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Investigation into competent teachers' choice and use of examples in teaching algebraic functions in Grade 11 in South African context: a case of two teachers.

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UNIVERSITY OF THE WITWATERSRAND

FACULTY OF SCIENCE

JOHANNESBURG

Title:

Investigation into competent teachers' choice and use of examples in teaching algebraic functions in grade 11 in South African context: A case of two teachers.

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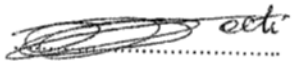
A research project submitted in partial fulfilment of the requirements for the degree of Masters in Science in Faculty of Science at University of the Witwatersrand in Johannesburg.

Date submitted : 16 July 2015

PLAGIARISM DECLARATION

I, Makhalanyane Phillip Moeti, declare that this research project is has not previously been submitted by me for a degree purposes at this university or any other and it is my own work in design and execution and no part of it has been copied from another source (unless indicated as quoted). All phrases, sentences and paragraphs taken directly from other materials and sources have been acknowledged.



M.P Moeti

ABSTRACT

The study focused on two competent, qualified, experienced secondary Mathematics teachers working in contrasting South African school contexts (fee-paying and no fee schools). The study investigated: on how teachers chose and used examples and how they explained their choices and usage; and what considerations were in play when these teachers chose and used examples. These teachers were purposely selected because we can learn more from their experiences as Mathematics teachers especially when they teach quadratic functions. Quadratic functions were used as unit of analysis to illuminate their choice and use of examples.

The contrasting socio-economic context of their schools (fee-paying and no-fee) provided these teachers with teaching and learning (re)sources. This study investigated further what and how (re)sources supported their choice and use of examples.

Examples are forever present in teaching mathematics and they are a central part for demonstrating, illustrating and communicating the critical aspects/features of mathematical concepts, methods, techniques and approaches. This implies that examples are used to facilitate the above skills but they are not the point of focus. This study, therefore pay attention on what and how examples were chosen and used by these teachers in contrasting schools context.

The study applied qualitative method to study the two teachers. The case study of each teacher reflected the teachers' decisions and considerations they made when they chose and used examples. Secondly, the interpretive approach assisted this study in describing and interpreting (Leedy & Ormrod, 2005) the teachers' choice and use of examples. This study applied cross-case synthesis as an analytical technique because this research studied two teachers from two contrasting contexts. The two teachers and their contexts are firstly treated as separate cases then meta-analysis was done in cross-case analysis.

For analytical purpose, different concepts were used. The teaching and learning (re)sources, namely; class size, time and teaching and learning aids illuminated the contrasting contexts of the two teachers. The studies by various researchers on nature of school functions provided this study with what actions and skills should be taught in functions examples and how teachers had undertaken those actions in their choice and use of examples. The role, purpose and types of examples illuminated what and how examples were chosen and used by these teachers. The

Rowland's categories, namely; variation, sequencing, learning objectives and representations illuminated what and how examples were chosen and used.

This study found that time and teaching and learning aids were important factors for generating and treating examples because they provided teachers with pacing, easy entry to and mediation of the task. They can constrain or afford the teacher with an opportunity to plan and use examples effectively during lesson presentation. This does not imply that teaching and learning aids have a bearing on a teacher to appropriately choose and use examples. The appropriate choice and use of examples, as espoused by Zodik and Zaslavsky (2008), and Sinclair et al. (2011), depend on a teacher's example space. Teachers' personal example spaces afford them to choose and use examples that had rigour.

The teaching and learning strategies have a bearing on teachers' choices and uses of examples. Competent, qualified, experienced teachers rely heavily on their experience to choose and use examples at the moment of teaching.

Key words: teachers' examples, quadratic functions and school contexts

Quotation;

"In virtually every subject areas, our knowledge is incomplete and problems are waiting to be solved. We can address the holes in our knowledge and those unresolved problems by asking relevant questions and then seeking answers through systematic research" (Leedy & Ormond, 2005, p. 1).

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List of acronyms and abbreviation

ATP	-	Annual Teaching Plan
CAPS	-	Curriculum and Assessment Policy Statement
DBE	-	Department of Basic Education
DoE	-	Department of Education
FET	-	Further Education and Training
LTSM	-	Learning and Teaching Support Material
MSc	-	Master's degree in Science
NCS	-	National Curriculum Statement
PCK	-	Pedagogic Content Knowledge
REQV	-	Required Education Qualification Value
SMK	-	Subject Matter Knowledge
TA	-	Teacher A
TB	-	Teacher B

Chapter 1: Introduction

1.1. Introduction

The problems and difficulties associated with teaching and learning school algebra are multifaceted and grounded in the nature of algebraic activities (Kieran, 2001; Kilpatrick, Swafford, & Findell, 2001; Hart, Brown, & Kuchmann, 2004; Drijvers, Goddijn & Kindt, 2011), the algebraic curriculum and its teaching implications (Herscovics & Linchevski, 1994; Sfard & Linchevski, 1994; Schmittau & Morris, 2004; Schliemann, Carraher & Brizuela, 2007). This study is based in the South African context and pays attention to teaching, particularly competent secondary mathematics teachers' choice of examples of functions before and during lessons, and the usage of these examples. The teachers' choice and use of examples may facilitate or hinder the opportunity to learn mathematical concepts (Watson & Mason, 2006; Goldenberg & Mason, 2008). The underlying assumption here is that teachers' knowledge and their instruction influence how and what learners learn. The study foregrounds instruction, particularly teachers' choice and use of examples.

The study focuses on what it is that selected, competent, qualified, experienced[1] South African secondary teachers of mathematics do and their rationales when dealing with examples of *school functions* backgrounding what teachers need to know when dealing with such examples. Teachers' knowledge (knowledge of mathematics, knowledge of learners' learning and pedagogical content knowledge, etc.) (Shulman, 1986; Ball & Bass, 2000) and beliefs are referred as teachers' personal example spaces (Alcock & Iglis, 2008; Goldenberg & Mason, 2008; Sinclair, Watson, Zaskis & Mason, 2011). The teachers' personal example spaces are therefore unique (an account is given in Chapter Two). At this point it suffices to state that the teachers' personal example spaces serve as reservoirs for teachers to generate examples in their instruction (Alcock & Inglis, 2008; Goldenberg & Mason, 2008; Rowland, 2008; Tsamir, Tirosh & Levenson, 2008). *My focus in this study is on what teachers come to use and their rationales for doing this*.

[1] The attributes of competent, qualified, experienced teachers are explained in Chapter 4 under *sampling*.

1.2. Purpose of the study.

The purpose of the study is to illuminate teachers' choices and use of examples. As I elaborate in Chapter 2, examples are ubiquitous in mathematics and in mathematics teaching (Adler & Venkat, 2014). They are a central part of any mathematics resource basket. Mathematical concepts and processes, abstract as they are, need to be exemplified. My study investigated two purposefully selected competent, experienced secondary teachers' choice of examples and the usage of these examples in their teaching of functions[2] (quadratic functions) in Grade 11. Goldenberg and Mason (2008) recommend that we can learn more from competent, experienced teachers' treatment and generation of examples in secondary schools. As described fully in later chapters, the study focuses on two competent, qualified, experienced secondary teachers working in contrasting school contexts: on how they chose examples and how they explained their choices; and what considerations were in play when these teachers chose and used examples? What I mean by this is what they thought is important and/or taken into account when choosing and using examples. It is thus important to background their personal example space (as the reservoirs of their knowledge) and foreground their choice and use of particular *function* examples apart from others. The assumption at work here is that teachers' choice and use of examples illuminate their instruction.

1.3. Research problem

Numerous studies (Zaslavsky & Lavie, 2005; Zaslavsky & Zodik, 2007; Zodik & Zaslavsky, 2007, 2008) have observed that choice, generation and treatment of examples in secondary mathematics are complex and involve a wide range of considerations. As stated above, the researchers articulate a position that in choosing, generating and using examples in their practice, secondary mathematics teachers face the challenge of taking multiple factors into account in their lessons. Such considerations should be done very carefully. The general research question is: what considerations are in play when secondary mathematics teachers in diverse school environments choose and use examples in the topic of functions?

[2] The definition of function is given in Chapter 2: Review of related literature and conceptual framing.

The specific research questions that underpin this exploratory study are:

1. What examples do teachers choose in the planning phase and during the lesson presentation?
2. How do they use the planned and additional spontaneous examples in their lessons?
3. How do they explain their choice and use of examples?
4. What patterns, if any, are there between and across the two contrasting teaching contexts, with respect to their choice, use and rationales for examples?

1.4. Rationale

Examples are frequently used by mathematics teachers to demonstrate, illustrate and communicate the critical aspects/features of mathematical concepts, methods, techniques and approaches (Zaskis & Leikin, 2008). Many researchers posit that examples may facilitate the understanding and meaning of concepts (Goldenberg & Mason, 2008; Rowland, 2008; Zaskis & Leikin, 2008; Zodik & Zaslavsky, 2008). Goldenberg and Mason (2008) emphasise that examples provide access to abstract ideas, scaffold critical aspects of the concept and thus provide access to that concept. For example, to understand the significance brought by "*a*" in the equation of a parabola, different values of "*a*" (range of examples where a > 1, a < 0 and "a" as fraction, e.g., $x^2, \pm 2x^2$ and $\pm \frac{1}{3}x^2$) are taken in order to understand its significance in a parabola and thus the quadratic function.

Examples are indices that provide contexts for illustration and communication of critical aspects (Watson & Mason, 2006; Goldenberg & Mason, 2008). For example, the speed of projectile motions (i.e. the parabolic motion) can be better understood when examples are used to display the projectile motion using a sketch and the instantaneous speed as time proceeds. The illustration and calculations on instantaneous speeds communicate critical aspects like stationary points and change of speed at different times. Given these analyses and exemplification, Zodik and Zaslavsky (2008) conclude that examples play a critical role in learning and teaching of mathematics. Secondly, examples may facilitate teaching and learning of mathematical concepts; and conversely, poor choice and use of examples may hinder teaching and learning of concepts.

There is a small, yet growing body of studies on exemplification and examplehood. The journal *Educational Studies in Mathematics* (2008), Issue number 69; has collected articles that deal

with exemplification from different perspectives and empirical settings. Zodik and Zaslavsky (2008) observed that in spite of the critical role examples play in teaching and learning of mathematics, there are a small number of studies focused on teachers' choice and use of examples. My study attempts to contribute to the growing body of knowledge related to exemplification in *Mathematics Education* through the in-depth study of two competent, experienced, qualified secondary teachers' choice and usage of examples in the South African context. The study is further motivated by Woods' (2003) observation that a limited number of studies "focus on secondary mathematics especially functions and graphs" (2003, p. 49). What is interesting is that literature on examples and example spaces does not focus on *functions*. This study, therefore investigates and explores competent teachers' choice and use of *function* examples in Grade 11 as stipulated in South African curriculum statement [namely, Curriculum and Assessment Policy Statement (CAPS) of 2011]..

Recently, teachers' workshops conducted by both the national and provincial Departments of Education in South Africa mainly dealt with curriculum issues and content knowledge in CAPS. In these workshops, examples were used to enhance and facilitate teachers' understanding of content knowledge. Interestingly, examples themselves were not the object of attention. In other words, teachers' choice and their use of examples[3] were not in focus. For example, the workshops on CAPS dealt with what topics are new in the curriculum, how the topics are assessed and what content should be covered by teachers and learners. My experiences as a secondary mathematics teacher and as Master Trainer (North West Department of Education and Training) who attended and trained teachers in those workshops is that we chose and used examples to discuss and develop content. Yet we did not draw attention to what and how examples could or should be chosen and used to teach the new content in CAPS. In the light of this study, this is an opportunity missed.

This study may thus also serve as reflection for education policy-makers and education programme developers to include teachers' generated examples as an important component in mathematics teaching in teachers' training in South Africa as noted.

[3] Examples have pedagogical significance which is ignored in teachers' capacity workshops. The teachers' choice and use of examples and sequencing of examples provide some of pedagogical importance examples have.

1.5. Conclusion

This study explores and investigates selected competent, qualified, experienced South African secondary mathematics teachers' choice and use of school *function* examples. The teachers are drawn from two contrasting schools, namely, a fee-paying school and a non-fee school.

In order to foreground teachers' choice and use of functions examples and background their example space, Chapter 2 reviews literature on the nature of functions in the curriculum; actions which should be undertaken when teaching *functions* in school; examples and teachers' examples; pedagogical usage of examples in teaching; teaching and learning (re)sources from which examples are drawn and that aide instruction. Lastly, conceptual constructs that underpin the conceptual framework are discussed.

The nature of functions in the curriculum depicts the definition of *function*, different representations of a *function* and properties of the school *functions*. For learners to understand the nature of *function* in the school context, the teachers' knowledge is paramount. The teachers' knowledge entails teachers' knowledge of *functions* as content, knowledge for teaching mathematics and knowledge of learning. As repeatedly noted, teachers' choice and their use of examples are a function of their knowledge in use therefore teachers' pedagogical content knowledge, as postulated by Zodik and Zaslavsky (2008) is considered in this study. Teachers' choice and use of examples direct the literature review on mathematical examples and teachers' examples. The roles and types of examples, the pedagogical considerations teachers are making when choosing and using examples in teaching *functions* are embedded in (re)sources teachers are using. Chapter 2 includes discussion of conceptual framework for the study and draws on Variation theory (Marton & Morris, 2002; Marton, Runesson & Tsui, 2004; Runesson, 2005; Watson & Mason, 2006) and Rowland's categories in his study of examples (Rowland, 2008).

Chapter 3 deals with documents which are key (re)sources that teachers draw on to direct their teaching. Analyses are made of Curriculum and Assessment Policy Statements (CAPS, DBE, 2011), official examination question papers and textbook. CAPS is the official policy document which stipulates knowledge and skills to be taught and learnt in the specific subject and grade and, what and how the content should be assessed. The examination question papers are instruments used to assess learners' skills and knowledge as enshrined in curriculum document

(CAPS). The previous examination question papers are analysed because they are (re)sources most teachers use for choosing and using examples for instruction and/or assessment. Textbooks are usually used by teachers as (re)sources for choosing and using examples in their teaching for concept understanding and exercises.

Chapter 4 justifies the methodology, selection of cases, data collection and instruments. The chapter further substantiates issues of how validity and reliability are established and maintained in data collection and data analysis. The analytical strategies and techniques are discussed in order to augment the procedures of data analysis. Lastly, issues of ethical consideration and limitations of data collection are discussed.

Data analyses are done in Chapters 5 and 6. Chapter 5 focuses on a teacher from the fee-paying school and Chapter 6 is about a teacher from the non-fee school. The data collected using different instruments and tools are used firstly to describe the events in each case (Chapter) and secondly, concepts discussed in Chapter 2 are employed to analyse data. Chapter 7 deals with cross–case analysis and synthesis of both teachers based on the analyses in Chapters 5 and 6. Furthermore, it concludes this research project by discussing, reflecting on and setting the limitations and the implications for Mathematics Education.

Chapter 2: Review of related literature and conceptual framing

2.1. Introduction

In this chapter, reviews of related literature and theory that frame the study are discussed. The study focuses on quadratic function in Grade 11 as stipulated in South African Curriculum and Assessment Policy Statement (CAPS) (DBE, 2011). Firstly, I discuss what a *function* is and what research literature offers about the teaching of functions. Secondly, I discuss the pedagogical importance (definitions, role, purpose, design, etc.) of examples; types of examples, research in teachers' considerations when they choose and use examples in both planning phase and in-the-moment of classroom interaction. Thirdly, since the study is focused on two contrasting socio-economic settings of teachers' schools, I also discuss the availability of (re)sources in so far as they inform example choice and usage in those contexts. Fourthly, the theoretical resources that frame my study are discussed.

2.2. Background on functions

2.2.1. Nature of functions in the curriculum

To understand the nature of functions in the curriculum, we need to understand fundamental characteristics, namely, definition, forms of representations, domain and range, and features/properties of functions. A function is *defined* as a special relation of correspondence of an independent variable (input value) and a dependent variable (output value) where there is a mapping of *one-to-one* between them and/or *many-to-one* mapping between them (Leinhardt, Zaslavsky & Stein, 1990; DBE, 2011; Nachlieli & Tabach, 2012). *One-to-one mapping* implies that for each value of the independent variable (input value/*x-value*) there is strictly and uniquely one value of the dependent variable (output value/*y-value*). For example, in $f(x) = x - 2$ (a linear function), if $x = 3$ then $f(3) = 1$.

A *many-to-one mapping* implies that many independent variables (input values/*x-values*) strictly map onto only one value of the dependent variable (output value/*y-value*). For example, in $y = x^2 - 4$, when $x = 3$, $y = (3)^2 - 4$ so $y = 5$. Similarly, when $x = -3$, $y = (-3)^2 - 4$ so $y = 5$. That is, two x-values (3 and -3) map strictly onto one y-value (5).

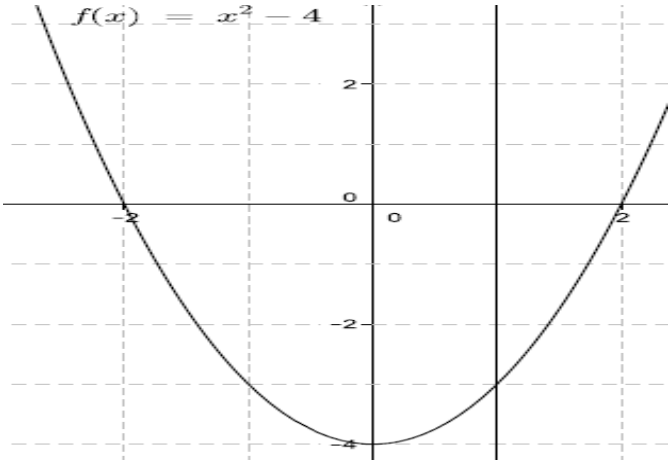

Fig. 2.1: parabola graph is many-to-one function.

Furthermore the *vertical line test*, as used in school mathematics, can illustrate whether or not a graph is a function. If a vertical line is drawn through a graph and that line intersects the graph only once, it shows the graph is a function (see fig. 2.1 above). The vertical line test holds for both one-to-one and many-to-one mappings, illustrating that each value of *x* maps onto only one value of *y*.

2.2.1.1 Types of functions

There are two types of functions, namely elementary functions and transcendental functions. Elementary functions are functions that can be expressed using a finite number of polynomials, involving only the algebraic operations (addition, subtraction, multiplication, division) where polynomials have constant coefficients (*Encyclopaedia of Math*, 2012). Examples of elementary functions are linear function and parabola. The exponential function, hyperbola, logarithm and trigonometric functions are not regarded as elementary functions but as transcendental functions. Transcendental functions are functions whose coefficients are not rational (Townsend, 2009). In the South African School Curriculum (CAPS, DBE, 2011), elementary functions and transcendental functions do not appear separately; instead they appear under Topic 1: Functions. Functions that are part of the Further Education and Training (FET) curriculum are: linear function, quadratic function, exponential function, hyperbola, trigonometric functions, logarithmic and cubic functions. For the purpose of this study only the parabola is studied because this is taught in Grade 11 (cf. Chapter 3 on document and content analyses).

2.2.1.2 Representations of functions

A function can be represented in different forms. A function can be described and represented as a set of ordered pairs, a mapping of elements of one set (the domain/input values) onto the elements of other set (the range/output values), in tabular form, graphical, in words (verbal) and in algebraic symbols (as a formula) in which the rule of one-to-one and/or many-to-one mapping are maintained (Carter, Govender & Heany., 2007; Carter & van der Lith, 2008, DBE, 2011; Nachlieli & Tabach, 2012). For example: the function where the output is 2 greater than the input can be represented in the following ways:

- As set of ordered pairs:

f = {(-3;-1); (-2;0); (-1;1); (0;2); (1;3); (2;4); (3;5)}

- Tabular form

X	-3	-2	-1	0	1	2	3
f(x)/y	-1	0	1	2	3	4	5

- As mapping of elements of one set (domain) onto the element of the other set (range)

  In f: x $\rightarrow x+2$

  $x \rightarrow y$

  $-3 \rightarrow -1$

  $-2 \rightarrow 0$

  $-1 \rightarrow 1$

  $0 \rightarrow 2$

  $1 \rightarrow 3$

- As a graph

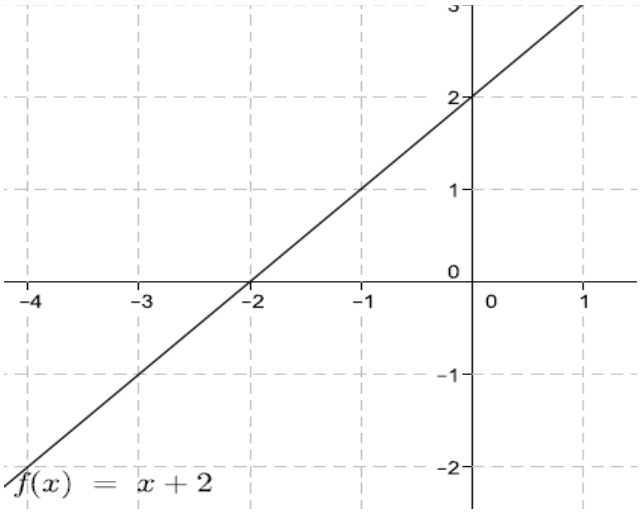

Fig. 2.2: A graphical representation

- As an algebraic expression: *f(x) = x+2*

The notation *f(x)* means the "function of *x*". The *f(x)* denotes the value(s) of the dependent variable *y* when the independent variable *x* assumes any value. Hence *y-values* are dependent on *x value(s).*

2.2.1.3 Properties of functions

All functions have specific properties. The properties differentiate types of functions. The properties of a linear function are different from a hyperbola. For example, the hyperbola has an asymptote and a linear function does not have an asymptote. The properties of functions are (a) increasing, decreasing and constant functions, (b) their symmetry, (c) translations, (d) reflection, (e) asymptotes, (f) intercepts on axes and (g) turning points. These properties are part of South African curriculum, that is, they appear in CAPS document. The change of parameters of functions of the same nature, that is, $y = ax^2 + q$ and $y = a(x - p)^2 + q$ has a bearing on the change in these properties. For example, the turning point in the former formula is $(0; q)$ and in the latter is $(p; q)$. This is due to the effect of parameter *p*. Parameters are constants that appear as variables in the standard form of functions and give a particular function a different form or shape from the parent function. For example, in $y = a(x - p)^2 + q$; *a, p* and *q* are parameters in the function of $y = x^2$, the parent function of the quadratic function. I discuss each of the properties below:

a. Defining increasing, decreasing and constant functions

Let x_1 and x_2 be any two points from an interval in the domain of a function $[x_1 < x < x_2]$ f (Exley & Smith, 1993):

If $x_1 < x_2$ implies $f(x_1) < f(x_2)$, then f is increasing on the interval. That is, a function is increasing when it is rising from left to right. For example, if $x_1 = -2$ and $x_2 = 3$, and $f(-2) < f(3)$ then the graph of f is increasing between the two points.

If $x_1 < x_2$ implies $f(x_1) > f(x_2)$, then f is decreasing on the interval. The function falls from left to right. If $f(x_1) = f(x_2)$, then f is constant on the interval.

b. Symmetry is the line that divides the graph into two mirror images. The distance between an object (point) and its image is equal from the line of symmetry.

c. Translations: The two types are vertical and horizontal translations/shifts.

If f is a function and $b > 0$, then

the graph of $y = f(x) + b$ is the graph of $y = f(x)$ shifts up b units

the graph of f $y = f(x) - b$ is the graph of $y = f(x)$ shifts down b units

the graph of $y = f(x + b)$ is the graph of $y = f(x)$ shifts b units to the left

the graph of $y = f(x - b)$ is the graph of $y = f(x)$ shifts b units to the right (Exley & Smith, 1993).

The first two translations are vertical shifts and the last two are horizontal shifts.

d. Reflection is a mirror image about a line of symmetry. Three types of reflection are reflection along *y-axis*/line $x = 0$, along *x-axis*/line $y = 0$ and along line $y = x$

Reflection along y-axis: $(x;y) \rightarrow (-x;y)$

Reflection along x-axis: $(x;y) \rightarrow (x;-y)$

Reflection along line $y = x$: $(x;y) \rightarrow (y:x)$ (Exley & Smith, 1993).

e. Asymptote: is a line, point or condition where the graph will never intersect/touch that condition because the domain is undefined or has no meaning in that condition. For example, in $y = \frac{1}{x-2}$; the graph does not intersect $x = 2$ because the denominator is 0 (zero) thus undefined. The asymptote is line $x = 2$.

f. Intercepts on axes. The graph can intersect the axes in either the y-axis or x-axis or both depending on the nature of a function. In the y-axis the value of x is $x = 0$ and in x-axis the value of y is $y = 0$. Therefore, when calculating the value of the y-intercept (intersection on y-axis), we let the value of x be zero; similarly with intersection on x-axis. In other words, the graph intersects the y-axis at line x = 0; and the x-axis at line $y = 0$.

g. Turning points are points where the gradient is 0 or where the function has the derivative of zero. This is the point where the graph changes from being increasing to decreasing or *vice versa*. Whether turning points are maximum or minimum depends on the parameter "a" in the quadratic function. If $a>0$, the graph has a minimum turning point and when $a<0$, a minimum turning point.

Furthermore, Even (1998) describes that school functions encompass symbolic and graphical representations. Working with the symbolic representation can involve an operational approach to a *function* (concept), that is, a process that needs algebraic manipulation by translating the algebraic expression into a graphical representation. This implies that in a move from a symbolic representation to a graphical one, algebraic manipulation is involved so as to be able to sketch the graph of, say, an algebraic formula (*viz.*, $f(x) = x^2 - 4$). To sketch the graph of this function, there is a need to determine the intersection on the axes, the axis of symmetry, translation, turning points, etc., which are determined through algebraic manipulations. The values are then translated into the Cartesian plane (system of axes) to have a global image of function. For example:

For x - intercept, let $f(x) = 0$:

$$x^2 - 4 = 0$$
$$x^2 = 4$$
$$x = \pm 2$$

For y - intercept, let $x = 0$

$y = 0^2 - 4$
$y = -4$

Translation:

The $f(x) = x^2 - 4$ is the function of $f(x) = x^2$ translated 4 units down.

Axis of symmetry:

$x = 0$

Turning points:

$(0;-4)$

The graphical representation entails construction of the graph and its interpretation. The interpretation and construction of functions entail reading the values of the graph, gaining meaning and understanding, and the plotting/sketching of the function (Leinhardt et al., 1990; Even, 1998) by taking into cognisance its properties (axis of symmetry, asymptotes, turning points, etc.). For understanding the concept "*function*" and its structures, Even (1998) posits that flexibility is needed for moving from symbolic representations to graphical ones and back.

The following Table 2.1 summarises the nature of functions in the school curriculum as drawn from cited literature:

Table 2.1: Nature of school functions.

Characteristics of functions	**Elements**
Definition	One-to-one rule Many-to-one rule
Representation/forms of functions	Ordered pairs Mapping Table of values Graphs Algebraic formula Words/verbal
Properties/features of a function	Increasing, decreasing and constant functions Symmetry Translations Reflection Asymptotes Intersection on axes and Turning points

To sum up, the review of nature of functions in the curriculum assists my study by enabling me to analyse characteristics of functions in teachers' choices and use of function examples; to examine how examples chosen and used afford the opportunity to learn these aspects of functions.

2.3 Teachers' knowledge aimed at teaching functions examples

This study mainly focuses on competent, qualified, experienced teachers' choice and use of *functions* examples. What they choose and what they use in the instruction (classroom interactions) is the function of their knowledge. While I have said that teachers' example spaces, which illuminate their knowledge, are backgrounded and their choice and use of examples foregrounded, it is necessary to discuss briefly their knowledge on choice and use of examples. I followed Zodik and Zaslavsky (2008) as they deal with teachers' knowledge in relation to teachers' choice and use of examples.

As noted, Zodik and Zaslavsky (2008) highlighted two aspects of teacher knowledge that are manifested in *teachers' example space*. They describe that teachers' personal example space in generating and treating examples is the reservoir of a teacher's content knowledge and pedagogical knowledge (Zodik & Zaslavsky, 2008). Sinclair et al. (2011) define personal example space as "repertoire of available examples, methods of examples construction for personal use" (p. 1). It therefore implies that the teachers' example space is teachers' knowledge that functions when the teachers choose and use examples in teaching a particular concept. Examples chosen and used are not necessarily a reflection of the total knowledge of the teacher about the subject and/or their teaching. Teachers face distractions during the choice and use of examples and other pragmatic challenges in both planning phase and classroom set-up (Rowland, 2008; Zodik & Zaslavsky, 2008).

The teachers' choice of examples and their treatment and usage of examples depend on the consideration teachers make (Zodik & Zaslavsky, 2008). The statement refers to, as alluded by Zodik and Zaslavsky, teachers' pedagogical content knowledge (PCK). PCK is a specialized knowledge that connects teachers' content knowledge with what to teach, how to teach and knowledge of how learners learn (Ball & Bass, 2000). In my study, the teachers are observed on what and how they select and use *functions examples*, from where and how they draw their examples, what considerations were in play when choosing and using examples. Therefore, as postulated by Zodik and Zaslavsky (2008), teachers' PCK is knowledge that is illuminated when teachers' choose and use examples in their instruction amidst distractions they encounter.

Zodik and Zaslavsky postulated that two aspects of teachers' PCK for choosing, treating and generating examples are mathematics content knowledge and pedagogical knowledge (cf. fig. 2.3). The content knowledge entails knowledge of the subject content and its organizing structures (Hill et al., 2008) and has do to with what is taught (Zodik & Zaslavsky, 2008) and how it can be exemplified. For teachers to teach functions, they should know key characteristics of functions, namely, the definition of the function, properties of a function, and different forms of representations of the function, types of function (linear, quadratic, exponential, hyperbola, etc.), construction of the function, interpretation and flexible movement between the representations (Leinhardt et al., 1990; Even, 1998; DBE, 2011).

Another facet of teachers' knowledge that is paramount for choosing and using examples is knowledge of the curriculum. Curricular knowledge involves knowing the prescripts and stipulations of the subject content to be covered in the curriculum, assessment designs and programmes pertaining to the subject and other related subjects (Shulman, 1986).

To sum up, Zaskis and Leikin (2008) argued that teachers need to consider alternatives when they plan a lesson and during instruction. They should consider strategies and methods of explaining or presenting an instruction to learners. The PCK, as specialized knowledge for teaching and in this study as knowledge that illuminate teachers' choice and use of examples, can be available to assist teachers with teaching content in logical and comprehensive way to learners (Shulman, 1986; Zaskis & Leikin, 2008). It should support teachers in choosing relevant materials, examples and connect curriculum prescripts/objectives with how learners can comprehend the subject content.

2.4. Research on the teaching of functions

Nachlieli and Tabach (2012) observed that *functions* are confusing and frustrate the learners because functions are not taught as the consolidation into one 'object', the seemingly unrelated representations: the algebraic expressions, tables and graphs. The difficulty of learning functions lies in the fact that functions are seen as formulae and graphs and not as a relationship between the two (*ibid.*).

Leinhardt et al. (1990) elaborated that teachers and learners should undertake the following actions, namely, interpretation and construction of functions in order to comprehend them. Interpretation refers to "action by which student makes sense or gains meaning from a graph (or a portion of a graph), a functional equation, or a situation." (Leinhardt et al., 1990, p. 8). The assertion includes but is not limited to the following actions, for example:

- when we derive an algebraic formula or other forms of representations/type of functions from a graph (either point by point or global sketch),
- describing what has changed in terms of the effect brought by parameters in the graph,
- reading the values of x *or* y from the sketched graphs. For example, determine the value(s) of x for which $f(x) \leq g(x)$.

Construction of a function entails identifying points in symbolic representations or other forms of representations of a function and plotting these to form graphical representations.

Researchers on teaching of school functions have found that learners and many textbooks, as well as many teachers, prefer to follow a route that starts from the algebraic formula and moves from this to graphical representations (Leinhardt et al., 1990; Even, 1998; Nachlieli and Tabach, 2012). Even (1998) believes that this routine normally does not address the flexible move from symbolic representation to graphical representation and vice versa, and therefore those other aspects of knowledge and understanding of functions are not addressed. Leinhardt et al. (1990) and Even (1998) found that the following route is being followed in teaching functions: as first entry to the introduction of functions, one finds the route of definition of a function, deriving order pairs by table method from the algebraic formula, translating these to sketch the graph and lastly, deriving symbolic expression of the graph from the sketched one. They claim that this route develops conceptualization and facilitates the comprehension of functions. The route develops construction action first and then later on interpretation. In construction, a point-wise approach is done first, followed by a global approach.

The flexible movement between representations addresses two distinctive yet interrelated approaches to functions namely, a pointwise approach and a global one (Even, 1998). A point-wise approach addresses the definition of a function and how to construct the graph from different forms of representations. The global approach demands both construction and interpretation of the function and its properties.

The following table, Table 2.2 summarises the actions that are required to teach and learn functions:

Table 2.2: Actions on teaching and learning functions.

	Action
Construction	• Identifying a function in different forms • Plotting/sketching point-by-point and global approaches • Matching the graph with families of equations
Interpretation	Gain meaning by: • Matching functions of same family • Identifying different types of functions in terms of shape and formula • Curve sketching and fitting
Flexibility between representations	By plotting/sketching Derive a different forms/representations of function from one given

Leinhardt et al. (1990) posit that when constructing a function, the interpretation of the said function occurs. When constructing functions of the same family, for example, if the quadratic functions vary in intersection on the y-axis (vertical shift from original, point (0;0)) then learners can gain meaning in interpreting the variation brought by change in parameter *q*. The lesson objective determines the focus on what will be taught and learned at any point in time.

Leinhardt et al. (1990) expressed the view that teachers' content knowledge is critical in teaching functions. While Nachlieli and Tabach (2012) expressed that teachers' ontological views about how and what to teach in function are also critical. To support the two assertions, a teacher's example space is a function of teachers' choice and use of examples in instruction, and thus teachers' pedagogical content knowledge is reflected in their choice and use of examples (Zodik & Zaslavsky, 2008). To sum up, teachers should have established knowledge of "concepts in functions, graphs and graphing" (Leinhardt, 1990, p 46) and be able to approach the *functions* as content in different perspectives to accommodate the patterns of learning.

Leinhardt et al. (1990) described issues on how to carry out actions elaborated above and how these may be taught effectively, drawing from different studies that dealt with teaching of functions and graphs. The constructs are: Entry or where to start, sequencing, explanation and example. These constructs inform one another. For example, when a teacher starts with a particular *function* example, she has to explain and elaborate and decide on the next example. The teacher's introduction and sequencing of examples should enhance and develop conceptual understanding by explanation and deal also with misconceptions (Leinhardt et al., 1990).

With respect to point of entry or where to start, Leinhardt et al. (1990) propose three competing views, namely, (a) discover the rule, (b) generate data and (c) interpret qualitative graphs of situation. In Grade 11, entry or where to start entails the previous knowledge of functions done in Grade 10 (see DBE, 2011) on $y = af(x) + q$ where activities suggested by Leinhardt et al. (1990) can be done. This trend suggests construction-to-interpretation tasks. The trend is in contradiction to Nachlieli and Tabach (2012) who posit that teachers should create a discourse where meta-learning occurs on discourse of the process to an object. That is, learners, through the discourse, experience the meta-discourse (function) by process (that is, by interacting both construction and interpretation in the discourse).

Sequencing of examples generally moves from the "less formal, less abstract, more global, and intuitive to the formal, notational rigorous system" (Leinhardt et al., 1990, p. 49). Rowland (2008) describes sequencing of examples and exercises as the combination of related examples that have connection between them in order to develop conceptual understanding and procedural fluency. In their study of functions, Leinhardt et al. (1990) posit that sequencing contains and builds on all significant properties and moves from one-to-one simple rule of correspondence to complex one in order to enhance and develop conceptual understanding. For example: in hyperbola, especially on construction of functions; sequencing of examples starts from tabulation of data from formula, plotting and reading values of the same family of hyperbolae from simple hyperbola of the form $f(x) = \frac{1}{x}$ to a complex one, $f(x) = \frac{-3}{x+4} - 1$. Most textbooks and curricula (see DBE, 2011, pp. 12 & 32) move from this perspective. Yerushalmy (1988) as quoted by Leinhardt et al. (1990) posits that under construction "sequencing starts with identification of graphs, moves to construction, and finally requires learners to match graphs

with families of equation" (p. 50). The last clause (*match graphs with families of equation*) has two meanings. The first one is that different equations (formulae) of the same function (e.g., quadratic) are matched with two or three sketched graphs of the form $y = ax^2 + q$ together by varying the values of both parameters *a* and *q*. The second meaning is that matching sketched graphs of the parabolas of the form $y = ax^2$, $g(x) = ax^2 + q$, $h(x) = a(x - p)^2$ and/or $j(x) = a(x - p)^2 + q$ together. This can be done with two or three graphs. The learning objective can be to illustrate both different properties in each form of the parabola against one/two graphs or what have varied and what has remain the same between or among the sketched graphs. This suggests another type of sequencing, moving from the first meaning then the last one.

Sequencing in interpretation of *function examples* start with matching functions of the same family (that is, quadratic function with another one where parameters and shapes are varied then interpreted), identification of different types of functions in terms of shape and formula (e.g., linear function and parabola) and finally curve sketching and fitting (Leinhardt et al., 1990). Another form of sequencing as cited by Leinhardt et al. is moving back and forth between qualitative (verbal/word representations) and quantitative (data from order pairs, tables or graphs) presentations.

Explanations are part of classroom interactions where a teacher and/or learners explain, demonstrate and describe characteristics of function (cf. Table 2.1). Explanations enhance meaning; functions are clarified and described in a logical way (Leinhardt et al., 1990). During explanation teachers may use *representations* (see definition of this word in 2.5.3. taking account of representations as cited by Rowland below) to increase learners access to abstract idea. Explanation should also be built from learners' knowledge (Leinhardt et al., 1990; Nachlieli & Tabach, 2012) and address learners' confusion of language especially, for example, of the meaning of scales and parameters (Leinhardt et al., 1990). The challenge in teachers' explanations is that teachers' explanations are mostly procedurally orientated and do not adequately address the specialised content as espoused by Adler and Venkat (2012). On specialised content in functions, Leinhardt et al. (1990) found that graphing and interpretation are difficult to explain by both teachers and learners.

Figure 2.3 below summarises the teachers' knowledge in teaching functions:

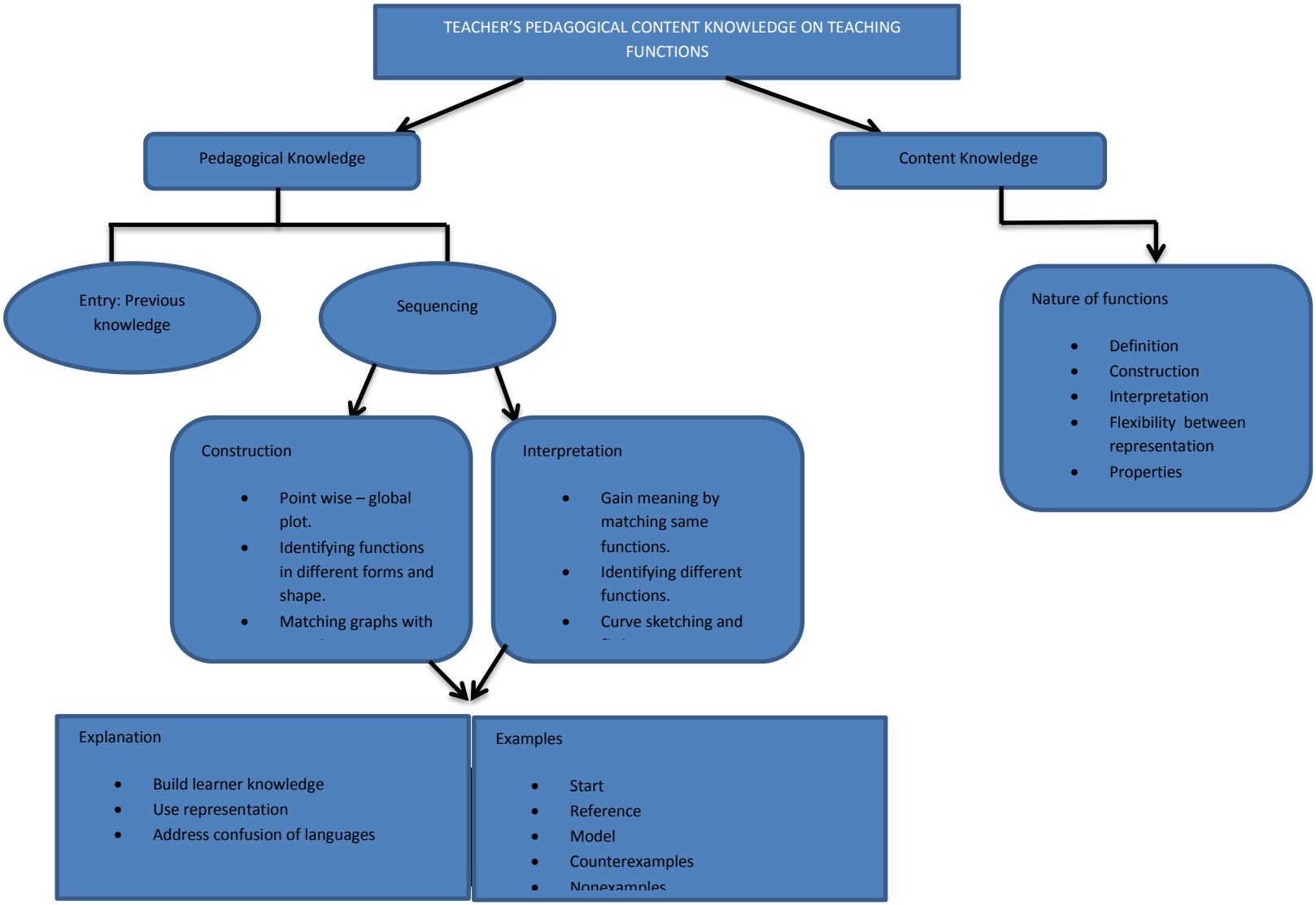

Fig 2.3: Teachers' PCK on teaching functions example

The teachers' knowledge of characteristics of functions (content) (namely, knowledge of real numbers, integer exponents, polynomials, factoring, graphing, definition of functions and their inverses, domain and ranges, quadratic theory, identities, etc.) which serve as reservoir of examples, should have a bearing on their knowledge on what and how to choose, generate and treat examples (pedagogical). Teachers should also be able to cater for different learning patterns of their learners (PCK). The teachers' PCK should provide them with knowledge of which function examples should be taught and learned first, how to sequence examples for easy access for understanding and procedural fluency and how to logically explain and address learners' misunderstanding and misconceptions. The teachers' PCK on functions is different from their PCK for teaching other concepts like analytical/coordinate geometry. What they know provides access to what and how to select, generate and treat functions. This study explores and investigates what and how examples were chosen, sequenced and explained during lesson presentation

2.5 Examples and teachers' examples

2.5.1 Role of examples

Rowland (2008) and, Zodik and Zaslavsky (2008) distinguish two roles and uses of examples in mathematics teaching as (1) *example of the general principle of a concept* and (2) *example for doing exercise or practice*. The role of example of a concept is to reflect a general principle of a concept by employing examples as particular instances of the general concept (Rowland, 2008, Zodik & Zaslavsky, 2008). Examples are used when the teacher wants to develop and enhance abstraction and conceptualization. For example, $f(x) = x^2$ and $g(x) = 2(x+3)^2 - 4$ are examples of the parabola and they denote and reflect properties of a parabola and they are solely limited to the parabola. The examples therefore provide instances of a general form of parabola and have the capacity to facilitate abstraction and concept understanding.

Teachers, when choosing an example of a parabola, become aware of different parabolic examples and their change in parameters that subscribe to "variation[4]" brought by parameters to develop conceptualisation of the parabola. Furthermore, they take note of not choosing examples

[4] Variation, dimensions of variation and associated range of permissible change are discussed in detail under section 2.5.1: taking account of variables.

of parameters which are not within "range of permissible change" (Watson & Mason, 2006) (that is, examples which are not parabolic in nature or variation do not define parabola) like asymptotes. The examples chosen and used should be within the scope and definition of the parabola for learners to experience variation and understand the concept (intended object of learning[5]).

Examples of doing exercises/practice are used as examples that facilitate procedural fluency (Alcock & Iglis; 2008; Rowland, 2008, Zodik & Zaslavsky, 2008) and enhancement of conceptual understanding (Rowland, 2008). Rowland (2008) posits that exercise examples "lead[s] to different kinds of awareness and comprehension" and are "instruments for assessment for the teacher" (p. 150-151). Exercises can be used for assessment purposes in classwork, homework, assignments, investigations and tests. Teachers should be aware when choosing and using examples of general principle of a concept and examples for doing exercises. Such awareness should be informed by their learning objectives, that is, what they want learners to learn. The learning objective can be to understand a concept, procedural fluency or both. That is, teachers' examples can afford the opportunity to discern both development of a concept and consolidation of that concept through practice at the same time. For example, in $y = -2x^2 + 1$, the $a < 0$ and $p = +1$ facilitate the significance brought by a (reflection along the x-axis and horizontal shrinking) and vertical shift by 1 unit up from the origin *(0;0)*. The three objects of learning facilitate the variation (concept development) from the graph of $y = x^2$. By repeatedly performing algebraic manipulations, the procedural fluency is facilitated which consolidate the concept understanding.

The teachers' choices of exercise examples are based on starting with routine examples and expanding to challenging ones (Goldenberg & Mason, 2008; Rowland, 2008) and consideration should also be based on sequencing (Watson & Mason; 2006, Rowland, 2008) to build understanding and facilitation of procedural fluency. Leinhardt et al. (1990) and Even (1998) postulated that the route of teaching functions starts from algebraic symbols to graphical representations. Leinhardt et al. (1990) posit that the sequencing of teaching function (as concept) and selecting examples for facilitation of procedural fluency should move from construction of a function to its interpretation. Although the two actions may be experienced

[5] Intended object of learning and other types of object of learning are discussed in section 2.6.1.

simultaneously, teachers discriminate on the two actions based on their intended object of learning/learning objective.

Examples may be chosen and used by the teacher for algebraic reasoning and representation of the concept (Alcock & Iglis, 2008). Counter-examples are normally employed to refute, to aid argumentation and for empirical generalization of the object of learning during reasoning (Alcock & Iglis, 2008; Zodik & Zaslavsky, 2008). Furthermore examples may play a role in the communication of concepts, argumentation and justifications (Alcock & Iglis, 2008; Goldenberg & Mason, 2008). Teachers should be aware of their choice of examples that elicit reasoning and which examples are suitable for that purpose. The awareness should be informed by dimensions of variation of parameters (which determine changes in representation of the graph and its critical features) to which reasoning may be elicited. Refutations, justifications and argumentations are dictated by the teachers' choice and use of examples that are restricted by what the teacher wants learners to discern and know or what have changed and what remain unchanged and why it is like that. For example, a teacher may select a parabola and hyperbola and discuss their properties so as to elicit reasoning on what is similar and different and why it is so.

The figure below (fig. 2.4) connects and locates the teachers' knowledge of the functions with the role of examples. The idea is that by knowing the characteristics of the function, what and how to teach them (characteristics of a function, cf. Table 2.1), teachers are in a better position to select and use appropriate examples of a concept and examples for exercises pertaining to functions.

Figure 2.4 summarises the role and use of function examples.

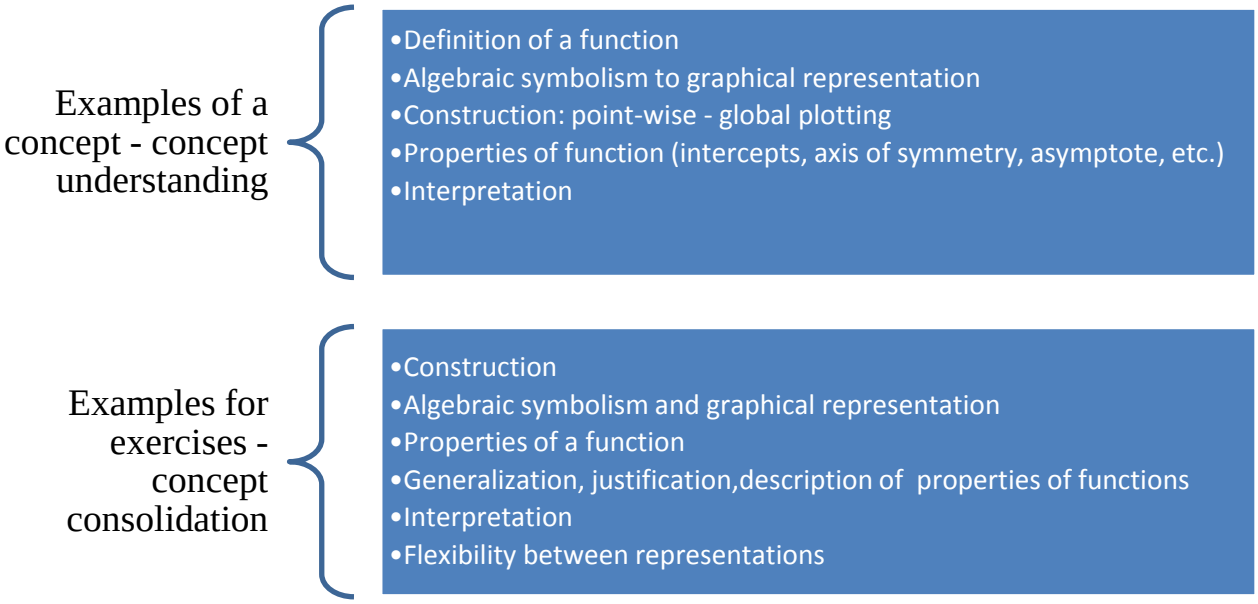

Fig 2.4: The role and use of function examples

The role and use of examples therefore is to develop concept understanding of the object of learning and consolidation of the concept of the object of learning (Tsamir et al., 2008). My study investigates and explores role and use of *function examples* (examples of a concept and examples for exercises) chosen and used by the two selected teachers.

2.5.2 Usage of examples

2.5.2.1 Planned examples versus spontaneous examples

Examples are chosen before the lesson and during the lesson. Examples chosen before the lesson are called planned examples and those that emanate during classroom interaction are termed spontaneous examples (Zodik & Zaslavsky, 2008). Spontaneous examples may be generated by both the teachers and learners. The planned examples are largely derived from teachers' knowledge and teaching (re)sources like textbooks, study guides, previous question papers and curriculum documents; and appear in teachers' notes and/or worksheets.

The spontaneous examples are largely derived from teachers' knowledge of the concept and rely "heavily on teachers' increasing awareness and on-going reflection" (Zodik & Zaslavsky, 2008,

p. 167) about the concept. Goldenberg and Mason (2008), and Zodik and Zaslavsky (2008) concur that the teachers' knowledge of recalling relevant examples in the moment of acting (spontaneously) is determined by their personal example space. The personal example space is enhanced and developed by teachers' knowledge and experience over time (Zodik & Zaslavsky, 2008) on teaching the concept. This suggests that teachers' personal example space illuminates teachers' elements of their specialised professional knowledge, and in this regard, pedagogical content knowledge. The spontaneous examples are therefore examples that are not planned by the teacher and these examples may be both from teachers' example space and teaching (re)sources within their immediate reach.

The spontaneous examples may be used in various contexts. The study by Zodik and Zaslavsky (2008) found the following contexts on spontaneous examples; firstly, spontaneous examples may be used by the teacher to respond to or elaborate on learner' question(s). Secondly, teachers need to present spontaneous examples when learners are making falsifiable claims to clarify those learners' utterances about a concept. Thirdly, spontaneous examples are used when teachers realise the limitation of their planned examples and therefore consider a "broader range of examples of the same kind" (p. 180). In their study, they (*ibid.*) observed that teachers' spontaneous examples are sometimes manifested as counter-examples[6].

The learners' generated examples (LGE) have pedagogical significance for teaching and learning (Watson & Mason, 2006; Watson & Shipman, 2008) and can be used as a teaching strategy (Watson & Shipman, 2008). For example, a teacher may request learners to give examples pertaining to the vertical shift and horizontal shift in a particular function. She can then evaluate those examples in terms of the object of learning (vertical and horizontal shift) and be able to clarify false claims or demonstrate the correctness of their claims. Watson and Shipman (2008) highlighted that learners' generated examples may "provide a good way to start understanding a new concept, with some caveats" (p. 105) and they (*ibid.*) found that learners with different cognitive levels may generate and learn from examples that they have generated. Zaskis and Leikin (2008) postulate that LGE reveal learners' understanding of the mathematical concept and meta-mathematical concept of a definition of that concept. Furthermore, Watson and Mason (2006) argued that LGE are complex yet motivating and are important for learners' conceptual

[6] Counter-examples and other types of examples are discussed in section 2.5.2.2 below.

understanding. Learners' generated examples, therefore, may be realised in any phase of lesson presentation (that is, at beginning of lesson and/or during the lesson) where learners may seek clarity on the concept and/or to demonstrate their understanding of the concept. So, in my study I investigated how and why teachers have chosen planned examples, and secondly, how teachers have chosen, treated and engaged with spontaneous examples both from themselves and learner-generated examples.

2.5.2.2 Types of examples

Various studies cite the types of examples from the seminal work of Michener (1978). The types of examples (namely, start-up examples, reference examples, model/generic examples and counterexamples) underpin their pedagogical usage before and during lesson presentation. Tsamir et al. (2008) postulated another type of example called non-examples.

Start-up examples are chosen before lesson presentation therefore they are planned examples and are used in the initial stages of lesson. It is not clear whether these examples are used in the introduction of new topic or during subsequent lessons of the same topic that was introduced the previous day(s). This study treats start-up examples as examples that introduce a new topic and/or introduction of an additional parameter on function. For example, the teacher may introduce the parabola of the form $y = ax^2$ and later introduces $y = a(x - p)^2$, both as start-up examples for the parabola. The start-up examples are then prototypical examples. The prototypical examples are defined as ideal examples which are often required first for the understanding of the concept (Tsamir et al., 2008). For example, the introduction of quadratic function in Grade 11 starts with $f(x) = x^2$ as a parent function of quadratic functions before introducing variation in terms of parameters in later stages (that is, $f(x) = ax^2$, $f(x) = a(x - p)^2$, $f(x) = a(x - p)^2 + q$) (DBE, 2011). CAPS prescribe that learners should firstly understand the attributes and features of the concept in its formative nature before experiencing complex features of concept. It suggests that start-up examples are examples of a concept and therefore are examples for developing and understanding the concept.

Tsamir et al. (2008) suggest that over-exposure of prototypical examples may impede the enhancement of concept acquisition (concept consolidation) because too much emphasis will be

on concept formation (concept understanding). Tsamir et al. posit that concept formation and concept acquisition can be experienced as a duality.

The reference examples are used to inform and develop the concept formation and concept acquisition (Tsamir et al., 2008) and are repeatedly used to link outcomes with concept (Alcock & Inglis, 2008). Reference examples are viewed as classical examples because they reflect features that are shared by all examples of the concept. Secondly, reference examples contain the critical and non-critical features for the concept. For example, $f(x) = 2x^2 + x$ contains critical feature of change of parameter of $g(x) = x^2$ in terms of variation brought by 2 (coefficient of x^2) and x. The non-critical features are the shape and how the intercept on axes is determined remain invariant for $g(x) = x^2$. Reference examples are examples for exercise and they can have a dual purpose as examples that are chosen and used to inform and develop concept understanding; and consolidate that development.

Model examples are generic examples that "suggest and summarise expectations and default assumptions about result and concept" (Alcock & Inglis, 2008, p. 113). Model examples may be used in reasoning to express variation on the function in terms of why and what have varied against what remains invariant. It suggests therefore that they are examples of a general principle that elicits concept understanding. Counter-examples serve to "show that the statement is not true and sharpen distinctions between concepts" (Alcock & Inglis, 2008, p. 113) and deepen the understanding of mathematical entities (Goldenberg & Mason, 2008). It suggests that counter-examples are examples that develop and enhance the understanding of a concept therefore they are examples of a concept. Counter-examples can be used in comparing critical features of the parabola and the hyperbola in terms of the characteristics and especially their properties.

The non-examples are examples that do not fall under a description of particular concept but they may be used to deepen the concept understanding of another concept under review. For example, the hyperbola is non-example of the parabola but the features and plotting of a hyperbola in the study of the parabola may highlight the differences of the features of the two functions and may facilitate the concept understanding and consolidation thereof of the parabola. Non-examples are examples of a concept.

This study investigates which types of examples are chosen and used in the planning phase and during classroom interaction (spontaneous examples), and secondly, how these examples have been sequenced and the frequency at which a particular type was used. These together reveal what pedagogical significance and roles these types of examples play, as espoused by Rowland (2008) and Tsamir et al. (2008). Thirdly, what is the rationale of teachers when choosing and using those particular types of examples in two phases?

2.5.2.3 Considerations teachers have in choosing and using examples

The teachers have reasons for choosing particular examples (Rowland, 2008) and choices of particular examples are examined in a context that is necessary to address a concept (Zodik & Zaslavsky, 2008). Rowland (2008) found that teachers consider viability of pragmatic situation for choosing particular examples for their lessons, namely, the learners' affective factors, the curriculum objectives and pedagogical considerations. The context of choosing and using examples are varied but Zodik and Zaslavsky (2008) postulate that generally teachers make pedagogical considerations on the choice and use of planned and spontaneous examples. The study limits itself to pedagogical consideration of examples postulated by Zodik and Zaslavsky because they are going to be used to analyse planned and spontaneous examples, and teachers' rationales for choosing and using the examples.

Zodik and Zaslavsky (2008) tabled pedagogical considerations for choosing and using examples from their study of five teachers as: (a) start with a simple or familiar case of examples, (b) choose and use examples that attend to learners' errors, (c) choose and use examples that draw attention to relevant features, (d) choose and use examples that convey generality by random choice, (e) choose and use examples that include uncommon cases and lastly (f) choose and use examples that keep unnecessary work to a minimum.

Zodik and Zaslavsky (2008) used an iterative and explorative approach to deduce these considerations. They (ibid.) inductively deduced that these considerations were based on teachers' pedagogical content knowledge, indications of sensitivity to learners and sequencing of treated concepts in subsequent lessons. Simon's (1995) Mathematical Teaching Cycle (MTC) served as Zodik and Zaslavsky' (2008) theoretical resource in which they reflected on these considerations. They argue that MTC helped them to think about the different roles and stages of

uses of planned and spontaneous examples in the classroom. The analyses of empirical evidence they collected through classroom observations (field notes) and conversations (audio transcripts) also inductively informed these pedagogical considerations. Some of the considerations were developed from guiding principles of teaching (e.g., moving from simple to complex examples), teachers' rule of thumb (e.g., drawing attention to relevant features) and teachers' accessible personal examples space.

Teachers' consideration of choosing and using simple or familiar examples is to build on learners understanding of the concept. They normally sequence examples and gradually increase the level of complexity (cognitive level) using variation in one feature at the time (Zodik & Zaslavsky, 2008). For example, in teaching parabola in Grade 11, teachers may introduce parabola of the form $f(x) = x^2$, $f(x) = ax^2$ then $f(x) = ax^2 + q$ as prior knowledge from Grade 10. They may then gradually increase level of complexity to parabola of the form $f(x) = a(x-p)^2$ then ultimately $f(x) = a(x-p)^2 + q$ of Grade 11. There are instances where the standard form of quadratic function: $f(x) = ax^2 + bx + c$ is algebraically manipulated by completing the square to be $f(x) = a(x-p)^2$ or $f(x) = a(x-p)^2 + q$. The gradual increase of cognitive level also includes determining the $\text{TP}(p;q)$ using $f(x) = a(x-p)^2 + q$ to $(-\frac{b}{2a}; \frac{b^2+4ac}{4a^2})$ using $f(x) = ax^2 + bx + c$. Zodik and Zaslavsky (2008) found that teachers used this first consideration more in spontaneous examples than in planned ones.

The teachers' choice and use of examples that attend to learners' errors mitigate common errors that learners are making or difficulties learners may encounter (Zodik & Zaslavsky, 2008). In drawing attention to relevant features, the teachers will make an attempt to avoid specific examples that can distract learners' attention from critical features of a concept (ibid.).

Zodik and Zaslavsky (2008) have observed that the choice of specific examples at random is "useful in conveying the idea of generality" (p. 175). They paid attention to Rowland's study of 2003 that emphasised that choice of random examples should be appropriate because they may mislead or miss a point. The random choice of example is manifested as quick selection of what comes first in the mind. To avoid inappropriate choice of specific random examples and to

randomly use examples that are quickly selected, they should be guided by varying the parameters. The study of Zodik and Zaslavsky (2008) found that the teachers' consideration of choosing and using examples that convey generality by random choice is associated only with spontaneous examples.

Zodik and Zaslavsky (2008) describe "inclusion of uncommon cases" as "either cases that are rather exceptional within mathematics or cases which are under-represented in teaching mathematics" (p. 177). The implication is that teachers should consider non-prototypical examples that are often overlooked in teaching functions. The study by Zodik and Zaslavsky (2008) found that 78% of these examples are manifested in planned examples.

Teachers' consideration of keeping unnecessary work to the minimum urges teachers to highlight relevant parts of the concept and avoid going into extra details that are understood by learners (Zodik & Zaslavsky, 2008). They (Zodik & Zaslavsky, 2008) further posit that the examples of this consideration intend to "illustrate a general solution strategy without completing all calculations" (p. 179). When teachers are making such considerations, they must be certain that learners have mastered procedural cues and their intention is to manifest concept understanding or to let learners fill in details. This consideration normally occurs in a global approach of sketching the function. The intention is not to concentrate per se on procedural cues but on getting sufficient data to sketch the graph. About 75% of observed teachers' examples appeared in planned examples.

The study of Zodik and Zaslavsky (2008) assisted me in categorising and analysing teachers taking into consideration for mathematics entities and objects (content) and reciprocal consideration for pedagogy.

The below table, Table 2.3 summarises the examples and teachers' examples*:*

Table 2.3: Summary of examples and examples usage.

Role of examples and their design	Examples of a concept – for concept understanding Examples for doing exercise – for concept consolidation (procedural fluency)
Phases of usage of example in teaching and learning	Planned examples Teacher's spontaneous examples and learners' generated examples (LGE)
Types of examples	Start – up Reference Model/generic Counter-examples Non-examples
Teachers' consideration when choosing and using examples	• Start with simple or familiar case of example • Choose and use examples that attend to learners' errors • Choose and use examples that draw attention to relevant features • Choose and use examples that convey generality by random choice • Choose and use examples that include uncommon cases • Choose and use examples that keep unnecessary work to a minimum

This study will probe teachers' rationale about their considerations when they choose and use examples. That is, what considerations did they keep in mind during the planning phase and/or spontaneously (classroom interactions), and type of examples they have chosen and used in each phase. The study further investigates and explores considerations the two teachers made in their contrasting contexts; namely, teaching in a fee-paying school and a no-fee school (see details in sampling section below). These contrasting socio-economic settings were selected to illuminate teachers' considerations of what, how and why they choose and use examples in planned and

spontaneous phases. The purpose here is to derive a deeper understanding of teachers' considerations when they choose and use examples.

2.6 Teaching and learning (re)sources

The study focuses on two mathematics teachers who teach in two contrasting socio-economic settings (a fee-paying school and a no-fee school). It is imperative to discuss the availability of learning and teaching support materials (LTSM), how they have been used, their transparency (as espoused by Adler, 2000) and what their significance is. In this study, LTSM are argued in line with Adler's (2000) description of (re)sources.

There is a sweeping statement that correlates quality of teaching and learning with the availability and usage of LTSM. Adler (2000) made an assertion that "more (re)sources do not necessarily lead to better practices" and that there are "wealthy schools that do not offer quality education to learners (amid enough resources) and there are impoverished schools that succeed against all odds" (p. 206). Given the contrasting South African socio-economic settings of the schools under study, the availability and usage of the (re)sources are therefore illuminating of the teachers' practices, namely, how and what (re)sources they draw examples from when they choose and use examples.

The word "(re)source" is viewed similarly to Adler's notion of (re)sources as both a verb and a noun. This study underpins (re)sources not only what are they but also how the (re)sources function as extension of teaching and learning. The discussion of (re)sources is limited to (re)sources that support mathematics teaching and learning practices. Adler identified such (re)sources as human resources, school mathematics materials, mathematical objects and cultural resources as (re)sources that are available in everyday teaching and learning process of mathematics. What remains critical, other than their availability (quantity) and quality, is their transparency (Adler, 2000). The transparency of (re)sources has the dual function of visibility and invisibility of resources. The (re)sources need to be seen (visible) and be used to support teaching and learning. Furthermore, Adler (2000) posits that (re)sources need to be invisible "so that they allow smooth entry into the practices" by been "seen through to illuminate mathematics (invisible)" (p. 214).

For example, in her study, she focused on the usage of the chalkboard where both or either the teacher and/or learners make writings on, it is a visible function of the chalkboard. That is, the chalkboard is seen and used to support the lesson. Although the chalkboard is visible, it must be invisible so it cannot be an object of attention for teaching and learning activities to occur but illuminate mathematical task. The chalkboard therefore has a dual function. Other (re)sources are intangible like time and language; and also have a dual function. For example, time has visible and invisible functions that support teaching and learning activities. The length of the period (say 40 minutes per period) is a visible function of time on teaching and learning activities in that moment in time. It regulates the length and occurrence of these activities. The invisible function of time as a (re)source that regulates and illuminates teaching and learning activities, it can be seen in planning of teaching and learning activities. How and when these activities should be taught (in terms of selection, sequencing and mediating the tasks). Therefore, time supports the lesson planning.

Human resources have two dimensions. The first dimension is teachers as human resources and secondly, the school human resources as workforce in the school. Teachers as human resources entail teachers' qualifications (see details in 4.3.2. in Chapter 4) and their knowledge structures as postulated by Shulman (1986), Ball and Bass (2000), Hill et al. (2008), Zodik and Zaslavsky (2008). Adler (2000) further posits that the human resources also entail teacher-learner ratios and class size.

School mathematics materials entail technologies (hardware and software), textbooks, charts and other materials specifically made for school mathematics. Mathematical objects entail objects emanating from mathematical content and context, and range from complex theorems to a simple number line and well established procedures (Adler, 2000).

Cultural resources entail language and time. Languages include the main languages that learners bring into the class and the relation of these languages to the language of instruction (ibid.). Time as cultural resource is used differently in urban and rural contexts (Adler, 2000) but in this study, time is illuminated by fee-paying school and no-fee school. Time encompasses the time-table, the length of a period, learners' attendance, time given on classwork and homework. Time also entails the pacing, selection and mediation of tasks (Adler, 2000). According to the Department of Basic Education (2011), the minimum teaching contact time in a week is 27.5 hours and time

allocated for teaching Mathematics in Further Education and Training Phase (Grade 10 – 12) is 4.5 hours.

The teaching and learning support materials (resources) are used to describe their availability and transparency in both fee-paying school and no-fee schools. That is, the availability of (re)sources underpin what and how many are they, and also how they have been used (transparency) to support teaching and learning of mathematical tasks. For all practical purposes and the relevance of this research; the school's human resources, school mathematics materials and time allocation are discussed. Some aspects of teachers as human resources (nature of their knowledge and knowledge structures) and mathematical objects are not explicitly discussed but they are grounded on teachers' choices and use of examples.

In conclusion, the teaching and learning (re)sources available in each school provide teachers with examples for planning, during classroom interaction more so as aid for teaching. Such (re)sources are reflected on so as to analyse whether and how LTSM (re)sources were used to choose and use examples in the two contrast setting. Lastly, we can ask what role (re)sources played in planning and during classroom interactions.

The following Table 2.4 summarises teaching and learning (re)sources**:**

Table 2.4: Teaching and learning (re)sources.

Aspects	Description
Human resource	Teacher's qualification Teacher's knowledge structure Teacher-learner ratio Class size
School mathematics material	Technology (hardware and software) Textbooks, charts, workbooks
Mathematical objects	Content, context, structure procedures
Cultural resource	Main languages spoken by learners Language of learning and teaching (LOLT) Time-table, length of a period, attendance registers, time for classwork and homework

2.7 Conceptual framing

This study draws its theoretical and analytical resources from the conceptual framework derived by Rowland in his 2008 study of teachers' practices. The Rowland study reports on categories that analyse teachers' choice and use of examples in their lesson. The four categories are derived from grounded theories of teachers' subject matter knowledge, pedagogical content knowledge and knowledge *for* and *in* mathematics teaching (Rowland, 2008). The Rowland study derived these concepts as factors that are rooted in teachers' awareness and what they take into account when choosing and using examples in their lesson. The four concepts/categorised are: variables, sequencing, representations and learning objectives. Taking account of variables should be understood in the context of aspects of object of learning (detailed account is given below) that changes not variables that deal with unknowns or a letter that stand for a number. To eradicate confusion and for clarity purposes, Rowland's "taking account of variables" will be referred to as taking account of variation.

I use Rowland's conceptual framework because my research project explores and investigates:

Teachers' choice and use of examples in their "lesson preparation and classroom interactions" (Rowland, 2008, p. 149).

2.7.1. Taking account of variation

Taking account of variation is derived from notions of Variation theory as espoused by Ference Marton and his colleagues. Variation theory has two notions, namely, dimensions of variation and its associated range of permissible changes (Watson & Mason, 2006). The variation theory provides tools for evaluating and analysing teaching because teachers become aware or take account of what learners should discern when they teach a concept during lesson presentation. Teachers' attempts, by using examples, bring variation in their examples so that learners should experience the difference rather than learners to recognise the similarities.

(a) Dimensions of variation

For Variation theorists, learning something is about discernment through what is varied and what remains invariant (Marton & Morris, 2002; Marton, Runesson & Tsui, 2004; Runesson, 2005; Marton & Pang, 2006; Runesson, 2006; Watson & Mason, 2006). By bringing the differences in

something into focus, learners experience and discern[7] critical features of object of learning. The object of learning is defined as skills, insights, proficiencies, competencies, capabilities that the learners are expected to develop during a lesson (Marton & Pang, 2006; Runesson, 2005 & 2006). The proponents of variation emphasise that learning is defined as change in the way "some-thing" (object of learning) is seen, experienced or understood by a learner. The implication is that what is varied by teachers when choosing and using examples are critical features and non-critical features of the object of learning. For example, when teaching the effect of "a" in a quadratic function, the critical feature becomes the concavity of the parabola (concave up when $a > 0$ and concave down when $a < 0$) and non-critical feature may be the actual turning points (TPs) and /or axis of symmetry. The concavity and TPs are experienced simultaneously but concavity becomes what should be discerned and learned at that moment and TP and axis of symmetry are discriminated. The dimension of variation, therefore, refers to what the learner discerns and discriminates within two different values or qualities of the same concept (Runesson & Marton, 2002; Runesson, 2005).

The three types of object of learning are: 1. Lived object of learning, which denotes what is actually learned by the learner; 2. Intended object of learning is capabilities the teacher, the textbooks or the curriculum wants the learners to develop; and lastly 3. Enacted object of learning is learning which is co-constituted by interaction between learners and the teacher or between learners (Runesson, 2005; Marton & Pang, 2006; Runesson, 2006).

The study is limited to intended object of learning and enacted object of learning because the teachers' choice and usage of examples are employed by the teachers to provide their learners with opportunities to discern the objects of learning in those examples. Secondly, the enacted object of learning is realised during classroom interaction between learners and teachers or among learners. The dimensions of variation of examples chosen and used by the teachers illuminate the possibilities for concept understanding and consolidation of the concept of intended object of learning and enacted object of learning. The meaning is that, in the intended object of learning, teachers discriminate purposely in their choice of examples that are intended for concept understanding or examples that are for exercise (consolidation of a concept). With

[7] Discernment is experiencing the difference between two things or between two parts of the same concept. People discern two qualities simultaneously and experience them in mutual exclusive way (Marton & Pang, 2006).

respect to the enacted object of learning, still teachers may discriminate on the purpose of choosing and using a particular example in the lesson presentation.

The two contexts in which each teacher teaches (fee-paying school and no-fee school), may enable or constrain teachers' choice and use of examples and enhance variation in their examples. For example, a teacher in a fee-paying school is likely to have access to a wider range of (re)sources (material and technology) and these could shape the kinds of examples teachers choose and use and thus the variation of examples that are possible to provide. It is interesting to thus explore what Adler (2000) claimed that "more resources do not necessarily lead to better practice" (p. 206).

(b) Associated range of permissible change

The proponents of variation theory argue that only what varies can be discerned, discriminated and experienced by the learner. In the process of learning; discernment, experiencing and discrimination creates notions of dimensions of variation and its association of range of permissible change (Watson & Mason, 2006). For example, in dealing with quadratic functions of the forms $f(x) = ax^2 + bx + c$ and $g(x) = a(x-p)^2 + q$, a learner can discern that the turning points (TP) of *f(x)* can be found by substituting constants in $x = -\frac{b}{2a}$ and the ordinate (y part) of TP can be found by substituting the value of *x* in *f(x)*. In *g(x)*, TP is found by substituting $(p;q)$ from *g(x)*.

Discrimination occurs when learners can realise that both equations are representations of the same function but finding TP is different. Marton and Pang (2006) assert that learners (read teachers) discern and discriminate differently hence they learn (teach) differently. The dimension of variation is modified as possible dimensions of variation because it is not always that teaches can discern, discriminate or experience concept the same and what is discerned is what is possible to the individual teacher. It is expected that the two competent, qualified, experience teachers will choose and use function examples differently. Therefore, the notion of range of permissible change is grounded and aligned to possible dimensions of variation (Marton & Pang, 2006). Teachers in choosing and using examples will be aware that learners can discern possible cues differently within the same concept due to changes in parameters or techniques of concept that is only possible and permissible to that concept. That is, the permissible changes can be

extended and be limited within the boundaries of the object of learning of that concept. The meaning is that what is possible to change or vary in the linear functions is the gradient and y-intersect, therefore what is possible and permissible to change will only occur in these parameters. Secondly, the turning points and asymptotes do not define or form part of a linear function and therefore they fall outside the scope/boundary defining it. Furthermore, we can assign the gradient and y-intercept different values that are possible and permissible to a linear function. Hence, the second notion is referred to as an associated range of permissible change of object of learning.

These notions are necessary for understanding teaching so that the teacher should be aware of certain patterns of variation to be able to choose and use examples effectively. Furthermore, teachers need to discern and discriminate a quality or value to be taught, for example, in the effect of change of parameters of the parabola the teacher can choose examples that illuminate the desired variation when different parameters are used. By so doing, teachers make a consideration on dimensions of variation and will also consider the range of possible change of parameters that are associated with parabola.

Taking account of variation, teachers will therefore illuminate possible dimensions of variation of functions and their associated range of permissible change when they choose and use examples in planning phase and in-the-moment of classroom interaction (spontaneously). Secondly, taking account of variation will provide theoretical (re)sources for considering and describing how teachers have structured their examples (Watson & Mason, 2006).

2.7.2. Taking account of sequencing

Sequencing of examples is defined as systematic combination of related examples of the same concept that are connected to each other to develop and facilitate conceptual understanding and procedural fluency on the concept (Leinhardt et al., 1990; Rowland, 2008). Teachers are assumed to be aware or take account of examples in planning phase or spontaneously about relevant examples that develop conceptual understanding (examples of a concept) and consolidation (examples for exercises) thereof. Examples are sequenced in the textbooks, worksheets even in teachers' lesson preparation and classroom interaction (Leinhardt et al., 1990; Rowland, 2008; Zodik & Zaslavsky, 2008). In the planning phase, teachers' sequencing of examples addresses their intentions of either or both developing concept understanding and

procedural fluency. In classroom interaction, there are instances where teachers randomly generate examples (as response from learners' questions and/or becoming aware of inadequacy of their planned examples) where sequencing of examples becomes unavoidable (Rowland, 2008; Zodik & Zaslavsky, 2008). Sequencing generally follows a trend of moving from simple or familiar cases to more complex examples (Leinhardt et al., 1990; Rowland, 2008; Zodik & Zaslavsky, 2008).

In functions, sequencing of examples follows two interwoven activities, firstly, from construction to interpretation (cf. figure 2.3). Here sequencing of examples moves from the definition of function, table method, and set of ordered pairs to construction of a function (Leinhardt, et al., 1990). This movement entails reading of values and plotting points (point-by-point approach) on the Cartesian plane (system of axes). Function is represented in different forms, namely, from algebraic symbols, table method, ordered pairs to a graph representation. From point-by-point plotting, a global approach of sketching graphs is introduced. The addition of parameters also provide for sequencing on the same function, for example in parabola, from $f(x) = x^2$, $f(x) = ax^2$, $f(x) = ax^2 + q, to f(x) = a(x - p)^2 + q$ for Grade 11 lessons.

Sequencing for construction, as espoused by Leinhardt et al. (1990), moves from identifying the graph ultimately to matching graphs with families of equations (cf. Table 2.3 and fig. 2.3). Flexible movement between different representations of a function, as espoused by Even (1998), is the last activity to be engaged in because it deals with enhancing construction and interpretation simultaneously in the same task.

Interpretation activity normally follows after construction activity (Leinhardt et al, 1990; Lichlieli & Tabach, 2012). Interpretation activities of function start with matching graphs of the same families, matching functions of different types in terms of shape and formulae to curve sketching (Leinhardt et al., 1990) (cf. Table 2.3. and fig. 2.3).

Taking account of sequencing provided a theoretical resource for describing and interpreting the selected teachers' choice and use of examples. The teachers' examples are analysed, firstly, based on how their examples have been sequenced taking into cognisance what Leinhardt et al. (1990) and Even (1990) posited on teaching of function (construction and interpretation

activities). Secondly, teachers are probed on their rationale for sequencing their examples in the manner that they did.

2.7.3. Taking account of representations

Taking account of representations helps to describe strategies used by teachers in order to represent an abstract mathematical idea in more accessible, visible and tangible manner to the learners so that learners may understand the concept/idea (Rowland, 2008).The different forms of representations of functions are algebraic formula, table method, sets of ordered pairs, words and graphs (Leinhardt et al., 1990; Even, 1998; DBE, 2011). These forms of representation of function provide access and mediate understanding of objects of function, namely; definition, properties, generalisation of the effect of parameters, and action to be taken in functions. For example, to understand the effect brought about by parameter *q*, a teacher may use the table method and sets of order pairs of two parabolas to demonstrate the vertical translation of the two graphs.

Taking account of representations is used to describe and interpret ways at which teachers have used examples to mediate object of learning of functions.

2.7.4. Taking account of learning objectives

Learning objectives are curriculum and/or teachers' intentions and goals of teaching a particular concept or topic. The learning objectives spell out skills, knowledge and values that learners should learn from the functions. Learning objectives are aligned with the intended object of learning and enacted object of learning. The intended objects of learning are capabilities, insights, proficiencies and competencies learners are expected to develop during a lesson (Marton & Pang, 2006; Runesson, 2005 & 2006). The enacted object of learning is capabilities, insights, proficiencies and competencies that arise from the interaction between the learners and the teacher (Marton & Pang, 2006; Runesson, 2005 & 2006). The core learning objectives of functions (as espoused by Leinhardt et al., 1990; Even, 1998; Nachlieli and Tabach, 2012) are definition, construction, interpretation and flexible movement between representations. Other learning objectives that emanate from the core ones are: generating many graphs by point-by-point and later by global plotting, generalise the effect of parameters and making conjectures, problem solving on graph work (CAPS, DBE, 2011) (cf. Table 2.1). Other skills which are

embedded in the previous statement include algebraic manipulation (determining intersections on axes), interpolation and extrapolation.

To sum up, the four categories of Rowland on the choice and use of examples are intertwined and inform one another. For example, if the learning objective of a lesson is to generalise the effect brought by a parameter of the same function by point-wise plotting (that is, construction); the teacher will take different examples that vary in terms of the parameters within permissible change of that function so that learners should discern variation on effect of parameters. The teacher will sequence his/her examples by matching graphs of the same function in the manner that develops generalisation of parameter so that learners can be able to discern the learning objective. S/he will use tables of values and ordered pairs (as representations) to mediate the learning objective by sequencing examples that provide conceptual understanding and/or procedural fluency where variation of what to be learned is discerned (see fig 2.5).

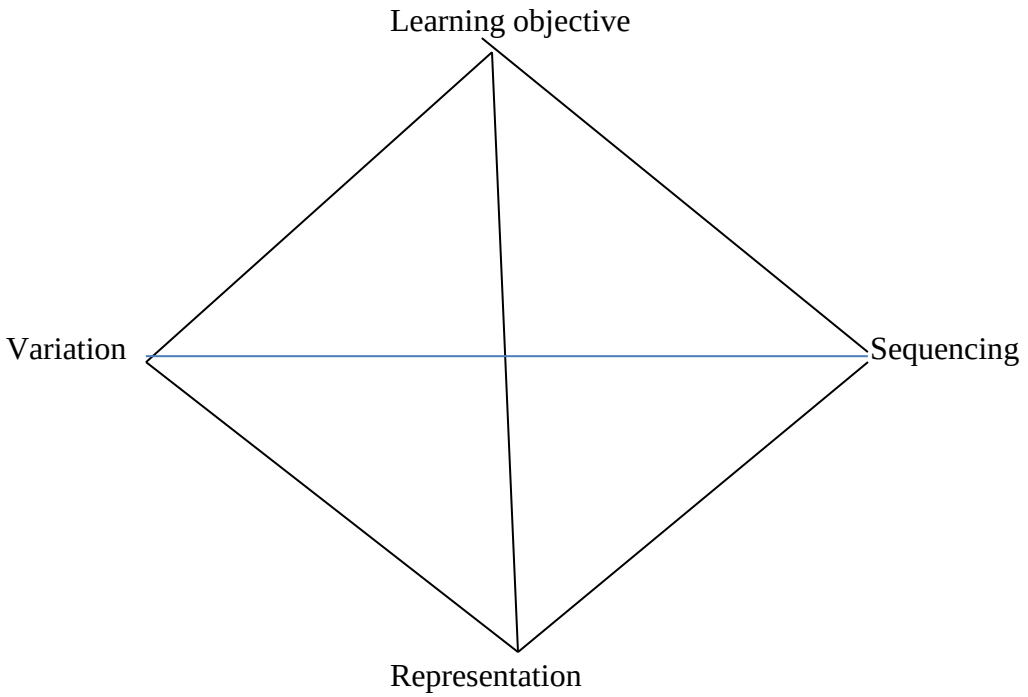

Fig. 2.5: Teachers' awareness on lesson preparation and interaction

2.8 Conclusion

The teachers' choices of examples and their uses thereof are manifested in the planning phase and during classroom interaction. Teachers' examples space serves as teachers' repertoire to

choose and use examples that illuminate the example of a concept (function) and examples for exercises and their proficiency (abstraction and conceptualization, and procedural fluency) during planning phase and in moment of classroom interaction. Teachers make many considerations when they choose and use examples. The below conceptual map (cf. figure 2.6) depicts and summarises the concepts that are the focus of this study.

The conceptual map assists in describing, interpreting and analysing documents and teachers' choice and use of examples in the subsequent chapters. In Chapter 3, the nature of school functions in the curriculum, as discussed earlier in the chapter, is employed to analyse the curriculum policy (CAPS) and previous examination question papers of Grades 10 and 11. The types of examples are described in previous question papers and textbooks. Taking account of learning objectives, variation, sequencing and representations is used to describe and interpret the function examples appearing in the textbook.

The conceptual map assisted with describing, interpreting and analysing choice and use of teachers' examples in Chapters 5 and 6. The pertinent issues emanating from these two chapters form the bases of meta-analysis in the cross-case synthesis of the two teachers.

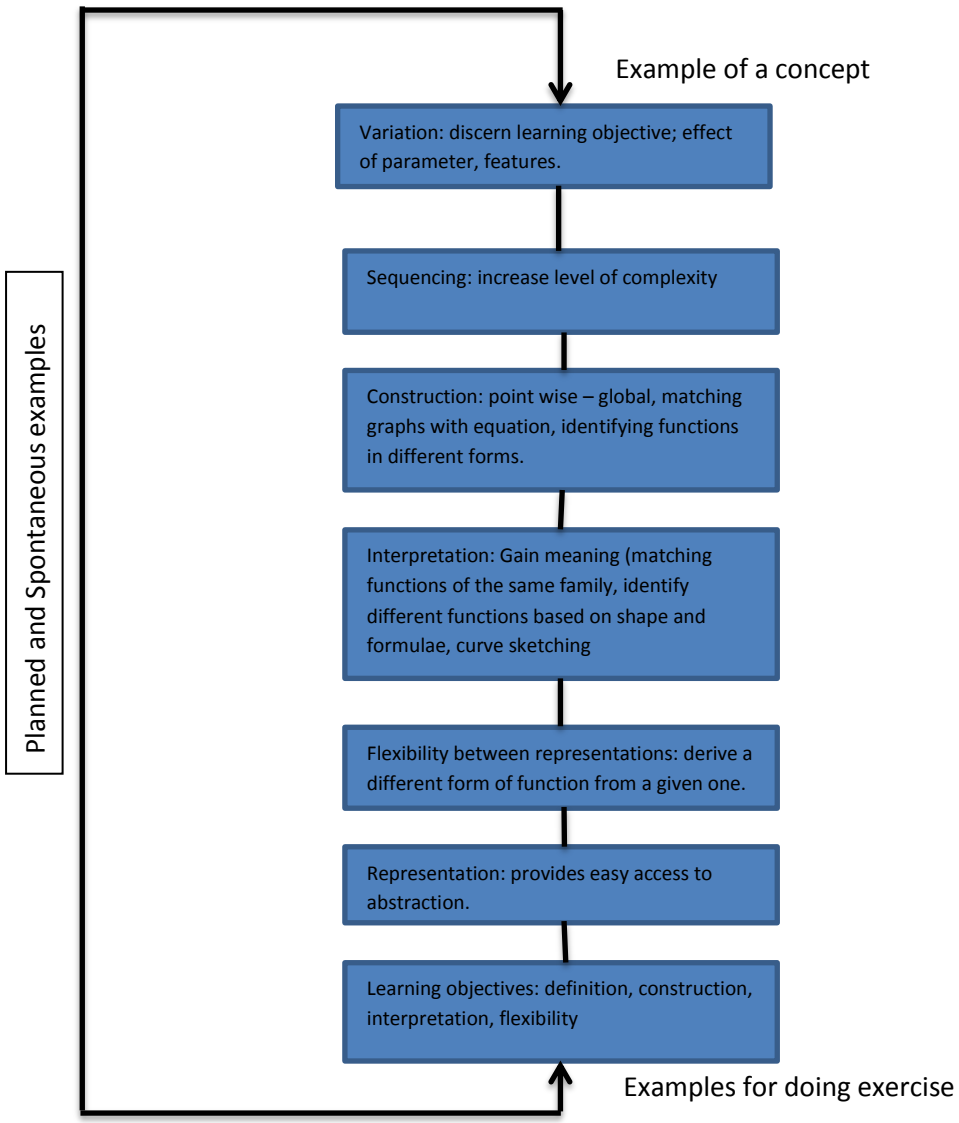

Fig. 2.6: Conceptual map

Chapter 3: Document and content analysis

3.1 Introduction

In this chapter I analyse three documents namely, (a) the Curriculum and Assessment Policy Statement (CAPS) in Mathematics Further Education and Training Phase (FET) (Grade 10 - Grade 12) (DBE, 2011), (b) the 2012 Grade 10 external question paper set by North West Provincial Department of Education and Training and 2013 National Exemplar question paper of Grade 11 and lastly (d) the Grade 11 textbook. The three documents are analysed because they are key (re)sources that structure and support teachers' work. That is, mathematics teachers normally draw from them for lesson preparation and planning and other related work. Secondly, the analysis will describe how the content with respect to *functions* is selected and sequenced and varied in each.

The analysis of the curriculum statement (CAPS, DBE, 2011) as an official document reveals what the curriculum stipulates. The intention of analysing the question papers is to describe how official assessment documents assess the functions, that is, what is valued and privileged. The analysis of the textbook is to see how functions are presented and sequenced particularly with respect to examples. In overview, the analyses of these documents provide a base on which to observe, describe and interpret the two teachers' lesson preparation, planning and presentation and how these teachers exemplify *functions*.

Table 3.1 (below) depicts how the documents are going to be *analysed* drawing from concepts cited in the literature. The reasons for deciding to use the concepts as depicted below are cited in each document analysis.

Table 3.1: Document and content analysis.

Document	Concepts
Curriculum and Assessment Policy Statement	▪ Progression across Grades 10-12 ▪ Nature of school functions: ✓ Definition ✓ Construction ✓ Interpretation ✓ Flexibility between representations
Examination question papers	Nature of school functions: ▪ Construction ▪ Interpretation ▪ Flexibility between representations ▪ Algebraic manipulations Types of examples
Textbooks	▪ Example of a concept: ✓ Type of examples ✓ Learning objective (cf. fig. 2.6) ✓ Sequencing ✓ Variation ✓ Representation ▪ Examples for exercise ✓ Type of examples ✓ Learning objective (cf. fig. 2.6) ✓ Sequencing ✓ Variation ✓ Representation

3.2 Curriculum and Assessment Policy Statement

The Curriculum Statement is the policy which dictates a framework and direction for teaching and learning. It provides what the state deems important in terms of content to be covered, what and which knowledge, skills and values need to be acquired by learners in the specific subject

and grade. It also provides teachers with what to teach in terms of content. The CAPS (DBE, 2011) stipulates critical outcomes and development outcomes as its main objectives for teaching and learning of mathematics in South African schools. These outcomes describe "knowledge, skills and values that should be acquired by the end of Further Education and Training band" (DBE, 2011, p. 8).

CAPS is analysed drawing from what literature imparts on the nature of school functions, namely, definition, construction, interpretation and flexible movement between representations. Firstly, an overview of progression of topics between grades is analysed, taking into cognizance what the literature denotes. Secondly, analyses are aimed at the definition of function; different forms of representation, construction and interpretations suggest knowledge and skills that should be located in the curriculum.

The progression between grades is on the content of functions that develop from simple to complex. The skills and knowledge espoused by researchers about functions are reflected in the curriculum. For example, as Even (1998) and Nachlieli and Tabach (2012) have cited, the skills and knowledge on functions are maintaining of the definition of a function, flexibility to move between different representations of functions, namely tables, graphs, words and algebraic expressions are listed in CAPS (DBE, 2011, p. 12). Other skills and knowledge envisaged in the curriculum are construction and interpretation (Leinhardt et al., 1990).

3.2.1. Definition of function

The following Table 3.2 assists in analysing the progression of definition of function. The depicted statements are taken from CAPS (DBE, 2011) (pp. 24, 32 & 40).

Table 3.2: Definition of a function.

Grade 10	Grade 11	Grade 12
The concept of a function, where a certain quantity (output value) uniquely depends on another quantity (input value). Page 24	The same definition is implied.	Same definition is implied and a general concept of the inverse of a function and how the domain in the function may need to be restricted (in order to obtain a one-to-one function) to ensure

		that the inverse is a function (page 40)

The one-to-one rule is described in all three grades. Definition of a function as a many-to-one rule is not vividly stated in Grades 10 and 11 but my take is that the many-to-one rule is implied in the different forms of representations (table of values, ordered pairs and graphing) especially on quadratic functions. In Grade 12, the definition of a function includes an inverse of a function and how it can qualify as a function. The inverse of function of many-to-one rule, for example, a parabola needs a restriction on its domain for it to be a function. Furthermore, the Grade 12 curriculum statement cites a formal definition. This may mean and include many-to-one rule.

On different forms of a function, namely, tables, graphs, words and formulae; flexibility between these representations as espoused by Even (1998) and, Nachlieli and Tabach (2012) appears in all three grades. The level of progression and the degree of difficulty between grades are brought about by the addition of parameters in Grades 10 and 11, and inverse functions (in Grade 12) (cf. Table 3.2).

Table 3.3: Addition of parameters.

Grade 10	Grade 11	Grade 12
3.Investigate the effect of *a* and *q* on the graphs defined by $y = af(x) + q$ where $f(x) = x$ (linear function), $f(x) = x^2$ (parabola), $f(x) = \frac{1}{x}$ (hyperbola) and $f(x) = b^x, b > 0, b \neq 1$ (Exponential function) (p. 24)	1.Revise the effect of the parameters *a and q* (from Grade 10) and investigate the effect of *p* on the graphs of the functions defined by: 1.1 $y = f(x) = a(x + p)^2 + q$ 1.2 $y = f(x) = \frac{a}{x + p} + q$ 1.3 $y = f(x) = a.b^{x+p} + q$ where $b > 0, b \neq 1$ (p. 32)	3.Determine and sketch graphs of the inverses of the functions defined by $y = ax + q$ (linear function); $y = ax^2$ (parabola); $y = b^x; (b > 0, b \neq 1)$ (exponential function) Focus on the following characteristics: domain and range, intercepts on axes, turning points, minima, maxima, asymptotes (horizontal and vertical), shape and symmetry,

		average gradient (average rate of change), intervals on which the function increases/decreases. (p. 40)

3.2.2. Construction

In construction of graphs, learners are expected to generate graphs, test conjectures, generalize the effects of the parameters, identify the features and apply them when learners sketch graphs of different functions (DBE, 2011, p. 12). In Grade 11, there is the addition of parameter *p* to Grade 10's parameters, namely, *a* and *q* (cf. Table 3.3). The effects of parameter *q* result in vertical shifts of the graph (either *q* units up or down from the origin, (0;0) or parent function). The effects of parameter *p* result in horizontal shifts. Reflection about the x-axis (line $y = 0$) appears in Grade 10 whereas in Grades 11 and 12 both reflection about the *x-axis* and *y-axis* (line $x = 0$) are studied. Furthermore, in Grade 12 different transformations of functions are expected to be done where different properties of graphs are dealt with. The inverse function (only in Grade 12) and the construction thereof include linear function and quadratic functions dealt with in previous grades except trigonometric functions. Furthermore, the inverse function of cubic functions is not studied in Grade 12.

A point-by-point approach is advised to be started first before generalization brought about by the effect of parameters can be addressed (Leinhardt et al., 1990). In the process of point-by-point sketching, a move between representations (formula, table of values, oreder pairs) is maintained. For example, point-by-point approach requires either the application of ordered pairs and/or table method or technology. The identification and the usage of applicable properties of the functions come next.

A point-by-point approach is realized in Grade 10 on linear function, quadratic function, hyperbola and exponential functions. There is no indication of point-by-point approach on the said functions in both Grades 11 and 12 but implication is that learners are expected to do both point-by-point sketching and properties of functions and global one.

3.2.3. Interpretation

Interpretation of functions occurs when learners are expected to investigate the effects of parameters as stipulated in the CAPS document. The investigations of these parameters across the grades implies, firstly, that learners may be required to make and test conjectures and generalize the effect brought about by these parameters between or among graphs of the same function. Secondly, learners are expected to identify two (Grades 10 and 11) (see DBE, 2011, pp. 24 & 32) or three (Grade 12) different/same types of functions in terms of shape, features, restrictions and formulae. Lastly, the reading of values from the graphs and translating of algebraic values into the graph (values of intersection on axes, asymptotes, axis of symmetry, etc.) are part of interpretation (Even, 1998) therefore they are embedded when sketching and/or interpreting the graphs. In all three Grades, two different parameters can be varied either in the same or different function(s), that is, one function having parameter p and another same/different function parameter q.

The flexible move between different forms of representations (tables, graphs, words, formula) provides an affordance to interpret the functions (Even, 1998; Nachlieli & Tabach, 2012). In Grade 10, the flexible move from one representation to another (for example, sketched graph to algebraic formula) occurs in linear, some of parabolas and hyperbolas, and exponential functions. In Grade 11 it includes all functions except hyperbola. In Grade 12, there is the addition of cubic functions, logarithmic functions and inverse functions.

The problem-solving and graph work requires evaluation and analysis of functions (Even, 1998; Nachlieli & Tabach, 2012). Before and/or in the process of solving a graph in any form, interpretation is needed.

Interpretation in Grade 10 is limited, firstly, to matching graphs of the same family, for example, quadratic function with another one where there is variation on parameters. Secondly, matching and identifying only two functions together, for example, linear function with a parabola, hyperbola or exponential function (see example, DBE, 2011, p. 24) in terms of shape and formula.

In Grade 11, interpretation is done by making deductions from graphs and real-life situations (see DBE, 2011, p. 32). Matching of same functions and/or different ones is limited on two

parameters at a time. In Grade 12, interpretations follow the one in Grade 11 and include two/three different functions and inverse functions.

CAPS is consistent with the nature of the school function as postulated by cited researchers in Chapter 2. The progression between grades in terms of increasing the level of complexity is maintained. Definitions of a function, construction, interpretation and flexible move between representations are part of the actions, skills and knowledge that are envisaged that learners should acquire in each grade (10–12).

3.3 Official examination papers

The official examination papers that are analysed are: 2012 Grade 10 external question paper set by North West Provincial Department of Education and Training (Appendix A) and 2013 National Exemplar question paper of Grade 11 (Appendix B). The two question papers are compliant with CAPS because CAPS was implemented in 2012 in Grade 10 and in 2013 in Grade 11. It is noted that the Grade 12 external question paper and/or its exemplar question papers is/are not considered because during this research project, that is 2013-2014, CAPS had not yet been implemented in Grade 12.

The question papers are assessment instruments that assess learners' skills and knowledge as enshrined in the curriculum document. The purpose of analysing these papers is that most teachers' use them for teaching and checking which content is highly valued (Luxomo, 2011) and privileged. Luxomo (2011) further elaborates that "teachers in most schools used assessment at national level *and provincial* as a guide to decide on the amount of time they spend to teach each topic" (p. 64, *my emphasis)*. Lastly, teachers refer to the externally set question papers as the guide on how to translate the curriculum objectives into assessment and how content is weighted in terms of cognitive levels (taxonomies).

Leinhardt et al. (1990) posit that construction and interpretation are skills and knowledge learners are expected to acquire. Furthermore, actions as espoused by Leinhardt et al. (1990), Even (1998) and Nachlieli and Tabach (2012) are skills and knowledge that will frame the analysis of the question papers/assessment instrument. The actions are construction, interpretation and flexible move between different representations. The algebraic manipulation is imbedded on the process of flexible move between representations or process of transferring

information from algebraic formula to sketching the graphs or *vice versa*, hence it is included as some of the skills and knowledge that need to be required by learners.

3.3.1 Construction

(a). ***The Grade 10*** question paper on functions begins with sketched graphs of $f(x) = 2^x + 1$ and $g(x) = \frac{a}{x} + q$ in question 4.1 (cf. Table 3.4 and Appendix A, pp. 4 - 5) then the sketching of the of the graph of $y - x + 3 = 0$ and $y = x^2 - 9$ in question 4.2 (p.5). Question 4.1 demands the interpretation of the sketched graphs. The approach, which is supported by Even (1998), is that learners may gain useful skills when we start with interpretation of functions and then their construction. The sketching of the graph (in 4.2) expected learners to show all intercepts with axes and points of intersection. These processes require algebraic manipulation. The sketching of these graphs implies that the global approach of plotting is expected because from algebraic workings learners may go directly to sketching without doing a point-by-point approach.

Table 3.4: Analysing official examination papers is depicted below.

Action		Grade 10 (cf. Appendix A)	Grade 11 (cf. Appendix B)
Construction	Global Plotting	4.2. sketch the graph of $y - x + 3 = 0$ and $y = x^2 - 9$ Show all intercepts with axes and points of intersection	9.1. sketch the graphs of $f(x) = -x^2 + 2x + 3$ and $g(x) = 1 - 2^x$
Interpretation	Gain meaning from the graph by matching function of the same family or different families	4.1.6 For which value(s) of *x* is $f(x) - g(x) = 0$? 4.1.7 for which value(s) of *x* is $g(x) - f(x) > 0$? 4.1.8 The graph of f(x) is shifted up two units. Give the new equation of *f(x)* in the form *h(x)* = …… 4.1.9 The graph of *g(x)* is reflected in the *x-axis*. Give the new equation in the form *k(x)* =……	8.5 For what values of *x* is $g(x) \geq f(x)$? 8.6 Determine an equation for the axis of symmetry of *f* which has a positive slope 9.3. For which value(s) of *x* is $f(x).g(x) \geq 0$? 9.4. Determine the value of *c* such that the x-axis will be a tangent to the graph of *h*, where $h(x) = f(x) + c$. 9.5 Determine the y-intercept of *t* if $t(x) = -g(x) + 1$ 9.6. the graph of *k* is a reflection of g about the y-axis. Write down the equation of k

	Reading values	4.1.1 write down the equation of the horizontal asymptote of f 4.1.2 determine the equation of g 4.1.3 give the range of f 4.1.4 give the domain of g 4.1.5 for which values of x is f increasing.	8.1 Write down the equation of the asymptote of *f*. 8.2 Write down the domain of *f*
Flexible move between representations			Question 10 Sketch the graph of $f(x) = ax^2 + bx + c$ if it is also given that: • the range of *f* is $(-\infty;7]$ • $a \neq 0$ • $b < 0$ • One root of *f* is positive and the other is negative
Algebraic manipulation		4.1.2. determine the equation of g 4.2. NB: show workings how you have determine those points	8.3 Determine the values of *d* and *e*, if the graph makes an angle of 76^0 with the x-axis. 8.4 Determine the coordinates of A and C. 9.2. Determine the average gradient of *f* between $x = -3$ and $x = 0$

(b). ***Grade 11***. The construction of the graphs of parabola and exponential graphs, namely $f(x) = -x^2 + 2x + 3$ and $g(x) = 1 - 2^x$ (cf. Table 3.4 and Appendix B, p. 7) are sketched on the same set of axes. A global approach may be used to sketch the graphs.

3.3.2 Interpretation

(a). ***Grade 10***: The interpretation starts with the reading of the values from the graphs (Appendix A, p. 5). The matching and identifying the sketched graphs with the symbols/formulae are demanded in 4.1.1 to 4.1.5. The matching of two different graphs (exponential function and hyperbola) demands that a learner to identify each function in terms of equation and shape. Secondly, the questions (see 4.1.6 & 4.1.7) require the interpretation in terms of curve sketching and fitting. Matching of the graphs of the same kind when undergoing transformation (vertical shift and reflection about x-axis) is observed in 4.1.8 and 4.1.9 and it is consistent with CAPS stipulations (CAPS, p. 24).

(b). ***Grade 11***: Interpretation starts with reading of values of x when $g(x) > f(x)$ (Appendix B, p. 6) between hyperbola and linear function. The reading of values occurs by knowing the properties of hyperbola, which is, reading the asymptote and domain without further manipulation (questions 8.1 and 8.2). Reading can be done from the formula of and/or the sketched hyperbola. This is similar to question 9.3 (Appendix B, p. 7) for parabola and exponential function. Questions 9.4, 9.5 and 9.6 (p. 7) also deal with gaining the meaning about a function where addition of parameters or transformation gives rise to a new function of the same family.

The algebraic manipulation is in questions 8.3, 8.4 and 9.2. Even (1998) posits that moving from symbolic representations to graphical representations and vice versa, algebraic manipulations are embedded. In the process of movement, interpretation is necessary. In question 8.3 and 9.4 (Appendix B, p. 7), it is not clear whether the type of interpretation is gaining meaning from the same function or is a combination of the process of moving from graphical representation to symbolic ones.

3.3.3 Flexible move between representations

This theme is not prevalent in ***Grade 10***. It does not imply that flexible movement between representations should not be taught or are not valued in Grade 10 and therefore it cannot be asked in future. In ***Grade 11***, a flexible move between representations starts from verbal descriptions of a parabola (question 10, Appendix B). Learners are expected to sketch the graph based on that description.

To sum up, both Grades 10 and 11 question papers prioritised interpretation more than construction action. The Grade 11 question paper further prioritised flexible movement between representations on which, as Even (1998) postulated, the flexible move between representations also facilitates interpretation action and other skills and knowledge. The algebraic proficiency was also prioritized in both papers.

3.4. Textbook analysis

The National Department of Basic Education (DBE) screens all mathematics textbooks for compliance with curriculum needs based on content coverage, readership, cognitive levels, progression; and knowledge, skills and values envisaged in that grade and FET band. The textbooks that comply with the said categories are then approved and catalogued for schools to select them from the list. Fee-paying schools have autonomy to select, order and purchase textbooks that appear on the catalogue. The no - fee school may only order or be sent books paid by the state if these have been approved. The state supports these schools by regulating their financial management. The effect is that textbooks that are sent to these schools may be restricted and many schools will be using the same textbook in the province/district.

The textbook is normally used by the teachers and learners as (re)source material for accessing skills and knowledge required of the concept. Teachers use textbooks during lesson preparation and planning, and also during lesson presentation (classroom interaction). Textbooks afford the teachers with an opportunity to translate curriculum objectives into classroom interaction and what should be taught and learned. Secondly, textbooks may make connections between concepts and content within the curriculum and beyond (Zaskis & Leikin, 2010). Significantly, textbooks provide both examples of a concept and examples for exercises for understanding and mastering a concept (Rowland, 2008).

In contrast, textbooks may limit the scope of the content coverage envisaged by the curriculum by putting more emphasis on certain topics/concepts and neglect (either by choice or not) some content and/or intended object of learning even when they have been screened. Secondly, textbooks may prioritise one type of example more than others therefore limit examples of a concept and/or examples for doing examples. Furthermore, as Zaskis and Leikin (2008) assert, they may not fully address a necessary specific content of concept. For example, some textbooks provide procedural shortcuts of the significance of *"a"* in quadratic function of the nature $y = a(x-p)^2 + q$. If $a > 0$, these textbooks (regrettably, some teachers too) say that the parabola is concave up (as a procedural shortcut) while the specific concept is that the parabola has a minimum turning point. Maybe textbooks, by so saying, leave teachers (therefore their content knowledge) to explain the specific content like significance of *"a"* during classroom interaction. This suggests that too much reliance on textbooks limits the necessary specific content of the concept to be learnt by learners. The teachers' knowledge coupled with necessary (re)source materials provides a meaningful knowledge and skills to be learned by learners.

The Grade 11 Mathematics textbook that I analysed here is Platinum Mathematics Grade 11 (Bradley et al., 2012) because both teachers understudy use this textbook for choosing examples. The textbook was chosen for analysis because it appeared in the catalogue and both teachers/schools that were observed for the purpose of this study used it. Secondly, Platinum Mathematics is widely used by schools in South Africa, and many teachers and learners referred to it for teaching and learning. Lastly, the structure of functions in this textbook is organized in terms of unit-topics, definition of key words, explanation of concepts, exercises, representations/illustrations, revision that integrate different topics related to functions.

I begin by looking at the types of examples as espoused by Michener (1978) and, Alcock and Inglis (2008) (start-up example, reference, model, counter-examples and non-examples). Secondly, I examine the examples using Rowland's (2008) categories: his distinction between examples of a concept and examples for exercises, and then how examples were varied (variation) across the set of examples, how examples were sequenced and represented and their related learning objectives. Thirdly, the definition, construction, interpretation and flexibility movement between the forms of representation of a function are discussed. These concepts are

not discussed in this order, but are integrated and intertwined to describe and analyse the examples appearing in the textbook (cf. Table 3.1).

Variation accounts for what remains a critical aspect and what remains a non-critical aspect of the functions when an example is taken. For example, when dealing with "the effect of parameter *a*" on the parabola (critical aspect by then), issues related to Axis of Symmetry (AS) and interception on axes become non-critical aspects. Sequencing of examples is based on Leinhardt et al.'s (1990) postulation on patterns of sequencing on construction and interpretation. Generally, sequencing in constructions of functions is from point wise approach to global approach (Leinhardt et al., 1990). Leinhardt et al. found that in that sequencing, a pattern of identifying of the graph, construction of the graph and matching the graph with families of equations is followed in construction.

The interpretation of functions follows matching functions of the same family, identification of different types of functions in terms of shape and formula, and curve sketching and filling. Representation accounts for numerical, graphical, verbal/word and algebraic expression/symbols (Even, 1998; Nachlieli & Tabach, 2012) and flexible movement between these representations.

Functions under review (parabola, hyperbola, exponential function) appear as stand-alone topics from page 85 to page 115 (Bradley et al., 2012). The analysis covers only quadratic functions because to cover all functions would render the exercise unnecessarily voluminous, as the analysis deals with how and what examples are captured. Secondly, textbooks have a tendency to follow the same approach when dealing with different types of functions. Platinum Mathematics follows this approach: effects of parameters *a, p and q* of different functions; worked examples, exercises and finding the equation of functions.

3.4.1 Examples of a concept – quadratic function

a. Types of examples

Unit 1 (Appendix C) deals with the effects of the parameters *a, p, and q* on the parabola of quadratic function of the nature of $y = a(x + p)^2 + q$ (Bradley et al., 2012, p. 82) and relevant features pertaining to it.

The first set of examples are: $f(x) = x^2; g(x) = 4x^2;\ h(x) = \frac{1}{4}x^2;$ and $y = -x^2; y = -4x^2$ and $y = -\frac{1}{4}x^2$. The six examples taken are prototypical examples for quadratic function and therefore start-up examples. Together they facilitate the understanding of and develop the effects brought about by the change in values of *"a"* from *a =1, a > 0, 0 < a < 1 to a < 0*. Secondly, these examples appear in the Grade 10 curriculum. They were used as start-up examples for understanding the significance of *a* in quadratic function.

The second set of examples deals with the effect of parameter *q* which also is prior knowledge from Grade 10. The examples are start-up examples because they explain the effect brought by *q* (vertical shift), the effect brought by *a* and reflection along the x-axis (see p. 83, worked example). Secondly, examples taken, through probing, demand the sketching of both vertical shift and reflection along the *x-axis* (due to the effect of *a)*. The vertical shift and reflection along the *x-axis* are simultaneously facilitated. The summary in the textbook explains the vertical shift and reflection.

The third set of examples deals with the horizontal shift, which is the effect brought by parameter *p*. The understanding of parameter *p* is developed from the parent function $f(x) = x^2$ where different values of *p* are taken (*p = 2, p = -1 and p = 4,* Appendix C, p. 85). The three examples are start-up examples because they cover the attributes and features brought by parameter *p* so as to understand parabola that have horizontal shift.

The fourth sets of examples are prototypical examples for understanding the quadratic function of the nature $y = a(x + p)^2 + q$, and also they are model or generic examples. The examples are: $f(x) = x^2, g(x) = (x+2)^2 - 4, h(x) = (x-1)^2 + 2, k(x) = (x+4)^2 - 1$. A table of values is generated from the four examples. The examples deal with the combination of horizontal and vertical shifts simultaneously (Appendix C, p. 86, see fig 3.1). They have most of the attributes and properties of a parabola and address its critical features, namely, axis of symmetry, turning point, minimum and maximum values, horizontal and vertical shifts. The last three examples are taken from the previous examples under horizontal shift where -4, +2 and -1 (values of parameter *q*) are added.

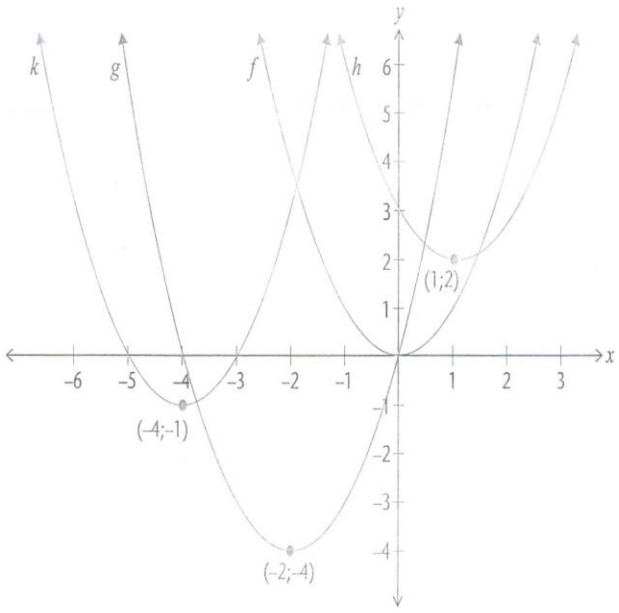

Figure 3.1: Model examples of parabola.

The last set of examples deals with "finding the equation of a parabola". Three forms of quadratic function are: $y = ax^2 + bx + c$ (standard form), $y = a(x - x_1)(x - x_2)$ (x-intercept formula) and $y = a(x - p)^2 + q$ (TP form) (Appendix C, p. 87). The three forms deal with procedural manipulation. The examples are reference examples because they link algebraic manipulation with functions and enhance interpretation of quadratic functions.

- "Sketching a parabola" elaborates on procedural cues to be followed when sketching parabolas. It starts with $a > 0$ (parabola 'smiles') and $a < 0$ (parabola 'frowns'). Although the significance of a is described in page 82 (Bradley at al., 2012), another specialised content about the significance of a is addressed when elaboration on TPs is discussed, that is, if $a > 0$ the parabola will have minimum turning point/value determined by an ordinate and if $a < 0$ the parabola will have maximum TP/value determine by an ordinate
- Determining the graph interception on axes. The graph interception on the y-axis (*y-intercept*) is elaborated using different forms of quadratic function. The *x-intercept* is followed where information on how to deal with quadratic equation that has rational

roots, factorise. If roots are irrational, use quadratic formula. ($y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$). In each case where roots are non-real, the graph does not intersect the x-axis. In all cases examples are taken to showcase the procedural cues.

- ❖ Every parabola has a TP and the abscissa (*x-coordinate*) of TP gives the equation of the line of symmetry.

All the examples taken are start-up examples for each cue to be taken under the features that are necessary to sketch the graph. The examples taken address the critical features under review and provide possibilities for conceptual understanding of the feature or attribute of the quadratic function.

The worked example on page 88 is a model example because it summarises most of the attributes/features of the quadratic function and demands both construction of the parabola and interpretation thereof.

b. Sequencing and variation: interpretation and construction

The sequencing moves from a parent quadratic function ($f(x) = x^2$) where simple example is taken to more complex ones. The values of *p* and *q* (both equal to zero) remain the same while "*a*" assumes different values that develop concept understanding. The values of *a* are sequenced so as to increase the level of complexity and understanding of reflection along the *x-axis* (Appendix C, p. 82). It is possible to generalise the effects brought by parameter *a* when it assumes different values by sequencing examples, that is, by varying values of parameter *a* it is possible to experience what has changed and possibly what can be discerned.

No table of values was drawn - instead $x = 1$ is arbitrarily taken to have coordinates of each function so as to sketch them. Therefore a global approach of sketching is done. The sketched graphs are colour coded to label each parabola so as to match graphs with equations and to illuminate variation in terms of parameter *a*.

Sequencing and variation are also observed in the "worked example" (p. 83). The significance of parameter *q* as vertical shift and parameter *a* for reflection along the *x-axis* are conceptualised through probing. The vertical shift is done by adding $q = 1$ and is integrated with $a = -2$. They have been dealt with in succession, that is, starting with *q* then *a*. In each case of their generation

(*p and a*), each parameter is possibly intended to be discerned discriminately from one another. I infer that through probing the parameter is discerned to facilitate concept understanding of quadratic function of the nature of $y = ax^2 + q$ and it is prior knowledge from Grade 10 (see DBE, 2011, p. 24).

For horizontal shift (p. 85), the sequencing of examples starts from $f(x) = x^2, g(x) = (x+2)^2, h(x) = (x-1)^2$ to $k(x) = (x+4)^2$. The table of data was drawn then global sketching. The colour coding of different parabolas matched each graph with its equation and to illuminate variation on the values of parameter *p*. The different values of *p* are varied to discern what have changed (effect of parameter *p*) therefore to generalise its effect. Values of parameter *p* are also sequenced to conceptualise equations/formula with related shape. That is, if *p > 0,* graph of *f(x)* (parent function) shifts *p* units to the left and if *p < 0,* the parent function shifts *p* units to the right.

Horizontal and vertical shifts are elaborated on page 86 (Bradley et al., 2012). The values of *p* and *q* do not demonstrate sequencing and variation because there is no increase level of complexity except on demonstrating the shifts (vertical and horizontal). However, the explanations given describe the properties of quadratic functions, namely, axis of symmetry, minimum and maximum values drawing from the examples taken. Values and effect of parameters *p* and *q* were discussed in previous examples (see pp. 83 & 85) so sequencing and variation are implied.

The worked example in page 88 moves from symbolic representations (algebraic formulae and manipulations) to construction. A global approach is used which emanates from intersection on axes and the TPs of those examples. Lastly the gaining of meaning between and reading of values of the two parabolas are probed to elicit these skills and knowledge. Different forms of quadratic functions [($f(x) = x^2 - 5x - 6$ and $h(x) = x^2 - 4x + 6$, standard form); and $g(x) = -\frac{1}{2}(x-4)^2 + 2$ (TP form)] are taken to understand sketching of the parabola from different forms. Furthermore, on the issue of construction, the graphs are matched with equations that described them. Question items 2 - 4 (p. 88), demand the interpretation of the graphs where values of *x* are determined by comparing two graphs $f(x) = g(x)$ in terms of the shape and

equations (standard and TP forms). The algebraic manipulation is required especially in item 3. By varying different forms of quadratic equation, we therefore discern how we can sketch each form.

The worked example in page 89 differs from the previous one because it compares quadratic function and linear function. The sketching of the two graphs is global and demands identifying different properties of the two functions. The interpretation action occurs on reading the values from the two function/graphs. So both construction and interpretation actions of two different functions are facilitated to understand their matching.

The sequencing of examples from page 88 to page 89 develops awareness about comparisons of same and different functions and through the global approach and the reading of values in the same way between two sets, construction and interpretation between functions were facilitated.

The progression of quadratic equation from Grade 10 to Grade 11 can enhance the conceptualisation of functions in Grade 11 by building on previous knowledge. The sequencing and variation can also build on graph construction and interpretation.

c. Learning objectives

Definitions of a function, effect of parameters, reflection on axes, intersection on axes, axis of symmetry, minimum and maximum TPs are given as key topics (object of learning) that needed to be taught and learned and they were facilitated to help learners understand the quadratic function.

In the effect of parameter *a*, the learning object appears in the conclusion as:

- the greater the value of *a*, the steeper the curve
- when *a* is negative the graph reflects in the x-axis
- changing the value of *a* has no impact on the y-intercept (Bradley et al., 2012, p. 82)

No mention of the significance of parameter *"a"* on TPs (maximum and minimum).

The learning objective of parameter *q* is that the function of *f(x)* moves/shifts *q* unit vertically. If $q > 0$, the y-coordinate (ordinate) will increase by q units therefore the graph will shift *q* units up

and if $q < 0$, the y-coordinate will decrease by q units and the graph will shift q units down. Lastly, negative function [$-f(x)$] reflects in the x-axis (see Appendix C, p. 83).

The learning objective of parameter p is: if $p > 0$, graph of $f(x)$ (parent function) shifts p units to the left and if $p < 0$, the parent function shifts p units to the right (p. 85)

The learning objectives in "sketching a parabola" (pp. 87-89) are to construct and interpret the graphs. It addresses properties that defined a parabola. The cues that deal with the move from symbolic representation to graphical representation and back are addressed. All necessary actions that need to be performed in the function are dealt with.

The learning objective of "finding the equation of a parabola" is to develop procedural fluency and better understanding of how to determine axis of symmetry, TP and intersection on axes.

To sum up, examples of a concept used two types of examples (start-up and model examples) that deal with the understanding of a concept (Alcock & Iglis, 2008). The generalisations of effects of parameters are facilitated through global sketching. Sketching the parabola facilitates construction and interpretation actions. The "different forms of quadratic equations" and "finding the equation of a parabola" are procedural in nature therefore are symbolic representations (Even, 1998). These topics mostly facilitated interpretation actions (Leinhardt et al., 1990; Even, 1998). The model examples facilitated flexible moves between representations and enhanced the understanding of properties of a quadratic function (and linear function).

3.4.2 Examples for exercises – quadratic function

a. Types of examples

Exercise 1 has four questions at which each has seven similar item questions (Appendix C, p. 84). The exercise examples given are (questions 1 to 4): $f(x) = -\frac{1}{2}x^2$, $f(x) = -8x^2$, $f(x) = \frac{1}{3}x^2$ and $f(x) = -3x^2$. The examples are reference examples because they develop procedural fluency and enhance conceptual understanding of parameter a in terms of constructions, effects brought by parameters a and reflection along the *x-axis*.

Questions in Exercise 2 (p. 85) are four and all of them require converting quadratic functions in standard form ($y = ax^2 + bx + c$) to turning point form [$y = a(x - p)^2 + q$] and hence describe the effect brought by parameter *p* (horizontal shift). The exercise examples are reference examples because they develop procedural proficiencies in converting standard form to turning point form and also require a description on how shifting has occurred from the original (0;0). They generalise the effect brought by parameter *p* through procedural manipulations therefore enhancing the effect of parameter *p*.

Examples in Exercise 3 (p. 86) are similar to examples in Exercise 2. Both Exercises deal with conversions from one form to another then describing the effects of parameters. The difference is that Exercise 3 requires conversion from turning point form to standard form and x-intersect form [$y = (x - x_1)(x - x_2)$]. Secondly, it requires the stating of TPs and description of the shift from the original (0;0). The exercise examples are reference examples because they require critical features (TP, horizontal shift and vertical shift) while the axis of symmetry and other features are not required. Secondly, these examples develop proficiency in moving from one form to others.

Examples in Exercise 4 (p. 89) are model examples because they summarise the properties of quadratic function and linear function. Secondly, exercise examples integrate both linear function and quadratic function where reasoning is required on how the linear differs from the quadratic function. The questions (1 – 3) also include procedural cues and what has varied between two functions.

Examples in Exercise 5 (p. 90) are reference examples that develop procedural cues when deriving the equation of the quadratic function.

Exercise 6 (p. 91) summarises the whole unit 1 by integrating different forms of quadratic functions, effect of parameters and procedural cues when determining the features of quadratic functions. The examples taken are therefore reference examples because they link the outcomes with procedure for understanding the properties of quadratic function.

B. Sequencing and variation: construction and interpretation

Examples for exercise are on pages 84, 85, 86, 89, 90 and 91 and are Exercises 1-6 respectively (Appendix C). Example 1 (p. 84) requires the integration of sketching of the graph, vertical shifting and reflection along the *x-axis*. Firstly, the global plotting is required by indicating three points. Secondly, the said function is sketched due to effect of parameter *q* to indicate vertical shift. Lastly, the reflection on the *x-axis* is sketched. The direction of events requires matching of the parabolas with their equation under the effect of parameters (vertical shifts) and reflection. By varying exercise examples in each question item, the intention is that when matching graphs with their equations that underwent both translation and reflection, we are able to gain an understanding of the translation and reflection.

The examples for exercise in Exercise 2 (p. 85) do not require construction of the graphs but deal with converting standard form of quadratic equation to another one (TP form). The procedural cue and description of the horizontal shift (effect of parameter *p*) (interpretation) are key. Similarly in Exercise 3 (p. 86) conversion from one form of parabola to another and description of vertical and horizontal shifts are required. The sequencing and variation of these exercises provide for development of the nature of quadratic equation and more so properties which are pertinent to quadratic function.

Examples for doing exercises in Exercise 4 (p. 89) deal with sketching of parabolas (by global approach) and gaining meaning by matching graphs of quadratic functions with linear functions. This purports what Leinhardt et al. (1990) asserted that, by sequencing examples of functions from construction to interpretation, we build and develop conceptual understanding of functions.

Exercise 5 (p. 90) requires that a quadratic function equation be determined under set condition(s). This is a flexible move from different form (points, TPs) of parabola to formula one. Learners may draw global images of the condition(s) given to get an idea of what the graph looks like. Even (1998) posit that learners gain a procedural fluency and develop conceptual understanding of functions when provided with these examples of flexible movement from one form to another.

Exercise 6 (p. 91) summarises attributes of quadratic function covered in Unit 1 but are limited. The examples of exercises demand the interpretation of the parabolas by requiring different

properties pertaining to quadratic function. It also includes procedural cues. Therefore symbolic representations are reflected. The skills and knowledge relating to graphical representation are not part of the examples of exercises in the summary. The skills and knowledge relating to graphical representation especially constructions have been ignored. Another action of importance which is not part of the summary is flexible movement between forms of representation of a function which is not included.

To sum-up, the exercise examples are mostly reference examples with few that are models. The emphasis on reference examples implies the facilitation and development of both concept understanding and procedural fluency (Alcock & Iglis, 2008).

3.5. Summary

The conceptual sources derived from the literature review guided me to analyse the CAPS, previous question papers and Platinum Mathematics Grade 11 textbook in a systematic way. The analyses of documents in this chapter assisted me, firstly, to reflect on what and how properties of functions and conceptual sources have been reflected in each document. It is of worth to note that the intention of analysing the previous question papers and the textbook is not to critique them but to describe what is available for teachers to draw from them. Secondly, they (documents) provided me with what to look for (which aspects to focus on) when I observe teachers' lesson plans and their classroom interactions. Lastly, they served as (re)sources with which to probe teachers during interviews on their choices and use of *function examples*.

The curriculum document (CAPS, DBE, 2011) dictated issues of progression and increased level of difficulty between/among Grade 10-12. These aspects purport connections/links from prior knowledge from Grade 10 to new knowledge and skills in Grades 11 and 12 in terms of definition of a function, construction, interpretation and a flexible move between representations from function of the nature $y = af(x) + q$ to $y = af(x-p)^2 + q$. These assisted me to be aware and observe what and how teachers' examples were chosen and used with respect to progression and degree of difficulty. CAPS stipulates the characteristics and properties of functions which are skills and knowledge to be acquired in Grade 11 (cf. Table 2.1). CAPS therefore frames the extent of what teachers' examples should include and the relevance of teachers' examples that can facilitate learners' conceptual development within the Grade.

The official question papers brought about an awareness of non-routine examples (Appendix B, Question 10, p. 7) that require flexible movement between representations and interpretation. Do teachers choose and use these examples in their instruction? Zodik and Zaslavsky (2008) deductively observed that some teachers consider the inclusion of uncommon (non-routine) examples when they teach mathematics especially in their lesson planning. Secondly, the question papers are more inclined towards interpretation seeking questions than construction one. Do teachers do the same? If yes, what is their rationale?

The textbook provides progression from Grade 10 to Grade 11. It sequenced and varied the examples from translation (vertical to horizontal shifts), to reflection (along x-axis then y-axis) and then flexible movement (from formula, table method, point-wise, global approach, verbal descriptions, sketched graphs and formula). In the process of undertaking these, construction, interpretation and algebraic manipulations are required. There is, firstly, more inclination in the global approach for sketching graphs than pointwise ones. Secondly, there is more inclination on symbolic representations than graphical ones in solving and determining of properties of quadratic functions, especially in exercise examples. The inclination reflects more emphasis on interpretation than construction (Even, 1998). The pattern is also observed in official question papers. Does this inclination towards symbolic representation have a bearing on teachers' choice and use of examples? What are the role of and types of examples that teachers had drawn from the textbook? Where do they draw most of their examples from in the planning phase and at the moment of teaching? What is their rationale for doing that?

3.6. Conclusion

CAPS (DBE, 2012) describes skills and knowledge that are necessary and relevant for understanding functions. The critical issues are definition of function, construction, interpretation and flexible move between different forms of functions. Another aspect consistent with any curriculum is progression of function concepts from Grade 10 to 12. CAPS provides progression in terms of increased degree of complexity from one grade to another.

The examination question papers that were analysed are consistent with provisions made in the CAPS and assessed key concepts of functions (parabola in this instant). The textbook (*Platinum*

Mathematics, Bradley at al., 2012) significantly addresses characteristics of quadratic function. It can provide its users with examples that address symbolic and graphical representations although to varying degrees.

Chapter 4: Research methodology and data collection.

4.1 Introduction

This chapter focuses on the methodology and how data were collected. I firstly justify the selection of research methodology and the research design and lastly how data were collected. Issues of rigour, trustworthiness, reliability and validity; and ethical consideration are also discussed. Furthermore the limitations on research paradigm are discussed.

4.2 Methodology

This research project studied two competent, qualified, experienced secondary teachers' choice and use of examples from two contrasting South African school contexts [i.e., urban (fee-paying school) and former informal settlement (no-fee school)] using qualitative methods. An interpretive approach assumes multiple perspectives (Leedy & Ormrod, 2005) held by teachers on their generation and treatment of examples in their respective classroom lessons. The multiple perspectives are reflected by the teachers' decisions and considerations they make when they choose and use examples. Secondly, the interpretive approach assisted this study in describing and interpreting (Leedy & Ormrod, 2005) the teachers' choice and use of examples before and during the lesson presentations, and what their rationale had been for choosing and using particular examples.

The qualitative approach is based on two cases and dictates my epistemological assumptions that knowledge is acquired and applied in social settings through social interactions (Opie, 2010). The data were collected from two teachers in their natural classrooms, albeit different socio-economic contexts when they were teaching school *functions*. My assumption was that we can learn from these teachers' choice and use of school function examples.

The selection of the case study method was based on the assertion made by Opie (2010) that the case study should be undertaken on "real situation, with real people in the environment often familiar to the researcher" and the "interactions of events, human relations and other factors are studied in the unique location" (p. 74). The two teachers are real people who are practising teachers like me; and the teachers' choice and use of examples are aspects which are studied in each teacher's respective school.

4.3 Selection of cases

A brief description is made on the purposive selection of schools and the teachers. An account on selection of functions of Grade 11 (secondary school) has been discussed in Chapters 1 and 2

4.3.1 The schools

The two contrasting South African school contexts [that is, urban (fee-paying school) and former informal settlement (no-fee school)] in which these teachers were teaching were purposely selected because (a) the two schools have acceptable functional administrations and high standards of governance (as determined by a Dr. Kenneth Kaunda District official of North West Department of Education and Training) and (b) the matriculation performance percentage was above the benchmark stipulated by North West Department of Education and Training, that is, 70%. The fee-paying school is situated in a mining area and was a former mining-sponsored school. The school fee is R700 per annum. The no-fee school is situated in a peri-mining area and it is in a former informal settlement and located on the outskirts of the main township in an extension.

The teaching and learning (re)sources (see Chapter 2) that are available in each school are described in the data analysis. The school and classroom observation checklist (cf. Table 2.4) reflect concepts underpinned in the literature review under teaching and learning (re)sources. The checklist informed what (re)sources were available and used in these two schools and which (re)sources supported teaching and learning of mathematics. Secondly, I also considered how these (re)sources regulated the teaching and learning activities and how teachers used them to select and use examples during lesson representations. As discussed in 2.6., teachers' knowledge structures (human resources) and mathematics objects (mathematical content, context and well-structured procedures) are embedded on teachers' choice and use of examples therefore are illuminated in teachers' data analysis.

4.3.2 The teachers

The two competent, qualified and experienced teachers were also purposively selected. Firstly, they both have "reasonable foundation of mathematical knowledge" (Kazima et al., 2008; p. 286) based on their experience and qualification (as human resource, Adler, 2000). By competent I

mean that the two teachers have achieved mathematics matriculation pass percentages of +70% since the inception of first National Curriculum Statement in 2008. Secondly, the selected teachers have more than 15 years of secondary teaching experience and are reasonably qualified, that is, individually, they have the Relative Education Qualification Value (REQV) 15 (cf. Table 3.1) or more. REQV 15 means that teachers have a five-year qualification in Mathematics Education training after their matriculation. The minimum REQV for teaching in South Africa is REQV 13 in Education (as a discipline) as stipulated in Educators Employment Act 76 of 1998 and, Norms and Standards for Educators (DoE, 2000). Thirdly, Goldenberg and Mason (2008) recommend that we can learn more from experienced teachers' treatment and generation of examples in secondary schools.

In other words, the two competent, qualified, experienced teachers and their respective schools were purposively selected to learn and develop a detailed understanding (Creswell, 2012) of their choice and use of examples in their different socio-economic settings of their schools, so as to illuminate the roles of examples in those contexts.

The two teachers were given pseudonyms to protect their identity and to respond to the ethical considerations discussed below. The teacher from the fee-paying school is named teacher A (TA) and teacher from no-fee school, teacher B (TB). No specific reasons or prejudice is attached to naming them as such. Note well: the abbreviations (in Table 4.1) are explained in the list of abbreviations.

The following Table 4.1 depicts both *teachers' profiles*:

Teacher	Qualification	REQV level	Experience	Current study
Teacher A	MSc (Mathematics) PGCE	16	20 years	None
Teacher B	STD ACE (Mathematics) ACE (Maths. Lit) B.Ed (Hons)	15	15 years	None

4.4 Data collection

Various instruments and tools were used to collect data to enhance the richness of case of data collected and triangulation. For this research, four instruments and tools were used, namely field notes, teachers written lesson plans, semi-structured interviews and classroom observation checklist. The usage and intentions of these instruments and tools are discussed below (cf. Table 4.2 for summary):

4.4.1 Teachers' written lesson plans and field notes

The teachers were requested to submit their written lesson plans (Appendices E & F) so as to have evidence of planned examples and learning objectives. The written lesson plans provided the opportunity to describe and interpret the types of examples intended to be taught, their sequencing and variation, and the frequency with which they appear. The field notes (Appendix G) taken during lesson observations captured the consistency and/or change of examples from written lesson plans and the types of examples chosen in the planning phase and during the lesson interaction. Firstly, field notes captured the lesson activities against their pacing, selection and mediating. That is, aspects of time taken to do a particular activity, time taken to select an activity and how long an activity was done. Secondly, field notes and written lesson plans were used to capture data that responded to the first two specific research questions, namely, 1. What examples do teachers choose in planning phase and during the lesson presentation? and 2. How do teachers use the examples in their lessons? Lastly, the two instruments were key to being able to describe and interpret the intended and enacted objects of learning.

4.4.2 School and classroom observation checklist

The classroom observation checklist served to report on teaching and learning materials available in each context. The checklist highlighted (re)sources available for planning the lesson, how (re)sources supported the teacher in choosing examples and how the teacher used them in lesson presentation (cf. Table 2.4).

4.4.3 Interviews

The semi-structured interviews were selected because they provided room to accommodate the teachers' responses (Posner & Gertzog, 1982; Opie, 2010) where the teachers gave accounts of

their "conscious reasons for action" (Leedy & Ormrod, 2005, p. 146). Lastly, the semi-structured interviews were interviews that let the teachers (interviewees) speak freely where I (interviewer) was checking their remarks against salient points that were emerging and noting what was new and revealing (Posner & Gertzog, 1982). The semi-structured interviews were open-ended questions (cf. Appendix D) so that the teachers can "best voice their experiences unconstrained by any perspectives by the researcher" (Creswell, 2012, p. 218). The one-on-one interview with each teacher was conducted for 20-30 minutes and duration of the interview depended mostly on teacher's responses. The process of interviews was audiotaped so as to give "accurate record of the conversation" (Creswell, 2012, p. 221) between the teacher and me.

The interviews were transcribed in order to capture instances or scenarios (excerpts) where inferences, interpretations and interpolation were necessary in terms of teachers' justifications or their voicing of their experiences on conscious actions they have taken.

The semi-structured interviews therefore enquired and probed teachers' choice and use of both planned and spontaneous examples. The semi-structured interviews were also used to enquire about teachers' (re)sources for generating examples. The schedule of semi-structured interviews (Appendix D) was informed by teachers' practice and aspects from literature.

4.4.4 When data were collected

The lesson observations occurred during early to mid-September 2013 because teachers were teaching functions by then as per North West Department of Education Annual Teaching Plan (work-schedule). The intention was not to distract or impose my request for the teachers do to function but to capture their activities in a natural way through their teaching of functions so as to study the teachers' choice and use of examples objectively.

4.4.5 Instrument and techniques of data collection

The first research question was divided into two in order to pay more attention to its analysis of data collected. The question, *what are the teachers' sources of examples?* (cf. Table 4.2) enquired on the (re)sources that supported teaching practice. The last research question, namely, *what patterns, if any, are there between and across the two contrasting teaching contexts, with respect to their choice, use and rationales for examples?* required the reflection from deductive

analysis after data had been described and interpreted using typologies. Therefore there is no specific instrument to collect data to answer the question.

The following Table 4.2 summarises instruments and techniques of data collection:

Table 4.2: Summary of instruments and techniques of data collection.

Research questions	Instrument/tool	How	What has been collected
1.1. What examples do teachers choose in planning phase?	Teachers' written lesson plans and interviews (audiotape).	Teachers will be requested to give a copy and interviews.	Learning objective, types of examples, examples of a concept, examples for exercises, variation, sequencing of examples and representation, and rationale or considerations
1.2. What examples do teachers choose during the lesson presentation?	Field notes (capturing teachers' spontaneous examples and interviews (audiotape)	Field notes on examples taken in the class and interviews	Types of examples, examples of a concept, examples for exercise, learning objective, variation, sequencing, representations and teachers' rationale and considerations
2. How teachers use the planned examples and additional spontaneous examples in their lessons?	Field notes, interviews (audiotape)	Selection of episodes that illuminate usage of examples and audiotape	Examples of a concept, examples for exercise, types of examples, variation, representations and sequencing of examples; and rationale or consideration. How resources supported teaching,
3. How do they explain their choice and use of examples?	Interviews (audiotape), field notes	Questions and answers.	Their reflection on their practice.
4. What patterns, if any, are there between and across the two contrasting teaching contexts, with respect to their choice, use and rationales for examples	Field notes, lesson plans, Interviews (audiotape)	Cross-case synthesis	Patterns, contrasts, contexts, synthesis

4.5 Validity and reliability

Yin (2003) argued that the most difficult aspects of qualitative case study are to establish and maintain reliability and validity in such research. This study follows the qualitative case study approach of investigating the work of two competent, qualified and experienced teachers in Grade 11. Opie (2010) asserts that qualitative studies should; through method, data collection and analysis, consistently address issues of reliability and validity in order to produce rigour, credible, trustworthy and quality research results. I discuss issues of reliability and validity to ascertain that this research has addressed issues of accurate data collection, rigour, credibility and trustworthiness therefore quality results.

4.6 Reliability

Reliability is established from data sources used and constructed categories by the researcher. Therefore, reliability deals with the extent to which data collected from instruments or category thereof remain stable and consistent when manipulated by different researchers. Firstly, different data sources (field notes, written lesson plans, interviews and classroom checklist) were used to establish triangulation. Creswell (2012) defines triangulation as the process of "corroborating evidence from different data sources" (p. 259) so as to establish and maintain reliability of data that has been collected.

Secondly, Phiri (2003) quoted Adler (1996) that she asserted that reliability is difficult to establish in qualitative research. Adler (1996) argued that there is a possibility that different researchers may not construct the same categories therefore it is important that constructed categories are recognizable to others to establish reliability. In this study, categories or themes (discussed in Chapters 5 & 6) were constructed based on the literature review and theoretical and inductive dispositions. Categories were verified by an expert (my supervisor) to check whether they are recognizable to others.

4.7 Validity

Validity is attained when the method and/or procedure measures what it is supposed to measure. Furthermore, Brinberg and McGrath (1985, p. 13) in Maxwell (1992, 281) describe validity as "integrity, character and quality to be assessed relative to purpose and circumstances". Therefore

the establishment and maintenance of validity in this research are critical and of paramount importance. Maxwell (1992) argues that different research paradigms view validity differently in terms of meaning, techniques and orientation. He lists five types of validity as descriptive validity, interpretive validity, theoretical validity, generalizability and evaluative validity. Maxwell (1992) postulated that the first three types of validity are mostly used in qualitative research paradigms by realistically orientated researchers. Based on my epistemological perspective, I followed the realist perspective that knowledge and acquisition thereof are inherent in the natural set-up through human actions. This study therefore explicitly and implicitly is grounded on the first three types of validity.

The purpose of this study has been to investigate and explore two competent, experienced secondary teachers' choice and use of school *functions examples* in two contrasting South African contexts. To establish and maintain validity the study has to describe, interpret data and provide theoretical orientation to the types of examples planned and used by teachers, the role and purposes of these examples, the nature of *school functions* and actions to be undertaken for conceptual understanding and development of *school functions*, teachers' considerations when choosing and using examples, and teaching and learning resources that aided the choice and usage of examples.

4.7.1 Descriptive validity

Descriptive validity accounts report factual accuracy from data sources and data collected (Maxwell, 1992). I describe what Maxwell called social acts by recording as accurately as possible what the teachers have done and said precisely in their own words. The teachers' written lesson plans, school and classroom observation checklist, field notes and transcripts derived from interviews assisted me in describing what happened and was said (see Chapters 5 & 6).

4.7.2 Interpretive validity

Interpretive validity was grounded in the inferences and explanations made by the researcher from the words and actions of the participants (teachers) without being judgmental or subjective (Maxwell, 1992). Interpretation was done based on empirical evidence collected from data sources and literature reviewed. The aspects of teachers' considerations, as espoused by Zodik and Zaslavsky (2008) and teachers' rationales, Rowland's (2008) role and purpose of examples,

types of examples by Tsamir et al. (2008), and Rowland's (2008) categories that framed usage of examples served to interpret the teachers' social acts. The literature reviewed aided the inference made from empirical evidence gathered from instruments and tools as discussed in 4.4 above.

4.7.3 Theoretical validity

Theoretical validity accounts link theoretical concepts with theoretical constructs emanating from empirical evidence. Maxwell (1992) describes theoretical validity as having two components, namely, the "concepts or categories that the theory employs, and the relationships that are thought to exist among these concepts" (p. 291). The discovered categories and theoretical constructs emerging from empirical evidence were checked and validated by my supervisor.

Table 4.3: Summary of accounts of validity.

Type of validity	Approaches
Descriptive	Describing data from written lesson plans, field notes, classroom observation checklist and transcripts of interviews
Interpretive	Interpreting empirical evidence based on literature reviews and salient points from description of data.
Theoretical	Making categories, theoretical constructs and theoretical concepts from described and interpreted empirical evidence.

The distinction between interpretive validity and theoretical validity may not be explicit because when interpreting empirical evidence, theoretical concepts and construct are implied implicitly (Maxwell, 1992). It therefore suggests that during data analysis the two types of validity may interplay and intertwine in the discussion of empirical data.

4.8. Data analysis

4.8.1. Typology of data analysis

Typological and inductive data analyses were both used to describe and interpret the data collected. Hatch (2002) advises that "studies that rely on interviews and have a focused purpose, narrow set of research questions, well-structured data set and consistent, guiding questions" (p.

160) may select typologies for analysing data. The typologies are framed from both reviews of related literature and concepts (from Chapter 2). These are: (a) teaching and learning resources (human resources, school mathematics resources and cultural resources) (b) the types of examples teachers have chosen and used both in planning phase and classroom interactions, (c) Rowland's examples of a concept and examples for doing exercises and, taking account of variation, sequencing, representation and lesson objectives. Typologies assisted me in describing, interpreting and reflecting on first three specific research questions. The above-mentioned concepts served as analytical concepts which describe and interpret teachers' choice and use of quadratic *function examples*. Table 4.4 depicts the *analytical concepts and indicators* (concepts and constructs used as lenses to analyse teachers' choice and use of examples).

4.8.2 Inductive data analysis

The specific questions three and four, namely, *how do they explain their choice and use of examples* and *what patterns, if any, are there between and across the two contrasting contexts, with respect to teachers' choice, use and rationales for examples*; require an exploration and emergence of patterns. Inductive data analysis is most appropriate for a search for patterns in data so that the general statement may illuminate salient aspects that are embedded in two contrasting contexts the teachers were teaching in (Hatch, 2002). In the exploration, data were identified that frame the analysis especially but not limited to what emanates from school and classroom observation checklists, and teachers' considerations when choosing and using examples as postulated by Zodik and Zaslavsky (2008).

Inductive data analysis assisted this study, firstly, to get an idea of what patterns, relationships and themes that might be present in the data (Hatch, 2002). Hatch (2002) further describes that patterns, themes and relationships may be realized in similarity, differences, frequency, sequence, correspondence and causation that exist between the two teachers when dealing with examples and their contrasting contexts. Comparisons and relations between the two contrasting contexts illuminated what and how (re)source materials have supported or not, teachers' choice and use of examples. Secondly, as data were re-read and coded/categorized, it became evident that not all excerpts from data collected fitted into the selected categories (Hatch, 2002).

It is apparent that descriptive validity, interpretive validity and theoretical validity were established and maintained, explicitly and implicitly, in the accounts of data analysis.

Table 4.4: Analytical concepts and indicators.

Analytical concepts	Indicators	Example
Human resources	Teacher-learner ratios	Number of learners per teacher
School mathematics material	Hardware, software, textbooks, technological gadgets	Computers, calculators, number of textbooks, overhead projectors, interactive boards, mind-sets
Cultural resources	Time-table, length of period, learners' attendance register	Number of mathematics periods per cycle, learners' attendance, pacing
Start-up example	• Introducing a new topic • Prototypical example	Parabola: $y = x^2$ Hyperbola: $y = \frac{1}{x}$ Exponential: $y = 2^x$
Reference example	• Contains critical and non-critical features of change in parameters of start-up examples • Repeatedly used to link outcomes with concept • Classical examples	Parabola: $y = x^2 - 4$ Hyperbola: $y = \frac{1}{x+2}$ Exponential: $y = 2^{x+3}$
Model/Generic example	• Used in reasoning in terms of why and what have varied • Used to make assumptions about results and concept • Has attributes that define a concept	Consider graph of $f(x) = x^2$ and $g(x) = (x-2)^2$: • Sketch the graphs of f and g on the same system of axes: show all points of intersection and important features. • describe the translation of f to g
Counter-examples	• Shows distinctions between concepts, • Compare critical features of different	In the graphs of $f(x) = 2^{x-2} + 3$ and $g(x) = \frac{1}{x-2} + 3$,

	concepts in terms of characteristics	• Sketch the graphs of *f* and *g*. • What are difference and similarities of the two graphs?
Nonexample	Example that do not fall under a description of particular concept but used in that concept	Does parabola $y = (x- 3)^2 + 4$ have asymptotes? Which features of the said parabola make it unique from hyperbolic functions and exponential functions
Examples for a concept	examples that explain the concept, deal with accessing abstraction for concept understanding	For understanding parameter *p:* $y = x^2 + 1$
Examples for exercise	examples that provide procedural fluency for concept consolidation and enhancement	Different exercises that provide procedural fluency
Learning objectives	• curriculum and teachers' lesson objectives • intended and enacted object of learning	Purpose of the lesson Skills, competences and proficiency learners should know
Sequence	increasing the degree of complexity between successive examples	After $y = x^2$ then $y = - x^2$ What has been sequenced and how has it been varied.
Variation	what is discerned and what is discriminated in an example	In $y = - x^2 + 2$ we discern both reflection of the graph along the x-axis and vertical shift by 2 units up from (0;0). We discriminate axis of symmetry.
Actions on functions	construction, interpretation and flexible movement between representations	Sketch the graph of $f(x) = 2x^2 + 5x + 3$ and $g(x) = x + 4$ in the same system of axes. Show all the necessary points. Determine the values of *x* for which: $f(x) > g(x)$.
Properties of the function	translation, reflection, domain, range, axis of symmetry, asymptote, intersection of axes	How and why they become learning objectives or object of learning, how sequencing and variation depicted them, how they been reflected in the examples,

4.9. Analytical strategies and techniques

It is imperative to describe, interpret and therefore analyse the data that were collected to reflect on the purpose of the research by answering research questions. The analysis of data results in research that is trustworthy, credible and reliable. Yin (2003) warns that it is difficult to analyse case study evidence because analysis of data thereof is not well-defined and does not have developed tools and techniques relative to other research designs. Analytical strategy and techniques are therefore paramount in assisting the researcher with "knowing what to do with evidence" and "what to look for" (Yin, 2003, p. 110) when analysing the empirical data. Analytical strategy is the strategy employed to prioritise what to analyse and why. Analytical technique is defined as tools/methods used to organized data in a manageable way so as to manipulate it objectively (Yin, 2003). Both analytical strategies and analytical techniques assisted the researcher in reducing the burden of analysing research (Yin, 2003) so as to be objective and the findings to be credible, trustworthy, and reliable and be of high quality.

Yin (2003) postulated that "developing a case description" and "relying on theoretical propositions" are commonly used strategies in case study research. The analytical technique used here is cross-case synthesis because this study is a multiple case study that focuses on two teachers from contrasting socio-economic settings in terms of their schools.

4.9.1. Analytical strategies

4.9.1.1. Case descriptions

The case description strategy is used to describe the empirical evidence, organize the cases in a manageable way, identify embedded unit of analysis, identify appropriate causal links to be analysed and to have overall pattern of complexity (Yin, 2003). I firstly described empirical evidence available from data sources, that is, from school and classroom observation checklists, teachers' written lesson plans and field notes by capturing examples teachers have chosen and used in the class. Secondly, I identified the unit of analysis and links between and among empirical evidence. The case description ascertains descriptive validity as discussed in 4.5.2.1 above and as espoused by Maxwell (1992).

4.9.1.2. Theoretical strategy

Yin (2003) postulates that theoretical strategy answers to "how" and "why" questions, it helps to focus attention on certain data and ignore others, and define the empirical evidence. I relied on theoretical concepts and constructs to interpret and reflect on the empirical evidence. The theoretical propositions have emanated from literature reviewed (see Chapter 2). I reviewed the literature that reflected on:

- examples of a concept and examples for exercise (Rowland, 2008),
- the nature of school functions and actions to be taken (Leinhardt et al., 1990; Even, 1998; DBE, 2011; Nachlieli & Tabach, 2012),
- types of examples (Michener, 1978 & Tsamir et al., 2008),
- taking of account of variation, sequencing, representations and lesson objectives (Rowland, 2008) and
- Considerations teachers are making when they choose and use examples (Zodik & Zaslavsky, 2008).

The theoretical strategy enhanced and ascertains interpretive validity and theoretical validity (Maxwell, 1992) as discussed in 3.5.2.2 and 3.5.2.3 above.

Table 4.5: Analytical strategies.

Case description	Theoretical concepts and constructs
Data from school and classroom observation checklist, teachers' written lesson plans and field notes	Typologies of data analysis: Examples for a concept, examples of exercise, types of examples, taking account of variation, sequencing, representation, learning objectives, considerations teachers are making, characteristics of functions, actions on functions
Data from the schools and the classrooms using the classroom observation checklist. Excerpts from interviews transcripts	Inductive data analysis: Patterns emerging

4.9.2. Analytical technique

The cross-case synthesis was applied on multiple cases of the case study research in order to strengthen the findings and have more robust findings (Yin, 2003). This study applies cross-case synthesis as an analytical technique because this research studied two teachers from two contrasting contexts. The two teachers and their contexts are firstly treated as separate cases, and then they are incorporated by applying "meta-analysis" (Yin, 2003, p. 135), that is, Chapters 5 and 6 deal with data analysis from each teacher, and Chapter 7 deals with cross-case synthesis of analysed data from both Chapters 5 and 6. The multiple case study approach was selected to either or both "predict similar results" or/and "predict contrasting results but for predictable reasons" (Yin, 2003, p.47) of two contrasting contexts of the two teachers. Yin (2003, p. 133-137) advises that:

- Create word tables that display the data from individual case according to analytical framework,
- Word tables should lead to the synthesis that will reflect patterns and outcomes of interest based on case descriptions and theoretical constructs/categories,
- Different cases will appear to share some similarities and/or differences,
- Analysis will further lead to the reflection on categories of the typologies of different cases, and
- Examinations of word tables will rely on argumentative interpretations that are supported by data.

The following flow diagram (fig. 4.1) depicts how data is going to be analysed

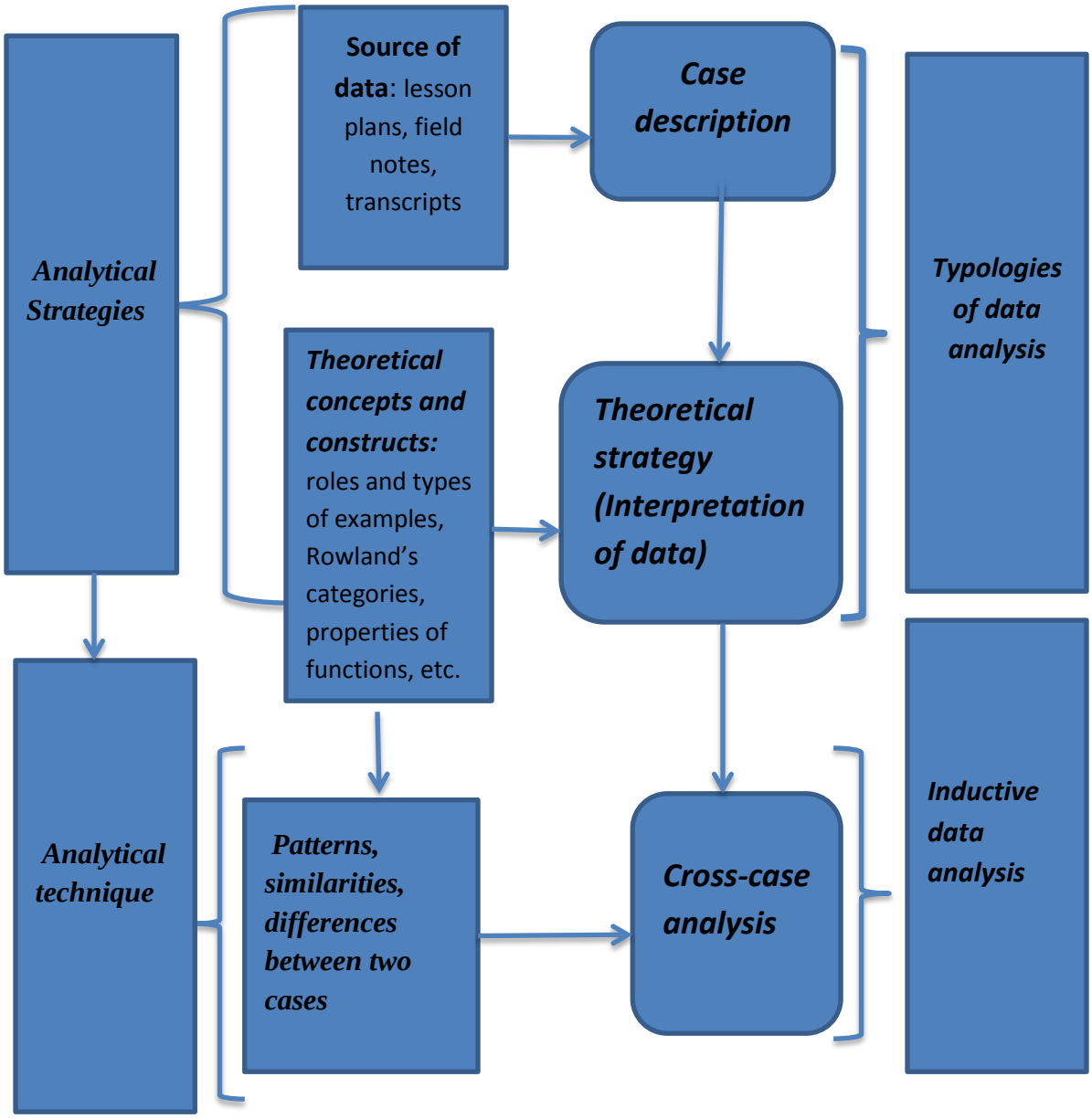

Figure 4.1: How data is going to be analysed.

4.10. Ethical consideration

Application for permission to undertake the research was made to and approved by the University of the Witwatersrand Human Research Ethics Committee (non-medical) for clearance on research involving human subjects(Protocol: 2012ECE143). Application for permission to observe and interview teachers was made to and granted by the North West Department of Education, Dr. Kenneth Kaunda District, to conduct research in the said district (cf. Appendix J).

The consent forms with information about the research project were given to the school principals and teachers concerned to conduct the research and were accepted. The copies of consent forms were also submitted during the application to Human Research Ethic committee. The teachers and I explained to the learners the purpose of the research. The information on the consent forms explains:

- the purpose of the research,
- what the research will do when collecting data and after they have been collected,
- how information is going to be used,
- the participants' risks and benefits and
- lastly what are participants' rights and how they may exercise these rights.

Permission to use audio recordings for the purpose of the interviews was separately attached to the consent forms.

The teachers' participation was voluntary and they were informed that they could discontinue their participation at any time without penalty or reproach. The teachers' real names and identities remained anonymous, that is pseudonyms are used to report on their actions. The teachers were informed up-front that they would not be paid and data collection did not disturb their normal teaching schedules.

4.11. Limitations

The research project is limited to only two teachers (cases) therefore the sample size has a bearing that the findings of the research cannot be generalized but they may be referred on the bases of relatebility to contexts (Opie, 2010) similar to the sampling of this research. Secondly, the study does not provide intervention strategies to the teachers on how to choose and use

examples and what considerations should be in play when they choose and use examples. However, the findings and recommendations may be used by other teachers to improve their practice in teaching functions. Teacher training institutions and mathematics textbooks writers can also use the information from the study for the benefit of Mathematics Education.

The study is conducted in partial fulfilment of MSc degree (coursework and research project). The research project should therefore be completed within six months as per the requirements for fulfilment of the course (MSc). The time for the research project limited and narrowed the focus of the research project.

I solicited assistance from two schools (fee-paying and no-fee school) that fell under the criteria I envisaged about competent, qualified and experienced teachers. The two schools have achieved +70% matric Mathematics since the inception of National Curriculum Statement. They have acceptable functional administrations and high standards in governance as determined by Dr Kenneth Kaunda District official of North West Department of Education and Training. The difference is that teacher A is alternating in teaching Grade 11s and 12s with other teachers therefore he may not be the sole contributor to his school Mathematics average of +70% in matriculation since the inception of NCS in 2008. Teacher B has been teaching from Grade 10 to 12 since the inception of NCS. The difference of two teachers in terms of their contribution towards matriculation Mathematics pass percentage alludes to inconsistencies between the selected cohorts.

Although much effort and care were taken to purposefully select the cohort, time was running out in terms of teaching functions as per Annual Teaching Plan (work-schedule) in Grade 11. The cohort satisfies the prescriptive of competent, qualified and experienced secondary school teachers (see Table 4.1).

Four lessons were observed for both Teacher A and Teacher B. These lessons were on quadratic function. I wish I could have observed all the lessons that dealt with other functions as well.

4.12. Conclusion

This chapter discussed the justification and relevancy of the research paradigm, methodology and design of this research. Issues of rigour, credibility and trustworthiness of data sources and analyses of data are grounded by aspects of reliability and validity. Discussion of ethical issues

and considerations were made to adhere to Human Research Ethics on research involving human subjects. The generalization of findings and the purpose of the research are discussed to cite the limitation of the research. The next two chapters, Chapters five and six, describe and analyse the data collected from each teacher.

Chapter 5: Data analysis of teacher A

5.1 Introduction

In this chapter I analyse choice and use of examples of TA. In each case (chapter) the research questions 1 to 3 were answered. Research question 4 is answered in Chapter 7. The research questions are:

1. What examples do teachers choose in the planning phase and during the lesson presentation? The evidence to describe and interpret this question is captured in the teacher's lesson plan (Appendices E and F) and field notes (Appendix G).
2. How do they use the planned and additional spontaneous examples in their lessons? Evidence is derived from lesson plan and field notes.
3. How do they explain their choice and use of examples? Evidence is derived from interview transcripts (Appendices H and I).
4. What patterns, if any, are there between and across the two teachers, in their contrasting teaching contexts, with respect to their choice, use and rationales for examples?

Analysis in both cases is as follows:

1. Evidence from data collected from data sources (cf. Table 4.2) is described and interpreted for each learning objective of the day/lesson. The units of analysis (examples) are analysed in terms of the types of examples chosen and used and whether these are examples of a concept and/or for doing exercises.
2. Rowland's categories (2008) are then applied to analyse examples chosen and used by the teacher by looking across each lesson presentation. Issues of how examples were varied, sequenced and what representations were used to provide for conceptualisation and for practice.
3. The research questions (questions 1 to 3) are answered and findings are given.
4. The analyses were done in three parts. Part A analyses teachers' choices and use of examples. Part B analyses teachers' lessons using the curriculum document (CAPS), question papers and textbook. Part C analyses teachers' considerations on their choice and use of examples.

The teachers' choice and use of examples are captured in teachers' written lesson plan/transparencies and my field notes. Their examples are described and interpreted interactively to corroborate teachers' social acts during their lesson preparation and presentations. They (written lesson plans and field notes) also provided evidence of types of examples and examples for a concept and/or examples for doing exercises. The school and classroom observation checklist provided evidence on (re)sources which regulate and support teachers' choice and use of examples. The transcript of interviews provided evidence on teachers' rationale for choosing and using examples in planning phase and in-moment of teaching.

Examples were analysed in terms of (a) types of examples (Michener, 1978; Tsamir et al., 2008) (cf. Table 2.3), (b) taking account of learning objectives, variation, sequencing and representations (Rowland, 2008) (Table 2.2). The characteristics of function (Leinhardt et al., 1990; Even, 1998; DBE, 2011; Nachlieli & Tabach, 2012) (cf. Table 2.1) are embedded in the description and interpretation of evidence (cf. fig. 2.6, conceptual map). (c) the considerations teachers are making for choosing and using examples are captured from teachers' rationale in the semi-structured interviews (Table 2.4). These exercises are intended to ascertain and enhance descriptive validity, interpretive validity and theoretical validity as espoused by Maxwell (1992).

The scenarios are chosen such that description and interpretation of data collected are manageable. Scenarios are informed by individual teacher's *object of learning* per lesson of the day which addresses the *learning objective* as per the curriculum. There are instances where scenarios are two or three in a lesson or address learning objective (effect of change of a parameter). This is informed by, for example, a teacher intends to do variation brought by a change in parameter using algebraic symbols (an object of learning by then). Subsequent to that scenario, the next scenario may address graphical representation (an object of learning) of change of that parameter.

The following table (Table 5.1) depicts how the analytical tools were used for each teacher:

Table 5.1: Analytical tools.

Data source	Theoretical concept and construct
Teacher's written lesson plan and field note provide texts of evidence on	1. Examples of a concept ➢ learning objective ➢ types of examples ➢ variation ➢ sequencing ➢ representation ➢ properties of a parabola 2. Examples for exercises ➢ learning objectives ➢ types of examples ➢ variation ➢ sequencing ➢ representations ➢ properties of a parabola
School and classroom observation	➢ materials (textbooks, Overhead projector) and cultural resources (time).
Interviews transcripts	Teachers rationale on their actions

5.2 Part A: Teachers' choice and use of examples

5.2.1 Review of examples of a concept, examples for doing exercises and types of examples

For analytical purposes, the definition of examples of a concept, examples for doing exercise and types of examples are defined as discussed in Chapter 2: *The review of related literature and conceptual framing*. I recapitulate these briefly here. The role of examples of a concept provides a general principle of a concept by employing examples as particular instances of the general concept (Rowland, 2008, Zodik & Zaslavsky, 2008). These examples are chosen and used to enhance and facilitate abstraction and conceptualization of a concept; therefore they are examples for concept understanding.

The role of examples for doing exercises/practice facilitates procedural fluency and enhancement of conceptual understanding (Rowland, 2008). The exercise examples "lead to different kinds of awareness and comprehension" and are "instruments for assessment for the teacher" (Rowland, 2008, pp. 150-151) in seat-works/classworks, homeworks, assignment, investigation and tests. These examples are either or both examples that facilitate concept understanding and/or develop the consolidation of concept understanding.

The types of examples are start-up, reference, model/generic, counterexamples and non-examples. Start-up examples are examples that are chosen and used in the introduction of a concept/topic. Start-up examples are therefore prototypical examples, that is, they are ideal examples which are often required first for the understanding of a concept. These examples are examples of a concept because they facilitate abstraction and concept understanding.

The reference examples are chosen and used to develop and enhance the understanding of a concept. The reference examples are repeatedly chosen and used to link understanding with mastery of the concept (Alcock & Inglis, 2008). They are therefore chosen and used to facilitate concept understanding and consolidation of that concept.

Model examples are generic examples of a concept. They are chosen and used to summarise all or most of the properties associated with a concept in a single example. These examples enhance and develop concept understanding.

Counter-examples serve to "show that the statement is not true and sharpen distinctions between concepts" (Alcock & Inglis, 2008, p. 113) and to deepen understanding of mathematical entities

(Goldenberg & Mason, 2008). The counter-examples can be chosen and used in comparing critical features of different functions in terms of their properties. Counter-examples therefore develop and enhance both concept understanding and consolidation thereof. The non-examples are examples that do not fall under a description of particular concept but they may be used to deepen the concept understanding of another concept under review.

As I describe TA's choice and use of examples of a concept and examples for exercise, and then what type of examples these are, it will be possible to interpret where he places emphasis, and consider the range of examples he selected and used.

5.2.2 Looking across TA's lessons using examples of a concept, examples for doing exercise and types of examples on choice and use of examples

5.2.2.1 Lesson 1 (40 minutes)

Scenario 1: The learning objective of the first observed day was significance of and effects of parameters *a* and *q*. Teacher A (TA) (cf. Appendix E) started with sketching of the graphs of $y = x^2$ and $y = 2x^2$ by drawing table of values where $x \in \{-4; -3; \ldots\ldots 3; 4\}$ on the chalkboard. The examples did not appear in the lesson plan but were in TA's transparencies, therefore they were planned examples. In his introduction, TA told learners that they needed to revise Grade 10 work on quadratic function. He personally sketched the graphs on the same set of axes. TA explained that the graph of $y = 2x^2$ is twice the graph of $y = x^2$. The value of $a > 0$, the graph becomes steeper and faces up (see fig. 5.1 below)

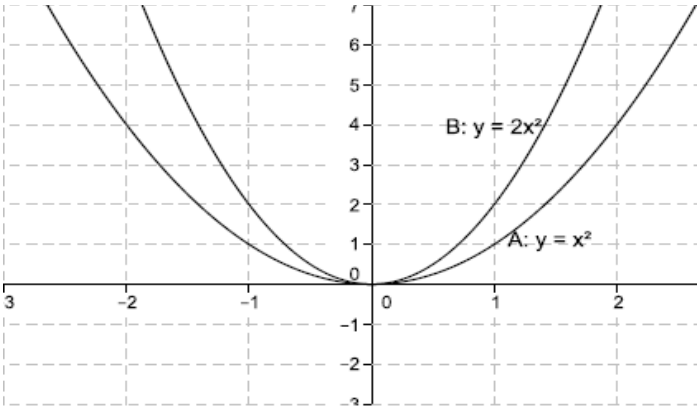

Fig 5.1

Interpretation: The two examples are prototypical examples that are ideal and required first to understand the quadratic function (Tsamir et al., 2008), therefore they are start-up examples and examples of a concept. Secondly, the examples are used as a prerequisite for understanding other forms of quadratic functions in Grade 11 and serve to revise prior knowledge from Grade 10 (cf. DBE, 2011, p. 24 & p. 32).

Scenario 2: The second set of examples were the graphs of $f(x) = x^2$ and $g(x) = -\frac{1}{2}x^2$ where another table of values was used to sketch these graphs (see fig. 5.2). These examples were written on the transparency (slides) for the overhead projector (OHP). The learning objective was to demonstrate the significances of $a < 0$ and a as fraction. TA explained that if $a < 0$ then the graph "faces downward". If $0 < a < 1$ (fraction) then the graph "open wider".

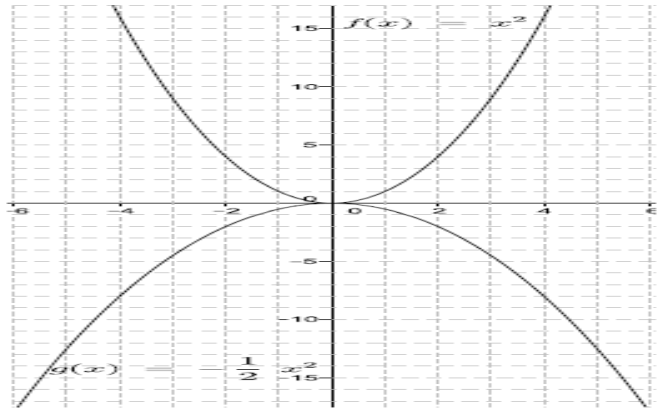

Fig. 5.2

Interpretation: It was not clear both in the sketch and in TA's explanation about "opening wider" since the two graphs were in opposite directions with the graph of *g(x)* along the *x-axis.* It is not easy to establish visually that the graph of *g(x)* is wider than the *f(x)* graph from the transparency. The *g(x)* can be considered as a reference example because a different value of *a* was taken. However, because *g(x)* addressed two features at the same time, namely, the reflection along the *x-axis* and the width of the graph, the latter in particular was obscured. Secondly, the procedure of plotting the parabola where table of values and point-wise plotting

were intended to be enhanced and facilitated. The examples were examples for understanding a concept (parabola), with potential for both concept understanding and consolidation of a parabola.

Scenario 3: The third sets of examples were $y = -\frac{1}{2}x^2$ and $y = -\frac{1}{2}x^2 + 2$ and were sketched on the same system of axes (see fig. 5.3). Learners were instructed to draw the table of values then sketch the graphs. After 7-10 minutes, TA put the transparency on the OHP which had the table of values and sketched graphs. TA explained the effect brought by parameter *q* that is 2, by pointing to the movement of the graph of $y = -\frac{1}{2}x^2$ two units up, and therefore a vertical shift. These examples appeared on the transparency and are examples from Grade 10.

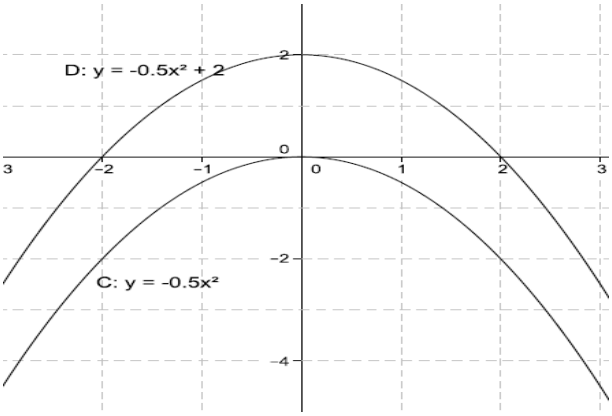

Fig. 5.3

Interpretation: The examples were examples of concept, therefore start-up examples, because they introduced a new feature (effect of parameter *q*) even though they served as prior knowledge from Grade 10. They were examples that, firstly, enhanced conceptual understanding of the effect of parameter *q* hence vertical shift. Secondly, they served as examples that develop acquisition of plotting the graphs by table of values and point-by-point plotting.

Scenario 4: The fourth example taken was the graph of $y = -\frac{1}{2}x^2 - 2$ which does not appear on the lesson plan. TA, together with learners, filled in data values on the table of values just under the graph of $y = -\frac{1}{2}x^2 + 2$. TA sketched the data values into the Cartesian plane of the drawn

graphs using a different colour. TA probed the learners to observe and explained the difference brought by change in values of parameters that is if $q > 0$ the graph shifts vertically up by q units and if q is negative then shifts vertically down by q units.

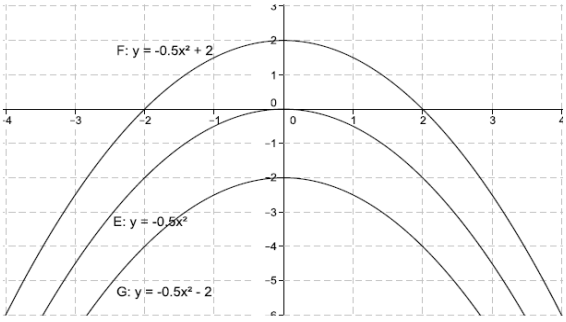

Fig. 5.4

Interpretation: The fourth example was a reference example because it developed the understanding of parameter q in different value (awareness of another value of q) therefore it was an example of doing an exercise. These examples together also serve to enhance procedural fluency of q in different value. Secondly, both TA and learners filled in the tables of values together, thus focusing on the graphical procedure through the table of values and the pointwise approach.

Discussion: TA's explanation covered both concept understanding and procedure of sketching the parabola from symbolic representations to graphical representations. The teacher's change of example from $y = \frac{1}{2}x^2 - 2$ to $y = -\frac{1}{2}x^2 - 2$ (a was changed) implies that the value of parameter a should remain unchanged and that of q becomes the object of change.

5.2.2.2. Lesson 2 (80 minutes).

Scenario 5: TA wrote "horizontal shift" and $y = (x - p)^2$ on the chalkboard. To recap from previous day's work, TA wrote $y = ax^2 + q$ and probed learners on the effect of a and p. He wrote the graphs of $y = x^2$ and $y = (x + 2)^2$ (on the chalkboard from lesson plan) and then drew a Cartesian plane with values of x from -2 to 2 on both axes (as appearing in the Cartesian plane)

(see field notes) and substituted them in the equation $y = (x+2)^2$ and plot *y-values* in correspondence with *x-values* as coordinates on the Cartesian plane. He did not write-up a table of values and plot points. TA sketched both graphs of $y = x^2$ and $y = (x+2)^2$ on the same system of axes (see fig. 5.5 below). He pointed to the graph of $y = (x+2)^2$ and probed learners about the points of intersection on both *x* and *y* axes, that is, *(-2;0)* and *(0;4)* respectively. He explained that the graph of "$y = (x+2)^2$ is the graph of $y = x^2$ which has moved 2 units to the left".

Interpretation: The set of examples formed was a start-up example because it introduced new knowledge, namely, the effect of parameter *p* and the horizontal shift of the graph. TA introduced global sketching when, collaboratively with learners; he matched *x-values* with corresponding *y-values.* This implies moving from symbolic representation to graphical representations without the aid of a table of values. Therefore the set of examples developed understanding of effect of parameter *p* and global sketching.

Scenario 6: TA instructed learners to sketch the graphs of $y = x^2$ and $y = (x-1)^2$ (lesson plan) on the same system of axes. After 7 – 10 minutes, TA sketched the graphs on the chalkboard (field notes) and probed learners on the intersection of the graph on axes and the horizontal shift to the right of the graph of $y = x^2$ brought by $p = -1$. TA then generalised the horizontal shift brought by parameter *p* from two sets of examples, that is between $y = (x+2)^2$ and $y = (x-1)^2$ (see fig.5.5). If $p > 0$, the graph shifts *p* units to the left and if $p < 0$ it then shifts *p* units to the right.

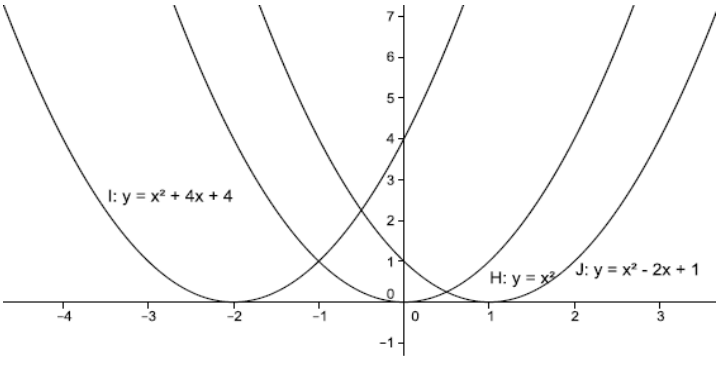

Fig. 5.5

Interpretation: The examples are reference examples because they inform and develop the effect of parameter *p*. TA also provided procedural cues as to how the graphs look when sketched. Secondly, the examples provided some means for learners to develop an understanding of the generalisation of the horizontal shift in two different directions based on $p > 0$ and $p < 0$. The understanding of the generalization of parameter *p* and global sketching were presented together, and can be considered as examples of a concept.

Scenario 7: TA wrote the "different forms of parabola" on the chalkboard and underneath wrote (lesson plan and field note):

- standard form: $y = ax^2 + bx + c$
- turning point form: $y = a(x - p)^2 + q$
- x – intercept form: $y = (x - x_1)(x - x_2)$.

TA put the transparency on the OHP which had the completing of the square of equation $y = x^2 - 4x + 4$. The procedural cues were factorising by completing the square. The example did not appear in the lesson plan but I consider it as a planned example because TA had already written it on the transparency. He explained each step taken until he reached the answer:

$y = (x - 2)^2$, a squared binomial.

Interpretation: The example was the start-up example because it introduced the conversion of the standard form of quadratic function to its turning point form applying factorisation by completing the square method. Although learners had previously done the completing of the square on *quadratic equations* of the form $ax^2 + bx + c = 0$ in Grade 11, that is, to find the solutions of the equation, it was the first time this was done on the *quadratic equation* of the form $y = ax^2 + bx + c$. The example is a perfect square or squared binomial.

Scenario 8: TA instructed learners to write $y = x^2 + 6x + 9$ and $y = -x^2 - 2x - 1$ in turning point forms and learners had to describe the shift of each from the origin in terms of "vertical" and "horizontal". The two examples appear in the transparency together with $y = x^2 - 8x + 16$ and $y = x^2 + 3x + 5$ which were discarded by TA.

Interpretation: The examples, $y = x^2 + 6x + 9$ and $y = -x^2 - 2x - 1$ are reference examples because they were repeatedly used to link standard form with turning point form. Secondly, the examples were for practice purposes, therefore the mastery of factorising by completing the square was required to be identified and enhanced. Thirdly, the final stages of procedural cues for these examples were perfect square/squared binomials like the previous examples.

Scenario 9: Homework was given: Determine the TP and sketch the graphs of showing intersections on axes.

1. $y = x^2 - 8x + 16$
2. $y = x^2 + 2x + 2$
3. $y = x^2 + 3x + 5$

The second and third examples are the ones which were discarded by the TA (see scenario 8).

Interpretation: TA discarded the second and third examples because they did not address squared binomials. He deferred these examples to homework.

5.2.2.3 Lesson 3 (80 minutes)

Scenario 10: The period started with the marking of the homework where three learners volunteered to do the exercises on chalkboard. This took 8 – 10 minutes. All exercise examples were correct but TA had a special interest in numbers 2 and 3. He explained the procedural cues undertaken for example $y = x^2 + 2x + 2$ to be $y = (x+1)^2 + 1$. He then did the same with $y = x^2 + 3x + 5$ to be $y = (x + \frac{3}{2})^2 + \frac{11}{4}$.

He wrote Horizontal and Vertical shifts: $y = a(x - p)^2 + q$ on the chalkboard. He first demonstrated that the first example $y = x^2 - 8x + 16$ [after factorising, it became $y = (x-4)^2$] does not have the vertical shift because its $q = 0$. In second and third exercises, he demonstrated the effect of parameters *p and q* both as units of shifts (horizontal and vertical, respectively) and as part of the turning points.

Interpretation: The three examples are start-up examples that introduce three critical features of the parabola, namely, vertical shift and horizontal shift and turning point simultaneously. Although in their representation the critical features were addressed individually by TA in his explanation, they were treated in a mutually exclusive way (Marton & Pang, 2006). Although the examples were given to the learners as exercise (homework), the last two examples were *not squared binomials,* which were more complex than the first one in terms of obtaining the value of *q*. Secondly, some learners were able to sketch these examples (three of them) in terms of indicating intersection on axes. Both concept understanding (vertical and horizontal, TP, sketching of intersection on axes) and procedural fluency (factorising by completing the square) were required in these examples.

Scenario 11: TA wrote the notes and explained them:

- $y = a(x-p)^2 + q$: TP *(p;q)*
- $y = a(x+p)^2 + q$: TP *(-p;q)*
- $y = a(x+p)^2 - q$: TP *(-p;-q)*

He let learners do seatwork (classwork) on: sketch the graphs of $y = x^2$, $y = (x-2)^2 - 4$ and $y = (x+1)^2 + 2$ on the same system of axes (appears in lesson plan). Learners were given 15-20 minutes to do the exercise. TA did corrections on the chalkboard while explaining and probing learners on key concepts, namely, shifts (vertical and horizontal), turning points and how to plot the parabolas. In the first example, learners indicated the TP was the origin, *(0;0),* the second one TP is *(2;-4)* and the third example's TP as *(-1;2).* The three parabolas were sketched.

Interpretation: The examples are reference examples because they are repeatedly used to link the critical features (shifts and TP) with global sketching of parabola which were done in the correction of the homework. The examples are examples of doing exercises and served to facilitate procedural fluency by translating properties of the parabola (horizontal and vertical shifts, TP) to the graphical representation (global plotting).

Scenario 12: After the corrections of the above exercise, TA wrote: sketch the graph of $y = ax^2 + bx + c$. He took the transparency that showed the following:

1. *TP = (p;q)* where $p = -\frac{b}{2a}$ and $q = \frac{4ac - b^2}{4a}$
2. x - intercept is given by: let *y = 0* then $x = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$
3. y – intercept is given by: let *x = 0* $\therefore$ *y = c*
4. Axis of symmetry: *x = p*

and explained the relation between standard form and turning point form in terms of properties of the parabola (TP, intersection on axes, axis of symmetry) using the above notes. No homework was given – instead, learners were referred to page 89 (see Appendix C), Exercise 4 (1) of *Platinum Mathematics Grade 11*(Bradley et al., 2012) to prepare for the next day's lesson.

Interpretation: The examples were start-up examples because they enable the discernment of how to determine the properties of parabola using two forms, namely, standard form and TP form, for the first time in Grade 11. The notes also provided information on the similarities and difference of the two forms of parabola based on the properties of the parabola. The examples were examples of understanding the similarities and the difference between the standard form and TP form.

5.2.2.4 Lesson 4 (40 minutes)

Scenario 13: The example in *Platinum Mathematics Grade 11*, page 89, Exercise 4 number 1 was given to the learners as seatwork (classwork). The instructions in the textbook were: Label each graph clearly and indicate the intercepts with the axes as well as any turning point(s).

1. 1.1. Sketch $f(x) = -x^2 + 4x + 12$ and $g(x) = 4x + 8$ on the same system of axes.
   1.2. For which value(s) of *x* is $f(x) = g(x)$?
   1.3. For which value(s) of *x* is $f(x) \geq g(x)$?

Learners worked this exercise for 10 – 15 minutes. The exercise appeared in TA's lesson plan. After confirming that learners had completed the exercise, the teacher put the transparency on the OHP where only the sketched graphs appeared (see fig. 5.6) that is the answer for 1.1.

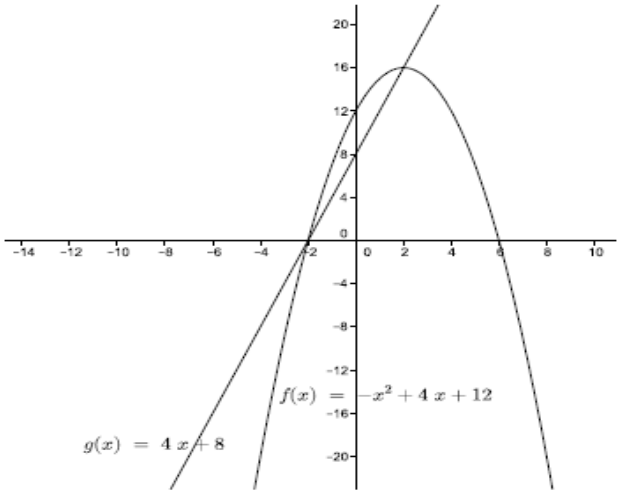

Fig. 5.6

TA explained while writing on the chalkboard (cf. Table 5.3).

Table 5.2: Procedure and explanation of two graphs.

Object	**Explanation and procedure**
a < 0	The parabola faces down
y- intercept	Value of c = 12
x-intercept	Let y = 0 then: $-x^2 + 4x + 12 = 0$ $x^2 - 4x - 12 = 0$ (multiply by -1 throughout) $(x - 6)(x + 2) = 0$ X = 6 or x = -2
Axis of symmetry	$x = -\frac{b}{2a}$ $x = -\frac{4}{2(-1)} = 2$

TP	Axis of symmetry passes through the TP, the value of x of TP is the x = 2. Substitute x = -2 in f(x) = y then $y = -(2)^2 + 4(2) + 12$ $= 16$ TP: (2;16)
In the graph of g(x) : the y –intercept	The value of c = 8
In straight line graph: The x – intercept	Let y = 0 then 4x + 8 = 0 4x = -8 x = - 2
For f(x) = g(x)	Is where the two graphs intersect/cross each other. From the sketched graphs learners were able to read only one point *(-2;0)* while others said another point is *(2;16)* therefore $x = \pm 2$. The teacher used the algebraic method to get the point. Since two graphs are equal then: $-x^2 + 4x + 12 = 4x + 8$ $-x^2 = 8 - 12$ $-x^2 = -4$ $x^2 = 4$ $x = \pm 2$
In f(x) ≥ g(x)	Read the interval of values of x where f graph is above the g graph, starting where they are equal or intersect one another. $x \in [-2;2]$ or $-2 \leq x \leq 2$. Most of the learners did not get it correct.

Interpretation: The two examples may appear to be start-up examples because it was the first time that two different kinds of functions were dealt with at the same time. The properties and attributes of the parabola and linear function had already been dealt with; therefore no new concept(s) were introduced. The example was a model example. The two functions were model examples because together they contained most of the properties and attributes of both functions (parabola and linear function) as per CAPS (DBE, 2011). Secondly, the example probed the

understanding of the parabola and linear function by comparing their critical features (TP, axis of symmetry). Their comparison demanded the discrimination between the two in terms of their properties and shape, therefore develop further their understanding through contrast.

The introduction of reading values from the sketched graphs demanded the conceptualisation of interpretation of functions. The interpretations on their relationship were probed [that is, for which value(s) of x is $f(x) = g(x)?$ and for which value(s) of x is $f(x) \geq g(x)?$] which demanded reading the values from the two functions. The probing was done to gain the understanding of that relationship and further enhance the procedural fluency of both functions. .

Having described all four lessons and scenarios in them, it is now possible to look across the set of examples chosen, their range and how these were used, and thus address research question 1. To facilitate this, I summarise the above descriptions in Table 5.3: Summary of TA's lesson.

Research question 1: What examples did TA choose in the planned phase and during the lesson presentation?

TA chose start-up examples in all new learning objectives related to the parabola (effects of parameter *a, p* and *q*; factorising by completing the square/converting standard form to TP form of a parabola) (cf. Table 5.3). All start-up examples were prototypical examples of a concept because they are ideal examples for the understanding of those concepts (Tsamir et al., 2008). The chosen and used examples probably served the development of an understanding of the objects of learning (Tsamir et al., 2008) of a parabola.

There were five reference examples that were both examples of a concept and that were examples for doing exercises (cf. Table 5.3). Reference examples that were examples of a concept required learners to discern and develop an understanding of a parabola and mastery thereof; and link the outcomes (effects of parameters, properties, different forms, plotting, etc.) with the concept (parabola) (Alcock & Inglis, 2008; Tsamir et al., 2008) through TA's explanations. The reference examples that were examples of doing exercises were more challenging than those done by TA's start-up examples (cf. Table 5.3). This suggests that TA's examples were aimed at affording learners an opportunity to have a deeper understanding of and procedural fluency on a parabola.

Only one model example was chosen and used by the teacher (cf. Table 5.3). The chosen and used example required that all different parameters and properties that define a parabola and linear function be matched for a deeper understanding and enhancement of procedural fluency of the two functions.

TA introduced one spontaneous example where he changed the planned example $y = \frac{1}{2}x^2 - 2$ to $y = -\frac{1}{2}x - 2$. His decision of changing the planned example to the spontaneous one is given in Part C. There were no learner-generated examples (LGE). This suggests that TA's teaching strategy did not afford learners an opportunity to generate their own examples. Watson and Shipman (2008) posit that LGE can be used as a teaching strategy.

My evaluation is that (cf. Table 5.3) in the generalisation of parameter *a* two objects of learning, namely *a < 0 and 0 < a < 1*, were addressed at the same time hence TA had difficulty in explaining these objects (scenario 2). Secondly, TA did not assess learners on the effect brought about by parameter *a*, that is, no examples for exercise on parameter *a*. It might be that the significance of parameter *a* was prior knowledge from Grade 10 and therefore only served as understanding of reflection along the *x*- axis in scenarios 2 & 3 (or vertical shift, see Table 5.3). Lastly, I cannot establish that only one model example was taken as example of a concept or no example for exercise was taken because my classroom observation captured only one model example as an example of a concept.

TA's choice and use of examples that were both examples of a concept and examples for doing exercise. His examples of a concept were aimed at developing the concept understanding of the learning objectives by employing start-up examples and a model example. In his introduction of the learning objectives (see scenarios 1 – 5, 7 & 11 & 12), TA chose and used start-up examples as examples of a concept. In scenario 13, he employed a model example as an example for a concept that facilitated both concept understanding of two graphs and algebraic manipulations. TA chose and used these types of examples to firstly inculcate concept understanding.

The examples for exercise were aimed at enhancing the understanding of concepts through procedural fluency (Rowland, 2008). Reference examples, in all scenarios, followed after start-up examples. Therefore TA employed reference examples as examples for exercise. Reference

examples enhance concept understanding and develop concept consolidation (Tsamir et al., 2008). Although there were more examples for exercises than examples of a concept (Table 5.3), Rowland (2008) found in his study that teachers' choices and use of examples that facilitated both concept understanding and procedural fluency were examples that were examples of exercises. These imply that TA wanted to develop and enhance both conceptualisation and procedural fluency. It further suggests that TA's choice and use of examples were balanced between examples that were intended for concept understanding and consolidation of it (concept understanding).

The next sub-section deals with an analysis of TA's examples by employing Rowland's categories. The sub-section illuminates how examples of a concept and examples of exercises were aimed at facilitating concept understanding and consolidation thereof by answering research question 2.

Table 5.3: Summary of teacher A's lessons.

Lesson	Learning objective	Examples of concept	Examples for exercise	Analysis
1	Effect of parameter *a*:	$y = x^2$ and $y = 2x^2$ (start-up example) $f(x) = x^2$ and $g(x) = -\frac{1}{2}x^2$ (reference)		Table of values, ordered-pairs and point-wise sketching of the functions were done. The values of parameter *a* were varied in order to discern the effect of *a*.
	Vertical shift: effect of parameter *q*	$y = -\frac{1}{2}x^2 + 2$ and $y = -\frac{1}{2}x^2 - 2$ (reference)	$y = -\frac{1}{2}x^2$ and $y = -\frac{1}{2}x^2 + 2$ $y = -\frac{1}{2}x^2 + 2$ and $y = -\frac{1}{2}x^2 - 2$ (both reference)	Table of values, ordered-pairs and sketching of the functions were done. The values of parameter *q* were varied in order to discern the effect of *q*.
2	Horizontal shift: effect of parameter *p*.	$y = x^2$ and $y = (x+2)^2$ (start-up) $y = x^2$ and $y = (x-1)^2$ (reference)	$y = x^2$ and $y = (x-1)^2$ (reference)	Connection from pointwise to global approach was made. The values of parameter p were varied to discern their effect.
	Different forms of quadratic function	$y = x^2 - 4x + 4$	$y = x^2 + 6x + 9$ $y = -x^2 - 2x - 1$ (reference) Homework on $y = x^2 - 8x + 16$	Completing the square is introduced therefore algebraic manipulation and moving from one form of quadratic function to another.

			$y = x^2 + 2x + 2$ $y = x^2 + 3x + 5$	
3	Vertical and horizontal shifts	$y = x^2 - 8x + 16$ to $y = (x-4)^2$ (start-up) $y = x^2 + 2x + 2$ to $y = (x+1)^2 + 1$ $y = x^2 + 3x + 5$ to $y = (x + \frac{3}{2})^2 + \frac{11}{2}$ (both reference)		Completing the square by algebraic manipulation. Converting different forms of parabola to another form.
	Sketching $y = ax^2 + bx + c$		$y = x^2, y = (x-2)^2 - 4, y = (x=1)^2 + 2$ (reference)	Enhancing completing the square, properties of parabola and global sketching.
4	Matching of quadratic function and linear function	$f(x) = -x^2 + 4x + 12$ and $g(x) = 4x + 8$ (Model)		Matching two functions and reading the values (interpretation) when compared

5.2.3 Review of Rowland's categories

Rowland (2008) used four categories, namely learning objective, variation, sequencing and representation to analyse the choice and use of teacher's examples. The categories are discussed in Chapter two, and again I review them here briefly. Rowland posits that the four categories are a teacher's awareness that s/he takes to account when s/he choose(s) and use(s) examples. The four categories are intertwined and inform one another (cf. fig. 2.5).

Learning objectives are curriculum and/or teachers' intentions and goals of teaching a particular concept or topic. The learning objectives spell out skills, knowledge and values learners should learn from the functions (cf. Tables 2.1, Table 2.2 & DBE, 2011, p. 12).

Taking account of variation is derived from Variation Theory which is a theory for learning and is useful for evaluating and analysing teaching. Proponents of variation theory argue that only what varies can be discerned, discriminated and experienced by the learner. Discernment is defined as a quality or qualities that are possible to be discerned through variation on those qualities (Runesson & Marton, 2004; Runesson, 2005). Discrimination is when the teacher/learner realises that a particular quality of a concept remains the same whereas another undergoes a change. Marton and Pang (2006) posit that both what is to be discerned and discriminated are experienced simultaneously although they are experienced in mutually exclusive ways.

Sequencing of examples is defined as a systematic combination of related examples of the same concept that are connected to each other to develop and facilitate conceptual understanding and procedural fluency of the concept (Leinhardt et al., 1990; Rowland, 2008). Sequencing in *functions* examples can follow two interwoven activities; namely, from construction to interpretation (cf. Table 2.2 & fig. 2.3). Sequencing of examples moves from definition of function, table method, and set of ordered pairs to construction of a function (Leinhardt, et al., 1990). This movement entails reading of values and plotting points (point-by-point and/or global sketching) in the Cartesian plane (system of axes). The properties of a parabola are embedded within these concepts.

Interpretation activity normally follows after construction activity (Leinhardt et al, 1990; Lichlieli & Tabach, 2012). Interpretation activities of function can start with matching graphs of

the same families, matching functions of different types in terms of shape and formulae to curve sketching (Leinhardt et al., 1990) (cf. Table 2.2. and fig. 2.3). The addition of parameters also provides for sequencing on the same function, for example in parabola, from $f(x) = x^2$, $f(x) = ax^2$, $f(x) = ax^2 + q, to f(x) = a(x - p)^2 + q$.

Taking account of representations refers to strategies used by teachers in order to represent an abstract mathematical idea in more accessible, visible and tangible ways to the learners so that learners may understand the concept/idea (Rowland, 2008). Learners have access to an abstract idea when teachers can use tables, graphs, procedural cues, flow charts, pictures, etc.

Table 5.4: ***Analysis using Rowland's categories*** below analyses TA's choice and use of examples. The table depicts how examples of a concept and examples of doing exercise were varied, sequenced and which representations were used throughout four observed lessons to facilitate concept understanding and concept consolidation in planning phase and during moment of teaching.

5.2.4 Looking across the TA's lessons using Rowland's categories

Table 5.4: Analysis on TA's examples using Rowland's categories.

Lesson	Example of a concept	Analysis	Examples for doing exercise	Analysis
1	$y = x^2$, $y = 2x^2$ *and* $g(x) = -\frac{1}{2}x^2$	Parameter *a* is varied (namely, a < 0 and 0 < a < 1) so they might be discerned in mutually exclusive ways		
	$y = -\frac{1}{2}x^2$ and $y = -\frac{1}{2}x^2 + 2$	Parameter *q* is a critical feature that is varied while $a = -\frac{1}{2}$ is discriminated (remains unchanged).		
	$y = -\frac{1}{2}x^2 - 2$ and $y = -\frac{1}{2}x^2 + 2$,	Parameter *q* is varied so as to discern the vertical shifts in terms of up and down. The values of *a* remain unchanged. The two parameters (*a* and *q*) served to be discerned simultaneously (Marton & Pang, 2006).	$y = -\frac{1}{2}x^2$ and $y = -\frac{1}{2}x^2 + 2$ $y = -\frac{1}{2}x^2 + 2$ $y = -\frac{1}{2}x^2 - 2$	Parameter *q* is varied while *a* is discriminated. Both parameters might be experienced in mutually exclusive ways
	Interpretation: 1. ***Variation and Representation:*** the choice and use of examples addressed variation in terms of parameters *a* and *q* (see scenarios 1-4). In the use of chosen examples, TA used the **representations** of tables of values and one-to-one relations to sketch the effects of parameters were both the definition and shape of the parabola and were possible to be discerned. The construction (plotting of the graphs due to variation on			

	parameters) and interpretation (what happens to the parabola when different values of parameters are used) were also possible to be discerned. The instruction moved from symbolic representation (algebraic manipulation/procedural cues) to a graphical one. 2. ***Sequencing:*** The variation of parameters afforded the increase in degree of complexity from simple to complex examples among examples of parameter *a* and also parameter *q*. **Sequencing in construction**: A point-by-point approach was the first to be done from the formulae using table of values and ordered pairs translated into the system of axes. The shape of the graphs and effects of the parameters (especially the effect of *a* and *q*) were possible to be enhanced by comparing two different parabolas varying in parameters. The vertical shifts and reflection along the x-axis (line y = 0) were done. **Sequencing in interpretation**: The effects of the variation of same parameter were matched in order to make the discernment of concept understanding of those variations possible. This made generalization of the parameters possible. Secondly, another action of interpretation, according to Leinhardt et al. (1990) occurred when translation of values (ordered pairs) from table of values to the sketched graphs using a pointwise approach to be possible. The sequencing moved from a symbolic representation to a graphical one (Even, 1998). 3. TA's variation and sequencing of parameters *a* and *q* served to generalize the effects of these parameter and reflection along the *x-axis*. This implies that the examples facilitated concept understanding. The example for exercise served to consolidate generalization of the effect of parameters and reflection along the *x-axis*. 4. ***Planned and spontaneous examples***: all the examples that TA planned and chose were used except one example. TA planned $y = \frac{1}{2}x^2 - 2$ but used $y = -\frac{1}{2}x^2 - 2$

<table>
<tr><td>2</td><td>$y = x^2$, $y = (x + 2)^2$ and
$y = (x -1)^2$</td><td>Parameter p was critical feature and varied from the parent parabola in to ascertain the horizontal shifts if $p > 0$ and $p < 0$ from the origin $(0;0)$ while a remain the same. The intersection on x-$axis$ was embedded.</td><td>$y = x^2 and y = (x-1)^2$</td><td>Parameter p is critical feature therefore varied.</td></tr>
<tr><td></td><td colspan="4">Interpretation:
1. Variation and representation: The values of parameter p were varied to elucidate the effect of it (horizontal shift). The representation used was global sketching where TA interactively probed learners to give him y-values when he was saying x-values (see scenario 5). TA was somehow introducing global sketching while addressing the generalization of the effect of parameter p.
2. Sequencing: the sequencing from a vertical shift to a horizontal one is progression from Grade 10 to Grade 11 and therefore ascertained the increase of degree of complexity. Another degree of complexity was realized when TA used $y = (x + 2)^2$ which has +2 as example of a concept and let $y = (x -1)^2$ to be example for exercise due to -1. It is generally perceived that learners find it easy to work with a + sign than - sign. Sequencing in construction moved from pointwise plotting to global one. Sequencing in interpretation moved from shifting from the left due to + sign to the right due to – sign then generalization of the effect of parameter p. The translation of values from algebraic symbols to graphical (global) was also sequenced.
3. The variation, different forms of representation and sequencing of both examples of a concept and examples for exercise served to generalise the effect of parameter p (horizontal shift) and consolidation thereof through procedural fluency.
4. Choice and used of examples: The planned examples were the same examples used in classroom interaction therefore no spontaneous examples or omitted examples.</td></tr>
</table>

	$y = x^2-4x+4$ to $y = (x-2)^2$ $y = x^2 - 8x + 16$ to $y = (x-4)^2$. $y = x^2 + 2x + 2$ and $y = x^2 + 3x + 5$ to $y = (x+1)^2 + 1$ and $y = (x+\frac{3}{2})^2 + \frac{11}{4}$.	Completing the square and determining the TP form. What varied were procedural cues, therefore degrees of complexity. Varying two forms and how each can be manipulated to get properties.	$y = x^2 + 6x + 9$ $y = -x^2 - 2x - 1$ Homework on $y = x^2 - 8x + 16$ $y = x^2 + 2x + 2$ $y = x^2 + 3x + 5$	Completing the square and determining the TP form. What varied was procedural cues based on having the value of *q* and fractions
	Interpretation: 1. ***Variation, sequencing and representation:*** Different manipulation of completing of the square of quadratic equation were varied in order to have understanding of similarity and differences between procedural cues of the two quadratic equations, namely, *quadratic equation* of the form $ax^2 + bx + c = 0$ and *quadratic equation* of the form $y = ax^2 + bx + c$ (see scenarios 7 – 9). The second variation occurs between different forms of quadratic functions, that is, moving from standard form to turning point form. The third is on TP form which have $q = 0$ (squared binomial/perfect square) while others had different values of *q* (see above, under examples of a concept). The fourth variation was to increase the degree of complexity by including fractions as values of both *p and q*. 2. Both examples of a concept and examples for doing exercises were manipulated algebraically. No construction of the quadratic functions because only manipulation of symbolic representations was learning objective. ***3. Planned and spontaneous examples:*** TA discarded the planned examples, $y = x^2 - 8x + 16$ and $y = x^2 + 3x + 5$ (his decision to omit these examples is analysed in Part C below).			

3	$y = x^2 - 8x + 16$ Sketch on the same system of axes: $y = x^2, y = (x-2)^2 - 4, y = (x=1)^2 + 2$	Varying two forms to get properties of quadratic functions.	$y = x^2 - 8x + 16$ $y = x^2 + 2x + 2$ $y = x^2 + 3x + 5$	
	Interpretation: ***1. Variation, sequencing and representation:*** From both examples of concept and exercise, TA varied his examples from squared binomials to non-squared binomials (see scenarios 10 & 11). This increased the degree of difficulty brought about by different values and fractional form of *p and q* in TP form. TA interlinked the plotting of TP form to global sketching by employing symbolic representations. He also linked both vertical and horizontal shifts of TP form with standard form so as illuminate similarities and difference between the two forms of quadratic functions. He assigned different properties from the two forms of quadratic functions. TA used global sketching to address the variation in terms of vertical, horizontal shifts and turning points. ***2. Planned and spontaneous examples:*** All planned examples were used during classroom interaction. No spontaneous example was chosen and used.			
4	*f(x) = -x² + 4x + 12* and *g(x) = 4x + 8* on the same system of axes. For which value(s) of *x* is *f(x) = g(x)*? For which value(s) of *x* is *f(x) ≥ g(x)?*	Two different functions with varying properties		Two different functions with varying properties
	Interpretation: 1. ***Variation, sequencing and representation***: Two different types of functions, namely parabola and linear function, were simultaneously manipulated (see scenario 12). The learning objective was to understand relationships of both functions. The two functions were sketched using a global approach and matched together in terms of their shapes and properties. Interpretation occurred in two ways. Firstly, by translating algebraic symbols (variables) from symbolic representations by determining their properties (AS, TP, intersection on axes, gradient, etc.) to the sketched graphs. Secondly, by matching and analysing the relation between the two			

	graphs when answering probed questions. This suggests that the two functions were chosen and used for concept understanding and procedural fluency. 2. Planned and spontaneous examples: Only one example was planned and used and no spontaneous example came up.

R***esearch question 2: How did TA use the planned and additional examples in his lessons?*** is answered.

TA varied his chosen and planned examples during his classroom interaction in ways that support learners' understanding and mastery of a quadratic function. In Lessons 1, 2 and 3, the parameters (*a, q and p*) were varied both in symbolic (table of values, order pairs, substitutions) and graphical representations (pointwise and global approaches) for the learners to discern the generalisation of their effects. There were instances where parameter *a* was invariant and parameters *q* and *p* were varied (see Lessons 1 and 2). Both symbolic and graphical representations were offered, though in mutually exclusive ways. By discriminating the values of parameters had made generalisation of their effects possible in the moment of teaching. Variation on chosen and used examples on factorising by completing the square provided for the discernment and discrimination of squared binomials and non-squared binomials, therefore procedural fluency on these was made possible.

TA's chosen and used examples provided sequencing in two ways. Firstly, TA sequenced his examples throughout the lessons by adding parameters from $f(x) = x^2$, $f(x) = ax^2$, $f(x) = ax^2 + q$, $f(x) = a(x - p)$ to $y = a(x - p)^2 + q$ where he increased degree of complexity in terms of adding parameters from the initial examples and sketching from a pointwise to a global approach. This was moving from known to unknown examples therefore increasing the complexity of examples. The second sequencing of examples was through the actions of construction and then interpretation as espoused by Leinhardt et al. (1990) and, Nachlieli and Tabach (2012). Sequencing in construction was, firstly, from pointwise to global (see scenarios 1 – 4). The point-by-point approach was the first to be done from the formulae using tables of values and ordered pairs into the system of axes as espoused by Leinhardt et al. (1990). The learning objectives were to develop an understanding of plotting of the graphs where shape and effect of the parameters were illuminated by comparing two different parabolas varying in values of parameters.

The global approach was used extensively especially in horizontal shift (effect of parameter *p,* lesson 2), vertical and horizontal shifts (both parameters *q* and *p,* lessons 2 & 3) to model example of parabola (Lesson 4). The global approach simplified the sketching of the parabola yet

keeping the learning objectives as focus of the lessons. The global approached was also used for generalization of the effects of parameters and discussion of the properties of parabola, namely, translation, axis of symmetry, intersection on axes and turning points. In so doing, parabolas were identified in terms of shape and formulae. Sequencing made the increase of complexity of examples possible.

Interpretation activity appeared in three ways as espoused by Leinhardt et al. (1990), Even (1998) and, Nachlieli and Tabach (2012). Firstly, Even posited that symbolic representations are translated to Cartesian plane, and interpretation occurs because we gain meaning by assigning these values to the graph. TA's examples also provided for translation of ordered pairs, values of intersection on axes, axis of symmetry and turning points (symbolic representations) to construct a parabola both in the point-by-point and global approaches. Secondly, in generalising the effects brought by variation of parameters from the graphs, it was interpretation action (DBE, 2011). Lastly, the reading of values from two different graphs (lesson 4, parabola and linear function) provided interpretation of the two graphs.

The flexible move between representations occurred in lesson 4. The example required the sketching of two graphs, that is, moving from symbolic representations to graphical ones by using the global approach. The readings of values (symbolic representations) were derived from the sketched graphs (graphical representation). Even (1998) posits that learners gain deeper knowledge, skills and understanding of functions from flexible moves between symbolic and graphical representations. Even further postulates that the global approach of plotting graphs demand flexible move between representations and these facilitate construction and interpretation actions of the function.

To sum up, Rowland's categories are intertwined and interrelated, so in my reflection, I am interrelating one category to another one. TA firstly varied parameters a, q and p by employing pointwise and global approaches to generalise the effects of these parameters (cf. Tables 5.3 & 5.4, lesson 1 & 2). By translating symbolic representations to graphical ones, both interpretation and construction actions were facilitated (Even, 1998). Secondly, he varied examples in terms of squared binomials and non-squared binomials. TA used algebraic manipulations to increase the level of complexity (see Lesson 2). The variation of algebraic expressions and sequencing thereof facilitated procedural proficiency. Thirdly, he compared different forms of quadratic

functions to illuminate the difference and similarities between these forms (Lesson 3). He algebraically manipulated one form to another and sketched the graphs. Both symbolic and graphical representations were facilitated. The global approach was used, therefore both construction and interpretation actions were facilitated. Lastly, two different functions, *viz*. linear and quadratic functions, were matched together (Lesson 4). I infer, as postulated by Runesson and Marton (2004), and Runesson (2005); the proficiencies, competencies and skills cited above increased the level of complexity and they were possible to be discerned. This implies that examples of a concept served to facilitate concept understanding and examples of doing exercises served to consolidate and master the concept understanding.

5.3 Part B: What were (re)sources for TA and his examples?

I am looking across the observed lessons by reflecting on (re)sources TA used to derive the examples and those that aided his teaching. The documents teachers normally draw their examples from are CAPS, question papers and textbooks which were discussed in Chapter 3. The LTSM are also analysed as (re)sources that supported TA's choice and use of examples.

5.3.1 Curriculum document (CAPS)

Curriculum and Assessment Policy Statement (CAPS) (DBE, 2011) provides the teachers with content to be covered and what knowledge, skills and values need to be acquired by the learner in each subject and grade. In Chapter 3, the curriculum was analysed based on the literature review in chapter 2. The following concepts were used: progression between grades, definition of the function, construction, interpretation and flexibility between representations (cf. Table 3.1). Only two categories of Rowland (2008) were used as analytical framework, namely, learning objectives and sequencing. Curriculum documents provide information on learning objectives the teacher should take into account as s/he prepare(s) and plan(s) for lesson presentation and these are taken from the curriculum. Taking account of sequencing is realised in the analysis on construction, interpretation and flexible movement between representations. Taking account representation was not dealt with at Chapter 3 because it only becomes clear from a teacher's classroom interaction. Taking account of variation requires an analysis on teachers' choices and use of examples and the CAPS document did not provide that. As cited in chapter 3, mathematics teachers normally draw from curriculum stipulations to plan and prepare

their lessons. Secondly, the CAPS document provides a base on which to observe, describe and interpret teachers' lesson preparation, planning and presentation. Thirdly, Both TA and TB had made a claim, during interviews, that they drew their examples from the curriculum.

Discussion

Definition of a function and curriculum progression. TA chose and used examples from Grade 10 curriculum examples in the first two sets of examples (cf. scenario 1 & 2) (DBE, 2011, p. 24). TA chose and used those examples to move from familiar examples done in Grade 10 the previous year so as to maintain progression and the definition of a function throughout the classroom interaction. The one-to-one rule that defines a function, though, does not appear in Grade 11; it is described by the use of formulae, table of values, ordered pairs and vertical line test on the graphs. Secondly, CAPS stipulates that:

1. Revise the effect of the parameters *a and q* (from Grade 10) and investigate the effect of *p* on the graphs of the functions defined by:

1.1 $y = f(x) = a(x+p)^2 + q$

1.2 $y = f(x) = \frac{a}{x+p} + q$

1.3 $y = f(x) = a.b^{x+p} + q$ where $b > 0, b \neq 1$ (DBE, 2001, p. 32).

TA abided by progression and the degree of complexity of quadratic function examples (in 1.1) from Grade 10 curriculum to Grade 11 (see scenario 1-13, above). The degree of complexity of TA chosen and used examples under parabola followed the sequence of examples from $f(x) = x^2$, $f(x) = ax^2$, $f(x) = ax^2 + q$, $f(x) = a(x-p) to f(x) = a(x-p)^2 + q$.

Construction, interpretation and flexible move between representations. TA chose and used examples that generate graphs, test conjectures, generalising the effects of the parameters, identifying the features/properties and applying them (see scenarios 1-13) (DBE, 2011, p. 12). TA chose and used two similar and different functions in terms of shape, properties, restrictions and formulae as per CAPS stipulation (DBE, 2011, pp. 24 & 32).

TA had made a claim that he used the Annual Teaching Plan (ATP) as a guide to plan for his lessons (see Part C below). This alluded to what source he consulted and considered for his examples. ATP is aligned with the prescripts and a stipulation of CAPS therefore CAPS somehow constrained TA's choice and use of *function examples*. The purpose of analysing CAPS based on TA's examples was to evaluate which prescripts hindered or afforded TA with an opportunity to prepare and plan examples he had chosen and used. Based on the analyses above, CAPS had constrained TA's choice and use of examples by his adhering to its prescripts.

5.3.2 Official examination question papers

The question papers are assessment instruments that assess learners' skills and knowledge as enshrined in the curriculum document. Most teachers use them for teaching and checking which content is highly valued (Luxomo, 2011) and privileged. Luxomo (2011) further elaborates that "teachers in most schools used assessment at national level *and provincial* as a guide to decide on the amount of time they spend to teach each topic" (p. 64, *my emphasis)*. Lastly, teachers refer to the externally set question papers as the guide on how to translate the curriculum objectives into assessment and how content is weighted in terms of cognitive levels (taxonomies).

TA had made a claim (see interviews in Part C) that he drew some of the examples from the previous question papers. There was no evidence of an example taken from the question paper. It suffices to reflect on what the content of the question papers (2012 externally set Grade 10 question paper and 2013 Grade 11 National question paper) did, and why it was highly valued against what TA valued in his choice and use of *function examples*.

Discussion

The question papers analysed in Chapter 3 (2012 externally set Grade 10 question paper and 2013 Grade 11 National question paper) valued and privileged interpretation action more than construction one. They assessed flexible move between representations and; between verbal descriptions and algebraic symbols. The movement require interpretation actions. The symbolic representations were assessed more than graphical representations, therefore algebraic proficiency was valued.

Construction: Firstly, TA used both pointwise (scenarios 1 - 4) and global (scenarios 5, 6 & 13) approaches to sketch the graphs. The question papers prioritised the global approach. It seems that TA kept a balance between pointwise and global approaches. I infer that TA used the pointwise approach to introduce and develop concept understanding on plotting the quadratic functions. He subsequently used the global approach to facilitate properties of quadratic functions as used in the question papers (cf. Appendix A: Grade 10, number 4.2; Appendix B: Grade 11, number 9.1).

Secondly, TA matched quadratic functions differing in values of parameters (Lessons 1 & 2) together and also quadratic function and linear function (Lesson 4). The question papers prioritized matching of two different functions, that is, linear function and quadratic function (Grade 10, number 4.2) and quadratic function and exponential function (Grade 11, number 9.1). I infer that TA through sequencing on construction, as posited by Leinhardt et al. (1990), matched quadratic functions together to facilitate concept understanding for the generalisation of parameters.

Interpretation: TA chose and used examples that required interpretation action by translating algebraic symbols to graphical representations, matched quadratic functions together for generalising the effect of parameter and matched different functions (linear and quadratic) together to read values from the sketched graphs. The question papers assessed interpretation action by translations verbal descriptions to algebraic formulae, transformation (reflections and translations) and properties (asymptote, axis of symmetry, interception on axes) (Grade 10, numbers 4.1.8 & 4.1.9; Grade 11, numbers 8.1, 8.6, 9.4, 9.5 & 9.6) and matching two different functions together so as to read the values from the graphs (Grade 10, numbers 4.1.1 – 4.1.5, 4.1.7; Grade 11, numbers 8.1 & 8.2). I infer that TA chose and used *function examples* that were almost similar to the ones in the question papers although the question papers' examples have more rigour.

5.3.3 Textbook

Teachers use textbooks during lesson preparation and planning, and also during lesson presentation (classroom interaction). Textbooks afford the teachers with an opportunity to translate curriculum objectives into classroom interaction and what should be taught and learned.

Secondly, textbooks may make connections between concepts and content within the curriculum and beyond (Zaski & Leikin, 2010). Significantly, textbooks provide both examples of a concept and examples for exercises (for practice and assessment) for understanding and mastering a concept (Rowland, 2008).

In contrast, textbooks may limit the scope of the content coverage envisaged by the curriculum by putting more emphasis on certain topics/concepts and neglecting (either by choice or not) some content. Secondly, textbooks may prioritize one type of example more than others therefore limit examples of a concept and/or examples for doing examples. Furthermore, as Zaski and Leikin (2010) assert, they may not fully address a necessary specific content of concept. This suggests that too much reliance on textbooks limits the necessary specific content of the concept to be learnt by learners. I reflect on the TA's examples that were drawn from *Platinum Mathematics Grade 11* (Bradley et al., 2012).

Discussion

Examples of a concept: TA drew examples from the textbook which were consistent with it both as examples of concept and start-up examples. The examples in scenario 5 [($y = x^2$ and $y = (x + 2)^2$] were similar to examples in the textbook (p.85). Both TA and textbook used the examples to facilitate the understanding of the effect of parameter *p* (horizontal shift). In both instances, the examples were start-up examples because they were ideal for understanding and introducing the effect of parameter *p*.

In scenarios 7 and 13, TA drew examples from the textbooks which were examples for exercise yet he chose and used them as examples of a concept. In scenario 7, $y = x^2 - 4x + 4$ was an example for exercise and a reference example (Bradley et al., 2012; page 85, exercise 2, number 3). TA chose and used the example for understanding of converting a standard form of quadratic function to turning point form by employing factorization (completing the square). It was a start-up example. The textbook used the example to enhance factorization and describing the horizontal shift from the original (0;0). In scenario 13, the example (Bradley et al., 2012, page 89, exercise 4 number 1) was also an example for exercise and a model example. Textbooks used the examples to develop the understanding of quadratic and linear functions by matching them together so as to consolidate their understanding in terms of construction and interpretation of

their properties. TA chose and used these examples for introducing and facilitating the understanding of same actions and skills as they were demanded in the textbook.

Examples for doing exercises: There was only one example of a concept, as it appeared in the textbook, which TA chose and used it as example for exercises. Example in scenario 6 (page 85), $y = (x - 1)^2$ was used in the textbook with other examples [$y = x^2$ and $y = (x + 2)^2$, see scenario 5] for understanding of the effect of parameter *p* when *p* assumes different value. It was a start-up example. TA chose and used the example to consolidate the understanding of the effect of parameter *p* when it assumed a different value.

TA drew a substantive number of examples from the textbook for exercise purposes. The examples TA drew from the textbooks were reference examples, therefore they enhanced and developed the consolidation of concept understanding through procedural cues. These examples are depicted in scenarios 8, 9 and 10 (page 85, exercise 2, numbers 1, 2 & 4).

I infer that the textbook provided TA with examples of a concept and examples for doing exercises irrespective of how the role and purpose of these examples were pitched in the textbook. I imply that TA had autonomy to choose and use examples from the textbook depending on what role and purpose he employed them.

5.3.4 Teaching and learning (re)sources

5.3.4.1 Review of teaching and learning (re)sources

As cited in the purpose of this study in Chapter 1, the two teachers under study teach in two contrasting South African contexts (fee-paying and no-fee schools). It is imperative to investigate and evaluate teaching and learning (re)sources (cf. Appendix D) available in their schools as (re)sources that aid their choice and use of examples. The following aspects pertaining to teaching and learning (re)sources as espoused by Adler (2000) provided what was available and used by the TA in his choice and use of examples.

The departmental policy on amount of mathematics learners' written work is a minimum of three classworks and/or homeworks in 4.5 hours of teaching contact time. TA was teaching 56 learners and I observed a class of 28 learners.

5.3.4.2 Teaching and learning aids

The school has separate mathematics and science classrooms where most of classrooms have Overhead Projectors (OHP), charts, placards and posters that are relevant to the subjects. Learners use *Platinum Mathematics Grade 11* (sponsored by the North West Department of Education) and a study guide that contained previous question papers of Grades 10 and 11. On enquiry about examples drawn from the guide, TA did choose and use examples drawn from it. TA had on several occasions used the transparency of OHP for lesson preparation and planning. The use of OHP had provided TA with the three important aspects of teaching and learning. TA was able to have smooth entry to the learning objectives because immediately as he started his lesson by switching on the OHP, learners paid attention. Secondly, TA's examples were sequenced in the transparency and this afforded him with logical presentation of his lessons. Thirdly, because his examples were already written in the transparency, TA was able to check and control learners' written work and therefore saved time on his teaching.

5.3.4.3 Time

Time as cultural resource is used in time-tables, length of the period and time given on classwork. Teacher A has eight mathematics periods in Grade 11 for each two groups he was teaching. The school used a time-table of eight 40-minute periods in a cycle of six days. This means TA had 5.3 hours of contact time with learners of each Grade 11 class he was teaching. This is more than DBE's (2011) stipulation on contact time which is 4.5 hours for Mathematics in the FET phase. There is no school policy on classwork and homework; therefore teachers may give learners these works at their own discretion. The 40-minute period provided the teacher with enough time to complete the lesson (learning objective) of the day as planned. Secondly, time allocated for the period helped TA with the pacing of his explanation during instruction and responding to learners questions of clarity.

5.3.4.4 TA's rationale about teaching and learning (re)sources

I Your school is well-resourced, what other resources are there?

TA Computer Labs for Mathematics, Geogebra to sketch and interpret graphs, to show different graphs.

I Do you use these resources and how often?

TA In Maths we don't completely use them regularly but we use them.

I What advantages do you get from these?

TA Save teaching time but takes time for preparing them. (Pause)……Learners are technologically inclined, they learn better.

I How do these resources help you to plan for a lesson?

TA In topics that need visuals like analytical geometry and geometry.

The teaching and learning (re)sources observed (those discussed above) had assisted TA with using time effectively by completing the planned examples, controlling learners' work and presenting his lessons in logical ways.

5.4 Part C: TA's considerations on his choice and use of examples, and his (re)sources

In this subsection, research question 3: ***How did TA explain his choice and use of examples?*** was analysed (cf. Appendix H). The semi-structured interviews were used to gather teachers' rationale for choosing and using their examples. Semi-structured interviews were audiotaped to ensure an accurate account of teachers' actions (Leedy & Ormrod, 2005; Creswell, 2012). The transcripts of the interviews were edited to clarity verbal prompts like "*this example, this one, etc"* with insertion of examples derived from the scenarios and/or lesson plans. Care was taken not to distort the actual responses.

The analyses were done to give an account of pedagogical considerations teachers have made in their choice and use of examples in the planning phase and in the moment of teaching. The Questions posed did not follow the Interview Schedule strictly in order or step by step (cf. Appendix E) because there were instances where further enquiry and probes demanded further clarity from teachers' responses.

Zodik and Zaslavsky (2008)'s study identified the following as pedagogical considerations for choosing and using examples: (a). start with a simple or familiar case of examples, (b). choose and use examples that attend to learners' errors, (c). choose and use examples that draw attention

to relevant features, (d). choose and use examples that convey generality by random choice, (e). choose and use examples that include uncommon cases and lastly (f). choose and use examples that keep unnecessary work to a minimum.

The focus in analysing their rationales was based on, firstly, what and how they chose their examples in the planning phase and during lesson presentations. The interview questions probed roles of examples and types of examples although teachers may not know these types of examples but they are embedded in their explanation. Secondly, interview questions probed teachers' rationale on how they used the planned and additional examples in their lessons. These gave an account of their rationale on how they have sequenced and varied their examples. Thirdly, interviews questions probed their rationale on what and how their social setting have supported or constrained their planned and use of examples based on (re)sources and other aspects.

5.4.1 TA's rationale on planning of examples

I where do you draw these examples (showing him his planned examples from Lesson plan and transparencies in scenarios 1 to 4)

TA For examples I didn't use the textbook because these examples are from Grade 10, I used my experience.

I so the rest are from textbook? And why?
TA It's a new topic therefore I need to take relevant examples to support learners' knowledge

I when you plan your lesson, there are things that you considered, which things you considered?
TA I considered ATP (Annual Teaching Plan) which guides me what to teach. You know …. I can't teach anything I want.

I I mean in terms of the topic and learners?
TA Ok, ok …. I consider the strength and weakness of the two groups. I take examples that will suit both groups. Learners are not the same, start with simple examples in all topics….we want all learners to know.

TA relied on his experience and textbook to draw his examples from. He considered the prescript of the Annual Teaching Plan which guided him on what to teach; therefore the

curriculum constrained his choice and use of examples. He considered his learners' capabilities (*strengths and weakness of two groups*). Lastly, his response *"start with simple examples in all topics"*, concur with Rowland (2008) and, Zodik and Zaslavsky (2008) postulations that s*tarting with simple or familiar examples* increase the level of complexity for the understanding and acquiring the concept.

I infer that TA's choices of examples in the planning phase were determined by curriculum stipulations, his learners' capability and sequencing of the examples. These factors also made his variation of examples possible.

5.4.2 TA's rationale on the use of both planned and spontaneous examples

I After $y = x^2$ and $y = 2x^2$, you chose and used $y = x^2$ and $y = -\frac{1}{2}x^2$. You seemed to have difficulty in explaining the width of the graph of $y = -\frac{1}{2}x^2$?

TA Yes (laughing).....I should have taken $y = \frac{1}{2}x^2$. I was hastening to show the reflection on the line *y = 0.* I was doing revision so learners know this.

I You decided to change $y = \frac{1}{2}x^2 - 2$ to $y = -\frac{1}{2}x^2 - 2$.

TA This problem ($y = \frac{1}{2}x^2 - 2$) will have confused learners because we have positive halve and this one half is negative (see scenario 3). (Pause). I wanted, wanted to compare +2 and -2 …. the vertical shift.

TA's assertions that *I should have taken* $y = \frac{1}{2}x^2$ and he was *"hastening to show the reflection"* suggest that his choice and use of examples were not properly varied and sequenced to address parameter *a*. I infer that by sequencing and varying examples support the teachers' explanation which is consistent with increasing the level of complexity of the content and what is necessary to be discerned from the concept. Secondly, TA's rationale of changing the planned example $y = \frac{1}{2}x^2 - 2$ to $y = -\frac{1}{2}x^2 - 2$ suggests that he realised the "limitation" of his planned example

(Zodik & Zaslavsky, 2008) therefore he made an attempt to avoid using a planned example which might *"confuse the learners"*.

I I see, if you have to change a planned example or add another one in the moment of teaching, what will inform your decision?

TA Learners may not understand, I add another example that is easier for them. Sometimes example that you have planned may confuse learners like this one (pointing to $y = \frac{1}{2}x^2 - 2$

I Where do you take them from?

TA From the textbooks, I create others. It depends on the topic, other topics you can't take examples haphazardly or from your head.

TA's responses indicate two aspects. The first aspect dealt with his assertion that he adds an example in the moment of teaching because it *is easy* or a *planned example may confuse the learners*. It concurs with what Zodik and Zaslavsky (2008) posit, *viz*. that spontaneous examples are taken when the teachers realised the limitation of their planned example. Secondly, Goldenberg and Mason (2008), and Zodik and Zaslavsky (2008) concur that teachers' recall of relevant examples in the moment of teaching are determined by their personal example space. TA's assertion suggests that his spontaneous examples are derived from both experience (*I create others*) and the textbooks (which is within his immediate reach). Teachers' example spaces therefore illuminate their *elements of specialised professional knowledge* (Zaski & Leikin, 2008; Sinclair et al., 2011).

I You used table of values to plot these parabolas (pointing on examples in scenarios 1 – 4) and you used global plotting in this one (scenario 5). Why?

TA Learners understood the table method. Actually the table method is basic method before substituting values into the formula then to Cartesian plan. It will be waste of time to do a long method when an easier and effective one is available.

The move from algebraic symbols (table of values and ordered pairs) to global sketching of the same function entails both construction and interpretation actions (Leinhardt et al., 1990). The claim concurs with what Leinhardt et al. (*ibid.*) posit about the sequencing under construction - that it can start from a pointwise to a global approach. This suggests that TA used global

sketching for learners to have easy access to the knowledge and skills of constructing a quadratic function.

I Why you left planned examples $y = x^2 - 8x + 16$ and $y = x^2 + 3x + 5$ and gave it to learners as Homework not classwork as planned?

TA I was doing vertical and horizontal shifts**.** $y = x^2 - 8x + 16$ does not have vertical shifts, other TP forms, you have seen that it didn't have it. I was also doing the completing of the square and wanted to compare TP forms with no vertical shifts and those that have both (meaning vertical and horizontal shifts). Learners must know the difference.

TA varied his examples based on critical features, that is, horizontal and vertical shifts. He deferred $y = x^2 - 8x + 16$ and $y = x^2 + 3x + 5$ which were classwork exercises to homework. The former example did not have vertical shift whereas the latter did have both vertical and horizontal shifts. By varying the two, he wanted learners to discern discrimination based on the squared binomial and non-square binomial. His rationale indicated that TA's examples drew learners' attention to the relevant features (Zodik & Zaslavsky, 2008).

I infer that TA's experience was critical for deciding on the use of examples in the moment of teaching. His knowledge of learners also determined the choice and use of appropriate examples that minimise or eradicate learners' confusion in the moment of teaching.

5.4.3 TA's knowledge of teaching and learning functions

I What do you find difficult for the learners to understand in the school functions?

TA Learners find working with inequalities between graphs to be difficult *(reading values between two graphs)*. (Pause)….I think also deriving equations from the graphs.

I According to your experience, which concepts or topics you think are important for teaching and learning functions.

TA All the topics in the curriculum are very important.

I In which manner?

TA Functions are important for teaching how to sketch graphs … to interpret information from the graphs like in other graphs of statistics.

Both reading the values between the graphs and deriving equations from the graphs are interpretation actions. Therefore TA's learners find interpretation action to be difficult. The

question papers discussed in chapter 3 valued interpretation actions more than construction ones. Secondly, TA's choice and use of examples are constrained by the curriculum.

I infer that TA valued interpretation actions more than construction ones although his examples that required interpretation did not facilitate in-depth rigour.

5.5 Findings

TA employed a substantive number of start-up and reference examples in his teaching. Only one example was classified as model examples. He sequenced examples of a concept with examples for doing exercises in order to develop concept understanding and consolidation thereof by procedural fluency. The examples of doing examples done by learners were more challenging than those by the teacher. The statement is supported by depth of reference examples done by the teacher against those done by the learners. Learners have also done the model example.

TA varied his examples for the learners to discern what critical features against what remained invariant one. Sequencing moved from simple/familiar cases to the complex one. Sequencing in construction activities, firstly, moved from a pointwise to a global approach. Secondly, it moved from matching parabolas together in terms of shape and lastly he matched parabolas with linear functions. In interpretation activities, sequencing moved from symbolic representations to graphical ones where meaning was gained on translations of algebraic cues and reading of the values from the graphs.

TA chose and used spontaneous example to avoid using a planned example that may *"confused the learners"*.

There were a substantive number of examples that paid more attention to construction than interpretation of the graphs that were chosen and used by the TA during classroom interactions. The previous externally set question papers analysed in Chapter three reflect more interpretation-seeking questions than construction ones. No non-routine or uncommon case of mathematical examples was observed in the lesson plan and/or during classroom interactions.

TA's choice and use of examples were constrained by the curriculum (ATP) hence his examples were consistent with the prescription of the curriculum.

The length of the period provided TA with enough time to complete his planned examples of the day. The time and use of transparencies facilitated TA's explanation and pacing of the learning objectives.

5.6 Conclusion

TA's choice and use of *function* examples addressed examples of a concept and examples for doing exercises. The lesson plan (with transparencies) was followed as planned with only three exceptions. He omitted one example in the moment of teaching, which might have "*confused the learners*". In terms of the second exception, he changed the planned examples because it could have not provided variation for understanding of an object of learning/concept. Thirdly, he deviated from the lesson plan when the planned examples were intended for classwork but deferred them to homework because the examples were over-emphasising squared binomials.

There were no learner-generated examples (LGE) and this was due to, as Watson and Shipman (2008) posit, TA's teaching strategies which might not accommodate LGE.

The teaching (re)sources available at TA's school assisted him to complete the task of the day and to logically present the lesson.

The next chapter, chapter six, analyses Teacher B's (TB) choice and use of examples. I followed the same approach of analysis as outlined above in the introduction of this chapter (5.1).

Chapter 6: Data analysis of Teacher B

6.1 Introduction

This chapter analysed data collected from teacher B (non-fee school teacher). The same approached used to analyse TA is applied to TB in order to provide consistence and compared them on the same items and processes undertaken by them. The processes and items observed in the classes are captured in Table 5.1.

There were two days used to observe TB but the number of observed classroom lessons was four lessons. In each two days observed, two lessons were observed in actual time allocated in the time-table and another two in the long afternoon classes. The long afternoon classes were meant for Mathematics, Physical Science, Accounting and English First Additional Language classes where teachers are supposed to teach, do remedial work and/or supervise studies.

6.2 Part A: Teacher B's choice and use of examples

6.2.1 Looking across TB's lessons using examples of a concept, examples for doing exercises and types of examples on choice and use of examples

6.2.1.1 Lesson 1 (60 minutes)

Scenario 1: The lesson started with marking the homework of the previous day (cf. Appendix F). Three learners did the corrections on the chalkboard. There were three sets of examples and each set had two examples. Each set of examples was sketched on the same system of axes using the table of values. The first set of examples was $y = x^2$ and $y = 2x^2$, the second set was $y = x^2$ and $y = \frac{1}{2}x^2$ and the last set was $y = x^2$ and $y = -x^2$. The corrections took about 10 – 12 minutes (see fig. 6.1).

He recapped by explaining that if $a > 0$ then the arms of graph $y = x^2$ "get nearer to the y- axis". He demonstrated by pointing at the graphs of $y = x^2$ and $y = \frac{1}{2}x^2$ that if $0 < a < 1$ then the "$y = x^2$ opens wider". If a < 0 then the parent graph "reflect along the *x-axis*".

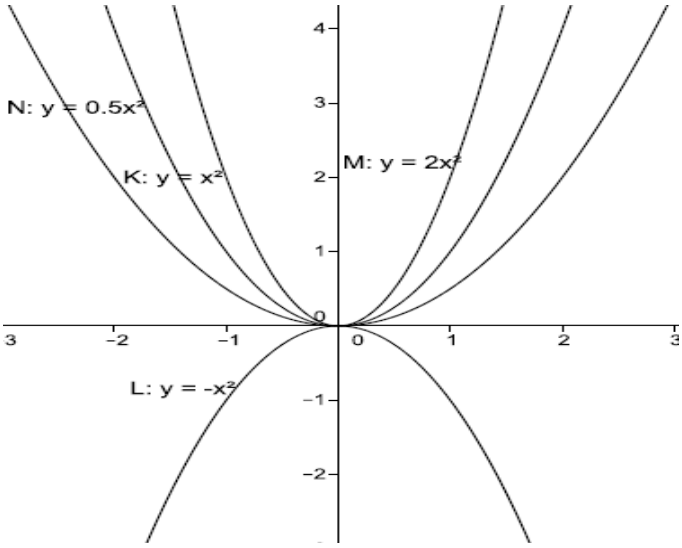

Fig. 6.1.

Interpretation: The three sets of examples were start-up examples because the learners are required first to understand parabola. The learning objective, though, was not written on the lesson plan and on the chalkboard, and had been the significance and effects brought by parameter *a*. TB's explanation generalised the effect brought by different values of parameter *a* therefore he introduced that generalisation of parameter *a*. These examples were examples of a concept although they were written as homework exercises by learners. They made the facilitation of the understanding of parabola possible. When probed about whether he taught learners the significance of parameter *a* and what he wanted learners to do in the previous day's homework, TB said he instructed learners to sketch the parabolas and he did not teach them. The exercises were sketched using tables of values, ordered pairs and translating order pairs into the system of axes, therefore there was movement of symbolic representation to graphical ones (Even, 1998).

Scenario 2: TB wrote $y = ax^2 + q$ and explained that the graph was the same as "$y = x^2$ which has move *q* units". Just beneath the formula, he wrote: the effect of parameter *q*. He wrote the graphs of $y = x^2 + 1$ and $y = -x^2 - 1$ (which are named first set of examples for the purpose of clarity) with corresponding sketched graphs (cf. fig 6.2) beneath the formulae. No table of values was used instead TB made points on the Cartesian plane and sketched the graphs, global sketching was introduced. He explained that in the first equation, $q = 1$ and $a > 0$ therefore the graph had "moved 1 unit up and is concave up". In the second equation $a < 0$, "the graph faces

down" and $q = -1$ therefor the "graph moves 1 unit down". Secondly, he demonstrated (by pointing at +1 and -1 on the equations and corresponding graphs) that "the value of q becomes the y value (ordinate)" in the TP that is $(0;q)$. He further elaborated that "if a and q" have the "same sign the graph will not touch the *x-axis*" (see fig. 6.2.). He probed learners to observe that in two given examples. This is a special proposition which I classify as special content (Adler & Venkat, 2012) and which does not appear in CAPS document (DoE, 2011) and textbook (Bradley et al., 2012) (see analyses in Chapter three).

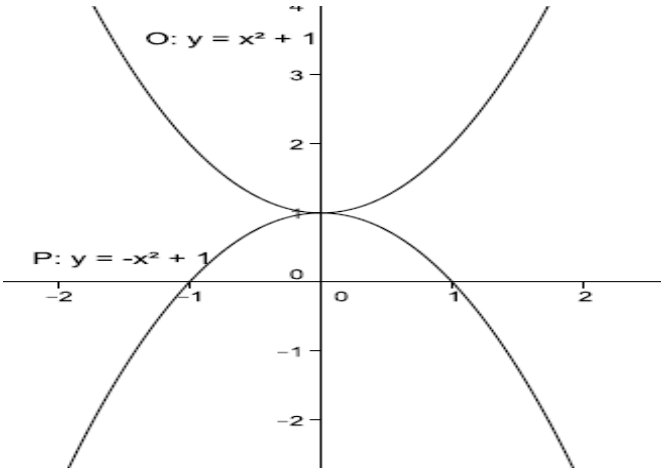

Fig. 6.2.

The second set of examples was $y = x^2 - 1$ and $y = -x^2 + 4$. He elaborated that "signs of a and q are not the same" therefore each parabola will have "the *x –intercept"*. To get *x-intercept*, $y = 0$ and solve for x.

$x^2 - 1 = 0$

$(x - 1)(x + 1) = 0$

$x = 1$ or $x = -1$

$-x^2 + 4 = 0$

$x^2 - 4 = 0$

$(x - 2)(x + 2) = 0$

$x = 2$ or $x = -2$

The two graphs were sketched to ascertain what the teacher was demonstrating. He elaborated on the values of parameters *a and q* in terms of "reflection along the *x-axis*" ($a < 0$) and translation (vertical shift brought by parameter q) for both graphs (cf. fig. 6.3). Lastly, he explained that the sign of a determines "the maximum/minimum value the graph will have" and q is the ordinate of TP on t*he y-axis*. He demonstrated by showing (in $y = x^2 - 1$) that $a = 1 > 0$ and $q = -1$ then the graph has minimum TP and minimum value is at $q = -1$ (cf. fig 6.3). He probed learners about

the TP and its value in $y = -x^2 + 4$. Learners responded that "it has maximum TP" (because $a < 0$) and "maximum value is at y = *4*".

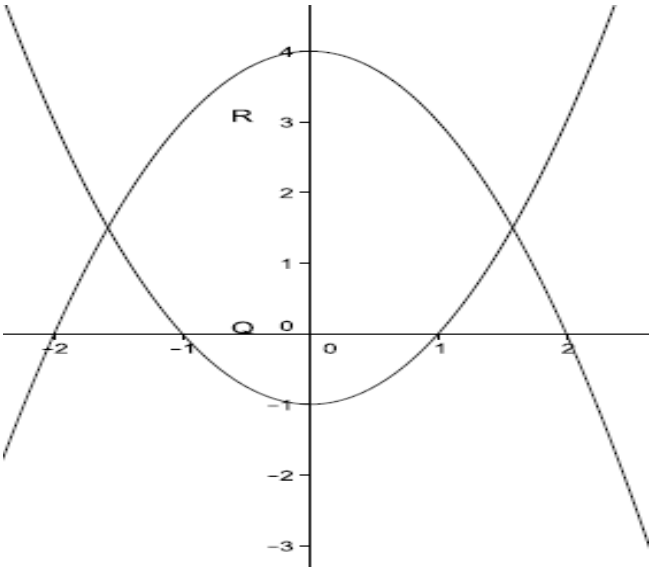

Fig. 6.3

Interpretation: The first set of examples ($y = x^2 + 1$ and $y = -x^2 - 1$) were start-up examples because they introduced the effect of parameter *q*. They develop the understanding of the generalisation of parameter *q*, the minimum and maximum turning points and special propositions. The special content dealt with signs of parameters *a and q* which may develop the understanding and procedural cue when constructing and interpreting parabola of the nature $y = ax^2 + q$.

The second set of examples ($y = x^2 - 1$ and $y = -x^2 + 4$) was reference examples because the effect of both *a and q* was repeated when *a and q* were assigned different values in terms of signs. Secondly, the examples ascertained the understanding of special proposition and minimum/maximum values introduced by TB in the first set of examples. Thirdly, global sketching was done to make its acquisition possible. Therefore, these examples were examples for exercise. They were chosen and used to make the consolidation of understanding a parabola possibly based on the skills and knowledge stated. The four skills were possible to be experienced simultaneously but discerned in a mutually exclusive way (Marton & Pang, 2006).

6.2.1.2 Lesson 2 (70 minutes)

Scenario 3: The lesson took place in the long afternoon and it took more than an hour.

Learners were given the seatwork (classwork) on the following: TB wrote them in the piece of paper.

1. Describe what happened to the graph of $y = x^2$ to be the graph of $y = -x^2 + 1$. Draw the graph.
2. Draw the graph of $y = -x^2 + 2$ and $y = -x^2 - 2$. Show the *interception* on axes and TPs.
3. Consider the graph of $y = x^2$ which has been moved three units up and two units to the left.

3.1. Sketch the graph

3.2. Write TP

3.3. Write y- intercept

The teacher went around and assisted learners who were struggling. The seatwork was written for 30 minutes and afterwards corrections were done on the chalkboard by learners.

There was interesting whole class debate on exercise example 3. Most of the learners, seemingly, got 3.1 (sketch the graph) correct (cf. fig 6.4 below) but they did not agree on the rest of the other exercises. The teacher worked out all exercises with learners where he probed and demonstrated how exercise example 3 was to be solved. Although confusion was observed from other learners, the teacher assured them that the next examples would deal with horizontal and vertical shifts. Learners were given ten minutes to do corrections in their classwork books.

Interpretation: The first two exercises were reference examples because they repeated the outcomes done in lesson 1 (scenario 2) (description and generalization of both parameters *a and q*, global sketching, intersection on axes and turning points). They were also examples for doing exercise because they made the enhancement of understanding the parabola of the nature $y = ax^2 + q$ and mastery thereof possible.

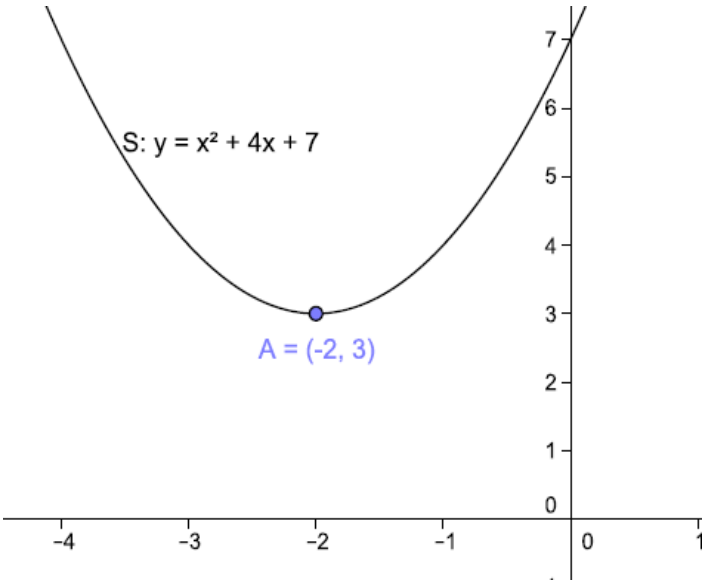

Fig. 6.4

The third exercise was a start-up example which introduced both vertical and horizontal shifts of the parabola. It was an example of a concept because it required learners to understand both vertical and horizontal shifts in a mutually exclusive way. Some of the learners got 3.1 (sketch the graph, cf. fig. 6.4) and 3.2 (write the TP) correct because, I infer, they required the translation of algebraic symbols to graphical representation which learners might have done them. TB introduced the next topic (horizontal and vertical shifts) where learners discovered some of the objects of learning for themselves.

Scenario 4: TB wrote horizontal shift: $y = (x - p)^2$ on the chalkboard. He wrote examples $f(x) = x^2$, $g(x) = (x - 2)^2$ and $h(x) = (x + 2)^2$ and they appeared in the lesson plan. TA used table of values, where $x \in \{-2;-1;0;1;2\}$ was substituted into the formulae to get ordered pairs. The three graphs were plotted by a pointwise approach. He labelled the graphs *f(x), g(x) and h(x)* which were on the same system of axes (cf. fig 6.5 below). He elaborated that the "*g(x) is f(x)* which is moved two units to the right" and "*h(x) is f(x)* which is moved two units to the left".

He demonstrated the TPs of *g(x)* and *h(x)* from the system of axes and TPs were *(2;0)* and *(-2;0)* respectively. The TPs were also the "*x-intercepts*". He proved *TPs, x-intercepts* and *y-intercept* in algebraic procedures. TB demonstrated that if $p > 0$ then the" parabola shifts *p* units to the left" and if $p < 0$ then it "shifts *p* units to right from the graph of $f(x) = x^2$". He further asserted that the TP was on the *x-axis* and becomes *(p;0)*.

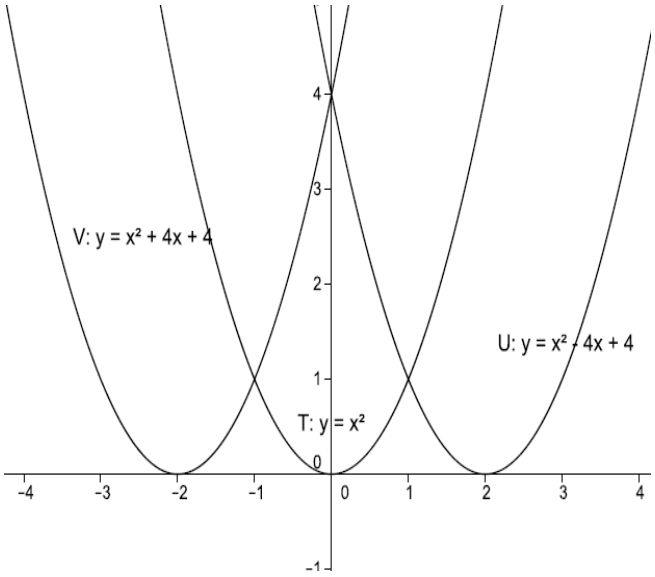

Fig. 6.5.

Interpretation: The graphs of *g(x) and h(x)* were start-up examples under the effect of parameter *p* (horizontal shifts). TB addressed the generalisation of the effect of parameter *p* in his usage of the two examples and the understanding of the effect of *p* was possible to be developed. The table of values and ordered pairs were used to plot the graphs. I infer that TB resorted to a pointwise approach in order to manifest the understanding of the effect of the parameter *p.* where learners may follow his explanation since others had a problem in the exercise examples above.

***Scenario 5*:** TA wrote horizontal and vertical shifts; $y = a(x - p)^2 + q$ (lesson plan) He explained while writing on the chalkboard that the graph of "$y = a(x - p)^2 + q$" is the graph of "$y = x^2$" which "has moved horizontally by *p* units and vertically by *q* units" and "TP = *(p;q)*". He went back to exercise example number 3 (above, see scenario 3) and where the equation was $y = (x + 2)^2 + 3$. He algebraically converted the TP form of quadratic equation to its standard form: $y = x^2 + 4x + 7$. The *y – intercept* was *y =7.* TB pointed in the sketched graph that there was "no *x – intercept"* therefore the quadratic equation had "*non-real roots"* and explained that "if a quadratic equation does not have roots, it means that there are no intersection on *x-axis"*. He demonstrated by using the quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(7)}}{2(1)}$$

$$x = \frac{-4 \pm \sqrt{-12}}{2}$$

Interpretation: Although example $y = (x + 2)^2 + 3$ was an exercise example in scenario 3 (cf. fig. 6.4), it was a start-up example because the vertical and horizontal shifts, the TP based on values of *p and q*; and intersections on axes were discussed to make their understanding possible. The converting of TP form to standard form was also explained to the learners. The link between non-real roots and the graphical representation were discussed for the first time.

Scenario 6*:* TB wrote *forms of parabola* and it appeared in TB's lesson plan.

- standard form: $y = ax^2 + bx + c$
- turning point: $y = a(x - p)^2 + q$
- x-intercept: $y = a(x - x_1)(x - x_2)$

TB explained that "converting the standard form to TP, we factorise". "If factorisation does not work, complete the square". He took $y = 2x^2 + 4x - 6$. Table 6.1 below depicts procedure of completing the square.

Table 6.1: Procedure of completing the square

Procedure	Explanation
$y = 2(x^2 + 2x - 3)$	Take 2 as common factor
$y = 2(x^2 + 2x + 1 - 3 - 1)$	Add half the coefficient of x and square it, then subtract it to maintain the value of the expression.
$y = 2[(x + 1)^2 - 4]$	Factorise
$y = 2(x + 1)^2 - 8$	Multiply throughout by 2

Learners were given the seatwork from *Platinum Mathematics Grade 11* textbook, page 85, Exercise 2, numbers 1 and 4; and page 86, Exercise 3, numbers 2 and 3 (cf. Appendix C). The exercise examples in page 85 were similar to TA's on the same learning objective (converting from one form to another). He instructed learners not to do *x-intercept form* of quadratic function for Exercise 3 in page 86; instead, they should sketch the graphs separately in the system of axes. Learners were allowed to take the exercises as homework because the period was over.

Interpretation: Conversion from a standard form of quadratic equation to a turning point form by factorising (employing completing the square) was done for the first time; therefore the example was a start-up example. The example was an example of a concept because it demanded the understanding of that conversion from one form to another. TB did not take an example that demanded factorisation of squared binomial but implied that it should be considered first before factorisation by completing the square when converting standard form to TP form of a parabola.

6.2.1.3 Lesson 3 (30 minutes)

Scenario 7: The homework was not marked but deferred to the long afternoon class. TB wrote *working with standard form* on the chalkboard (appeared in lesson plan). He took $y = ax^2 + bx + c$ and completed the square and got $y = a(x + \frac{b}{2a})^2 + \frac{4ac - b^2}{4a}$. He compared the parameters from the turning point form $y = a(x - p)^2 + q$ with formula from standard form: "$p = -\frac{b}{2a}$ and is Axis of symmetry and also an *x-part* of TP , $q = \frac{4ac - b^2}{4a}$ and is *y-part* of TP". He took the example $y = 2x^2 - x - 6$ and did the following:

Axis of symmetry: $x = -\frac{b}{2a}$

$$x = -\frac{(-1)}{2(2)}$$

$$x = \frac{1}{4}$$

TP: Substitute value of AS in $y = 2x^2 - x - 6$ and also use $q = \frac{4ac - b^2}{4a}$. TP: $(\frac{1}{4}; -\frac{49}{8})$

y - intercept: y = -6

$$2x^2 - x - 6 = 0$$

x – intercept: $(2x + 3)(x - 2) = 0$

$$x = -\frac{3}{2} or x = 2$$

He then sketched the graph. Learners were given sketch $y = x^2 + 4x$ and had to show all interceptions, TP and AS. The exercise example was not corrected because the period was over.

Interpretation: The comparison between the standard form and TP form in terms of parameters and formulae suggests that the example was a reference example. The example in scenario 6 demanded converting standard form to TP form by factorisation (completing the square) whereas this one dealt with comparing parameters with formulae from standard TP form $y = a(x + \frac{b}{2a})^2 + \frac{4ac - b^2}{4a}$ which was not written in parameters *p and q*. Therefore this example in scenario 7 manifested further the understanding of converting the standard form to TP form by aligning the formulae with their similar parameters of TP form. Through this association between the two related forms, procedural cues were made possible. The concept understanding (association between the two forms) and procedural fluency (converting by factorization) were experienced in duality (Tsamir et al., 2008).

6.2.1.4 Lesson 4 (80 minutes)

Scenario 8: The lesson started with the marking of page 85, Exercise 2, numbers 1 and 4. The Exercise 3, number 2 and 3 on page 86 were second set which was marked. The last exercise example was: *sketch* $y = x^2 + 4x$. Show all interceptions, TP and AS. The corrections took about 35-40 minutes. The learners did these example exercises on the chalkboard while the teacher was moving around controlling the work by picking learners randomly.

He took an example from the textbook, page 89, Exercise 4 number 1. This example did not appear in the lesson plan. The example was similar to the one done by TA:

1.1. Sketch $f(x) = -x^2 + 4x + 12$ and $g(x) = 4x + 8$ on the same system of axes.

1.2. For which value(s) of x is $f(x) = g(x)$?

1.3.For which value(s) of x is $f(x) \geq g(x)$?

The procedure and explanations are illustrated in Table 5.2.

Learners were given the homework on page 89, Exercise 4 number 2.

Interpretation: The example exercises in Exercise 2 (p. 85), Exercise 3 (p. 86) and $y = x^2 + 4x$ were all examples for doing exercises because they demanded the practice, therefore mastery of procedure how a parabola in different forms can be sketched and how the properties can be translate to global sketching. Examples in Exercise 2 (p. 85) were reference examples because they demanded to link the understanding of converting standard forms to TP form and procedures to do that. Example $y = x^2 + 4x$ was also a reference example because it demanded to repeatedly (from previous example done by TB in scenarios 6 & 7) link the understanding of global sketching by translating properties of parabola from algebraic symbols to graphical representation.

Examples in Exercise 3 (p. 86) and Exercise 4 (number 1, p. 89) were model examples for a parabola because they required the summary of a parabola where axis of symmetry, intersection on axes, minimum/maximum values, TPs, converting from one form to another and the description on shifts were probed. All the said aspects define a parabola as a unique function from other functions. Examples in Exercise 3 (p. 86) were examples for doing exercises because they demanded the mastery of parabola by practice while understandings of all properties of parabola are dealt with. Exercise 4 (number 1, p. 89), although done after Examples in Exercise 3 (p. 86), was an example of a concept because it matched two different functions (parabola and linear function) (Leinhardt et al., 1990) and it dealt with a flexible move between symbolic and graphical representations as espoused by Even (1998).

The following Table 6.2 summarises TB's lessons:

Lesson	Learning objective	Examples of concept	Examples for exercise	Analysis
1	Effect of parameter *a*:	$y = x^2 and y = 2x^2$ $y = x^2$ and $y = \frac{1}{2}x^2$ $y = x^2$ and $y = -x^2$ (start-up)		Table of values, ordered-pairs and point-wise sketching of the functions were done. The values of parameter *a* were varied in order to generalize the effect of *a*.
	Vertical shift: effect of parameter a and *q*	$y = ax^2 + q$ *and* $y = x^2$ $y = x^2 + 1$ and $y = -x^2 - 1$(start-up)	$y = x^2 - 1$ and $y = -x^2 + 4$) (reference)	No table of values. Global sketching, minimum/maximum TP *(0;q)* were introduced. Generalizations on the effect of both *a and q (special content)*.
2	Repeat of vertical shift: effect of parameter *q* (scenario 2). Introducing vertical and horizontal shifts (example exercise 3)	3. Consider the graph of $y = x^2$ which has been moved three units up and two units to the left (start-up)	1. Describe $y = x^2$ to be the graph of $y = -x^2 + 1$. Draw the graphs. 2. Draw the graph of $y = -x^2 + 2$ and $y = -x^2 - 2$. Show the *interception* on axes and TPs. (reference)	1 & 2: Generalization of effect of *q*, global plotting, TPs. Examples of understanding and mastery of effect of *q*. 3. Global approach, generalization of both parameters *p and q*, TP *(p,q)* and intersection on axes. Example for concept understanding and procedural fluency.
	Horizontal shift: effect of parameter *p*.	$f(x) = x^2$, $g(x) = (x - 2)^2$ and $h(x) = (x + 2)^2$ (start-up)		Pointwise approach, generalization of parameter *p*. TP and intersection on axes. Example of a concept for concept understanding.

	Vertical and horizontal shifts	Example exercise 3 (scenario 3): Consider the graph of $y = x^2$ which has been moved three units up and two units to the left (start-up)		Conversion from TP form to standard form algebraically, TP *(p;q)*, linking algebraic symbols with graphical representation (non-real roots, intersection on axes).
	Forms of parabola	*1. $y = 2x^2 + 4x - 6$ (start-up).*	2. p. 85 Exercise 2, no. 1 & 4. p.86, Exercise 3, no. 2 & 3 (reference) (cf. Appendix C).	1. Convert from standard form to TP another, (factorization by completing the square), example of a concept therefore concept understanding 2. Examples for exercise for converting from one form to another (enhancing the completing of the square), description of shifts (generalization and testing conjectures). Examples for concept understanding and procedural fluency.
3			*From $y = ax^2 + bx + c$* to $y = a(x + \frac{b}{2a})^2 + \frac{4ac - b^2}{4a}$.	Comparing standard form and TP form in terms of parameters and formulae, The concept understanding (association between the two forms) and procedural

			$p = -\frac{b}{2a}$ and $q = \frac{4ac-b^2}{4a}$ (reference)	fluency (converting by factorization) were experienced in duality
4			Marking of p. 85 Exercise 2, no. 1 & 4. (reference) p.86, Exercise 3, no. 2 & 3 (model) and *Sketch* $y = x^2 + 4x$. Show all interceptions (reference)	Examples for practice therefore mastery of procedure how a parabola in different forms can be sketched and how the properties can be translated to global sketching.
	Matching of quadratic function and linear function	$f(x) = -x^2 + 4x + 12$ and $g(x) = 4x + 8$ (Model)		Matching two functions and reading the values (interpretation) when compared. Example for concept understanding.

Discussion

There were six start-up examples (scenarios 1- 6), six of reference examples (scenarios 2, 3, 7 & 8) and two were model examples (scenario 8) (cf. Table 6.2). There were equal numbers of start-up examples and reference examples. This implies that there was a balance between TB's chosen and used examples that demanded concept understanding and procedural fluency of a parabola. The model examples suggest that TB chose and used examples for learners to have a deeper understanding of parabola.

Research question 1: What examples did TB choose in the planning phase and during the lesson presentation?

TB chose start-up examples in all lesson introductions of a new learning objective of parabola (effects of parameter *a*, *p* and *q*; factorising by completing the square/converting standard form to TP form of a parabola) (cf. Table 6.2). In all start-up examples, the examples made concept understanding possible. TB's chosen and used examples in scenario 2 is a special content which required learners to have a deeper understanding of parabola in terms of being aware of signs *(+/-)* of *a and q* for TPs and graph sketching (graphical representation) (cf. fig. 6.2 & fig. 6.3).
All reference examples chosen and used by TB were examples for doing exercises (cf. Table 6.2). TB chose and used reference examples in two ways. Firstly, his reference examples required the outcomes of start-up examples (that is concept understanding) to be repeated in order to enhance procedural fluency and consolidate the concept understanding (see scenarios 2 & 4). Secondly, reference examples were chosen and used to link two learning objectives together (see scenarios 3, 5 & 6). For examples, in scenarios 3, 5 and 6 he successively linked vertical shift, vertical and horizontal shifts and conversion of forms of quadratic equations together. It suggests that TB's chosen and used reference examples made concept understanding and consolidation thereof to be possibly acquired in duality (Tsamir et al., 2008).

TB chose and used two model examples which differed in scope (cf. Table 6.2). The first exercise examples [Examples in Exercise 3 (p. 86)] were examples for doing exercises which required the matching parabolas together where all properties that define a parabola were dealt

with. The second examples (Exercise 4, no 1, p. 89) were examples of a concept that required matching of two different functions (quadratic and linear functions) where the properties of each and flexible move between the representations were focus areas.

My reflection on TB's choice and use of examples was based on the role and use of examples and his way of teaching them. TB had two approaches when choosing and using start-up examples. The first approach was a conventional approach where he chose and used start-up examples that he personally did and explained to the learners to facilitate the understanding of a parabola. The second approached (see scenarios 1 & 3), TB chose start-up examples that encouraged learners to discover and develop on their own the understanding of a parabola. TB explained the concepts after the learners were required to develop concept understanding to manifest that concept understanding. Similarly, learners' algebraic proficiencies were possible to be manifested in his explanations and demonstrations. In so doing, TB infused examples of a concept with example for exercises. I classified such examples (scenarios 1 & 3) as examples of a concept because they introduced a concept and made the concept understanding to be possible.

TB's start-up examples were examples of a concept therefore facilitated concept understanding. The reference examples were examples for exercise therefore facilitated both concept understanding and consolidation thereof through procedural cues. One would expect that reference examples should follow start-up examples but in scenarios 1 & 3, reference examples were coupled with start-up examples. Like in TA, there were more examples for exercises than examples of a concept. The same reason stated for TA holds true for TB.

The choice and use of two model examples, although differing in scope, implies that TB facilitated the understanding of quadratic functions in different focus areas; therefore learners were able to understand and acquire quadratic functions in those foci.

6.2.2 Looking across the TB's lessons using Rowland's categories

The Rowland's categories were defined in Chapter 5 (see 5.2). This sub-section analyses and discusses TB's choice and use of examples using Rowland's categories.

Table 6.3: Analysis on TB's examples using Rowland's categories.

Lesson	Example of a concept	Analysis	Examples for doing exercises	Analysis
1	$y = x^2 and y = 2x^2$ $y = x^2$ and $y = \frac{1}{2}x^2$ $y = x^2$ and $y = -x^2$	Parameter *a* was varied. Three qualities of parameter *a* namely (namely, $a > 0$, $0 < a < 1$ and $a < 0$) each were compared with parent parabola to generalize the parameter for concept understanding and procedural fluency.		
	$y = ax^2 + q$ *and* $y = x^2$	Parameters *a and q* were critical features and varied; were experienced simultaneously without discrimination (Marton & Pang, 2006). TB's choice and use of those examples required generalization and understanding of special content.	$y = x^2 + 1$ and $y = -x^2 - 1$	*Parameters a and q* were critical features and varied, required understanding on generalization of parameters and special content; and procedural fluency thereof.
	Interpretation: ***1. Variation and Representation:*** The different values of parameter *a* were varied while the parent parabola $y = x^2$ remained invariant. TB's chosen and used examples made the generalization of the effect of parameter *a* to be possible. **Representations** were tables of values, ordered pairs and a one-to-one relation ultimately pointwise approach where the shape of the parabolas signified the generalization brought by parameter *a*.			

<table>
<tr><td></td><td colspan="4">2. In examples defined by $y = ax^2 + q$ (effects of a and q combined), the use of global sketching established two forms of generalization. The first one was generalization brought about by values assigned to parameters (vertical shift) and secondly, generalization brought by the same or different operational signs (+/-) on parameter a and q (special content) on intersection on axes. The construction (sketching of the graphs due to variation on parameters) and interpretation (what happens to the parabola when different values of parameters were used and operational signs) were used for generalisation. The instruction moved from symbolic representations (algebraic manipulation/procedural cues) to graphical ones.
3. Sequencing: The variation of parameters afforded the increase in degree of complexity from simple to complex examples among examples of parameters a and q. Sequencing in construction: The point-by-point approach was the first to be done from the formulae using tables of values and ordered pairs into the system of axes. The shape of parabolas illuminated the generalization of the effect of the parameter a. The global approach illuminated those two generalizations discussed in 2 above. Sequencing in interpretation: TB chose and used examples that matched parabolas with the same parameters (a and q) although values were varied. That accorded with the generalization of the parameters. Secondly, another action of interpretation, according to Leinhardt et al. (1990); occurred when translation of algebraic symbols (table of values, ordered pairs) to sketch graphs using pointwise approached was done. Thirdly, as cited by Leinhardt et al. (1990) and Even (1998) that a global approach facilitates interpretation of the graph. The sequencing moved from symbolic representations to graphical ones (Even, 1998).</td></tr>
<tr><td>2</td><td>3. Consider the graph of $y = x^2$ which has been moved three units up and two units to the left</td><td>The example exercise introduced parabola in the TP form $y = a(x - p)^2 + q$ where vertical and horizontal shifts were critical features. Conversion of TP form to standard form was also a critical feature to be discerned.</td><td>1. Describe $y = x^2$ to be the graph of $y = -x^2 + 1$. Draw the graphs.

2. Draw the graph of $y = -x^2 + 2$ and $y = -x^2 - 2$. Show the interception on axes and TPs.</td><td>The parameters a and q were varied to enhance the effect brought by changes in parameters and special content. The global approach illuminated the generalization of the parameter and enhancement of the special content.</td></tr>
<tr><td></td><td colspan="4">Interpretation:
1. Variation and representation in example exercise 1 & 2: the questions probed the formation of graphs from verbal representations to formulae to have a new parabola based on those descriptions. In example exercise 1, both parameters a and q were varied and they</td></tr>
</table>

	illuminated the global approach. Reflection along the *x-axis* and vertical move was intended to be discerned (learners' written work on the chalkboard). In example exercise 2, parameter *q* was varied while parameter *a* remained unchanged. The purpose was to enhance the discernment of the special content because special content illustrates what happens to -interception on *x-axes* when parameters *a and q* have the same or different operational signs (+/-). The global approach illuminated special content. Lastly, the example exercises made the discernment of generalization of parameters *a and q*, special content and; vertical and horizontal shifts to be possible. Global sketching depicted the verbal descriptions -so as to comprehend and master the concepts. 2. ***Sequencing:*** the sequencing occurred in matching a parental parabola with parabola of the form $y = ax^2 + q$ (example exercise 1), matching parabolas with the same parameters differing in values (example exercise 2) to matching parental parabola with parabola that demanded both vertical and horizontal shifts (example exercise 3). In exercise example 2, there sketching occurred after symbolic representations (determining the interception on axes, etc.) to graphical representations. There was an increase in the level of difficulty. ***Variation, sequencing and representation on example exercise 3***: In verbal description, vertical and horizontal movements of the parent graph of $y = x^2$ resulted in $y = (x + 2)^2 + 3$. The global image, determining the TP and getting the y- intercept of the new graph were required. **Sequencing**: There was movement from verbal description to graphical representations ultimately to symbolic representation (write TP, write y – intercept). The example exercise, therefore afforded interpretation action then construction. According to Even (1998) such move affords the learners to acquire different skills from when the move is from construction to interpretation. Sequence also increases the level of difficulty (hence learners had difficulty in answering subsequent questions).			
2	$f(x) = x^2$, $g(x) = (x - 2)^2$ and $h(x) = (x + 2)^2$	The parameter p $(p > 0$ *and* $p < 0)$ was varied to illuminate generalization of its effect on the parental function. A pointwise approach was used to demonstrate how variation of the parameters had informed generalization.		
	Variation, sequencing and representation: The values of parameter p were varied to illuminate that if $p > 0$, the graph of $y = x^2$ shifted to the right and if $p < 0$, the parent graph shifted to the left. Pointwise approach was used so as to discern that generalization of parameter p when p assume different value (+ and -). The significance of p in determining the intersection on the axes and TPs was also explained by TB. In the parabola of the			

	nature $y = (x - p)^2$, parameter becomes abscissa in the TP that is, TP*(0;p)* and it is also part of the intersection on the *x-axis*. The *y-intercept* was derived from converting the TP form of the parabola $y = (x - p)^2$ to standard form of the parabola where *constant c* is the value of ordinate (y) in the y-intercept.			
3	*1.* $y = 2x^2 + 4x - 6$	Converting standard form to its equivalent TP form by factoring through by completing the square. The procedural cues were varied in order to illuminate properties of a parabola in these forms of a parabola	2. p. 85 Exercise 2, no. 1 & 4. p.86, Exercise 3, no. 2 & 3 (cf. Appendix C).	Conversion from standard form to TP form by factorising through completing the square. Example exercises 1 & 4 were varied where *a = 1 and a = -1* to illuminate the degree of complexity in executing the completing the square. The description of vertical shift and horizontal one also brought about complexity.
	Interpretation: ***Variation, Sequencing and representation:*** Table 6.1 in scenario 6 provided algebraic cues in the converting the standard form to TP form. TB, in scenario 5, converted TP to standard form and in scenario 6 he moved from standard form to TP. By varying the algebraic cues on conversions of forms of quadratic functions from one form to another; these made discernment possible and increased the degree of complexity. The properties of a parabola (TP, axis of symmetry, intersection on axes) (see scenario 7) were compared from these two forms to illuminate how each form depicted properties. The symbolic representations (conversion between the forms and assigned values to the properties) were described in terms of vertical and horizontal shifts and what were TPs. The quadratic theory (use of discriminant: $\Delta = b^2 - 4ac$) was used to ascertain that if $\Delta < 0$ (roots are non-real) there are no x-intercepts. This links algebraic symbols with graphical representation therefore interpretation action was made possible (Even, 1998).			
	2. $f(x) = -x^2 + 4x + 12$	Two different functions with varying properties	1. p.86, Exercise 3, no. 2 & 3 and *sketch* $y = x^2 + 4x$. Show all	Convert from TP to standard forms, description of shifts and

	and $g(x) = 4x + 8$		interceptions.	properties of a parabola
	Interpretation: ***Variation, sequencing and representation:*** 1. The exercise examples in 1 (examples of doing exercise) were varied in terms of *a > 1* (in no. 2) and a *< 0* (in no. 3) so as to discern complexity brought about by those values. The sequencing started with a > 1 to a < 0 to increase complexity in algebraic manipulation in the conversion of TP to standard forms. The probed questions required that algebraic cues should illuminate the properties of the parabola and therefore shifts could be described. 2. In 2 above, two different types of functions, namely parabola and linear function were both algebraically manipulated and sketching of the graphs made the understanding of both functions possible. The two functions were sketched using global approach and matched together in terms of their shapes and properties. Interpretation occurs in two ways. Firstly, by translating algebraic symbols (variables) from symbolic representations by determining the properties (AS, TP, intersection on axes, gradient, etc.) to the sketched graphs. Secondly, by matching and analysing the relation between the two graphs when answering probed questions the two functions made concept and understanding and consolidation thereof to be possible.			

Discussion

In this sub-section, ***research question 2: How did TB use the planned and additional examples in his lessons?*** is answered.

TB varied his choice and use of examples to illuminate the discernment of critical features by varying the parameters. TB did this in two approaches. Firstly, TB discriminated between critical features from non-critical features so as to discern conceptual understanding of the learning objectives (for example, see scenarios 1 & 4). In these scenarios (1 & 4) the parent parabola remained invariant while parameters *a* (scenario 1) and *p* (scenario 4) were varied for conceptual understanding of effect brought by parameters (generalizations) and procedural fluency. Secondly, TB varied two critical features simultaneously (Runesson & Marton, 2002; Runesson, 2005) (for example, see scenario 2) as objects of learning to emphasise the learning objectives, namely, effect brought about by parameter and proposition of the special content. There were instances where examples of the same concept (e.g., example exercise, see scenarios 6, 7, 8) to increase the degree of complexity. Those examples illuminated the concept understanding and procedural fluency thereof.

Sequencing of examples chosen and used by TB was from easy/familiar examples to complex ones in terms of construction and interpretation actions undertaken to illuminate conceptual understanding and procedural fluency. In scenarios 1, & 4, TB chose and used examples that move from symbolic/algebraic representations (table of values, ordered pairs) to point-by-point plotting (graphical representations). Such moves between representations, as cited by Even (1998), facilitate understanding of the definition, conceptual understanding of the learning objectives (that is, the effects/generalisation brought by parameters) and enhancement of construction action. The construction action moved from comparing parent parabola with the same parabolas that underwent variation in terms of parameters to illuminate both the effects brought by parameter. In scenario 8, construction action illuminated matching of two different functions (Leinhardt et al., 1990).

In scenarios 2, 3, 5, 7 & 8; TB chose and used a global approach for the sketching of the functions. Leinhardt et al. (1990) and Even (1998) posit that a global approach facilitates interpretation of the graphs as we translate algebraic symbols (values of intersection on axes,

TPs, axis of symmetry) to the graphs. Sequencing moved from symbolic/algebraic representations to graphical ones. In scenarios 3 and 8, there was a flexible move from symbolic representation to graphical ones and back. Scenario 3 facilitated verbal descriptions to graphical and then symbolic representations. Those movements required construction based on verbal description and interpretation ensued in the process of translating verbal descriptions to graphical representations. Sequencing in interpretation action also moved from algebraic symbols, global approach to reading of the values from the sketched graphs (see scenario 8).

TB sequenced his examples through-out the lessons by adding parameters from $f(x) = x^2$, $f(x) = ax^2$, $f(x) = ax^2 + q$, $f(x) = a(x - p)$ to $f(x) = a(x - p)^2 + q$ where he increases the degree of complexity in terms of adding parameters from parent parabola (see scenarios 1, 2, 3, 4, 7 & 8). Both construction and interpretation actions also increase in complexity in the process of translating algebraic symbols, verbal description to graphical representation and vice versa.

I reflect on TB's choice and use of examples based on Rowland's categories. The same focus done on TA's choice and use of examples based on Rowland's categories is applied. TB, firstly, varied parameters *a*, *q* and *p* by employing pointwise and global approaches to generalise the effects of these parameters (cf. Tables 6.2 & 6.3, lesson 1 & 2). TB translated symbolic representations to graphical ones and both interpretation and construction actions were facilitated (Even, 1998). Secondly, TB varied the operational signs on parameters *a and q* to facilitate the understanding of interception on axes (lesson 1, special content) and consolidated the understating of the special content (lesson 2). He employed both algebraic manipulations and graphical representations to manifest the special content. He sequenced his examples to increase the level of complexity (see lessons 1 & 2). Thirdly, he compared different forms of quadratic functions to illuminate the differences and similarities between these forms (lesson 3). He algebraically manipulated one form to another and globally sketched the graphs. Both symbolic and graphical representations were facilitated. A global approached was therefore used and both construction and interpretation actions were facilitated (Even, 1998). Lastly, in two model examples, TB matched a quadratic function with another and, in another model example he matched two different functions (linear and quadratic) together (lesson 4). I infer that

construction and interpretation actions were possible to be discerned and the levels of complexity increased. This implies that examples of a concept served to facilitate concept understanding and examples of doing exercises served to consolidate and master the concept understanding.

6.3 Part B: What were (re)sources for TB and his examples?

6.3.1 The curriculum document (CAPS)

Definition of a function and curriculum progression. TB chose and used examples from Grade 10 curriculum examples in scenarios 1-3 (DBE, 2011, p. 24). TB chose and used those examples to move from familiar examples done in grade 10 the previous year so as to illuminate the definition of a function throughout the classroom interaction and therefore facilitated progression. The one-to-one rule that defines a function, although it does not appear in Grade 11; was described by the use of formulae, table of values, ordered pairs and graphs. Secondly, CAPS stipulates that:

1 Revise the effect of the parameters *a and q* (from Grade 10) and investigate the effect of *p* on the graphs of the functions defined by:

1.1 $y = f(x) = a(x+p)^2 + q$

1.2 $y = f(x) = \frac{a}{x+p} + q$

1.3 $y = f(x) = a.b^{x+p} + q$ where $b > 0, b \neq 1$ (DBE, 2001, p. 32).

TB conformed to progression and the degree of difficulty of quadratic *functions examples* (in 1.1 above) from Grade 10 curriculum to Grade 11 (see scenarios 1-8, above). The degree of complexity of TB's chosen and used examples under parabola followed the sequence of examples from: $f(x) = x^2$, $f(x) = ax^2$, $f(x) = ax^2 + q$, $f(x) = a(x-p)$ to $f(x) = a(x-p)^2 + q$

Construction, interpretation and flexible move between representations: TB chose and used examples that generated graphs, generalised the effects of the parameters, identified the features/properties and applied them (see scenarios 1-8) (DBE, 2011, p. 12). TB chose and used two same and different functions in terms of shape, properties, restrictions and formulae as per CAPS stipulation (DBE, 2011, pp. 24 & 32).

TB's reliance on the prescription of the curriculum somehow did not constrain his choice and use of the examples. TB used the special content which did not appear in the curriculum or in the textbook (*Platinum Mathematics Grade 11,* Bradley et al., 2012). I infer that the special content was relevant to the teaching and learning of parabolas in Grade 11.

6.3.2 Official examination question papers

TB asserted that (see interviews in Part C) he drew some of the examples from the *exemplar question papers* (see examples in scenario 3). I reflect, firstly, on what content of *function examples* did those examples (in scenario 3) prioritised and valued. Secondly, I reflect on what content other examples of TB prioritised and valued against what content the externally set question papers (2012 externally set Grade 10 question paper and 2013 Grade 11 National question paper) had prioritized and valued. Like I reflected in 5.3.2 in chapter 5, these question papers valued and privileged interpretation action as more important than a construction one. They assessed flexible movement between verbal descriptions and algebraic symbols. The symbolic representations were assessed more than graphical representations therefore algebraic proficiency was enhanced.

Discussion

Construction: Firstly, TB used both pointwise (scenarios 1 & 4) and global (scenarios 2, 3, 5, 6, & 8) approaches to sketch the graphs. TB chose and used examples that elucidated global approach more than pointwise one. The question papers prioritized global approached. TB used both pointwise and global approaches for concept understanding, but global approach was used for both concept understanding and consolidation thereof. I infer that TB chose and used global approach to facilitate properties of quadratic functions the same as in the question papers (cf. Appendix A: Grade 10, number 4.2; Appendix B: Grade 11, number 9.1).

Secondly, TB matched quadratic function and linear function (scenario 8). The question papers prioritized matching of two different functions, that is, linear function and quadratic function (Grade 10, number 4.2) and quadratic function and exponential function (Grade 11, number 9.1). In scenario 8, the exercise example: *sketch* $y = x^2 + 4x$*, show all interception, TP and AS* was similar to examples in question papers (cf. Appendix A, number 4.2; Appendix B, number 9.1).

The sketching of functions where properties had to be shown was prioritized by TB and question papers to make the development and enhancement of construction of quadratic functions possible.

Thirdly, in scenario 3, TB chose and used, as he claimed, examples from exemplar question paper. The examples number 1 & 3 used verbal description to "sketch the graph". These examples did not appear in the question paper. The exemplar question paper probed the flexible move among verbal descriptions, symbolic and graphical representations. I infer that these skills/actions had made the development and enhancement of translating one representation to others possible for the learners to acquire construction of functions especially the quadratic in different representations.

Interpretation: TB chose and used examples that required interpretation actions by translating algebraic symbols, verbal and graphical representations (flexible moves between representations, scenario 3); matched quadratic functions together for generalising the effect of parameter and matched different functions (linear and quadratic) together to read values from the sketched graphs. The question papers assessed interpretation action by translating verbal descriptions into algebraic formulae, transformation (reflections and translations) and properties (asymptote, axis of symmetry, interception on axes) (Grade 10, numbers 4.1.8 & 4.1.9; Grade 11, numbers 8.1, 8.6, 9.4, 9.5 & 9.6) and matching two different functions together so as to read the values from the graphs (Grade 10, numbers 4.1.1 – 4.1.5, 4.1.7; Grade 11, numbers 8.1 & 8.2). I infer that TB chose and used *function examples* that were almost similar to the ones in the question papers and both examples had rigour.

6.3.3 Textbook

Examples of a concept: TB drew one example of a concept from the *Platinum Mathematics Grade 11* (Bradley et al., 2012). The example in the textbook was an exercise example (ibid., p. 86, exercise 4, number 1) and a model example but TB used it as an example of a concept. Textbooks used the examples to develop the understanding of quadratic and linear functions by matching them together so as to consolidate their understanding in terms of construction and interpretation of their properties. TB chose and used these examples for introducing and developing an understanding of those actions and skills as they were required in the textbook.

Examples for exercise: TB drew a number of examples for doing exercises from the textbook for exercise purposes. Examples that were drawn from the textbooks were all reference examples except the one in scenario 8 which was a model example on purpose; therefore they enhanced and developed the consolidation of concept understanding through procedural cues. These examples were depicted in scenario 6 (p. 85, Exercise 2, number 1 & 4; p. 86, Exercise 3, number 2 & 3) and scenario 8 (p. 89, Exercise 4, numbers 1). I infer that TB drew examples from the textbook as examples for exercise to consolidate the concept understanding.

6.3.4 Teaching and learning (re)sources

6.3.4.1 Teaching and learning aids

I What teaching and learning resources are available in your school?

TB Textbooks, workbooks and calculators**.**

I How did lack of other resources affect your teaching?

TB As you can see, most of children are relying on me for their support. Textbooks are not enough. I have to photocopy books and previous question papers.

Teacher B, according to my observation, only had textbooks as teaching and learning support material. The classroom did not have mathematical materials like charts, posters and placards. The reason might be the classroom was being used for different subjects because teachers rotate during period changes, not learners, or it might be lack of finances to purchase them. All Grade 10 - 12 mathematics learners use two textbooks, namely, *Platinum Mathematics* (issued by the North West Department of Education) and Siyavula (sponsored by the Shuttleworth Foundation). Lack of other teaching and learning aids constrained TB's choice and use of examples because TB's other learning objectives could not be mediated especially where he had to draw the graphs on the chalkboard. He deferred some of planned examples to long afternoon classes because much time was used for the drawing of the graphs. If he could have used other teaching and learning aids, I suggest that these could have been avoided.

6.3.4.2 Time

The six-day cycle time-table of nine periods a day was used to regulate teaching contact time. TB had nine periods of 30 minutes in each period in Grade 11. He therefore had 4.5 hours of teaching contact time. This is in compliance with CAPS (DBE, 2011).

I You are also having extra classes in the afternoon. Why do you have them?

TB It helps children to understand mathematics more; there is an outcry in our province that there is a problem in Maths so children must practice Maths under supervised condition because during the day I teach basically new topics and extra classes I use them for marking and practising.

I Can you say the examples you take during the day and extra-classes are the same?

TB They are the same but in the afternoon they are intensive. Afternoons I use more question papers than anything else.

School B had extra classes (they call them long afternoon classes) for Mathematics, Physical Science and Accounting from Grades 10–12. The purpose is to supplement teaching and learning time and improve learners' pass percentages in those subjects. TB had three extra classes of one hour each in a six-day cycle. TB used the long afternoon periods to cover the Annual Teaching Plan and was able to do corrections. The 30-minute period constrained TB's completion of the intended learning objectives of the day and some of the work (lesson 3) were deferred to the long afternoon. TB took more time than stipulated (that is more than one hour) in the long afternoon periods (see lessons 2 & 4). Most of the examples that were chosen and used by TB in the long afternoon periods were examples for doing exercises. There were instances where TB assisted learners who were struggling and randomly controlled learners' work/exercise books because of the limited time and class size (52 learners), therefore he might not know their progress. He managed to keep learners focused. I infer that the official time allocation of school B (length of time) constrained some of TB's chosen and used examples while the long afternoon classes afforded him an opportunity to do examples for doing exercises.

6.4 Part C: TB's considerations on his choice and use of examples, and his (re)sources

Research question 3: ***How did TB explain his choice and use of examples?*** is here answered (cf. Appendix I).

6.4.1 TB's rationale on planning of examples

I Where do you draw most of your examples when planning your lesson? Which resources are you using to draw you examples?

TB These ones (pointing to scenario 1, 2 & 4) are from me and the rest from different books.

I I mean in any topics?

TB Ok, eish, from the textbooks, myself sometimes question papers.

I Which examples in this planning are from you and which ones are from the question papers?

TB These ones (pointing to scenarios 1, 2 & 4) are from me. These ones (scenario 3 examples) are from the exemplar question papers.

I Why you have to use different resources to plan your lesson?

TB It is important to pick-up exercises that will make my children understand the topics well. When I use different sources I think of their background and how I can help them understand.

I What things do you consider when you prepare your lesson?

TB I consider different factors in time of preparation, level of understanding of my children that I teach as well as not forgetting to develop those learners who are weak. When I prepare, I prepare from easy to moderate and more difficult examples. I consider the time for teaching.

TB's response indicated that he was aware of different (re)sources and factors that were needed for his planning of the lessons. He indicated that he drew most of his examples for planning from "*his experience, textbooks and question papers"*. He also indicated that he had learners' consideration *(their background and how I can help them understand)* when planning the lesson. This concurs with Rowland's (2008) assertion that teachers consider many factors which include pragmatic and learners' affections. The assertions *"easy to moderate and more difficult examples)* concur with Rowland (2008) and, Zodik and Zaslavsky (2008*)* assertion that "starting with simple or familiar cases of examples" increases the level of complexity of a concept. The

consideration of *time (I consider the time for teaching)* is an important factor for planning a lesson because it regulates the number of examples and classroom activities.

I infer that, in his planning phase TB drew examples from his experience, textbook and question papers. He considered learners' capabilities and sequencing of examples to facilitate the increase level of complexity therefore variation of examples is implied.

6.4.2 TB's rationale on the use of both planned and spontaneous examples

I In these ones (pointing to scenario 2)? The textbooks do not show that?

TB Yes, that is true. What is important here is showing that *a and q* have different or same signs how to draw them even conjecturing.

I I must admit that was unique way of showing the effect of parameters *a and q*. Where did you really learn about this, because I checked in the textbooks and ATP, such information is not there?

TB (smiling) From ACE (Advanced Certificate in Education).

The examples in scenario 2 dealt with the effect of parameter *a and q*. TB demonstrated the vertical shift brought about by parameter *q* and the special content (Adler & Venkat, 2012) which is content that is overlooked in teaching functions. The latter content concurs with what Zodik and Zaslavsky (2008) referred as non-prototypical examples which are "either cases that are rather exceptional within mathematics or cases which are under-represented in teaching mathematics" (ibid., p. 177). Such cases were described by Zodik and Zaslavsky as *inclusion of uncommon cases* or non-routine example.

I Let us go back to sequencing of your examples. Why you chose and used exercise 3 in your classwork (scenario 3) because learners haven't done combination of vertical and horizontal shifts?

TB We called that discovery by learning, bro (meaning brother). Learners should be able to investigate and discover for themselves and I can say my children did well. Questions were leading them. By the way number 1 and 2 are vertical shifts so I was assessing them.

Exercise example 3 in scenario 3 served as introduction of the horizontal and vertical shifts. TB chose and used those examples for the learners *(my children)* to discover the content of the next topic (horizontal and vertical shifts) which dealt with generalising the effect brought by both *p and q*. Two aspects become salient for the choice and use of those examples. The first aspect is that sequencing of examples did not follow the stipulated pattern of sequencing from vertical shift to horizontal shift which is:

> *Revise the effect of the parameters a and q (from Grade 10) and investigate the effect of p on the graphs of the functions.* (DBE, 2011, p. 32).

Learners were expected to ***discover*** the horizontal and vertical shifts without being taught, of which a reflective method of investigating the effect of parameters was used to discover/investigate them. The second aspect is that he knew his learners (my children) would have attempted the exercise as a challenge. The discovery/investigation method is not uncommon in teaching and learning of Mathematics - Zaski and Leikin (2010) argue that teachers need to consider alternative strategies and methods when they plan a lesson and during instruction *so as to accommodate learners with different learning patterns* (my emphasis). The discovery method was more on teaching and learning strategy than choice and use of examples.

I $y=(x+1)^2-2$ appeared in your lesson plan but decided not to use it for vertical and horizontal shifts, instead you opted for $y=(x+2)^2+3$ which comes from exercise 3. Why?

TB This problem (pointing to $y=(x+2)^2+3$ exercise 3 from scenario 3 but dealt with in scenario 5) some of my children did not understand it well, you have seen that. I was elaborating more (on it) since this one (pointing to $y=(x+1)^2-2$, planned example) has same effect with this one. (Pause)….it is important to link from that.

I When did you realise that exercise 3 has non-real roots?

TB Another reason for taking this problem I have seen it during planning that it has non-real roots and no x-intercepts. Linking discriminant to the graphs (meaning parabolas). You see, I told them before that the discriminant tells us about the nature of x-intercepts.

TB was aware learners did not understand example exercise $y=(x+2)^2+3$ from scenario 3 (exercise 3). He changed the planned one [$y=(x+1)^2-2$] because both examples required the facilitation of the vertical and horizontal shifts. This concurs, to some extent, with what Zodik

and Zaslavsky (2008) found in their study, *viz*. that spontaneous examples are used when the teacher realised the limitation of their planned examples therefore teachers consider a range of examples of the same kind. TB realised that exercise example 3, $y = (x+2)^2 + 3$ was not thoroughly understood by some learners and it served the same purpose with the omitted planned example (on vertical and horizontal shifts) therefore he chose and used exercise example 3. He asserted that the exercise example "linked" one concept (quadratic theory: discriminant, nature of roots) with *x-interception*. I infer that TB chose and used this exercise example to attend to learners' errors. Zodik and Zaslavsky (2008) posit that teachers are considering the examples that attend to learners' errors when teachers are aware that learners are encountering difficulties.

I Why you instructed learners not do x-intercept form of quadratic function in Exercise 3 (see scenario 6)?

TB I don't want them to do many thinks at the same time they get confused

I You converted the standard form $y = ax^2 + bx + c$ to TP (pointing in the lesson plan, see scenario 7) form by completing the square after you have done $y = 2x^2 + 4x - 6$. Why did you sequence them in that order?

TB Bro, variables confuse my children. The standard form was meant to link its variables to the same parameters of parabola in TP. I can't do completing the square and linking features at the same time. If I could have done that first, my fear is that children could not comprehend the link.

TB's responses allude to two aspects of teaching and learning in mathematics. TB has the knowledge of his learners' learning style which he repeated twice *(I don't want them to do many things at the same time they get confused and variables confuse my children).* Although he sequenced his examples to increase the level of complexity of examples, he was aware not to create confusion in the process of sequencing his examples. Secondly, in his choice and use of examples, he sequenced his examples in the manner that paid attention to relevant features (Zodik & Zaslavsky, 2008) *(I can't do completing the square and linking features at the same time)* without creating confusion.

I Generally speaking, if you have to change a planned example or add another one in the moment of teaching, what will inform your decision?

TB Pause….many things bro. When my children do not understand I change or modify the problem. Sometimes you took a problem which is difficult and can take an easier one. You know, you see it in the class (laughing). Mnn, sometimes you have few minutes for the period to be over and you didn't do some topics you planned and want to give your children homework; I change some of the exercises.

I Normally, where do you draw examples in the moment of teaching?

TB Two things, my textbook is always with me. I use it, I am an experienced Maths teacher so I take them from my heart, I know them.

TB asserted that he considers learners' understanding and time constraints when he had to add or change the planned examples. TB asserted that he drew examples from his experience (*I take them from my heart*) and the textbook in the moment of teaching. Goldenberg and Mason (2008) and, Zodik and Zaslavsky (2008) concur that teachers' recall of relevant examples in the moment of teaching is determined by their example space. Personal example space is a function of ones' experience on the concept (Sinclair et al., 2011).

I infer that TB used his experience to choose and use examples that were non-routine (special content) and for deciding of an example in the moment of teaching. He used discovery method to accommodate different patterns of learning.

6.4.3. TA's knowledge on teaching and learning functions.

I What do you think learners are struggling with in functions?

TB Most cases learners struggle with interpretation of the graph like most of us we teach learners to draw and not to interpret the graph.

I What do you mean by interpretation? Can you take an example?

TB Like I can give a learner this graph:

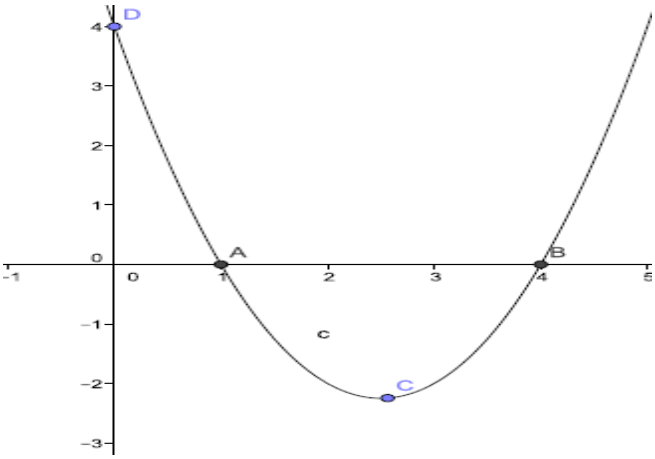

Fig. 6.6

I can give a learner this point here (A) and this point here (B). From here I can ask the learner to give me the formula, children can work from the formula to the graph not from graph to the formula.

I So learners have difficulty to move from graph to formula. Another one?

TB Ok, another one whereby we have two types of graphs like linear and parabola that I took (referring to exercise example in scenario 8). Children have to tell us if you compare linear and parabola at which values of *x* will one graph be greater or smaller than another one.

TB's assertion concurs with an observation made by Nachlieli and Tabach (2012) that the difficulty of learning functions is that functions are seen as algebraic formulae and graphs and not as a relationship between the two. The relationships attest to construction and interpretation actions. Even (1998) also posits that the way functions are taught and learnt does not address the flexible move between representations and this stymies other aspects of knowledge and skills associated with functions. Leinhardt et al. (1990) cited that interpretation action include reading values of *x and y* when comparing the same and different functions.

I infer that TB valued and prioritized interpretation action more than construction. His choice and use of examples that required flexible moves between representations facilitated interpretation action.

6.5 Findings

TB chose and used an equal number of start-up and reference examples. Two model examples were chosen and used. Firstly, there were more examples of exercises than examples of a concept. Secondly, examples of doing exercise done by learners were more challenging than those done by TB. Thirdly, TB's choice and use of two model examples and examples from previous (exemplar) question paper attested to rigour in understanding and consolidation of quadratic functions. This implied that TB paid attention to examples that facilitated both concept understanding and procedural fluency.

TB varied his examples for the learners who could discern what was critical against what was invariant. The sequencing of examples moved from familiar cases to complex examples. He firstly matched parabolas varying in both the values of the parameters and shapes in order to generalise the effects of the parabolas. He matched two different functions so that learners might discern the differences and gain meaning in those differences. He used different representations to provide an opportunity to access abstraction and for gaining understanding of the concept.

TB's chosen and used examples paid more attention to interpretation actions of the graphs than the construction one. The use of exemplar question paper's examples and reference examples done by learners attests to this. Examples in scenarios 2 also attest to uncommon cases of teaching and learning parabolas. This suggested TB's inclination on interpretation action as the skill that should be advanced in learning quadratic function.

TB's choice and use of spontaneous examples paid attention on learners' errors. TB changed the planned example because he realised that learners got it wrong or was difficult for the learners.

The curriculum somehow did not constrain TB's choice and use of examples. The choice and use of special content indicated his limited reliance on the curriculum prescriptions. Most of his examples for exercises were from the textbook. The use of previous question papers afforded TB with the choice and use of examples that facilitated verbal descriptions, symbolic and graphical representations therefore flexible move between representations was facilitated.

The (re)sources had obstacles in terms of three teaching and learning aspects. The first one is the number of learners (52) taught at same time. TB was unable to move around and assisted weak/struggling learners. He managed to keep learners focused and disciplined but was not able

to control most of learners' written work. He therefore might not know their progress. Secondly, there was the time allocated for the period, 30 minutes per period. There was an instance where TB did not complete activities which were meant for that period and he had to defer the activities to the next period. This stymied the usage of example exercises which facilitate procedural fluency. However, TB used officially allocated periods to teach examples of a concept and used the extra (long afternoon) classes to do examples exercises. The third one was scarcity of teaching and teaching aids which affected the mediation of the tasks by taking more time to write on the chalkboard.

6.6 Conclusion

TB's choice and use of *function examples* addressed examples of a concept and examples for doing exercises. The concept understanding and procedural fluency were facilitated. The lesson plan was followed as planned with only two exceptions. Firstly, he deferred some of the planned activities to long afternoon classes. Second exception, he substituted the planned example with a similar one (scenario 3 & 4) because he claimed that some of the learners did not understand the first one although it was also a planned one.

There were no learner generated examples (LGE) and this was due to, as Watson and Shipman (2008) posit, TB's teaching strategies which did not accommodate LGE.

Chapter 7: Cross-case synthesis

7.1 Introduction

This chapter synthesises the data analysed in Chapter 5 (Teacher A) and Chapter 6 (Teacher B). The cross-case synthesis was used to draw similarities and contrasts between the two teachers. Yin (2003) posits that cross-case synthesis is used to "predict similar and/or predict contrasting results but for predictable reasons" (p. 47). The purpose of cross-case analysis is to inductively explore and draw patterns that emerge from similarities, differences and relationships between these teachers' choice and use of examples (Hatch, 2002).

The cross-case synthesis was drawn from the recommendations cited by Yin (2003, pp. 133-137). The procedures for undertaking cross-case synthesis are:

- Create word tables that display the data from individual case according to an analytical framework,
- Word tables should lead to the synthesis that will reflect patterns and outcomes of interest based on case descriptions and theoretical constructs/categories,
- Different cases will appear to share some similarities and/or differences,
- Analysis will further lead to the reflection on categories of the typologies of different cases,
- Examinations of word tables will rely on argumentative interpretations that are supported by data.

The cross-case synthesis was used to answer the fourth research question:

What patterns, if any, are there between and across the two contrasting teaching contexts, with respect to their choice, use and rationales for examples?

7.2 Procedure

The following procedure was undertaken for synthesising. Firstly, the analytical framework was drawn to highlight themes which are synthesised and what to look for. Secondly, meta-analysis

was done with each teacher under the same theme in the next sub-section 7.3. In the meta-analysis, similarities, differences and patterns or trends can emerge. Lastly, the overall examination of the synthesis was made.

Table 7.1 below depicts *analytical framework of cross-case synthesis*.

Theme	**What to look for**
Teaching and learning resources	Qualifications, class size, time and teaching and learning aids
Learning objectives	Curriculum (CAPS document, DBE, 2011), types of examples (frequency, role and purpose) and synthesis of examples of a concept and examples for doing exercise
Variation, sequencing and representation	What was varied. Sequencing in construction, interpretation and flexible movement between the representations. What was used to create access to abstraction.
Teachers' rationale	Considerations.

The following subsections answer the research question: ***What patterns, if any, are there between and across the two contrasting teaching contexts, with respect to their choice, use and rationales for examples?***

7.3 Teaching and learning (re)sources

7.3.1 Qualifications and experience

TA has upper leverage in qualifications and experience (REQV 16, 20 years of teaching) than TB (REQV 15, 15 years) (cf. Table 4.1). One would have expected TA to dispense uncommon cases and more rigorous examples than TB. Instead TB chose and used uncommon case (special content) of *function examples*. TB claimed that he learnt about that uncommon case of an example when he was studying for ACE.

7.3.2 Class size

The class size of TA was 28 learners of two groups each and TB was 52 in one group. TB's class size constrained his usage of example exercises because he cannot assist the majority of learners who were struggling because of their numbers in class and his mobility in the classroom.

7.3.3 Time

TA had eight periods of 40 minutes in a six-day cycle. The time allocation was 5.3 hours of contact time with learners. The length of the period accorded him ample time to do both examples of a concept and examples for doing exercise with learners. TB had nine periods of 30 minutes each in a six-day cycle. The time allocation was 4.5 hours of contact time with learners. The official contact time for Mathematics in Further Education and Training phase (Grade 10 - 11) is 4.5 hours in a five-day cycle. Somehow officially TB had a shortage of contact time. TB's school had extra classes (long afternoon classes) where he was allocated three hours in a six-day cycle. All in all, TB had 7.5 hours of contact time with learners per cycle. My observation and his own words, TB used afternoon class for examples for doing exercises whereas during normal teaching and learning time he did both examples of a concept and examples for doing exercises.

7.3.4 Teaching and learning aids

The use of transparencies for planning and teaching afforded TA with an easy entry to teaching, pacing in his explanation, logical presentation and completion of his lesson of the day. TA also used the chalkboard. TB's incomplete offering of lessons can be attributed to his over-explanation and lack of other teaching and learning aids except the chalkboard. The availability of other (re)sources could have provided TB with an opportunity to use and complete his planned examples.

7.4 Learning objectives

The curriculum statement on Grade 11 functions stipulates:

1 Revise the effect of the parameters *a and q* (from Grade 10) and investigate the effect of *p* on the graphs of the functions defined by:

1.1 $y = f(x) = a(x+p)^2 + q$

1.2 $y = f(x) = \frac{a}{x+p} + q$

1.3 $y = f(x) = a.b^{x+p} + q$ where $b > 0, b \neq 1$

(DBE, 2011, p. 32).

The actions that need to be taken, as stipulated in the curriculum and other researchers, are:

Extend Grade 10 work on the relationships between variables in terms of numerical, graphical, verbal and symbolic representations of functions and convert flexibly between these representations (tables, graphs, words and formulae). Include linear and quadratic polynomial functions, exponential functions, some rational functions and trigonometric functions. Generate as many graphs as necessary, initially by means of point-by-point plotting, supported by available technology, to make and test conjectures and hence generalise the effects of the parameter which results in a horizontal shift and that which results in a horizontal stretch and/or reflection about the y axis. (DBE, 2011, p. 13)

7.4.1 Learning objectives and CAPS (cf. Table 7.2).

The two teachers chose and used examples that addressed the curriculum objectives but differing in the sequencing of the topics. The curriculum statement stipulates that Grade 10 revision needs to be done. The differences were two, namely, the examples chosen and used by TA and TB in teaching parameter *q* (vertical shift). TB did not teach the effect of parameter *q* as a stand-alone topic but cited that *p* affect the vertical shift. Instead TB fused the effect of parameter *a and q* together where two special types of content were observed. The special content used by TB was based on the operational signs (+/-) of parameters *a and q*. The first one was about the minimum and maximum values vis-à-vis signs of parameter *a,* and parameter *q* being an ordinate in TP. If $a > 0$ then the parabola has minimum value of *q* on the TP alternatively, if $a < 0$ then the parabola has a maximum value of q at TP. The second special content was the proposition that if parameters *a and q* have the same signs, the parabola does not intersect the *x-axis* in the parabola $y = ax^2 + q$ and its converse statement. In those, the TB facilitated the concept understanding and procedural fluency of special content while addressing the effect of parameters *a and q*.

Secondly, TA, in his second set of examples $f(x) = x^2$ and g(x) = $-\frac{1}{2}x^2$ it was not clear, both in explanation and visually, that if $0 < a < 1$ then g(x) opens wider than $f(x)$ and if $a < 0$ then $g(x)$ becomes the mirror image (reflects) of $f(x)$ along *x-axis*. He could have either taken $y = x^2$ and $y = \frac{1}{2}x^2$ or $y = -x^2$. By trying to address the two significances of parameter *a*, that is $a < 0$ and $0 < a < 1$, simultaneously did not facilitate concept understanding of parameter *a* on $a < 0$ and $0 < a < 1$.

TB made an early entry in the introduction of global sketching when he was teaching the effect of parameters *a and q*. The demonstration of signs of parameters *a and q* required the global sketching because the pointwise plotting could have delayed the entry into the special content. The meaning is that, although pointwise plotting is an entry in understanding global sketching, too much emphasis on it (prototypical) impedes development of procedural fluency (Tsamir et al., 2008).

Conversion from standard form to TP form required the usage of factorization of trinomials which were squared binomials or perfect squares then after factorization by completing the square. Both teachers demonstrated this. The difference is that TB integrated the applicability of discriminate ($\Delta = b^2 - 4ac$) into quadratic functions. He elaborated that if $\Delta < 0$ then roots (*x – values/intercepts*) are non-real therefore the parabola cannot intersect the *x-axis*. The example chosen and used by TB made connection between quadratic theory and quadratic functions therefore it develops both concept understanding and procedural fluency of two concepts.

There were both choice and use of examples of a concept carried out by the two teachers because both taught learning objectives that were reflected in the curriculum and Annual Teaching Plan of the North West Department of Education and Training. Secondly, they drew their examples from the *Platinum Mathematics Grade 11* (Bradley et al., 2012) textbook. TB had more leverage than TA on examples for a concept. Firstly TB chose and used examples that suggested special content. Secondly, drawing examples from the previous question papers, TB used examples that dealt with verbal descriptions, symbolic and graphical representations which did not appear in TA's examples.

The learning objectives on Grade 10 revision are parameters *a and q*, as stipulated in the curriculum statement. The learning objectives addressed four distinctive yet related objects of learning, namely, a pointwise approach, graph constrain/stretch, reflection along the *x-axis* and vertical shift. In his exercises for this learning objective, TA's objects of learning distinctively addressed vertical shift (in $y = -\frac{1}{2}x^2 + 2$ and $y = -\frac{1}{2}x^2 - 2$) and pointwise plotting.

TB's choice and use of examples from Grade 10 revision demanded the facilitation of eight objects of learning, namely, a pointwise approach, a global approach, graph constraint/stretch, reflection along the x – axis, vertical shift, interception on axes, minimum and maximum TPs, and verbal description of the parabola to its graphical form.

7.4.2 Types of examples (cf. Table 7.2)

TA had chosen and used five start-up examples that were examples of a concept (cf. Table 7.2). TB had chosen four start-up examples that required concept understanding and two start-up examples that were exercises yet all six of them were examples of a concept (cf. Table 7.2). It implies that TA's example of a concept illuminated concept understanding while TB's two examples of a concept illuminated concept understanding and procedural fluency which were acquired in duality (Tsamir et al., 2008).

The two model examples chosen and used by TB imply that his examples illuminated the generic nature of quadratic functions (cf. Table 7.2). The first model example of TB (cf. Appendix C, p.86, number 2 & 3) seems to be similar to TA's $y = x^2$, $y = (x-2)^2 - 4$ and $y = (x+1)^2 + 2$. Examples are classified as a particular type due to how it has been used. That is, TB's examples were classified as model examples because they summarised many properties that define a parabola (min/max, AS, TP, interception on axes), skills and actions (symbolic, graphical and flexible move between representations, construction and interpretation) in one example. TA's example was classified as a reference example because it required the global approach and only construction and interpretation actions were required (cf. Table 7.2).

Types of examples as examples of a concept and examples for doing exercise.

Table 7.2: Types of examples and their role.

Learning objective	Examples of a concept		Examples for exercise	
	TA	TB	TA	TB
Effect of *a*	$y = x^2 and y = 2x^2$ $f(x) = x^2 and g(x) = -\frac{1}{2}x^2$ (start-up). **Action:** pointwise	*1.y = x^2 and y = $2x^2$* *2. y = x^2 and y =* $\frac{1}{2}x^2$ *3. y = x^2 and y = - x^2* (start-up, **taken as exercise for learners**). **Action**: pointwise		
Effect of *q*: vertical shift	$y = -\frac{1}{2}x^2 and y = -\frac{1}{2}x^2 + 2$ Start up, taken as exercise by learners	*y = ax^2 + q and y = x^2* start-up, no algebraic symbols and graphical representations	$y = -\frac{1}{2}x^2 + 2 and y = -\frac{1}{2}x^2 - 2$ Reference, pointwise	
Effect of both *a and q* (address two parameters at same time)		*y = x^2 +1* and *y = $-x^2$ – 1* : start-up, global plotting (**special content**)		*y = x^2 -1* and *y = $-x^2$ +4*: reference, algebraic procedures, global plotting. Scenario 3: from exemplar question paper (number 1 & 2)
Effect of *p*: horizontal shift	$y = x^2 and y = (x+2)^2$ Start-up, global plotting	*f(x) = x^2, g(x) = $(x – 2)^2$* and *h(x) = $(x + 2)^2$*. Start up, pointwise	$y = x^2 and y = (x-1)^2$ Reference, global plotting	Seatwork from *Platinum Mathematics Grade 11* textbook, p. 85, Exercise 2 number 1 & 4: dealt with converting standard form to TP form, standing TP and

				describing horizontal shift (reference, algebraic symbols and verbal descriptions)
Vertical and horizontal shifts	$y = x^2 - 4x + 4$ to $y = (x-2)^2$ start–up, completing the square. $y = x^2 + 2x + 2 to y = (x+1)^2 + 1$ $y = x^2 + 3x + 5 to y = (x + \frac{3}{2})^2 + \frac{11}{4}$ Completing the square of non-squared binomials	Example exercise 3 (scenario 3): Consider the graph of y = x^2 which has been moved three units up and two units to the left. **Start-up, taken as exercise for learners** (scenario 3). Verbal description to global sketching then formula $y = 2x^2 + 4x - 6$ Start-up, completing the square	$y = x^2 + 6x + 9$ $y = -x^2 - 2x - 1$ (reference) $y = x^2 - 8x + 16$ $y = x^2 + 2x + 2$ $y = x^2 + 3x + 5$	p. 86, Exercise 3 number 2 & 3. Dealt with converting TP to standard form, stating the TPs and describing vertical and horizontal shifts (model, algebraic symbols and verbal descriptions, summaries most of the characteristics of a parabola in a single examples) Sketch
Forms of parabola	Notes on parabola, scenarios 11 and 12	$y = x^2, y = (x-2)^2 - 4, y = (x=1)^2 + 2$ (reference)	TB dealt with forms of parabola in vertical and horizontal shifts	*From* $y = ax^2 + bx + c$ to $y = a(x + \frac{b}{2a})^2 + \frac{4ac - b^2}{4a}$. $p = -\frac{b}{2a}$ and $q = \frac{4ac - b^2}{4a}$ (reference)
Matching parabola with linear function		$f(x) = -x^2 + 4x + 12$ and $g(x) = 4x + 8$ (Model)		$f(x) = -x^2 + 4x + 12$ and $g(x) = 4x + 8$ (Model)

7.5 Rowland's categories

Table 7.3: Rowland's categories.

Learning objectives	TA's examples of concept	TB's examples of a concept	TA's examples for exercises	TB's examples for exercise
Parameter a	Parameter *a* is varied. If two qualities of parabola namely (namely, $a < 0$, $0 < a < 1$ and $a < 0$) are intended to be discerned in a mutually exclusive way. Table of values, ordered-pairs and point-wise sketching of the functions were done.	Parameter *a* was varied. Two qualities of parabola namely (namely, $a > 0$, $0 < a < 1$ and $a < 0$) were compared with parent parabola to discern generalization. Table of values, ordered-pairs and point-wise sketching of the functions were done. The values of parameter a were varied in order to generalize the effect of a.		
Parameter q	Parameter *q* is a critical feature that is varied while $a = -\frac{1}{2}$ is discriminate (remaining unchanged). Table of values, ordered-pairs and sketching of the functions were done.		Parameter *q* is varied while *a* is discriminated. Both parameters were experienced in a mutually exclusive way	
Parameter *a and q*		Parameters *a and q* were		*Parameters a and q* were

		critical features and varied; were experienced simultaneously without discrimination (Marton & Pang, 2006). TB's choice and use demanded generalization and special contented.		critical features and varied, demanded enhancement and acquisition of generalization on parameters and special content. Global sketching, minimum/maximum TP (0;q) were introduced. Generalizations on the effect of both a and q (special content) 1 & 2: Generalization of effect of q, global sketching, TPs.
Parameter *p*	Parameter *p* is critical feature and varied from the parent parabola in order to ascertain the horizontal shifts if $p > 0$ and $p < 0$ from the origin *(0;0)* while *a* remain the same. The intention was to discern the object of learning which was parameter *p*. The intersection on *x-axis* is embedded.	The parameter p $(p > 0$ *and* $p < 0)$ was varied to illuminate generalization of its effect on the parental function. Pointwise approach was used to demonstrate how variation of the parameters has informed generalization. Pointwise approach, generalization of parameter p. TP and intersection on axes.	Parameter *p* is a critical feature therefore varied.	Pointwise approach, generalization of parameter *p*. TP and intersection on axes.
Parameter p and q and forms of parabola	Completing the square is introduced therefore algebraic manipulation and	Conversion from TP form to standard form algebraically, TP *(p;q)*, linking algebraic	Enhancing completing the square, properties of parabola and global sketching	Conversion from standard form to TP form by factorising through completing the

	moving from one form of quadratic function to another.	symbols with graphical representation (non-real roots, intersection on axes). Convert from standard form to TP another, (factorization by completing the square)		square. Example exercises 1 & 4 were varied where *a = 1 and a = -1* to illuminate degree of complexity in executing the completing the square. The description of vertical shift and horizontal one also brought complexity and for converting from one form to another (enhancing the completing of the square), description of shifts (generalization and testing conjectures)
Matching parabola and linear function		Matching two functions and reading the values (interpretation) when compared		Matching two functions and reading the values (interpretation) when compared

Representations: Both teachers used different representations to present the abstraction or conceptualisation in an easy way to develop and enhance conceptual understanding and procedural fluency. They used algebraic symbols (tables of values, ordered pairs, procedural cues) to sketch the graphs (pointwise, global) (cf. Table 7.3). They also used the graphs to read the values that pertain to parabola and linear functions. TB further used, apart from representations observed above, verbal description as a form of representation (cf. Table 7.3).

Sequence: They both sequenced their examples by matching parabolas together in order to generalise the effect of parameters. Their sequencing in constructions moved from pointwise to global approach. The interpretation actions moved from matching parabolas together for generalisation of parameters to matching different functions together. Interpretation was observed from translating algebraic symbols to the Cartesian plane by both pointwise and global approach. Lastly, interpretation occurred in model example(s) where reading of values from the graphs was done. TB further used verbal descriptions to plot the parabolas then used the graphical representations to deduce algebraic symbol (formula). On flexible moves between representations, TB had better usage of it than TA.

Variation: Both teachers varied their choice and use of examples to discern the critical features while the non-critical features were discriminated and therefore remained invariant. TB chose and used examples (scenario 2) and varied two critical features (parameters *a and q)* simultaneously to discern the uncommon case (special content) for development of understanding of parabola.

7.6 Findings

Time as cultural (re)source was an important factor for generating and treating examples. It can constrain or afford the teacher with an opportunity to plan and use examples effectively during lesson presentation.

Teaching and learning aids provided TA with pacing and mediation of the task. It also provided him with easy entry for teaching the lesson in logical ways. The use of transparencies provided TA with a better leverage for systematically planning his lessons. This does not imply that

teaching and learning aids have a bearing on a teacher to appropriately choose and use examples, for instance, in TA's examples in scenario 4 where not appropriate as he wished. The appropriate choice and use of examples, as espoused by Zodik and Zaslavsky (2008), and Sinclair et al. (2011), depend on a teacher's example space and the content prescriptions stipulated in the curriculum. Rowland (2008) and, Zodik and Zaslavsky (2008) assert that teachers' choices and uses of examples do not necessary reflect teachers' knowledge at any moment in time. Teacher can encounter distraction in their planning and use of examples. TB's examples, amidst constrained and limited teaching and learning (re)sources in his school, chose and use examples that had rigour (special content, verbal descriptions, flexible move between representations, model examples). My finding somehow concur with what Adler (2000) found in her study that "more resources do not necessarily lead to better practice *(examples that have rigour)*" (p. 206) (*my emphasis*).

Both TA and TB chose and used spontaneous examples. TA changed a planned example that might have "confuse the learners". TB chose and used a spontaneous example because the learners got the planned example wrong or was difficult for the learners. He therefore chose and used an example to attend to learners' errors. This example was not a planned example but chose and used example that had the same object of learning with the difficult one.

7.8 Reflection

Competent, qualified, experienced teachers rely heavily on their experience and textbooks to choose and use examples at the moment of teaching.

The different socio-economic settings of the two teachers (fee-paying and no-fee school) understudy seemingly do not have a bearing on their choice and use of examples. The time-table and time allocation (the length of a period) may constrain or afford the choice and use of examples. The length of the period is not affected by school's socio- economic setting but the length of a period is determined by official approaches to teaching and learning contact time.

The teaching and learning strategies have a bearing on teachers' choices and uses of examples. For example, the discovery method affords and accommodates different patterns of learning. The strategies determine how examples are treated and used.

7.8 Limitation

1 I had difficulty in classifying TA's $y = x^2$, $y = (x-2)^2 - 4$ and $y = (x+1)^2 + 2$ as reference examples.

2 I wish I could have collected data from other types of functions (hyperbola, exponential function and trigonometric functions) to observe how the teachers would have chosen and used *function examples*. The observation could have supported or not the findings that I made. It is not entirely convincing to draw findings from four lessons observed.

3 Given that only two teachers were observed, the findings cannot be generalised to other teachers' choices and use of examples but they may be referred on the bases of relatebility (Opie, 2010) to sampling similar to this research study. The findings may be used by other teachers to improve their practise in teaching functions.

7.9 Implication for Mathematics Education

Mathematics teachers and textbook authors have been varying and sequencing examples using different representations. This study revealed that teachers and textbook authors need to be conscious and critical, firstly, in how and what needs to vary and what should be invariant in the choice and use of examples. The discernment of the object of learning should be a critical feature to be varied. Secondly, sequencing, apart from being the guiding principle of teaching of different mathematical concepts (Zodik & Zaslavsky, 2008) (e.g., functions, algebraic expressions, trigonometry, analytical geometry, etc.), has its distinctive ways to address the content. This has a bearing on how we choose and use the examples.

Examples are important indices and concept for mathematics teaching and learning. This study should serve as a reflection for education policy-makers and education programme developers to

include teachers' treatment and generation of examples as an important component in mathematics teaching both in teachers' training at tertiary education and also on practising teachers' on going professional development.

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Appendix A. Grade 10 : Question paper.

QUESTION 3

3.1 Solve for x:

3.1.1 $x-2(2x+3)=0$ (3)

3.1.2 $5^x=1$ (2)

3.1.3 $-3<7-2x\leq 11$ (3)

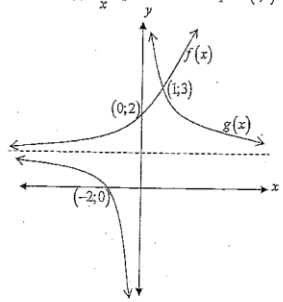
3.2 Mr. Modise has a rectangular garage which measures $6m$ by $7m$. He wants to increase the area by 30 square metres by altering the length and width by the same distance. Calculate the new dimensions for the garage. (5)

[13]

QUESTION 4:

4.1 The diagram below shows the graphs of f and g.

$f(x)=2^x+1$ and $g(x)=\frac{a}{x}+q$ intersect at the point $(1;3)$.



4.1.1 Write down the equation of the horizontal asymptote of f. (1)

4.1.2 Determine the equation of g. (3)

4.1.3 Give the range of f. (1)

4.1.4 Give the domain of g (2)

4.1.5 For which values of x is f increasing? (2)

4.1.6 For which value(s) of x is $f(x)-g(x)=0$? (1)

4.1.7 For which values of x is $g(x)-f(x)>0$? (1)

4.1.8 The graph of $f(x)$ is shifted up two units. Give the new equation of $f(x)$ in the form $h(x)=...$ (1)

4.1.9 The graph of $g(x)$ is reflected in the $x-axis$. Give the new equation in the form $k(x)=...$ (2)

4.2 Sketch on the same set of axes the graphs of $y-x+3=0$ and $y=x^2-9$. Show on the graphs all intercepts with the axes and points of intersection.
N.B: Show workings how you have determined those points. (15)

[29]

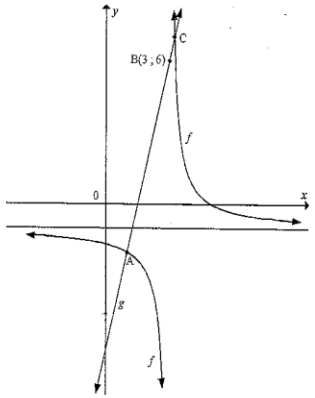
APPENDIX B: Grade 11 Question paper

QUESTION 7

A given quadratic pattern $T_n = an^2 + bn + c$ has $T_2 = T_4 = 0$ and a second difference of 12.
Determine the value of the 3rd term of the pattern. **[6]**

QUESTION 8

The sketch below represents the graphs of $f(x) = \frac{2}{x-3} - 1$ and $g(x) = dx + e$.
Point B (3 ; 6) lies on the graph of g and the two graphs intersect at points A and C.



8.1 Write down the equations of the asymptotes of f. (2)

8.2 Write down the domain of f. (2)

8.3 Determine the values of d and e, correct to the nearest integer, if the graph of g makes an angle of 76° with the x-axis. (3)

8.4 Determine the coordinates of A and C. (6)

8.5 For what values of x is $g(x) \geq f(x)$? (3)

8.6 Determine an equation for the axis of symmetry of f which has a positive slope. (3)
[19]

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QUESTION 9

Given: $f(x) = -x^2 + 2x + 3$ and $g(x) = 1 - 2^x$

9.1 Sketch the graphs of f and g on the same set of axes. (9)

9.2 Determine the average gradient of f between $x = -3$ and $x = 0$. (3)

9.3 For which value(s) of x is $f(x).g(x) \geq 0$? (3)

9.4 Determine the value of c such that the x-axis will be a tangent to the graph of h, where $h(x) = f(x) + c$. (2)

9.5 Determine the y-intercept of t if $t(x) = -g(x) + 1$ (2)

9.6 The graph of k is a reflection of g about the y-axis. Write down the equation of k. (1)
[20]

QUESTION 10

Sketch the graph of $f(x) = ax^2 + bx + c$ if it is also given that:

- The range of f is $(-\infty; 7]$
- $a \neq 0$
- $b < 0$
- One root of f is positive and the other root of f is negative. **[4]**

QUESTION 11

Given: $P(W) = 0,4$
$P(T) = 0,35$
$P(T \text{ and } W) = 0,14$

11.1 Are the events W and T mutually exclusive? Give reasons for your answer. (2)

11.2 Are the events W and T independent? Give reasons for your answer. (3)
[5]

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APPENDIX C: Textbook

TOPIC 5

Functions: Effects of parameters

KEY WORDS

parameter – a variable that restricts or gives a particular form or shape to the equation it characterises
function – a relationship between two variables, usually x and y, where for every value of x there is one corresponding value for y
increase – to grow bigger
decrease – to grow smaller
reflect – create a mirror image or reflection
y-intercept – the point where the graph cuts the x-axis (where $x = 0$)

Unit 1: The effects of the parameters a, p and q on parabolas

We define trigonometric graphs by parametric equations which group the graphs into families. When we assign values to the parameters and substitute them into the equation, it becomes a specific equation with a specific shape that is restricted by these values. Parameters do not change the type of graph, but define the characteristics of a particular graph within a family of graphs.

We will investigate the effect of the parameters a, p and q on the graph of the function defined by $y = f(x) = a(x + p)^2 + q$, but firstly, we will revise the effects of a and q on the parabola given by $y = f(x) = ax^2 + q$ by comparing the following graphs:

$f(x) = x^2$; $g(x) = 4x^2$, $h(x) = \frac{1}{4}x^2$ as well as $y = -x^2$; $y = -4x^2$ and $y = -\frac{1}{4}x^2$.

$f(1) = 1^2 = 1 \Rightarrow (1;1)$ $g(2) = 2^2 = 4 \Rightarrow (2;4)$ $g(3) = 3^2 = 9 \Rightarrow (3;9)$

$g(x) = 4x^2$

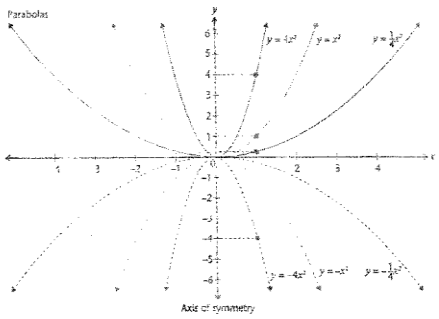
$g(1) = 4(1)^2 \Rightarrow (1;4)$ $g(2) = 4(2)^2 = 16 \Rightarrow (2;16)$ $g(3) = 4(3)^2 = 36 \Rightarrow (3;36)$

$h(x) = \frac{1}{4}x^2$

$h(1) = \frac{1}{4}(1)^2 = \frac{1}{4} (1;0,25)$ $h(2) = \frac{1}{4}(2)^2 = 1 (2;1)$ $h(3) = \frac{1}{4}(3)^2 = \frac{9}{4} (3;2,25)$

We can conclude that:

- the greater the value of a, the steeper the curve
- when a is negative the graph reflects in the x-axis
- changing the value of a has no impact on the y-intercept.



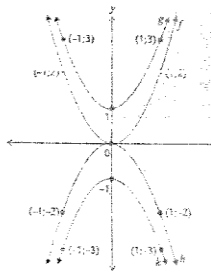
WORKED EXAMPLE

Consider the function: $f(x) = 2x^2$

1 Sketch f, showing the coordinates of three points.
2 If f is shifted up 1 unit, the translated graph is g. Draw g on the same system of axes.
3 State the equation of g.
4 Sketch h, the reflection of f in the x-axis, on the same system of axes.
5 State the equation of h.
6 Sketch k, the reflection of g in the x-axis, on the same system of axes.
7 State the equation of k.

SOLUTIONS

1 $f(0) = 0 \Rightarrow (0;0)$
$f(1) = 2(1)^2 = 2 \Rightarrow (1;2)$
$f(-1) = 2(-1)^2 = 2 \Rightarrow (-1;2)$
2 Add 1 to each y-value:
$(0;0) \rightarrow (0;1)$
$(1;2) \rightarrow (1;3)$
$(-1;2) \rightarrow (-1;3)$
3 $g(x) = f(x) + 1 = 2x^2 + 1$
4 Leave the x-coordinates unchanged, but change the sign of each y-value:
$(0;0) \rightarrow (0;0)$
$(1;2) \rightarrow (1;-2)$
$(-1;2) \rightarrow (-1;-2)$
5 $h(x) = -f(x) = -2x^2$
6 Leave the x-coordinates unchanged, but change the sign of each y-value:
$(0;1) \rightarrow (0;-1)$ $(1;3) \rightarrow (1;-3)$ $(-1;3) \rightarrow (-1;-3)$
7 $k(x) = -g(x) = -2x^2 - 1$



KEY WORDS

reflection – a mirror image about a line of symmetry

REMEMBER

The vertical axis is the y-axis.
The equation of the y-axis is given by $x = 0$.
The horizontal axis is the x-axis.
The equation of the x-axis is given by $y = 0$.
The axes cross at the origin, **(0;0)**.
Coordinates
Give both the x- and y-coordinates in the form $(x;y)$.
If the first coordinate is 0, then you have a y-intercept: $(0;3)$ cuts the y-axis at 3.
If the second coordinate is 0, then you have an x-intercept. $(2;0)$ cuts the x-axis at 2.

Vertical shifts require movement up or down only and do not change the x-coordinates.

Reflections in the x-axis do not change x-coordinates, but change the signs of the y-coordinates.

- $f(x) + 2$ will increase each y-coordinate by 2 and the graph will shift up 2 units.
- $f(x) - 3$ will decrease each y-coordinate by 3 and the graph will shift down 3 units.
- $-f(x)$ will reflect f in the x-axis.

REMEMBER

On the 'same set of axes' means you will draw *more than one graph* on the same sketch.

REMEMBER

To find an *x*-intercept, substitute $y = 0$.
To find a *y*-intercept, substitute $x = 0$.

EXERCISE 1

1 Consider the function: $f(x) = -\frac{1}{4}x^2$
- 1.1 Sketch the graph of f, showing the coordinates of three points.
- 1.2 If g is obtained by shifting f up 2 units, sketch g on the same system of axes as f.
- 1.3 State the equation of g.
- 1.4 Sketch h, the reflection of f in the x-axis, on the same system of axes as f and g.
- 1.5 State the equation of h.
- 1.6 Sketch k, the reflection of g in the x-axis, on the same system of axes as f, g and h.
- 1.7 State the equation of k.

2 Consider the function: $f(x) = -8x^2$
- 2.1 Sketch the graph of f, showing the coordinates of three points.
- 2.2 If g is obtained by shifting f down 1 unit, sketch g on the same system of axes as f.
- 2.3 State the equation of g.
- 2.4 Sketch h, the reflection of f in the x-axis, on the same system of axes as f and g.
- 2.5 State the equation of h.
- 2.6 Sketch k, the reflection of g in the x-axis, on the same system of axes as f, g and h.
- 2.7 State the equation of k.

3 Consider the function: $f(x) = \frac{1}{8}x^2$
- 3.1 Sketch the graph of f, showing the coordinates of three points.
- 3.2 If g is obtained by shifting f down three units, sketch g on the same system of axes as f.
- 3.3 State the equation of g.
- 3.4 Sketch h, the reflection of f in the x-axis, on the same system of axes as f and g.
- 3.5 State the equation of h.
- 3.6 Sketch k, the reflection of g in the x-axis, on the same system of axes as f, g and h.
- 3.7 State the equation of k.

4 Consider the function: $f(x) = -3x^2$
- 4.1 Sketch the graph of f, showing the coordinates of three points.
- 4.2 If g is obtained by shifting f down 2 units, sketch g on the same system of axes as f.
- 4.3 State the equation of g.
- 4.4 Sketch h, the reflection of f in the x-axis, on the same system of axes as f and g.
- 4.5 State the equation of h.
- 4.6 Sketch k, the reflection of g in the x-axis, on the same system of axes.
- 4.7 State the equation of k.

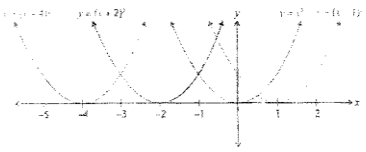
Horizontal shifts of the parabola $y = x^2$

x		−5	−4	−3	−2	−1	0	1	2	3	Shift
$y = x^2$		25	16	9	4	1	0	1	4	9	0
$y = (x+2)^2$		9	4	1	0	1	4	9	16	25	2 left
$y = (x-1)^2$	49	36	25	16	9	4	1	0	1	4	1 right
$y = (x-3)^2$	64	49	36	25	16	9	4	1	0	1	3 right
$y = (x+5)^2$	0	1	4	9	16	25	36	49	64	81	5 left

- *Adding* to x *increases* its value so the pattern of answers *shifts* to the *left* by the amount added.
- *Subtracting from* x *reduces* its value so the pattern *shifts* to the *right* by the amount subtracted.
- This pattern of *horizontal shifting* applies to all graphs, not only to the parabola.

The graphs in the table are sketched in the figure below. Notice that they are all identical in shape.

The *x*-intercept in each graph clearly indicates how the graph has shifted.



KEY WORD

x-intercept – the point where the graph cuts the *x*-axis (where $y = 0$)

Each graph has equal roots and touches the *x*-axis, without crossing it.
There are three forms of each graph:
- $y = (x + 2)^2$ is the turning point form
- $y = (x + 2)(x + 2)$ is the *x*-intercept form
- $y = x^2 + 4x + 4$ is standard form.

EXERCISE 2

Write each equation in the turning point form. State the turning point and then describe the shift of each parabola from the origin (0;0).

1 $y = x^2 + 6x + 9$
2 $y = x^2 - 8x + 16$
3 $y = x^2 - 4x + 4$
4 $y = -x^2 - 2x - 1$

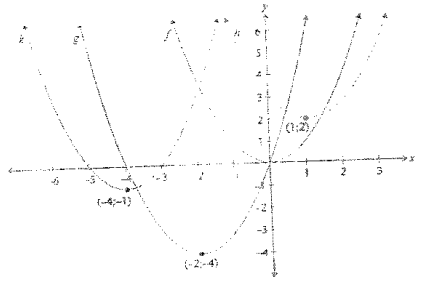
Horizontal and vertical shifts

Consider the parabola equation $y = (x + p)^2 + q$:

- changing p results in a **horizontal shift**
- changing q results in a **vertical shift**
- p and q do not influence each other.

x	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	Shift
$y = x^2$	36	25	16	9	4	1	0	1	4	9	16	25	36	None
$y = (x + 2)^2$	16	9	4	1	0	1	4	9	16	25	36	49	64	2 left
$y = (x - 1)^2$	49	36	25	16	9	4	1	0	1	4	9	16	25	1 right
$y = (x + 4)^2$	4	1	0	1	4	9	16	25	36	49	64	81	100	4 left
$y = x^2 + 2$	38	27	18	11	6	3	2	3	6	11	18	27	38	2 up
$y = x^2 - 1$	35	24	15	8	3	0	-1	0	3	8	15	24	35	1 down
$y = (x + 2)^2 - 4$	12	5												
$y = (x - 1)^2 + 2$	51	38	27	18	11	6	3	2	3	6	11	18	27	

The graphs in the figure are drawn from the table. Notice that all the graphs are identical in shape.



KEY WORD

vertical shift – with straight up or down, no sideways movement
horizontal shift – with sideways, with no up or down movement
axis of symmetry – a line that divides a graph into two identical halves and about which a function is symmetrical

- $y = x^2$ has a minimum value of 0 at (0;0) and an axis of symmetry given by $x = 0$.
- $y = (x + 2)^2 - 4$ has a minimum value of -4 at $(-2;-4)$ and an axis of symmetry $x = -2$.
- $y = (x - 1)^2 + 2$ has a minimum value of 2 at (1;2) and an axis of symmetry $x = 1$.
- $y = (x + 4)^2 - 1$ has a minimum value of -1 at $(-4;-1)$ and an axis of symmetry $x = -4$.

EXERCISE 3

Write each equation in standard form and in the x-intercept form. State the turning point of each graph and then describe the shift of each parabola from the origin (0;0).

1 $y = (x - 4)^2 - 9$
2 $y = 4(x + 2)^2 - 1$
3 $y = -(x + 5)^2 + 16$
4 $y = -(x - 3)^2 + 4$

Sketching a parabola

Check the sign of a:

- If $a > 0$, then the parabola 'smiles' ☺ and the arms go up.
- If $a < 0$, then the parabola 'frowns' ☹ and the arms go down.

Find the y-intercept by making $x = 0$ and solving for y:

- If $y = 2x^2 - 3x - 5$, then the constant will be the y-intercept, so $y = -5$.
- If $y = (x - 5)^2 - 4$, then $y = (0 - 5)^2 - 4 = 25 - 4 = 21$.

Find the x-intercepts, make $y = 0$ and then solve the equation.

- Most graphs will have rational roots and so we can find the roots by factorisation:
  $y = 2x^2 - 3x - 5 \Rightarrow 0 = (2x - 5)(x + 1)$ and $x = 2,5$ or $x = -1$
  x-intercepts $(-1;0)$ and $(2,5;0)$
- Use the quadratic formula for graphs with irrational roots:
  $y = x^2 - 2x - 1 \Rightarrow x^2 - 2x - 4 = 0$ cannot be solved by factors.
  quadratic formula: $a = 1$, $b = -2$ and $c = -4 \Rightarrow b^2 - 4ac = (-2) - 4(1)(-4) = 20$
  $x = \frac{-(-2) \pm \sqrt{20}}{2(1)} = 3,24$ or $-1,24$
  x-intercepts $(-1,24;0)$ and $(3,24;0)$
- Parabolas do not necessarily have x-intercepts and in this case we cannot find roots:
  $y = x^2 - x + 2 \Rightarrow x^2 - x + 2 = 0$ cannot be solved by factors.
  Test $b^2 - 4ac$ from the quadratic formula: $a = 1$, $b = -1$ and $c = 2 \Rightarrow b^2 - 4ac = (-1)^2 - 4(1)(2) = -7$
  No x-intercepts because $b^2 - 4ac < 0$

Find the turning point:

- Every parabola has a turning point.
  The x-coordinate is the axis of symmetry, given by $x = -\frac{b}{2a}$.
  The y-coordinate is the minimum value if $a > 0$, then the parabola 'smiles' ☺
  The y-coordinate is the maximum value if $a < 0$, then the parabola 'frowns' ☹
  Find the y-coordinate by substituting the x-coordinate.
- If you know the x-intercepts, then the axis of symmetry is $\frac{x_1 + x_2}{2}$
  x-intercepts $(-1;0)$ and $(2,5;0) \Rightarrow$ axis of symmetry is $x = \frac{-1 + 2,5}{2} = \frac{1,5}{2} = 0,75$
- If the roots are irrational or if there are no roots, then use the formula $x = -\frac{b}{2a}$
  $y = x^2 - 2x - 4$ has $a = 1$ and $b = -2$, so the axis of symmetry is $x = \frac{-b}{2a} = \frac{-(-2)}{2(1)} = 1$
  $y = x^2 - x + 2$ has $a = 1$ and $b = -1$, so the axis is symmetry is $x = \frac{-(-1)}{2(1)} = \frac{1}{2}$

REMEMBER

$a > 0$ means that a is positive.
$a < 0$ means that a is negative.
minimum – smallest possible value
maximum – largest possible value
axis of symmetry – divides graph into two identical halves

WORKED EXAMPLE

1 Sketch $f(x) = x^2 - 5x - 6$, $g(x) = -\frac{1}{2}(x-4)^2 + 2$ and $h(x) = x^2 - 4x + 6$ on the same system of axes, clearly indicating the intercepts with the axes and the coordinates of the turning points.
2 Use your graphs to determine two values of x for which $f(x) = g(x)$.
3 Solve algebraically for x if $f(x) = g(x)$.
4 Determine $h(4) - g(4)$.
5 Use your graph to determine two values of x for which $g(x) - f(x) = 0$.

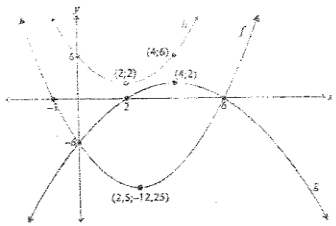
SOLUTIONS

1 $f(x) = x^2 - 5x - 6$ ☺
y-intercept: $(0; -6)$
x-intercepts: $(6;0)$ and $(-1;0)$
$x^2 - 5x - 6 = 0$
$(x - 6)(x + 1) = 0$
$x = 6$ or $x = -1$

Turning point: $(2,5;-12,25)$
$x = \frac{-b}{2a} = \frac{5}{2} = 2,5$
$y = (2,5)^2 - 5(2,5) - 6 = -12,25$

$g(x) = -\frac{1}{2}(x-4)^2 + 2$ ☹
y-intercept: $(0;-6)$
$-\frac{1}{2}(0-4)^2 + 2 = -6$
x-intercepts: $(0;0)$ and $(4;0)$
$-\frac{1}{2}(x-4)^2 + 2 = 0$
$(x-4)^2 = 4$
$x - 2 = 2$ or $x - 2 = -2$
$x = 4$ or $x = 0$
Turning point: $(4;2)$

$h(x) = x^2 - 4x + 6$
y-intercept: $(0;6)$
x-intercepts: None
No factors ⇒ check $b^2 - 4ac$
$a = 1$, $b = -4$ and $c = 6$
$b - 4ac = (-4)^2 - 4(1)(6) = -8$
No roots
Turning point: $(2;2)$
$x = -\frac{b}{2a} = -\frac{(-4)}{2(1)} = 2$
$y = (2)^2 - 4(2) + 6 = 2$



2 $x = 0$ or $x = 6$
3 $x^2 - 5x - 6 = -\frac{1}{2}(x-4)^2 + 2 \Rightarrow -2x^2 + 10x + 12 = x^2 - 8x + 16 - 4$
$3x^2 - 18x = 0 \Rightarrow 3x(x-6) = 0$ and $x = 0$ or $x = 6$
4 $h(4) - g(4) = 6 - 2 = 4$ units
5 $x = 0$ or $x = 6$

WORKED EXAMPLE

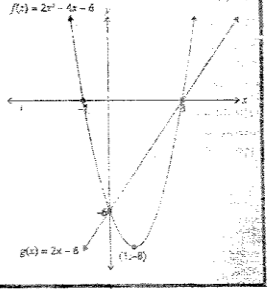
1 $f(x) = 2x^2 - 4x - 6$ and $g(x) = 2x - 6$
1.1 Draw neat sketch graphs of f and g on the same system of axes. Indicate the intercepts with the axes, as well as the coordinates of any turning points.
1.2 Use your graphs to answer the questions:
1.2.1 For which value(s) of x is $f(x) = g(x)$?
1.2.2 For which value(s) of x is $f(x) < g(x)$?
1.2.3 For which value(s) of x is $f(x).g(x) \geq 0$?

SOLUTIONS

1 $f(x) = 2x^2 - 4x - 6$ ☺
y-intercept: $(0;-6)$
x-intercepts: $(-1;0)$ and $(3;0)$

1.1 $2x^2 - 4x - 6 = 0$
$\div 2$: $x^2 - 2x - 3 = 0$
$(x - 3)(x + 1) = 0$,
so $x = 3$ or $x = -1$

Turning point: $(1;-8)$
$x = \frac{3 + (-1)}{2} = \frac{2}{2} = 1$
$y = 2(1)^2 - 4(1) - 6 = -8$



$g(x) = 2x - 6$
y-intercept: $(0;-6)$
x-intercept: $(3;0)$
$2x - 6 = 0$
$2x = 6 \Rightarrow x = 3$

1.2 1.2.1 $x = 0$ and $x = 3$
1.2.2 $0 < x < 3$
1.2.3 $x \in [-1;\infty)$

REMEMBER

If $A \times B \geq 0$, then A and B are both positive or both negative.
If $C \times D < 0$, then C and D must be opposite in sign.
$f(x) = g(x)$ means that the y-value of f is equal to the y-value of g.
$f(x) < g(x)$ means that the y-value of f is less than the y-value of g.
$f(x).g(x) \geq 0$ means that the product of the y-values of f and g is positive or 0.
If $a > 0$, the arms of the parabola go up ☺ and the parabola has a minimum y-value.
If $a < 0$, the arms of the parabola go down ☹ and the parabola has a maximum y-value.

REMEMBER

x-intercepts and roots represent the point where the graph cuts the x-axis (where $y = 0$)

EXERCISE 4

Label each graph clearly and indicate the intercepts with the axes as well as any turning point(s).

1 1.1 Sketch $f(x) = -x^2 + 4x + 12$ and $g(x) = 4x + 8$ on the same system of axes.
1.2 For which value(s) of x is $f(x) = g(x)$?
1.3 For which value(s) of x is $f(x) \geq g(x)$?
2 2.1 Sketch $f(x) = \frac{1}{2}x^2 - x - 4$ and $g(x) = x - 4$ on the same system of axes.
2.2 For which value(s) of x is $g(x) - f(x) = 2$?
2.3 For which value(s) of x is $f(x) \leq g(x)$?
3 3.1 Sketch $f(x) = -(x + 1)^2 + 9$ and $g(x) = -3x + 6$ on the same system of axes.
3.2 For which value(s) of x is $f(x).g(x) \leq 0$?

Finding the equation of a parabola

We can represent a parabola in three forms:

- $y = ax^2 + bx + c$ | Standard form
- $y = a(x - x_1)(x - x_2)$ | x-intercept formula, with x_1 and x_2 being the x-intercepts
- $y = a(x - p)^2 + q$ | TP formula, with TP $(p;q)$

If the x-intercepts and a random point are given, use the x-intercept formula.

WORKED EXAMPLE 1

Determine the equation of a parabola which has points (2;0), (–4;0) and (4;8).

SOLUTION

$y = a(x - x_1)(x - x_2)$ where $x_1 = 2$, $x_2 = -4$, and the substitution point is (4;8)

Substitute x-intercepts: $y = a(x - 2)(x - (-4)) \Rightarrow y = a(x - 2)(x + 4)$

Now substitute the point: $8 = a(4 - 2)(4 + 4) \Rightarrow 8 = 16a$ and $a = \frac{1}{2}$

The equation is $y = \frac{1}{2}(x - 2)(x + 4)$ in x-intercept form or $y = \frac{1}{2}x^2 + x - 4$ in standard form.

If the turning point and a random point are given, use the turning point formula.

WORKED EXAMPLE 2

Determine the equation of the parabola if it has a turning point (1;2) and passes through (2;–1).

SOLUTION

$y = a(x - p)^2 + q$ where the turning point is (1;2) and the substitution point is (2;–1).

Substitute the turning point: (1;2): $y = a(x - 1)^2 + 2$

Now substitute the point: (2;–1) $\Rightarrow -1 = a(2 - 1)^2 + 2$ and $a = -3$

The equation is $y = -3(x - 1)^2 + 2$ in turning point form or $y = -3x^2 + 6x - 1$ in standard form.

EXERCISE 5

1 Determine the equation of the parabola which passes through the points.

1.1 (–6;0), (1;0) and (0;12)

1.2 $\left(\frac{1}{2};0\right)$, (–2;0) and $\left(-\frac{1}{2};6\right)$

1.3 $\left(\frac{1}{3};0\right)$, (3;0) and (0;1)

1.4 $\left(\frac{3}{2};0\right)$, (–4;0) and (2;–3)

2 2.1 Determine the equation of the parabola which has a turning point (3;4) and passes through the point (4;2).

2.2 Determine the equation of the parabola which has a turning point (1;–2) and passes through the point (2;1).

2.3 Determine the equation of the parabola with a turning point (–2;4) and a y-intercept of 2.

2.4 Determine the equation of the parabola which has a turning point $\left(\frac{5}{2};-\frac{9}{8}\right)$ and passes through the point (5;2).

EXERCISE 6

Complete the table and then draw each graph on its own system of axes.

- State each function in the three ways: $y = ax^2 + bx + c$, $y = a(x - x_1)(x - x_2)$ and $y = a(x - p)^2 + q$
- p is associated with the horizontal shift and q indicates the vertical shift.
- The first two rows have been completed for you as Worked examples with the given information high-lighted in blue.
- Roots (x-intercepts) tell us where a graph passes through the x-axis and $y = 0$.
- Not all parabolas have x-intercepts.

	Functions	Shift	Turning point	Min/Max	Axis of symmetry	y-intercept	x-int
1	$y = x^2 - 6x + 5$ $y = (x - 1)(x - 5)$ $y = (x - 3)^2 - 4$	3 right 4 down	(3;–4)	Minimum $y = -4$	$x = 3$	(0;5)	(1;0) (5;0)
2	$y = 2x^2 + 4x + 6$ No factorised form $y = 2(x + 1)^2 + 4$	1 left 4 up	(–1;4)	Minimum $y = 4$	$x = -1$	(0;6)	None
3			(–4;9)			(0;–7)	
4							
	$y = -(x + 5)^2 + 16$						
5							
	$y = 3(x - 4)^2 - 12$						
6		3 right 8 down				(0;10)	

Appendix B

SCHEDULE OF INTERVIEWS

1. Where do you draw most of your examples in planning phase? and Why?
2. Which and what consideration you made when you chose examples in the planning phase?
3. Why have you chosen those examples?
4. You started with example....., what you wanted learners to learn?
5. What informed your sequencing of planned examples?
6. By varying your examples, what is the purpose? What you want learners to learn?
7. In the moment of teaching, what could be the reasons for taking examples in the moment of teaching?
8. Where do you draw examples from in the moment of teaching?

Refer from the lesson plan and field notes.

9. What other teaching and learning resources are available at your school? How often/frequency you use them? Do they assist in planning a lesson and how?
10. What topics or concepts should be taught in school function? What are fundamental concepts in understanding functions that are appear in curriculum?

Appendix E: TA lesson plan

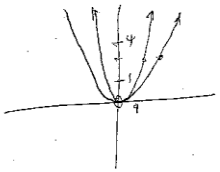
NB I have to re-write it because transparencies are not clear.

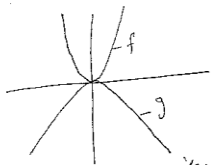
$y = x^2$

$y = 2x^2$

x	-4	-3	-2	-1	0	1	2
x^2	16	9	4		0	1	4
$2x^2$	32	18	8		0	2	8

$a = 2 > 0$





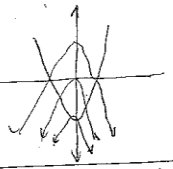
$0 < a < 1$ – wide

$a < 0$ – down

$y = -\frac{1}{2}x^2$

$y = +\frac{1}{2}x^2 - 2$

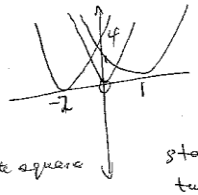
$y = -\frac{1}{2}x^2 - 2$



Horizontal shift: $y = (x - p)^2$

$p > 0$ left

$p < 0$ right



Complete square

$y = x^2 - 4x + 4$

$y = (x - 2)^2$.

standard: $y = ax^2 + bx + c$

turning point: $y = a(x - p)^2 + q$

x-intercept: $y = (x - x_1)(x - x_2)$

Exercise: $y = x^2 + 6x + 9$ } describe shift and TP:

$y = -x^2 - 2x - 1$ }

HW: $y = x^2 - 8x + 16$

$y = x^2 + 2x + 2$

$y = x^2 + 3x + 5$

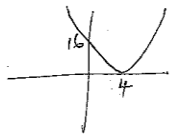
Vertical and Horizontal shifts

$y = x^2 + 2x + 2 \longrightarrow y = (x+1)^2 + 1$

$y = x^2 + 3x + 5 \longrightarrow y = (x + \frac{3}{2})^2 + \frac{11}{4}$

$y = x^2 - 8x + 16 \longrightarrow y = (x-4)^2$

No vertical: TP(4,0)



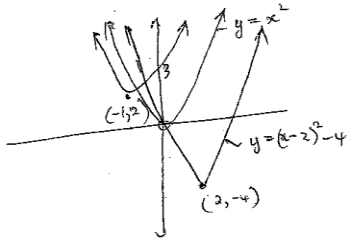
$y = a(x-p)^2 + q$: TP$(p;q)$

$y = a(x+p)^2 + q$: TP$(-p;q)$

$y = a(x+p)^2 - q$: TP$(-p;-q)$

Classwork.

Sketch: $y = x^2$, $y = (x-2)^2 - 4$ and $y = (x+1)^2 + 2$



Sketch : $y = ax^2 + bx + c$.

TP $= (p, q)$: $p = \frac{-b}{2a}$, $q = \frac{4ac - b^2}{4a}$

x - intercept is given by : let $y = 0$ then $x = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$

y - intercept is given by : let $x = 0 \therefore y = 0$

axis of symmetry : $x = p$, $x = \frac{-b}{2a}$.

Check : pg 69 Ex 4 number 1

Appendix 4 F

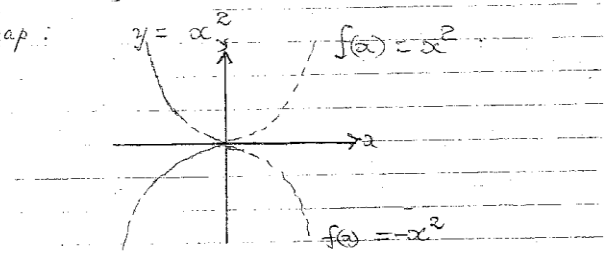
TB LESSON PLAN.

* Lesson plan

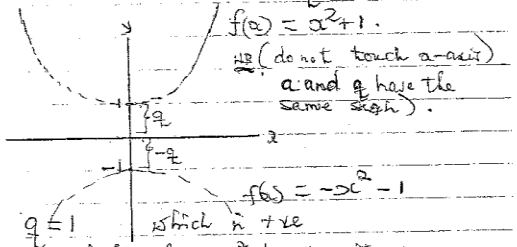
The graph of $y = ax^2 + q$

Basic shape - same as the graph of $y = x^2$

Recap: $y = x^2$ / $f(x) = x^2$.



Now $f(x) = x^2 + 1$ i.e. $a = 1$, $q = 1$.



Effects: $q = 1$ which is +ve

$\Rightarrow$ shifts $f(x) = x^2$ by 1 unit up.

Similarly shifts down if $q < 0$

NB: $a > 0$ $\therefore$ the graph is cup-shaped.
- Parabolic

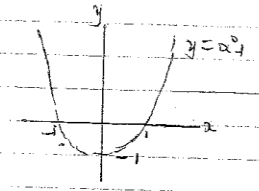
$(0,0) \rightarrow (0, q)$ on $f(x) = x^2 + 1$.

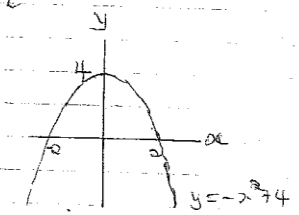
$(0,0) \rightarrow (0, -q)$ on $f(x) = -x^2 - 1$

Example 2:

$y = x^2 - 1$ and $y = -x^2 + 4$.

$\begin{cases} a = 1 \\ q = -1 \end{cases}$ $\begin{cases} a = -1 \\ q = 4 \end{cases}$





Learners to calculate x-axis intercept

NB at x-intercept $y = 0$

$\therefore$ Solve $f(x) = 0$.

$x^2 - 1 = 0$

$(x-1)(x+1) = 0$

$x = 1$ or $x = -1$

$-x^2 + 4 = 0$.

$x^2 - 4 = 0$

$(x-2)(x+2) = 0$

$x = 2$ or $x = -2$

* Maximum and min value.

If $\begin{cases} a > 0 & f(x) \text{ has min value.} \\ a < 0 & f(x) \text{ has max value.} \end{cases}$

Note: $f(x) = x^2 - 1$ has min value $y = -1$

and $f(x) = -x^2 + 4$ has max value $y = 4$.

Lesson Plan : Continuation

Class work

- Learners to attempt the following in 30 mins.
- Tr. to assist having challenges.

1. Describe what happened to the graph of $y = x^2$ to be the graph of $y = -x^2 + 1$.
   Draw the graph.

2. Draw the graph of $y = \div x^2 + 2$ and $y = -x^2 - 2$
   Show the interception on axis and TPs.

3. Consider the graph of $y = x^2$ which has been moved three units up and 2 units to the left:
   3.1. Sketch the graph
   3.2 Wrote the TP.
   3.3 Write y-intercept.

Remediation:

- Tr to assist learner do all corrections
- All solution to be written in class work book.
- Interesting comments from learner are made and conclusion are drawn.

FIELD NOTES : APPENDIX G.

TA.

Day 1 : Period : 40 mins.

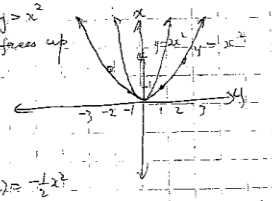
$y = x^2$

$y = 2x^2$

x	-4	-3	-2	-1	0	1	2	3	4
x^2	16	9	4	1	0	1	4	9	16
$2x^2$	32	18	8	2	0	2	8	18	32

$y = 2x^2$ is twice $y = x^2$

$a > 0$, steeper faces up

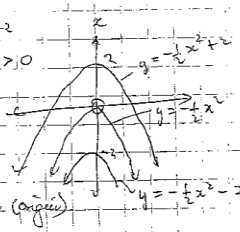


$f(x) = x^2$; $g(x) = -\frac{1}{2}x^2$

$0 < a < 1$ faces down because $a < 0$ ($a = -\frac{1}{2}$)

$y = -\frac{1}{2}x^2$ and $y = -\frac{1}{2}x^2 + 2$

moved 2 units up $q = 2 > 0$



$y = -\frac{1}{2}x^2 - 2$

$q = -2 < 0$

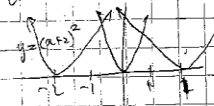
moved 2 units down (again)

Day 2 : 80 minutes

Horizontal stuff : $y = (x-p)^2$

$y = x^2$

$y = (x+2)^2$



horizontal : 2

$p > 0$: left

$p < 0$: right

Forms of parabola

Standard form

TP form form!

x - intercept form!

Why discard?

$y = x^2 + 2x + 2$

$y = x^2 + 3x + 5$

Complete the square

$y = x^2 - 4x + 7$

$= x^2 - (4x + (-2) + 4 - (-2))$ procedural

$= (x^2 - 2)^2 + 0$.

Horizontal and vertical stuffy.

$y = x^2 - 8x + 6$

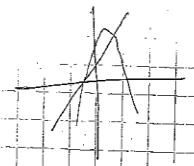
$y = (x-4)^2$ global property

classwork.

$y = x^2$, $y = (x-2)^2 - 4$ and $y = (x+1)^2 + 2$

global, algebraic manipulations.

Day 4 (40 minutes).



Two graphs.

Mixed

global

TB

Day 1 (60min) H/W.

$y = x^2$

$y = 2x^2$

$y = \frac{1}{2}x^2$

$y = -x^2$

x	
$2x^2$	
$\frac{1}{2}x^2$	
$-x^2$	

Start-up. pointwise.

$a > 0$ nearer y-axis

$0 < a < 1$ opens wider

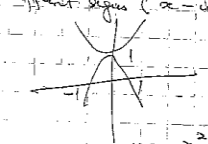
$a < 0$ reflect x-axis

$y = ax^2 + q$

Start up $\begin{cases} y = x^2 + 1 & \text{global} \quad TP(0,1) \\ y = -x^2 - 1 & \text{beneath the formula} \quad TP(0,-1) \end{cases}$

a and q : same signs (No x-intercept)

a different signs (x-intercept.



$y = x^2 - 1$ $\qquad$ $y = -x^2 + 4$ $\qquad$ global

x-intercept : $x = 1$ or $x = -1$ $\qquad$ $x = 2$ or $x = -2$

$a = 1 > 0$ min TP $\quad$ min $p = -1$

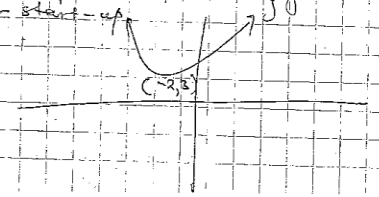
$a = -1 < 0$ max TP $\quad$ max $p = 4$ $\qquad$ Ref.

Lesson 2 (70 min).

Examples

No 1 & 2 Examples – reference } global

3 New – start-up

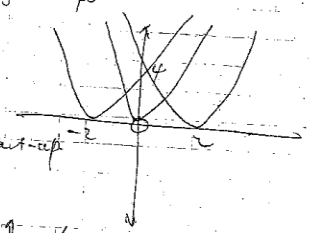


Horizontal shift : $y = (x - p)^2$

$f(x) = y = x^2$

$g(x) = (x-2)^2$

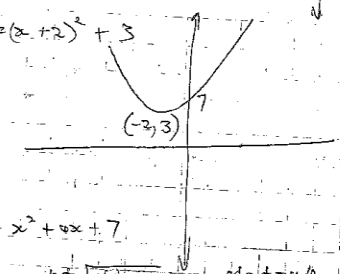
$h(x) = (x+2)^2$



$p > 0$ left

$p < 0$ right } start-up

$y = (x+2)^2 + 3$



$y = x^2 + 4x + 7$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $\qquad$ start-up combining functions with quadratic theory

$= \frac{-4 \pm \sqrt{4^2 - 4(1)(7)}}{2(1)}$

$= \frac{-4 \pm \sqrt{-12}}{2}$

Complete the square : $y = 2x^2 + 4x - 6$

$y = 2(x^2 + 2x - 3)$ $\qquad$ take 2 as common factor

$y = 2(x^2 + 2x + 1 - 3 - 1)$ $\qquad$ add half of coefficient of x and square it, subtract

$y = 2(x+1)^2 - 8$

Lesson 3

$y = 2x^2 - x - 6$

AS : $x = \frac{-b}{2a}$

$= \frac{-(1)}{2(2)}$

$= \frac{1}{4}$

TP : substitute AS

or use $q = \frac{4ac - b^2}{4a}$

value of x of AS is part of TP.

y- intercept : $y = c = -6$.

x - intercept : Solve for x

$2x^2 - x - 6 = 0$

$x = -\frac{3}{2}$ or $x = 2$.

Lesson 4.

Pg 85 Exercise 8.2 No 1 and 4 (35 - 40 minutes)

Appendix H

(I = MOETI & TA)

I : Where do you draw most of your examples when planning your lesson? Where do you take your examples from, which source are you using to plan examples.

TA : Sometimes from textbook, sometimes from my head.

I : Where do you draw these examples (showing him his planned examples from Lesson plan and transparencies in scenarios 1 to 6)

TA : For examples I didn't use the textbook because these examples are from Grade 10, I used my experience.

I : So the rest are from textbook? And why?

TA : It's a new topic therefore I need to take relevant examples to support learners' knowledge

I : When you plan your lesson, there are things that you considered, which things you considered?

TA : I considered ATP (Annual Teaching Plan) which guides me what to teach. You know.... I can't teach anything I want.

I : I mean in terms of the topic and learners?

TA : Ok, ok.... I consider the strength and weakness of the two groups. I take examples that will suit both groups. Learners are not the same, start with simple examples in all topics....we want all learners to know.

I : after $y = x^2$ and $y = 2x^2$, you chose and used $y = x^3$ and $y = -\frac{1}{2}x^2$. You seemed to have difficulty in explaining the wideness of the graph of $y = -\frac{1}{2}x^2$?

TA : Yes (laughing).....I should have taken $y = \frac{1}{2}x^2$. I was hasten to show the reflection on the line $y = 0$. I was doing revision so learners know this.

I : You decided to change $y = \frac{1}{2}x^2 - 2$ to $y = -\frac{1}{2}x^2 - 2$.

TA : This problem ($y = \frac{1}{2}x^2 - 2$) will have confused learners because we have positive halve and this one half is negative (see scenario 3). (Pause). I wanted, wanted to compare +2 and -2 .... the vertical shift.

I : You used table method to plot these parabolas (pointing on examples in scenario 1 – 4) and you used global plotting in this one (scenario 5). Why?

TA : Learners understood the table method. Actually table method is basic method before substituting values into the formula then to Cartesian plan. It will be waste of time to do a long method when and easier and effective one is available.

I : I see, if you have to change a planned example or add another one in the moment of teaching, what will inform your decision?

TA : Learners may not understand, I add another example that is easier for them. Sometimes example that you have planned may confuse learners like this one (pointing to $y = \frac{1}{2}x^2 - 2$

I : Where do you take them from?

TA : From the textbooks, I create others. It depends on the topic, other topics you can't take examples haphazardly or from your head.

I : Let's see these examples these ones (pointing to examples in scenario 7), I see that you are dealing with converting from standard form to TP form, you have chosen and used examples that are easy to factorise; why you sequence them in this order?

TA : I started with normal factorization which is not difficult. We did these in quadratic equations (meaning $ax^2 + bx + c = 0$). Completing the square is more challenging, I know in quadratic equation. Take simple examples before doing challenging ones.

I : Why you left planned examples $y = x^2 - 8x + 16$ and $y = x^2 + 3x + 5$ and gave it to learners as Homework not classwork?

TA : I was doing vertical and horizontal shifts. $y = x^2 - 8x + 16$ does not have vertical shifts, other TP forms, you have seen that it didn't have it. I was also doing the completing of the square and wanted to compare TP forms with no vertical shifts and those that have both (meaning vertical and horizontal shifts). Learners must know the difference.

I : Where do you derived your notes (see scenario 11)? What is you source?

TA : From the textbook (Platinum Mathematics for Grade 11). They are compressively written, I liked that.

I : You compared the graphs of $y = x^2$, $y = (x-2)^2 - 4$ and $y = (x+1)^2 + 2$ and learners have to draw them in same system of axis. What was the purpose of doing that?

TA : Yes, in that case, they can see TP is going to change according to the notes I have written. Eer, to see that *p and q* are also the coordinates of TP

I : You have used slides – transparency extensively, how do they help you? I can see you have 1,2,3,4 (I counted them) slides in all lessons I have observed.

TA : I am teaching two groups, I can use them with all groups. I need not to write it again and again one thing. I can hide this one and speak about this.

I : The way you teach these groups you teach them using the same examples?

TA : Same examples, Yes, yes! I can change a little bit depending on their understanding. Two groups are not the same.

I : Let me take back to these examples where you were teaching learners to sketch:

$f(x) = -x^2 + 4x + 12$ *and* $g(x) = 4x+8$. Why did you take these examples?

TA : Straight line they did study in grade 10. Parabola they are doing it now in grade 11. We were trying to find the point of intersection of two graphs, they can apply algebraic method and graphical method, so from these examples we were trying to find points of interception and show the difference of the two graphs. Inequalities (referring to reading the values) are very important also for grade 12.

I : What do you find difficult for the learners to understand in the school functions?

TA : Learners find working with inequalities between graphs to be difficult *(reading values between two graphs)*. (Pause)....I think also deriving equations from the graphs.

I : According to your experience, which concepts or topics you think are important for teaching and learning functions.

TA : All the topics in the curriculum are very important.

I : In which manner?

TA : Functions are important for teaching how to sketch graphs.....to interpret information from the graphs like in other graphs of statistics.

I : Your school is well-resourced, what other resources are there?

TA : Computer Labs for Mathematics, Geogebra to sketch and interpret graphs, to show different graphs.

I : Do you use these resources and how often?

TA : In Maths we don't completely use them regularly but we use them.

I : What advantages do you get from these?

TA : Save teaching time but takes time for preparing them. (Pause).....Learners are technologically inclined, they learn better.

I : These resources assist in planning a lesson and how they help you?

TA : In topics that need visuals like analytical geometry and Geometry

I : Thank you for your time and allowing me to observe you.

TA : Thanks

Appendix I

INTERVIEW

(I = MOETI & TB)

I : Where do you draw most of your examples when planning you lesson? Which resources are you using to draw you examples

TB : These ones (pointing to scenario 1) are from me and the rest from different books.

I : I mean in any topics?

TB : Ok, eish, from the textbooks, myself sometimes question papers.

I : Which examples in these planning are from you and question papers

TB : These ones (pointing to scenario 1) are from me, its grade 10 stuff. These ones (scenario 3 examples) are from the exemplar question papers.

I : Why you have to use different resources to plan your lesson?

TB : It is important to pick-up exercises that will make my children understand the topics well. When I use different sources I think of their background and how I can help them understand.

I : What things do you consider when you prepare your lesson?

TB : I consider different factors in time of preparation, level of understanding of my children that I teach as well as not forgetting to develop those learners who are weak. When I prepare, I prepare from easy to moderate and more difficult examples. I consider the time for teaching.

I : Why you sequenced these examples (scenario 1) like that.

TB : Check grade 10, they are sequenced that way.

I : meaning the curriculum and/or textbooks shows them that way?

TB : The textbooks and ATP (Annual Teaching Plan)

I : In these ones (pointing to scenario 2)? The textbooks do not show that?

TB : Yes, that is true. What is important here is showing that *a and q* have different or same signs how to draw them even conjecture.

I : I must admit that was unique way of showing the effect of parameters *a and q*. Where do you really learnt about this, because I checked in the textbooks and ATP, such information is not there?

TB : (smiling)...From ACE (Advanced Certificate in Education).

I : Let us go back to sequencing of your examples. Why you chose and used exercise 3 in your classwork (scenario 3) because learners haven't done combination of vertical and horizontal shifts?

TB : we called that discovery by learning broar (brother). Learners should be able to investigate and discover for themselves and I can say my children did well. Questions were leading them. By the way number 1 and 2 are vertical shifts so was assessing them.

I : $y=(x+1)^2-2$ appeared in your lesson plan but decided not to use it for vertical and horizontal shifts, instead you opted for $y=(x+2)^2+3$ which comes from exercise 3. Why?

TB : This problem (pointing to $y=(x+2)^2+3$ exercise 3) some of my children did not understand it well, you have seen that. I was elaborating more (on it) since this one (pointing to $y=(x+1)^2-2$, planned example) has same effect with this one. (Pause)....it is important to link from that.

I : When did you realise that exercise 3 has non-real roots?

TB : Another reason for taking this problem I have seen it during planning that it has non-real roots and no x-intercepts. Linking discriminant to the graphs (meaning parabolas). You see, I told them before about discriminant tells us about the nature of x-intercepts.

I : Why you instructed learners not do x-intercept form of quadratic function in Exercise 3 (see scenario 6)?

TB : I don't want them to do many thinks at the same time they get confused

I : You converted the standard form $y=ax^2+bx+c$ to TP (pointing in the lesson plan, see scenario 7) form by completing the square after you have done $y=2x^2+4x-6$. Why you sequence them in that order?

TB : Broer, variables confuses my children. The standard form was meant to link its variables to the same parameters of parabola in TP. I can't do completing the square and linking features at the same time. If I could have done that first, my fear is that children could not comprehend the link.

I : generally speaking, if you have to change a planned example or add another one in the moment of teaching, what will inform your decision?

TB : Pause....many things broar. When my children do not understand I change or modify the problem. Sometimes you took a problem which is difficult and can take an easier one. You know, you see it in the class (laughing). Mnn, sometimes you have few minutes for the period to be over and you didn't do some topics you planned and want to give your children homework; I change some of the exercises.

I : Normally, where do you draw examples in the moment of teaching?

TB : Two things, my textbook is always with me. I use it, I am experience Maths teacher so I take them from my heart, I know them.

I : You are also having extra-classes in the afternoon. Why do you have them?

TB : It helps children to understand mathematics more; there is outcry in our province that there is problem in Maths so children must practice Maths under supervised condition because during the day I teach basically new topics and extra-classes I use them for marking and practising.

I : Can you say the examples you take during the day and extra-classes are the same?

TB : They are the same but in the afternoon they are intensive. Afternoons I use more question papers and anything else.

I : You emphasise more on the graphing of functions? You explain different functions. What do you think should be done in function except sketching the graph?

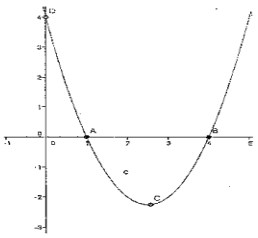
TB : Functions are symbols in Mathematics because sometimes you give learners a graph and ask learners to interpret, to calculate x-axis, the TP and the like. It is not always that you want learners to draw. Sometimes you give learners a graph and ask something in graph you know Domain and Range even the equation of the graph. When I start with the drawing the graph, they can see how the equation of the graph is sketched and back.

I : What do you think learners are struggling with in function?

TB : Most cases learners struggle with interpretation of the graph like most of us we teach learners to draw and not to interpret the graph.

I : What do you mean by interpretation? Can you take an example?

TB : Like I can give a learner this graph



I can give a learner this point here and this point here. From here I can ask the learner to give me the formula, children can work from the graph to the formula not from formula to the graph.

I : So learners have difficulty to move from graph to formula. Another one?

TB : Ok, another one whereby we have two types of graphs like linear and parabola that I took (referring to examples exercise in scenario 8). Children have to tell us if you compare linear and parabola at which values of x will one graph be greater or smaller than another one.

I : What teaching and learning resources are available in your school?

TB : Textbooks, wookbooks and calculators.

I : How lack of resources affect your teaching?

TB : As you can see, most of children are relying on me for their support. Textbooks are not enough. I have to photostat books and previous question papers.

Appendix J: North West Department of Education and Training Approval letter

education and training
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DR KENNETH KAUNDA DISTRICT

OFFICE OF THE EXECUTIVE MANAGER

To : Mr M.P Moeti
Educator - Vuyani Mawethu Secondary School

From : Dr S.H Mvula
Executive District Manager

Date : 03 May 2012

PERMISSION TO CONDUCT RESEARCH

We hereby acknowledge receipt of your e-mail dated 19 April 2012. Please note that your request to the above subject has been granted under the following provisions:

1. The activities you undertake at schools should not tamper with the normal process of learning and teaching;
2. You inform the principals of your identified schools of your impending visit and activity; and
3. You obtain prior permission from this office before availing your findings for public or media consumption.

Wishing you well in your endeavour.

Thanking you

DR S.H MVULA
EXECUTIVE DISTRICT MANAGER