A Study of Financial Models and their Symmetry Driven Analytical Solutions

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Declaration

I declare that the contents of this thesis are original, except where due references have been made. It is submitted for the degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It was not submitted before for any degree or examination to any other institution.

R. Maphanga

Signed at Johannesburg on the 8^{th} day of July 2024.

Abstract

The theory of financial models play a crucial role in understanding and predicting the behaviour of various financial instruments. In this thesis, we explore the application of Lie symmetries and boundary conditions in four prominent financial models: the Black-Scholes, a generalized bond-pricing, a CEV type, and an option-pricing model. These models revolutionized the field of mathematical finance by introducing a framework for valuing options or bonds. We investigate the Lie symmetries underlying these equations and explore their implications in financial mathematics. By employing Lie symmetries, we are able to identify invariant solutions, leading to a deeper understanding of the dynamics and behaviour of the equations. Furthermore, the thesis delves into the role of boundary conditions in financial models. Boundary conditions play a vital role in defining the behaviour of financial instruments, and their accurate specification is essential for obtaining meaningful results. We analyze the impact of different boundary or terminal conditions on option and bond pricing models. By examining the effects of boundary conditions, we enhance our understanding of the limitations and nuances of these models in different financial scenarios. Bond pricing models are vital in the valuation and risk management of fixed-income securities and their investigation provides insights into the behaviour of bond prices and yields. By uncovering the underlying symmetries and understanding the implications of boundary conditions, we aim to enhance the accuracy and predictive power of bond and option pricing models.

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Dedication

In profound gratitude and with immense love, I dedicate this doctoral thesis to the constellation of stars in my life—my extraordinary family, whose unwavering support and love have been the driving force behind my academic journey. Each member has played a unique and vital role, contributing to the tapestry of my academic journey.

To my late mother, whose indomitable spirit and boundless love continue to inspire me every day. Your brilliance continues to illuminate my path. Your sacrifices and encouragement have shaped my character and instilled in me a deep sense of resilience and determination. Though you are no longer physically present, your memory lives on in every page of this thesis. You called me "Dokotela" even before I became one. That proclamation is within reach moms, continue to RIP.

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This work is a testament to the collective influence of these remarkable individuals in my life, and I am profoundly thankful for the role each has played in shaping my academic and personal growth. May this thesis be a star map, reflecting the brilliance of my family's influence. Each chapter is a constellation, formed by the luminosity of their love, guidance, and unwavering support. As I navigate the academic cosmos, I am profoundly grateful for the gravitational pull of my family, keeping me grounded and propelling me toward new intellectual horizons.

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List of Publications

Details of contributions to publications that form part of the research presented in this thesis are:

- A Terminal Condition in Linear Bond-Pricing under Symmetry Invariance, published in the Journal of Nonlinear Mathematical Physics, 2023 [35].
- A progressive approach to solving a generalized CEV type model via symmetry invariant surface conditions, published in the AIMS Mathematics, 2024 [44].
- An approach to solve parabolic financial equations by point symmetries that incorporate boundary conditions, submitted, 2024.

List of Abbreviations

PDEs	Partial Differential Equations
CEV	Constant Elasticity of Variance
SDEs	Stochastic Differential Equations
GBM	Geometric Brownian Motion
ISC	Invariant Surface Condition

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Chapter 1

Introduction

Preliminaries

The theory of option pricing has a long history. It dates back to the late 20^{th} century, accredited to the French Mathematician Louis Jean-Baptiste Alphonse Bachelier, the first person to model the stochastic process [1]. Bachelier is also known as a pioneer of the mathematics of finance, as his work formed or laid a base to the most widely used mathematical models [2]. He was working on his PhD thesis when he first had this phenomenal discovery - he deduced an option pricing formula based on the assumption that stock prices follow a Brownian motion with zero drift [2]. Since then, numerous mathematicians and researchers have contributed to his theory [2].

His work was and still is very influential in the development of other widely used mathematical models, including the Black-Scholes model, attributed to economists Fischer Black and Myron Scholes (1973) [2]. The Black-Scholes model is commonly

defined as a pricing model used to determine the fair price or theoretical value for a call or a put option based on six variables such as volatility, type of option, underlying stock price, time, strike price, and risk-free rate [2, 3]. Below is the 3-D representation of the Black-Scholes model, showing the relationship between time to expiration, spot price, and the option value [4].

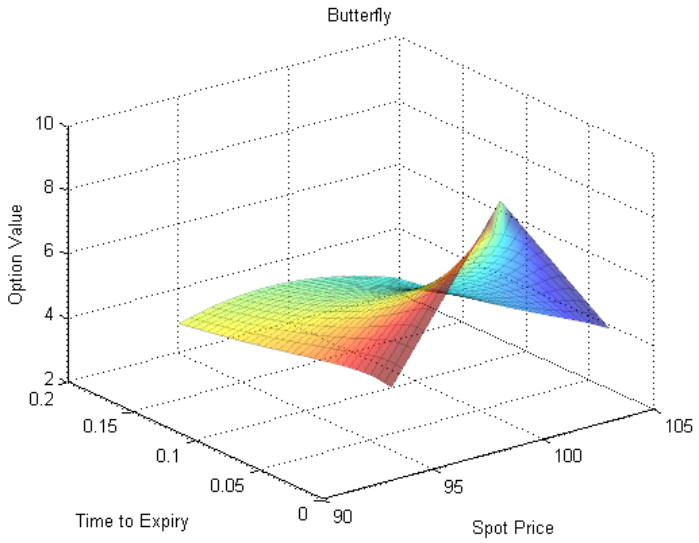

Figure 1.1: The Black-Scholes 3-D model

Mathematically, the Black-Scholes model can be described as a second-order PDE, which originates from SDEs [5]:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{1.1}$$

where S_t is the stochastic process which follows the GBM, W_t is the Wiener process, and μ is the percentage drift and σ is the percentage volatility and are constants.

A more generalised definition of the Black-Scholes PDE follows.

Definition 1.0.1 *The Black-Scholes equation is a PDE that governs the price evolution of a European call or European put under the Black-Scholes model. It is commonly given by:*

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{1.2}$$

where V is the price of the option as a function of stock price S and time t, r is the risk-free interest rate, and σ is the volatility of the stock, with the exception that the European call or put on an underlying stock pays no dividends [2, 5, 6].

The **European call** is the class into which an option gives the owner the right to acquire the stock or bond at expiry [2, 3].

The **European put** is the class into which an option allows the holder the right to sell the stock or bond at expiry [2, 3].

The Black-Scholes PDE (1.2) is derived using the SDE and Ito's Lemma [2, 6]. It is imperative to know and understand this derivation as it is of interest. Consider the option value $V(S,t)$ at maturity, and assume that the Black-Scholes model follows the general geometric Brownian motion [2], that is,

$$dS = \mu S dt + \sigma S dW, \tag{1.3}$$

with discrete form:

$$\delta S = \mu S \delta t + \sigma S \delta W. \tag{1.4}$$

Employing Talyor series on this option value results in the following series expansion for $V(S,t)$ [2]:

$$\delta V = V_s \delta S + V_t \delta t + \frac{1}{2!} V_{ss} \delta S^2 + \frac{1}{2!} 2V_{st} \delta S \delta t + \frac{1}{2!} V_{tt} \delta t^2 + \cdots \tag{1.5}$$

Substituting equation (1.4) into equation (1.5) and cancelling all the insignificant terms we get:

$$\delta V \approx V_s(\mu S \delta t + \sigma S \delta W) + V_t \delta t + \frac{1}{2!} V_{ss}(\mu S \delta t + \sigma S \delta W)^2. \tag{1.6}$$

Taking the limits $\delta S \to 0$, $\delta W^2 \to \delta t$ as $\delta t \to 0$ gives (1.6) in the following form:

$$dV = V_s(\mu S dt + \sigma S dW) + V_t dt + \frac{1}{2!} V_{ss}(\mu S dt + \sigma S dW)^2. \tag{1.7}$$

From equation (1.3), we have

$$(dS)^2 = (\mu S dt + \sigma S dW)^2 = \mu^2 S^2 dt^2 + 2\sigma\mu S^2 dW dt + \sigma^2 S^2 dW^2. \tag{1.8}$$

Applying Ito's Lemma to equation (1.8) and assuming that $dW^2 \to dt$ as $dt \to 0$, we get

$$dS^2 \to \sigma^2 S^2 dt. \tag{1.9}$$

Remark 1.0.1 *Itô's lemma is an identity used in Itô calculus to find the differential of a time-dependent function of a stochastic process. It is mathematically defined as: Let W_t, $t \in \mathbb{R}^+$ be an Itô process $W : \omega \cdot \mathbb{R}^+ \to \mathbb{R}$ and $f = C^2(\mathbb{R} \cdot \mathbb{R}^+ \cdot \mathbb{R}^+)$. Then the stochastic process $f_t = f(W_t, t)$ is also an Itô which satisfies [2],*

$$df_t = \left(\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 f}{\partial t^2}\right)dt + \frac{\partial f}{\partial x}dW_t. \tag{1.10}$$

Substituting (1.9) into (1.7) results in

$$dV = \frac{\partial V}{\partial S}(\mu S dt + \sigma S dW) + \frac{\partial V}{\partial t}dt + \frac{1}{2!}\frac{\partial^2 V}{\partial S^2}(\sigma^2 S^2 dt). \tag{1.11}$$

Therefore, by Itô's lemma for two variables we have

$$dV = \left(\mu S\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt + \sigma S\frac{\partial V}{\partial S}dW. \tag{1.12}$$

Now consider a certain portfolio consisting of one option with value $V(S,t)$ and the number $\hat{\Delta}$ of the underlying assets [2]. The value of these holdings will be:

$$\Gamma = V - \hat{\Delta}S, \tag{1.13}$$

and the change in portfolio will be

$$d\Gamma = dV - \hat{\Delta}dS. \tag{1.14}$$

Now, substituting equations (1.3) and (1.12) into (1.14) we get

$$d\Gamma = \left(\mu S\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - \mu\hat{\Delta}S\right)dt + \sigma S\left(\frac{\partial V}{\partial S} - \hat{\Delta}\right)dW. \tag{1.15}$$

We then choose

$$\hat{\Delta} = \frac{\partial V}{\partial S}, \tag{1.16}$$

to eliminate the main contribution of randomness.

We note that $\hat{\Delta}$ is chosen such that the portfolio (1.13) will be deterministic. The rate of return on this portfolio must be equal to the rate of return on any other risk-less instrument; otherwise, there would be opportunities for arbitrage [2, 6]. Now, assuming the risk-free rate of return is r, we must have (after simplifying)

$$\mathrm{d}\Gamma = r\Gamma dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt, \tag{1.17}$$

over the time period $[t, t + \hat{\Delta}t]$[2]. Finally, substituting the value of Γ from (1.14) into (1.17) and dividing by dt throughout we obtain the Black-Scholes equation (1.2) [2]. The rest of the models considered here have a similar derivation, but rely on many transformations to get to its particular form, and hence the details are omitted.

For a long time, the Black-Scholes equation was commonly known for it's flexibility in governing the price of the option over time [2]. However, recent progressive studies has proven that the Black-Scholes equation has many more functions or uses due to its simplicity and clarity to obtain the price option call [2, 5]. This second-order PDE holds for any type of option as long as its price function V is twice differentiable with respect to S and once with respect to t. Different pricing formulae for various options will arise from the choice of payoff function at expiry and appropriate boundary conditions [6].

In this thesis, we examine other several financial models, which we find as indispensable tools in quantitative finance, enabling the valuation and risk assessment of various financial instruments. Over the years, researchers have sought to develop efficient and accurate analytical solutions for these models to better understand their

underlying dynamics. This thesis provides an approach, on how to get these analytical solutions, focusing on Lie symmetries, terminal conditions and convolution techniques.

Lie symmetries, named after the mathematician Sophus Lie, form the basis of this study. By exploring the symmetries present in the model or equation understudy, we can derive analytical solutions that remain invariant under certain transformations. These symmetry driven analytical solutions then provide deeper insights into the pricing and risk management of options, illuminating their sensitivities and dynamics.

Boundary conditions on the other hand play a crucial role in financial models, ensuring that the solutions remain valid within specific domains. In this regard, we investigate the impact of a particular boundary condition on the model or the PDE understudy. By imposing appropriate boundary conditions, we can obtain meaningful results that align with market dynamics and constraints. Understanding the effect of boundary conditions on option pricing enhances the accuracy of valuation and aids in decision-making.

Linear bond pricing models are vital for the valuation and risk assessment of fixed-income securities. Terminal conditions, which represent the value of a bond at its maturity, are of utmost importance in these models. We examine symmetry invariance for suitable terminal conditions of linear bond pricing models. By utilizing symmetry driven approaches, we aim to derive new symmetries inherent in the underlying mathematical equations, leading to more accurate bond pricing and risk assessment.

The Cauchy problem, which involves determining a solution based on an initial condition, arises in various financial instruments. We explore the Cauchy problem of

financial instruments and its associated Lie symmetries, which enable the derivation of analytical solutions. To this end, we make use of convolution integrals which are also powerful tools. We investigate the application of convolution and symmetry invariance in financial modelling. By employing convolution, we can solve complex equations efficiently and derive analytical solutions that exhibit symmetry invariance. This approach enhances the accuracy of our solutions.

Through a combination of theoretical analysis, mathematical derivations, and numerical simulations, this thesis aims to contribute to the field of quantitative finance by exploring analytical solutions in financial models. By investigating Lie symmetries, terminal conditions, the Cauchy problem, and convolutions, we seek to enhance the accuracy, efficiency, and interpretability of financial modeling. The insights gained from this research have the potential to improve pricing accuracy, risk management strategies, and decision-making processes in quantitative finance, benefiting both academia and industry practitioners.

1.1 Outline of Chapters

In Chapter 2, we highlight the basic notation used throughout this thesis, and outline our methods.

In Chapter 3, we investigate a general bond-pricing model in terms of its solutions that meet a specified terminal condition. We start by deriving reversible transformations that convert the model into the one-dimensional heat equation. Furthermore, we demonstrate that the terminal condition undergoes a transformation into a non-smooth initial condition. A significant outcome is revealed wherein Lie symmetries are employed to address the equation within the framework of its distinctive set of

initial conditions.

Chapter 3 has been published in [35].

In Chapter 4, we analyze a CEV model in relation to its terminal condition. We establish that specific transformations can be employed to simplify the equation, revealing known solution methods. Here, four distinct cases emerge, each contributing to the determination of numerous unknown parameters within the model. The associated terminal condition is converted into an initial condition, and we illustrate the resolution of this Cauchy problem using Lie symmetries and convolution. Ultimately, we study the behaviour of the solutions obtained.

Chapter 4 has been published in [44].

In Chapter 5, we demonstrate the application of transformation techniques and convolution integrals to solve a generalized option-pricing model, encompassing the Black-Scholes model, with a specified terminal condition. We also notice that the convolutions stemming from the initial-value problem with Lie symmetries and an invariant surface condition are intricate. The key finding from this investigation is that point symmetries effectively generate solutions meeting the prescribed terminal condition.

Chapter 5 has been submitted for publication.

Chapter 2

Notation and Theory

Consider a PDE for $u(x,t)$, whose domain is $(x,t) \in \mathbb{R}^2$ [7]. It often happens that a transformation of variables gives a new solution to the equation [7]. In general, by a symmetry of such a differential equation we mean a transformation which takes solutions to solutions [8]. A general approach to solving PDEs uses the symmetry property of differential equations, that is, the continuous limiting form of small transformation of solutions to solutions [7, 8]. For instance, if $\mu(x,t)$ is a solution to some differential equation, say, the heat equation $u_t = u_{xx}$, it is easy to show that $\mu(x - x_0, t - t_0)$ is also a solution [7]. In this regard, symmetry methods are very useful as they allow the construction of new solutions from the ones already known.

Given a function $u(x,t)$, the independent and dependent variables can be transformed by a mapping $(x,t,u) \to (\hat{x},\hat{t},\hat{u})$ [7].

We define point symmetries of an n-th order PDE [9] as

$$G(x,t,u,u_x,u_t,\dots) = 0 \tag{2.1}$$

The point transformation Γ is a point symmetry of (2.1) if

$$G(\hat{x}, \hat{t}, \hat{u}, \hat{x}_{\hat{x}}, \hat{u}_{\hat{t}}, \dots) = 0 \tag{2.2}$$

when (2.1) holds [9]. In most cases, the symmetry condition (2.1) is solvable by automated software. However, it is quite simple to check if a given point transformation is a symmetry of a PDE or not [9].

Just as for Lie point transformations of the plane, each one-parameter (local) Lie group of point transformations is obtained by exponentiating its infinitesimal generator [9], which is

$$X = \xi \partial_x + \tau \partial_t + \eta \partial_u. \tag{2.3}$$

Equivalently, we can obtain $(\hat{x}, \hat{t}, \hat{u})$ by solving

$$\begin{aligned} \frac{d\hat{x}}{d\varepsilon} &= \xi(\hat{x}, \hat{t}, \hat{u}), \\ \frac{d\hat{t}}{d\varepsilon} &= \tau(\hat{x}, \hat{t}, \hat{u}), \\ \frac{d\hat{u}}{d\varepsilon} &= \eta(\hat{x}, \hat{t}, \hat{u}) \end{aligned} \tag{2.4}$$

subject to the initial conditions

$$(\hat{x}, \hat{t}, \hat{u})|_{\varepsilon=0} = (x, t, u).$$

Lie point symmetries are obtained by differentiating the symmetry condition (2.2) with respect to ε at $\varepsilon = 0$ [9]. That is to say, we may obtain the linearized symmetry condition

$$XG = 0 \tag{2.5}$$

when $G = 0$.

We have mentioned that the concept of Lie symmetries originated from the work of the Norwegian mathematician Sophus Lie. This occurred in the late 19th century

and the Lie symmetries refer to transformations that leave a given mathematical object or system invariant. These transformations form a group, known as a Lie group, which possesses certain algebraic and differential properties. Lie groups provide a powerful framework for studying continuous symmetries in particular, and have found applications in a wide range of fields, including differential equations, differential geometry, quantum mechanics, and more.

One of the key insights derived from the theory of Lie symmetries is the concept of symmetry invariance. Symmetry invariance refers to the property that certain quantities or equations retain their form under transformations generated by Lie symmetries. Invariance under these transformations provides deep insights into the structure and behaviour of systems and allows for the derivation of analytical solutions.

In mathematics, the study of invariance has had profound implications in the analysis of differential equations. By identifying the symmetries inherent in a differential equation, one can reduce the complexity of the problem and derive solutions that exhibit the same symmetries. This approach, known as symmetry analysis or Lie symmetry analysis, has been successfully applied to a wide range of differential equations, including ODEs, PDEs and integral equations.

If a group transformation maps a solution into itself, we arrive at what is called a self-similar or group invariant solution [10]. The search for this type of solution reduces the number of independent variables of the equation in question. Namely, the invariance with respect to the one-parameter group reduces the number the variables by one. Further reduction can be achieved by considering an invariance under symmetry groups with two or more parameters. For example, the construction of these particular solutions is reduced, in the case of the one-dimensional Black-Scholes model, either to ordinary differential equations if the solution is invariant

under a one-parameter group, or to an algebraic relation if the solution is invariant with respect to a multi-parameter group [1, 10]. The construction of invariant solutions under one-parameter groups is widely known in the literature.

The operator in Eq. (2.3) defines a Lagrange system or characteristic equation

$$\frac{dx}{\xi} = \frac{du}{\eta},$$

whose solution provides the zero-order invariants

$$W^{[0]}(x, u), \tag{2.6}$$

that may be used to transform or reduce a given equation.

In physics, the application of Lie symmetries has been instrumental in understanding the fundamental laws governing the universe. Noether's theorem, formulated by mathematician Emmy Noether, establishes a deep connection between symmetries and conservation laws in physical systems. According to this theorem, for every continuous symmetry of a physical system, there is a corresponding conserved quantity. This profound result has played a pivotal role in the development of modern physics, from classical mechanics to quantum field theory.

Furthermore, in the realm of quantitative finance, Lie symmetries have emerged as powerful tools for pricing and risk management. The application of symmetry approaches in option pricing models, bond pricing models, and risk assessment has provided valuable insights into the behaviour and sensitivities of financial instruments. By exploiting the symmetries inherent in these models, one can derive analytical solutions, improve pricing accuracy, and gain a deeper understanding of market dynamics.

As we see, Lie symmetries are central concepts in mathematics, physics, and quantitative finance. Their applications span a wide range of disciplines and by exploring

these concepts in this thesis, we aim to uncover new insights, derive analytical solutions, and enhance our understanding of complex systems. The insights gained from this research have the potential to advance various fields of study and provide practical implications for real-world applications.

2.1 Group Classification of Differential Equations

Mathematical models often contain unspecified parameters and/or arbitrary functions of specific variables. Usually, these elements (parameters or functions) are either deduced experimentally or selected based on simplicity criteria. Lie group theory offers a systematic approach for identifying arbitrary elements from a symmetry point of view. This investigation area is now recognized as the Lie group classification of differential equations. The Lie group classification of differential equations furnishes a theoretical foundation for what can be termed as group-theoretic modeling. In this framework, equations exhibiting greater symmetries are considered more desirable, often leading to equations with notable physical characteristics. When dealing with a set of differential equations, the process of Lie group classification begins by identifying the primary Lie group of this set or principal Lie algebra. This primary group is applicable to any equation within the specified family [10].

2.2 Transformation to the Heat Equation

Let us recall Lie's result of group classification of linear second-order PDEs with two independent variables. In the case of evolutionary parabolic equations this result is

formulate as follows. Consider the family of linear parabolic equations:

$$P(t,x)u_t + Q(t,x)u_x + R(t,x)u_{xx} + S(t,x)u = 0, \quad P \neq 0, \quad R \neq 0. \tag{2.7}$$

The principal Lie algebra is spanned by the generators:

$$\begin{aligned} X_1 &= u\frac{\partial}{\partial u}, \\ X_\phi &= \phi(t,x)\frac{\partial}{\partial u}, \end{aligned} \tag{2.8}$$

of trivial symmetries. Any equation (2.7) can be reduced to the form

$$v_\tau = v_{yy} + Z(\tau, y)v \tag{2.9}$$

by a transformation - Lie's equivalence transformation:

$$\begin{aligned} y &= \alpha(t,x), \\ \tau &= \beta(t), \\ v &= \gamma(t,x)u, \\ & \alpha_x \neq 0, \quad \beta_t \neq 0, \end{aligned} \tag{2.10}$$

obtained with the help of two quadratures.

If equation (2.7) admits an extension of the principal Lie algebra by one additional symmetry operator then it is reduced to the form:

$$v_\tau = v_{yy} + Z(y)v \tag{2.11}$$

for which the additional operator is

$$X = \frac{\partial}{\partial \tau} \tag{2.12}$$

Likewise, if the Lie algebra extends by three additional operators, equation (2.7) is reduced to the form:

$$v_\tau = v_{yy} + \frac{A}{y^2}v \tag{2.13}$$

the three additional operators being:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial \tau}, \\ X_2 &= 2\tau\frac{\partial}{\partial \tau} + y\frac{\partial}{\partial y}, \\ X_3 &= \tau^2\frac{\partial}{\partial \tau} + \tau y\frac{\partial}{\partial y} - \left(\frac{1}{4}y^2 + \frac{1}{2}\tau\right)v\frac{\partial}{\partial v} \end{aligned} \tag{2.14}$$

Lastly, if the Lie algebra extends by five additional operators, equation (2.7) is reduced to the heat equation:

$$v_\tau = v_{yy}, \tag{2.15}$$

with the five additional operators being:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial y}, \\ X_2 &= \frac{\partial}{\partial \tau}, \\ X_3 &= 2\tau\frac{\partial}{\partial y} - yv\frac{\partial}{\partial v}, \\ X_4 &= 2\tau\frac{\partial}{\partial \tau} + y\frac{\partial}{\partial y}, \\ X_5 &= \tau^2\frac{\partial}{\partial \tau} + \tau y - \left(\frac{1}{4}y^2 + \frac{1}{2}\tau\right)v\frac{\partial}{\partial v}. \end{aligned} \tag{2.16}$$

Equations (2.11) to (2.15) provide the canonical forms of all linear parabolic second-order equations (2.7) that admit non-trivial symmetries, that is, extensions of the principal Lie algebra. We apply the above theory in determining when it is possible to find transformations to the heat equation.

2.3 Heat Equation: Fundamental Solution

We study the heat equation in the form

$$u_t - \Delta u = 0 \tag{2.17}$$

subject to appropriate initial and boundary conditions [11]. Here, $t > 0$ and $x \in U$ where $U \subset \mathbb{R}^n$ is open. The unknown is u, defined as

$$u : \hat{U} \times [0, \infty) \to \mathbb{R}, \tag{2.18}$$

where $u = (x, t)$, and the Laplacian $\hat{\Delta}$ is taken with respect to the spatial variables

$$x = (x_1, \ldots, x_n) : \Delta u = \Delta_x u = \sum_{i=1}^{n} u_{x_i x_i} \tag{2.19}$$

2.3.1 Derivation of the fundamental solution

An important first step in studying any PDE is often to come up with some specific solutions. We observe that the heat equation involves one derivative with respect time variable t, but two derivatives with respect to the space variables x_i $(i = 1, \ldots, n)$. Consequently we see that if u solves the heat equation (2.17), then so does $u(\lambda x, \lambda^2 t)$ for $\lambda \in \mathbb{R}$. This scaling indicates the ratio $\frac{r^2}{t}$ $(r = |x|)$ is important for the heat equation and suggests that we search for a solution of the heat equation (2.17) having the form

$$u(x, t) = v\left(\frac{r^2}{t}\right) = v\left(\frac{|x|^2}{t}\right) \quad (t > 0, x \in \mathbb{R}^2),$$

for some function v as yet undetermined.

Although this approach eventually leads to what we want, it is quicker to seek a solution u having a special structure.

$$u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right) \quad (x \in \mathbb{R}^n, t > 0), \tag{2.20}$$

where the constants α, β and the function $v : \mathbb{R}^n \to \mathbb{R}$ must be found. We come to equation (2.20) if we look for a solution u of the heat equation invariant under the dilation scaling

$$u(x,t) \mapsto \lambda^\alpha u(\lambda^\beta x, \lambda t). \tag{2.21}$$

Setting $\lambda = t^{-1}$, we derive equation (2.20) for $v(y) := u(y,1)$. Let us insert equation (2.20) into equation (2.17) and compute

$$\alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} y \cdot Dv(y) + t^{-(\alpha+2\beta)} \Delta v(y) = 0 \tag{2.22}$$

for $y := t^{-\beta} x$. In order to transform (2.22) into an expression involving the variable y alone, we take $\beta = \frac{1}{2}$. Then the terms with t are identical, therefore equation (2.22) reduces to

$$\alpha v + \frac{1}{2} y \cdot Dv + \Delta v = 0. \tag{2.23}$$

We simply further by guessing v to be radial: that is, $v(y) = \omega(|y|)$ for some $\omega : \mathbb{R} \to \mathbb{R}$. Thereupon, equation (2.23) becomes

$$\alpha\omega + \frac{1}{2} r\omega' + \omega'' + \frac{n-1}{r}\omega' = 0, \tag{2.24}$$

for $r = |y|$, $' = \frac{d}{dr}$. Now if we set $\alpha = \frac{n}{2}$, this simplifies to read as follows:

$$(r^{n-1}\omega')' + \frac{1}{2}(r^n \omega)' = 0. \tag{2.25}$$

Thus

$$r^{n-1}\omega' + \frac{1}{2} r^n \omega = \alpha, \tag{2.26}$$

for some constant α. Assuming $\lim_{r\to\infty}\omega,\ \omega' = 0$, we conclude $\alpha = 0$; whence

$$\omega' = -\frac{1}{2}r\omega. \tag{2.27}$$

It follows then for some constant b, we have

$$\omega = be^{-\frac{r^2}{2}}. \tag{2.28}$$

Combining equations (2.20), (2.28) and our choices for α, β, we conclude that $\frac{b}{t^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}}$ solves the heat equation (2.17). This computation motivates the following definition

Definition 2.3.1 *The function*

$$\Phi(x,t) := \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}}, & (x\in\mathbb{R}^n, t>0), \\ 0, & (x\in\mathbb{R}^n, t<0) \end{cases} \tag{2.29}$$

is called the fundamental solution of the heat equation [11].

Remark 2.3.1 *Note that Φ is singular at the point $(0,0)$. We sometimes write $\Phi(x,t) = \Phi(|x|,t)$ to emphasize that the fundamental solution is radial in the variable x [11].*

The choice of normalizing the constant $(4\pi)^{-\frac{n}{2}}$ is dictated by the following lemma.

Lemma 2.3.1 *(Integral of fundamental solution). For each $t>0$ [11],*

$$\int_{\mathbb{R}^n}\Phi(x,t)\,dx = 1. \tag{2.30}$$

Proof 2.3.1 *We calculate*

$$\begin{aligned}\int_{\mathbb{R}^n} \Phi(x,t)\,dx &= \frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}}\,dx \qquad (2.31)\\ &= \frac{1}{(\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-|z|}\,dz\\ &= \frac{1}{(\pi)^{\frac{n}{2}}}\prod_{i=1}^{n}\int_{-\infty}^{\infty} e^{-z_i^2}\,dz_i\\ &= 1\end{aligned}$$

2.3.2 The Cauchy Problem

We now employ Φ to fashion a solution to the initial-value (or Cauchy) problem [11]

$$\begin{cases} u_t - \Delta u = 0, & \text{in} \quad \mathbb{R}^n \times (0,\infty),\\ u = g, & \text{on} \quad \mathbb{R}^n \times \{t=0\}. \end{cases} \qquad (2.32)$$

Remark 2.3.2 *We note that the function $(x,t) \mapsto \Phi(x,t)$ solves the heat equation away from the singularity at $(0,0)$, and thus so does $(x,t) \mapsto \Phi(x-y,t)$ for each fixed $y \in \mathbb{R}^n$ [11].*

Consequently the convolution

$$\begin{aligned}u(x,t) &= \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)\,dy \qquad (2.33)\\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy \quad (x \in \mathbb{R}^n, t>0)\end{aligned}$$

should also be a solution.

Theorem 2.3.2 *Solution of the initial-value problem [11]: Assume* $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, *and define* u *by (2.33). Then*

(a) $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$,

(b) $u_t(x,t) - \Delta u(x,t) = 0 \quad (x \in \mathbb{R}^n, t > 0)$

(c) $\lim_{(x,t)\mapsto(x^0,0)} u(x,t) = g(x^0)$ *for each point* $x^0 \in \mathbb{R}^n$, $\quad (x \in \mathbb{R}^n, t > 0)$

where C^∞ *and* L^∞ *represent spaces of functions defined on* $\mathbb{R}^n$ *with certain properties.* $C^\infty(\mathbb{R}^n)$ *denotes the space of smooth functions on* $\mathbb{R}^n$. $L^\infty(\mathbb{R}^n)$ *denotes the space of essentially bounded functions on* $\mathbb{R}^n$.

Proof 2.3.3 *1. Since the function* $\frac{1}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$ *is infinitely differentiable, with uniformly bounded derivatives of all orders, on* $\mathbb{R}^n \times [\sigma,\infty)$ *for each* $\sigma > 0$, *we see that* $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$. *Furthermore,*

$$u_t(x,t) - \Delta u(x,t) = \int_{\mathbb{R}^n} [(\Phi_t - \Delta_x \Phi)(x-y,t)] g(y)\, dy \tag{2.34}$$

$$= 0 \quad (x \in \mathbb{R}^n, t > 0)$$

since Φ *itself solves the heat equation.*

2. Fix $x^0 \in \mathbb{R}^n$, $\epsilon > 0$. *Choose* $\sigma > 0$ *such that*

$$|g(y) - g(x^0)| < \epsilon \quad \text{if} \quad |y - x^0| < \sigma, \quad y \in \mathbb{R}^n. \tag{2.35}$$

Then if $|x - x^0| < \frac{\sigma}{2}$, *we have, according to lemma (2.3.1),*

$$\begin{aligned} |u(x,t) - g(x^0)| &= |\int_{\mathbb{R}^n} \Phi(x-y,t)[g(y) - g(x^0)]\, dy| \qquad (2.36) \\ &\leq \int_{B(x^0,\sigma)} \Phi(x-y,t)|g(y) - g(x^0)|\, dy \\ &+ \int_{\mathbb{R}^n - B(x^0,\sigma)} \Phi(x-y,t)|g(y) - g(x^0)|\, dy \\ &=: I + J \end{aligned}$$

Now

$$I \leq \epsilon \int_{\mathbb{R}^n} \Phi(x-y,t)\, dy = \epsilon \qquad (2.37)$$

owing to equation (2.35) and lemma (2.3.1). Furthermore, if $|x - x^0| \leq \frac{\sigma}{2}$ *and* $|y - x^0| \geq \sigma$, *then*

$$|y - x^0| \leq |y - x| + \frac{\sigma}{2} \leq |y - x| + \frac{1}{2}|y - x^0|. \qquad (2.38)$$

Thus $|y - x| \geq \frac{1}{2}|y - x^0|$. *Consequently*

$$\begin{aligned} J &\leq 2||g||_{L^\infty} \int_{\mathbb{R}^n - B(x^0,\sigma)} \Phi(x-y,t)\, dy \\ &\leq \frac{C}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n - B(x^0,\sigma)} e^{-\frac{|x-y|^2}{4t}} g(y), dy \\ &\leq \frac{C}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n - B(x^0,\sigma)} e^{-\frac{|y-x^0|^2}{16t}} g(y), dy \\ &= \frac{C}{t^{\frac{n}{2}}} \int_{\sigma}^{\infty} e^{-\frac{r^2}{16t}} r^{n-1}\, dr \to 0 \quad as \quad t \to 0^+. \qquad (2.39) \end{aligned}$$

Hence if $|x - x^0| < \frac{\sigma}{2}$ *and* $t > 0$ *is small enough,* $|u(x,t) - g(x^0)| < 2\epsilon$ *[11].*

Remark 2.3.3 *(a) In view of Theorem (2.3.2) we sometimes write*

$$\begin{cases} \Phi_t - \Delta\Phi = 0, & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ \Phi = \sigma_0, & \text{on} \quad \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{2.40}$$

σ_0 denoting the Dirac measure on $\mathbb{R}^n$ giving unit mass to the point 0.

(b) Notice that g is bounded, continuous, $g \geq 0$, $g \not\equiv 0$, then

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy \quad (x \in \mathbb{R}^n, t > 0) \tag{2.41}$$

is in fact positive for all points $x \in \mathbb{R}^n$ and times $t > 0$.

2.4 Symmetry Incorprating Boundary Conditions

Our primary solution technique contained in this thesis is outlined below.

Consider the 1+1 heat equation of the form

$$\omega_{\hat{t}} - \omega_{yy} = 0, \tag{2.42}$$

where $\omega = \omega(y, \hat{t})$. Given the theory above, let the fundamental solution of the heat equation be denoted as

$$A(y,\hat{t}) := \begin{cases} \frac{1}{\sqrt{4\pi\hat{t}}} e^{-\frac{y^2}{4\hat{t}}}, & (y \in \mathbb{R}, \hat{t} > 0), \\ 0, & (y \in \mathbb{R}, \hat{t} < 0) \end{cases} \tag{2.43}$$

with convolution

$$\omega(y,\hat{t}) = \int_{\mathbb{R}} A(y-m,\hat{t})\omega(m,0)dm \quad (y \in \mathbb{R}, \hat{t} > 0). \tag{2.44}$$

The symmetries of the heat equation (2.42) are denoted by

$$X = \xi(y,\hat{t},\omega)\frac{\partial}{\partial y} + \eta(y,\hat{t},\omega)\frac{\partial}{\partial \hat{t}} + \phi(y,\hat{t},\omega)\frac{\partial}{\partial \omega}, \tag{2.45}$$

or, equivalently

$$\begin{aligned} X &= \left(4c_1\hat{t}^2 + 2c_2\hat{t} + c_6\right)\frac{\partial}{\partial \hat{t}} + \left(4c_1 y\hat{t} + c_2 y + 2c_3\hat{t} + c_4\right)\frac{\partial}{\partial y} \\ &\quad \left(\left(c_1(-2\hat{t} - y^2) - c_3 y + c_5\right)\omega + \alpha(y,\hat{t})\right)\frac{\partial}{\partial \omega}, \end{aligned} \tag{2.46}$$

where $\alpha(y,\hat{t})$ is an arbitrary solution of the heat equation, i.e. $\alpha_{\hat{t}} = \alpha_{yy}$.

Therefore, we may consider the initial-value problem:

$$\begin{cases} \omega_{\hat{t}} - \omega_{yy} = 0, & \text{in } \mathbb{R}\times(0,\infty), \\ \omega(y,0) = F(y), & \text{on } \mathbb{R}\times\{\hat{t}=0\}. \end{cases} \tag{2.47}$$

Next, we recall the the ISC

$$\xi(y,\hat{t},\omega)\omega_y + \eta(y,\hat{t},\omega)\omega_{\hat{t}} - \phi(y,\hat{t},\omega) = 0, \tag{2.48}$$

which holds at the boundary where we impose $\hat{t} = 0$ and $\omega(y,0)$, to obtain the condition [28]

$$\frac{\phi(y,0,F(y)) - \xi(y,0,F(y))F'(y)}{\eta(y,0,F(y))} = \omega_{yy}(y,0). \tag{2.49}$$

From condition (2.49), we have that

$$\alpha(y,0) = \left(c_1 y^2 + c_3 y - c_5\right)F(y) + \left(c_2 y + c_4\right)F'(y) + c_6 F''(y). \tag{2.50}$$

The function $\omega(y,0)$ will be found from a transformation of some given terminal condition. Notice that (2.47) can be constructed for the function $\alpha(y,\hat{t})$ as well. Our problem lies in finding $\alpha(y,t)$ subject to $\alpha(y,0)$. This process would create a symmetry that incorporates the given initial-values, and we highlight that it applies to boundary value problems too.

Chapter 3

A Terminal Condition in Linear Bond-Pricing Under Symmetry Invariance.

In this chapter, we examine a general bond-pricing model with respect to its solutions that satisfy a given terminal condition. Firstly, we obtain reversible transformations that change the model to a classical and well-known PDE, the one-dimensional heat equation. We further show that the terminal condition is transformed into a nonsmooth initial condition. The important result that emerges is that the Lie symmetries are adopted to solve the equation subject to its unique configuration of initial conditions.

3.1 Introduction

Consider the general bond-pricing equation

$$\frac{\partial}{\partial t}u(x,t)+\frac{1}{2}\rho^2x^{2\gamma}\frac{\partial^2}{\partial x\,\partial x}u(x,t)+(\kappa-\lambda\rho x^{\gamma}+\beta x)\frac{\partial}{\partial x}u(x,t)-xu(x,t)=0, \quad (3.1)$$

where $\kappa,\beta,\rho,\lambda$ (the market price of risk) and γ are constants, t is time, x is the stock price or instantaneous short-term interest rate at current time t and $u(x,t)$ is the current value of the bond. This model, like many other related bond-pricing models, exhibits a terminal condition.

The terminal condition is

$$u(x,T)=\begin{cases}x-k & \text{if} \quad x>k\\ 0 & \text{if} \quad x\leq k,\end{cases} \quad (3.2)$$

where k is the strike price and T is the time of expiry of the option. Noting that this terminal condition is continuous but not smooth at $x=k$.

Bond-pricing originated in the remarkable paper by Bachelier [12] that connected stock prices to Brownian motion. This later led to major developments by Merton and Black and Scholes [3, 13] in mathematical finance. Eq. (3.1) is a linear parabolic PDE, that is formulated from stochastic differential equations. Various modified versions of equations of this type exist, for example, the Vasicek model ($\gamma=0$), Cox-Ingersoll model ($\gamma=1/2$, $\lambda=0$), and Dothan model ($\kappa=\beta=0,\gamma=1$), see [14, 15, 16, 17]. In [17], the above model was considered in order to render it into Lie canonical forms and obtain solutions.

The key difference with this study is that the ISC is invoked to find a symmetry generator that incorporates a transformed version of the terminal condition. This itself is a nontrivial task and it is the first time that symmetry invariant solutions

that satisfy the terminal condition are presented for the model under study. Additionally, we consider novel transformations and different cases of the model that yield unique boundary conditions and hence new solutions.

The theory of Lie's classification of linear second-order PDEs, translated in English, can be found in Ibragimov's remarkable contribution [10]. Subsequently, Ibragimov showed that the Black-Scholes model, which has a similar symmetry group to the heat equation, is then included in Lie's classification, via equivalence transformation formulae [10]. Lie's formulae may then be extended to any model that satisfies these classification properties.

The model in this paper is first transformed and then its initial value problem is derived. The method of Lie symmetries plays a pivotal role in constructing solutions to the initial conditions. We emphasize that symmetries are not at the forefront when solving initial or boundary conditions. This method is categorically avoided for PDEs with nonsmooth boundary conditions, where derivatives may not exist at certain points and/or the imposed conditions result in asymptotic behavior. However, we show how to engage with such problems in light of the equation under study.

The plan of the chapter follows. In Section 3.2, we introduce the parameters and transformations required to manipulate the model into the classical heat equation. Section 3.3 contains the main results, whereby we detail certain features of the boundary condition, and go on, to provide exact solutions of the model subject to the condition (3.2). Finally, we conclude in Section 3.4.

3.2 Transformations

In this section, we establish successive transformations that convert the model (3.1) to the well-known heat equation. The change of variables is necessary to apply some of the forthcoming initial value results to the model (3.1). It turns out that this model is transformable under precise forms of its arbitrary parameters. Hence we prove the following result.

Theorem 3.2.1 *The option pricing model Eq. (3.1), is reducible to the heat equation (2.42), under the following specific parameter settings:* $\gamma = 2, \quad \kappa = \beta = 0, \quad \lambda = -\frac{1}{\rho}, \quad$ *and* $\quad \rho \quad$ *is arbitrary. The transformations are*

$$\hat{t} = T - t, \tag{3.3}$$

$$y = -\frac{\sqrt{2}}{x \cdot \rho}. \tag{3.4}$$

Proof 3.2.2 *An invertible change of the independent variables of the form* y *and* $\hat{t}$*, reduces Eq. (3.1) to the form*

$$a(y, \hat{t})u_y + c(y, \hat{t})u + u_{\hat{t}} - u_{yy} = 0, \tag{3.5}$$

where $u(y, \hat{t})$*,* $a(y, \hat{t}) = y\left(-\frac{2}{y^2} - \frac{\sqrt{2}}{y \cdot \rho}\right)$ *and* $c(y, \hat{t}) = -\frac{\sqrt{2}}{y \cdot \rho}$.

Thereafter, under another transformation, the dependent variable $u(y, \hat{t})$ *undergoes a transformation*

$$u(y, \hat{t}) = \omega(y, \hat{t}) \cdot e^{-\phi(y, \hat{t})}, \tag{3.6}$$

where

$$\phi(y, \hat{t}) = \frac{y}{\sqrt{2} \cdot \rho} + \frac{\hat{t}}{2\rho^2} + \ln|y|. \tag{3.7}$$

This transformation procedure converts Eq. (3.5) to the heat equation (2.42).□

It is also possible to convert the PDE into the heat equation, under the parameters $\gamma = \frac{1}{2}, \quad \kappa = \frac{\rho^2}{4}, \quad \beta = \sqrt{2}i\rho, \quad \rho = i \quad$ and $\quad \lambda = 0$, with transformations

$$\hat{t} = T - t, \tag{3.8}$$

$$y = \frac{2\sqrt{2}\sqrt{x}}{\rho}, \tag{3.9}$$

$$\phi(y, \hat{t}) = -\frac{\hat{t}}{2\sqrt{2}} - \frac{y^2}{4\sqrt{2}}, \tag{3.10}$$

following the same procedure as above. However, this is inconvenient due to the imagrinary terms.

Naturally, the above result gives rise to the following lemma.

Lemma 3.2.1 *The corresponding terminal condition (3.2) is transformed into a boundary condition of the heat equation, viz.*

$$\omega(y, \hat{t}) = max\Big\{ - e^{\frac{y}{\sqrt{2}\cdot\rho} + \frac{\hat{t}}{2\rho^2} + \ln|y|} \left(\frac{\sqrt{2}}{y \cdot \rho} + k \right), 0 \Big\}. \tag{3.11}$$

Proof 3.2.3 *The terminal condition (3.2) may be expressed as* $u = max\{x - k, 0\}$, *and with the application of the transformations* (3.3), (3.4), (3.6) *and* (3.7) *from Theorem* (3.2.1), *a boundary condition for the heat equation follows for Case I. Similarly, the transformations* (3.8), (3.9), (3.6) *and* (3.10) *yield a boundary condition for Case II.* □

3.3 The Initial-Value Problem

The heat equation is often bench-marked as one of the most useful PDEs. It admits the Gaussian function as a fundamental solution, which leads to several analytical applications. We note that many studies of parabolic PDEs involve the heat

equation, we refer the reader to [20, 21, 22, 23, 24, 25, 26, 27]. In our context, we showcase that the heat equation is an easy facilitator of boundary conditions.

From Lemma 3.2.1, we construct the initial-value problem for the heat equation as described in Section 2.4.

To proceed, we consider the well known symmetries of (2.42), that is, (2.46). Note that equation (2.50) is an ODE which when solved yields the most general initial condition. This condition gives less restrictive conditions on the initial conditions such that they be defined by a particular symmetry. We shall demonstrate several special conditions and solutions below. Now, the known the fundamental solution of the heat equation is (2.43), which is singular at the point (0,0). The knowledge of $A(y,\hat{t})$ allows the construction of a solution to the initial value problem of the heat equation. Since $A(y-m,\hat{t})$ is also a solution for each fixed $y \in \mathbb{R}$, then consequently the convolution (2.44) is a solution to the one dimensional heat equation as well [11].

Next, we apply the theory above to explore various solutions to the heat equation and subsequently, the general bond-pricing equation. All solutions are subject to the initial conditions given above, while solutions to the general bond-pricing equation is subject to the terminal condition (3.2).

For simplicity, let $c_4 = 1, c_1 = c_2 = c_3 = c_5 = c_6 = 0$. Therefore, from (2.50) we have

$$\alpha(y,0) = F'(y),$$

and the symmetry we apply is

$$X = \frac{\partial}{\partial y} + \alpha(y,\hat{t})\frac{\partial}{\partial \omega}.$$

Hence, with the ISC (2.48), we find that the solution to the heat equation is

$$\omega(y,\hat{t}) = \int \alpha(y,\hat{t})dy + H(\hat{t}).$$

Without loss of generality, we let $H(\hat{t}) = 0$, and we are now faced with the challenge of finding $\alpha(y, \hat{t})$. Since $\alpha(y, \hat{t})$ is a solution to the heat equation, it will also satisfy the convolution property (2.44). Thus by integration with Maple, we obtain

$$\alpha(y, \hat{t}) = -\frac{e^{\frac{t+\sqrt{2}\rho x}{2\rho^2}} \left(2kt + k\rho\left(2\rho + \sqrt{2}x\right) + 2\right)}{2\rho^2}. \tag{3.12}$$

We let $x = y$ and $t = \tau$, then we integrate again with respect to y to find

$$\omega(y, \hat{t}) = -\frac{e^{\frac{\tau+\sqrt{2}\rho y}{2\rho^2}} \left(\sqrt{2}k\tau + k\rho y + \sqrt{2}\right)}{\rho} \tag{3.13}$$

which is a solution to the heat equation (2.42).

Now recall the transformations from Theorem 3.2.1 to reverse the substitutions and yield the solution to the general bond-pricing model (3.1), viz.

$$u(x,t) = -\frac{\left(\sqrt{2}k(T-t) - \frac{\sqrt{2}k}{x} + \sqrt{2}\right) e^{\frac{-t+T-\frac{2}{x}}{2\rho^2}} e^{-\frac{T-t}{2\rho^2} + \frac{1}{\rho^2 x} - \log\left(-\frac{\sqrt{2}}{\rho x}\right)}}{\rho} \tag{3.14}$$

If we let $t = T$ the time of expiry, we see that (3.14) simplifies to

$$u(x, T) = x - k,$$

which shows the solution is subject to the terminal condition.

Below is the graphical illustration of our solution.

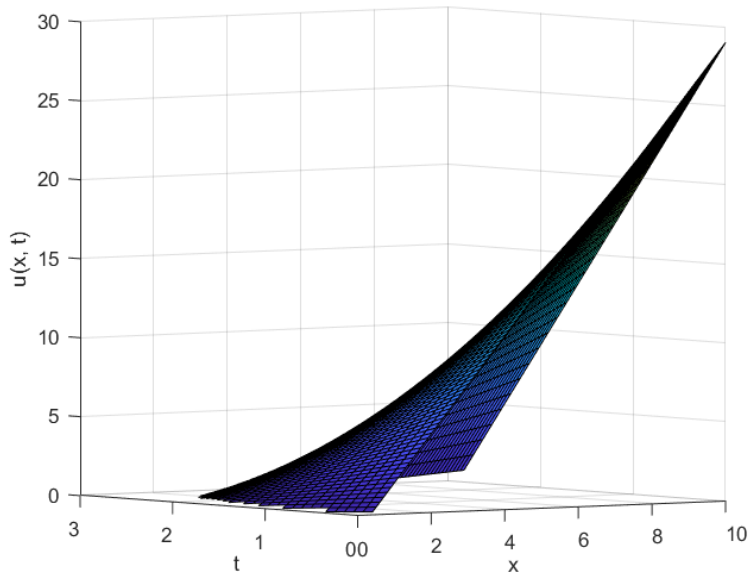

Figure 3.1: 3D illustration of our analytical solution (3.14), with $k = 1$.

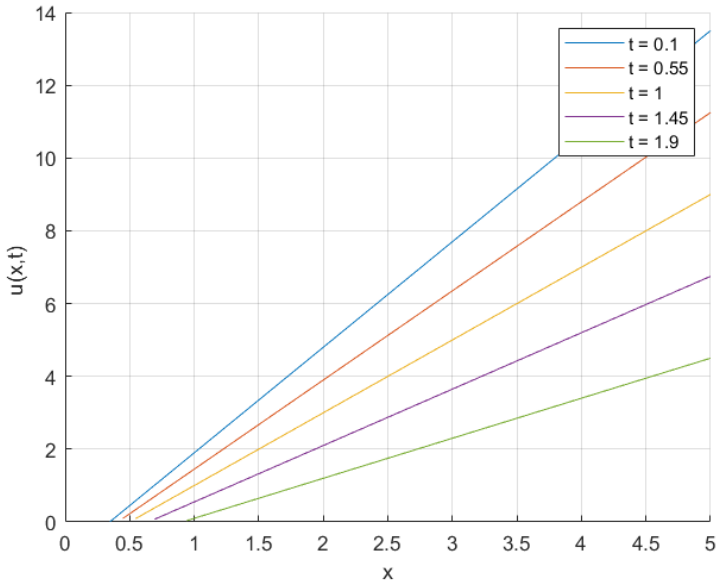

Figure 3.2: 2D plot of (3.14), with $k = 1$.

3.4 Conclusion

A general equation linked to bond-pricing has been solved subject to its terminal condition, for several parameter values. We have transferred the problem of solving this equation to a different problem corresponding to the heat equation. Specifically, we have employed symmetry generators to solve for invariant solutions that satisfy some resulting initial values. In fact, the analytical solutions at the boundary involved the nontrivial computations of functions that solve the heat equation. Note that these conditions may be badly behaved but the technique used is up to the challenge. In the wider context of other applications, one may encounter integration difficulties for the solutions itself.

The heat equation possesses other well-known solutions aside from (2.43). As an example, take the infinite series solution [29]. However, such solutions also admit other boundary conditions. Hence, we remark that in principle, it is the fundamental solution or heat kernel defined above that fits perfectly when we transform the terminal condition into a specific initial condition, and thereby set up the convolution and any ensuing integrations required to solve the equation.

For nonlinear equations, it is possible to apply the above approach, in one of two ways, namely: either by first linearising the equation and then constructing transformations to the 1+1 heat equation, or secondly if one can find direct transformations to the heat equation. Both scenarios would certainly pose interesting challenges to pursue.

Chapter 4

A progressive approach to solving a generalized CEV type model via symmetry invariant surface conditions.

In this chapter, we examine a type of constant elasticity of variance (CEV) model subject to its terminal condition. We prove that certain transformations may be applied to obtain a simpler equation that has known solution processes. Four cases are obtained that play a role in specifying the many unknown parameters of the model. The corresponding terminal condition is transformed into an initial condition, and we then demonstrate how to solve this Cauchy problem with Lie symmetries and Poisson's formula. Finally, we examine the behaviour of the obtained solutions.

4.1 Introduction

In the theory of PDEs of a financial nature, the Black-Scholes model [3] is arguably the most researched equation in mathematical finance. Most importantly, in [12] the equation was reduced to the classical heat equation. Other mathematical interest in such equations is because stock prices are often linked to Brownian motion [3]. However, more complicated models exist, that may not be reduced to simple models with known solutions.

A CEV or constant elasticity of variance model [13] is commonly used to investigate option and asset pricing. Consider a type of CEV model that has been modified for a portfolio optimisation problem and that contains unknown constants [30], viz.

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) &= -\left(\frac{\alpha\lambda-\delta}{1-\lambda}+\frac{\lambda x(\beta-\alpha)^2}{K^2\left(2(\lambda-1)^2\right)}\right)u(x,t)-\left(\frac{2\gamma\lambda(\beta-\alpha)}{\lambda-1}-2\beta\gamma\right)x\frac{\partial}{\partial x}u(x,t)- \\ &\quad \gamma(2\gamma+1)K^2\frac{\partial u(x,t)}{\partial x}-2\gamma^2K^2x\frac{\partial^2}{\partial x\,\partial x}u(x,t), \quad x>0,\ t\in[0,T] \qquad (4.1)\end{aligned}$$

where x is the stock/asset price at current time t and $u(x,t)$ is the value function for the portfolio optimisation problem. The constants $\alpha, \beta, \gamma, \delta, \lambda$ and K represent; the interest rate, the expected instantaneous return rate of stock, elasticity parameter, a subjective discount, a parameter linked to a utility function assumed by the model for optimal investment and consumption, and a constant related to the instantaneous volatility of the stock, respectively.

This model must satisfy a terminal or boundary condition, namely [30]

$$u(x,T)=1, \qquad (4.2)$$

where T is the time of expiry. This approach will work for other types of terminal conditions.

Eq. (4.1) is a linear parabolic equation, which may be studied from various perspectives [31, 32, 23], but finding a solution is nontrivial.

Within the remarkable theory of Lie's classification of linear second-order PDEs [19] is a nontrivial and rarely used mechanism to transform certain models into a heat transfer equation. This is our approach in the present study. We find new transformations that convert the above model into the classical heat equation. This provides us with selection criteria for the numerous free parameters of the model, resulting in four cases of interest.

The solution of Cauchy problems for parabolic PDEs, such as (4.1) can be manipulated into a particular solution with specific singularities or so-called fundamental solutions [33]. In some cases, the fundamental solution is an invariant solution [34]. We shall prove this property for the model under study.

As a major achievement in this study, the ISC is exploited to find a symmetry generator that will solve the model subject to the terminal condition. This is also a nontrivial task and it is the first time that such solutions are pursued. As we shall demonstrate, these new symmetries are extremely involved, but nonetheless invariant solutions are possible.

The technique of Lie symmetries is rarely used in evaluating PDEs that are subject to initial or boundary conditions. In most of the literature, initial conditions are ignored under symmetry analysis, however, in this study, we highlight the need for symmetries in tackling such problems.

The plan of the chapter follows. In Section 4.2, we prove the transformations of the

model that we require. Section 4.3 describes the initial conditions and the theory in approaching the solutions of the model. Finally, we conclude in Section 4.4.

4.2 The Construction of Transformations

In what follows, we use the heat equation as a tool in solving a Cauchy problem. Hence, we first need to transform the PDE (4.1) and its terminal condition. It turns out that this model is transformable under specific restrictions on its arbitrary parameters. Hence we prove the following result.

Theorem 4.2.1 *The model Eq. (4.1), is reducible to the 1+1 heat equation (2.42) under the following cases.*

(a) Case I: $\alpha = \sqrt{2}, \quad \lambda = \frac{1}{2}, \quad \beta = 1, \quad K = \frac{1}{\gamma}, \quad \gamma = -1$ *and* δ *is arbitrary.*

The transformations are

$$\hat{t} = T - t, \tag{4.3}$$

$$y = \sqrt{2x}. \tag{4.4}$$

(b) Case II: $\alpha = \sqrt{\lambda + 1}, \quad \lambda = 2, \quad \beta = \sqrt{3\lambda}, \quad K = -1 \quad \gamma = 1$ *and* δ *is arbitrary.*

The transformations are

$$\hat{t} = T - t, \tag{4.5}$$

$$y = \sqrt{2x}. \tag{4.6}$$

(c) Case III: $\alpha = \frac{1}{\beta}, \quad \lambda = 4, \quad \beta = \sqrt{2}, \quad \gamma = -1$ *and* $K, \quad \delta$ *are arbitrary.*

The transformations are

$$\hat{t} = T - t, \tag{4.7}$$

$$y = \frac{\sqrt{2x}}{K}. \tag{4.8}$$

(d) Case IV: $\beta = \alpha\sqrt{\lambda}, \quad \gamma = 1 \quad$ *and* $\quad \alpha, \quad \lambda, \quad K, \quad \delta \quad$ *are arbitrary.*

The transformations are

$$\hat{t} = T - t, \tag{4.9}$$

$$y = \frac{\sqrt{2x}}{\gamma K}. \tag{4.10}$$

Proof 4.2.2 *Consider Case I. An invertible change of the independent variables of the form* y *and* $\hat{t}$*, reduces Eq. (4.1) to equation (3.5) where* $u(y,\hat{t})$*,* $a(y,\hat{t}) = \left(\sqrt{2}-2\right)y$ *and* $c(y,\hat{t}) = 2\delta + \left(\sqrt{2}-\frac{3}{2}\right)y^2 - \sqrt{2}$*.*

Thereafter, under another transformation, the dependent variable $u(y,\hat{t})$ *undergoes a transformation (3.6), where*

$$\phi(y,\hat{t}) = 2\delta\hat{t} + \hat{t} - \frac{3\hat{t}}{\sqrt{2}} - \frac{1}{4}\left(\sqrt{2}-2\right)y^2. \tag{4.11}$$

This transformation procedure converts Eq. (3.5) to the heat equation (2.42).

The proofs for Case II, Case III and Case IV are analogous, but where

$$a(y,\hat{t}) = -\frac{2\left(\frac{1}{2}\left(\sqrt{6}-2\sqrt{3}\right)y^2+1\right)}{y} \quad \text{and} \quad c(y,\hat{t}) = \frac{1}{2}\left(2\left(2\sqrt{3}-\delta\right) - \left(\sqrt{3}-\sqrt{6}\right)^2 y^2\right),$$

for Case II,

$$a(y,\hat{t}) = -\frac{1}{3}\left(\sqrt{2}y\right) \quad \text{and} \quad c(y,\hat{t}) = \frac{1}{3}\left(2\sqrt{2}-\delta\right) - \frac{1}{9}\left(\frac{1}{\sqrt{2}} - \sqrt{2}\right)^2 y^2,$$

for Case III, and

$$a(y,\hat{t}) = \frac{\sqrt{\lambda}\left(\alpha y^2 - 2\right) - 2}{\left(\sqrt{\lambda}+1\right)y} \quad \text{and} \quad c(y,\hat{t}) = \frac{\alpha\lambda - \delta}{\lambda - 1} - \frac{\alpha^2\lambda y^2}{4\left(\sqrt{\lambda}+1\right)^2},$$

for Case IV.

Similarly, we define the transformation,

$$\phi(y,\hat{t}) = -\delta\hat{t} - \sqrt{3}\hat{t} + 3\sqrt{\frac{3}{2}}\hat{t} + \frac{1}{4}\left(\sqrt{2}-2\right)\sqrt{3}y^2 + \log(y), \tag{4.12}$$

for Case II,

$$\phi(y,\hat{t}) = -\frac{\delta\hat{t}}{3} + \frac{5\hat{t}}{3\sqrt{2}} + \frac{y^2}{6\sqrt{2}}, \tag{4.13}$$

for Case III, and

$$\phi(y,\hat{t}) = \frac{3\alpha\sqrt{\lambda}\hat{t}}{2(\lambda-1)} - \frac{\alpha\lambda\hat{t}}{2(\lambda-1)} - \frac{\delta\hat{t}}{\lambda-1} - \frac{\alpha\sqrt{\lambda}y^2}{4\left(\sqrt{\lambda}+1\right)} + \log(y), \tag{4.14}$$

for Case IV.

All of the restrictions on the above parameter are mathematical requirements, but do not violate any financial interpretations of the model. The elasticity parameter γ is often negative, but special situations do exist for positive elasticity. It follows that the same transformation are applicable to the terminal condition, hence we have the following lemma.

Lemma 4.2.1 *The terminal condition (3.2) is transformed via the transformations (4.3)-(4.14), such that*

$$\omega(y,\hat{t}) = e^{2\delta\hat{t}+\hat{t}-\frac{3\hat{t}}{\sqrt{2}}-\frac{1}{4}\left(\sqrt{2}-2\right)y^2}, \tag{4.15}$$

for Case I,

$$\omega(y,\hat{t}) = e^{-\delta\hat{t}-\sqrt{3}\hat{t}+3\sqrt{\frac{3}{2}}\hat{t}+\frac{1}{4}\left(\sqrt{2}-2\right)\sqrt{3}y^2+\log(y)}, \tag{4.16}$$

for Case II,

$$\omega(y,\hat{t}) = e^{-\frac{\delta\hat{t}}{3}+\frac{5\hat{t}}{3\sqrt{2}}+\frac{y^2}{6\sqrt{2}}}, \tag{4.17}$$

for Case III, and

$$\omega(y,\hat{t}) = e^{\frac{3\alpha\sqrt{\lambda}\hat{t}}{2(\lambda-1)}-\frac{\alpha\lambda\hat{t}}{2(\lambda-1)}-\frac{\delta\hat{t}}{\lambda-1}-\frac{\alpha\sqrt{\lambda}y^2}{4(\sqrt{\lambda}+1)}+\log(y)}, \tag{4.18}$$

for Case IV.

The proof is rudimentary and therefore left to the reader.

4.3 The Initial Problem

It is well-known that the heat equation (2.42) is an important PDE that has fuelled many analytical studies of the past [35, 36, 22, 37, 38]. Significantly, it admits a fundamental solution, which is pivotal to our solution process.

From the Lemma 4.2.1, let $\omega(y,0) = F(y)$. Next, the Lie point symmetries of (2.42) are of the form (2.46), where $\alpha(y,\hat{t})$ is an arbitrary solution of the heat equation, $\alpha_{\hat{t}} = \alpha_{yy}$. Hence, the ISC (2.48) holds at the boundary where $\omega(y,0)$, such that we have condition (2.49), [28].

Consider the initial-value problem for the heat equation (2.42), that is, (2.47). The Cauchy problem (2.47) is well-posed. For uniqueness of the solution, see Tikhonov [39]. Now, the fundamental solution of the equation (2.42) is (2.43), which is singular at the point (0,0). The function $A(y-m,\hat{t})$ is also a solution for each fixed $y \in \mathbb{R}$, then consequently the convolution (2.44) is a solution to the one-dimensional heat equation (2.42) [11]. This integral is also called Poisson's formula.

As mentioned before, we may now apply the above theoretical considerations to solve the heat equation (2.42) subject to its initial values. Subsequently, we reverse all the transformations and ultimately solve the original PDE, the CEV model. By virtue of the above formulation, these solutions will satisfy the terminal condition (3.2). Note that these solutions are lengthy and we have not provided an exhaustive list of obtainable solutions, but rather, we have depicted selected solutions for each of the four cases. More solutions do exist.

Suppose we begin with Case I and let $c_4 = 1, c_1 = c_2 = c_3 = c_5 = c_6 = 0$ for simplicity. Therefore $\alpha(y,0) = F'(y)$, and the resultant symmetry that incorporates the initial condition is

$$X = \frac{\partial}{\partial y} + \alpha(y,\hat{t})\frac{\partial}{\partial \omega}. \tag{4.19}$$

Hence, with the condition (2.48), we find that the solution to the equation (2.42) is $\omega(y,\hat{t}) = \int \alpha(y,\hat{t})dy + H(\hat{t})$. Without loss of generality, we let $H(\hat{t}) = 0$, and we must then determine $\alpha(y,\hat{t})$. Since the α function is a solution to the heat equation (2.42), it will also satisfy the convolution (2.44). Thus by integration with Mathematica, we obtain

$$\alpha(y,\hat{t}) \;=\; -\frac{\left(\sqrt{2}-2\right) y e^{-\frac{(\sqrt{2}-2)y^2}{4(\sqrt{2}-2)\hat{t}+4}}}{2\sqrt{\frac{1}{\hat{t}}+\sqrt{2}-2}\sqrt{\hat{t}}\left(\left(\sqrt{2}-2\right)\hat{t}+1\right)}, \tag{4.20}$$

and we can integrate (4.20) to find $\omega(y,\hat{t})$, which is a solution to the heat equation (2.42), viz.

$$\omega(y,\hat{t}) \;=\; \frac{e^{-\frac{(\sqrt{2}-2)y^2}{4(\sqrt{2}-2)\hat{t}+4}}}{\sqrt{\frac{1}{\hat{t}}+\sqrt{2}-2}\sqrt{\hat{t}}}. \tag{4.21}$$

Now the transformations from Theorem (4.2.1) are used to reverse the substitutions and yield the solution to the model (4.1), viz. $u(x,t)$.

$$u(x,t) \;=\; \frac{\exp\left(2\delta(t-T) - \frac{2\left(\sqrt{2}-2\right)x}{4\left(\sqrt{2}-2\right)(T-t)+4} + \frac{3(T-t)}{\sqrt{2}} + t - T + \frac{1}{2}\left(\sqrt{2}-2\right)x\right)}{\sqrt{\left(\sqrt{2}-2\right)(T-t)+1}}. \tag{4.22}$$

If we let $t = T$ the time of expiry, we see that (4.22) simplifies to

$$u(x, T) = 1,$$

which shows the solution is subject to the terminal condition. All solutions below satisfy this relation.

Similarly for Case II, the same procedure is followed and we obtain the functions

$$\begin{aligned}\alpha(y,\hat{t}) &= \sqrt{\frac{1}{\hat{t}}+2\sqrt{3}-\sqrt{6}}\sqrt{\hat{t}}e^{\frac{\sqrt{3}(\sqrt{2}-2)y^2}{4-4\sqrt{3}(\sqrt{2}-2)\hat{t}}}\times \\ &\quad \frac{\left(\left(36-24\sqrt{2}\right)\hat{t}^2-2\hat{t}\left(\left(9-6\sqrt{2}\right)y^2+2\left(\sqrt{2}-2\right)\sqrt{3}\right)+\left(\sqrt{2}-2\right)\sqrt{3}y^2+2\right)}{2\left(\sqrt{3}\left(\sqrt{2}-2\right)\hat{t}-1\right)\left(6\left(7\sqrt{2}-10\right)\sqrt{3}\hat{t}^3+18\left(2\sqrt{2}-3\right)\hat{t}^2+3\left(\sqrt{2}-2\right)\sqrt{3}\hat{t}-1\right)}.\end{aligned} \tag{4.23}$$

We then integrate (4.23) to find

$$\omega(y,\hat{t}) = -\frac{\sqrt{\frac{1}{\hat{t}}+2\sqrt{3}-\sqrt{6}}\sqrt{\hat{t}}\left(6\left(2\sqrt{2}-3\right)\hat{t}+\left(\sqrt{2}-2\right)\sqrt{3}\right)ye^{\frac{\sqrt{3}(\sqrt{2}-2)y^2}{4-4\sqrt{3}(\sqrt{2}-2)\hat{t}}}}{\sqrt{3}\left(\sqrt{2}-2\right)\left(6\left(7\sqrt{2}-10\right)\sqrt{3}\hat{t}^3+18\left(2\sqrt{2}-3\right)\hat{t}^2+3\left(\sqrt{2}-2\right)\sqrt{3}\hat{t}-1\right)}, \tag{4.24}$$

and it follows that the solution to the original PDE is

$$\begin{aligned}u(x,t) &= -\exp\left(\delta(T-t)+\frac{2\left(\sqrt{2}-2\right)\sqrt{3}x}{4\left(\sqrt{2}-2\right)\sqrt{3}(t-T)+4}\right)\times \\ &\quad \exp\left(\sqrt{3}(T-t)+3\sqrt{\frac{3}{2}}(t-T)-\frac{1}{2}\sqrt{3}\left(\sqrt{2}-2\right)x\right)\times \\ &\quad \frac{\sqrt{\left(2\sqrt{3}-\sqrt{6}\right)(T-t)+1}\left(6\left(2\sqrt{2}-3\right)(T-t)+\left(\sqrt{2}-2\right)\sqrt{3}\right)}{\sqrt{3}\left(\sqrt{2}-2\right)}\times \\ &\quad \frac{1}{\left(6\left(7\sqrt{2}-10\right)\sqrt{3}(T-t)^3+3\left(\sqrt{2}-2\right)\sqrt{3}(T-t)+18\left(2\sqrt{2}-3\right)(t-T)^2-1\right)}.\end{aligned} \tag{4.25}$$

For Case III, we have the functions

$$\alpha(y,\hat{t}) = \frac{\sqrt{\frac{3}{2}}ye^{\frac{y^2}{6\sqrt{2}-4\hat{t}}}}{\left(3-\sqrt{2}\hat{t}\right)^{3/2}}, \tag{4.26}$$

and

$$\omega(y,\hat{t}) = \frac{\sqrt{\frac{3}{2}}\left(3\sqrt{2}-2\hat{t}\right)e^{\frac{y^2}{6\sqrt{2}-4\hat{t}}}}{\left(3-\sqrt{2}\hat{t}\right)^{3/2}}, \tag{4.27}$$

and it follows that the solution to (4.1) is

$$u(x,t) = \frac{\sqrt{\frac{3}{2}}\left(2t-2T+3\sqrt{2}\right)\exp\left(\frac{1}{6}\left(-2\delta(t-T)+\frac{6x}{K^2\left(2t-2T+3\sqrt{2}\right)}+5\sqrt{2}(t-T)-\frac{\sqrt{2}x}{K^2}\right)\right)}{\left(\sqrt{2}t-\sqrt{2}T+3\right)^{3/2}}. \tag{4.28}$$

For Case IV, we have the functions

$$\alpha(y,\hat{t}) = \frac{\left(\sqrt{\lambda}+1\right)^2\sqrt{\hat{t}}\sqrt{-\frac{\alpha}{\sqrt{\lambda}+1}+\alpha+\frac{1}{\hat{t}}}e^{-\frac{\alpha\sqrt{\lambda}y^2}{4\sqrt{\lambda}(\alpha\hat{t}+1)+4}}\left(\sqrt{\lambda}\left(2\alpha\hat{t}-\alpha y^2+2\right)+2\right)}{2\left(\sqrt{\lambda}(\alpha\hat{t}+1)+1\right)^3}, \tag{4.29}$$

and

$$\omega(y,\hat{t}) = \frac{\left(\sqrt{\lambda}+1\right)^2\sqrt{\hat{t}}y\sqrt{-\frac{\alpha}{\sqrt{\lambda}+1}+\alpha+\frac{1}{\hat{t}}}e^{-\frac{\alpha\sqrt{\lambda}y^2}{4\sqrt{\lambda}(\alpha\hat{t}+1)+4}}}{\left(\sqrt{\lambda}(\alpha\hat{t}+1)+1\right)^2}, \tag{4.30}$$

and it follows that another solution to the original equation (4.1) is

$$\begin{aligned} u(x,t) &= \exp\left(\frac{1}{2}\left(-\frac{t\left(\alpha\lambda-3\alpha\sqrt{\lambda}+2\delta\right)}{\lambda-1}+\frac{\alpha\sqrt{\lambda}x\left(\frac{1}{\sqrt{\lambda}+1}+\frac{1}{\sqrt{\lambda}(\alpha t-\alpha T-1)-1}\right)}{K^2}\right)\right) \\ &\quad \exp\left(\frac{1}{2}\left(\frac{T\left(\alpha\lambda-3\alpha\sqrt{\lambda}+2\delta\right)}{\lambda-1}\right)\right) \\ &\quad \frac{\left(\sqrt{\lambda}+1\right)^2\sqrt{\left(\alpha-\frac{\alpha}{\sqrt{\lambda}+1}\right)(T-t)+1}}{\left(\sqrt{\lambda}(\alpha t-\alpha T-1)-1\right)^2}. \end{aligned} \tag{4.31}$$

We have recorded four unique solutions for (4.1) as seen above, obtainable through convolution and invariance, and restrictions on the parameters $K, \alpha, \beta, \gamma, \delta$, and λ. Theorem (4.2.1) highlights how these parameters were selected.

Table 4.1 shows the arbitrary parameters for various solutions, as per Figures 4.1-4.12.

Table 4.1: Values of the arbitrary constants across all four solutions.

Parameter	**Solution 1** (4.22)	**Solution 2** (4.25)	**Solution 3** (4.28)	**Solution 4** (4.31)
T	2	2	2	2
δ	0.5	0.5	0.5	0.5
K			2	2
α				1.5
λ				0.75

Below are the graphical illustrations of the analytical solutions (4.22), (4.25), (4.28)

and (4.31), respectively.

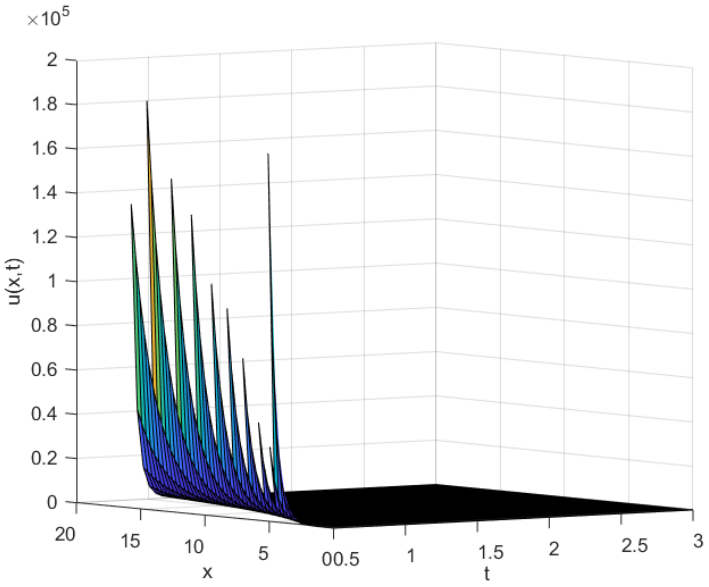

Figure 4.1: 3D illustration of our analytical solution (4.22).

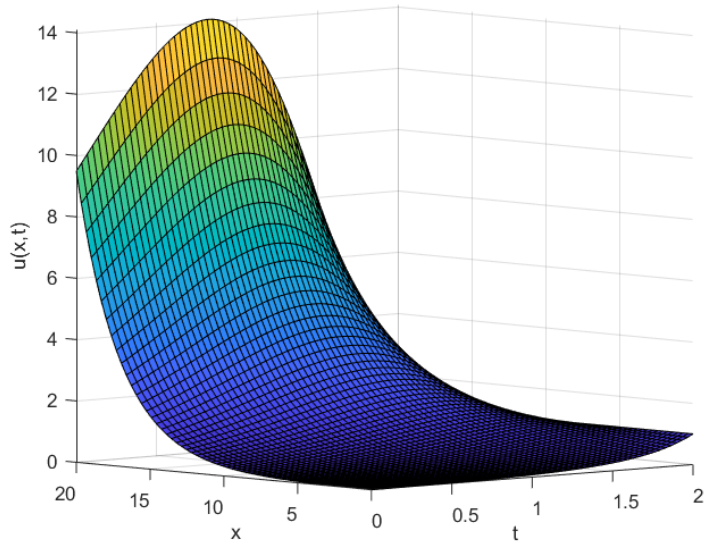

Figure 4.2: 3D illustration of our analytical solution (4.25).

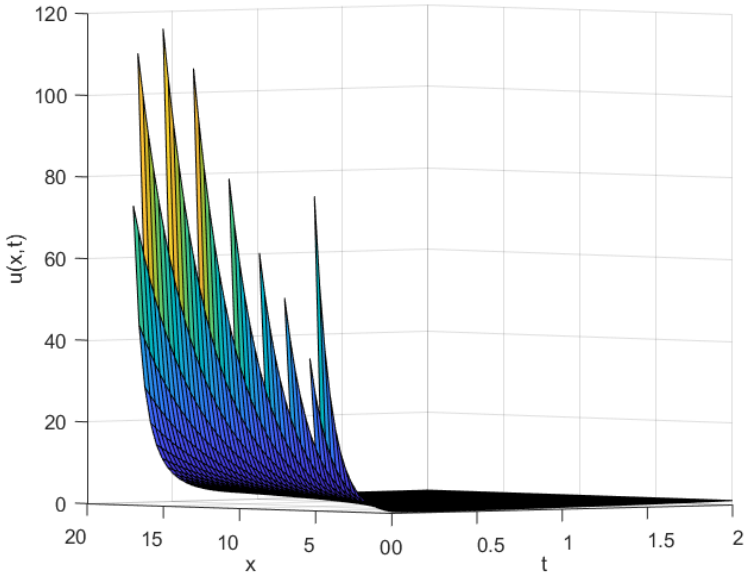

Figure 4.3: 3D illustration of our analytical solution (4.28).

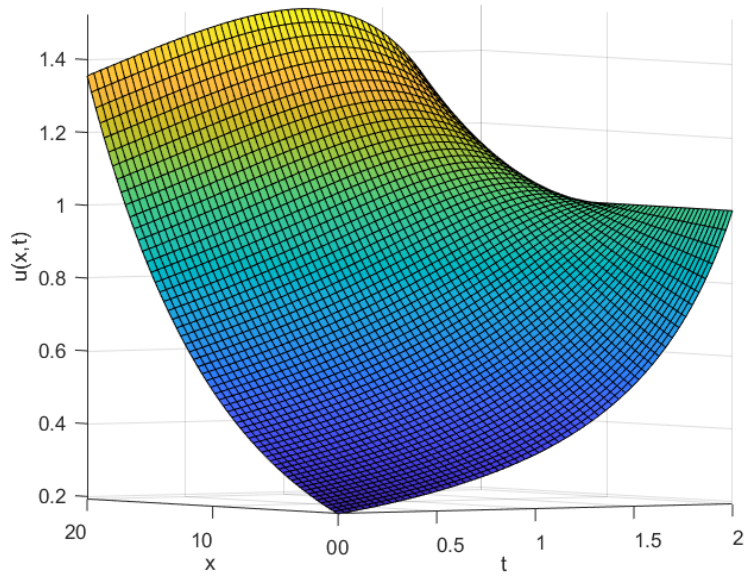

Figure 4.4: 3D illustration of our analytical solution (4.31).

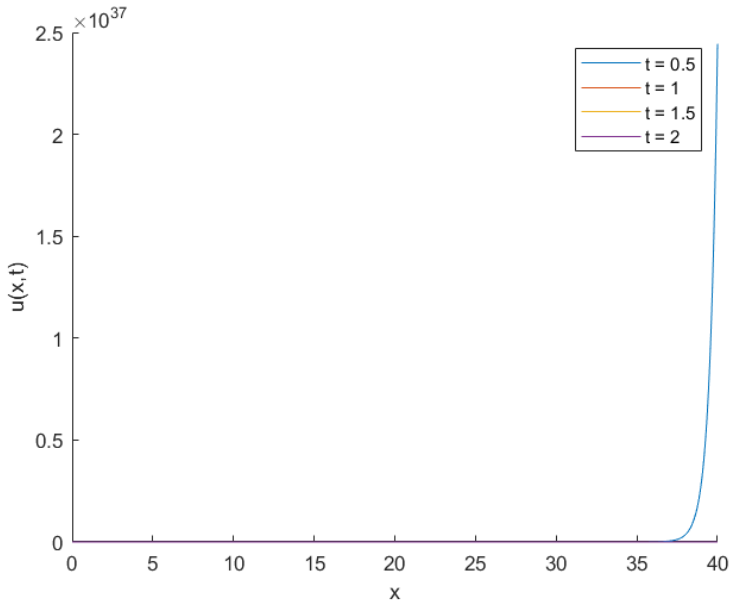

Figure 4.5: 2D illustration of our analytical solution (4.22) for $0 < t \leq 2$

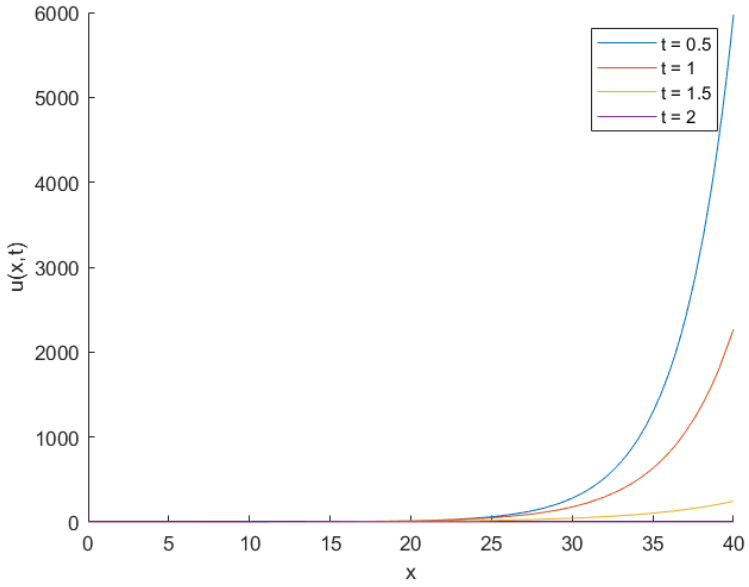

Figure 4.6: 2D illustration of our analytical solution (4.25) for $0 < t \leq 2$

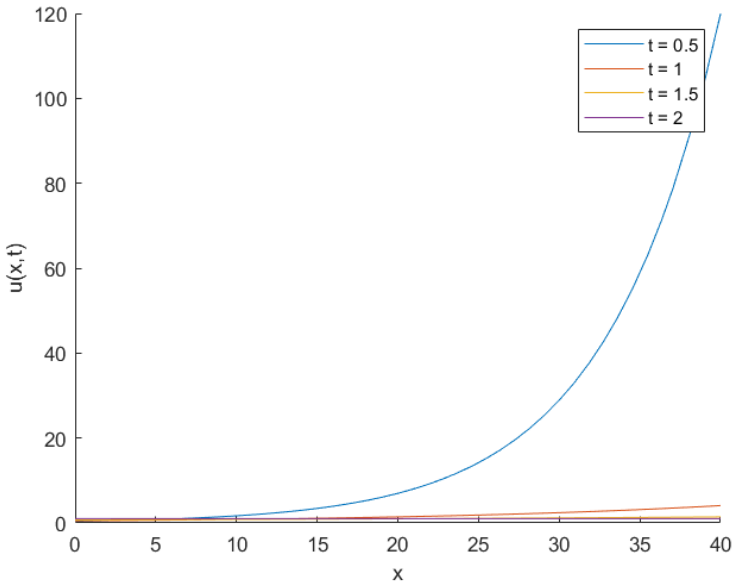

Figure 4.7: 2D illustration of our analytical solution (4.28) for $0 < t \leq 2$

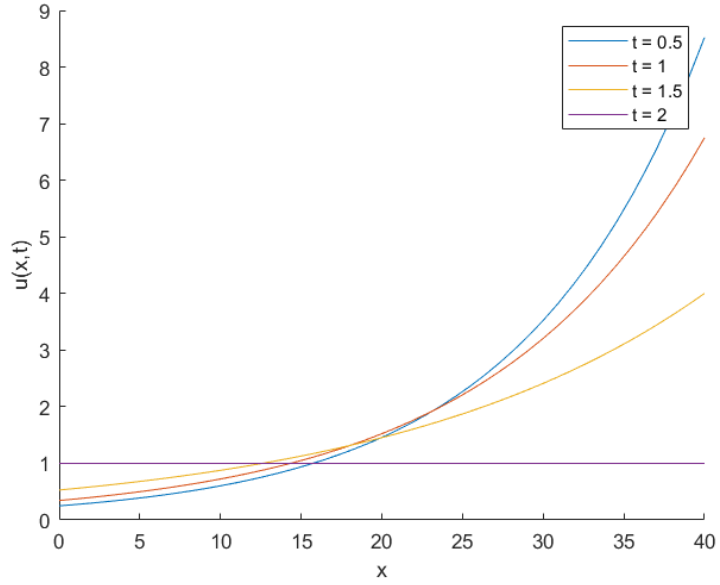

Figure 4.8: 2D illustration of our analytical solution (4.31) for $0 < t \leq 2$

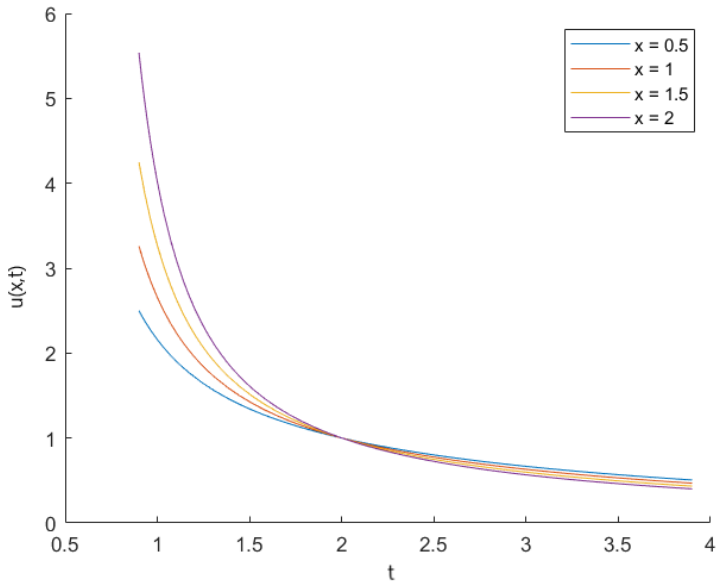

Figure 4.9: 2D illustration of our analytical solution (4.22) for $0 < x \leq 2$

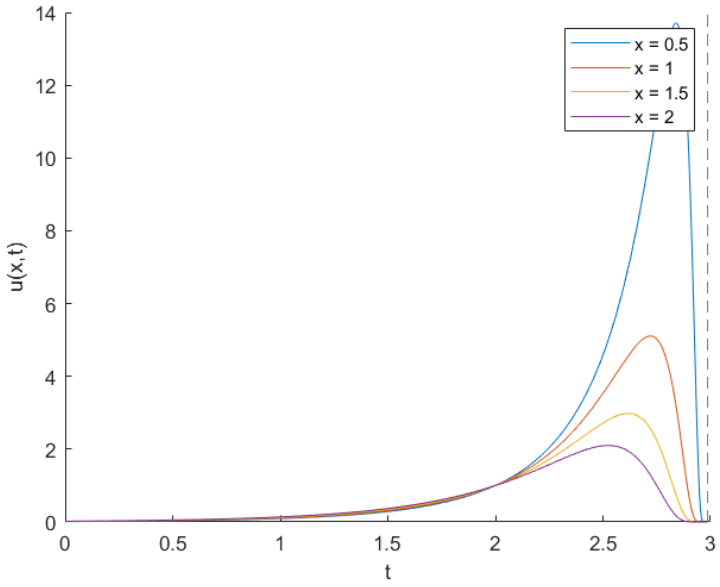

Figure 4.10: 2D illustration of our analytical solution (4.25) for $0 < x \leq 2$

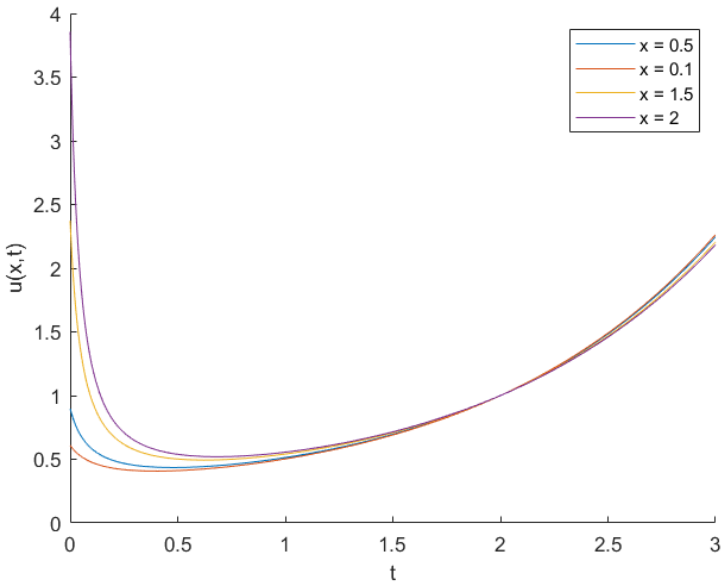

Figure 4.11: 2D illustration of our analytical solution (4.28) for $0 < x \leq 2$

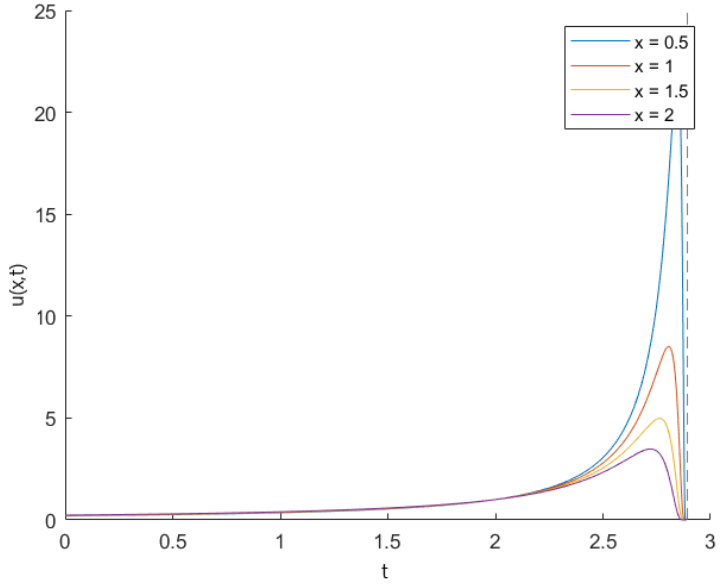

Figure 4.12: 2D illustration of our analytical solution (4.31) for $0 < x \leq 2$

4.4 Graphical Results

The obtained solutions help us deduce the relationship between the value, time and the asset price as shown in figures 4.1-4.4. From these plots, we see that as time increases, the asset price decreases and the value increases. We note the direct relation between time and the value, as opposed to time and the asset price. We also note that at an early stage, the asset price is very high, and this is where one buys an asset at an amount the market assigned to it. But as time passes, the value of the asset depreciates and so does the price.

Figures (4.5-4.8) show the relation between the value and the asset price. From these plots, we see a direct relationship between the two variables. As asset price increases, so does the value. This relationship is not linear, but follows a convex curve. The plots for solutions (4.22) and (4.28) are not far from being linear, meaning the rate of increase for the asset price is not too much higher than that of the value, unlike in solutions (4.25) and (4.31). We also take into account that solutions (4.25) and (4.31) are impacted by the extra values K, σ, α and λ. These factors have a notable impact on the value. Lastly, from these figures, we also see that at $t = 2$, the value $u(x, t) = 1$ (very clear on plot (4.8), again proving our solutions to indeed satisfy our boundary condition $u(x, T) = 1$.

4.5 Conclusion

Investment and consumption problems are an active area of study, although not many findings report exact solutions subject to the terminal condition. In this work, we have successfully derived such solutions for several parameter values. This

approach may be applied to similar types of models, however, the convolution integrals may sometimes be challenging to solve.

We have derived new symmetry generators, viz. (4.19) for each case with the α coefficient (4.20), (4.23), (4.26) and (4.29). These symmetries with the α coefficients are extremely involved but are essential in the sense that they incorporate the important initial conditions. Hence, these are novel solutions and we have presented a substantial advancement in the application of symmetry groups to models in the mathematics of finance.

Chapter 5

An approach to solve parabolic financial equations by point symmetries that incorporate boundary conditions

In this chapter, we show how transformation techniques coupled with a convolution integral can be used to solve a generalised option-pricing model, including the Black-Scholes model, and a given terminal condition. The special convolutions are extremely involved as they arise from the initial-value problem with Lie symmetries and an ISC. The main outcome is that the point symmetries are effective in producing solutions that satisfy a given terminal condition.

5.1 Introduction

In this study, we consider a general pricing equation, given by

$$\frac{\partial u(x,t)}{\partial t} + \kappa x^p \frac{\partial^2}{\partial x\,\partial x} u(x,t) + \lambda(\beta - x)\frac{\partial}{\partial x} u(x,t) + \gamma x^q u(x,t) = 0, \qquad x > 0,\ t \in [0,T], \tag{5.1}$$

where $p \geq 0$, $q \geq 0$, are arbitrary constants. Other constants, in various contexts, may be linked to the volatility of the stock (κ), the risk free interest rate (λ and β) and elasticity (γ). Also, x is the stock price at time t, T is the time of expiry and $u(x,t)$ is the value function, where $u : (0,\infty) \times [0,T] \to \mathbb{R}$. This approach will work for many types of terminal conditions. Suppose we consider a European call-option, that is

$$u(x,T) = \begin{cases} x - k & \text{if} \quad x > k \\ 0 & \text{if} \quad x \leq k, \end{cases} \tag{5.2}$$

where k is the strike price. This terminal equation is continuous and unsmooth at $x = k$.

Eq. (5.1) is a generalisation of several popular models in financial mathematical applications.

Originally, many financial models such as the one above, appear as evolutionary PDEs with unknown coefficients. Black-Scholes [3] were the first to reduce a financial equation to the classical heat equation. In another study [40], Kolmogorov related equations which were transformable to the backward heat model were studied. However, along with Eq. (5.1), more complex models exist in the known literature, but many cannot be reduced to simple equations with known solutions.

In fact, in the nineteenth century, Lie [19] devised a way to classify certain classes

of equations that reduce to the heat equation. However, this particular application of his work is certainly not trivial and not well-known. This is our approach in the present paper.

The equations that model financial instruments have long been an important research area. A parabolic model of this type was previously considered in [49, 45] to find solutions based on symmetry invariance, but did not consider the terminal condition. Symmetry analysis is commonly performed without any consideration of initial or boundary values. However, in this work we apply the symmetry generators in such a way that the above terminal condition is satisfied for the obtained solutions. We consider two cases, one of which is related to the popular equations mentioned above but the other case is introduced here, from certain assumptions of the free parameters. As a consequence of this study, we offer a mathematical way of determining how such free parameters of the model should be determined. This is a significant step in formulating equations for mathematical modelling.

We note that many studies of equations, in finance or otherwise, involve the Black-Scholes model or the famous heat transfer equation [10, 22, 23, 35, 38, 41, 43, 44, 47, 48], to mention a few. As we will uncover below, Lie symmetries are a versatile tool for manipulating initial or boundary conditions. As previously mentioned, the solution of initial data problems for parabolic models, such as (5.1) can be reduced to the construction of a particular solution with specific singularities known in the literature as fundamental solutions [33, 42] According to [34], for certain families of equations, the fundamental solution is an invariant solution.

In Section 5.2, we define the unknown coefficients and specify the transformations to reduce the above equation to the one dimensional heat model. Section 5.3 describes the primary results of this investigation. We establish certain characteristics of the Cauchy problem, and thereafter provide analytical solutions of (5.1) given some

terminal condition. We conclude our results in Section 5.4.

5.2 Transformations and Parameters

We first prove that it is possible to transform (5.1) to the classical heat equation, under specific restrictions on the free parameters of the model. The change of variables is important to set up the Cauchy problem and manipulate the solutions such that they satisfy the terminal condition (5.2). This idea provides a mathematical tool for choosing the parameters and formulate experimental models for further study. Therefore we will prove the following.

Theorem 5.2.1 *Eq. (5.1) is converted to (2.42), which is the 1+1 heat equation, with the parameter choices:*

(a) Case I: $p = 2,\ q = 0, \quad \kappa = \frac{A^2}{2}, \quad \beta = 0, \quad \gamma = -B \quad$ *and* $\quad \lambda = -B$, *(A, B are arbitrary constants). The transformations are*

$$\hat{t} = T - t, \tag{5.3}$$

$$y = \frac{\sqrt{2}\log(x)}{A}. \tag{5.4}$$

(b) Case II: $p = 0,\ q = 2, \quad \kappa = \frac{1}{4}, \quad \beta = 0, \quad \gamma = 4 \quad$ *and* $\quad \lambda = 2$. *The transformations are*

$$\hat{t} = T - t, \tag{5.5}$$

$$y = 2x. \tag{5.6}$$

Proof 5.2.2 *Case I: application of y and $\hat{t}$, reduces Eq. (5.1) to the form (3.5), where $u(y, \hat{t})$, and the functions are $a(y, \hat{t}) = \dfrac{A^2 - 2B}{\sqrt{2}A}$ and $c(y, \hat{t}) = B$.*

Thereafter, the variable $u(y,\hat{t})$, is defined as (3.6) and

$$\phi(y,\hat{t}) = \frac{B^2\hat{t}}{2A^2} - \frac{y\,(A^2-2B)}{2\sqrt{2}A} + \frac{A^2\hat{t}}{8} + \frac{B\hat{t}}{2}. \tag{5.7}$$

This transformation procedure converts Eq. (3.5) to the heat equation (2.42).

Case II is analogous, but with $a(y,\hat{t}) = -2y$ and $c(y,\hat{t}) = -y^2$ and similarly, we define (3.6) with

$$\phi(y,\hat{t}) = \hat{t} + \frac{y^2}{2}. \tag{5.8}$$

Case I is equivalent to the Black-Scholes call option model while Case II can be viewed as a modified pricing model. We note that the transformation of the Black-Scholes model to the 1+1 heat model is well known but we include it under our study to validate our approach below. Theorem 5.2.1 induces the following lemma involving the terminal condition.

Lemma 5.2.1 *The corresponding condition (5.2) is, via the transformations (5.3)-(5.8),*

$$\omega(y,\hat{t}) = max\Big\{\left(e^{\frac{Ay}{\sqrt{2}}} - k\right)\exp\left(\frac{B^2\hat{t}}{2A^2} - \frac{y\,(A^2-2B)}{2\sqrt{2}A} + \frac{A^2\hat{t}}{8} + \frac{B\hat{t}}{2}\right), 0\Big\}, \tag{5.9}$$

for Case I, and

$$\omega(y,\hat{t}) = max\Big\{e^{\hat{t}+\frac{y^2}{2}}\left(\frac{y}{2} - k\right), 0\Big\}, \tag{5.10}$$

for Case II.

The proof follows easily from the transformations given in Theorem 5.2.1.

5.3 The Cauchy Problem

The 1+1 heat equation is one of the most unique models, whose fundamental solution is remarkable. In the work that follows, we illustrate how the fundamental solution, combined with symmetry properties, solves a Cauchy problem for Eq. (2.42), the heat equation. This process, once the transformations of Theorem 5.2.1 are reversed, leads to the solution of (5.1) subject to the terminal condition.

From Lemma 5.2.1, let $\omega(y,0) = F(y)$, where $F(y)$ is the max function.

We now proceed to the application of Lie point symmetries This is an intricate process and often requires one to evaluate complicated integrals. Recall that the symmetry generators of (2.42), are given in the most general form as (2.45) or, equivalently (2.46). In this symmetry form, $\alpha(y,\hat{t})$ is an arbitrary solution that satisfies the heat equation, i.e. $\alpha_{\hat{t}} = \alpha_{yy}$. We apply the invariant surface condition (ISC) (2.48) which holds at the boundary where it is imposed that $\hat{t} = 0$ [46]. This leads to the condition (2.49) [28]. If we evaluate condition (2.49), we have that (2.50). Equation (2.50) is an ODE which is connected to a general initial condition and it provides less restrictive criteria on the initial data so that they may be obtained by a specific symmetry. We now illustrate some solutions, where we do not provide an exhaustive list of solutions that can be found by the method described above, but rather we focus on several special conditions and solutions.

As mentioned previously, the above theory gives rise to various solutions to (2.42) and consequently, Eq. (5.1). We emphasise that all solutions are subject to the Cauchy problem, while solutions to (5.1) satisfy the terminal equation (5.2).

Noting that it is rare to find a symmetry study involving initial conditions, we start

with Case I. Let $c_1 = c_2 = c_3 = c_4 = c_5 = 0$, $c_6 = 1$ to obtain the symmetry

$$X = \frac{\partial}{\partial \hat{t}} + \alpha(y, \hat{t}) \frac{\partial}{\partial \omega}$$

and function

$$\alpha(y, 0) = F''(y).$$

Thus, with (2.48), the solution to (2.42), the heat model is

$$\omega(y, \hat{t}) = \int \alpha(y, \hat{t}) d\hat{t} + H(y).$$

Without loss of generality, the arbitrary function may vanish, i.e. $H(y) = 0$ but $\alpha(y, \hat{t})$ is still unknown. However, we recall something remarkable about $\alpha(y, \hat{t})$ - it is also a solution to (2.42).

Hence form integrating the convolution, we find

$$\alpha(y, \hat{t}) = \frac{\exp\left(\frac{\left(A^2-2B\right)\left(A^2\hat{t}-2\sqrt{2}Ay-2B\hat{t}\right)}{8A^2}\right)\left(\left(A^2+2B\right)^2 e^{\frac{Ay}{\sqrt{2}}+B\hat{t}} - k\left(A^2-2B\right)^2\right)}{8A^2}, \tag{5.11}$$

and another integration to find $\omega(y, \hat{t})$ subject to (2.47), we calculate that

$$\omega(y, \hat{t}) = \exp\left(\frac{\left(A^2-2B\right)\left(A^2\hat{t}-2\sqrt{2}Ay-2B\hat{t}\right)}{8A^2}\right)\left(e^{\frac{Ay}{\sqrt{2}}+B\hat{t}} - k\right). \tag{5.12}$$

To construct the solution to the original model (5.1), we induce the transformations from Theorem 5.2.1 to find that

$$u(x, t) = x - ke^{B(t-T)}, \tag{5.13}$$

subject to (5.2).

As a final illustration, consider Case II. For simplicity, let $c_4 = 1, c_1 = c_2 = c_3 = c_5 = c_6 = 0$. Hence

$$\alpha(y, 0) = F'(y),$$

and

$$X = \frac{\partial}{\partial y} + \alpha(y, \hat{t})\frac{\partial}{\partial \omega}.$$

Thus, via the ISC (2.48), we obtain that the solution to (2.42) is

$$\omega(y, \hat{t}) = \int \alpha(y, \hat{t})dy + H(\hat{t}).$$

We let $H(\hat{t}) = 0$, and determine $\alpha(y, \hat{t})$.

Thus, repeating the procedure, we obtain

$$\alpha(y, \hat{t}) = -\frac{e^{-\frac{y^2}{4\hat{t}+2}}\left(-2k(2\hat{t}y + y) - 2\hat{t} + y^2 - 1\right)}{2\sqrt{\frac{1}{\hat{t}} + 2}\sqrt{\hat{t}}(2\hat{t} + 1)^2}, \tag{5.14}$$

the heat equation solution subject to (2.47) is

$$\omega(y, \hat{t}) = \frac{e^{-\frac{y^2}{4\hat{t}+2}}(y - 2k(2\hat{t} + 1))}{2\sqrt{\frac{1}{\hat{t}} + 2}\sqrt{\hat{t}}(2\hat{t} + 1)} \tag{5.15}$$

and the pricing equation has solution, subject to (5.2), as

$$u(x, t) = \frac{e^{-\frac{4x^2}{4(T-t)+2} - t + T + 2x^2}(2x - 2k(2(T - t) + 1))}{2\sqrt{2(T - t) + 1}(2(T - t) + 1)}. \tag{5.16}$$

More solution can be found in this manner. If we let $t = T$ the time of expiry, we see that (5.13) and (5.16) simplifies to

$$u(x, t) = x - k,$$

which shows that the solutions are subject to the terminal condition. We plot the solution of (5.13) which shows a steady growth in option value, but (5.16) predicts a steep option value as asset price rises to a particular value. We also see the following solution arising:

$$u(x,t) = e^{C(t-T)}\left(xe^{B(T-t)} - k\right) \tag{5.17}$$

which was also tested and found to be indeed satisfying the PDE (5.1). It was also found to be subject to the terminal condition (5.2) at $t = T$.

Below are the graphical illustrations of the analytical solution (5.13):

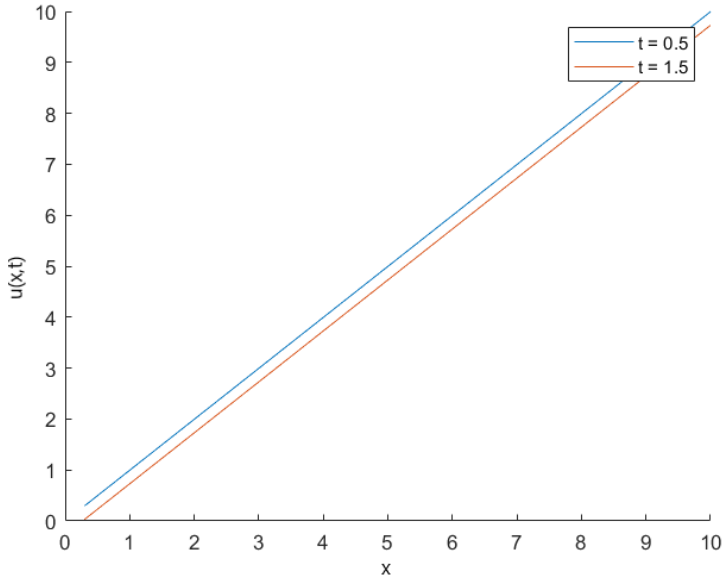

Figure 5.1: 2D Solution (5.13) for $0 < t < 2$, with $k = 2, T = 2, B = 4$.

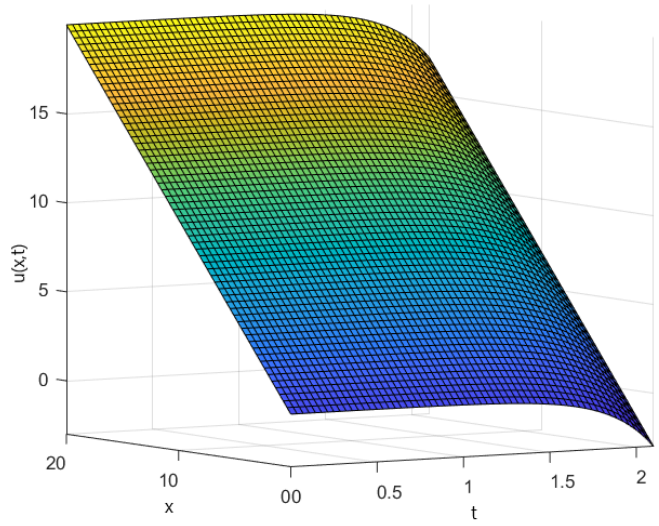

Figure 5.2: 3D plot of solution (5.13), with $k = 2, T = 2, B = 4$.

Below are the graphical illustrations of the analytical solution (5.17):

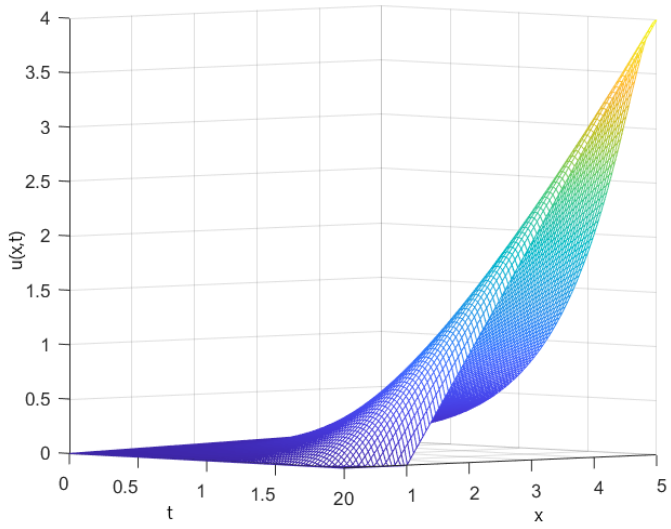

Figure 5.3: 3D illustration of our analytical solution (5.17)

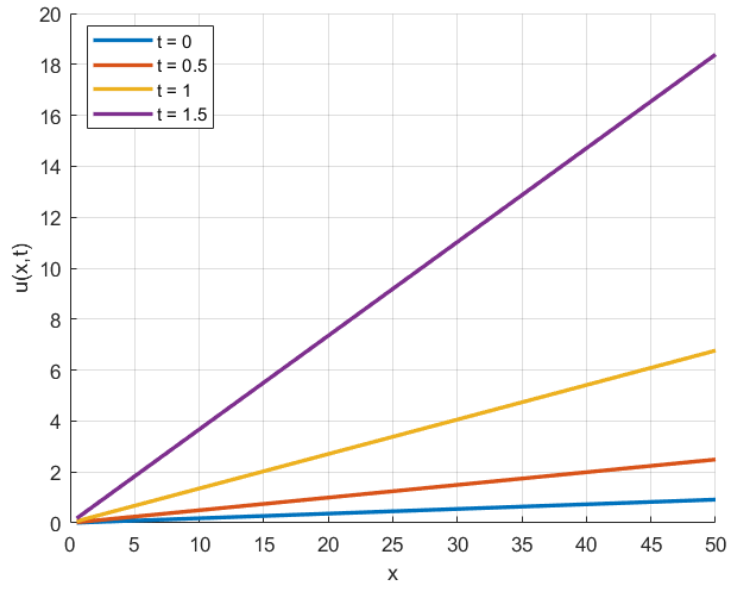

Figure 5.4: 2D illustration of our analytical solution (5.17), with $0 \leq t < 2$.

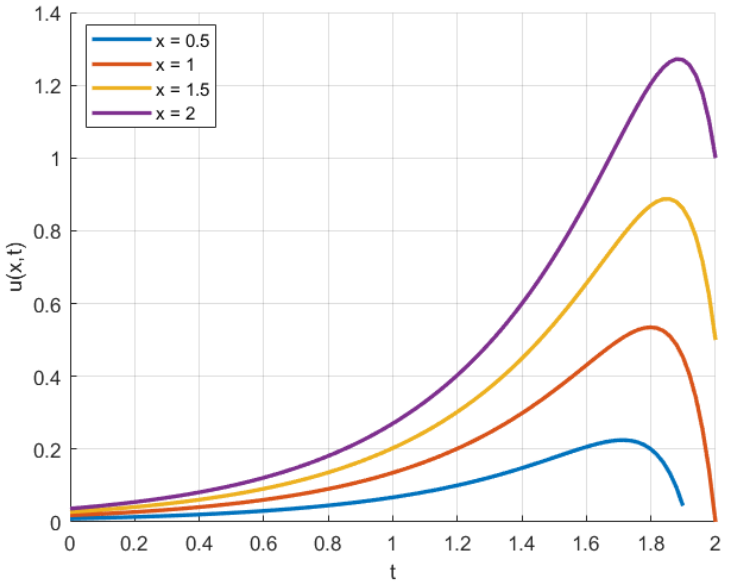

Figure 5.5: 2D illustration of our analytical solution (5.17), with $0 < x \leq 2$.

Below are the graphical illustrations of the analytical solution (5.16):

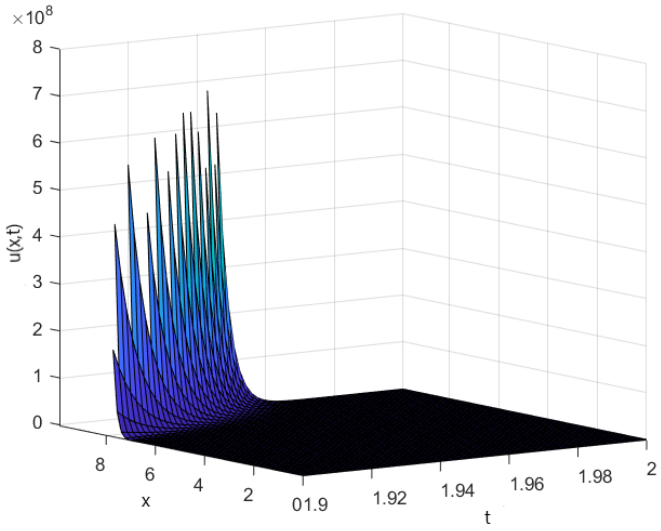

Figure 5.6: 3D illustration of our analytical solution (5.16), when $k = 6$.

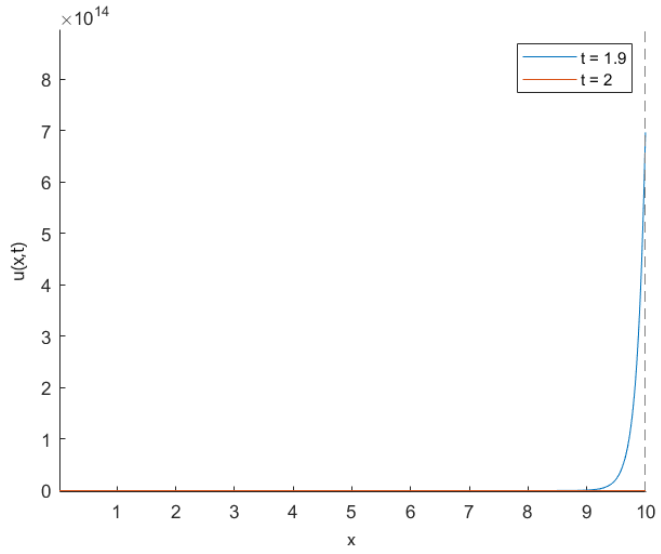

Figure 5.7: 2D illustration of our analytical solution (5.16), when $k = 6$.

5.4 Conclusion

A general equation composed of European options has been solved for certain parameters. The unknown parameters were chosen to effect a transformation to the heat equation. Consequently the main pricing equation was analysed subject to its terminal condition. We have set up a Cauchy problem to the heat equation and used Lie symmetry generators to obtain new and non-trivial solutions. Most of the equations that model financial derivatives are of the type considered here, and therefore this approach may be adopted for many of the models. Such models are often plagued with difficulties in finding solutions, however we have demonstrated how effective transformations are in simplifying these models in order to obtain solutions.

Conclusion

Through a combination of theoretical analysis and numerical simulations, this thesis contributes to the existing body of knowledge on financial modelling by incorporating Lie symmetries, boundary conditions and convolution into the analysis of the Black-Scholes model and the pricing models. The insights gained from this research have the potential to improve pricing accuracy, risk management strategies, and decision-making in the field of finance, benefiting both academia and industry practitioners.

By employing symmetry analysis, we uncovered the symmetries inherent in the transformed model's governing equation, the heat equation, which allowed us to derive invariant solutions and gain deeper insights into the behaviour of options. Furthermore, we explored the impact of boundary conditions on financial models, and have discovered that accurate specification of boundary conditions is crucial for obtaining meaningful results in option pricing and bond pricing models. We also examined the effects of different boundary conditions, on these models, enhancing our understanding of their limitations and providing insights into their behaviour in various financial scenarios.

We have also studied the convolution technique, a powerful mathematical tool which

play a significant role in solving heat models. We investigated the application of initial or Cauchy problems, in option pricing and bond pricing models. These techniques allowed us to solve complex equations efficiently, leading to solutions offering more accurate pricing and risk assessment. Through these techniques, we were able to effectively derived insights on the relation between the value function $u(x,t)$ and the asset or stock price, as well as the relation between the value function and time.

Moreover, we explored the role of the heat equation in financial modelling. The heat equation emerges as a fundamental equation in many financial models due to its diffusion-like properties. We analyzed the connection between the heat equation and the Black-Scholes model, option pricing models, and bond pricing models, highlighting the heat equation's role in solving these models and understanding their underlying dynamics.

In further applications, we plan to extend this study to other models belonging to different fields.

Appendix

Chapter 3: Solution (3.14)

Verification that the solution (3.14) is indeed a solution to the PDE (3.1), and it's subject to the terminal condition (3.2).

Our solution is:

```
u[x, t] := -(E^(-(-t+T)/(2 ρ^2) + 1/(x ρ^2) - Log[-Sqrt[2]/(x ρ)]) E^((-t+T-2/x)/(2 ρ^2)) (Sqrt[2] + Sqrt[2] k (-t + T) - Sqrt[2] k/x))/ρ
```

So we have that:

```
γ = 2; κ = β = 0; λ = -1/ρ;
∂t u[x, t] + 1/2 ρ^2 * x^(2+γ) * ∂x,x u[x, t] + (κ + β * x - λ * ρ * x^γ) * ∂x u[x, t] - x * u[x, t] // Simplify
```

```
0
```

Now, lets see what happens to u[x,t] at expiry, t=T:

```
t := T
Simplify[-(E^(-(-t+T)/(2 ρ^2) + 1/(x ρ^2) - Log[-Sqrt[2]/(x ρ)]) E^((-t+T-2/x)/(2 ρ^2)) (Sqrt[2] + Sqrt[2] k (-t + T) - Sqrt[2] k/x))/ρ]
```

```
-k + x
```

This proves the solution is subject to the terminal condition.

Chapter 4: Solution (4.22)

Verification that the solution (4.22) is indeed a solution to the PDE (4.1), and it's subject to the terminal condition (4.2).

Our solution is:

$$u[x,t] := \frac{e^{t-T+\frac{3(-t+T)}{\sqrt{2}}+\frac{1}{2}(-2+\sqrt{2})x-\frac{2(-2+\sqrt{2})x}{4+4(-2+\sqrt{2})(-t+T)}+2(t-T)\delta}}{\sqrt{(-2+\sqrt{2})*(-t+T)+1}}$$

So we have that:

$$\alpha := \sqrt{2};\ \lambda := \frac{1}{2};\ \beta := 1;\ K := \frac{1}{\gamma};\ \gamma := -1;$$

Hence:

$$\partial_t u[x,t] + \left(\frac{2\lambda}{-1+\lambda}*\gamma*(\beta-\alpha) - 2\gamma*\beta\right)*x*\partial_x u[x,t] + \left(\frac{-\delta+\alpha\lambda}{1-\lambda} + \frac{\lambda}{2(-1+\lambda)^2}*\left(\frac{(-\alpha+\beta)^2}{K^2}\right)*x\right)*u[x,t] + \gamma*(2\gamma+1)*K^2*\partial_x u[x,t] + 2\gamma^2*K^2*x*\partial_{x,x}u[x,t] \text{ // Simplify}$$

$$0$$

Now, lets see what happens to u[x,t] at expiry, t=T:

$$t := T$$

$$\text{Simplify}\left[\frac{e^{t-T+\frac{3(-t+T)}{\sqrt{2}}+\frac{1}{2}(-2+\sqrt{2})x-\frac{2(-2+\sqrt{2})x}{4+4(-2+\sqrt{2})(-t+T)}+2(t-T)\delta}}{\sqrt{(-2+\sqrt{2})*(-t+T)+1}}\right]$$

$$1$$

This proves the solution is subject to the terminal condition.

Chapter 4: Solution (4.25)

Verification that the solution (4.25) is indeed a solution to the PDE (4.1), and it's subject to the terminal condition (4.2).

Our Solution is:

```
u[x_, t_] := -((3 Sqrt[2] (t - T) + Sqrt[3] (-t + T) - (1/2) Sqrt[3] (-2 + Sqrt[2]) x - (2 Sqrt[3] (-2 + Sqrt[2]) x)/(4 + 4 Sqrt[3] (-2 + Sqrt[2]) (t - T)) + (-t + T) K)
  Sqrt[(2 Sqrt[3] - Sqrt[6]) (-t + T) + 1] (Sqrt[3] (-2 + Sqrt[2]) + 6 (-3 + 2 Sqrt[2]) (-t + T)))/
 (Sqrt[3] (-2 + Sqrt[2]) (-1 + 18 (-3 + 2 Sqrt[2]) (t - T)^2 + 3 Sqrt[3] (-2 + Sqrt[2]) (-t + T) + 6 Sqrt[3] (-10 + 7 Sqrt[2]) (-t + T)^3))
```

So we have:

```
α := Sqrt[1 + λ]; λ := 2; β := Sqrt[3 + λ]; K := -1; γ := 1;
```

Hence:

```
∂t u[x, t] + ((2 λ)/(-1 + λ) + γ (β - α) - 2 γ β) x ∂x u[x, t] + (((-δ + α λ)/(1 - λ) + λ/(2 (-1 + λ)^2)) + ((-α + β)^2/K^2) x) u[x, t] + γ (2 γ + 1) K^2 ∂x u[x, t] + 2 γ^2 K^2 x ∂x,x u[x, t] // Simplify
```

```
0
```

Now, lets see what happens to u[x,t] at expiry, t=T:

```
t := T
```

```
Simplify[-((3 Sqrt[2] (t - T) + Sqrt[3] (-t + T) - (1/2) Sqrt[3] (-2 + Sqrt[2]) x - (2 Sqrt[3] (-2 + Sqrt[2]) x)/(4 + 4 Sqrt[3] (-2 + Sqrt[2]) (t - T)) + (-t + T) K)
  Sqrt[(2 Sqrt[3] - Sqrt[6]) (-t + T) + 1] (Sqrt[3] (-2 + Sqrt[2]) + 6 (-3 + 2 Sqrt[2]) (-t + T)))/
 (Sqrt[3] (-2 + Sqrt[2]) (-1 + 18 (-3 + 2 Sqrt[2]) (t - T)^2 + 3 Sqrt[3] (-2 + Sqrt[2]) (-t + T) + 6 Sqrt[3] (-10 + 7 Sqrt[2]) (-t + T)^3))]
```

```
1
```

This proves the solution is subject to the terminal condition.

Chapter 4: Solution (4.28)

Verification that the solution (4.28) is indeed a solution to the PDE (4.1), and it's subject to the terminal condition (4.2).

Our solution is:

```
In[•]:= u[x, t] := Sqrt[2/3] e^(1/6 (5 Sqrt[2] (t-T) - 2 (t-T) δ - (Sqrt[2] x)/κ^2 + (6 x)/((3 Sqrt[2] + 2 t - 2 T) κ^2))) (3 Sqrt[2] + 2 t - 2 T) / (3 + Sqrt[2] t - Sqrt[2] T)^(3/2)
```

So we have:

```
In[•]:= α := 1/β; λ := 4; β := Sqrt[2]; γ := -1;
```

Hence:

```
In[•]:= ∂t u[x, t] + ((2 λ)/(-1 + λ) * γ * (β - α) - 2 γ * β) * x * ∂x u[x, t] + ((-δ + α λ)/(1 - λ) + λ/(2 (-1 + λ)^2) * ((-α + β)^2/κ^2)) * x) * u[x, t] + γ * (2 γ + 1) * κ^2 * ∂x u[x, t] + 2 γ^2 * κ^2 * x * ∂x,x u[x, t] // Simplify
Out[•]= 0
```

Now, lets see what happens to u[x,t] at expiry, t=T:

```
In[•]:= t := T
In[•]:= Simplify[Sqrt[2/3] e^(1/6 (5 Sqrt[2] (t-T) - 2 (t-T) δ - (Sqrt[2] x)/κ^2 + (6 x)/((3 Sqrt[2] + 2 t - 2 T) κ^2))) (3 Sqrt[2] + 2 t - 2 T) / (3 + Sqrt[2] t - Sqrt[2] T)^(3/2)]
Out[•]= 1
```

This proves the solution is subject to the terminal condition.

Chapter 4: Solution (4.31)

Proof that the solution (4.31) is indeed a solution to the PDE (5.1), and it's subject to the terminal condition (4.2).

Our solution is:

```
In[•]:= u[x, t] := e^( 1/2 ( x α ( 1/(1+√λ) + 1/(-1+(-1+t α-T α) √λ) ) √λ / E^2 - t (2 δ-3 α √λ+α λ)/(-1+λ) + T (2 δ-3 α √λ+α λ)/(-1+λ) ) ) √(1 + (α - α/(1+√λ)) (-t+T)) (1+√λ)^2 / (-1+(-1+t α-T α) √λ)^2
```

Other constraints on (1) for case IV:

```
In[•]:= β := α √λ; γ := 1;
```

Hence:

```
In[•]:= ∂t u[x, t] + ((2 λ)/(-1+λ) γ (β - α) - 2 γ β) x ∂x u[x, t] + (-(δ + α λ)/(1 - λ) + λ/(2 (-1+λ)^2) ((-α+β)^2/E^2) x) u[x, t] + γ (2 γ + 1) E^2 ∂x u[x, t] + 2 γ^2 E^2 x ∂x,x u[x, t] // Simplify
Out[•]= 0
```

Now, lets see what happens to u[x,t] at expiry, t=T:

```
In[•]:= t := T
In[•]:= Simplify[ e^( 1/2 ( x α ( 1/(1+√λ) + 1/(-1+(-1+t α-T α) √λ) ) √λ / E^2 - t (2 δ-3 α √λ+α λ)/(-1+λ) + T (2 δ-3 α √λ+α λ)/(-1+λ) ) ) √(1 + (α - α/(1+√λ)) (-t+T)) (1+√λ)^2 / (-1+(-1+t α-T α) √λ)^2 ]
Out[•]= 1
```

This proves the solution is subject to the terminal condition.

Chapter 5: Solution (5.13)

Verification that the solution (5.13) is indeed a solution to the PDE (5.1), and it's subject to the terminal condition (5.2).

Our solution is:

```
In[1]:= u[x, t] := x - k e^(B (t-T));
```

So we have:

```
p = 2; q = 0; κ = A^2/2; β = 0; λ = γ = -B;
```

Hence

```
In[3]:= ∂t u[x, t] + κ * x^p * ∂x,x u[x, t] + λ * (β - x) * ∂x u[x, t] + γ * x^q * u[x, t] // Simplify
Out[3]= 0
```

Now, lets see what happens to u[x,t] at expiry, t=T:

```
In[4]:= t := T
In[5]:= Simplify[x - k e^(B (t-T))]
Out[5]= -k + x
```

This proves the solution is subject to the terminal condition.

Chapter 5: Solution (5.16)

Proof that the solution (5.16) is indeed a solution to the PDE (5.1), and it's subject to the terminal condition (5.2).

Our solution is:

In[4]:=
$$u[x,t] := \frac{e^{-\frac{4x^2}{4(T-t)+2}-t+T+2x^2}(2x-2k(2(T-t)+1))}{2\sqrt{2(T-t)+1}\,(2(T-t)+1)};$$

So we have:

In[5]:=
$$p=0;\ q=2;\ \kappa=\frac{1}{4};\ \beta=0;\ \lambda=2;\ \gamma=4;$$

Hence

In[6]:=
$$\partial_t u[x,t] + \kappa * x^p * \partial_{x,x} u[x,t] + \lambda * (\beta - x) * \partial_x u[x,t] + \gamma * x^q * u[x,t] \text{ // Simplify}$$

Out[6]=
$$0$$

Now, lets see what happens to u[x,t] at expiry, t=T:

In[7]:=
$$t := T$$

In[8]:=
$$\text{Simplify}\left[\frac{e^{-\frac{4x^2}{4(T-t)+2}-t+T+2x^2}(2x-2k(2(T-t)+1))}{2\sqrt{2(T-t)+1}\,(2(T-t)+1)}\right]$$

Out[8]=
$$-k+x$$

This proves the solution is subject to the terminal condition.

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