{aligned}
\omega_1^1 &= \widetilde{D}_t(\mu^1) - \mathcal{P}_\alpha \widetilde{D}_t(\eta^1) \equiv \mu_t^1 - \mathcal{P}_\alpha \eta_t^1, && (2.15)\\
\omega_2^1 &= \widetilde{D}_x(\mu^1) - \mathcal{P}_\alpha \widetilde{D}_x(\eta^1) \equiv \mu_x^1 - \mathcal{P}_\alpha \eta_x^1, && (2.16)\\
\omega_3^1 &= \widetilde{D}_\alpha(\mu^1) - \mathcal{P}_\alpha \widetilde{D}_\alpha(\eta^1) \equiv \mu_\alpha^1 - \mathcal{P}_\alpha \eta_{\mathcal{P}}^1 - \mathcal{P}_\alpha \eta_\alpha^1, && (2.17)\\
\omega_4^1 &= \widetilde{D}_u(\mu^1) - \mathcal{P}_\alpha \widetilde{D}_u(\eta^1) \equiv \mu_u^1 - R_u \eta_R^1 - \mathcal{P}_\alpha \eta_u^1, && (2.18)\\
\omega_1^2 &= \widetilde{D}_t(\mu^2) - R_\alpha \widetilde{D}_t(\eta^1) - R_u \widetilde{D}_t(\eta^2) \equiv \mu_t^2 - R_\alpha \eta_t^1 - R_u \eta_t^2, && (2.19)\\
\omega_2^2 &= \widetilde{D}_x(\mu^2) - R_\alpha \widetilde{D}_x(\eta^1) - R_u \widetilde{D}_x(\eta^2) \equiv \mu_x^2 - R_\alpha \eta_x^1 - R_u \eta_x^2, && (2.20)
\end{aligned}$$

respectively, where

$$D_t = \partial_t + \alpha_t \partial_\alpha + u_t \partial_u \quad \text{and} \quad D_x = \partial_x + \alpha_x \partial_\alpha + u_x \partial_u$$

are the total derivative operators whereas

$$\widetilde{D}_t = \partial_t, \quad \widetilde{D}_x = \partial_x, \quad \widetilde{D}_\alpha = \partial_\alpha + \mathcal{P}_\alpha \partial_{\mathcal{P}} + R_\alpha \partial_R \quad \text{and} \quad \widetilde{D}_u = \partial_u + R_u \partial_R$$

are the total derivative operators for the extended system (2.8)–(2.9). The action of the prolonged operator (2.10) upon Eqs. (2.8)–(2.9) yields the determining equations

$$\begin{aligned}
\zeta_1^1 + u_x \eta^1 + \alpha \zeta_2^2 + \alpha_x \eta^2 + u \zeta_2^1 = 0, && (2.21)\\
\rho(\zeta_1^2 + u_x \eta^2 + u \zeta_2^2) + \mathcal{P}_\alpha \zeta_2^1 + \alpha_x \omega_3^1 + \mu^2 = 0, && (2.22)\\
\omega_1^1 = 0, \quad \omega_2^1 = 0, \quad \omega_4^1 = 0, \quad \omega_1^2 = 0, \quad \omega_2^2 = 0. && (2.23)
\end{aligned}$$

Firstly we consider Eq. (2.23) together with Eqs (2.15), (2.16), (2.18)–(2.20). Assuming that the arbitrary elements and their derivatives are arbitrary functions we obtain

$$\begin{aligned}
\omega_1^1 = \mu_t^1 - \mathcal{P}_\alpha \eta_t^1 = 0 \ &\Rightarrow \mu_t^1 = 0 \text{ and } \eta_t^1 = 0,\\
\omega_2^1 = \mu_x^1 - \mathcal{P}_\alpha \eta_x^1 = 0 \ &\Rightarrow \mu_x^1 = 0 \text{ and } \eta_x^1 = 0,\\
\omega_4^1 = \mu_u^1 - R_u \eta_R^1 - \mathcal{P}_\alpha \eta_u^1 = 0 \ &\Rightarrow \mu_u^1 = 0,\ \mu_R^1 = 0 \text{ and } \eta_u^1 = 0,\\
\omega_1^2 = \mu_t^2 - R_\alpha \eta_t^1 - R_u \eta_t^2 = 0 \ &\Rightarrow \mu_t^2 = 0,\ \eta_t^1 = 0 \text{ and } \eta_t^2 = 0,\\
\omega_2^2 = \mu_x^2 - R_\alpha \eta_x^1 - R_u \eta_x^2 = 0 \ &\Rightarrow \mu_x^2 = 0,\ \eta_x^1 = 0 \text{ and } \eta_x^2 = 0.
\end{aligned}$$

Therefore

$$\eta^1 = \eta^1(\alpha), \quad \eta^2 = \eta^2(\alpha, u), \quad \mu^1 = \mu^1(\alpha, \mathcal{P}), \quad \mu^2 = \mu^2(\alpha, u, \mathcal{P}, R). \qquad (2.24)$$

Next we consider Eq. (2.21). Taking into account Eqs (2.24) we substitute (2.11), (2.12) and (2.14) into Eq. (2.21). This is followed by equating to zero the coefficients of $\mathcal{P}_\alpha$ and the terms without $\mathcal{P}_\alpha$. Thereafter separating by α_x, u_x and their powers we obtain

$$\xi^1_x = 0, \quad \xi^2_u + \alpha\xi^1_\alpha - u\xi^1_u = 0, \tag{2.25}$$

$$\eta^1 - \alpha\eta^1_\alpha + \alpha(\eta^2_u - \xi^2_x + \xi^1_t) = 0, \quad \eta^2 + \alpha\eta^2_\alpha - u(\xi^2_x - \xi^1_t) - \xi^2_t = 0. \tag{2.26}$$

Now we consider Eq. (2.22). We substitute (2.12), (2.13), (2.14) and (2.17) into Eq. (2.22) taking into account Eqs (2.24). By equating to zero the coefficients of $\mathcal{P}_\alpha$ and $\mathcal{P}^2_\alpha$ and separating by α_x and α^2_x we obtain

$$\xi^1_u = 0, \quad \mu^1_{\mathcal{P}} = \eta^2_u + \xi^2_x - \xi^1_t, \quad \xi^2_\alpha - u\xi^1_\alpha = 0. \tag{2.27}$$

When we separate by α_x, u_x and their powers, the terms without $\mathcal{P}_\alpha$ result in

$$\mu^1_\alpha = 0, \tag{2.28}$$

$$\rho\left[\eta^2 - \alpha\eta^2_\alpha - u(\xi^2_x - \xi^1_t) - \xi^2_t\right] + (\xi^2_u - \alpha\xi^1_\alpha)R = 0, \tag{2.29}$$

$$\mu^2 = (\eta^2_u - \xi^1_t)R. \tag{2.30}$$

The first equations in (2.25) and (2.27) imply that $\xi^1 = \xi^1(t, \alpha)$. Hence the second equation in (2.25) reduces to

$$\xi^2_u + \alpha\xi^1_\alpha = 0. \tag{2.31}$$

In summary we have

$$\xi^1 = \xi^1(t, x), \quad \xi^2 = \xi^2(t, x, \alpha, u), \quad \eta^1 = \eta^1(\alpha), \quad \eta^2 = \eta^2(\alpha, u), \tag{2.32}$$

$$\mu^1 = \mu^1(\mathcal{P}), \quad \mu^2 = \mu^2(\alpha, u, R), \tag{2.33}$$

$$\xi^2_\alpha - u\xi^1_\alpha = 0, \tag{2.34}$$

$$\xi^2_u + \alpha\xi^1_\alpha = 0, \tag{2.35}$$

$$\eta^1 - \alpha\eta^1_\alpha + \alpha(\eta^2_u - \xi^2_x + \xi^1_t) = 0, \tag{2.36}$$

$$\eta^2 + \alpha\eta^2_\alpha - u(\xi^2_x - \xi^1_t) - \xi^2_t = 0, \tag{2.37}$$

$$\rho\left[\eta^2 - \alpha\eta^2_\alpha - u(\xi^2_x - \xi^1_t) - \xi^2_t\right] + (\xi^2_u - \alpha\xi^1_\alpha)R = 0, \tag{2.38}$$

$$\mu^1_{\mathcal{P}} = \eta^2_u + \xi^2_x - \xi^1_t, \tag{2.39}$$

$$\mu^2 = (\eta^2_u - \xi^1_t)R. \tag{2.40}$$

According to (2.32) and (2.33), Eqs (2.34)–(2.40) can be differentiated with respect to relevant variables to yield

$$\xi^1_{tt} = 0, \tag{2.41}$$

$$\xi^2_{tt} = \xi^2_{tx} = \xi^2_{xx} = \xi^2_{x\alpha} = \xi^2_{xu} = \xi^2_{uu} = 0, \tag{2.42}$$

$$\xi^2_{xt} - \xi^1_{tt} = 0, \quad \xi^2_{ut} - \alpha\xi^1_{\alpha t} = 0, \tag{2.43}$$

$$\eta^2_{uu} = 0, \quad \eta^2_{u\alpha} - \xi^1_{t\alpha} = 0. \tag{2.44}$$

The solution of Eqs (2.34)–(2.44) yields the general solution of the determining equations (2.21)–(2.23) given by

$$\xi^1 = C_1 t + C_2, \quad \xi^2 = C_3 x + C_4 t + C_5, \quad \eta^1 = C_6\alpha, \quad \eta^2 = (C_3 - C_1)u + C_4,$$
$$\mu^1 = 2(C_3 - C_1)\mathcal{P} + C_7, \quad \mu^2 = (C_3 - 2C_1)R, \tag{2.45}$$

where $C_1,\ C_2, \ldots,\ C_7$ are the arbitrary constants of integration. Thus the generator of the equivalence group is

$$Y = (C_1 t + C_2)\partial_t + [C_3 x + C_4 t + C_5]\partial_x + C_6\alpha\partial_\alpha + [(C_3 - C_1)u + C_4]\partial_u$$
$$+[2(C_3 - C_1)\mathcal{P} + C_7]\partial_{\mathcal{P}} + (C_3 - 2C_1)R\partial_R. \tag{2.46}$$

Therefore the underlying system (2.5) has a seven-dimensional equivalence Lie algebra spanned by the operators

$$Y_1 = \partial_t, \quad Y_2 = \partial_x, \quad Y_3 = \partial_{\mathcal{P}}, \quad Y_4 = t\partial_t - u\partial_u - 2\mathcal{P}\partial_{\mathcal{P}} - 2R\partial_R,$$
$$Y_5 = x\partial_x + u\partial_u + 2\mathcal{P}\partial\mathcal{P} + R\partial_R, \quad Y_6 = t\partial_x + \partial_u, \quad Y_7 = \alpha\partial_\alpha. \tag{2.47}$$

The group of equivalence transformations includes the following discrete transformations:

$$t \to -t, \quad \alpha \to -\alpha, \quad u \to -u, \tag{2.48}$$

$$t \to -t, \quad u \to -u, \quad \mathcal{P} \to -\mathcal{P}, \quad R \to -R, \tag{2.49}$$

$$\alpha \to -\alpha. \tag{2.50}$$

We use the theorem on projections of equivalence Lie algebras [30, 31, 34] to find the principal Lie algebra admitted by system (2.5).

Consider the following projections of the equivalence generator (2.7):

$$X = \mathrm{pr}_{(t,x,\alpha,u)}(Y) \equiv \xi^1\partial_t + \xi^2\partial_x + \eta^1\partial_\alpha + \eta^2\partial_u, \tag{2.51}$$

$$Z = \mathrm{pr}_{(\alpha,u,\mathcal{P},R)}(Y) \equiv \eta^1\partial_\alpha + \eta^2\partial_u + \mu^1\partial_{\mathcal{P}} + \mu^2\partial_R, \tag{2.52}$$

where $\mathrm{pr}_{(\alpha,u,\alpha,u)}$ denotes projection onto the space (t, x, α, u) and $\mathrm{pr}_{(\alpha,u,\mathcal{P},R)}$ onto the space $(\alpha, u, \mathcal{P}, R)$.

An operator X belongs to the principal Lie algebra if and only if

$$Z = \mathrm{pr}_{(\alpha,u,\mathcal{P},R)}(Y) = 0. \tag{2.53}$$

According to (2.46) and (2.52), Eq. (2.53) is written

$$C_6\alpha\partial_\alpha + [(C_3 - C_1)u + C_4]\partial_u + [2(C_3 - C_1)\mathcal{P} + C_7]\partial_{\mathcal{P}} + (C_3 - 2C_1)R\partial_R = 0.$$

Thus

$$C_1 = 0, \quad C_3 = 0, \quad C_4 = 0, \quad C_6 = 0, \quad C_7 = 0.$$

Hence in view of (2.46) and (2.51)

$$Y = C_2\partial_t + C_5\partial_x$$

and we conclude that the principal Lie algebra of system (2.5) is two-dimensional and spanned by the operators

$$X_1 = \partial_t, \quad X_2 = \partial_x\cdot \tag{2.54}$$

2.3 Classical Lie's algorithm

According to Lie's algorithm [10, 65, 67], the infinitesimal generator of the maximal symmetry group admitted by (2.5) is

$$X = \xi^1(t, x, \alpha, u)\partial_t + \xi^2(t, x, \alpha, u)\partial_x + \eta^1(t, x, \alpha, u)\partial_\alpha + \eta^2(t, x, \alpha, u)\partial_u \tag{2.55}$$

if and only if the invariance conditions of system (2.5) are

$$\begin{gathered}\widetilde{X}\,(\alpha_t + \alpha u_x + u\alpha_x) = 0,\\ \widetilde{X}\Big(\rho(u_t + uu_x) + \alpha_x\mathcal{P}_\alpha + R(\alpha, u)\Big) = 0,\end{gathered} \tag{2.56}$$

where

$$\widetilde{X} = X + \zeta_1^1\partial_{\alpha_t} + \zeta_2^1\partial_{\alpha_x} + \zeta_1^2\partial_{u_t} + \zeta_2^2\partial_{u_x}$$

is the prolongation of the operator (2.55). The variables ζ_j^i are as given in Eqs. (2.11)–(2.14).

The coefficients ξ^i and η^i of the symmetry generator (2.55) do not involve the derivatives of the dependent variables. Thus we can separate (2.56) with respect to the derivatives of the dependent variables and their powers to obtain an over-determined system of linear homogeneous PDEs. The following over-determined system of equations was generated manually and confirmed by the software package `YaLie`.

$$u\xi_u^1 - \xi_u^2 - \alpha\xi_\alpha^1 = 0, \tag{2.57}$$

$$\rho(\eta_t^1 + u\eta_x^1 + \alpha\eta_x^2) + (\alpha\xi_x^1 - \eta_u^1)R = 0, \tag{2.58}$$

$$\alpha\eta_u^2 + \eta^1 - \alpha\eta_\alpha^1 - \alpha\xi_x^2 + 2\alpha u\xi_x^1 + \alpha\xi_t^1 = 0, \tag{2.59}$$

$$\rho(u\xi_t^1 + u^2\xi_x^1 - \xi_t^2 - u\xi_x^2 + \eta^2 + \alpha\eta_\alpha^2) + \left(\alpha\xi_x^1 - \eta_u^1\right)\mathcal{P}' = 0, \tag{2.60}$$

$$\xi_u^1\mathcal{P}' + \rho(\xi_\alpha^2 - u\xi_\alpha^1) = 0, \tag{2.61}$$

$$\rho^2(\eta_t^2 + u\eta_x^2) + \rho\left[\eta_x^1\mathcal{P}' + \eta^1 R_\alpha + \eta^2 R_u + (\xi_t^1 + u\xi_x^1 - \eta_u^2)R\right] - \xi_u^1 R^2 = 0, \tag{2.62}$$

$$\rho\eta^1\mathcal{P}'' + \left[\rho(\xi_t^1 + 2u\xi_x^1 - \xi_x^2 + \eta_\alpha^1 - \eta_u^2) - 2\xi_u^1 R\right]\mathcal{P}' = 0, \tag{2.63}$$

$$\begin{aligned}\rho(u\xi_t^1 + u^2\xi_x^1 - \xi_t^2 - u\xi_x^2 + \eta^2 - \alpha\eta_\alpha^2) + \left(\alpha\xi_x^1 + \eta_u^1\right)\mathcal{P}' \\ -(u\xi_u^1 - \xi_u^2 + \alpha\xi_\alpha^1)R = 0,\end{aligned} \tag{2.64}$$

where a prime denotes differentiation with respect to α and subscripts denote partial differentiation.

From Eqs. (2.57), (2.60) and (2.64) we obtain

$$\alpha(\rho\eta_\alpha^2 + \xi_\alpha^1 R) - \eta_u^1\mathcal{P}' = 0. \tag{2.65}$$

Since $\rho > 0$ and we require $R > 0$, Eq. (2.65) holds provided $\xi_\alpha^1 = 0$, $\eta_\alpha^2 = 0$ and $\eta_u^1 = 0$ or $\mathcal{P}'(\alpha) = 0$.

We proceed with the analysis of the above determining equations by considering the cases: $\mathcal{P}'(\alpha) \neq 0$ and $\mathcal{P}'(\alpha) = 0$.

Case I. $\mathcal{P}'(\alpha) \neq 0$.

From Eq. (2.65) we obtain

$$\xi^1 = \xi^1(t,x,u), \quad \eta^1 = \eta^1(t,x,\alpha), \quad \eta^2 = \eta^2(t,x,u). \tag{2.66}$$

According to the first equation in (2.66), Eqs. (2.57) and (2.61) imply that

$$\xi^1 = \xi^1(t,x), \quad \xi^2 = \xi^2(t,x). \tag{2.67}$$

When we take into account the last two equations in (2.66) and Eq. (2.67), the determining equations are

$$\rho(\eta^1_t + u\eta^1_x + \alpha\eta^2_x) + \alpha\xi^1_x R = 0, \tag{2.68}$$

$$\alpha\eta^2_u + \eta^1 - \alpha\eta^1_\alpha - \alpha\xi^2_x + 2\alpha u\xi^1_x + \alpha\xi^1_t = 0, \tag{2.69}$$

$$\rho(u\xi^1_t + u^2\xi^1_x - \xi^2_t - u\xi^2_x + \eta^2) + \alpha\xi^1_x\mathcal{P}' = 0, \tag{2.70}$$

$$\rho(\eta^2_t + u\eta^2_x) + \eta^1_x\mathcal{P}' + \eta^1 R_\alpha + \eta^2 R_u + (\xi^1_t + u\xi^1_x - \eta^2_u)R = 0, \tag{2.71}$$

$$\eta^1\mathcal{P}'' + \left(\xi^1_t + 2u\xi^1_x - \xi^2_x + \eta^1_\alpha - \eta^2_u\right)\mathcal{P}' = 0. \tag{2.72}$$

When we using reasoning similar to the above, Eq. (2.68) implies that $\eta^1_t + u\eta^1_x + \alpha\eta^2_x = 0$ and $\xi^1_x = 0$. Thus the remaining determining equations (2.68)–(2.72) are

$$\eta^1_t + u\eta^1_x + \alpha\eta^2_x = 0, \tag{2.73}$$

$$\alpha\eta^2_u + \eta^1 - \alpha\eta^1_\alpha - \alpha\xi^2_x + \alpha\xi^1_t = 0, \tag{2.74}$$

$$u\xi^1_t - \xi^2_t - u\xi^2_x + \eta^2 = 0, \tag{2.75}$$

$$\rho(\eta^2_t + u\eta^2_x) + \eta^1_x\mathcal{P}' + \eta^1 R_\alpha + \eta^2 R_u + (\xi^1_t - \eta^2_u)R = 0, \tag{2.76}$$

$$\eta^1\mathcal{P}'' + \left(\xi^1_t - \xi^2_x + \eta^1_\alpha - \eta^2_u\right)\mathcal{P}' = 0. \tag{2.77}$$

Solving Eqs. (2.100) and (2.101) we obtain

$$\eta^1 = C(t,x)\alpha, \quad \eta^2 = (\xi^2_x - \xi^1_t)u + \xi^2_t, \tag{2.78}$$

where C is an arbitrary function. Therefore in summary we have

$$\xi^1 = \xi^1(t), \quad \xi^2 = \xi^2(t,x), \quad \eta^1 = C(t,x)\alpha, \quad \eta^2 = (\xi^2_x - \xi^1_t)u + \xi^2_t \tag{2.79}$$

and the determining equations are

$$C_t + u\left(C_x + \xi^2_{xx}\right) + \xi^2_{tx} = 0, \tag{2.80}$$

$$C\alpha\mathcal{P}'' + \left[C + 2(\xi^1_t - \xi^2_x)\right]\mathcal{P}' = 0, \tag{2.81}$$

$$\rho\left[(2\xi^2_{xt} - \xi^1_{tt})u + \xi^2_{tt} + u^2\xi^2_{xx}\right] + C_x\alpha\mathcal{P}' + C\alpha R_\alpha + \left[(\xi^2_x - \xi^1_t)u + \xi^2_t\right]R_u + (2\xi^1_t - \xi^2_x)R = 0. \tag{2.82}$$

Therefore the symmetry generator (2.55) takes the form

$$X = \xi^1(t)\partial_t + \xi^2(t,x)\partial_x + C(t,x)\alpha\partial_\alpha + \left[(\xi^2_x - \xi^1_t)u + \xi^2_t\right]\partial_u. \tag{2.83}$$

In order to satisfy the remaining determining equations (2.80)–(2.82), since R depends upon more than one variable, it becomes difficult to employ the procedure discussed in [32, 67]. However, this procedure can be used to solve for $\mathcal{P}(\alpha)$ in Eq. (2.81) and hence leaving only $R(\alpha, u)$ to be solved from Eq. (2.82). Since the direct Lie symmetry classification is difficult to implement, instead we make use of classification with respect to three- and four-dimensional real Lie algebras by Patera and Winternitz [69] to solve Eqs. (2.80)–(2.82).

2.4 Classification with respect to three-dimensional Lie algebras

We consider the three-dimensional (3D) Lie algebras having the basis (e_1, e_2, e_3). The two operators are those that span the principal Lie algebra and a third operator (which is to be obtained) is in form of the symmetry generator (2.83). Thus in order to obtain the third operator we consider the following possible realizations:

(a) $e_1 = \partial_t, \quad e_2 = \partial_x, \quad e_3 = X_{tn}$ (b) $e_1 = \partial_t, \quad e_2 = X_{tn}, \quad e_3 = \partial_x$

(c) $e_1 = \partial_x, \quad e_2 = \partial_t, \quad e_3 = X_{tn}$ (d) $e_1 = \partial_x, \quad e_2 = X_{tn}, \quad e_3 = \partial_t$

(e) $e_1 = X_{tn}, \quad e_2 = \partial_t, \quad e_3 = \partial_x$ (f) $e_1 = X_{tn}, \quad e_2 = \partial_x, \quad e_3 = \partial_t$

where X_{tn} is the extension of the symmetry Lie algebra of the form

$$X_{tn} = A(t)\partial_t + B(t,x)\partial_x + C(t,x)\alpha\partial_\alpha + [(B_x - A_t)u + B_t]\partial_u. \tag{2.84}$$

Our goal is to find the functions A, B, C for which X_{tn} is a symmetry generator of the underlying system and thus obtaining the corresponding forms of $\mathcal{P}$ and R.

We illustrate the calculations involved by considering the possibilities (a)–(f) for the algebras $3A_1$ and $A_1 \oplus A_2$.

Firstly we consider $3A_1$ with the Lie Brackets:
$[e_1, e_2] = 0,\ [e_1, e_3] = 0,\ [e_2, e_3] = 0.$

(a) The first Lie Bracket is identically satisfied. The second Lie Bracket implies that

$$e_3 = c_1\partial_t + B(x)\partial_x + C(x)\alpha\partial_\alpha + B_x u\partial_u$$

for c_1 arbitrary. The last Lie Bracket yields

$$e_3 = c_1\partial_t + c_2\partial_x + c_3\alpha\partial_\alpha,$$

where c_2 and c_3 are arbitrary constants.

Therefore the determining equations (2.80)–(2.82) reduce to

$$c_3(\alpha\mathcal{P}'' + \mathcal{P}') = 0, \quad c_3\alpha R_\alpha = 0. \tag{2.85}$$

We take $e_3 = \alpha\partial_\alpha$ and hence the corresponding forms of the arbitrary functions from the classifying equations (2.85) are

$$\mathcal{P} = k_1 \ln\alpha + k_2, \quad R = f(u), \tag{2.86}$$

where $k_1 \neq 0,\ k_2$ are arbitrary constants and f is an arbitrary function of u.

We reach the same conclusion for the remaining possibilities (b)–(f).

Secondly consider $A_1 \oplus A_2$ with the Lie Brackets:
$[e_1, e_2] = e_2,\ [e_1, e_3] = 0,\ [e_2, e_3] = 0.$

The first Lie Bracket implies that (b), (d), (e) and (f) are the only possible realizations.

(b) $[e_1, e_2] = e_2$ implies that

$$e_2 = e^t[\bar{c}_1\partial_t + b(x)\partial_x + c(x)\alpha\partial_\alpha + [(b_x - \bar{c}_1)u + b]\partial_u].$$

The second Lie Bracket is identically satisfied. The third Lie Bracket yields

$$e_2 = e^t[\bar{c}_1\partial_t + \bar{c}_2\partial_x + \bar{c}_3\alpha\partial_\alpha + (\bar{c}_2 - \bar{c}_1 u)\partial_u],$$

where $\bar{c}_1$, $\bar{c}_2$ and $\bar{c}_3$ are arbitrary constants.

The determining equations (2.80)–(2.82) reduce to

$$\bar{c}_3 e^t \alpha = 0, \tag{2.87}$$

$$e^t[\bar{c}_3\alpha\mathcal{P}'' + (2\bar{c}_1 + \bar{c}_3)\mathcal{P}'] = 0, \tag{2.88}$$

$$e^t[\bar{c}_3\alpha R_\alpha + (\bar{c}_2 - \bar{c}_1 u)R_u + 2\bar{c}_1 R + \rho(\bar{c}_2 - \bar{c}_1 u)] = 0. \tag{2.89}$$

With $\bar{c}_3 = 0$ Eq. (2.88) implies that $\bar{c}_1 = 0$ and $\mathcal{P}(\alpha)$ is arbitrary since we require that $\mathcal{P}'(\alpha) \neq 0$. Thus $e_2 = e^t(\partial_x + \partial_u)$ and solving Eq. (2.89) we obtain

$$R = g(\alpha) - \rho u, \tag{2.90}$$

where g is an arbitrary function of α.

(e) $[e_1, e_2] = e_2$ and $[e_1, e_3] = 0$ imply that

$$e_1 = -t\partial_t + \tilde{c}_3\alpha\partial_\alpha + u\partial_u$$

for arbitrary $\tilde{c}_3$. The determining equations (2.80)–(2.82) become

$$\tilde{c}_3\alpha\mathcal{P}'' + (\tilde{c}_3 - 2)\mathcal{P}' = 0 \quad \text{and} \quad \tilde{c}_3\alpha R_\alpha + uR_u - 2R = 0. \tag{2.91}$$

Solving (2.91) for nonzero $\tilde{c}_3$ we obtain

$$\mathcal{P} = \frac{1}{2}\tilde{k}_1\tilde{c}_3\alpha^{2/\tilde{c}_3} + \tilde{k}_2, \quad R = \alpha^{2/\tilde{c}_3}\tilde{f}(u\alpha^{-1/\tilde{c}_3}), \tag{2.92}$$

where $\tilde{k}_1 \neq 0$, $\tilde{k}_2$ are arbitrary constants and $\tilde{f}$ is an arbitrary function of its argument.

The complete results are presented in Table 2.1, excluding the cases which yield $R = 0$.

In addition to the possible realizations (a)–(f) we also consider the 3D Lie algebras which contain the two-dimensional subalgebras $2A_1$ and A_2 (c.f. Table I [69]). The 2D subalgebra, $2A_1$ yields the same results as those presented in Table 2.1.

Table 2.1: Classification with respect to three-dimensional Lie algebras for $\mathcal{P}'(\alpha) \neq 0$.

Algebra	$\mathcal{P}$	R	Condition on consts.	Extra operator
$3A_1$	$k_1 \ln\alpha + k_2$	$\Gamma_1(u)$	$k_1 \neq 0$	$\alpha\partial_\alpha$
$A_1 \oplus A_2$	Arbitrary	$\Gamma_2(\alpha) - \rho u$		$e^t(\partial_x + \partial_u)$
	$\bar{k}_2 - \dfrac{\bar{k}_1}{2\alpha^2}$	$\dfrac{\Gamma_3(\alpha u)}{\alpha} + \dfrac{\bar{k}_1}{\alpha^2} - \rho u^2$	$\bar{k}_1 \neq 0$	$e^x(\partial_x - \alpha\partial_\alpha + u\partial_u)$
	$\dfrac{\bar{k}_3}{2} k\alpha^{2/k} + \bar{k}_4$	$\alpha^{2/k}\Gamma_4(u\alpha^{-1/k})$	$\bar{k}_3, k \neq 0$	$t\partial_t - k\alpha\partial_\alpha - u\partial_u$
	$\bar{k}_6 - \dfrac{\bar{k}_5}{2}\lambda\alpha^{-2/\lambda}$	$\alpha^{-1/\lambda}\Gamma_5(u\alpha^{1/\lambda})$	$\bar{k}_5, \lambda \neq 0$	$x\partial_x - \lambda\alpha\partial_\alpha + u\partial_u$
$A_{3,1}$	$k_3 \ln\alpha + k_4$	$\Gamma_6\left(u - \frac{\ln\alpha}{h}\right)$	$k_3, h \neq 0$	$t\partial_x + h\alpha\partial_\alpha + \partial_u$
	Arbitrary	$\Gamma_7(\alpha)$	$h = 0$	$t\partial_x + \partial_u$
$A_{3,2}$	$k_5 \ln\alpha + k_6$	$\alpha^{-1/\kappa}\Gamma_8\left(u - \frac{\ln\alpha}{\kappa}\right)$	$k_5, \kappa \neq 0$	$t\partial_t + (t+x)\partial_x + \kappa\alpha\partial_\alpha + \partial_u$
	Arbitrary	$e^{-u}\Gamma_9(\alpha)$	$\kappa = 0$	$t\partial_t + (t+x)\partial_x + \partial_u$
$A_{3,3}$	$k_7 \ln\alpha + k_8$	$\alpha^{-1/\ell}\Gamma_{10}(u)$	$k_7, \ell \neq 0$	$t\partial_t + x\partial_x + \ell\alpha\partial_\alpha$
$A_{3,4}$	$\tilde{k}_2 - \dfrac{\tilde{k}_1}{4} m\alpha^{-4/m}$	$\alpha^{-3/m}\Gamma_{11}(u\alpha^{2/m})$	$\tilde{k}_1, m \neq 0$	$t\partial_t - x\partial_x + m\alpha\partial_\alpha - 2u\partial_u$
$A^a_{3,5}$	$\dfrac{\hat{k}_1 n}{2(a-1)}\alpha^{\frac{2(a-1)}{n}} + \hat{k}_2$	$\alpha^{\frac{a-2}{n}}\Gamma_{12}(u\alpha^{\frac{1-a}{n}})$	$\hat{k}_1, n \neq 0$	$t\partial_t + ax\partial_x + n\alpha\partial_\alpha + (a-1)u\partial_u$
	$\dfrac{\hat{k}_3\bar{n}}{2(1-a)}\alpha^{\frac{2(1-a)}{\bar{n}}} + \hat{k}_4$	$\alpha^{\frac{1-2a}{\bar{n}}}\Gamma_{13}(u\alpha^{\frac{a-1}{\bar{n}}})$	$\hat{k}_5, \bar{n} \neq 0$	$at\partial_t + x\partial_x + \bar{n}\alpha\partial_\alpha + (1-a)u\partial_u$

The Γ_i are arbitrary functions and $0 < |a| < 1$.

We show some calculations for the 3D Lie algebra $A_1 \oplus A_2$ containing A_2 with the basis $(e_1 + pe_3; e_2)$, where $p \neq 0$. The case $p = 0$ leads to the cases which are already treated.

Let $\hat{e}_1 = e_1 + pe_3$, $\hat{e}_2 = e_2$, $\hat{e}_3 = e_1$ or e_3, where e_1, e_3 can be either ∂_t or ∂_x and $\hat{e}_2$ is of the form (2.84). Likewise, we can have $\hat{e}_1 = e_1$ or e_3, $\hat{e}_2 = e_2$, $\hat{e}_3 = e_1 + pe_3$.

Consider $\hat{e}_1 = \partial_t + p\partial_x$, $\hat{e}_2 = e_2$, $\hat{e}_3 = \partial_t$. The Lie Brackets $[\hat{e}_1, \hat{e}_2] = \hat{e}_2$ and $[\hat{e}_2, \hat{e}_3] = 0$ yield

$$e_2 = e^{x/p}\left[k_2\partial_x + k_3\alpha\partial_\alpha + (k_2u/p)\partial_u\right]$$

for arbitrary constants k_2 and k_3.

Eventually e_2 takes the form $e_2 = e^{x/p}\left(p\partial_x - \alpha\partial_\alpha + u\partial_u\right)$ and the corresponding arbitrary elements are

$$R = \frac{\hat{k}_1 - \rho\alpha^2u^2 + p\alpha\Gamma(\alpha u)}{p\alpha^2}, \quad \mathcal{P} = \hat{k}_2 - \frac{\hat{k}_1}{2\alpha^2}, \tag{2.93}$$

where $\hat{k}_1$, $\hat{k}_2$ are arbitrary constants and $\Gamma(\alpha u)$ is an arbitrary function.

The summary of results is given in Table 2.2.

2.5 Classification with respect to four-dimensional Lie algebras

Since each four-dimensional (4D) Lie algebra contains a 3D subalgebra, equations admitting a 4D symmetry Lie algebra stem from specification of arbitrary functions in equations admitting a 3D symmetry Lie algebra. Thus we have to examine all 4D Lie algebras admitting the previously obtained 3D symmetry Lie algebras (given in Table 2.1) as subalgebras.

We follow the same procedure used in the classification with respect to 3D Lie algebra and the cases which lead to $R = 0$ are discarded. Below we show some calculations for the 4D Lie algebra containing 3D Lie algebra $A_1 \oplus A_2$ as subalgebra.

The algebra $2A_2$ with the nonzero Lie Brackets $[e_1, e_2] = e_2$ and $[e_3, e_4] = e_4$ is the only 4D Lie algebra containing the 3D Lie algebra $A_1 \oplus A_2$.

Consider the possible realization $(\partial_x, e_2, -t\partial_t + k\alpha\partial_\alpha + u\partial_u, \partial_t)$: the Lie Brackets $[e_1, e_2] = e_2$ and $[e_2, e_4] = 0$ of the algebra $2A_2$ give $e_2 = e^x(c_2\partial_x + c_3\alpha\partial_\alpha + c_2u\partial_u)$, for arbitrary

Table 2.2: Classification with respect to two-dimensional subalgebra of $A_1 \oplus A_2$ for $\mathcal{P}'(\alpha) \neq 0$.

$\mathcal{P}$	R	Condition on consts.	Operators
$\hat{k}_2 - \dfrac{\hat{k}_1}{2\alpha^2}$	$\dfrac{\hat{k}_1 - \rho\alpha^2 u^2 + p\alpha\Gamma(\alpha u)}{p\alpha^2}$	$\hat{k}_1 \neq 0$	$\partial_t,\ \partial_t + p\partial_x,$ $e^{x/p}\,(p\partial_x - \alpha\partial_\alpha + u\partial_u)$
Arbitrary	$\Gamma(\alpha) - \dfrac{\rho u}{p}$		$\partial_x,\ p\partial_t + \partial_x,\ e^{t/p}\,(p\partial_x + \partial_u)$
$\hat{k}_4 - \dfrac{\hat{k}_3}{2\alpha^2}$	$\dfrac{\rho\alpha^2(u-p)^2 - \hat{k}_3 + p\alpha\Gamma(\alpha u - p\alpha)}{p\alpha^2}$	$\hat{k}_3 \neq 0$	$\partial_t,\ \partial_t + p\partial_x,$ $e^{t-x/p}\,[p\partial_x + \alpha\partial_\alpha + (p-u)\partial_u]$
$\hat{k}_6 - \dfrac{\hat{k}_5}{2\alpha^2}$	$\dfrac{\hat{k}_5 - \rho\alpha^2(u-p)^2 + \alpha\Gamma(\alpha u - p\alpha)}{\alpha^2}$	$\hat{k}_5 \neq 0$	$\partial_x,\ \partial_t + p\partial_x,$ $e^{x-pt}\,[\partial_x - \alpha\partial_\alpha + (u-p)\partial_u]$
$\hat{k}_8 - \dfrac{\hat{k}_7}{2\alpha^2}$	$\dfrac{\rho\alpha^2(1-pu)^2 - \hat{k}_7 p^2 + p\alpha\Gamma(\alpha u - \alpha/p)}{p\alpha^2}$	$\hat{k}_7 \neq 0$	$\partial_t,\ p\partial_t + \partial_x,$ $e^{t-px}\,[\partial_x + p\alpha\partial_\alpha + (1-pu)\partial_u]$
$\hat{k}_{10} - \dfrac{\hat{k}_9}{2\alpha^2}$	$\dfrac{\hat{k}_9 p^2 - \rho\alpha^2(1-pu)^2 + p^2\alpha\Gamma(\alpha u - \alpha/p)}{p^2\alpha^2}$	$\hat{k}_9 \neq 0$	$\partial_x,\ p\partial_t + \partial_x,$ $e^{x-t/p}\,[p\partial_x - p\alpha\partial_\alpha + (pu-1)\partial_u]$

The arbitrary functions, Γ are in general different and $p \neq 0$.

constants c_2 and c_3. The determining equations (2.80)–(2.82) become

$$e^x \alpha u(c_2 + c_3) = 0, \tag{2.94}$$

$$\bar{k}_3 e^x \alpha^{2/k}(kc_2 - c_3) = 0, \tag{2.95}$$

$$e^x[k(c_2\rho u^2 + c_3\bar{k}_3\alpha^{2/k}) - (kc_2 - 2c_3)\alpha^{2/k}\Gamma_4(u\alpha^{-1/k}) + (kc_2 - c_3)u\alpha^{1/k}\Gamma_4'(u\alpha^{-1/k})] = 0. \tag{2.96}$$

The first two equations (2.94) and (2.95) imply that $c_3 = -c_2$ and thus $k = -1$. Therefore $e_2 = e^x(\partial_x - \alpha\partial_\alpha + u\partial_u)$, $e_3 = -t\partial_t - \alpha\partial_\alpha + u\partial_u$ and $\Gamma_4 = \bar{k}_3 - \rho\alpha^2 u^2$, i.e., $R = \bar{k}_3\alpha^{-2} - \rho u^2$.

The complete results are presented in Table 2.3.

Table 2.3: Classification with respect to four-dimensional Lie algebras for $\mathcal{P}'(\alpha) \neq 0$.

$\mathcal{P}$	R	Condition on consts.	Operators
$k_3 \ln\alpha + k_4$	$\ell_1 \alpha^{-1/\bar{\ell}} \exp[hu/\bar{\ell}\,]$	$\bar{\ell}, h \neq 0$	$t\partial_x + h\alpha\partial_\alpha + \partial_u,\ t\partial_t + x\partial_x + \bar{\ell}\alpha\partial_\alpha$
$k_7 \ln\alpha + k_8$	$\ell_2 \alpha^{-1/\ell}$	$k_7, \ell \neq 0$	$t\partial_x + \partial_u,\ t\partial_t + x\partial_x + \ell\alpha\partial_\alpha$
$\bar{k}_4 - \dfrac{\bar{k}_3}{2\alpha^2}$	$\rho u^2 - \dfrac{\bar{k}_3}{\alpha^2}$	$\bar{k}_3 \neq 0$	$e^x(\partial_x - \alpha\partial_\alpha + u\partial_u),\ t\partial_t + \alpha\partial_\alpha - u\partial_u$
$\bar{k}_8 - \dfrac{\bar{k}_7}{2}\lambda\alpha^{-2/\lambda}$	$\ell_3 \alpha^{-1/\lambda} - \rho u$	$\bar{k}_7, \lambda \neq 0$	$e^t(\partial_x + \partial_u),\ x\partial_x - \lambda\alpha\partial_\alpha + u\partial_u$
$\lambda_2 - \dfrac{\lambda_1}{2}\gamma\alpha^{-2/\gamma}$	$\ell_4 \alpha^{-2/\gamma}$	$\lambda_1, \gamma \neq 0$	$t\partial_x + \partial_u,\ t\partial_t + \gamma\alpha\partial_\alpha - u\partial_u$
$\dfrac{\lambda_3}{2b}\nu\alpha^{\frac{2b}{\nu}} + \lambda_4$	$\ell_5 \alpha^{\frac{b-1}{\nu}}$	$\lambda_3, \nu \neq 0$	$t\partial_x + \partial_u,\ t\partial_t + (1+b)x\partial_x + \nu\alpha\partial_\alpha + bu\partial_u$
$\dfrac{\lambda_5}{2}\sigma\alpha^{2/\sigma} + \lambda_6$	ℓ_6	$\lambda_5, \sigma \neq 0$	$t\partial_x + \partial_u,\ t\partial_t + 2x\partial_x + \sigma\alpha\partial_\alpha + u\partial_u$
$\dfrac{\hat{k}_1 n}{2(a-1)}\alpha^{\frac{2(a-1)}{n}} + \hat{k}_2$	$\ell_7 \alpha^{\frac{a-2}{n}}$	$\hat{k}_1, n \neq 0$	$t\partial_x + \partial_u,\ t\partial_t + ax\partial_x + n\alpha\partial_\alpha + (a-1)u\partial_u$
$\dfrac{\hat{k}_3 \bar{n}}{2(1-a)}\alpha^{\frac{2(1-a)}{\bar{n}}} + \hat{k}_4$	$\ell_8 \alpha^{\frac{1-2a}{\bar{n}}}$	$\hat{k}_3, \bar{n} \neq 0$	$t\partial_x + \partial_u,\ at\partial_t + x\partial_x + \bar{n}\alpha\partial_\alpha + (1-a)u\partial_u$

The ℓ_i are arbitrary nonzero constants and $0 < |b| < 1$. The symmetry Lie algebra for each case includes the operators ∂_t and ∂_x.

It can be shown by direct calculation that there are no realizations of the 4D Lie algebras containing the 3D Lie algebras from Table 2.2.

Case II. $\mathcal{P}'(\alpha) = 0$, i.e., $\mathcal{P} = \mathcal{P}_0 \equiv$ constant.

According to Eq (2.65) the symmetry generator (2.55) is given by

$$X = \xi^1(t,x,u)\partial_t + \xi^2(t,x,u)\partial_x + \eta^1(t,x,\alpha,u)\partial_\alpha + \eta^2(t,x,u)\partial_u. \tag{2.97}$$

The corresponding determining equations are

$$u\xi^1_u - \xi^2_u = 0, \tag{2.98}$$

$$\rho(\eta^1_t + u\eta^1_x + \alpha\eta^2_x) + (\alpha\xi^1_x - \eta^1_u)R = 0, \tag{2.99}$$

$$\alpha\eta^2_u + \eta^1 - \alpha\eta^1_\alpha - \alpha\xi^2_x + 2\alpha u\xi^1_x + \alpha\xi^1_t = 0, \tag{2.100}$$

$$u\xi^1_t + u^2\xi^1_x - \xi^2_t - u\xi^2_x + \eta^2 = 0, \tag{2.101}$$

$$\rho^2(\eta^2_t + u\eta^2_x) + \rho\left[\eta^1 R_\alpha + \eta^2 R_u + (\xi^1_t + u\xi^1_x - \eta^2_u)R\right] - \xi^1_u R^2 = 0, \tag{2.102}$$

$$\rho(u\xi^1_t + u^2\xi^1_x - \xi^2_t - u\xi^2_x + \eta^2) - (u\xi^1_u - \xi^2_u)R = 0. \tag{2.103}$$

Proceeding as in the previous case our aim is to find an extra operator in the form of (2.97). Hence we obtain the corresponding form of R whenever the extra operator exists.

We give an illustration of some calculations for the 3D Lie algebra $A_1 \oplus A_2$ and the 4D Lie algebra containing this algebra.

Consider firstly the 3D Lie algebra having the realization $(e_1, \partial_x, \partial_t)$. The Lie Brackets $[e_1, e_2] = e_2$ and $[e_1, e_3] = 0$ yield

$$e_1 = a(u)\partial_t + (-x + b(u))\partial_x + C(\alpha, u)\partial_\alpha - d(u)\partial_u. \tag{2.104}$$

Substituting the forms of ξs and ηs from (2.104) into the determining equations (2.98)–(2.103) we have

$$e_1 = a(u)\partial_t + \left(-x + a(u)\,u - \int a(u)\,du + C_2\right)\partial_x + C_3\alpha\partial_\alpha - u\partial_u, \tag{2.105}$$

where C_2 and C_3 are arbitrary constants.

From (2.105) we investigate the cases: $a'(u) \neq 0$ and $a'(u) = 0$. Firstly for $a'(u) \neq 0$ and $C_3 \neq 0$ we take

$$e_1 = u\partial_t + \left(-x + \frac{u^2}{2}\right)\partial_x + C_3\alpha\partial_\alpha - u\partial_u. \tag{2.106}$$

Thus $R = -\rho/(\alpha^{1/C_3}\exp\left[\rho\Phi(u\alpha^{1/C_3})\right] - 1)$, where Φ is an arbitrary function of its argument.

For $C_3 = 0$, $R = -\rho u/(\exp\left[\rho\bar{\Phi}(\alpha)\right] - u)$, where $\bar{\Phi}$ is an arbitrary function of α.

Secondly for $a'(u) = 0$ and $C_3 \neq 0$ the operator (2.105) takes the form

$$e_1 = -x\partial_x + C_3\alpha\partial_\alpha - u\partial_u \tag{2.107}$$

and thus $R = \alpha^{-1/C_3}\Lambda(u\alpha^{1/C_3})$, where Λ is an arbitrary function of its argument.

For $C_3 = 0$, $R = u\bar{\Lambda}(\alpha)$ and $\tilde{e}_1 = -x\partial_x - u\partial_u$, where $\bar{\Lambda}$ is an arbitrary function of α.

We now turn to the 4D Lie algebra containing $A_1 \oplus A_2$ for the extended operator (2.107) and show calculations for the possible realizations $(\partial_t, \bar{e}_2, -x\partial_x + C_3\alpha\partial_\alpha - u\partial_u, \partial_x)$ and $(-x\partial_x + C_3\alpha\partial_\alpha - u\partial_u, \partial_x, \bar{e}_3, \partial_t)$.

When we take the first realization, the Lie Brackets $[\bar{e}_1, \bar{e}_2] = \bar{e}_2$, $[\bar{e}_2, \bar{e}_3] = 0$, $[\bar{e}_2, \bar{e}_4] = 0$ yield

$$\bar{e}_2 = e^t[C_1\partial_t + C_2 u\partial_x + \bar{C}(u\alpha^{1/C_3})\alpha\partial_\alpha + C_4 u\partial_u],$$

where C_1, C_2, C_4 are arbitrary constants and $\bar{C}$ is an arbitrary function. Substituting the forms of ξs and ηs in $\bar{e}_2$ into the determining equations (2.98)–(2.103) we obtain

$$\bar{e}_2 = e^t(\partial_t - u\partial_u) \quad \text{and} \quad G = \rho u\alpha^{1/C_3} + K_1 u^2\alpha^{2/C_3}$$

for arbitrary constant K_1. Therefore $R = u\left(\rho + K_1 u\alpha^{1/C_3}\right)$ provided $C_3 \neq 0$.

For the second realization we have

$$\bar{e}_3 = -t\partial_t + \bar{C}_3\alpha\partial_\alpha + u\partial_u \quad \text{and} \quad R = K_2 u(C_3 + \bar{C}_3)\left[(C_3 + \bar{C}_3)u\alpha^{1/C_3}\right]^{\frac{C_3}{C_3+\bar{C}_3}},$$

where $C_3 \neq -\bar{C}_3$ and $K_2 \neq 0$ are arbitrary constants.

Considering $C_3 = 0$. The first realization gives $\bar{e}_2 = e^t(\partial_t - u\partial_u)$ and $R = \rho u$. The second realization yields

$$\bar{e}_3 = t\partial_t - \bar{C}_3\alpha\partial_\alpha - u\partial_u \quad \text{and} \quad R = \bar{K}_2 u\alpha^{1/\bar{C}_3},$$

where $\bar{C}_3$ and $\bar{K}_2$ are nonzero arbitrary constants.

The complete results for both the 3D Lie algebras and 4D Lie algebras are summarized in Tables 2.4 and 2.5 respectively. We also present without calculations the results involving the 2D subalgebra A_2 in Table 2.6.

Remark: The details of some calculations for cases $\mathcal{P}'(\alpha) \neq 0$ and $\mathcal{P}'(\alpha) = 0$ using `YaLie` are given in the appendix.

2.6 Conclusion

We obtained the forms of the arbitrary elements that appear in the one-dimensional collapsible tube model via the classification with respect to subalgebras of real three and four-dimensional Lie algebras.

The procedure employed in finding extra operators and hence the arbitrary elements does not guarantee the full symmetry Lie algebra. It only provides a subalgebra of the symmetry Lie algebra, that is, there may be extra operators for the equations admitting 3D and 4D Lie algebras. The guarantee for obtaining the maximal symmetry Lie algebra would be reached only after performing symmetry analysis of individual cases from Tables 2.1 and 2.4 for 3D Lie algebras and Tables 2.3 and 2.5 for 4D Lie algebras.

Table 2.4: Classification with respect to three-dimensional Lie algebras for $\mathcal{P}'(\alpha) = 0$.

Algebra	R	Condition on consts.	Extra operator
$3A_1$	$-\dfrac{\rho\bar{c}}{\ln\alpha + \rho\bar{c}G(u)}$	$\bar{c} \neq 0$	$u\partial_t + \frac{u^2}{2}\partial_x + \bar{c}\alpha\partial_\alpha$
	$G(u)$		$\alpha\partial_\alpha$
$A_1 \oplus A_2$	ρu		$e^t u\alpha\partial_\alpha$
	$\rho(u + i\sqrt{2c_1}) \times \tan\left[2i \arctan\left[\frac{u}{\sqrt{2c_1}}\right] + G(\alpha)\right]$	$c_1 \neq 0$	$e^t\left[u\partial_t + \left(\frac{u^2}{2} + c_1\right)\partial_x + \left(c_1 - \frac{u^2}{2}\right)\partial_u\right]$
	$\dfrac{\rho u\,(2u + \exp[\rho G(\alpha)])}{2\,(u + \exp[\rho G(\alpha)])}$	$c_1 = 0$	$e^t(u\partial_t + \frac{u^2}{2}\partial_x - \frac{u^2}{2}\partial_u)$
	$\rho(u - c_2) + (u - c_2)^2 G(\alpha)$		$e^t[\partial_t + c_2\partial_x + (c_2 - u)\partial_u]$
	$G(\alpha) - \rho u$		$e^t(\partial_x + \partial_u)$
	$\rho\left(\dfrac{u^2}{2} - c_3\right)$		$e^x\left[u\partial_t + \left(\frac{u^2}{2} + c_3\right)\partial_x + \left(c_3 - \frac{u^2}{2}\right)u\partial_u\right]$
	ρu^2		$e^x u\alpha\partial_\alpha$
	$\rho u(u - c_4)$		$e^x[\partial_t + c_4\partial_x + (c_4 - u)u\partial_u]$
	$\alpha^{-1}G(\alpha u) - \rho u^2$		$e^x(\partial_x - \alpha\partial_\alpha + u\partial_u)$
	$\dfrac{2\rho\alpha^{2/c_5}\exp\left[2\rho G(u\alpha^{-1/c_5})\right]}{1 - \alpha^{2/c_5}\exp\left[2\rho G(u\alpha^{-1/c_5})\right]}$	$c_5 \neq 0$	$(u - t)\partial_t + \frac{u^2}{2}\partial_x + c_5\alpha\partial_\alpha + u\partial_u$
	$\dfrac{2\rho u^2\exp[2\rho G(\alpha)]}{1 - u^2\exp[2\rho G(\alpha)]}$	$c_5 = 0$	$(u - t)\partial_t + \frac{u^2}{2}\partial_x + u\partial_u$

The G are (different) arbitrary functions of their arguments.

Table 2.4 (*Continued*).

Algebra	R	Condition on consts.	Extra operator
	$\alpha^{2/c_6} G(u\alpha^{-1/c_6})$	$c_6 \neq 0$	$t\partial_t - c_6\alpha\partial_\alpha - u\partial_u$
	$u^2 G(\alpha)$	$c_6 = 0$	$t\partial_t - u\partial_u$
	$\dfrac{\rho}{1-\alpha^{1/c_7}\exp\left[\rho G(u\alpha^{1/c_7})\right]}$	$c_7 \neq 0$	$u\partial_t + \left(\frac{u^2}{2} - x\right)\partial_x + c_7\alpha\partial_\alpha - u\partial_u$
	$\dfrac{\rho u}{u-\exp\left[\rho G(\alpha)\right]}$	$c_7 = 0$	$u\partial_t + \left(\frac{u^2}{2} - x\right)\partial_x - u\partial_u$
	$\alpha^{-1/c_8} G(u\alpha^{1/c_8})$	$c_8 \neq 0$	$x\partial_x - c_8\alpha\partial_\alpha + u\partial_u$
	$uG(\alpha)$	$c_8 = 0$	$x\partial_x + u\partial_u$
$A_{3,1}$	$\dfrac{2\rho u}{1+2u\alpha^3\rho G(u\alpha)}$		$(x+u)\partial_t + \frac{u^2}{2}\partial_x + u\alpha\partial_\alpha - u^2\partial_u$
	$\alpha^{-3} G(\alpha u)$		$x\partial_t + u\alpha\partial_\alpha - u^2\partial_u$
	$\dfrac{\rho\alpha^{1/c_9}\exp\left[\rho G(u\alpha^{-1/c_9})\right]}{1-\alpha^{1/c_9}\exp\left[\rho G(u\alpha^{-1/c_9})\right]}$	$c_9 \neq 0$	$u\partial_t + (x + \frac{u^2}{2})\partial_x + c_9\alpha\partial_\alpha + u\partial_u$
	$\dfrac{\rho u\exp\left[\rho G(\alpha)\right]}{1-u\exp[\rho G(\alpha)]}$	$c_9 = 0$	$u\partial_t + (x + \frac{u^2}{2})\partial_x + u\partial_u$
	$\alpha^{1/c_{10}} G(u\alpha^{-1/c_{10}})$	$c_{10} \neq 0$	$x\partial_x + c_{10}\alpha\partial_\alpha + u\partial_u$
	$uG(\alpha)$	$c_{10} = 0$	$x\partial_x + u\partial_u$

Table 2.4 (*Continued*).

Algebra	R	Condition on consts.	Extra operator
$A_{3,2}$	$\frac{2\rho u^3}{u^2-u+\left[2\rho c_{11}^3 G\left(\ln[u\alpha]-\frac{c_{11}}{u}\right)-\text{ExpIntegralEi}\left[-\frac{1}{u}\right]\right]e^{1/u}}$	$c_{11}\neq 0$	$(t+x+u)\partial_t+\left(x+\frac{u^2}{2}\right)\partial_x+(u+c_{11})\alpha\partial_\alpha-u^2\partial_u$
	$27u^3e^{-1/u}G\left(\ln[u\alpha]-\frac{1}{3u}\right)$		$(t+x)\partial_t+x\partial_x+\left(u+\frac{1}{3}\right)\alpha\partial_\alpha-u^2\partial_u$
	$\alpha^{-3}e^{-1/3u}G(u\alpha)$		$(t+x)\partial_t+x\partial_x+u\alpha\partial_\alpha-u^2\partial_u$
	$\frac{\rho}{1-\alpha^{1/c_{12}}\exp\left[\rho G\left(u-\frac{\ln\alpha}{c_{12}}\right)\right]}$	$c_{12}\neq 0$	$(t+u)\partial_t+\left(t+x+\frac{u^2}{2}\right)\partial_x+c_{12}\alpha\partial_\alpha+\partial_u$
	$\frac{\rho}{1-\exp\left[u+\rho G(\alpha)\right]}$	$c_{12}=0$	$(t+u)\partial_t+\left(t+x+\frac{u^2}{2}\right)\partial_x+\partial_u$
	$\alpha^{-1/c_{13}}G\left(u-\frac{\ln\alpha}{c_{13}}\right)$	$c_{13}\neq 0$	$t\partial_t+(t+x)\partial_x+c_{13}\alpha\partial_\alpha+\partial_u$
	$e^{-u}G(\alpha)$	$c_{13}=0$	$t\partial_t+(t+x)\partial_x+\partial_u$
$A_{3,3}$	$\frac{\rho}{1-\alpha^{1/c_{14}}\exp[\rho G(u)]}$	$c_{14}\neq 0$	$(t+u)\partial_t+\left(x+\frac{u^2}{2}\right)\partial_x+c_{14}\alpha\partial_\alpha$
	ρ	$c_{14}=0$	$(t+u)\partial_t+\left(x+\frac{u^2}{2}\right)\partial_x$

Table note: In Mathematica `ExpIntegralEi`$[\tau]$ gives an integral function $\mathrm{Ei}(\tau)$, where $\mathrm{Ei}(\tau)=-\int_{-\tau}^{\infty}(e^{-t}/t)dt$.

Table 2.4 (*Continued*).

Algebra	R	Condition on consts.	Extra operator
	$\alpha^{-1/c_{15}}G(u)$	$c_{15}\neq 0$	$t\partial_t + x\partial_x + c_{15}\alpha\partial_\alpha$
$A_{3,4}$	$\dfrac{3\rho}{1-\alpha^{3/c_{16}}\exp\left[3\rho G(u\alpha^{2/c_{16}})\right]}$	$c_{16}\neq 0$	$(t+u)\partial_t + \left(\frac{u^2}{2}-x\right)\partial_x + c_{16}\alpha\partial_\alpha - 2u\partial_u$
	$\dfrac{3\rho u^{3/2}}{u^{3/2}-\exp[3\rho G(\alpha)]}$	$c_{16}=0$	$(t+u)\partial_t + \left(\frac{u^2}{2}-x\right)\partial_x - 2u\partial_u$
	$\alpha^{-3/c_{17}}G(u\alpha^{2/c_{17}})$	$c_{17}\neq 0$	$t\partial_t - x\partial_x + c_{17}\alpha\partial_\alpha - 2u\partial_u$
	$u^{3/2}G(\alpha)$	$c_{17}=0$	$t\partial_t - x\partial_x - 2u\partial_u$
$A^a_{3,5}$	$\dfrac{\rho(2-a)}{1-\alpha^{\frac{2-a}{c_{18}}}\exp\left[\rho(2-a)G\left(u\alpha^{\frac{1-a}{c_{18}}}\right)\right]}$	$c_{18}\neq 0$	$(t+u)\partial_t + \left(ax+\frac{u^2}{2}\right)\partial_x + c_{18}\alpha\partial_\alpha + (a-1)u\partial_u$
	$\text{InverseFunction}\left[\frac{\ln\left[\frac{\#1}{\rho(a-2)+\#1}\right]}{\rho(a-2)}\&\right]\left[\frac{\ln u}{\rho(a-1)}+G(\alpha)\right]$	$c_{18}=0$	$(t+u)\partial_t + \left(ax+\frac{u^2}{2}\right)\partial_x + (a-1)u\partial_u$
	$\alpha^{\frac{a-2}{c_{19}}}G\left(u\alpha^{\frac{1-a}{c_{19}}}\right)$	$c_{19}\neq 0$	$t\partial_t + ax\partial_x + c_{19}\alpha\partial_\alpha + (a-1)u\partial_u$
	$u^{\frac{a-2}{a-1}}G(\alpha)$	$c_{19}=0$	$t\partial_t + ax\partial_x + (a-1)u\partial_u$

Table note: In Mathematica `InverseFunction[f][y]` gives the value of x for which $f(x)=y$.

Table 2.4 (*Continued*).

Algebra	R	Condition on consts.	Extra operator
	$\dfrac{\rho(2a-1)}{1-\alpha^{\frac{2a-1}{c_{20}}}\exp\left[\rho(2a-1)G\left(u\alpha^{\frac{a-1}{c_{20}}}\right)\right]}$	$c_{20}\neq 0$	$(at+u)\partial_t+\left(x+\frac{u^2}{2}\right)\partial_x+c_{20}\alpha\partial_\alpha+(1-a)u\partial_u$
	$\text{InverseFunction}\left[\frac{\ln\left[\frac{\#1}{\rho(1-2a)+\#1}\right]}{\rho(2a-1)}\&\right]\left[\frac{\ln u}{\rho(a-1)}+G(\alpha)\right]$	$c_{20}=0$	$(at+u)\partial_t+\left(x+\frac{u^2}{2}\right)\partial_x+(1-a)u\partial_u$
	$\alpha^{\frac{1-2a}{c_{21}}}G\left(u\alpha^{\frac{a-1}{c_{21}}}\right)$	$c_{21}\neq 0$	$at\partial_t+x\partial_x+c_{21}\alpha\partial_\alpha+(1-a)u\partial_u$
	$[(1-a)u]^{\frac{1-2a}{1-a}}G(\alpha)$	$c_{21}=0$	$at\partial_t+x\partial_x+(1-a)u\partial_u$
$A_{3,6}$	$\dfrac{2\rho(1+u^2)^{3/2}}{u\sqrt{1+u^2}+arc\sinh u+2\rho G\left(c_{22}\arctan u+\ln\left[\alpha\sqrt{1+u^2}\right]\right)}$		$(x+u)\partial_t+\left(\frac{u^2}{2}-t\right)\partial_x+(u+c_{22})\alpha\partial_\alpha-(1+u^2)\partial_u$
	$\left(1+u^2\right)^{3/2}G\left(c_{23}\arctan u+\ln[\alpha\sqrt{1+u^2}]\right)$		$x\partial_t-t\partial_x+(u+c_{23})\alpha\partial_\alpha-(1+u^2)\partial_u$
$A^a_{3,7}$	$\left(1-au+u^2\right)^{3/2}\exp\left[\frac{a\arctan[\frac{2u-a}{\sqrt{4-a^2}}]}{\sqrt{4-a^2}}\right]$ $\times G\left(-\frac{(a+2c_{24})\arctan[\frac{2u-a}{\sqrt{4-a^2}}]}{\sqrt{4-a^2}}-\ln[\alpha\sqrt{1-au+u^2}]\right)$		$(at+x)\partial_t+(ax-t)\partial_x+(u+c_{24})\alpha\partial_\alpha+(au-u^2-1)\partial_u$
	$\dfrac{\left(1+u^2\right)^{3/2}}{\exp[a\arctan u]}G\left(\ln\left[\alpha\sqrt{1+u^2}\right]-c_{25}\arctan u\right)$		$(at-x)\partial_t+(t+ax)\partial_x+(c_{25}-u)\alpha\partial_\alpha+(1+u^2)\partial_u$

Table 2.5: Classification with respect to four-dimensional Lie algebras for $\mathcal{P}'(\alpha) = 0$.

R	Condition on consts.	Operators
ρu		$\alpha\partial_\alpha,\ e^t u\alpha\partial_\alpha,\ (x+u)\alpha\partial_\alpha,\ x\partial_x + u\partial_u,\ e^t(\partial_t - u\partial_u)$
$\rho(u + i\sqrt{2\bar{c}_1})\tan\left[2i\operatorname{arc\,tanh}\left[\frac{u}{\sqrt{2\bar{c}_1}}\right] + K\right]$	$\bar{c}_1 \neq 0$	$\alpha\partial_\alpha,\ e^t\left[u\partial_t + \left(\bar{c}_1 + \frac{u^2}{2}\right)\partial_x + \left(\bar{c}_1 - \frac{u^2}{2}\right)\partial_u\right]$
$\frac{\rho u(2u + \exp[\rho K])}{2(u + \exp[\rho K])}$	$\bar{c}_1 = 0$	$\alpha\partial_\alpha,\ e^t\left(u\partial_t + \frac{u^2}{2}\partial_x - \frac{u^2}{2}\partial_u\right)$
$\rho(u - \bar{c}_2) + K(u - \bar{c}_2)^2$		$\alpha\partial_\alpha,\ e^t[\partial_t + \bar{c}_2\partial_x + (\bar{c}_2 - u)\partial_u]$
$K - \rho u$		$\alpha\partial_\alpha,\ e^t(\partial_x + \partial_u)$
ρu^2		$\alpha\partial_\alpha,\ e^x u\alpha\partial_\alpha,\ e^x(\partial_x - \alpha\partial_\alpha + u\partial_u),\ e^x(\partial_t - u^2\partial_u),\ e^{t-\frac{1}{u}}\alpha\partial_\alpha,\ t\partial_t + \partial_x - u\partial_u$
$Ku - \rho u^2$		$\alpha\partial_\alpha,\ e^x\left(\partial_x - \alpha\partial_\alpha + u\partial_u\right)$
$\frac{2\rho u^2 \exp[2\rho K]}{1 - u^2\exp[2\rho K]}$		$\alpha\partial_\alpha,\ (u-t)\partial_t + \frac{u^2}{2}\partial_x + u\partial_u$

The K are arbitrary nonzero constants which in general are different. The symmetry Lie algebra for each case includes the operators ∂_t and ∂_x. This is the same for the Tables that follow.

Table 2.5 (*Continued*).

R	Condition on consts.	Operators
Ku^2		$\alpha\partial_\alpha,\ t\partial_t - u\partial_u$
$\dfrac{\rho u}{u - \exp[\rho K]}$		$\alpha\partial_\alpha,\ u\partial_t + \left(\frac{u^2}{2} - x\right)\partial_x - u\partial_u$
Ku		$\alpha\partial_\alpha,\ x\partial_x + u\partial_u$
$\dfrac{2\rho u^3}{u^2 + 2\rho K}$		$\alpha\partial_\alpha,\ (x+u)\partial_t + \frac{u^2}{2}\partial_x + u\alpha\partial_\alpha - u^2\partial_u$
Ku^3		$\alpha\partial_\alpha,\ x\partial_t + u\alpha\partial_\alpha - u^2\partial_u,\ 2t\partial_t + x\partial_x - u\partial_u$
$-\dfrac{\rho}{u + \rho K}$		$\alpha\partial_\alpha,\ u\partial_t + \left(t + \frac{u^2}{2}\right)\partial_x + \partial_u$
K		$\alpha\partial_\alpha,\ t\partial_x + \partial_u$
ρ		$\alpha\partial_\alpha,\ (t+u)\alpha\partial_\alpha,\ (t+u)\partial_t + \left(x + \frac{u^2}{2}\right)\partial_x,\ \left(x + \frac{3}{2}u^2\right)\partial_t + u^3\partial_x + u\alpha\partial_\alpha - u^2\partial_u,$ $\left(x + 1 + \frac{3}{2}u^2\right)\partial_t + (u^3 - t)\partial_x + u\alpha\partial_\alpha - (1+u^2)\partial_u$
$\dfrac{2\rho(1+u^2)^{3/2}}{u\sqrt{1+u^2} + arc\sinh u + 2\rho K}$		$\alpha\partial_\alpha,\ (x+u)\partial_t + \left(\frac{u^2}{2} - t\right)\partial_x + u\alpha\partial_\alpha - (1+u^2)\partial_u$

Table 2.5 (*Continued*).

R	Condition on consts.	Operators
$K\left(1+u^2\right)^{3/2}$		$\alpha\partial_\alpha,\ x\partial_t - t\partial_x + u\alpha\partial_\alpha - (1+u^2)\partial_u$
$\dfrac{2\rho\left(1+u^2\right)^{3/2}\exp[-i\arctan u]}{(u-a)\sqrt{1+u^2}+\frac{2\rho K}{\exp[a\arctan u]}+2(a-i)\ {}_2F_1}$		$\alpha\partial_\alpha,\ (at+x+u)\partial_t + (ax-t+\frac{u^2}{2})\partial_x + u\alpha\partial_\alpha - (1+u^2)\partial_u$
$K\left(1+u^2\right)^{3/2}\exp[a\arctan u]$		$\alpha\partial_\alpha,\ (at+x)\partial_t + (ax-t)\partial_x + u\alpha\partial_\alpha - (1+u^2)\partial_u$
$\rho(1-au)$		$\alpha\partial_\alpha,$ $\left(at+x+\frac{(1+a^2)\left[\frac{1}{1-au}-ln[au-1]\right]-2au}{a^2}\right)\partial_t + \left(\frac{a+u}{1-au}\right)u^2\partial_x + (ax-t)\alpha\partial_\alpha - (a+u)u\partial_u$
$\dfrac{\rho u\left[(u-1)\exp[2\rho K]-u\alpha\right]}{(u-2)\exp[2\rho K]-u\alpha}$	$c_1=0$	$e^t\left(u\partial_t+\frac{u^2}{2}\partial_x-\frac{u^2}{2}\partial_u\right),\ e^{(x+u)}[\partial_t+(u-1)\partial_x+\alpha\partial_\alpha-u\partial_u]$
$\dfrac{\rho u}{2}\left[2-\dfrac{1}{1+u\alpha^{1/m_1}\exp[-\rho K]}\right]$	$c_1=0,\ m_1\neq 0$	$e^t\left(u\partial_t+\frac{u^2}{2}\partial_x-\frac{u^2}{2}\partial_u\right),\ \partial_t-\partial_x+m_1\alpha\partial_\alpha-u\partial_u$
$\dfrac{\rho(c_2-u)\left[1+u\left(\alpha\exp[\rho K]-1\right)\right]}{c_2(1-\alpha\exp[\rho K])-1}$		$e^t[\partial_t+c_2\partial_x+(c_2-u)\partial_u],\ e^x(c_2-u)^{c_2}\left[\frac{c_2}{c_2-u}\partial_t+\frac{c_2+u(c_2-1)}{c_2-u}\partial_x-\alpha\partial_\alpha+u\partial_u\right]$
$\rho u\left(\dfrac{u}{c_2}-1\right)$	$c_2\neq 0$	$e^t[\partial_t+c_2\partial_x+(c_2-u)\partial_u],\ e^x(c_2-u)^{c_2}\alpha\partial_\alpha,$ $\frac{c_2}{u-c_2}\partial_t-\left[x+c_2\left(\ln[u-c_2]-\frac{c_2}{u-c_2}\right)\right]\partial_x-u\partial_u$

Table note: ${}_2F_1$ is the Hypergeometric2F1$\left[\frac{1-ia}{2},1,\frac{3-ia}{2},-\exp[2i\arctan u]\right]$.

Table 2.5 (*Continued*).

R	Condition on consts.	Operators
$\rho\left(\frac{1}{1-\alpha\exp[\rho K]}-u\right)$		$e^t(\partial_x+\partial_u),\ e^{(x-u)}[\partial_t+(u+1)\partial_x-\alpha\partial_\alpha+u\partial_u]$
$\frac{\rho(c_2-u)\left[u\alpha^{1/m_2}-\exp[\rho K]\right]}{\exp[\rho K]-c_2\alpha^{1/m_2}}$	$m_2\neq 0$	$e^t[\partial_t+c_2\partial_x+(c_2-u)\partial_u],\ \frac{c_2}{u-c_2}\partial_t-\left[x+c_2\left(\ln[u-c_2]-\frac{c_2}{u-c_2}\right)\right]\partial_x+m_2\alpha\partial_\alpha-u\partial_u$
$\rho(u-c_2)$	$m_2=0$	$e^t[\partial_t+c_2\partial_x+(c_2-u)\partial_u],\ \frac{c_2}{u-c_2}\partial_t-\left[x+c_2\left(\ln[u-c_2]-\frac{c_2}{u-c_2}\right)\right]\partial_x-u\partial_u$
$K\alpha^{-1/m_3}-\rho u$	$m_3\neq 0$	$e^t(\partial_x+\partial_u),\ \partial_t+x\partial_x-m_3\alpha\partial_\alpha+u\partial_u$
$-\rho u$	$m_3=0$	$e^t(\partial_x+\partial_u),\ \partial_t+x\partial_x+u\partial_u$
$\rho(\frac{u^2}{2}-c_3)$	$c_3\neq 0$	$e^x\left[u\partial_t+\left(\frac{u^2}{2}+c_3\right)\partial_x+\left(c_3-\frac{u^2}{2}\right)u\partial_u\right],\ \exp\left[t-\frac{\sqrt{2}\ arc\tanh[\frac{u}{\sqrt{2c_3}}]}{\sqrt{c_3}}\right]\alpha\partial_\alpha$
$\rho u^2/2$	$c_3=0$	$e^x\left(u\partial_t+\frac{u^2}{2}\partial_x-\frac{u^3}{2}\partial_u\right),\ e^{(t-2/u)}\alpha\partial_\alpha$
$\rho u(u-c_4)$	$c_4\neq 0$	$e^x[\partial_t+c_4\partial_x+(c_4-u)\partial_u],\ e^t\left(\frac{u-c_4}{u}\right)^{\frac{1}{c_4}}\alpha\partial_\alpha,$
$u\left[K(1+m_4)[(1+m_4)u\alpha]^{\frac{1}{1+m_4}}-\rho u\right]$	$m_4\neq -1$	$e^x(\partial_x-\alpha\partial_\alpha+u\partial_u),\ t\partial_t-m_4\alpha\partial_\alpha-u\partial_u$
$-\rho u^2$	$m_4=-1$	$e^x(\partial_x-\alpha\partial_\alpha+u\partial_u),\ t\partial_t+\alpha\partial_\alpha-u\partial_u$

Table 2.5 (*Continued*).

R	Condition on consts.	Operators
$\rho\left[\frac{4}{2+u^2-K(1+c_5)^2u^2\left[(1+c_5)u\alpha^{-1/c_5}\right]^{-\frac{c_5}{1+c_5}}}-2\right]$	$c_5\neq 0,-1$	$(u-t)\partial_t+\frac{u^2}{2}\partial_x+c_5\alpha\partial_\alpha+u\partial_u,\ e^{\left(x-\frac{u^2}{2}\right)}\left[\frac{u}{2}\partial_t+\left(1+\frac{u^2}{2}\right)\partial_x-\alpha\partial_\alpha+u\partial_u\right]$
$-\frac{2\rho u^2}{u^2+2}$	$c_5=-1$	$(u-t)\partial_t+\frac{u^2}{2}\partial_x-\alpha\partial_\alpha+u\partial_u,\ e^{\left(x-\frac{u^2}{2}\right)}\left[\frac{u}{2}\partial_t+\left(1+\frac{u^2}{2}\right)\partial_x-\alpha\partial_\alpha+u\partial_u\right]$
$\rho\left[\frac{4}{2+u^2\left(1+\alpha\exp[4\rho K]\right)}-2\right]$	$c_5=0$	$(u-t)\partial_t+\frac{u^2}{2}\partial_x+u\partial_u,\ e^{\left(x-\frac{u^2}{2}\right)}\left[\frac{u}{2}\partial_t+\left(1+\frac{u^2}{2}\right)\partial_x-\alpha\partial_\alpha+u\partial_u\right]$
$2\rho\left[\frac{1}{1-\alpha^{2/c_5}\exp[2\rho K]\left[2\rho(c_5+m_5)u\alpha^{-1/c_5}\right]^{\frac{c_5+2m_5}{c_5+m_5}}}-1\right]$	$c_5\neq 0,-m_5$	$(u-t)\partial_t+\frac{u^2}{2}\partial_x+c_5\alpha\partial_\alpha+u\partial_u,\ \frac{u}{2}\partial_t+\left(x+\frac{u^2}{4}\right)\partial_x-m_5\alpha\partial_\alpha+u\partial_u$
$2\left[\frac{1}{1-u^2\alpha^{1/m_5}\exp[2\rho K]}-1\right]$	$c_5=0,\ m_5\neq 0$	$(u-t)\partial_t+\frac{u^2}{2}\partial_x+u\partial_u,\ \frac{u}{2}\partial_t+\left(x+\frac{u^2}{4}\right)\partial_x-m_5\alpha\partial_\alpha+u\partial_u$
$K\alpha^{2/c_6}\left[(1+c_6)u\alpha^{-1/c_6}\right]^{\frac{c_4+2}{c_4+1}}-\rho u^2$	$c_6\neq 0,-1$	$t\partial_t-c_6\alpha\partial_\alpha-u\partial_u,\ e^x(\partial_x-\alpha\partial_\alpha+u\partial_u)$
$(\alpha K-\rho)u^2$	$c_6=0$	$t\partial_t-u\partial_u,\ e^x(\partial_x-\alpha\partial_\alpha+u\partial_u)$
$K(c_6+m_6)^2u^2\left[(c_6+m_6)u\alpha^{-1/c_6}\right]^{-\frac{c_6}{c_6+m_6}}$	$c_6\neq 0,-m_6$	$t\partial_t-c_6\alpha\partial_\alpha-u\partial_u,\ x\partial_x-m_6\alpha\partial_\alpha+u\partial_u$
$Ku^2\alpha^{1/m_6}$	$c_6,m_6\neq 0$	$t\partial_t-u\partial_u,\ x\partial_x-m_6\alpha\partial_\alpha+u\partial_u$

Table 2.5 (*Continued*).

R	Condition on consts.	Operators
$\rho\left[1+\dfrac{1}{u\left(u\alpha^{1/c_7}\exp[\rho K]-1\right)-1}\right]$	$c_7\neq 0$	$u\partial_t+\left(\frac{u^2}{2}-x\right)\partial_x+c_7\alpha\partial_\alpha-u\partial_u,\ e^{(t+u)}[(u+1)\partial_t+u^2\partial_x-u\partial_u]$
$\dfrac{\rho u}{u+1}$	$c_7=0$	$u\partial_t+\left(\frac{u^2}{2}-x\right)\partial_x-u\partial_u,\ e^{(t+u)}[(u+1)\partial_t+u^2\partial_x-u\partial_u]$
$\dfrac{\rho}{1-\alpha^{1/c_7}\left[\rho(c_7+m_7)u\alpha^{1/c_7}\right]^{-\frac{2c_7+m_7}{c_7+m_7}}\exp[\rho K]}$	$c_7\neq 0,-m_7$	$u\partial_t+\left(\frac{u^2}{2}-x\right)\partial_x+c_7\alpha\partial_\alpha-u\partial_u,\ (t+2u)\partial_t+u^2\partial_x-m_7\alpha\partial_\alpha-u\partial_u$
$\dfrac{\rho u}{u-\alpha^{-1/m_7}\exp[\rho K]}$	$c_7=0,\ m_7\neq 0$	$u\partial_t+\left(\frac{u^2}{2}-x\right)\partial_x-u\partial_u,\ (t+2u)\partial_t+u^2\partial_x-m_7\alpha\partial_\alpha-u\partial_u$
$u\left(\rho+Ku\alpha^{1/c_8}\right)$	$c_8\neq 0$	$x\partial_x-c_8\alpha\partial_\alpha+u\partial_u,\ e^t(\partial_t-u\partial_u)$
$K(c_8+m_8)u\left[(c_8+m_8)u\alpha^{1/c_8}\right]^{\frac{c_8}{c_8+m_8}}$	$c_8\neq 0,-m_8$	$x\partial_x-c_8\alpha\partial_\alpha+u\partial_u,\ t\partial_t-m_8\alpha\partial_\alpha-u\partial_u$
$Ku\alpha^{1/m_8}$	$c_8=0,m_8\neq 0$	$x\partial_x+u\partial_u,\ t\partial_t-m_8\alpha\partial_\alpha-u\partial_u$
$\dfrac{2\rho u^3}{2\rho K+u^2}$		$(x+u)\partial_t+\frac{u^2}{2}\partial_x+u\alpha\partial_\alpha-u^2\partial_u,\ \alpha\partial_\alpha$
$\dfrac{\rho p_1u^3}{\rho p_1K-\ln[u\alpha]}$	$p_1\neq 0$	$x\partial_t+u\alpha\partial_\alpha-u^2\partial_u,\ \frac{1}{2u^2}\partial_t+\frac{1}{u}\partial_x-p_1\alpha\partial_\alpha$
$\dfrac{2\rho p_2u^3}{p_2(2\rho K+u^2)-2\ln[u\alpha]}$	$p_2\neq 0$	$(x+u)\partial_t+\frac{u^2}{2}\partial_x+u\alpha\partial_\alpha-u^2\partial_u,\ \frac{1}{2u^2}\partial_t+\frac{1}{u}\partial_x-p_2\alpha\partial_\alpha$

Table 2.5 (*Continued*).

R	Condition on consts.	Operators
$\dfrac{\rho p_3 u}{c_9\left(K-\ln[u\alpha^{-1/c_9}]\right)+p_3 u}$	$c_9, p_3 \neq 0$	$u\partial_t+\left(x+\frac{u^2}{2}\right)\partial_x+c_9\alpha\partial_\alpha+u\partial_u,\ \ln u\partial_t+u\partial_x-p_3\alpha\partial_\alpha$
$\dfrac{\rho p_3 u}{\ln\alpha-p_3(\rho K+u)}$	$c_9=0,\ p_3\neq 0$	$u\partial_t+\left(x+\frac{u^2}{2}\right)\partial_x+u\partial_u,\ \ln u\partial_t+u\partial_x-p_3\alpha\partial_\alpha$
$\dfrac{2\rho u^3}{1+2\rho K u^3\alpha^3[(1-p_4)u\alpha]^{\frac{1-3p_4}{p_4-1}}}$	$p_4\neq 1$	$x\partial_t+u\alpha\partial_\alpha-u^2\partial_u,\ \frac{1}{2u^2}\partial_t+\left(\frac{1}{u}-x\right)\partial_x+p_4\alpha\partial_\alpha-u\partial_u$
$2\rho u^3$	$p_4=1$	$x\partial_t+u\alpha\partial_\alpha-u^2\partial_u,\ \frac{1}{2u^2}\partial_t+\left(\frac{1}{u}-x\right)\partial_x+\alpha\partial_\alpha-u\partial_u$
$K\alpha^{-3}[(1+p_5)u\alpha]^{\frac{1+3p_5}{1+p_5}}$	$p_5\neq -1$	$x\partial_t+u\alpha\partial_\alpha-u^2\partial_u,\ x\partial_x+p_5\alpha\partial_\alpha+u\partial_u$
$\dfrac{2\rho u}{1+2\rho K u\alpha^3[(1+p_6)u\alpha]^{-\frac{1+3p_6}{1+p_6}}}$	$p_6\neq -1$	$(x+u)\partial_t+\frac{u^2}{2}\partial_x+u\alpha\partial_\alpha-u^2\partial_u,\ x\partial_x+p_6\alpha\partial_\alpha+u\partial_u$
$\dfrac{\rho u}{p_7-u+\left(u\alpha^{-1/c_9}\right)^{-\frac{2c_9}{1+c_9}}\exp\left[\frac{2Kc_9}{1+c_9}\right]}$	$c_9\neq 0,-1$	$u\partial_t+\left(x+\frac{u^2}{2}\right)\partial_x+c_9\alpha\partial_\alpha+u\partial_u,\ \left(x+2p_7u-\frac{3}{2}u^2\right)\partial_t+\left(p_7u^2-u^3\right)\partial_x+u\alpha\partial_\alpha-u^2\partial_u$
$\dfrac{\rho u}{p_7-u}$	$c_9=-1$	$u\partial_t+\left(x+\frac{u^2}{2}\right)\partial_x-\alpha\partial_\alpha+u\partial_u,\ \left(x+2p_7u-\frac{3}{2}u^2\right)\partial_t+\left(p_7u^2-u^3\right)\partial_x+u\alpha\partial_\alpha-u^2\partial_u$
$\dfrac{\rho u}{p_7-u+\alpha^2\exp[\rho K]}$	$c_9=0$	$u\partial_t+\left(x+\frac{u^2}{2}\right)\partial_x+u\partial_u,\ \left(x+2p_7u-\frac{3}{2}u^2\right)\partial_t+\left(p_7u^2-u^3\right)\partial_x+u\alpha\partial_\alpha-u^2\partial_u$

Table 2.5 (*Continued*).

R	Condition on consts.	Operators
$\dfrac{\rho u(1+c_{10})}{p_8(1+c_{10})+\rho K\left[(1+c_{10})u\alpha^{-1/c_{10}}\right]^{-\frac{2c_{10}}{1+c_{10}}}}$	$c_{10}\neq 0,-1$	$x\partial_x+c_{10}\alpha\partial_\alpha+u\partial_u,\ (x+2p_8u)\partial_t+p_8u^2\partial_x+u\alpha\partial_\alpha-u^2\partial_u$
$\rho u/p_8$	$c_{10}=-1,\ p_8\neq 0$	$x\partial_x-\alpha\partial_\alpha+u\partial_u,\ (x+2p_8u)\partial_t+p_8u^2\partial_x+u\alpha\partial_\alpha-u^2\partial_u$
$\dfrac{\rho}{p_8-\alpha^2\exp[\rho K]}$	$c_{10}=0$	$x\partial_x+u\partial_u,\ (x+2p_8u)\partial_t+p_8u^2\partial_x+u\alpha\partial_\alpha-u^2\partial_u$
$K\alpha^{-3}[(b-p_9)u\alpha]^{\frac{1+2b-3p_9}{p_4-b}}$	$p_9\neq b$	$x\partial_t+u\alpha\partial_\alpha-u^2\partial_u,\ (b+1)t\partial_t+x\partial_x+p_9\alpha\partial_\alpha-bu\partial_u$
$\dfrac{2\rho u}{1+2\rho Ku\alpha^3[(b-p_{10})u\alpha]^{-\frac{1+2b-3p_{10}}{b-p_{10}}}}$	$p_{10}\neq b$	$(x+u)\partial_t+\frac{u^2}{2}\partial_x+u\alpha\partial_\alpha-u^2\partial_u,\ (b+1)\left(t+\frac{1}{2}\ln u\right)\partial_t+\left[x+(b+1)\frac{u}{2}\right]\partial_x+p_{10}\alpha\partial_\alpha-bu\partial_u$
$2\rho u$	$p_{10}=b$	$(x+u)\partial_t+\frac{u^2}{2}\partial_x+u\alpha\partial_\alpha-u^2\partial_u,\ (b+1)\left(t+\frac{1}{2}\ln u\right)\partial_t+\left[x+(b+1)\frac{u}{2}\right]\partial_x+b\alpha\partial_\alpha-bu\partial_u,$ $\left(t+\frac{\ln u}{2}\right)\partial_t+\left(x+\frac{u}{2}\right)\partial_x,\ x\partial_x-\alpha\partial_\alpha+u\partial_u$
$Ku^{\frac{3p_{11}-1}{p_{11}}}\alpha^{-1/p_{11}}$	$p_{11}\neq 0$	$x\partial_t+u\alpha\partial_\alpha-u^2\partial_u,\ t\partial_t+x\partial_x+p_{11}\alpha\partial_\alpha$
$\dfrac{2\rho u^3}{u^2+2\rho K(u\alpha)^{1/p_{12}}}$	$p_{12}\neq 0$	$(x+u)\partial_t+\frac{u^2}{2}\partial_x+u\alpha\partial_\alpha-u^2\partial_u,\ \left(t+\frac{\ln u}{2}\right)\partial_t+\left(x+\frac{u}{2}\right)\partial_x+p_{12}\alpha\partial_\alpha$

Table 2.5 (*Continued*).

R	Condition on consts.	Operators
$\frac{2\rho q_1 u^3}{e^{1/u}\left[2\left(\ln[u\alpha]-\frac{c_{11}}{u}+\rho K c_{11}^3 q_1\right)-q_1\text{ExpIntegralEi}\left[-\frac{1}{u}\right]\right]+q_1(u-1)u}$	$c_{11}, q_1 \neq 0$	$(t+x+u)\partial_t+\left(x+\frac{u^2}{2}\right)x\partial_x+(c_{11}+u)\alpha\partial_\alpha-u^2\partial_u$, $e^{\frac{1}{u}}\left(\frac{1}{u}-1\right)\partial_t+e^{\frac{1}{u}}\partial_x+q_1\alpha\partial_\alpha$
$\frac{2\rho u^3}{u^2-u+e^{1/u}\left(2\rho K c_{11}^3-\text{ExpIntegralEi}\left[-\frac{1}{u}\right]\right)}$		$(t+x+u)\partial_t+\left(x+\frac{u^2}{2}\right)x\partial_x+(c_{11}+u)\alpha\partial_\alpha-u^2\partial_u,\ \alpha\partial_\alpha$
$\frac{2\rho u^3}{u^2-u-\text{ExpIntegralEi}\left[-\frac{1}{u}\right]e^{1/u}}$	$c_{11}=0$	$(t+x+u)\partial_t+\left(x+\frac{u^2}{2}\right)x\partial_x+u\alpha\partial_\alpha-u^2\partial_u,\ \alpha\partial_\alpha$
$\frac{81\rho q_2 u^4 e^{-1/u}}{1+(81\ln[u\alpha]-3\rho q_2 K)u}$	$q_2\neq 0$	$(t+x)\partial_t+x\partial_x+\left(u+\frac{1}{3}\right)\alpha\partial_\alpha-u^2\partial_u,\ e^{\frac{1}{u}}\left(\frac{1}{u}-1\right)\partial_t+e^{\frac{1}{u}}\partial_x+q_2\alpha\partial_\alpha$
$27Ku^3e^{-1/u}$		$(t+x)\partial_t+x\partial_x+\left(u+\frac{1}{3}\right)\alpha\partial_\alpha-u^2\partial_u,\ \alpha\partial_\alpha$
$\frac{\rho q_3 u^3 e^{-1/u}}{\ln[u\alpha]+\rho q_3 K}$	$q_3\neq 0$	$(t+x)\partial_t+x\partial_x+u\alpha\partial_\alpha-u^2\partial_u,\ e^{\frac{1}{u}}\left(\frac{1}{u}-1\right)\partial_t+e^{\frac{1}{u}}\partial_x+q_3\alpha\partial_\alpha$
$Ku^3e^{-1/u}$		$(t+x)\partial_t+x\partial_x+u\alpha\partial_\alpha-u^2\partial_u,\ \alpha\partial_\alpha$
$\frac{\rho q_4}{q_4-e^u(K+\ln\alpha-c_{12}u)}$	$c_{12}, q_4\neq 0$	$(t+u)\partial_t+\left(t+x+\frac{u^2}{2}\right)x\partial_x+c_{12}\alpha\partial_\alpha+\partial_u$, $e^{\frac{1}{u}}\left(\frac{1}{u}-1\right)\partial_t+e^{\frac{1}{u}}\partial_x+q_4\alpha\partial_\alpha$

Table 2.5 (*Continued*).

R	Condition on consts.	Operators
$\dfrac{\rho q_4}{q_4 - e^u(\ln\alpha + \rho q_4 K)}$	$c_{12}=0,\ q_4\neq 0$	$(t+u)\partial_t + \left(t+x+\frac{u^2}{2}\right)x\partial_x + \partial_u,\ e^u\partial_t + e^u(u-1)\partial_x + q_4\alpha\partial_\alpha$
$\dfrac{\rho}{1-\exp[u+\rho K]}$	$c_{12}=0$	$(t+u)\partial_t + \left(t+x+\frac{u^2}{2}\right)x\partial_x + \partial_u,\ \alpha\partial_\alpha$
$\dfrac{\rho q_5 e^{-u}}{c_{13}u - \ln\alpha + \rho q_5 K}$	$c_{13}, q_5\neq 0$	$t\partial_t + (t+x)\partial_x + c_{13}\alpha\partial_\alpha + \partial_u,\ e^u\partial_t + e^u(u-1)\partial_x + q_5\alpha\partial_\alpha$
$-\dfrac{\rho q_5 e^{-u}}{\rho q_5 K + \ln\alpha}$	$c_{13}=0$	$t\partial_t + (t+x)\partial_x + \partial_u,\ e^u\partial_t + e^u(u-1)\partial_x + q_5\alpha\partial_\alpha$
Ke^{-u}	$c_{13}=0$	$t\partial_t + (t+x)\partial_x + \partial_u,\ \alpha\partial_\alpha$
$\dfrac{\rho}{1-u^{\frac{1-3c_{14}}{c_{14}}}\alpha^{1/c_{14}}\exp[\rho K - \frac{r_1}{c_{14}u}]}$	$c_{14}\neq 0$	$(t+u)\partial_t + \left(x+\frac{u^2}{2}\right)\partial_x + c_{14}\alpha\partial_\alpha,\ \left(x+\frac{3}{2}u^2\right)\partial_t + u^3\partial_x + (r_1+u)\alpha\partial_\alpha - u^2\partial_u$
$\dfrac{\rho}{1-(1+u^2)^{\frac{1-3c_{14}}{2c_{14}}}\alpha^{1/c_{14}}\exp[\frac{r_2\arctan u}{c_{14}} + \rho K]}$	$c_{14}\neq 0$	$(t+u)\partial_t + \left(x+\frac{u^2}{2}\right)\partial_x + c_{14}\alpha\partial_\alpha,$ $\left(x+1+\frac{3}{2}u^2\right)\partial_t + (u^3-t)\partial_x + (r_2+u)\alpha\partial_\alpha - (1+u^2)\partial_u$
$Ku^{\frac{3c_{15}-1}{c_{15}}}\alpha^{-1/c_{15}}\exp[\frac{r_3}{c_{15}u}]$	$c_{15}\neq 0$	$t\partial_t + x\partial_x + c_{15}\alpha\partial_\alpha,\ x\partial_t + (r_3+u)\alpha\partial_\alpha - u^2\partial_u$
$K\alpha^{-1/c_{15}}\exp[\frac{r_4 u}{c_{15}}]$	$c_{15}\neq 0$	$t\partial_t + x\partial_x + c_{15}\alpha\partial_\alpha,\ t\partial_x + r_4\alpha\partial_\alpha + \partial_u$

Table 2.5 (*Continued*).

R	Condition on consts.	Operators
$K\alpha^{1/c_{15}}(1+u^2)^{\frac{3c_{15}-1}{2c_{15}}}\exp[-\frac{r_5\arctan u}{c_{15}}]$	$c_{15}\neq 0$	$t\partial_t+x\partial_x+c_{15}\alpha\partial_\alpha,\ x\partial_t-t\partial_x+(r_5+u)\alpha\partial_\alpha-(1+u^2)\partial_u$
$\dfrac{\rho u(2+\ln u)}{t+u}$	$c_{16}=0$	$(t+u)\partial_t+\left(\frac{u^2}{2}-x\right)\partial_x-2u\partial_u,\ e^{(t+\frac{u}{2})}u^{t/2}\alpha\partial_\alpha$
$\dfrac{3\rho u^{3/2}}{u^{3/2}+3s_1-K\left[2(c_{16}+2s_2)u\alpha^{2/c_{16}}\right]^{-\frac{c_{16}}{2c_{16}+4s_2}}}$	$c_{16}\neq 0,-2s_2$	$(t+u)\partial_t+\left(\frac{u^2}{2}-x\right)\partial_x+c_{16}\alpha\partial_\alpha-2u\partial_u,$ $\left(\frac{s_1}{\sqrt{u}}-\frac{2}{3}u-t\right)\partial_t-\left(\frac{1}{3}u^2+s_1\sqrt{u}\right)\partial_x+s_2\alpha\partial_\alpha+u\partial_u$
$\dfrac{3\rho u^{3/2}}{u^{3/2}-3s_1+\alpha^{-1/2s_2}\exp[3\rho K]}$	$c_{16}=0$ $s_2\neq 0$	$(t+u)\partial_t+\left(\frac{u^2}{2}-x\right)\partial_x-2u\partial_u,$ $\left(\frac{s_1}{\sqrt{u}}-\frac{2}{3}u-t\right)\partial_t-\left(\frac{1}{3}u^2+s_1\sqrt{u}\right)\partial_x+s_2\alpha\partial_\alpha+u\partial_u$
$\dfrac{3\rho u^{3/2}}{u^{3/2}-3s_1}$	$c_{16},s_2=0$	$(t+u)\partial_t+\left(\frac{u^2}{2}-x\right)\partial_x-2u\partial_u,\ \left(\frac{s_1}{\sqrt{u}}-\frac{2}{3}u-t\right)\partial_t-\left(\frac{1}{3}u^2+s_1\sqrt{u}\right)\partial_x+u\partial_u$
$\dfrac{12\rho u^{3/2}}{4s_3u^{3/2}-3\left(K-c_{16}\ln\left[u\alpha^{2/c_{16}}\right]\right)}$	$c_{16}\neq 0$	$(t+u)\partial_t+\left(\frac{u^2}{2}-x\right)\partial_x+c_{16}\alpha\partial_\alpha-2u\partial_u,\ \frac{1}{\sqrt{u}}\partial_t-\sqrt{u}\,\partial_x+s_3\alpha\partial_\alpha$
$\dfrac{6\rho s_3u^{3/2}}{2s_3(u^{3/2}-3\rho K)-\ln\alpha}$	$c_{16}=0,\ s_3\neq 0$	$(t+u)\partial_t+\left(\frac{u^2}{2}-x\right)\partial_x-2u\partial_u,\ \frac{1}{\sqrt{u}}\partial_t-\sqrt{u}\,\partial_x+s_3\alpha\partial_\alpha$
$\dfrac{\rho\sqrt{u}\,\alpha^{-2/c_{17}}\left[(c_{17}+2s_5)u\alpha^{2/c_{17}}\right]^{\frac{2c_{17}+3s_5}{c_{17}+2s_5}}}{\rho K\sqrt{u}\,\alpha^{1/c_{17}}+s_4(c_{17}+2s_5)\left[(c_{17}+2s_5)u\alpha^{2/c_{17}}\right]^{\frac{c_{17}+s_5}{c_{17}+2s_5}}}$	$c_{17}\neq 0,-2s_5$	$t\partial_t-x\partial_x+c_{17}\alpha\partial_\alpha-2u\partial_u,\ \left(\frac{s_4}{\sqrt{u}}-t\right)\partial_t-s_4\sqrt{u}\,\partial_x+s_5\alpha\partial_\alpha+u\partial_u$

Table 2.5 (*Continued*).

R	Condition on consts.	Operators
$\dfrac{\rho u^{3/2}\alpha^{1/2s_5}}{s_4\alpha^{1/2s_4}-\exp[\rho K]}$	$c_{17}=0,\ s_5\neq 0$	$t\partial_t - x\partial_x - 2u\partial_u,\ \left(\frac{s_4}{\sqrt{u}}-t\right)\partial_t - s_4\sqrt{u}\,\partial_x + s_5\alpha\partial_\alpha + u\partial_u$
$\dfrac{\rho}{s_4}u^{3/2}$	$c_{17}, s_5=0,\ s_4\neq 0$	$t\partial_t - x\partial_x - 2u\partial_u,\ \left(\frac{s_4}{\sqrt{u}}-t\right)\partial_t - s_4\sqrt{u}\partial_x + u\partial_u$
$\dfrac{4\rho s_6 u^{3/2}}{c_{17}\ln\left[u\alpha^{2/c_{17}}\right]+4K\rho s_6}$	$c_{17}, s_6\neq 0$	$t\partial_t - x\partial_x + c_{17}\alpha\partial_\alpha - 2u\partial_u,\ \frac{1}{\sqrt{u}}\partial_t - \sqrt{u}\,\partial_x + s_6\alpha\partial_\alpha$
$\dfrac{2\rho s_6 u^{3/2}}{\ln\alpha - 2\rho s_6 K}$	$c_{17}=0,\ s_6\neq 0$	$t\partial_t - x\partial_x - 2u\partial_u,\ \frac{1}{\sqrt{u}}\partial_t - \sqrt{u}\,\partial_x + s_6\alpha\partial_\alpha$
$Ku^{3/2}$	$c_{17}=0$	$t\partial_t - x\partial_x - 2u\partial_u,\ \alpha\partial_\alpha$
$\dfrac{\rho(a-2)}{u^{\frac{2-a}{a-1}}\exp[\rho(2-a)K]-1}$	$c_{18}\neq 0$	$(t+u)\partial_t + \left(ax+\frac{u^2}{2}\right)\partial_x + c_{18}\alpha\partial_\alpha + (a-1)u\partial_u,\ \alpha\partial_\alpha$
$\dfrac{\rho y_1(a-1)^2 u^{\frac{a-2}{a-1}}}{\rho y_1 K(a-1)^2 + ac_{19}\ln\left[u\alpha^{\frac{1-a}{c_{19}}}\right]}$	$c_{19}, y_1\neq 0$	$t\partial_t + ax\partial_x + c_{19}\alpha\partial_\alpha + (a-1)u\partial_u,\ au^{\frac{1}{a-1}}\partial_t + u^{\frac{a}{a-1}}\partial_x + y_1\alpha\partial_\alpha$
$-\dfrac{\rho y_1(a-1)u^{\frac{a-2}{a-1}}}{\rho y_1 K(a-1)+a\ln[\rho y_1(1-a)\alpha]}$	$c_{19}=0,\ y_1\neq 0$	$t\partial_t + ax\partial_x + (a-1)u\partial_u,\ au^{\frac{1}{a-1}}\partial_t + u^{\frac{a}{a-1}}\partial_x + y_1\alpha\partial_\alpha$
$Ku^{\frac{a-2}{a-1}}$	$c_{19}=0$	$t\partial_t + ax\partial_x + (a-1)u\partial_u,\ \alpha\partial_\alpha$

Table 2.5 (*Continued*).

R	Condition on consts.	Operators
$\dfrac{\rho(a-2)}{y_2}u^{\frac{a-2}{a-1}}$	$c_{19}=0,\ y_2\neq 0$	$t\partial_t+ax\partial_x+(a-1)u\partial_u,\ \left(\frac{y_2}{2-a}\right)u^{\frac{2-a}{a-1}}\partial_t+\left(t+y_2u^{\frac{1}{a-1}}\right)\partial_x+\partial_u$
$\dfrac{\rho(2a-1)}{1-[\rho(1-a)u]^{\frac{2a-1}{1-a}}\exp[\rho(2a-1)K]}$	$c_{20}\neq 0$	$(at+u)\partial_t+\left(x+\frac{u^2}{2}\right)\partial_x+c_{20}\alpha\partial_\alpha+(1-a)u\partial_u,\ \alpha\partial_\alpha$
$K\left[(1-a)u\right]^{\frac{2a-1}{a-1}}$	$c_{21}=0$	$at\partial_t+x\partial_x+(1-a)u\partial_u,\ \alpha\partial_\alpha$
$\dfrac{2\rho z_1(1+u^2)^{3/2}}{z_1\left(2\rho K+arc\sinh u+u\sqrt{1+u^2}\right)-2c_{22}\arctan u-\ln\left[\alpha^2(1+u^2)\right]}$	$c_{22},z_1\neq 0$	$(x+u)\partial_t+\left(\frac{u^2}{2}-t\right)\partial_x+(c_{22}+u)\alpha\partial_\alpha-(1+u^2)\partial_u,$ $\frac{u}{\sqrt{1+u^2}}\partial_t-\frac{1}{\sqrt{1+u^2}}\partial_x+z_1\alpha\partial_\alpha$
$-\dfrac{2\rho z_2(1+u^2)^{3/2}}{2c_{23}\arctan u+\ln\left[\alpha^2(1+u^2)\right]+2\rho z_2K}$	$c_{23}\neq 0$	$x\partial_t-t\partial_x+(c_{23}+u)\alpha\partial_\alpha-(1+u^2)\partial_u,\ \frac{u}{\sqrt{1+u^2}}\partial_t-\frac{1}{\sqrt{1+u^2}}\partial_x+z_2\alpha\partial_\alpha$
$\dfrac{\rho(1+u^2)^{3/2}}{z_3-\left(\alpha\sqrt{1+u^2}\right)^{1/z_4}\exp\left[\rho K+\frac{c_{23}\arctan u}{z_4}\right]}$	$c_{23}\neq 0$	$x\partial_t-t\partial_x+(c_{23}+u)\alpha\partial_\alpha-(1+u^2)\partial_u,$ $\left(t+\frac{z_3u}{\sqrt{1+u^2}}\right)\partial_t+\left(x-\frac{z_3}{\sqrt{1+u^2}}\right)\partial_x+z_4\alpha\partial_\alpha$
$\dfrac{\rho}{z_3}(1+u^2)^{3/2}$	$z_3\neq 0,\ z_4=0$	$x\partial_t-t\partial_x+u\alpha\partial_\alpha-(1+u^2)\partial_u,$
$K(1-au+u^2)^{3/2}\exp\left[\frac{a\arctan[\frac{2u-a}{\sqrt{4-a^2}}]}{\sqrt{4-a^2}}\right]$		$(at+x)\partial_t+(ax-t)\partial_x+u\alpha\partial_\alpha+(au-u^2-1)\partial_u,\ \alpha\partial_\alpha$

Table 2.6: Classification with respect to two-dimensional subalgebra of $A_1 \oplus A_2$ for $\mathcal{P}'(\alpha) = 0$.

R	Condition on consts.	Operators
$\rho u^2/p$	$p \neq 0, 1$	$\partial_t,\ \partial_t + p\partial_x,\ e^{x/p} u\alpha\partial_\alpha$
$\rho\left(u^2/2 - \bar{k}\right)/p$	$p \neq 0, 1$	$\partial_t,\ \partial_t + p\partial_x,\ e^{x/p}\left[pu\partial_t + p\left(\frac{u^2}{2} + \bar{k}\right)\partial_x + u\left(\bar{k} - \frac{u^2}{2}\right)\partial_u\right]$
$\rho u(u-r)/p$	$p \neq 0, 1$	$\partial_t,\ \partial_t + p\partial_x,\ e^{x/p}\left[p\partial_t + pr\partial_x + (r-u)u\partial_u\right]$
$\rho u/p$	$p \neq 0, 1$	$\partial_x,\ p\partial_t + \partial_x,\ e^{t/p} u\alpha\partial_\alpha,\ e^{t/p}\left(2pu\partial_t + pu^2\partial_x + 2pu\alpha\partial_\alpha - u^2\partial_u\right)$
$\left\{\rho\left(u + i\sqrt{2\bar{k}p}\right)\tan\left[2i\arctan\left[\frac{u}{\sqrt{2\bar{k}}}\right] + \bar{G}(\alpha)\right]\right\}/p$	$\bar{k} \neq 0,\ p \neq 0, 1$	$\partial_x,\ p\partial_t + \partial_x,\ e^{t/p}\left[pu\partial_t + p\left(\frac{u^2}{2} + \bar{k}\right)\partial_x + \left(\bar{k} - \frac{u^2}{2}\right)\partial_u\right]$
$(u-s)^2\bar{G}(\alpha) + \rho(u-s)/p$	$p \neq 0, 1$	$\partial_x,\ p\partial_t + \partial_x,\ e^{t/p}[p\partial_t + ps\partial_x + (s-u)\partial_u]$
$\rho u(p-u)/p$	$p \neq 0$	$\partial_t,\ \partial_t + p\partial_x,\ e^{t-x/p} u\alpha\partial_\alpha$
$\frac{\rho}{2pu}(p-u)(u^2 - 2\bar{k})$	$p \neq 0$	$\partial_t,\ \partial_t + p\partial_x,\ e^{t-x/p}\left[\partial_t + \left(\frac{u^2}{2} + \bar{k}\right)\partial_x - \frac{p-u}{2p}\left(u^2 - 2\bar{k}\right)\partial_u\right]$
$\rho(p-u)(u-y)/p$	$p \neq 0$	$\partial_t,\ \partial_t + p\partial_x,\ e^{t-x/p}\left[p\partial_t + py\partial_x - (p-u)(u-y)\partial_u\right]$
$\rho u(u-p)$	$p \neq 0$	$\partial_x,\ \partial_t + p\partial_x,\ e^{x-pt} u\alpha\partial_\alpha$

The arbitrary functions $\bar{G}$ are in general different and $0 \leq \phi < \pi$.

Table 2.6 (*Continued*).

R	Condition on consts.	Operators
$\frac{\rho}{2u}(u-p)(u^2-2\bar{k})$	$p\neq 0$	$\partial_x,\ \partial_t+p\partial_x,\ e^{x-pt}\left[u\partial_t+\left(\frac{u^2}{2}+\bar{k}\right)\partial_x+(p-u)\left(\frac{u^2}{2}-\bar{k}\right)\partial_u\right]$
$\rho(p-u)(z-u)$	$p\neq 0$	$\partial_x,\ \partial_t+p\partial_x,\ e^{x-pt}\left[\partial_t+z\partial_x+(u-z)(p-u)\partial_u\right]$
$\rho u(1-pu)$	$p\neq 0$	$\partial_t,\ p\partial_t+\partial_x,\ e^{t-px}u\alpha\partial_\alpha$
$\frac{\rho}{2u}(1-pu)(u^2-2\bar{k})$	$p\neq 0$	$\partial_t,\ p\partial_t+\partial_x,\ e^{t-px}\left[u\partial_t+\left(\frac{u^2}{2}+\bar{k}\right)\partial_x+(pu-1)\left(\frac{u^2}{2}-\bar{k}\right)\partial_u\right]$
$\rho(pu-1)(\tau-u)$	$p\neq 0$	$\partial_t,\ p\partial_t+\partial_x,\ e^{t-px}\left[\partial_t+\tau\partial_x+(\tau u-1)(u-\tau)\partial_u\right]$
$\rho u(pu-1)/p$	$p\neq 0$	$\partial_x,\ p\partial_t+\partial_x,\ e^{x-t/p}u\alpha\partial_\alpha$
$\frac{\rho}{2pu}(pu-1)(u^2-2\bar{k})$	$p\neq 0$	$\partial_x,\ p\partial_t+\partial_x,\ e^{x-t/p}\left[u\partial_t+\left(\frac{u^2}{2}+\bar{k}\right)\partial_x-\frac{pu-1}{2p}\left(u^2-2\bar{k}\right)\partial_u\right]$
$\rho(pu-1)(u-\omega)/p$	$p\neq 0$	$\partial_x,\ p\partial_t+\partial_x,\ ^{x-t/p}\left[p\partial_t+p\omega\partial_x+(pu-1)(u-\omega)\partial_u\right]$
$-\rho u^2$	$\phi\neq\pi/2$	$\partial_x,\ e^{-x}u\alpha\partial_\alpha,\ \cos\phi\, e^{-x}u\alpha\partial_\alpha$
$\rho\left(\bar{k}-u^2/2\right)$	$\phi\neq\pi/2$	$\partial_x,\ \cos\phi\, e^{-x}\left[u\partial_t+\left(\frac{u^2}{2}+\bar{k}\right)\partial_x+u\left(\frac{u^2}{2}-\bar{k}\right)\partial_u\right],\ e^{-x}\left[u\partial_t+\left(\frac{u^2}{2}+\bar{k}\right)\partial_x+u\left(\frac{u^2}{2}-\bar{k}\right)\partial_u\right]$
$\rho u(\mu-u)$	$\phi\neq\pi/2$	$\partial_x,\ \cos\phi\, e^{-x}\left[\partial_t+\mu\partial_x+(u-\mu)u\partial_u\right],\ e^{-x}\left[\partial_t+\mu\partial_x+(u-\mu)u\partial_u\right]$
$(u-\sigma)^2\bar{G}(\alpha)-\rho(u-\sigma)$	$\phi\neq\pi/2$	$\partial_t,\ \cos\phi\, e^{-t}\left[\partial_t+\sigma\partial_x+(u-\sigma)\partial_u\right],\ e^{-t}\left[\partial_t+\sigma\partial_x+(u-\sigma)\partial_u\right]$

Chapter 3

Symmetry Classification of Collapsible Tube Model incorporating Tension

We perform a complete symmetry classification of a collapsible tube model investigated in the previous chapter by incorporating the constant longitudinal tension and curvature of the tube wall. The first part of classification involves the same procedure employed in the previous chapter, i.e., we perform the classification with respect to three- and four-dimensional Lie algebras. Thereafter we analyze individual cases arising from both classifications in order to establish the maximal symmetry Lie algebra. The classification procedure specifies the arbitrary elements or parameters appearing in an equation or a system under study, hence yielding new models and those that are analogous to the existing ones. The one-dimensional model [25] to be investigated is governed by the system of equations

$$\alpha_t + (\alpha u)_x = 0, \tag{3.1}$$

$$\rho(u_t + uu_x) = -p_x - R(\alpha, u), \tag{3.2}$$

$$p - p_e = \mathcal{P}(\alpha) - T\alpha_{xx}. \tag{3.3}$$

The tube law (3.3) includes the tension term $T\alpha_{xx}$, in which the effect of longitudinal tension is represented by $T > 0$ and α_{xx} approximates the curvature of the tube wall.

The importance of the tension term is to match the order of the PDEs to the four boundary conditions required to describe flow in the Starling Resistor [2, 25]. We put the restriction $T > 0$ in order to exclude the model for which $T = 0$ (already treated in Chapter 2). There is a number of special cases of the underlying model such as: "A model of a snorer's upper airway" which has been investigated numerically in [2].

The nomenclature follows from that of the model in Chapter 2.

We eliminate the pressure from Eqs. (3.1)–(3.3) and obtain the system

$$\begin{aligned} &\alpha_t + \alpha u_x + u\alpha_x = 0, \\ &\rho(u_t + uu_x) + \alpha_x \mathcal{P}_\alpha - T\alpha_{xxx} + R(\alpha, u) = 0. \end{aligned} \tag{3.4}$$

The principal symmetry Lie algebra of the above system is spanned by at least two operators (to be established later), namely

$$X_1 = \partial_t \quad \text{and} \quad X_2 = \partial_x.$$

3.1 Determining equations for classification

Generating the determining equations manually can be very tedious especially when the order of the PDEs is two or more. However, the method of Lie is algorithmical and nowadays has been implemented using packages for symbolic computation. We use the software package `YaLie` for generating and manipulating determining equations for the maximal symmetry Lie algebra.

We seek the generator of symmetry admitted by system (3.4) having the form

$$X = \xi^1(t, x, \alpha, u)\partial_t + \xi^2(t, x, \alpha, u)\partial_x + \eta^1(t, x, \alpha, u)\partial_\alpha + \eta^2(t, x, \alpha, u)\partial_u. \tag{3.5}$$

We solve the following determining equations for the coefficients ξ^i and η^i of the symmetry generator (3.5):

$$\eta^1_u = 0,\ \eta^1_{\alpha\alpha} = 0,\ \xi^2_u = 0,\ \xi^2_\alpha = 0,\ \xi^1_u = 0,\ \xi^1_\alpha = 0,\ \xi^1_x = 0, \tag{3.6}$$

$$u\eta^1_x + \alpha\eta^2_x + \eta^1_t = 0, \tag{3.7}$$

$$\eta^1 + \alpha\eta^2_u - \alpha\eta^1_\alpha - \alpha\xi^2_x + \alpha\xi^1_t = 0, \tag{3.8}$$

$$\eta^2 + \alpha\eta^2_\alpha - u\xi^2_x + u\xi^1_t - \xi^2_t = 0, \tag{3.9}$$

$$\eta^1_{x\alpha} - \xi^2_{xx} = 0, \tag{3.10}$$

$$\eta^2_u - \eta^1_\alpha + 3\xi^2_x - \xi^1_t = 0, \tag{3.11}$$

$$\eta^2 R_u + \eta^1 R_\alpha - R\eta^2_u + \mathcal{P}'\eta^1_x + u\rho\eta^2_x - T\eta^1_{xxx} + \rho\eta^2_t + R\xi^1_t = 0, \tag{3.12}$$

$$\eta^1\mathcal{P}'' - \mathcal{P}'\eta^2_u + \mathcal{P}'\eta^1_\alpha - \mathcal{P}'\xi^2_x - 3T\eta^1_{xx\alpha} + T\xi^2_{xxx} + \mathcal{P}'\xi^1_t = 0, \tag{3.13}$$

$$\eta^2 - \alpha\eta^2_\alpha - u\xi^2_x + u\xi^1_t - \xi^2_t = 0, \tag{3.14}$$

where subscripts denote differentiation with respect to the indicated variables and prime indicates derivative with respect to α.

From Eqs. (3.6), (3.9) and (3.14) we have

$$\xi^1 = \xi^1(t),\ \xi^2 = \xi^2(t,x),\ \eta^1 = c(t,x)\alpha + d(t,x),\ \eta^2 = (\xi^2_x - \xi^1_t)u + \xi^2_t, \tag{3.15}$$

where c and d are arbitrary functions of t and x.

Upon substitution of the forms for ξ^i and η^i from (3.15) into Eqs. (3.7), (3.8) and (3.10)–(3.13) we obtain

$$u\alpha c_x + u\alpha\xi^2_{xx} + \alpha c_t + \alpha\xi^2_{tx} = 0, \tag{3.16}$$

$$c_x - \xi^2_{xx} = 0, \tag{3.17}$$

$$c + 2\xi^1_t - 4\xi^2_x = 0, \tag{3.18}$$

$$\begin{aligned}\mathcal{P}'\alpha c_x + R\left(2\xi^1_t - \xi^2_x\right) + uR_u\xi^2_x + u^2\rho\xi^2_{xx} + \alpha cR_\alpha + R_u\xi^2_t + 2u\rho\xi^2_{tx} + \rho\xi^2_{tt}\\ -u\rho\xi^1_{tt} - u\xi^1_t R_u - T\alpha c_{xxx} = 0,\end{aligned} \tag{3.19}$$

$$c\left(\mathcal{P}' + \alpha\mathcal{P}''\right) + 2\mathcal{P}'\left(\xi^1_t - \xi^2_x\right) + T\xi^2_{xxx} - 3Tc_{xx} = 0. \tag{3.20}$$

We use Eq. (3.18) to eliminate c from the other equations and get

$$5\left(u\xi^2_{xx} + \xi^2_{tx}\right) - 2\xi^1_{tt} = 0, \tag{3.21}$$

$$\xi^2_{xx} = 0, \tag{3.22}$$

$$\begin{aligned}R\left(2\xi^1_t - \xi^2_x\right) + u^2\rho\xi^2_{xx} + 4\alpha\mathcal{P}'\xi^2_{xx} + 4\alpha\xi^2_x R_\alpha + R_u\left(u\xi^2_x + \xi^2_t\right) + 2u\rho\xi^2_{tx} + \rho\xi^2_{tt}\\ -u\rho\xi^1_{tt} - 4T\alpha\xi^2_{xxxx} - \xi^1_t\left(uR_u + 2\alpha R_\alpha\right) = 0,\end{aligned} \tag{3.23}$$

$$2\alpha\xi^1_t\mathcal{P}'' - 2\mathcal{P}'\xi^2_x + 11T\xi^2_{xxx} - 4\alpha\mathcal{P}''\xi^2_x = 0. \tag{3.24}$$

Solving Eq. (3.22) we have

$$\xi^2(t,x) = A(t)x + B(t), \tag{3.25}$$

where A and B are arbitrary functions of t.

When we take into account Eq. (3.25), the remaining determining equations become

$$5B_t - 2A_{tt} = 0, \tag{3.26}$$

$$\begin{aligned} 2RA_t + 2u\rho B_t + x\rho B_{tt} + \rho C_{tt} + xB_tR_u + C_tR_u + B\left(uR_u + 4\alpha R_\alpha - R\right) \\ -u\rho A_{tt} - A_t\left(uR_u + 2\alpha R_\alpha\right) = 0, \end{aligned} \tag{3.27}$$

$$\alpha A_t\mathcal{P}'' - B\left(\mathcal{P}' + 2\alpha\mathcal{P}''\right) = 0. \tag{3.28}$$

Upon integration of Eq. (3.26) we obtain

$$B(t) = c_1 + \frac{2}{5}A_t, \tag{3.29}$$

where c_1 is an arbitrary constant.

The substitution of Eq. (3.29) into Eqs. (3.27)–(3.28) yields the compatibility conditions

$$\begin{aligned} \left[2xA_{tt} + 5C_t + (5c_1 - 3A_t)u\right]R_u + 2(10c_1 - A_t)\alpha R_\alpha + \left(8A_t - 5c_1\right)R \\ +\rho\left(2xA_{ttt} + 5C_{tt} - A_{tt}u\right) = 0, \end{aligned} \tag{3.30}$$

$$\left(5c_1 + 2A_t\right)\mathcal{P}' + \left(10c_1 - A_t\right)\alpha\mathcal{P}'' = 0. \tag{3.31}$$

Therefore the symmetry generator (3.5) takes the form

$$\begin{aligned} X = A(t)\partial_t + \left[\left(c_1 + \frac{2}{5}A_t\right)x + C(t)\right]\partial_x + \left(4c_1 - \frac{2}{5}A_t\right)\alpha\partial_\alpha \\ + \left[C_t + \frac{2}{5}xA_{tt} + \left(c_1 - \frac{3}{5}A_t\right)u\right]\partial_u. \end{aligned} \tag{3.32}$$

If we assume that $R(\alpha, u)$ and $\mathcal{P}(\alpha)$ are arbitrary functions of their arguments, then from the classifying equations (3.30) and (3.30) we should have

$$5c_1 + 2A_t = 0,\ 10c_1 - A_t = 0,\ 8A_t - 5c_1 = 0,\ 2xA_{tt} + 5C_t + (5c_1 - 3A_t)u = 0. \tag{3.33}$$

Solving (3.33) for the unknowns we obtain $A(t) = c_2$ and $C(t) = c_3$. This leads to the symmetry generators

$$X_1 = \partial_t, \quad X_2 = \partial_x. \tag{3.34}$$

Thus the operators given by (3.34) span the principal symmetry Lie algebra.

Similarly to Chapter 2 we analyze the classifying relations (3.30) and (3.31) via classification with respect to three- and four-dimensional real Lie algebras by Patera and Winternitz [69]. This procedure does not guarantee the maximal symmetry Lie algebra. It only provides a subalgebra of the symmetry Lie algebra. Hence we go further by carrying out symmetry analysis of each case arising from the 3D and 4D Lie algebra classifications.

3.2 Classification with respect to 3D Lie algebras

The principal Lie algebra admitted by the underlying system (3.4) is two-dimensional and spanned by the operators ∂_t, ∂_x. In order to extend the principal symmetry Lie algebra we find the third operator in the form (3.32). This operator is a symmetry generator of the system provided it satisfies the compatibility conditions for the arbitrary parameters $\mathcal{P}(\alpha)$ and $R(\alpha, u)$. We consider the following possible realizations with the basis (e_1, e_2, e_3).

(a) $e_1 = \partial_t, \quad e_2 = \partial_x, \quad e_3 = X_{tn}$ (b) $e_1 = \partial_t, \quad e_2 = X_{tn}, \quad e_3 = \partial_x$

(c) $e_1 = \partial_x, \quad e_2 = \partial_t, \quad e_3 = X_{tn}$ (d) $e_1 = \partial_x, \quad e_2 = X_{tn}, \quad e_3 = \partial_t$

(e) $e_1 = X_{tn}, \quad e_2 = \partial_t, \quad e_3 = \partial_x$ (f) $e_1 = X_{tn}, \quad e_2 = \partial_x, \quad e_3 = \partial_t$

where X_{tn} is the extension of the symmetry Lie algebra of the form (3.32).

We give illustration of the calculations involved by considering the possible realizations (a)–(f) for the algebras $3A_1$ and $A_1 \oplus A_2$.

The algebra $3A_1$ has zero Lie Brackets.

(a) $[e_1, e_2] = 0$ is identically satisfied. The Lie Bracket: $[e_1, e_3] = 0$ implies that

$$e_3 = c_2\partial_t + (c_1 x + c_3)\partial_x + 4c_1\alpha\partial_\alpha + c_1 u\partial_u$$

for arbitrary constants c_2 and c_3. We take $e_3 = c_1 x\partial_x + 4c_1\alpha\partial_\alpha + c_1 u\partial_u$. Lastly the Lie

Bracket: $[e_2, e_3] = 0$ yields $c_1 = 0$ and thus there is no realization. Likewise there are no realizations for (b)–(f).

Next we consider $A_1 \oplus A_2$ with the nonzero Lie Bracket: $[e_1, e_2] = e_2$. The possible realizations (a) and (c) do not satisfy this relation.

(b) $[e_2, e_3] = 0$ yields

$$e_2 \approx -\frac{5}{2}c_1 t\partial_t + C(t)\partial_x + 5c_1\alpha\partial_\alpha + \left(C_t + \frac{5}{2}c_1 u\right)\partial_u.$$

We invoke $[e_1, e_2] = e_2$. This implies that $c_1 = 0$ and $C(t) = c_3 e^t$. Thus $e_2 = e^t(\partial_x + \partial_u)$. With this form of e_2 the compatibility conditions (3.30)–(3.31) imply that $\mathcal{P}(\alpha)$ is arbitrary and $R(\alpha, u) = F_1(\alpha) - \rho u$ for an arbitrary smooth function $F_1(\alpha)$.

(d) $[e_2, e_3] = 0$ yields $e_2 = c_1 x\partial_x + 4c_1\alpha\partial_\alpha + c_1 u\partial_u$ and $[e_1, e_2] = e_2$ implies that $c_1 = 0$. Clearly e_2 cannot be zero. Therefore the case is discarded.

(e) $[e_1, e_3] = 0$ yields

$$e_1 \approx -\frac{5}{2}c_1 t\partial_t + C(t)\partial_x + 5c_1\alpha\partial_\alpha + \left(C_t + \frac{5}{2}c_1 u\right)\partial_u.$$

The Lie Bracket $[e_1, e_2] = e_2$ implies that

$$e_1 = t\partial_t + c_3\partial_x - 2\alpha\partial_\alpha - u\partial_u \approx t\partial_t - 2\alpha\partial_\alpha - u\partial_u.$$

The classifying relations (3.30) and (3.31) reduce to

$$uR_u + 2\alpha R_\alpha - 2R \quad \text{and} \quad \mathcal{P}'' = 0. \tag{3.35}$$

Therefore solving (3.35) we have

$$R(\alpha, u) = \alpha F_2(u/\sqrt{\alpha}), \quad \mathcal{P}(\alpha) = k_1\alpha + k_2,$$

where k_1, k_2 are arbitrary constants and F_2 is an arbitrary function of its argument.

(f) $[e_1, e_3] = 0$ yields $e_1 = c_1 x\partial_x + 4c_1\alpha\partial_\alpha + c_1 u\partial_u$ and invoking $[e_1, e_2] = e_2$ we get $e_1 = -x\partial_x - 4\alpha\partial_\alpha - u\partial_u$. The classifying relations (3.30) and (3.31) simplify to

$$uR_u + 4\alpha R_\alpha - R = 0 \quad \text{and} \quad \mathcal{P}_\alpha + 2\alpha\mathcal{P}'' = 0. \tag{3.36}$$

From (3.36) we obtain

$$R(\alpha,u)=\alpha^{1/4}F_3(u\alpha^{-1/4}),\quad \mathcal{P}(\alpha)=2k_3\sqrt{\alpha}+k_4$$

for arbitrary constants k_3, k_4 and F_3 an arbitrary function of its argument.

All the results are presented in Table 3.1.

Table 3.1: Classification with respect to 3D Lie algebras.

Algebra	$\mathcal{P}$	R	Extra operator
$A_1\oplus A_2$	Arbitrary	$F_1(\alpha)-\rho u$	$e^t(\partial_x+\partial_u)$
	$k_1\alpha+k_2$	$\alpha F_2(u/\sqrt{\alpha})$	$t\partial_t-2\alpha\partial_\alpha-u\partial_u$
	$2k_3\sqrt{\alpha}+k_4$	$\alpha^{1/4}F_3(u\alpha^{-1/4})$	$x\partial_x+4\alpha\partial_\alpha+u\partial_u$
$A_{3,1}$	Arbitrary	$F_4(\alpha)$	$t\partial_x+\partial_u$
$A_{3,2}$	$k_5\ln\alpha+k_6$	$F_5(u-\frac{1}{2}\ln\alpha)/\sqrt{\alpha}$	$t\partial_t+(t+x)\partial_x+2\alpha\partial_\alpha+\partial_u$
$A_{3,3}$	$k_7\ln\alpha+k_8$	$F_6(u)/\sqrt{\alpha}$	$t\partial_t+x\partial_x+2\alpha\partial_\alpha$
$A_{3,4}$	$\frac{3}{2}k_9\alpha^{2/3}+k_{10}$	$\sqrt{\alpha}\,F_7(u\alpha^{-1/3})$	$t\partial_t-x\partial_x-6\alpha\partial_\alpha-2u\partial_u$
$A^a_{3,5}$	$k_{11}\dfrac{(2a-1)[(1-2a)\alpha]^{-\frac{a}{2a-1}}}{a-1}+k_{12}$	$[2(2a-1)\alpha]^{\frac{a-2}{2(2a-1)}}$ $\times F_8\left(u[2(1-2a)\alpha]^{\frac{a-1}{2(1-2a)}}\right)$	$t\partial_t+ax\partial_x+2(2a-1)\alpha\partial_\alpha$ $+(a-1)u\partial_u$
	$k_{13}\dfrac{(a-2)\alpha^{\frac{a-1}{a-2}}}{a-1}+k_{14}$	$[2(2-a)\alpha]^{\frac{1-2a}{2(2-a)}}$ $\times F_9\left(u[2(2-a)\alpha]^{\frac{a-1}{2(2-a)}}\right)$	$at\partial_t+x\partial_x+2(2-a)\alpha\partial_\alpha$ $+(1-a)u\partial_u$

The F_i are arbitrary functions of the indicated arguments and $0<|a|<1$.

3.3 Classification with respect to 4D Lie algebras

We examine all the 4D Lie algebras containing as subalgebras the 3D symmetry Lie algebras obtained in the previous section (Table 3.1). We show calculations involved for the 3D Lie algebra $A_1 \oplus A_2$.

The 4D Lie algebra $2A_2$ contains only $A_1 \oplus A_2$ as a subalgebra and the corresponding nonzero Lie Brackets are $[e_1, e_2] = e_2$ and $[e_3, e_4] = e_4$.

Firstly we consider the possible realization: $(e_1, \partial_x, \partial_t, e^t(\partial_x + \partial_u))$. From the Lie Bracket $[e_1, e_3] = 0$, we have $e_1 = c_1 x\partial_x + 4c_1\alpha\partial_\alpha + c_1 u\partial_u$. The Lie bracket $[e_1, e_4] = 0$ is satisfied. Finally, $[e_1, e_2] = e_2$ implies that $c_1 = -1$ and hence, $e_1 = -x\partial_x - 4\alpha\partial_\alpha - u\partial_u$. The classifying relations (3.30) and (3.31) reduce to

$$4\alpha F_1' - F_1 = 0 \quad \text{and} \quad 2\alpha\mathcal{P}'' + \mathcal{P}' = 0, \tag{3.37}$$

The solution to (3.37) is given by

$$F_1(\alpha) = K_1\, \alpha^{1/4}, \quad \mathcal{P}(\alpha) = 2\ell_1\sqrt{\alpha} + \ell_2,$$

where ℓ_1, ℓ_2 and K_1 are arbitrary constants.

Next we consider the possible realization: $(-t\partial_t + 2\alpha\partial_\alpha + u\partial_u, \partial_t, e_3, \partial_x)$. The Lie Bracket $[e_2, e_3] = 0$ yields $e_3 = c_1 x\partial_x + 4c_1\alpha\partial_\alpha + c_1 u\partial_u$. The Lie Bracket $[e_1, e_3] = 0$ is satisfied identically. Lastly $[e_3, e_4] = e_4$ implies that $c_1 = -1$ and therefore $e_3 = -x\partial_x - 4\alpha\partial_\alpha - u\partial_u$. The compatibility conditions (3.30)–(3.31) yield

$$k_1 = 0, \quad zF_2'(z) - 3\alpha F_2(z) = 0, \tag{3.38}$$

where prime represents derivative with respect to $z = u/\sqrt{\alpha}$. The first condition in (3.38) implies that $\mathcal{P}(\alpha) = k_2 \equiv$ constant and solving the second equation we have

$$F_2(z) = K_2 z^3$$

for arbitrary constant K_2. Thus $R(\alpha, u) = K_2 u^3/\sqrt{\alpha}$.

Finally we obtain $e_3 = -t\partial_t + 2\alpha\partial_\alpha + u\partial_u$ corresponding to the possible realization: $(-x\partial_x - 4\alpha\partial_\alpha - u\partial_u, \partial_x, e_3, \partial_t)$. This is the same realization as the latter and hence redundant.

All the results are given in Table 3.2.

Table 3.2: Classification with respect to 4D Lie algebras.

$\mathcal{P}$	R	Operators
$2\ell_1\sqrt{\alpha}+\ell_2$	$K_1\alpha^{1/4}-\rho u$	$\partial_t,\ \partial_x,\ e^t(\partial_x+\partial_u),\ x\partial_x+4\alpha\partial_\alpha+u\partial_u$
$\mathcal{P}_0$	$K_2u^3/\sqrt{\alpha}$	$\partial_t,\ \partial_x,\ t\partial_t-2\alpha\partial_\alpha-u\partial_u,\ x\partial_x+4\alpha\partial_\alpha+u\partial_u$
$\ell_3\alpha+\ell_4$	$K_3\alpha$	$\partial_t,\ \partial_x,\ t\partial_x+\partial_u,\ t\partial_t-2\alpha\partial_\alpha-u\partial_u$
$\ell_5\ln\alpha+\ell_6$	$K_4/\sqrt{\alpha}$	$\partial_t,\ \partial_x,\ t\partial_x+\partial_u,\ t\partial_t+x\partial_x+2\alpha\partial_\alpha$
$3\ell_7\alpha^{1/3}+\ell_8$	K_5	$\partial_t,\ \partial_x,\ t\partial_x+\partial_u,\ t\partial_t+2x\partial_x+6\alpha\partial_\alpha+u\partial_u$
$\ell_9\frac{[(1+2b)\alpha]^{\frac{b}{1+2b}}}{b}+\ell_{10}$	$K_6[(1+2b)\alpha]^{\frac{b-1}{2(2b+1)}}$	$\partial_t,\ \partial_x,\ t\partial_x+\partial_u,\ t\partial_t+(1+b)x\partial_x+2(2b+1)\alpha\partial_\alpha+bu\partial_u$

The K_i are arbitrary nonzero constants and $0<|b|<1$.

It can be shown by direct calculation that there are no realizations for the 3D Lie algebras which contain the two-dimensional subalgebras $2A_1$ and A_2. This is the same for the 4D Lie algebras which contain as subalgebras the 3D Lie algebras corresponding to $2A_1$ and A_2 (see Table II [69]).

3.4 Complete classification

The classification performed so far does not provide the full symmetry Lie algebra, that is, there may be extra operators for the equations admitting 3D and 4D Lie algebras. As a result we continue classification by performing symmetry analysis of every case from Tables 3.1 and 3.2 to achieve complete group classification.

We illustrate calculations by taking one case from Tables 3.1 & 3.2 and present the rest

of the calculations in tabular form.

We consider the first case in Table 3.1. The substitution of the form for R into the original system (3.4) and execution of the Lie's algorithm result in the classifying relations

$$\begin{aligned}(5c_1 - 8A_t)\,F_1 + (5u\rho + 2\alpha F_1')\,A_t + \rho\,[5C_t + (u+2x)A_{tt}] & \\ -20\alpha c_1 F_1' - 5\rho C_{tt} - 2x\rho A_{ttt} &= 0, \qquad (3.39)\\ (5c_1 + 2A_t)\,\mathcal{P}' + (10c_1 - A_t)\,\alpha\mathcal{P}'' &= 0. \qquad (3.40)\end{aligned}$$

The corresponding symmetry generator is the same as the one given by (3.32).

Upon separation of Eq. (3.39) with respect to u and then x, we obtain $A(t) = -\frac{1}{5}c_2 e^{-5t} + c_3$ and $c_2 = 0$ respectively for arbitrary constants c_2, c_3. Thus $A(t) = c_3$ and the classifying relations (3.39)–(3.40) simplify to

$$\begin{aligned}c_1\,(F_1 - 4\alpha F_1') + \rho\,(C_t - C_{tt}) &= 0, \qquad (3.41)\\ c_1\,(\mathcal{P}' + 2\alpha\mathcal{P}'') &= 0. \qquad (3.42)\end{aligned}$$

The differentiation of Eq. (3.41) with respect to t yields

$$\begin{aligned}c_1\,(F_1 - 4\alpha F_1') + \rho c_5 = 0, \qquad (3.43)\\ C(t) = c_4 e^t + c_5 t + c_6, \qquad (3.44)\end{aligned}$$

where c_4, c_5 and c_6 are the arbitrary constants. Definitely c_1 cannot be zero in Eq. (3.42) and (3.43). If we assume that $c_1 \neq 0$, then the solution to these equations is given by

$$F_1(\alpha) = K\alpha^{1/4} - \rho\beta, \quad \mathcal{P}(\alpha) = 2\kappa_1\sqrt{\alpha} + \kappa_2,$$

where $\beta = c_5/c_1$, κ_1, κ_2 and K are arbitrary constants. Therefore the generator of symmetry takes the form

$$X = c_3\partial_t + \left[c_6 + c_4 e^t + c_1(\beta t + x)\right]\partial_x + 4c_1\alpha\partial_\alpha + \left[c_4 e^t + c_1(\beta + u)\right]\partial_u.$$

The full symmetry Lie algebra is given by

$$X_1 = \partial_t,\ X_2 = \partial_x,\ X_3 = e^t(\partial_x + \partial_u),\ X_4 = (\beta t + x)\partial_x + 4\alpha\partial_\alpha + (\beta + u)\partial_u.$$

Next we consider the first case in Table 3.2. Substituting the forms of $\mathcal{P}$ and R into the original system (3.4) and generating the determining equations, we have

$$5B_t - 2A_{tt} = 0, \tag{3.45}$$

$$\left(2u\rho - 3K_1\alpha^{1/4}\right) A_t + 2\rho\left[(x-2u)B_t + C_t + uA_{tt} - xB_{tt} - C_{tt}\right] = 0, \tag{3.46}$$

$$\frac{\ell_1 A_t}{\sqrt{\alpha}} = 0, \tag{3.47}$$

provided

$$X = A(t)\partial_t + \left(B(t)x + C(t)\right)\partial_x + (4B - 2A_t)\alpha\partial_\alpha + \left[C_t + xB_t + (B - A_t)u\right]\partial_u. \tag{3.48}$$

The last equation, (3.47), prompts consideration of the cases: $\ell_1 \neq 0$ and $\ell_1 = 0$. The case $\ell_1 \neq 0$ implies that $A_t = 0$ and hence $A(t) = c_1$. In turn from Eqs. (3.45) and (3.46) we get $B(t) = c_2$ and $C(t) = c_3 e^t + c_4$ for arbitrary constants c_1, c_2, c_3 and c_4. Thus the symmetry generator (3.48) becomes

$$X = c_1\partial_t + \left(c_4 + c_3 e^t + c_2 x\right)\partial_x + 4c_2\alpha\partial_\alpha + \left(c_3 e^t + c_2 u\right)\partial_u. \tag{3.49}$$

Therefore the complete symmetry Lie algebra is given by

$$X_1 = \partial_t,\ X_2 = \partial_x,\ X_3 = e^t(\partial_x + \partial_u),\ X_4 = x\partial_x + 4\alpha\partial_\alpha + u\partial_u.$$

For the second case Eq. (3.47) is identically satisfied and $\mathcal{P}(\alpha) = \ell_2$. From Eq. (3.45) we have $B(t) = c_1 + \frac{2}{5}A_t$ and consequently Eq. (3.46) simplifies to

$$5\left(2u\rho - 3K_1\alpha^{1/4}\right) A_t + 2\rho\left[5C_t + (u+2x)A_{tt} - 5C_{tt} - 2xA_{ttt}\right] = 0, \tag{3.50}$$

where c_1 is an arbitrary constant.

Separating Eq. (3.50) by α and u we get $A(t) = c_2$, $C(t) = c_3 e^t + c_4$ for arbitrary c_2, c_3, c_4. We eventually obtain the same symmetry generator as (3.49) and hence conclude that the first case in Table 3.2 is possible provided $\mathcal{P}(\alpha) = \mathcal{P}_0 \equiv$ constant.

The complete classification results are summarized in the following tables according to the dimension of the symmetry Lie algebra.

Table 3.3: Classification results: 3D Lie algebra.

$\mathcal{P}$	R	Condition on consts.	Operators
$\mathcal{P}_0$	$K\alpha+\rho m_1 u$	$k_1=0, m_1\neq 0$	$\partial_t,\ \partial_x,\ e^{-m_1 t}(\partial_x-m_1\partial_u)$
	$\frac{K}{\sqrt{\alpha}}+\rho m_3\left(u-\frac{1}{2}\ln\alpha\right)$	$k_5=0, m_3\neq 0$	$\partial_t,\ \partial_x,\ e^{-m_3 t}(\partial_x-m_3\partial_u)$
	$\frac{K}{\sqrt{\alpha}}+\rho m_4 u$	$k_9=0, m_4\neq 0$	$\partial_t,\ \partial_x,\ e^{-m_4 t}(\partial_x-m_4\partial_u)$
	$K[2(2a-1)\alpha]^{\frac{a-2}{2(2a-1)}}+\rho m_5 u$	$k_{11}=0, m_5\neq 0$	$\partial_t,\ \partial_x,\ e^{-m_5 t}(\partial_x-m_5\partial_u)$
	$K[2(2-a)\alpha]^{\frac{1-2a}{2(2-a)}}+\rho m_6 u$	$k_{13}=0, m_6\neq 0$	$\partial_t,\ \partial_x,\ e^{-m_6 t}(\partial_x-m_6\partial_u)$

Table 3.4: Classification results: 4D Lie algebra.

$\mathcal{P}$	R	Condition on consts.	Operators
$2\kappa_1\sqrt{\alpha}+\kappa_2$	$K\alpha^{1/4}-\rho(\beta+u)$	$\kappa_1\neq 0$	$\partial_t,\ \partial_x,\ e^t(\partial_x+\partial_u)$, $(\beta t+x)\partial_x+4\alpha\partial_\alpha+(\beta+u)\partial_u$
$\mathcal{P}_0$	$K\alpha^{1/4}+\rho m_2 u$	$k_3=0$ $m_2\neq 0,-1$	$\partial_t,\ \partial_x,\ x\partial_x+4\alpha\partial_\alpha+u\partial_u$, $e^{-m_2 t}(\partial_x-m_2\partial_u)$
	$K\alpha^{1/4}$	$k_3=0$ $m_2=0$	$\partial_t,\ \partial_x,\ x\partial_x+4\alpha\partial_\alpha+u\partial_u,\ t\partial_x+\partial_u$
$\frac{(\lambda-10)\alpha^{\frac{3\lambda-5}{\lambda-10}}}{3\lambda-5}\kappa_3+\kappa_4$	$\frac{K(8\lambda-5)[2(10-\lambda)\alpha]^{\frac{8\lambda-5}{2(\lambda-10)}}-10\rho\mu}{8\lambda-5}$	$\kappa_3\neq 0$ $\lambda\neq\frac{5}{8},\frac{5}{3},10$	$\partial_t,\ \partial_x,\ t\partial_x+\partial_u$, $\lambda t\partial_t+\left[\mu t^2+\left(\frac{2\lambda}{5}+1\right)x\right]\partial_x$ $-\frac{2}{5}(\lambda-10)\alpha\partial_\alpha+\left[2\mu t+\left(1-\frac{3\lambda}{5}\right)u\right]\partial_u$

Table 3.4 continued.

$\mathcal{P}$	R	Condition on consts.	Operators (including $\partial_t,\ \partial_x$)
$3\kappa_5\alpha^{1/3}+\kappa_6$	$K-\frac{8}{15}\rho\mu\ln\alpha$	$\kappa_5\neq 0,\ \lambda=5/8$	$t\partial_x+\partial_u,\ \frac{5}{8}t\partial_t+\left(\mu t^2+\frac{5}{4}x\right)\partial_x+\frac{15}{4}\alpha\partial_\alpha+\left(2\mu t+\frac{5}{8}u\right)\partial_u$
$\kappa_7\ln\alpha+\kappa_8$	$\frac{K}{\sqrt{\alpha}}-\frac{6}{5}\rho\mu$	$\kappa_7\neq 0,\ \lambda=5/3$	$t\partial_x+\partial_u,\ \frac{5}{3}t\partial_t+\left(\mu t^2+\frac{5}{3}x\right)\partial_x+\frac{10}{3}\alpha\partial_\alpha+2\mu t\partial_u$
$\mathcal{P}_0$	$-\frac{2}{15}\rho\mu$	$\lambda=10$	$t\partial_x+\partial_u,\ 10t\partial_t+(\mu t^2+5x)\partial_x+2\mu t\partial_u$
$\frac{\kappa_9}{3}\alpha^3+\kappa_{10}$	$K\alpha^4-\frac{5}{4}\rho\nu$	$\kappa_9\neq 0$	$t\partial_x+\partial_u,\ t\partial_t+\left(\nu t^2+\frac{2}{5}x\right)\partial_x-\frac{2}{5}\alpha\partial_\alpha+\left(2\nu t-\frac{3}{5}u\right)\partial_u$
$\mathcal{P}_0$	$\frac{K[u(5-3\tau)+5\omega]^3}{\sqrt{\alpha}}$	$k_7=0$	$t\partial_t+x\partial_x+2\alpha\partial_\alpha,\ \tau t\partial_t+\left[\omega t+\left(\frac{2\tau}{5}+1\right)x\right]\partial_x-\frac{2}{5}(\tau-10)\alpha\partial_\alpha+\left[\omega+\left(\frac{3\tau}{5}+1\right)u\right]\partial_u$
	$K\sqrt{\alpha}$	$k_9,m_4=0$	$t\partial_t-x\partial_x-6\alpha\partial_\alpha-2u\partial_u,\ \frac{5}{7}t\partial_t-\left(nt+\frac{5}{7}x\right)\partial_x-\frac{30}{7}\alpha\partial_\alpha-\left(n+\frac{10}{7}u\right)\partial_u$
	$K[2(2a-1)\alpha]^{\frac{a-2}{2(2a-1)}}$	$k_{11},m_5=0$	$t\partial_t+ax\partial_x+2(2a-1)\alpha\partial_\alpha+(a-1)u\partial_u,\ \frac{5}{5a-2}t\partial_t+\left(pt+\frac{5a}{5a-2}x\right)\partial_x+\frac{10(2a-1)}{5a-2}\alpha\partial_\alpha+\left[p+\frac{5(a-1)}{5a-2}u\right]\partial_u$
$\mathcal{P}_0$	$K[2(2-a)\alpha]^{\frac{1-2a}{2(2-a)}}$	$k_{13},m_6=0$	$at\partial_t+x\partial_x+2(2-a)\alpha\partial_\alpha+(1-a)u\partial_u,\ \frac{5}{5-2a}t\partial_t+\left(qt+\frac{5a}{5-2a}x\right)\partial_x+\frac{10(a-2)}{2a-5}\alpha\partial_\alpha+\left[q+\frac{5(a-1)}{2a-5}u\right]\partial_u$
	$K\alpha^{1/4}-\rho u$	$\ell_1=0$	$e^t(\partial_x+\partial_u),\ x\partial_x+4\alpha\partial_\alpha+u\partial_u$
$3\ell_7\alpha^{1/3}+\ell_8$	K	$\ell_7\neq 0$	$t\partial_x+\partial_u,\ t\partial_t+2x\partial_x+6\alpha\partial_\alpha+u\partial_u$
$\mathcal{P}_0$	$K[(1+2b)\alpha]^{\frac{b-1}{2(2b+1)}}$	$\ell_9=0$	$t\partial_x+\partial_u,\ t\partial_t+(1+b)x\partial_x+2(2b+1)\alpha\partial_\alpha+bu\partial_u$

Table 3.5: Classification results: 5D Lie algebra.

$\mathcal{P}$	R	Condition on consts.	Operators
$\mathcal{P}_0$	$K\alpha$	$k_1=0, m_1=0$	$\partial_t,\ \partial_x,\ t\partial_t-2\alpha\partial_\alpha-u\partial_u, t\partial_x+\partial_u$ $\frac{5}{2}t\partial_t-\gamma t\partial_x-5\alpha\partial_\alpha-\left(\gamma+\frac{5}{2}u\right)\partial_u$
	$K/\sqrt{\alpha}$	$k_5=0, m_3=0$	$\partial_t,\ \partial_x,\ t\partial_x+\partial_u,\ t\partial_t+x\partial_x+2\alpha\partial_\alpha,$ $\frac{5}{3}t\partial_t+\left(\sigma t+\frac{5}{3}x\right)\partial_x+\frac{10}{3}\alpha\partial_\alpha+\sigma\partial_u$
	$Ku^3/\sqrt{\alpha}$	$k_1=0$	$\partial_t,\ \partial_x,\ t\partial_t-2\alpha\partial_\alpha-u\partial_u,\ x\partial_x+4\alpha\partial_\alpha+u\partial_u,$ $t\partial_t+\frac{2}{5}x\partial_x-\frac{2}{5}\alpha\partial_\alpha-\frac{3}{5}u\partial_u$
	K	$\ell_7=0$	$\partial_t,\ \partial_x,\ t\partial_x+\partial_u,$ $t\partial_t+\frac{2}{5}\left(x-\frac{2Kt^2}{\rho}\right)\partial_x-\frac{2}{5}\alpha\partial_\alpha-\frac{1}{5}\left(\frac{8Kt}{\rho}+3u\right)\partial_u,$ $\left(x+\frac{2Kt^2}{5\rho}\right)\partial_x+4\alpha\partial_\alpha+\left(\frac{Kt}{\rho}+u\right)\partial_u$

The arbitrary nonzero constants K are in general different.

3.5 Similarity reductions

In many real-life applications models are expressed in terms of PDEs as opposed to ODEs which are relatively easier to solve. Fortunately symmetries can be used to construct invariant solutions (i.e., solutions that are unchanged under the action of a subgroup of the symmetry group) which in turn allows the reduction of PDEs to ODEs.

Invariant solutions of a given equation satisfy an equation with fewer variables. Thus the search for invariant solutions can be viewed as a sort of dimensional reduction. One of the procedures commonly used is dimensional analysis. Dimensional analysis is reminiscent of scaling symmetries and dimensionless variables are simply invariants of the scaling symmetry group. By using symmetries more general than scaling symmetries it is possible to reduce the number of independent variables by introducing the invariants

of these symmetries as new independent variables. Therefore the existence of symmetries for PDEs allows a kind of dimensional reduction. The systematic way of constructing nonequivalent classes of invariant solutions is to determine the optimal system.

3.5.1 Optimal system of subalgebras

There are two methods frequently used to construct the optimal system of subalgebras: the method suggested by Ovsiannikov [66, 67] is based on finding the matrix of inner automorphism corresponding to the operators of the adjoint group of a given Lie algebra. Olver [65] suggested a slightly different approach which involves construction of the adjoint action on Lie algebras. There is also another method which is not quite common due to Patera *et al* [68]. It involves reduction procedure by the ideals of Lie algebras.

A given group-invariant solution S arising from a subgroup of the symmetry group can be transformed by means of symmetry transformations to another solution $\bar{S}$. The two group-invariant solutions are said to be *nonsimilar* if they cannot be mapped to each other by means of invertible symmetry transformations. By definition of an invariant solution, a solution S is invariant under a subgroup H of the symmetry group G if $S = hS$ for all $h \in H$. We find under which subgroup K of the group G is S transformed to $\bar{S}$, i.e., $\bar{S} = gS$ for all $g \in G$. Invariance under K means that $\bar{S} = k\bar{S}$ for all $k \in K$. Now $\bar{S} = gS = ghS = ghg^{-1}gS = ghg^{-1}\bar{S} = k\bar{S}$, that is, if S is H-invariant, then gS is K-invariant with $K = gHg^{-1}$. The group K is known as the *conjugate* or *adjoint* subgroup to H under the symmetry group G. Conjugacy provides information on how solutions transform under the action of other subgroups of the symmetry group. Therefore the search for nonequivalent group-invariant solutions reduces to the construction of nonsimilar subgroup of the symmetry group under the adjoint action. This is known as *optimal system of subalgebras* (see also [83]).

As an illustrative example we consider from Table 3.4 the case: $\mathcal{P} = 3\ell_7\alpha^{1/3} + \ell_8$; $R = K$; $\quad X_1 = \partial_t$, $X_2 = \partial_x$, $X_3 = t\partial_x + \partial_u$, $X_4 = t\partial_t + 2x\partial_x + 6\alpha\partial_\alpha + u\partial_u$.

Taking into account Table 3.6 of commutators, we find the adjoint representation of X_i

Table 3.6: Table of commutators.

$\nearrow [X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	X_2	X_1
X_2	0	0	0	$2X_2$
X_3	$-X_2$	0	0	X_3
X_4	$-X_1$	$-2X_2$	$-X_3$	0

using

$$\mathrm{Ad}\left(e^{\epsilon X_i}\right) X_j = X_j - \epsilon[X_i, X_j] + \frac{1}{2!}\epsilon^2 \left[X_i, [X_i, X_j]\right] - \cdots , \tag{3.51}$$

where i, j take values from 1 to 4.

We show calculations for the following:

$\mathrm{Ad}\left(e^{\epsilon X_1}\right) X_3 = X_3 - \epsilon[X_1, X_3] + \frac{1}{2!}\epsilon^2 \left[X_1, [X_1, X_3]\right] - \cdots = X_3 - \epsilon X_2.$

$\mathrm{Ad}\left(e^{\epsilon X_4}\right) X_1 = X_1 + \epsilon X_1 + \frac{1}{2!}\epsilon^2 X_1 - \cdots = \left(1 + \epsilon + \frac{1}{2!}\epsilon^2 - \cdots\right) X_1 = e^{\epsilon} X_1.$

All the calculations are summarized in the following table.

Table 3.7: Table of adjoint representation.

Ad	X_1	X_2	X_3	X_4
X_1	X_1	X_2	$X_3 - \epsilon X_2$	$X_4 - \epsilon X_1$
X_2	X_1	X_2	X_3	$X_4 - 2\epsilon X_2$
X_3	$X_1 + \epsilon X_2$	X_2	X_3	$X_4 - \epsilon X_3$
X_4	$e^{\epsilon} X_1$	$e^{2\epsilon} X_2$	$e^{\epsilon} X_3$	X_4

In constructing the one-dimensional optimal system of the symmetry group $\langle X_1, X_2, X_3, X_4 \rangle$, we consider the general operator

$$X = a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4, \tag{3.52}$$

where $a_1, \ldots, a_4$ are arbitrary constants and investigate whether the application of the adjoint action to (3.52) can transform it to a new operator having simpler form.

Firstly assume that $a_4 \neq 0$ (take $a_4 = 1$). From (3.52) we have

$$X = a_1X_1 + a_2X_2 + a_3X_3 + X_4. \tag{3.53}$$

Referring to Table 3.7 we act on (3.53) by $\mathrm{Ad}\big(e^{a_3X_3}\big)$ in order to eliminate a_3X_3 and obtain

$$X' = a_1'X_1 + a_2'X_2 + X_4, \tag{3.54}$$

for some constants a_1', a_2'.

On elimination of $a_2'X_2$, we act on (3.54) by $\mathrm{Ad}\Big(e^{\frac{1}{2}a_2'X_2}\Big)$ which yields

$$X'' = a_1'X_1 + X_4. \tag{3.55}$$

Likewise, acting on (3.55) by $\mathrm{Ad}\big(e^{a_1'X_1}\big)$ we eliminate $a_1'X_1$ and thus, obtain $X''' = X_4$.

Next assume that $a_4 = 0$ with $a_3 \neq 0$. We take $a_3 = 1$ and operator (3.52) becomes

$$X = a_1X_1 + a_2X_2 + X_3. \tag{3.56}$$

In order to eliminate a_2X_2, we act on (3.56) by $\mathrm{Ad}\big(e^{a_2X_1}\big)$ to get

$$X' = a_1'X_1 + X_3 \tag{3.57}$$

for some a_1'.

Now acting on (3.57) by $\mathrm{Ad}\big(e^{\epsilon X_4}\big)$, i.e. the group generated by X_4, we obtain

$$X'' = a_1'e^{\epsilon}X_1 + e^{\epsilon}X_3. \tag{3.58}$$

The operator (3.58) is a scalar multiple of

$$X''' = a_1'X_1 + X_3. \tag{3.59}$$

Depending upon the sign of a_1' from (3.59), we can make the coefficient of X_1 to be $+1$, -1, 0. Thus for $a_1' > 0$: $X''' = X_1 + X_3$, $a_1' < 0$: $X''' = -X_1 + X_3$ and $a_1' = 0$: $X''' = X_3$.

Finally, we assume that $a_3 = a_4 = 0$ and the operator (3.52) simplifies to

$$X = a_1X_1 + a_2X_2. \tag{3.60}$$

From (3.60) $a_2 \neq 0$ yields $X = X_2$ and otherwise, $X = X_1$.

Therefore an optimal system of one-dimensional subalgebras is spanned by

$$\{X_1,\ X_2,\ X_1 + X_3,\ X_3 - X_1,\ X_3,\ X_4\}.$$

3.5.2 Symmetry reductions and invariant solutions

We perform symmetry reductions of the case treated in the previous section for the optimal system. Invariance under time and space translations (X_1 and X_2) is trivial and hence not considered. However, a linear combination of these symmetries can be used when we seek the travelling-wave solutions.

(a) Invariance under X_3: The characteristic equations for the invariants of X_3 are

$$\frac{dt}{0} = \frac{dx}{t} = \frac{d\alpha}{0} = \frac{du}{1}. \tag{3.61}$$

Solving (3.61) we find that the invariant solutions assume the form

$$\alpha(t,x) = F(t), \tag{3.62}$$

$$u(t,x) = G(t) + \frac{x}{t}, \tag{3.63}$$

where F and G are arbitrary differentiable functions of t.

Upon substitution of (3.62) and (3.63) into system (3.4) we obtain the reduced system of ODEs

$$F + t\dot{F} = 0, \tag{3.64}$$

$$Kt + \rho\left(G + t\dot{G}\right) = 0. \tag{3.65}$$

In the above system and the subsequent systems an overdot denotes differentiation with respect to t.

From Eqs. (3.64) and (3.65) we obtain

$$F(t) = \frac{C_1}{t}, \quad G(t) = \frac{C_2}{t} - \frac{Kt}{2\rho}, \tag{3.66}$$

where $C_1 \neq 0$ and C_2 are arbitrary constants.

In view of (3.66) the invariant solutions (3.62) and (3.63) are given respectively by

$$\alpha(t,x) = \frac{C_1}{t}, \quad u(t,x) = \frac{x + C_2}{t} - \frac{Kt}{2\rho}. \tag{3.67}$$

(b) Invariance under $X_1 + X_3$: The corresponding characteristic system for the invariants of $X_1 + X_3$ is

$$\frac{dt}{1} = \frac{dx}{t} = \frac{d\alpha}{0} = \frac{du}{1}. \tag{3.68}$$

The invariant solutions are

$$\alpha(t,x) = F(\gamma), \tag{3.69}$$

$$u(t,x) = t + G(\gamma), \tag{3.70}$$

where F and G are arbitrary functions of the similarity variable $\gamma = x - t^2/2$.

The Substitution of (3.69) and (3.70) into system (3.4) yields

$$GF' + FG' = 0, \tag{3.71}$$

$$K + \rho\,(1 + GG') + \frac{\ell_7 F'}{F^{2/3}} - TF''' = 0, \tag{3.72}$$

where a 'prime' in the above system and those that follow denote differentiation with respect to the relevant similarity variable.

Therefore the invariant solutions for the operator $X_1 + X_3$ are of the form (3.69) and (3.70), where the functions $F(\gamma)$ and $G(\gamma)$ are defined by the system (3.71)–(3.72).

(c) The solutions invariant under $-X_1 + X_3$ have the form

$$\alpha(t,x) = F(\gamma), \tag{3.73}$$

$$u(t,x) = G(\gamma) - t, \tag{3.74}$$

where the similarity variable is $\gamma = x + t^2/2$ and $F(\gamma)$, $G(\gamma)$ are differentiable functions satisfying the reduced system

$$GF' + FG' = 0, \tag{3.75}$$

$$K + \rho\,(GG' - 1) + \frac{\ell_7 F'}{F^{2/3}} - TF''' = 0. \tag{3.76}$$

(d) The characteristic system for the invariants of X_4 is

$$\frac{dt}{t} = \frac{dx}{2x} = \frac{d\alpha}{6\alpha} = \frac{du}{u}\cdot \tag{3.77}$$

From (3.77) we infer that the invariant solutions assume the form

$$\alpha(t,x) = t^6 F(\gamma), \tag{3.78}$$

$$u(t,x) = tG(\gamma), \tag{3.79}$$

where F and G are arbitrary functions of the similarity variable $\gamma = xt^{-2}$ defined by the system

$$(G - 2\gamma)\, F' + (6 + G')\, F = 0, \tag{3.80}$$

$$K + \frac{\ell_7 F'}{F^{2/3}} + \rho\,(G + GG' - 2\gamma G') - TF''' = 0. \tag{3.81}$$

Next we consider the remaining cases from Table 3.4 for which the form of the tube law is a nonlinear function of the cross-sectional area, α. For easy reference we summarize these cases in Table 3.8.

Table 3.8: Summary of results considered for similarity reductions.

$\mathcal{P}$	R	Condition on consts.	Operators $X_1 = \partial_t,\ X_2 = \partial_x$
$2\kappa_1\sqrt{\alpha} + \kappa_2$	$K\alpha^{1/4} - \rho(\beta + u)$	$\kappa_1 \neq 0$	$X_3 = e^t(\partial_x + \partial_u)$, $X_4 = (\beta t + x)\partial_x + 4\alpha\partial_\alpha + (\beta + u)\partial_u$
$\frac{(\lambda-10)\alpha^{\frac{3\lambda-5}{\lambda-10}}}{3\lambda-5}\kappa_3 + \kappa_4$	$\frac{K(8\lambda-5)[2(10-\lambda)\alpha]^{\frac{8\lambda-5}{2(\lambda-10)}} - 10\rho\mu}{8\lambda-5}$	$\kappa_3 \neq 0$ $\lambda \neq \frac{5}{8}, \frac{5}{3}, 10$	$X_3 = t\partial_x + \partial_u$, $X_4 = \lambda t\partial_t + \left[\mu t^2 + \left(\frac{2\lambda}{5} + 1\right)x\right]\partial_x$ $-\frac{2}{5}(\lambda - 10)\alpha\partial_\alpha + \left[2\mu t + \left(1 - \frac{3\lambda}{5}\right)u\right]\partial_u$
$3\kappa_5\alpha^{1/3} + \kappa_6$	$K - \frac{8}{15}\rho\mu\ln\alpha$	$\kappa_5 \neq 0$ $\lambda = 5/8$	$X_3 = t\partial_x + \partial_u$, $X_4 = \frac{5}{8}t\partial_t + \left(\mu t^2 + \frac{5}{4}x\right)\partial_x + \frac{15}{4}\alpha\partial_\alpha$ $+ \left(2\mu t + \frac{5}{8}u\right)\partial_u$
$\kappa_7\ln\alpha + \kappa_8$	$\frac{K}{\sqrt{\alpha}} - \frac{6}{5}\rho\mu$	$\kappa_7 \neq 0$ $\lambda = 5/3$	$X_3 = t\partial_x + \partial_u$, $X_4 = \frac{5}{3}t\partial_t + \left(\mu t^2 + \frac{5}{3}x\right)\partial_x$ $+ \frac{10}{3}\alpha\partial_\alpha + 2\mu t\partial_u$
$\frac{1}{3}\kappa_9\alpha^3 + \kappa_{10}$	$K\alpha^4 - \frac{5}{4}\rho\nu$	$\kappa_9 \neq 0$	$X_3 = t\partial_x + \partial_u$, $X_4 = t\partial_t + \left(\nu t^2 + \frac{2}{5}x\right)\partial_x - \frac{2}{5}\alpha\partial_\alpha$ $+ \left(2\nu t - \frac{3}{5}u\right)\partial_u$

Proceeding likewise with X_3, its linear combinations: $X_3 + X_1$, $X_3 - X_1$ and X_4 we have the following similarity reductions.

1. (a) X_3: Invariant solutions are given by

$$\alpha(t,x) = C_3 e^{-t}, \quad u(t,x) = x + \beta t + \frac{4K\left(C_3 e^{-t}\right)^{1/4}}{\rho} + C_4, \tag{3.82}$$

where $C_3 \neq 0$ and C_4 are arbitrary constants.

(b) $X_1 + X_3$: Invariant solutions take the form

$$\alpha(t,x) = F(\gamma), \quad u(t,x) = e^t + G(\gamma), \tag{3.83}$$

where F and G are smooth functions of the similarity variable $\gamma = x - e^t$ satisfying

$$GF' + FG' = 0, \tag{3.84}$$

$$KF^{1/4} + \frac{\kappa_1 F'}{\sqrt{F}} + \rho\left(GG' - G - \beta\right) - TF''' = 0. \tag{3.85}$$

(c) $X_3 - X_1$: Same as (c) by means of the discrete transformations $x \to -x$, $u \to -u$.

(d) X_4: Invariant solutions assume the form

$$\alpha(t,x) = (\beta t + x)^4 F(t), \quad u(t,x) = (\beta t + x)G(t) - \beta \tag{3.86}$$

for arbitrary functions $F(t)$ and $G(t)$ defined by

$$5FG + \dot{F} = 0, \tag{3.87}$$

$$KF^{1/4} + \kappa_1\sqrt{F} + \rho\left(G^2 - G + \dot{G}\right) - 24TF = 0. \tag{3.88}$$

2. (a) X_3: Invariant solutions are

$$\alpha(t,x) = \frac{C_5}{t}, \tag{3.89}$$

$$u(t,x) = \frac{x + C_6}{t} + \frac{5\mu t}{8\lambda - 5} + \frac{2^{\frac{10\lambda-15}{2(\lambda-10)}} C_5^4 K(\lambda-10)^5 \left[C_5 \frac{(\lambda-10)}{t}\right]^{\frac{75}{2(\lambda-10)}}}{\rho t^3(4\lambda+35)}, \tag{3.90}$$

where $C_5 \neq 0$ and C_6 are arbitrary constants.

(b) $X_1 + X_3$: Invariant solutions assume the form

$$\alpha(t,x) = F(\gamma), \quad u(t,x) = t + G(\gamma), \tag{3.91}$$

where $\gamma = x - t^2/2$ is the similarity variable and the arbitrary functions $F(\gamma)$, $G(\gamma)$ satisfy

$$\begin{aligned} GF' + FG' &= 0, && (3.92)\\ \frac{K(8\lambda-5)\left[2(10-\lambda)F\right]^{\frac{8\lambda-5}{2(\lambda-10)}} - 10\mu\rho}{8\lambda-5} + \kappa_3 F^{\frac{2\lambda+5}{\lambda-10}}F' - TF''' & \\ +\rho\left(1+GG'\right) &= 0. && (3.93) \end{aligned}$$

(c) $X_3 - X_1$: Invariant solutions have the form

$$\alpha(t,x) = F(\gamma), \quad u(t,x) = G(\gamma) - t \tag{3.94}$$

for similarity variable $\gamma = x + t^2/2$. The arbitrary functions $F(\gamma)$ and $G(\gamma)$ are defined by

$$\begin{aligned} GF' + FG' &= 0, && (3.95)\\ \frac{K(8\lambda-5)\left[2(10-\lambda)F\right]^{\frac{8\lambda-5}{2(\lambda-10)}} - 10\mu\rho}{8\lambda-5} + \kappa_3 F^{\frac{2\lambda+5}{\lambda-10}}F' - TF''' & \\ +\rho\left(GG'-1\right) &= 0. && (3.96) \end{aligned}$$

(d) X_4: The characteristic system for the invariants of X_4 yields the similarity variable

$$\gamma = \frac{x}{t^{\frac{1}{\lambda}+\frac{2}{5}}} - \frac{5\mu t^{(8-5)/5\lambda}}{8\lambda-5}.$$

The invariant solutions are of the form

$$\alpha(t,x) = t^{\frac{4}{\lambda}-\frac{2}{5}}F(\gamma), \quad u(t,x) = \frac{10\mu t}{8\lambda-5} + t^{\frac{1}{\lambda}-\frac{3}{5}}G(\gamma), \tag{3.97}$$

where $F(\gamma)$ and $G(\gamma)$ satisfy the system

$$\begin{aligned} \left[5\lambda G - (5+2\lambda)\gamma\right]F' + \left(20 - 2\lambda + 5\lambda G'\right)F &= 0, && (3.98)\\ \frac{K(8\lambda-5)[2(10-\lambda)F]^{\frac{8\lambda-5}{2(\lambda-10)}}}{8\lambda-5} + \kappa_3 F^{\frac{2\lambda+5}{\lambda-10}}F' - TF''' & \\ +\rho\left[\left(\frac{1}{\lambda}-\frac{3}{5}\right)G + GG' - \left(\frac{2}{5}+\frac{1}{\lambda}\right)\gamma G'\right] &= 0. && (3.99) \end{aligned}$$

3. (a) X_3: We have the following invariant solutions

$$\alpha(t,x) = \frac{C_7}{t}, \quad u(t,x) = t\left(\frac{2\mu}{15} - \frac{K}{2\rho}\right) + \frac{x+C_8}{t} + \frac{4\mu t}{15}\ln\left[\frac{C_7}{t}\right], \tag{3.100}$$

where $C_7 \neq 0$ and C_8 are arbitrary constants.

In the sequel the solutions invariant under $X_1 + X_3$ and $X_3 - X_1$ have the same form as those in case 2. (b) and (c) respectively. However, the reduced systems of ODEs are different and hence the only ones presented.

(b) $X_1 + X_3$:

$$GF' + FG' = 0, \tag{3.101}$$

$$K + \rho\left(1 + GG' - \frac{8\mu}{15}\ln F\right) + \frac{\kappa_5 F'}{F^{2/3}} - TF''' = 0. \tag{3.102}$$

(c) $X_3 - X_1$:

$$GF' + FG' = 0, \tag{3.103}$$

$$K + \rho\left(GG' - \frac{8\mu}{15}\ln F - 1\right) + \frac{\kappa_5 F'}{F^{2/3}} - TF''' = 0. \tag{3.104}$$

(d) X_4: From the characteristic system for the invariants of X_4 we obtain the invariant solutions

$$\alpha(t,x) = t^6 F(\gamma), \quad u(t,x) = \frac{16\mu}{5} t\ln t + tG(\gamma), \tag{3.105}$$

where similarity variable is given by

$$\gamma = \frac{x}{t^2} - \frac{8\mu}{5}\ln t.$$

The differentiable functions $F(\gamma)$ and $G(\gamma)$ are defined by the system

$$\left(\frac{4\mu}{5} + \gamma - \frac{1}{2}G\right)F' - \left(3 + \frac{1}{2}G'\right) = 0, \tag{3.106}$$

$$K + \rho\left[\frac{16\mu}{5} - \frac{8}{15}\mu\ln F + G + \left(G - 2\gamma - \frac{8\mu}{5}\right)G'\right] + \frac{\kappa_5 F'}{F^{2/3}} - TF''' = 0. \tag{3.107}$$

4. (a) X_3: The solution of the characteristic equations for the invariants of X_3 results in the following invariant solutions

$$\alpha(t,x) = \frac{C_9}{t}, \quad u(t,x) = \frac{3\mu t}{5} - \frac{2Kt}{5\rho\sqrt{\frac{C_9}{t}}} + \frac{x + C_{10}}{t} \tag{3.108}$$

for $C_9 \neq 0$ and C_{10} arbitrary.

(b) $X_1 + X_3$:

$$GF' + FG' = 0, \tag{3.109}$$

$$\rho\left(1 + GG' - \frac{6\mu}{5}\right) + \frac{K}{\sqrt{F}} + \frac{\kappa_7 F'}{F} - TF''' = 0. \tag{3.110}$$

(c) $X_3 - X_1$:

$$GF' + FG' = 0, \tag{3.111}$$

$$\rho\left(GG' - \frac{6\mu}{5} - 1\right) + \frac{K}{\sqrt{F}} + \frac{\kappa_7 F'}{F} + \rho GG' - \frac{6\mu\rho}{5} - TF''' = 0. \tag{3.112}$$

(d) X_4: Invariant solutions assume the form

$$\alpha(t,x) = t^2 F(\gamma), \quad u(t,x) = \frac{6\mu}{5}t + G(\gamma), \tag{3.113}$$

where $\gamma = xt^{-1} - 3\mu t/5$ and F, G are arbitrary differentiable functions of γ satisfying

$$(\gamma - 5G)\,F' - 5\,(2 + G')\,F = 0, \tag{3.114}$$

$$\rho\,(\gamma - G)\,G' - \frac{5}{F}\left(K\sqrt{F} + \kappa_7 F'\right) + TF''' = 0. \tag{3.115}$$

5. (a) X_3: The solutions invariant under X_3 take the form

$$\alpha(t,x) = \frac{C_{11}}{t}, \quad u(t,x) = \frac{5\nu t}{8} + \frac{KC_{11}^4}{2\rho t^3} + \frac{x + C_{12}}{t}, \tag{3.116}$$

where $C_{11} \neq 0$ and C_{12} are arbitrary constants.

(b) $X_1 + X_3$:

$$GF' + FG' = 0, \tag{3.117}$$

$$\rho\left(1 + GG' - \frac{5\nu}{4}\right) + KF^4 + \kappa_9 F^2 F' + TF''' = 0. \tag{3.118}$$

(c) $X_3 - X_1$:

$$GF' + FG' = 0, \tag{3.119}$$

$$\rho\left(GG' - \frac{5\nu}{4} - 1\right) + KF^4 + \kappa_9 F^2 F' + TF''' = 0. \tag{3.120}$$

(d) Invariance under X_4: the invariant solutions are

$$\alpha(t,x) = t^{-2/5}F(\gamma), \quad u(t,x) = \frac{5\nu}{4}t + t^{-3/5}G(\gamma), \tag{3.121}$$

where F and G are arbitrary functions of the similarity variable $\gamma = xt^{-2/5} - (5\nu t^{8/5})/8$ defined by

$$(5G - 2\gamma)\,F' + (5G' - 2)\,F = 0, \tag{3.122}$$

$$\rho\,(5GG' - 2\gamma G' - 3G) + 5KF^4 + 5\kappa_9 F^2 F' - 5TF''' = 0. \tag{3.123}$$

Some of the invariant solutions may fail to explain realistic situations if the physics of problem being investigated is taken into account.

3.6 Conclusion

We have performed the complete group classification of the one-dimensional collapsible tube model which incorporates the tension and curvature of the tube wall. This has been achieved by firstly determining the extra symmetry operators and hence the arbitrary elements via classification with respect to subalgebras of real three- and four-dimensional Lie algebras. As a guarantee for obtaining the maximal symmetry Lie algebra, symmetry analysis of individual cases arising from classification with respect to 3D and 4D Lie algebras was performed.

We have also exploited the symmetries of the submodels to perform similarity reductions. It should be noted that some of the reduced submodels can be challenging to solve analytically. The next logical step would be to look at numerical investigation of the reduced submodels and relate the results to the similarity (invariant) solutions. This investigation might be of great importance in understanding the complex dynamics involved for flow in collapsible tubes and channels.

Part II

Coupled Diffusion System

In diffusion processes a chemical or material migrates from an area initially of high concentration to an area of low concentration. Likewise in process of conduction heat is transfered by molecular motion from regions of high temperature to regions of low heat. The mathematical description of both diffusion and conduction is the same and hence the heat equation (which models the conduction of heat) is also known as the diffusion equation. The diffusion equation or its modification has many important applications in mathematical physics, mathematical finance, chemistry, biology and environmental science.

In some research areas of practical interest, where the diffusion equation is used to describe a certain physical process, the one-dimensional analysis of the problem fails to explain the processes involved. However, in order to understand fully and explain the processes involved, the study of one-dimensional model becomes a foundation upon which multi dimensional models can be built.

Much research has been done on the linear and nonlinear diffusion equation together with its modified forms and systems of diffusion equations in which Lie group analysis is used. One cannot exhaust all these investigations on diffusion processes by use of the Lie symmetry approach and other procedures. The results of earlier work on the Lie symmetry approach are compiled in [32] and thereafter different forms of the diffusion equation were studied by various authors [12, 17, 85, 86], just to mention a few.

In the case of diffusion systems Torrisi *et al* [78] considered a second-order system of equations that describe the diffusion of solvent into a polymer slab:

$$u_t = h(u, v), \quad v_t = \left[f(v)u_x + g(v)v_x\right]_x .$$

The diffusion systems with applications in extended thermodynamics and drift-diffusion phenomena are investigated in [72, 79]. Also the study of energy-transport model in different coordinates can be found in [51, 73]. Wiltshire *et al* [4, 84] considered a simplified form of a nonlinear coupled diffusion model by Jury *et al* [37] which describes heat, moisture and solute transport, viz.,

Wiltshire *et al*:

$$y_t = \left[\Lambda(y)y_x\right]_x , \; y = \{y_i\} \quad (i = 1, 2, \ldots, n).$$

Jury *et al*:

$$T_t = [\kappa_T(T,\theta)T_x + \kappa_\theta(T,\theta)\theta_x]_x\,, \quad \theta_t = [D_T(T,\theta)T_x + D_\theta(T,\theta)\theta_x]_x\,.$$

The linearisation and potential symmetry classification of the Jury *et al* model was performed in [74]. By 'potential' we mean that group classification is achieved via potential symmetries.

A detailed account of using both the classical and non classical procedures on theory of transformation groups to derive the similarity solutions for a one-dimensional coupled system of reaction-diffusion equations is given in [15]. More recent works on reaction-diffusion equations can be found in [62, 63]. The numerical and experimental analyses of the diffusion process of heat transfer have been extensively investigated and are accessible from relevant journals.

Chapter 4

A Group Classification of General Second-Order Diffusion System

The purpose of this Chapter is to classify systems of coupled diffusion equations. It is aimed at laying a foundation for solving a group classification problem for a general second-order system of diffusion equations based upon Lie algebras of low dimension. Briefly the procedure begins at the search for realizations of low-dimensional Lie algebras and then moves to realizations of Lie algebras of higher dimension until the arbitrary functions are completely specified. Even though this procedure was suggested in [5, 86] for PDEs, a much earlier work on ODEs using these ideas was done in [47]. We use the results on classification of solvable Lie algebras by Mubarakzyanov [57] reported in Basarab–Horwarth [5].

We consider the generalized coupled diffusion (heat-conductivity) system of the form

$$\begin{aligned} u_t &= F^1(t,x,u,v,u_x,v_x)u_{xx} + F^2(t,x,u,v,u_x,v_x)v_{xx} + G^1(t,x,u,v,u_x,v_x), \\ v_t &= F^3(t,x,u,v,u_x,v_x)u_{xx} + F^4(t,x,u,v,u_x,v_x)v_{xx} + G^2(t,x,u,v,u_x,v_x), \end{aligned} \tag{4.1}$$

where $u(t,x)$ and $v(t,x)$ are dependent variables which represent the type of diffusion process being modelled. The arbitrary functions F^1, F^2, F^3 and F^4 are the nonzero diffusion coefficients (thermal conductivities in the case of heat conduction) which depend

upon the particular type of diffusion process involved. The arbitrary functions G^1 and G^2 are the source terms (heat transfer coefficients).

4.1 Symmetry generator and determining equations

Using the basic Lie technique [10, 65, 67] we seek a symmetry generator of the form

$$\Gamma = \xi^1(t,x,u,v)\partial_t + \xi^2(t,x,u,v)\partial_x + \eta^1(t,x,u,v)\partial_u + \eta^2(t,x,u,v)\partial_v. \tag{4.2}$$

The operator Γ is a generator of symmetry group of system (4.1) if and only if

$$\begin{gathered}\Gamma^{[2]}(u_t - F^1 u_{xx} - F^2 v_{xx} - G^1) = 0,\\ \Gamma^{[2]}(v_t - F^3 u_{xx} - F^4 v_{xx} - G^2) = 0\end{gathered} \tag{4.3}$$

whenever system (4.1) is satisfied and where

$$\Gamma^{[2]} \;=\; \Gamma + \zeta_1^1\partial_{u_t} + \zeta_2^1\partial_{u_x} + \zeta_1^2\partial_{v_t} + \zeta_2^2\partial_{v_x} + \zeta_{22}^1\partial_{u_{xx}} + \zeta_{22}^2\partial_{v_{xx}}.$$

The variables ζ_j^i are given by the prolongation formulae

$$\begin{aligned}
\zeta_1^1 &= D_t(\eta^1) - u_t D_t(\xi^1) - u_x D_t(\xi^2), && (4.4)\\
\zeta_2^1 &= D_x(\eta^1) - u_t D_x(\xi^1) - u_x D_x(\xi^2), && (4.5)\\
\zeta_1^2 &= D_t(\eta^2) - v_t D_t(\xi^1) - v_x D_t(\xi^2), && (4.6)\\
\zeta_2^2 &= D_x(\eta^2) - v_t D_x(\xi^1) - v_x D_x(\xi^2), && (4.7)\\
\zeta_{22}^1 &= D_x(\zeta_2^1) - u_{tx} D_x(\xi^1) - u_{xx} D_x(\xi^2), && (4.8)\\
\zeta_{22}^2 &= D_x(\zeta_2^2) - v_{tx} D_x(\xi^1) - v_{xx} D_x(\xi^2), && (4.9)
\end{aligned}$$

where D_t and D_x are the total derivative operators defined as follows

$$D_t = \partial_t + u_t\partial_u + v_t\partial_v + \cdots, \quad D_x = \partial_x + u_x\partial_u + v_x\partial_v + \cdots. \tag{4.10}$$

With the help of program `YaLie` the generator of symmetry group for system (4.1) is of the form

$$\Gamma = a(t)\partial_t + b(t,x)\partial_x + c(t,x,u,v)\partial_u + d(t,x,u,v)\partial_v, \tag{4.11}$$

where $a, \dots, d$ are the smooth functions satisfying the determining equations

$$c_{vv}F^1 + d_{vv}F^2 = 0, \tag{4.12}$$
$$c_{uv}F^1 + d_{uv}F^2 = 0, \tag{4.13}$$
$$c_{uu}F^1 + d_{uu}F^2 = 0, \tag{4.14}$$
$$c_{vv}F^3 + d_{vv}F^4 = 0, \tag{4.15}$$
$$c_{uv}F^3 + d_{uv}F^4 = 0, \tag{4.16}$$
$$c_{uu}F^3 + d_{uu}F^4 = 0, \tag{4.17}$$
$$(d_v - b_x)\, F^1_{v_x} + c_v F^1_{u_x} = 0, \tag{4.18}$$
$$(b_x - c_u)\, F^1_{u_x} - d_u F^1_{v_x} = 0, \tag{4.19}$$
$$(d_v - b_x)\, F^2_{v_x} + c_v F^2_{u_x} = 0, \tag{4.20}$$
$$(b_x - c_u)\, F^2_{u_x} - d_u F^2_{v_x} = 0, \tag{4.21}$$
$$(d_v - b_x)\, F^3_{v_x} + c_v F^3_{u_x} = 0, \tag{4.22}$$
$$(b_x - c_u)\, F^3_{u_x} - d_u F^3_{v_x} = 0, \tag{4.23}$$
$$(d_v - b_x)\, F^4_{v_x} + c_v F^4_{u_x} = 0, \tag{4.24}$$
$$(b_x - c_u)\, F^4_{u_x} - d_u F^4_{v_x} = 0, \tag{4.25}$$

$$2c_{xv}F^1 + (d_v - b_x)\, G^1_{v_x} + c_v G^1_{u_x} - (b_{xx} - 2d_{xv})\, F^2 = 0, \tag{4.26}$$
$$b_t + 2d_{xu}F^2 + d_u G^1_{v_x} + (c_u - b_x)\, G^1_{u_x} - (b_{xx} - 2c_{xu})\, F^1 = 0, \tag{4.27}$$
$$b_t + 2c_{xv}F^3 + (d_v - b_x)\, G^2_{v_x} + c_v G^2_{u_x} - (b_{xx} - 2d_{xv})\, F^4 = 0, \tag{4.28}$$
$$2d_{xu}F^4 + d_u G^2_{v_x} + (c_u - b_x)\, G^2_{u_x} - (b_{xx} - 2c_{xu})\, F^3 = 0, \tag{4.29}$$

$$(\dot{a} - 2b_x)\, F^1 + d_x F^1_{v_x} + c_x F^1_{u_x} + dF^1_v + cF^1_u + bF^1_x + aF^1_t + d_u F^2 - c_v F^3 = 0, \tag{4.30}$$
$$c_v F^1 + (\dot{a} - 2b_x + d_v - c_u)\, F^2 + d_x F^2_{v_x} + c_x F^2_{u_x} + dF^2_v + cF^2_u + bF^2_x + aF^2_t - c_v F^4 = 0, \tag{4.31}$$
$$d_u F^4 + (\dot{a} - 2b_x - d_v + c_u)\, F^3 + d_x F^3_{v_x} + c_x F^3_{u_x} + dF^3_v + cF^3_u + bF^3_x + aF^3_t - d_u F^1 = 0, \tag{4.32}$$
$$(\dot{a} - 2b_x)\, F^4 + d_x F^4_{v_x} + c_x F^4_{u_x} + dF^4_v + cF^4_u + bF^4_x + aF^4_t + c_v F^3 - d_u F^2 = 0, \tag{4.33}$$
$$c_{xx}F^1 + d_{xx}F^2 + d_x G^1_{v_x} + c_x G^1_{u_x} + dG^1_v + cG^1_u + bG^1_x + aG^1_t - (c_u - \dot{a})\, G^1 - c_v G^2 - c_t = 0, \tag{4.34}$$

$$c_{xx}F^3 + d_{xx}F^4 + (\dot{a} - d_v)\,G^2 + d_x G^2_{v_x} + c_x G^2_{u_x} + dG^2_v + cG^2_u + bG^2_x + aG^2_t$$
$$-d_u G^1 - d_t = 0. \quad (4.35)$$

In the determining equations above the overdot represents differentiation with respect to t and the subscripts denote partial differentiation.

4.2 Equivalence group

We use the direct approach [5, 39] instead of the infinitesimal method for calculating the equivalence group. The infinitesimal approach is best executed when the arbitrary functions do not depend on many variables.

Let the invertible transformations taking the original system (4.1) into a system of the same form be

$$\bar{t} = T(t, x, u, v), \quad \bar{x} = X(t, x, u, v), \quad \bar{u} = U(t, x, u, v), \quad \bar{v} = V(t, x, u, v), \qquad (4.36)$$

where the functions T, X, U and V satisfy the Jacobian

$$\frac{D(T, X, U, V)}{D(t, x, u, v)} \neq 0.$$

The operators of differentiation transform as follows

$$D_t = D_t(T)D_{\bar{t}} + D_t(X)D_{\bar{x}}, \quad D_x = D_x(T)D_{\bar{t}} + D_x(X)D_{\bar{x}}.$$

Firstly we find u_x and v_x using (4.36). Thus for some solutions h_1, h_2 of $\bar{u} = h_1(\bar{t}, \bar{x})$ and $\bar{v} = h_2(\bar{t}, \bar{x})$ we have

$$U(x, u) = h_1\left(T(x, u), X(x, u)\right), \quad V(x, v) = h_2\left(T(x, v), X(x, v)\right). \qquad (4.37)$$

Now from the first equation in (4.37) we get

$$U_x + U_u u_x = \bar{u}_{\bar{t}}\left(T_x + T_u u_x\right) + \bar{u}_{\bar{x}}\left(X_x + X_u u_x\right), \qquad (4.38)$$

i.e.,

$$U_x + u_x(U_u - T_u\bar{u}_{\bar{t}} - X_u\bar{u}_{\bar{x}}) = T_x\bar{u}_{\bar{t}} - X_x\bar{u}_{\bar{x}},$$

Hence

$$u_x = \frac{T_x \bar{u}_{\bar{t}} - X_x \bar{u}_{\bar{x}} - U_x}{U_u - T_u \bar{u}_{\bar{t}} - X_u \bar{u}_{\bar{x}}}. \tag{4.39}$$

From (4.39) we require $u_x \to g_1(\bar{t}, \bar{x}, \bar{u}, \bar{u}_{\bar{x}})$ for some g_1. This implies that the coefficients of $\bar{u}_{\bar{t}}$ must vanish, i.e.,

$$T_x = T_u = 0. \tag{4.40}$$

In similar fashion $v_x \to g_2(\bar{t}, \bar{x}, \bar{v}, \bar{v}_{\bar{x}})$ for some g_2. Hence

$$T_x = T_v = 0. \tag{4.41}$$

From Eqs. (4.40) and (4.41) we obtain $T = T(t)$ for arbitrary function $T(t)$ satisfying $\dot{T} \neq 0$ where the dot denotes the total time derivative.

Therefore the equivalence group is given by

$$\bar{t} = T(t), \quad \bar{x} = X(t, x, u, v), \quad \bar{u} = U(t, x, u, v), \quad \bar{v} = V(t, x, u, v), \tag{4.42}$$

where $\dot{T} \neq 0$ and $\dfrac{D(X, U, V)}{D(x, u, v)} \neq 0$.

Next we find u_t and v_t. We begin with u_t and have

$$U_t + U_u u_t = \bar{u}_{\bar{t}} \left(T_t + T_u u_t\right) + \bar{u}_{\bar{x}} \left(X_t + X_u u_t\right). \tag{4.43}$$

Thus taking into account the fact that $T = T(t)$,

$$\begin{aligned} u_t &= \frac{\dot{T} \bar{u}_{\bar{t}} + X_t \bar{u}_{\bar{x}} - U_t}{U_u - X_u \bar{u}_{\bar{x}}}, \\ &= \dot{T} \bar{u}_{\bar{t}} \left(U_u - X_u \bar{u}_{\bar{x}}\right)^{-1} + \theta_1(\bar{t}, \bar{x}, \bar{u}, \bar{u}_{\bar{x}}) \end{aligned} \tag{4.44}$$

for some arbitrary function $\theta_1(\bar{t}, \bar{x}, \bar{v}, \bar{v}_{\bar{x}})$.

Likewise we obtain v_t which has the form

$$v_t = \dot{T} \bar{v}_{\bar{t}} \left(V_v - X_v \bar{v}_{\bar{x}}\right)^{-1} + \theta_2(\bar{t}, \bar{x}, \bar{v}, \bar{v}_{\bar{x}}) \tag{4.45}$$

for θ_2 an arbitrary function of its arguments.

To find u_{xx} we rewrite Eq. (4.38) in the form

$$D_x(U) = \bar{u}_{\bar{t}} D_x(T) + \bar{u}_{\bar{x}} D_x(X). \tag{4.46}$$

In view of Eqs. (4.40) and (4.41) the above equation (4.46) becomes

$$D_x(U) = \bar{u}_{\bar{x}} D_x(X). \tag{4.47}$$

By the application of D_x on Eq. (4.47) we obtain

$$D_x^2(U) = \bar{u}_{\bar{x}} D_x^2(X) + \bar{u}_{\bar{x}\bar{x}} \left(D_x(X)\right)^2. \tag{4.48}$$

We consider the left hand side of Eq. (4.48), that is

$$\begin{aligned} D_x^2(U) &= \partial_x(D_x(U)) = \partial_x\left(U_x + U_u u_x\right) = U_{xx} + U_u u_{xx} + u_x \partial_x\left(U_u\right), \\ &= U_{xx} + U_u u_{xx} + u_x\left[U_{xu} + \partial_u\left(U_x + U_u u_x\right)\right], \\ &= U_{xx} + U_u u_{xx} + 2U_{xu} u_x + U_{uu} u_x^2. \end{aligned}$$

Similarly by consideration of the right hand side of Eq. (4.48) we deduce the expression

$$\left(X_{xx} + X_u u_{xx} + 2X_{xu} u_x + X_{uu} u_x^2\right) \bar{u}_{\bar{x}} + \left(X_x^2 + 2X_x X_u u_x + X_u^2 u_x^2\right) \bar{u}_{\bar{x}\bar{x}}.$$

Therefore

$$\begin{aligned} u_{xx} = [&\left(X_x^2 + 2X_x X_u u_x + X_u^2 u_x^2\right) \bar{u}_{\bar{x}\bar{x}} + \left(X_{xx} + 2X_{xu} u_x + X_{uu} u_x^2\right) \bar{u}_{\bar{x}} \\ &- \left(U_{xx} + 2U_{xu} u_x + U_{uu} u_x^2\right)](U_u - X_u \bar{u}_{\bar{x}})^{-1}. \end{aligned} \tag{4.49}$$

The substitution of (4.39) into (4.49) eventually results in

$$u_{xx} = \theta_3(\bar{t}, \bar{x}, \bar{u}, \bar{u}_{\bar{x}}) \bar{u}_{\bar{x}\bar{x}} + \theta_4(\bar{t}, \bar{x}, \bar{u}, \bar{u}_{\bar{x}}). \tag{4.50}$$

In the same manner

$$v_{xx} = \theta_5(\bar{t}, \bar{x}, \bar{u}, \bar{u}_{\bar{x}}) \bar{v}_{\bar{x}\bar{x}} + \theta_6(\bar{t}, \bar{x}, \bar{v}, \bar{v}_{\bar{x}}). \tag{4.51}$$

The θ_i are arbitrary functions of their respective arguments.

The substitution of (4.44), (4.45), (4.50) and (4.51) into the underlying system (4.1) yields the system

$$\begin{aligned} \bar{u}_{\bar{t}} &= \dot{T}^{-1}\left(U_u - X_u \bar{u}_{\bar{x}}\right)\left(\theta_3 F^1 \bar{u}_{\bar{x}\bar{x}} + \theta_5 F^1 \bar{v}_{\bar{x}\bar{x}} + \theta_4 F^1 + \theta_6 F^2 + G^1 - \theta_1\right), \\ \bar{v}_{\bar{t}} &= \dot{T}^{-1}\left(V_v - X_v \bar{v}_{\bar{x}}\right)\left(\theta_3 F^3 \bar{u}_{\bar{x}\bar{x}} + \theta_5 F^4 \bar{v}_{\bar{x}\bar{x}} + \theta_4 F^3 + \theta_6 F^3 + G^2 - \theta_2\right). \end{aligned} \tag{4.52}$$

Therefore under the change of variables (4.42) the original system (4.1) is transformed into the system of the same form

$$\bar{u}_{\bar{t}} = \bar{F}^1(\bar{t},\bar{x},\bar{u},\bar{v},\bar{u}_{\bar{x}},\bar{v}_{\bar{x}})\bar{u}_{\bar{x}\bar{x}} + \bar{F}^2(\bar{t},\bar{x},\bar{u},\bar{v},\bar{u}_{\bar{x}},\bar{v}_{\bar{x}})\bar{v}_{\bar{x}\bar{x}} + \bar{G}^1(\bar{t},\bar{x},\bar{u},\bar{v},\bar{u}_{\bar{x}},\bar{v}_{\bar{x}}),$$
$$v_t = \bar{F}^3(\bar{t},\bar{x},\bar{u},\bar{v},\bar{u}_{\bar{x}},\bar{v}_{\bar{x}})\bar{u}_{\bar{x}\bar{x}} + \bar{F}^4(\bar{t},\bar{x},\bar{u},\bar{v},\bar{u}_{\bar{x}},\bar{v}_{\bar{x}})\bar{v}_{\bar{x}\bar{x}} + \bar{G}^2(\bar{t},\bar{x},\bar{u},\bar{v},\bar{u}_{\bar{x}},\bar{v}_{\bar{x}}). \tag{4.53}$$

The arbitrary functions $\bar{F}^1,\dots,\ \bar{F}^4$, $\bar{G}^1$ and $\bar{G}^2$ can be identified from (4.52) as follows

$$\begin{aligned}
\bar{F}^1 &= \dot{T}^{-1}\left(U_u - X_u\bar{u}_{\bar{x}}\right)\theta_3 F^1,\\
\bar{F}^2 &= \dot{T}^{-1}\left(U_u - X_u\bar{u}_{\bar{x}}\right)\theta_5 F^2,\\
\bar{F}^3 &= \dot{T}^{-1}\left(V_v - X_v\bar{v}_{\bar{x}}\right)\theta_3 F^3,\\
\bar{F}^4 &= \dot{T}^{-1}\left(V_v - X_v\bar{v}_{\bar{x}}\right)\theta_5 F^4,\\
\bar{G}^1 &= \dot{T}^{-1}\left(U_u - X_u\bar{u}_{\bar{x}}\right)\left(\theta_4 F^1 + \theta_6 F^2 + G^1 - \theta_1\right),\\
\bar{G}^2 &= \dot{T}^{-1}\left(V_v - X_v\bar{v}_{\bar{x}}\right)\left(\theta_4 F^3 + \theta_6 F^4 + G^2 - \theta_2\right).
\end{aligned}$$

4.3 Classification with respect to low-dimensional Lie algebras

4.3.1 One-dimensional Lie algebras

A symmetry generator (4.11) can be transformed by the change of variables (4.42) into the form

$$\bar{\Gamma} = a\dot{T}(t)\partial_{\bar{t}} + (aX_t + bX_x + cX_u + dX_v)\partial_{\bar{x}} + (aU_t + bU_x + cU_u + dU_v)\partial_{\bar{u}} + (aV_t + bV_x + cV_u + dV_v)\partial_{\bar{v}}. \tag{4.54}$$

We follow an argument similar to that used in [5, 6]. We consider the cases $a(t) \neq 0$ and $a(t) = 0$. If $a \neq 0$, then the choice in (4.54) of a function $T(t)$ such that $a(t)\dot{T}(t) = 1$ and functions X, U, V of the fundamental solutions of the system of PDEs

$aX_t + bX_x + cX_u + dX_v = 0$, $aU_t + bU_x + cU_u + dU_v = 0$, $aV_t + bV_x + cV_u + dV_v = 0$,

gives rise to $\Gamma \to \bar{\Gamma} = \partial_{\bar{t}}$.

If $a = 0$, then the choice in (4.54) of a particular solution of the PDE $bX_x + cX_u + dX_v = 1$ as a function of X and fundamental solutions of the PDEs $bU_x + cU_u + dU_v = 0,\ bV_x + cV_u + dV_v = 0$ as functions of U and V respectively, we have $\Gamma \to \bar{\Gamma} = \partial_{\bar{x}}$.

Therefore when we drop the bars, the symmetry operator (4.11) reduces to one of the canonical operators

$$\Gamma = \partial_t, \quad \Gamma = \partial_x.$$

Consequently there are two inequivalent realizations of one-dimensional Lie algebras denoted: $A_1^1 = \langle \partial_t \rangle$ and $A_1^2 = \langle \partial_x \rangle$. The superscript distinguishes one realization from the other while the subscript denotes the dimension of the Lie algebra.

4.3.2 Two-dimensional solvable Lie algebras

There are two inequivalent two-dimensional solvable Lie algebras:

$A_{2,1} : [e_1, e_2] = 0, \quad A_{2,2} : [e_1, e_2] = e_2.$

These algebras contain the one-dimensional Lie algebras $A_1^1 = \langle \partial_t \rangle$ and $A_1^2 = \langle \partial_x \rangle$. Hence in search for realizations of the two-dimensional Lie algebras there are two cases to consider: $\langle \partial_t, e_2 \rangle$ and $\langle \partial_x, e_2 \rangle$, where e_2 is of the form (4.11).

Consider firstly the Lie algebra $A_{2,1}$.

(a) $A = \langle \partial_t, e_2 \rangle$: The Lie Bracket $[e_1, e_2] = 0$ yields

$$e_2 = b(x)\partial_x + c(x, u, v)\partial_u + d(x, u, v)\partial_v.$$

We use the equivalence transformations (4.42) to simplify e_2. Also we need the equivalence group of $A_1^1 = \langle \partial_t \rangle$, i.e.,

$$\begin{aligned} \partial_{\bar{t}} &= \partial_t(\bar{t})\partial_{\bar{t}} + \partial_t(\bar{x})\partial_{\bar{x}} + \partial_t(\bar{u})\partial_{\bar{u}} + \partial_t(\bar{v})\partial_{\bar{v}}, \\ &= \dot{T}\partial_{\bar{t}} + X_t\partial_{\bar{x}} + U_t\partial_{\bar{u}} + V_t\partial_{\bar{v}}. \end{aligned}$$

Thus the equivalence transformations for A_1^1 are given by

$$\mathcal{E}(\partial_t) = \{\bar{t} = t + a_1,\ \bar{x} = X(x,u,v),\ \bar{u} = U(x,u,v),\ \bar{v} = V(x,u,v)\}. \tag{4.55}$$

By the use of these transformations e_2 is transformed into

$$\bar{e}_2 = (bX_x + cX_u + dX_v)\partial_{\bar{x}} + (bU_x + cU_u + dU_v)\partial_{\bar{u}} + (bV_x + cV_u + dV_v)\partial_{\bar{v}}. \tag{4.56}$$

According to (4.56) we have the following realizations of two-dimensional Lie algebras:

$A = \langle \partial_t, \partial_x \rangle, \quad A = \langle \partial_t, \partial_u \rangle, \quad A = \langle \partial_t, \partial_v \rangle.$

(b) Likewise for the case $A = \langle \partial_x, e_2 \rangle$ we obtain the realizations $\langle \partial_x, \partial_u \rangle$ and $\langle \partial_x, \partial_v \rangle$.

Therefore the realizations of the two-dimensional Lie algebra $A_{2,1}$ are

$$A_{2,1}^1 = \langle \partial_t, \partial_x \rangle, \quad A_{2,1}^2 = \langle \partial_t, \partial_u \rangle, \quad A_{2,1}^3 = \langle \partial_t, \partial_v \rangle, \quad A_{2,1}^4 = \langle \partial_x, \partial_u \rangle, \quad A_{2,1}^5 = \langle \partial_x, \partial_v \rangle.$$

The corresponding forms of the arbitrary elements are such that they are independent of the variables appearing in each realization. For instance all the arbitrary functions corresponding to $A_{2,1}^1$ are independent of t and x.

If we proceed in the same manner, the two-dimensional Lie algebra $A_{2,2}$ has the realizations

$$A_{2,2}^1 = \langle -t\partial_t - x\partial_x, \partial_t \rangle, \quad A_{2,2}^2 = \langle -t\partial_t - x\partial_x, \partial_x \rangle, \quad A_{2,2}^3 = \langle -x\partial_x - u\partial_u, \partial_x \rangle, \quad A_{2,2}^4 = \langle -x\partial_x - v\partial_v, \partial_x \rangle.$$

Functional forms of the arbitrary elements are given as follows.

$$\begin{aligned}
A_{2,2}^1 :&\quad F^i = x\tilde{F}^i(u,v), \quad G^j = x^{-1}\tilde{G}^j(u,v), \quad i = 1,2,3,4;\ j = 1,2.\\
A_{2,2}^2 :&\quad F^i = t\tilde{F}^i(u,v), \quad G^j = t^{-1}\tilde{G}^j(u,v).\\
A_{2,2}^3 :&\quad F^1 = u^2\tilde{F}^1(t,v,u_x), \quad F^2 = u^3\tilde{F}^2(t,v,u_x), \quad F^3 = u\tilde{F}^3(t,v,u_x),\\
&\quad F^4 = u^2\tilde{F}^4(t,v,u_x), \quad G^1 = u\tilde{G}^1(t,v,u_x), \quad G^2 = \tilde{G}^2(t,v,u_x).\\
A_{2,2}^4 :&\quad F^1 = v^2\tilde{F}^1(t,u,v_x), \quad F^2 = v^3\tilde{F}^2(t,u,v_x), \quad F^3 = v\tilde{F}^3(t,u,v_x),\\
&\quad F^4 = v^2\tilde{F}^4(t,u,v_x), \quad G^1 = \tilde{G}^1(t,u,v_x), \quad G^2 = v\tilde{G}^2(t,u,v_x).
\end{aligned}$$

In general $\tilde{F}^1$, $\tilde{F}^2$, $\tilde{F}^3$, $\tilde{F}^4$, $\tilde{G}^1$ and $\tilde{G}^2$ are different arbitrary functions of their respective arguments.

4.3.3 Three-dimensional solvable Lie algebras

The solvable Lie algebras can be classified into decomposable and non decomposable Lie algebras [5].

1. Decomposable Lie algebras

There are two decomposable three-dimensional solvable Lie algebras:

$A_{3,1} = 3A_1$: $[e_i, e_j] = 0$, $i, j = 1, 2, 3$, contains $A^k_{2,1}$ $(k = 1 : 5)$.

$A_{3,2} = A_{2,2} \oplus A_1$: $[e_1, e_2] = e_2$ contains $A^k_{2,2}$ $(k = 1 : 4)$.

We show some calculations for $A_{3,2}$ containing $A^1_{2,2} = \langle -t\partial_t - x\partial_x, \partial_t \rangle$ and the rest are presented without detailed calculations. The corresponding equivalence group is given by

$$\mathcal{E}(A^1_{2,2}) = \{\bar{t} = t + a_1,\ \bar{x} = xY(u, v),\ \bar{u} = U(u, v),\ \bar{v} = V(u, v)\}. \tag{4.57}$$

The Lie Brackets: $[e_1, e_3] = 0$ and $[e_2, e_3] = 0$ imply that

$$e_3 = c_2 x\partial_x + c(u, v)\partial_u + d(u, v)\partial_v,\ c_2 \neq 0.$$

Now, by the use of (4.57), e_3 is transformed into

$$\bar{e}_3 = \bar{x}\left(c_2 + c\frac{Y_u}{Y} + d\frac{Y_v}{Y}\right)\partial_{\bar{x}} + (cU_u + dU_v)\partial_{\bar{u}} + (cV_u + dV_v)\partial_{\bar{v}}. \tag{4.58}$$

The simplification of (4.58) results in the realizations

$\langle -t\partial_t - x\partial_x, \partial_t, x\partial_x + u\partial_u \rangle$ and $\langle -t\partial_t - x\partial_x, \partial_t, x\partial_x + v\partial_v \rangle$.

Therefore in summary the realizations of the three-dimensional decomposable solvable Lie algebras are:

$A^1_{3,1} = \langle \partial_t, \partial_x, \partial_u \rangle$, $\quad \tilde{A}^1_{3,1} = \langle \partial_t, \partial_x, \partial_v \rangle$.

$A^1_{3,2} = \langle -t\partial_t - x\partial_x, \partial_t, x\partial_x + u\partial_u \rangle$, $\quad \tilde{A}^1_{3,2} = \langle -t\partial_t - x\partial_x, \partial_t, x\partial_x + v\partial_v \rangle$.

$A^2_{3,2} = \langle -t\partial_t - x\partial_x, \partial_x, t\partial_t + u\partial_u \rangle$, $\quad \tilde{A}^2_{3,2} = \langle -t\partial_t - x\partial_x, \partial_x, t\partial_t + v\partial_v \rangle$.

$A^3_{3,2} = \langle -x\partial_x - u\partial_u, \partial_x, t\partial_t + u\partial_u \rangle$, $\quad \tilde{A}^3_{3,2} = \langle -x\partial_x - u\partial_u, \partial_x, u\partial_u + v\partial_v \rangle$.

$A^4_{3,2} = \langle -x\partial_x - v\partial_v, \partial_x, t\partial_t + v\partial_v\rangle, \quad \tilde{A}^4_{3,2} = \langle -x\partial_x - v\partial_v, \partial_x, u\partial_u + v\partial_v\rangle.$

The corresponding functional forms of arbitrary functions are:

$$
\begin{aligned}
A^1_{3,1}: &\quad F^i = F^i(v, u_x, v_x),\ i = 1:4; \quad G^j = G^j(v, u_x, v_x),\ j = 1, 2.\\
\tilde{A}^1_{3,1}: &\quad F^i = F^i(u, u_x, v_x), \quad G^j = G^j(u, u_x, v_x).\\
A^1_{3,2}: &\quad F^1 = xu\hat{F}^1(v), \quad F^2 = xu^2\hat{F}^2(v), \quad F^3 = x\hat{F}^3(v), \quad F^4 = xu\hat{F}^4(v),\\
&\quad G^1 = x^{-1}u^2\hat{G}^1(v), \quad G^2 = x^{-1}u\hat{G}^2(v).\\
\tilde{A}^1_{3,2}: &\quad F^1 = xv\hat{F}^1(u), \quad F^2 = x\hat{F}^2(u), \quad F^3 = xv^2\hat{F}^3(u), \quad F^4 = xv\hat{F}^4(u),\\
&\quad G^1 = x^{-1}v\hat{G}^1(u), \quad G^2 = x^{-1}v^2\hat{G}^2(u).\\
A^2_{3,2}: &\quad F^1 = tu^{-2}\hat{F}^1(v), \quad F^2 = tu^{-1}\hat{F}^2(v), \quad F^3 = tu^{-3}\hat{F}^3(v), \quad F^4 = tu^{-2}\hat{F}^4(v),\\
&\quad G^1 = t^{-1}u\hat{G}^1(v), \quad G^2 = t^{-1}\hat{G}^2(v).\\
\tilde{A}^2_{3,2}: &\quad F^1 = tv^{-2}\hat{F}^1(u), \quad F^2 = tv^{-3}\hat{F}^2(u), \quad F^3 = tv^{-1}\hat{F}^3(u), \quad F^4 = tv^{-2}\hat{F}^4(u),\\
&\quad G^1 = t^{-1}\hat{G}^1(u), \quad G^2 = t^{-1}v\hat{G}^2(u).\\
A^3_{3,2}: &\quad F^1 = t^{-3}u^2\hat{F}^1(v), \quad F^2 = t^{-3}u^3\hat{F}^2(v), \quad F^3 = t^{-3}u\hat{F}^3(v), \quad F^4 = t^{-3}u^2\hat{F}^4(v),\\
&\quad G^1 = t^{-1}u\hat{G}^1(v), \quad G^2 = t^{-1}\hat{G}^2(v).\\
\tilde{A}^3_{3,2}: &\quad F^1 = u^2v^{-2}\hat{F}^1(t), \quad F^2 = u^3v^{-3}\hat{F}^2(t), \quad F^3 = uv^{-1}\hat{F}^3(t), \quad F^4 = u^2v^{-2}\hat{F}^4(t),\\
&\quad G^1 = u\hat{G}^1(t), \quad G^2 = v\hat{G}^2(t).\\
A^4_{3,2}: &\quad F^1 = t^{-3}v^2\hat{F}^1(u), \quad F^2 = t^{-3}v\hat{F}^2(u), \quad F^3 = t^{-3}v^3\hat{F}^3(u), \quad F^4 = t^{-3}v^2\hat{F}^4(u),\\
&\quad G^1 = t^{-1}\hat{G}^1(u), \quad G^2 = t^{-1}v\hat{G}^2(u).\\
\tilde{A}^4_{3,2}: &\quad F^1 = u^{-2}v^2\hat{F}^1(t), \quad F^2 = uv\hat{F}^2(t), \quad F^3 = u^{-3}v^3\hat{F}^3(t), \quad F^4 = u^{-2}v^2\hat{F}^4(t),\\
&\quad G^1 = u\hat{G}^1(t), \quad G^2 = v\hat{G}^2(t).
\end{aligned}
$$

In general the unknows $\hat{F}^1$, $\hat{F}^2$, $\hat{F}^3$, $\hat{F}^4$, $\hat{G}^1$ and $\hat{G}^2$ are different arbitrary functions of their arguments.

2. Non decomposable Lie algebras

Below we proceed as in the previous sections. We make use of the non decomposable three-dimensional solvable Lie algebras [5, 57]:

$$
\begin{aligned}
A_{3,3}: &\quad [e_2, e_3] = e_1;\\
A_{3,4}: &\quad [e_1, e_3] = e_1, \quad [e_2, e_3] = e_1 + e_2;
\end{aligned}
$$

$$
\begin{aligned}
A_{3,5}: &\quad [e_1,e_3]=e_1, \quad [e_2,e_3]=e_2;\\
A_{3,6}: &\quad [e_1,e_3]=e_1, \quad [e_2,e_3]=-e_2;\\
A_{3,7}: &\quad [e_1,e_3]=e_1, \quad [e_2,e_3]=qe_2 \quad (0<|q|<1);\\
A_{3,8}: &\quad [e_1,e_3]=-e_2, \quad [e_2,e_3]=e_1;\\
A_{3,9}: &\quad [e_1,e_3]=qe_1-e_2, \quad [e_2,e_3]=e_1+qe_2 \quad (q>0).
\end{aligned}
$$

All these algebras contain $A^k_{2,1}$ $(k=1:5)$.

The illustration by an example of the calculations for the algebra $A_{3,3}$ shows that there are no realizations for $A_{3,3}$ containing $A^1_{2,1}=\langle\partial_t,\partial_x\rangle$. Now we turn to $A^2_{2,1}=\langle\partial_t,\partial_u\rangle$. The corresponding equivalence transformations are of the form

$$
\mathcal{E}(A^2_{2,1})=\{\bar t=t+a_1,\ \bar x=X(x,v),\ \bar u=u+Z(x,v),\ \bar v=V(x,v)\}. \tag{4.59}
$$

The Lie Brackets: $[e_1,e_3]=0$ and $[e_2,e_3]=e_1$ yield

$$
e_3=b(x)\partial_x+(t+c_3(x,v))\partial_u+d(x,v)\partial_v.
$$

Under the change of variables (4.59) e_3 is transformed to

$$
\bar e_3=(bX_x+dX_v)\partial_{\bar x}+(\bar t-a_1+c_3+bZ_x+dZ_v)\partial_{\bar u}+(bV_x+dV_v)\partial_{\bar v}. \tag{4.60}
$$

The resulting realizations are:

$\langle\partial_u,\partial_t,x\partial_x+t\partial_u\rangle$, and $\langle\partial_u,\partial_t,t\partial_u+v\partial_v\rangle$.

Therefore the realizations of all the three-dimensional non decomposable solvable Lie algebras are:

$A^2_{3,3}=\langle\partial_u,\partial_t,x\partial_x+t\partial_u\rangle$, $\tilde A^2_{3,3}=\langle\partial_u,\partial_t,t\partial_u+v\partial_v\rangle$,

$A^3_{3,3}=\langle\partial_v,\partial_t,x\partial_x+t\partial_v\rangle$, $\tilde A^3_{3,3}=\langle\partial_v,\partial_t,u\partial_u+t\partial_v\rangle$,

$A^4_{3,3}=\langle\partial_u,\partial_x,t\partial_t+x\partial_u\rangle$, $\tilde A^4_{3,3}=\langle\partial_u,\partial_x,x\partial_u+v\partial_v\rangle$.

$A^5_{3,3}=\langle\partial_v,\partial_x,t\partial_t+x\partial_v\rangle$, $\tilde A^5_{3,3}=\langle\partial_u,\partial_x,u\partial_u+x\partial_v\rangle$.

$A^2_{3,4}=\langle\partial_u,\partial_t,t\partial_t+x\partial_x+(t+u)\partial_u\rangle$, $\tilde A^2_{3,4}=\langle\partial_u,\partial_t,t\partial_t+(t+u)\partial_u+v\partial_v\rangle$.

$A^3_{3,4} = \langle \partial_v, \partial_t, t\partial_t + x\partial_x + (t+v)\partial_v \rangle$, $\tilde{A}^3_{3,4} = \langle \partial_v, \partial_t, t\partial_t + u\partial_u + (t+v)\partial_v \rangle$.

$A^4_{3,4} = \langle \partial_u, \partial_x, t\partial_t + x\partial_x + (x+u)\partial_u \rangle$, $\tilde{A}^4_{3,4} = \langle \partial_u, \partial_x, x\partial_x + (x+u)\partial_u + v\partial_v \rangle$,
$\hat{A}^4_{3,4} = \langle \partial_u, \partial_x, x\partial_x + (x+u)\partial_u \rangle$.

$A^5_{3,4} = \langle \partial_v, \partial_x, t\partial_t + x\partial_x + (x+v)\partial_v \rangle$, $\tilde{A}^5_{3,4} = \langle \partial_v, \partial_x, x\partial_x + u\partial_u + (x+v)\partial_v \rangle$,
$\hat{A}^5_{3,4} = \langle \partial_v, \partial_x, x\partial_x + (x+v)\partial_v \rangle$.

$A^1_{3,5} = \langle \partial_t, \partial_x, t\partial_t + x\partial_x + u\partial_u \rangle$, $\tilde{A}^1_{3,5} = \langle \partial_t, \partial_x, t\partial_t + x\partial_x + v\partial_v \rangle$.

$A^2_{3,5} = \langle \partial_t, \partial_u, t\partial_t + x\partial_x + u\partial_u \rangle$, $\tilde{A}^2_{3,5} = \langle \partial_t, \partial_u, t\partial_t + u\partial_u + v\partial_v \rangle$.

$A^3_{3,5} = \langle \partial_t, \partial_v, t\partial_t + x\partial_x + v\partial_v \rangle$, $\tilde{A}^3_{3,5} = \langle \partial_t, \partial_v, t\partial_t + u\partial_u + v\partial_v \rangle$.

$A^4_{3,5} = \langle \partial_x, \partial_u, t\partial_t + x\partial_x + u\partial_u \rangle$, $\tilde{A}^4_{3,5} = \langle \partial_x, \partial_u, x\partial_x + u\partial_u + v\partial_v \rangle$.

$A^5_{3,5} = \langle \partial_x, \partial_v, t\partial_t + x\partial_x + v\partial_v \rangle$, $\tilde{A}^5_{3,5} = \langle \partial_x, \partial_v, x\partial_x + u\partial_u + v\partial_v \rangle$.

$A^1_{3,6} = \langle \partial_t, \partial_x, t\partial_t - x\partial_x + u\partial_u \rangle$, $\tilde{A}^1_{3,6} = \langle \partial_t, \partial_x, t\partial_t - x\partial_x - v\partial_v \rangle$.

$A^2_{3,6} = \langle \partial_t, \partial_u, t\partial_t + x\partial_x - u\partial_u \rangle$, $\tilde{A}^2_{3,6} = \langle \partial_t, \partial_u, t\partial_t - u\partial_u + v\partial_v \rangle$.

$A^3_{3,6} = \langle \partial_t, \partial_v, t\partial_t + x\partial_x - v\partial_v \rangle$, $\tilde{A}^3_{3,6} = \langle \partial_t, \partial_v, t\partial_t + u\partial_u - v\partial_v \rangle$.

$A^4_{3,6} = \langle \partial_x, \partial_u, t\partial_t + x\partial_x - u\partial_u \rangle$, $\tilde{A}^4_{3,6} = \langle \partial_x, \partial_u, x\partial_x - u\partial_u + v\partial_v \rangle$.

$A^5_{3,6} = \langle \partial_x, \partial_v, t\partial_t + x\partial_x - v\partial_v \rangle$, $\tilde{A}^5_{3,6} = \langle \partial_x, \partial_v, x\partial_x + u\partial_u - v\partial_v \rangle$.

$A^1_{3,7} = \langle \partial_t, \partial_x, t\partial_t + qx\partial_x + u\partial_u \rangle$, $\tilde{A}^1_{3,7} = \langle \partial_t, \partial_x, t\partial_t + qx\partial_x + v\partial_v \rangle$,
$\hat{A}^1_{3,7} = \langle \partial_x, \partial_t, qt\partial_t + x\partial_x + u\partial_u \rangle$, $\bar{A}^1_{3,7} = \langle \partial_x, \partial_t, qt\partial_t + x\partial_x + v\partial_v \rangle$.

$A^2_{3,7} = \langle \partial_t, \partial_u, t\partial_t + x\partial_x + qu\partial_u \rangle$, $\tilde{A}^2_{3,7} = \langle \partial_t, \partial_u, t\partial_t + qu\partial_u + v\partial_v \rangle$,
$\hat{A}^2_{3,7} = \langle \partial_u, \partial_t, qt\partial_t + x\partial_x + u\partial_u \rangle$, $\bar{A}^2_{3,7} = \langle \partial_u, \partial_t, qt\partial_t + u\partial_u + v\partial_v \rangle$.

$A^3_{3,7} = \langle \partial_t, \partial_v, t\partial_t + x\partial_x + qv\partial_v \rangle$, $\tilde{A}^3_{3,7} = \langle \partial_t, \partial_v, t\partial_t + u\partial_u + qv\partial_v \rangle$,
$\hat{A}^3_{3,7} = \langle \partial_v, \partial_t, qt\partial_t + x\partial_x + v\partial_v \rangle$, $\bar{A}^3_{3,7} = \langle \partial_v, \partial_t, qt\partial_t + u\partial_u + v\partial_v \rangle$.

$A^4_{3,7} = \langle \partial_x, \partial_u, t\partial_t + x\partial_x + qu\partial_u \rangle$, $\tilde{A}^4_{3,7} = \langle \partial_x, \partial_u, x\partial_x + qu\partial_u + v\partial_v \rangle$,
$\hat{A}^4_{3,7} = \langle \partial_u, \partial_x, t\partial_t + qx\partial_x + u\partial_u \rangle$, $\bar{A}^4_{3,7} = \langle \partial_u, \partial_x, qx\partial_x + u\partial_u + v\partial_v \rangle$.

$A^5_{3,7} = \langle \partial_x, \partial_v, t\partial_t + x\partial_x + qv\partial_v \rangle$, $\tilde{A}^5_{3,7} = \langle \partial_x, \partial_v, x\partial_x + u\partial_u + qv\partial_v \rangle$,
$\hat{A}^5_{3,7} = \langle \partial_v, \partial_x, t\partial_t + qx\partial_x + v\partial_v \rangle$, $\bar{A}^5_{3,7} = \langle \partial_v, \partial_x, qx\partial_x + u\partial_u + v\partial_v \rangle$.

The notation used in the above sets of realizations simply means that all the realizations in each set are obtained from the same two-dimensional realization.

The corresponding forms of arbitrary elements are summarized in Table 4.1.

4.4 Classification with respect to four-dimensional solvable Lie algebras

1. Decomposable Lie algebras

There are ten decomposable four-dimensional solvable Lie algebras [5]:

$$4A_1 = A_{3,1} \oplus A_1, \quad A_{3,2} \oplus A_1, \quad 2A_{2,2} = A_{2,2} \oplus A_{2,2}, \quad A_{3,i} \oplus A_1 \quad (i = 3:9).$$

Consider the four-dimensional Lie algebra, $A_{3,2} \oplus A_1$, which contain the three-dimensional Lie algebra, $A^1_{3,2}$. From the Lie Brackets: $[e_1, e_4] = 0$, $[e_2, e_4] = 0$ and $[e_3, e_4] = 0$ we have

$$e_4 = c_3(v)u\partial_u + d(v)\partial_v.$$

The equivalence transformations for $A^1_{3,2}$ transform e_4 to the form

$$\bar{e}_4 = \bar{x}\left(d\frac{Z_v}{Z}\right)\partial_{\bar{x}} + \bar{u}\left(c_3 + d\frac{H_v}{H}\right)\partial_{\bar{u}} + dV_v\partial_{\bar{v}}. \tag{4.61}$$

Eventually we obtain the realization

$$A_{3,2} \oplus A^1_1 = \langle -t\partial_t - x\partial_x, \partial_t, x\partial_x + u\partial_u, x\partial_x + u\partial_u + v\partial_v \rangle.$$

The invariant system for this realization is given by

$$\begin{aligned} u_t &= \bar{K}_1 xuu_{xx} + \bar{K}_2 xuv^{-1}v_{xx} + \bar{K}_5 x^{-1}u, \\ v_t &= \bar{K}_3 xu_{xx} + \bar{K}_4 xuv_{xx} + \bar{K}_6 x^{-1}uv, \end{aligned} \tag{4.62}$$

where $\bar{K}_1, \bar{K}_2, \ldots, \bar{K}_6$ are arbitrary constants.

Table 4.1: 3D non decomposable solvable Lie algebras.

Algebra	F^1	F^2	F^3	F^4	G^1	G^2
$A^2_{3,3}$	$x^2\Phi^1(v)$	$x^2\Phi^2(v)$	$x^2\Phi^3(v)$	$x^2\Phi^4(v)$	$\ln x+\Psi^1(v)$	$\Psi^2(v)$
$\tilde{A}^2_{3,3}$	$\Phi^1(x,u_x)$	$v^{-1}\Phi^2(x,u_x)$	$v\Phi^3(x,u_x)$	$\Phi^4(x,u_x)$	$\ln v+\Psi^1(x,u_x)$	$v\Psi^2(x,u_x)$
$A^3_{3,3}$	$x^2\Phi^1(u)$	$x^2\Phi^2(u)$	$x^2\Phi^3(u)$	$x^2\Phi^4(u)$	$\Psi^1(u)$	$\ln x+\Psi^2(u)$
$\tilde{A}^3_{3,3}$	$\Phi^1(x,v_x)$	$u\Phi^2(x,v_x)$	$u^{-1}\Phi^3(x,v_x)$	$\Phi^4(x,v_x)$	$u\Psi^1(x,v_x)$	$\ln u+\Psi^2(x,v_x)$
$A^4_{3,3}$	$t^{-1}\Phi^1(\mu)$	$t^{-1}\Phi^2(\mu)$	$t^{-1}\Phi^3(\mu)$	$t^{-1}\Phi^4(\mu)$	$t^{-1}\Psi^1(\mu)$	$t^{-1}\Psi^2(\mu)$
$\tilde{A}^4_{3,3}$	$\Phi^1(\nu)$	$v^{-1}\Phi^2(\nu)$	$v\Phi^3(\nu)$	$\Phi^4(\nu)$	$\Psi^1(\nu)$	$v\Psi^2(\nu)$
$A^5_{3,3}$	$t^{-1}\Phi^1(\tau)$	$t^{-1}\Phi^2(\tau)$	$t^{-1}\Phi^3(\tau)$	$t^{-1}\Phi^4(\tau)$	$t^{-1}\Psi^j(\tau)$	$t^{-1}\Psi^2(\tau)$
$\tilde{A}^5_{3,3}$	$\Phi^1(\omega)$	$u\Phi^2(\omega)$	$u^{-1}\Phi^3(\omega)$	$\Phi^4(\omega)$	$u\Psi^1(\omega)$	$\Psi^2(\omega)$
$A^2_{3,4}$	$x\Phi^1(v,u_x)$	$x^2\Phi^2(v,u_x)$	$\Phi^3(v,u_x)$	$x\Phi^4(v,u_x)$	$\ln x+\Psi^1(v,u_x)$	$x^{-1}\Psi^2(v,u_x)$
$\tilde{A}^2_{3,4}$	$v^{-1}\Phi^1(x)$	$v^{-1}\Phi^2(x)$	$v^{-1}\Phi^3(x)$	$v^{-1}\Phi^4(x)$	$\ln v+\Psi^1(x)$	$\Psi^2(x)$
$A^3_{3,4}$	$x\Phi^1(u,v_x)$	$\Phi^2(u,v_x)$	$x^2\Phi^3(u,v_x)$	$x\Phi^4(u,v_x)$	$x^{-1}\Psi^1(u,v_x)$	$\ln x+\Psi^2(u,v_x)$
$\tilde{A}^3_{3,4}$	$u^{-1}\Phi^1(x)$	$u^{-1}\Phi^2(x)$	$u^{-1}\Phi^3(x)$	$u^{-1}\Phi^4(x)$	$\Psi^1(x)$	$\ln u+\Psi^2(x)$
$A^4_{3,4}$	$t\Phi^1(\mu)$	$t^2\Phi^2(\mu)$	$\Phi^3(\mu)$	$t\Phi^4(\mu)$	$\Psi^1(\mu)$	$t^{-1}\Psi^2(\mu)$
$\tilde{A}^4_{3,4}$	$v^2\Phi^1(\nu)$	$v^2\Phi^2(\nu)$	$v^2\Phi^3(\nu)$	$v^2\Phi^4(\nu)$	$v\Psi^1(\nu)$	$v\Psi^2(\nu)$
$\hat{A}^4_{3,4}$	$e^{2u_x}\Phi^1(t,v)$	$e^{3u_x}\Phi^2(t,v)$	$e^{u_x}\Phi^3(t,v)$	$e^{2u_x}\Phi^4(t,v)$	$e^{u_x}\Psi^1(t,v)$	$\Psi^2(t,v)$
$A^5_{3,4}$	$t\Phi^1(\tau)$	$\Phi^2(\tau)$	$t^2\Phi^3(\tau)$	$t\Phi^4(\tau)$	$t^{-1}\Psi^1(\tau)$	$\Psi^2(\tau)$
$\tilde{A}^5_{3,4}$	$u^2\Phi^1(\omega)$	$u^2\Phi^2(\omega)$	$u^2\Phi^3(\omega)$	$u^2\Phi^4(\omega)$	$u\Psi^1(\omega)$	$u\Psi^2(\omega)$
$A^1_{3,5}$	$u\Phi^1(v,u_x)$	$u^2\Phi^2(v,u_x)$	$\Phi^3(v,u_x)$	$u\Phi^4(v,u_x)$	$\Psi^1(v,u_x)$	$u^{-1}\Psi^2(v,u_x)$
$\tilde{A}^1_{3,5}$	$v\Phi^1(u,v_x)$	$\Phi^2(u,v_x)$	$v^2\Phi^3(u,v_x)$	$v\Phi^4(u,v_x)$	$v^{-1}\Psi^1(u,v_x)$	$\Psi^2(u,v_x)$
$A^2_{3,5}$	$x\Phi^1(v,u_x)$	$x^2\Phi^2(v,u_x)$	$\Phi^3(v,u_x)$	$x\Phi^4(v,u_x)$	$\Psi^1(v,u_x)$	$x^{-1}\Psi^2(v,u_x)$
$\tilde{A}^2_{3,5}$	$v^{-1}\Phi^1(x)$	$v^{-1}\Phi^2(x)$	$v^{-1}\Phi^3(x)$	$v^{-1}\Phi^4(x)$	$\Psi^1(x)$	$\Psi^2(x)$
$A^3_{3,5}$	$x\Phi^1(u,v_x)$	$\Phi^2(u,v_x)$	$x^2\Phi^3(u,v_x)$	$x\Phi^4(u,v_x)$	$x^{-1}\Psi^1(u,v_x)$	$\Psi^2(u,v_x)$
$\tilde{A}^3_{3,5}$	$u^{-1}\Phi^1(x)$	$u^{-1}\Phi^2(x)$	$u^{-1}\Phi^3(x)$	$u^{-1}\Phi^4(x)$	$\Psi^1(x)$	$\Psi^2(x)$
$A^4_{3,5}$	$t\Phi^1(v,u_x)$	$t^2\Phi^2(v,u_x)$	$\Phi^3(v,u_x)$	$t\Phi^4(v,u_x)$	$\Psi^1(v,u_x)$	$t^{-1}\Psi^2(v,u_x)$
$\tilde{A}^4_{3,5}$	$v^2\Phi^1(t,u_x)$	$v^2\Phi^2(t,u_x)$	$v^2\Phi^3(t,u_x)$	$v^2\Phi^4(t,u_x)$	$v\Psi^1(x)$	$v\Psi^2(x)$
$A^5_{3,5}$	$t\Phi^1(u,v_x)$	$\Phi^2(u,v_x)$	$t^2\Phi^3(u,v_x)$	$\Phi^4(u,v_x)$	$t^{-1}\Psi^1(u,v_x)$	$\Psi^2(u,v_x)$
$\tilde{A}^5_{3,5}$	$u^2\Phi^1(t,u_x,v_x)$	$u^2\Phi^2(t,u_x,v_x)$	$u^2\Phi^3(t,u_x,v_x)$	$u^2\Phi^4(t,u_x,v_x)$	$u\Psi^1(t,u_x,v_x)$	$u\Psi^2(t,u_x,v_x)$
$A^1_{3,6}$	$u^{-3}\Phi^1(v)$	$u^{-2}\Phi^2(v)$	$u^{-4}\Phi^3(v)$	$u^{-3}\Phi^4(v)$	$\Psi^1(v)$	$u^{-1}\Psi^2(v)$
$\tilde{A}^1_{3,6}$	$v^3\Phi^1(u,v_x)$	$v^2\Phi^2(u,v_x)$	$v^4\Phi^3(u,v_x)$	$v^3\Phi^4(u,v_x)$	$v\Psi^1(u,v_x)$	$v^2\Psi^2(u,v_x)$

The Φ^i and Ψ^i are arbitrary functions of their arguments; $\mu = (v)(u_x - \ln t)$, $\nu = (t)(u_x - \ln v)$, $\tau = (u)(v_x - \ln t)$, $\omega = (t)(v_x - \ln u)$.

Table 4.1 (Continued).

Algebra	F^1	F^2	F^3	F^4	G^1	G^2
$A^2_{3,6}$	$x\Phi^1(v)$	$\Phi^2(v)$	$x^2\Phi^3(v)$	$x\Phi^4(v)$	$x^{-2}\Psi^1(v)$	$x^{-1}\Psi^2(v)$
$\tilde{A}^2_{3,6}$	$v^{-1}\Phi^1(x)$	$v^{-3}\Phi^2(x)$	$v\Phi^3(x)$	$v^{-1}\Phi^4(x)$	$v^{-2}\Psi^1(x)$	$\Psi^2(x)$
$A^3_{3,6}$	$x\Phi^1(u)$	$x^2\Phi^2(u)$	$\Phi^3(u)$	$x\Phi^4(u)$	$x^{-1}\Psi^1(u)$	$x^{-2}\Psi^2(u)$
$\tilde{A}^3_{3,6}$	$u^{-1}\Phi^1(x)$	$u\Phi^2(x)$	$u^{-3}\Phi^3(x)$	$u^{-1}\Phi^4(x)$	$\Psi^1(x)$	$u^{-2}\Psi^2(x)$
$A^4_{3,6}$	$t\Phi^1(v)$	$\Phi^2(v)$	$t^2\Phi^3(v)$	$t\Phi^4(v)$	$t^{-2}\Psi^1(v)$	$t^{-1}\Psi^2(v)$
$\tilde{A}^4_{3,6}$	$v^2\Phi^1(t,v_x)$	$\Phi^2(t,v_x)$	$v^4\Phi^3(t,v_x)$	$v^2\Phi^4(t,v_x)$	$v^{-1}\Psi^1(t,v_x)$	$v\Psi^2(t,v_x)$
$A^5_{3,6}$	$t\Phi^1(u)$	$t^2\Phi^2(u)$	$\Phi^3(u)$	$t\Phi^4(u)$	$t^{-1}\Psi^1(u)$	$t^{-2}\Psi^2(u)$
$\tilde{A}^5_{3,6}$	$u^2\Phi^1(t,u_x)$	$u^4\Phi^2(t,u_x)$	$\Phi^3(t,u_x)$	$u^2\Phi^4(t,u_x)$	$u\Psi^1(t,u_x)$	$u^{-1}\Psi^2(t,u_x)$
$A^1_{3,7}$	$u^{2q-1}\Phi^1(v)$	$u^{2q}\Phi^2(v)$	$u^{2(q-1)}\Phi^3(v)$	$u^{2q-1}\Phi^4(v)$	$\Psi^1(v)$	$u^{-1}\Psi^2(v)$
$\tilde{A}^1_{3,7}$	$v^{2q-1}\Phi^1(u)$	$v^{2(q-2)}\Phi^2(u)$	$v^{2q}\Phi^3(u)$	$v^{2q-1}\Phi^4(u)$	$v^{-1}\Psi^1(u)$	$\Psi^2(u)$
$\hat{A}^1_{3,7}$	$u^{2-q}\Phi^1(v,u_x)$	$u^{3-q}\Phi^2(v,u_x)$	$u^{1-q}\Phi^3(v,u_x)$	$u^{2-q}\Phi^4(v,u_x)$	$u^{1-q}\Psi^1(v,u_x)$	$u^{-q}\Psi^2(v,u_x)$
$\bar{A}^1_{3,7}$	$v^{2-q}\Phi^1(u,v_x)$	$v^{1-q}\Phi^2(u,v_x)$	$v^{3-q}\Phi^3(u,v_x)$	$v^{2-q}\Phi^4(u,v_x)$	$v^{-q}\Psi^1(u,v_x)$	$v^{1-q}\Psi^2(u,v_x)$
$A^2_{3,7}$	$x\Phi^1(v)$	$x^{q+1}\Phi^2(v)$	$x^{1-q}\Phi^3(v)$	$x\Phi^4(v)$	$x^{q-1}\Psi^1(v)$	$x^{-1}\Psi^2(v)$
$\tilde{A}^2_{3,7}$	$v^{-1}\Phi^1(x)$	$v^{q-2}\Phi^2(x)$	$v^{-q}\Phi^3(x)$	$v^{-1}\Phi^4(x)$	$v^{q-1}\Psi^1(x)$	$\Psi^2(x)$
$\hat{A}^2_{3,7}$	$x^{2-q}\Phi^1(v,u_x)$	$x^{3-q}\Phi^2(v,u_x)$	$x^{1-q}\Phi^3(v,u_x)$	$x^{2-q}\Phi^4(v,u_x)$	$x^{1-q}\Psi^1(v,u_x)$	$x^{-q}\Psi^2(v,u_x)$
$\bar{A}^2_{3,7}$	$v^{-q}\Phi^2(x)$	$v^{-q}\Phi^2(x)$	$v^{-q}\Phi^3(x)$	$v^{-q}\Phi^4(x)$	$v^{1-q}\Psi^1(x)$	$v^{1-q}\Psi^2(x)$
$A^3_{3,7}$	$x\Phi^1(u)$	$x^{1-q}\Phi^2(u)$	$x^{1+q}\Phi^3(u)$	$x\Phi^4(u)$	$x^{-1}\Psi^1(u)$	$x^{q-1}\Psi^2(u)$
$\tilde{A}^3_{3,7}$	$u^{-1}\Phi^1(x)$	$u^{-q}\Phi^2(x)$	$u^{q-2}\Phi^3(x)$	$u^{-1}\Phi^4(x)$	$\Psi^1(x)$	$u^{q-1}\Psi^2(x)$
$\hat{A}^3_{3,7}$	$x^{2-q}\Phi^1(u,v_x)$	$x^{1-q}\Phi^2(u,v_x)$	$x^{3-q}\Phi^3(u,v_x)$	$x^{2-q}\Phi^4(u,v_x)$	$x^{-q}\Psi^1(u,v_x)$	$x^{1-q}\Psi^2(u,v_x)$
$\bar{A}^3_{3,7}$	$u^{-q}\Phi^1(x)$	$u^{-q}\Phi^2(x)$	$u^{-q}\Phi^3(x)$	$u^{-q}\Phi^4(x)$	$u^{1-q}\Psi^1(x)$	$u^{1-q}\Psi^2(x)$
$A^4_{3,7}$	$t\Phi^1(v)$	$t^{q+1}\Phi^2(v)$	$t^{1-q}\Phi^3(v)$	$t\Phi^4(v)$	$t^{q-1}\Psi^1(v)$	$t^{-1}\Psi^2(v)$
$\tilde{A}^4_{3,7}$	$v^2\Phi^1(t,v_x)$	$v^{q+1}\Phi^2(t,v_x)$	$v^{3-q}\Phi^3(t,v_x)$	$v^2\Phi^4(t,v_x)$	$v^q\Psi^1(t,v_x)$	$v\Psi^2(t,v_x)$
$\hat{A}^4_{3,7}$	$t^{2q-1}\Phi^1(v)$	$t^{2q}\Phi^2(v)$	$t^{2(q-1)}\Phi^3(v)$	$t^{2q-1}\Phi^4(v)$	$\Psi^1(v)$	$t^{-1}\Psi^2(v)$
$\bar{A}^4_{3,7}$	$v^{2q}\Phi^1(t)$	$v^{2q}\Phi^2(t)$	$v^{2q}\Phi^3(t)$	$v^{2q}\Phi^4(t)$	$v\Psi^1(t)$	$v\Psi^2(t)$
$A^5_{3,7}$	$t\Phi^1(u)$	$t^{1-q}\Phi^2(u)$	$t^{q+1}\Phi^3(u)$	$t\Phi^4(u)$	$t^{-1}\Psi^1(u)$	$t^{q-1}\Psi^2(u)$
$\tilde{A}^5_{3,7}$	$u^2\Phi^1(t,u_x)$	$u^{3-q}\Phi^2(t,u_x)$	$u^{q+1}\Phi^3(t,u_x)$	$u^2\Phi^4(t,u_x)$	$u\Psi^1(t,u_x)$	$u^q\Psi^2(t,u_x)$
$\hat{A}^5_{3,7}$	$t^{2q-1}\Phi^1(u)$	$t^{2(q-1)}\Phi^2(u)$	$t^{2q}\Phi^3(u)$	$t^{2q-1}\Phi^4(u)$	$t^{-1}\Psi^1(u)$	$\Psi^2(u)$
$\bar{A}^4_{3,7}$	$u^{2q}\Phi^1(t)$	$u^{2q}\Phi^2(t)$	$u^{2q}\Phi^3(t)$	$u^{2q}\Phi^4(t)$	$u\Psi^1(t)$	$u\Psi^2(t)$

The rest of the realizations of the four-dimensional decomposable solvable Lie algebras and their corresponding invariant systems are:

$4A^1_1 = \langle \partial_t, \partial_x, \partial_u, v\partial_v \rangle$:

$$\begin{aligned} u_t &= \bar{\mathcal{F}}^1(u_x)u_{xx} + v^{-1}\bar{\mathcal{F}}^2(u_x)v_{xx} + \bar{\mathcal{G}}^1(u_x), \\ v_t &= v\bar{\mathcal{F}}^3(u_x)u_{xx} + \bar{\mathcal{F}}^4(u_x)v_{xx} + \bar{\mathcal{G}}^2(u_x). \end{aligned}$$

$4A_1^2 = \langle \partial_t, \partial_x, u\partial_u, \partial_v \rangle$:

$$\begin{aligned} u_t &= \tilde{\mathcal{F}}^1(v_x)u_{xx} + u\tilde{\mathcal{F}}^2(v_x)v_{xx} + u\tilde{\mathcal{G}}^1(v_x), \\ v_t &= u^{-1}\tilde{\mathcal{F}}^3(v_x)u_{xx} + \tilde{\mathcal{F}}^4(v_x)v_{xx} + \tilde{\mathcal{G}}^2(v_x). \end{aligned}$$

$\tilde{A}_{3,2} \oplus \tilde{A}_1^1 = \langle -t\partial_t - x\partial_x, \partial_t, x\partial_x + v\partial_v, x\partial_x + u\partial_u + v\partial_v \rangle$:

$$\begin{aligned} u_t &= \tilde{K}_1 xvu_{xx} + \tilde{K}_2 xuv_{xx} + \tilde{K}_5 x^{-1}uv, \\ v_t &= \tilde{K}_3 xv^2u_{xx} + \tilde{K}_4 xvv_{xx} + \tilde{K}_6 x^{-1}v^2. \end{aligned}$$

The unknowns $\bar{\mathcal{F}}^1, \ldots, \bar{\mathcal{F}}^4$, $\tilde{\mathcal{F}}^1, \ldots, \tilde{\mathcal{F}}^4$, $\bar{\mathcal{G}}^1$, $\tilde{\mathcal{G}}^1$, $\bar{\mathcal{G}}^2$ and $\tilde{\mathcal{G}}^2$ are arbitrary functions of their respective arguments and $\tilde{K}_1, \ldots, \tilde{K}_6$ are arbitrary constants.

2. Non decomposable Lie algebras

These Lie algebras are such that $A_{4,i} = \langle \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \rangle$ for $i = 1, 2, \ldots, 10$. They are solvable and can be written in terms of a one-dimensional Lie algebra $\langle \Gamma_4 \rangle$ and a three-dimensional ideal $\langle \Gamma_1, \Gamma_2, \Gamma_3 \rangle$ which is already known. For $A_{4,i}$, $i = 1, 2, \ldots, 6$, the three-dimensional ideal is of abelian type. It is of type $A_{3,3}$ (nilpotent) for $i = 7, 8, 9$ and finally for $i = 10$ of type $A_{3,5}$; (see also [5, 6]).

$A_{4,1}$: $[e_2, e_4] = e_1, \quad [e_3, e_4] = e_2;$

$A_{4,2}$: $[e_1, e_4] = qe_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_2 + e_3, \quad q \neq 0;$

$A_{4,3}$: $[e_1, e_4] = e_1, \quad [e_3, e_4] = e_2;$

$A_{4,4}$: $[e_1, e_4] = e_1, \quad [e_2, e_4] = e_1 + e_2, \quad [e_3, e_4] = e_2 + e_3;$

$A_{4,5}$: $[e_1, e_4] = e_1, \quad [e_2, e_4] = qe_2, \quad [e_3, e_4] = pe_3, \quad -1 \leq p \leq q \leq 1, \ p \cdot q \neq 0;$

$A_{4,6}$: $[e_1, e_4] = qe_1, \quad [e_2, e_4] = pe_2 - e_3, \quad [e_3, e_4] = e_2 + pe_3, \quad q \neq 0, \ p \geq 0;$

$A_{4,7}$: $[e_1, e_4] = 2e_1, \quad [e_2, e_3] = e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_2 + e_3;$

$A_{4,8}$: $[e_1, e_4] = (1 + q)e_1, \quad [e_2, e_3] = e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = qe_3, \quad |q| \leq 1;$

$A_{4,9}$: $[e_1, e_4] = 2qe_1, \quad [e_2, e_3] = e_1, \quad [e_2, e_4] = qe_2 - e_3, \quad [e_3, e_4] = e_2 + qe_3, \quad q \geq 0;$

$A_{4,10}$: $[e_1, e_3] = e_1, \quad [e_1, e_4] = -e_2, \quad [e_2, e_3] = e_2, \quad [e_2, e_4] = e_1.$

The use of the above classification of four-dimensional non decomposable solvable Lie algebras enables the calculations for the search of possible realizations to follow the same

route as in the previous sections. We present only the realizations and their respective invariant systems.

$A^1_{4,2} = \langle \partial_t, \partial_u, \partial_x, qt\partial_t + x\partial_x + (x+u)\partial_u + v\partial_v \rangle$:

$$\begin{aligned} u_t &= v^{2-q}\mathcal{F}^1(\gamma)u_{xx} + v^{2-q}\mathcal{F}^2(\gamma)v_{xx} + v^{1-q}\mathcal{G}^1(\gamma), \\ v_t &= v^{2-q}\mathcal{F}^3(\gamma)u_{xx} + v^{2-q}\mathcal{F}^4(\gamma)v_{xx} + v^{1-q}\mathcal{G}^2(\gamma), \quad \gamma = u_x - \ln v. \end{aligned}$$

$\tilde{A}^1_{4,2} = \langle \partial_t, \partial_u, \partial_x, qt\partial_t + x\partial_x + (x+u)\partial_u \rangle$:

$$\begin{aligned} u_t &= e^{(2-q)u_x}\mathcal{F}^1(v)u_{xx} + e^{(3-q)u_x}\mathcal{F}^2(v)v_{xx} + e^{(1-q)u_x}\mathcal{G}^1(v), \\ v_t &= e^{(1-q)u_x}\mathcal{F}^3(v)u_{xx} + e^{(2-q)u_x}\mathcal{F}^4(v)v_{xx} + e^{-qu_x}\mathcal{G}^2(v). \end{aligned}$$

$\hat{A}^1_{4,2} = \langle \partial_x, \partial_u, \partial_t, t\partial_t + qx\partial_x + (t+u)\partial_u + v\partial_v \rangle$:

$$\begin{aligned} u_t &= K_1 v^{2q-1}u_{xx} + K_2 v^{2q-1}v_{xx} + \ln v + K_5, \\ v_t &= K_3 v^{2q-1}u_{xx} + K_4 v^{2q-1}v_{xx} + K_6. \end{aligned}$$

$A^2_{4,2} = \langle \partial_t, \partial_v, \partial_x, qt\partial_t + x\partial_x + u\partial_u + (x+v)\partial_v \rangle$:

$$\begin{aligned} u_t &= u^{2-q}\mathcal{F}^1(\lambda)u_{xx} + u^{2-q}\mathcal{F}^2(\lambda)v_{xx} + u^{1-q}\mathcal{G}^1(\lambda), \\ v_t &= u^{2-q}\mathcal{F}^3(\lambda)u_{xx} + u^{2-q}\mathcal{F}^4(\lambda)v_{xx} + u^{1-q}\mathcal{G}^2(\lambda), \quad \lambda = v_x - \ln u. \end{aligned}$$

$\tilde{A}^2_{4,2} = \langle \partial_t, \partial_v, \partial_x, qt\partial_t + x\partial_x + (x+v)\partial_v \rangle$:

$$\begin{aligned} u_t &= e^{(2-q)v_x}\mathcal{F}^1(u)u_{xx} + e^{(3-q)v_x}\mathcal{F}^2(u)v_{xx} + e^{(1-q)v_x}\mathcal{G}^1(u), \\ v_t &= e^{(1-q)v_x}\mathcal{F}^3(u)u_{xx} + e^{(2-q)v_x}\mathcal{F}^4(u)v_{xx} + e^{(1-q)v_x}\mathcal{G}^2(u). \end{aligned}$$

$\hat{A}^2_{4,2} = \langle \partial_x, \partial_v, \partial_t, t\partial_t + qx\partial_x + u\partial_u + (t+v)\partial_v \rangle$:

$$\begin{aligned} u_t &= K_1 u^{2q-1}u_{xx} + K_2 u^{2q-1}v_{xx} + K_5, \\ v_t &= K_3 u^{2q-1}u_{xx} + K_4 u^{2q-1}v_{xx} + \ln u + K_6. \end{aligned}$$

$A^1_{4,3} = \langle \partial_t, \partial_u, \partial_x, t\partial_t + x\partial_u + v\partial_v \rangle$:

$$\begin{aligned} u_t &= v^{-1}\mathcal{F}^1(\gamma)u_{xx} + v^{-2}\mathcal{F}^2(\gamma)v_{xx} + v^{-1}\mathcal{G}^1(\gamma), \\ v_t &= \mathcal{F}^3(\gamma)u_{xx} + v^{-1}\mathcal{F}^4(\gamma)v_{xx} + \mathcal{G}^2(\gamma), \quad \gamma = u_x - \ln v. \end{aligned}$$

$\tilde{A}^1_{4,3} = \langle \partial_t, \partial_u, \partial_x, t\partial_t + x\partial_u \rangle$:

$$\begin{aligned} u_t &= e^{-u_x}\mathcal{F}^1(v, v_x)u_{xx} + e^{-u_x}\mathcal{F}^2(v, v_x)v_{xx} + e^{-u_x}\mathcal{G}^1(v, v_x), \\ v_t &= e^{-u_x}\mathcal{F}^3(v, v_x)u_{xx} + e^{-u_x}\mathcal{F}^4(v, v_x)v_{xx} + e^{-u_x}\mathcal{G}^2(v, v_x). \end{aligned}$$

$\hat{A}^1_{4,3} = \langle \partial_x, \partial_u, \partial_t, x\partial_x + t\partial_u + v\partial_v \rangle$:

$$\begin{aligned} u_t &= v^2\mathcal{F}^1(v_x)u_{xx} + v\mathcal{F}^2(v_x)v_{xx} + \mathcal{G}^1(v_x) + \ln v, \\ v_t &= v^3\mathcal{F}^3(v_x)u_{xx} + v^2\mathcal{F}^4(v_x)v_{xx} + v\mathcal{G}^2(v_x). \end{aligned}$$

$A^2_{4,3} = \langle \partial_t, \partial_v, \partial_x, t\partial_t + u\partial_u + x\partial_v \rangle$:

$$\begin{aligned} u_t &= u^{-1}\mathcal{F}^1(\lambda)u_{xx} + \mathcal{F}^2(\lambda)v_{xx} + \mathcal{G}^1(\lambda), \\ v_t &= u^{-2}\mathcal{F}^3(\lambda)u_{xx} + u^{-1}\mathcal{F}^4(\lambda)v_{xx} + u^{-1}\mathcal{G}^2(\lambda), \quad \lambda = v_x - \ln u. \end{aligned}$$

$\tilde{A}^2_{4,3} = \langle \partial_t, \partial_v, \partial_x, t\partial_t + x\partial_v \rangle$:

$$\begin{aligned} u_t &= e^{-v_x}\mathcal{F}^1(u, u_x)u_{xx} + e^{-v_x}\mathcal{F}^2(u, u_x)v_{xx} + e^{-v_x}\mathcal{G}^1(u, u_x), \\ v_t &= e^{-v_x}\mathcal{F}^3(u, u_x)u_{xx} + e^{-v_x}\mathcal{F}^4(u, u_x)v_{xx} + e^{-v_x}\mathcal{G}^2(u, u_x). \end{aligned}$$

$\hat{A}^2_{4,3} = \langle \partial_x, \partial_v, \partial_t, x\partial_x + u\partial_u + t\partial_v \rangle$:

$$\begin{aligned} u_t &= u^2\mathcal{F}^1(u_x)u_{xx} + u^3\mathcal{F}^2(u_x)v_{xx} + u\mathcal{G}^1(u_x), \\ v_t &= u\mathcal{F}^3(u_x)u_{xx} + u^2\mathcal{F}^4(u_x)v_{xx} + \mathcal{G}^2(u_x) + \ln u. \end{aligned}$$

$A^1_{4,5} = \langle \partial_t, \partial_x, \partial_u, t\partial_t + qx\partial_x + pu\partial_u + v\partial_v \rangle$:

$$\begin{aligned} u_t &= K_1 v^{2q-1}u_{xx} + K_2 v^{p+2q-2}v_{xx} + K_5 v^{p-1}, \\ v_t &= K_3 v^{2q-p}u_{xx} + K_4 v^{2q-1}v_{xx} + K_6. \end{aligned}$$

$\tilde{A}^1_{4,5} = \langle \partial_t, \partial_u, \partial_x, t\partial_t + px\partial_x + qu\partial_u + v\partial_v \rangle$:

$$\begin{aligned} u_t &= K_1 v^{2p-1}u_{xx} + K_2 v^{2p+q-2}v_{xx} + K_5 v^{q-1}, \\ v_t &= K_3 v^{2p-q}u_{xx} + K_4 v^{2p-1}v_{xx} + K_6. \end{aligned}$$

$\hat{A}^1_{4,5} = \langle \partial_x, \partial_t, \partial_u, qt\partial_t + x\partial_x + pu\partial_u + v\partial_v \rangle$:

$$\begin{aligned} u_t &= v^{2-q}\mathcal{F}^1(v_x)u_{xx} + v^{p-q+1}\mathcal{F}^2(v_x)v_{xx} + v^{p-q}\mathcal{G}^1(v_x), \\ v_t &= v^{3-p-q}\mathcal{F}^3(v_x)u_{xx} + v^{2-q}\mathcal{F}^4(v_x)v_{xx} + v^{1-q}\mathcal{G}^2(v_x). \end{aligned}$$

$\bar{A}^1_{4,5} = \langle \partial_x, \partial_u, \partial_t, pt\partial_t + x\partial_x + qu\partial_u + v\partial_v \rangle$:

$$\begin{aligned} u_t &= v^{2-p}\mathcal{F}^1(v_x)u_{xx} + v^{1-p+q}\mathcal{F}^2(v_x)v_{xx} + v^{q-p}\mathcal{G}^1(v_x), \\ v_t &= v^{3-p-q}\mathcal{F}^3(v_x)u_{xx} + v^{2-p}\mathcal{F}^4(v_x)v_{xx} + v^{1-p}\mathcal{G}^2(v_x). \end{aligned}$$

$\check{A}^1_{4,5} = \langle \partial_u, \partial_t, \partial_x, qt\partial_t + px\partial_x + u\partial_u + v\partial_v \rangle$:

$$\begin{aligned} u_t &= K_1 v^{2p-q}u_{xx} + K_2 v^{2p-q}v_{xx} + K_5 v^{1-q}, \\ v_t &= K_3 v^{2p-q}u_{xx} + K_4 v^{2p-q}v_{xx} + k_6 v^{1-q}. \end{aligned}$$

$\grave{A}^1_{4,5} = \langle \partial_u, \partial_x, \partial_t, pt\partial_t + qx\partial_x + u\partial_u + v\partial_v \rangle$:

$$\begin{aligned} u_t &= K_1 v^{2q-p}u_{xx} + K_2 v^{2q-p}v_{xx} + K_5 v^{1-p}, \\ v_t &= K_3 v^{2q-p}u_{xx} + K_4 v^{2q-p}v_{xx} + k_6 v^{1-p}. \end{aligned}$$

$A^2_{4,5} = \langle \partial_t, \partial_x, \partial_v, t\partial_t + qx\partial_x + u\partial_u + pv\partial_v \rangle$:

$$\begin{aligned} u_t &= K_1 u^{2q-1}u_{xx} + K_2 u^{2q-p}v_{xx} + K_5, \\ v_t &= K_3 u^{p+2q-2}u_{xx} + K_4 u^{2q-1}v_{xx} + k_6 u^{p-1}. \end{aligned}$$

$\tilde{A}^2_{4,5} = \langle \partial_t, \partial_v, \partial_x, t\partial_t + px\partial_x + u\partial_u + qv\partial_v \rangle$:

$$\begin{aligned} u_t &= K_1 u^{2p-1}u_{xx} + K_2 u^{2p-q}v_{xx} + K_5, \\ v_t &= K_3 u^{2p+q-2}u_{xx} + K_4 u^{2p-1}v_{xx} + k_6 u^{q-1}. \end{aligned}$$

$\hat{A}^2_{4,5} = \langle \partial_x, \partial_t, \partial_v, qt\partial_t + x\partial_x + u\partial_u + pv\partial_v \rangle$:

$$\begin{aligned} u_t &= u^{2-q}\mathcal{F}^1(u_x)u_{xx} + u^{3-p-q}\mathcal{F}^2(u_x)v_{xx} + u^{1-q}\mathcal{G}^1(u_x), \\ v_t &= u^{p-q+1}\mathcal{F}^3(u_x)u_{xx} + u^{2-q}\mathcal{F}^4(u_x)v_{xx} + u^{p-q}\mathcal{G}^2(u_x). \end{aligned}$$

$\bar{A}^2_{4,5} = \langle \partial_x, \partial_v, \partial_t, pt\partial_t + x\partial_x + u\partial_u + qv\partial_v \rangle$:

$$\begin{aligned} u_t &= u^{2-p}\mathcal{F}^1(u_x)u_{xx} + u^{3-p-q}\mathcal{F}^2(u_x)v_{xx} + u^{1-p}\mathcal{G}^1(u_x), \\ v_t &= u^{1-p+q}\mathcal{F}^3(u_x)u_{xx} + u^{2-p}\mathcal{F}^4(u_x)v_{xx} + u^{q-p}\mathcal{G}^2(u_x). \end{aligned}$$

$\check{A}^2_{4,5} = \langle \partial_v, \partial_t, \partial_x, qt\partial_t + px\partial_x + u\partial_u + v\partial_v \rangle$:

$$\begin{aligned} u_t &= K_1 u^{2p-q}u_{xx} + K_2 u^{2p-q}v_{xx} + K_5 u^{1-q}, \\ v_t &= K_3 u^{2p-q}u_{xx} + K_4 u^{2p-q}v_{xx} + k_6 u^{1-q}. \end{aligned}$$

$\grave{A}^2_{4,5} = \langle \partial_v, \partial_x, \partial_t, pt\partial_t + qx\partial_x + u\partial_u + v\partial_v \rangle$:

$$\begin{aligned} u_t &= K_1 u^{2q-p}u_{xx} + K_2 u^{2q-p}v_{xx} + K_5 u^{1-p}, \\ v_t &= K_3 u^{2q-p}u_{xx} + K_4 u^{2q-p}v_{xx} + k_6 u^{1-p}. \end{aligned}$$

$A^2_{4,8} = \langle \partial_u, \partial_t, t\partial_u + v\partial_v, t\partial_t + x\partial_x + (1+q)u\partial_u + qv\partial_v \rangle$:

$$\begin{aligned} u_t &= K_1 x u_{xx} + K_2 x^{q+2}v^{-1}v_{xx} + K_5 x^q, \\ v_t &= K_3 x^{-q}v u_{xx} + K_4 x v_{xx} + k_6 x^{-1}v. \end{aligned}$$

$A^4_{4,8} = \langle \partial_u, \partial_x, t\partial_t + x\partial_u, qt\ln t\partial_t + x\partial_x + (1+q)u\partial_u + v\partial_v \rangle$:

$$\begin{aligned} u_t &= K_1 t^{-1}v^{2-q}u_{xx} + K_2 t^{-1}v^2 v_{xx} + K_5 t^{-1}v, \\ v_t &= K_3 t^{-1}v^{2(1-q)}u_{xx} + K_4 t^{-1}v^{2-q}v_{xx} + k_6 t^{-1}v^{1-q}. \end{aligned}$$

$A^5_{4,8} = \langle \partial_v, \partial_x, t\partial_t + x\partial_v, qt\ln t\partial_t + x\partial_x + u\partial_u + (1+q)v\partial_v \rangle$:

$$\begin{aligned} u_t &= K_1 t^{-1}u^{2-q}u_{xx} + K_2 t^{-1}u^{2(1-q)}v_{xx} + K_5 t^{-1}u^{1-q}, \\ v_t &= K_3 t^{-1}u^2 u_{xx} + K_4 t^{-1}u^{2-q}v_{xx} + k_6 t^{-1}u. \end{aligned}$$

In general the arbitrary functions, $\mathcal{F}^1, \ldots, \mathcal{F}^4$, $\mathcal{G}^1$ and $\mathcal{G}^2$, of their respective arguments are different. Also the arbitrary constants, $K_1,\ K_2, \ldots, K_6$, are generally different. It should be noted that similar cases can be regarded as one case if a suitable choice of equivalence transformations exists for transforming one into another and when $p = q$.

4.5 Conclusion

In this Chapter we have performed a group classification of the general second-order system of diffusion equations with respect to solvable Lie algebras, that is, up to equivalence transformations, the realizations of Lie algebras of dimension up to and including four were determined and hence the specification of the corresponding functional forms for the arbitrary functions. In order to specify completely the arbitrary functions, a further algebraic study of realizations of dimension five and more is required. Otherwise symmetry analysis of individual cases for the four-dimensional realizations can be performed provided the manipulation of the resulting determining equations is manageable. Either one of these alone comprises another investigation which would be reported in future work. Moreover it would be a sort of complete analysis of group classification if the other types of Lie algebras are considered. These are semi simple Lie algebras and the algebras which are semi direct sum of semi simple algebras and solvable Lie algebras [5].

Chapter 5

Group Classification of Coupled Diffusion System with Applications in Soil Science

The theory of coupled diffusion appears in the works of Philip and De Vries [70], De Vries [18] in the late fifties and later by many authors, such as Jury *et al* [37]. As an illustration of how the results of the previous chapter can be utilized, we perform a complete group classification of a coupled system of diffusion equations with applications in soil science [4, 37, 75], i.e.,

$$\begin{aligned} u_t &= \left[f^1(u,v)u_x + f^2(u,v)v_x\right]_x, \\ v_t &= \left[f^3(u,v)u_x + f^4(u,v)v_x\right]_x, \end{aligned} \tag{5.1}$$

where $u(t,x)$ and $v(t,x)$ are the soil temperature and volumetric water content respectively. The variable x measures the depth of the soil and t is the time. The arbitrary functions $f^1, \ldots, f^4$ are the diffusion coefficients.

Wiltshire *et al* [4, 84] investigated the Lie symmetries of a simplified model of the coupled diffusion system (5.1) written in the form

$$y_t = \left[\Lambda(y)y_x\right]_x, \; y = \{y_i\} \quad (i = 1, 2, \ldots, n), \tag{5.2}$$

where $\Lambda(y)$ is a square matrix of diffusion coefficients. These investigations involve rewriting a system of coupled diffusion equations into a single equation in which the diffusion coefficients are written in a matrix form. Thus the resulting determining equations to be solved for symmetries are in terms of the systems of matrix equations. The matrix diffusion equation (5.2) is a generalized form of the extensively studied one-dimensional nonlinear heat conduction equation the group properties of which were first considered in [66]. Therefore the group classification procedure on Eq. (5.2) is analogous to the analysis performed in the cited reference and those that followed.

In this Chapter our goal is to derive the Lie symmetries of system (5.1) as it is and see whether a comparison with those of the simplified model (5.2) can be established. The non classical (potential) symmetries of the coupled system (5.1) were generated in [74]. As in the previous chapters the generation of the determining equations and the manipulation of them are with the aid of the `YaLie` software package [19].

5.1 Equivalence group and determining equations for classification

The equivalence group for system (5.1) was obtained in [75] using the infinitesimal approach. It can shown that the direct method yields the same result. However, instead of using these equivalence transformations we choose the equivalence group of the form

$$\bar{t} = T(t), \quad \bar{x} = X(x), \quad \bar{u} = U(u, v), \quad \bar{v} = V(u, v), \tag{5.3}$$

where the functions T, X, U and V satisfy the Jacobian

$$\frac{D(T, X, U, V)}{D(t, x, u, v)} \neq 0.$$

The execution of the classical Lie algorithm requires that the symmetry generator for the underlying system (5.1) take the form

$$\Gamma = a(t)\partial_t + b(t, x)\partial_x + c(t, x, u, v)\partial_u + d(t, x, u, v)\partial_v, \tag{5.4}$$

where $a, \ldots, d$ are the smooth differentiable functions satisfying the determining equations

$$
\begin{aligned}
c_{xx}f^1 + d_{xx}f^2 - c_t &= 0, && (5.5)\\
c_{xx}f^3 + d_{xx}f^4 - d_t &= 0, && (5.6)\\
c_x\left(f^1_v + f^2_u\right) + 2d_x f^2_v + 2c_{xv}f^1 - \left(b_{xx} - 2d_{xv}\right)f^2 &= 0, && (5.7)\\
\left(\dot{a} - 2b_x\right)f^1 + df^1_v + cf^1_u + d_u f^2 - c_v f^3 &= 0, && (5.8)\\
df^2_v + cf^2_u + c_v f^1 + \left(\dot{a} - 2b_x + d_v - c_u\right)f^2 - c_v f^4 &= 0, && (5.9)\\
\dot{a}f^2_v + df^2_{vv} + cf^2_{uv} + c_v\left(f^1_v + f^2_u - f^4_v\right) + 2d_v f^2_v + c_{vv}f^1 + d_{vv}f^2 & \\
-2b_x f^2_v - c_u f^2_v &= 0, && (5.10)\\
\dot{a}f^1_u + df^1_{uv} + cf^1_{uu} + c_u f^1_u + d_u\left(f^1_v + f^2_u\right) + c_{uu}f^1 + d_{uu}f^2 & \\
-2b_x f^1_u - c_v f^3_u &= 0, && (5.11)\\
\dot{a}\left(f^1_v + f^2_u\right) + c\left(f^1_{uv} + f^2_{uu}\right) + d\left(f^1_{vv} + f^2_{uv}\right) + d_v\left(f^1_v + f^2_u\right) + 2d_u f^2_v & \\
+2c_{uv}f^1 + 2d_{uv}f^2 - 2b_x\left(f^1_v + f^2_u\right) - c_v\left(2f^1_u + f^3_v + f^4_u\right) &= 0, && (5.12)\\
b_t - b_{xx}f^1 + 2c_x f^1_u + d_x\left(f^1_v + f^2_u\right) + 2c_{xu}f^1 + 2d_{xu}f^2 &= 0, && (5.13)\\
2c_x f^3_u + d_x\left(f^3_v + f^4_u\right) + 2d_{xu}f^4 - \left(b_{xx} - 2c_{xu}\right)f^3 &= 0, && (5.14)\\
df^3_v + cf^3_u + \left(\dot{a} - 2b_x - d_v + c_u\right)f^3 + d_u\left(f^4 - f^1\right) &= 0, && (5.15)\\
\left(\dot{a} - 2b_x\right)f^4 + df^4_v + cf^4_u + c_v f^3 - d_u f^2 &= 0, && (5.16)\\
\dot{a}f^4_v + df^4_{vv} + cf^4_{uv} + c_v\left(f^3_v + f^4_u\right) + d_v f^4_v + c_{vv}f^3 + d_{vv}f^4 & \\
-2b_x f^4_v - d_u f^2_v &= 0, && (5.17)\\
\dot{a}f^3_u + df^3_{uv} + cf^3_{uu} + 2c_u f^3_u + d_u\left(f^3_v + f^4_u - f^1_u\right) + c_{uu}f^3 + d_{uu}f^4 & \\
-2b_x f^3_u - d_v f^3_u &= 0, && (5.18)\\
\dot{a}\left(f^3_v + f^4_u\right) + c\left(f^3_{uv} + f^4_{uu}\right) + d\left(f^3_{vv} + f^4_{uv}\right) + 2f^3_u c_v + c_u\left(f^3_v + f^4_u\right) & \\
+2f^3 c_{uv} + 2f^4 d_{uv} - 2b_x\left(f^3_v + f^4_u\right) - d_u\left(f^1_v + f^2_u + 2f^4_v\right) &= 0, && (5.19)\\
b_t - b_{xx}f^4 + c_x\left(f^3_v + f^4_u\right) + 2d_x f^4_v + 2c_{xv}f^3 + 2d_{xv}f^4 &= 0. && (5.20)
\end{aligned}
$$

In the above equations and throughout this Chapter the overdot represents total differentiation with respect to t and the subscripts denote partial differentiation.

It is not easy to employ the group classification procedure discussed in [32, 67] to solve completely these determining equations. Therefore we opt for the approach used in the

previous chapter to perform a complete group classification up to the equivalence group (5.3).

5.2 Classification with respect to low-dimensional Lie algebras

It has been shown in Sections 4.3 and 4.4 of Chapter 4 how the classification procedure works. The search for realization of low dimension upto that of higher dimension is obtained by simplifying a given symmetry generator by making use of the equivalence transformations. Even though the equivalence group (5.3) for the system (5.1) under study is slightly different compared to the one in the previous chapter, the general symmetry generator (5.4) has the same form. Hence we proceed as in Sections 4.3 and 4.4 in order to carry out the complete group classification of the coupled diffusion system (5.1). We use the same notation for the realizations of the Lie algebras.

We begin with the one-dimensional Lie algebras. There are two inequivalent realizations of one-dimensional Lie algebras given by $A_1^1 = \langle \partial_t \rangle$ and $A_1^2 = \langle \partial_x \rangle$.

Next we have the five inequivalent realizations for the two-dimensional Lie algebra $A_{2,1}$, viz,

$$A_{2,1}^1 = \langle \partial_t, \partial_x \rangle, \quad A_{2,1}^2 = \langle \partial_t, \partial_u \rangle, \quad A_{2,1}^3 = \langle \partial_t, \partial_v \rangle, \quad A_{2,1}^4 = \langle \partial_x, \partial_u \rangle, \quad A_{2,1}^5 = \langle \partial_x, \partial_v \rangle.$$

The Lie algebras $A_{2,1}^2$, $A_{2,1}^4$ and $A_{2,1}^3$, $A_{2,1}^5$ imply that the arbitrary functions $f^1, \ldots, f^4$ are respectively independent of u and v.

For the two-dimensional Lie algebra, $A_{2,2}$, there are two inequivalent realizations, $A_{2,2}^3 = \langle -x\partial_x - u\partial_u, \partial_x \rangle$ and $A_{2,2}^4 = \langle -x\partial_x - v\partial_v, \partial_x \rangle$. Their corresponding functional forms for the arbitrary functions are given by

$A_{2,2}^3$: $f^1 = u^2 F(v), \quad f^2 = u^3 F(v), \quad f^3 = uF(v), \quad f^4 = u^2 F(v).$

$A_{2,2}^4$: $f^1 = v^2 F(u), \quad f^2 = vF(u), \quad f^3 = v^3 F(u), \quad f^4 = v^2 F(u).$

In general the arbitrary functions F of their respective arguments are different.

Now we turn to the three-dimensional solvable Lie algebras. Firstly we consider the decomposable Lie algebras and seek their corresponding realizations. There are two such realizations obtained from the realization of the two-dimensional Lie algebra $A^1_{2,1}$. They are $A^1_{3,1} = \langle \partial_t, \partial_x, \partial_u \rangle$ and $\tilde{A}^1_{3,1} = \langle \partial_t, \partial_x, \partial_v \rangle$. These realizations satisfy the determining equations (5.5)–(5.20) provided the diffusion coefficients are functions of v and u respectively.

There are also two realizations for the three-dimensional Lie algebra $A_{3,2}$. This Lie algebra contains the two-dimensional Lie algebras $A^3_{2,2}$ and $A^4_{2,2}$ from which we obtain $\tilde{A}^3_{3,2} = \langle -x\partial_x - u\partial_u, \partial_x, u\partial_u + v\partial_v \rangle$ and $\tilde{A}^4_{3,2} = \langle -x\partial_x - v\partial_v, \partial_x, u\partial_u + v\partial_v \rangle$ respectively. Therefore the respective forms of the diffusion coefficients are

$$\tilde{A}^3_{3,2}\colon\ f^1 = k_1\left(\frac{u}{v}\right)^2, \quad f^2 = k_2\left(\frac{u}{v}\right)^3, \quad f^3 = k_3\left(\frac{u}{v}\right), \quad f^4 = k_4\left(\frac{u}{v}\right)^2.$$

$$\tilde{A}^4_{3,2}\colon\ f^1 = \bar{k}_1\left(\frac{v}{u}\right)^2, \quad f^2 = \bar{k}_2\left(\frac{v}{u}\right), \quad f^3 = \bar{k}_3\left(\frac{v}{u}\right)^3, \quad f^4 = \bar{k}_4\left(\frac{v}{u}\right)^2.$$

The nonzero constants, $k_1, \ldots,\ k_4$ and $\bar{k}_1, \ldots,\ \bar{k}_4$, are arbitrary.

Secondly we look at the three-dimensional non decomposable solvable Lie algebras. The results are summarized in Table 5.1. The Lie algebras that do not feature in this table of results imply that there are no realizations for such Lie algebras which satisfy the determining equations (5.5)–(5.20).

5.3 Complete group classification

We proceed to the classification with respect to four-dimensional solvable Lie algebras, that is, we extend the realizations of the three-dimensional Lie algebras obtained in the previous section. After this classification the functional forms of the arbitrary functions are in terms of the arbitrary constants only (c.f. Table 5.1). However, there is a need to investigate whether or not the realizations of both the three- and four-dimensional Lie algebras give the maximal symmetry Lie algebra.

Table 5.1: 3D non decomposable solvable Lie algebras.

Algebra	f^1	f^2	f^3	f^4	Realization
$A^1_{3,5}$	$uF(v)$	$u^2F(v)$	$F(v)$	$uF(v)$	$\langle\partial_t,\partial_x,t\partial_t+x\partial_x+u\partial_u\rangle$
$\tilde{A}^1_{3,5}$	$vF(u)$	$F(u)$	$v^2F(u)$	$vF(u)$	$\langle\partial_t,\partial_x,t\partial_t+x\partial_x+v\partial_v\rangle$
$\tilde{A}^2_{3,5}$	K_1v^{-1}	K_2v^{-1}	K_3v^{-1}	K_4v^{-1}	$\langle\partial_t,\partial_u,t\partial_t+u\partial_u+v\partial_v\rangle$
$\tilde{A}^3_{3,5}$	K_1u^{-1}	K_2u^{-1}	K_3u^{-1}	K_4u^{-1}	$\langle\partial_t,\partial_v,t\partial_t+u\partial_u+v\partial_v\rangle$
$\tilde{A}^4_{3,5}$	K_1v^2	K_2v^2	K_3v^2	K_4v^2	$\langle\partial_x,\partial_u,x\partial_x+u\partial_u+v\partial_v\rangle$
$\tilde{A}^5_{3,5}$	K_1u^2	K_2u^2	K_3u^2	K_4u^2	$\langle\partial_x,\partial_v,x\partial_x+u\partial_u+v\partial_v\rangle$
$A^1_{3,6}$	$u^{-3}F(v)$	$u^{-2}F(v)$	$u^{-4}F(v)$	$u^{-3}F(v)$	$\langle\partial_t,\partial_x,t\partial_t-x\partial_x+u\partial_u\rangle$
$\tilde{A}^1_{3,6}$	$v^3F(u)$	$v^2F(u)$	$v^4F(u)$	$v^3F(u)$	$\langle\partial_t,\partial_x,t\partial_t-x\partial_x-v\partial_v\rangle$
$\tilde{A}^2_{3,6}$	K_1v^{-1}	K_2v^{-3}	K_3v	K_4v^{-1}	$\langle\partial_t,\partial_u,t\partial_t-u\partial_u+v\partial_v\rangle$
$\tilde{A}^3_{3,6}$	K_1u^{-1}	K_2u	K_3u^{-3}	K_4u^{-1}	$\langle\partial_t,\partial_v,t\partial_t+u\partial_u-v\partial_v\rangle$
$\tilde{A}^4_{3,6}$	K_1v^2	K_2	K_3v^4	K_4v^2	$\langle\partial_x,\partial_u,x\partial_x-u\partial_u+v\partial_v\rangle$
$\tilde{A}^5_{3,6}$	K_1u^2	K_2u^4	K_3	K_4u^2	$\langle\partial_x,\partial_v,x\partial_x+u\partial_u-v\partial_v\rangle$
$A^1_{3,7}$	$u^{2q-1}F(v)$	$u^{2q}F(v)$	$u^{2(q-1)}F(v)$	$u^{2q-1}F(v)$	$\langle\partial_t,\partial_x,t\partial_t+qx\partial_x+u\partial_u\rangle$
$\tilde{A}^1_{3,7}$	$v^{2q-1}F(u)$	$v^{2(q-2)}F(u)$	$v^{2q}F(u)$	$v^{2q-1}F(u)$	$\langle\partial_t,\partial_x,t\partial_t+qx\partial_x+v\partial_v\rangle$
$\hat{A}^1_{3,7}$	$u^{2-q}F(v)$	$u^{3-q}F(v)$	$u^{1-q}F(v)$	$u^{2-q}F(v)$	$\langle\partial_x,\partial_t,qt\partial_t+x\partial_x+u\partial_u\rangle$
$\bar{A}^1_{3,7}$	$v^{2-q}F(u)$	$v^{1-q}F(u)$	$v^{3-q}F(u)$	$v^{2-q}F(u)$	$\langle\partial_x,\partial_t,qt\partial_t+x\partial_x+v\partial_v\rangle$
$\tilde{A}^2_{3,7}$	K_1v^{-1}	K_2v^{q-2}	K_3v^{-q}	K_4v^{-1}	$\langle\partial_t,\partial_u,t\partial_t+qu\partial_u+v\partial_v\rangle$
$\bar{A}^2_{3,7}$	K_1v^{-q}	K_2v^{-q}	K_3v^{-q}	K_4v^{-q}	$\langle\partial_u,\partial_t,qt\partial_t+u\partial_u+v\partial_v\rangle$
$\tilde{A}^3_{3,7}$	K_1u^{-1}	K_2u^{-q}	K_3u^{q-2}	K_4u^{-1}	$\langle\partial_t,\partial_v,t\partial_t+u\partial_u+qv\partial_v\rangle$
$\bar{A}^3_{3,7}$	K_1u^{-q}	K_2u^{-q}	K_3u^{-q}	K_4u^{-q}	$\langle\partial_v,\partial_t,qt\partial_t+u\partial_u+v\partial_v\rangle$
$\tilde{A}^4_{3,7}$	K_1v^2	K_2v^{q+1}	K_3v^{3-q}	K_4v^2	$\langle\partial_x,\partial_u,x\partial_x+qu\partial_u+v\partial_v\rangle$
$\bar{A}^4_{3,7}$	K_1v^{2q}	K_2v^{2q}	K_3v^{2q}	K_4v^{2q}	$\langle\partial_u,\partial_x,qx\partial_x+u\partial_u+v\partial_v\rangle$
$\tilde{A}^5_{3,7}$	K_1u^2	K_2u^{3-q}	K_3u^{q+1}	K_4u^2	$\langle\partial_x,\partial_v,x\partial_x+u\partial_u+qv\partial_v\rangle$
$\bar{A}^4_{3,7}$	K_1u^{2q}	K_2u^{2q}	K_3u^{2q}	K_4u^{2q}	$\langle\partial_v,\partial_x,qx\partial_x+u\partial_u+v\partial_v\rangle$

The unknowns F are (in general different) arbitrary functions of their arguments and K_i are (generally different) arbitrary constants.

Firstly we consider the decomposable four-dimensional solvable Lie algebras. The extension of the realizations is possible for the four-dimensional Lie algebra $4A_1$ in the following cases:

$4A_1^1 = \langle \partial_t, \partial_x, \partial_u, v\partial_v \rangle$: $f^1 = \ell_1, \quad f^2 = \ell_2 v^{-1}, \quad f^3 = \ell_3 v, \quad f^4 = \ell_4.$

$4A_1^2 = \langle \partial_t, \partial_x, u\partial_u, \partial_v \rangle$: $f^1 = \bar{\ell}_1, \quad f^2 = \bar{\ell}_2 u, \quad f^3 = \bar{\ell}_3 u^{-1}, \quad f^4 = \bar{\ell}_4.$

The nonzero constants, $\ell_1, \ldots, \ell_4$ and $\bar{\ell}_1, \ldots, \bar{\ell}_4$, are arbitrary.

Now in the following table we present the realizations of four-dimensional non decomposable solvable Lie algebras which contain the three-dimensional realizations and their corresponding forms of the diffusion coefficients.

Table 5.2: 4D non decomposable solvable Lie algebras.

Algebra	f^1	f^2	f^3	f^4	Realization
$A_{4,5}^1$	$\lambda_1 v^{2q-1}$	$\lambda_2 v^{p+2q-2}$	$\lambda_3 v^{2q-p}$	$\lambda_4 v^{2q-1}$	$\langle \partial_t, \partial_x, \partial_u, t\partial_t + qx\partial_x + pu\partial_u + v\partial_v \rangle$
$\tilde{A}_{4,5}^1$	$\lambda_1 v^{2p-1}$	$\lambda_2 v^{2p+q-2}$	$\lambda_3 v^{2p-q}$	$\lambda_4 v^{2p-1}$	$\langle \partial_t, \partial_u, \partial_x, t\partial_t + px\partial_x + qu\partial_u + v\partial_v \rangle$
$\hat{A}_{4,5}^1$	$\lambda_1 v^{2-q}$	$\lambda_2 v^{p-q+1}$	$\lambda_3 v^{3-p-q}$	$\lambda_4 v^{2-q}$	$\langle \partial_x, \partial_t, \partial_u, qt\partial_t + x\partial_x + pu\partial_u + v\partial_v \rangle$
$\bar{A}_{4,5}^1$	$\lambda_1 v^{2-p}$	$\lambda_2 v^{1-p+q}$	$\lambda_3 v^{3-p-q}$	$\lambda_4 v^{2-p}$	$\langle \partial_x, \partial_u, \partial_t, pt\partial_t + x\partial_x + qu\partial_u + v\partial_v \rangle$
$\check{A}_{4,5}^1$	$\lambda_1 v^{2p-q}$	$\lambda_2 v^{2p-q}$	$\lambda_3 v^{2p-q}$	$\lambda_4 v^{2p-q}$	$\langle \partial_u, \partial_t, \partial_x, qt\partial_t + px\partial_x + u\partial_u + v\partial_v \rangle$
$\grave{A}_{4,5}^1$	$\lambda_1 v^{2q-p}$	$\lambda_2 v^{2q-p}$	$\lambda_3 v^{2q-p}$	$\lambda_4 v^{2q-p}$	$\langle \partial_u, \partial_x, \partial_t, pt\partial_t + qx\partial_x + u\partial_u + v\partial_v \rangle$
$A_{4,5}^2$	$\lambda_1 u^{2q-1}$	$\lambda_2 u^{2q-p}$	$\lambda_3 u^{p+2q-2}$	$\lambda_4 u^{2q-1}$	$\langle \partial_t, \partial_x, \partial_v, t\partial_t + qx\partial_x + u\partial_u + pv\partial_v \rangle$
$\tilde{A}_{4,5}^2$	$\lambda_1 u^{2p-1}$	$\lambda_2 u^{2p-q}$	$\lambda_3 u^{2p+q-2}$	$\lambda_4 u^{2p-1}$	$\langle \partial_t, \partial_v, \partial_x, t\partial_t + px\partial_x + u\partial_u + qv\partial_v \rangle$
$\hat{A}_{4,5}^2$	$\lambda_1 u^{2-q}$	$\lambda_2 u^{3-p-q}$	$\lambda_3 u^{p-q+1}$	$\lambda_4 u^{2-q}$	$\langle \partial_x, \partial_t, \partial_v, qt\partial_t + x\partial_x + u\partial_u + pv\partial_v \rangle$
$\bar{A}_{4,5}^2$	$\lambda_1 u^{2-p}$	$\lambda_2 u^{3-p-q}$	$\lambda_3 u^{1-p+q}$	$\lambda_4 u^{2-p}$	$\langle \partial_x, \partial_v, \partial_t, pt\partial_t + x\partial_x + u\partial_u + qv\partial_v \rangle$
$\check{A}_{4,5}^2$	$\lambda_1 u^{2p-q}$	$\lambda_2 u^{2p-q}$	$\lambda_3 u^{2p-q}$	$\lambda_4 u^{2p-q}$	$\langle \partial_v, \partial_t, \partial_x, qt\partial_t + px\partial_x + u\partial_u + v\partial_v \rangle$
$\grave{A}_{4,5}^2$	$\lambda_1 u^{2q-p}$	$\lambda_2 u^{2q-p}$	$\lambda_3 u^{2q-p}$	$\lambda_4 u^{2q-p}$	$\langle \partial_v, \partial_x, \partial_t, pt\partial_t + qx\partial_x + u\partial_u + v\partial_v \rangle$

The λ_i are arbitrary constants which are in general different.

5.4 Further analysis

In order to investigate the maximal symmetry Lie algebra, we perform symmetry analysis of all the cases in Tables 5.1 and 5.2 for which the arbitrary functions are completely specified. The cases arising from the decomposable solvable Lie algebras are also taken into account.

We show the detailed calculations of one case for both the classification with respect to three- and four-dimensional Lie algebras and present the results only for the rest.

Firstly we consider the three-dimensional realization $\tilde{A}^2_{3,7} = \langle \partial_t, \partial_u, t\partial_t + qu\partial_u + v\partial_v \rangle$ from Table 5.1. We proceed by substituting the corresponding forms of the arbitrary functions into the system (5.1) under consideration. Lie's algorithm yields the symmetry generator (5.4) the coefficients of which are of the form $a = a(t)$, $b = b(t,x)$, $c = e(t,x)u + f(t,x)$, $d = d(t,x,v)$ for arbitrary functions $e(t,x)$ and $f(t,x)$. The smooth differentiable functions, a, b, d, e and f, satisfy the determining equations

$$
\begin{aligned}
K_1 v\left(u e_x + f_x\right) + v^q K_2\left[v b_{xx} - 2\left((q-2)d_x + v d_{xv}\right)\right] &= 0, && (5.21)\\
K_1\left(u e_{xx} + f_{xx}\right) + K_2 v^{q-1} d_{xx} - v\left(u e_t + f_t\right) &= 0, && (5.22)\\
v\left(\dot{a} - 2b_x + d_v\right) - 2d &= 0, && (5.23)\\
d - v\dot{a} + 2v b_x &= 0, && (5.24)\\
(q-2)d + v\left(-e + \dot{a} - 2b_x + d_v\right) &= 0, && (5.25)\\
v\left[(q-2)\dot{a} - (q-2)e + 2(2-q)b_x + 2(q-2)d_v + v d_{vv}\right] & \\
+ \left(6 - 5q + q^2\right) d &= 0, && (5.26)\\
K_1\left(v b_{xx} + d_x - 2v e_x\right) - v^2 b_t &= 0, && (5.27)\\
v b_{xx} + q d_x - 2v e_x &= 0, && (5.28)\\
v^{-(q+1)}\left[K_3 v\left(u e_{xx} + f_{xx}\right) + K_4 v^q d_{xx}\right] - d_t &= 0, && (5.29)\\
v\left(\dot{a} - 2b_x + d_v - v d_{vv}\right) - 2d &= 0, && (5.30)\\
(1+q)d - v\left(e + \dot{a} - 2b_x\right) &= 0, && (5.31)\\
v\left(e + \dot{a} - 2b_x - d_v\right) - q d &= 0, && (5.32)\\
K_3 q v^{1-q}\left(u e_x + f_x\right) + K_4\left(v b_{xx} + 2d_x - 2v d_{xv}\right) - v^2 b_t &= 0. && (5.33)
\end{aligned}
$$

From Eqs. (5.23)–(5.26) and (5.30)–(5.32) we obtain

$$d = (\dot{a} - 2b_x)v, \quad e = q(\dot{a} - 2b_x). \tag{5.34}$$

In view of (5.34) the remaining determining equations simplify to

$$\begin{aligned}
K_1 f_x - (2quK_1 + 3v^q K_2 - 4qv^q K_2)\, b_{xx} &= 0, && (5.35)\\
quv\ddot{a} - K_1 f_{xx} + 2quK_1 b_{xxx} + 2v^q K_2 b_{xxx} + v f_t - 2quv b_{tx} &= 0, && (5.36)\\
K_1(1-4q)b_{xx} + v b_t &= 0, && (5.37)\\
K_3(1+2q)v^{-q} b_{xx} &= 0, && (5.38)\\
v\ddot{a} + 2v^{-q}\,(quK_3 + v^q K_4)\, b_{xxx} - v^{-q} K_3 f_{xx} - 2v b_{tx} &= 0, && (5.39)\\
qK_3 f_x + \left(-2q^2 uK_3 + v^q K_4\right) b_{xx} - v^{1+q} b_t &= 0. && (5.40)
\end{aligned}$$

The solution of Eqs. (5.35)–(5.40) is given by

$$a = C_3 t + C_4, \quad b = C_1 x + C_2, \quad c = q(C_3 - 2C_1)u + C_5, \quad d = (C_3 - 2C_1)v, \tag{5.41}$$

where $C_1, \ldots, C_5$ are the arbitrary constants of integration.

Thus the symmetry Lie algebra for this case is spanned by the operators

$$\Gamma_1 = \partial_t,\ \Gamma_2 = \partial_x,\ \Gamma_3 = \partial_u,\ \Gamma_4 = t\partial_t + qu\partial_u + v\partial_v,\ \Gamma_5 = x\partial_x - 2qu\partial_u - 2v\partial_v. \tag{5.42}$$

The full symmetry Lie algebra for the rest of the cases that arise from symmetry analysis are presented below.

$\tilde{A}^2_{3,5}$: $\Gamma_1 = \partial_t$, $\Gamma_2 = \partial_x$, $\Gamma_3 = \partial_u$, $\Gamma_4 = t\partial_t + u\partial_u + v\partial_v$, $\Gamma_5 = x\partial_x - 2u\partial_u - 2v\partial_v$.

$\tilde{A}^3_{3,5}$: $\Gamma_1 = \partial_t$, $\Gamma_2 = \partial_x$, $\Gamma_3 = \partial_v$, $\Gamma_4 = t\partial_t + u\partial_u + v\partial_v$, $\Gamma_5 = x\partial_x - 2u\partial_u - 2v\partial_v$.

$\tilde{A}^4_{3,5}$: $\Gamma_1 = \partial_t$, $\Gamma_2 = \partial_x$, $\Gamma_3 = \partial_u$, $\Gamma_4 = 2t\partial_t - u\partial_u - v\partial_v$, $\Gamma_5 = x\partial_x + u\partial_u + v\partial_v$.

$\tilde{A}^5_{3,5}$: $\Gamma_1 = \partial_t$, $\Gamma_2 = \partial_x$, $\Gamma_3 = \partial_v$, $\Gamma_4 = 2t\partial_t - u\partial_u - v\partial_v$, $\Gamma_5 = x\partial_x + u\partial_u + v\partial_v$.

$\tilde{A}^2_{3,6}$: $\Gamma_1 = \partial_t$, $\Gamma_2 = \partial_x$, $\Gamma_3 = \partial_u$, $\Gamma_4 = t\partial_t - u\partial_u + v\partial_v$, $\Gamma_5 = x\partial_x + 2u\partial_u - 2v\partial_v$.

$\tilde{A}^3_{3,6}$: $\Gamma_1 = \partial_t$, $\Gamma_2 = \partial_x$, $\Gamma_3 = \partial_v$, $\Gamma_4 = t\partial_t + u\partial_u - v\partial_v$, $\Gamma_5 = x\partial_x - 2u\partial_u + 2v\partial_v$.

$\tilde{A}^4_{3,6}$: $\Gamma_1 = \partial_t$, $\Gamma_2 = \partial_x$, $\Gamma_3 = \partial_u$, $\Gamma_4 = 2t\partial_t + u\partial_u - v\partial_v$, $\Gamma_5 = x\partial_x - u\partial_u + v\partial_v$.

$\tilde{A}^5_{3,6}$: $\Gamma_1 = \partial_t$, $\Gamma_2 = \partial_x$, $\Gamma_3 = \partial_v$, $\Gamma_4 = 2t\partial_t - u\partial_u + v\partial_v$, $\Gamma_5 = x\partial_x + u\partial_u - v\partial_v$.

$\bar{A}^2_{3,7}$: $\Gamma_1 = \partial_t$, $\Gamma_2 = \partial_x$, $\Gamma_3 = \partial_u$, $\Gamma_4 = qt\partial_t + u\partial_u + v\partial_v$, $\Gamma_5 = qx\partial_x - 2u\partial_u - 2v\partial_v$.

$\tilde{A}^3_{3,7}$: $\Gamma_1 = \partial_t$, $\Gamma_2 = \partial_x$, $\Gamma_3 = \partial_v$, $\Gamma_4 = t\partial_t + u\partial_u + qv\partial_v$, $\Gamma_5 = x\partial_x - 2u\partial_u - 2qv\partial_v$.

$\bar{A}^3_{3,7}$: $\Gamma_1 = \partial_t$, $\Gamma_2 = \partial_x$, $\Gamma_3 = \partial_v$, $\Gamma_4 = qt\partial_t + u\partial_u + v\partial_v$, $\Gamma_5 = qx\partial_x - 2u\partial_u - 2v\partial_v$.

$\tilde{A}^4_{3,7}$: $\Gamma_1 = \partial_t$, $\Gamma_2 = \partial_x$, $\Gamma_3 = \partial_u$, $\Gamma_4 = 2t\partial_t - qu\partial_u - v\partial_v$, $\Gamma_5 = x\partial_x + qu\partial_u + v\partial_v$.

$\bar{A}^4_{3,7}$: $\Gamma_1 = \partial_t$, $\Gamma_2 = \partial_x$, $\Gamma_3 = \partial_u$, $\Gamma_4 = 2qt\partial_t - u\partial_u - v\partial_v$, $\Gamma_5 = qx\partial_x + u\partial_u + v\partial_v$.

$\tilde{A}^5_{3,7}$: $\Gamma_1 = \partial_t$, $\Gamma_2 = \partial_x$, $\Gamma_3 = \partial_v$, $\Gamma_4 = 2t\partial_t - u\partial_u - qv\partial_v$, $\Gamma_5 = x\partial_x + u\partial_u + qv\partial_v$.

$\tilde{A}^5_{3,7}$: $\Gamma_1 = \partial_t$, $\Gamma_2 = \partial_x$, $\Gamma_3 = \partial_v$, $\Gamma_4 = 2qt\partial_t - u\partial_u - v\partial_v$, $\Gamma_5 = qx\partial_x + u\partial_u + v\partial_v$.

For every case above the symmetry Lie algebra is spanned by the three operators from the three-dimensional realization and two additional operators that arise from symmetry analysis. When we look at the pairs of cases $\tilde{A}^2_{3,5}$ and $\tilde{A}^3_{3,5}$, $\tilde{A}^4_{3,5}$ and $\tilde{A}^5_{3,5}$, ..., $\tilde{A}^4_{3,6}$ and $\tilde{A}^5_{3,6}$ respectively, it should be pointed out that each pair can be regarded as one case provided a suitable choice of equivalence transformations exists for transforming one into another.

Next we show the details of symmetry analysis for the case of four-dimensional decomposable Lie algebra $4A_1$ with the realization: $4A^1_1 = \langle \partial_t, \partial_x, \partial_u, v\partial_v \rangle$.

We proceed as in the previous symmetry analysis with the substitution of the corresponding forms of the arbitrary functions into the underlying system (5.1). Following the usual Lie's algorithm we obtain the symmetry generator (5.4) the coefficients of which $a = a(t)$, $b = b(t,x)$, $c = c(t,x,u,v)$, $d = d(t,x,u,v)$ satisfy the determining equations

$$\ell_1 v c_{uu} + \ell_2 d_{uu} = 0, \tag{5.43}$$

$$2\ell_1 v^2 c_{uv} + \ell_2 \left(2v d_{uv} - 2d_u\right) - \ell_3 v^2 c_v = 0, \tag{5.44}$$

$$\ell_2 \left(v b_{xx} + 2d_x - 2v d_{xv}\right) - 2\ell_1 v^2 c_{xv} = 0, \tag{5.45}$$

$$\ell_1 v c_{xx} + \ell_2 d_{xx} - c_t v = 0, \tag{5.46}$$

$$\ell_1 \left(\dot{a} v - 2b_x v\right) + \ell_2 d_u - \ell_3 v^2 c_v = 0, \tag{5.47}$$

$$\ell_1 v^3 c_{vv} + \ell_2 \left(2d + 2b_x v + v^2 d_{vv} + c_u v - v\dot{a} - 2v d_v\right) = 0, \tag{5.48}$$

$$\ell_4 v^2 c_v + \ell_2 \left(d + 2b_x v + c_u v - \dot{a} v - v d_v\right) - \ell_1 v^2 c_v = 0, \tag{5.49}$$

$$b_t v + \ell_1 \left(2c_{xu} v - b_{xx} v\right) + 2\ell_2 d_{xu} = 0, \tag{5.50}$$

$$\ell_3 \left(d_u + v c_{uu}\right) + \ell_4 d_{uu} = 0, \tag{5.51}$$

$$\ell_2 d_u + \ell_3 \left(v^2 c_v + v^3 c_{vv}\right) + \ell_4 v^2 d_{vv} = 0, \tag{5.52}$$

$$\ell_3 \left(d_x + 2v c_{xu} - v b_{xx}\right) + 2\ell_4 d_{xu} = 0, \tag{5.53}$$

$$\ell_3 c_{xx} v + \ell_4 d_{xx} - d_t = 0, \tag{5.54}$$

$$\ell_3 \left(\dot{a} + c_u + 2v c_{uv} - 2b_x\right) + 2\ell_4 d_{uv} = 0, \tag{5.55}$$

$$\ell_4 \left(\dot{a} v - 2b_x v\right) + \ell_3 v^2 c_v - \ell_2 d_u = 0, \tag{5.56}$$

$$\ell_4 d_u + \ell_3 \left(d + \dot{a} v + c_u v - 2b_x v - v d_v\right) - \ell_1 d_u = 0, \tag{5.57}$$

$$b_t + \ell_3 \left(c_x + 2v c_{xv}\right) + \ell_4 \left(2d_{xv} - b_{xx}\right) = 0. \tag{5.58}$$

Since we require that $\ell_i \neq 0$, from Eqs. (5.43) and (5.47) we obtain

$$b = \frac{1}{2}\dot{a} x + \bar{e}(t), \quad c = \bar{f}(t,x) u + \bar{g}(t,x), \quad d = d(t,x,v), \tag{5.59}$$

where $\bar{e}(t)$, $\bar{f}(t,x)$, $\bar{g}(t,x)$ are the arbitrary functions. Taking into account Eq. (5.59), the above determining equations reduce to

$$v d_{xv} - d_x = 0, \tag{5.60}$$

$$\ell_1 v \left(u \bar{f}_{xx} + \bar{g}_{xx}\right) + \ell_2 d_{xx} - v \left(u \bar{f}_t + \bar{g}_t\right) = 0, \tag{5.61}$$

$$2d + v \left(\bar{f} - 2d_v + v d_{vv}\right) = 0, \tag{5.62}$$

$$d + v \left(\bar{f} - d_v\right) = 0, \tag{5.63}$$

$$2\bar{e}_t + x\ddot{a} + 4\ell_1 \bar{f}_x = 0, \tag{5.64}$$

$$\ell_4 d_{vv} = 0, \tag{5.65}$$

$$2v \bar{f}_x + d_x = 0, \tag{5.66}$$

$$\ell_3 v \left(u \bar{f}_{xx} + \bar{g}_{xx}\right) + \ell_4 d_{xx} - d_t = 0, \tag{5.67}$$

$$\ell_3 \bar{f} = 0, \tag{5.68}$$

$$2\bar{e}_t + x\ddot{a} + 2\ell_3 \left(u \bar{f}_x + \bar{g}_x\right) + 4\ell_4 d_{xv} = 0. \tag{5.69}$$

Eventually the solution of the determining equations (5.60)–(5.69) is given by

$$a = 2\bar{C}_1 t + \bar{C}_2, \quad b = \bar{C}_1 x + \bar{C}_3, \quad c = \bar{C}_4, \quad d = \bar{C}_5 v, \tag{5.70}$$

where $\bar{C}_1, \ldots, \bar{C}_5$ are arbitrary constants of integration.

Therefore for this case the symmetry Lie algebra is spanned by the operators

$\Gamma_1 = \partial_t$, $\Gamma_2 = \partial_x$, $\Gamma_3 = \partial_u$, $\Gamma_4 = v\partial_v$, $\Gamma_5 = 2t\partial_t + x\partial_x$.

The symmetry Lie algebra for the other case of the four-dimensional decomposable Lie algebra $4A_1^2$ is spanned by

$\Gamma_1 = \partial_t$, $\Gamma_2 = \partial_x$, $\Gamma_3 = u\partial_u$, $\Gamma_4 = \partial_v$, $\Gamma_5 = 2t\partial_t + x\partial_x$.

The maximal symmetry Lie algebras for every case in Table 5.2 of the four-dimensional non decomposable Lie algebras are summarized as follows:

$A^1_{4,5} \cup \langle\, (2q-1)t\partial_t - pu\partial_u - v\partial_v,\ (2q-1)x\partial_x + 2pu\partial_u + 2v\partial_v \rangle$.

$\tilde{A}^1_{4,5} \cup \langle\, (2p-1)t\partial_t - qu\partial_u - v\partial_v,\ (2p-1)x\partial_x + 2qu\partial_u + 2v\partial_v \rangle$.

$\hat{A}^1_{4,5} \cup \langle\, (q-2)t\partial_t + pu\partial_u + v\partial_v,\ (q-2)x\partial_x - 2pu\partial_u - 2v\partial_v \rangle$.

$\bar{A}^1_{4,5} \cup \langle\, (p-2)t\partial_t + qu\partial_u + v\partial_v,\ (p-2)x\partial_x - 2qu\partial_u - 2v\partial_v \rangle$.

$\check{A}^1_{4,5} \cup \langle\, (2p-q)t\partial_t - u\partial_u - v\partial_v,\ (2p-q)x\partial_x + 2u\partial_u + 2v\partial_v \rangle$.

$\grave{A}^1_{4,5} \cup \langle\, (p-2q)t\partial_t + u\partial_u + v\partial_v,\ (p-2q)x\partial_x - 2u\partial_u - 2v\partial_v \rangle$.

$A^2_{4,5} \cup \langle\, (2q-1)t\partial_t - u\partial_u - pv\partial_v,\ (2q-1)x\partial_x + 2u\partial_u + 2pv\partial_v \rangle$.

$\tilde{A}^2_{4,5} \cup \langle\, (2p-1)t\partial_t - u\partial_u - qv\partial_v,\ (2p-1)x\partial_x + 2u\partial_u + 2qv\partial_v \rangle$.

$\hat{A}^2_{4,5} \cup \langle\, (q-2)t\partial_t + u\partial_u + pv\partial_v,\ (q-2)x\partial_x - 2u\partial_u - 2pv\partial_v \rangle$.

$\bar{A}^2_{4,5} \cup \langle\, (p-2)t\partial_t + u\partial_u + qv\partial_v,\ (p-2)x\partial_x - 2u\partial_u - 2qv\partial_v \rangle$.

$\check{A}^2_{4,5} \cup \langle\, (2p-q)t\partial_t - u\partial_u - v\partial_v,\ (2p-q)x\partial_x + 2u\partial_u + 2v\partial_v \rangle$.

$\grave{A}^2_{4,5} \cup \langle\, (p-2q)t\partial_t + u\partial_u + v\partial_v,\ (p-2q)x\partial_x - 2u\partial_u - 2v\partial_v \rangle$.

In all the above cases the symmetry Lie algebra is six-dimensional, four operators from the classification with respect to four-dimensional Lie algebras and the additional two operators obtained via symmetry analysis. Moreover it can be seen that if $p = q$, then a pair of similar cases is reduced to one case.

5.5 Conclusion

We have managed to specify completely the functional forms of the diffusion coefficients through the classification with respect to the solvable Lie algebras of dimension up to four. In order to determine the maximal symmetry Lie algebra, we performed the symmetry analysis of individual cases for the realizations of the three- and four-dimensional Lie algebras in which the functional forms of the unknown functions are in terms of the arbitrary constants. In short the full classification has been achieved. The comparison with the previous studies of the model considered in [4, 84] can hardly be established. It should be noted that group classification using the other types of Lie algebras has not been considered in this work. The next step will be to find both the analytical and numerical solutions of the submodels.

CONCLUSIONS

This piece of work is another contribution in the area of group classification of systems of partial differential equations. Traditionally the procedure for solving a group classification problem involves the construction of the equivalence group and then the classification is performed up to equivalence group. However, in some instances the calculation of the equivalence transformations may not be necessary as the direct approach of Lie is easy to implement.

Two models of great importance in real-life applications were considered for group classification. The model of flow in collapsible channel and the coupled diffusion model. In carrying out the group classification, the two approaches were used which combined the standard Lie technique with the approach which exploits the classification of real Lie algebras. The first approach used in Chapters 2 and 3 required no use of the equivalence transformations. The classification is performed by the utilization of the subalgebras of 3D and 4D Lie algebras. In Chapters 4 and 5 the second approach was used which deals with the use of the equivalence transformations to simplify the forms of the symmetry operators. In both procedures the Lie bracket of the low-dimensional Lie algebras and the Lie algebras of higher dimension is invoked to find the operators which satisfy the given model. This in turn provides the specification of the functional forms of the arbitrary elements that appear in the model under study.

A group classification problem is said to be completely solved provided the forms of the unknown functions are in terms of the arbitrary constants and the symmetry Lie algebra is maximal. The arbitrary elements may be completely specified, but this does not give a guarantee that the symmetry Lie algebra is maximal. The guarantee for the full symmetry Lie algebra is achieved via the symmetry analysis for individual cases that arise from the classification with respect to the Lie algebras of low dimensions upto those of higher dimensions. The group analysis of individual cases may not be necessary provided one is determined to handle many tedious though easy calculations especially when dealing with the Lie algebras of dimensions greater than four.

Chapters 2 and 4 respectively dealt with 'a group classification' of the coupled systems of partial differential equations with applications in the models of flow in collapsible channels and diffusion phenomena. In Chapter 2 the unknown functions were completely solved via the classification with respect to the subalgebras of 4D Lie algebras whereas for the subalgebras of 3D Lie algebras they remain to be fully specified. In Chapter 4 there was no complete specification of the arbitrary functions for the classification with respect to the 3D solvable Lie algebras. There were as many fully solved unknown functions as those not completely solved for the classification with respect to the 4D solvable Lie algebras. As complement towards the complete symmetry classification, the special cases of the models investigated in these chapters were considered in Chapters 3 and 5 respectively. In Chapter 3 we performed the full symmetry classification of the collapsible tube model which incorporates the tension and curvature of the tube wall. All the cases that arise from the classification with respect to the subalgebras of 3D and 4D Lie algebras were analyzed for the maximal Lie symmetry algebra. We obtained the cases with 3, 4 and 5 symmetries. The optimal system of the subalgebras was used to construct the invariant solutions for the cases which can be of practical importance. These invariant solutions were in turn used to obtain the reduced submodels. In Chapter 5 we achieved the complete group classification of the coupled system of diffusion equations which modelled the diffusion of soil temperature and water content in dry areas. We obtained the cases which have 5 and 6 symmetries.

Finally we hint on further investigations. In Chapter 2 the complete symmetry classification will be reached by performance of group analysis of all the cases obtained from the classification with respect to subalgebras of 3D and 4D Lie algebras. For a given practical situation it will be of vital importance to investigate numerically the realistic submodels in Chapter 3. In a move to obtain the complete group classification in Chapter 4 it may be easier to perform symmetry analysis of all the cases in which the diffusion coefficients are completely solved and proceed to the classification with respect to 5D solvable Lie algebras for the other cases. The next logical step in Chapter 5 will be to identify the realistic submodels and construct the solutions thereof. In Chapters 4 and 5 it is only the solvable Lie algebras which were considered for classification. The investigation that utilizes the other types of Lie algebras is subject of future work.

Appendix

A Brief Introduction to YaLie

We give a brief explanation accompanied by an illustrative example on how to use the program `YaLie`. This Mathematica package was developed by Díaz [19]. We used version 2.0 for our computations. There is no software package that is the best, it depends upon the user being familiar with the particular package. However, the better software package should have the following advantages: fast in executing the commands, totally interactive, easy to install and maintain. Below we go through step by step installation and utilization of `YaLie`. For an advanced but concise guide to `YaLie`, consult [19].

Installation

In the sequel we assume that the reader is familiar with basic Mathematica syntax. Copy `YaLie.m` file into your working directory. In order to start a `YaLie` session, open a new Mathematica notebook and type in the line

`Get["c:\YaLie.m"]` (assuming that your working directory is c:).

Compiling this single line yields

```
====================================
                YaLie
          Version 2.0 (2000)
          @ JM Díaz
     Departamento de Matemáticas
       Universidad de Cádiz
====================================
```

If you obtain the above output after compilation, you have properly installed `YaLie`. Now you are ready to use `YaLie`.

Using YaLie

Here we illustrate with an example how the YaLie program works. Consider the example of heat equation $u_t - u_{xx} = 0$. Open a Mathematica notebook in your working directory and type in the following lines:

```
Get["c:\YaLie.m"]

Heat = D[u[t,x],t]- D[u[t,x],{x,2}] == 0;

VD = {u};

VIs = {t,x};

VP = {D[u[t,x],t]};

EDs = YaLie[{Heat},VD,VIs,VP]
```

Before proceeding further we explain the above command lines. The first line loads the `YaLie` package. The second line defines the equation of interest. The third line specifies the dependent variable. The fourth line defines the independent variables. The fifth line tells us which derivative to substitute into the determining equations. The sixth command line generates the determining equations. Compiling the above six command lines (step[1]) yields the determining equations

$$\begin{aligned}
\texttt{Xi[1]}^{(0,1,0)}[\texttt{t},\texttt{x},\texttt{u}] &= 0,\\
\texttt{Xi[1]}^{(0,0,1)}[\texttt{t},\texttt{x},\texttt{u}] &= 0,\\
\texttt{Xi[2]}^{(0,0,1)}[\texttt{t},\texttt{x},\texttt{u}] &= 0,\\
\texttt{Phi[1]}^{(0,0,2)}[\texttt{t},\texttt{x},\texttt{u}] &= 0,\\
\texttt{Phi[1]}^{(\texttt{t},0,0)}[\texttt{t},\texttt{x},\texttt{u}] - \texttt{Phi[1]}^{(0,2,0)}[\texttt{t},\texttt{x},\texttt{u}] &= 0,\\
2\texttt{Xi[2]}^{(0,1,0)}[\texttt{t},\texttt{x},\texttt{u}] - \texttt{Xi[1]}^{(1,0,0)}[\texttt{t},\texttt{x},\texttt{u}] &= 0,\\
\texttt{Xi[2]}^{(0,2,0)}[\texttt{t},\texttt{x},\texttt{u}] - \texttt{Xi[1]}^{(1,0,0)}[\texttt{t},\texttt{x},\texttt{u}] - 2\texttt{Phi[1]}^{(0,1,1)}[\texttt{t},\texttt{x},\texttt{u}] &= 0.
\end{aligned}$$

In the above determining equations the coordinates of the symmetry generator corresponding to the independent variables `t` and `x` are `Xi[1][t,x,u]` and `Xi[2][t,x,u]` respectively and the coordinate corresponding to the dependent variable `u` is `Phi[1][t,x,u]`.

Adding the command lines `DFormat[True]` and `EDs` leads to the determining equations being displayed in traditional notation. i.e.,

$$\begin{aligned}
\texttt{Xi}[1]_{\texttt{x}} &= 0,\\
\texttt{Xi}[1]_{\texttt{u}} &= 0,\\
\texttt{Xi}[2]_{\texttt{u}} &= 0,\\
\texttt{Phi}[1]_{\texttt{uu}} &= 0,\\
\texttt{Phi}[1]_{\texttt{t}} - \texttt{Phi}[1]_{\texttt{xx}} &= 0,\\
2\texttt{Xi}[2]_{\texttt{x}} - \texttt{Xi}[1]_{\texttt{t}} &= 0,\\
\texttt{Xi}[2]_{\texttt{xx}} - \texttt{Xi}[2]_{\texttt{t}} - 2\texttt{Phi}[1]_{\texttt{xu}} &= 0.
\end{aligned}$$

From the determining equations above, we find that

$$\begin{aligned}
\xi^1 &= \xi^1(t),\\
\xi^2 &= a(t)x + b(t),\\
\phi^1 &= c(t,x)u + d(t,x).
\end{aligned}$$

We can now type this information on our notebook. This is step[2]:

$$\begin{aligned}
\texttt{LXi} &= \{\texttt{Xi[1][t]}, \texttt{a[t]x} + \texttt{b[t]}\};\\
\texttt{LPhi} &= \{\texttt{c[t,x]u} + \texttt{d[t,x]}\};\\
\texttt{EDs} &= \texttt{Simplify[SInfinitesimales[EDs, LXi, LPhi]]}
\end{aligned}$$

The first command line is a list of ξ^i while the second command line is a list of ϕ^i. The number of ξ^i and ϕ^i to be listed should correspond to the number of independent variables and dependent variables respectively. The last command line simplifies the determining equations using the new forms of ξ^i and ϕ^i deduced from the output of step[1]. The equations which are satisfied are replaced by the value `True`. After executing step[2] we get the following equations

$$\begin{aligned}
\texttt{uc}_{\texttt{xx}} + \texttt{d}_{\texttt{xx}} &= \texttt{uc}_{\texttt{t}} + \texttt{d}_{\texttt{t}},\\
2\texttt{a[t]} &= \texttt{Xi}[1]_{\texttt{t}},\\
\texttt{xa}_{\texttt{t}} + \texttt{b}_{\texttt{t}} + 2\texttt{c}_{\texttt{x}} &= 0.
\end{aligned}$$

The first four equations in the output of step[1] are satisfied. Solving the last equation we obtain

$$c(t,x) = -\frac{1}{4}a_t x^2 - \frac{1}{2}b_t x + e(t).$$

With this form of $c(t,x)$ we update `LPhi` whereas `LXi` remains unchanged. Thus

$$\begin{aligned}
\texttt{LXi} &= \{\texttt{Xi[1][t]}, \texttt{a[t]x} + \texttt{b[t]}\};\\
\texttt{LPhi} &= \{(-\texttt{x}^2\texttt{D[a[t],t]}/4 - \texttt{xD[b[t],t]}/2 + \texttt{e[t]})\texttt{u} + \texttt{d[t,x]}\};\\
\texttt{EDs} &= \texttt{Simplify[SInfinitesimales[EDs,LXi,LPhi]]}
\end{aligned}$$

We must compile step[1] again before compiling this step. This is supposed to be done in the subsequent steps also, i.e., before executing every updated step we must firstly generate the determining equations. Therefore step[1] becomes step[3] and the updated step becomes step[4]. After executing step[4] the remaining determining equations are

$$\begin{aligned}
\texttt{2ua}_\texttt{t} + \texttt{4ue}_\texttt{t} + \texttt{4d}_\texttt{t} &= \texttt{ux}^2\texttt{a}_\texttt{tt} + \texttt{2uxb}_\texttt{tt} + \texttt{4d}_\texttt{xx},\\
\texttt{2a[t]} &= \texttt{Xi[1]}_\texttt{t}.
\end{aligned}$$

From the above equations we have

$$a(t) = k_1 t + k_2, \quad b(t) = k_3 t + k_4, \quad e(t) = -\frac{1}{2}k_1 t + k_6.$$

Thus,

$$c(t,x) = -\frac{1}{4}k_1 x^2 - \frac{1}{2}k_3 x - \frac{1}{2}k_1 t + k_6.$$

We update both `LXi` and `LPhi`, generate the determining equations and execute step[6],

$$\begin{aligned}
\texttt{LXi} &= \{\texttt{k[1]t}^2 + \texttt{2k[2]t} + \texttt{k[5]}, (\texttt{k[1]t} + \texttt{k[2]})\texttt{x} + \texttt{k[3]t} + \texttt{k[4]}\};\\
\texttt{LPhi} &= \{(-\texttt{x}^2\texttt{k[1]}/4 - \texttt{xk[3]}/2 - \texttt{k[1]t}/2 + \texttt{k[6]})\texttt{u} + \texttt{d[t,x]}\};\\
\texttt{EDs} &= \texttt{Simplify[SInfinitesimales[EDs,LXi,LPhi]]}
\end{aligned}$$

to get the output

$$\texttt{d}_\texttt{xx} = \texttt{d}_\texttt{t}.$$

Therefore we deduce that

$$\begin{aligned}
\xi^1 &= k_1 t^2 + 2k_2 t + k_5,\\
\xi^2 &= k_1 t x + k_2 x + k_3 t + k_4,\\
\phi^1 &= \Big(k_6 - \frac{1}{4}k_1 x^2 - \frac{1}{2}k_3 x - \frac{1}{2}k_1 t\Big)u + d(t,x),
\end{aligned}$$

where the k_i are arbitrary constants and $d(t, x)$ is an arbitrary solution of the heat equation. It is easy to show with identification of constants that this is indeed the forms of ξ and η for symmetries given in the illustrative example of the heat equation treated in Example 1.4.5.

YaLie Calculations

We present the details of some calculations which were done interactively using `YaLie`. They are the calculations which appear on pages 52 & 54 for Case I and pages 57–58 for Case II. The rest of the calculations of Chapter 2 together with those of the other chapters in which `YaLie` was used for simplifying purpose are summarized in the relevant tables. It should be noted that `YaLie` is used only to simplify the determining equations for the given coordinates of the symmetry generator, that is, the necessary calculations of the coefficients of symmetry generator have to be done first.

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