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ORIGINAL PAPER

The Diophantine equations $P_n^x + P_{n+1}^y = P_m^x$ or $P_n^y + P_{n+1}^x = P_m^x$

Bernadette Faye[1] · Florian Luca[2,3,4] · Salah Eddine Rihane[5] · Alain Togbé[6]

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Abstract
Here, we find all positive integer solutions of the Diophantine equations in the title, where $(P_n)_{n\geq 0}$ is the Pell sequence $P_0 = 0,\ \ P_1 = 1$ and $P_{n+2} = 2P_{n+1} + P_n$ for all $n \geq 0$.

Keywords Pell numbers · Linear form in logarithms · Reduction method

Mathematics Subject Classification 11B39 · 11J86

1 Introduction

We look at the title equation in positive integer variables (n, m, x, y), where $(P_k)_{k\geq 0}$ is the Pell sequence given by $P_0 = 0,\ \ P_1 = 1$, and $P_{k+2} = 2P_{k+1} + P_k$ for all $k \geq 0$. We have the following theorem.

✉ Salah Eddine Rihane
salahrihane@hotmail.fr; salaheddine.rihane@nhsm.edu.dz

Bernadette Faye
bernadette.fayee@gmail.com

Florian Luca
Florian.Luca@wits.ac.za

Alain Togbé
atogbe@pnw.edu

1 UFR SAT, Section Maths Appliquées, University Gaston Berger of Saint-Louis (UBG), Saint-Louis, Senegal

2 School of Mathematics, University of the Witwatersrand, Private Bag 3, Wits, Johannesburg 2050, South Africa

3 Research Group in Algebraic Structures and Applications, King Abdulaziz University, Jeddah, Saudi Arabia

4 Centro de Ciencias Matemáticas UNAM, Morelia, Mexico

5 National Higher School of Mathematics, P.O. Box 75, Mahelma, Sidi Abdellah, 16093 Algiers, Algeria

6 Department of Mathematics and Statistics, Purdue University Northwest, 1401 S, U.S. 421, Westville, IN 46391, USA

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Theorem 1.1 *The only solution in positive integers* (n, m, x, y) *of either of the equations*

$$P_n^x + P_{n+1}^y = P_m^x \quad or \quad P_n^y + P_{n+1}^x = P_m^x \tag{1.1}$$

is $(n, m, x, y) = (1, 3, 1, 2)$*, for which*

$$1 + 2^2 = 5 = P_3. \tag{1.2}$$

Our work complements recent results on ternary equations $a^x + b^y = c^z$, where a, b, c are positive integers belonging to either the Fibonacci or the Pell sequence and x, y, z are positive integers [5, 7, 9, 10, 13]. Our proof uses Carmichael's theorem on primitive divisors for Lucas sequences, lower bounds for linear forms in two or three logarithms, both complex and p-adic, as well as the LLL reduction algorithm and reductions based on continued fractions. We will recall such results when we need them.

2 The proof

Since $\gcd(P_n, P_{n+1}) = 1$, it follows that in Eq. 1.1 the two summands in the left-hand side are coprime. By Fermat's last theorem proved by Wiles, $\gcd(x, y) \in \{1, 2\}$. Either one of Eq. 1.1 implies $P_m^x > P_n^x$, so $m > n$. The case $m = n + 1$ is impossible since P_n and P_{n+1} are coprime. Thus, $m \geq n + 2$. If $y \leq x$, then

$$P_m^x \leq P_n^x + P_{n+1}^x \leq (P_n + P_{n+1})^x < P_{n+2}^x,$$

so $m < n + 2$, which is impossible. So, we have the first lemma.

Lemma 2.1 *Assume that* (n, m, x, y) *is a solution in positive integers of equation 1.1. Then:*

(i) $m \geq n + 2$;
(ii) $y > x$;
(iii) $\gcd(x, y) \in \{1, 2\}$.

Let $(Q_k)_{k\geq 0}$ be the companion Pell numbers given by $Q_0 = 2,\ Q_1 = 2$ and $Q_{k+2} = 2Q_{k+1} + Q_k$ for all $k \geq 0$. We have the following well-known property.

Lemma 2.2 *Assume* $m \equiv n \pmod 2$. *Then, we have*

$$P_m - P_n = P_{(m-\varepsilon n)/2} Q_{(m+\varepsilon n)/2}$$

where $\varepsilon = 1,\ -1$*, respectively according to* $m - n \equiv 0,\ 2 \pmod 4$.

Lemma 2.3 *The only solution* (n, m, x, y) *in positive integers of Eq. 1.1 with* $n \in \{1, 2\}$ *is* $(n, m, x, y) = (1, 3, 1, 2)$.

Proof If $n = 1$, the equation is one of

$$1 + 2^y = P_m^x \quad \text{or} \quad 1 + 2^x = P_m^x.$$

The second one is impossible for $x > 1$ since

$$P_m^x - 2^x = (P_m - 2)(P_m^{x-1} + \cdots + 2^{x-1}) > 1$$

and also for $x = 1$ since 3 is not a Pell number. The first one is impossible for $x > 1$ by what is known about the Catalan equation solved by Mihăilescu [12]. So, $x = 1$. Since $P_n \equiv n \pmod 2$, it follows that m is odd. Hence, we obtain

$$2^y = P_m - 1 = P_m - P_1 = P_{(m\pm 1)/2} Q_{(m\mp 1)/2}.$$

The primitive divisor theorem for Lucas sequences with real roots (see [3]) implies that P_k has a primitive prime factor for all positive integers k, except eventually, $k \in \{1, 2, 3, 4, 6\}$. The primitive prime factor of P_k does not divide P_ℓ for any $\ell \in [1, k-1]$. In the case of the Pell sequence one checks by inspecting the cases $k \in \{1, 2, 3, 4, 6\}$ that P_k has a primitive divisor for all $k \geq 2$. Since $P_2 = 2$, we get that $(m \pm 1)/2 \leq 2$, which shows that $m \leq 5$ and one checks that only $m = 3$ giving $y = 2$ is convenient.

Assume next that $n = 2$. We then need to solve

$$2^x + 5^y = P_m^x \quad \text{or} \quad 2^y + 5^x = P_m^x.$$

The second one implies that

$$2^y = P_m^x - 5^x = (P_m - 5)\left(\frac{P_m^x - 5^x}{P_m - 5}\right).$$

Thus, $P_m - 5$ is a power of 2. Since 6 is not a Pell number, it follows that $P_m - 5$ is even so m is even. Thus, $P_m - 5 = P_m - P_3 = P_{(m\pm3)/2}Q_{(m\mp3)/2}$. Since this is a power of 2, again by the primitive divisors theorem, we get that $(m \pm 3)/2 \leq 2$, so $m \leq 7$. One now checks $P_m - 5$ for $m \in \{5, 7\}$ and one does not find a number of this form which is a power of 2. The first one implies

$$5^y = P_m^x - 2^x = (P_m - 2)\left(\frac{P_m^x - 2^x}{P_m - 2}\right).$$

Thus, $P_m - 2$ is a power of 5. Since 3 is not a member of the Pell sequence, we get that $P_m \equiv 2 \pmod 5$. The period of the Pell sequence modulo 5 is 12 and the only residues $k \pmod{12}$ such that $P_k \equiv 2 \pmod 5$ are $k \in \{2, 4\}$. But then $m \equiv 2, 4 \pmod{12}$, so P_m is even, which is impossible. This shows that the only solutions with $n \in \{1, 2\}$ is the one shown at Eq. 1.2. □

From now on, we assume $n \geq 3$.

Lemma 2.4 *Assume that (n, m, x, y) is a solution of Eq. 1.1 with $n \geq 3$. Then $m \geq 5$ and*

$$(n+1)y > (m-2)x \quad \text{and} \quad (n-2)y < (m-1)x. \tag{2.1}$$

Proof The fact that $m \geq 5$ follows from (i) of Lemma 2.1. For Eq. 2.1, we use that

$$\alpha^{k-2} \leq P_k \leq \alpha^{k-1}$$

where $\alpha = 1 + \sqrt{2}$ for all $k \geq 1$, where both inequalities are strict for $k \geq 3$. We thus get

$$\begin{aligned}\alpha^{(m-2)x} &< P_m^x \leq \max\{P_n^x + P_{n+1}^y, P_n^y + P_{n+1}^x\} < (P_n + P_{n+1})^y < P_{n+2}^y \\ &< \alpha^{(n+1)y},\end{aligned}$$

and

$$\alpha^{x(m-1)} > P_m^x \geq \min\{P_n^x + P_{n+1}^y, P_n^y + P_{n+1}^x\} > P_n^y > \alpha^{y(n-2)},$$

from where we get the desired inequalities. □

Now note that since one of P_n, P_{n+1} is even and the other is odd it follows that m is odd. Further, in the left–hand side exactly one of the two summands is even.

Lemma 2.5 *Assume that (n, m, x, y) is a solution in positive integers of Eq. 1.1 with $n \geq 3$. Then the even number in the left-hand side has exponent x.*

Proof Say the equation is $P_n^x + P_{n+1}^y = P_m^x$, and P_{n+1} is even. Then n is odd and we get

$$P_{n+1}^y = P_m^x - P_n^x = (P_m - P_n)\left(\frac{P_m^x - P_n^x}{P_m - P_n}\right) = P_{(m\pm n)/2}Q_{(m\mp n)/2}\left(\frac{P_m^x - P_n^x}{P_m - P_n}\right).$$

In the left, both $P_{(m\pm n)/2}$ and $Q_{(m\mp n)/2}$ have primitive divisors (by abuse we say that 1 is the primitive divisor of P_1). Since these primitive divisors are prime factors of P_{n+1} as well, we get that $(m \pm n)/2$ and $(m \mp n)/2$ both divide $n + 1$. These two numbers are coprime, since if d divides both of them, then d divides their sum m and their difference n, so P_d divides both P_n^x and P_m^x; hence P_{n+1}^y, which is impossible unless $d = 1$. Thus, $(m^2 - n^2)/4 \mid n + 1$. Since $m \equiv n \pmod 2$, we get $m \geq n + 2$. If $m \geq n + 4$, then $(m^2 - n^2)/4 > n + 1$, a contradiction. Thus, $m = n + 2$, and so we get

$$P_{n+1}^y = 2P_{n+1}\left(\frac{P_{n+2}^x - P_n^x}{P_{n+2} - P_n}\right).$$

The numbers $u_k := (a^k - b^k)/(a - b)$ with $(a, b) = (P_{n+2}, P_n)$ form a Lucas sequence with integer roots whose discriminant is $(a - b)^2 = 4P_{n+1}^2$. The above equation shows that u_x does not have primitive divisors, so $x \in \{1, 2, 3, 4, 6\}$. If $x = 1$, we get $P_{n+1}^y = 2P_{n+1}$, so $P_{n+1}^{y-1} = 2$, showing that $n = 2$, $y = 2$, which contradicts $n \geq 3$. If x is even, then

$$\begin{aligned} P_{n+1}^y &= (P_{n+2}^2 - P_n^2)\left(\frac{P_{n+2}^x - P_n^x}{P_{n+2}^2 - P_n^2}\right) \\ &= (P_{n+2} - P_n)(P_{n+2} + P_n)\left(\frac{P_{n+2}^x - P_n^x}{P_{n+2}^2 - P_n^2}\right) \\ &= 4P_{n+1}(P_{n+1} + P_n)\left(\frac{P_{n+2}^x - P_n^x}{P_{n+2}^2 - P_n^2}\right), \end{aligned}$$

and the third factor $P_{n+1} + P_n$ in the right-hand side above is coprime to P_{n+1}, a contradiction. Finally, if $x = 3$, then

$$P_{n+1}^y = 2P_{n+1}(P_{n+2}^2 + P_{n+2}P_n + P_n^2) = 2P_{n+1}(4P_{n+1}^2 + 6P_{n+1}P_n + 3P_n^2).$$

Thus, any prime p dividing $4P_{n+1}^2 + 6P_{n+1}P_n + 3P_n^2$ divides also P_{n+1} so it can be only 3. Since 9 does not divide a number of the form $a^2 + ab + b^2$ for coprime a and b, it follows that $(4P_{n+1}^2 + 6P_{n+1}P_n + 3P_n^2)/\gcd(3, P_{n+1}) > 1$ is both an integer coprime to P_{n+1} and a divisor of P_{n+1}^y, contradiction.

Assume next that P_n^y is the even number in the left. We then have $m \equiv n+1 \equiv 1 \pmod 2$ and

$$\begin{aligned} P_n^y &= P_m^x - P_{n+1}^x = (P_m - P_{n+1})\left(\frac{P_m^x - P_{n+1}^x}{P_m - P_{n+1}}\right) \\ &= P_{(m\pm(n+1))/2}Q_{(m\mp(n+1))/2}\left(\frac{P_m^x - P_{n+1}^x}{P_m - P_{n+1}}\right). \end{aligned}$$

A similar argument as before based on the existence of primitive divisors shows that $(m\pm(n+1))/2$ and $(m\mp(n+1))/2$ are coprime and divide n so $(m^2-(n+1)^2)/4 \mid n$. Since $m \equiv n+1 \pmod 2$, we get that $m \geq n+3$, so $(m^2-(n+1)^2)/4 \geq ((n+3)^2-(n+1)^2)/4 = n+2 > n$, a contradiction. □

From now on, the even number in the left-hand side of our equation has x in the exponent.

Lemma 2.6 *Assume (n, m, x, y) is a solution of Eq. 1.1 with $n \geq 3$. Then $m > n + 2$.*

Proof Assume $m = n + 2$. Since m is odd, it follows that $n + 1$ is even, so the equation is

$$P_{n+1}^x + P_n^y = P_{n+2}^x.$$

Thus, we obtain

$$\begin{aligned} P_n^y &= P_{n+2}^x - P_{n+1}^x = (P_{n+2} - P_{n+1}) \left(\frac{P_{n+2}^x - P_{n+1}^x}{P_{n+2} - P_{n+1}} \right) \\ &= (P_{n+1} + P_n) \left(\frac{P_{n+2}^x - P_{n+1}^x}{P_{n+2} - P_{n+1}} \right), \end{aligned}$$

and we see that $P_{n+1} + P_n$ is coprime to P_n, so it cannot divide a power of it. $\square$

Lemma 2.7 *We have*

$$\textit{either} \quad x < 162mn \left(\log \left(\frac{x}{n} + \frac{y}{m} \right) \right)^2 \quad \textit{or} \quad \frac{x}{n} + \frac{y}{m} < 25000.$$

Proof For a nonzero rational number r let $\nu_2(r)$ be the exponent of 2 in the factorization of r. Let $c \in \{0, 1\}$ be such that P_{n+c}^x is even. We then have

$$P_{n+c}^x = P_m^x - P_{n+1-c}^y = P_m^x (1 - P_m^{-x} P_{n+1-c}^y),$$

so

$$x \leq \nu_2(P_{n+c}^x) = \nu_2(1 - P_m^{-x} P_{n+1-c}^y).$$

We will apply a linear form in two 2-adic logarithms. A convenient one for us is Corollary 1 of Bugeaud-Laurent [1]. We take

$$\alpha_1 = P_{n+1-c}, \quad \alpha_2 = P_m.$$

In the notation of [1], we have $D = 1,\ p = 2,\ g = 1$. Since $P_k < \alpha^{k-1}$ holds for all positive integers k, we can take $\log A_1 = n \log \alpha$ and $\log A_2 = m \log \alpha$. Then

$$\frac{x}{n \log \alpha} + \frac{y}{m \log \alpha} < 1.14 \left(\frac{x}{n} + \frac{y}{m} \right) =: b'.$$

We get

$$x \leq \frac{24 \cdot 2}{(2-1)(\log 2)^4} \left(\max \left\{ \log b' + \log \log 2 + 0.14, 10 \right\} \right)^2 mn (\log \alpha)^2,$$

so

$$x < 162mnM^2,$$

where M is the max. If $M = 10$, then $\log b' + \log \log 2 + 0.14 \leq 10$, which implies

$$\frac{x}{n} + \frac{y}{m} < \frac{e^{9.86}}{1.14 \times \log 2} < 25000.$$

Otherwise, since $1.14 \times \log 2 \times e^{0.14} < 0.91 < 1$, we get

$$x < 162mn \left(\log \left(\frac{x}{n} + \frac{y}{m} \right) \right)^2.$$

$\square$

Remark Observe that by Lemma 2.4, we have

$$\frac{x}{n} < \left(\frac{n+1}{n}\right)\frac{y}{m}\left(\frac{m}{m-2}\right) \leq \left(\frac{20}{9}\right)\frac{y}{m} < \frac{3y}{m},$$

and

$$\frac{y}{m} < \left(\frac{m-1}{m}\right)\frac{x}{n}\left(\frac{n}{n-2}\right) < \frac{3x}{n}.$$

the following lemma, which is Lemma 7 in [6], is used often in the sequel.

Lemma 2.8 *If $s \geq 1$ is an integer and $T > (4s^2)^s$ and $T > x/(\log x)^s$, then*

$$x < 2^s T(\log T)^s.$$

Lemma 2.9 *We have*

$$y < 66000m^2(\log m)^2.$$

Proof We use Lemma 2.7. If we are in the second case, then $y < 25000m$, a much better inequality. So, let us treat the first case. Then, one can see that

$$\frac{x}{n} < 162m\left(\log\left(\frac{x}{n}+\frac{y}{m}\right)\right)^2 < 162\left(\log\left(\frac{4y}{m}\right)\right)^2$$

(see the Remark after Lemma 2.7). Thus, we obtain

$$\frac{y}{m} < \frac{3x}{n} < 3 \times 162m(\log(4y/m))^2 < 500(\log(4y/m))^2.$$

Hence, with $z := 4y/m$, we get

$$z < 2000m(\log z)^2.$$

We apply Lemma 2.8 with $s = 2$ and $T := 2000m$ and get

$$\begin{aligned} z &< 4 \times 2000m(\log(2000m))^2 = 8000m(\log m)^2\left(\frac{\log(2000)}{\log 5}+1\right)^2 \\ &< 262000m(\log m)^2, \end{aligned}$$

which yields

$$y < 66000m^2(\log m)^2.$$

□

Lemma 2.10 *Let k, ℓ be positive integers. We have*

(i) $P_{k+\ell} \geq 2^\ell P_k$;
(ii) $P_{k+\ell}^2 \equiv \pm P_\ell \pmod{P_k}$.

Proof (i) For $\ell = 1$, we get $P_{k+1} = 2P_k + P_{k-1} \geq 2P_k$. The statement now follows by induction on $\ell \geq 1$.

(ii) In the formula

$$2P_{k+\ell} = P_k Q_\ell + P_\ell Q_k,$$

we divide both sides by 2 (note that Q_k is always even) and square to get

$$P_{k+\ell}^2 = (P_k(Q_\ell/2) + P_\ell(Q_k/2))^2 \equiv P_\ell^2(Q_k/2)^2 \equiv \pm P_\ell^2 \pmod{P_k},$$

where we used the fact that $(X, Y) = (Q_k/2, P_k)$ are solutions of the Pell equation $X^2 - 2Y^2 = \pm 1$.

□

Lemma 2.11 *Let $c \in \{0, 1\}$ such that P_{n+c}^x is even. We then have*

$$|x \log P_m - y \log P_{n+1-c}| < \frac{2}{2^{(m-(n+1))x}}.$$

Additionally, the right–hand side is less that 1.

Proof The right-hand side is less than 1 by Lemma 2.6. Next, the equation

$$P_{n+c}^x + P_{n+1-c}^y = P_m^x,$$

leads to

$$1 - P_{n+1-c}^y P_m^{-x} = \left(\frac{P_{n+c}}{P_m}\right)^x \leq \left(\frac{P_{n+1}}{P_m}\right)^x \leq \frac{1}{2^{(m-(n+1))x}}, \tag{2.2}$$

where the last inequality follows from Lemma 2.10. The right-hand side is at most 1/4. Putting

$$\Gamma := y \log P_{n+1-c} - x \log P_m,$$

we get

$$|e^\Gamma - 1| < \frac{1}{2^{(m-(n+1))x}}.$$

This leads to

$$|\Gamma| < \frac{2}{2^{(m-(n+1))x}},$$

which is what we wanted. □

Remark Note that from 2.2 we get that $\Gamma < 0$.

Lemma 2.12 *There are no solutions with $n \geq 3$ and $m \leq 2500$.*

Proof Assume $m \leq 2500$. By Lemma 2.9, we get

$$y < 66000m^2(\log m)^2 < 66000(2500)^2(\log(2500))^2 < 3 \times 10^{13}.$$

Lemma 2.11 gives us that

$$\left|\frac{\log P_{n+1-c}}{\log P_m} - \frac{x}{y}\right| < \frac{2}{y(\log P_m)2^{(m-(n+1))x}} < \frac{0.6}{y2^{(m-(n+1))x}}. \tag{2.3}$$

There are two scenarios. The first one is when

$$\frac{0.6}{2^{(m-(n+1))x}} < \frac{1}{2(3 \times 10^{13})} \left(< \frac{1}{2y}\right). \tag{2.4}$$

In this case, the right–hand side of Eq. 2.3 is less than $1/(2y^2)$, so by Legendre's criterion x/y is a convergent of $\log P_{n+1-c}/\log P_m$. Here is a way to generate the potential candidates (x, y). Loop over all odd $n' := n+1-c \in [3, 2497]$ and odd m with $n'+1 < m \leq 2499$. For each such pair, generate convergents p_k/q_k with $q_k < 3\times 10^{13}$ of $\log P_{n'}/\log P_m$. Then $(x, y) \in \{(p_k, q_k), (2p_k, 2q_k)\}$. These are candidates for solutions to our equations in this range. Then simply check, for example, whether $P_{n'}^{2q_k} \equiv 0 \pmod{P_m - P_{n'\pm 1}}$. In Mathematica, we used the PowerMod command. This computation took a couple of hours on a desk computer and no solution was found. Another case is when Eq. 2.4 fails. Then $2^{(m-(n+1))x} < 0.6 \times 6 \times 10^{12}$, so $(m-(n+1))x < 46$. This shows that $x \leq 45$ and $m \leq n+1+45/x$. This suggests to fix $x \leq 45$ and $n \leq 2497$. Then take odd $m \in [n+3, n+1+45/x)$. Then check whether y arising from one of the two equations $P_{n+1}^y = P_m^x - P_n^x$ (so $y = \lceil \log(P_m^x - P_n^x)/\log P_{n+1}\rceil$) verifies $P_{n+1}^y \equiv 0 \pmod{P_m - P_n}$, or $P_n^y = P_m^x - P_{n+1}^x$ (so, $y = \lceil \log(P_m^x - P_{n+1}^x)/\log P_n\rceil$) verifies $P_n^y \equiv 0 \pmod{P_m - P_{n+1}}$. Again we used the PowerMod function in Mathematica. This computation took less than a minute and no solution was found. □

From now on, we assume that $m > 2500$. In this case, we have

$$\begin{aligned}\log P_m &= \log(\alpha^m - \beta^m) - \log(2\sqrt{2})\\ &= m\log\alpha - \log(2\sqrt{2}) + \log\left(1 \pm \frac{1}{\alpha^{2m}}\right)\\ &= m\log\alpha - \log(2\sqrt{2}) + \zeta_m,\end{aligned}$$

where

$$|\zeta_m| < \frac{1}{\alpha^{2m}}.$$

Inserting the above calculation into the inequality of Lemma 2.11, we get

$$\left| y\log P_{n+1-c} - xm\log\alpha - x\log(2\sqrt{2})\right| < \frac{2}{2^{(m-(n+1))x}} + \frac{x}{\alpha^{2m}}. \tag{2.5}$$

Since

$$x < y < 66000m^2(\log m)^2$$

and $m > 2500$, it follows that $x < \alpha^m$. We record Eq. 2.5 for future use.

Lemma 2.13 *Assume (n, m, x, y) is a positive integer solution of equation 1.1 with $n \geq 3$ and $m > 2500$. Then, we obtain*

$$\left| y\log P_{n+1-c} - xm\log\alpha - x\log(2\sqrt{2})\right| < \frac{2}{2^{(m-(n+1))x}} + \frac{1}{\alpha^{m}}. \tag{2.6}$$

Lemma 2.14 *Assume (n, m, x, y) is a solution of Eq. 1.1 with $n \geq 3$ and $m > 2500$. Then, we get*

$$y < 10^{40}n^2(\log n)^4.$$

Proof If $y \leq 2x$, then Eq. 2.1 shows that

$$(m-2)x < (n+1)y \leq 2(n+1)x,$$

so $m < 2n+5 < 4n$. Then, Lemma 2.9 gives

$$y < 66000(4n)^2\log(4n)^2 < (66000)\times 16n^2\log(n^3)^2 < 10^7n^2(\log n)^2,$$

a much better inequality than the one aimed for. So, in the rest of this proof we may assume that $y > 2x$. Lemma 2.4 shows that

$$(m-1)x > (n-2)y > 2(n-2)x$$

so $m > 2n - 3$. Since m is odd, we get $m \geq 2n - 1$, so $n \leq (m+1)/2$. Thus,

$$m - (n+1) \geq m - ((m+1)/2 + 1) \geq (m-3)/2 \geq 0.49m,$$

where this last inequality holds since $m > 2500$. Invoking Lemma 2.13, we get

$$\left| y \log P_{n+1-c} - xm \log \alpha - x \log(2\sqrt{2}) \right| < \frac{2}{2^{(m-(n+1))x}} + \frac{1}{\alpha^m} < \frac{3}{2^{0.49m}}.$$

The right–hand side is a linear form in three logarithms. It is nonzero, since $P_{n+1-c},\ \alpha,\ 2\sqrt{2}$ are multiplicatively independent. Indeed, assume they were. Since α is a unit and $P_{n+1-c}^2,\ (2\sqrt{2})^2$ are integers, it follows that P_{n+1-c} and 2 should be multiplicatively dependent, which would then make P_{n+1-c} a power of 2, which is not the case by the primitive divisor theorem and the fact that $n \geq 3$. We apply a lower bound for linear forms in logarithms due to Matveev [11]. Here, we use the version due to Bugeaud–Mignotte–Siksek [2]. We get that

$$|P_{n+1-c}^y \alpha^{mx} (2\sqrt{2})^x - 1| = |e^\Lambda - 1| < 2|\Gamma| < \frac{6}{2^{0.49m}},$$

where Λ is the linear form in logarithms appearing in the left-hand side of Eq. 2.5. We take $\alpha_1 = P_{n+1-c},\ \alpha_2 = \alpha,\ \alpha_3 = 2\sqrt{2}$. We have $D = 2$, We take $A_1 = 2n \log \alpha$ ($> D \log(P_{n+1-c})$), $A_2 = \log \alpha$ ($= \log(Dh(\alpha_2))$), and $A_3 = 4 \log(2\sqrt{2}) = 2 \log(8)$ ($= Dh(\alpha_3)$). We take $b_1 = y,\ b_2 = xm,\ b_3 = x$. Since $n \geq 3$, Lemma 2.4 gives $mx > (m-1)x > y(n-2) \geq y$, so we can take $B = mx$. Applying Matveev's inequality, we get

$$\begin{aligned}
&0.49m \log 2 - \log 6 \\
&\quad < 1.4 \times 30^6 \times 3^{4.5} \times 2^2 (1 + \log 2)(1 + \log(mx))(2n \log \alpha)(\log \alpha)(2 \log 8) \\
&\quad < 6.3 \times 10^{12} n \log(emx),
\end{aligned}$$

and then

$$m < \frac{7 \times 10^{12} n \log(emx)}{0.49 \log \alpha} < 1.63 \times 10^{13} n \log(emx).$$

By Lemma 2.9, we have

$$ex < ey/2 < 33000em^2 (\log m)^2 < 100000m^2 (\log m)^2 < m^5,$$

where the last inequality holds since $m > 2500$. We get

$$m < 1.63 \times 10^{13} n \log(emx) < 1.63 \times 10^{13} n \log(m^6) < 10^{14} n \log m.$$

Thus, Lemma 2.8 with $s = 1$ and $T := 10^{14} n$ gives

$$\begin{aligned}
m &< 2 \times 10^{14} n \log(10^{14} n) = 2 \times 10^{14} n (\log n) \left(\frac{\log(10^{14})}{\log n} + 1 \right) \\
&< 7 \times 10^{15} n \log n.
\end{aligned}$$

This together with Lemma 2.9 gives

$$\begin{aligned} y &< 66000m^2(\log m)^2 < 66000\left(7\times 10^{15} n\log n\right)^2\left(\log(7\times 10^{15} n\log n)\right)^2 \\ &< 3.3\times 10^{36} n^2(\log n)^2(\log n)^2\left(\frac{\log(7\times 10^{15})}{\log n}+2\right)^2 < 10^{40} n^2(\log n)^4, \end{aligned}$$

which is what we wanted to prove. $\square$

We now write

$$\begin{aligned} \log P_{n+1-c} &= \log(\alpha^{n+1-c}-\beta^{n+1-c}) - \log(2\sqrt{2}) \\ &= (n+1-c)\log\alpha - \log(2\sqrt{2}) + \log\left(1\pm\frac{1}{\alpha^{2n}}\right) \\ &= (n+1-c)\log\alpha - \log(2\sqrt{2}) + \zeta_n, \end{aligned}$$

where

$$|\zeta_n| < \frac{1}{\alpha^{2n}}.$$

We get

$$y\log(P_{n+1-c}) = y(n+1-c)\log\alpha - y\log(2\sqrt{2}) + y\zeta_n,$$

where

$$|y\zeta_n| < \frac{y}{\alpha^{2n}} < \frac{10^{40} n^2(\log n)^4}{\alpha^{2n}}.$$

Assume next that $n > 400$. Then $10^{40} n^2(\log n)^4 < \alpha^n$, so the above fraction is smaller than $1/\alpha^n$. Inserting this into Lemma 2.13, we get the following result.

Lemma 2.15 *Assume that (n, m, x, y) is a solution of Eq. 1.1 with $n > 400$ and $m > 2500$. Then*

$$\left|((n+1-c)y - mx)\log\alpha - (y-x)\log(2\sqrt{2})\right| < \frac{2}{2^{(m-(n+1))x}} + \frac{2}{\alpha^n}.$$

To continue, we need the following fact.

Lemma 2.16 *We have $(m-(n+1)) \geq (n-2)/2$.*

Proof We put $\ell = m - n$, so

$$P_{n+c}^x + P_{n+1-c}^y = P_{n+\ell}^x.$$

We square both sides and reduce them modulo P_n getting

$$P_{n+1}^{2z} \equiv P_{n+\ell}^{2x} \pmod{P_n},$$

for $z \in \{x, y\}$. Now Lemma 2.10, (ii) gives $\pm 1 \equiv \pm P_\ell^{2x} \pmod{P_n}$, so

$$P_n \mid P_\ell^{2x} \pm 1.$$

The above expression is not zero since $\ell \geq 3$. Hence, we obtain

$$\alpha^{n-2} < P_n \leq P_\ell^{cx} + 1 < \alpha^{2(\ell-1)x} + 1 < 2\alpha^{2(\ell-1)x} < \alpha^{2(\ell-1)x+1},$$

and then $n - 2 < 2(\ell - 1)x + 1$. This shows that

$$(m - (n+1))x = (\ell - 1)x \geq (n-2)/2,$$

which is what we wanted. □

Using Lemma 2.16 into the right–hand side of Lemma 2.15, we get the following.

Lemma 2.17 *Assume* (n, m, x, y) *is a solution of Eq. 1.1 in positive integers with* $n > 400$ *and* $m > 2500$. *Then, we get*

$$\left|((n+1-c)y - mx)\log\alpha - (y-x)\log(2\sqrt{2})\right| < \frac{5}{2^{n/2}}. \tag{2.7}$$

now we are ready to find an upper bound for n.

Lemma 2.18 *We have* $n < 10^7$.

Proof We may assume that $n > 6000$. The left-hand side of Eq. 2.7 is nonzero since α and $2\sqrt{2}$ are multiplicatively independent. To get a lower bound on it we use Corollaire 2 in Laurent–Mignotte–Nesterenko [8]. We have $\alpha_1 = \alpha$, $\alpha_2 = 2\sqrt{2}$, $D = 2$. We take $\log A_1 = h(\alpha)/2 = (1/2)\log\alpha$ $(= h(\alpha_1))$ and $\log A_2 = \log(2\sqrt{2})$ $(= h(\alpha_2))$. Since n is large, the left-hand side of Eq. 2.7 is smaller than 0.01, so

$$(n+1-c)y - mx < \frac{(y-x)\log(2\sqrt{2}) + 0.01}{\log\alpha} < 1.18(y-x) + 0.02 < 1.2y. \tag{2.8}$$

We put

$$b' = \frac{((n+1-c)y - mx}{\log A_2} + \frac{y-x}{\log A_1} < \left(\frac{1.2}{\log(2\sqrt{2})} + \frac{1}{\log\alpha}\right) y < 2.3y.$$

Then Corollaire 2 in [8] together with inequality Eq. 2.7 gives

$$\begin{aligned}(n/2)\log 2 - \log 5 &< 24.34 \times 2^4\,(\max\{\log(2.3y) + 0.14, 10.5\})^2\\ &\quad \times (1/2)(\log\alpha)\log(2\sqrt{2})\\ &< 180M^2,\end{aligned}$$

where M is the above max. This gives

$$n < \frac{181M^2}{(\log 2)/2} < 523M^2.$$

If $M = 10.5$, we then get $n < 60000$. Otherwise, we get

$$n < 523(\log(2.13y) + 0.14))^2 < 523\log(2.5y)^2.$$

Using Lemma 2.14, we get

$$n < 523\left(\log\left(2.5 \times 10^{40} n^2 (\log n)^4\right)\right)^2$$

which gives $n < 10^7$. □

We can now show that there are no solutions with $n \geq 420$.

Lemma 2.19 *We have* $n < 420$.

Proof Assume $n \geq 420$. We know that $n < 10^7$. By Lemma 2.14, we get that

$$y < 10^{40} n^2 (\log n)^4 < 7 \times 10^{58},$$

and inequality Eq. 2.8 shows that $(n+1-c)y - mx < 1.23y < 10^{59}$. So, putting $a := (n+1-c)y - mx,\ b := y - x$, we have

$$\left| \frac{\log \alpha}{\log 2\sqrt{2}} - \frac{b}{a} \right| < \frac{5}{a \log(2\sqrt{2}) 2^{n/2}} < \frac{5}{a \cdot 2^{n/2}}. \tag{2.9}$$

We know that $\max\{a, b\} < 10^{59}$. Since $F_{300} > 10^{60}$, it suffices to compute the first 300 partial quotients of $\tau := \log \alpha / \log 2\sqrt{2}$. Writing the continued fraction of τ as $\tau = [a_0, a_, \ldots, a_{300}]$, we have that

$$\max\{a_i : 0 \leq i \leq 300\} = 1136.$$

Thus, we get that the left–hand side of Eq. 2.9 is bounded below by $1/(1138a^2)$. So, we get

$$\frac{1}{1138a^2} < \frac{5}{a 2^{n/2}},$$

which yields

$$2^{n/2} \leq 1138 \times 5 \times a < 10^{63},$$

showing that $n < 2 \times 63(\log 10)/\log 2 < 420$, a contradiction. □

So, we have $n \leq 420$ and $m > 2500$. If $y \leq 2x$, then as in the beginning of the proof of Lemma 2.14 using Eq. 2.1, we get

$$(m-2)x < (n+1)y \leq 2x(n+1),$$

so $m \leq 2n + 5 \leq 945$, which is not the case since $m > 2500$. So, we have $y > 2x$. Then, as in the proof of Lemma 2.14, or using again inequalities Eq. 2.1, we get

$$(m-1)x > (n-2)y > 2(n-2)x,$$

so $m > 2n - 3$ and since m is odd we get $m \geq 2n - 1$. Thus, $n \leq (m+1)/2$, so that

$$m - (n+1) \geq m - ((m+1)/2 + 1) \geq (m-3)/2 > 0.49m \quad \text{since} \quad m > 2500.$$

Inequality Eq. 2.6 gives

$$\left| y \log P_{n+1-c} - mx \log \alpha - x \log(2\sqrt{2}) \right| < \frac{2}{2^{(m-3)x/2}} + \frac{1}{\alpha^m} < \frac{3}{2^{0.49m}}.$$

Now $y < mx$. Lemma 2.14 gives

$$y < 10^{40} n^2 (\log n)^4 \leq 10^{40} (420)^2 (\log 420)^4 < 2.35 \times 10^{49}.$$

and Eq. 2.1 gives

$$0.99mx < (m-2)x < (n+1)y < 2.35 \times 421 \times 10^{49} < 0.99 \times 10^{53},$$

showing that $mx < 10^{53}$. So, we take $\alpha_1 = P_{n+1-c}$ for some $n+1-c \leq 420$, $\alpha_2 = \alpha$, $\alpha_3 = 2\sqrt{2}$. We use LLL (see Proposition 2.3.20 in [4, Section 2.3.5]) to compute a lower bound on

$$|x_1 \log \alpha_1 + x_2 \log \alpha_2 + x_3 \log \alpha_3| > \frac{1}{T}, \tag{2.10}$$

for $\mathbf{x} = (x_1, x_2, x_2) \in \mathbb{Z}^3$ with $\max\{|x_i| : 1 \le i \le 3\} \le 10^{62} := X$. Here, $n + 1 - c$ is an odd number in [3, 419]. Taking $n + 1 - c = 3$, $C = (10X)^3$ was enough and we got

$$\sqrt{d^2 - 3X^2} - (1 + 3X)/2 > 6 \cdot 10^{62}, \quad \text{so 2.10 holds with} \quad T = 10^{127}.$$

But this is only for $n + 1 - c = 3$. As $n + 1 - c$ became larger in [3, 219], larger and larger values of C were needed. At the end, $C = X^3 \times 10^{220}$ worked for all odd values of $n+1-c$ in [3, 419]. The smallest

$$\sqrt{d^2 - 3X^2} - (1 + 3X)/2$$

computed was greater than 8.3×10^{69} and was obtained for $n + 1 - c = 219$. At any rate, this shows that one can take $T = X^3 \times 10^{220}/10^{69} = 10^{337}$ in Eq. 2.10. Thus, we get

$$\frac{1}{10^{337}} < \frac{3}{2^{0.49m}},$$

so $m < 2300$, which is a contradiction and finishes the proof of Theorem 1.1.

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