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Potential Symmetries and Conservation Laws

for p.d.e.s including Perturbations

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Declaration

I declare that the contents of this research report are original except where due references have been made. It has not been submitted before for any degree to any other institution.

R.I. Kiguwa
30 April 2004

Abstract

Relationships between symmetries and conservation laws of perturbed partial differential equations are reviewed. Potential symmetries and their applications to perturbed partial differential equations and conservation laws are presented in detail. An example of a perturbed wave equation for an inhomogeneous medium is solved in detail. Proofs of some of the lesser-known theorems are outlined. A wide range of examples is given to further explain these concepts.

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Contents

Introduction **5**

1 Preliminaries **7**

1.1 Introduction 7

1.2 Notation and Preliminaries 7

1.3 Potential Symmetries 11

1.4 Approximate Symmetries 12

2 Approximate Potential Symmetries **15**

2.1 Introduction 15

2.2 Approximate Potential Symmetries of PDEs 15

2.3 A Perturbed Wave Equation for an Inhomogeneous Medium 18

2.4 Approximate Invariant Solutions 28

3 Potential Symmetries and Associated Conservation Laws **32**

3.1 Introduction 32

3.2 Relating Symmetries to Conservation Laws 33

3.3 Wave Equation with Variable speed Example 37

3.4 Approximate Potential Conservation Laws 45

Conclusion **50**

References 52

Introduction

This report will look at partial differential equations and their solutions. The study of partial differential equations (PDE's) is a fundamental subject area in mathematics which links important strands of pure mathematics to applied and computational mathematics. PDE's are important in almost all of the applications of mathematics where they provide a natural mathematical description of phenomena in the physical, natural and social sciences.

Sophus Lie invented the idea of a continuous group called a Lie group. His invention unified the various solution methods for ordinary differential equations (ODEs). Lie showed that if an ODE is invariant under a one parameter Lie group of point transformations, its order can be reduced, therefore simplifying and making it easier to solve.

A German mathematician Emmy Noether proved her theorem in 1918, which explained a justification for conservation laws. Her theorem explains that conservation laws follow from the symmetry properties of nature. She proved that for Euler-Lagrange differential equations there exists a Noether symmetry associated with a Lagrangian that corresponded to a conservation law.

Bluman and Kumei [4] takes a look at classes of symmetries of differential equations by considering non-local symmetries, called potential symmetries. These potential symmetries can be computed by Lie's algorithm.

The concept of potential symmetries is further investigated by extending it to approximate symmetries. The idea of approximate symmetries was developed by Baikov et al [2], which is the application of Lie group methods on differential equations with small parameters. The symmetries obtained are used to construct invariant solutions of the partial differential equation.

Kara and Mahomed [10] investigated the association of symmetries with conservation laws in the absence of a Lagrangian, which was not possible with Noether's theorem.

All these ideas and concepts were looked at in isolated reports. The aim of this report is to put these isolated works into one complete report.

In chapter 1, the notation and operators used throughout the report is given. An introduction to potential and approximate symmetries is given.

In chapter 2, the notion of approximate potential symmetries for a partial differential equation with a small parameter is discussed. Examples on how to construct approximate potential symmetries are shown and their corresponding approximate group invariant solutions are also obtained.

In chapter 3, a relationship between symmetries and conservations laws is shown by a link between components of a conserved vector of a partial diferential equation and the Lie-Bäcklund symmetry generator of the equation. We are then able to construct conservation laws from a potential symmetry for a partial differential equation, irrespective of the existence of a Lagrangian. We further extended and combined the concepts of approximate potential symmetries and conservation laws to generate approximate potential conservation laws.

Chapter 1

Preliminaries

1.1 Introduction

This chapter describes the notation and operators used in the report. Most of the literature was obtained from Kara and Mahomed [10], Kara and Qu [13]. A brief description on potential and approximate symmetries is given at end of the chapter.

1.2 Notation and Preliminaries

The convention that repeated indices imply summation is used in the report. Let $x = (x^1, x^2, \ldots, x^n) \in \mathbb{R}^n$ be the independent variable with coordinates x^i and $u = (u^1, u^2, \ldots, u^m) \in \mathbb{R}^m$ be the dependent variable with coordinates u^α. Let $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be the projection map $\pi(x, u) = x$. Let $s : \mathcal{X} \subset \mathbb{R}^n \to \mathcal{U} \subset \mathbb{R}^{n+m}$ be a smooth map such that $\pi \circ s = 1_\chi$, where 1_χ is the identity map on χ. The partial derivatives of u with respect to x are connected by the operator of total differentation

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, \ldots, n, \tag{1.1}$$

as

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j D_i(u^\alpha), \quad \ldots. \tag{1.2}$$

We let $u_{(1)}$ denote the collection of all first-order derivatives u_i^α. Similarly, we let $u_{(2)}$, $u_{(3)}$,... denote the collection of all higher order derivatives. The r-jet bundle $J^r(\mathcal{U})$ is given by the equivalence classes of sections of $\mathcal{U}$, with coordinates denoted

by $(x^i, u^\alpha, \ldots, u^\alpha_{i_1 \ldots i_r})$, where $1 \leq i_1 \leq \ldots \leq i_r \leq n$. The r-jet bundle on $\mathcal{U}$ will be written as

$$J^r(\mathcal{U}) = \{(x, u, u_{(1)}, \ldots, u_{(r)}) : (x, u) \in \mathcal{U}\}.$$

We now look at the space of the diferential forms on $J^r(\mathcal{U})$. Let $\Omega^r_k(\mathcal{U})$ be the vector space of *differential k-forms* on $J^r(\mathcal{U})$ with *differential* d. A smooth differential k-form on $J^r(\mathcal{U})$ is given by

$$\omega = f_{i_1 i_2 \ldots i_k} \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \ldots \wedge \mathrm{d}x^{i_k},$$

where each component $f_{i_1 i_2 \ldots i_k} \in \Omega^r_0(\mathcal{U})$, i.e.,

$$f_{i_1 i_2 \ldots i_k} = f_{i_1 i_2 \ldots i_k}(x, u, u_{(1)}, \ldots, u_{(r)}).$$

For differentiable functions $f \in \Omega^r_0(\mathcal{U})$

$$Df = D_j f \mathrm{d}x^j, \tag{1.3}$$

where D is the *total differential* or the *total exterior derivative*. In addition, the total exterior derivative of ω is

$$D\omega = Df_{i_1 i_2 \ldots i_k} \wedge \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \ldots \wedge \mathrm{d}x^{i_k},$$

and we can use (1.3) to obtain

$$D\omega = Df_{i_1 i_2 \ldots i_k} \mathrm{d}x^j \wedge \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \ldots \wedge \mathrm{d}x^{i_k}.$$

The exterior derivative d has properties similar to the algeraic properties of the total derivative D, that is

$$D(\omega \wedge \nu) = D\omega \wedge \nu + (-1)^k \omega \wedge D\nu,$$

for ω a k-form and ν an l-form, and also $D(D\omega) = 0$. Also, a known result is that, if $D\omega = 0$, then ω is a locally exact k-form, i.e., $\omega = D\nu$ for some $(k-1)$-form ν.

Let us consider an rth-order system of partial differential equations of n independent and m dependent variables

$$F^\beta(x, u, u_{(1)}, \ldots, u_{(r)}) = 0, \quad \beta = 1, \ldots, m\,. \tag{1.4}$$

Definition 1.1 A *conserved form* of (1.4) is a differential $(n-1)$-form

$$\omega = T^i(x, u, u_{(1)}, \ldots, u_{(r-1)}) \left(\frac{\partial}{\partial x^i} \lrcorner (\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n) \right), \tag{1.5}$$

defined on $J^{r-1}(\mathcal{U})$ if

$$D\omega = 0, \tag{1.6}$$

is satisfied for all solutions of (1.4).

When Definition 1.1 is satisfied, (1.6) is called a *conservation law* for (1.4). Also, when (1.6) is evaluated on the surface of (1.4) we obtain

$$D_i T^i = 0. \tag{1.7}$$

The tuple $T = (T^1, \ldots, T^n)$, $T^j \in \Omega_0^{r-1}(\mathcal{U})$, $j = 1, \ldots, n$, is called a *conserved vector* of (1.4).

Let

$$\mathcal{A} = \bigcup_{r=0}^{p} \Omega_0^r(\mathcal{U}), \text{ for some} p < \infty.$$

Then $\mathcal{A}$ is the universal space of differential function of infinite order.

Definition 1.2 The Lie-Bäcklund operator is given by the infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \cdots, \tag{1.8}$$

where $\xi^i, \eta^\alpha \in \mathcal{A}$ and the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} \zeta_i^\alpha &= D_i(W^\alpha) + \xi^j u_{ij}^\alpha, \\ \zeta_{i_1 i_2}^\alpha &= D_{i_1} D_{i_2}(W^\alpha) + \xi^j u_{j i_1 i_2}^\alpha, \\ &\vdots \\ \zeta_{i_1 \ldots i_s}^\alpha &= D_{i_1} \ldots D_{i_s}(W^\alpha) + \xi^j u_{j i_1 \ldots i_s}^\alpha, \quad s = 1, 2, \ldots . \end{aligned} \tag{1.9}$$

In (1.9), W^α is the Lie characteristic function given by

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha. \tag{1.10}$$

If X is a Lie-Bäcklund operator, ω a k-form and ν an l-form, e.g

$$X(\omega \wedge \nu) = X(\omega) \wedge \nu + \omega \wedge X(\nu).$$

In the special cases where there are four or less independent variables (x_1, x_2, x_3, x_4), and one dependent variable u, the independent variables will be denoted by (t, x, y, z). Then the operators D_1, D_2,... will be denoted by D_t, D_x, ..., and the Lie-Bäcklund operator $X = \xi^i \partial/\partial x^i + \eta^\alpha \partial/\partial u^\alpha$ will be written as follows

$$X = \tau \frac{\partial}{\partial t} + \xi^x \frac{\partial}{\partial x} + \xi^y \frac{\partial}{\partial y} + \xi^z \frac{\partial}{\partial z} + \eta \frac{\partial}{\partial u},$$

with prolongation coefficients denoted by ζ_t, ζ_x, ζ_y, ζ_z, ζ_{tt}, ζ_{tx}, ...

Definition 1.3 The Lie-Bäcklund operators $\tilde{X}$ and X are said to be *equivalent* if

$$X - \tilde{X} = \lambda^i D_i, \quad \lambda^i \in \mathcal{A}.$$

If $\lambda^i = \xi^i$ then $\tilde{X}$ is called *canonical* operator.

Definition 1.4 The Euler-Lagrange operator is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 \cdots i_s}}, \quad \alpha = 1, \ldots, m. \tag{1.11}$$

The Euler-Lagrange operator (1.11) is used in the multi-dimensional case. For the one-dimensional case the operator

$$\frac{\partial}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial u'}.$$

is used. These two operators were introduced by Lagrange(1762) and Euler(1744), respectively.

Definition 1.5 A Lie-Bäcklund operator X is said to be a *Noether symmetry* generator associated with a Lagrangian $L \in \mathcal{A}$ if there exists a vector $B = (B^1, \ldots, B^n)$, $B^i \in \mathcal{A}$, such that

$$X(L) + L D_i(\xi^i) = D_i(B^i). \tag{1.12}$$

If, in equation (1.12), $B^i = 0$, $i = 1, \ldots, n$, then X is called a *strict Noether symmetry* generator associated with a Lagrangian $L \in \mathcal{A}$.

Theorem 1.6 For any Noether symmetry X corresponding to a given Lagrangian $L \in \mathcal{A}$, there corresponds a vector $T = (T^1, \ldots, T^n)$, $T^i \in \mathcal{A}$, defined by

$$T^i = N^i(L) - B^i, \quad i = 1, \ldots, n, \tag{1.13}$$

which is a conserved vector of the Euler-Lagrange equations which are in differential form as

$$\frac{\delta L}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m, \tag{1.14}$$

where $\delta/\delta u^\alpha$ is the Euler-Lagrange operator defined by

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 \cdots i_s}} \qquad \alpha = 1, \ldots, m\,. \tag{1.15}$$

Definition 1.7 The Noether operator associated with a Lie-Bäcklund operator X is defined by

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s \geq 1} D_{i_1} \cdots D_{i_s}(W^\alpha) \frac{\delta}{\delta u^\alpha_{i i_1 \cdots i_s}}, \quad i = 1, \ldots, n, \tag{1.16}$$

where the Euler-Lagrange operators with respect to derivatives of u^α are obtained from (1.15) by replacing u^α by the corresponding derivatives, for example,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s \geq 1} (-1)^s D_{j_1} \cdots D_{j_s} \frac{\partial}{\partial u^\alpha_{i j_1 \cdots j_s}} \quad i = 1, \ldots, n, \qquad \alpha = 1, \ldots, m\,. \tag{1.17}$$

1.3 Potential Symmetries

The reduction procedure for a PDE in two variables is based on a similarity variable which will allow to get solutions of the original PDE by integration of an ordinary differential equation (ODE). Reduction procedures usually search solutions which are invariant under local symmetries (see Pucci and Saccomandi [15]). Bluman et al

[3] introduced a method that finds a new class of symmetries for PDE's written in conservative form. They introduced a potential as an unknown function to analyse the Lie symmetries of the system that is obtained. Bluman and Kumei [4], define non-local symmetries as symmetries whose infinitesimal at any point x depend on the global behaviour of $u(x)$. This implies that the infinitestimals may depend on the integrals of dependent variables.

In determining potential symmetries admitted by a partial differential equation, we find local symmetries of an auxiliary system of the PDE. The auxiliary system is obtained using the conserved form of the PDE. This gives an auxiliary dependent variable v which is a potential to an auxiliary system of PDE's $S\{x, u, v\}$. Let $R\{x, y\}$ be a system of equations which are in a conserved form. We note that $R\{x, y\}$ is embedded in $S\{x, u, v\}$. This implies that any solution $(u(x), v(x))$ of $S\{x, u, v\}$ will define a $u(x)$ of $R\{x, y\}$, also for any solution of $u(x)$ of $R\{x, y\}$ there is an associated function $v(x)$ such that $(u(x), v(x))$ defines a solution of $S\{x, u, v\}$.

Furthermore, if there are local symmetries defining a group G_s which are admitted by $S\{x, u, v\}$ then any symmetry of G_s maps any solution of $S\{x, u, v\}$ into another solution of $S\{x, u, v\}$. This implies that any solution $R\{x, y\}$ maps into another solution of $R\{x, y\}$. A non-local symmetry is thus admitted by $R\{x, y\}$ if it is induced by a local symmetry in G_s, as a result the infinitestimals of variables (x, u) of $S\{x, u, v\}$ depend explicitly on the potential variable v. The non-local symmetry of $R\{x, y\}$ is called a potential symmetry.

In summary potential symmeties admitted by $R\{x, y\}$ are solved or computed by Lie's algorithm using the fact that they are local symmetries admitted by an auxiliary system $S\{x, u, v\}$.

1.4 Approximate Symmetries

It was observed that most differential equations that arise in mathematical modelling are invariably approximate, hence approximate symmetries should be considered. Therefore, the theory of approximate symmetries associated with a differential equation with a small parameter is looked at. This will lead to an understanding of how approximate conservation laws can be associated with approximate symmetries.

It is important to note that no matter the type of infinitesimal transformation

under consideration it is always possible to introduce the approximate form of that particular transformation. If the differential equation being looked at has a small constant or parameter then the equation is a perturbation of the equation when the constant is set to zero.

Definition 1.8 ([6]) For any equation

$$F(x, u, \ldots, \epsilon) \equiv F_o(x, u, \ldots) + \epsilon F_1(x, u, \ldots) = 0 \tag{1.18}$$

with a small parameter ϵ, we can define an approximate equation as the class of equations $G(x, u, \ldots, \epsilon) = 0$ with functions G such that $G(x, u, \ldots, \epsilon) \approx F(x, u, \ldots, \epsilon)$.

Definition 1.9 ([6]) The class of transformation of the form

$$\bar{x}^i = \phi^i(x, u, \varepsilon), \bar{u}^k = \psi^k(x, u, \varepsilon), i = 1, \ldots, n; k = 1, \ldots, m \tag{1.19}$$

with the functions ϕ, ψ satisfying the conditions

$$\phi^i(x, u, \varepsilon) \approx \phi_0^i(x, u,) + \varepsilon\phi_1^i(x, u,)$$
$$\psi^k(x, u, \varepsilon) \approx \psi_0^k(x, u) + \varepsilon\psi_1^k(x, u,)$$

is called an approximate transformation.

We can conclude that equation (1.19) is an approximate symmetry transformation of equation (1.18) if it approximately conserves the corresponding approximate equation.

Similarly just like the theory of Lie symmetry transformation, the calculation of approximate symmetry transformation are obtained through the approximate symmetry group.

Definition 1.10 ([12]) An operator $\mathcal{X}$ is a k^{th} - order approximate symmetry of equation (1.18) if

$$\mathcal{X}(E^\beta) \mid_{E^\beta=0} = O(\varepsilon^{k+1}), \tag{1.20}$$

where

$$\mathcal{X} = X_0 + \varepsilon X_1 + \ldots + \varepsilon^k X_k, \tag{1.21}$$

and X_i are Lie-Bäcklund operators . In the case $X_0 \neq 0$, X_0 is a stable symmetry and vice-versa.

Theorem 1.7 ([6]) Let the equation in (1.18) be approximately invariant under the approximate group with the generator

$$\mathcal{X} = X_0 + \varepsilon X_1 + \ldots + \varepsilon^k X_k,$$

such that $X_0 \neq 0$, is the generator of the equation

$$F_0(x, u, \ldots) = 0. \tag{1.22}$$

For calculating first order approximate symmetries of equations with a small parameter, the following 3-step algorithm can be used [6].

Step 1 Find the exact symmetry generators X_0 of the unperturbed equation. We can do this by solving the determining equation for exact symmetries

$$X_0 F_0(x, u, \ldots) \mid_{F_0(x,u,\ldots)=0} = 0. \tag{1.23}$$

Step 2 Using the X_0 obtained from step 1 and a perturbation $\varepsilon F_1(x, u, \ldots)$, calculate the *auxiliary function* H given by

$$H \approx \frac{1}{\varepsilon}[X_0(F_0 + \varepsilon F_1) \mid_{F_0+\varepsilon F_1=0}]. \tag{1.24}$$

Step 3 Find the first-order deformations (i.e. the operators X_1) from the determining equation for deformations

$$X_1 F_0 \mid_{F_0=0} + H = 0. \tag{1.25}$$

Chapter 2

Approximate Potential Symmetries

2.1 Introduction

In the last chapter, a brief outline of potential symmetries and approximate symmetries was given. This was a necessary introduction to approximate potential symmetries, which combines both concepts. The aim of this chapter is to discuss the method of approximate potential symmetries for PDEs, and then solve some examples to illustrate how the method works. At the end of the chapter, we show how to obtain approximate invariant solutions.

2.2 Approximate Potential Symmetries of PDEs

In general, the method of finding approximate potential symmetries involves writing a given perturbed partial diffential equation R in a conserved form. From this conserved form an associated system S of potential variable is obtained. This implies that approximate Lie point symmetries admitted by S induce approximate potential symmetries of R.

The method of approximate symmetries is developed as follows [11].

Consider a scalar k-th order perturbed partial differential equation $R\{x, u, \varepsilon\}$, which

is written in conserved form

$$D_i[T_0^i(x,u,u_{(1)},\ldots,u_{(k-1)})+\varepsilon T_1^i(x,u,u_{(1)},\ldots,u_{(k-1)})]=0. \tag{2.1}$$

The PDE (2.1) is in conserved form, therefore there exist $n(n-1)/2$ functions $\psi^{i,j}$ components of an antisymmetric tensor $(i<j)$, such that (2.1) can be shown in the form

$$\begin{aligned} T_0^i(x,u,u_{(1)},\ldots,u_{(k-1)})+\varepsilon T_1^i(x,u,u_{(1)},\ldots,u_{(k-1)}) \\ =\sum_{i<j}(-1)^j\frac{\partial}{\partial x_j}\psi^{i,j}+\sum_{j<i}(-1)^{i-1}\frac{\partial}{\partial x_j}\psi^{j,i}, \end{aligned} \tag{2.2}$$

where $i,j=1,2,\ldots,n$. Setting $\psi^{i,j}=0$ for $j\neq i+1$ and introducing

$$v_i=\psi^{i,i+1},\quad i=1,2,\ldots,n-1\,,$$

the system (2.2) associated with $R\{x,u,\varepsilon\}$ given by (2.1) becomes the following auxiliary system of partial differential equations, $S\{x,u,v,\varepsilon\}$

$$\begin{aligned} &T_0^1+\varepsilon T_1^1=\frac{\partial}{\partial x_2}v^1,\\ &T_0^j+\varepsilon T_1^j=(-1)^{j-1}\Big[\frac{\partial}{\partial x_{j+1}}v^j+\frac{\partial}{\partial x_{j-1}}v^{j-1}\Big],\quad 1<j<n,\\ &T_0^n+\varepsilon T_1^n=(-1)^{n-1}\frac{\partial}{\partial x_{n-1}}v^{n-1}. \end{aligned} \tag{2.3}$$

In paricular,[8] for $n=2$ (let $x^1=t$ and $x^2=x$) we can cast (1.18) as an auxiliary system $S\{x,t,u,v\}$ so that

$$\begin{aligned} T_0^1+\varepsilon T_1^1&=v_x,\\ T_0^2+\varepsilon T_1^2&=-v_t. \end{aligned} \tag{2.4}$$

Definition 2.1[8]. An approximate Lie point symmetry generator

$$\begin{aligned} X&=X_0+\varepsilon X_1\\ &=\tau_0\frac{\partial}{\partial t}+\xi_0\frac{\partial}{\partial x}+\phi_0\frac{\partial}{\partial u}+\zeta_0\frac{\partial}{\partial v}\\ &+\varepsilon\left[\tau_1\frac{\partial}{\partial t}+\xi_1\frac{\partial}{\partial x}+\phi_1\frac{\partial}{\partial u}+\zeta_1\frac{\partial}{\partial v}\right], \end{aligned} \tag{2.5}$$

of (2.4) is an approximate potential symmetry of (1.18) if and only if the coefficients τ_i, ξ_i and ϕ_i depend explicitly on v. In general, all the coefficients depend on x, t, u and v.

The algorithm developed by Baikov et al [2] for determining approximate symmetries $X = X_0 + \epsilon X_1$ is used on (2.4), where X_0 is a Lie point symmetry generator of the unperturbed form of (2.4). This implies that X_0 leaves the system $v_x - T_0^1 = 0$ and $v_t + T_0^2 = 0$ invariant, then we can calculate the auxilliary functions H_1 and H_2 by

$$\begin{aligned} H_1 &= \varepsilon^{-1} X_0(v_x - T_0^1 - \varepsilon T_1^1) \mid_{(2.4)}, \\ H_2 &= \varepsilon^{-1} X_0(v_t + T_0^2 + \varepsilon T_1^2) \mid_{(2.4)} . \end{aligned} \tag{2.6}$$

Then, X_1 can be determined from

$$\begin{aligned} X_1(v_x - T_0^1) + H_1 &= o(\varepsilon), \\ X_1(v_t + T_0^2) + H_2 &= o(\varepsilon), \end{aligned} \tag{2.7}$$

along the unperturbed system $v_x - T_0^1 = 0$ and $v_t + T_0^2 = 0$.

It is important to note that X_0 may not be a potential symmetry of the unperturbed pde, but the overall symmetry X may turn out to be an approximate potential symmetry for that particular or respective perturbed PDE. Kara,[8] gives an example of the wave equation $u_{tt} = e^{2x} u_{xx}$ which has an auxilliary form $v_x = e^{-2x} u_t$ and $v_t = -u_x$ with a Lie point symmetry generator $\partial/\partial x - t\partial/\partial t - v\partial/\partial v$ which is not a potential symmetry of the equation. However the equation is perturbed to $u_{tt} = e^{2x} u_{xx} - \epsilon u_t$, it then will have the approximate potential symmetry $\partial/\partial x - t\partial/\partial t - v\partial/\partial v + \epsilon[\frac{1}{2} v\partial/\partial u + \frac{1}{2} e^{-2x} u\partial/\partial v]$.

In the next sections, we will give examples of solving different types of equations to obtain approximate potential symmetries. These examples in Kara, [8], include Burger's equation, and some shallow water wave equations.

In the last section, we will show how to obtain approximate invariant solutions using these approximate potential symmetries.

2.3 A Perturbed Wave Equation for an Inhomogeneous Medium

This example illustrates how a symmetry which is not potential symmetry of the unperturbed PDE is used, but the resulting symmetry turns out to be an approximate potential symmetry of the PDE in question.

The wave equation with a first-order damping term,

$$u_{tt} + \epsilon u_t = e^{2x} u_{xx} \tag{2.8}$$

has an auxiliary system

$$\begin{aligned} u_t + \epsilon u = e^{2x} v_x \quad &\Longrightarrow v_x = e^{-2x}(u_t + \epsilon u), \\ u_x &= v_t. \end{aligned} \tag{2.9}$$

The unperturbed equation $u_{tt} = e^{2x} u_{xx}$ has a symmetry of

$$X_0 = \frac{\partial}{\partial x} - t\frac{\partial}{\partial t} - v\frac{\partial}{\partial v}$$

which is not a potential symmetry.

X_0 is prolonged to the first order to become

$$X_0 = X_0 + \eta_u^t \frac{\partial}{\partial u_t} + \eta_v^x \frac{\partial}{\partial v_x},$$

where

$$\eta_u^t = D_t(0 - u_x + t u_t) = u_t$$

and

$$\eta_v^t = D_x(-v - v_x + t v_t) + v_{xx} - t v_{xt}.$$

Then substitute for η_u^t and η_v^t in X_0, i.e,

$$X_0 = \frac{\partial}{\partial x} - t\frac{\partial}{\partial t} - v\frac{\partial}{\partial v} + u_t\frac{\partial}{\partial u_t} - v_x\frac{\partial}{\partial v_x}.$$

Using the algorithm developed by Baikov et al, equations (2.6) becomes

$$H_1 = \varepsilon^{-1} X_0(u_t - \varepsilon u - e^{2x} v_x) \mid_{u_x = v_t, u_t = e^{2x} v_x - \varepsilon u},$$

$$H_1 = \varepsilon^{-1}(u_t - 2e^{2x} v_x + e^{2x} v_x) \mid_{u_x = v_t, u_t = e^{2x} v_x - \varepsilon u} .$$

That is,

$$H_1 = \varepsilon^{-1}(e^{2x} v_x - \varepsilon u - 2e^{2x} v_x + e^{2x} v_x),$$

$$H_1 = \varepsilon^{-1}(-\varepsilon u) = -u.$$

X_1 can be determined from equation (2.7),

$$X_1 = \xi(x,t)\frac{\partial}{\partial x} + \tau(x,t)\frac{\partial}{\partial t} + \phi(x,t,u,v)\frac{\partial}{\partial u} + \psi(x,t,u,v)\frac{\partial}{\partial v} + \phi^x \frac{\partial}{\partial u_x} + \phi^t \frac{\partial}{\partial u_t} + \psi^x \frac{\partial}{\partial v_x} + \psi^t \frac{\partial}{\partial v_t},$$

on equation 2.7

$$X_1(u_t - e^{2x} v_x) \mid_{u_x = v_t, u_t = e^{2x} v_x - \varepsilon u} = 0,$$

that is,

$$\phi^t - 2e^{2x} v_x . \xi - e^{2x} \psi^x - u = 0,$$

or

$$\phi_t + u_t(\phi_u - \tau_t) - u_x \xi_t - u_t u_x \xi_u + v_t \phi_v - u_x v_t \xi_v - u_t v_t \tau_v - u_t^2 \tau_u - 2e^{2x} v_x \xi$$

$$-e^{2x}[\psi_x + u_x \psi_u + v_x(\psi_v - \xi_x) - v_t \tau_x - u_x v_x \xi_u - v_x^2 \xi_v - u_x v_t \tau_u - v_x v_t \tau_v] - u = 0. \quad (2.10)$$

Substituting for u_t and v_t in (2.10) we obtain

$$\phi_t + e^{2x}(\phi_u - \tau_t)v_x + (\phi_v - \xi_t)u_x - e^{2x}(\tau_v + \xi_u)u_x v_x - \xi_v u_x^2 - e^{4x}v_x^2\tau_u - 2e^{2x}v_x\xi - e^{2x}[\psi_x + (\psi_u - \tau_x)u_x + (\psi_v - \xi_x)v_x - (\tau_v + \xi_u)u_x v_x - v_x^2\xi - \tau_u u_x^2] - u = 0.$$

Separating by derivatives of u and v yields

$$u_x^2 : \qquad -\xi_v + e^{2x}\tau_u = 0$$

$$v_x^2 : \qquad -e^{4x}\tau_u + e^{2x}\xi_v = 0$$

$$u_x : \qquad \phi_v - \xi_t - e^{2x}(\psi_u - \tau_x) = 0$$

$$v_x : \qquad e^{2x}(\phi_u - \tau_t) - 2e^{2x}\xi - e^{2x}(\psi_v - \xi_x) = 0$$

$$u_x v_x : \qquad -e^{2x}(\tau_v + \xi_u) + e^{2x}(\tau_v + \xi_u) = 0$$

$$Rest : \qquad \phi_t - e^{2x}\psi_x - u = 0.$$

The second auxilliary function H_2 is

$$H_2 = \varepsilon^{-1}X_0(u_x - v_t) \mid_{u_x = v_t, u_t = e^{2x}v_x - \varepsilon u} = 0$$

then X_1 acting on equation (2.7) gives,

$$X_1(u_x - v_t) \mid_{u_x = v_t, u_t = e^{2x}v_x - \varepsilon u} -0 = 0,$$

or

$$\phi^x - \psi^t = 0. \tag{2.11}$$

Substituting for u_t and v_t in (2.11) we obtain

$$\phi_x + (\phi_u - \xi_x)u_x + (\phi_v - e^{2x}\tau_x)v_x - u_x^2\xi_u - (\xi_v + e^{2x}\tau_u)u_x v_x - e^{2x}\tau_v v_x^2$$
$$-\psi_t - (\psi_u e^{2x} - \xi_t)v_x - (\psi_v - \tau_t)u_x + \xi_u e^{2x}v_x^2 + u_x v_x \xi_v + \tau_u e^{2x}v_x u_x + \tau_v u_x^2 = 0.$$

Separating by derivatives of u and v

$$u_x^2: \qquad -\xi_u + \tau_v = 0 \quad \Longrightarrow \xi_u = \tau_v$$

$$v_x^2: \qquad -e^{2x}\tau_v + e^{2x}\xi_u = 0 \quad \Longrightarrow \xi_u = \tau_v$$

$$u_x v_x: \qquad -\xi_v - e^{2x}\tau_u + e^{2x}\tau_u + \xi_v = 0$$

$$u_x: \qquad \phi_u - \xi_x - \psi_v + \tau_t = 0$$

$$v_x: \qquad \phi_v - e^{2x}\tau_x - \psi_u e^{2x} + \xi_t = 0$$

$$Rest: \qquad \phi_x - \psi_t = 0.$$

Summary of all equations obtained are

$$\xi_v = e^{2x}\tau_u, \tag{2.12}$$

$$\xi_u = \tau_v, \tag{2.13}$$

$$\phi_v - \xi_t - e^{2x}(\psi_u - \tau_x) = 0, \tag{2.14}$$

$$\phi_u - \tau_t - \psi_v + \xi_x - 2\xi = 0, \tag{2.15}$$

$$\phi_t - e^{2x}\psi_x = u, \tag{2.16}$$

$$\xi_x + \psi_v + \phi_u - \tau_t = 0, \tag{2.17}$$

$$e^{2x}\tau_x + e^{2x}\psi_u - \phi_v - \xi_t = 0, \tag{2.18}$$

$$\phi_x = \psi_t. \tag{2.19}$$

Combining (2.15) and (2.17) gives

$$\tau_t = \xi_x - \xi \tag{2.20}$$

and

$$\phi_u = \psi_v + \xi. \tag{2.21}$$

Combining (2.14) and (2.18) gives

$$\tau_x = e^{-2x}\xi_t \tag{2.22}$$

and

$$\phi_v = \psi_u e^{2x}. \tag{2.23}$$

Thus,

$$\begin{aligned} \phi &= f_1(x,t)u + g_1(x,t)v + h_1(x,t), \\ \psi &= f_2(x,t)u + g_2(x,t)v + h_2(x,t), \end{aligned} \tag{2.24}$$

$$\tau = \tau(x,t),$$

$$\xi = \xi(x,t).$$

From (2.21)

$$f_1 = g_2 + \xi \tag{2.25}$$

and by (2.23)

$$g_1 = f_2 e^{2x}. \tag{2.26}$$

From (2.16)

$$f_{1t}u + g_{1t}v + h_{1t} = e^{2x}(f_{2x}u + g_{2x}v + h_{2x}) + u. \tag{2.27}$$

Separation of variable for the above equation,

u : $\quad f_{1t} = e^{2x} f_{2x} + 1$

v : $\quad g_{1t} = e^{2x} g_{2x}$

Rest : $\quad h_{1t} = e^{2x} h_{2x}.$

From (2.19)

$$f_{1x}u + g_{1x}v + h_{1x} = f_{2t}u + g_{2t}v + h_{2t} \tag{2.28}$$

and separation gives

$$u: \quad f_{1x} = f_{2t}$$

$$v: \quad g_{1x} = g_{2t}$$

$$Rest: \quad h_{1x} = h_{2t}.$$

As a result of the above we obtain $h_1 = h_2 = 0$.

From (2.20) we obtain

$$\tau_{tx} = \xi_{xx} - \xi_x.$$

From (2.23) we obtain

$$\tau_{xt} = e^{-2x}\xi_{tt}$$

$$\Longrightarrow \xi_{xx} - \xi_x = e^{-2x}\xi_{tt}$$
$$\Longrightarrow e^{2x}(\xi_{xx} - \xi_x) = \xi_{tt}.$$

From (2.25) we obtain

$$f_{1t} = g_{2t} + \xi_t$$

$$\Longrightarrow g_{2t} + \xi_t = e^{2x} f_{2x} + 1.$$

From (2.26) we obtain

$$g_{1t} = e^{2x} f_{2t}$$

$$\Longrightarrow e^{2x} f_{2t} = e^{2x} g_{2x} \Longrightarrow f_{2t} = g_{2x}$$
$$\Longrightarrow f_{2t} = f_{1x} = g_{2x}$$

$$\Longrightarrow f_1 = g_2 + a(t). \tag{2.29}$$

Since

$$f_1 = g_2 + \xi, \quad \xi = a(t).$$

From (2.26)

$$g_{1x} = 2e^{2x} f_2 + f_{2x} e^{2x}.$$

From (2.29)

$$g_2 = f_1 - a(t).$$

As

$$g_{2t} = g_{1x},$$

$$g_{2t} = f_{1t} - a^{'} = e^{2x} f_{2x} + 1 - a^{'}.$$

This gives

$$f_2 = \frac{1}{2} e^{-2x}(1 - a^{'}). \tag{2.30}$$

Using equation (2.30)

$$g_1 = f_2 e^{2x} \quad \Longrightarrow g_1 = \frac{1}{2}(1 - a^{'})$$

and

$$g_{2t} = 0 \quad \Longrightarrow g_2 = b(x).$$

Hence from (2.29)

$$f_1 = b(x) + a(t).$$

Then from Equation (2.20)

$$\tau_t = -a.$$

$$\tau = -at + c(x),$$

using

$$a^{'} = c^{'} e^{2x}$$

we obtain

$$a^{'} = 0 \quad \Longrightarrow a = c_1$$
$$c^{'} = 0 \quad \Longrightarrow c = c_2.$$

Then

$$\tau = -c_1 t + c_2.$$

Thus,

$$g_1 = \frac{1}{2},$$
$$f_2 = \frac{1}{2} e^{-2x},$$
$$\tau = -c_1 t + c_2,$$
$$\xi = a(t) = c_1,$$
$$b = c_3.$$

Substituting these new variables into equation (2.24) gives

$$\phi = (c_1 + c_3)u + \frac{1}{2}v,$$
$$\psi = \frac{1}{2} e^{-2x} u + c_3 v.$$

The following symmetries are obtained,

$$u\frac{\partial}{\partial u} - t\frac{\partial}{\partial t} + \frac{\partial}{\partial x},$$
$$\frac{\partial}{\partial t},$$

$$\frac{1}{2}v\frac{\partial}{\partial u}+\frac{1}{2}e^{-2x}u\frac{\partial}{\partial v}.$$

Hence, we obtain the following approximate potential symmetry of equation (2.8). Note that even though the unperturbed part is not a potential symmetry but the perturbed term is

$$X=\frac{\partial}{\partial x}-t\frac{\partial}{\partial t}-v\frac{\partial}{\partial v}+\varepsilon(\frac{1}{2}v\frac{\partial}{\partial u}+\frac{1}{2}e^{-2x}u\frac{\partial}{\partial v}). \tag{2.31}$$

We can further illustrate how to obtain an approximate potential symmetry of the same equation, by using another potential symmetry.

Let us again consider the equation (2.8).

With its auxiliary system(2.9).

The following potential symmetry of the unperturbed equation (2.8) is used

$$X_0=-4t\frac{\partial}{\partial x}+2(t^2+e^{-2x})\frac{\partial}{\partial t}+2(v-2tu)\frac{\partial}{\partial u}+2e^{-2x}u\frac{\partial}{\partial v}+(2v_t-4u+4u_x-8tu_t)\frac{\partial}{\partial u_t}$$
$$-(4e^{-2x}u-2e^{-2x}u_x-4e^{-2x}v_t)\frac{\partial}{\partial v_x}.$$

Finding the functions H_1 and H_2 we obtain

$$\begin{aligned}H_1&=\varepsilon^{-1}X_0(u_t-\varepsilon u-e^{2x}v_x)\\&=2v+4tu\end{aligned}$$

and

$$\begin{aligned}H_2&=\varepsilon^{-1}X_0(u_x-v_t)\\&=0.\end{aligned}$$

We can determine X_1 using equation (2.7), i.e.,

$$X_1(u_t - e^{2x}v_x)\mid_{u_x=v_t,u_t=e^{2x}v_x} +2v + 4tu = 0.$$

This equation yields

$$\phi^t - 2e^{2x}v_x\xi - e^{2x}\psi^x + 2v + 4tu = 0.$$

Also,

$$X_1(u_x - v_t)\mid_{u_x=v_t,u_t=e^{2x}v_x} = 0,$$

which gives

$$\phi^x - \psi^t = 0.$$

We again make assumptions that ξ and τ are functions of x and t
and
$\phi = f_1(x,t)u + g_1(x,t)v + h_1(x,t), \quad \psi = f_2(x,t)u + g_2(x,t)v + h_2(x,t).$

We separate by derivatives of u and v to obtain the following system of equations.

$$f_1 = g_2 + \xi,$$
$$g_1 = e^{2x}f_2,$$
$$f_{1t} = e^{2x}f_{2x} - 4t,$$
$$g_{1t} = e^{2x}g_{2x} - 2,$$
$$f_{2t} = f_{1x},$$
$$g_{2t} = g_{1x},$$
$$\tau_t = \xi_x - \xi,$$
$$\xi_t = e^{2x}\tau_x.$$

After solving, we obtain

$$\xi = e^{-2x} + 3t^2 + c_3t + c_4,$$

$$\tau = -3e^{-2x}t - t^3 - \frac{1}{2}c_3t^2 - c_4t - \frac{1}{2}c_3e^{-2x} + c_5,$$
$$f_1 = \frac{5}{2}e^{-2x} + 3t^2 + c_3t + c_1 + c_4,$$
$$f_2 = -5te^{-2x} - \frac{1}{2}c_3e^{-2x},$$
$$g_1 = -5t - \frac{1}{2}c_3,$$
$$g_2 = \frac{3}{2}e^{-2x} + c_1.$$

Here c_1,c_3,c_4 and c_5 are constants. Making c_1=c_3=c_4=c_5= 0 will give us the approximate potential symmetry

$$X = -4t\frac{\partial}{\partial x} + 2(t^2 + e^{-2x})\frac{\partial}{\partial t} + 2(v - 2tu)\frac{\partial}{\partial u} + 2e^{-2x}u\frac{\partial}{\partial v}$$
$$+\varepsilon[(e^{-2x}+3t^2)\frac{\partial}{\partial x}+(-3e^{-2x}t-t^3)\frac{\partial}{\partial t}+((\frac{5}{2}e^{-2x}+3t^2)u-5tv)\frac{\partial}{\partial u}+(-5te^{-2x}u+\frac{3}{2}e^{-2x}v)\frac{\partial}{\partial v}].$$

2.4 Approximate Invariant Solutions

We now find an approximate invariant solution up to the first order in ε using the approximate potential symmetry obtained for the first example.

Let us consider equation (2.8). We are going to use the approximate potential symmetry (2.31) to obtain approximate invariant solutions of (2.8)of up to order ε. Then (2.31) becomes

$$X = \frac{\partial}{\partial x} - t\frac{\partial}{\partial t} + \varepsilon\frac{v}{2}\frac{\partial}{\partial u} + (\frac{\varepsilon}{2}e^{-2x}u - v)\frac{\partial}{\partial v}.$$

The characteristics equation is

$$\frac{dx}{1} = \frac{dt}{-t} = \frac{du}{\frac{\varepsilon v}{2}} = \frac{dv}{\frac{\varepsilon e^{-2x}u}{2} - v}. \tag{2.32}$$

We solve the first equation of (2.32) to obtain an invariant of the potential symmetry being used, viz.,

$$x = -\ln t + \tilde{\lambda}, \quad \Longrightarrow \tilde{\lambda} = x + \ln t.$$

Let $\lambda = e^{\tilde{\lambda}} = e^x t$,
the other equations are

$$\frac{du}{dt} = -\frac{\varepsilon v}{2t} \quad \Longrightarrow v = -\frac{2t}{\varepsilon}u',$$

$$\begin{aligned}\frac{dv}{dt} = -\frac{\frac{\varepsilon}{2}e^{-2x}u - v}{t} &= -\frac{1}{t}[\frac{\varepsilon}{2}t^2\lambda^{-2}u - v] \\ &= -\frac{2}{\varepsilon}u' + (-\frac{2t}{\varepsilon}u''),\end{aligned}$$

$$\frac{d^2u}{dt^2} = -\frac{\varepsilon}{2}[-\frac{v}{t^2} + \frac{1}{t}\frac{dv}{dt}],$$

$$\begin{aligned}\frac{2t}{\varepsilon}u'' - \frac{2}{\varepsilon}u' &= -\frac{\varepsilon}{2}t\lambda^{-2}u + \frac{2}{\varepsilon}u' \\ \Longrightarrow u'' - \frac{\varepsilon^2}{4}\lambda^{-2}u &= 0.\end{aligned}$$

After solving, we obtain

$$\begin{aligned}u &= u_1(\lambda)e^{\frac{\varepsilon}{2\lambda}t} + u_2(\lambda)e^{-\frac{\varepsilon}{2\lambda}t} \\ &= A(\lambda) + \varepsilon B(\lambda)e^{-x}.\end{aligned}$$

We now solve for u_t,u_{tt} and u_{xx} and substitute into our given equation (2.8) then

solve for $A(\lambda)$ and $B(\lambda)$.

$$
\begin{aligned}
u_t &= A^{'}e^x + \varepsilon B^{'}, \\
u_{tt} &= A^{''}e^{2x} + \varepsilon B^{''}e^x, \\
u_x &= A^{'}te^x + \varepsilon B^{'}t - \varepsilon Be^{-x}, \\
u_{xx} &= A^{''}t^2e^{2x} + tA^{'}e^x + \varepsilon[B^{''}t^2e^x - tB^{'} + Be^{-x}].
\end{aligned}
$$

$$A^{''}e^{2x} + \varepsilon B^{''}e^x + \varepsilon[A^{'}e^x + \varepsilon B^{'}] = e^{2x}(A^{''}t^2e^{2x} + tA^{'}e^x + \varepsilon[B^{''}t^2e^x - tB^{'} + Be^{-x}]),$$

$$A^{''} = A^{''}\lambda^2 + \lambda A^{'},$$

$$B^{''}e^x + \varepsilon A^{'}e^x = (B^{''}\lambda^2 - \lambda B^{'} + B)e^x,$$

$$A^{''}(1-\lambda^2) = \lambda A^{'}, \tag{2.33}$$

$$(1-\lambda^2)B^{''} - \lambda B^{'} + B = A^{'}. \tag{2.34}$$

Now we solve (2.33) to obtain

$$A = k_1 \operatorname{arccosh}(te^x) + k_2$$

and

$$A^{'} = \frac{k_1}{\sqrt{\lambda^2 - 1}} \ .$$

From equation (2.34) we substitute for $A^{'}$and solve for B to obtain,

$$B = k_1\lambda + k_2\sqrt{\lambda^2-1} - \frac{1}{2}\sqrt{\lambda^2-1}\ln(\lambda^2-1) + \lambda\operatorname{arccosh}\lambda$$

Hence

$$u = k_1\operatorname{arccosh}(te^x) + k_2 + \varepsilon[k_1\lambda + k_2\sqrt{\lambda^2-1} - \frac{1}{2}\sqrt{\lambda^2-1}\ln(\lambda^2-1) + \lambda\operatorname{arccosh}\lambda]e^{-x} \ .$$

This is an approximate invariant solution of order ε. In conclusion we have discussed the approximate potential symmetry method of PDE's. The method was shown using an example of a perturbed wave equation with a variable speed.

Chapter 3

Potential Symmetries and Associated Conservation Laws

3.1 Introduction

Emmy Noether's 1918 paper [14] on symmetries and conservation laws, proves that for Euler-Lagrange differential equations and each Noether symmetry associated with a Lagrangian, there exists a corresponding conservation law which can be determined explicitly by means of Theorem 1.6. What if a differential equation does not have a Lagrangian, can a similar result be obtained? This has been pursued by Anco and Bluman [1] where an identity was derived which yields a correspondence between symmetries and conservation laws for self adjoint differentials. This identity does not depend on a Lagrangian as needed by Noether's theorem.

In this chapter a relationship between symmetries and conservation laws will be derived showing that there is a link between the components of a conserved vector of a partial differential equation and the Lie-Bäcklund symmetries generator of the equation.

When there is no Lagrangian for a given equation, the direct method can be employed to construct conservation laws. The method we are investigating uses the direct method and uses a natural symmetry condition to generate a conservation law for the equation in question.

Then we will apply the identity derived by Kara and Mahomed [10] to construct potential conservation laws from potential symmetries for classes of waves equations

with variable wave speeds.

3.2 Relating Symmetries to Conservation Laws

Before we can show how to construct non-local conservation laws from non-local symmetries, a review of some definitions leading to the proof of the identity is given. Ibragimov et al [7] gives an important relation between a Noether symmetry X and its corresponding conserved vector $T = (T^1 ...., T^n)$ where T^i satisfies Theorem 1.6. This is shown by Theorem 3.1. This derivation is obtained from [10].

Theorem 3.1 [7]. The components of the Noether conserved vector T, given by Theorem 1.6, associated with the Lie-Backlund operator X, which is a generator of a Noether symmetries, satisfy

$$X(T^i) + D_k(\xi^k)T^i - T^k D_k(\xi^i) = N^i\big(D_k(B^k)\big) + B^k D_k(\xi^i) \qquad (3.1)$$
$$-D_k(\xi^k)B^i - X(B^i), \qquad i = 1, \ldots, n\,.$$

Ibragimov *et al* [7], proved that any Noether symmetry is equavalent to a strict Noether symmetry. This leads to the next theorem.

Theorem 3.2 [7]. If a Lie-Bäcklund operator X satisfies (1.12), then its equivalent operator

$$\tilde{X} = X - \frac{1}{L}B^i D_i = (\xi^i - \frac{1}{L}B^i)\frac{\partial}{\partial x^i} + (\eta^\alpha - \frac{1}{L}B^i u_i^\alpha)\frac{\partial}{\partial u^\alpha} + ...$$

satisfies

$$\tilde{X}L + LD_i\tilde{\xi}^i = 0,$$

where $\tilde{\xi}^i = \xi^i - \frac{1}{L}B^i$ for $i = 1, \ldots, n$.

If we write Theorem 3.1 independent of B^i and N^i, then we will obtain the following result below

$$X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0, \qquad i = 1, \ldots, n. \tag{3.2}$$

This result (3.2) connects a Noether symmetry generator to components of a conserved vector given by Theorem 1.6. It is important to note that this formulation is obtained from a Lagrangian formulation. What if a differential equation does not have a Lagrangian, will equation (3.2) still hold? This is discussed in Kara and Mahomed [10] where they show a formula relating a conserved vector of a differential equation and its associated Lie-Bäcklund symmetry generator. This is shown as follows using a definition and lemma to prove the formula in the main theorem.

Definition 3.1 The differential k-form

$$\omega = f_{i_1 i_2 \ldots i_k}(x, u, u_{(1)}, \ldots, u_{(r)}) \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k},$$

is called an *invariant form of order k* with respect to the Lie-Bäcklund transformation group

$$\begin{aligned} \bar{x}^i &= \exp(\varepsilon X)(x^i), \\ \bar{u}^\alpha &= \exp(\varepsilon X)(u^\alpha), \\ \bar{u}^\alpha_i &= \exp(\varepsilon X)(u^\alpha_i), \\ &\vdots \end{aligned}$$

for $i = 1, \ldots, n$, $\alpha = 1, \ldots, m$ on $\mathcal{A}$, where ε is the group parameter,

$$\exp(\varepsilon X) = 1 + \varepsilon X + \frac{\varepsilon^2}{2!} X^2 + \frac{\varepsilon^3}{3!} X^3 + \cdots,$$

and X is a Lie-Bäcklund operator (1.8), if

$$f_{i_1 i_2 \ldots i_k}(\bar{x}, \bar{u}, \bar{u}_{(1)}, \ldots, \bar{u}_{(r)}) \mathrm{d}\bar{x}^{i_1} \wedge \ldots \wedge \mathrm{d}\bar{x}^{i_k} = f_{i_1 i_2 \ldots i_k}(x, u, u_{(1)}, \ldots, u_{(r)}) \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}.$$

Lemma 3.1 The differential form

$$\omega = f_{i_1 i_2 \ldots i_k}(x, u, u_{(1)}, \ldots, u_{(r)}) \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k},$$

is an invariant form of the Lie-Bäcklund group with generator X if and only if

$$X(\omega) = 0\,.$$

Proof. In the proof we will use Definition 3.1. To indicate that ω is in terms of transformed quantities we write $\bar{\omega}$. By using the infinitesimal form of the Lie-Bäcklund group and the Taylor expansion, we obtain

$$\begin{aligned}\bar{\omega} &= f_{i_1 i_2 \ldots i_k}(\bar{x}, \bar{u}, \bar{u}_{(1)}, \ldots, \bar{u}_{(r)}) d\bar{x}^{i_1} \wedge \ldots \wedge d\bar{x}^{i_k} \\ &= [f_{i_1 i_2 \ldots i_k} + \varepsilon X(f_{i_1 i_2 \ldots i_k})] d(x^{i_1} + \varepsilon \xi^{i_1}) \wedge \ldots \wedge d(x^{i_k} + \varepsilon \xi^{i_k}) + O(\varepsilon^2) \\ &= \omega + \varepsilon X(\omega) + O(\varepsilon^2).\end{aligned}$$

Therefore if $\bar{\omega} = \omega$,then $X(\omega) = 0$ for small ε. Conversely, suppose that $X(\omega) = 0$. Then $\bar{X}(\bar{\omega}) = 0$,

$$\frac{d}{d\varepsilon}\bar{\omega} = X(\omega) = \bar{X}(\bar{\omega}) = 0,$$

this implies that $\bar{\omega}$ is constant, hence by using the identity property of the group $\bar{\omega} = \omega$.

Theorem 3.3 (*Main Theorem*) ([10]) Suppose that X is a Lie-Bäcklund operator (1.8) such that the form ω, given by (1.5), is invariant under X. Then

$$X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0, \qquad i = 1, \ldots, n. \tag{3.3}$$

A summary of the proof is given below, a detailed proof is shown in Kara and Mahomed [10].

Proof We write the $(n-1)$-form (1.5) as

$$\omega = T^{i_k}\frac{\partial}{\partial x^{i_k}} \lrcorner (\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_n}). \tag{3.4}$$

Therefore,

$$X(\omega) = X(T^{i_k})\frac{\partial}{\partial x^{i_k}} \lrcorner (\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_n})$$

$$+ T^{i_k}(-1)^{k-1}X(\mathrm{d}x^{i_1} \wedge \ldots \wedge \widehat{\mathrm{d}x^{i_k}} \wedge \ldots \wedge \mathrm{d}x^{i_n})$$

where $\widehat{\ }$ denotes omission. Now for fixed l,

$$D\xi^{i_l} = D_{i_k}\xi^{i_l}\mathrm{d}x^{i_k}$$

by (1.3), so we have

$$X(\omega) = X(T^{i_k})\frac{\partial}{\partial x^{i_k}} \lrcorner (\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_n}) + T^{i_k}D_{i_l}\xi^{i_l}\frac{\partial}{\partial x^{i_k}} \lrcorner (\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_n})$$

$$- T^{i_l}D_{i_l}\xi^{i_k}\frac{\partial}{\partial x^{i_k}} \lrcorner (\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_n}).$$

Thus, $X(\omega) = 0$ gives

$$X(T^{i_k}) + T^{i_k}D_{i_l}(\xi^{i_l}) - T^{i_l}D_{i_l}(\xi^{i_k}) = 0, \qquad k = 1, \ldots, n,$$

which is (3.3).

The folowing definition and corollary further shows the implication of the above result.

Definition 3.4 A Lie-Bäcklund symmetry generator X is said to be associated with a conserved vector T (or its corresponding conserved form ω) of the system (3.1) if X and T satisfy the relations (3.3)(or equivalently if $X(\omega) = 0$).

Corollary 3.5 Suppose that X is a canonical Lie-Bäcklund symmetry generator of the system (1.4) such that the conserved form ω of (1.4), given by (1.5), is invariant under X. Then

$$X(T^i) = 0, \quad i = 1, \ldots, n.$$

The following example will show how to construct non-local conservation laws from non-local symmetries for some classes of wave equations with variable wave speeds.

3.3 Wave Equation with Variable speed Example

Example 2.8 Let us consider the wave equation

$$u_{tt} = e^{2x} u_{xx}, \tag{3.5}$$

which has a auxiliary form

$$\begin{aligned} u_t &= e^{2x} v_x, \\ u_x &= v_t. \end{aligned} \tag{3.6}$$

One of the Lie point symmetry generator of (3.6) is

$$X = \frac{\partial}{\partial x} - t\frac{\partial}{\partial t} - v\frac{\partial}{\partial v}.$$

We can now use X to construct an associated (non-local) conservation law for (3.5) by applying the identity (3.3) and the conserved form

$$D_i T^i = 0$$

to the system of auxilliary equations (3.6).

Applying equation (3.3) gives us

$$X(T^1) + D_1\tau T^1 + D_2\xi T^1 - T^1 D_1\tau - T^2 D_2\tau = 0,$$

$$X(T^2) + D_1\tau T^2 + D_2\xi T^2 - T^1 D_1\xi - T^2 D_2\xi = 0.$$

This simplifies to a system

$$X(T^1) + T^1 D_x \xi - T^2 D_x \tau = 0, \quad X(T^2) + T^2 D_t \tau - T^1 D_t \xi = 0. \tag{3.7}$$

From the symmetry above we obtain $\xi = 1$ and $\tau = -t$ and applying the conservation law

$$D_t T^1 + D_x T^2 = 0 \tag{3.8}$$

along the solutions of (3.6), we see that the conserved vector (T^1, T^2) is dependent on x, t, u, v and independent of derivatives of u and v. As a result, (3.7) becomes

$$0\frac{\partial T^1}{\partial u} + \frac{\partial T^1}{\partial x} - t\frac{\partial T^1}{\partial t} - v\frac{\partial T^1}{\partial v} = 0, \quad 0\frac{\partial T^2}{\partial u} + \frac{\partial T^2}{\partial x} - t\frac{\partial T^2}{\partial t} - v\frac{\partial T^2}{\partial v} = T^2, \tag{3.9}$$

which has characteristic form of

$$\frac{\mathrm{d}u}{0} = \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}t}{-t} = \frac{\mathrm{d}v}{-v} = \frac{\mathrm{d}T^1}{0}, \quad \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}t}{-t} = \frac{\mathrm{d}v}{-v} = \frac{\mathrm{d}T^2}{T^2}. \tag{3.10}$$

The two characteristic equations above have invariants $c_1 = x + \ln t, c_2 = u$, and $c_3 = x + \ln v$, which gives $T^1 = f_1(c_1, c_2, c_3)$ and $tT^2 = f_2(c_1, c_2, c_3)$.

The conserved form (3.8) was expanded to obtain

$$\frac{\partial T^1}{\partial t} + u_t\frac{\partial T^1}{\partial u} + v_t\frac{\partial T^1}{\partial v} + \frac{\partial T^2}{\partial x} + u_x\frac{\partial T^2}{\partial u} + v_x\frac{\partial T^2}{\partial v} = 0, \tag{3.11}$$

which along the solutions of (3.6) become

$$\frac{\partial T^1}{\partial t} + e^{2x}v_x\frac{\partial T^1}{\partial u} + u_x\frac{\partial T^1}{\partial v} + \frac{\partial T^2}{\partial x} + u_x\frac{\partial T^2}{\partial u} + v_x\frac{\partial T^2}{\partial v} = 0. \tag{3.12}$$

We can separate by derivatives of u and v, to obtain

$$
\begin{aligned}
u_x: \quad & \frac{\partial T^1}{\partial v} + \frac{\partial T^2}{\partial u} = 0, \\
v_x: \quad & e^{2x}\frac{\partial T^1}{\partial u} + \frac{\partial T^2}{\partial v} = 0, \\
rest: \quad & \frac{\partial T^1}{\partial t} + \frac{\partial T^2}{\partial x} = 0,
\end{aligned}
\tag{3.13}
$$

which in terms of the invariants obtained above becomes

$$
\begin{aligned}
& e^{c_1}\frac{\partial f_1}{\partial c_3} + e^{c_3}\frac{\partial f_2}{\partial c_2} = 0, \\
& e^{c_1+c_3}\frac{\partial f_1}{\partial c_2} + \frac{\partial f_2}{\partial c_3} = 0, \\
& \frac{\partial f_1}{\partial c_1} + \frac{\partial f_2}{\partial c_1} + \frac{\partial f_2}{\partial c_3} = 0.
\end{aligned}
\tag{3.14}
$$

We can now solve (3.14) using some ad hoc methods.

Let $y = Y(c_3)$, This implies that $\dfrac{\partial f_1}{\partial c_3} = \dfrac{\partial f_1}{\partial y}.Y'$,
then

$$
\begin{aligned}
\frac{\partial^2 f_1}{\partial {c_3}^2} &= \frac{\partial f_1}{\partial y}.Y'' + \frac{\partial^2 f_1}{\partial y^2}.Y'^2, \\
e^{2x}\frac{\partial^2 f_2}{\partial {c_2}^2} &= \frac{1}{v^2}[\frac{\partial^2 f_2}{\partial {c_3}^2} - \frac{\partial f_2}{\partial c_3}] = 0, \\
e^{2c_3}\frac{\partial^2 f_2}{\partial {c_2}^2} &= \frac{\partial^2 f_2}{\partial {c_3}^2} - \frac{\partial f_2}{\partial c_3} = 0,
\end{aligned}
$$

so that

$$
\frac{\partial^2 f_2}{\partial {c_2}^2} = e^{-2c_3}[\frac{\partial f_1}{\partial y}.Y'' + \frac{\partial^2 f_1}{\partial y^2}.Y'^2 - \frac{\partial f_1}{\partial y}.Y']
$$

but

$$f_1 = f_1(c_2 + y) + g_1(c_2 - y).$$

This implies that

$$f_1 = f(c_2 + e^{c_3}, c_1) + g(c_2 - e^{c_3}, c_1),$$

and

$$f_2 = \tilde{f}(c_2 + e^{c_3}, c_1) + \tilde{g}(c_2 - e^{c_3}, c_1). \tag{3.15}$$

We let $\lambda = c_2 + e^{c_3}$ and $\mu = c_2 - e^{c_3}$,

$$e^{c_1}[\frac{\partial f}{\partial \lambda}.e^{c_3} + \frac{\partial g}{\partial \mu}.(-e^{c_3})] + e^{c_3}[\frac{\partial \tilde{f}}{\partial \lambda} + \frac{\partial \tilde{g}}{\partial \mu}] = 0,$$

$$\frac{\partial \tilde{f}}{\partial \lambda} + \frac{\partial \tilde{g}}{\partial \mu} + e^{c_1}\frac{\partial f}{\partial \lambda} - e^{c_1}\frac{\partial g}{\partial \mu} = 0,$$

$$e^{c_1}[\frac{\partial f}{\partial \lambda} + \frac{\partial g}{\partial \mu}] + \frac{\partial \tilde{f}}{\partial \lambda} - \frac{\partial \tilde{g}}{\partial \mu} = 0.$$

Separating by coefficients gives

$$\frac{\partial \tilde{f}}{\partial \lambda} + e^{c_1}\frac{\partial f}{\partial \lambda} = 0,$$

$$\tilde{f} + e^{c_1} f = A(c_1) \implies \tilde{f} = A(c_1) - e^{c_1} f,$$

$$\frac{\partial \tilde{g}}{\partial \mu} - e^{c_1}\frac{\partial g}{\partial \mu} = 0,$$

$$\tilde{g} - e^{c_1} g = B(c_1) \implies \tilde{g} = B(c_1) - e^{c_1} g.$$

Substitute above equations into (3.15) to obtain,

$$f_2 = A(c_1) - e^{c_1} f + B(c_1) - e^{c_1} g$$

$$= (A+B)c_1 + e^{c_1}(g-f)$$

$$= H(c_1) + e^{'}(g-f) + e^{c_1}[\frac{\partial g}{\partial \mu}.(-e^{c_3}) - \frac{\partial f}{\partial \lambda}e^{c_3}]$$

$$= (1+e^{c_1})\frac{\partial g}{\partial c_1} = (1-e^{c^1})\frac{\partial f}{\partial c_1} + H^{'} + e^{c^1}(g-f) + e^{c_1+C_3}[\frac{\partial g}{\partial \mu} - \frac{\partial f}{\partial \lambda}] = 0.$$

Since $\lambda = c_2 + e^{c_3}$ and $\mu = c_2 - e^{c_3}$

then $\quad e^{C_3} = \frac{\lambda-\mu}{2}$,

$$\frac{\partial g}{\partial c_1} + \frac{\partial f}{\partial c_1} + H^{'} = 0,$$

$$\Longrightarrow f + g = -H(c_1) + D(\lambda, \mu)$$

$$H - D = -g - f$$

$$\Longrightarrow \frac{\partial g}{\partial c_1} - \frac{\partial f}{\partial c_1} + (g-f) + \frac{\lambda-\mu}{2}[\frac{\partial g}{\partial \mu} - \frac{\partial f}{\partial \lambda}] = 0,$$

$$\frac{\partial g}{\partial c_1} + H^{'} + \frac{\partial f}{\partial c_1} + g - H + g - D + \frac{\lambda-\mu}{2}[\frac{\partial g}{\partial \mu} - \frac{\partial f}{\partial \lambda}] = 0,$$

$$2\frac{\partial g}{\partial c_1} + 2g + H^{'} + H - D + \frac{\lambda-\mu}{2}[\frac{\partial g}{\partial \mu} - \frac{\partial f}{\partial \lambda}] = 0.$$

Case 1

If $\frac{\partial g}{\partial \mu} = 0$,

$$2\frac{\partial g}{\partial c_1} + 2g + H' + H = const = \varepsilon,$$

$$\frac{\lambda - \mu}{2}\frac{\partial D}{\partial \lambda} + D = const = \varepsilon,$$

Also if $H = c_1$, then

$$2\frac{\partial g}{\partial c_1} + 2g + 1 + c_1 = 0,$$

$$\frac{\partial g}{\partial c_1} + g = -\frac{1}{2}(1 + c_1),$$

$$\frac{\partial (e^{c_1} g)}{\partial c_1} = -\frac{1}{2}(1 + c_1)e^{c_1},$$

$$ge^{c_1} = -\frac{1}{2}[e^{c_1}(1 + c_1) - e^{c_1}] + k \quad , where \quad k = const$$

$$g = -\frac{1}{2}c_1 + ke^{c_1}.$$

From

$$\frac{\lambda - \mu}{2}\frac{\partial D}{\partial \lambda} + D = const = \varepsilon,$$

$$\Longrightarrow \frac{\partial D}{D} = \frac{2\partial \lambda}{\lambda - \mu},$$

then

$$\ln D = -2\ln(\lambda - \mu) + \ln L(\mu)$$

$$\Longrightarrow D = \frac{L(\mu)}{(\lambda - \mu)^2}.$$

Thus

$$f = -\frac{1}{2}c_1 - ke^{-c_1} + \frac{L(\mu)}{(\lambda - \mu)^2},$$

$$f_1 = -c_1 + \frac{L(\mu)}{(\lambda - \mu)^2},$$

$$f_2 = c_1 - e^{c_1}(2ke^{-c_1} - \frac{L(\mu)}{(\lambda - \mu)^2}).$$

Since $T^1 = f_1(c_1, c_3, c_3)$ and $tT^2 = f_2(c_1, c_3, c_3)$

$$T^1 = -(x + lnt) + L(c_2 - e^{c_3})(2e^{c_3})^{-2},$$

$$T^2 = \frac{1}{t}[(x + lnt) + te^x(\frac{k}{e^x t} - \frac{L}{2e^x v}],$$

so that

$$D_t T^1 = -\tfrac{1}{t} + [e^{-2x}\tfrac{1}{2}v^{-3}v_t L + \tfrac{1}{4}e^{-2x}v^{-2}L'(u_t - v_t e^x)],$$

$$D_t T^2 = \tfrac{1}{t} + te^x(2k\tfrac{e^{-x}}{t} - \tfrac{L}{4}e^{-2x}v^{-2}) - (-\tfrac{k}{t}e^{-x} - \tfrac{1}{4}L'(u_x - v_x e^x - ve^x)e^{-2}v^{-2})$$

$$+\tfrac{2}{4}L'e^{2x}v - 2 + \tfrac{L}{4}e^{-2x}v^{-3}v_x.$$

Then substituting for $v_t = u_x$ and $u_t = e^{2x}v_x$ gives $v_x = e^{-2x}u_t$, we then obtain

$$D_t T^1 + D_t T^2 = \frac{L'}{4v^2}[e^{-2x}u_t - v_x - e^{-x}(v_t - u_x)],$$

provided

$$\frac{L}{2v^3}[e^{-x}v_x - e^{-2x}(v_t] - \frac{L}{4e^x v^2} - \frac{L'}{4v} = 0.$$

Case 2

$g = \mu, \quad H' + H = 0 \quad$ if $\quad H = Ke{-c_1}, \quad k = constant, \quad \dfrac{\partial g}{\partial \mu} = 1$

$2\mu - D + \frac{\lambda - \mu}{2}(1 - \dfrac{\partial D}{\partial \lambda}) = 0,$

$$\frac{3\mu}{2} - D + \frac{\lambda}{2} - \frac{\lambda-\mu}{2}\frac{\partial D}{\partial \lambda} = 0,$$

$$\frac{\lambda-\mu}{2}\frac{\partial D}{\partial \lambda} + D = \frac{3\mu+\lambda}{2},$$

$$\frac{\partial[(\lambda-\mu)^2.D]}{\partial \lambda} = (3\mu+\lambda)(\lambda-\mu),$$

$$D(\lambda-\mu)^2 = \tfrac{1}{3}\lambda^3 + \mu\lambda^2 - 3\mu^2\lambda + L(\mu).$$

Since $\lambda - \mu = 2e^{c_3}$,

$$D4e^{2c_3} = \tfrac{1}{3}[c_2^3 + 3c_2e^{c_3} + 3c_2e^{2c_3} + e^{2c_3}] + (c_2^2 + 2c_2e^{c_3} + e^{2c_3})(c_2 - e^{c_3})$$

$$-3(c_2^2 - 2c_2e^{c_3} + e^{2c_3})(c_2 + e^{c_3}) + L(\mu),$$

then

$$D = \tfrac{1}{12}[-c_2^3e^{-2c_3} + 7c_2^2e^{-c_3} + 5c_2 - 5e^{c_3}] + \tfrac{L}{4}e^{-2c_3}.$$

From previous result

$$f = -g - H + D,$$

so that

$f = -(c_2 - e^{c_3}) - (ke^{-c_1}) + D$, where $g = c_2 - e^{c_3}$ and ke^{-c_1}

$$f_1 = f(\lambda, c_1) + g(\mu, c_1)$$

$$= -ke^{-c_1} + D,$$

$$f_2 = H(c_1) + e^{c_1}(g - f)$$

$$= Ke^{-c_1} - e^{c_1}[Ke^{-c_1} + c_2 - e^{c_3} - D].$$

Since

$$T^1 = f_1(c_1, c_3, c_3) \text{ and } T^2 = \tfrac{1}{t}f_2(c_1, c_3, c_3)$$

then

$$T^1 = -Ke^{-c_1} + D,$$

which implies

$$T^1 = -Ke^{-c_1} + \frac{1}{12}[-c_2^3 e^{-2c_3} + 7c_2^2 e^{-c_3} + 5c_2 - 5e^{c_3}] + \frac{L}{4}e^{-2c_3},$$

then

$$T^1 = \frac{-K}{e^x t} + \frac{1}{12}[\frac{-u^3}{v^2 e^{2x}} + \frac{7u^2}{ve^x} + 5u - 5ve^x] + \frac{L(u - ve^x)}{4v^2 e^{2x}}.$$

Also

$$T^2 = \frac{1}{t}[Ke^{-c_1} + e^{-c_1}(Ke^{-c_1} + c_2 - e^{c_3} - D)],$$

$$T^2 = \frac{K}{t^2 e^x} + e^x(\frac{k}{te^x} + u - e^x v - \frac{1}{12}[\frac{-u^3}{v^2 e^{2x}} + \frac{7u^2}{ve^x} + 5u - 5ve^x] + \frac{L(u - ve^x)}{4v^2 e^{2x}}).$$

If

$$L' + \frac{L}{ve^x}[\frac{2v_t}{v} + \frac{2v_x}{v} + 1] = 0,$$

then (T^1, T^2) is a non-local conserved vector associated with the symmetry X.

3.4 Approximate Potential Conservation Laws

In the previous chapters, we have discussed and shown that potential symmeties can be used to construct approximate potential symmetries. We later showed an association between potential symmetries and conservation laws.

We now further extend and combine these concepts to construct approximate potential conservation laws. This method is shown using an example from previous

chapters. It has been discussed in Kara and Davison [9].

The perturbed wave equation (2.8) is written in auxilliary form (2.9),

the unperturbed form ($\varepsilon = 0$) of this system in potential form is,

$$u_x = v_t,$$

$$u_t = e^{2x} v_x,$$

which has a symmetry

$$X_0 = \frac{\partial}{\partial x} - t\frac{\partial}{\partial t} - v\frac{\partial}{\partial v}.$$

From the previous chapter, a conserved vector $T_0 = (T_0^1, T_0^2)$ associated with the symmetry X_0, is

$$T_0^1 = \frac{-K}{e^x t} + \frac{1}{12}[\frac{-u^3}{v^2 e^{2x}} + \frac{7u^2}{ve^x} + 5u - 5ve^x],$$

and

$$T_0^2 = \frac{K}{t^2 e^x} + \frac{k}{t} + 2ue^x - 2ve^{2x} - \frac{e^x}{12}[\frac{-u^3}{v^2 e^{2x}} + \frac{7u^2}{ve^x} + 5u - 5ve^x].$$

The symmetry of the above perturbed equation from the previous chapter is given by $X = X_0 + \varepsilon X_1$ where

$$X_1 = \frac{v}{2}\frac{\partial}{\partial u} + \frac{1}{2}e^{-2x}u\frac{\partial}{\partial v}.$$

Our aim is to find a conserved vector T of the perturbed equation, which is $T = T_0 + \varepsilon T_1$, where $T_1 = (T_1^1, T_1^2)$.

To find the approximate potential conservation law, we again apply identity (3.3) and the conservation form $D_i T^i = 0$ to obtain following two equations.

$$X_0(T_1^1) + D_x(\xi_0)T_1^1 - T_1^2 D_x(\tau_0) = -X_1(T_0^1) - D_x(\xi_1)T_0^1 + T_0^2 D_x(\tau_1),$$
$$X_0(T_1^2) + D_t(\tau_0)T_1^2 - T_1^1 D_t(\xi_0) = -X_1(T_0^2) - D_t(\tau_1)T_0^2 + T_0^1 D_t(\xi_1). \tag{3.16}$$

For X_0, $\xi_0 = 1$ and $\tau_0 = -t$, and for X_1, $\xi_1 = 0$ and $\tau_1 = 0$.

This implies that,

$$X_0(T_1^1) = -X_1(T_0^1),$$

which gives

$$\frac{\partial T_1^1}{\partial x} - t\frac{\partial T_1^1}{\partial t} - v\frac{\partial T_1^1}{\partial v} = -X_1(\frac{-K}{e^x t} + \frac{1}{12}[\frac{-u^3}{v^2e^{2x}} + \frac{7u^2}{ve^x} + 5u - 5ve^x]).$$

This gives the characteristic equation

$$\frac{du}{0} = \frac{dx}{1} = -\frac{dt}{t} = -\frac{dv}{v} = \frac{e^x dT_1^1}{\frac{u^4}{12v^3e^{3x}} - \frac{7}{24}\frac{u^3}{v^2e^{2x}} - \frac{1}{8}\frac{u^2}{ve^x} + \frac{9}{24}u + \frac{5}{24}ve^x}.$$

From $\frac{du}{0}$ we obtain $u = a$ where u is an invariant, using $\frac{dx}{1} = -\frac{dt}{t}$,we obtain an invariant $b = te^x$. Then from $-\frac{dt}{t} = -\frac{dv}{v}$, we obtain $c = \frac{t}{v}$.

Now we substitute a, b and c into RHS of the characteristic equation,viz.,

$$\frac{e^x dT_1^1}{\frac{u^4}{12v^3e^{3x}} - \frac{7}{24}\frac{u^3}{v^2e^{2x}} - \frac{1}{8}\frac{u^2}{ve^x} + \frac{9}{24}u + \frac{5}{24}ve^x} = \frac{\frac{b}{t}dT_1^1}{\frac{1}{12}a^4(\frac{c}{b})^3 - \frac{7}{24}a^3(\frac{c}{b})^2 - \frac{1}{8}a^2\frac{c}{b} + \frac{9}{24}a + \frac{5}{24}\frac{b}{c}}.$$

Solving the system,

$$\frac{\frac{b}{t}dT_1^1}{\frac{1}{12}a^4(\frac{c}{b})^3 - \frac{7}{24}a^3(\frac{c}{b})^2 - \frac{1}{8}a^2\frac{c}{b} + \frac{9}{24}a + \frac{5}{24}\frac{b}{c}} = -\frac{dt}{t},$$

to obtain

$$T_1^1 = \frac{1}{b}(\frac{1}{12}a^4(\frac{c}{b})^3 - \frac{7}{24}a^3(\frac{c}{b})^2 - \frac{1}{8}a^2\frac{c}{b} + \frac{9}{24}a + \frac{5}{24}\frac{b}{c})t + \kappa(a,b,c).$$

We now solve the second equation (3.16b)

$$X_0(T_1^2) - T_1^2 = -X_1(T_0^2)$$

to obtain

$$\frac{\partial T_1^2}{\partial x} - t\frac{\partial T_1^2}{\partial t} - v\frac{\partial T_1^2}{\partial v} = T_1^2 - \frac{v}{2}(2e^x + \frac{1}{4}\frac{u^2}{v^2e^x} - \frac{7u}{6v} - \frac{5}{12}e^x) - \frac{e^{-2x}}{2}(-2e^{2x} - \frac{1}{6}\frac{u^3}{v^3e^x} + 7\frac{u^2}{v^2} + 5e^{2x}).$$

This becomes in characteristic form,

$$\frac{dx}{1} = \frac{du}{0} = -\frac{dt}{t} = -\frac{dv}{v} = \frac{dT_1^2}{T_1^2 - \frac{v}{2}(2e^x + \frac{1}{4}\frac{u^2}{v^2e^x} - \frac{7u}{6v} - \frac{5}{12}e^x) - \frac{e^{-2x}}{2}(-2e^{2x} - \frac{1}{6}\frac{u^3}{v^3e^x} + 7\frac{u^2}{v^2} + 5e^{2x})}.$$

From $\frac{du}{0}$ we obtain $u = a$ as an invariant, and using $\frac{dx}{1} = -\frac{dt}{t}$ we obtain $b = te^x$ as an invariant, also using $\frac{dt}{t} = \frac{dv}{v}$ we obtain $c = \frac{t}{v}$ as an invariant.

We again substitute for a, b and c, using $(\frac{dx}{1})$ to solve for T_1^2

$$\frac{dx}{1} = \frac{dT_1^2}{T_1^2 - \frac{v}{2}(2e^x + \frac{1}{4}\frac{u^2}{v^2e^x} - \frac{7u}{6v} - \frac{5}{12}e^x) - \frac{e^{-2x}}{2}(-2e^{2x} - \frac{1}{6}\frac{u^3}{v^3e^x} + 7\frac{u^2}{v^2} + 5e^{2x})}.$$

This gives,

$$x = \int \frac{dT_1^2}{T_1^2 - \frac{v}{2}(2e^x + \frac{1}{4}\frac{u^2}{v^2e^x} - \frac{7u}{6v} - \frac{5}{12}e^x) - \frac{e^{-2x}}{2}(-2e^{2x} - \frac{1}{6}\frac{u^3}{v^3e^x} + 7\frac{u^2}{v^2} + 5e^{2x})} + \Re(a,b,c),$$

where $\Re$ is an arbitrary function of the invariants a, b and c.

Thus,

$$x = \ln[T_1^2 - \frac{v}{2}(2e^x + \frac{1}{4}\frac{u^2}{v^2e^x} - \frac{7u}{6v} - \frac{5}{12}e^x) - \frac{e^{-2x}}{2}(-2e^{2x} - \frac{1}{6}\frac{u^3}{v^3e^x} + 7\frac{u^2}{v^2} + 5e^{2x})] + \Re(a, b, c),$$

then

$$T_1^2 = e^{(x-\Re)} + \frac{v}{2}(2e^x + \frac{1}{4}\frac{u^2}{v^2e^x} - \frac{7u}{6v} - \frac{5}{12}e^x) - \frac{e^{-2x}}{2}(-2e^{2x} - \frac{1}{6}\frac{u^3}{v^3e^x} + 7\frac{u^2}{v^2} + 5e^{2x}),$$

subject to the conditions

$$\frac{\partial T_1^1}{\partial t} + \frac{\partial T_1^2}{\partial x} + \frac{u}{e^{2x}}\frac{\partial T_0^2}{\partial v} = 0,$$

$$\frac{\partial T_1^1}{\partial v} + \frac{\partial T_1^2}{\partial u} = 0,$$

$$\frac{\partial T_1^1}{\partial u} + e^{-2x}\frac{\partial T_1^2}{\partial v} = 0.$$

We have shown that a class of conservation laws for wave equations with variable speeds can be constructed from potential symmetries and approximate potential symmetries. However, it is important to note that in constructing conservation laws in the examples, we have used the direct method as this equation does not have a Lagrangian. The direct method does not assume a correspondence between the symmetries and conservation laws. Hence, we use equation (3.3) to provide this relation. In conclusion, what we have shown is that equation (3.3) together with a known symmetry X and using the conservation law $D_iT^i = 0$, we are able to obtain a system of linear PDEs, which are solved for components T^i of the conserved vector T (see Kara and Mahomed [10]).

Conclusion

In this report, we have investigated and reviewed the theory of the application of potential symmetries to study of PDEs. We have looked at their application to approximate symmetries and conservation laws. The importance of potential symmetries is necessary because of the obvious limitation of group-theoretic methods based on local symmetries, where many PDEs do not have local symmetries. As a result it turns out that PDEs can admit non-local symmetries whose infinitesimal generators depend on the integrals of the dependent variable in some specific manner (see Gandarias et al [5]).

Perturbation theory is used in many areas of science, therefore a study of differential equations with a small parameter (perturbed equations) is of importance. Finding approximate symmetries of perturbed equations using potential symmetries is discussed. This was successfully achieved using an algorithm by Baikov et al [2]. We also investigated the existence of approximate conserved forms of perturbed equations.

Conservation laws play an important role in science because they are central to the analysis of physical field equations where they provide conserved quantities such as energy, momentum and angular momentum. Therefore, we addressed the computation of conservation laws in the absence of a Lagrangian, which is not possible using Noether's famous theorem. We showed a result that relates symmetries and conservation laws. In finding the conservation law of a PDE using a known potential symmetry of the PDE, the direct method and a natural symmetry condition were used. These ideas and concepts were further extended to compute approximate potential conservation laws using the same procedure.

For further research we could investigate other types of non-local symmetries different from potential symmetries, and use them to generate approximate symmetries and conservation laws of PDEs. These may give rise to different invariant solutions.

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