The *Mincut* Graph: Intersection Graph and Graph Operator

by

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DECLARATION

I declare that this Thesis is my own, unaided work. It is being submitted for the Degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

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(Signature of candidate)

20th day of April 2021 in JOHANNESBURG.

The *Mincut* Graph: Intersection Graph and Graph Operator.

Abstract: In this work we introduce and study an intersection graph of a graph G, with vertex set the minimum edge-cuts of G. We start our study of this new graph by finding the minimum cut-set graphs of some well-known families of graphs. We study the minimum cut-set graph, henceforth called a *mincut graph*, as a graph operator, using the well known line graph operator as a model. In doing so we follow the research programme on graph operators, as introduced by Prisner in the 1995 monograph "Graph Dynamics". Thus we answer questions such as 'Which graphs appear as images of graphs?'; 'Which graphs are fixed under the operator?'; 'What happens if the operator is iterated?'. We show that every graph is a mincut graph, and that every graph has an infinite number of roots under the operator. Furthermore, we show that every graph is part of an infinite sequence of mincut graphs under iteration of the operator. We characterise graphs that are fixed under the mincut operator and examine the properties of graphs that have a periodicity higher than one. We show that no graph diverges under iteration of the operator and we conjecture that most graphs converge to the null graph, K_0. Finally we link the two fields of connectivity and graph colouring by using the bad colouring polynomial to count the number of mincuts of a graph, that is, the order of the mincut graph of a graph.

— C.W. Kriel

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Chapter 1

Introduction

In this chapter we give a brief background to graph theory and connectivity problems as well as graph operators, specifically the line graph operator, and intersection graphs. These are the general areas in graph theory under which the problems discussed in this thesis fall. Then we give a motivation for our problem of study. The line graph, both as graph operator and intersection graph, guides our study of the *mincut graph* introduced in this thesis. We follow this with an overview of the thesis, some basic definitions and then a detailed introduction to graph operators and intersection graphs.

1.1 Background

The subject matter of this thesis falls, broadly speaking, under the umbrella of connectivity problems. The short introduction to graph theory, connectivity, and line and intersection graphs that follows, was compiled from [6, 16, 23, 29, 32, 33].

In the late eighteenth century the city of Königsberg, in Prussia, was built on two sides of the river Pregel and two islands in the river. Seven bridges connected the different parts of the city as shown in Figure 1.1.

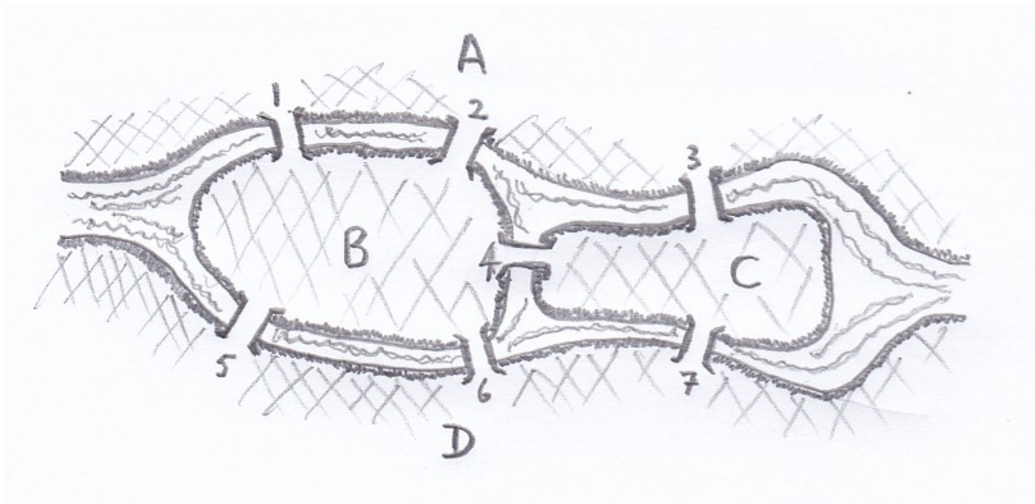

Figure 1.1: Eighteenth century Königsberg and its seven bridges over the river Pregel.

It was a well known problem amongst the citizens to determine whether it is possible to walk across each of the seven bridges exactly once and end up where you started from. The problem came to the attention of Leonhard Euler who proved that it was impossible. In the process he started a new branch of geometry where the size of objects and the distance between them do not matter, only their position in relation to each other. Today we call this *graph theory*.

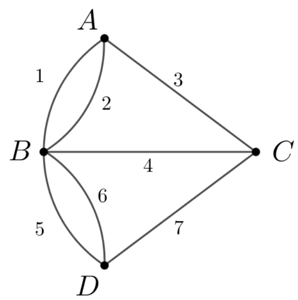

Figure 1.2: A graph G, representing Königsberg and its seven bridges.

Although Euler did not actually use a graph in his proof, we would today model

Königsberg with a graph as shown in Figure 1.2. The different areas of the city are represented as dots (vertices) and the connections between them, the bridges, as lines (edges). In order for it to be possible to cross every bridge exactly once and end up where you started, every vertex should have an even number of edges incident on it – every time you "enter" a vertex along an edge you must be able to "leave" along a different edge. An odd number of edges means that eventually you will visit a vertex along an edge and not be able to exit on an unused edge. Today we call this a problem of finding an *Eulerian cycle* on the graph.

Now, suppose a fire breaks out on island B. Luckily everyone escapes to one of the other parts of the city. Is it possible to reunite families who escaped to different parts? Yes, as there is still at least one path connecting areas A, C and D. However, should the fire spread to island C, the bank at A will not be accessible from D and vice versa. Today we say that removing the vertices B and C and their incident edges from the graph G will disconnect the graph and the *vertex connectivity* of G is two.

Similarly, if the Pregel should flood and wash away any two of the bridges, the citizens would still be able to access the different parts of the city. However, if bridges 1, 2 and 3 are all washed away, bank A would be inaccessible from the other parts of the city. Today we say the minimum number of edges that need to be removed in order to disconnect the graph, the *edge connectivity* of G, is three and the set of edges $\{1, 2, 3\}$ is a minimum edge-cut.

In today's connected world, graphs are used to model various types of networks: power and electricity grids; data and communication networks; water and the pipe network supplying various parts of a city; the way parts flow through a production facility. The robustness of the network becomes very important – how easy is it to disconnect one part of the network from another and can the network still continue to function?

The *line graph* was introduced by Whitney (1935) and Krausz (1943). The line graph, $L(G)$, of a graph G has a vertex corresponding to every edge of G and two vertices in $L(G)$ are connected if the two edges in G are incident upon the same vertex. Thus the line graph of a graph G reflects the mutual positions of the edges, see [33].

A graph operator is a mapping ϕ which maps every graph G from some class of graphs to a new graph $\phi(G)$. The *operator* L maps every graph G to its line graph $L(G)$.

The line graph operator was the first to receive wide attention, starting in 1962 with Ore [30]. Since that time numerous papers have appeared on "graph operators" or "graph-valued functions" such as the line graph, the clique graph, the complement, and powers, see [32].

The line graph of a graph can also be characterised in terms of *intersection graphs*. Suppose we have a family of sets $\mathcal{F}$. If we represent each set with a vertex and join two vertices if the intersection of their respective sets is non-empty, then we have the intersection graph, $\Omega(\mathcal{F})$, of the family of sets $\mathcal{F}$.

In 1945 Szpilrajn-Marczewski proved that every graph is an intersection graph, see [12]. As with graph operators, one of the first class of intersection graphs to be widely studied was the line graph. In the 1970's chordal graphs were first characterised in terms of intersection graphs, while other intersection graphs that are studied intensively are interval and circular-arc graphs, competition graphs, p-intersection and tolerance graphs, to name but a few, see [15, 23, 31]. Problems involving intersection graphs often have real world applications in topics like biology, computing, matrix analysis and statistics, see [15, 17, 23].

Vertex- and edge-connectivity are two basic and very important measures of the connectedness of a graph. In this thesis we will investigate the connectedness of a graph by looking at the mutual positions of the sets of minimum edge-cuts. Hence, we introduce the *mincut graph*, $X(G)$ of a graph G. The minimum edge-cuts of G are the vertices of $X(G)$ and two such vertices are adjacent whenever the corresponding edge sets have non-empty intersection. Thus the mincut graph reflects the mutual positions of the minimum edge-cuts of a graph G. We will use the study of the line graph and line graph operator as a template to start our study of the mincut graph. Furthermore, we characterise the mincut graph of a graph G in terms of intersection graphs. This brings new questions to the fore and new properties of the mincut graph that can be studied.

1.2 Motivation

Networks of various types play an increasingly important role in the modern world. Graphs are used to model, for example, data and communication networks such as the internet; the way users of social media networks interact; the way that people's interactions facilitate or impede the spread of viruses; power and electricity grids; water and the pipe network supplying various parts of a city; the way parts flow through a production facility. How easy it is to disconnect one part of the network from another becomes an important issue, as well as whether the network can still continue to function if certain parts become disconnected.

Two important parameters that give us information on the connectedness of a graph are its vertex and edge connectivity numbers, that is, the minimum number of vertices or edges, respectively, that need to be removed before the graph becomes disconnected. In this thesis we introduce an intersection graph of a graph G, called a *mincut graph*, with vertex set the minimum edge-cuts of G and vertices adjacent whenever the corresponding edge sets have non-empty intersection. As the line graph of a graph G reflects the mutual positions of the edges, so the mincut graph reflects the mutual positions of the minimum edge-cuts. It is our hope that this new structure can be helpful in examining the effect of certain graph parameters on connectivity and possibly also lead to the identification of some new connectivity parameters.

Our study of the mincut intersection graph is further motivated by the fact that every graph is an intersection graph, that is, a graph with vertices v_i corresponding to each of the subsets S_i of a family of subsets of a set S, and two vertices v_i and v_j are adjacent if $S_i \cap S_j \neq \emptyset$. One of the first class of intersection graphs to be widely studied was the line graph.

The line graph *operator* was also the first graph operator to receive wide-spread attention, starting in 1962 with Ore. Since that time numerous papers have appeared on "graph operators" or "graph-valued functions". Thus we are motivated to study the mincut graph as a graph operator, using the study of the line graph operator as a model.

1.3 Overview

In Chapter 2 we define a new intersection graph, the *mincut graph* of a graph G, with vertex set the minimum edge-cuts of G and edges between those vertices whose corresponding edge-cuts have non-empty intersection. We show that the mincut graph of a graph is unique and derive the mincut graphs of some well-known families of graphs, including Trees, Cycles, Wheels, Complete graphs, Line graphs of Complete graphs, and Bipartite graphs. The null graph, K_0, the graph with no vertices and no edges, is an important graph also included here that, although not often encountered in graph theory texts, plays an important part in the theory of the mincut graph operator. We note that in some texts this is called the empty graph, but in this thesis we will call the graph on n vertices and no edges the empty graph on n vertices. The properties of the families of graphs and their mincut graphs, studied in this chapter, lead us to introduce and study the properties of the mincut *operator* in Chapters 3 and 4.

In Chapter 3 we introduce and study the properties of the mincut *operator*, as per the research program introduced by Prisner in [32], where the study of graph operators is unified under such questions as ‘Which graphs appear as images of graphs?’; ‘Which graphs are fixed under the operator?’; ‘What happens if the operator is iterated?’. In this chapter we address the first two questions:

1. When is a graph a mincut graph?
2. When is a graph isomorphic to its mincut graph?

Specifically we show, in Section 3.2, that, unlike the line graph, every graph is a mincut graph, that is, every graph lies in the image of the mincut operator. We start Section 3.3 by examining the structure of minimum edge cuts of a graph in more detail, and introduce the important concepts of nested and crossing minimum edge cuts, as well as vertices of attachment. Building on these ideas, we show that, unless G is the null graph with no vertices or edges, the property of being super-λ and r-regular is both necessary and sufficient for a graph to be fixed under the operator. In other words, a graph is isomorphic to its mincut graph if, and only if, it is both super-λ and r-regular.

Chapter 4 addresses Prisner’s third question: “What happens when the mincut operator is applied repeatedly to a graph?” If a graph is fixed under the operator then iteration of the operator just yields the same graph. We characterise such graphs for the X-operator in Chapter 3. But in general, when does a graph “grow bigger” or “smaller” or re-appear after more than one iteration? Is there a unique graph that will yield the graph after a number of iterations? These questions concern divergence, convergence, periodicity, roots and depth of a graph under the operator. In Section

4.2 we show that every graph not only has an infinite number of X-roots, but also has infinite depth under the X-operator, $X(\cdot)$. That is, for every graph G we can find a graph H such that G is the mincut graph of H. In Section 4.3, we show that no graph diverges and we make the conjecture that most graphs converge to the null graph, K_0, with no vertices or edges, after repeated iteration of $X(\cdot)$. Furthermore, in Section 4.4, we identify two classes of graphs that are 2-periodic under the X-operator, in other words, graphs where the original graph reappears after two iterations of the operator, or, $X^2(G) \cong G$.

In Chapter 5 we make explicit a relationship between two central concepts in the areas of graph colouring and graph connectivity, namely, improper colouring and the minimum edge-cut number of a graph. We show that the number of minimum edge-cuts, and the number of sets of bad edges in an improper 2-colouring of a graph with $|E| - \lambda$ bad edges are the same. This equivalence can be used to study aspects of improper colouring by using minimum edge cuts, as well as minimum edge cuts by using concepts from the study of improper colouring of graphs. For example, we use this equivalence, in Section 5.4, to prove Lemma 5.4.1 on values of k for which the k-defect polynomials of a graph are zero. By exploiting the equivalence further, we are also able to determine the number of mincuts of a graph, that is, the order of its mincut graph, through an evaluation of the *bad colouring polynomial*, itself an evaluation of the well-known Tutte Polynomial.

In Chapter 6 we conclude the thesis and we outline some further questions on the mincut operator that still need to be addressed. These fall under the areas of determining what we can conclude about a graph given the mincut graph, and characteristics of a graph that will ensure a connected mincut graph. Furthermore, we conjecture that most graphs will converge to the null graph. We also raise a question related to the intersection number of a graph, the minimum cardinality of a set S such that G is the intersection graph of a family of subsets of S, and ask whether we can define a new graph invariant called the mincut intersection number of a graph.

1.4 Basic definitions

In this section we start by defining a graph G, the main structure of investigation, and some of its properties and related concepts which are useful to this work. For basic definitions and properties of graphs we will follow the details and notation as given in [6].

Definition 1.4.1. A *graph* G is a finite nonempty set V of objects called vertices together with a possibly empty set E of 2-element subsets of vertices called edges.

Graphically, we represent the vertices as dots or points and the edges are the lines that join them. That is, if two vertices are in the same 2-element subset, then the dots representing them are joined by a line. We refer to two such vertices as the *end points* of the edge. We say that these vertices are *adjacent* and will call two adjacent vertices *neighbours* of each other. The set of all vertices adjacent to a vertex v in a graph G is called the *neighbourhood of* v in G and is denoted $N_G(v)$, or simply $N(v)$ if the graph is understood to be G.

If a vertex v is the endpoint of an edge e we say that v is *incident* on e. If $E(G)$ is a multiset, a set containing more than one copy of a 2-element subset, we have multiple edges between vertices and we call such edges *parallel edges*. If the 2-element subsets themselves are multisets, that is, we allow an element v to be repeated in the subset, $\{v, v\}$, then we have an edge that has the same vertex as endpoints. An edge with the same vertex, v, as endpoints is called a *loop*. In this work we will deal only with *simple* graphs, that is, we allow only one edge between two vertices, and an edge must be incident on two vertices, that is, it cannot be a loop.

The *order* of a graph G is the number of vertices of G, denoted $|V(G)|$, and the *size* of a graph G is the number of edges of G, denoted $|E(G)|$. The degree of a vertex v, denoted $deg(v)$, is the number of edges incident on it. If every vertex of a graph has the same degree, r, we say the graph is r-regular.

Definition 1.4.2. A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In this case we write $H \subseteq G$.

For a nonempty subset $S \subset V(G)$ of vertices of G, the *vertex induced subgraph*, $G[S]$, has S as its vertex set and $u, v \in S$ are adjacent if and only if they are adjacent in G. For a nonempty subset $X \subset E(G)$ of edges of G, the *edge induced subgraph*, $G[X]$, has X as its edge set and a vertex $v \in V(G)$ belongs to $G[X]$ if v is incident with at least one edge in X.

In this work we will regularly encounter certain *classes* of graphs, that is, graphs that share certain properties by definition. Some of the classes that we will encounter again in Chapter 2 are complete graphs, K_n, cycles, C_n, trees, T_n, and wheels, W_n, where n is the number of vertices of the graph. The complete graph on n vertices has every vertex adjacent to every other vertex and hence is $(n-1)$-regular. A cycle is a 2-regular graph on n vertices. That is, every vertex is of degree two. The vertices of a cycle can be labelled $v_1, \ldots, v_n$ with edges $v_n v_1$ and $v_i v_{i+1}$ for $i \in \{1, 2, \ldots, n-1\}$. A tree is a graph with no cycles, that is, a graph on n vertices with $n-1$ edges. The wheel on n vertices is a cycle on $n-1$ vertices (the "rim") to which is added an n-th vertex (the "hub") that is joined to each of the vertices on the rim by an edge (the "spokes").

Example 1.4.3. The graphs shown in Figure 1.3 are examples of some of the well known classes of graphs. They are, respectively, a tree, a complete graph, a cycle and a wheel. We note that the star in the figure is not the only example of a tree on six vertices. Any connected *acyclic* graph on n vertices is an example of a tree.

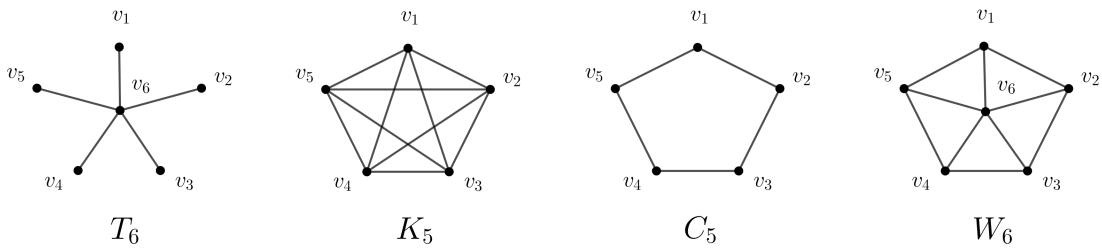

Figure 1.3: T_6, K_5, C_5 and W_6.

The graphs in the other three classes are uniquely determined, up to isomorphism, in their class by the number of their vertices.

Definition 1.4.4. Two graphs, G and H, are *isomorphic* if there is a bijective function $\psi : V(G) \to V(H)$, such that two vertices, u and v, are adjacent in G, if and only if $\psi(u)$ and $\psi(v)$ are adjacent in H. We denote the isomorphism of the two graphs by $G \cong H$.

Definition 1.4.5. The *complement* , $\overline{G}$, of a graph G is that graph with vertex set $V(G)$ and $v_iv_j \in E(\overline{G})$ if and only if $v_iv_j \notin E(G)$.

Since $v_iv_j \in E(K_n)$ for all $v \in V(K_n)$, we have the complement of K_n, that is $\overline{K}_n$, being the empty graph on n vertices.

1.5 Connectivity

A *u-v path* P in a graph G is a sequence of non-repeated vertices in G, starting at u and ending at v, such that consecutive vertices are adjacent in P. Two vertices $u, v \in V(G)$ are *connected* in G if there exists a u-v path in G.

Definition 1.5.1. A graph G is *connected* if there exists a u-v path for every two vertices u and v of G. If G is not connected, it is a *disconnected graph*.

A *component* of a graph G is a connected subgraph H such that no subgraph of G that properly contains H is connected. In this work the number of components in a graph G will be denoted by $c(G)$ and hence G is connected if and only if $c(G) = 1$, see [6].

The operation of *deleting a vertex* u from a graph G not only removes the vertex u but also removes every edge with u as an endpoint. The resulting graph is denoted $G - u$ or $G\backslash u$. The operation of *deleting an edge* e from a graph G removes only that edge. The resulting graph is denoted $G - e$ or $G\backslash e$.

A *vertex-cut* is a set S of vertices of G such that $G - S$ (or $G\backslash S$) is a disconnected graph. A vertex-cut of minimum cardinality is called a *minimum vertex-cut*. The cardinality of a minimum vertex-cut is called the *connectivity* of G and is denoted $\kappa(G)$.

Definition 1.5.2. Let G be a simple, connected graph. An *edge-cut* is a subset, X, of $E(G)$ such that $G - X$ (or $G\backslash X$) is a disconnected graph. An edge-cut of minimum cardinality is called a *minimum edge-cut*. The cardinality of a minimum edge-cut is called the *edge-connectivity* of G and is denoted $\lambda(G)$.

The maximum number of vertices adjacent to a vertex in a graph of order n is $n - 1$. Therefore the maximum number of edges needed to be removed in order to disconnect G is $n - 1$, disconnecting a single vertex from the graph. The trivial graph K_1 does not have an edge-cut but we define $\lambda(K_1) = 0$. Hence we have the following inequality:

$$0 \leq \lambda(G) \leq n - 1,$$

and we note that $\lambda(G) = 0$ if $G \cong K_1$ or G is disconnected.

Example 1.5.3. The diagram shown in Figure 1.4 is an example of a connected graph G, and two disconnected graphs G_1 and G_2, with $c(G_1) = c(G_2) = 2$.

In graph G, the vertex set $\{v_1, v_3, v_4\}$ is a vertex-cut while $\{v_1, v_3\}$ and $\{v_4, v_5\}$ are two minimum vertex cuts. Similarly, $\{e_1, e_2, e_3\}$ is an edge-cut while $\{e_1, e_2\}$ and

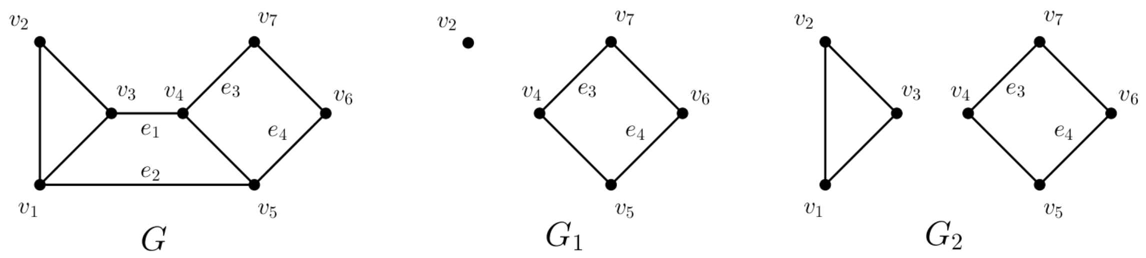

Figure 1.4: A graph G with $c(G_1) = 2$ and $c(G_2) = 2$.

$\{e_3, e_4\}$ are minimum edge-cuts. G_1 is the disconnected graph $G - \{v_1, v_3\}$, while G_2 is the disconnected graph $G - \{e_1, e_2\}$. Graph G has equal (vertex) connectivity and edge connectivity, $\kappa(G) = \lambda(G) = 2$.

We state the following important fact about minimum edge-cuts as a lemma. What it says, is that deleting a minimum edge-cut from a graph results in exactly two components, as we saw in graph G_2 in Example 1.5.3.

Lemma 1.5.4. *Let G be a simple connected graph and $X \in E(G)$ a minimum edge-cut. Then*

$$c(G \backslash X) = 2.$$

Proof. Let X be a minimum edge-cut of G and suppose $c(G \backslash X) > 2$. Without loss of generality we let $c(G \backslash X) = 3$ with disjoint components A, B and C. The cardinality of X is λ, $|X| = \lambda$, and X can be characterised as the union of the edge sets between the different components, $X = \langle A, B \rangle \cup \langle A, C \rangle \cup \langle B, C \rangle$.

Let $X_1 = \langle A, B \rangle \cup \langle A, C \rangle$ be the edge sets between components A and B and components A and C respectively, then $|X_1| < |X|$. But $G \backslash X_1$ is disconnected. Deleting X_1 from $E(G)$ disconnects component A from G. Therefore X cannot be a minimum edge-cut. $\square$

The following theorem which relates the connectivity, edge-connectivity and minimum vertex degree of a graph, is well known in the literature, see [6, 16].

Theorem 1.5.5 (Whitney)**.** *Let G be a graph with connectivity $\kappa(G)$, edge-connectivity $\lambda(G)$ and minimum vertex degree $\delta(G)$. Then*

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

It is known that the maximum number of minimum edge-cuts has different order of magnitude for graphs with odd and even edge-connectivity, see [19]. For example, let G be a graph with minimum edge-cut number λ and n vertices, with $X = \{X_1, X_2, \ldots, X_i\}$ the set of minimum edge-cuts of G, then the following upper bounds for $|X|$, the number of minimum edge-cuts, are known, see [5, 21].

$$|X| \leq \begin{cases} \frac{2n^2}{(\lambda+1)^2} + \frac{(\lambda-1)n}{\lambda+1}, & \text{if } \lambda \geq 4 \text{ and } \lambda \text{ is an even integer,} \\ (1 + \frac{4}{\lambda+5})n, & \text{if } \lambda > 5 \text{ and } \lambda \text{ is an odd integer.} \end{cases}$$

1.5.1 Super edge-connected graphs

For more on the following definitions and proposition see [16].

Definition 1.5.6. A graph G is *maximally edge connected* if $\lambda = \delta$, where λ is the cardinality of the minimum edge-cut and δ is the minimum vertex degree of G.

Definition 1.5.7. A minimum edge-cut, X_i, is called *trivial* if it consists of the edges incident on a vertex v of minimum degree. That is, $G - X_i$ is the graph with two components $G - v$ and v.

Definition 1.5.8. A maximally edge-connected graph is *super-λ* if every minimum edge-cut set is *trivial.*

Example 1.5.9. The Peterson graph, shown in Figure 1.5, is a 3-regular or *cubic* graph. The set of dotted edges $\{e_1, \ldots, e_5\}$ is an edge-cut set. However, it is not a minimum edge-cut since the edge connectivity of G, $\lambda(G) = 3$. The only minimum edge-cuts are the trivial edge-cut sets incident on each vertex. We say the Peterson graph is 3-regular and super-λ.

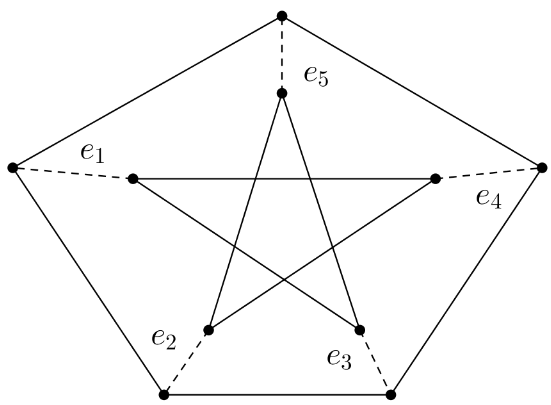

Figure 1.5: The Peterson graph is 3-regular and super-λ.

The following proposition states five sufficient conditions for a graph to be super-λ, see [16]. The first two conditions were proved by Lesniak in 1974, see [22].

Proposition 1.5.10. *Let G be a graph of order n, minimum degree δ and maximum degree Δ. Then G is super-λ if any of the following conditions are satisfied:*

1. *Let $G \neq K_{n/2} \times K_2$. If for any non-adjacent $u, v \in V(G)$, $deg(u) + deg(v) \geq n$, then G is super-λ.*
2. *If for any non-adjacent $u, v \in V(G)$, $deg(u) + deg(v) \geq n+1$, then G is super-λ. This is a corollary to the preceding condition.*
3. *If $\delta \geq \lfloor \frac{n}{2} \rfloor + 1$, then G is super-λ.*

4. If G has diameter 2 and contains no complete subgraph H on δ vertices with $deg_G v = \delta$ for all $v \in V(H)$, then G is super-λ.

5. If G has diameter 2 and $n > 2\delta + \Delta - 1$, then G is super-λ.

Example 1.5.11. The Peterson graph in Figure 1.5 is an illustration of Proposition 1.5.10, condition 4. In a graph G, the distance, $d(u,v)$, between two vertices is the length of the shortest path between u and v. The diameter of a graph is the maximum of the distances between any two vertices in G. The diameter of the Peterson graph is two (the distance between any two vertices is either one or two) and $\delta = 3$. A complete subgraph on three vertices such that the vertices are of degree three in G implies that we should have at least one triangle, K_3, in the graph. But the graph is triangle free. Thus we conclude that the Peterson graph is super-λ by condition 4. Similarly, we have $|V(G)| = n = 10$ and $\delta = \Delta = 3$ giving $n > 2\delta + \Delta - 1$ and hence the Peterson graph also satisfies condition 5. We note, however, that the conditions in Proposition 1.5.10 are sufficient only — the Peterson graph satisfies only conditions 4 and 5.

Example 1.5.12. In a complete graph, K_n, every vertex is adjacent to every other vertex and we have $\delta(K_n) = n - 1$. For $n \geq 3$, complete graphs on n vertices satisfy condition 3, $\delta = n - 1 \geq \lfloor \frac{n}{2} \rfloor + 1$, and we conclude that complete graphs on $n \geq 3$ are super-λ. K_2 satisfies none of the conditions, but is clearly super-λ.

In [18], Harary defines the *conditional edge-connectivity*, $\lambda(G : P)$ of a graph G to be the minimum cardinality of a set S of edges, such that $G - S$ is disconnected and every component of $G - S$ has property P. Hence, if we define P to be the property that a graph has more than one vertex, the minimum cuts will be nontrivial, since the components of G will all have more than one vertex. A graph where none of the minimum cuts are trivial will then not be super-λ and the inequality on λ and δ will be strict.

1.6 Graph operators

We recall the following definition from Section 1.1.

Definition 1.6.1. A *graph operator* is a mapping ϕ which maps every graph G from some class of graphs to a new graph $\phi(G)$.

We define the operator recursively, such that, if ϕ is an operator and k a positive integer, then $\phi^1 = \phi$ and $\phi^k(G) = \phi(\phi^{k-1}(G))$, for $k \geq 2$, see [40].

As we saw in Section 1.1, the line graph was introduced by Whitney (1935) and Krausz (1943). The line graph operator L maps every graph G to its line graph $L(G)$. $L(G)$ has a vertex corresponding to every edge of G and two vertices in $L(G)$ are connected if the two edges in G are incident upon the same vertex. We demonstrate the concept of a line graph operator in Figure 1.6.

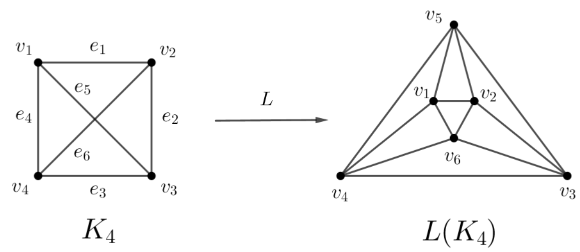

Figure 1.6: K_4 and $L(K_4)$.

Given the complete graph on four vertices, K_4, L maps each of the edges e_i, $i \in \{1, 2, \ldots, 6\}$ of K_4 to a vertex v_i of $L(K_4)$. Thus, for example, $e_1 \to v_1$ and $e_2 \to v_2$. Now both e_1 and e_2 are incident on vertex v_2 in K_4 and hence they are joined by an edge in $L(K_4)$.

In 1962 Ore raised several questions around this graph operator, see [30]. When is $L(G)$ isomorphic to G? When the line graph is given, is the original graph G uniquely determined? How does one describe the sequence of iterated line graphs of a graph? These questions were addressed, amongst others, by van Rooij and Wilf in 1965, [37], where the line graph was called the *interchange graph* of a graph, and Beineke in 1970, [2].

Some of their answers to Ore's questions were that:

1. Graphs isomorphic to their line graphs are all cyclic.

2. A graph is isomorphic to the line graph of some graph unless it contains an induced subgraph isomorphic to one of nine graphs, see Figure 3.1.

3. Repeated application of L leads to steadily increasing numbers of vertices unless a finite number of conditions apply.

Since that time numerous papers have appeared on "graph operators" or "graph-valued functions" such as the line graph, the clique graph, the complement, and powers, see [32]. Also, new concepts were brought into play such as *periodicity*.

Definition 1.6.2. Let ϕ be an operator on a class of graphs Γ. A graph $G \in \Gamma$ is *periodic* in ϕ if there is some natural number n with $G \cong \phi^n(G)$. The smallest such number is called the *period* of G.

In [32], Prisner generalises Ore's questions to the field of graph operators and unifies their study under such questions as 'Which graphs are fixed under the operator?'; 'Which graphs appear as images of graphs?'; 'What happens if the operator is iterated?'

1.7 Intersection graphs

Definition 1.7.1. Let $\mathcal{F} = \{S_1, S_2, \ldots, S_n\}$ be any family of sets. The *intersection graph* of $\mathcal{F}$, denoted $\Omega(\mathcal{F})$ is the graph having $\mathcal{F}$ as vertex set with S_i adjacent to S_j if $S_i \cap S_j \neq \emptyset$. A graph, G, is an *intersection graph* if there exists a family of sets $\mathcal{F}$ such that $G \cong \Omega(\mathcal{F})$.

In 1945 Szpilrajn-Marczewski proved that every graph is an intersection graph, see [12]. As with graph operators, one of the first class of intersection graphs to be widely studied was the line graph, as outlined in Section 1.6. Let $E = \{e_1, \ldots, e_m\}$ be the edge set of a graph G. We can write each edge as a vertex pair of its endpoints, $\{v_i, v_j\}$, and we get $\mathcal{F} = \{\{v_i, v_j\}$ such that $v_i v_j \in E(G)\}$. Hence, by Definition 1.7.1, we see that $\Omega(\mathcal{F}) \cong L(G)$.

Example 1.7.2. Let $\mathcal{F} = \{\{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}, \{2,4\}\}$, then $\Omega(\mathcal{F})$, the intersection graph of $\mathcal{F}$, is as shown in Figure 1.7. If we consider the graph G with vertex set $V(G) = \{1,2,3,4\}$ and edge set $E(G) = \{\{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}, \{2,4\}\}$, we note that $\mathcal{F} = E(G)$. Since the line graph $L(G)$ has edges of G (vertex pairs) as vertices, with vertices adjacent if their corresponding edges are incident upon a common vertex in G (the two vertex-pair sets intersect) we see that $L(G) \cong \Omega(\mathcal{F})$ and hence is an intersection graph.

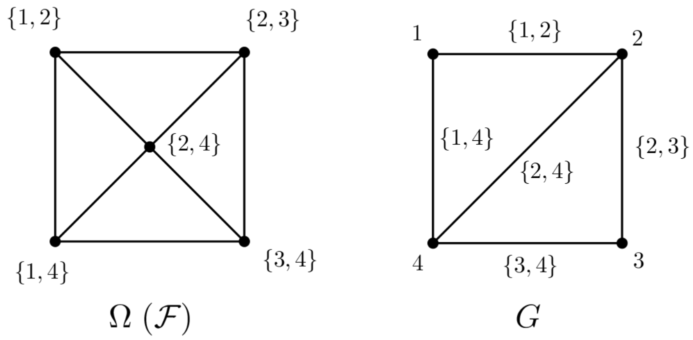

Figure 1.7: G and $\Omega(\mathcal{F}) \cong L(G)$.

Line graphs were generalised as (X, Y)-intersection graphs by Cai *et al* in 1996 as follows, see [4].

Definition 1.7.3. Given a pair (X, Y) of fixed graphs X and Y, the (X, Y)-intersection graph of a graph G, denoted $I_{X,Y}$, is a graph in which:

1. each vertex corresponds to a distinct induced subgraph of G that is isomorphic to Y and
2. two vertices are adjacent if and only if the intersection of their corresponding subgraphs contains an induced subgraph isomorphic to X.

Given Definition 1.7.3, we see that the line graph as defined in Section 1.6 is a special case of the general definition. Every edge in G induces a subgraph of G isomorphic to K_2. If two edges are incident on a shared vertex then the intersection of their corresponding induced K_2 graphs is isomorphic to K_1. Hence, we see that the line graph of G is the (K_1, K_2)-intersection graph of G.

In the 1970's chordal graphs were first characterised in terms of intersection graphs, while other intersection graphs that are studied intensively are interval and circular-arc graphs, competition graphs, p-intersection and tolerance graphs, to name but a few, see [15, 23, 31]. As noted before, problems involving intersection graphs often have real world applications in topics like biology, computing, matrix analysis and statistics, see [15, 17, 23].

Chapter 2

The mincut graph

2.1 Introduction

In this chapter we start by defining a new intersection graph of a graph, the *mincut graph* of a graph G. We then derive the *mincut* graphs of some well-known families of graphs. The properties of these families of graphs and their mincut graphs lead us to introduce and study the properties of the mincut *operator* in Chapters 3 and 4, as per the research program introduced by Prisner in [32], see Section 1.6.

We recall from Definition 1.5.2, that an *edge-cut* is a subset, X, of $E(G)$ such that $G - X$ is a disconnected graph. An edge-cut of minimum cardinality is called a *minimum edge-cut*. In this thesis we will from now on refer to a minimum edge-cut as a *mincut*.

Definition 2.1.1. Let $X = \{X_1, X_2, \ldots, X_i\}$ be the set of all mincuts of a simple connected graph G. Represent each of the X_i with a vertex v_i such that two vertices v_i and v_j are adjacent if $X_i \cap X_j \neq \emptyset$, and call this intersection graph the *mincut graph* of G, denoted by $X(G)$.

Lemma 2.1.2. *Let G be a graph and $\lambda(G) = 1$. Then $X(G)$ is disconnected.*

Proof. If $\lambda = 1$ then each mincut set is a singleton set and hence has empty intersection with any other mincut set. Therefore, no vertex $v \in V(X(G))$ will be adjacent to any other vertex in the vertex set of $X(G)$. $\square$

Lemma 2.1.3. *Let G be a disconnected graph. Then $X(G)$ is the null graph , K_0, with no vertices or edges.*

Proof. Since G is disconnected, $\lambda(G) = 0$ and G has no edge-cut. Hence $X(G)$ has no vertices and consequently no edges. $\square$

Lemma 2.1.4. *Let $G \cong K_0$. Then $X(G) \cong K_0$.*

Proof. Since K_0 has no edges, it has no mincuts. Therefore, the mincut graph of K_0 has no vertices and, hence, no edges. $\square$

Theorem 2.1.5. *Let G be a connected graph with n vertices and $\lambda(G) = k$, $k \geq 1$. Then $X(G)$ is unique.*

Proof. Since $k \geq 1$ we have to remove at least one edge to disconnect the graph. Thus we know there is at least one edge-cut and hence at least one mincut. Thus, by Definition 2.1.1, $X(G)$ exists and has at least one vertex.

Let $X = \{X_1, X_2, \ldots, X_i\}$ be the set of all mincuts of a simple connected graph G. Since $k \geq 1$ we know that X is non-empty. Let $X'(G)$ and $X''(G)$ be two mincut graphs of G. By the definition of a mincut graph we have $V(X'(G)) = V(X''(G))$ corresponding to X. Since two vertices in the mincut graph of G are adjacent if their corresponding mincuts have non-empty intersection, we must have $v_iv_j \in E(X'(G))$ if and only if $v_iv_j \in E(X''(G))$ for all $v_i, v_j \in V(X'(G))$ and $v_i, v_j \in V(X''(G))$. Therefore $X'(G) \cong X''(G)$ and we conclude that $X(G)$ is unique. $\square$

2.2 Mincut graphs of some families of graphs

In this section we characterise the mincut graphs of some of the well-known graph families described in Example 1.4.3 and the line graph described in Section 1.6. We recall Definition 1.4.5 of the empty graph on n vertices, $\overline{K}_n$, with no edges. Some of the results in this section are that $X(T_n) \cong \overline{K}_{n-1}$, $X(K_n) \cong K_n$, $X(K_{n,n}) \cong K_{n,n}$, $X(K_{m,n}) \cong \overline{K}_n$ for $n > m$, $X(L(K_n)) \cong L(K_n)$, $X(C_n) \cong L(K_n)$, and $X(W_n) \cong C_{n-1}$.

Proposition 2.2.1. *Let T_n be a tree on n vertices. Then the mincut graph of T_n*

$$X(T_n) \cong \overline{K}_{n-1}.$$

Proof. Every edge of T_n is a bridge and the edge connectivity of T_n is $\lambda(T_n) = 1$. Each of the $n-1$ edges of T_n is a mincut and hence $X(T_n)$ has $n-1$ vertices. But none of the singleton mincuts intersect and thus $X(T_n)$ is the empty graph on $n-1$ vertices. □

We recall from Definition 1.5.8, that a maximally connected graph G, that is $\lambda(G) = \delta(G)$, is called *super-λ* if every mincut is trivial, in other words, consists only of the edges incident on a single vertex. Also, from Proposition 1.5.10, we know that a graph is super-λ if $\delta \geq \lfloor \frac{n}{2} \rfloor + 1$.

Lemma 2.2.2. *Let $n \in \mathbb{N}$. Then $n - 1 \geq \lfloor \frac{n}{2} \rfloor + 1$ for $n \geq 3$.*

Proof. We proceed by induction on n.

For $n = 3$, $3 - 1 = 2$ and $\lfloor \frac{3}{2} \rfloor + 1 = 1 + 1 = 2$.

Suppose $k - 1 \geq \lfloor \frac{k}{2} \rfloor + 1$.

Now, $\lfloor \frac{k+1}{2} \rfloor = \lfloor \frac{k}{2} + \frac{1}{2} \rfloor$ and $\frac{k}{2}$ is an integer for even k. Hence, by the definition of the floor function, $\lfloor \frac{k+1}{2} \rfloor = \lfloor \frac{k}{2} \rfloor$ for even k and $\lfloor \frac{k+1}{2} \rfloor = \lfloor \frac{k}{2} \rfloor + 1$ for odd k.

Hence, for even k we have

$$\begin{aligned} k > k - 1 &\geq \lfloor \frac{k}{2} \rfloor + 1 \\ &= \lfloor \frac{k+1}{2} \rfloor + 1, \end{aligned}$$

and for odd k

$$\begin{aligned} k-1 \geq \lfloor \frac{k}{2} \rfloor + 1 \implies k &\geq \lfloor \frac{k}{2} \rfloor + 2 \\ &= \lfloor \frac{k+1}{2} \rfloor + 1. \end{aligned}$$

□

Proposition 2.2.3. *Let K_n be a complete graph on n vertices, $n > 2$. Then the mincut graph of K_n*

$$X(K_n) \cong K_n.$$

Proof. We exclude K_2, since $K_2 \cong T_2$ and $X(T_2) \cong K_1$, by Proposition 2.2.1.

In a complete graph, K_n, each of the n vertices is adjacent to every other vertex. Hence K_n is $(n-1)$-regular and $\delta = n - 1$.

By Lemma 2.2.2 we have $\delta = n - 1 \geq \lfloor \frac{n}{2} \rfloor + 1$ for $n \geq 3$, so K_n is super-λ, and hence all mincuts are trivial. But K_n is $(n-1)$-regular, which implies that all trivial cuts are mincuts. So we have $|V(X(K_n))| = |V(K_n)|$.

Every two vertices in K_n are adjacent and hence their incident edge-sets (all of them are mincuts) intersect. Therefore every mincut of K_n intersects every other mincut of K_n and we can conclude that

$$v_i v_j \in E(K_n) \implies X_i \cap X_j \neq \emptyset \implies v_i v_j \in E(X(K_n)).$$

Therefore $X(K_n) \cong K_n$. □

Definition 2.2.4. A graph G is *bipartite* if $V(G)$ can be partitioned into two sets U and W such that every edge $e \in E(G)$ joins a vertex in U to a vertex in W. A graph G is a *complete* bipartite graph if $V(G)$ can be partitioned into two sets U and W so that $uw \in E(G)$ if and only if $u \in U$ and $w \in W$. We denote the complete bipartite graph $K_{m,n}$ where $|U| = m$ and $|W| = n$.

Example 2.2.5. The graphs in Figure 2.1 are $K_{1,3}$, $K_{2,4}$ and $K_{3,3}$ respectively.

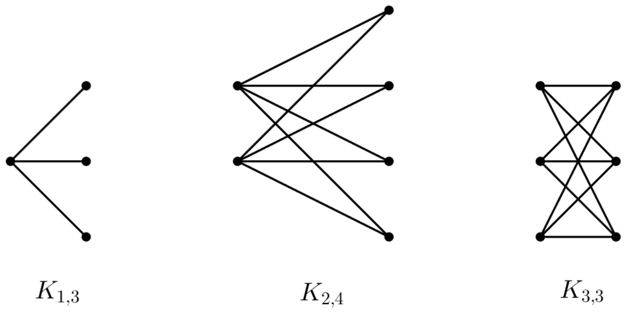

Figure 2.1: Three complete bipartite graphs

Recall from Example 1.5.11 that the distance, $d(u,v)$, between two vertices is the length of a shortest path between u and v. The diameter of a graph is the maximum of the distances between any two vertices in G. Proposition 1.5.10 states that if G is a graph with $diam(G) = 2$ and contains no complete subgraph H on δ vertices with $\deg_G v = \delta$ for all $v \in V(H)$, then G is super-λ.

Proposition 2.2.6. *For two positive integers $n \geq m$, let $K_{m,n}$ be a complete bipartite graph with vertex partition sets of size m and n respectively. Then the mincut graph of $K_{m,n}$ is*

$$X(K_{m,n}) \cong \begin{cases} \overline{K}_n & \text{if } n > m, \\ K_{n,n} & \text{if } n = m,\ m > 2. \end{cases}$$

Proof. Let $K_{m,n}$ be a complete bipartite graph with vertex partition sets U and W such that $|U| = m$ and $|W| = n$.

If $n > m$ and $m = 1$, $K_{1,n}$ is a star and the proof follows from Proposition 2.2.1 since $K_{1,n}$ is a tree on $n+1$ vertices.

Let $n > m$ and $m > 1$. Every vertex $u \in U$ is adjacent to every vertex $w \in W$, therefore $deg(u) = n$ for every $u \in U$ and $deg(w) = m$ for all $w \in W$. Therefore we conclude $\delta(K_{m,n}) = m$.

Every $u \in U$ is adjacent to every $w \in W$ but to no other $u \in U$. We conclude that $diam(K_{m,n}) = 2$.

The only vertices of degree δ in $K_{m,n}$ are the n vertices in W. But not any of these are adjacent to each other, therefore there is no complete subgraph $K_\delta \subset K_{m,n}$ on δ vertices and we conclude that $K_{m,n}$ is super-λ. The trivial cuts are the incident edge-sets on the n vertices of W. Since not any of these intersect, we have $K_{m,n} \cong \overline{K}_n$.

If we let $n = m$, $m > 2$, then we have $K_{n,n}$ with $\delta = n$. As before we have $diam(K_{n,n}) = 2$. Choose any two adjacent vertices $u \in U$ and $w \in W$. Each of the vertices is adjacent to $n-1$ other vertices that are not adjacent to the second vertex. Therefore there can be no $K_n \subset K_{n,n}$ and $K_{n,n}$ is super-λ. But $K_{n,n}$ is also r-regular and hence the trivial sets are the incident edge sets on *all* $2n$ vertices. Thus the mincut on each vertex intersects the mincuts on the n vertices to which it is adjacent and we can conclude that $X(K_{n,n}) \cong K_{n,n}$.

We exclude the graph $K_{2,2}$ since $K_{2,2} \cong C_4$. Mincuts of cycles are dealt with in Proposition 2.2.10. □

Recall from Section 1.6 that the line graph, $L(G)$, of a graph G has the edges of G as its vertices, such that two vertices in $L(G)$ are adjacent if their corresponding edges in G have a vertex in common.

Definition 2.2.7. A *strongly regular* graph is a regular graph G with parameters $srg(v, r, \lambda, \mu)$, where v is the order and r the regularity of G, such that every two adjacent vertices have λ neighbours in common and every two non-adjacent vertices have μ neighbours in common.

Lemma 2.2.8. *The line graph of a complete graph $L(K_n)$ is a strongly regular graph with parameters $srg(\binom{n}{2}, 2(n-2), n-2, 4)$.*

Proof. Let K_n be a complete graph on n vertices and $L(K_n)$ its associated line graph. Since every vertex in K_n is connected to every other vertex, the size of K_n is $\binom{n}{2}$ and hence the order of $L(K_n)$ is $v = \binom{n}{2}$. Let u and v be vertices of K_n and uv, the edge that joins them, be the corresponding vertex in $L(K_n)$. Vertex u is incident on $n-2$ edges other than uv and so is vertex v, which implies that vertex uv in

$L(K_n)$ is adjacent to $2(n-2)$ other vertices and hence $L(K_n)$ is $2(n-2)$-regular. Let u, v, $w \in V(K_n)$ with edges uv, vw, $uw \in V(L(K_n))$. Vertex v is incident on $n-3$ edges other than uv and vw. Hence, uv and vw share $n-3$ neighbours in $L(K_n)$. But uw is incident on both u and w and hence is also a neighbour of uv and vw in $L(K_n)$. Thus, two adjacent vertices in $L(K_n)$, uv and vw, have $\lambda = n-2$ common neighbours. Let u, v, w, $z \in V(K_n)$ with edges uv and wz not adjacent in $L(K_n)$. Any choice of four vertices in K_n induces a complete subgraph $K_4 \subset K_n$ and hence uw, uz, vw, vz are four common neighbours to uv and wz in $L(K_n)$, see Figure 2.2. Suppose we add a vertex x to our vertex subset in K_n, then edge xu is incident upon u but not on any of the remaining vertices of the subset and hence is not adjacent to wz in $L(K_n)$. Thus, any two non-adjacent vertices, uv and wz, in $L(K_n)$ have exactly $\mu = 4$ common neighbours. $\square$

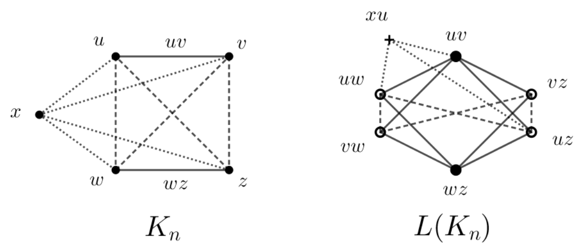

Figure 2.2: Neighbours of non-adjacent vertices in $L(K_n)$.

Proposition 2.2.9. *Let $L(K_n)$ be the line graph of the complete graph K_n, $n > 2$. Then*

$$X(L(K_n)) \cong L(K_n).$$

Proof. We exclude K_2 since $L(K_2) \cong K_1$ and $X(K_1) \cong K_0$ by Lemma 2.1.3.

Recall from Proposition 1.5.10 that a graph G is super-λ if G has diameter 2 and contains no complete subgraph H on δ vertices with $\deg_G v = \delta$ for all $v \in V(H)$.

We know from Lemma 2.2.8 that $L(K_n)$ is a strongly regular graph and that any two non-adjacent vertices have 4 neighbours in common, that is, any two vertices are at a maximum distance of 2 and hence $diam(L(K_n)) = 2$. Also, any two adjacent vertices have $n-2$ common neighbours. Therefore, given any two adjacent vertices, the largest possible induced complete graph in $L(K_n)$ is K_n. $L(K_n)$ is $2(n-2)$ regular and therefore has minimum degree $\delta = 2(n-2)$. But, $2(n-2) > n$ for all $n > 4$ and hence there is no induced subgraph $K_{2(n-2)}$ on $2(n-2)$ vertices of $L(K_n)$. Therefore, $L(K_n)$ is super-λ.

$L(K_n)$ is super-λ, and hence all mincuts are trivial. However, $L(K_n)$ is also $2(n-2)$-regular, which implies that all trivial cuts are mincuts and that $|V(L(K_n))| = |V(X(L(K_n)))|$. This also means that the mincut incident on a vertex intersects with the mincuts of the vertices adjacent to that vertex and hence adjacency of the vertices is preserved.

We conclude that for $n > 4$ we have $X(L(K_n)) \cong L(K_n)$.

Similarly, by Lemma 2.2.8, $L(K_4)$ is 4-regular and has a diameter of 2. We verify by looking at the diagram in Figure 1.6 that $L(K_4)$ is K_4 free and conclude that it is super-λ and 4-regular. By the same reasoning as above we have $L(K_4) \cong X(L(K_4))$.

We recall from Section 1.6 that cyclic graphs are isomorphic to their line graphs. Thus we have $L(C_3) \cong C_3$. But $K_3 \cong C_3$ and hence $K_3 \cong L(K_3) \cong X(L(K_3)$, by Proposition 2.2.3.

Therefore, we can conclude that $X(L(K_n)) \cong L(K_n)$ for $n > 2$. □

Proposition 2.2.10. *Let C_n be the cycle on n vertices and $L(K_n)$ the line graph of K_n. Then the mincut graph of C_n is*

$$X(C_n) \cong L(K_n).$$

Proof. The edge connectivity of C_n, $\lambda(C_n) = 2$, and any choice of two edges is a mincut. Thus the set of all mincuts, X, of C_n is the set of all two element subsets of

$E(C_n)$, where $|E(C_n)| = n$. But the vertex set of $L(K_n)$ is the set of all two element subsets of the n-element vertex set of K_n, the edges of K_n. Thus $|V(X(C_n))| = |V(L(K_n))|$.

Furthermore, two vertices in $X(C_n)$ are adjacent if their corresponding two element subsets have non-empty intersection, and two vertices in $L(K_n)$ are adjacent if their corresponding two element subsets (edges in K_n) have non-empty intersection.

Therefore $X(C_n) \cong L(K_n)$. $\square$

Proposition 2.2.11. *Let W_n be the wheel on n vertices and set $n > 4$. Then the mincut graph of W_n*

$$X(W_n) \cong C_{n-1}.$$

Proof. For $n = 4$, $W_4 \cong K_4$. We set $n > 4$, since complete graphs are dealt with in Proposition 2.2.3. Then, for any W_n, $\lambda(W_n) = 3$ and the mincuts are exactly the three edges incident on every vertex v_i on the "rim" of the wheel. By labeling the $n - 1$ vertices on the rim in sequence, we see that every mincut X_i has non-empty intersection with X_{i-1} and X_{i+1}, thus giving the cycle C_{n-1} as the mincut graph. $\square$

Example 2.2.12. In Figure 2.3 we give examples of the mincut graphs of some of the graphs dealt with in the preceding section. We note that $X(T_6)$, the mincut of a tree on six vertices is $\overline{K}_5$, the empty graph on five vertices, $X(K_5)$ is isomorphic to K_5, the mincut graph of W_6, the wheel on six vertices, is a cycle, C_5, on five vertices, and $X(C_5) \cong L(K_5)$, the line graph of the complete graph, K_5.

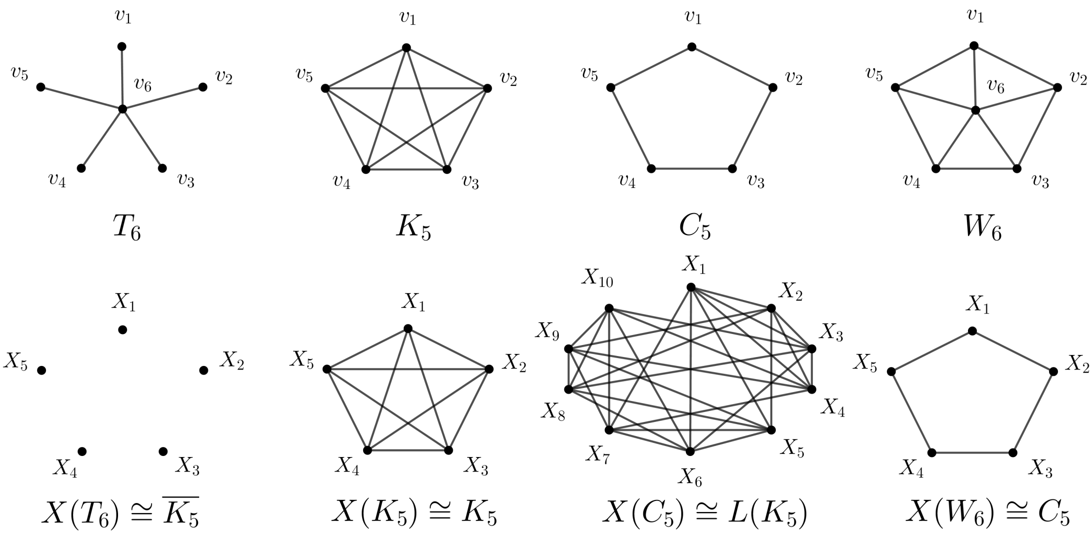

Figure 2.3: Mincut graphs of some well-known classes of graphs.

Chapter 3

The mincut operator

3.1 Introduction

In this chapter we introduce and study the properties of the mincut *operator*, as per the research program introduced by Prisner in [32], see Section 1.6. Specifically we show that every graph is a mincut graph and we show that, apart from the null graph, K_0, the property of being super-λ and r-regular is both necessary and sufficient for a graph to be fixed under the operator.

We recall Definition 1.6.1, that a graph operator is a mapping ϕ which maps every graph G from some class of graphs to a new graph $\phi(G)$. We define the operator recursively, such that, if ϕ is an operator and k a positive integer, then $\phi^1 = \phi$ and $\phi^k(G) = \phi(\phi^{k-1}(G))$, for $k \geq 2$, see [40].

In [32], Prisner unifies the study of graph operators under such questions as 'Which graphs appear as images of graphs?'; 'Which graphs are fixed under the operator?'; 'What happens if the operator is iterated?' In this section we study the mincut operator as per this research program outlined by Prisner.

We investigate Prisner's three questions as follows:

1. When is a graph a mincut graph?

2. When is a graph isomorphic to its mincut graph?

3. What happens when the mincut operator is iterated?

Iteration of the operator is dealt with in Chapter 4.

3.2 When is a graph a mincut graph?

Since an operator, ϕ, is a mapping on the set of finite graphs into itself, it is natural to ask which graphs lie in the image of ϕ. And, given $\phi(G)$, are we able to determine G?

Beineke showed that a graph cannot be a line graph if it has one of the nine graphs shown in Figure 3.1 as an induced subgraph, see [2].

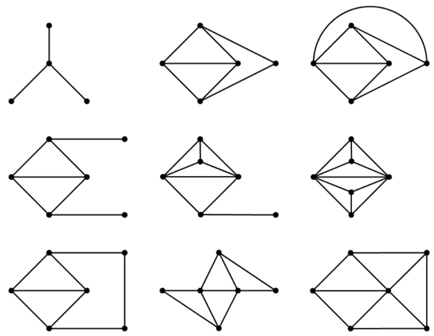

Figure 3.1: Nine forbidden induced subgraphs if a graph is a line graph.

Furthermore, Whitney's theorem, see [33, 38], implies that a graph is uniquely determined by its line graph, up to isolated vertices, except in the case $K_{1,3}$ and C_3. We see in Figure 3.2 that $K_{1,3}$ has three edges all incident on one vertex, giving the cycle C_3 as its line graph. We also note that C_3 has three edges that are pairwise incident on a shared vertex which means that $C_3 \cong L(C_3)$. In fact, Wilf and Van Rooij showed in [37] that a graph is isomorphic to its line graph if and only if it is a cycle.

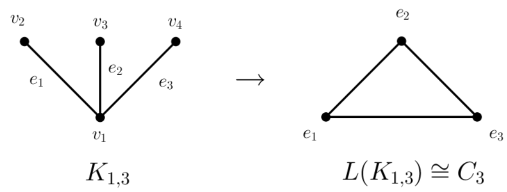

Figure 3.2: $K_{1,3}$ and C_3 have isomorphic line graphs.

Definition 3.2.1. For a given graph operator ϕ, a *ϕ-root* of a graph G is any graph H with $\phi(H) \cong G$.

Whitney's theorem tells us that unless a graph is $K_{1,3}$ or C_3 it has a unique *connected* L-root.

In this section we address the question of which graphs are mincut graphs, that is, which graphs lie in the image of $X(\cdot)$. We show that every graph is a mincut graph, by constructing a super-graph G^* from a given graph G such that G is the mincut graph of G^*. Furthermore, we show that every graph has an infinite number of X-roots.

Lemma 3.2.2. *Let G be a super-λ graph with $H \subseteq G$ the induced subgraph on the set of vertices of G such that $V(H) = \{v \in V(G) | deg(v) = \delta(G)\}$. Then $H \cong X(G)$.*

Proof. Let $x_i \in V(X(G))$ be the vertex in $X(G)$ corresponding to the mincut $X_i \in X$, the set of all mincuts in G. The mincuts in G are exactly the incident edge-sets on the vertices $v \in V(H)$ and hence $|X| = |V(H)|$. Thus for every $v_i \in V(H)$ there is exactly one $x_i \in V(X(G))$. Furthermore, if $v_iv_j \in E(H)$ then $X_i \cap X_j \neq \emptyset$, where X_i and X_j are the incident edge sets on v_i and v_j. If $v_iv_j \notin E(H)$ then $X_i \cap X_j = \emptyset$, since they are the incident edge-sets on v_i and v_j. Hence $x_ix_j \in E(X(G))$ if and only if $v_iv_j \in E(H)$ and isomorphism follows. $\square$

Example 3.2.3. The graph G in Figure 3.3 is super-λ with vertices of minimum degree v_1, v_2, v_3, and v_4 and $X(G) \cong K_{1,3}$ is the induced subgraph on vertices $v_1 \ldots v_4$.

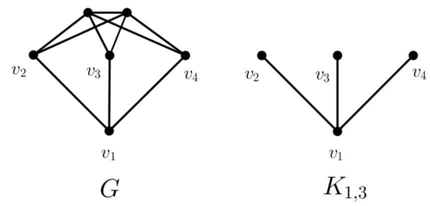

Figure 3.3: A super-λ graph G with induced subgraph on vertices of minimum degree

Corollary 3.2.4. *If G is super-λ and r-regular, $G \ncong K_2$, then $X(G) \cong G$.*

Proof. If G is r-regular and super-λ, then $V(H) = V(G)$ where $V(H) = \{v \in V(G) | deg(v) = \delta(G)\}$ and the edge-set incident on *every* vertex is a mincut. Hence, if $v_iv_j \in E(G)$, then $X_i \cap X_j \neq \emptyset$ and $x_ix_j \in E(X(G))$. We exclude K_2 since $K_2 \cong T_2$ and $X(T_2) \cong K_1$ by Proposition 2.2.1. □

Definition 3.2.5. The *join* $G = G_1 \vee G_2$ of two graphs G_1 and G_2 has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1),\ v \in V(G_2)\}$.

In other words, the join of two graphs joins every vertex in the one graph to every vertex in the other graph. Thus, for example, we can say $K_n = K_1 \vee K_{n-1}$ and $K_{m,n} = \overline{K}_m \vee \overline{K}_n$. If one of the graphs is K_1 the operation is also called the *vertex join* of the graph.

Lemma 3.2.6. *Let G be an r-regular graph of order $n > r + 1$ and let $G^* = K_1 \vee G$. Then $X(G^*) \cong G$.*

Proof. The join $K_1 \vee G$ yields a graph with one vertex of degree $deg(v) > r + 1$ connected to every other vertex which is now of degree $r + 1$. If G is super-λ then G^* is super-λ, since we have added one vertex of higher degree than all the others, and $G^*[V(G)]$ is the induced subgraph on vertices of degree δ. Then by Lemma 3.2.2 $X(G^*) \cong G$. Suppose G is not super-λ and X is a non-trivial mincut of G with $\langle Y, \bar{Y} \rangle$ the two components of $G - X$. In G^* every vertex in Y and $\bar{Y}$ is now adjacent to the added vertex and hence $G - X$ is no longer disconnected. Hence X is no longer a mincut and we conclude that G^* is super-λ with $G^*[V(G)]$ the induced subgraph on vertices of degree δ and by Lemma 3.2.2 $X(G^*) \cong G$. $\square$

Every graph is an intersection graph, see [12, 17]. We show in the following theorem that every graph is also the mincut graph of not just one graph, but a number of other, not necessarily isomorphic, graphs.

Theorem 3.2.7. *Every graph G is the mincut graph of an infinite family of graphs.*

Proof. Let G be a graph of order p with minimum vertex degree $\delta(G)$ and maximum vertex degree $\Delta(G)$. We construct a super-λ super graph G^* such that $G \subset G^*$ is the induced subgraph on the vertices of G^* of minimum degree, that is, such that $V(G) = \{v \in V(G^*) | deg(v) = \delta(G^*)\}$. Then, we conclude by Lemma 3.2.2, $G \cong X(G^*)$.

Suppose G is not r-regular. We start the construction with $G \cup K_m$, $m = \Delta(G) - \delta(G)$, the disjoint union of G and a complete graph K_m. The choice of m is the minimum number of vertices necessary to add edges between the two graphs so that $deg_{G^*}(v) = \Delta(G)$ for all $v \in V(G)$. We add edges between the graphs until the following three conditions are met. If we need to add more than m edges to any of the vertices in G, we can always increase m, the number of vertices of K_m, see Example 3.2.9.

1. $deg_{G^*}(v) = k$ for all $v \in V(G)$,

2. $deg_{G^*}(v) > k$ for all $v \in K_m$,

3. $deg(u) + deg(v) \geq n + 1$ for all non-adjacent vertices $u, v \in V(G^*)$, where $|V(G^*)| = n$.

Then G^* is super-λ by condition 2 of Proposition 1.5.10 and G is the induced subgraph on vertices of minimum degree in G^*. That is, $V(G) = \{v \in V(G^*) | deg(v) =$

$\delta(G^*)\}$ and $\lambda(G^*) = deg_{G^*}(v)$, $v \in V(G)$. Hence, by Lemma 3.2.2 $G \cong X(G^*)$. We note that, by condition 1 of Proposition 1.5.10, if $G^* \ncong K_{\frac{n}{2}} \times K_2$ we can have $deg(u) + deg(v) \geq n$ for all non-adjacent vertices u and v.

If $k+k < n+1$ add edges between G and K_m until $2k \geq n+1$. This ensures that condition 3 is met for all non-adjacent $u, v \in V(G)$. If $k+m-1 < n+1$ increase k, and m if necessary, until we know that the graph is super-λ. Then increase m to ensure that condition 2 eventually applies, see Example 3.2.9.

If G is r-regular of order $p > r+1$ then our construction simply needs $G^* = K_1 \vee G$, then $X(G^*) \cong G$, by Lemma 3.2.6.

If G is r-regular, but $p \leq r+1$ then $p-1 \leq r$. But the maximum degree of a vertex in any simple graph of order p is $p-1$ so G must be complete. We start with $m = 2$ and join all the vertices of G to K_m. Then simply increase m until $m > p+2$ without adding any new edges between G and K_m. Both G and K_m are super-λ and we have added $2p$, more than $\delta(G^*)$, edges between them.

If G is not connected we follow the same procedure, connecting each component of G to K_m.

Once we have G^* super-λ with $V(G) = \{v \in V(G^*) | deg(v) = \delta(G^*)\}$ we can keep increasing m and still have a graph that fulfils the conditions of Lemma 3.2.2 since we are only adding trivial edge cuts but they are not mincuts. Thus for any graph G we can construct an infinite family of graphs G^* such that $X(G^*) \cong G$. $\square$

Example 3.2.8. The graph $K_{1,3}$ is the mincut graph of a family of super-λ graphs $\mathscr{F}(G^*)$ such that $G \subseteq G^*$ and $V(G) = \{v \in V(G^*) | deg(v) = \delta(G^*)\}$. We have $\Delta(G) = 3$, $\delta(G) = 1$ and $p = 4$. We let $m = 3 - 1 = 2$ and add edges between $K_{1,3}$ and K_m until the vertices $\{v_1 \ldots v_4\}$ are of degree 3, as shown in Figure 3.4. Now we have $n = 6$ and $k + k = 6$. However, since $G \neq K_3 \times K_2$, the Cartesian product of the two graphs, condition 1 of Proposition 1.5.10 tells us that it is sufficient for $deg(u) + deg(v) \geq n$ for G^* to be super-λ in this case. Hence, by now increasing m, for any $m \geq 5$, G^* would be super-λ, since the minimum edge cuts in a complete graph are only the trivial cuts and $deg_{G^*}(v) = 3$ for all $v \in V(G)$.

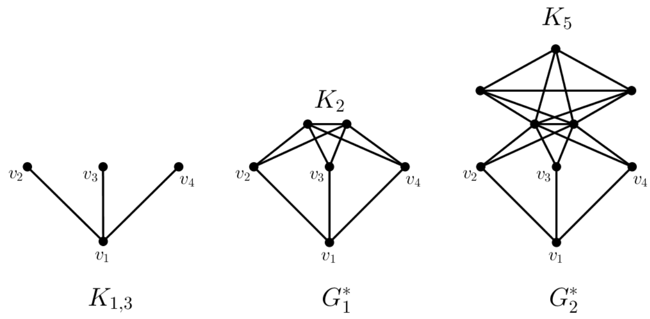

Figure 3.4: Graphs G_1^* and G_2^*, constructed such that $X(G_1^*) \cong X(G_2^*) \cong K_{1,3}$.

In the following example we demonstrate a case where it is necessary to increase k and consequently m to ensure that G^* is super-λ such that $V(G) = \{v \in V(G^*) | deg(v) = \delta(G^*)\}$.

Example 3.2.9. The graph G formed by the one point join of C_3 and P_2 is sometimes called "the Paw". We have $\Delta(G) = 3$, $\delta(G) = 1$ and $n = 4$. We proceed as in Example 3.2.8 and let $m = 2$ and add edges until $deg_{G^*}(v) = 3$ for all $v \in V(G)$. We obtain the graph G_1^* in Figure 3.5 with $n = 6$ and $deg(u) + deg(v) = 6$ for any two non-adjacent vertices. Condition 1 of Proposition 1.5.10 does not apply here, however, since $G_1^* \cong K_3 \times K_2$ and in fact the edge set $\{e_1, e_2, e_3\}$ is a non-trivial mincut. Hence we increase k, the degree of each vertex of G in G^*, by one in G_2^* which also necessitates increasing m by one. Now G_2^* is super-λ since $deg(u) + deg(v) = 8$ or 9 for any two non-adjacent vertices and $n = 7$. The problem remains that one of the vertices in K_3 also has degree 4 and hence is also a mincut, implying $X(G^*) \ncong G$. We increase m to $k + 2$ in G_3^* which means that all vertices of K_m in G^* now have degree at least 5. Also we have added only trivial cuts, since K_6 is complete and so we still have a super-λ graph with vertices of minimum degree exactly the vertices of G. Clearly increasing m so that $m \geq 6$ will have the same result.

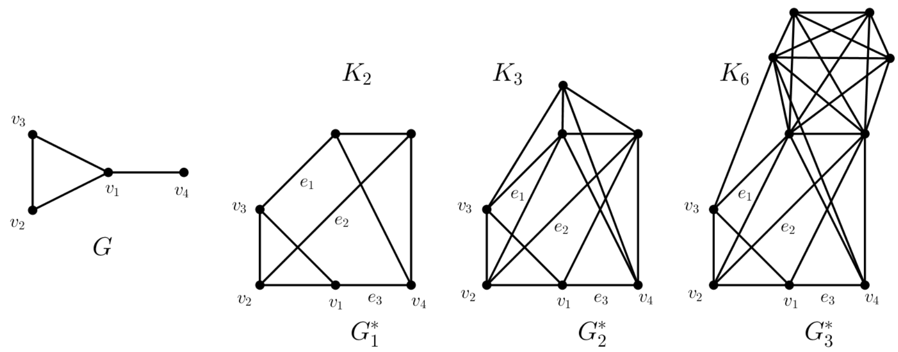

Figure 3.5: The "Paw" is the mincut graph of a family of graphs G_i^* for $i \geq 3$.

Corollary 3.2.10. *For any connected graph G, there is a graph H that is non-isomorphic to G such that $X(G) \cong X(H)$.*

3.3 When is a graph isomorphic to its mincut graph?

In this section we address the question of which graphs are fixed under the mincut operator, that is, remain unchanged under the operator. We know, from Lemma 2.1.4, that $X(K_0) \cong K_0$ and is therefore fixed under the X-operator. In Theorem 3.3.16 we show that, for all other graphs, the property of a graph G being super-λ and r-regular is not only sufficient, but also necessary to have $X(G) \cong G$. We already saw the sufficiency of this property illustrated in Section 2.2 when we showed that K_n, $K_{n,n}$ and $L(K_n)$, $n > 2$, all super-λ and r-regular, are isomorphic to their mincut graphs.

We start by examining the structure of mincuts in more detail.

We recall from Lemma 1.5.4 that if X is a mincut of a graph G, then $G \backslash X$ is a disconnected graph with exactly two components, A and $\bar{A}$. We will also refer to the vertex sets of the two components as A and $\bar{A}$, respectively, and write $X = \langle A, \bar{A}\rangle$ to make clear that X is the set of edges connecting the two components. For the following definition see [5, 21].

Definition 3.3.1. Let $X = \langle A, \bar{A}\rangle$ and $Y = \langle B, \bar{B}\rangle$ be two mincuts of a graph G. If their vertex sets have non-empty intersection, that is $A \cap B \neq \emptyset$, then they are either *nested*, i.e. $A \subset B$ or $B \subset A$, or they *overlap* (also called *crossing mincuts*), i.e. $A \cap B$, $\bar{A} \cap B$ and $A \cap \bar{B}$ are *non-empty.*

We also note from [21] that two mincuts can overlap only if λ is even.

Proposition 3.3.2 (Lehel et al, [21])**.** *Let $X = \langle A, \bar{A}\rangle$ and $Y = \langle B, \bar{B}\rangle$ be two overlapping mincuts of a connected graph G. Then $|\langle \bar{A}\cap\bar{B}, A\cap\bar{B}\rangle| = |\langle \bar{A}\cap\bar{B}, \bar{A}\cap B\rangle| = |\langle A\cap B, A\cap\bar{B}\rangle| = |\langle A\cap B, \bar{A}\cap B\rangle| = \dfrac{\lambda}{2}$. Consequently $A \cup B$, $A \cap B$, $A \cap \bar{B}$, $\bar{A} \cap B$ are mincuts. Moreover, $|\langle \bar{A}\cap\bar{B}, A\cap B\rangle| = |\langle \bar{A}\cap B, A\cap\bar{B}\rangle| = 0$.*

Lemma 3.3.3 (Chandran & Ram, [5])**.** *If $X = \langle A, \bar{A}\rangle$ and $Y = \langle B, \bar{B}\rangle$ are a pair of crossing mincuts, then $X \cap Y = \emptyset$.*

We illustrate the concepts of nested and crossing mincuts in the following two examples.

Example 3.3.4. The graph G in Figure 3.6 is a simple connected graph with $\lambda = 3$. We note that G has four non-trivial nested mincuts. The diagrams in the figure highlight the four non-trivial mincuts $Z = \langle A, \bar{A}\rangle = \{e_3, e_6, e_7\}$, $X = \langle B, \bar{B}\rangle = \{e_1, e_2, e_3\}$, $Y = \langle C, \bar{C}\rangle = \{e_1, e_4, e_5\}$ and $W = \langle D, \bar{D}\rangle = \{e_5, e_9, e_{10}\}$. The intersections $Z \cap X$, $X \cap Y$ and $Y \cap W$ are all nonemtpy.

We note $A = \{b, c, d, e\}$, $B = \{a, b, c, d, e\}$, $C = \{a, b, c, d, e, g\}$ and $D = \{a, b, c, d, e, f, g\}$ and the intersections of the respective vertex sets are non-empty. We also have λ odd which means that the vertex sets are nested, i.e. $A \subset B \subset C \subset D$.

Equivalently we can look at the complementary vertex sets in each partition and we see that their intersections are non-empty and $\bar{D} \subset \bar{C} \subset \bar{B} \subset \bar{A}$.

The subgraph induced by the four vertices corresponding to these non-trivial mincuts in $X(G)$ is a path $Z \to X \to Y \to W$.

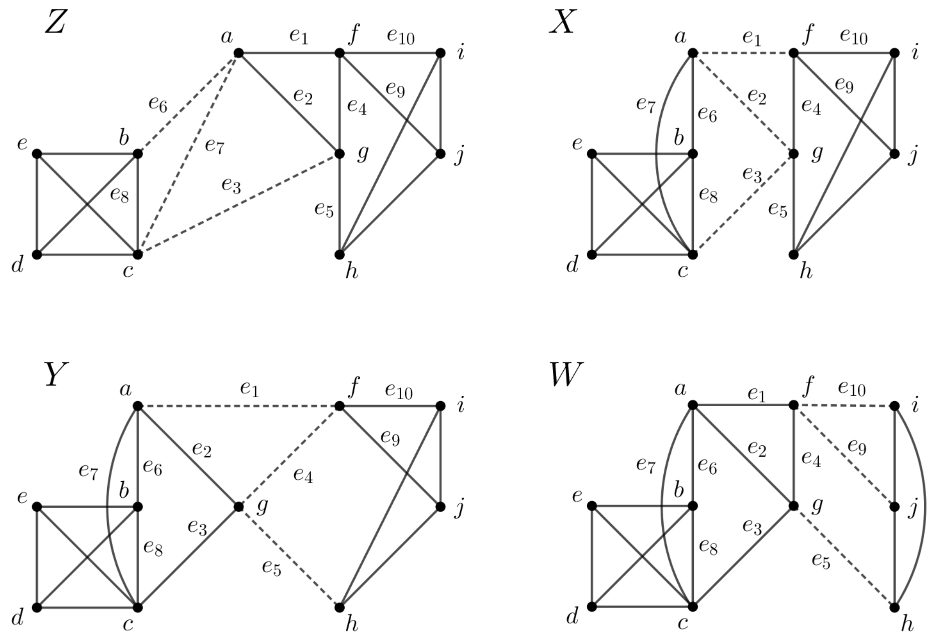

Figure 3.6: Nested mincuts where λ is odd.

Example 3.3.5. The graph G in Figure 3.7 is a simple connected graph with $\lambda = 4$. We note that G has five non-trivial mincuts. The diagrams in the figure highlight the five non-trivial mincuts $Z = \langle A, \bar{A} \rangle$, $X = \langle B, \bar{B} \rangle$, $Y = \langle C, \bar{C} \rangle$, $V = \langle B \cap C, \bar{B} \cup \bar{C} \rangle$ and $W = \langle B \cup C, \bar{B} \cap \bar{C} \rangle$.

We look at X and Y and note that $B = \{a, b, c, d, k, l, m\}$, $\bar{B} = \{e, f, g, h, i, j\}$, $C = \{a, b, c, e, k, l, m\}$ and $\bar{C} = \{d, f, g, h, i, j\}$. Thus we have $B \cap C$, $\bar{B} \cap C$, $B \cap \bar{C}$ and $\bar{B} \cap \bar{C}$ all non-empty and, by Definition 3.3.1, X and Y are overlapping or crossing mincuts.

Since X and Y overlap, Proposition 3.3.2 indicates that we should also have $\langle B \cap C, \bar{B} \cup \bar{C} \rangle$ and $\langle B \cup C, \bar{B} \cap \bar{C} \rangle$ as mincuts and these are respectively V and W.

Furthermore, $B \cap \bar{C} = \{d\}$, and $\bar{B} \cap C = \{e\}$ are both vertices of degree four and hence their incident edge sets are mincuts.

By Lemma 3.3.3, if X and Y are crossing mincuts then we should have $X \cap Y = \emptyset$ and indeed we see that $X = \{e_3, e_4, e_5, e_6\}$ and $Y = \{e_1, e_2, e_7, e_8\}$.

Mincut Z is a nested mincut since $A \subset (B \cap C)$. We also have $A \subset B$ and $A \subset C$. Similarly V is nested since $(B \cap C) \subset B, C$. We also have $B, C \subset (B \cup C)$ so X and Y are nested with respect to W, but $B \not\subset C$ and $C \not\subset B$ since X and Y overlap. Looking at the edge sets we also have $Z = \{e_1, e_4, e_9, e_{10}\}$, $V = \{e_1, e_2, e_3, e_4\}$ and $W = \{e_5, e_6, e_7, e_8\}$. Hence, the subgraph induced by the vertices in $X(G)$ corresponding to V, X, Y, W is a cycle and the vertex corresponding to Z is adjacent to the vertices V, X, Y.

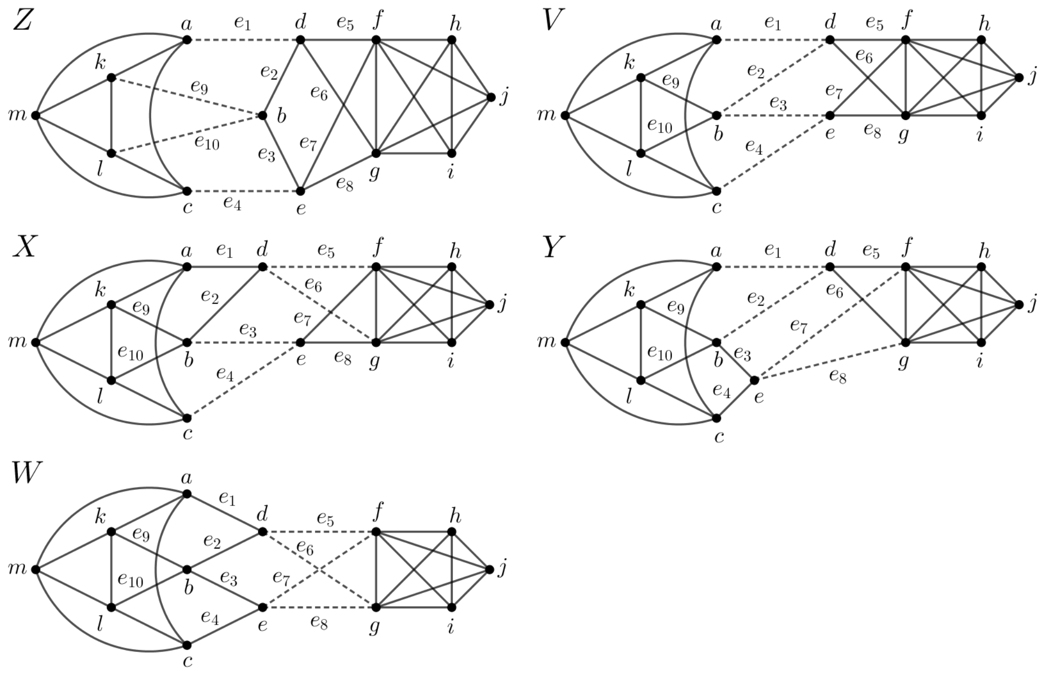

Figure 3.7: Nested and overlapping mincuts where λ is even.

When we look at a mincut X in terms of the vertex sets of the two components of

$G\backslash X$, then there is an important subset of vertices of each of the components, namely the vertices incident on the edges of X. We follow Tutte, [35, 36], and define vertices of attachment as follows, specifically with regard to mincuts, although the definition can be generalised for any subgraph of G.

Definition 3.3.6. Let X be a mincut of G with A and $\bar{A}$ the two components of $G\backslash X$. A *vertex of attachment* of A (resp. $\bar{A}$) in G is a vertex of A (resp. $\bar{A}$) that is incident with some edge of G that is not an edge of A (resp. $\bar{A}$).

We will denote the vertices of attachment of the two components A and $\bar{A}$ of $G\backslash X$ as V_A and $V_{\bar{A}}$ respectively. Also, we will denote the neighbourhood of a vertex v restricted to the neighbours in one of the two components as $N_A(v)$ or $N_{\bar{A}}(v)$ respectively.

Example 3.3.7. If we look at $Z = \langle A, \bar{A}\rangle = \{e_1,\, e_4,\, e_9,\, e_{10}\}$ in Figure 3.7, we have $V_A = \{a,\, c,\, k,\, l\}$ and $V_{\bar{A}} = \{b,\, d,\, e\}$.

Definition 3.3.8. Let X be a trivial mincut. Then X is the set of edges incident on a single vertex v such that $deg(v) = \lambda$. We define v to be a *trivial* vertex .

Lemma 3.3.9. *Let $X = \langle A,\, \bar{A}\rangle$ and $Y = \langle B,\, \bar{B}\rangle$ be two non-trivial mincuts of a graph G with A, $\bar{A}$ and B, $\bar{B}$ the vertex sets of the components of $G\backslash X$ and $G\backslash Y$ respectively. If $X \cap Y \neq \emptyset$ then either $A \cap B \neq \emptyset$ or $A \cap \bar{B} \neq \emptyset$.*

Proof. Let $e = uv \in X \cap Y$ be an edge belonging to both X and Y, then u and v are vertices of attachment, one of A and the other of $\bar{A}$, since $e \in X$. Now $e \in Y$ and hence u and v are also vertices of attachment of B and $\bar{B}$. Therefore we must have $A \cap B \neq \emptyset$ or $A \cap \bar{B} \neq \emptyset$. □

Note that the converse to Lemma 3.3.9 is not necessarily true.

We recall Definition 1.4.4 on the isomorphism of two graphs by rewriting it in terms of G and $X(G)$.

Two graphs, G and $X(G)$, are isomorphic if there is a bijective function $\psi : V(G) \to V(X(G))$ such that two vertices v_i and v_j are adjacent in G if and only if $\psi(v_i) = x_i$ and $\psi(v_j) = x_j$ are adjacent in $X(G)$.

The X-operator maps a mincut $X_i \in E(G)$ to a vertex $x_i \in V(X(G))$. We will say that x_i corresponds to X_i. But if $G \cong X(G)$ then there is a vertex v_i mapped to x_i and thus we can say that this v_i corresponds to X_i and to x_i. In the rest of this section we will use v for vertices in G, X for mincuts in G and x for vertices in $X(G)$ and use the same subscript if two objects correspond. So, for example, we will write $v_i \sim x_i$ or $v_i \sim X_i$.

The key behind the isomorphism mapping $G \to X(G)$ is that if $v_j \sim x_j$ and v_j is adjacent to v_i in G, $v_j \in N_G(v_i)$, then $x_j \in N_{X(G)}(x_i)$ and $X_j \cap X_i \neq \emptyset$.

Lemma 3.3.10. *Let v_i, $v_j \in V(G)$ be trivial vertices with incident edge sets X_i and X_j and $v_iv_j \in E(G)$. Then $x_i \sim X_i$ and $x_j \sim X_j$ are vertices of $X(G)$ such that $x_ix_j \in E(X(G))$. If $v_iv_j \notin E(G)$ then $x_ix_j \notin E(X(G))$.*

Proof. X_i and X_j are the edge sets incident on v_i and v_j respectively and are mincuts since the two vertices are both trivial. Let e be an edge incident on both v_i and v_j then $v_iv_j \in E(G)$ and e is an edge belonging to both X_i and X_j, so $X_i \cap X_j \neq \emptyset$ and $x_ix_j \in E(X(G))$. Suppose $v_iv_j \notin E(G)$ and e is an edge incident on v_i, then e is not incident on v_j and hence $e \notin X_j$. Therefore, $X_i \cap X_j = \emptyset$ and $x_ix_j \notin E(X(G))$. □

Lemma 3.3.11. *Let X_k be a non-trivial mincut with A and $\bar{A}$ the components of $G \backslash X_k$ and $x_k \sim X_k$ where $x_k \in V(X(G))$. If v_i is a trivial vertex and $v_i \in V_A$ (or $V_{\bar{A}}$), then $X_i \cap X_k \neq \emptyset$ and $x_ix_k \in E(X(G))$. If $v_i \notin V_A$, $V_{\bar{A}}$ then $X_i \cap X_k = \emptyset$ and $x_ix_k \notin E(X(G))$.*

Proof. If $v_i \in V_A$ then $v_iv_j \in E(G)$ for some $v_j \in V_{\bar{A}}$ which means there is an edge $e \in X_k$ incident on v_i and v_j. But every edge incident on v_i is in X_i and X_i is a mincut. So $X_i \cap X_k \neq \emptyset$ and $x_ix_k \in E(X(G))$. Let $v_i \in A$ and $v_i \notin V_A$. Then v_i

is adjacent to no $v_j \in V_{\bar{A}}$ and hence there is no $e \in X_i$ such that $e \in X_k$. Hence $X_i \cap X_k = \emptyset$ and $x_i x_k \notin E(X(G))$. $\square$

Example 3.3.12. The diagram in Figure 3.7 shows the non-trivial mincut $Z = \langle A, \bar{A} \rangle = \{e_1, e_4, e_9, e_{10}\}$. Vertex $a \in V_A$ is trivial and we see that X_a shares edge e_1 with Z. However, the trivial vertex $m \notin V_A$ and we note that X_m shares no edge with Z.

Lemma 3.3.13. *Let $G \cong X(G)$ and X_k be a non-trivial mincut. Then there is $v_k \in V(G)$ such that $deg(v_k) > \lambda$.*

Proof. Suppose there is no such v_k, then all vertices of G have degree λ and their incident edge-sets are mincuts. Since X_k is a non-trivial mincut this gives $|V(G)|+1$ mincuts in G and hence $|V(X(G))| > |V(G)|$ so $G \ncong X(G)$. $\square$

Lemma 3.3.14. *Let $G \cong X(G)$. Then a trivial mincut $X_i \in E(G)$ corresponds to a (trivial) vertex $v_i \in V(G)$ such that $deg(v_i) = \lambda$ and a non-trivial mincut $X_j \in E(G)$ corresponds to a (non-trivial) vertex $v_j \in V(G)$ such that $deg(v_j) > \lambda$.*

Proof. Let v_i, v_j, $v_k \in V(G)$ be trivial vertices and $v_l \in V(G)$ be a non-trivial vertex with $v_i v_j$, $v_j v_l \in E(G)$.

Suppose $v_l \sim X_k$ where X_k is the mincut incident on v_k. Since $v_l v_j \in E(G)$ we must have $x_l x_j \in E(X(G))$. But $v_k v_j \notin E(G)$ and hence, by Lemma 3.3.10 $X_k \cap X_j = \emptyset$. So, if $v_l \sim X_k$ we cannot have $x_l x_j \in E(X(G))$.

So, suppose $v_l \sim X_j$ where X_j is the mincut incident on v_j. Since $v_i v_j \in E(G)$ we have $X_j \cap X_i \neq \emptyset$. But that would imply $x_l x_i \in E(X(G))$ and hence $X(G) \ncong G$. $\square$

Lemma 3.3.15. *Let $X_k = \langle A, \bar{A} \rangle$ be a non-trivial mincut of a connected graph G with edge connectivity λ. If either $|V_A|$ or $|V_{\bar{A}}| = 1$ then $G \ncong X(G)$.*

Proof. Let $v_k \in A$ be the only vertex of attachment of A. Since X_k is a mincut we must have $|N_{\bar{A}} v_k| = \lambda$. If $N_A(v_k) = \emptyset$ then X_k is a trivial cut. Therefore we must have some $v_l \in N_A(v_k)$ and v_k is non-trivial. But $|N_A(v_k)| \geq \lambda$, otherwise we have an edge set with cardinality less than λ that will disconnect v_k from A, contradicting the minimality of λ.

If $N_A(v_k) = \lambda$ then there exists a mincut, $X_l = \langle B, \bar{B} \rangle$ say, such that $v_k \in \bar{B}$ and $|V_{\bar{B}}| = 1$. Hence $X_l \cap X_k = \emptyset$ and $X(G)$ will be disconnected. In fact no $X_i \in E(G[A])$, the edge set of the induced subgraph on A, can intersect X_k and $X(G)$ will be disconnected.

Suppose there is no mincut, trivial or otherwise, of G in the subgraph induced by A, $G[A]$. Then X_k is mapped to $x_k \in V(X(G))$ but none of the other vertices in $N_A(v_k)$ are mapped to a corresponding component in $X(G)$ and $X(G) \ncong G$. $\square$

Theorem 3.3.16. *Let $G \ncong K_0$. Then $G \cong X(G)$ if, and only if, G is super-λ and r-regular.*

Proof. By Lemma 2.1.4, we know that K_0 is fixed under the operator.

By Corollary 3.2.4, if G is super-λ and r-regular then $G \cong X(G)$.

To prove necessity we need to show that if $G \cong X(G)$ then G is super-λ and r-regular. We will prove the contrapositive, that is, if G does not have the property of being *both* super-λ and r-regular then $G \ncong X(G)$. Hence, we need to prove that if G is not super-λ or not r-regular then $G \ncong X(G)$.

Suppose G is super-λ but not r-regular.

If G is super-λ then all mincuts are trivial, that is only edge sets incident on trivial vertices are mincuts. But G is not r-regular and we conclude that there must be vertices $v \in V(G)$ such that $deg(v) > \lambda$. Hence, we have $|X| < |V(G)|$, the number of mincuts is less than the number of vertices and consequently $|V(X(G))| < |V(G)|$, so $G \ncong X(G)$.

It remains to show that if G is not super-λ but r-regular or G is not super-λ and not r-regular, then $G \ncong X(G)$. We show that G being not super-λ is sufficient for $G \ncong X(G)$.

Suppose $G \cong X(G)$ and G is not super-λ.

If G is not super-λ then not all mincuts are trivial. Let X_k be a non-trivial mincut with A and $\bar{A}$ the two components of $G\backslash X_k$ and V_A and $V_{\bar{A}}$ their respective sets of vertices of attachment. By Lemmas 3.3.13 and 3.3.14 there exists a non-trivial vertex $v_k \in V(G)$ such that $v_k \sim X_k$. We have a number of cases of how the position of v and $N(v)$ relate to the sets of vertices of attachment corresponding to X_k. We start with the supposition that $\lambda = \delta$ and *all* the neighbours of v_k are trivial.

Case I: Suppose $v_k \in V_A$ and we have $v_i \in N_A(v_k)$, but $v_i \notin V_A$. Then, by Lemma 3.3.11 $X_i \cap X_k = \emptyset$ and $x_i \notin N(x_k)$ in $X(G)$. The same holds if $v_k \notin V_A$ and $v_i \notin V_A$. We conclude that any $v_i \in N_A(v_k)$ must be in V_A.

Case II: Suppose $v_k \notin V_A$ is adjacent to $v_i \in V_A$ and $v_i v_j \in E(G)$ where $v_j \in V_{\bar{A}}$ is trivial. Then $v_k v_j \notin E(G)$ but $X_k \cap X_j \neq \emptyset$ and hence $x_k x_j \in E(X(G))$.

We conclude that we must have $v_k \in V_A$ and all trivial vertices $v \in N_A(v_k)$ must be in V_A. Furthermore this implies that all other neighbours of v_k are in $V_{\bar{A}}$.

Case III: Suppose $v_k \in V_A$, and all $v \in N(v_k)$ are trivial and are either in V_A or $V_{\bar{A}}$. All $v_i \in V_A$, $V_{\bar{A}}$ are neighbours of v_k and, since their mincuts intersect X_k, their corresponding vertices x_i are neighbours of x_k.

X_k is a mincut and hence $|X_k| = \lambda$. Consider the neighbours of v_k. Each vertex $v \in N_{\bar{A}}(v_k)$ contributes at least one edge to X_k (edges incident on v_k) and each vertex $v \in N_A(v_k)$ contributes at least one edge to X_k since they are in V_A. Therefore

$$\lambda = |X_k| \geq |N_{\bar{A}}(v_k)| + |N_A(v_k)|.$$

But $deg(v_k) = |N_{\bar{A}}(v_k)| + |N_A(v_k)|$ contradicting the non-triviality of v_k.

From cases I to III, if all $v \in N(v_k)$ are trivial, then $G \ncong X(G)$. So, suppose we have some $v_l \in N(v_k)$ such that $deg(v_l) > \lambda$.

Case IV: Suppose $v_l \in V_A$, then we have the same situation as in the previous case. The neighbours of v_k contribute at least one edge to X_k regardless of whether they are trivial or not as long as they are in V_A or $V_{\bar{A}}$ and since $deg(v_k) > \lambda$ we have $|X_k| > \lambda$ and then X_k cannot be a mincut.

From Cases I to IV we can now conclude the following:

Firstly, if v_i and v_j are trivial and v_k is non-trivial with $v_k \in A$, $v_i \in N_A(v_k)$ and $v_j \in V_{\bar{A}}$, then v_i, $v_k \in V_A$. Furthermore, if all $v \in N(v_k)$ are trivial then X_k is not a mincut.

Secondly, if all $v \in N(v_k)$ are in V_A or $V_{\bar{A}}$, then X_k is not a mincut. Importantly, this also applies if $V_A = A$. So there must be some $v_l \in N(v_k)$ such that $v_l \in A$ but $v_l \notin V_A$. If v_l is trivial then $v_l \in V_A$, so v_l must be non-trivial.

Case V: Let $v_l \in N_A(v_k)$ but $v_l \notin V_A$ and $deg(v_l) > \lambda$. This is different to the first case since here $v_l \in N_A(v_k)$, but v_l is not trivial. By Lemmas 3.3.13 and 3.3.14 there exists a non-trivial mincut $X_l = \langle B, \bar{B} \rangle$ such that $X_l \sim v_l$. If $G \cong X(G)$ then $X_l \cap X_k \neq \emptyset$. Let V_B and $V_{\bar{B}}$ be the respective sets of vertices of attachment of B and $\bar{B}$. Assume that there are some $v \in N_B(v_l)$ and $v \in V_{\bar{B}}$ that are trivial, so $v_l \in V_B$.

From Lemma 3.3.3 we know that if X_k and X_l are crossing mincuts, then $X_k \cap X_l = \emptyset$, so X_l is nested with respect to X_k and $B \subset A$, so $|B| < |A|$. If $A \subset B$ we simply interchange v_k and v_l and the reasoning is the same.

Applying the same reasoning following cases I to IV to v_l, we must have some non-trivial $v_m \in N_B(v_l)$ such that $v_m \notin V_B$ and by Lemma 3.3.13 we have a non-trivial mincut $v_m \sim X_m = \langle C, \bar{C} \rangle$. Also $X_m \cap X_l \neq \emptyset$ which implies that $C \subset B$ with $|C| < |B|$. Again, assume that there are some $v \in N_C(v_k)$ and $v \in V_{\bar{C}}$ that are trivial, so $v_m \in V_C$.

We continue the process of identifying some non-trivial, adjacent $v \notin V_{(\cdot)}$, which implies the existence of a nested non-trivial $X_{(\cdot)}$ and each successive vertex set is smaller than the previous one.

Eventually we must get to a non-trivial $v_z \sim X_z = \langle Z, \bar{Z} \rangle$, such that $v_z \in N_Y(v_y)$, for some $v_y \sim X_y = \langle Y, \bar{Y} \rangle$, with $v_z \notin V_Y$, where $Z = V_Z$. If v_z is trivial then it must be in V_Y. Recall from Lemma 3.3.15 that if X_z is non-trivial then $V_Z > 1$. Since $Z = V_z$, all neighbours of v_z are in V_Z or $V_{\bar{Z}}$ and hence X_z cannot be a mincut.

Consequently we cannot have a non-trivial mincut for every non-trivial vertex and we conclude that $|V(G)| > |V(X(G))|$ so $G \ncong X(G)$.

Case VI: Finally it remains to show that if G is not super-λ and G has no trivial mincuts, then $G \ncong X(G)$. Let $X_k = \langle A, \bar{A} \rangle$ be a nontrivial mincut of G where $\lambda(G) < \delta$ so G has no trivial mincuts. We know from Lemma 3.3.15 that $|V_A| > 1$. Let v_k be one of the vertices in V_A and $v_l \in N_A(v_k)$ such that $v_l \notin V_A$. There must be such a v_l otherwise all neighbours of v_k are in V_A or $V_{\bar{A}}$ which would mean that X_k is not a mincut. Hence we also have $v_l \sim X_l = \langle B, \bar{B} \rangle$.

If $G \cong X(G)$ we must have $X_l \cap X_k \neq \emptyset$ so X_l is nested and $B \subset A$. As before, continuing the process of identifying adjacent vertices and corresponding non-trivial cuts leads to successively smaller vertex sets and eventually we will have a situation where $X_z = \langle Z, \bar{Z} \rangle$ is not a mincut, leading to the conclusion that $|X| < V(G)$ and hence $G \ncong X(G)$.

We conclude that if G is not super-λ we cannot have $G \cong X(G)$.

□

By Lemma 2.1.4 and Theorem 3.3.16 we have shown that the only graphs fixed under the operator are K_0 and graphs that are super-λ and r-regular. These graphs are, therefore, 1-periodic since $X^n(G) \cong G$ for $n = 1$. This fact is very important when we examine the question of convergence and divergence under the operator in Chapter 4. We note, once again, that we need to exclude K_2, and graphs isomorphic to it, from the class of graphs fixed under the operator, since $X(K_2) \cong K_1$ and $X(K_1) \cong K_0$.

Chapter 4

Iteration of the mincut operator

4.1 Introduction

In this chapter, we address the question of what happens when the mincut operator is applied repeatedly to a graph. If a graph is fixed under the operator then iteration of the operator just yields the same graph. We characterised such graphs for the X-operator in Section 3.3. But in general, when does a graph "grow bigger" or "smaller" or re-appear after more than one iteration? Is there a unique graph that will yield the graph after a number of iterations? These questions concern divergence, convergence, periodicity, roots and depth of a graph under the operator. In this chapter we show that every graph has infinite depth under the X-operator, $X(\cdot)$, that no graph diverges, and we make the conjecture that most graphs converge to the null graph, K_0, with no vertices and no edges after repeated iteration of $X(\cdot)$. Furthermore, we identify two classes of graphs of period 2 under the X-operator.

We recall from Definition 1.6.1 that an operator is a mapping ϕ of every graph G from some class of graphs to a new graph $\phi(G)$ and that we define the operator recursively such that $\phi^1 = \phi$ and $\phi^k(G) = \phi(\phi^{k-1}(G))$ for some positive integer $k \geq 2$. Thus, we will denote n iterations of an operator ϕ by $\phi^n(G)$.

In [37], Wilf and van Rooij proved the following theorem on what happens when the line graph of G, $L(G)$, is iterated.

Theorem 4.1.1. *The sequence of graphs G, $L(G)$, $L^2(G)$, $\ldots$, where G is connected,*

1. *has ultimately steadily increasing numbers of vertices if G has at least one vertex with $deg(v) \geq 3$ and $G \not\cong K_{1,3}$,*
2. *is of the form G, G, G, $\ldots$, if G is regular of degree 2,*
3. *is of the form G, H, H, H, $\ldots$, if $G \cong K_{1,3}$ and hence $H \cong C_3$,*
4. *is of the form G_1, G_2, $\ldots$, G_n, $\emptyset$, $\emptyset$, $\ldots$, if G has all $deg(v) \leq 2$ and some $deg(v) < 2$.*

Example 4.1.2. We illustrate the iteration of the line graph in Figure 4.1.

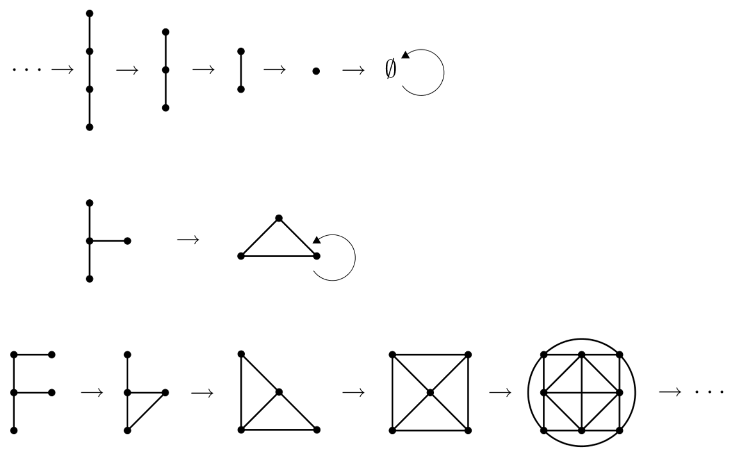

Figure 4.1: Effect of the iteration of the line graph operator on some graphs.

The first graph in the figure is a path, that is, a graph with all vertices $deg(v) \leq 2$ and some vertices $deg(v) < 2$ and repeated application of the line graph operator eventually leads to the null graph. The second graph is $K_{1,3}$ and $L(K_{1,3}) \cong C_3$. Since C_3 is a cycle and hence regular of degree 2, all further iterations of the operator are congruent to C_3. Cycles are fixed under the line graph operator. The third graph, also called the F graph, has one vertex with $deg(v) = 3$ and is not $K_{1,3}$. Iteration of the operator at first yields a decrease in vertices, but ultimately has steadily increasing numbers of vertices. We note that $K_{1,3}$ is not a line graph and since the F graph has an induced $K_{1,3}$, it is also not the line graph of any graph. Refer to Figure 3.1, where we showed the nine forbidden subgraphs for a graph to be a line graph.

Example 4.1.3. We examine the iteration of the mincut operator on some graphs in Figure 4.2. Iteration of the X-operator on the first graph leads to the null graph K_0. We note that any graph where the set of mincuts can be partitioned into two or more disjoint sets, will give a disconnected graph and hence iteration will lead to K_0. Iteration of the operator on the second graph leads through the wheel W_6 to the cycle C_5 and hence to the line graph of the complete graph $L(K_5)$ as per Proposition 2.2.10, which is fixed under the operator. In the third graph we have the *vertex join* of a graph, $K_1 \vee G$, recall Definition 3.2.5, which gives a graph that once again has the same join as its mincut graph and hence iteration of the operator forms a cycle between the two graphs.

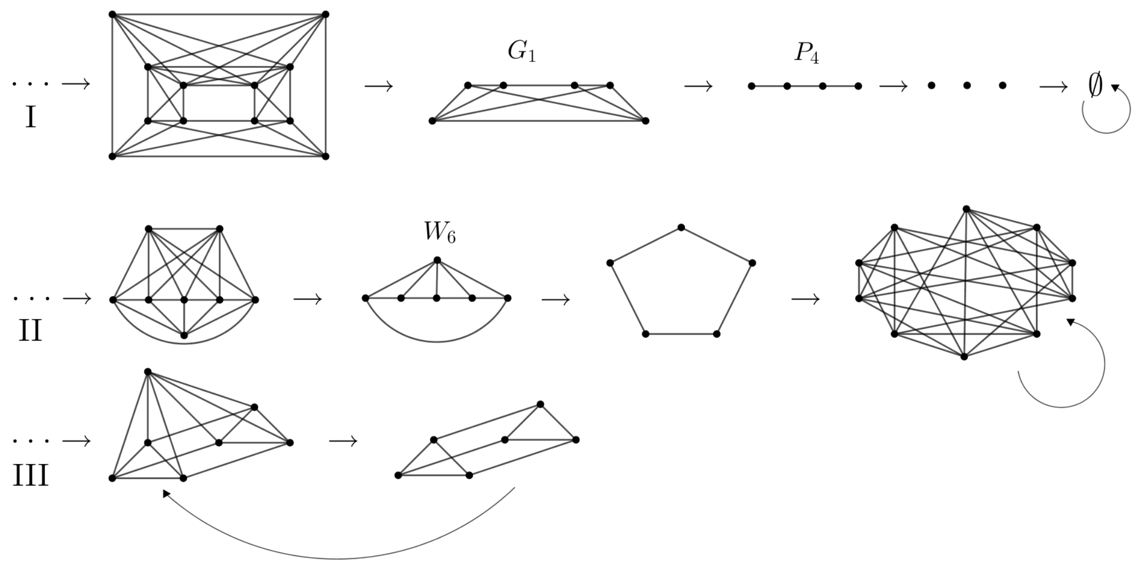

Figure 4.2: Effect of the iteration of the mincut operator on some graphs.

The definitions on *roots* and *depth*, *convergence/ divergence*, etc. in the following sections are from [32].

4.2 Roots and depth

Recall Definition 3.2.1, that a graph G is a ϕ-root of a graph H if $\phi(G) \cong H$. With respect to roots and iteration we have the following definition of the depth of a graph under some operator.

Definition 4.2.1. For a graph G and an operator ϕ we define the ϕ-depth, $depth(G)$, as the largest integer (if there is one, otherwise ∞) for which there is some graph H such that $\phi^d(H) \cong G$.

Proposition 4.2.2. *Every graph G has an infinite number of X-roots.*

Proof. The proof follows directly from the construction in Theorem 3.2.7. Since our construction starts with $G \cup K_m$ with appropriate choice of m, it follows that any choice of K_n with $n > m$ will have the desired mincut graph. $\square$

Proposition 4.2.3. *Every graph G has infinite X-depth.*

Proof. By Theorem 3.2.7, every graph is the mincut graph of a family of graphs. Therefore, given a graph G_0, there is some graph $G_{(-1)}$ such that $X(G_{(-1)}) \cong G_0$. By the same reasoning there is a graph $G_{(-2)}$ such that $X(G_{(-2)}) \cong G_{(-1)}$. Thus we can construct a set $\{\ldots, G_{(-3)}, G_{(-2)}, G_{(-1)}, G_0\}$ where every $X(G_{(-n)}) \cong G_{(-n+1)})$ for all $n \geq 1$ and $X^n(G_{(-n)}) \cong G_0$. $\square$

Example 4.2.4. We refer to the graphs in Figure 4.2. Each sequence represents a number of iterations under $X(\cdot)$. We know that $X(P_4) \cong \overline{K}_3$ since it is basically a tree on four vertices, and since $\overline{K}_3$ is disconnected $X(\overline{K}_3) \cong K_0$ by Lemma 2.1.3. The graph G_1 was obtained from P_4 by the construction in Theorem 3.2.7 using $P_4 \cup K_2$. We note that any graph with exactly three mincuts that do not intersect, is also a root of $\overline{K}_3$ and any disconnected graph is a root of K_0. In the second iteration sequence, we have C_5 as a root of $L(K_5)$. We could have constructed a root using our construction method from Theorem 3.2.7, or, since line graphs of complete graphs are fixed under $X(\cdot)$, we could simply have $L(K_5)$. W_6 is a root of C_5 since $W_6 \cong K_1 \vee C_5$ and C_5 is 2-regular with $n = 5 > 2 + 1$ (recall Lemma 3.2.6). We applied our construction technique again to G_1 and W_6 respectively to continue each sequence in reverse. We can continue to apply the construction from Theorem 3.2.7 and generate an infinite sequence of graphs such that each is an X-root of the next. Admittedly our graphs will become very large very soon but it is possible in principle.

4.2.1 Determination problem

We know from Whitney that a graph is uniquely determined by its line graph up to isolated vertices, unless it is C_3 or $K_{1,3}$, see [33]. Although our construction in Theorem 3.2.7 gives the interesting result that every graph is a mincut graph, it, unfortunately, does not shed much light on what the mincut graph $X(G)$ tells us about the mincuts of G if G is not a graph of the family obtained by our construction. In this section we explore this problem in a little more detail.

We know, from Definition 3.3.1, that if $X = \langle A, \bar{A}\rangle$ and $Y = \langle B, \bar{B}\rangle$ are two mincuts of G and $A \cap B \neq \emptyset$ then either $A \subset B$ (or $B \subset A$), that is, they are *nested*, or they *overlap*. If X and Y overlap (also called *crossing mincuts*), then $A \cap B$, $\bar{A} \cap B$ and $A \cap \bar{B}$ are *non-empty*. We also note from [21] that two mincuts can overlap only if λ, the minimum edge connectivity number of the graph, is even and recall from Lemma 3.3.3 that overlapping mincuts do not share any edges.

Let $X = \langle A, \bar{A}\rangle$ and $Y = \langle B, \bar{B}\rangle$ be overlapping mincuts of a graph G with non-trivial mincuts $W = \langle A \cap B, \bar{A} \cup \bar{B}\rangle$, $Z = \langle A \cup B, \bar{A} \cap \bar{B}\rangle$ as per Proposition 3.3.2. Suppose that $W \cap X$, Y and X, $Y \cap Z$ are non-empty, then we have a corresponding induced cycle C_4 in $X(G)$. Given what we know from Proposition 3.3.2, it would seem that we should at least be able to construct an induced subgraph in G corresponding to the cycle in $X(G)$ without resorting to the construction in Theorem 3.2.7.

Similarly, let $X = \langle A, \bar{A} \rangle$, $Y = \langle B, \bar{B} \rangle$ and $Z = \langle C, \bar{C} \rangle$ be non-trivial nested mincuts of a graph G with $A \subset B \subset C$. If the induced subgraph in $X(G)$ corresponding to the mincuts X, Y and Z is connected then we should have at least an induced path P_3 in $X(G)$, or, possibly, a cycle C_3.

It would also be interesting to know whether we can have induced cycles larger than C_4 in $X(G)$ such that the corresponding mincuts in G are not of the form as per our construction in Theorem 3.2.7.

4.3 Convergence

We know from Section 4.2 that every graph G has infinitely many X-roots and is of infinite X-depth. We note in Figure 4.2 that iteration of the first graph leads to K_0. Recall Theorem 2.1.5 that the mincut graph of a graph is unique. Hence, each of the infinite roots of the first graph in sequence I will lead to K_0. Similarly, each of the infinite roots of the first graph in sequence II will lead to $L(K_5)$. In sequence III each of the infinite roots of the first graph will lead to the circuit $K_1 \vee K_3 \times K_2 \leftrightarrow K_3 \times K_2$. We say that these graphs *converge* to the respective graphs and/or circuits. In this section, we clarify what we mean by convergence or divergence under an operator. We show that no graph diverges under $X(\cdot)$ and we make the conjecture that, in most cases, iteration of the operator leads to the null graph K_0.

Recall Definition 1.6.2 that a graph G is *ϕ-periodic* under an operator ϕ if $G \cong \phi^p(G)$ for some integer $p \geq 1$. The smallest such integer p is called the *ϕ-period* of G.

Definition 4.3.1. Let G be a graph and ϕ an operator on G. G is *convergent* if $\{\phi^n(G) | n \in \mathbb{N}\}$ is finite. That is, G is ϕ-convergent if and only if some iterated ϕ-graph is ϕ-periodic. If G is not convergent then it is *divergent*.

We recall from Lemma 2.1.4 that K_0 is fixed under the X-operator and, hence, is 1-periodic by Definition 1.6.2.

Definition 4.3.2. A *circuit* is any set of the form

$$\{G, \phi(G), \phi^2(G), \ldots, \phi^{p-1}(G), \phi^p(G) \cong G\},$$

where G is a graph and ϕ an operator on G.

Let G be some convergent graph. Then there is a unique circuit M such that $\phi^n(G) \in M$ for sufficiently large n, see [32].

Do any graphs diverge under $X(\cdot)$? We usually distinguish ϕ-convergent from ϕ-divergent graphs by means of some parameter that increases under ϕ, see [33]. With respect to the X-operator we consider the order, $|V|$, and size, $|E|$, of successive graphs under iteration of $X(\cdot)$ and the implications of increasing these numbers with respect to the number of mincuts of a graph, $|X|$, and the number of non-empty intersections, $|\cap X_i|$, of edge sets belonging to X, the set of mincuts of a graph.

Definition 4.3.3. The *edge density* , D, of a graph is the ratio of edges to the total number of edges possible, that is,

$$D = \frac{|E|}{\binom{n}{2}} = \frac{2|E|}{n(n-1)}.$$

We recall from the definition of the mincut graph that the number of mincuts $|X|$ in G is the number of vertices in $X(G)$ and the number of intersections of the mincuts in G is the number of edges in $X(G)$. Now, the maximum number of edges in a simple connected graph is $\binom{n}{2}$, where every choice of two vertices are adjacent, and when this happens the graph is complete, in other words, fixed under the X-operator and the graph converges. Furthermore we know that the maximum number of mincuts in a graph of order n is $\binom{n}{2}$ and this bound is tight for simple graphs when $G \cong C_n$, see [8, 21], and $X(C_n) \cong L(K_n)$, which is fixed. We also have other upper bounds on the number of mincuts of a graph depending on whether λ is even or odd, see [21],

$$|X| \leq \begin{cases} \frac{2n^2}{(\lambda+1)^2} + \frac{(\lambda-1)n}{\lambda+1}, & \text{if } \lambda \geq 4 \text{ and } \lambda \text{ is an even integer,} \\ (1 + \frac{4}{\lambda+5})n, & \text{if } \lambda > 5 \text{ and } \lambda \text{ is an odd integer.} \end{cases}$$

Considering these bounds we should expect that, for large λ, $|X|$ approaches n and we have an upper limit on the number of vertices of $X(G)$ and that $X(G)$ must at some point either become fixed, periodic or disconnected. We note, with reference to this last case, that if at any stage of the iteration we have a mincut (subset of mincuts) that does not intersect with any other mincut (shares no element with any other subset of mincuts), then $X(G)$ is disconnected and converges to the null graph K_0.

We recall from Proposition 1.5.10 that if $deg(u) + deg(v) \geq n$ and $G \ncong K_{n/2} \times K_2$ then G is super-λ. In the light of convergence we will use this sufficient condition since we know that $K_{n/2} \times K_2$ is 2-periodic, see for example graph III in Figure 4.2, and hence converges by Definitions 4.3.1 and 4.3.2. We prove this periodicity in Section 4.4.

Theorem 4.3.4. *Let G be a simple graph of order n and size m, minimum degree δ, and minimum edge-connectivity λ. Then G converges under iteration of the X-operator, that is, $X^i(G)$ converges for sufficiently large i.*

Proof. We proceed by showing that no graph diverges under iteration of $X(\cdot)$. In order for G to diverge we must have an increase in n, m or both.

Suppose n increases but m does not. If we get to a point where $n = m$ we have a cycle and the next iteration is $L(K_n)$, fixed under the operator. If $m = n - 1$ we have a tree and the next iteration is disconnected leading to the null graph. If $m < n - 1$ then G is disconnected and $X(G) \cong K_0$. In other words, if the graph becomes sufficiently *sparse* then $X^i(G)$ converges to the null graph.

Suppose m increases but n does not. We get to $m = \binom{n}{2}$, the maximum number of edges possible. But then G is complete and fixed under the operator. In other words, if the graph becomes sufficiently *dense* $X^i(G)$ becomes fixed.

Hence, we need both n and m to increase under iteration of the operator in order for the graph to diverge. In order for n to increase we need $|X|$ to increase, and in order for m to increase we need the number of intersecting mincuts $|\cap X_i|$ to increase as we iterate the operator. In short, for each G_i in the iteration sequence, we need $|X| \geq n$ and $|\cap X_i| \geq m$, where one of the inequalities has to be strict, but for this to happen we need λ to increase. If λ does not increase, the size of each X_i stays the same (or decreases) and eventually n increases but m does not, since the number of possible intersections must depend on the size of the mincuts. If λ increases, δ increases since $\kappa \leq \lambda \leq \delta$, as originally proved by Whitney. Now if $\delta \geq \frac{n}{2}$, or equivalently, $deg(u) + deg(v) \geq n$ for any $u, v \in V(G)$, then G is super-λ or $G \cong K_{n/2} \times K_2$, by Proposition 1.5.10.

If $G \cong K_{n/2} \times K_2$ then it is not diffcult to show that G is 2-periodic and hence converges.

If G is super-λ and r-regular then G is fixed, by Theorem 3.3.16.

If G is super-λ but not r-regular, then $|X| < |V(G)|$, see the relevant part of the proof to Theorem 3.3.16. That is, $|V(X(G))| < |V(G)|$, so n decreases and the graph cannot diverge. □

4.3.1 Iteration to the null graph

If $X(G)$ is disconnected, $X(X(G))$ is the null graph. That is, if there is any of the X_i, or subset of intersecting X_i, that does not intersect with any other of the X_i, or subset of intersecting X_i, then the operator will yield a disconnected graph and, hence, iteration leads to the null graph. In general, it would therefore seem that, in most cases, repeated application of X should at some stage yield a disconnected graph G with $\lambda(G) = 0$ and hence $X(G)$ is the null graph with no vertices and no edges.

Conjecture 4.3.5 (Convergence to null graph)**.** Let G be a simple connected graph and $X(\cdot)$ the mincut operator. Then $X^k(G) \to K_0$ except under a finite number of conditions.

4.3.2 Connected mincut graphs

A question that naturally arises and that is closely connected to the previous question is, under what conditions will $X(G)$ be connected. That is, what (if any) are the properties of G that will ensure that every X_i, or subset of intersecting X_i, intersects with some other X_i, or intersecting subset of X_i. Furthermore, does the edge-connectivity of $X(G)$ tell us anything about the connectivity of G?

What leads to G not converging to the null graph? We recall from Lemmas 3.3.10 and 3.3.11, that if adjacent vertices are trivial then their mincuts intersect and if a trivial vertex is an element of the vertices of attachment of a non-trivial cut then their intersections are nonempty. It would therefore seem that a high proportion of trivial vertices to the total number of vertices guarantees a connected mincut graph. See for example the cycle: large cycles have sparse edge density but their mincuts have high edge density and every vertex is trivial.

4.4 Periodicity

Recall from Definition 1.6.2 that a graph G is n-periodic in an operator ϕ if $\phi^n(G) \cong G$, for some $n \in \mathbb{N}$. Thus graphs fixed under the operator are graphs of period 1 and by Theorem 3.3.16 the only such graphs fixed under $X(\cdot)$ are either K_0 or otherwise super-λ and r-regular. We note that a graph G is *convergent* if and only if G is periodic or there is some integer $n \geq 1$ with $\phi^n(G)$ periodic, see [32]. In this section we identify two classes of graphs that are 2-periodic under $X(\cdot)$.

4.4.1 The Cartesian product $K_n \times K_2$

The first class of graphs we examine is $K_n \times K_2$, the Cartesian product of any complete graph, K_n, and K_2.

Definition 4.4.1. The *Cartesian product*, $G = G_1 \times G_2$, of two graphs G_1 and G_2

has vertex set $V(G_1) \times V(G_2)$. Two distinct vertices (u, v) and (w, z), $u, w \in V(G_1)$ and $v, z \in V(G_2)$, are adjacent if $u = w$ and $vz \in E(G_2)$ or $v = z$ and $uw \in E(G_1)$.

Example 4.4.2. We illustrate the Cartesian product of two graphs, P_3 and K_2, in Figure 4.3. We can think of drawing the graph by replacing each vertex in K_2 with a copy of P_3 and joining the corresponding vertices in the two copies of P_3 by an edge.

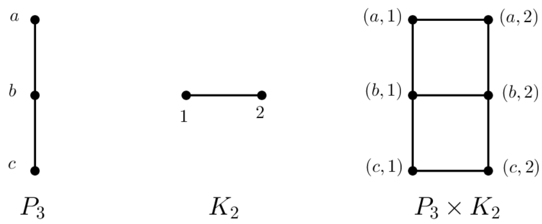

Figure 4.3: $P_3 \times K_2$.

Theorem 4.4.4 shows that the Cartesian product $K_n \times K_2$ is a 2-periodic graph under the mincut operator.

Lemma 4.4.3. *Let $K_n \times K_2$ be the Cartesian product of K_n and K_2 and $K_1 \vee K_n \times K_2$ be the vertex join of $K_n \times K_2$. Then the mincut graph of $K_n \times K_2$ is*

$$X(K_n \times K_2) \cong K_1 \vee K_n \times K_2,\ n > 2.$$

Proof. From Proposition 2.2.3, $\lambda(K_n) = n - 1$ and each mincut of K_n is exactly the set of $n - 1$ edges incident on every vertex. Now, in the Cartesian product, each vertex is joined to a corresponding vertex in a copy of K_n. Thus $\lambda(K_n \times K_2) = n$ and the vertices of $X(K_n \times K_2)$ are the vertices corresponding to vertices of $K_n \times K_2$, all trivial, with one additional vertex representing the mincut on the n edges between the two copies of K_n. Since each of the mincuts in the two individual copies of K_n contains an element from the mincut joining the two copies, the additional mincut intersects with all the other mincuts.

Without loss of generality, let $n = 4$, then $K_4 \times K_2$ is as shown in Figure 4.4.

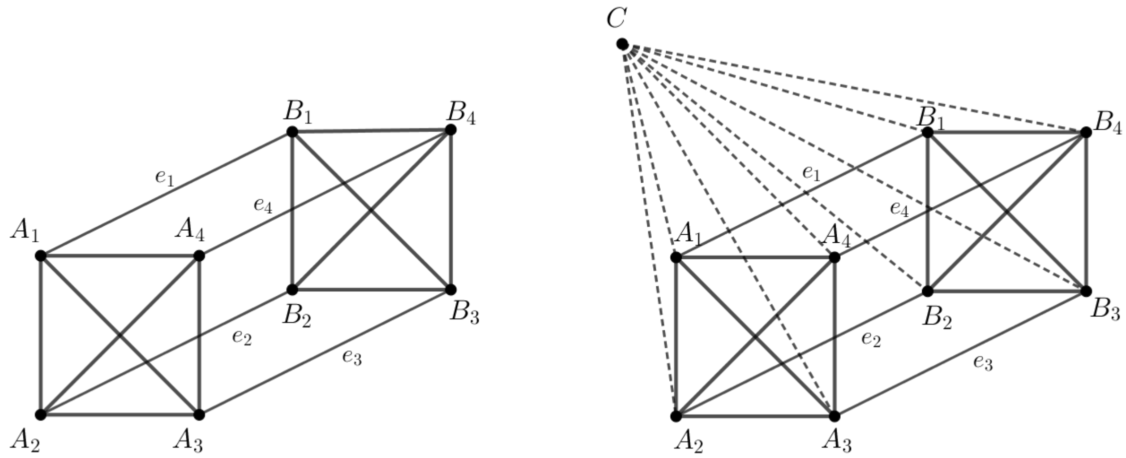

Figure 4.4: $K_4 \times K_2$ and $K_1 \vee K_4 \times K_2$.

The edges incident on each of the vertices $A_1 \ldots A_4$ and $B_1 \ldots B_4$ are mincuts and preserve the adjacencies of $K_4 \times K_2$. In addition, the edge set $\{e_1, e_2, e_3, e_4\}$ is a mincut and it intersects all eight the other mincuts. Thus we have a vertex join of the product graph. We set $n > 2$ since $K_1 \times K_2 \cong K_2$ with $X(K_2) \cong K_1$, and $K_2 \times K_2 \cong C_4$ with $X(C_4) \cong L(K_4)$. $\square$

Theorem 4.4.4. *The Cartesian product $K_n \times K_2$ is 2-periodic under the mincut operator $X(\cdot)$ for $n > 2$.*

Proof. By Lemma 4.4.3 we have $X(K_n \times K_2) \cong K_1 \vee K_n \times K_2$.

By definition the Cartesian product $K_n \times K_2$ has $|V(K_n \times K_2)| = 2n$ and $\delta(K_n \times K_2) = n$. Also, $K_n \times K_2$ is n-regular. Thus we have an n-regular graph of order $2n > n + 1$. Hence, by Lemma 3.2.6, $X(K_1 \vee K_n \times K_2) \cong K_n \times K_2$.

Thus we have a class of graphs where $X^i(G) \cong G$ for $i = 2$ but not for $i = 1$. $\square$

4.4.2 Complete component cycles

In this section we construct a graph that we will call a complete component cycle, and show that it is 2-periodic under the X-operator.

Definition 4.4.5. Let G be a cycle on n vertices and replace each vertex with K_m such that m is even and $m > 2$. Connect $m/2$ vertices from each complete component to $m/2$ corresponding vertices in each of the two adjacent complete components and delete the original edges of the cycle such that each vertex in the new graph has degree m. We call this new graph an (m, n)-*complete component cycle* and denote it by $C_{m,n}$.

Example 4.4.6. Figure 4.5 shows $C_{4,4}$, $C_{4,5}$ and $C_{6,4}$ replacing each vertex in a C_4 with K_4, each vertex in a C_5 with K_4 and each vertex in C_4 with K_6, respectively.

We have labelled each complete component $\{A,\ B,\ C, \ldots\}$ and each of the component-connecting sets of edges $\{1,\ 2,\ 3\ldots\}$. We note, for example, that the $C_{4,4}$ graph is four regular with $\lambda = 4$ and that each vertex is therefore a trivial vertex. The graph is not super-λ, however, since we also have six non-trivial mincuts, $W = 1 \cup 4 = \{e_1,\ e_2,\ e_7,\ e_8\}$, $X = 1 \cup 2 = \{e_1,\ e_2,\ e_3,\ e_4\}$, $Y = 2 \cup 3 = \{e_3,\ e_4,\ e_5,\ e_6\}$, $Z = 3 \cup 4 = \{e_5,\ e_6,\ e_7,\ e_8\}$, $S = 1 \cup 3 = \{e_1,\ e_2,\ e_5,\ e_6\}$ and $T = 2 \cup 4 = \{e_3,\ e_4,\ e_7,\ e_8\}$. The mincuts W, X, Y and Z each disconnects one of the complete components, while mincuts S and T each separates two of the complete components from the other two. If we compare the intersections of these six mincuts we note that we get $L(K_4) \cong X(C_4)$ as should be expected. Hence, $X(C_{4,4})$ will have an induced subgraph isomorphic to $L(K_4)$. Also, since each of the vertices is trivial, that is, $deg(v) = \lambda$, the corresponding trivial mincuts will be adjacent and hence we will also have an induced subgraph isomorphic to $C_{4,4}$ in $X(C_{4,4})$.

Lemma 4.4.7. *$C_{m,n}$ is m-regular, has order $|V| = mn$, all trivial vertices, and $\binom{n}{2}$ non-trivial mincuts.*

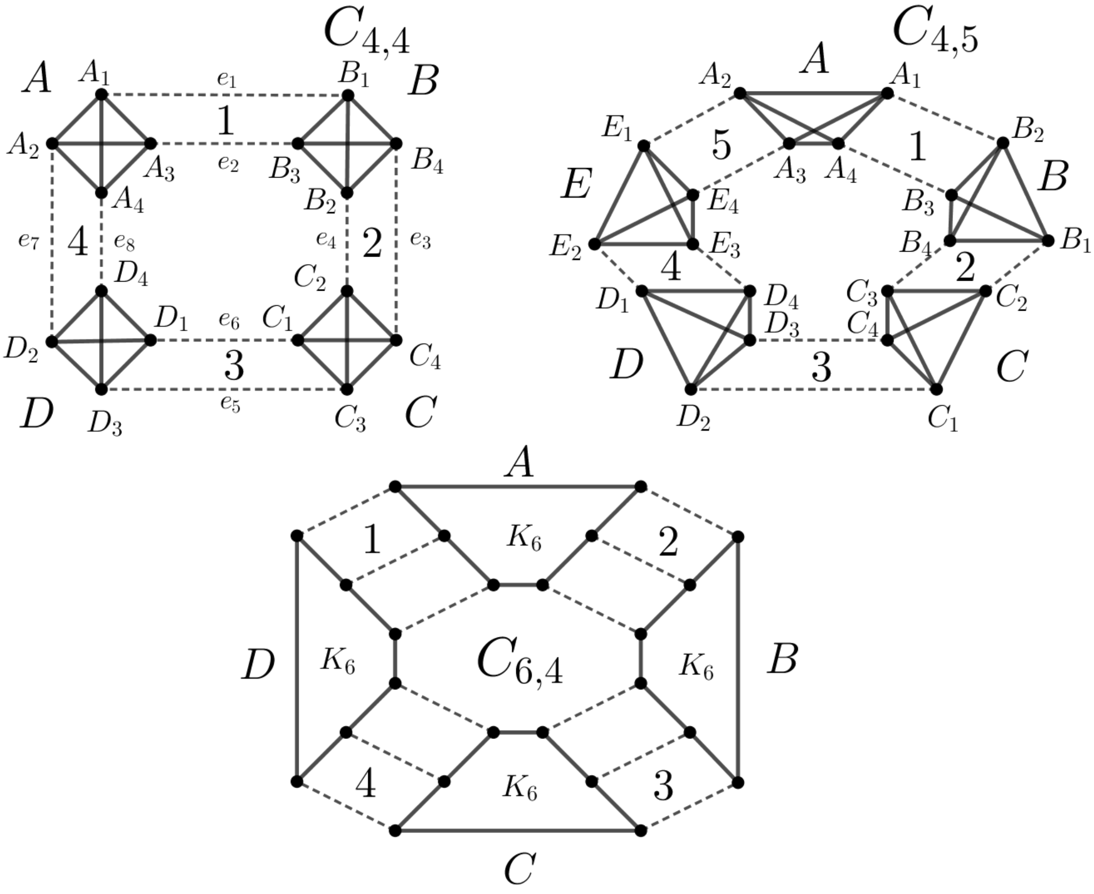

Figure 4.5: $C_{4,4}$, $C_{4,5}$ and $C_{6,4}$.

Proof. Every vertex of a K_m component is connected to one other vertex belonging to one of the two adjacent complete components in $C_{m,n}$. Since K_m is $(m-1)$-regular, $C_{m,n}$ must be m-regular. Also, in the construction of $C_{m,n}$ each of the n vertices of C_n is replaced by the m vertices of a complete component and hence $C_{m,n}$ has order mn.

Each of the edge sets connecting the complete components is of size $m/2$. There are n such subsets replacing the n edges of C_n. Any choice of two such subsets will disconnect the graph. Without loss of generality we refer to $C_{4,4}$ in Figure 4.5. There are four edge sets of size 2 connecting the K_4 complete components and we labelled them 1, 2, 3 and 4. The union of any two of these edge sets gives an edge set of size 4 which disconnects the graph. Hence, there are $\binom{4}{2} = 6$ such edge sets.

The K_m components are complete and the only mincuts in these components are trivial of size m. Thus we can conclude that the unions of the component-connecting edge sets are mincuts and we have $\lambda(C_{m,n}) = m$. The degree of each vertex in $C_{m,n}$ is m and we conclude that the edge set incident on each vertex is a trivial mincut, giving mn trivial mincuts in $C_{m,n}$. □

Lemma 4.4.8. *Let $G \cong C_{m,n}$ and G not a cycle. Then $n \geq 3$ and $m \geq 4$.*

Proof. By construction, we need C_n to be a cycle and hence $n \geq 3$. If $m = 2$ we replace each vertex in C_n with K_2 and each of the vertices of these K_2 components is adjacent to one vertex in each of the adjacent components, giving a cycle, C_{2n}, of size $2n$ and order $2n$. Therefore, we need $m > 2$, but since m is even, we have $m \geq 4$. □

Lemma 4.4.9. *$X(C_{m,n})$ has $\binom{n}{2} + mn$ vertices and has two induced subgraphs $G_1 \cong C_{m,n}$ and $G_2 \cong L(K_n)$.*

Proof. By Lemma 4.4.7, $C_{m,n}$ has mn trivial mincuts and $\binom{n}{2}$ non-trivial mincuts, which implies that $X(C_{m,n})$ has order $\binom{n}{2} + mn$.

We recall from Lemma 3.3.10 that if two trivial vertices are adjacent in G, then their incident edge sets have non-empty intersection and hence their corresponding vertices in $X(G)$ will be adjacent. Since all the vertices in $C_{m,n}$ are trivial, we have an induced subgraph in $X(C_{m,n})$ isomorphic to $C_{m,n}$.

Any choice of two component-connecting edge sets is a mincut which gives an induced subgraph on $\binom{n}{2}$ vertices congruent to $L(K_n)$, by the same reasoning that we used in Proposition 2.2.10 to show that $X(C_n) \cong L(K_n)$. □

Lemma 4.4.10. *Given $X(C_{m,n})$, $\Delta = 2m + 2(n-2)$ and $\delta = m + n - 1$.*

Proof. Let $[C_{m,n}]$ and $[L(K_n)]$ denote the respective induced subgraphs from Lemma 4.4.9. Each of the vertices of $[C_{m,n}]$ is adjacent to m other vertices in $[C_{m,n}]$. Furthermore, the trivial mincut in $C_{m,n}$ corresponding to each vertex in $X(C_{m,n})$ intersects one component-connecting edge set and hence intersects each of the $n-1$ non-trivial mincuts formed with that component-connecting edge set as element. Thus, the degree of each vertex of $[C_{m,n}]$ in $X(C_{m,n})$ is $m+n-1$.

Each component-connecting edge set intersects $m/2 \times 2$ trivial mincuts, the mincuts on the vertices of attachment in each component incident on the component-connecting set. Thus each non-trivial mincut intersects $2m$ trivial mincuts. Therefore the corresponding vertex in $[L(K_n)]$ is adjacent to $2m$ vertices in $[C_{m,n}]$. Furthermore, we recall from Lemma 2.2.8 that $L(K_n)$ is $2(n-2)$-regular. Thus the degree of each vertex of $[L(K_n)]$ in $X(C_{m,n})$ is $2m+2(n-2)$. $\square$

Example 4.4.11. We refer to $C_{4,4}$ in Figure 4.5. The trivial vertex A_1 in component A is adjacent to four trivial vertices, three in component A and one in component B. The trivial edge cut incident on A_1 shares edge e_1 with component-connecting edge set 1. Therefore it intersects three non-trivial mincuts $1 \cup 2$, $1 \cup 3$ and $1 \cup 4$.

Component-connecting edge set 1 intersects four trivial mincuts, those incident on the vertices of attachment A_1, A_3, B_1 and B_3. Similarly component-connecting edge sets 2, 3 and 4 intersect four trivial mincuts each. Thus the non-trivial mincut $1 \cup 2$, for example, intersects $2 \times 4 = 8$ trivial mincuts and $2(4-2) = 4$ non-trivial mincuts, namely $1 \cup 3$, $1 \cup 4$, $2 \cup 3$ and $2 \cup 4$.

Proposition 4.4.12. *$X(C_{m,n})$ is super-λ.*

Proof. All vertices in $X(C_{m,n})$ are either in $[C_{m,n}]$ with degree $\delta = m+n-1$ or $[L(K_n)]$ with degree $\Delta = 2m+2(n-2)$. These are the only trivial cuts and we show that all other disconnecting edge cuts will, like those of size Δ, also have greater cardinality than δ. This implies that all and only the trivial cuts on the vertices belonging to $[C_{m,n}]$ are mincuts.

$[L(K_n)]$ is super-λ and hence has only trivial mincuts of size $2(n-2)$. $[C_{m,n}]$ has $\binom{n}{2}$ non-trivial mincuts, any choice of two component-connecting edges sets.

To disconnect one vertex in $V[L(K_n)]$ from $X(C_{m,n})$, we need to delete $2(n-2)$ edges from $E[L(K_n)]$ and $2m$ edges connecting the vertex to vertices in $V[C_{m,n}]$. Thus we need to delete $2m + 2(n-2)$ edges, and $2m + 2(n-2) > m + n - 1$ if $m + n > 3$, which is true, by Lemma 4.4.8, since $m+n \geq 7$. To disconnect k vertices in $V[L(K_n)]$ we need to delete $p \geq 2(n-2)$ edges from $E[L(K_n)]$ and $2km$ edges connecting the vertices to $[C_{m,n}]$. Hence, we have to delete $2km + p > 2m + 2(n-2) > m + n - 1$ edges to disconnect the graph in this manner.

Suppose we disconnect two adjacent vertices, one in $V[L(K_n)]$ and one in $V[C_{m,n}]$. We need to delete $2(n-2)$ edges in $E[L(K_n)]$, m edges in $E[C_{m,n}]$, and $(2m-1)+(n-2)$ edges joining the two respective subgraphs to each other through the two chosen vertices. That is, we need to delete $3m + 3n - 7$ edges, which is greater than δ if $m + n > 3$. If we choose more than two vertices, say k vertices, in the two induced graphs we must delete more edges than $3m + 3n - 7$, since m and $2(n-2)$ are the minimum edges to delete from $E[C_{m,n}]$ and $E[L(K_n)]$, respectively, while the edges between the two graphs increase by a factor of k, depending on where the other $k-2$ vertices lie.

We proceed to disconnect one vertex in $V[C_{m,n}]$ from the graph. We need to delete m edges from $E([C_{m,n}])$ and $n-1$ edges connecting the vertex to $[L(K_n)]$. Thus we need to delete $\delta = m+n-1$ edges to disconnect the graph in this manner. If we choose k vertices in $V[C_{m,n}]$ we need to delete at least m edges in $E[C_{m,n}]$ to disconnect the k vertices from $[C_{m,n}]$, since $\lambda(C_{m,n}) = m$, by Lemma 4.4.7. Furthermore, we need to delete $k(n-1)$ edges connecting the vertices to $[L(K_n)]$ and we conclude that this number of edges must be strictly greater than $m + n - 1$.

We have $\delta(X(C_{m,n})) = m + n - 1$ and we have shown that only the trivial edge cuts incident on the vertices of $[C_{m,n}]$ have this same size. All other edge cuts are of greater cardinality. Hence, we conclude that $X(C_{m,n})$ is super-λ. $\square$

Theorem 4.4.13. *$C_{m,n}$ is 2-periodic under the mincut operator.*

Proof. By Proposition 4.4.12 $X(C_{m,n})$ is super-λ and the mincuts are the trivial edge sets incident on vertices of degree $m + n = 1$. By Lemma 4.4.10, these are exactly the vertices belonging to $V[C_{m,n}]$. Recall, from Lemma 3.2.2, that if a graph G is super-λ, then $X(G)$ is congruent to the induced subgraph on the set of vertices of minimum degree. We conclude that $X(X(C_{m,n})) \cong C_{m,n}$ and, hence, that $C_{m,n}$ is 2-periodic under the X-operator. $\square$

Chapter 5

Improper colouring and mincuts of a graph

5.1 Introduction

Graph colouring and graph connectivity are two areas in graph theory that are widely studied. In this chapter, we make explicit a relationship between two central concepts in these areas, improper colouring and the minimum edge-cut number of a graph. We show that the number of minimum edge-cuts and the number of possible improper 2-colourings of a graph with $|E| - \lambda$ bad edges are the same. Hence, we are able to determine the number of mincuts of a graph, that is, the order of its mincut graph, through an evaluation of the *bad colouring polynomial*, an evaluation of the well-known Tutte Polynomial.

Recall from Chapter 1, that a graph $G = (V, E)$ is *connected* if there exists a $u - v$ path in G for every pair of vertices $u, v \in V$. A graph that is not connected, is *disconnected*. Also, an *edge-cut* is a set X of edges of G such that $G - X$ is a disconnected graph. An edge-cut of minimum cardinality is called a *minimum edge-cut*. This cardinality is called the *edge-connectivity*, or *minimum edge-cut number*, of G and in previous chapters we have denoted this cardinality with $\lambda(G)$. Since we will be using graph polynomials to count the number of bad colourings, we will use α_1 in this chapter to denote the mincut number of a graph. This is to avoid confusion with the traditional use of λ in the area of graph colouring. Furthermore, we will use $|E|$ to denote the size of a graph, that is, the number of edges.

For more information on the vertex- and edge-connectivity of graphs we refer the reader to Chapter 4 on *Connectivity and Traversability* by Fàbrega and Fiol in [16] and the survey by Oellermann in [29].

We recall the upper bounds on the mincut number from Section 4.3 and that it is known that the maximum number of mincuts has different order of magnitude for graphs with odd and even edge connectivity, see [19]. Let G be a graph with minimum edge-cut number α_1 and n vertices, with $X = \{X_1, X_2, \ldots, X_i\}$ the set of mincuts of G. In [21], Lehel *et al* determine the following upper bounds for $|X|$, the number of minimum cuts, in simple graphs:

$$|X| \leq \begin{cases} \frac{2n^2}{(\alpha_1+1)^2} + \frac{(\alpha_1-1)n}{\alpha_1+1}, & \text{if } \alpha_1 \geq 4 \text{ and } \alpha_1 \text{ is an even integer,} \\ (1 + \frac{4}{\alpha_1+5})n, & \text{if } \alpha_1 > 5 \text{ and } \alpha_1 \text{ is an odd integer.} \end{cases}$$

The problem of counting the number of mincuts of a graph is considered by many authors and various data structures are created to represent all these mincuts, see for example [8, 13, 14].

In this chapter, one of the results is that we count the number of minimum edge-cut sets of G, using the concept of *improper colouring* developed in the area of graph colouring. Using graph polynomials to count edge-cuts of any size is not new, see for example, Theorems 1.3 and 1.4 in [28], for an evaluation of the Tutte Polynomial to get the number of edge-cuts of any size, as well as the “bipartition polynomial” in Chapter 10 of [1], a generating function for the set of bipartitions of a graph, also making use of an evaluation of the Tutte Polynomial. By making the equivalence of an improper 2-colouring on $|E| - \alpha_1$ bad edges and minimum edge cuts explicit, we can use concepts from graph colouring to explore graph connectivity *and* vice versa. We illustrate this in Section 5.4 by using mincuts to show that certain k-defect polynomials of a graph are zero, as well as by using the *bad colouring* polynomial to count the number of mincuts. The sections on Graph Colouring and the Tutte and Bad Colouring polynomials is a summary of the material in [20] with some new examples.

5.1.1 Graph colouring

A *colouring* of a graph G is a mapping $\sigma : V(G) \to C$, from its vertex set to a set C of colours. A *proper colouring* of a graph G is a colouring of the vertices of G in such a way that no two adjacent vertices are coloured with the same colour. The minimum number of colours required for such a colouring is called the *chromatic number* of the graph, denoted $\chi(G)$. A colouring for which this is not true is called an *improper colouring*. To be precise, if we allow an improper colouring of a graph, we allow certain adjacent vertices to have the same colour.

Definition 5.1.1. In an improper colouring of a graph G, a *bad* edge is an edge $e \in E(G)$ with endpoints $u, v \in V(G)$ such that u and v are assigned the same colour. We denote the number of bad edges allowed in an improper λ-colouring of a graph G with k, where k is a non-negative integer.

Example 5.1.2. In Figure 5.1, we show different colourings of the three vertices of K_3 with 0, 1, 2 and 3 bad edges, where the bad edges are solid. For a proper colouring of K_3, that is, no adjacent vertices the same colour, we clearly need a minimum of three colours. If we colour two vertices with colour 1 and the third with colour 2, then we have an improper colouring with one bad edge. Colouring all three vertices with the same colour gives all three bad edges. It should be clear that it is not possible to colour K_3 with two bad edges, since, if the third vertex is also coloured with 1 in order to have a second bad edge, then the third edge must also be bad.

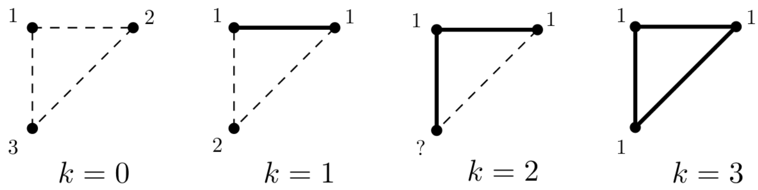

Figure 5.1: Colourings of K_3 with 0, 1, 2 and 3 bad edges. Clearly it is not possible to colour K_3 in such a way that we have two bad edges.

5.1.2 Proper colouring and chromatic polynomials

In this section we define the chromatic polynomial of a graph, a polynomial that allows us to count the number of ways to properly colour a graph given λ colours.

Definition 5.1.3. The number of distinct proper λ-colourings of G, $\lambda \in \mathbb{N}$, which we denote by $\chi(G;\lambda)$, is a polynomial in λ called the *chromatic polynomial* .

The smallest integer value k for which $\chi(G;\lambda) > 0$, is the chromatic number $\chi(G)$ and, by convention, $\chi(G;0) = 0$, see [6]. For more details on the proper colouring of a graph G and an introduction to chromatic polynomials, see [34], as well as the monograph, [9], by Dong, Koh and Teo for a detailed exposition on chromatic polynomials and chromaticity of graphs.

For example, if we want a proper colouring of the complete graph K_3 in Figure 5.1 with λ colours, we can see that there are λ ways to colour the first vertex, $\lambda - 1$ ways to colour the second vertex and $\lambda - 2$ ways to colour the remaining vertex. So there are $\lambda(\lambda-1)(\lambda-2)$ ways to colour K_3 using λ colours and the chromatic polynomial of K_3 is $\lambda^3 - 3\lambda^2 + 2\lambda$. Any values of λ giving values such that $\chi(G,\lambda) \leq 0$ implies that there is no proper colouring of the graph using that number of colours. That is, we need at least 3 colours for a proper colouring of K_3.

We note that we regard graphs as if their nodes were fixed in space, see [34]. In other words that a colouring with colours a, b, c of nodes labeled 1, 2, 3, given by the ordered pairs $(1,a)$, $(2,b)$, $(3,c)$ is different to $(1,c)$, $(2,a)$, $(3,b)$, even though the two colourings may be regarded as differing only by a cyclic permutation and hence equivalent.

5.1.3 k-defect polynomials

In this section, we define k-defect polynomials, that is, polynomials of improper colourings.

Definition 5.1.4. The k-defect polynomial, $\phi_k(G)$, of a graph G, is a polynomial that counts the number of improper λ-colourings of G with k bad edges.

For more information on the k-defect polynomial we refer the reader to a recent introductory paper by Mphako-Banda, [26], and the section on *two variable colouring* by Brylawski and Oxley in [3], where an edge is called *monochromatic* if its two endpoints are the same colour.

If we allow an improper colouring of a graph we are allowing certain adjacent vertices to have the same colour. Suppose we allow one bad edge in a colouring of K_3, see Figure 5.1. There are three edges, so there are three ways in which we can choose one bad edge. This gives λ ways to colour the two adjacent vertices of the bad edge and $\lambda - 1$ ways to colour the remaining vertex. So the 1-defect polynomial for K_3 is $3\lambda(\lambda-1) = 3\lambda^2 - 3\lambda$. There is no way to choose two bad edges since this would imply all three vertices the same colour and hence the third edge has to be bad too, since it joins two vertices of the same colour. The 3-defect polynomial for K_3 would then simply be λ.

We can see from the definition of the k-defect polynomial that the chromatic polynomial is the 0-defect polynomial and, hence, that the idea of k-defect polynomials is a generalisation of chromatic polynomials.

5.2 Mincuts and sets of $|E| - \alpha_1$ bad edges

In this section, we show that the number of mincuts of a graph G is equal to the number of good-edge sets of cardinality α_1 in a two coloring of G.

Lemma 5.2.1. *Let $P_v = \{V_1, V_2\}$ be a 2-partition of the vertex set $V(G)$ of a simple connected graph G. Then the set of edges, $E_3 \subset E(G)$, incident on the vertices of attachment of the two induced subgraphs, $G[V_1]$ and $G[V_2]$, is a set of good edges in a 2-colouring of G. Moreover, $G - E_3$ is disconnected.*

Proof. Let E_1, $E_2 \subset E(G)$ be the two edge sets of $G[V_1]$ and $G[V_2]$, respectively. Then every $e \in E_1$ has both endpoints in V_1, every $e \in E_2$ has both endpoints in V_2 and every $e \in E_3$ has one endpoint in V_1 and one in V_2. Colour every $v \in V_1$ one colour and every $v \in V_2$ another colour, then E_1 and E_2 are sets of bad edges and E_3 is a set of good edges. Since E_3 is the set of edges joining the two induced subgraphs, $G - E_3$ is disconnected and E_3 is an edge cut of G. □

Theorem 5.2.2. *Let G be a simple connected graph with minimum edge-cut number α_1. Then every set of α_1 good edges in a 2-colouring of G with $|E| - \alpha_1$ bad edges is a minimum edge-cut of G.*

Proof. Partition the vertex set, $V(G)$ into two sets, R and B, with red vertices in R and blue vertices in B. Let $G[R]$ and $G[B]$ be the vertex induced subgraphs on R and B respectively. All edges of $E(G[R])$ have both endpoints red and, similarly, all edges of $E(G[B])$ have both endpoints blue. Thus the edges of $G[R]$ and $G[B]$ are all bad edges and we have $|E(G[R]) \cup E(G[B])| = |E| - \alpha_1$ by our choice of the number of bad edges. The remaining α_1 good edges have one endpoint in each $G[R]$ and $G[B]$ and they are the only such edges.

Thus, $G - S_{\alpha_1}$, where S_{α_1} is the corresponding set of α_1 good edges, is disconnected, by Lemma 5.2.1. Hence we have $c(G - S_{\alpha_1}) = 2$. Since α_1 is the minimum edge-cut number of G, S_{α_1} is a minimum edge-cut. It is not possible to have three or more disjoint vertex sets on two colours and α_1 good edges, since this would imply a minimum edge-cut number for G of less than α_1, by the same reasoning used in Lemma 1.5.4. $\square$

Theorem 5.2.3. *Let G be a simple connected graph with minimum edge-cut number α_1. If X is the set of mincuts and S the set of good-edge sets on a 2-colouring of G with each $|S_i| = \alpha_1$, then $X = S$.*

Proof. We know from Theorem 5.2.2 that every set of α_1 good edges in a two colouring of G with $|E| - \alpha_1$ bad edges is a minimum cut, so $S \subseteq X$.

By Lemma 1.5.4, every minimum edge-cut partitions the vertex set, $V(G)$, into two disjoint sets, such that the induced subgraph on each set is a connected subgraph of G. Let $X =< Y, \bar{Y} >$ be the minimum edge-cut on α_1 edges that partitions $V(G)$ into two vertex subsets, Y and $\bar{Y}$, with $G[Y]$ and $G[\bar{Y}]$ the respective induced subgraphs on Y and $\bar{Y}$. Then every edge $e = y_i y_j \in E(G[Y])$ is such that y_i, $y_j \in V(G[Y])$. Similarly for every edge $e = \bar{y}_i \bar{y}_j \in E(G[\bar{Y}])$ we have $\bar{y}_i$, $\bar{y}_j \in V(G[\bar{Y}])$. In order for $< Y, \bar{Y} >$ to be an edge-cut we must have, for every edge $e = y\bar{y} \in< Y, \bar{Y} >$, $y \in V(G[Y])$ and $\bar{y} \in V(G[\bar{Y}])$. So, if we colour all $y \in Y$ red and all $\bar{y} \in \bar{Y}$ blue, X is a set of good edges and $X \subseteq S$. Hence, we conclude $X = S$. $\square$

5.3 The Tutte and bad colouring polynomials

In this section, we introduce two further graph polynomials, the Tutte polynomial and the bad colouring or *co-boundary* polynomial.

Following Ellis-Monaghan and Merino, in [11], we define a graph polynomial as follows:

Definition 5.3.1. A *graph polynomial* is an algebraic object associated with a graph that is usually invariant at least under graph isomorphism. As such it encodes information about the graph and enables algebraic methods for extracting this information.

5.3.1 The Tutte polynomial

In this section, we give a definition and a brief overview of the well known Tutte polynomial. We illustrate how to calculate the Tutte polynomial of a graph, using the definition as well as by using the method of deletion and contraction. For a more complete introduction to the Tutte polynomial, we refer the reader to [10].

Definition 5.3.2. The *rank* of a graph G is

$$r(G) = |V(G)| - c(G),$$

where $|V(G)|$ is the order and $c(G)$ the number of components of G.

We recall that if a graph is connected then $c(G) = 1$. We can also use $r(E)$ to denote the rank of G in order to emphasise that it is the graph induced by the edge set E.

Definition 5.3.3. The *nullity* or *cycle rank* of a graph G is

$$\mu(G) = |E(G)| - |V(G)| + c(G),$$

where $|E(G)|$ is the size, $|V(G)|$ the order and $c(G)$ the number of components of G.

One can think of the nullity of a graph as the minimum number of edges that must be deleted in order to break all the cycles of the graph, see [16].

Definition 5.3.4. The *Tutte polynomial* of a graph G is

$$T(G; x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{\mu(A)}$$

where $r(E)$ denotes the rank of G and $r(A)$ the rank of the subgraph induced by the edge set A.

Example 5.3.5. In Figure 5.2, we see all the edge subsets $A \subseteq E(K_3)$ of K_3. The rank of K_3 is $3 - 1 = 2$. Applying Definition 5.3.4 we get

$$\begin{aligned} T(K_3; x, y) &= (x-1)^2(y-1)^0 + 3(x-1)^1(y-1)^0 + 3(x-1)^0(y-1)^0 + (x-1)^0(y-1)^1 \\ &= x^2 + x + y. \end{aligned}$$

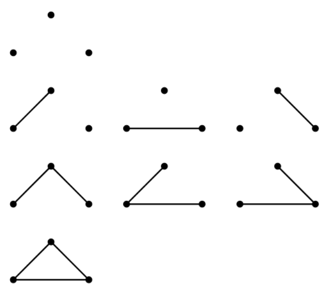

Figure 5.2: Edge subsets of K_3.

The Tutte polynomial of a graph can also be calculated using a method of deletion and contraction as stated in the following proposition, see [10].

Proposition 5.3.6. *The Tutte Polynomial of a graph G is a two variable polynomial $T(G; x, y)$. If G is a graph and e is an edge, then*

$$T(G; x, y) = \begin{cases} yT(G\backslash e; x, y) & \text{if } e \text{ is a loop} \\ xT(G/e; x, y) & \text{if } e \text{ is a bridge} \\ T(G\backslash e; x, y) + T(G/e; x, y) & \text{if } e \text{ is neither} \\ x^i y^j & \text{if } G \text{ consists of } i \text{ bridges and } j \text{ loops.} \end{cases}$$

Example 5.3.7. In Figure 5.3, we calculate the Tutte polynomial using deletion and contraction. In each step the graph on the left is formed by deleting an edge and the graph on the right is formed by contracting the same edge.

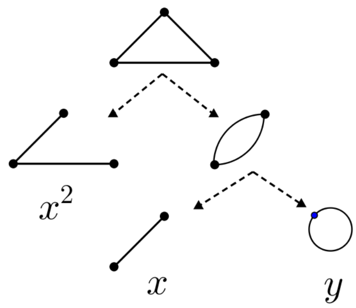

Figure 5.3: $T(K_3; x, y)$ found by the method of deletion and contraction.

As an example of the power of the Tutte polynomial we give the following theorem from [10], illustrating how to extract information about the graph, by evaluating $T(G; x, y)$ at certain values of x and y.

Theorem 5.3.8. *If $G = (V, E)$ is a connected graph then:*

1. $T(G; 1, 1) = \tau(G)$*, the number of spanning trees of* G.
2. $T(G; 2, 1)$ *equals the number of spanning forests of* G.
3. $T(G; 1, 2)$ *equals the number of spanning connected subgraphs of* G.
4. $T(G; 2, 2) = 2^{|E|}$.

In [10], we also find the universality property of the Tutte polynomial as one of its most powerful aspects. This property says that any graph invariant that is multiplicative on disjoint unions and one-point joins of graphs and that has a deletion/contraction reduction must be an evaluation of the Tutte polynomial.

5.3.2 The bad colouring polynomial

In 1969, the concept of bad edges was discussed by Wilf, [39], where a set of bad edges is called a *bond* of the colouring. In the same year the *co-boundary polynomial* of a matroid, $\bar{B}(M;\lambda,S)$, or *bad colouring polynomial* of a graph G, $B(G;\lambda,S)$, was originally defined and studied by Crapo as a generating function in S, see [7, 27], where the polynomial coefficients of the S^k are the k-defect polynomials of a graph G.

Definition 5.3.9. The *bad colouring polynomial* of a graph G is a polynomial in two independent variables and is defined as

$$B(G;\lambda,S) = \lambda^{c(G)} \sum_{A\subseteq E} (S-1)^{|A|}\lambda^{r(E)-r(A)}$$

where $|A|$ is the size of the subgraph induced by the subset A of edges.

Example 5.3.10. We calculate the bad colouring polynomial of K_3 using Definition 5.3.9 and the edge subsets shown in Figure 5.2. Thus we have

$$B(K_3;\lambda,S) = \lambda[(S-1)^0\lambda^2 + 3(S-1)^1\lambda^1 + 3(S-1)^2\lambda^0 + (S-1)^3\lambda^0],$$

which simplifies to

$$B(K_3;\lambda,S) = \lambda(\lambda-1)(\lambda-2)S^0 + 3\lambda(\lambda-1)S^1 + \lambda S^3.$$

If we recall the chromatic polynomial and the k-defect polynomials of K_3 from Subsections 5.1.2 and 5.1.3, we note that the coefficients of the S^k are indeed the 0, 1, 2 and 3-defect polynomials of K_3. Since it is not possible to have two bad edges in K_3, the coefficient of S^2 is 0.

The Tutte polynomial and bad colouring polynomial of a graph are equivalent, as is shown in [25]. The following proposition states the equivalence between $T(G;x,y)$ and $B(G;\lambda,S)$.

Proposition 5.3.11. *The bad colouring polynomial of a graph G is*

$$B(G;\lambda,S) = \lambda(S-1)^{r(G)}T(G;\frac{S+\lambda-1}{S-1},S),$$

where $r(G)$ is the rank of G. If G is connected $r(G) = n - 1$, where n is the order of G. Vice versa, given the bad colouring polynomial of G, we can calculate the Tutte polynomial,

$$T(G; x, y) = \frac{1}{(x-1)(y-1)(y-1)^{r(G)}} B(G; (x-1)(y-1), y).$$

Example 5.3.12. As an example of the equivalence we calculate $B(K_3; \lambda, S)$ using $T(K_3; x, y) = x^2 + x + y$. Using the equivalence we have

$$B(K_3; \lambda, S) = \lambda(S-1)^2 [\frac{(S+\lambda-1)^2}{(S-1)^2} + \frac{S+\lambda-1}{S-1} + S],$$

which, when simplified, gives us $\lambda(\lambda-1)(\lambda-2)S^0 + 3\lambda(\lambda-1)S^1 + \lambda S^3$.

As mentioned at the beginning of this section, the *bad colouring polynomial* of a graph G was originally defined and studied by Crapo as a generating function in S, see [7, 27].

Proposition 5.3.13. *The bad colouring polynomial, $B(G; \lambda, S)$, of a graph G is given by*

$$B(G; \lambda, S) = \sum S^k \phi_k(G; \lambda),$$

where $\phi_k(G; \lambda)$ is the k-defect polynomial of the graph G.

Hence, given the equivalence of the Tutte and bad colouring polynomials, if we have the Tutte polynomial of a graph we can find the bad colouring polynomial and thus we will have the $(|E| - \alpha_1)$-defect polynomial. Given the bijection between the set of mincuts and the set of good edges of size α_1 in a 2-colouring of G with $(|E| - \alpha_1)$ bad edges, we can evaluate the Tutte polynomial to count the number of mincuts.

5.4 Counting the number of mincuts of G

In this section we give two examples of exploiting the equivalence of bad colouring and mincuts, proven in Theorem 5.2.2, to get results on bad colouring and on mincuts.

We use mincuts to show that certain k-defect polynomials of a graph are zero and we count the number of mincuts of a graph by evaluating the bad colouring polynomial.

The following lemma was proved in [24] for matroids. We prove it here for graphs. We use the equivalence proven in Theorem 5.2.2. What the lemma implies, is that there is no bad colouring of G with $|E| - \alpha_1 < k < |E|$ bad edges, where the mincut number of G is α_1.

Lemma 5.4.1. *Let G be a simple connected graph with minimum edge-cut number α_1. Then the k-defect polynomials for $|E| - \alpha_1 < k < |E|$ are all zero.*

Proof. We know a bad colouring is possible for $k = |E| - \alpha_1$, since any mincut of G partitions the vertex set into two, each of which can be coloured with one of two colours, respectively, giving $|E| - \alpha_1$ bad edges and the corresponding α_1 good edges belong to the mincut. Similarly, colouring all the vertices the same colour will give a bad colouring with $|E|$ bad edges.

Suppose we have $S \subseteq E(G)$ such that $|S| < \alpha_1$ and S is a set of good edges, that is, it is possible to have a bad colouring of G with $|E| - \alpha_1 < k < |E|$ bad edges. Then it is possible to partition $V(G)$ into two (or more, the argument is the same) disjoint vertex sets B and R such that all $v \in B$ are the same colour and $v \in R$ the same colour, giving a bad colouring of G with $k > |E| - \alpha_1$ bad edges. But then $G - S$ is disconnected, contradicting the assumption of the mincut number of G. □

We recall from Proposition 5.3.13 that $B(G; \lambda, S)$, the bad colouring polynomial of a graph G with minimum edge-cut number α_1, is

$$\begin{aligned} B(G;\lambda,S) &= \sum_{k=0}^{|E|} \phi_k(G;\lambda) S^k \\ &= \phi_0(G;\lambda)S^0 + \phi_1(G;\lambda)S^1 + \ldots + \phi_{|E|-\alpha_1}(G;\lambda)S^{|E|-\alpha_1} + \ldots + \phi_{|E|}(G;\lambda)S^{|E|} \end{aligned}$$

where $\phi_k(G, \lambda)$ is the k-defect polynomial of G.

Hence, following directly from Lemma 5.4.1, we have the following lemma.

Lemma 5.4.2. *The exponent of S in the second last term of the bad colouring polynomial in ascending powers of S, gives the minimum edge-cut number of the graph G, since it is $|E| - \alpha_1$.*

Theorem 5.4.3. *Let G be a simple, connected graph with minimum edge-cut number α_1. Then the $(|E| - \alpha_1)$-defect polynomial evaluated at $\lambda = 2$, counts the number of minimum edge-cuts of G.*

Proof. Since G is connected we have $\alpha_1 \geq 1$. Hence we have at least one good edge. That is, we have at least two vertices that have to be coloured differently and hence $\phi_{|E|-\alpha_1}(G; 2) > 0$.

We know from Theorem 5.2.3 that there is a bijection between the set of mincuts and the set of α_1 good-edge sets in a 2-colouring of G with $|E| - \alpha_1$ bad edges. By Definition 5.1.4, $\phi_{|E|-\alpha_1}(G; 2)$ counts the number of ways to colour G with two colours and $|E| - \alpha_1$ bad edges, or, conversely, α_1 good edges. Hence we can say, $\phi_{|E|-\alpha_1}$ with $\lambda = 2$ counts all and only the sets of edges that are minimum cuts. □

We note that, since we are using two colours, the k-defect polynomial will actually give us twice the number of sets of bad edges, since it counts colouring each of the two partitions with each of the two colours.

Recall, from Proposition 5.3.11, that the bad colouring polynomial can be found as an evaluation of the Tutte polynomial:

$$B(G; \lambda, S) = \lambda(S - 1)^{|V|-c(G)} T(G; \frac{S - 1 + \lambda}{S - 1}, S).$$

Since we are interested in the bad colouring polynomial evaluated at $\lambda = 2$ we will use the following evaluation of $T(G; x, y)$ to generate a simplified 2-colouring polynomial in y^k that gives the number of k bad edges possible with a 2 colouring of G.

Theorem 5.4.4. *Let G be graph with order $|V|$, size $|E|$, Tutte polynomial $T(G; x, y)$*

and component number $c(G)$. Then

$$(y-1)^{|V|-c(G)}T(G;\frac{y+1}{y-1},y)=\sum_{k=0}^{k=|E|}a_k y^k,$$

is a polynomial in y, where a_k is the number of sets of k bad edges, given a 2-colouring of G.

Proof. From Proposition 5.3.13 we have $B(G;\lambda,S)=\sum S^k\phi_k(G;\lambda)$, where $\phi_k(G;\lambda)$ is the k-defect polynomial of G. By definition the k-defect polynomial counts the number of ways we can colour a graph with λ colours, given k bad edges. Setting $\lambda=2$ will then give the number of 2-colourings of G, given k bad edges.

Setting $\lambda=2$ and $S=y$, the equivalence of the bad colouring and Tutte polynomials given in Proposition 5.3.11 yields

$$B(G;2,y)=2(y-1)^{r(G)}T(G;\frac{y+1}{y-1},y).$$

This will give twice the number of sets of k bad edges, since the k-defect polynomial will count two colourings for each set, one with colour R assigned to one set of vertices, and one with colour B assigned to the same set of vertices. Hence, we divide the expression by 2 to get

$$(y-1)^{|V|-c(G)}T(G;\frac{y+1}{y-1},y)=\sum_{k=0}^{k=|E|}a_k y^k.$$

□

Corollary 5.4.5. *Let G be graph with order $|V|$, size $|E|$, Tutte polynomial $T(G;x,y)$ and component number $c(G)$. Then the coefficient of $y^{|E|-\alpha_1}$ in*

$$(y-1)^{|V|-c(G)}T(G;\frac{y+1}{y-1},y)=\sum_{k=0}^{k=|E|}a_k y^k,$$

is the number of mincuts of G.

Proof. The proof follows directly from the Theorems 5.4.3 and 5.4.4. □

We use some well known classes of graphs on 5 vertices to illustrate the 2-colouring polynomial. Recall that all trees on n vertices have the same Tutte polynomial.

Example 5.4.6. Consider a tree which is a star graph, S_n, the cycle graph, C_n, and the complete graph, K_n. The number of mincuts on a tree with n vertices is $n-1$, for a cycle it is $\binom{n}{2}$, since any choice of two edges will disconnect the graph, and for complete graphs it is simply n. For complete graphs the only mincuts are the trivial cuts incident on every vertex. We will use the specific cases of S_5, C_5 and K_5 to illustrate. The respective Tutte polynomials are as follows.

Since every edge of T_n is a bridge and the Tutte polynomial is multiplicative on one point joins, we have, by Proposition 5.3.6, $T(T_n; x, y) = x^{n-1}$ and, hence, $T(S_5; x, y) = x^4$.

Given a cycle C_n, deleting an edge gives a Path (same Tutte polynomial as a Tree), P_{n-1}, and contracting the edge gives a cycle, C_{n-1}. Continuing the process we see $T(C_n; x, y) = x^{n-1} + x^{n-2} + \ldots + x + y$. The last graph in the sequence of edge contractions is a loop, as we saw in Figure 5.3, hence the last term is y. Therefore we have $T(C_5; x, y) = x^4 + x^3 + x^2 + x + y$.

Finding the Tutte polynomial for a complete graph, K_n, is not as straight forward, unfortunately, and many attempts have been made to get a direct formula for its calculation. See for example [20], where triangular number partitions are used to generate the k-defect polynomials of complete graphs and then using the bad colouring polynomial to calculate the Tutte polynomial. Hence, we give here the Tutte polynomial as generated using Mathematica: $T(K_5; x, y) = 6x + 11x^2 + 6x^3 + x^4 + 6y + 20xy + 10x^2y + 15y^2 + 15xy^2 + 15y^3 + 5xy^3 + 10y^4 + 4y^5 + y^6$.

Using the evaluation of the bad colouring polynomial given in Theorem 5.4.4 and simplifying we have:

$$(y-1)^{|V|-c(G)}T(G;\frac{y+1}{y-1},y) = \begin{cases} 1+4y+6y^2+4y^3+y^4, & \text{for } G \cong S_5, \\ 5y+10y^3+y^5, & \text{for } G \cong C_5, \\ 10y^4+5y^6+y^{10}, & \text{for } G \cong K_5. \end{cases}$$

The edge-connectivity for each of the three graphs is, respectively, 1, 2 and 4. The value of $|E| - \alpha_1$ is, respectively 3, 3 and 6. We note that the coefficient of $y^{|E|-\alpha_1}$ in each case gives the number of mincuts of the graph. The coefficient of y^0 for S_5 is 1, that is, there is one set of zero bad edges given two colours, whereas there is no such set for C_5, since it is a cycle of odd order and $\chi(C_5) = 3$.

Chapter 6

Conclusion

In this thesis we introduced a new intersection graph, the mincut graph of a graph, and set out by looking at the mincut graphs of certain well known classes of graphs. Our study of the mincut graph was modelled on the study of the well-known line graph, since the line graph of a graph reflects the mutual positions of a graph's edges, and the mincut graph reflects the mutual positions of the minimum edge-cuts. The line graph operator was one of the first graph operators to receive widespread attention and, hence, we also introduced the mincut operator. We studied the mincut operator on the basis of the research program on graph operators introduced by Prisner in the 1995 monograph *Graph Dynamics*.

In Chapter 2 we defined the mincut graph of a graph G to be the graph with vertex set the minimum edge cuts of G, with two vertices adjacent if the intersection of the two edge sets is non-empty. We showed that the mincut graph of a graph is unique and that the mincut graph of a disconnected graph is the null graph, K_0, the graph with no vertices or edges. By generating the mincut graphs of some well-known classes of graphs we saw, for example, that the mincut graphs of Trees are the empty graphs with no edges, that Complete Graphs are isomorphic to their mincut graphs and that the mincut graphs of Cycles are line graphs of complete graphs, which in turn are also isomorphic to their mincut graphs.

The interesting properties of the mincut graphs of the graphs studied in Chapter 2 motivated us to define and look deeper into the properties of the mincut operator. Thus, in Chapter 3, we showed that all graphs lie in the image of the mincut operator and that every graph is the mincut graph of an infinite family of graphs that can be obtained by constructing a super-λ super graph with the graph as induced subgraph on the set of vertices of minimum degree. Furthermore, we showed that a graph is isomorphic to its mincut graph, that is, fixed under the operator, if and only if it is K_0 or otherwise both super-λ and r-regular. What this also implies, is that these are the only 1-periodic graphs under the X-operator.

In Chapter 4 we studied the effect of iterating the operator, that is, we looked at what happens when we apply the operator repeatedly to a graph. Thus we showed that not only is every graph the mincut graph of an infinite family of graphs, but also that we can continue the process of constructing super graphs indefinitely and that every graph, therefore, has infinite depth under the X-operator. Furthermore we showed that, unlike the line graph operator, no graph diverges under the mincut operator. We also identified two classes of graphs that are 2-periodic under $X(\cdot)$, namely the Cartesian product of a complete graph with K_2, as well as complete component cycles.

The number of mincuts of a graph determines the order of its mincut graph and the number of intersections of these edge-sets determine the size of the mincut graph. In Chapter 5 we showed that the number of mincuts of a graph is equal to the number of sets of bad edges in an improper 2-colouring of the graph, given $|E|-\alpha_1$ bad edges. We also showed how this equivalence can be exploited to generate results about improper colouring as well as connectivity, by proving that the k-defect polynomials of a graph for $|E| - \alpha_1 < k < |E|$ are all zero and by using the bad colouring polynomial of a graph to count the number of mincuts, that is, to determine the order of its mincut graph.

From this study it has also emerged some further problems that may be interesting to look at and merit further investigation.

By construction we found the interesting result that every graph is a mincut graph. Unfortunately, this does not tell us much about the root graph if it is not of the class of super graphs we used in the construction. This leads us to the determination problem. What can we conclude about the mincuts of the root graph given certain induced subgraphs of the mincut graph. For example, does the existence of a 4-cycle necessarily imply that we have crossing mincuts in the root graph and that we can therefore also conclude that the root graph has even edge connectivity? Similarly, can we conclude that the existence of a path implies that we have nested edge cuts?

In Chapter 4 we identified two classes of graphs that are 2-periodic under $X(\cdot)$. It would be interesting to know whether there are other graphs of period 2 and whether there are shared characteristics that can help us to identify these classes of graphs. Furthermore, are there graphs of periodicity higher than 2 under the mincut operator?

We also showed in Chapter 4 that no graph diverges under the mincut operator. We conjectured that, since the mincut graph of a disconnected graph is the null graph, most graphs will converge to the null graph. That is because we need every mincut to intersect with at least every other mincut in order for the mincut graph to be connected, which, intuitively, is unlikely to happen very often under repeated application of the mincut operator. This also leads to further questions on the connectivity of mincut graphs. For example, are there parameters that will indicate whether the mincut graph will be a connected graph?

Lastly, in 1945 Szpilrajn-Marczewski proved that every graph is an intersection graph. Thus we can meaningfully define a graph invariant, the *intersection number* of a graph G, denoted $i(G)$, to be the minimum cardinality of a set S such that G is the intersection graph of a family of subsets of S, see [16, 17]. In [12] it is proved that $i(G) \leq \lfloor \frac{n^2}{4} \rfloor$, where $|V(G)| = n$. In view of Theorem 3.2.7, is it then possible to define a graph invariant similar to $i(G)$ from the mincut graph, say the *mincut intersection number*, $i_X(G)$? That is, what is the smallest set S, such that there is a family of subsets that represents G as a mincut graph of some graph H? One of the conditions that needs to be set on the set of subsets of S is, of course, that they must all have cardinality λ, since they must all be minimum edge cuts.

Bibliography

[1] M. Aigner. *A Course in Enumeration.* Springer, Berlin, 2007.

[2] L.W. Beineke. Characterizations of derived graphs. *Journal of Combinatorial Theory*, 9:129–135, 1970.

[3] T. Brylawski and J. Oxley. The Tutte Polynomial and its Applications. In N. White, editor, *Matroid Applications (Encyclopedia of Mathematics and its Applications)*, pages 123–225. Cambridge University Press, Cambridge, 1992.

[4] L. Cai, D. Corneil, and A. Proskurowski. A generalization of line graphs: (X,Y)-intersection graphs. *Journal of Graph Theory*, 21(3):267–287, 1996.

[5] L.S. Chandran and L.S. Ram. On the number of minimum cuts in a graph. *SIAM Journal on Discrete Mathematics*, 18(1):177–194, 2004.

[6] G. Chartrand, L. Lesniak, and P. Zhang. *Graphs and Digraphs.* CRC Press, Boca Raton, 5th edition, 2011.

[7] H.H. Crapo. The tutte polynomial. *Aequationes Mathematicae*, 3(3):211–229, 1969.

[8] E. Dinic, A. Karzanov, and M. Lomosonov. The structure of a system of minimal edge cuts of a graph. In A.A. Fridman, editor, *Studies in Discrete Optimization*, pages 290–306. Nauka, Moscow, 1976. Translation at `http://alexander-karzanov.net/ScannedOld/76_cactus_transl.pdf`.

[9] F.M. Dong, K.M. Koh, and K.L. Teo. *Chromatic Polynomials and Chromaticity of Graphs*. World Scientific, New Jersey, 2005.

[10] J.A. Ellis-Monaghan and C. Merino. Graph Polynomials and Their Applications I: The Tutte Polynomial. In M. Dehmer, editor, *Structural Analysis of Complex Networks*, pages 219–255. Birkhäuser Boston, Boston, 2011.

[11] Joanna A. Ellis-Monaghan and C. Merino. Graph polynomials and their applications II: Interrelations and interpretations. In M. Dehmer, editor, *Structural Analysis of Complex Networks*, pages 257–292. Birkhäuser Boston, Boston, 2011.

[12] P. Erdos, A. W. Goodman, and L. Posa. The representation of a graph by set intersections. *Canadian Journal of Mathematics*, 18:106–112, 1966.

[13] L. Fleischer. Building chain and cactus representations of all minimum cuts from hao-orlin in the same asymptotic run time. *Journal of Algorithms*, 33:51–72, 1999.

[14] H.N. Gabow. A matroid approach to finding edge connectivity and packing arborescences. *Journal of Computer and System Sciences*, 50:259–273, 1995.

[15] M.C. Golumbic. *Algorithmic Graph Theory and Perfect Graphs*. Elsevier, Amsterdam, 2nd edition, 2004.

[16] J.L. Gross and J. Yellen. *Handbook of Graph Theory*. CRC Press, Boca Raton, 2004.

[17] F. Harary. *Graph Theory*. Addison-Wesley, Reading, 1969.

[18] F. Harary. Conditional connectivity. *Networks*, 13:347–357, 1983.

[19] A Kanevsky. Graphs with odd and even edge connectivity are inherently different. Technical report, Tech. report, TAMU-89-10, 1989.

[20] C.W. Kriel. Graphs and graph polynomials. Master's thesis, University of the Witwatersrand, Johannesburg, October 2017.

[21] J. Lehel, F. Maffray, and M. Preissmann. Graphs with largest number of minimum cuts. *Discrete Applied Mathematics*, 65:387–407, 1996.

[22] L. Lesniak. Results on the edge-connectivity of graphs. *Discrete Mathematics*, 8:351–354, 1974.

[23] T.A. McKee and F.R. McMorris. *Topics in Intersection Graph Theory*. SIAM, Philadelphia, 1999.

[24] E.G. Mphako. *Tutte polynomials, chromatic polynomials and matroids.* PhD thesis, Victoria University of Wellington, Wellington, 2001.

[25] E. Mphako-Banda. A Case of Chromatic Equivalence Implying Tutte-equivalence. *British Journal of Applied Science and Technology*, 15(1):1–7, 2016.

[26] E. Mphako-Banda. An introduction to the k-defect polynomials. *Quaestiones Mathematicae*, 42(2):207–216, 2019.

[27] E. Mphako-Banda and T. Mansour. Defect polynomials and Tutte polynomials of some asymmetric graphs. *Turkish Journal of Mathematics*, 39:706–718, 2015.

[28] S. Negami. Polynomial invariants of graphs. *Transactions of the American of Mathematical Society*, 299(2):601–622, 1987.

[29] O.R. Oellermann. Connectivity and Edge-connectivity in Graphs: A Survey. *Congressus Numerantium*, 116:231–252, 1996.

[30] O. Ore. *Theory of Graphs.* American Mathematical Society colloquium publications. American Mathematical Society, 1962.

[31] M. Pal. Intersection graphs: An introduction. *Annals of Pure and Applied Mathematics*, 4(1):43–91, 2013.

[32] E. Prisner. *Graph Dynamics*. Longman, Harlow, 1995.

[33] E. Prisner. Line graphs and generalizations – a survey. In G. Chartrand and M. Jacobson, editors, *Surveys in Graph Theory*, pages 193–229. Utilitas Mathematica, Winnipeg, 1996. Special volume, 116, Congressus Numerantium.

[34] R.C. Read. An Introduction to Chromatic Polynomials. *Journal of Combinatorial Theory*, 4:52–71, 1968.

[35] W.T. Tutte. *Connectivity in Graphs.* Oxford University Press, London, 1966.

[36] W.T. Tutte and C.S.J.A. Nash-Williams. *Graph Theory.* Encyclopedia of mathematics and its applications. Addison-Wesley Publishing Company, Advanced Book Program, 1984.

[37] A.C.M. Van Rooi and H.S. Wilf. The interchange graph of a finite graph. *Acta Mathematica Hungarica*, 16(3–4):263–269, 1965.

[38] H. Whitney. Congruent graphs and the connectivity of graphs. *American Journal of Mathematics*, 54:150–168, 1932.

[39] H.S. Wilf. The Möbius Function in Combinatorial Analysis and Chromatic Graph Theory. In F. Harary, editor, *Proof Techniques in Graph Theory*, pages 179–192. Academic Press, New York, 1969.

[40] B. Zelinka. Periodicity of graph operators. *Discrete Mathematics*, 235:349–351, 2001.

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