Mixing Processes in Riesz Spaces and their Ergodic Properties

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October 2021

ABSTRACT

The ergodic theorems of Hopf, Wiener and Birkhoff were extended to the context of Riesz spaces with a weak order unit and conditional expectation operator by Kuo, Labuschagne and Watson in [*Ergodic Theory and the Strong Law of Large Numbers on Riesz Spaces.* Journal of Mathematical Analysis and Applications, **325**, (2007), 422–437.]. However, the precise concept of what constitutes ergodicity in Riesz spaces was not considered. This omission will be filled in and some explanations of the choices made will be given. In addition, the interplay between mixing and ergodicity in the Riesz space setting is considered. In order for this to happen, the Koopman-von Neumann convergence condition on the Cesàro mean must be be extended to the context of a Dedekind complete Riesz space with weak order unit. As a consequence, a characterisation of conditional weak mixing is given in the Riesz space setting. The results are applied to convergence in L^1.

PREFACE

Chapters and a section of this PhD. thesis have been submitted for publication. The articles in which these will appear are listed and numbered as per the bibliography.

[10], Homann, J.M., Kuo, W.-C., Watson, B.A., Characterisation of conditional weak mixing via ergodicity of the tensor product in Riesz Spaces. in preparation, arXiv: 2101.00207v1 **[math.FA]** ,

[11], Homann, J., Kuo WC., Watson B.A., Ergodicity in Riesz Spaces, pp. 193-201. In: Kikianty E., Mabula M., Messerschmidt M., van der Walt J.H., Wortel M. (eds) *Positivity and its Applications.* Trends in Mathematics. Birkhäuser, Cham. 2021.
arXiv: 2010.07206v2 **[math.DS]**

and

[12] Homann, J.M., Kuo, W-C., Watson, B.A., A Koopman-von Neumann type theorem on the convergence of Cesàro means in Riesz spaces, *Proc. Amer. Math. Soc., Series B*, **8**, 75-85. , arXiv: 2010.12215v1 **[math.FA]**.

The chapters and section are as follows.

Chapter 3 is submitted in [12]. Chapter 4 is submitted in [11]. Finally, Section 5.1 of Chapter 5 appears in [12].

The following three results are presented in Chapter 2, but these results appear in [12]: Lemma 2.17, Theorem 2.18 and Corollary 2.19 appear in [12] as Lemma 2.1, Theorem 2.2 and Corollary 2.3, respectively.

ACKNOWLEDGEMENTS

I would like to express my gratitude to my supervisors, The Prof. (Prof. Watson) and Dr. Kuo. The knowledge I have gained extends beyond the realm of mathematics.

DECLARATION

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Doctor of Philosophy in the University of the Witwatersrand, Johannesburg.

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Signed on the 11th day of October 2021, at Johannesburg, South Africa.

Contents

Chapter 1

Introduction

1.1 A Brief History

As a loose and physical interpretation, a dynamical system studies the movement of a particle in space. In other words, a dynamical system studies what happens to a particle under a transformation. We can then say that if Ω is a set and $\tau : \Omega \to \Omega$ is a map, a dynamical system studies what happens to $x \in \Omega$ under τ. We will have that $\tau^0(x) := x$ and $\tau^n(x)$ is the n-th function composition of τ applied to x for some $n \in \mathbb{N}_0$. As per [25], we can studies various types of dynamical systems, but the one of interest for this thesis is the case of measure preserving dynamical systems.

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $\tau : \Omega \to \Omega$, then we say that $(\Omega, \mathcal{A}, \mu, \tau)$ is a measure preserving dynamical system if $\mu(\tau^{-1}(A)) = \mu(A)$ for each $A \in \mathcal{A}$. where $\tau^{-1}(A) := \{x \in \Omega | \, \tau(x) \in A\}$. As an intuitive example to the meaning of this, suppose that we have a bathtub with a certain volume of water in it and we add some paint to the water (we will also assume some ideal conditions such as none of the liquid being lost to evaporation and such). We then have a set, Ω, which is all of the liquid in the bathtub (the water and the paint, where we are treating paint as a liquid). If our transformation, τ, is to stir the liquid and we require that we do not let any of the liquid leave the bathtub (no spilling, no splashing, no evaporation, etc.), then we see that $\tau : \Omega \to \Omega$. Furthermore, since none of the liquid is being lost, we have that any chosen volume of liquid has that its volume is unaffected by the transformation; i.e., if we consider a "patch" of liquid then this "patch" is definitely a subset of all of the liquid and the volume of this patch is the same after stirring as it was before stirring, then we that $\mu(\tau^{-1}(A)) = \mu(A)$. If we can do this for all "patches" of liquid, then we have $\mu(\tau^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{A}$, and so this system is a measure preserving dynamical system. A more relevant example would be the case of allowing gases to mix. This is more relevant as it relates to thermodynamics and thermodynamics was a key influence in the development of ergodic theory.

We recall the symmetric difference of sets, $\triangle$: let $A, B \subseteq \Omega$ be sets then $A \triangle B := (A \setminus B) \cup (B \setminus A)$. Furthermore, if there is a mapping $\tau : \Omega \to \Omega$, then we define $\tau(A) := \{\tau(x) | x \in A\}$. If $(\Omega, \mathcal{A}, \mu, \tau)$ is a measure preserving dynamical system, then we say that $A \in \mathcal{A}$ is invariant under τ if $\mu(\tau^{-1}(A) \triangle A) = 0$.

We say that a measure preserving dynamical system, $(\Omega, \mathcal{A}, \mu, \tau)$, with $\mu(\Omega) = 1$, is ergodic if every $A \in \mathcal{A}$ which is invariant under τ satisfies that $\mu(A) \in \{0, 1\}$.

In the study of ergodic theory, a crucial result is developed, see [5, 16, 25, 26], which asserts that $(\Omega, \mathcal{A}, \mu, \tau)$ is ergodic if and only if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu\left(\tau^{-k}(A) \cap B\right) = \mu(A)\, \mu(B), \tag{1.1}$$

for all $A, B \in \mathcal{A}$. This result is then used to define two more processes: strong mixing and weak mixing. This thesis will not study strong mixing in the setting of Riesz spaces, but the classical definition will be provided. Weak mixing is defined as follows: the measure preserving dynamical system $(\Omega, \mathcal{A}, \mu, \tau)$ with $\mu(\Omega) = 1$ is said to be weakly mixing if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\left|\mu\left(\tau^{-k}(A)\cap B\right)-\mu(A)\mu(B)\right|=0, \tag{1.2}$$

for all $A, B \in \mathcal{A}$. The definition of strong mixing is given: the measure preserving dynamical system $(\Omega, \mathcal{A}, \mu, \tau)$ with $\mu(X) = 1$ is said to be strongly mixing if

$$\lim_{n\to\infty}\left|\mu\left(\tau^{-n}(A)\cap B\right)-\mu(A)\mu(B)\right|=0, \tag{1.3}$$

for all $A, B \in \mathcal{A}$.

In order to study such systems in Riesz spaces, a generalisation of the idea of a measure preserving system must be defined and, on this system, the transformation will be a Riesz homomorphism . Recall from [34, Definition 19.1] that if E and F are Riesz spaces and $S : E \to F$ is a linear operator, then S is said to be a Riesz homomorphism if $S(f \vee g) = S(f) \vee S(g)$ holds for all $f, g \in E$, where $f \vee g$ is the supremum of f and g in E and $S(f) \vee S(g)$ is the supremum of $S(f)$ and $S(g)$ in F. Furthermore, it must be noted that S is a positive operator (that is, $S(f) \geq 0$ in F for all $0 \leq f \in E$).

The advantage of generalising ergodicity and weak mixing to Riesz spaces is that a conditional version of these processes is developed, as will be shown.

1.2 Outline of this Thesis

This thesis extends the study of ergodic theory and weak mixing found in dynamical systems to the case of a Riesz space setting. A survey of the contents of this thesis is provided.

Chapter 2 presents the necessary preliminaries of Riesz spaces, f-algebras, bands and band projections and convergence results.

Chapter 3 presents the Koopman-von Neumann Theorem for the Riesz space setting. This result is key for further work, but it is an interesting enough result as it stands. To demonstrate this, an application to measure theory is presented in Section 5.1.

Chapter 4 presents the Riesz space definition of ergodicity and, as an application, defines conditional weak mixing in a Riesz space. An application to measure theory is presented in Section 5.2.

Chapter 5 contains the aforementioned applications to measure theory.

Chapter 6 provides the concluding remarks to this thesis as well as the outline of potential areas of further research.

Chapter 2

Preliminaries

Preliminary results on Riesz spaces and f-algebras are presented. For details see Aliprantis and Border [1], Fremlin [7], Meyer-Nieberg [24], and Zaanen [33] and [34]. Some key definitions and results are presented.

2.1 Riesz Spaces

Definition 2.1 Let V be a real vector space with a partial ordering, $\leq$, such that, if $f, g, h \in V$ with $f \leq g$, then $f + h \leq g + h$ and if $f \geq 0$ and $0 \leq \alpha \in \mathbb{R}$, then $\alpha f \geq 0$. The pair $(V, \leq)$ is called a partially ordered vector space.

Definition 2.2 Let $(V, \leq)$ be a partially ordered vector space and let $D \subset V$. The supremum of D (respectively, infimum of D), written as $\sup(D)$ (resp. $\inf(D)$), if it exists, is defined to be $f \in V$ such that $f \geq g$ (resp. $f \leq g$) for all $g \in D$ and if $h \geq g$ (resp. $h \leq g$) for all $g \in D$, then $h \geq f$ (resp. $h \leq f$).

When dealing with pairs (say, of elements $f, g \in D$), then the supremum of f and g (typically written $\sup\{f, g\}$), will be written as $f \vee g$. Similarly, the infimum of f and g (typically written as $\inf\{f, g\}$), will be written as $f \wedge g$. Furthermore, suppose that there is a collection of n elements, $f_1, f_2, ...f_n$, then the supremum (sup), of these elements shall be written as $\bigvee_{k=1}^{n} f_k$. Similarly, the infimum (inf), will be $\bigwedge_{k=1}^{n} f_k$ (the typical notation is $\sup\{f_1, f_2, ...f_n\}$ and $\inf\{f_1, f_2, ..., f_n\}$).

Definition 2.3 Let $(V, \leq)$ be a partially ordered vector space in which each pair of elements has a sup or inf compatible with $\leq$. In this case, $(V, \leq)$ is called a Riesz space .

Remark 2.4 A Riesz space will typically be represented by a single character, such as E, rather than by the pair $(E, \leq)$.

Subsets of a Riesz space are assumed to inherit the ordering of the Riesz space.

A subset of a Riesz space, say, D, is called "order bounded from above" if there is a vector that dominates each element of D. Subsets that are "order bounded from below" are defined similarly. A subset that is both order bounded from above and order bounded from below is called "order bounded" .

Definition 2.5 Let E be a Riesz space, let S be a non-empty subset of E and let V be a vector subspace of E, then we have the following definitions.

i. V is a Riesz subspace of E if, for all $f, g \in V$ we have that $f \vee g \in V$ and $f \wedge g \in V$.

ii. S is solid whenever $f \in S$ and $|g| \leq |f|$ implies $g \in S$.

iii. I is said to be an ideal in E whenever I is a vector subspace of E and I is solid.

iv. B is said to be a band in E whenever B is an ideal in E and any subset of B with a supremum has that this supremum is in B.

Definition 2.6 Let E be a Riesz space and let A be a non-empty subset of E. The smallest ideal in E which contains A is said to be the ideal generated by A and the smallest band in E which contains A is said to be the band generated by A . Furthermore, if $f \in E$, then the ideal generated by f (that is, the ideal generated by $\{f\}$), is called the principal ideal generated by f and the band generated by f (that is, the band generated by $\{f\}$), is called the principal band generated by f.

Definition 2.7 Let E be a Riesz space, then any $0 < e \in E_+$ is said to be a weak order unit of E whenever $B_e = E$, where B_e is the band generated by e.

Definition 2.8 Given a Riesz space, E, if there is a band , say, M, such that $E = M \oplus M^{\mathrm{d}}$, then M is called a projection band . Any $f \in E$ can be uniquely written as $f = f_1 + f_2$, where $f_1 \in M$ and $f_2 \in M^{\mathrm{d}}$. A band projection , say, P, satisfies $P\colon E \to E$ and $Pf = f_1$.

Definition 2.9 A Riesz space is called Archimedean if $\inf \left\{ \frac{1}{n}u \,\middle|\, n \in \mathbb{N} \right\} = 0$ holds for all $u \in E_+$.

Definition 2.10 A Riesz space is called Dedekind complete if every non-empty subset that is order bounded from above has a sup (equivalently, if every non-empty subset that is order bounded from below has an inf).

Theorem 2.11 In a Dedekind complete Riesz space, every band is a projection band.

We recall that if E and F are Riesz spaces then $T : E \to F$ is said to be a linear operator from E to F whenever $T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$ for all $f, g \in E$ and for all $\alpha, \beta \in \mathbb{R}$. If $F = E$, then we say that T is a linear operator on E. We now define the following operators.

Definition 2.12 Let E and F be a Riesz spaces and let $T : E \to F$ be a linear operator.

i. T is said to be positive whenever $T(|f|) \geq 0$ in F for all $f \in E$.

ii. T is said to be strictly positive whenever T is positive and $Tf = 0$ in F implies $f = 0$ in E for $f \in E_+$.

iii. T is said to be order continuous whenever, for each $\emptyset \neq A \subseteq E$ with $A \downarrow 0$ in E we have $\inf \{|T(f)| \,|\, f \in A\} = 0$ in F.

The definition of a conditional expectation operator is recalled from [17].

Definition 2.13 Let E be a Riesz space with weak order unit. A positive order continuous projection $T\colon E \to E$, with range, $\mathcal{R}(T)$, a Dedekind complete Riesz subspace of E, is called a conditional expectation if Te is a weak order unit of E for each weak order unit e of E.

From [4], we recall the definition of an f-algebra.

Definition 2.14 Let E be a Riesz space, then E is said to be an f-algebra if there exists an associative multiplication, $\cdot$, such that

i. if $f, g \in E_+$ then $f \cdot g \in E_+$ and

ii. if $f, g, h \in E_+$ with $f \wedge g = 0$ then $(f \cdot h) \wedge g = (h \cdot f) \wedge g = 0$.

We recall that, for $e \in E_+$ a weak order unit, $E_e = \{f \in E | \, |f| \leq ke \text{ for some } k \in \mathbb{R}_+\}$, the subspace of E of e bounded elements of E, is an f-algebra , see [2, 31, 33]. The f-algebra structure on E_e is generated by setting $Pe \cdot Qe = PQe$ for all band projections P and Q on E (here, $\cdot$ represents the f-algebra multiplication on E_e). The linear extension of this multiplication and use of order limits extends this multiplication to E_e. We refer the reader to Azouzi and Trabelsi [2], Venter and van Eldik [31] and Zaanen [33, Chapter 20] for further background on f-algebras.

If T is a conditional expectation operator on E with $Te = e$, then T is also a conditional expectation operator on E_e, since, if $f \in E_e$, then $|f| \leq ke$, giving $|Tf| \leq T\,|f| \leq Tke = ke$. Further T is an averaging operator on E_e, that is

$$T(f \cdot g) = f \cdot Tg$$

for $f, g \in E_e$ with $f \in \mathcal{R}\,(T)$, see [17]

2.2 Convergence in Riesz Spaces

We recall the definition of order convergence of a sequence from [34].

Definition 2.15 Let E be a Riesz space and let $(f_n)_{n\in\mathbb{N}} \subseteq E$ be a sequence in E, then the sequence is increasing if $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$, denoted by $f_n \uparrow$, and decreasing if $f_n \geq f_{n+1}$ for all $n \in \mathbb{N}$, denoted by $f_n \downarrow$.

Furthermore, if $f_n \uparrow$ and $f = \bigvee_{n\in\mathbb{N}} f_n$ exists, we write $f_n \uparrow f$ and say that f_n converges (monotonely), to f. If $f_n \downarrow$ and $f = \bigwedge_{n\in\mathbb{N}} f_n$, then we write $f_n \downarrow f$ and say that f_n converges (monotonely) to f.

Finally, if $(f_n)_{n\in\mathbb{N}} \subseteq E$ is a sequence and $f \in E$, then we say that f_n converges, in order, to f, denoted $f_n \to f$, in order, or $f_n \to f$, in order, as $n \to \infty$, whenever there exists a sequence $u_n \downarrow 0$ in E such that $|f - f_n| \leq u_n$ for all $n \in \mathbb{N}$.

Some convergence results are presented. By an application of [23, Theorem 16.1, page 78], we have the following lemma.

Lemma 2.16 Let E be a Riesz space and $(f_n)_{n\in\mathbb{N}_0}$ a sequence in E, then, as order limits,

$$\lim_{n\to\infty} |f_n| = 0 \Leftrightarrow \lim_{n\to\infty} f_n = 0.$$

In real analysis, one would be familiar with the idea of absolute convergence of a sum and conditional convergence of a sum and, in particular, the result that absolute convergence implies conditional convergence. This property holds in Riesz spaces. We will first note the following: if $(f_n)_{n\in\mathbb{N}_0}$ is a sequence in a Riesz space E, then, as an order limit,

$$\sum_{k=0}^{\infty} f_n := \lim_{n\to\infty} \sum_{k=0}^{n} f_n.$$

Lemma 2.17 Let $(f_n)_{n\in\mathbb{N}_0}$ be a sequence in E, a Dedekind complete Riesz space, then order convergence of $\sum_{k=0}^{\infty} |f_k|$ implies the order convergence of $\sum_{k=0}^{\infty} f_k$.

Proof. Suppose that $\sum_{k=0}^{n-1} |f_k| \to \ell$, in order, as $n \to \infty$. Then ℓ is an upper bound for the increasing sequences $\left(\sum_{k=0}^{n-1} f_k^{\pm}\right)$, which, from the Dedekind completeness of E, have order limits, say, $f^{\pm}$. Thus, $\sum_{k=0}^{n-1} f_k = \sum_{k=0}^{n-1} f_k^{+} - \sum_{k=0}^{n-1} f_k^{-} \to f^{+} - f^{-}$, in order, as $n \to \infty$. $\square$

From Lemma 2.17 and [19, Lemma 2.1], we have the following theorem.

Theorem 2.18 Let E be a Dedekind complete Riesz space and $(f_n)_{n\in\mathbb{N}_0}$ a sequence in E with $f_n \to 0$, in order, as $n \to \infty$, then $\frac{1}{n}\sum_{k=0}^{n-1} |f_k| \to 0$, in order, as $n \to \infty$.

Corollary 2.19 Let E be a Dedekind complete Riesz space and $(f_n)_{n\in\mathbb{N}_0}$ a sequence in E with order limit f, then, as an order limit, we have $\frac{1}{n}\sum_{k=0}^{n-1} f_k \to f$ in order, as $n \to \infty$.

Proof. Let $g_n := f_n - f$ for each $n \in \mathbb{N}_0$, then $(|g_n|)_{n\in\mathbb{N}_0}$ is order convergent to 0, by assumption. Thus, by Theorem 2.18,

$$0 \leq \left| f - \frac{1}{n}\sum_{k=0}^{n-1} f_k \right| \leq \frac{1}{n}\sum_{k=0}^{n-1} |g_k| \to 0,$$

in order, as $n \to \infty$, and the result follows as E is Archimedean. $\square$

Finally, the idea of the convergence of a sequence of band projections is defined as follows: if E is a Riesz space with weak order unit, say, e, and $(P_n)_{n\in\mathbb{N}0}$ is a sequence of band projections in E, then the sequence is convergent if the sequence $(P_n e)_{n\in\mathbb{N}0}$ is convergent.

Chapter 3

The Koopman-von Neumann Condition

3.1 Density Zero Sequence of Band Projections

In [25], a subset N of $\mathbb{N}_0$ is said to be of density zero if $\frac{1}{n}\sum_{k=0}^{n-1}\chi_N(k) \to 0$ as $n \to \infty$, where $\chi_N(k) = 0$ if $k \in \mathbb{N}_0 \setminus N$ and $\chi_N(k) = 1$ if $k \in N$. The Koopman-von Neumann Lemma [25, Lemma 6.2] asserts that if a sequence $(a_n)_{n\in\mathbb{N}_0}$ of non-negative real numbers is bounded, then $\frac{1}{n}\sum_{k=0}^{n-1} a_k \to 0$ as $n \to \infty$ if and only if there is N, a subset of $\mathbb{N}_0$, of density zero, such that $a_n \to 0$ as $n \to \infty$, $n \in \mathbb{N}_0 \setminus N$.

We recall that $E_e = \{f \in E \,|\, |f| \leq ke \text{ for some } k \in \mathbb{R}_+\}$, the subspace of E consisting of the e bounded elements of E is an f-algebra , see [2, 31, 33]. For all band projections P and Q on E, we set $Pe \cdot Qe = PQe$. Here, $\cdot$ represents the f-algebra multiplication on E_e. The linear extension of this multiplication and use of order limits extends this multiplication to the f-algebra multiplication on E_e. Further, the weak order unit, e, is the multiplicative unit of the f-algebra.

In a Riesz space, E, for $u \in E_+$, $p \in E_+$ is called a component of u if $u \wedge (u - p) = 0$, see [34, pg. 213]. Furthermore, if E has the principal projection property , then p is a component of u if and only if $p = P_f u$, for some principal projection P_f, [34, Theorem 32.7]. If E has a weak order unit, say e, then p is a component of e if and only if there is a band projection P on E with $Pe = p$. Further, $e - p = (I - P)e$ and $p \cdot f = Pf$ for each $f \in E$. We note here that E is an E_e module since the f-algebra multiplication in E is a restriction of that in E_u and if $f \in E_e$ with $|f| \leq ke$ and $g \in E$ then $|f \cdot g| \leq k\,|g|$ so $f \cdot g \in E$. Any $s \in E$ for which there exist pairwise disjoint components $p_1, ..., p_n$ of e and real numbers $\alpha_1, ..., \alpha_n$ such that $s = \sum_{k=1}^{n} \alpha_k p_k$ is called an e-step function . Notice that if E has the principal projection property, then there exist principal band projections $P_1, ..., P_n$ such that $s = \sum_{k=0}^{n} \alpha_k P_k e$, where the band projections are pairwise disjoint. Consequently, if E is a Dedekind complete Riesz space with weak order unit, say e, then $B_e = E \supseteq E_e$ (where B_e is the principal band generated by e), and any e-step function $s \in E$ can be represented by $s = \sum_{k=1}^{n} \alpha_k P_k e$.

In order to extend the Koopman-von Neumann Lemma to sequences in a Riesz space, we define a density zero sequence of band projections as follows.

Definition 3.1 (Density Zero Sequence of Band Projections) A sequence $(P_n)_{n\in\mathbb{N}_0}$ of band projections in a Riesz space E with weak order unit e is said to be of density zero if $\frac{1}{n}\sum_{k=0}^{n-1} P_k e \to 0$, in order as $n \to \infty$.

The above definition could be rephrased as saying that a sequence of components of e is said to be of density zero if its Cesàro mean tends to zero in order.

3.2 The Koopman-von Neumann Lemma

Having defined what a sequence of density zero band projections is, it is now possible to give an analogue of the Koopman-von Neumann Lemma in Riesz spaces.

Theorem 3.2 (Koopman-von Neumann) Let E be a Dedekind complete Riesz space with weak order unit, say e, and let $(f_n)_{n\in\mathbb{N}_0}$ be an order bounded sequence in the positive cone, E_+, of E, then $\frac{1}{n}\sum_{k=0}^{n-1} f_k \to 0$, in order, as $n \to \infty$, if and only if there exists a density zero sequence of band projections $(P_n)_{n\in\mathbb{N}_0}$ on E such that $(I-P_n)\,f_n \to 0$, in order, as $n \to \infty$.

Proof. Suppose that $(f_n)_{n\in\mathbb{N}_0} \subset E_+$ and there exists $g \in E_+$ with $f_n \le g$, for all $n \in \mathbb{N}_0$.

If $(P_n)_{n\in\mathbb{N}_0}$ is a density zero sequence of band projections with $(I-P_n)f_n \to 0$, in order, as $n \to \infty$, then

$$0 \le \frac{1}{n}\sum_{k=0}^{n-1} f_k = \frac{1}{n}\sum_{k=0}^{n-1} P_k e \cdot f_k + \frac{1}{n}\sum_{k=0}^{n-1} (I-P_k)\,f_k \tag{3.1a}$$

$$\le \left(\frac{1}{n}\sum_{k=0}^{n-1} P_k e\right)\cdot g + \frac{1}{n}\sum_{k=0}^{n-1} (I-P_k)\,f_k. \tag{3.1b}$$

Since $(P_n)_{n\in\mathbb{N}_0}$ is of density zero, $\frac{1}{n}\sum_{k=0}^{n-1} P_k e \to 0$, in order, as $n \to \infty$, giving $\left(\frac{1}{n}\sum_{k=0}^{n-1} P_k e\right)\cdot g \to 0$, in order, as $n \to \infty$. Furthermore, by Theorem 2.18, as $(I-P_n)f_n \to 0$, in order, as $n \to \infty$ and $(I-P_n)f_n \ge 0$, we have $\frac{1}{n}\sum_{k=0}^{n-1} (I-P_k)\,f_k \to 0$, in order, as $n \to \infty$. Thus, by (3.1a)-(3.1b), as E is Archimedean, $\frac{1}{n}\sum_{k=0}^{n-1} f_k \to 0$, in order, as $n \to \infty$.

Conversely, suppose that $\frac{1}{n}\sum_{k=0}^{n-1} f_k \to 0$, in order, as $n \to \infty$. Let $P_{m,i}$ be the band projection onto the band generated by $\left(f_i - \frac{1}{m}e\right)^+$ and $p_{m,i} = P_{m,i}e$. Let $u_{m,j} := \sup_{k\geq j} \frac{1}{k}\sum_{i=0}^{k-1} p_{m,i}$. Let $R_{m,j}$ be the band projection onto the band generated by $\left(u_{m,j} - \frac{1}{m}e\right)^+$ and $r_{m,j} = R_{m,j}e$. As $0 \leq p_{m,i} \leq e$, we have that $0 \leq u_{m,j} \leq e$. Further, since $p_{m,i}$ is increasing in m for fixed i, it follows that $u_{m,j}$ is increasing in m for fixed j and hence $u_{m,j} - \frac{1}{m}e$ is increasing in m for fixed j, giving that $r_{m,j}$ is increasing in m for fixed j.

Since $\{k \in \mathbb{Z} \,|\, k \geq j+1\} \subset \{k \in \mathbb{Z} \,|\, k \geq j\}$, it follows that

$$u_{m,j+1} = \sup_{k\geq j+1} \frac{1}{k}\sum_{i=0}^{k-1} p_{m,i} \leq \sup_{k\geq j} \frac{1}{k}\sum_{i=0}^{k-1} p_{m,i} = u_{m,j},$$

giving that $u_{m,j}$ is decreasing in j. Hence, $r_{m,j}$ is decreasing in j, for fixed m. We now show that, for fixed m, $r_{m,j} \downarrow 0$, in order, as $j \to \infty$. Since $f_i \geq P_{m,i}f_i \geq \frac{1}{m}p_{m,i}$ (as $P_{m,i}$ is the projection onto the band generated by $\left(f_i - \frac{1}{m}e\right)^+$), we have

$$\sup_{k\geq j} \frac{1}{k}\sum_{i=0}^{k-1} f_i \geq \frac{1}{m}\sup_{k\geq j} \frac{1}{k}\sum_{i=0}^{k-1} p_{m,i} = \frac{1}{m}u_{m,j}. \tag{3.2}$$

However,

$$u_{m,j} \geq R_{m,j}u_{m,j} \geq \frac{1}{m}r_{m,j}, \tag{3.3}$$

(since $R_{m,j}$ is the band projection onto the band generated by $\left(u_{m,j} - \frac{1}{m}e\right)^+$), so, by (3.2) and (3.3), we have

$$0 \leq \frac{1}{m^2}r_{m,j} \leq \frac{1}{m}u_{m,j} \leq \sup_{k\geq j} \frac{1}{k}\sum_{i=0}^{k-1} f_i \to 0, \tag{3.4}$$

in order, as $j \to \infty$. Thus, for fixed m, $r_{m,j} \downarrow 0$, in order, as $j \to \infty$.

Observe that $R_{1,j} = 0$, since $R_{1,j}$ is the band projection onto the band generated by $(u_{1,j} - e)^+ = 0$. Let $H_j := \sup_{m \in \mathbb{N}} R_{m,j}$, for $j = 0, 1, \ldots$, then $H_j \downarrow H$, say, in order, as $j \to \infty$. Here, the band projections H_j and H can be explicitly determined from their components $h_j := H_j e = \sup_{m \in \mathbb{N}} r_{m,j}$, for $j = 0, 1, \ldots$, and $h_j \downarrow h := He$, in order, as $j \to \infty$. Further $r_{m+1,j} \cdot (e - r_{m,j})$, $m \in \mathbb{N}$, is a partition of h_j for each $j = 0, 1, \ldots$, that is,

$$(r_{m+1,j} \cdot (e - r_{m,j})) \wedge (r_{n+1,j} \cdot (e - r_{n,j})) = 0, \text{ for } m \neq n, \tag{3.5}$$

and

$$\sum_{m=1}^{M-1} r_{m+1,j} \cdot (e - r_{m,j}) = r_{M,j} \uparrow_M h_j, \tag{3.6}$$

in order, as $M \to \infty$, giving

$$\sum_{m \in \mathbb{N}} r_{m+1,j} \cdot (e - r_{m,j}) = h_j. \tag{3.7}$$

Let

$$Q_j := \bigvee_{m \in \mathbb{N}} P_{m,j} R_{m+1,j} (I - R_{m,j}) = \sum_{m \in \mathbb{N}} P_{m,j} R_{m+1,j} (I - R_{m,j}), \tag{3.8}$$

by (3.5), and in terms of components we have

$$q_j := Q_j e = \bigvee_{m \in \mathbb{N}} p_{m,j} \cdot r_{m+1,j} \cdot (e - r_{m,j}) = \sum_{m \in \mathbb{N}} p_{m,j} \cdot r_{m+1,j} \cdot (e - r_{m,j}). \tag{3.9}$$

Further, as $0 \leq p_{m,j} \leq e$, by (3.8),

$$q_j = \sum_{m \in \mathbb{N}} p_{m,j} \cdot r_{m+1,j} \cdot (e - r_{m,j}) \leq \sum_{m \in \mathbb{N}} r_{m+1,j} \cdot (e - r_{m,j}) = h_j. \tag{3.10}$$

Hence, subtracting the left-hand side of (3.10) from the right-hand side of (3.10), we obtain

$$(e - q_j) = (e - h_j) + (h_j - q_j) = (e - h_j) + \sum_{m \in \mathbb{N}} (e - p_{m,j}) \cdot r_{m+1,j} \cdot (e - r_{m,j}) \,. \quad (3.11)$$

For $m > j+1$, $j \in \mathbb{N}_0$, we have that $\frac{1}{j+1} > \frac{1}{m}$. Now, from the definition of $u_{m,j}$, we have

$$u_{m,j} = \sup_{k \geq j} \frac{1}{k} \sum_{i=0}^{k-1} p_{m,i} \geq \frac{1}{j+1} \sum_{i=0}^{j} p_{m,i} \geq \frac{1}{j+1} p_{m,j},$$

thus $P_{m,j} u_{m,j} \geq \frac{1}{j+1} p_{m,j}$, giving

$$P_{m,j} \left(u_{m,j} - \frac{1}{m} e \right)^+ \geq P_{m,j} \left(u_{m,j} - \frac{1}{m} e \right) \geq \left(\frac{1}{j+1} - \frac{1}{m} \right) p_{m,j}.$$

Hence, $\left(u_{m,j} - \frac{1}{m} e\right)^+ \geq \left(\frac{1}{j+1} - \frac{1}{m} \right) p_{m,j}$. Further, $\frac{1}{j+1} - \frac{1}{m} > 0$, so, for $m > j+1$, $p_{m,j}$ is in the band generated by $\left(u_{m,j} - \frac{1}{m} e\right)^+$, giving $R_{m,j} \geq P_{m,j}$. Applying the above to f_j, we have

$$r_{m,j} \cdot f_j \geq p_{m,j} \cdot f_j. \quad (3.12)$$

Taking the supremum over $m \in \mathbb{N}$ in (3.12) gives $h_j \cdot f_j \geq f_j$, since $\sup\limits_{m \in \mathbb{N}} r_{m,j} = h_j$ and $\sup\limits_{m \in \mathbb{N}} p_{m,j} \cdot f_j = f_j$. Further, $h_j \leq e$, hence, $(e - h_j) \cdot f_j = 0$, so multiplying (3.11) by f_j gives

$$(e - q_j) \cdot f_j = \sum_{m \in \mathbb{N}} r_{m+1,j} \cdot (e - r_{m,j}) \cdot (I - P_{m,j}) f_j \quad (3.13a)$$

$$\leq \sum_{m \in \mathbb{N}} \frac{1}{m} r_{m+1,j} \cdot (e - r_{m,j}) \,, \quad (3.13b)$$

where we have used that $(I - P_{m,j}) f_j \leq \dfrac{1}{m} e$. Now, from (3.6), (3.7) and (3.13a)-(3.13b), for $k \leq j$,

$$(e - q_j) \cdot f_j \leq \sum_{m=1}^{k-1} \frac{1}{m} r_{m+1,j} \cdot (e - r_{m,j}) + \sum_{m=k}^{\infty} \frac{1}{m} r_{m+1,j} \cdot (e - r_{m,j})$$

$$\begin{aligned}&\leq \sum_{m=1}^{k-1} r_{m+1,j} \cdot (e - r_{m,j}) + \frac{1}{k} h_j \\ &\leq r_{k,j} e + \frac{1}{k} e.\end{aligned}$$

So, taking the limit supremum as $j \to \infty$, we obtain

$$0 \leq \limsup_{j\to\infty} (e - q_j) \cdot f_j \leq \frac{1}{k} e + \limsup_{j\to\infty} r_{k,j} = \frac{1}{k} e, \tag{3.15}$$

for each $k \in \mathbb{N}$. Thus, as E is Archimedean, $\limsup_{j\to\infty} (I - Q_j) f_j = 0$ and $(I - Q_j) f_j \to 0$, in order, as $j \to \infty$.

It now remains to show that $\frac{1}{n}\sum_{j=0}^{n-1} q_j \to 0$, in order, as $n \to \infty$. For $j < n$ and $m \geq M+1$, we have $r_{M+1,n} \leq r_{m,j}$, giving $r_{M+1,n} \cdot (e - r_{m,j}) = 0$. Thus, for $M, n \in \mathbb{N}$, we have

$$\begin{aligned}& r_{M+1,n} \cdot (e - r_{M,n}) \cdot \left(\frac{1}{n} \sum_{j=0}^{n-1} q_j \right) \\ &= \frac{1}{n} \sum_{j=0}^{n-1} r_{M+1,n} \cdot (e - r_{M,n}) \cdot \left(\bigvee_{m\in\mathbb{N}} p_{m,j} \cdot r_{m+1,j} \cdot (e - r_{m,j}) \right) \\ &= \frac{1}{n} \sum_{j=0}^{n-1} \bigvee_{m\leq M} r_{M+1,n} \cdot (e - r_{M,n}) \cdot p_{m,j} \cdot r_{m+1,j} \cdot (e - r_{m,j}) \\ &\leq \frac{1}{n} \sum_{j=0}^{n-1} r_{M+1,n} \cdot (e - r_{M,n}) \cdot p_{M,j} \\ &= r_{M+1,n} \cdot (e - r_{M,n}) \cdot \left(\frac{1}{n} \sum_{j=0}^{n-1} p_{M,j} \right) \\ &\leq r_{M+1,n} \cdot (I - R_{M,n}) u_{M,n} \\ &\leq \frac{1}{M} r_{M+1,n} \cdot (I - R_{M,n}) e.\end{aligned}$$

Summing the above over $M \geq K$ gives

$$(h_n - r_{K,n}) \cdot \left(\frac{1}{n}\sum_{j=0}^{n-1} q_j\right) \leq \sum_{M \geq K} \frac{1}{M} r_{M+1,n} \cdot (e - r_{M,n}) \tag{3.16a}$$

$$\leq \frac{1}{K}(h_n - r_{K,n}) \leq \frac{1}{K}e. \tag{3.16b}$$

We recall, from (3.10), that $Q_j \leq H_j$, so if $j \geq \lfloor\sqrt{n}\rfloor$ then $Q_j \leq H_j \leq H_{\lfloor\sqrt{n}\rfloor}$ and $Q_j H_{\lfloor\sqrt{n}\rfloor} = Q_j$, that is, $(e - h_{\lfloor\sqrt{n}\rfloor}) \cdot q_j = 0$. Hence,

$$(e - h_{\lfloor\sqrt{n}\rfloor}) \cdot \left(\frac{1}{n}\sum_{j=0}^{n-1} q_j\right) = \frac{1}{n}\sum_{j=0}^{\lfloor\sqrt{n}\rfloor-1} q_j \leq \frac{1}{\sqrt{n}}e. \tag{3.17}$$

Combining (3.16a)-(3.16b) and (3.17), we have, for each $K \in \mathbb{N}$,

$$\frac{1}{n}\sum_{j=0}^{n-1} q_j = \left(\left(e - h_{\lfloor\sqrt{n}\rfloor}\right) + \left(h_{\lfloor\sqrt{n}\rfloor} - h_n\right) + (h_n - r_{K,n}) + r_{K,n}\right) \cdot \left(\frac{1}{n}\sum_{j=0}^{n-1} q_j\right) \tag{3.18a}$$

$$\leq \frac{1}{\sqrt{n}}e + (h_{\lfloor\sqrt{n}\rfloor} - h_n) + \frac{1}{K}e + r_{K,n}. \tag{3.18b}$$

Here, h_n and $h_{\lfloor\sqrt{n}\rfloor}$ both converge, in order, to h, so $h_{\lfloor\sqrt{n}\rfloor} - h_n$ converges to 0, in order, as $n \to \infty$, as does $r_{K,n}$. Thus, taking the limit supremum as $n \to \infty$ in (3.18b) gives

$$0 \leq \limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} q_j \leq \frac{1}{K}e \tag{3.19}$$

for each $K \in \mathbb{N}$. Hence,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} q_j = 0,$$

in order, as E is Archimedean.

Note 3.3 If $(f_n)_{n\in\mathbb{N}_0}$ in Theorem 3.2 is not assumed to be order bounded, but $\frac{1}{n}\sum_{k=0}^{n-1} f_k \to 0$, in order, as $n \to \infty$, then it still follows that there is a density zero sequence of band projections, $(P_n)_{n\in\mathbb{N}_0}$, such that $(I - P_n)f_n \to 0$, in order, as $n \to \infty$, but one cannot conclude boundedness of $(P_n f_n)_{n\in\mathbb{N}_0}$. Further, the converse need not hold. In particular, if $(f_n)_{n\in\mathbb{N}_0}$ is not bounded and there is a density zero sequence of band projections $(P_n)_{n\in\mathbb{N}_0}$ such that $(I-P_n)f_n \to 0$, in order, as $n \to \infty$, then one cannot conclude that $\frac{1}{n}\sum_{k=0}^{n-1} f_k$ is order convergent to 0 as $n \to \infty$.

For example, working in the classical case of $E = \mathbb{R}$, for $1 < p < \infty$, we have

$$g_j^p = \begin{cases} n, & j = \lfloor n^p \rfloor, n \in \mathbb{N}_0, \\ 0, & \text{otherwise} \end{cases}$$

is unbounded and $N = \{\lfloor n^p \rfloor \mid n \in \mathbb{N}_0\}$ is a set of density zero. Here $g_j^p = 0 \to 0$ for $j \in \mathbb{N}_0 \backslash N$ as $j \to \infty$ but $\frac{1}{n}\sum_{k=0}^{n-1} g_k^p$ is convergent to 0 for $p > 2$, convergent to a non-zero value for $p = 2$ and divergent to ∞ for $1 < p < 2$.

Chapter 4

Ergodicity

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space , that is, Ω is a set, $\mathcal{A}$ is a σ-algebra of subsets of Ω and μ is a measure on $\mathcal{A}$ with $\mu(\Omega) = 1$. The mapping $\tau : \Omega \to \Omega$ is called a measure preserving transformation if $\mu(\tau^{-1}A) = \mu(A)$, for each $A \in \mathcal{A}$, in which case $(\Omega, \mathcal{A}, \mu, \tau)$ is called a measure preserving system . Further details may be found in [5, 16, 25].

The measure preserving system $(\Omega, \mathcal{A}, \mu, \tau)$ is said to be weakly mixing if

$$\frac{1}{n}\sum_{k=0}^{n-1} \left|\mu\left(\tau^{-1}(A) \cap B\right) - \mu(A)\,\mu(B)\right| \to 0 \tag{4.1}$$

as $n \to \infty$, for each $A, B \in \mathcal{A}$. To give a Riesz space analogue of a measure preserving system and weak mixing, we recall from [17] the definition of a conditional expectation operator on a Riesz space.

We recall Definition 2.13, the definition of a conditional expectation operator on a Riesz space. Let E be a Riesz space with weak order unit. A positive order continuous projection $T\colon E \to E$, with range, $\mathcal{R}(T)$, a Dedekind complete Riesz subspace of E, is called a conditional expectation operator if Te is a weak order unit of E for each weak order unit e of E.

If T is a conditional expectation operator on E with $Te = e$, then T is also a conditional expectation operator on E_e, since, if $f \in E_e$, then $|f| \le ke$, giving $|Tf| \le T|f| \le Tke = ke$.

The Riesz space analogue of a measure preserving system is introduced in the following definition.

Definition 4.1 Let E be a Dedekind complete Riesz space with weak order unit, say, e, and T be a conditional expectation operator on E with $Te = e$. If S is an order continuous Riesz homomorphism on E with $Se = e$ and $TSPe = TPe$ for each band projection P on E, then (E, T, S, e) is called a conditional expectation preserving system .

By Freudenthal's Spectral Theorem, see [34, Theorem 33.2], the condition $TSPe = TPe$ for each band projection P on E in the above definition is equivalent to $TSf = Tf$ for all $f \in E$. In [18, Theorems 3.7 and 3.9] various generalisations of Birkhoff's ergodic theorem to conditional expectation preserving systems (E, T, S, e) were given, resulting in convergence conditions for $\left(\frac{1}{n}\sum_{k=0}^{n-1} S^k f\right)$, $f \in E$.

We are now in a position to define conditional weak mixing on a Riesz space with a conditional expectation operator and weak order unit.

Definition 4.2 (Conditional Weak Mixing) The conditional expectation preserving system (E, T, S, e) is said to be conditionally weak mixing if, for all band projections P and Q on E,

$$\frac{1}{n}\sum_{k=0}^{n-1} \left|T\left(\left(S^k Pe\right) \cdot Qe\right) - TPe \cdot TQe\right| \to 0, \tag{4.2}$$

in order, as $n \to \infty$.

We note that for $E = L^1(\Omega, \mathcal{A}, \mu)$, the band projections on E are of the form $P_A f = \chi_A f$, for $f \in E$ and $A \in \mathcal{A}$. Definition 4.2 now gives a conditional weak mixing condition on E, conditioned by $T = \mathbb{E}[\cdot \mid \mathcal{B}]$, for $\mathcal{B}$ a sub-σ-algebra of $\mathcal{A}$. If $\mathcal{B} = \{\Omega \setminus C | C \in \mathcal{A}, \mu(C) = 0\} \cup \{C \in \mathcal{A} | \mu(C) = 0\}$, then conditional weak mixing coincides with the weak mixing on $(\Omega, \mathcal{A}, \mu)$.

We recall, [34, pg. 49], that if E is a Riesz space and $(x_n)_{n\in\mathbb{N}} \subset E$ converges to $x \in E$, we say that x_n converges to x u-uniformly for given $0 < u \in E$ if for each $0 < \epsilon \in \mathbb{R}$, there is some $N_\epsilon \in \mathbb{N}$ such that $|x_n - x| \le \epsilon u$ whenever $n \ge N_\epsilon$.

Theorem 4.3 Given the conditional expectation preserving system (E, T, S, e), then the following statements are equivalent.

i. (E, T, S, e) is conditionally weak mixing.

ii. For all $f, g \in E_e$, we have that

$$\frac{1}{n}\sum_{k=0}^{n-1}\left|T\left(\left(S^k f\right)\cdot g\right) - Tf\cdot Tg\right| \to 0,$$

in order, as $n \to \infty$.

iii. For each pair of band projections P and Q on E, there is a sequence of density zero band projections, $(R_n)_{n\in\mathbb{N}_0}$, in E such that

$$(I - R_n)\left|T\left(\left(S^n Pe\right)\cdot Qe\right) - T(Pe)\cdot T(Qe)\right| \to 0,$$

in order, as $n \to \infty$.

Proof. (i)⇒(ii): Suppose that (E, T, S, e) is conditionally weak mixing. Let $s, t \in E$ be e-step functions with $s = \sum_{i=1}^{m} \alpha_i P_i e$ and $t = \sum_{j=1}^{r} \beta_j Q_j e$, where P_i and Q_j are band projections on E and α_i and β_j are real numbers, $i = 1, \ldots, m$, $j = 1, \ldots, r$, then

$$\frac{1}{n}\sum_{k=0}^{n-1}\left|T\left(\left(S^k(s)\right)\cdot t\right) - T(s)\cdot T(t)\right| \tag{4.3a}$$

$$\leq \sum_{i=1}^{m}\sum_{j=1}^{r}|\alpha_i\beta_j|\,\frac{1}{n}\sum_{k=0}^{n-1}\left|T\left(\left(S^k(P_i e)\right)\cdot Q_j e\right) - T(P_i e)\cdot T(Q_j e)\right| \to 0, \tag{4.3b}$$

in order, as $n \to \infty$.

By Freudenthal's Spectral Theorem, [34, Theorem 33.2], $f, g \in E_e$ can be expressed as e-uniform order limits of sequences, say $(s_i)_{i\in\mathbb{N}}, (t_j)_{j\in\mathbb{N}}$, of e-step functions in E_e and there is $K > 0$ so that $|s_i|, |t_j|, |f|, |g| \leq Ke$, for all $i, j \in \mathbb{N}$. This implies that $T|s_i|, T|t_j|, T|f|, T|g| \leq Ke$, for each $i, j \in \mathbb{N}$. For each $\epsilon > 0$ there is $N_\epsilon \in \mathbb{N}$ so that $|s_i - f| \leq \epsilon e$ and $|t_j - g| \leq \epsilon e$ for each $i, j \geq N_\epsilon \in \mathbb{N}$. Hence, $T|s_i - f| \leq \epsilon e$ and $T|t_j - g| \leq \epsilon e$ for each $i, j \geq N_\epsilon$. Let $b_k := \left|T\left(S^k(f)\cdot g\right) - Tf\cdot Tg\right|$ and $b_{k,i,j} := \left|T\left(S^k(s_i)\cdot t_j\right) - Ts_i\cdot Tt_j\right|$, then

$$\begin{aligned}
&|b_{k,i,j} - b_k| \\
&\quad = \left|\left|T\left(S^k(s_i)\cdot t_j\right) - Ts_i\cdot Tt_j\right| - \left|T\left(S^k(f)\cdot g\right) - Tf\cdot Tg\right|\right| \\
&\quad \leq \left|T\left(S^k(s_i)\cdot t_j\right) - Ts_i\cdot Tt_j - T\left(S^k(f)\cdot g\right) + Tf\cdot Tg\right| \\
&\quad \leq \left|T\left(S^k(f)\cdot g - S^k(s_i)\cdot t_j\right)\right| + |Tf\cdot Tg - Ts_i\cdot Tt_j| \\
&\quad \leq T\left|\left(S^k(f - s_i)\right)\cdot g\right| + T\left|S^k(s_i)\cdot(g - t_j)\right| + T|f - s_i|\cdot T|g| \\
&\qquad + T|s_i|\cdot T|g - t_j| \\
&\quad \leq Ke\cdot\left(T\left|S^k(f - s_i)\right| + T|g - t_j| + T|f - s_i| + T|g - t_j|\right) \\
&\quad \leq 4K\epsilon e.
\end{aligned}$$

Hence, $b_k \leq b_{k,i,j} + 4K\epsilon e$, for all $i, j \in \mathbb{N}$ with $i, j \geq N_\epsilon$, so

$$0 \leq \frac{1}{n}\sum_{k=0}^{n-1} b_k \leq \frac{1}{n}\sum_{k=0}^{n-1} b_{k,N_\epsilon,N_\epsilon} + 4K\epsilon e,$$

for all $n \in \mathbb{N}$. By (4.3a)-(4.3b), $\frac{1}{n}\sum_{k=0}^{n-1} b_{k,N_\epsilon,N_\epsilon} \to 0$, in order, as $n \to \infty$, so

$$0 \le \limsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} b_k \le 4K\epsilon e,$$

for all $\epsilon > 0$, implying that $\frac{1}{n}\sum_{k=0}^{n-1} b_k \to 0$, in order.

(ii) $\Rightarrow$ (i): Choosing $f = Pe$ and $g = Qe$, the result follows directly.

(i)$\Leftrightarrow$(iii): Taking $f_n = |T((S^n Pe) \cdot Qe) - TPe \cdot TQe|$, the result follows from Theorem 3.2.

The reader is referred to Aliprantis and Border [1, pages 263-300], Fremlin [7, Chapter 35, pages 219-274], Meyer-Nieberg [24], and Zaanen [34] for background in Riesz spaces.

Let (E, T, S, e) be a conditional expectation preserving system. For $f \in E$ and $n \in \mathbb{N}$ we denote

$$S_n f := \frac{1}{n}\sum_{k=0}^{n-1} S^k f, \tag{4.5a}$$

$$L_S f := \lim_{n\to\infty} S_n f, \tag{4.5b}$$

where the above limit is the order limit, if it exits. We say that $f \in E$ is S-invariant if $Sf = f$. The set of all S-invariant $f \in E$ will be denoted $\mathcal{I}_S := \{f \in E \,|\, Sf = f\}$. The set of $f \in E$ for which $L_S f$ exists will be denoted $\mathcal{E}_S$ and thus $L_S : \mathcal{E}_S \to E$.

Lemma 4.4 Let (E, T, S, e) be a conditional expectation preserving system and define $\mathcal{I}_S$ and $\mathcal{E}_S$ as before, then $\mathcal{I}_S \subset \mathcal{E}_S$ and $L_S f = f$, for all $f \in \mathcal{I}_S$, so $\mathcal{I}_S \subset R(L_S)$.

Proof. If $f \in \mathcal{I}_S$, then $S^k f = f$, for all $k \in \mathbb{N}_0$. Thus $S_n f = \frac{1}{n}\sum_{k=0}^{n-1} f = f$, for all $n \in \mathbb{N}$, giving $S_n f \to f$, in order, as $n \to \infty$. Hence $f \in \mathcal{E}_S$ and $L_S f = f$, thus $\mathcal{I}_S \subset \mathcal{E}_S$. $\square$

We recall Birkhoff's bounded ergodic theorem from [18, Theorem 3.7].

Theorem 4.5 (Birkhoff's (Bounded) Ergodic Theorem) Let (E, T, S, e) be a conditional expectation preserving system. For $f \in E$, the sequence $(S_n f)_{n\in\mathbb{N}}$ is order bounded in E if and only if $f \in \mathcal{E}_S$. For each $f \in \mathcal{E}_S$ we have $L_S f = SL_S f$ and $TL_S f = Tf$. If $E = \mathcal{E}_S$ then L_S is a conditional expectation operator on E with $L_S e = e$.

Note 4.6 If we restrict our attention to E_e, then $(S_n f)_{n\in\mathbb{N}}$ is order bounded in E_e for each $f \in E_e$, so $E_e \subset \mathcal{E}_S$. Furthermore, $L_S f \in E_e$, giving that $L_S|_{E_e}$ is a conditional expectation on E_e.

E is said to be universally complete with respect to T (T-universally complete) , if, for each increasing net (f_α) in E_+ with (Tf_α) order bounded, we have that (f_α) is order convergent in E.

We recall Birkhoff's ergodic theorem for a T-universally complete Riesz space from [18, Theorem 3.9].

Theorem 4.7 (Birkhoff's (Complete) Ergodic Theorem) Let (E, T, S, e) be a conditional expectation preserving system with E T-universally complete then $E = \mathcal{E}_S$ and hence $L_S = SL_S$. In addition, $TL_S = T$ and L_S is a conditional expectation operator on E.

4.1 Ergodicity on Riesz Spaces

Consider the Riesz space $E = L^1(\Omega, \mathcal{A}, \mu)$, where μ is a probability measure and T is the expectation operator $T = \mathbb{E}[\cdot]\mathbf{1}$, where $\mathbf{1}$ is the equivalence class, in $L^1(\Omega, \mathcal{A}, \mu)$, of function with value 1 almost everywhere. For τ a measure preserving transformation on Ω setting $Sf = f \circ \tau$ we have that $(E, T, S, \mathbf{1})$ is a conditional expectation preserving system. The definition of ergodicity as $f \circ \tau = f$ for measurable f only if f is almost everywhere constant, can now be written as $Sf = f$ only if $f \in R(T)$. This leads naturally to the following definition in Riesz spaces, for conditional expectation preserving systems.

Definition 4.8 (Ergodicity) The conditional expectation preserving system (E, T, S, e) is said to be ergodic if $L_S f \in \mathcal{R}(T)$ for all $f \in \mathcal{I}_S$.

As will be seen later, this definition preserves the original philosophy of an ergodic process as one in which the time mean (the Cesàro mean) is equal to the spatial mean (conditional expectation) in the limit.

We note that E_e is a $R(T)$-module, see [20], an hence, when dealing with conditioned systems, the role of the scalars ($\mathbb{R}$ or $\{ke \mid k \in \mathbb{R}\}$) is replaced by the ring $R(T)$.

Theorem 4.9 The conditional expectation preserving system (E, T, S, e) is ergodic if and only $L_S f = Tf$ for all $f \in \mathcal{I}_S$.

Proof. Let (E, T, S, e) be ergodic and $f \in \mathcal{I}_S$, then, by Theorem 4.5, $L_S f$ exists and $TL_S f = Tf$. As (E, T, S, e) is ergodic $L_S f \in R(T)$, so there exists $g \in E$ such that $L_S f = Tg$. As T is a projection $TTg = Tg$, so $Tf = TL_S f = TTg = Tg = L_S f$.

Conversely, if $f \in \mathcal{I}_S$ then, as $\mathcal{I}_S \subset \mathcal{E}_S$, $L_S f$ exists and if we assume $L_S f = Tf$ then $L_S f \in R(T)$. Thus (E, T, S, e) is ergodic. $\square$

For (E, T, S, e) a conditional expectation preserving system we denote by $\mathcal{B}$ the band projections on E, by $\mathcal{P} := \{P \in \mathcal{B} | \, TPe = PTe = Pe\}$ the band projections on E which commute with T and by $\mathcal{A} := \{P \in \mathcal{B} | \, SPe = Pe\}$ the set of S invariant band projections on E. Since S is a positive operator with $Se = e$ and P is positive and below the identity, we have that $SPe, Pe \in E_e$, for all $P \in \mathcal{B}$. Hence, L_S can always be applied to SPe and Pe. We now show that $\mathcal{A}$ characterises L_S.

Theorem 4.10 For (E, T, S, e) a conditional expectation preserving system we have that $\mathcal{P} \subset \mathcal{A} = \mathcal{F}$ where $\mathcal{F} := \{P \in \mathcal{B} | \, Pe = L_S Pe\}$ is the set of band projections on E which commute with L_S.

Proof. For the purpose of this theorem we can assume that $E = E_e$ as the sets $\mathcal{A}, \mathcal{B}, \mathcal{P}$ do not change when E is replaced by E_e. Further, by Theorem 4.5, L_S restricted to E_e is everywhere defined and a conditional expectation on E_e with $SL_S = L_S$.

If $P \in \mathcal{A}$ then $Pe \in \mathcal{I}_S$, so, by Lemma 4.4, $L_S Pe = Pe$, hence $P \in \mathcal{F}$. Conversely, if $P \in \mathcal{F}$ then $L_S Pe = PL_S e = Pe$, so $SPe = SL_S Pe = L_S Pe = Pe$, giving $P \in \mathcal{A}$. Hence, $\mathcal{A} = \mathcal{F}$.

From Theorem 4.5, $L_S T = T = TL_S$ giving that if $P \in \mathcal{P}$ then $TPe = Pe$ so $L_S Pe = L_S TPe = TPe = Pe$ and $P \in \mathcal{F}$. $\square$

The following corollary to Theorem 4.10 is critical when applying the concepts of ergodicity in Riesz spaces, further to this, it shows that ergodicity is equivalent to the time and spatial means coinciding in the limit.

Corollary 4.11 The conditional expectation preserving system (E, T, S, e) is ergodic if and only if $T = L_S$, where $E = \mathcal{E}_S$.

Proof. Theorem 4.5 gives that $SL_S = L_S$, hence $R(L_S) \subset \mathcal{I}_S$. However, by Lemma 4.4, $\mathcal{I}_S \subset R(L_S)$. Hence $\mathcal{I}_S = R(L_S)$.

If (E, T, S, e) is ergodic, then $Tf = L_S f = f$, for all $f \in \mathcal{I}_S$, so $f \in \mathcal{R}(T)$, giving $\mathcal{R}(L_S) = \mathcal{I}_S \subset \mathcal{R}(T)$. Hence, $\mathcal{R}(L_S) = \mathcal{R}(T)$, but $L_S T = T = TL_S$ so, from [32], $L_S = T$.

Conversely, suppose that $T = L_S$, then by Theorem 4.9, (E, T, S, e) is ergodic. $\square$

The following theorem will be seen to be fundamental in linking the concepts of ergodicity and conditional weak mixing in Riesz spaces.

Theorem 4.12 A conditional expectation preserving system (E, T, S, e), where T being strictly positive, is ergodic if and only if

$$\frac{1}{n}\sum_{k=0}^{n-1} T\left(S^k f \cdot g\right) \to Tf \cdot Tg, \tag{4.6}$$

in order, as $n \to \infty$, for $f, g \in E_e$.

Proof. For $f, g \in E_e$, we note that

$$\frac{1}{n}\sum_{k=0}^{n-1} T\left(\left(S^k f\right) \cdot g\right) = T\left(\left(\frac{1}{n}\sum_{k=0}^{n-1} S^k f\right) \cdot g\right) = T\left(S_n f \cdot g\right),$$

and in E_e, $S_n f \to L_S f$, in order, as $n \to \infty$. Thus (4.6) is equivalent to

$$T(L_S f \cdot g) = Tf \cdot Tg. \tag{4.7}$$

Suppose that (E, T, S, e) is ergodic. As $L_S f \in \mathcal{R}(T)$ and T is an averaging operator $T(L_S f \cdot g) = L_S f \cdot Tg$. Again from the ergodicity of (E, T, S, e) we have $L_S f = Tf$. Thus (4.7) holds.

Conversely, suppose that (4.6) or equivalently (4.7) holds. As T is an averaging operator , $Tf \cdot Tg = T(Tf \cdot g)$, which combined with (4.7) gives

$$T((L_S f - Tf) \cdot g) = 0. \tag{4.8}$$

Taking $g = P_{\pm} e$ where $P_{\pm}$ are the band projections onto the bands generated by $(L_S f - Tf)^{\pm}$ in (4.9) gives

$$0 = T((L_S f - Tf) \cdot P_{\pm} e) = T(P_{\pm}(L_S f - Tf)) = T((L_S f - Tf)^{\pm}). \tag{4.9}$$

The strict positivity of T now gives that $(L_S f - Tf)^{\pm} = 0$, hence $L_S f = Tf$. $\square$

We then have the following corollary.

Corollary 4.13 The conditional expectation preserving system (E, T, S, e) is ergodic if and only if

$$\frac{1}{n}\sum_{k=0}^{n-1} T\left(\left(S^k Pe\right) \cdot Qe\right) \to TPe \cdot TQe, \tag{4.10}$$

in order as $n \to \infty$, for all band projections P and Q on E.

4.2 Application to Conditional Weak Mixing

We recall from Chapter 3 the definition of conditional weak mixing in Riesz spaces. Various other types of mixing in Riesz spaces were studied in [20].

Definition 4.14 (Weak Mixing) The conditional expectation preserving system (E, T, S, e) is said to be weakly mixing if, for all band projections P and Q on E,

$$\frac{1}{n}\sum_{k=0}^{n-1}\left|T\left(\left(S^k Pe\right)\cdot Qe\right) - TPe\cdot TQe\right| \to 0, \tag{4.11}$$

in order, as $n \to \infty$.

From Theorem 4.3 we have (4.11) is equivalent to

$$\frac{1}{n}\sum_{k=0}^{n-1}\left|T\left(\left(S^k f\right)\cdot g\right) - Tf\cdot Tg\right| \to 0 \tag{4.12}$$

for all $f, g \in E_e$, in order as $n \to \infty$.

Combining Theorem 4.12 with Lemma 2.17 and (4.12), we have the following.

Corollary 4.15 If the conditional expectation preserving system (E, T, S, e), with T strictly positive, is conditionally weak mixing then it is ergodic.

Proof. If (E, T, S, e) is conditionally weak mixing, then by (4.12),

$$\frac{1}{n}\sum_{k=0}^{n-1}\left|T\left(\left(S^k f\right)\cdot g\right) - Tf\cdot Tg\right| \to 0,$$

in order, as $n \to \infty$, for all $f, g \in E_e$, which, by Lemma 2.17 gives

$$\frac{1}{n}\sum_{k=0}^{n-1}T\left(\left(S^k f\right)\cdot g\right) - Tf\cdot Tg = \frac{1}{n}\sum_{k=0}^{n-1}\left(T\left(\left(S^k f\right)\cdot g\right) - Tf\cdot Tg\right) \to 0,$$

in order, as $n \to \infty$, for all $f, g \in E_e$. That is

$$\frac{1}{n}\sum_{k=0}^{n-1}T\left(\left(S^k f\right)\cdot g\right) \to Tf\cdot Tg,$$

in order, as $n \to \infty$, for all $f, g \in E$, giving that (E, T, S, e) is ergodic, by Theorem 4.12. □

Chapter 5

Application to Measure Theory

5.1 Application of Theorem 3.2

If we consider the Riesz space E of equivalence classes of almost everywhere identical functions in $L^1(\Omega, \mathcal{A}, \mu)$, where μ is a finite measure (the case of μ σ-finite is an easy extension of this case), then E is a Dedekind complete Riesz space under a.e. pointwise ordering and $\mathbf{1}$, the a.e. equivalence class of the constant function with value 1, is a weak order for E. Here the band projections, P, on E are multiplication by the characteristic functions of measurable sets, i.e., P is of the form $Pf = \chi_A f, f \in E$, for $A \in \mathcal{A}$. We recall from [1, Lemma 8.17] and [24, pg. 9, Example (ii)] that a sequence $(g_n)_{n\in\mathbb{N}} \subset L^1(\Omega, \mathcal{A}, \mu)$ is order convergent in $L^1(\Omega, \mathcal{A}, \mu)$ if and only if $g_n \to g$ a.e. pointwise and there exists $h \in L^1(\Omega, \mathcal{A}, \mu)$ for which $|g_n| \leq h$ a.e. for all $n \in \mathbb{N}$.

If $(f_n)_{n\in\mathbb{N}_0}$ is a non-negative order bounded sequence in E, i.e., $f_n \geq 0$ a.e. and there exists $g \in L^1(\Omega, \mathcal{A}, \mu)$ so that $f_n \leq g$ a.e. for all n, then, by Theorem 3.2,

$$s_n := \frac{1}{n}\sum_{j=0}^{n-1} f_j \to 0, \quad \text{in order, as} \quad n \to \infty, \tag{5.1}$$

if and only if there is a density zero sequence of band projections $(P_n)_{n\in\mathbb{N}_0}$ such that $(I - P_n)f_n \to 0$, in order, as $n \to \infty$, i.e., there is a sequence of measurable sets

$(A_n)_{n\in\mathbb{N}_0}$ with

$$c_n := \frac{1}{n}\sum_{j=0}^{n-1}\chi_{A_j} \to 0 \quad \text{in order as} \quad n \to \infty, \tag{5.2}$$

with $\chi_{\Omega\setminus A_n} f_n \to 0$ in order as $n \to \infty$.

In the above, $0 \le s_n \le g$ and $0 \le c_n \le \mathbf{1}$, for each n, so $(s_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ are order bounded and thus order convergent if and only if they are a.e. pointwise convergent. Further, Lebesgue's Dominated Convergence Theorem is applicable. Thus we have the following.

Corollary 5.1 If $(f_n)_{n\in\mathbb{N}_0}$ is a non-negative sequence in $L^1(\Omega, \mathcal{A}, \mu)$, where μ is a finite measure, and there exists $g \in L^1(\Omega, \mathcal{A}, \mu)$ so that $f_n \le g$ a.e. for all $n \in \mathbb{N}_0$, then $\frac{1}{n}\sum_{j=0}^{n-1} f_j \to 0$ as $n \to \infty$, if and only if there is a sequence of measurable sets $(A_n)_{n\in\mathbb{N}_0}$ with $\frac{1}{n}\sum_{j=0}^{n-1}\chi_{A_j} \to 0$ and $\chi_{\Omega\setminus A_n} f_n \to 0$ as $n \to \infty$. Here, the limits can be taken as either a.e. pointwise or in norm.

Remark 5.2 The comment preceding Corollary 5.1 coupled with Lebesgue's Dominated Convergence Theorem permits the limits in Corollary 5.1 to be taken a.e. pointwise or in norm.

5.2 Conditional Weak Mixing Application

Proceeding as in [20, Section 5], the conditional weak mixing of Section 4.2 can be carried over to $L^1(\Omega, \mathcal{A}, \mu)$ to give a characterisation of conditional weak mixing in measure spaces. As per [20, Section 5], let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space and let $\mu(\Omega) = \infty$. Let $(\Omega_i)_{i\in\mathbb{N}}$ be a μ-measurable partition of Ω into sets of finite positive measure and let $\mathcal{A}_0$ be the sub-σ-algebra of $\mathcal{A}$ generated by $(\Omega_i)_{i\in\mathbb{N}}$. We can then set $E = L^1(\Omega, \mathcal{A}, \mu)$. In this case, E is a Dedekind complete Riesz space. We construct the conditional expectation operator

$$T(f(\omega)) = \frac{1}{\mu(\Omega_i)} \int_{\Omega_i} f \, \mathrm{d}\mu, \text{ for } \omega \in \Omega_i. \tag{5.3}$$

Let $\tau : \Omega \to \Omega$ be measure preserving, that is, $\mu(\tau^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{A}$, then we have that (E, T, S, e) is a conditional expectation preserving system, where $E = L^1(\Omega, \mathcal{A}, \mu)$, T is as constructed, $S : \Omega \to \Omega$ satisfies $Sf = f \circ \tau$ and e is the constant 1 on Ω.

Consider $\mathcal{B}$, a sub-σ-algebra of $\mathcal{A}$ which contains $\mathcal{A}_0$. Then we can consider the conditional expectation $\mathbb{E}_i[f|\mathcal{B}] = \mathbb{E}_i\left[\chi_{\Omega_i}\middle|\mathcal{B}\right]$, the conditional expectation on Ω_i of f restricted to Ω_i, where we use the probability measure

$$\mu_i(A) := \frac{\mu(A \cap \Omega_i)}{\mu(\Omega_i)}$$

and the σ-algebra $\{A \cap \Omega_i | A \in \mathcal{B}\}$.

In the system, band projections are of the form χ_A, where $A \in \mathcal{A}$ and χ_A has been defined as in Section 3.1. We then have that, for $P = \chi_A$ and $Q = \chi_B$, $T(Pe) = \mathbb{E}_i[\chi_A|\mathcal{B}] = \mu_i(A \mid \mathcal{B})$.

Consequently, we have the following condition for weak mixing condition, for the conditional case:

$$\frac{1}{n}\sum_{k=0}^{n-1} \left|\mu_i\left(\tau^k(A) \cap B \mid \mathcal{B}\right) - \mu_i(A \mid \mathcal{B})\,\mu_i(B \mid \mathcal{B})\right| \to 0$$

as $n \to \infty$, for $A, B \in \mathcal{B}$. Furthermore, we have the condition for ergodicity as follows:

$$\frac{1}{n}\sum_{k=0}^{n-1}\mu_i\left(\tau^k\left(A\right)\cap B\mid\mathcal{B}\right)\to\mu_i\left(A\mid\mathcal{B}\right)\mu_i\left(B\mid\mathcal{B}\right)$$

as $n\to\infty$, for $A,B\in\mathcal{B}$.

Chapter 6

Conclusion

The consideration of this thesis concluded with linking weak mixing and ergodicity in Riesz spaces, however, it managed to define conditional versions of these processes. A characterisation of weak mixing by means of the tensor product is presented in texts covering ergodic theory, such as [5, 16, 25, 26]. The Riesz/Fremlin tensor product as developed in [3, 4, 6, 15, 29, 30] requires further work in order to generalise tensor product results from ergodic theory and weak mixing to Riesz spaces. This will be considered in future work.

In texts covering ergodic theory, such as [5, 16, 25, 26], strong mixing processes are studied. Recall from Section 1.1 that a measure preserving system $(\Omega, \mathcal{A}, \mu, \tau)$ with $\mu(\Omega) = 1$ is called strongly mixing if (1.3) holds; i.e., if

$$\lim_{n\to\infty} \left|\mu\left(\tau^{-n}(A) \cap B\right) - \mu(A)\,\mu(B)\right| = 0, \tag{6.1}$$

for all $A, B \in \mathcal{A}$.

The topic of strong mixing was not considered in this thesis. This will be considered in future work. Further to this, the strong mixing processes characterised in [20] are related to strong mixing of stochastic processes. This link between the strong mixing of dynamical systems (defined by (6.1), and the strong mixing of stochastic processes would be an interesting topic for further research.

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