

https://dspace.org/

Pricing of double barrier options from a symmetry group approach

Sidogi, Thendo

2014-07-02
http://hdl.handle.net10539/14847

Downloaded from a DSpace repository. https://dspace.org/

Pricing of Double Barrier Options from a Symmetry Group Approach

Thendo Sidogi

December 11, 2013

DECLARATION

I declare that the contents of this thesis are original except where due references have been made. It has not been submitted before for any degree to any other institution.

T Sidogi

Acknowledgements

I acknowledge the continued support and advice of my supervisors, Dr A.H Davison and Dr J.C Ndogmo. Their guidance and technical assistance has been of great value to me and is very much appreciated

I am grateful for all the love and support from my family and friends over the past year and a half as I worked on this research report.

Abstract

In this research report we explore some applications of symmetry methods for boundary value problems in the pricing of barrier options. Various financial instruments satisfy the Black-Scholes partial differential equation (PDE) but with different domain, maturity date and boundary conditions. We find Lie symmetries that leave the Black-Scholes (PDE) invariant and will guarantee that the relevant solutions satisfy the boundary conditions. Using these symmetries, we can thus generate group-invariant solutions to the boundary value problem.

Contents

Introduction **6**

1 Preliminaries **9**

1.1 Introduction 9

1.2 Notation and Preliminaries 9

1.2.1 Lie Symmetry Analysis on Initial and Boundary Value Problem 13

2 Lie symmetry groups and initial and boundary value problems **14**

2.1 Introduction 14

2.2 Lie's Algorithm 15

2.3 Boundary Conditions 15

3 Financial derivatives and the Black-Scholes equation **19**

3.1 Introduction 19

3.1.1 Derivation of Black-Scholes PDE 21

4 The Black-Scholes and the Heat equation **24**

4.1 Introduction 24

4.1.1 Setting up the problem as a boundary value problem 24

4.2 Transformation to Heat equation and dimensionless variables 25

4.3 Finding the exact solution for the BVP 27

4.4 Calculations 28

4.5 Invariant solutions 34

4.5.1 Symmetry generator $\widetilde{X_1}$ 34

4.5.2 Symmetry generator $\widetilde{X_2}$ 36

4.5.3 Symmetry generator $\widetilde{X_3}$ 37

4.5.4 Symmetry generator $\widetilde{X}$ 38

5 Conclusion **42**

References 43

Introduction

A barrier option is a path-dependent financial derivative whose monetary value depends on the particular path followed by the underlying asset price during the option's life, as well as on some specified asset values that are commonly called barriers. The barrier options market has been expanding rapidly over the past two decades because barriers can be added virtually to any existing type of standard or exotic options; and given that the additional constraints imposed on options by the barriers makes them a much cheaper and more attractive product [14, 6, 25].

However, as is well known, the pricing of options has led to mathematical models which are often challenging to solve. Nevertheless, analytical formulas have been obtained for most of the standard barrier options. In particular, Rubinstein and Reiner [24] and Kunitomo and Ikeda [17] have derived analytical formulas for a variety of standard knock-in and knock-out European options with full barriers, and similar results have been obtained for double barrier options [17, 11, 23]. Broadie and Detemple [4], Gao *et al.* [9] and Haug [13] also considered the extension of these formulas to the more complex case of American style options, although all formulas obtained in this case are only closed-form approximations.

Methods used for the derivation of pricing formulas for barrier options are generally probability-based, especially in the case of analytical formulas. However, numerical methods are also often used, especially for those cases where no analytical formulas exists such as for discretely monitored barrier options [19, 7, 5, 8, 1]. Another important method that has yet not been used often in literature for the derivation of

analytical formulas for options is the Lie groups analysis method applied to boundary value problems [3, 18, 15].

Lie presented different integration techniques that depended upon the invariance of a system of differential equations for a particular changes in co-ordinates, with these changes in co-ordinates forming a continuous group, called a Lie group. Such symmetries are described using elements of the tangent space to the group at the identity (Lie Algebra). If we exponentiate elements of the Lie algebra, this helps us categorize the co-ordinate transformation. The elements of the Lie algebra are called the Lie symmetries of system of equations. In simple terms, a Lie symmetry is a vector field in the space of independent and dependent variables which carries solutions of the system to other solutions.

An important application of Lie's work is that the symmetry properties of a system can be used to reduce the number of independent variables and eventually get a solvable ordinary differential equation to yield particular solutions for the initial partial differential equation (PDE). Lie symmetries can also be used to reduce the order of a differential equation. In the case of partial differential equations (PDEs) which are generally used in the pricing problems for options, symmetry analysis will offer only a partial solution, and not the most general solution [21, 22, 3]. However, there are efficient methods for finding particular solutions when no general solutions are available, and such particular solutions may turn out to be the exact solution in the case of boundary value problems [15, 18].

In this report, a review of some recent advances in this area of research is presented, along with several examples.

In Chapter 1, the notation that will be used is described, and the operators that will be used throughout the report are defined.

In Chapter 2, we give a more explicit description of the aims and objectives of the research project. We describe the application of Lie symmetries to boundary value and initial conditions problems.

Chapter 3 offers an insight into options as financial derivatives and the derivation and ideas behind the Black-Scholes equation. We do this considering the strong correlation between the Black-Scholes equation and the Heat equation.

Chapter 4, contains brief discussions and applications of symmetry methods and group-invariant solutions for differential equations, as well as some applications of symmetries of boundary value problems. We also formulate the pricing of barrier options as a boundary value problem in this chapter.

Chapter 5 discusses possibilities of improvements to the model and research questions.

Chapter 1

Preliminaries

1.1 Introduction

The notation that will be used in this report will be described in this chapter, along with the definition of the main operators will be defined.

1.2 Notation and Preliminaries

Let $X = (x^1, x^2, \ldots, x^n) \in \mathbb{R}^n$ be the independent variable, and let
$U = (u^1, u^2, \ldots, u^m) \in \mathbb{R}^m$ be the dependent variable, with co-ordinates x^i and u^α respectively.

Let the space consisting of our dependent and independent variables mentioned above be denoted by $X \times U \in \mathbb{R}^{nm}$.

Let u^α_i denote the partial derivative of u^α with respect to x^i, and similarly let $u^\alpha_{i_1,i_2,\ldots,i_r}$ denote the mixed partial derivatives of u^α with respect to $x^{i_1}, x^{i_2}, \ldots, x^{i_r}$.

With one exception, namely the prolongation coefficients $\zeta^{\alpha}_{i_1 \ldots i_n}$ that will be reviewed later, subscripts will be used to indicate partial differentiation.

Let $u_{(1)}$ denote the collection of all first-order derivatives, u^{α}_{i}. Similarly, let $u_{(2)}$, $u_{(3)}$, and so on, denote the collections of higher order derivatives. The r-jet bundle $J^r(U)$ is the equivalence class of sections of U, with co-ordinates $(x^i, u^{\alpha}, u^{\alpha}_{i}, u^{\alpha}_{i_1,i_2}, \ldots, u^{\alpha}_{i_1,\cdots,i_r})$ where $1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq n$. The r-jet bundle will be written $J^r(U) = \{(x, u, u_{(1)}, \ldots, u_{(r)}) \ : (x, u) \in U\}$. It will be expedient to work in $J^r(U)$, where u and its derivatives are treated as independent variables.

Since the Black-Scholes equation is a second order differential equation with two independent variables, for much of this report we will consider $X = (x, t)$ and the jet-space where $r = 2$ which is a section of U containing derivatives of u of order at most 2 with respect to x and t which has coordinates of the form

$$(u; u_x, u_t, u_{tt}, u_{tx}, u_{xx}) = (u, u_{(1)}, u_{(2)}).$$

We will consider a particular system of PDE written in the form

$$\Delta(x, t, u, u_{(1)}, u_{(2)}) = 0. \tag{1.1}$$

In this case our differential operator is defined by

$$\Delta(x, t; u_{(1)}, u_{(2)}) := u_t - A(x, t)u_{xx} - B(x, t)u_x - C(x, t)u. \tag{1.2}$$

Our differential operator is continuous map from the jet space to $\mathbb{R}$.
Next, we consider a generalized kth order PDE given by

$$F(x, u, u_{(1)}, u_{(2)}, ..., u_{(k)}) = 0. \tag{1.3}$$

A symmetry is a one-parameter group of transformation that maps solutions to other solutions of equation.

Infinitesimal generator

We first define some variables.

Consider a system denoted by (1.3) with a dependent variable u and independent variables $x = (x^1, x^2, ..., x^n)$ with $u = u(x)$. Consider also the transformations of (x, u) to $(\overline{x}, \overline{u}, \overline{u}_{(1)}, ..., \overline{u}_{(k)})$. Let equation (1.3) be associated with a one-parameter Lie group of point transformation given by

$$\begin{aligned}
\overline{x}^i &= x^i + \epsilon\tau^i(x,u) + O(\epsilon^2),\\
\overline{u} &= u + \epsilon\zeta(x,u) + O(\epsilon^2),\\
\overline{u}_i &= u_i + \epsilon\zeta_i(x,u,u_{(1)}) + O(\epsilon^2),\\
&..\\
&..\\
\overline{u}_{i_1,i_2,...,i_k} &= u_{i_1,i_2,...,i_k} + \epsilon\zeta_{i_1,i_2,...,i_k}(x,u,u_{(1)},u_{(2)},...,u_{(k)}) + O(\epsilon^2),
\end{aligned}$$

where ϵ is the group parameter with the corresponding generator with $k \geq 1$ and $i_l = 1, ..., n$.
Let

$$\Gamma = \textstyle\sum_{i=1}^{n} \tau^i(x,u)\partial_{x^i} + \zeta(x,u)\partial_u,$$

and

$$\tau^i = \frac{\partial\overline{x}^i}{\partial\epsilon}|_{\epsilon=0}, \quad \zeta = \frac{\partial\overline{u}}{\partial\epsilon}|_{\epsilon=0}.$$

where $i = 1, ..., n$.

The corresponding (k^{th} extension) infinitesimal generator is given by

$$\Gamma^{[k]} = \Gamma + \zeta_i(x,u)\partial_{u_i} + ... + \zeta_{i_1,i_2,...,i_k}(x,u,u_{(1)},u_{(2)},...,u_{(k)})\partial_{u_{i_1,i_2,...,i_k}}. \quad (1.4)$$

where $\zeta_i = \frac{\partial \overline{u}_i}{\partial \epsilon}|_{\epsilon=0}$. It can be shown [21] that

$$\begin{aligned} \zeta_i &= D_i\zeta - u_j(D_i\tau^j), & (1.5)\\ \zeta_{i_1,i_2,...,i_k} &= D_{i_k}\zeta_{i_1,i_2,...,i_{k-1}} - u_{i_1,i_2,...,i_{k-1},j}(D_i\tau^j), & (1.6) \end{aligned}$$

with the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial_{x^i}} + u_i^\alpha \frac{\partial}{\partial_{u^\alpha}} + u_{ij}^\alpha \frac{\partial}{\partial_{u_j^\alpha}} + ... \quad i = 1, ..., n.$$

The point symmetries $\Gamma_1, ..., \Gamma_n$ form a Lie algebra under the operation of a Lie Bracket.

The symmetry Γ leaves equation (1.3) invariant if

$$\Gamma(F)|_{F=0} = 0. \quad (1.7)$$

1.2.1 Lie Symmetry Analysis on Initial and Boundary Value Problem

A Boundary Value Problem (BVP) of kth order can be defined as before with $(k \geq 2)$ written in the form

$$F(x, u, u_{(1)}, u_{(2)}, ..., u_{(k)}) = 0,$$

involves finding a solution $u(x)$ defined on a domain Ω_x in x-space $x = (x_1, x_2, ..., x_n)$ that satisfies additional boundary conditions

$$B_\alpha(x, u, u_{(1)}, ..., u_{(k-1)}) = 0, \text{when} \quad w_\alpha(x) = 0 \tag{1.8}$$

for $\alpha = 1, 2, ..., s$. Essentially, this will mean that solutions satisfying our PDE should coincide with B_α boundary above when $w_\alpha(x) = 0$. If $w_\alpha = t$, the problem is sometimes called an Initial Value Problem (IVP).

Throughout the paper we will consider a case where $\alpha = 2$ i.e $B_\alpha = B$ and $w_\alpha = w$.

Chapter 2

Lie symmetry groups and initial and boundary value problems

2.1 Introduction

In the nineteenth century Sophus Lie discovered that the most notable methods to solve particular types of differential equations were special cases of integration based on the invariance of differential equations. This extension of possible integration methods persuaded Lie to develop and use the theory of continuous groups. We have seen application of Lie groups in most mathematical-oriented sciences. Classic examples of such transformation include a scalar transformation or rotations of solutions to solutions.

In the past two decades of the twentieth century, we have seen research activity in this field spanning from tangible physical systems to finance and economic applications. Gazizov and Ibragimov [10] calculated continuous symmetries of the Jacoby-Jones and Black-Scholes equations. There has been less work on Lie symmetry methods applied to boundary value problems. In this research project we will

refer to work done by Joanna Goard [12] in applying Lie's methods to boundary value problems.

2.2 Lie's Algorithm

Lie's classical method to find symmetry reductions of an nth order partial differential equation (PDE) where u is the only dependent variable and $\bar{x}$ is a k-tuple of independent variables i.e. $\bar{x} = (x^1, \ldots, x^k)$ is based on the fact that the PDE is invariant under a one-parameter Lie group of transformations. Consider the use of a Lie symmetry to reduce the order of an equation.

Example: The ordinary differential equation $y''(x) + y(x) - y^{-3}(x) = 0$ has a symmetry given by $X = \frac{\partial}{\partial x} + 0\frac{\partial}{\partial t} + 0\frac{\partial}{\partial u}$. By the invariance condition

$$\frac{\partial F}{\partial x} + 0\frac{\partial F}{\partial t} + 0\frac{\partial F}{\partial u} = 0$$

In characteristic form then

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dy'}{0},$$

The pair $\frac{dx}{1} = \frac{dy}{0}$ will yield $u = y$ being a zero order invariant solution. The pair given by $\frac{dx}{1} = \frac{dy'}{0}$ will yield $v = y'$ being a first order invariant solution.

2.3 Boundary Conditions

Symmetry analysis has been used to obtain exact solutions of differential equations but there has been little application to initial value problems or boundary value problems. We revisit the requirement for a IVP or BVP to remain invariant after a group transformation from a symmetry generator.

Three criteria are traditionally required of a symmetry; It should be

- a symmetry of the governing differential equation;
- a smooth bijective mapping of the domain to itself;
- a mapping of the set of boundary data to itself.

i.e $\Gamma w(x) = 0$ when $w_\alpha(x) = 0$ and $\Gamma^{(k-1)} B(x, u, \ldots, u_{k-1}) = 0$ when $w_\alpha(x) = 0$.

This can be unnecessarily restrictive. In a paper by Joanna Goard [12], solutions found by the classical method can satisfies boundary conditions which themselves are not invariant under the same symmetry generator which generated the solution.

For example, we would like to find out if a solution generated by a symmetry satisfy the boundary condition while the same symmetry does not leave the boundary conditions invariant. We consider the Heat equation $u_t = u_{xx}$ with condition $x = 1$, $u(x,t) = \text{erf}(1/2\sqrt{t})$ remains invariant under the symmetry $\Gamma = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$. The solution generated by the symmetry is given by $u(x,t) = \text{erf}(x/2\sqrt{t})$. It is evident that the solution does satisfy the boundary condition but the boundary condition is not invariant under the same symmetry generator.

Invariant surface condition (ISC)
We start by deriving the invariant surface condition. Consider partial differential equation (PDE) (1.3) in one dependent variable u and n independent variables with symmetry generator (1.2). (1.3) can be written in the form

$$u - F(x, u, u_{(1)}, u_{(2)}, ..., u_{(k)}) = 0. \tag{2.1}$$

Apply the symmetry generator on the above equation i.e.

$$\Gamma(u - F(x, u, u_{(1)}, u_{(2)}, ..., u_{(k)})) = 0 \tag{2.2}$$

$$\sum_i X_i(x,u)\frac{\partial u}{\partial x_i} - U(x,u)\frac{\partial u}{\partial u} = 0 \tag{2.3}$$

It is clear that for a suitable generator the invariant surface condition (ISC) is given by

$$\nabla = \sum_i X_i(x,u)u_{x_i} - U(x,u) = 0. \tag{2.4}$$

We then get equation (2.4) as a solution in some functional form. Usually we would substitute our solution in the PDE to solve for the arbitrary function and get a solution satisfying the PDE. Joanna Goard [12] illustrated that after getting the solution in functional form, we can immediately substitute it into our boundary condition to determine the arbitrary function(s).

For the ISC to hold at boundary for a general moving boundary $u(x,t) = f(t)$ at $x = L(t)$ will then satisfy

$$X(L(t),t,f(t))u_x + T(L(t),t,f(t))u_t = U(L(t),t,f(t)). \tag{2.5}$$

Consider the following example:

$$u(x,0) = F(x), \tag{2.6}$$

that has a symmetry generator

$$\gamma = X(x,t,u)\tfrac{\partial}{\partial x} + T(x,t,u)\tfrac{\partial}{\partial t} + U(x,t,u)\tfrac{\partial}{\partial x},$$

admitted by the PDE. Solution corresponding to the above symmetry generator will then satisfy the invariant surface condition i.e.

$$X(x,t,u)\tfrac{\partial u}{\partial x} + T(x,t,u)\tfrac{\partial u}{\partial t} = U(x,t,u).$$

Also, the ISC at boundary for a general initial value problem ($t = 0$), $u = F(x)$ will then satisfy

$$X(x,0,F(x))F'(x) + T(x,0,F(x))u_t(x,0) = U(x,0,F(x)). \tag{2.7}$$

So if the symmetry generator leaves the initial value problem invariant then the boundary conditions will also be invariant when transformed same generator. This implies that

$$T(x,0,u) = 0, \tag{2.8}$$

$$U(x,0,F(x)) - X(x,0,F(x))F'(x) = 0. \tag{2.9}$$

The invariance surface condition plays a role of a constrain to a system of PDE. When looking for a solution to a system of PDE with an unknown constrains system we seek the symmetry that will leave the PDE system invariant such that the invariance surface condition becomes the unknown constrain system. Essentially, this is mapping the boundary conditions to the invariance surface. To generate the invariant solution and solve the boundary value problem, one will then solve the ISC.

Conversely we may use the ISC condition to find symmetries that admit a given boundary conditions or to find boundary conditions admitted by given symmetries.

Chapter 3

Financial derivatives and the Black-Scholes equation

3.1 Introduction

In recent times, we have seen a large increase in the innovation of new financial products in order to cater for different financial positions and different speculative positions. The most innovative of them all is possible a financial derivative. A derivative may be defined as a security whose value is derived from the underlying variables, which may be any market prices. (for example, prices of traded equities or interest rate and stock indices). Finding a pricing formula for a fair valued financial derivative like an option was a top priority for most academic communities during the 1960s. Creating of the theoretical framework for the pricing of new derivatives is one of the major challenges in the field of financial engineering. In 1973, Black, Scholes [2] and Merton published their seminal papers on the theory of option pricing. Ever since then we have seen an exponential growth in the field of derivative securities. The Black-Scholes general equilibrium formulation of the option pricing theory is attractive because the final valuation formula deduced from their model

is a function of a few observable variables (excluding the volatility parameter) such that the precision of the model can be ascertained by direct empirical tests with market data. With certain assumptions (discussed below) the Black-Scholes model has the ability to explain or replicate the empirical data and hence the option pricing theory is widely acclaimed to be the most successful theory not only in finance, but in all areas of economics. Black, Scholes and Merton were further awarded a nobel prize for this achievement.

Black and Scholes made the following assumptions on the financial market:

- Trading takes place all the time i.e trading in continuous time;
- Risk free rate r is observable and constant over time;
- No dividend on the derivative;
- Short-selling is permitted;
- The derivative are perfectly divisible;
- No tax and transaction cost;
- The market is complete;
- No risk-free arbitrage opportunities.

These assumptions have been critically examined by later works and are seen to be unrealistic at times [4]. For instance, it is common for the interest rate to fluctuate over time in an irregular manner, rather than being constant and deterministic. In the finance community, the Black-Scholes model is still fundamental in derivative pricing theory, although various forms of modification to this basic model have been proposed to accommodate the above shortcomings. These include a stochastic varying interest rate and volatility. However, the last assumption about the absence of arbitrage opportunities remains unchanged and represents a cornerstone of the entire theory. Black and Scholes built a partial differential equation whose solutions

are the valuations of several derivative instruments in terms of the underlying stock's price. Under the standard assumptions listed above, the value $u = u(t, x)$ of the derivative will depend only on the price x of the stock, on time t and on variables that are taken to be known constants.

3.1.1 Derivation of Black-Scholes PDE

Let S represent a stock price. Then the dynamics of S can be represented by an Ito process given by

$$dS = \mu S dt + \sigma S dW \tag{3.1}$$

where μ is the mean return of the stock,
σ is volatility of stock price,
W is a Wiener process.

Now that the dynamics of the stock over time are known, the Black-Scholes formulation then uses Ito's lemma to calculate the dynamics of an option price over time. Ito's lemma allows continuous replication of a portfolio in order for the option price to be fair valued. From Ito's lemma [16], it follows that the process followed by a function u of S and t is

$$\Delta u = \left(\frac{\partial u}{\partial S} \mu S + \frac{\partial u}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} \right) \Delta t + \frac{\partial u}{\partial S} \sigma S \Delta W. \tag{3.2}$$

Black-Scholes used a replication methodology to price a fairly valued option. They created a theoretical portfolio such that the payoff would mimic that of a financial derivative so that the two portfolio's have equal prices (assuming no risk-free arbitrage opportunities). The appropriate portfolio to eliminate risk is short one derivative and long an amount $\frac{\partial u}{\partial S}$ of shares. Let Π be the value of the portfolio.

The dynamics of this portfolio can be shown by

$$\begin{aligned} d\Pi &= -du + \frac{\partial u}{\partial S} dS, \\ &= \left(-\frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} \right) dt. \end{aligned}$$

As a result of no arbitrage opportunities, there is an over time increment dt

$$d\Pi = r\Pi dt. \tag{3.3}$$

The equation for Π is just an ordinary differential equation ODE with solution $\Pi = \Pi e^{rt}$. The general formula for an option price is discussed below.

Let

$u(x,t)$ be the price of a European call option,
x price of the underlying asset, $x \geq 0$,
t time,
σ asset price volatility and r risk-free interest rate.

Then

$$\tfrac{\partial u}{\partial t} + \tfrac{1}{2}\sigma^2 r^2 \tfrac{\partial^2 u}{\partial x^2} + rx\tfrac{\partial u}{\partial x} - ru = 0.$$

The PDE is a backwards diffusion process i.e. that the initial value problem is well-posed in backwards time. This means for a prescribed future value of an option we can get the price now. The particular derivative that is obtained when the equation is solved depends on the boundary conditions that are used. The most popular financial instruments satisfying the Black-Scholes equation are option contracts and this is what Black and Scholes set out to achieve. Over the years since

the Black-Scholes model, we have seen ordinary European options being modified to meet investors needs, for example barrier options. Barrier options were created for investors willing to speculate for an asset price to move within certain limits in the future [25, 24]. An up-down-and-out double barrier option is financial contract wherein the contract is only terminated if the price of the underlying asset is hit. Thus the boundary conditions of a barrier option are:

$$\begin{aligned} u(x,T) &= \max\{x-K,0\}, \\ u(x,t) &= 0, \text{for } x = L(t) \text{ or } x = H(t) \text{ with } L(t) \leq K \leq H(t). \end{aligned}$$

Chapter 4

The Black-Scholes and the Heat equation

4.1 Introduction

In this research project, we will solve the Black-Scholes equation as a boundary value problem by transforming it into the Heat equation. This technique has been explored in the literature by Gazizov and Ibragimov [10]. Since the Heat equation is known to have fewer parameters and simpler Lie symmetries; we conclude that this equation will work better with our transformed boundary value problem.

4.1.1 Setting up the problem as a boundary value problem

As before, if $u(x,t)$ is the price of a European call option, then

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 r^2 \frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru = 0. \tag{4.1}$$

subject to the following boundary conditions

$$\begin{aligned} u(x,T) &= \max\{x-K, 0\}, \\ u(x,t) &= 0, \text{for } x = L(t) \text{ or } x = H(t) \text{ with } L(t) \leq K \leq H(t). \end{aligned}$$

with K strike price of the option and self-absorbing barriers $L(t)$ and $H(t).0 \leq t \leq T$ with T as maturity date of the option, where x price of the underlying asset, $x \geq 0$, t time and σ asset price volatility and r risk-free interest rate.

4.2 Transformation to Heat equation and dimensionless variables

Following the method of James Nolen [20] we will transform our BVP into the Heat equation using a sequence of simple transformations. We know by Lie's equivalent transformation that if the principal Lie algebra extends by five additional generators, equation (4.1) transforms to

$$v_\tau = v_{yy}. \tag{4.2}$$

Concurrently we will reduce the number of parameters in the model by transforming to "dimensionless variables". This helps to eliminate the degrees of freedom.

Below is the five part transformation of the Black-Scholes equation (4.1) to the Heat equation (4.2) where v_i is the i^{th} transform of the European call option u.

We start by defining the following variables

$$\begin{aligned}\delta &= \frac{2}{\sigma^2},\\ \tau &= \delta^{-1}(T-t),\\ y &= log(\lambda^{-1}x), \lambda \in (0,1).\end{aligned}$$

The transformation will be defined as follows:-

1. $v_1(x,\tau) = u(x, T-\delta\tau)$,
   i.e. Time is reversed
2. $v_2(y,\tau) = v_1(x,\tau)$,
   Dimensionless variable y solves $x = \lambda e^y$
3. $v_3(y,\tau) = e^{\alpha y} v_2(y,\tau)$, $\alpha = \frac{1}{2}(\frac{2r}{\sigma^2} - 1)$
4. $v_4(y,\tau) = e^{\beta\tau} v_3(y,\tau)$, $\beta = \frac{1}{4}(\frac{2r}{\sigma^2} - 1)^2$,

Let the v be the fifth transform i.e $v = v_5$ $v(y,\tau)$ satisfies

$$v_\tau = v_{yy}.$$

It is evident that after this transformation, we get the Heat equation. In fact we may scale v by p such that $v_5(y,\tau) = pv_4(y,\tau)$, where p is a scaling factor, without changing the equation.

Under this transformation, the boundary condition $u(x,T) = \max\{x-K, 0\}$ where $y \in \mathbb{R}$ and $\tau = \delta^{-1}(T-t), \tau > 0$ becomes

$$v(y,0) = v_0(y) := \max(e^{\frac{1}{2}(k+1)y} - e^{\frac{1}{2}(k-1)y}, 0),\ y \in \mathbb{R}\ \ v(y,\tau) = 0 \text{ for } y = \ln(\lambda^{-1}L) \text{ or } y = \ln(\lambda^{-1}H),$$

where $k = \frac{2r}{\sigma^2}$. As we will see, our barriers will be time dependent and not ordinary constant barrier which are usually found in the literature.(see, e.g. [9],[11],[24])

4.3 Finding the exact solution for the BVP

The Lie algebra of the infinitesimal generators of the Heat equation has been largely studied in previous literature (see, e.g. [21])

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial y}, \\
X_2 &= \frac{\partial}{\partial \tau}, \\
X_3 &= v\frac{\partial}{\partial v}, \\
X_4 &= y\frac{\partial}{\partial y} + 2\tau\frac{\partial}{\partial \tau}, \\
X_5 &= 2\tau\frac{\partial}{\partial y} - yv\frac{\partial}{\partial v}, \\
X_6 &= 4\tau y\frac{\partial}{\partial y} + 4\tau^2\frac{\partial}{\partial \tau} - (y^2 + 2\tau)v\frac{\partial}{\partial v}, \\
X_\infty &= g(y,\tau)\frac{\partial}{\partial v}.
\end{aligned}$$

where $g_{yy} = g_\tau$

We note that our initial condition is left invariant under any symmetry generators that has no infinitesimal $\frac{\partial}{\partial y}$ component. This condition is sufficient but not necessary. We use the non-invariant boundary condition of Joanna Goard [12] next.

4.4 Calculations

From the symmetries listed above, the most general finite dimensional symmetry of the Heat equation is given by

$$\mathbf{X} = [(c_1 4y\tau + c_2 y + c_3 2\tau + c_4)]\,\tfrac{\partial}{\partial y} + [\phi_2(c_1 4\tau^2 + c_2 2\tau + c_6)]\,\tfrac{\partial}{\partial \tau} + [v(c_1(-2\tau - y^2) - c_3 y + c_5)]\,\tfrac{\partial}{\partial v}.$$

Our transformed initial condition is

$$v(y,0) = F(y) = e^{\frac{1}{2}(k+1)y} - e^{\frac{1}{2}(k-1)y}, \tag{4.3}$$

for $y \geq 0$ which we substitute into the requirement given by the invariance surface condition (2.7)

$$Y(y,0,F(y))F'(y) + T(y,0,F(y))v_\tau(y,0) = V(y,0,F(y)),$$

along $v_\tau = v_{yy}$ to get

$$\begin{aligned}(e^{\frac{1}{2}(k+1)y} - e^{\frac{1}{2}(k-1)y})(-c_1 y^2 - c_3 y + c_5) - (\frac{1}{2}(k+1)e^{\frac{1}{2}(k+1)y} - \frac{1}{2}(k-1)e^{\frac{1}{2}(k-1)y})(c_2 y + c_4)\\ = (\frac{1}{4}(k+1)^2 e^{\frac{1}{2}(k+1)y} - \frac{1}{4}(k-1)^2 e^{\frac{1}{2}(k-1)y})c_6.\end{aligned}$$

We equate coefficients of $e^{\frac{1}{2}(k+1)y}$ and $e^{\frac{1}{2}(k-1)y}$ to get

$$\begin{aligned} -c_1 &= 0,\\ -c_3 - \frac{1}{2}(c_2)(k+1) &= 0,\\ c_3 + \frac{1}{2}(c_2)(k-1) &= 0,\\ c_5 - \frac{1}{2}(k+1)c_4 &= \frac{1}{4}(k+1)^2 c_6,\\ -c_5 + \frac{1}{2}(k-1)c_4 &= -\frac{1}{4}(k-1)^2 c_6,\end{aligned}$$

and solve simultaneously to find that $c_1 = c_2 = c_3 = 0$ (if $c_2 \neq 0$ then the interest rate must be zero) and

$$\begin{aligned} c_6 &= -\frac{c_4}{k}, \\ c_5 &= \left(\frac{k^2-1}{4k}\right) c_4. \end{aligned}$$

The most general symmetry that allows the maturity condition is

$$\widetilde{X} = \tfrac{\partial}{\partial y} - \tfrac{1}{k}\tfrac{\partial}{\partial \tau} + \tfrac{k^2-1}{(4k)} v \tfrac{\partial}{\partial v},$$

where for the sake of convenience we set $c_4 = 1$.
Now we solve the system

$$dy = -k d\tau = \tfrac{4k dv}{(k^2-1)v}.$$

The left-hand pair gives $I_1 = y + k\tau$, while the right-hand pair after a few steps gives

$$\begin{aligned} v &= e^{-(\frac{k^2-1}{4})\tau} I_2, \\ &= e^{-(\frac{k^2-1}{4})\tau} h(I_1). \end{aligned}$$

Thus

$$\begin{aligned} v_\tau &= -(\frac{k^2-1}{4}) e^{-(\frac{k^2-1}{4})\tau} h'(I_1), \\ v_{yy} &= e^{-(\frac{k^2-1}{4})\tau} h''(I_1). \end{aligned}$$

Substitute these into the Heat equation $v_\tau = v_{yy}$ to get

$$\begin{aligned} -(\frac{k^2-1}{4})h + kh' &= h'', \\ \Rightarrow \left[D^2 - kD + (\frac{k^2-1}{4})\right] &= 0, \\ \Rightarrow h(I_1) &= A e^{\frac{1}{2}(k+1)I_1} + B e^{\frac{1}{2}(k-1)I_1}. \end{aligned}$$

Hence

$$\begin{aligned} v &= e^{-(\frac{k^2-1}{4})\tau}(Ae^{\frac{1}{2}(k+1)I_1} + Be^{\frac{1}{2}(k-1)I_1}), \\ &= e^{-(\frac{k^2-1}{4})\tau}(Ae^{\frac{1}{2}(k+1)(y+k\tau)} + Be^{\frac{1}{2}(k-1)(y+k\tau)}), \\ &= e^{-(\frac{k^2-1}{4})\tau}(e^{\frac{1}{2}(k+1)(y+k\tau)} - e^{\frac{1}{2}(k-1)(y+k\tau)}). \end{aligned}$$

It is clear to see that for $y \geq 0$, the initial condition is satisfied when $\tau = 0$ but not for $y < 0$. This is not a problem if we have a down-out barrier $L(t)$ such that $L(\tau) = -k\tau$. When back-transformed, we get $L(t) = Ke^{\frac{-2r^2(T-t)}{\sigma^2}}$.

To find the price of the barrier option $u(x,t)$ we back transform to get

$$u(x,t) = e^{-\frac{r(T-t)}{2}(\frac{2r}{\sigma^2}-3)}\left[\frac{e^{\frac{2r^2(T-t)}{\sigma^2}}x}{K} - 1\right].$$

The maturity condition changes definition when $y = 0$ so it is necessary to have a barrier condition that has $v = 0$ when $y = \tau = 0$. The following figure will show the path of Heat equation together with the maturity condition.

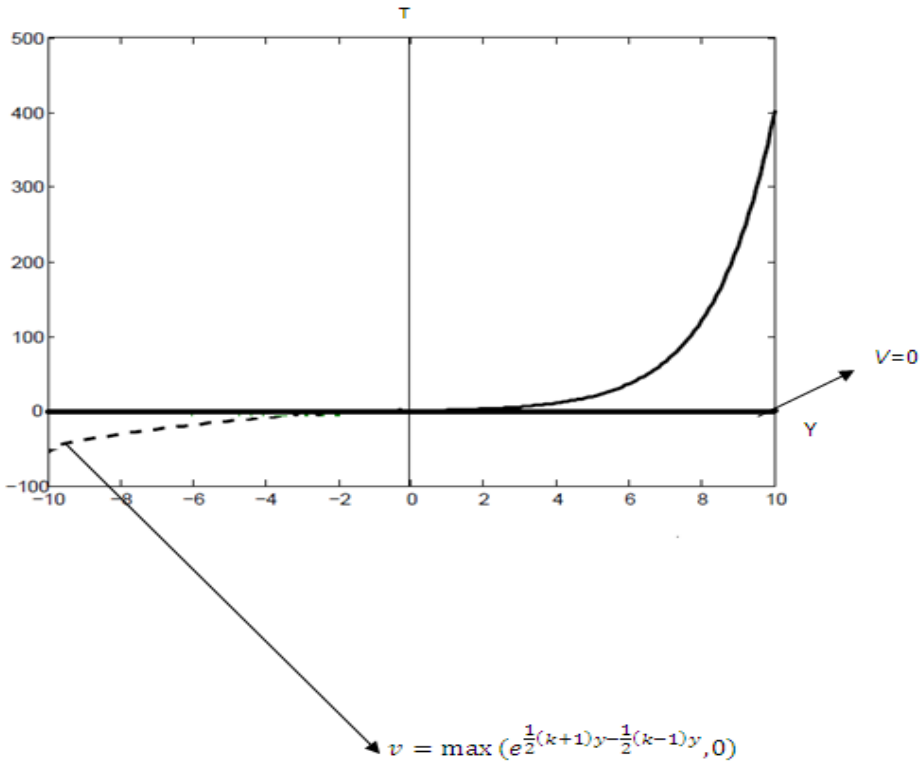
Symmetry generator and ISC at the barrier

We now look at the ISC in equation at a barrier. If $\widetilde{X}$ is admitted by equation (4.2), then according to the invariant surface condition, invariant solutions that satisfy our barrier condition will then satisfy

$$Y\left(L(\tau),\tau,f(\tau)\right)v_y + T\left(L(\tau),\tau,f(\tau)\right)v_t = V\left(L(\tau),\tau,f(\tau)\right).$$

Using the analysis in [12] the ISC condition at the maturity boundary becomes

Figure 4.1: *This figure of the transformed shows a path where the two boundary conditions meet.*



$$c_4 v_y - \frac{c_4}{k} v_\tau = v c_4 (\frac{k^2 - 1}{4k}) \tag{4.4}$$

We notice that our boundary conditions are not satisfied since equation (4.4) is discontinuous at boundary. We see that there are limited types of barriers which our boundary value problem will allow.

Through observation, only time dependent barriers will satisfy the above solution e.g. with $L(t) = \frac{-4t}{k}$. Let us now consider a more generalized symmetry

Infinite Dimensional Symmetry

If we include the infinite dimensional generator, namely $g(y,\tau)\frac{\partial}{\partial v}$ we get the following symmetry

$$\begin{aligned} \widetilde{X} = &(c_1 4y\tau + c_2 y + c_3 2\tau + c_4)\frac{\partial}{\partial y} + \phi_2(c_1 4\tau^2 + c_2 2\tau + c_6)\frac{\partial}{\partial \tau} \\ &+ (v(c_1(-2\tau - y^2) - c_3 y + c_5) + g(y,\tau))\frac{\partial}{\partial v}, \end{aligned} \tag{4.5}$$

where g is a solution to the Heat equation; we can choose a g such that $g_\tau = g_{yy}$. We consider a g of the form $g(y,\tau) = A(\tau)e^{\frac{1}{2}(k-1)y} + B(\tau)e^{\frac{1}{2}(k+1)y}$.

Solving the Heat equation gives

$$\begin{aligned} A(\tau) &= s_1 e^{\frac{1}{4}(k-1)^2\tau}, \\ B(\tau) &= s_2 e^{\frac{1}{4}(k+1)^2\tau}. \end{aligned}$$

so that g is given by

$$g(y,\tau) = s_1 e^{\frac{1}{4}(k-1)^2\tau + \frac{1}{2}(k-1)y} + s_2 e^{\frac{1}{4}(k+1)^2\tau + \frac{1}{2}(k+1)y}.$$

Let us find a symmetry $\widetilde{X}$ of the Heat equation with g as above, that generates solutions that satisfy our initial condition

$$v(y,0) = F(y) = e^{\frac{1}{2}(k+1)y} - e^{\frac{1}{2}(k-1)y}.$$

We require that the invariant surface condition be satisfied, i.e.

$$Y(y,0,F(y))F'(y) + T(y,0,F(y))v_\tau(y,0) = V(y,0,F(y)).$$

Along $v_\tau = v_{yy}$ we get

$$Y(y,0,F(y))F'(y) + T(y,0,F(y))v_{yy}(y,0) = V(y,0,F(y)),$$

so that equation (2.7) after simplification becomes

$$\begin{aligned} e^{\frac{1}{2}(k+1)y}(\frac{1}{4}(k+1)^2c_6) - e^{\frac{1}{2}(k-1)y}(-\frac{1}{4}(k-1)^2c_6) = {} & e^{\frac{1}{2}(k+1)y}(\frac{1}{2}(k+1)(c_2y+c_4) + c_1y^2) \\ & + e^{\frac{1}{2}(k+1)y}(c_3y - c_5 - s_2) \\ & + e^{\frac{1}{2}(k-1)y}(\frac{1}{2}(k-1)(c_2y+c_4)) \\ & + e^{\frac{1}{2}(k-1)y}(c_1y^2 + c_3y - c_5 + s_1). \end{aligned}$$

Equating the coefficients of $e^{\frac{1}{2}(k+1)y}$ and $e^{\frac{1}{2}(k-1)y}$ we get

$$\begin{aligned} \frac{1}{4}(k+1)^2c_6 &= -\frac{1}{2}(k+1)(c_2y+c_4) - c_1y^2 - c_3y + c_5 + s_2, \\ -\frac{1}{4}(k-1)^2c_6 &= \frac{1}{2}(k-1)(c_2y+c_4) + c_1y^2 + c_3y - c_5 + s_1. \end{aligned}$$

Separation by powers of y and solving for $c_1, c_2, c_3, c_4, c_5, c_6, s_1, s_2$ gives

$$
\begin{aligned}
c_1 &= 0,\\
c_2 &= 0,\\
c_3 &= 0,\\
s_2 + c_5 - \frac{1}{2}(k+1)c_4 &= \frac{1}{4}(k+1)^2 c_6,\\
s_1 - c_5 + \frac{1}{2}(k-1)c_4 &= -\frac{1}{4}(k-1)^2 c_6.
\end{aligned}
$$

This gives

$$
\begin{aligned}
\widetilde{X} = c_4\frac{\partial}{\partial y} + c_6\frac{\partial}{\partial \tau} + (c_5 v + (c_5 - \frac{1}{4}(k-1)^2 c_6 - \frac{1}{2}(k-1)c_4)e^{\frac{1}{4}(k-1)^2\tau + \frac{1}{2}(k-1)y}\\
+ (-c_5 + \frac{1}{4}(k+1)^2 c_6 + \frac{1}{2}(k+1)c_4)e^{\frac{1}{4}(k+1)^2\tau + \frac{1}{2}(k+1)y})\frac{\partial}{\partial v}.
\end{aligned}
$$

The Lie algebra is then generated by

$$
\begin{aligned}
\widetilde{X_1} &= \frac{\partial}{\partial y} + (-\frac{1}{2}(k-1)e^{\frac{1}{4}(k-1)^2\tau + \frac{1}{2}(k-1)y} + \frac{1}{2}(k+1)e^{\frac{1}{4}(k+1)^2\tau + \frac{1}{2}(k+1)y})\frac{\partial}{\partial v},\\
\widetilde{X_2} &= (v + e^{\frac{1}{4}(k-1)^2\tau + \frac{1}{2}(k-1)y} - e^{\frac{1}{4}(k+1)^2\tau + \frac{1}{2}(k+1)y})\frac{\partial}{\partial v},\\
\widetilde{X_3} &= \frac{\partial}{\partial \tau} + (-\frac{1}{4}(k-1)^2 e^{\frac{1}{4}(k-1)^2\tau + \frac{1}{2}(k-1)y} + \frac{1}{4}(k+1)^2 e^{\frac{1}{4}(k+1)^2\tau + \frac{1}{2}(k+1)y})\frac{\partial}{\partial v}.
\end{aligned}
$$

4.5 Invariant solutions

4.5.1 Symmetry generator $\widetilde{X_1}$

We use the symmetry generator below to generate an invariant solution

$$\widetilde{X_1} = \frac{\partial}{\partial y} + (-\tfrac{1}{2}(k-1)e^{\frac{1}{4}(k-1)^2\tau+\frac{1}{2}(k-1)y} + \tfrac{1}{2}(k+1)e^{\frac{1}{4}(k+1)^2\tau+\frac{1}{2}(k+1)y})\frac{\partial}{\partial v}$$

to reduce the Heat equation. The corresponding characteristic equation is

$$\frac{dy}{1} = \frac{dv}{\frac{1}{2}(k+1)e^{\frac{1}{4}(k+1)^2\tau+\frac{1}{2}(k+1)y} - \frac{1}{2}(k-1)e^{\frac{1}{4}(k-1)^2\tau+\frac{1}{2}(k-1)y}}.$$

with τ being an invariant i.e let $I_1 = \tau$. Then

$$\begin{aligned} \frac{dv}{dy} &= \frac{1}{2}(k+1)e^{\frac{1}{4}(k+1)^2\tau+\frac{1}{2}(k+1)y} - \frac{1}{2}(k-1)e^{\frac{1}{4}(k-1)^2\tau+\frac{1}{2}(k-1)y} \\ \Rightarrow v &= e^{\frac{1}{4}(k+1)^2\tau+\frac{1}{2}(k+1)y} - e^{\frac{1}{4}(k-1)^2\tau+\frac{1}{2}(k-1)y} + I_2. \end{aligned}$$

Let $h(I_1) = I_2$ i.e $h(I_1) = h(\tau)$, so that

$$v = e^{\frac{1}{4}(k+1)^2\tau+\frac{1}{2}(k+1)y} - e^{\frac{1}{4}(k-1)^2\tau+\frac{1}{2}(k-1)y} + h(\tau).$$

Now we find an h such that v satisfies the Heat equation.

$$\begin{aligned} v_\tau &= \frac{1}{4}(k+1)^2 e^{\frac{1}{4}(k+1)^2\tau+\frac{1}{2}(k+1)y} - \frac{1}{4}(k-1)^2 e^{\frac{1}{4}(k-1)^2\tau+\frac{1}{2}(k-1)y} + h'(\tau), \\ v_y &= \frac{1}{2}(k+1)e^{\frac{1}{4}(k+1)^2\tau+\frac{1}{2}(k+1)y} - \frac{1}{2}(k-1)e^{\frac{1}{4}(k-1)^2\tau+\frac{1}{2}(k-1)y}, \\ v_{yy} &= \frac{1}{4}(k+1)^2 e^{\frac{1}{4}(k+1)^2\tau+\frac{1}{2}(k+1)y} - \frac{1}{4}(k-1)^2 e^{\frac{1}{4}(k-1)^2\tau+\frac{1}{2}(k-1)y}. \end{aligned}$$

When $v_\tau = v_{yy}$

$$\frac{1}{4}(k+1)^2 e^{\frac{1}{4}(k+1)^2\tau+\frac{1}{2}(k+1)y} - \frac{1}{4}(k-1)^2 e^{\frac{1}{4}(k-1)^2\tau+\frac{1}{2}(k-1)y} + h'(\tau) = \frac{1}{4}(k+1)^2 e^{\frac{1}{4}(k+1)^2\tau+\frac{1}{2}(k+1)y} - \frac{1}{4}(k-1)^2 e^{\frac{1}{4}(k-1)^2\tau+\frac{1}{2}(k-1)y}.$$

Almost everything cancels, leaving $h'(\tau) = 0$ so that $h = A$ for A an arbitrary constant. Thus

$$v(y,\tau) = e^{\frac{1}{4}(k+1)^2\tau+\frac{1}{2}(k+1)y} - e^{\frac{1}{4}(k-1)^2\tau+\frac{1}{2}(k-1)y} + A.$$

Now we test the above solution at boundary. The maturity condition is easily satisfied by choosing an appropriate A. If we let $A = 0$ and choose a specific barrier $L(\tau) = -k\tau$ the barrier condition will be satisfied. We are restricted to this type of barrier as the symmetry generator will also have to satisfy the maturity condition. When back transformed, the specific barrier is given by $L(t) = Ke^{-r(T-t)}$

To back transform to Black-Scholes equation using the five step transformation mentioned in section 4.2 we get

$$u(x,t) = \frac{e^{r(T-t)}x}{K} - 1 \tag{4.6}$$

4.5.2 Symmetry generator $\widetilde{X_2}$

Next we consider the second generator

$$\widetilde{X_2} = (v + e^{\frac{1}{4}(k-1)^2\tau+\frac{1}{2}(k-1)y} - e^{\frac{1}{4}(k+1)^2\tau+\frac{1}{2}(k+1)y})\frac{\partial}{\partial v}.$$

The corresponding characteristic equation is

$$\frac{dy}{0} = \frac{d\tau}{0} = \frac{dv}{v + e^{\frac{1}{4}(k-1)^2\tau + \frac{1}{2}(k-1)y} - e^{\frac{1}{4}(k+1)^2\tau + \frac{1}{2}(k+1)y}}.$$

Here y and τ are invariants. An invariant solution is given by

$$v(y,\tau) = e^{\frac{1}{4}(k-1)^2\tau + \frac{1}{2}(k-1)y} - e^{\frac{1}{4}(k+1)^2\tau + \frac{1}{2}(k+1)y}. \tag{4.7}$$

We see that $\widetilde{X_2}$ generates similar solution to $\widetilde{X_1}$ hence we will get same back-transformed solution

4.5.3 Symmetry generator $\widetilde{X_3}$

Lastly, we generate invariants using

$$\widetilde{X_3} = \frac{\partial}{\partial\tau} + (-\frac{1}{4}(k-1)^2 e^{\frac{1}{4}(k-1)^2\tau + \frac{1}{2}(k-1)y} + \frac{1}{4}(k+1)^2 e^{\frac{1}{4}(k+1)^2\tau + \frac{1}{2}(k+1)y})\frac{\partial}{\partial v},$$

the corresponding characteristic equation is

$$\frac{d\tau}{1} = \frac{dv}{-\frac{1}{4}(k-1)^2 e^{\frac{1}{4}(k-1)^2\tau + \frac{1}{2}(k-1)y} + \frac{1}{4}(k+1)^2 e^{\frac{1}{4}(k+1)^2\tau + \frac{1}{2}(k+1)y}},$$

with y being another invariant.

$$\frac{dv}{d\tau} = -\frac{1}{4}(k-1)^2 e^{\frac{1}{4}(k-1)^2\tau + \frac{1}{2}(k-1)y} + \frac{1}{4}(k+1)^2 e^{\frac{1}{4}(k+1)^2\tau + \frac{1}{2}(k+1)y},$$

Using Mathematica to solve the ODE $\frac{dv}{d\tau} = ...$, we get

$$\begin{aligned} v(y,\tau) &= e^{\frac{1}{4}(k+1)^2\tau + \frac{1}{2}(k+1)y} - e^{\frac{1}{4}(k-1)^2\tau + \frac{1}{2}(k-1)y} + I_2 \\ v(y,\tau) &= -e^{\frac{1}{4}(k-1)^2\tau + \frac{1}{2}(k-1)y} + e^{\frac{1}{4}(k+1)^2\tau + \frac{1}{2}(k+1)y} + h(y) \end{aligned}$$

since $I_2 = h(I_1) = h(y)$.

$\Rightarrow v(y,\tau) = -e^{\frac{1}{4}(k-1)^2\tau+\frac{1}{2}(k-1)y} + e^{\frac{1}{4}(k+1)^2\tau+\frac{1}{2}(k+1)y} + h(y).$
Now we find an h such that v satisfies the Heat equation.

$$\begin{aligned} v_\tau &= -\frac{1}{4}(k-1)^2 e^{\frac{1}{4}(k-1)^2\tau+\frac{1}{2}(k-1)y} + \frac{1}{4}(k+1)^2 e^{\frac{1}{4}(k+1)^2\tau+\frac{1}{2}(k+1)y}, \\ v_y &= -\frac{1}{2}(k-1) e^{\frac{1}{4}(k-1)^2\tau+\frac{1}{2}(k-1)y} + \frac{1}{2}(k+1) e^{\frac{1}{4}(k+1)^2\tau+\frac{1}{2}(k+1)y} + h'(y), \\ v_{yy} &= -\frac{1}{4}(k-1)^2 e^{\frac{1}{4}(k-1)^2\tau+\frac{1}{2}(k-1)y} + \frac{1}{4}(k+1)^2 e^{\frac{1}{4}(k+1)^2\tau+\frac{1}{2}(k+1)y} + h''(y). \end{aligned}$$

Along $v_{yy} = v_\tau$, almost everything cancels, leaving

$$\begin{aligned} h''(y) &= 0, \\ h(y) &= A(\tau) + B(\tau)y. \end{aligned}$$

Hence

$$v(y,\tau) = -e^{\frac{1}{4}(k-1)^2\tau+\frac{1}{2}(k-1)y} + e^{\frac{1}{4}(k+1)^2\tau+\frac{1}{2}(k+1)y} + A(\tau) + B(\tau)y,$$

Setting $A = B = 0$ will generate solution satisfying the maturity condition. This solution however is similar to the one previously found. In principle it may be possible to find barriers that admit the full solution, but it seems difficult, and any such barriers are unlikely to be of much practical importance.

4.5.4 Symmetry generator $\widetilde{X}$

Now we generate invariants using the following general form of our generator

$$\widetilde{X} = c_4\frac{\partial}{\partial y} + c_6\frac{\partial}{\partial \tau} + (c_5 v + (c_5 - \tfrac{1}{4}(k-1)^2 c_6 - \tfrac{1}{2}(k-1)c_4)e^{\frac{1}{4}(k-1)^2\tau + \frac{1}{2}(k-1)y} + (-c_5 + \tfrac{1}{4}(k+1)^2 c_6 + \tfrac{1}{2}(k+1)c_4)e^{\frac{1}{4}(k+1)^2\tau + \frac{1}{2}(k+1)y})\frac{\partial}{\partial v}.$$

The corresponding characteristic equation is given by

$$\frac{dy}{c_4} = \frac{d\tau}{c_6} = \frac{dv}{w},$$

where

$$\begin{aligned} w =& c_5 v + (c_5 - \frac{1}{4}(k-1)^2 c_6 \\ & - \frac{1}{2}(k-1)c_4)e^{\frac{1}{4}(k-1)^2\tau + \frac{1}{2}(k-1)y} \\ & + (-c_5 + \frac{1}{4}(k+1)^2 c_6 + \frac{1}{2}(k+1)c_4)e^{\frac{1}{4}(k+1)^2\tau + \frac{1}{2}(k+1)y}. \end{aligned}$$

With the below pair

$$\frac{dy}{c_4} = \frac{d\tau}{c_6},$$

then $y = \frac{c_4}{c_6}\tau + I_1$

With the pair

$$\frac{dy}{c_4} = \frac{dv}{w}$$

we get

$$\begin{aligned} v(y,\tau) = \frac{1}{2}e^{c_5 y + \frac{1}{4}((k-1)^2\tau + 2(k-2c_5-1)y)} \Bigg(& \frac{-2c_4(k-1) + 4c_5 - (k-1)^2 c_6}{k - 1 - 2c_5} \\ & + \frac{e^{k\tau + y}(2c_4(k+1) - 4c_5 + (k+1)^2 c_6)}{k + 1 - 2c_5} \Bigg) + e^{c_5 y} I_2, \end{aligned}$$

Since $I_2 = h(I_1)$, we can write

$$v(y,\tau) = \frac{1}{2}e^{c_5 y+\frac{1}{4}((k-1)^2\tau+2(k-2c_5-1)y)}\left(\frac{-2c_4(k-1)+4c_5-(k-1)^2c_6}{k-1-2c_5}\right)$$
$$+\frac{1}{2}e^{c_5 y+\frac{1}{4}((k-1)^2\tau+2(k-2c_5-1)y)}\left(\frac{e^{k\tau+y}(2c_4(k+1)-4c_5+(k+1)^2c_6)}{k+1-2c_5}\right)+e^{c_5 y}h(I_1),$$

$$v_\tau = \frac{ke^{k\tau+y+c_5 y+\frac{1}{4}((k-1)^2\tau+2(k-2c_5-1)y)}(2c_4(k+1)-4c_5+(k+1)^2c_6)}{2(k+1-2c_5)}$$
$$+\frac{1}{8}e^{c_5 y+\frac{1}{4}((k-1)^2\tau+2(k-2c_5-1)y)}(k-1)^2(\frac{-2c_4(k-1)+4c_5-(k-1)^2c_6}{k-1-2c_5})$$
$$+\frac{1}{8}e^{c_5 y+\frac{1}{4}((k-1)^2\tau+2(k-2c_5-1)y)}(k-1)^2(\frac{e^{k\tau+y}(2c_4(k+1)-4c_5+(k+1)^2c_6)}{k+1-2c_5})$$
$$+e^{c_5 y}h'(I_1),$$

$$v_{yy} = \frac{e^{k\tau+y+c_5 y+\frac{1}{4}((k-1)^2\tau+2(k-2c_5-1)y)}(\frac{1}{2}(k-1-2c_5)+c_5)(2c_4(k+1)-4c_5+(k+1)^2c_6)}{2(k+1-c_5)}$$
$$+\frac{e^{k\tau+y+c_5 y+\frac{1}{4}((k-1)^2\tau+2(k-2c_5-1)y)}(1+\frac{1}{2}(k-1-2c_5)+c_5)(2c_4(k+1)-4c_5+(k+1)^2c_6)}{2(k+1-c_5)}$$
$$+\frac{1}{2}e^{c_5 y+\frac{1}{4}((k-1)^2\tau+2(k-2c_5-1)y)}(\frac{1}{2}(k-1-2c_5)+c_5)^2(\frac{-2c_4(k-1)+4c_5-(k-1)^2c_6}{k-2c_5-1})$$
$$+\frac{1}{2}e^{c_5 y+\frac{1}{4}((k-1)^2\tau+2(k-2c_5-1)y)}(\frac{1}{2}(k-1-2c_5)+c_5)^2(\frac{e^{k\tau+y}(2c_4(k+1)-4c_5+(k+1)^2c_6)}{k+1-2c_5})$$
$$+e^{c_5 y}c_5^2h(I_1)+2e^{c_5 y}h'(I_1)+e^{c_5 y}h''(I_1).$$

The Heat equation

$$v_\tau = v_{yy},$$

becomes

$$e^{c_5 y}(c_5^2h-h'+2c_5h'+h'')=0,$$

which can be solved to get

$$h = Ce^{\frac{1}{2}(1-\sqrt{1-4c_5}-2c_5)(y-\frac{c_4}{c_6}\tau)}+De^{\frac{1}{2}(1+\sqrt{1-4c_5}-2c_5)(y-\frac{c_4}{c_6}\tau)},$$

where C and D are unknown constants

Hence

$v(y,\tau) = \frac{1}{2}e^{c_5 y+\frac{1}{4}((k-1)^2\tau+2(k-2c_5-1)y)}\left(\frac{-2c_4(k-1)+4c_5-(k-1)^2c_6}{k-1-2c_5}+\frac{e^{k\tau+y}(2c_4(k+1)-4c_5+(k+1)^2c_6)}{k+1-2c_5}\right)+$
$e^{c_5 y}\left(Ce^{\frac{1}{2}(1-\sqrt{1-4c_5}-2c_5)(y-\frac{c_4}{c_6}\tau)}+De^{\frac{1}{2}(1+\sqrt{1-4c_5}-2c_5)(y-\frac{c_4}{c_6}\tau)}\right).$

Choosing the boundary specific constants of the above solution has proven to be impractical and hence we did not generate the back-transformed solution.

Chapter 5

Conclusion

It is evident that Lie symmetries can successfully solve the Black-Scholes equation with certain boundary conditions, particulary barrier conditions. The barriers are however limited to those mentioned in this thesis. Barriers which are widely used within the industry are those which are arbitrarily constant throughout the life of the option. However, Lie symmetries are mostly applicable on partial differential equations with smooth boundaries and we have not found a constant barrier condition admitted by the symmetries. This does not mean there are no solutions to a constant barrier; we made certain limiting assumptions, for example, about the form of $g(y, \tau)$ in $\widetilde{X}$. Other forms of g have not been considered, so that may be something for future research.

This research has shown that using symmetries to solve the pricing problem of European options with barriers can be very difficult since Lie symmetries would usually work on smooth functions. Pricing a barrier European option requires more boundary conditions to be satisfied. In the research we only concentrated on down-Out double barrier option. When the barriers are hit, the contracts will be nullified an investor will lose the 'option' that he/she purchased at the beginning of the contract. As we have illustrated in chapter 4, the self absorbing barriers will enable the

maturity condition of the option to be governed by the symmetry for values greater or equal to 0. Using the calculated Lie symmetries and the analysis of [12], we could generate invariant solutions that would satisfy both the maturity condition and the barrier condition. The solution to the pricing problem was one which only had a certain type of barrier i.e a time dependent barrier. In the industry there are various types of double barrier option but time-dependent, barriers are seldom used, if ever. This observation is very much dependent on investor's interest at the time. For future purposes, we will investigate whether time dependent barriers are practical within the industry and whether investors will be interested in these types of financial products e.g barrier options which will change sometime in the future if investors expect asset prices to be more volatile. Across the industry various methods have been used to price a double barrier option. In both literature and in practice, these methods include a probabilistic approach [11] and using Laplace transforms [23]. Although these methods do not generate a closed form solution, they can price other types of barrier options including the ones that this research paper did not touch on.

In this research, we have assumed that the barrier options will be continuously monitored throughout the life of the financial contract. This means that within certain time intervals the asset price will be evaluated to check if the barriers ever equated to the asset price at any particular point in time. This is how all barrier options are monitored within the market. For future research (similar to that of [4]) might focus on finding attempt to find a fair valued price of a double barrier option that will also incorporate a discretely monitored barrier throughout the life of the financial contract. Other possible research topics will be investigating the price of different types of double barrier options.

References

[1] M. Bertoli and M. Bianchetti. Monte carlo simulation of discrete barrier options. *Financial engineering-Derivatives Modelling, Caboto SIM S.p.a., Banca Intesa Group, Milan, Italy*, 25:1, 2003.

[2] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:637–659, 1973.

[3] George W. Bluman and Sukeyuki Kumei. *Symmetries and differential equations*. Springer-Verlag, 1989.

[4] M. Broadie and J. Detemple. American capped call options on dividend-paying assets. *Rev. Financial Stud.*, 8:161–192, 1995.

[5] M. Broadie, P. Glasserman, and S. Kou. A continuity correction for barrier options. *Math. Finance*, 7:325–349, 1997.

[6] P. Carr. Two extensions to barrier option valuation. *Appl. Math. Finance*, 2:173–209, 1995.

[7] G. Fusai, I. Abrahams, and C. Sgarra. An exact analytical solution for barrier options. *Finance Stoch*, 10:1–26, 2006.

[8] G. Fusai and M. Recchioni. Analysis of quadrature methods for pricing discrete barrier options. *J. Econ. Dyn. Control*, 31:826–860, 2007.

[9] B. Gao, J. Huang, and M. Subrahmanyam. The valuation of american barrier options using the decomposition technique. *J. Econ. Dyn. Control*, 24:1783–1827, 2000.

[10] R. K. Gazizov and N. H. Ibragimov. Lie symmetry analysis of differential equations in finance. *Nonlinear Dynamics*, 17:387–407, 1998.

[11] H. Geman and M. Yor. Pricing and hedging double-barrier options: a probabilistics approach. *Math. Finance*, 6:365–378, 1996.

[12] Joanna Goard. Noninvariant boundary conditions. *Applicable Analysis: An International Journal*, 82(5):473–481, 2003.

[13] E. Haug. Closed form valuation of american barrier options. *Int. J. Theor. Appl. Finance*, 4:355–359, 2001.

[14] H. Hsu. Surprised parties. *RISK*, 10:27–29, 1997.

[15] Peter E. Hydon. Symmetry analysis of initial-value problems. *J. Math. Anal. Appl.*, 309:103–116, 2005.

[16] Kiyoshi Ito. Stochastic integral. *Proc. Imperial Acad. Tokyo*, 20:519–524, 1944.

[17] N. Kunitomo and M. Ikeda. Pricing options with curved boundaries. *Math. Finance*, 2:275–298, 1992.

[18] G. Metin, I. Habibullin, and K. Zheltukhin. Integrable boundary value problems for elliptic type toda lattice in a disk. *J. Math. Anal. Appl.*, 48:102702, 2007.

[19] J.C. Ndogmo and D.B. Ntwiga. High-order accurate implicit methods for barrier option pricing. *Appl. Math. Comp.*, 218:2210–2224, 2011.

[20] James Nolen. *Diffusion Process*. Stanford, Department of Mathematics, 2000.

[21] Peter J Olver. *Applications of Lie Groups to Differential equations*. Springer, 1986.

[22] Peter J Olver. *Equivalence, Invariants, and Symmetry*. Springer, 1995.

[23] A. Pelseer. Pricing double barrier options using laplace transforms. *Finance Stoch*, 4:95–104, 2000.

[24] M. Rubinstein and E. Reiner. Breaking down the barries. *RISK*, 4:28–35, 1994.

[25] R. Zvan, K. Vetzal, and P. Forsyth. Pde methods for pricing barrier options. *Journal Econ. Dyn. Control*, 24:1563–1590, 2000.