# RESOLVABILITY OF GROUPS

by

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# Abstract

ABSTRACT. A topological group is called resolvable if it can be partitioned into two dense subsets. A group is absolutely resolvable if it can be partitioned into two subsets dense in any nondiscrete group topology. The aim of this dissertation is to give a unified exposition of some major results about resolvability of groups. In particular, we show that;

1. Every countable nondiscrete topological group not containing an open Boolean subgroup is resolvable,

2. Every infinite Abelian group not containing an infinite Boolean subgroup is absolutely resolvable.

# Declaration

I declare that this Dissertation is my own, unaided work. It is submitted to the University of The Witwatersrand, Johannesburg, in conformity with the requirement for Master of Science. It has not been submitted before for any degree or examination at any other university.

Student Name: ........................................

Signature: ..................................................

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# Acknowledgments

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# Contents

# Chapter 1

# Introduction

A topological space is called *resolvabe* ($\omega$-*resolvable*) or *irresolvabe* ($\omega$-*irresolvable*) depending on whether or not it can be partitioned into two (into countably infinite many) dense subsets. A space is called maximal if it has no isolated points but has an isolated point in any stronger topology. All spaces are assumed to be Hausdorff. Irresolvable and maximal spaces were introduced by E. Hewitt [18], and M. Katétov who showed that all usual spaces are resolvable and not maximal.

The study of resolvability for topological groups was initiated by W. Comfort and J. van Mill [11]. They showed that if $G$ is an infinite Abelian group not containing an infinite Boolean subgroup, then every nondiscrete group topology on $G$ is resolvable. Later on, in that direction the following results were in particular established.

Every nondiscrete $\omega$-irresolvable topological Abelian group contains an open countable Boolean subgroup. This result, in some sense, reduces the investigation of nondiscrete $\omega$-irresolvable group topologies on Abelian groups to their investigation on a countable Boolean group $\mathbb{B}$. The fundamental relations holding for abstract groups and topological spaces are more or less bodily carried over onto topological groups. For this reason, the study inherits the concepts of subgroups, normal subgroups, the factor groups, etc. Historically, the concept of a topological group arose in connection with the consideration of groups of continuous transformations. A deeper study of this was made by D. van Dantzig [41, 42, 44].

Assuming Martin's Axiom, there is a maximal (and consequently, irresolvable) group topology on the countably infinite Boolean group [25]. However, the existence of a maximal topological group implies the existence of a $P$-point in $\omega^*$ [32]. $P$-points were invented by W. Rudin [35] who used them to show that under CH, $\omega^*$ is not homogeneous. There are models of Zermelo-Fraenkel set-theory with the axiom of choice included (ZFC), without $P$-points in $\omega^*$ [37]. Thus, the existence of a maximal topological group cannot be established in ZFC.

A topological space is called *extremally disconnected* if closures of disjoint open subsets are disjoint. We shall discuss a well-known difficult question investigating whether there exists in ZFC a nondiscrete extremally disconnected topological group, formulated by A. Arhangel'skíi [3]. We will also mention the first consistent example of such a group that was constructed by S. Sirota [38]. This is of course with the aid of having defined first the important set-theoretic principle. In particular, a theorem under $\mathfrak{p} = \mathfrak{c}$ which is the consequence of Martin's Axiom which in turn is a consequence of Continuum Hypothesis. As a consequence of the mentioned principle, V. Malykhin [25] showed that there exists a maximal topological group.

A group $G$ is *absolutely resolvable* if it can be partitioned into two dense subsets of arbitrary nondiscrete topological group. The case of *absolutely $\omega$-resolvable* follows immediately if the group can be partitioned into $w$-dense subsets of arbitrary nondiscrete topological group. Indeed otherwise, if the group $G$ does not admit a nondiscrete group topology, then, by definition, it is absolutely $\omega$-resolvable. It is also known that each countable Abelian group with no infinite Boolean subgroup is absolutely $\omega$-resolvable. Furthermore, each Abelian group with no infinite Boolean subgroup is absolutely resolvable [56].

One of the aims is to present two major theorems. The first one states that every countable $\omega$-irresolvable group topology contains an open Boolean subgroup. It follows from this that the existence of any countable nondiscrete $\omega$-irresolvable topological group implies the existence of a $P$-point in $\omega^*$. The second theorem states that every countable group with finitely many elements of order 2, that can be embedded in a compact topological group, is absolutely $\omega$-resolvable. The proofs of the above mentioned theorems is based on the inversion's automorphization and the development of the approach from [56].

# Chapter 2

# Topological Groups

In this chapter we give some basic concepts and results about topological groups. In general, one can say that a *topological group* is a set endowed with two structures: that of a group and that of a topological space. The standard reference for topological groups are [29] and [19]. More in-depth understanding about topological groups can be found also in [10], [6], and [4]. We also look at some basic application of Markov's Criteria [26] of topologizability of countable groups.

## 2.1 Notion of Topological Group

**Definition 2.1.1** (Topological Group)**.** Suppose $G$ is a group with endowed topology, then $G$ is a topological group if the multiplication

$$\mu : G \times G \ni (x, y) \mapsto xy \in G$$

and the inversion

$$\iota : G \ni x \mapsto x^{-1} \in G$$

are continuous mappings. A topology which makes a group into topological group is called a *group topology*.

In simple terms, topological group can be thought off two homogeneous joined entities, one with a topological structure and another with a group structure. That is, a *set* with *binary operations* * such that it satisfies group properties and the product map satisfies topological properties [11].

**Example 2.1.1.** *A topological vector space, such as Banach space or Hilbert space is an* ***abelian topological group*** *under addition.*

**Example 2.1.2.** *A set of real numbers* $\mathbb{R}$ *, with usual topology form a topological group.*

**Example 2.1.3.** *The general linear group* $GL(n, \mathbb{R})$ *of all* $n \times n$ *matrices with determinant different from zero and real entries can be a topological group with* *the topology defined by viewing $GL(n,\mathbb{R})$ as a subspace of Euclidean space $\mathbb{R}^{n\times n}$. This is classified as classical group.*

**Example 2.1.4.** *Orthogonal group $O(n)$ is another example of a classical group, the group of all **linear maps** from $\mathbb{R}^n$ to itself preserving the length of all vectors.*

The continuity of the multiplication and the inversion mentioned in Definition 2.1.1 is equivalent to the continuity of the mapping $\mu' : G \times G \mapsto G$ defined by the function $\mu'(x,y) = xy^{-1}$

In fact, for each $x, y \in G$;

$$\mu'(x,y) = xy^{-1} = x \cdot \iota(y) = \mu(x, \iota(y))$$

$$\iota(x) = x^{-1} = 1 \cdot x^{-1} = \mu'(1,x)$$

and also, $\mu(x,y) = x \cdot y = x(y^{-1})^{-1} = x(\iota(y))^{-1} = \mu'(x, \iota(y))$.

In particularly, the continuity of $\mu'$ means that for any $a, b \in G$ and $U$ a neighborhood of $ab$, there exist neighborhoods $V$ and $W$ of $a$ and $b$, respectively, such that

$$VW^{-1} \subseteq U.$$

Here $VW = \{xy : x \in V, y \in W\}$ and $W^{-1} = \{y^{-1} : y \in W\}$. It follows that whenever $a_i \in G$, $k_i \in \mathbb{Z}$, $\forall i = 1, ..., n$ and $U$ is a neighborhood of $a_1^{k_1} ... a_n^{k_n} \in G$, there are neighborhoods $V_i$ of $a_i$, $\forall i = 1, ..., n$, respectively, such that

$$V_1^{k_1} \cdots V_n^{k_n} \subseteq U.$$

Here $V^k = \{x_1 \cdots x_k : x_1, ..., x_k \in V\}$.

Also, the translations and the inversion of a topological group $G$ are homeomorphisms. In fact, for each $a \in G$, the *left translation*

$$\lambda_a : G \ni x \mapsto ax \in G$$

and the *right translation*

$$\rho_a : G \ni x \mapsto xa \in G$$

are continuous mappings, being restrictions of the multiplication. We know by Definition 2.1.1 the inversion $\iota$ is continuous. Since we also have that $(\lambda_a)^{-1} = \lambda_{a^{-1}}$, $(\rho_a)^{-1} = \rho_{a^{-1}}$ and $\iota^{-1} = \iota$, then all the translations are homeomorphisms.

**Definition 2.1.2** (Homogeneous)**.** Let $X$ be a topological space. Then $X$ is *homogeneous* if for each $a, b \in X$, there is a homeomorphism $f : X \to X$ such that $f(a) = b$. If $G$ is a topological group and $a, b \in G$, then $\lambda_{ba^{-1}} : G \mapsto G$ is a homeomorphism and $\lambda_{ba^{-1}}(a) = ba^{-1}a = b$.

We will show later that topological groups satisfying the first separation axiom are regular. To illustrate this concept we look at some separation axioms.

In particular, these axioms denote that for each element that can be separated in a weak sense, that element must also be separated in a stronger sense.

To establish the separation properties, we make use of Definition 2.1.2 and the lemma below.

**Lemma 2.1.1.** *The space of a topological group is homogeneous.*

In particular, one can interpret Lemma 2.1.1 as, for each $x, y \in G$, there exist a homeomorphism $f : G \to G$ such that $f(x) = y$.

**Definition 2.1.3** ($T_0$-separation axiom)**.** A space $X$ is a $T_0$-space if for each pair of distinct points there is an open set containing exactly one of the points.

**Definition 2.1.4** ($T_1$-separation axiom)**.** A space $X$ is a $T_1$-space if for each pair of distinct points, say $x, y \in X$ there is an open set $U \subset X$ such that $x \in U$, and $y \notin U$.

**Lemma 2.1.2.** *Every topological group satisfying the $T_0$-separation axiom is regular and hence Hausdorff.*

*Proof.* Let $G$ be a $T_0$ topological group. We firstly show that $G$ is a $T_1$-space. By Lemma 2.1.1 we see that $G$ is homogeneous and thus, it suffices to show that for each $x \in G \backslash \{1\}$, there is a neighborhood $U$ of $1$ not containing $x$. By Definition 2.1.3, there is an open set $U$ containing exactly one of the two points $1, x$. If $1 \in U$, we are done since $U$ can only contain one of the points. Else, $xU^{-1}$ is a neighborhood of $1$ not containing $x$.

Now, we show that for each neighborhood $U$ of $1$, there exist a closed neighborhood of $1$ contained in $U$. Suppose $V$ is a neighborhood of $1$ such that $VV^{-1} \subseteq U$. Then for each $x \in G \backslash U$, one has $xV \cap V = \emptyset$. Indeed, otherwise for some $a, b \in V$, $xa = b$, which gives us that $x = ba^{-1} \in VV^{-1} \subseteq U$, a contradiction. Hence $\operatorname{cl} V \subseteq U$. $\square$

The best possible general separation result can be given by the below theorem. However, the theorem can be improved when dealing with extended cases of topological groups like when dealing with countable topological groups.

**Theorem 2.1.1.** *Every Hausdorff topological group ($T_2$-space) is completely regular.*

It follows from the consideration of this section that the continuity of $G$ establishes a very close connection between Algebraic operations and Topological operations in a topological group $G$.

## 2.2 Neighborhood Filter of the Identity

In general, a filter is defined as a subset of a partially ordered set. Filters were introduced by Henri Cartan [7] in 1937.

**Definition 2.2.1.** Let $X$ be a given nonempty set. A *filter* on $X$ is a family $\mathcal{F} \subseteq \mathcal{P}(X)$ with the following properties:

1. $X \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$,

2. if $A, B \in \mathcal{F}$, then $A \bigcap B \in \mathcal{F}$, and

3. if $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$.

In particular, a filter on $X$ is a nonempty family $\mathcal{F}$ of nonempty subsets of $X$ closed under finite intersections and supersets.

**Definition 2.2.2** (Neighborhood)**.** Suppose $X$ is a topological space and $p$ is a point in $X$. A *neighborhood* of $p$ is a subset $V \subset X$ such that the relationship $p \in U$ holds, for any subset $U \subset V$. That is, $V$ is a neighborhood of point $p$ if its interior $U$, also contains point $p$.

A classical example of a filter and one that we will use very often, is of the set $\mathcal{N}_x$ of all neighborhoods of a point $x$ in a topological space $X$ called the *neighborhood filter* of $x$.

Furthermore, we define a *neighborhood system* of $X$, $\{\mathcal{N}_x : x \in X\}$, as a system of all neighborhood filters on $X$.

Note that a neighborhood $U$ of the identity element $1$ of a topological group $G$ is said to be *symmetric* if $U = U^{-1}$. Now, suppose that $U$ a neighborhood of the identity element $1$ in topological group $G$ is defined, and let $V = U \bigcap U^{-1}$. It follows that $V = U \bigcap U^{-1} = (U \bigcap U^{-1})^{-1} = V^{-1}$, and $V \subseteq U$ is the neighborhood of the identity. Hence, every neighborhood of the identity contains a symmetric neighborhood.

Same as we have the notion of a Base in topology, we have a similar notion with filters.

**Definition 2.2.3.** Let $X$ be a nonempty set. A *filter base* on $X$ is a nonempty family $\mathcal{B} \subseteq \mathcal{P}(X)$ with the following properties:

i) $\emptyset \notin \mathcal{B}$, and

ii) for each set $A, B \in \mathcal{B}$ there is $C \in \mathcal{B}$ such that $C \subseteq A \cap B$.

Subsequently, $\mathcal{B} \subseteq \mathcal{P}(X)$ is a filter base if

$$\mathcal{F} = \{A \subseteq X : A \supseteq B \text{ for some } B \in \mathcal{B}\}$$

is a filter , and we thus say $\mathcal{B}$ is a *base* for $\mathcal{F}$. One can simply see that if $\mathcal{F}$ is a filter, then $\mathcal{B} \subseteq \mathcal{F}$ is a base for $\mathcal{F}$ if and only if for each $A \in \mathcal{F}$ there is $B \in \mathcal{B}$ such that $B \subseteq A$.

We conclude this section by giving the following Corollary:

**Corollary 2.2.1.** *Let $\mathcal{B}$ be a filter base on $G$ satisfying the following conditions:*

*1) for each set $U \in \mathcal{B}$, there is $V \in \mathcal{B}$ such that $VV \subseteq U$,*

*2) for each $U \in \mathcal{B}$, $U^{-1} \in \mathcal{B}$, and*

*3) for each $U \in \mathcal{B}$ and element $x \in G$, $xUx^{-1} \in \mathcal{B}$.*

*Then, there exists a unique topological group $\mathcal{T}$ on $G$ for which $\mathcal{B}$ is a neighborhood base at $1$. Also, $\mathcal{T}$ is Hausdorff if and only if*

*4)* $\bigcap \mathcal{B} = \{1\}$.

## 2.3 Topologizing a Group

In this section we discuss a theorem that is due to A. Markov [26] and an example of a nontopologizable group. The first example of a nontopologizable group was produced by S. Shelah [36]. It was an uncountable group and its construction used the Continuum Hypothesis.

**Definition 2.3.1.** Let $G$ be a countably infinite group. Enumerate $G$ as $\{g_n : n < \omega\}$ without repetitions and with $g_0 = 1$.

1. For each infinite sequence $(a_n)_{n=1}^{\infty}$ in $G$, define the subset $U((a_n)_{n=1}^{\infty}) \subseteq G$ by
$$U((a_n)_{n=1}^{\infty}) = \bigcup_{n=1}^{\infty} \bigcup_{\pi \in S_n} \prod_{i=1}^{n} B_{\pi(i)}$$
where $B_i = \bigcup_{j=0}^{\infty} g_j \{1, a_{i+j}^{\pm 1}, a_{i+j+1}^{\pm 1}, ...\} g_j^{-1}$.

2. For each finite sequence $a_1, ..., a_n$ in $G$, define $U(a_1, ..., a_n) \subseteq G$ by
$$U(a_1, ..., a_n) = \bigcup_{\pi \in S_n} \prod_{i=1}^{n} B_{\pi(i)}^{n}$$

where $B_i^n = \bigcup_{j=0}^{n-1} g_j\{1, a_{i+j}^{\pm 1}, a_{i+j+1}^{\pm 1}, ..., a_n^{\pm}\}g_j^{-1}$. That is, $U(a_1, ..., a_n)$ consists of all elements of the form

$$g_{j_1} c_1 g_{j_1}^{-1} \cdots g_{j_n} c_n g_{j_n}^{-1},$$

where $j_i \in \{0, ..., n - \pi(i)\}$ and $c_i \in \{1, a_{\pi(i)+j_i}^{\pm 1}, ..., a_n^{\pm}\}$ for every $i \in \{1, \ldots, n\}$ and $\pi \in S_n$. In particular, $U(a_1) = \{1, a_1^{\pm}\}$. Also, put $U(\emptyset) = \{1\}$.

3. For each finite sequence $a_1, ..., a_{n-1}$ in $G$, let $W(a_1, ..., a_{n-1}, x)$ denote the set of group words $f(x)$ in the alphabet $G \cup \{x\}$ in which variable $x$ occurs and which have the form

$$f(x) = g_{j_1} c_1 g_{j_1}^{-1} \cdots g_{j_n} c_n g_{j_n}^{-1},$$

where $j_i \in \{0, ..., n - \pi(i)\}$ and $c_i \in \{1, a_{\pi(i)+j_i}^{\pm 1}, \cdots, a_{n-1}^{\pm 1}, x^{\pm 1}\}$ for each $i \in \{1, ..., n\}$, and $\pi \in S_n$. In particular, $W(x)$ consists of two group words $x$ and $x^{-1}$.

However, in a case where $G$ is Abelian, all the above become straightforward. In particular,

$$B_i^n = \{0, \pm a_i, \ldots, \pm a_n\} \quad \text{and} \quad U(a_1, ..., a_n) = \sum_{i=1}^{n} B_i^n.$$

We can now give a theorem that gives us a method of topologizing a countable group since we also know Definition 2.3.1.

**Theorem 2.3.1.** *For every sequence $(a_n)_{n=1}^{\infty}$ in $G$, the following statements hold:*

1. *$U((a_n)_{n=1}^{\infty})$ is a neighborhood of $1$ in $\mathcal{T}((a_n)_{n=1}^{\infty})$,*

2. *$U((a_n)_{n=1}^{\infty}) = \bigcup_{n=1}^{\infty} U(a_1, ..., a_n)$,*

3. *$U(a_1, ..., a_n) = U(a_1, ..., a_{n-1}) \cup \{f(a_n) : f(x) \in W(a_1, ..., a_{n-1}, x)\}$ for each $n \in \mathbb{N}$, and*

4. *for each $n \in \mathbb{N}$ and $f(x) \in W(a_1, ..., a_{n-1}, x)$, $f(1) \in U(a_1, ..., a_{n-1})$.*

*Proof.* See (Theorem 1.19 , [56]) □

**Theorem 2.3.2** (Markov's Criterion)**.** *A countable group $G$ is topologizable if and only if every finite system of inequalities over $G$ having a solution has also another solution.*

A. Markov asked a big question about every infinite group being topologizable. We will now show in detail how this idea is true for both infinite Abelian groups and free groups. We will also answer A. Markov's question by giving a sufficient example of infinite nontopologizable group.

**Theorem 2.3.3.** *For every Abelian group $G$ and for each $a \in G\backslash\{0\}$, there is a homomorphism $f : G \to \mathbb{T}$ such that $f(a) \neq 1$.*

*Proof.* Firstly, we define a homomorphism $f_0 : \langle a \rangle \to \mathbb{T}$ such that $f_0(a) \neq 1$. If $a$ has finite order, say $n$, define

$$f(ka) = e^{\frac{2\pi i k}{n}}$$

for each $k = 1, ..., n$. If $a$ has infinite order, choose any $x \in \mathbb{T} \backslash \{1\}$ and put

$$f(ka) = x^k$$

for all $k \in \mathbb{Z}$.

Then for $H$ a subgroup of $G$ containing $\langle a \rangle$, consider the set of all pairs $(H, g)$ and $g : H \to \mathbb{T}$ a homomorphism extending $f_0$, ordered by

$$(H_1, g_1) \leq (H_2, g_2) \quad \text{if and only if} \quad H_1 \subseteq H_2 \text{ and } g_2|_{H_1} = g_1.$$

By Zorn's Lemma, there is a maximal pair $(H, f)$. We claim that $H = G$. Indeed, assume the contrary that there is $c \in G \backslash H$. To derive a contradiction, let $H' = H + \langle c \rangle$ and define a homomorphism $f' : H' \to \mathbb{T}$ extending $f$. If there is $n \in \mathbb{N}$ with the product $nc \in H$, choose the smallest such $n$ and then $z \in \mathbb{T}$ such that $z^n = f(nc)$, and if there is no such $n$, choose arbitrary $z \in \mathbb{T}$. Define

$$f'(b + kc) = f(b + kc) = f(b) \cdot z^k$$

for every $b \in H$ and $k \in Z$. □

**Definition 2.3.2.** Let $G$ be a topological group. Then $G$ is called *totally bounded* if it is Hausdorff and for each nonempty open $U \subseteq G$ there is a finite $F \subseteq G$ such that $FU = G$.

Also, a topological group is totally bounded if and only if it can be topologically and algebraically embedded into a compact group.

**Corollary 2.3.1.** *Every infinite Abelian group admits a totally bounded group topology.*

*Proof.* Let $G$ be an infinite Abelian group. By Theorem 2.3.3, for each $a \in G\backslash\{0\}$ there is a homomorphism $f_a : G \to \mathbb{T}$ with $f_a(a) \neq 1$. Let $K = \prod_{a \in G\backslash\{0\}} \mathbb{T}_a$ where $\mathbb{T}_a = \mathbb{T}$. Now define the mapping $f : G \to K$ by $(f(x))_a = f_a(x)$. Clearly $f$ is an injective homomorphism. Being a subgroup of a compact group, $f(G)$ is totally bounded. Hence, the topology on $G$ consisting of subsets $f^{-1}(U)$, where $U$ ranges over open subsets of $f(G)$, is as required.

□

**Corollary 2.3.2.** *Every free group admits a totally bounded group topology*

**Example 2.3.1.** *Let $m, n \in \mathbb{N}$ such that $m \geq 2$, $n \geq 665$ and $n$ is odd. Consider the Adian group $A(m.n)$. This is a torsion free $m$-generate group whose center is an infinite cyclic group $\langle c \rangle$ and the quotient $A(m, n)/\langle c \rangle$ is an infinite group of* *period $n$. In fact, the quotient $A(m,n)/\langle c\rangle$ is the Burnside group $B(m,n)$, i.e the largest group on $m$-generators satisfying the identity $x^n = 1$.*

*Let $x \in A(m,n)/\langle c\rangle$. It is clear that $x^n \in \langle c\rangle$, since $A(m,n)/\langle c\rangle$ has period $n$.*

*We claim that $x^n \notin \langle c^n\rangle$. To see this, assume the contrary. So $x^n = (c^n)^k = (c^k)^n$ for some $k \in \mathbb{Z}$. Let $z = xc^{-k}$. Then $z \notin \langle c\rangle$ and $z^n = x^n c^{-kn} = 1$, since $\langle c\rangle$ is the center. But this contradicts that $A(m,n)$ is torsion free. Now let $G = A(m,n)/\langle c^n\rangle$ and let $S = \langle c\rangle/\langle c^n\rangle$. Then $G$ is infinite,*

$$S = \{1, s_1, \ldots, s_{n-1}\} \subset G$$

*and for each $x \in G\backslash S$, one has $c^n \in \{s_1, \ldots, s_{n-1}\}$. It follows that for every $T_1$-topology (Definition 2.1.4) on $G$ in which $1$ is not an isolated point, the mapping*

$$G \ni x \mapsto x^n \in G$$

*is discontinuous at $1$. Thus, $G$ is nontopologizable.*

The above example is due to A. Ol' śanskií [28]. This example serves as sufficient proof that, not all infinite groups can be topologizable. The Adian group see S. Adian [1] and the quotient of Adian group (Burnside group) $B(m,n)$ was proved by P. Novikov and S. Adian [27].

# Chapter 3

# Ultrafilters

In this chapter we introduce and give some basic facts about ultrafilters and the Stone-Ćech compactification of discrete spaces. We will also show how the Ultrafilter Theorem is equivalent to the Prime Ideal Theorem, which is a well known *weaker* form of Axiom of Choice (see T. Jech [23]).

The studies of Ultrafilters were firstly introduced by F. Riesz [34] and S. Ulam [40]. In the perspective of compact Hausdorff space, each one of these ultrafilters converges to a single point.

## 3.1 Notion of Ultrafilter

Using the concept of filters (Definition 2.2.1) studied in section 2.2, we give the following definition on nonempty set $D$.

**Definition 3.1.1.** An *ultrafilter* on nonempty set $D$ is a filter on $D$ which is not properly contained in any other filter on $D$.

In particular, an ultrafilter can be thought of as a maximal filter.

**Definition 3.1.2.** A family $\mathcal{A} \subseteq \mathcal{P}(D)$ is said to have *finite intersection property* if $\bigcap \mathcal{B} \neq \emptyset$ for all $\mathcal{B} \subseteq \mathcal{A}$.

From Definition 3.1.2, we can clearly see that every filter has the finite intersection property. Conversely, every family $\mathcal{A} \subseteq \mathcal{P}(D)$ with the finite intersection property generates a filter flt($\mathcal{A}$) on $D$ by

$$\text{flt}(\mathcal{A}) = \{A \subseteq D : A \supseteq \bigcap \mathcal{B} \text{ for some finite } \mathcal{B} \subseteq \mathcal{A}\}.$$

**Example 3.1.1.** *A proper filter on a set has finite intersection property.*

**Proposition 3.1.1.** *Let $\mathcal{A} \subseteq \mathcal{P}(D)$. Then the following statements are equivalent:*

*1) $\mathcal{A}$ is an ultrafilter,*

*2) $\mathcal{A}$ is a maximal family with the finite intersection property,*

*3) $\mathcal{A}$ is a filter and for every $A \subseteq D$, either $A \in \mathcal{A}$ or $D \backslash A \in \mathcal{A}$.*

*Proof.* $1) \implies 3)$

Suppose $\mathcal{A}$ is an ultrafilter. We consider two cases.

Case 1: there is a set $B \in \mathcal{A}$ such that $B \cap A = \emptyset$. That is, $B$ and $A$ are *disjoint*. Then clearly $B \subseteq D \backslash A$ and, since $\mathcal{A}$ is closed under supersets, $D \backslash A \in \mathcal{A}$.

Case 2: for every $B \in \mathcal{A}$, $B \cap A \neq \emptyset$. That is, $B$ and $A$ are *adjoint*. Then

$$\{B \cap A : B \in \mathcal{A}\}$$

is a family of nonempty sets closed under finite intersections, since $\mathcal{A}$ is closed under finite intersections. Consequently,

$$\mathcal{F} = \{C \subseteq D : C \supseteq B \cap A \text{ for some } B \in \mathcal{A}\}$$

is a filter. We see that $\mathcal{A} \subseteq \mathcal{F}$ and $A \in \mathcal{F}$, and thus since $\mathcal{A}$ is an ultrafilter, then $\mathcal{A} = \mathcal{F}$. Hence $A \in \mathcal{A}$.

$3) \implies 2)$

Suppose the contrary, that is, $\mathcal{A}$ is a filter on D and for each $A \subseteq D$, $A \notin \mathcal{A}$. Therefore, $D \backslash A \in \mathcal{A}$ and $A \cap (D \backslash A) = \emptyset$. Since the latter holds, $\mathcal{A} \cup \{A\}$ has no finite intersection property.

$2) \implies 1)$

Suppose $\mathcal{A}$ is a maximal family with the finite intersection property. Since $\mathcal{A}$ has finite intersection property, so does flt($\mathcal{A}$), and since $\mathcal{A}$ is maximal, flt($\mathcal{A}$) $=$ $\mathcal{A}$. It follows that $\mathcal{A}$ is a filter, and consequently, an ultrafilter. □

Furthermore, these ultrafilters can be classified in two ways, depending on the properties they have. We can see from Definition 3.1.2 and Proposition 3.1.1 that for each $x \in D$, $\{A \subseteq D : x \in A\}$ is an *ultrafilter* and such ultrafilters are called *principal*. Ultrafilters which are not principal are called *nonprincipal*.

From these we can state the below lemma without proof.

**Lemma 3.1.1.** *Let $\mathcal{U}$ be an ultrafilter on nonempty set $D$. Then the following statements are equivalent:*

*1) $\mathcal{U}$ is a nonprincipal ultrafilter,*

*2) $\bigcap \mathcal{U} = \emptyset$,*

*3) for every $A \in \mathcal{U}$, $|A| \geq \omega$.*

**Lemma 3.1.2** (Zorn's Lemma)**.** *Let $P$ be a partially ordered set, and let $C$ be a family chains in $P$ with upper bounds. Then the set $P$ contains at least one maximal element.*

**Proposition 3.1.2** (Ultrafilter Theorem)**.** *Every filter on $D$ can be extended to an ultrafilter on $D$.*

*Proof.* Suppose $\mathcal{F}$ is a filter on D and let

$$P = \{\mathcal{G} \subseteq \mathcal{P}(D) : \mathcal{F} \subseteq \mathcal{G} \text{ and } \mathcal{G} \text{ is a filter on } D\}.$$

If we are given any chain $C$ in $P$, then $\bigcup C$ is a filter, and so $\bigcup C$ is an upper bound of $C$. Hence, by Zorn's Lemma 3.1.2, $P$ has a maximal element $\mathcal{U}$. Clearly $\mathcal{U}$ is a maximal filter. $\square$

We say that an ultrafilter $\mathcal{U}$ on nonempty set $D$ is *uniform* if for every $A \in \mathcal{U}$, $|A| = |D|$. We can further show that there are *uniform ultrafilters* on any *infinite* set. To show this, let $\mathcal{F} = \{A \subseteq D : |D \backslash A| < |D|\}$ be a filter, and $D$ be any infinite set. Then it follows by Proposition 3.1.1 that there is an ultrafilter $\mathcal{U}$ on $D$ containing $\mathcal{F}$. If $A \subseteq D$ and $|A| < |D|$, then $D \backslash A \in \mathcal{F} \subseteq \mathcal{U}$, so $A \notin \mathcal{U}$. Thus, $\mathcal{U}$ is uniform.

**Definition 3.1.3.** Let $\mathcal{F}$ be a filter on $D$. If $C \subseteq D$ and $C \cap A \neq \emptyset$ for every $A \in \mathcal{F}$, then the *trace* of $\mathcal{F}$ on $C$,

$$\mathcal{F}|_C = \{C \cap A : A \in \mathcal{F}\}$$

is a filter on $C$. If $f : D \to E$, then

$$f(\mathcal{F}) = \{f(A) : A \in \mathcal{F}\}$$

is a filter base on $E$ called the *image* of $\mathcal{F}$ with respect to $f$.

We see that if $f$ is surjective, then $f(\mathcal{F})$ is a filter and it is quite obvious that if $\mathcal{F}$ is an ultrafilter, so is $\mathcal{F}|_C$.

**Definition 3.1.4.** Let $X$ be a space, then $X$ is *compact* if every open cover of $X$ has a finite sub-cover.

Equivalently, $X$ is compact if every family of closed subsets of $X$ with the finite intersection property has a nonempty intersection.

Recall the definition of a filter base $\mathcal{B}$, Definition 2.2.3. A filter base $\mathcal{B}$ on a space $X$ *converges* to a point $x \in X$ if for each neighborhood $U_x$ of $x \in X$, there is $B \in \mathcal{B}$ such that $B \subseteq U_x$. Furthermore, in a case where $\mathcal{B}$ is a filter, $\mathcal{B}$ converges to $x$ if and only if $\mathcal{B}$ contains the neighborhood filter of $x$.

**Lemma 3.1.3.** *A space $X$ is Hausdorff if and only if every ultrafilter converges to at most one point.*

**Proposition 3.1.3.** *A space $X$ is compact if and only if every ultrafilter on $X$ is convergent.*

*Proof.* Let $X$ be a compact space and let $\mathcal{U}$ be an ultrafilter on $X$. Suppose the contrary, that is, for each point $x \in X$, there is a neighborhood $U_x$ of $x$ such that $U_x \notin \mathcal{U}$. Clearly one can choose $U_x$ to be open. Then, since $\mathcal{U}$ is an ultrafilter on $X$, we have $X \backslash U_x \in \mathcal{U}$. The open sets $U_x$, where $x \in X$, cover $X$. That is, $X \subseteq \bigcup\{U_x : x \in X\}$. Since $X$ is compact, there is a finite subcover $\{U_{x_i} : i < n\}$ of $\{U_x : x \in X\}$. But then $\emptyset = \bigcap_{i<n}(X \backslash U_{x_i}) \in \mathcal{U}$, contradiction.

Conversely, suppose that every ultrafilter on $X$ is convergent and let $\mathcal{A}$ be a family of closed subsets of $X$ with the finite intersection property. Suppose the contrary, that is, $\bigcap \mathcal{A} = \emptyset$. By Proposition 3.1.2, there is an ultrafilter $\mathcal{U}$ on $X$ such that $\mathcal{A} \subseteq \mathcal{U}$. We claim that $\mathcal{U}$ is not convergent. Indeed, let $x \in X$. Since $\bigcap \mathcal{A} = \emptyset$, there is $F_x \in \mathcal{A}$ such that $x \notin F_x$. Then $U_x = X \backslash F_x$ is a neighborhood of $x$ and $U_x \notin \mathcal{U}$. Hence $\mathcal{U}$ is not convergent, a contradiction. $\square$

## 3.2 The Space $\beta D$

In this section we will be looking at the results of Stone-Ćech compactification obtained by M. H Stone [39] and E. Ćech [8] independently. M. H. Stone used the principles of algebraic topology through the application of Boolean rings to come up with his results. While on the other hand E. Ćech, demonstrated the existence of the Stone-Ćech compactification of a space and used it to investigate properties of that space by simply embedding it into a product of lines. We will use the modern method to define the Stone-Ćech compactificaion of a discrete space which is similar to that of H. Wallman [46].

**Definition 3.2.1.** Let $D$ be a nonempty set and let $\beta D$ denote the set of all ultrafilters on $D$. For every $A \subseteq D$, define $\overline{A} \subseteq \beta D$ by

$$\overline{A} = \{p \in \beta D : A \in p\}.$$

This implies that, for each $A \subseteq D$ and $p \in \beta D$, $p \in \overline{A}$ if and only if $A \in p$. We have that

$$\overline{A} \cap \overline{B} = \overline{A \cap B}$$

for all $A, B \subseteq D$, so the family $\{\overline{A} : A \in \mathcal{P}(D)\}$ is closed under finite intersections. Consequently, this family is a base for a topology on $\beta D$. In simple terms, for each pair of different $p, q \in \beta D$, and $p \neq q$, there are $A \in p$ and $B \in q$ such that $A \cap B = \emptyset$, and so $\overline{A} \cap \overline{B} = \emptyset$. Thus, the topology is Hausdorff. Note that in the context of this topology, the set of principal ultrafilters on $D$ is a discrete open dense subset of $\beta D$.

**Definition 3.2.2.** Let $D$ be a discrete space, then $\beta D$ is the space of ultrafilters on $D$ with the topology generated by taking as a base subsets $\overline{A}$, where $A \in \mathcal{P}(D)$. The principal ultrafilters are being identified with the points of $D$. For every $A \subseteq D$, $A^* = \overline{A} \backslash A$.

We have that

$$\overline{D \backslash A} = \beta D \backslash \overline{A}$$

for every $A \subseteq D$. Consequently, the subset $\overline{A} \subseteq \beta D$ is closed as well as open. Since also $A$ is dense in $\overline{A}$, it follows that

$$\mathrm{cl}_{\beta D} A = \overline{A}.$$

Recall that a subset $A$ is dense in space $D$ if every point $a \in D$ either belongs in $A$ or is a limit point of $A$.

We now give an explicit definition of Stone-Ćech Compactification.

**Definition 3.2.3.** The *Stone-Ćech compactification* of a discrete space $D$ is a compact Hausdorff space, say $Y$, containing a dense subspace $D$ such that every mapping, $f : D \to Z$, from subspace $D$ into any compact Hausdorff space $Z$,can be extended to a continuous mapping $\overline{f} : Y \to Z$.

It is also important to note that any discrete topological space $D$ is metrizable using the discrete metric, and this implies that it is Hausdorff and completely regular. Thus the space $D$ has a Stone-Ćech compactification $\beta D$.

It is easy to see that Stone-Ćech compactification of $D$ is unique up to a homeomorphism leaving $D$ pointwise fixed.

We need some necessary properties before we can attempt to prove that $\beta D$ is a Stone-Ćech compactification of $D$. First approach is to introduce the idea of the limits of continuous functions on Hausdorff spaces. Suppose that $X$ is a dense subset of a space $Y$ and the mapping, $f : X \to Z$, is continuous from $X$ to a Hausdorff space $Z$. For each $p \in Y$, denote the trace of the neighborhood filter of $p \in Y$ on $X$ by $\mathcal{F}_p$ and let $\lim_{X \ni x \to p} f(x)$ denote the limit of the filter base $f(\mathcal{F}_p)$ in $Z$, if exists. From this it is clear that if $f$ has a continuous extension $\overline{f} : Y \to Z$, then it is unique and

$$\overline{f}(p) = \lim_{X \ni x \to p} f(x).$$

for each $p \in Y$.

We also need the following lemma to show the continuous extension of $f$.

**Lemma 3.2.1.** *Suppose that $Z$ is regular and for each $p \in Y$, there is $\lim_{X \ni x \to p} f(x)$. If we define $\overline{f} : Y \to Z$ by*

$$\overline{f}(p) = \lim_{X \ni x \to p} f(x).$$

*Then $\overline{f}$ is the continuous extension of $f$.*

*Proof.* To show this we let $q \in \mathrm{cl}_Y A$ and let $W$ be a neighborhood of $\overline{f}(q) \in Z$. Since $\overline{f}(q) = \lim_{x \to q f(x)}$, there is $B \in \mathcal{F}_q$ such that $f(B) \subseteq W$. We have that $B \cap A \neq \emptyset$ and $f(B \cap A) \subseteq W$. Hence, $\overline{f}(q) \in \mathrm{cl}_Z f(A)$. We need to show that $\overline{f}$ is continuous. Let $p \in Y$ and let $U$ be a neighborhood of $\overline{f}(p) \in Z$. Since $Z$ is regular, one may suppose that $U$ is closed. Choose $A \in \mathcal{F}_p$ such that $f(A) \subseteq U$. Put $V = \mathrm{cl}_Y A$, then $V$ is a neighborhood of $p \in Y$ and $\overline{f}(V) \subseteq U$. $\square$

Now we have sufficient tools to prove the theorem.

**Theorem 3.2.1.** *$\beta D$ is the Stone-Ćech compactification of $D$.*

*Proof.* Now, since $\overline{D \backslash A} = \beta D \backslash \overline{A}$, the sets $\overline{A}$ form also a base for the closed sets (that is, every closed set in $\beta D$ is the intersection of some sets $\overline{A}$). Consequently, in order to show that $\beta D$ is compact, it suffices to show that every family $\mathcal{A}$ of sets of the form $\overline{A}$ with the finite intersection property has a nonempty intersection. Let

$$\mathcal{B} = \{A \subseteq D : \overline{A} \in \mathcal{A}\}.$$

Then it is clear that $\mathcal{B}$ has the finite intersection property. Then by Proposition 3.1.2, there is $p \in \beta D$ such that $\mathcal{B} \subseteq p$. But then $p \in \bigcap \overline{A}$. Now let $f : D \to Z$ be any mapping with $Z$ a compact Hausdorff space and let $p \in \beta D$. The trace of the neighborhood filter of $p \in \beta D$ on $D$ is the ultrafilter of $p$. Since $Z$ is compact, the ultrafilter base $f(p)$ converges, so $\lim_{D \ni x \to p} f(x)$ exists. Thus, by Lemma 3.2.1.

$$\overline{f} : \beta D \ni p \mapsto \lim_{D \ni x \to p} f(x) \in Z$$

is the continuous extension of $f$. □

**Definition 3.2.4.** For every mapping $f : D \mapsto Z$ of a discrete space $D$ into a compact Hausdorff space $Z$, let $\overline{f} : \beta D \to Z$ denote the continuous extension of $f$.

## 3.3 Martin's Axiom

In this section we shall introduce the set-theoretic axiom, which is known as Martin's Axiom (MA). This set-theoretic axiom is a consequence of Continuum Hypothesis (CH), which we shall also discuss in this section.

As a form of standard notation in this section, we will use $P = (P, \leq)$ to define a partially ordered set. A $filter$ in $P$ is a nonempty subset $G \subseteq P$ such that; i) for each $a, b \in G$, $\exists\ c \in G$ such that $c \leq a$ and $c \leq b$, and ii) for each $a \in G$ and $b \in P$, $a \leq b$ implies $b \in G$.

We shall denote the cardinal number $|\mathbb{N}|$ by $\omega$. A set is said to be *countable* if it is either finite or has cordiality $\mathcal{N}_0$.

We say that the elements $p_1, p_2 \in P$ are *incompatible* if there is no $q \in P$ such that $q \leq p_1$ and $q \leq_2$. Otherwise it is *compatible*. A subset of pairwise incompatible elements in $P$ is called *antichain*. $P$ has the *countable chain condition* if every antichain in $P$ is countable. Also, $D \subseteq P$ is a *dense* subset if for every $p \in P$ there is $d \in D$ such that $d \leq p$.

Also, we denote the cardinal number $|\mathcal{P}(\mathcal{N})|$ by $\mathfrak{c}$ or $2^\omega$ and call it *continuum*.

**Definition 3.3.1** (Martin's Axiom)**.** Whenever $(P, \leq)$ is a partially ordered set with the countable chain condition and $\mathcal{D}$ is a family of $< 2^\omega$ subsets of $P$, there is a filter $G$ in $P$ such that $G \cap D \notin \emptyset$ for all $D \in \mathcal{D}$.

We now show how MA follows from the Continuum Hypothesis.

**Lemma 3.3.1.** *Continuum Hypothesis implies Martin's Axiom.*

*Proof.* Let $(P, \leq)$ be a partially ordered set with the countable chain condition and let $\mathcal{D}$ be a family of $< 2^\omega$ dense subsets of $P$. Then by $CH$, $\mathcal{D}$ is countable. Now, enumerate $\mathcal{D}$ as $\{D_n : n < \omega\}$. Construct inductively a sequence $(a_n)_{n<\omega}$ in $P$ such that $a_n \in D_n$ and $a_{n+1} \leq a_n$. Then the filter $G = \{b \in P : b \geq a_n \text{ for some } n < \omega\}$ generated by $(a_n)_{n<\omega}$ is as required. $\square$

We say that a family $\mathcal{F} \subseteq \mathcal{P}(X)$ has a *strong finite intersection property* if for each finite $\mathcal{H} \subseteq \mathcal{F}$, $\bigcap \mathcal{H}$ is infinite. Furthermore, a subset $A \subseteq X$ is a *pseudo-intersection* of a family $\mathcal{F} \subseteq \mathcal{P}(X)$ if $A \backslash B$ is finite for every $B \in \mathcal{F}$.

**Definition 3.3.2.** The *pseudo-intersection* number $\mathfrak{p}$ is the minimum cardinality of a family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ having the strong finite intersection property but no infinite pseudo-intersection.

From Definition 3.3.2 we can easily see that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{c}$.

**Theorem 3.3.1.** *MA implies* $\mathfrak{p} = \mathfrak{c}$.

*Proof.* Let $\mathcal{F} \subseteq \mathcal{P}(\omega)$ be a family having the strong finite intersection property with $|\mathcal{F}| < \mathfrak{c}$. We need to find an infinite $A \subseteq \omega$ such that $A \backslash B$ is finite for all $B \in \mathcal{F}$. Let $P$ be the set of all pairs $(F, \mathcal{H})$ where $F$ is a finite subset of $\omega$ and $\mathcal{H}$ is a finite subfamily of $\mathcal{F}$. Define the relation $\leq$ on $P$ by

$$(F', \mathcal{H}') \leq (F, \mathcal{H}) \quad \text{if and only if} \quad F \subseteq F', \mathcal{H} \subseteq \mathcal{H}' \text{ and } F' \backslash F \subseteq \bigcap \mathcal{H}.$$

Note that if $(F, \mathcal{H})$ and $(F', \mathcal{H}')$ are incompatible elements of $P$, then $F \neq F'$. Indeed, otherwise $(F, \mathcal{H} \cup \mathcal{H}') \leq (F, \mathcal{H})$ and $(F, \mathcal{H} \cup \mathcal{H}') \leq (F, \mathcal{H}')$. Since the set of finite subsets of $\omega$ is countable, it follows that $P$ has the countable chain condition. Now for every $B \in \mathcal{F}$, let

$$D_B = \{(F, \mathcal{H}) \in P : B \in \mathcal{H}\}.$$

Given any $(F, \mathcal{H}) \in P$, one has $(F, \mathcal{H} \cup \{B\}) \in D_B$ is dense. Also for each $n < \omega$, let

$$D_n = \{(F, \mathcal{H}) \in P : \max F \geq n\}.$$

Given any $(F, \mathcal{H}) \in P$, there is $m \in \bigcap \mathcal{H}$ such that $m \geq n$. Then $(F \cup \{m\}, \mathcal{H}) \in D_n$ and $(F \cup \{m\}, \mathcal{H}) \leq (F, \mathcal{H})$. So $D_n$ is dense.

By MA, there is a filter $G$ in $P$ such that $G \cap D_B \neq \emptyset \; \forall B \in \mathcal{F}$ and $G \cap D_n \neq \emptyset$, $\forall \; n < \omega$. Let

$$A = \bigcup_{(F,\mathcal{H}) \in G} F.$$

It follows that $A$ is infinite since $G \cap D_n \neq \emptyset$ for every $n < \omega$. To show that $A \backslash B$ is finite for every $B \in \mathcal{F}$, we pick $(F, \mathcal{H}) \in G \cap D_B$. We claim that $A \backslash B \subseteq F$. In fact, let $x \in A \backslash B$, then there is $(F', \mathcal{H}') \in G$ such that $x \in F'$. Since $G$ is a filter, there is $(F'', \mathcal{H}'')$ in $G$ such that $(F'', \mathcal{H}'') \leq (F, \mathcal{H})$ and $(F'', \mathcal{H}'') \leq (F', \mathcal{H}')$. That is, $x \in F' \subseteq F''$, $F'' \backslash F \subseteq \bigcap \mathcal{H}$ and $B \in \mathcal{H}$. Hence $x \in F$.

□

## 3.4 Ramsey Ultrafilters and $P$-points

In this section we look into the results of Frank P. Ramsey [33]. We will look into these results in two perspectives, one in the perspective of graphs and the other in set notations. We will begin by examining Ramsey's Theorem for graphs and then later on shift to a more general perspective of sets. We begin by defining a few fundamental concepts and notations.

We need to understand what is meant by a colored graph.

**Definition 3.4.1.** A $2$-colored graph is a graph whose edges have been colored with 2 different colors.

**Example 3.4.1.** *In the below figure we show 2 ways in which a* $^1K_4$ *graph can be* $2$*-colored*

**(a)** **(b)**

**Figure 3.1:** 2-colored $^1K_4$

**Definition 3.4.2.** A graph is $r$-colored if we color each edge of the graph with one of $r$ colors.

**Definition 3.4.3.** For some set, $X$, and $k \in \mathbb{N}$, the subsets of $X$ of size $k$ are are called $k$-tuples and the set of all $k$-tuples in $X$ is $X^{(k)}$.

**Example 3.4.2.** *The* $3$*-tuples of the set* $X = \{1, 2, 3, 4\}$ *are the sets* $\{1, 2, 3\}$*,* $\{1, 2, 4\}$*,* $\{1, 3, 4\}$*,* $\{2, 3, 4\}$*. The set* $X^{(3)} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$*. A*

*$3$-coloring of $X^{(3)}$ is $\{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$.*

A more detailed content about Ramsey's theory for graphs can be found in [15]. We now jump to Ramsey's theory for sets.

Given a set $X$ and a cardinal $k$,

$$[X]^k = \{Y \subseteq X : |Y| = k\}.$$

Ramsey's Theorem says that whenever $k, r \in \mathbb{N}$ and $[\omega]^k$ is $r$-colored, there exists an infinite $A \subseteq \omega$ such that $[A]^k$ is monochrome.

**Definition 3.4.4.** An ultrafilter $p$ on $\omega$ is Ramsey if whenever $k, r \in \mathbb{N}$ and $[\omega]^k$ is $r$-colored, there exist $A \in p$ such that $[A]^k$ is monochrome ($1$-colored).

**Theorem 3.4.1.** *Let $p$ be an ultrafilter on $\omega$. Then the following statements are equivalent*

*1) $p$ is Ramsey,*

*2) for every $2$-coloring of $[\omega]^2$, there exists $A \in p$ such that $[A]^2$ is monochrome, and*

*3) for every partition $\{A_n : n < \omega\}$ of $\omega$, either the partition $A_n \in p$ for some $n < \omega$ or there exist $A \in p$ such that $|A \cap A_n| \leq 1$ for all $n < \omega$.*

*Proof.* 1) $\implies$ 2)

It is trivial.

2) $\Longrightarrow$ 3)

Suppose that for every 2-coloring of $[\omega]^2$, there exist $A \in p$ such that $[A]^2$ is monochrome. Define $\xi : [\omega]^2 \to \{0,1\}$ by

$$\xi(i,j) = \begin{cases} 0 & \text{if } \{i,j\} \subseteq A_n \text{ for some } n < \omega \\ 1 & \text{otherwise.} \end{cases}$$

By our assumption, $A \in p$ such that $[A]^2$ is monochrome. Suppose now that $A_n \notin p$ for all $n < \omega$. By above defined function we have that $\xi([A]^2) = \{1\}$, and consequently, $|A \cap A_n| \leq 1$ for all $n < \omega$.

3) $\Longrightarrow$ 1)

Suppose that for every partition $\{A_n : n < \omega\}$ of $\omega$, either $A_n \in p$ for some $n < \omega$ or there exists $A \in p$ such that $|A \cap A_n| \leq 1$ for all $n < \omega$. We prove this for fixed $r$ by induction on $k$. For $k = 1$, it is straightforward. Suppose now that 1) holds for some $k$ and let $\xi : [\omega]^{k+1} \to \{0, 1, \ldots, r-1\}$ be given. For each $i < \omega$, define $\xi_i : [\omega]^k \to \{0, 1, \ldots, r-1\}$

$$\xi_i(x) = \begin{cases} \xi(\{1\} \cup x) & \text{if } \min x > i \\ 0 & \text{otherwise.} \end{cases}$$

By the inductive assumption, for each $i < \omega$, there are $B_i \in p$ and $l_i < r$ such that $\xi([B_i]^k) = \{l_i\}$. Clearly one may suppose that min $B_i > i$ and the sequence

$(B_i)_{i<\omega}$ is strictly decreasing, so $\{B_i \backslash B_{i+1} : i < \omega\}$ is a partition of $B_0$ into nonempty subsets. By applying our assumption (3)), we obtain that there is a sequence $(b_i)_{i<\omega}$ with $b_i \in B_i \backslash B_{i+1}$ such that $B = \{b_i : i < \omega\} \in p$. We then have that whenever $j_0 < j_1 < \cdots < j_k < \omega$ and $j_1 \geq b_{j_0}$,

$$\xi(\{b_{j_0}, b_{j_1}, \ldots, b_{j_k}\}) = \xi_{b_{j_0}}(\{b_{j_1}, \ldots, b_{j_k}\}) = l_{b_{j_0}}.$$

Define recursively an increasing sequence $(i_n)$ in $\omega$ by $i_0 = 0$ and

$$i_{n+1} = \max\{i_n + 1, \max\{b_i : i < i_n\}\}.$$

By applying 3) once again, we get that there exist $I \subseteq \omega$ such that $\{|I \cap [i_n, i_{n+1})| \leq 1$ for each $n < \omega$ and $\{b_i : i \in I\} \in p$. Let

$$I_0 = I \cap \bigcup_{n<\omega} [i_{2_n}, i_{2n+1}) \quad \text{and} \quad I_1 = I \cap \bigcup_{n<\omega} [i_{2_n+1}, i_{2n+2}).$$

Then $\exists\ s < 2$ such that $\{b_i : i \in I_s\} \in p$. Then whenever $i, j \in I_s$ and $i < j$, there is $n < \omega$ such that $i < i_n < i_{n+1} \leq j$, and so $j \geq b_i$. Finally, there is $l < r$, such that

$$\{b_i : i \in I_s \quad \text{and} \quad l_{b_i} = l\} \in p.$$

Denote this member of $p$ by $A$. Then $\xi([A]^{k+1}) = \{l\}$. $\square$

**Theorem 3.4.2.** *Assume $\mathfrak{p} = \mathfrak{c}$. Then there exists a nonprincipal Ramsey ultra-*

*filter on $\omega$.*

*Proof.* Let $\{X_\alpha : \alpha < 2^\omega\}$ be an enumeration of $\mathcal{P}(\omega)$ and let $\{\chi_\alpha : \alpha < 2^\omega\}$ be an enumeration of 2-colorings $[\omega]^2 \to \{0,1\}$. We show by constructing inductively a $2^\omega$-sequence $(A_\alpha)_{\alpha<2^\omega}$ of infinite subsets of $\omega$ such that for every $\alpha < 2^\omega$, the following conditions are satisfied:

i) for every $\beta < \alpha$, $|A_\alpha \backslash A_\beta| < \omega$,

ii) $[A_\alpha]^2$ is monochrome with respect to $\chi_\alpha$, and

iii) either $A_\alpha \subseteq X_\alpha$ or $A_\alpha \subseteq \omega \backslash X_\alpha$.

By Ramsey's Theorem, there is an infinite $C_0 \subseteq \omega$ such that $[C_0]^2$ is monochrome with respect to $\chi_0$. If $|C_0 \cap X_0| = \omega$, let $A_0 = C_0 \cap X_0$. Otherwise put $A_0 = C_0 \cap (\omega \backslash X_0)$. Now fix $\gamma < 2^\omega$ and suppose the second step of mathematical induction has been shown, that is, we have already constructed a sequence $(A_\alpha)_{\alpha<\gamma}$ satisfying conditions i) - iii) for all $\alpha < \gamma$. Since $\mathfrak{p} = \mathfrak{c}$, there is $B_\gamma \subseteq \omega$ such that $|B_\gamma \backslash A_\omega| < \omega$ for all $\alpha < \gamma$. Choose an infinite $C_\gamma \subseteq B_\gamma$ such that $[C_\gamma]^2$ is monochrome with respect to $\chi_\gamma$ and put

$$A_\gamma = \begin{cases} C_\gamma \cap X_\gamma & \text{if } |C_\gamma \cap X_\gamma| = \omega \\ C_\gamma \cap (\omega \backslash X_\gamma) & \text{otherwise.} \end{cases}$$

Having constructed $(A_\alpha)_{\alpha<2^\omega}$, let

$$p = \{A \subseteq \omega : |A_\omega \backslash A| < \omega \text{ for some } \alpha < 2^\omega\}.$$

Then $p$ is a nonprincipal Ramsey ultrafilter.

$\square$

We will now look at the nonprinciple Ramsey ultrafilters in the perspective of $P$-point.

**Definition 3.4.5** ($P$-point)**.** A point $x$ in a topological space $X$ is a $P$-point if the intersection of countably many neighborhoods of $x$ is again a neighborhood of $x$.

Suppose we have already constructed $\beta X$ as the set of all ultrafilters on $X$ and identified the principal ultrafilters with points of $X$. As a consequence, the purely set-theoretic construction of $X$ leads to a fairly detailed description of the space $\omega^* = \beta X \backslash X$ (which was shown by W. Rudin [35]) which is also found to be a compact Hausdorff space with $2^\omega$ points where there are exactly $\omega$ open-closed subsets. These sets form a basis for the topology $\omega^*$, and any two nonempty open and closed subsets are homeomorphic.

**Definition 3.4.6.** We say that a set of nonprincipal ultrafilters $p$ on $\omega$ is a $P$-point if it is a $P$-point in $\omega^*$.

**Lemma 3.4.1.** *Let $p$ be a nonprincipal ultrafilter on $\omega$. Then $p$ is a $P$-point if and only if whenever $\{A_n : n < \omega\}$ is a partition on $\omega$ and $A_n \notin p$ for every $n < \omega$, there is $A \in p$ such that $A \cap A_n$ is finite for every $n < \omega$.*

*Proof.* $:\Rightarrow$

Suppose $p$ is a $P$-point. Let $\{A_n : n < \omega\}$ be a partition of $\omega$ and $A_n \notin p$ for every $n < \omega$. Define the sequence $\{B_n : n < \omega\}$ of members of $p$ by $B_n = \bigcup_{n<i<\omega} A_i$. We have that $\bigcap_{n<\omega} B_n^*$ is a neighborhood of $p \in \omega^*$ since $p$ is a $P$-point. Consequently, there is $A \in p$ such that $A^* \subseteq \bigcap_{n<\omega} B_n^*$. It follows that for each $n < \omega$, $A \backslash B_n$ is finite. Thus $A \cap A_n$ is finite.

$\Leftarrow$:

Suppose there is $A \in p$ such that $A \cap A_n$ is finite for every $n < \omega$ whenever $\{A_n : n < \omega\}$ is a partition of $\omega$ and $A_n \notin p$ for every $n < \omega$. Let $\{U_n : n < \omega\}$ be a sequence of neighborhoods of $p \in \omega^*$. for each $n < \omega$, pick $B_n \in p$ such that $B_n^* \subseteq U_n$. Without loss of generality one may assume that the sequence $\{B_n : n < \omega\}$ is decreasing and $\bigcap_{n<\omega} B_n = \emptyset$. Define the partition $\{A_n : n < \omega\}$ of $\omega$ by letting $A_0 = \omega \backslash B_0$ and $A_{n+1} = B_n \backslash B_{n+1}$. We see that $A_n \notin p$ for every $n < \omega$. Then for each $n < \omega$, $A \backslash B_n$ is finite, and therefore $A^* \subseteq B_n^*$. We have that $p \in A^* \subseteq \bigcap_{n<\omega} B_n^* \subseteq \bigcap_{n<\omega} U_n$. Hence $\bigcap_{n<\omega} U_n$ is a neighborhood of $p \in \omega^*$ □

**Corollary 3.4.1.** *Every nonprincipal Ramsey ultrafilter on $\omega$ is a $P$-point.*

*Proof.* By Theorem 3.4.2, $\mathfrak{p} = \mathfrak{c}$ implies the existence of a Ramsey ultrafilter, and consequently, a $P$-point. By assuming $\mathfrak{p} = \mathfrak{c}$, we can also construct a $P$-point not being a Ramsey ultrafilter. However, neither of such ultrafilters can be constructed in ZFC, the system of usual axioms of set theory.

$\square$

**Theorem 3.4.3.** *It is consistent with ZFC that there is no $P$-point in $\omega^*$.*

*Proof.* See [VI, Section 4, [36]] $\square$

An ultrafilter $p$ on $X$ with cardinality $\kappa$ is said to be *selective* if for each partition $\{A_\alpha : \alpha < \kappa\}$ of $X$, either $A_\alpha \in p$ for some $\alpha$ or there is $A \in p$ such that $|A \cap A_\alpha| \leq 1$ for all $\alpha$. Furthermore, we say that an ultrafilter $p$ on $X$ is *partially selective* if there is $E \subseteq [X]^2$ such that $\{x \in X : \{y \in X : \{x, y\} \in E\} \in p\}$ for each partition $\{A_\alpha : \alpha < \kappa\}$ of $X$, either $A_\alpha \in p$ for some $\alpha$ or there is $A \in p$ such that $[A \cap A_\alpha]^2 \cap E = \emptyset$ for all $\alpha$.

As a consequence, if $E = [X]^2$ witness that an ultrafilter $p$ on $X$ is partially selective, then in fact $p$ is selective.

Recall: For any given set $X$, $\beta X$ is the Stone-Ćech compactification of $X$ as a discrete space. So, in the topology $\beta X$, every mapping $f : X \to Y$ has a unique continuous extension given by the mapping $f : X \to \beta Y$. For each $p \in \beta X$, $f^\beta(p)$ is an ultrafilter on $Y$ with a base consisting of subsets of the form $f(A)$, with $A \in p$

**Lemma 3.4.2.** *Let $X$ be a countably infinite set and let $p$ be a nonprincipal* *partially selective ultrafilter on $X$. Then there is a mapping $f : X \to \omega$ such that $f^\beta(p)$ is a $P$-point in $\omega^*$.*

*Proof.* Without loss of generality, we assume that for each $x \in X$, $\{y \in X : \{x, y\} \in E\} \in p$. We choose a decreasing sequence $(X_n)_{n<\omega}$ of subsets from $p$ with $X_0 = X$ and a mapping $\nu : X \to \omega$, such that

$$x \notin X_{\nu(x)} \subseteq \{y \in X : \{x, y\} \in E\}.$$

Define the mapping $f : X \to \omega$ by

$$f(x) = n, \quad \text{if } x \in X_n\ X_{n+1}.$$

Consequently $f^\beta(p)$ is a nonprincipal ultrafilter on $\omega$. To prove that $f^\beta(p)$ is a $P$-point, let $\{A_n : n < \omega\}$ be any partition of $\omega$ with $A_n \notin f^\beta(p)$. Then $\{f^{-1}(A_n) : n < \omega\}$ is a partition of $X$ with $f^1(A_n) \notin p$. Since $p$ is partially selective, there is $A \in p$ such that

$$\Big[A \cap f^{-1}(A_n)\Big]^2 \cap E = \emptyset.$$

We claim that $f(A) \cap A_n$ is finite. Indeed, otherwise there is $x \in A \cap f^{-1}(A_n)$ and $y \in A \cap f^{-1}(A_n) \cap X_{\nu(x)}$, with $\{x, y\} \in [A \cap f^{-1}(A_n)]^2 \cap E = \emptyset$. Contraction. $\square$

# Chapter 4

# Spaces and Groups with Extremal properties

The study of both maximal space and irresolvable spaces was initiated by E. Hewitt [18] and M. Katetov [24]. However, a great number of fundamental results were obtained by A. El'kin [14], E. van Douwen [45], I. Protasov [32], V. Malykhin [25] and Y. Zelenyuk [56].

## 4.1 Topological Spaces with Extremal Properties

In this section we discuss in general, spaces with extremal properties as they form part of the building blocks of irresolvable spaces. We start by defining basic concepts of filters and Ultrafilters on topological spaces then move to general concepts of spaces with extremal properties.

One key pre-requisite when dealing with topological spaces is that the neighborhoods need to be open. We will frequently denote the *interior* of a set $A$, and *closure* of a set $A$ by $\overset{o}{A}$ (or int $A$) and $\overline{A}$ respectively.

**Definition 4.1.1.** Let $\mathbb{F}$ be a filter on a topological space.

1. $\mathcal{F}$ is *open (closed)* if it has a base of open (closed) sets.
2. $\mathcal{F}$ is *nowhere dense* if there is $A \in \mathcal{F}$ such that int$\overline{A} = \emptyset$
3. $\mathcal{F}$ is *dense* if for all $A \in \mathcal{F}$, int$\overline{A} \in \mathcal{F}$

**Lemma 4.1.1.** *Every ultrafilter on a space is either dense or nowhere dense.*

**Corollary 4.1.1.** *An ultrafilter, say $\mathcal{U}$ on a space $X$ is dense if and only if for every $A \in \mathcal{U}$, $\mathrm{int}\overline{A} \neq \emptyset$.*

Filters $\mathcal{F}$ and $\mathcal{G}$ are said to be *incompatible* if $A \cap B = \emptyset$ for some $A \in \mathcal{F}$ and $B \in \mathcal{G}$.

**Lemma 4.1.2.** *Every open filter is contained in a maximal open filter. Different maximal open filters are pairwise incompatible.*

**Corollary 4.1.2.** *Every ultrafilter containing a maximal open filter is dense.*

**Definition 4.1.2.** A topological space $X$ is *nodec* if every nowhere dense subset it is closed.

In fact, one may say that a space is nodec if every nowhere dense subset is discrete.

**Proposition 4.1.1.** *A $T_1$-space $X$ is nodec if and only if every nonprincipal converging ultrafilter on $X$ is dense.*

*Proof.* A $T_1$ space $X$ contains no nonclosed nowhere dense set if and only if there is no nonprincipal converging nowhere dense ultrafilter on $X$, consequently by Lemma 4.1.1, if and only if every nonpricipal converging ultrafilter on $X$ is dense. □

**Proposition 4.1.2.** *A space $X$ is extremally disconnected if and only if for every $x \in X$, there is exactly one maximal open filter on $X$ converging to $x$.*

*Proof.* :⇒

Suppose that $X$ is not extremally disconnected. Then there are disjoint open sets $U, V \subset X$ and $x \in X$ such that $x \in \overline{U} \cap \overline{V}$. Let $\mathcal{F}$ be the neighborhood filter of $x$ and let

$$\mathcal{F}_U = \text{flt}(\mathcal{F} \cup \{U\}) \quad \text{and} \quad \mathcal{F}_V = \text{flt}(\mathcal{F} \cup \{V\}).$$

Then clearly $\mathcal{F}_U$ and $\mathcal{F}_V$ are incompatible open filters on $X$ converging to $x$. Therefore, the maximal open filters extending $\mathcal{F}_U$ and $\mathcal{F}_V$ are different( from Lemma 4.1.2).

⇐:

Suppose that there are at least two different maximal open filters on $X$, say $\mathcal{F}_1$ and $\mathcal{F}_2$, converging to some $x \in X$. By Lemma 4.1.2 we have that $\mathcal{F}_1$ and $\mathcal{F}_2$ are incompatible, hence there are disjoint open sets $U \in \mathcal{F}_1$ and $V \in \mathcal{F}_2$. Clearly $x \in \overline{U} \cap \overline{V}$. Thus, $X$ is not extremally disconnected.

□

**Theorem 4.1.1.** *Let $X$ be an extremally disconnected Hausdorff space and let $f : X \to X$ be a homeomorphism. Then the set $M = \{x \in X : h(x) = x\}$ of all fixed points of $f$ is clopen.*

*Proof.* By Lemma 3.1.2, there is a maximal open set $U \subset X$ such that $f(U) \cap U = \emptyset$. Since $X$ is extremally disconnected, $U$ is clopen. Let

$$F = U \cup f(U) \cup f^1(U).$$

Then by this we have $F$ as clopen and $M \cap F = \emptyset$. We claim that $M = X \backslash F$. In fact, suppose the contrary that $f(x) \neq x$ for some $x \in X \backslash F$. Pick an open neighborhood $V$ of $x$ such that $V \cap F = \emptyset$ and $f(V) \cap V = \emptyset$. It follows that $f(V) \cap U = \emptyset$, since $V \cap f^{-1}(U) = \emptyset$. But then $f(U \cup V) \cap (U \cup V) = \emptyset$, which contradicts maximality of $U$. □

**Definition 4.1.3.** A space $X$ is *strongly extremally disconnected* if for each open nonclosed $U \subset X$, there is $x \in X \backslash U$ such that $U \cup \{x\}$ is open.

The following Proposition holds good concepts which helps in proving that a dense in itself space is strongly disconnected iff every perfect subset of it is open.

**Proposition 4.1.3.** *A space $X$ is strongly extremally disconnected if and only if it extremally disconnected and each nonempty nowhere dense subset of $X$ has an isolated point.*

**Corollary 4.1.3.** *A dense in itself space $X$ is strongly extremally disconnected if and only if each perfect subset of $X$ is open.*

*Proof.* Necessity: Let $F$ be a perfect subset of $X$ and let $U = \text{int } F$. To prove that that $U = F$, we assume the contrary that $F \backslash U \neq \emptyset$. We see that $U$ is nonclosed. In fact, otherwise $F \backslash U$ is perfect, therefore by Proposition 4.1.3, int $(F \backslash U) \neq \emptyset$. This contradicts the idea that $U = \text{int } F$. Since $U$ is nonclosed, there is $x \in F \backslash U$ such that $U \cup \{x\}$ is open, a contradiction with $U = \text{int } F$.

Sufficiently: Suppose that every perfect subset of $X$ is open. By Proposition 4.1.3 we only need to check if $X$ is extremally disconnected. Let $U$ be a nonempty open subset of $X$. Since $X$ is dense in itself, so is $U$. Consequently, $\overline{U}$ is perfect and so by our assumption it is open. □

In general, a space is said to be *resolvable* or *irresolvable* depending on whether or not it can be partitioned into two dense subsets.

**Definition 4.1.4.** We say that a space $X$ is *open-hereditarily irresolvable* if every nonempty open subset of $X$ is irresolvable.

**Proposition 4.1.4.** *A space $X$ is open-hereditarily irresolvable if and only if every (converging) maximal open filter on $X$ is an ultrafilter.*

*Proof.* Let $\mathcal{F}$ be a maximal open filter on $X$ and suppose that $\mathcal{F}$ is not an ultrafilter. Then there are two different ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $X$ containing $\mathcal{F}$. Choose pairwise disjoint $A \in \mathcal{U}$ and $B \in \mathcal{V}$. We see that following from Corollary 4.1.2 both $\mathcal{U}$, and $\mathcal{V}$ are dense. Therefore we get int $\overline{A} \in \mathcal{F}$ and int $\overline{B} \in \mathcal{F}$. Now, define an open $U \in \mathcal{F}$ by

$$U = (\text{int}\,\overline{A}) \cap (\text{int}\,\overline{B}).$$

Which implies that both $A \cap U$ and $B \cap U$ are dense in $U$, consequently $U$ is resolvable. Hence, $X$ is not open-hereditarily irresolvable.

Conversely, suppose that every converging maximal open filter on $X$ is an ultrafilter. Let $U$ be any nonempty open subset of $X$ and let $\{A_0, A_1\}$ be a partition of $U$. Choose any converging maximal open filter $\mathcal{F}$ on $X$ containing $U$. This implies that $\mathcal{F}$ is an ultrafilter. So $A_i \in \mathcal{F}$ for some $i < 2$, say $A_0 \in \mathcal{F}$. Since $\mathcal{F}$ is open, it follows that $A_0 \neq \emptyset$. Consequently, $A_1$ is not dense in $U$. Thus, $X$ in not open-hereditarily irresolvable. $\square$

**Definition 4.1.5.** A space $(X, \mathcal{T})$ is maximal if $\mathcal{T}$ is maximal among all dense in itself tolopogies on $X$.

Also, one can say a space is maximal if it has no isolated point but has it in any stronger topology.

**Theorem 4.1.2.** *If $X$ is a dense in itself Hausdorff space, then the following statements are equivalent:*

*1) $X$ is maximal,*

*2) $X$ is extremally disconnected, open-hereditarily irresolvable and nodec,*

*3) for each $x \in X$, there is exactly one nonprincipal ultrafilter on $X$ converging to $x$.*

*Proof.* See (Theorem 3.22, [56])

$\square$

The following closing definition is justified by the equivalence 1) $\Leftrightarrow$ 3) in Theorem 4.1.2

**Definition 4.1.6.** A space $X$ is *almost maximal* if it is dense in itself and for every $x \in X$ there are only finitely many ultrafilters on $X$ converging to $x$.

## 4.2 Irresolvability

Let $X$ be a topological space and suppose there is a homeomorphism $f : X \to X$ such that $f(x) = y$, for every pair of points $x, y \in X$. Then $X$ is a homogeneous topological space.

**Definition 4.2.1.** Given a cardinal $k \geq 2$, a space is $k$-resolvable or $k$-irresolvable depending on whether or not it can be partitioned into $k$-many dense subsets. A space $X$ is hereditarily $\kappa$-irresolvable (open-hereditarily $\kappa$-irresolvable) if every nonempty subset of $X$ (every nonempty open subset of $X$) is $\kappa$-irresolvable.

**Lemma 4.2.1.** *For every $\kappa \geq 2$, the following statements hold:*

1) *an open subset of a $\kappa$-resolvable space is $k$-resolvable,*

2) *the closure of a $\kappa$-resolvable subset of a space is $\kappa$-resolvable, and*

3) *the union of a family of $\kappa$-resolvable subsets of a space is $\kappa$-resolvable.*

*Proof.* It is clear that statements 1) and 2) hold, so it is only fair and sufficient to show 3). Let $\Omega$ be a family of $\kappa$-resolvable subsets of a space $X$ and let $Y = \bigcup \Omega$. We have to show that $Y$ is $\kappa$-resolvable.

Consider the set $P$ of all $\kappa$-sequences $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ consisting of pairwise disjoint nonempty subsets $A_\alpha \subseteq Y$ dense in $\bigcup \mathcal{A}$. Define the order on $P$ by

$$\mathcal{A} \leq \mathcal{B} \quad \text{if and only if} \quad A_\alpha \subseteq B_\alpha \text{ for all } \alpha < \kappa.$$

Every chain $(\mathcal{A}_i)_{i \in I}$ in $P$, where $\mathcal{A}_i = \{A^i_\alpha : \alpha < \kappa\}$, has an upper bound

$$\left\{ \bigcup_{i \in I} A^i_\alpha : \alpha < \kappa \right\}.$$

Consequently by Zorn's Lemma, there is a maximal element $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ in $P$. Let $Z$ be the collection of all unions of $A_\alpha$ for all $\alpha < \kappa$, that is, $Z = \bigcup_{\alpha<\kappa} A_\alpha$. It is obviouse that $Z$ is $\kappa$-resolvable. Thus, $Z$ is closed. Indeed, otherwise the element $\mathcal{B} = \{B_\alpha : \alpha < \kappa\}$ in $P$ defined by

$$B_\alpha = \begin{cases} A_0 \cup \overline{Z} \backslash Z & \text{if } \alpha = 0 \\ \\ A_\alpha & otherwise \end{cases}$$

is greater than $\mathcal{A}$. We claim that $Z = Y$. To see this claim, suppose the contrary. Then there exists $M \in \Omega$ such that $M \backslash Z \neq \emptyset$. Since $Z$ is closed, $M \backslash Z \in \Omega$ as well. It follows that there is $\mathcal{C} = \{C_\alpha : \alpha < \kappa\}$ in $P$ such that $\bigcup \mathcal{C} \cap Z = \emptyset$. Then the element $\mathcal{D} = \{D_\alpha : \alpha < \kappa\}$ in $P$ defined by

$$D_\alpha = A_\alpha \cup C_\alpha$$

is greater than $\mathcal{A}$, a contradiction.

$\square$

**Lemma 4.2.2.** *Let $X$ be a given space and $k \geq 2$. Let $M_\kappa(X)$ denote the union of all $\kappa$-resolvable subsets of space $X$. Then*

*i) $M_\kappa(X)$ is the largest $\kappa$-resolvable subset of $X$,*

*ii) $M_\kappa(X)$ is closed,*

*iii) $X$ is -resolvable if and only if $M_\kappa(X) = X$, and*

*iv) if $X$ is $\kappa$-irresolvable, then $X \backslash M_\kappa(X)$ is hereditary $\kappa$-irresolvable.*

The following Corollary is important as it will also help in later chapters when proving $\omega$-resolvable spaces.

**Corollary 4.2.1.** *A homogeneous space is $k$-irresolvable if and only if it is hereditarily $k$-irresolvable.*

*Proof.* See [ [56], Corollary 3.27] □

**Corollary 4.2.2.** *Every almost maximal space is $\omega$-irresolvable.*

**Lemma 4.2.3.** *Every countable discrete subset of a regular space is strongly discrete.*

*Proof.* Let $X$ be a regular space and let $D$ be a countably infinite discrete subset of $X$. Enumerate $D$ without repetitions as $\{x_n : n < \omega\}$. For each $n < \omega$, pick a closed neighborhood $U_n$ of $x_n \in X$ such that $U_n \cap D = \{x_n\}$. For each $n < \omega$, define a neighborhood $V_n$ of $x_n \in X$ by

$$V_n = U_n \backslash \bigcup_{i<n} U_i.$$

Then the subsets $V_n$, where $n < \omega$, are pairwise disjoint. □

Before we start with the next theorem, we need to establish the following simple fact. We call a subset $D$ of a space $X$ *Strongly discrete* if for each $x \in D$, there is a neighborhood $U_x$ of $x \in X$ such that the subsets $U_x$, where $x \in D$, are pairwise disjoint.

**Theorem 4.2.1.** *Let $X$ be a homogeneous regular space and suppose that $X$ contains a countable discrete nonclosed set. Then $X$ is $\omega$-resolvable.*

*Proof.* Let $D$ be a countable discrete nonclosed subset of $X$ and let $a \in \overline{D} \setminus D$. For each $x \in X$, there is a homeomorphism $f_x : X \to X$ with $f_x(a) = x$. Construct inductively a sequence $(D_n)_{n=1}^{\infty}$ of discrete subsets of $X$ such that

$$D_n \subseteq \overline{D}_{n+1} \setminus D_{n+1}.$$

Put $D_1 = D$. Fix $k \in \mathbb{N}$ and suppose that the sequence $(D_n)_{n=1}^{k}$ has already been constructed satisfying that condition. Denote

$$Y_{k-1} = \begin{cases} \bigcup_{n=1}^{k-1} \overline{D_n} & \text{if } k > 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

It follows from the condition that $D_k \cap Y_{k-1} = \emptyset$. By Lemma 4.2.3, for each $x \in D_k$, there is an open neighborhood $U_x$ of $x \in X \setminus Y_{k-1}$ such that the sets $U_x$, where $x \in D_k$, are pairwise disjoint. Then the set

$$D_{k+1} = \bigcup_{x \in D_k} f_x(D \cap f{-}1_x(U_x))$$

is as required.

Now define $Z \subseteq X$ by

$$Z = \bigcup_{n=1}^{\infty} D_n.$$

We claim that $Z$ is $\omega$-resolvable.

Indeed, let $\{I_n : n \in \mathbb{N}\}$ be any partition of $\mathbb{N}$ into infinite sets. Define the partition $\{A_n : n \in \mathbb{N}\}$ of $Z$ by

$$A_n = \bigcup_{i \in I_n} D_i.$$

Then each $A_n$ is dense in $Z$.

Finally, since $X$ contains a nonempty $\omega$-resolvable subset, $X$ itself is $\omega$-resolvable by Corollary 4.2.1. $\square$

The above mentioned theorem (Theorem 4.2.1) tells us that in a homogeneous regular $\omega$-irresolvable space each countable discrete subset is closed.

We will conclude this chapter by showing that $\omega$-irresolvable spaces are finitely irresolvable.

**Lemma 4.2.4.** *Given a space $X$, let $W = W(X)$ denote the union of all open sets in $X$ containing a dense hereditarily irresolvable subset. Then*

1. *$W$ is the largest open set in $X$ containing a dense open-hereditarily irresolvable subset, and*

2. *every dense subset of $X \backslash \overline{W}$ is resolvable and consequently, $\omega$-resolvable.*

*Proof.* By Zorn's Lemma 3.1.2, there is a maximal family $\mathcal{U}$ of pairwise open sets in $X$ containing a dense hereditarily irresolvable subset. It is obvious that $\bigcup \mathcal{U}$ is dense in $W$. For each $U \in \mathcal{U}$, we pick a dense hereditarily irresolvable $D_U \subseteq U$. Then the collection $D = \bigcup_{U \in \mathcal{U}} D_U$ is a dense open-hereditarily irresolvable subset of $W$.

Now let $A$ be a dense subset of $X \backslash \overline{W}$. Suppose the contrary that $A$ is irresolvable. It then follows from Lemma 4.2.2 that there is a nonempty open set, say, $V$ in $X \backslash \overline{W}$ such that $V \cap A$ is hereditarily irresolvable. Clearly, $V \cap A$ is also closed in $V$. But this contradicts maximality of $\mathcal{U}$. $\square$

**Lemma 4.2.5.** *Let $D$ be an open-heriditarily irresolvable subset of a space $X$ and suppose that $X$ is $(n+1)$-resolvable for some $n$. Then $X \backslash D$ is dense in $X$ and $n$-resolvable.*

*Proof.* By Lemma 4.2.2, it is sufficient to only show that for every nonempty open set $U \subseteq X$, $U \backslash D$ contains a nonempty $n$-resolvable subset. One may suppose that $D \cap U \neq \emptyset$. Partition $X$ into dense subsets $A_1, \ldots, A_{n+1}$. Then $D \cap U$ is partitioned into subsets $A_1 \cap D \cap U, ..., A_{n+1} \cap D \cap U$. Since no space is the union of finitely many nowhere dense sets, at least one of those sets, say $A_{n+1} \cap D \cap U$, is not nowhere dense in $D \cap U$. So there is an open $U_0 \subseteq U$ such that

$$\theta \neq D \cap U_0 \subseteq \overline{(A_{n+1} \cap D \cap U)}_{D \cap U}.$$

It follows that $A_{n+1} \cap D \cap U_0$ is dense in $D \cap U_0$. Then, since $D \cap U_0$ is irresolvable $(D \cap U_0) \backslash A_{n+1}$ is not dense in $D \cap U_0$. Consequently, there is an open $U_1 \subseteq U_0$ such that $\emptyset \neq D \cap U_1 \subseteq A_{n+1} \cap D \cap U_0$, and so $\emptyset \neq D \cap U_1 \subseteq A_{n+1} \cap U_1$. It follows that $A_1 \cap U_1, ..., A_n \cap U_1$ are pairwise disjoint dense subsets of $U_1 \backslash D$. $\square$

The above two Lemmas are a pre-requisite in proving the below theorem which is due to A. Illanes [22].

**Theorem 4.2.2.** *If a space is $n$-resolvable for every $n < \omega$, then it is $\omega$-resolvable.*

*Proof.* By Lemma 4.2.2, it suffices to show that every nonempty open set $X_0 \subseteq X$ contains a nonempty $\omega$-resolvable subset. Being open subset of $X$, $X_0$ is $n$-resolvable for each $n$. Consider the subset $W(X_0) \subseteq X_0$ given by Lemma 4.2.4. If it is not dense, then $X_0 \backslash \overline{W(X_0)}$ is $\omega$-resolvable and we are done. Otherwise pick a dense open-hereditarily irresolvable subset $D_1 \subset W(X_0)$ and put $X_1 = W(X_0) \backslash D_1$. Being an open subset of $X_0$, $W(X_0)$ is $n$-resolvable for each $n$, and then by Lemma 4.2.5, $X_1$ is $n$-resolvable for each $n$. If $W(x_1) \subseteq X_1$ is not dense, then $X_1 \backslash \overline{W(X_1)}$ is $\omega$-resolvable. Otherwise pick a dense open-hereditarily irresolvable subset $D_2 \subset W(X_1)$ and put $X_2 = W(X_1) \backslash D_2$, and so on.

If at some $m$, $W(X_m) \subseteq X_m$ is not dense, $X_m \backslash \overline{W(X_m)}$ is the required nonempty

$\omega$-resolvable subset of $X_0$. Otherwise we construct an infinite sequence $(D_m)_{m=1}^{\infty}$ of pairwise disjoint dense subsets of $X_0$, and then $X_0$ is itself $\omega$-resolvable. $\square$

## 4.3 Topological Groups with Extremal properties

In this section we show that some topological groups with extremal properties cannot be constructed in ZFC.

The question whether there is in ZFC a nondiscrete extremally disconnected topological group was raised by A. Arhangel'skiì [4].

We assume and employ the basic concepts behind Left Invariant Topologies which in detail can be found in [56]. A topology $\mathcal{T}$ on a semigroup $S$ is left invariant if and only if for each $a \in S$ and $U \in \mathcal{T}$, $a^{-1}U \in \mathcal{T}$ where

$$a^{-1}U = \lambda_a^{-1}(U) = \{x \in S : ax \in U\},$$

where, $\lambda_a$ is the left translation.

**Definition 4.3.1.** Let $S$ be a semigroup with identity. For each filter $\mathcal{F}$ on $S$, let $\mathcal{T}[\mathcal{F}]$ denote the largest left invariant topology on $S$ in which $\mathcal{F}$ converges to 1.

Recall Definition 4.1.3. Similarly, the topology $\mathcal{T}[\mathcal{F}]$ is *strongly extremally* *disconnected* if for each open nonclosed $U$ subset of $\mathcal{T}[\mathcal{F}]$, there is $x \in \mathcal{T}[\mathcal{F}]\backslash U$ such that $U \cup \{x\}$ is open.

**Theorem 4.3.1.** *For every nonprincipal ultrafilter $\mathcal{F}$ on $S$, $\mathcal{T}[\mathcal{F}]$ is strongly extremally disconnected.*

*Proof.* See ( [56], Theorem 4.11) □

**Theorem 4.3.2.** *For every $n < \omega$, let $G_n$ be a nontrivial finite group written additively, and let $G = \bigoplus_{n<\omega} G_n$. Let $p$ be a nonprincipal ultrafilter on $\omega$, let $\mathcal{T}$ be the group topology on $G$ with a neighborhood base at $0$ consisting of subgroups*

$$H_A = \{x \in G : \operatorname{supp}(x) \subset A\}$$

*where $A \in p$, let $\mathcal{F}$ be the filter on $G$ with a base consisting of subsets*

$$D_A\{x \in G : \operatorname{supp}(x) \subset A \textit{ and } |\operatorname{supp}(x)| = 1\}$$

*where $A \in p$, and let $D = D_\omega$. Then the following statements are equivalent:*

*1) $p$ is Ramsey,*

*2) $\mathcal{T} = \mathcal{T}[\mathcal{F}]$,*

*3) $D$ is a locally maximal discrete subset of $(G, \mathcal{T})$ with respect to $0$.*

*Furthermore, if $G = \bigoplus_\omega \mathbb{Z}_2$, statements 1) - 3) are equivalent to each of the following:*

*4) $\mathcal{T}$ is extremally disconnected.*

*5) every nonempty nowhere dense subset of $(G, \mathcal{T})$ has an isolated point.*

*Proof.* 1) $\implies$ 2)

Since $\mathcal{F}$ converges to $0$ in $\mathcal{T}$, $\mathcal{T} \subseteq \mathcal{T}[\mathcal{F}]$. To show that $\mathcal{T}[\mathcal{F}] \subseteq \mathcal{T}$, let $M : G \to \mathcal{F}$. Choose $B_n \in p$, for each $n < \omega$, such that for every $x \in G$ with $\max \operatorname{supp}(x) = n$, one has $D_{B_n} \subseteq M(x)$. In addition, choose $B_0$ so that also $D_{B_0} \subseteq M(0)$. Define the coloring $\chi : [\omega]^2 \to \{0,1\}$ by putting for every $m, k \in [\omega]^2$ with $m < k$,

$$\chi(\{m,k\}) = \begin{cases} 1 & \text{if } k \in B_m \\ 0 & \text{otherwise.} \end{cases}$$

Since $p$ is Ramsey by assumption, then $\exists\ A \in p$ such that $A \subseteq B_0$ and $[A]^2$ is monochrome. Observe that $\chi([A]^2) = \{1\}$. We claim that $H_A \subseteq [M]_0$. To see this, let $0 \neq x \in H_A$ and let $\operatorname{supp}(x) = \{n_1, ..., n_k\}$, where $n_1 < \cdots < n_k$. We have that $n_1 \in B_0$ and $n_{i+1} \in B_{n_i}$ for each $i = 1, \ldots, k-1$. Write $x$ as $x = x_1 + \cdots + x_k$, where $\operatorname{supp}(x_i) = \{n_i\}$. Then $x_1 \in D_{B_0} \subseteq M(0)$ and

$$x_{i+1} \in D_{B_{n_i}} \subseteq M(x_1 + \cdots + x_i).$$

Hence $x \in [M]_0$.

2) $\implies$ 3)

Let $E$ be a discrete subset of $(g, \mathcal{T})$ such that $D \cap U \subseteq E \cap U$ for some neighborhood $U$ of $0$. One may suppose that $U$ is open. For each $x \in D \cap U$, choose a neighborhood $V_x$ of $0$ such that $x + V_x \subseteq U$ and $(x + V_x) \cap E = \{x\}$. Let

$$V = \bigcup_{x \in D \cap U} (x + V_x) \cup \{0\}.$$

$V$ is a neighborhood of $0$ and by the construction, $D \cap V = E \cap V$.

3) $\implies 1)$

Let $\{A_n : n < \omega\}$ be any partition of $\omega$ such that $A_n \notin p$ for all $n < \omega$. Define $E \subseteq G$ by

$$E = \{x \in G : \mathrm{supp}(x) \subseteq A_n \text{ for some } n < \omega \text{ and } |\mathrm{supp}(x)| = 2\}.$$

Notice that every point from $E$ is isolated in $D \cup E$. Since $A_n \notin p$ for all $n < \omega$, it follows that also every point from $D$ is isolated in $D \cup E$. Consequently $D \cup E$ is discrete. Since $D$ is a locally maximal discrete subset with respect to $0$, there is a neighborhood $U$ of $0$ such that $U \cap E = \emptyset$. Choose $A \in p$ such that $H_A \subset U$. Then $|A \cap A_n| \leq 1$ for all $n < \omega$.

Now let $G = \bigoplus_\omega \mathbb{Z}_2$. Then $\mathcal{F}$ is an ultrafilter. 2) $\implies$ 4) and 2) $\implies$ 5) follows immediately from Theorem 4.3.1 and Proposition 4.1.3.

4) $\implies 3)$

To show this, let $E$ be a discrete subset of $G = (G, \mathcal{T})$ such that $D \cap U \subseteq E \cap U$

for some neighborhood $U$ of $0$. For every $x \in E$, choose an open neighborhood $U_x$ of $0$ such that the subsets $x + U_x$ are pairwise disjoint. Put

$$U_D = \bigcup_{x \in D} (x + U_x) \quad \text{and} \quad U_E = \bigcup_{x \in E \setminus D} (x + U_x).$$

Then $U_D$ and $U_{E \setminus D}$ are disjoint open subsets and, obviously, $0 \in \overline{U_D}$. However, since $G$ is extremally disconnected, $0 \notin \overline{U_{E \setminus D}}$. Hence, $V = U \setminus U_{E \setminus D}$ is a neighborhood of 0 and $D \cap V = E \cap V$.

5) $\Longrightarrow$ 1)

Suppose that $p$ is not Ramsey. Then there is a partition $\{A_n : n < \omega\}$ of $\omega$ such that $A_n \notin p$ for all $n < \omega$ and for every $A \in p$, $|A \cap A_n| \geq 2$ for some $n < \omega$. Now, define $F$ a subset of $(G, \mathcal{T})$ by

$$F = \{x \in G : |\mathrm{supp}(x) \cap A_n| \leq 1 \text{ for all } n < \omega\}.$$

We claim that $F$ is perfect and nowhere dense. We see that $F$ is closed. To show that $F$ is dense in itself we let $x \in F$. Pick $n_0 < \omega$ such that

$$\max \mathrm{supp}(x) < \min A_n \text{ for all } n \geq n_0$$

and define $A \in p$ by, $A = \bigcup_{n_0 \leq n < \omega} A_n$.

Then $x + D_A \subset F$ and $x$ is an accumulation point of $x + D_A$. Finally, to show that $F$ is nowhere dense, let $x \in G$ and let $U$ be a neighborhood of 0 in $\mathcal{T}$. Pick $A \in p$ such that

$$\max \operatorname{supp}(x) < \min A$$

and $H_A \subseteq U$. Then $|A \cap A_n| \geq 2$ for some $n$. Consequently, there is $y \in H_A$ such that $\operatorname{supp}(y)$ is a 2-element subset of $A_n$, and thus $x + y \notin F$. Since $F$ is closed, this shows that $F$ is nowhere dense. □

**Example 4.3.1** (S. Sirota [38]). *The first consistent example of a group in ZFC that is nondiscrete extremally disconnected was constructed by S. Sirota, the implication* 1) $\implies$ 4) *in Theorem 4.3.2 for* $\bigoplus_\omega \mathbb{Z}_2$.

**Corollary 4.3.1.** *Assume* $\mathfrak{p} = \mathfrak{c}$. *Then there exists a countable nondiscrete extremally disconnected topological group.*

A group of exponent 2 is called *Boolean*. Furthermore, if $G$ is a Boolean group then for each $x, y \in G$, we have $xy = (xy)^{-1} = y^{-1}x^{-1} = yx$, so $G$ is Abelian. It follows that $G$ is isomorphic to $\bigoplus_\kappa \mathbb{Z}_2$ for some cardinal $\kappa$. For each such group $G$, let

$$B(G) = \{x \in G : x^2 = 1\}.$$

Note that if $G$ is Abelian, then $B(G)$ is the largest Boolean subgroup $G$. However, this concept of Boolean groups will be used frequently in the next chapter.

**Lemma 4.3.1.** *Let $G$ be a topological group. If $B(G)$ is a neighborhood of $1$, then $G$ contains an open Boolean subgroup.*

*Proof.* Let $B = B(G)$. Since $B$ is a neighborhood of $1$, there is a neighborhood $W$ of $1$ such that $W^2 \subseteq B$. Then for every $x, y \in W$, $xy \in B$ and consequently $xy = (xy)^{-1} = y^{-1}x^{-1} = yx$. It then follows that for every $x_1, ..., x_n \in W$, $(x_1 \cdots x_n)^2 = x_1^2 \cdots x_n^2 = 1$ and $x_1 \cdots x_n \in B$. Hence, $\langle W \rangle$ is an open Boolean subgroup of $G$. □

**Definition 4.3.2.** Let $S$ be a semigroup and let $(x_n)_{n=1}^{\infty}$ be a sequence in $S$. A sequence $(y_n)_{n=1}^{\infty}$ is a *product subsystem* of $(x_n)_{n=1}^{\infty}$ if there is a sequence $(H_n)_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that $\max H_n < \min H_{n+1}$ and $y_n = \prod_{i \in H_n} x_i$ for every $n \in \mathbb{N}$. If $S$ is an additive semigroup, then we call it a *sum subsystem* instead of product subsystem.

**Theorem 4.3.3.** *Let $S$ be a semigroup and let $(x_n)_{n=1}^{\infty}$ be a sequence in $S$. Whenever $\mathrm{FP}((x_n)_{n=1}^{\infty})$ is partitioned into finitely many subsets, there exists a product subsystem $(y_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $\mathcal{FP}((y_n)_{n=1}^{\infty})$ is contained in one subset of the partition.*

The above mentioned theorem (Theorem 4.3.3) which is due to N. Hindman [20], is stated for the purpose of helping to prove the theorem showing the existence of maximal topological group.

We now state and prove a theorem showing that under $\mathfrak{p} = \mathfrak{c}$, there is a maximal group topology, which is irresolvable. This is the results of V. Malykhin [25].

**Theorem 4.3.4.** *Assume $\mathfrak{p} = \mathfrak{c}$. Then there exists a maximal topological group.*

*Proof.* Let $G = \bigoplus_\omega \mathbb{Z}_2$. Enumerate the subsets of $G$ as $\{Z_\alpha : \alpha < \mathfrak{c}\}$ with $Z_0 = G$. For each $\alpha < \mathfrak{c}$, we construct a sequence $(x_{\alpha,n})_{n<\omega}$ in $G\backslash\{0\}$ such that;

i) $\max \text{supp}(x_{\alpha,n}) < \min \text{supp}((x_{\alpha,n+1})$ for all $n < \omega$,

ii) for every $\gamma < \alpha$, $\{x_{\alpha,n} : n < \omega\}\backslash \text{FS}((x_{\gamma,n})_{n<\omega})$ is finite, and

iii) either $\text{FS}((x_{\alpha,n})_{n<\omega}) \subseteq Z_\alpha$ or $\text{FS}((x_{\alpha,n})_{n<\omega}) \subseteq G\backslash Z_\alpha$.

From i) and ii), it follows that

v) whenever $k < \omega$, $\gamma_0 < \cdots < \gamma_k < \alpha$ and $n_0, \ldots, n_{k-1} \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $\text{FS}((x_{\gamma k,n})_{n_k \le n<\omega}) \subseteq \text{FS}((x_{\gamma i,n})_{n_i \le n<\omega})$ for all $i < k$.

For $(x_{o,n})_{n<\omega}$ pick any sequence in $G\backslash\{0\}$ satisfying i) (we see that ii) and iii) are satisfied trivially).

Fix $\alpha < \mathfrak{c}$ and suppose that we have constructed sequences $(x_{\gamma,n})_{n<\omega}$, $\gamma < \alpha$, satisfying i) - iii).

Since $\mathfrak{p} = \mathfrak{c}$ and $\alpha < \mathfrak{c}$, there is $Y_\alpha \subseteq G\backslash\{0\}$ such that $Y_\alpha\backslash \text{FS}((x_{\gamma,n})_{m\le n<\omega})$ is finite for all $\gamma < \alpha$ and $m < \omega$. Pick a sequence $(y_{\alpha,n})_{n<\omega}$ in $Y_\alpha$ such that

$$\max \text{supp}(y_{\alpha,n}) < \min \text{supp}(y_{\alpha,n+1}), \text{ for all } n < \omega.$$

By Theorem 4.3.3, there is a sum subsystem $(x_{\alpha,n})_{n<\omega}$ of $(y_{\alpha,n})_{n<\omega}$ satisfying condition iii). Then conditions i) and ii) are satisfied as well.

Now, define the group topology on $G$ by taking as a neighborhood base at 0 the subgroups $\mathrm{FS}((x_{\alpha,n})_{m\leq n<\omega}) \cup \{0\}$, where $\alpha < \mathfrak{c}$ and $m < \omega$, of course having firstly constructed the sequences $(x_{\alpha,n})_{n<\omega}$, $\alpha < \omega$. That this topology is maximal follows from iii).

$\square$

**Definition 4.3.3.** Define the functions $\theta, \phi : (\bigoplus_\omega \mathbb{Z}_2)\backslash\{0\} \to \omega$ by

$$\theta(x) = \text{min supp}(x) \quad \text{and} \quad \phi(x) = \text{max supp } (x).$$

**Theorem 4.3.5.** *Let $\mathcal{T}$ be a nondiscrete group topology on $\bigoplus_\omega \mathbb{Z}_2$ finer than that induced by the product topology on $\prod_\omega \mathbb{Z}_2$ and let $\mathcal{F}$ denote the neighborhood filter of $0$ in $\mathcal{T}$. Suppose that $(\bigoplus_\omega \mathbb{Z}_2, \mathcal{T})$ contains no subset with exactly one accumulation point. Then $\phi(\mathcal{F})$ is a $P$-set. In particular, if $\phi(\mathcal{F})$ is finite, then each of its points is a $P$-point.*

We conclude this section by showing the results of I. Prostasov which shows that the existence of a countable nondiscrete $\omega$-irresolvable topological group implies a $P$-point (whose existence by S. Shelah [36] cannot be established in ZFC).

**Theorem 4.3.6.** *The existence of a countable nondiscrete $\omega$-irresolvable topological group implies the existence of a $P$-point in $\omega^*$.*

*Proof.* Let $(G, \mathcal{T})$ be a countable $\omega$-irresolvable topological group. Suppose that $G = \bigoplus_\omega \mathbb{Z}_2$. Let $\mathcal{T}_0$ denote the topology on $G$ induced by the product topology on $\prod_\omega \mathbb{Z}_2$. Then $\mathcal{T} \vee \mathcal{T}_0$ is a nondiscrete $\omega$-irresolvable group topology. Therefore, one may suppose that $\mathcal{T}_0 \subseteq \mathcal{T}$. By Theorem 4.2.1, every discrete subset of $(G, \mathcal{T})$ is closed. Let $\mathcal{F}$ be the neighborhood filter of $0$ in $\mathcal{T}$. It follows immediately that $\{\overline{\phi}^{-1}(u) : u \in \overline{\phi(\mathcal{F})}\}$ is a partition of $\mathrm{Ult}(\mathcal{T})$ into right ideals. From Theorem 4.2.2 it follows that $\overline{\phi(\mathcal{F})}$ is finite. Thus by Theorem 4.3.5, each point of $\overline{\phi(\mathcal{F})}$ is a $P$-point. $\square$

The following arises as a consequence of combining Theorem 4.3.6 and Theorem 3.4.3.

**Corollary 4.3.2.** *It is consistent with ZFC that there is no countable nondiscrete $\omega$-irresolvable topological group.*

# Chapter 5

# Resolvability of Topological Groups

In this chapter we prove a base theorem for a wide group of homeomorphisms of finite order on countable regular spaces. By this concept we prove that every countable topological group not containing an open Boolean subgroup can be partitioned into countably many dense subsets. In particular, every countable topological group not containing an open Boolean subgroup is $\omega$-resolvable. We also prove that every countable group with finitely many elements of order 2, that can be embedded in a compact topological group, can be partitioned into countably many subsets dense in any nondiscrete group topology.

## 5.1 Local Left Topological Groups and Local Automorphisms

In this section we start by showing that if $G$ is a torsion free group, then $\beta G$ contains no nontrivial finite groups. We also extend in some sense this result to the case where $G$ is an arbitrary countable group. Two fundamental theorems that we shall prove in this section were proved in [53]. To this extend, we develop a special technique based on the concept of a local left group and a local homomorphism. The exposition of this section is a base of the treatment in [56].

**Definition 5.1.1.** A *local left group* is a $T_1-$space $X$ with a partial binary operation $\cdot$ and a distinguished point $1 \in X$ such that

1. $x \cdot 1 = x$ for all $x \in X$,
2. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ whenever all the products in the equality are defined (associative), and
3. for each $x \in X \backslash \{1\}$, there is a neighborhood $U$ of $1$ such that $x \cdot y$ is defined for all $y \in U$, $x \cdot U$ is a neighborhood of $x$ and $\lambda_x : U \ni y \mapsto x \cdot y \in x \cdot U$ is a homeomorphism.

From this point on, for a Local left topological group, when we write $x \cdot$

$y$ we assume that $y \in U_x$ and when we write $x \cdot U$, for $U$ a neighborhood of distinguished point $1$, we assume that $U \subseteq U_x$.

**Example 5.1.1.** *A basic example of a local left group is an open neighborhood of the identity of a $T_1$ left topological group.*

**Definition 5.1.2** (Local Isomorphism)**.** For any two local left topological groups, $X$, and $Y$, a mapping $f : X \to Y$ is said to be a local isomorphism if $f$ is a bijection, $f(1_X) = 1_Y$ and for each $x \in X$, and $x \neq 1$, there is a neighborhood of the identity $V_x$ such that $f(xy) = f(x)f(y)$ for any $y \in V_x$.

Furthermore, a topological local isomorphism of a local left topological group on itself is a *local automorphism*. A local automorphism is trivial if it is the identity map on some neighborhood of the identity. From here going forward we will define a bijection $f$ on a set $X$ as having a finite order $n$, if $f^n = id_x$ and $n$ is the least integer satisfying this property.

**Example 5.1.2.** *Let $\tau$ be a topological group on $G$ with continuous shifts. Then for each $x \in G$ $x$-conjugation, given by, $x \mapsto x^{-1}yx$ on $(G, \tau)$ is a topological automorphism. This is obvious (as a local automorphism) if and only if the centralizer of $x \in G$ is open. Also, if $|x| = n$, then $x$-conjugation has order $\leq n$.*

We call a *Boolean part* of a group $G$ a subset

$$B(G) = \{x \in G : x^2 = 1\}.$$

In general, a Boolean part $B(G)$ is not necessarily a subgroup. However, if $G$ itself is Abelian, then the Boolean Part is a subgroup. In particular, for a set $A(G)$ and $xy = yx$, where $x, y \in A$. A subgroup generated by $A$, is contained in the Boolean part $B(G)$. Indeed, for any $x_1, \dots, x_n \in A(x_1 \cdots x_n)^2 = x_1^2 \cdots x_n^2 = 1$.

**Example 5.1.3.** *If $\tau$ is a topology on Abelian group $G$ with continuous shifts and inversion. Then an inversion on $(G, \tau)$ is a topological automorphism of order $\leq 2$. As a local automorphism, it is trivial if and only if $B(G)$ is a neighbourhood of the identity of $(G, \tau)$ or, equivalently there exist an open Boolean subgroup in $(G, \tau)$.*

**Example 5.1.4.** *Let $\tau$ be a topology on $G$ with continuous shifts and inversion. In general, inversion is not a local automorphism. However, it is a local automorphism if each conjugation on $(G, \tau)$ is trivial. Indeed, for each $a \in G$ there is a neighborhood $U$ of the identity such that $ax = xa$ for any $x \in U$. It follows that $x^{-1}a^{-1} = a^{-1}x^{-1}$ and then $(ax)^{-1} = x^{-1}a^{-1} = a^{-1}x^{-1}$. Furthermore, an inversion is trivial if and only if the Boolean part $B(G)$ is a neighborhood of the identity of $(G, \tau)$. This is equivalent to the existence of an open Boolean subgroup in $(G, \tau)$ in the case where $\tau$ is a group topology. Indeed, pick a neighborhood $U$ of the identity such that $U^2 \subseteq B(G)$. Then for any $x, y \in U$, we have $(xy)^2 = 1$ and $xyyx = 1$, and thus follows that $xy = yx$. But the subgroup, generated by $U$, is contained in $B(G)$.*

Let $X$ be a local left group and let $X_d$ be the partial semigroup $X$ endowed with the discrete topology. As in the case of $\beta S$, the partial operation of $X_d$ can be naturally extended to $\beta X_d$ by

$$pq = \lim_{x \to p} \lim_{y \to q} xy,$$

where $x, y \in X$, making $\beta X_d$ a right topological partial semigroup. The product $pq$ is defined if and only if

$$x \in X : \{y \in X : xy \text{ is defined}\} \in q\} \in p,$$

in particular, if the ultrafilter $q$ converges to $1 \in X$.

**Definition 5.1.3.** Given a local left group X,

$$\operatorname{Ult}(X) = \{p \in X_d^* : p \text{ converges to } 1 \in X\}.$$

It can easily be shown that $\operatorname{Ult}(X)$ is a closed subsemigroup in $X_d^*$. In particular, if $X$ is an open neighborhood of the identity of a left topological group $(G, \mathcal{T})$, we identify $\operatorname{Ult}(X)$ with $\operatorname{Ult}(\mathcal{T})$.

**Definition 5.1.4.** Let $X$ and $Y$ be local left groups. A mapping $f : X \to Y$ is a local homomorphism if $f(1_X) = 1_Y$ and for each $x \in X \backslash \{1\}$, there is a neighborhood $U$ of $1 \in X$ such that $f(xy) = f(x)f(y)$ for all $y \in U$. We say that $f : X \to Y$ is a *local isomorphism* if $f$ is a local homomorphism and homeomorphism.

Note, since an isomorphic function is both bijective and continuous, then if $f : X \to Y$ is a local isomorphism, so is $f^{-1} : Y \to X$.

**Definition 5.1.5.** Let $X$ be a local left group and let $S$ be a semigroup. A mapping $f : X \to S$ is a *local homomorphism* if for every $x \in X\backslash\{1\}$, there is a neighborhood $U$ of $1$ such that $f(xy) = f(x)f(y)$ for all $y \in U\backslash\{1\}$.

**Definition 5.1.6.** We say that an element $a \in \bigoplus_\omega \mathbb{Z}_2$ is *basic* if $\operatorname{supp}(a)$ is a nonempty interval in $\omega$. Each nonempty $a \in \bigoplus_\omega \mathbb{Z}_2$ can be uniquely defined as

$$a = a_0 + \cdots + a_k$$

where

1. for each $i \le k$, $a_i$ is basic, and

2. for each $i \le k-1$, $\max \operatorname{supp}(a_i) + 2 \le \min \operatorname{supp}(a_{i+1})$.

We call such a decomposition *canonical*.

Endow $\bigoplus_\omega \mathbb{Z}_2$ with the group topology by taking as a neighborhood base at $0$ the subgroups

$$H_n = \{x \in \bigoplus_\omega \mathbb{Z}_2 : x(i) = 0 \text{ for all } i < n\}$$

where $n < w$.

**Lemma 5.1.1.** *Let $S$ be a semigroup and let $f : \bigoplus_\omega \mathbb{Z}_2 \to S$. The first two of the following statements are equivalent and imply the third:*

*1)* $f(a) = f(a_0)\cdots f(a_k)$ *whenever* $a = a_0 + \cdots + a_k$ *is a canonical decomposition,*

*2)* $f(a+b) = f(a)f(b)$ *whenever* $\max \operatorname{supp}(a) + 2 \leq \min \operatorname{supp}(b)$*,*

*3)* $f$ *is a local homomorphism.*

*Proof.* 1) $\implies$ 2)

Let $\max \operatorname{supp}(a) + 2 \leq \min \operatorname{supp}(b)$ and let $a = a_0 + \cdots a_k$ and $b = b_0 + \cdots + b_l$ be the canonical decompositions. Since $\max \operatorname{supp}(a) = \max \operatorname{supp}(a_k)$ and $\min \operatorname{supp}(b) = \min \operatorname{supp}(b_0)$, one has $\max \operatorname{supp}(a_k) + 2 \leq \min \operatorname{supp}(b_0)$, so $a + b = a_0 + \cdots + a_k + b_0 + \cdots b_l$ is the canonical decomposition. Hence

$$\begin{aligned} f(a+b) &= f(a_0 + \cdots + a_k + b_0 + \cdots b_l) \\ &= f(a_0)\cdots f(a_k) f(b_0)\cdots f(b_l) = f(a)f(b). \end{aligned}$$

2) $\implies$ 1)

If $a = a_0 + \cdots + a_k$ is a canonical decomposition, then

$$f(a) = f(a_0 + \cdots + a_{k-1}) f(a_k) = \cdots = f(a)\cdots f(a_k).$$

2) $\implies$ 3)

Let $0 \neq a \in \bigoplus_\omega \mathbb{Z}_2$ and let $n = \max \operatorname{supp}(a) + 2$. Then for every $b \in H_n \backslash \{0\}$, max supp$(a) + 2 \leq \min \operatorname{supp}(b)$, and so $f(a+b) = f(a)f(b)$. $\square$

Before we can come to one of the main results of this section, we need one more definition of a canonical decomposition which will help us prove the closing theorem.

**Definition 5.1.7.** Given $m \geq 2$, let $W = W(\mathbb{Z}_m)$ denote the set of all words on the alphabet $\mathbb{Z}_m$ including the empty word $\emptyset$. A nonempty word $w \in W$ is *basic* if all nonzero letters in $w$ form a final subword. In particular, every nonempty *zero* word (= all the letters are zeros) is basic. For every $v, w \in W$ to e the result of substituting $v$ for the initial subword of length $|v|$ in $w$. Each nonempty $w \in W$ can be uniquely written in the form $w = w_0 + \cdots + w_k$ where

a) for each $i \leq k$, $w_i$ is basic, and

b) for each $i < k$, $w_i$ is nonzero.

**Theorem 5.1.1.** *Let $X$ be a countable regular local left group and let $\{U_x : x \in X \backslash \{1\}\}$ be a family of neighborhoods of $1 \in X$. Then there is a continuous bijection $h : X \to \bigoplus_\omega \mathbb{Z}_2$ with $h(1) = 0$ such that*

*1)* $h^{-1}(H_n) \subseteq U_x$ *whenever* $\max \operatorname{supp}(h(x)) + 2 \leq n$, *and*

*2)* $h(xy) = h(x) + h(y)$ *whenever* $\max \operatorname{supp}(h(x)) + 2 \leq \min \operatorname{supp}(h(y))$.

Notice that condition 2) in Theorem 5.1.1 may be rewritten as

2') $h^{-1}(ab) = h^{-1}(a) + h^{-1}(b)$ whenever $\max \operatorname{supp}(a) + 2 \leq \min \operatorname{supp}(b)$, so $h^{-1} : \bigoplus_\omega \mathbb{Z}_2 \to X$ is a local homomorphism, by Lemma 5.1.1. Since $h$ is continuous, it follows that $h$ also is a local homomorphism. Finally, if $\{U_x : x \in X\backslash\{1\}\}$ is a neighborhood base at $1 \in X$, then $h$ is open, and so $h$ is a local isomorphism. To see that $h$ is open, let $x \in X$. Put $n = \max \operatorname{supp}(h(x)) + 2$ (if $x = 1$, put $n = 0$). Since $h$ is continuous, there is a neighborhood $V$ of $1$ such that $h(V) \subseteq H_n$. Now let $U$ be any neighborhood of $1$ contained in $V$. Then $h(xU) = h(x) + h(U)$ by 2). Pick $y \in X\backslash\{1\}$ such that $U_y \subseteq U$ and put $m = \max \operatorname{supp}(h(y)) + 2$. Then $h(U_y) \supseteq H_m$ by 1). Hence, $h(xU) \supseteq h(x) + H_m$.

*Proof.* Enumerate $X$ without repetitions as $\{1, x_1, x_2, ...\}$ and let $W = W(\mathbb{Z}_2)$. We shall assign to each $w \in W$ a point $x(w) \in X$ and a clopen neighborhood $X(w)$ of $x(w)$ such that

i) $x(0^n) = 1$, $X(\emptyset) = X$, and $X(0^n) \subseteq U_{x(v)}$ for all $v \in W$ with $|v| \leq n - 2$,

ii) $X(w^\frown 0) \cap X(w^\frown 1) = \emptyset$ and $X(w^\frown 0) \cup X(w^\frown 1) = X(w)$,

iii) $x(w) = x(w_0) \cdots x(w_k)$ and $X(w) = x(w_0) \cdots x(w_{k-1}) X(w_k)$ where $w = w_0 + \cdots + w_k$ is the canonical decomposition, and

iv) $x_n \in \{x(v) : v \in W \text{and} |v| = n\}$.

Choose as $X(0)$ a clopen neighborhood of $1$ such that $x_1 \notin X(0)$. Put $X(1) =$

$X \backslash X(0)$, $x(0) = 1$ and $x(1) = x_1$. Fix $n \geq 2$ and suppose that $X(w)$ and $x(w)$ have been constructed for all $w$ with $|w| < n$ such that conditions i)-iv) hold.

Notice, that the subsets $X(w)$, $|w| = n-1$, form a partition of $X$. So, one of them, say $X(u)$, contains $x_n$. Let $u = u_0 + \cdots + u_r$ be the canonical decomposition. Then $X(u) = x(u_0) \cdots x(u_{r-1})(X(u_r))$ and $x_n = x(u_0) \cdots x(u_{r-1}) y_n$ for some $y_n \in X(u_r)$. If $y_n = x(u_r)$, choose as $X(0^n)$ a clopen neighborhood of $1$ such that

a) $X(0^n) \subseteq U_{x(w)}$ for all $w \in W$ with $|w| \leq n-2$, and

b) for each basic $w$ with $|w| = n-1$, $X(w)x(w)X(0^n) \neq \emptyset$.

For each basic $w$ with $|w| = n-1$, put

$$X(w^\frown 1) = X(w) \backslash x(w) X(0^n)$$

and pick as $x(w^\frown 1)$ any element of $X(w^\frown 1)$. Also put $x(0^n) = 1$ If $y_n \neq x(u_r)$, choose $X(0^n)$ in addition so that

c) $y_n \notin x(u_r) X(0^n)$

and put $x(u_r^\frown 1) = y_n$.

For all nonbasic $w$ with $|w| = n$, define $X(w)$ and $x(w)$ by condition iii). Then

$$x(w) = x(w_0) \cdots x(w_k) \in x(w_0) \cdots x(w_{k-1}) X(w_k) = X(w)$$

and if $x_n \notin \{x(w) : |w| = n-1\}$.

$$x_n = x(u_0) \cdots x(u_{r-1}) x(u_r^\frown 1) = x(u^\frown 1) \in \{x(w) : |w| = n\}.$$

To check ii), let $|w| = n-1$. If $w$ is basic, $X(w^\frown 0) = x(w)X(0^n) \subset X(w)$ and $X(w^\frown 1) = X(w) \backslash x(w)X(0^n)$. Suppose that $w$ is nonbasic and let $w = w_0 + \cdots + w_k$ be the canonical decomposition. If $w_k$ is zero, then $w^\frown 0 = w_0 + \cdots + w_{k-1} + 0^n$ is the canonical decomposition, and consequently,

$$X(w^\frown 0) = x(w_0) \cdots x(w_{k-1})(X(0^n)) = x(w_0) \cdots x(w_{k-1}) X(w_k^\frown 0).$$

Otherwise $w^\frown 0 = w_0 + \cdots w_{k-1} + w_k + 0^n$ is the canonical decomposition, and then

$$X(w^\frown 0) = x(w_0) \cdots x(w_{k-1}) x(w_k) X(0^n) = x(w_0) \cdots x(w_{k-1}) X(w_k^\frown 0).$$

In any case,

$$X(w^\frown 0) = x(w_0) \cdots x(w_{k-1}) X(w_k^\frown 0).$$

Next, since $w^\frown 1 = w_0 + \cdots + w_{k-1} + w_k^\frown 1$ is the canonical decomposition,

$$X(w^\frown 1) = x(w_0) \cdots x(w_{k-1}) X(w_k^\frown 1).$$

It follows that

$$\begin{aligned} X(w^\frown 0) \cup X(w^\frown 1) &= x(w_0)\cdots x(w_{k-1})[X(w_k^\frown 0) \cup X(w_k^\frown 1)] \\ &= x(w_0)\cdots x(w_{k-1})X(w_k) \\ &= X(w) \end{aligned}$$

and $X(w^\frown 0) \cap X(w^\frown 1) = \emptyset$.

Now for every $x \in X$, there is $w \in W$ with nonzero last letter such that $x = x(w)$, so $\{v \in W : x = x(v)\} = \{w^\frown 0^n : n < \omega\}$. Consequently, we can define $h : X \to \bigoplus_\omega \mathbb{Z}_2$ by putting for every $w = \xi \cdots \xi_n \in W$,

$$h(x(w)) = \overline{w} = (\xi_0, \ldots, \xi_n, 0, 0, \ldots).$$

It is obvious that $h$ is bijective and $h(1) = 0$. Since for each $z = (\xi_i)_{i<\omega} \in \bigoplus_\omega \mathbb{Z}_2$,

$$h^{-1}(z + H_n) = X(\xi_0 \cdots \xi_{n-1}),$$

then $h$ is continuous. And since

$$h^{-1}(H_n) = X(0^n) \subseteq U_{x(w)},$$

foll all $w \in W$ with $|w| \leq n-2$, then 1) is satisfied.

To see 2), let $\max \operatorname{supp}(h(x)) + 2 \leq \min \operatorname{supp}(h(y))$. Pick $w, v \in W$ with nonzero last letters such that $x = x(w)$ and $y = x(v)$. Let $w = w_0 + \cdots + w_k$, consequently $w + v = w_0 + \cdots + w_k + v_0 + \cdots + v_t$ is the canonical decomposition, and so

$$xy = x(w_0) \cdots x(w_k) x(v_0) \cdots x(v_t) = x(w+v).$$

Hence,

$$\begin{aligned} h(xy) &= h(x(w+v)) \\ &= \overline{w+v} \\ &= \overline{w} + \overline{v} \\ &= h(x(w)) + h(x(v)) \\ &= h(x) + h(y). \end{aligned}$$

□

**Corollary 5.1.1.** *Let $X$ and $Y$ be countable nondiscrete regular local left groups. Then there is a bijection $f : X \to Y$ such that both $f$ and $f^{-1}$ are local homomorphisms.*

*Proof.* Let $h : X \to \bigoplus_\omega \mathbb{Z}_2$ and $g : Y \to_\omega \mathbb{Z}_2$ be bijections guaranteed by Theorem 5.1.1. If we put $f = g^{-1} \circ h$, then the corollary holds. □

## 5.2 Structure of a local automorphism

Let $X$ be a set and let $f : X \to X$. For each $x \in X$ we denote the $f$-orbit $\{f^n(x) : n < \omega\}$ of the element $x$ by $O(x)$. If $|O(x)| = \omega$, then $f^n(x)f^m(x)$ for any distinct $n, m < \omega$. If $|O(x)| = n$, then $f^n(x) = x$.

A subset $Y \subseteq X$ is *invariant* (with respect to $f$) if $f(Y) \subseteq Y$. We say that a family $\mathcal{F}$ of subsets of $X$ is *invariant* if for every $Y \in \mathcal{F}$, $f(Y) \in \mathcal{F}$. For every $x \in X$, let $O(x) = \{f^n(x) : n < \omega\}$.

**Lemma 5.2.1.** *Let $X$ be a space, $f$ be a homeomorphism on $X$, $x \in X$, $f(x) = x$ and $U$ a neighborhood of $x$. If there exists $n \in \mathbb{N}$ such that $|O(y)| \leq n$ for any $y \in U$, then there exists an open $f$-invariant neighborhood $V \subseteq U$ of $x$. If $X$ is zero-dimensional, then $V$ can be chosen clopen.*

*Proof.* Take an open neighborhood $S$ of $x$ such that $f^j(S) \subset U$, $j = 0, \ldots, n-1$ and $V = \bigcup_{j=0}^{n-1} f^j(S)$. □

**Lemma 5.2.2.** *Let $X$ be a space, let $f : X \to X$ be a homeomorphism, and let $x \in X$ with $|O(x)| = s \in \mathbb{N}$. Let $U$ be a neighborhood of $x$ such that the family $\{f^j(U) : j < s\}$ is disjoint and suppose that there is $n \in \mathbb{N}$ such that $f^n|_U = id_U$. Then there is an open neighborhood $V$ of $x$ contained in $U$ such that the family $\{f^j(V) : j < s\}$ is invariant. If $X$ is zero-dimensional, then $V$ can be chosen to be clopen.*

*Proof.* Clearly, $n = sl$ for some $l \in \mathbb{N}$. Choose an open neighborhood $W$ of $x$ such that $f^{j+is}(W) \subseteq f^j(U)$ for all $j < s$ and $i < l$, in particular, $f^{is}(W) \subseteq U$ for all $i < l$. This can be done because $f^s(x) = x$. Now let

$$V = \bigcup_{i<l} f^{is}(W).$$

Then $V$ is an open neighborhood of $x$ contained in $U$ and

$$f^s(V) = \bigcup_{i<l} f^{(i+1)s}(W).$$

Since $f^{ls} = f^0$,

$$f^s(V) = \bigcup_{i<l} f^{is}(W) = V.$$

It follows that $\{f^j(V) : j < s\}$ is invariant. $\square$

A bijection $f : X \to X$ has a *finite order* if there is $n \in \mathbb{N}$ such that $f^n = id_X$, and the smallest such $n$ is the *order* of $f$.

**Definition 5.2.1.** Let $X$ be a space with a distinguished point $1 \in X$ and let $f : X \to X$ be a homeomorphism of finite order with $f(1) = 1$. We define the *spectrum* of $f$ by

$$\operatorname{spec}(f) = \{|O(x)| : x \in X \backslash \{1\}\},$$

and more generally, for any subset $Y \subseteq X$,

$$\mathrm{spec}(f, Y) = \{|O(x)| : x \in Y \backslash \{1\}\}.$$

Then $f$ is *spectrally irreducible* if for every neighborhood $U$ of $1$,

$$\mathrm{spec}(f, U) = \mathrm{spec}(f).$$

Also, a neighborhood $U$ of a point $x \in X$ is *spectrally minimal* if for every neighborhood $V$ of $x$ contained in $U$,

$$\mathrm{spec}(f, V) = \mathrm{spec}(f, U).$$

If $X$ is a spectrally minimal neighborhood of $1$ and $f$ is a homeomorphism on $X$. Then $f$ is called *stable*. From Lemma 5.2.1 it follows that

**Proposition 5.2.1.** *Let $(G, \tau)$ be a local left topological group and let $f$ be a local automorphism on $(G, \tau)$. Then one of the following statements holds:*

1. *for any neighborhood $U$ of the identity and for any $n \in \mathbb{N}$ there is $x \in U$ such that $|O(f, x)| > n$;*

2. *there is an open $f$ invariant neighborhood $U$ of the distinguished element such that $f|_U$ is a stable local automorphism of finite order on $U$.*

**Corollary 5.2.1.** *Let $X$ be a space with a distinguished point $1 \in X$ and let $f : X \to X$ be a homeomorphism of finite order with $f(1) = 1$. Then there is an open invariant neighborhood $U$ of $1$ such that $f|_U$ is spectrally irreducible.*

This Corollary tells us that we can restrict ourselves in the study of homeomorphisms of finite order in a neighborhood of a fixed point to considering spectrally irreducible ones.

**Definition 5.2.2.** A *local automorphism* of a local left group $X$ is a local isomorphism of $X$ onto itself.

In particular, one may say, a mapping $f : X \to X$ is a local automorphism if $f$ is both a homeomorphism with $f(1) = 1$ and a local homomorphism, that is, for every $x \in X\backslash\{1\}$, there is a neighborhood $U$ of $1$ such that $f(xy) = f(x)f(y)$ for all $y \in U$.

**Lemma 5.2.3.** *Let $X$ be a Hausdorff local left group and let $f : X \to X$ be a spectrally irreducible local automorphism of finite order. Let $x_0 \in X\backslash\{1\}$ with $|O(x_0)| = s$ and let $U$ be a spectrally minimal neighborhood of $x_0$. Then*

$$\operatorname{spec}(f, U) = \{\operatorname{lcm}(s, t) : t \in \{1\} \cup \operatorname{spec}(f)\},$$

*where lcm$(s, t)$ is the least common multiple of $s$ and $t$.*

*Proof.* For each $x \in O(x_0)$, let $V_x$ be a neighborhood of $1$ such that $V_x \ni y \mapsto xy \in xV_x$ is a homeomorphism and $f(xy) = f(x)f(y)$ for all $y \in V_x$. Choose a neighborhood $V$ of $1$ such that

$$V \subseteq \bigcap_{x \in O(x_0)} V_x,$$

$x_0V \subseteq U$, and the subsets $xV$, where $x \in O(x_0)$, are pairwise disjoint. Let $n$ be the order of $f$. Choose a neighborhood $W$ of $1$ such that $f^i(W) \subseteq V$ for all $i < n$. Then clearly this inclusion holds for all $i < \omega$, and furthermore, for each $y \in W$, $f^i(x_0y) = f^i(x_0)f^i(y)$. Indeed, it is trivial for $i = 0$, and by induction, we further obtain that

$$f^i(x_0y) = ff^{i-1}(x_0y) = f(f^{i-1}(x_0)f^{i-1}(y)) = f^i(x_0)f^i(y).$$

Now, let $y \in W$, $|O(y)| = t$ and $k = \operatorname{lcm}(s,t)$. We claim that $|O(x_0y)| = k$. Indeed,

$$f^k(x_0y) = f^k(x_0)f^k(y) = x_0y.$$

On the other hand, suppose that $f^i(x_0y) = x_0y$ for some $i$. Then $f^i(x_0)f^i(y) = x_0y$. Since the subsets $xV$, $x \in O(x_0)$, are pairwise disjoint, it follows from this that $f^i(x_0) = x_0$, so $s|i$. But then also $f^i(y) = y$, and so $t|i$. Hence $k|i$. $\square$

Now, given any finite subset of $\mathbb{N}$ closed under $\operatorname{lcm}$, we produce a spectrally irreducible local automorphism of the corresponding spectrum.

**Corollary 5.2.2.** *Let $X$ be a local left topological group and let $f$ be a stable* *local automorphism of finite order on the local left topological group $X$. Then spec$(f)$ is a finite subset of $\mathbb{N}$ containing the LCM of any two of its elements.*

**Example 5.2.1.** *Let $S$ be finite subset of $\mathbb{N}$ closed under* $\mathrm{lcm}$ *and let $m = 1 + \sum_{s \in S} s$. Consider the direct sum $\bigoplus_\omega \mathbb{Z}_m$ of $\omega$ copies of the group $\mathbb{Z}_m$. Endow $\bigoplus_\omega \mathbb{Z}_m$ with the group topology by taking as a neighborhood base at $0$ the subgroups*

$$H_n = \{x \in \bigoplus_\omega \mathbb{Z}_m : x(i) = 0 \textit{ for all } i < n\}$$

*where $n < \omega$. Write the elements of $S$ as $s_1 < \cdots < s_t$. Define the permutation $\pi_0$ on $\mathbb{Z}_m$ by the product of disjoint cycles*

$$\pi_0 = (1, ..., s_1)(s_1 + 1, ..., s_1 + s_2) \cdots (s_1 + \cdots + s_{t-1} + 1, ..., s_1 + \cdots + s_t).$$

*Let $\pi$ be the coordinatewise permutation on $\bigoplus_\omega \mathbb{Z}_m$ induced by $\pi_0$. That is, $\pi(x)(n) = \pi_0(x(n))$. Then $\pi$ is a homeomorphism with $\pi(0) = 0$, $\mathrm{spec}(\pi, H_n) = S$ for each $n < \omega$, and $\pi(x + y) = \pi(x) + \pi(y)$ whenever $\max \mathrm{supp}(x) < \min \mathrm{supp}(y)$. Hence, $\pi$ is a spectrally irreducible local automorphism of spectrum $S$. We call $\pi$ the **standard** permutation of spectrum $S$.*

**Lemma 5.2.4.** *Let $X$ be a countable nondiscrete regular space with a distinguished point $1 \in X$ and let $f : X \to X$ be a homeomorphism of finite order* *with* $f(1) = 1$*. Let* $U$ *be a clopen invariant partition of* $U$ *such that for each* $C \in \mathcal{P}$*,* $\operatorname{spec}(f, K \cap C) = \operatorname{spec}(f, C)$*. Then there is a clopen invariant partition* $\{U(x) : x \in K\}$ *of* $U$ *inscribed into* $\mathcal{P}$ *such that for each* $x \in K$*,* $U(x)$ *is a spectrally minimal neighborhood of* $x$*.*

*Proof.* Enumerate $U$ as $\{x_n : n < \omega\}$ with $x_o \in K$. For each $x \in K$, we shall construct an increasing sequence $(U_n(x))_{n<\omega}$ of clopen spectrally minimal neighborhoods of $x$ such that for all $n < \omega$, the family $\{U_n(x) : x \in K\}$ is disjoint, inscribed into $\mathcal{P}$ and invariant, and $x_n \in U_n = \bigcup_{x \in K} U_n(x)$. Then the subsets $U(x) = \bigcup_{n<\omega} U_n(x)$, $x \in K$, will be as required. We proceed by induction on $n$. Let $y_i$, $i < l$, be representatives of all orbits in $K$ and let $|O(y_i)| = s_i$. For each $i < l$, choose a neighborhood $W_i$ of $y_i$ such that $f^j(W_i)$ is a spectrally minimal neighborhood of $f^j(y_i)$ for all $j < s_i$, and the family $\{f^j(W_i) : i < l, j < s_i\}$ is disjoint and inscribed into $\mathcal{P}$. By Lemma 5.2.2, for each $i < l$, there is a clopen neighborhood $V_i$ of $y_i$ contained in $W_i$ such that the family $\{f^j(V_i) : j < s_i\}$ is invariant. Put $U_0(f^j(y_i)) = f^j(V_i)$. Fix $n > 0$ and suppose that we have constructed required $U_{n-1}(x)$, $x \in K$. Without loss of generality one may suppose also that $x_n \notin U_{n-1}$. Let $|O(x_n)| = s$ and let $x_n \in C_n \in \mathcal{P}$. Using Lemma 5.2.2, choose a clopen neighborhood $V_n$ of $x_n$ such that for each $j < s$, $f^j(V_n)$ is a spectrally minimal neighborhood of $f^j(x_n)$, and the family

$$\{f^j(V_n) : j < s\} \cup \{U_{n-1}(x) : x \in K\}$$

is disjoint, inscribed into $\mathcal{P}$ and invariant. Pick $z_n \in K \cap C_n$ with $|O(z_n)| = s$. For each $j < s$, put

$$U_n(f^j(z_n)) = U_{n-1}(f^j(z_n)) \cup f^j(V_n).$$

For each $x \in K \backslash O(z_n)$, put $U_n(x) = U_{n-1}(x)$.

$\square$

**Corollary 5.2.3.** *Let $X$ be a space with a distinguished point $1 \in X$ and let $f : X \to X$ be a homeomorphism of finite order with $f(1) = 1$. Then there is an open invariant neighborhood $U$ of $1$ such that $f|_U$ is spectrally irreducible.*

**Theorem 5.2.1.** *Let $X$ be a countable nondiscrete regular local left group, let $f : X \to X$ be a spectrally irreducible local automorphism of finite order, let $S = \operatorname{spec}(f)$, and let $m = 1 + \sum_{s \in S} s$. Let $\pi$ be the standard permutation on $\bigoplus_\omega \mathbb{Z}_m$ of spectrum $S$. Then there is a continuous bijection $h : X \to \bigoplus_\omega \mathbb{Z}_m$ with $h(1) = 0$ such that*

1. *$f = h^{-1}\pi h$, and*

2. *$h(xy) = h(x) + h(y)$ whenever $\max \operatorname{supp}(h(x)) + 2 \le \min \operatorname{supp}(h(y))$.*

*If $X$ is first countable, then $h$ can be chosen to be a homeomorphism.*

*Proof.* The proof is similar to that of Theorem 5.1.1 Enumerate $X$ as $\{x_n : n < \omega\}$ with $x_0 = 1$ and let $W = (\mathbb{Z}_m)$. The permutation $\pi_0$ on $\mathbb{Z}_m$, which induces the standard permutation $\pi$ on $\bigoplus_\omega \mathbb{Z}_m$, also induces the permutation $\pi_1$ on $W$. If $w = \xi_0 \cdots \xi_n$, then $\pi_1(w) = \pi_0(\xi_0) \cdots \pi_0(\xi_n)$. We will write $\pi$ instead of $\pi_0$ and $\pi_1$.

We shall assign to each $w \in W$ a point $x(w) \in X$ and a clopen spectrally minimal neighborhood $X(w)$ of $x(w)$ such that

i) $x(0^n) = 1$ and $X(\emptyset) = X$,

ii) $\{X(w^\frown \xi) : \xi \in \mathbb{Z}_m\}$ is a partition of $X(w)$,

iii) $x(w) = x(w_0) \cdots x(w_{k-1}) x(w_k)$ and $X(w) = x(w_0) \cdots x(w_{k-1}) X(w_k)$ where $w = w_0 + \cdots + w_k$ is the canonical decomposition,

iv) $f(x(w)) = x(\pi(w))$ and $f(X(w)) = X(\pi(w))$, and

v) $x_n \in \{x(v) : v \in W \text{ and } |v| = n\}$.

Using Lemma 5.2.4, we can write the elements of $S$ as $s_1 < \cdots < s_t$. For each $i = i, ..., t$, pick a representative $\xi_i$ of the orbit in $\mathbb{Z}_m \backslash \{0\}$ of lengths $s_i$. Choose a clopen invariant neighborhood $U_1$ of $1 \in X$ such that $x_1 \notin U_1$ and $\operatorname{spec}(f, X \backslash U_1) = \operatorname{spec}(f)$. Put $x(0) = 1$ and $X(0) = U_1$. Then pick points $a_i \in X \backslash U_1$, for $i = 1, ..., t$, with pairwise disjoint orbits of lengths $s_i$ such that $x_1 \in \bigcup_{i=1}^t O(a_i)$. Now, for each $i = 1, ..., t$ and $f < s_i$, put

$$x(\pi^j(\xi_i)) = f^j(a_i).$$

By Lemma 5.2.4, there is an invariant partition $\{X(\xi) : \xi \in \mathbb{Z}_m\backslash\{0\}\}$ of $X\backslash U_1$ such that $X(\xi)$ is a clopen spectrally minimal neighborhood of $x(\xi)$.

Fix $n > 1$ and suppose that $X(w)$ and $x(w)$ have been constructed for all $w \in W$ with $|w| < n$ so that conditions i)-v) are satisfied.

Notice that the subsets $X(w)$, $|w| = n-1$, form a partition of $X$. So one of them, say $X(u)$, contains $x_n$. Let $u = u_0+\cdots+u_q$ be the canonical decomposition. Then $X(u) = x(u_0)\cdots x(u_{q-1})X(u_q)$ and $x_n = x(u_0)\cdots x(u_{q-1})y_n$ for some $y_n \in X(u_q)$. Choose a clopen invariant neighborhood $U_n$ of $1$ such that for all basic $w$ with $|w| = n-q$,

a) $x(w)U_n \subset X(w)$,

b) $f(xy) = f(x)f(y)$ for all $y \in U_n$, and

c) $\mathrm{spec}(f, X(w)\backslash x(w)U_n) = \mathrm{spec}(X(w))$.

If $y_n \notin x(u_q)$, choose $U_n$ in addition so that

d) $y_n \notin x(u_q)U_n$.

Put $x(0^n) = 1$ and $X(0^n) = U_n$.

Let $w \in W$ be an arbitrary nonzero basic word with $|w| = n-1$ and let $O(w) = \{w_j : j < s\}$, where $w_{j+1} = \pi(w_j)$ for $j < s-1$ and $\pi(w_{s-1}) = w_0$. Put $Y_j = X(w_j)\backslash x(w_j)U_n$. Using Lemma 5.2.3, choose points $b_i \in Y_0$, $i = 1, ..., t$, with pairwise disjoint orbits of lengths $\mathrm{lcm}(s_i, s)$. If $u_q \in O(w)$, choose $b_i$ in addition so that

$$y_n \in \bigcup_{i=1}^{t} O(b_i).$$

For each $i = 1, ..., t$ and $j < s_i$, put

$$x(\pi^j(w^\frown \xi_i)) = f^j(b_i).$$

Then, using Lemma 5.2.4, inscribe an invariant partition

$$\{X(v^\frown \xi) : v \in O(w), \xi \in \mathbb{Z}_m \backslash \{0\}\}$$

into the partition $\{Y_j : j < s\}$ such that $X(v^\frown \xi)$ is a clopen spectrally minimal neighborhood of $x(v^\frown \xi)$. For nonbasic $w \in W$ with $|w| = n$, define $x(w)$ and $X(w)$ by condition iii). Now to check the other two conditions, that is, condition ii) and iv), let $|w| = n - 1$ and let $w = w_0 + \cdots + w_k$ be the canonical decomposition. Then

$$X(w^\frown 0) = x(w_0) \cdots x(w_k)(X(0^n)) = x(w_{k-1})x(w_k)(X(0^n))$$

and

$$X(w^\frown \xi) = x(w_0) \cdots x(w_{k-1})X(w_k^\frown \xi),$$

so ii) is satisfied. Next,

$$
\begin{aligned}
f(x(w)) &= f(x(w_0)\cdots x(w_{k-1})(x(w_k))) \\
&= f(x(w_0))f(x(w_1)\cdots x(w_{k-1})(x(w_k))) \\
&\vdots \\
&= f(x(w_0))\cdots f(x(w_{k-1}))f(x(w_k)) \\
&= x(\pi(w_0))\cdots x(\pi(w_{k-1}))(x(\pi(w_k)) \\
&= x(\pi(w_0)\cdots \pi(w_{k-1})\pi(w_k)) \\
&= x(\pi(w)),
\end{aligned}
$$

so iv) is satisfied as well.

Now to check v), suppose that $x_n \notin \{x(w) : |w| = n-1\}$. Then

$$
\begin{aligned}
x_n &= x(u_0)\cdots x(u_{q-1})(y_n) \\
&= x(u_0)\cdots x(u_{q-1})x(u_q^\frown \xi) \\
&= x(u^\frown \xi).
\end{aligned}
$$

Now, for every $x \in X$, there is $w \in W$ with nonzero last letter such that $x = x(w)$, so $\{v \in W : x = x(v)\} = \{w^\frown 0^n : n < \omega\}$. Hence, we can define $h : X \to$

$\bigoplus_\omega \mathbb{Z}_m$ by putting for every $w = \xi_0 \cdots \xi_n \in W$,

$$h(x(w)) = \overline{w} = (\xi_0, \ldots, \xi_n, 0, 0, \ldots).$$

It is obvious that $h$ is bijective and $h(1) = 0$. Since for each $z = (\xi_i)_{i<\omega} \in \bigoplus_\omega \mathbb{Z}_m$,

$$h^{-1}(z + H_n) = X(\xi_0 \cdots \xi_{n-1}),$$

then $h$ is continuous. Now we can easily show the three conditions of the theorem having $h$ as a continuous mapping.

To see 1), let $x = x(w)$. Then

$$\begin{aligned} h(f(x(w))) &= h(x(\pi(w))) \\ &= \overline{\pi(w)} \\ &= \pi(\overline{w}) \\ &= \pi(h(x(w))). \end{aligned}$$

To see 2), let $x = x(w)$, $w = w_0 + \cdots + w_k$ and $n = \max \operatorname{supp}(h(x)) + 2$. Let

$y \in h^{-1}(H_n)$, $y = x(v)$ and $v = v_0 + \cdots + v_l$. Then

$$\begin{aligned}
h(xy) &= h(x(w_0)\cdots x(w_k)x(v_0)\cdots x(v_{l-1})(x(v_l)) \\
&= h(x(w+v)) \\
&= \overline{w+v} \\
&= \overline{w} + \overline{v} \\
&= h(x(w)) + h(x(v)) \\
&= h(x) + h(y).
\end{aligned}$$

Finally, if $X$ is first countable, $\{X(0^n); n < \omega\}$ can be chosen to be a neighborhood base at 1 and then $h$ will be a homeomorphism.

□

## 5.3 Resolving by Automorphism

**Theorem 5.3.1.** *Let $(X, \tau)$ be a countable nondiscrete regular local left topological group, let $f$ be a nontrivial stable local automorphism on $(X, \tau)$ of finite order and let $s$ be the least number of* $\operatorname{spec}(f)\backslash\{1\}$. *Then $X$ can be partitioned into countably many subsets dense in any nondiscrete topology $\rho$ on $X$ such that*

*i) $(X, \rho)$ is a local left topological group,*

*ii) $f$ is a homeomorphism on $(X, \rho)$*

*iii)* $s \in \operatorname{spec}(f, U \cap V)$ *for any neighborhoods* $U, V$ *of the identity in topologies* $\rho$ *and* $\tau$.

*Proof.* Let $h : (X, \tau) \to \bigoplus_\omega \mathbb{Z}(m)$ be a continuous local isomorphism given by Theorem 5.2.1 and let $S$ be a $s$-element and $g$-orbit in $\mathbb{Z}(m)$. For an arbitrary $x \in X$ we consider a sequence of coordinates $h(x)$ belonging to $S$. We shall denote the first element by $L(x)$ and the last by $R(x)$. Let $\nu(x)$ be the number of pairs of neighboring distinct elements in this sequence. Note, if $|O(x)| = s$, then $h(x)$ has coordinates from $S$ and $\nu(x) = \nu(f(x))$. If $x, y \in G$ and $r(h(x)) + 1 < l(h(y))$, then

$$\nu(xy) = \begin{cases} \nu(x) + \nu(y) & \text{if } R(x) = L(y) \\ \\ \nu(x) + \nu(y) + 1 & \text{otherwise}. \end{cases}$$

And if, besides, $|O(x)| = |O(y)| = s$, then there is $i < s$ with the property that

$$\nu\big(x \cdot f^i(y)\big) = \nu(x) + \nu(y),$$

$$\nu\big(x \cdot f^{i+1}(y)\big) = \nu(x) + \nu(y) + 1.$$

Given $\eta < \omega$ we put $X_\eta = \{x \in X : l(\nu(x)) = \eta\}$. Trivially, the family of subsets $X_\eta$, $\eta < \omega$ is disjoint. We need to prove that every $X_\eta$ is dense in $(X, \rho)$. Let $x \in X$ and let $U$ be a neighborhood of the identity of $(X, \rho)$. Put $k = 2^{\eta+1}$ and pick elements $x_1, \ldots, x_k$ in $U$ such that

1. $|O(x_j)| = s$
2. $r(h(x)) + 1 < l(h(x_1)), r(h(x_j)) + 1 < l(h(x_{j+1}))$
3. $y_1 \cdots y_k \in U$ for any $y_i \in O(x_j)$.

Then there is an element of $X_\eta$ among elements $x \cdot y_1 \cdots y_k \in xU$ for $y_j \in O(x_j)$. $\square$

**Theorem 5.3.2.** *If a countable regular space admits a nontrivial regular homeomorphism of finite order, then it is $\omega-$resolvable.*

*Proof.* Let $X$ be a a countable regular space with a distinguished point $1 \in X$ and let $f : X \to X$ be a nontrivial regular homeomorphism of finite order. By Corollary 5.2.1 and Corollary 4.2.1, one may suppose that $f$ is spectrally irreducible. Let $h : X \backslash \bigoplus_\omega \mathbb{Z}_m$ with $h(1) = 0$ be a bijection guaranteed by;

1. $f = h^{-1}\pi h$, and
2. for each $x \in X$, $\lambda_x = h^{-1}\sigma_{h(x)}h$ is homeomorphism of $X$ onto itself.

Furthermore, if $X$ is a local left group and $f$ is a local automorphism, then $h$ can be chosen so that

3. $\lambda_x(y) = xy$, whenever $\max \operatorname{supp}(h(x)) + 1 < \min \operatorname{supp}(h(y))$.

where, $\pi$ is the standard permutation on $\bigoplus_\omega \mathbb{Z}_m$ of spectrum of $f$.

Now, pick an orbit $C$ in $\mathbb{Z}_m$ (with respect to $\pi_0$) of the smallest possible length $s > 1$. For every $x \in X$, consider the sequence of coordinates of $h(x)$ belonging to $C$ and define $\nu(x)$ to be the number of pairs of distinct neighbouring elements in this sequence. Denote also by $\alpha(x)$ and $\gamma(x)$ the first and last elements in the sequence (if nonempty). Then whenever $x, y \in X$ and $\max \operatorname{supp}(h(x)) + 1 < \min \operatorname{supp}(h(y))$,

$$\nu(\lambda_x(y)) = \begin{cases} \nu(x) + \nu(y) & \text{if } \gamma(x) = \alpha(y) \\ \nu(x) + \nu(y) + 1 & \text{otherwise}. \end{cases}$$

We define a disjoint family $\{X_n : n < \omega\}$ of subsets of $X$ by

$$X_n = \{x \in X : \nu(x) \equiv 2^n (\bmod 2^{n+1})\}.$$

To see that every $X_n$ is dense in $X$, let $x \in X$ and let $U$ be an open neighborhood of $1$. We have to show that $\lambda_x(U) \cap X_n \neq \emptyset$. Put $k = 2^{n+1}$ and choose inductively $x_1, \ldots, x_k \in U$ such that

i) $|O(x_j)| = s$,

ii) $\max \operatorname{supp}(h(x_j)) + 1 < \min \operatorname{supp}(h(x_{j+1}))$,

and if $x \neq 0$, then

$\max \operatorname{supp}(h(x)) + 1 < \min \operatorname{supp}(h(x_1))$

iii) $\lambda_{y_1}...\lambda_{y_k}(1) \in U$ whenever $y_i \in O(x_j)$.

Suppose (without loss of generality) that $\gamma(x_j) = \alpha(x_{j+1})$, and that if $x \in X$, then $\gamma(x) = \alpha(x_1)$. For every $l = 0, 1, ..., k-1$, define $z_l \in U$ by

$$z_l = \lambda_{x_1}\lambda_{f(x_2)} \cdots \lambda_{f^l(x_{l+1})}\lambda_{f^l(x_{l+2})} \cdots \lambda_{f^l(x_k)}(1)$$

(in particular, $z_0 = \lambda_{x_1}\lambda_{x_2} \cdots \lambda_{x_k}(1)$). Then

$$h(\lambda_x(z_l)) = h(x) + h(x_1) + \pi h(x_2) + \cdots + \pi^l h(x_{l+1}) + \pi^l h(x_{l+2}) + \cdots + \pi^l h(x_k).$$

It follows that $\nu(\lambda_x(z_0)) = \nu(x) + (X_1) + \cdots + \nu(x_k)$ and $\nu(\lambda_x(z_l)) = \nu(z_0) + l$. Hence, for some $l$, $\nu(\lambda_x(z_l)) \equiv 2_n(\text{mod} 2^{n+1})$, so $\lambda_x(z_l) \in X_n$. $\square$

**Corollary 5.3.1.** *In a countable group endowed with a regular $\omega$-irresolvable topology with continuous shifts, the centralizer of any element of finite order is open.*

**Theorem 5.3.3.** *Let $G$ be a countable group and let $\tau$ be a nondiscrete regular topology on $G$ with continuous shifts and inversion. Suppose $B(G)$ is not a neighborhood of the distinguished element of $(G, \tau)$. Let $B = B(G)\backslash\{1\}$ and let $X$ be an open symmetric neighborhood of the identity of $(G, \tau)$ satistying one of the following conditions:*

*1. $X \cap B = \emptyset$,*

*2. The centralizer of each element of $B \cap X$ is open and $1 \in \overline{B}$*

*Then there is a partial operation $+$ on $X$ such that, for every topology $\rho \supseteq \tau$ on $G$ with continuous shifts and inversion, $(X, \tau|_X, +)$ is a local left topological group and inversion on it is a local automorphism.*

*Proof.* To prove we start by case 2). Let $X$ be a semigroup of words on the alphabet $\mathbb{Z}(4)$. Enumerate the elements of $X$ as a sequence $1, x_1, x_2, \dots$ and put $X(\emptyset) = X$ and $x(\emptyset) = 1$. For each $n \in \mathbb{N}$ we will define a clopen partition $\{X(w) : w \in W_n\}$ of $X$, a subset $\{x(w) \in X(w) : w \in W_n\}$ and a partial operation, $x(w) + y, w \in W_{n-1}, y \in X(\emptyset_n)$ such that

**(1)$_n$** $\{X(w^\frown i) : i \in \mathbb{Z}(4)\}$ is a partition of $X(w)$, and $x(w^\frown 0) = x(w)$, where $w \in W_{n-1}$,

**(2)$_n$** $X(w)$ is a spectrally minimal neighborhood of $x(w)$, where $w \in B_n$, and $X(\emptyset_n)$ is contained in the centralizer of each element of order $2$ from the set $\{x(w) : w \in B_{n-1}\}$,

**(3)$_n$** $(X(w))^{-1} = X(w^{-1})$, and $(x(w))^{-1} = x(w^{-1})$, where $w \in B_n$,

**(4)$_n$** for each $w \in B_{n-1}$, $y \in X(\emptyset_n)$ one has $x(w) + y = y \cdot x(w)$ if the last letter from the set $\{1, 3\}$ is 3 in the word $w$, otherwise $x(w) + y = x(w) \cdot y$,

**(5)$_n$** $x(w) + y = x(w') + (x(w^*) + y)$, where $w \in W_{-1}$, $y \in X(\emptyset_n)$,

**(6)$_n$** $X(w) = x(w') + X(w^*)$, and $x(w) = x(w') + x(w^*)$, where $w \in W_n$, and

**(7)$_n$** $x_n \in \{x(w) : w \in W_n\}$.

We choose a clopen symmetric neighborhood of $1$ as $X(0)$ such that the set $U = X \setminus X(0)$ has elements of order 2 and $x_1 \notin X(0)$. Let $x(0) = 1$. Pick the elements $x(1)$, $x(2)$, $x(3)$ in $U$ such that $(x(1))^{-1} = x(3), (x(2))^{-1} = x(2)$, $x_1 \in \{x(i) : i = 1, 2, 3\}$. Then choose a clopen invariant partition $\{X(i) : i = 1, 2, 3\}$ of $U$ such that $X(i)$ is a spectrally minimal neighborhood of $x(i)$.

Suppose that we have already defined the partition $\{X(w) : w \in W_n\}$, the subset $\{x(w) \in X(w) : w \in W_n\}$ and the partial operation $x(w) + y$, for $w \in W_{n-1}, y \in X(\emptyset_n)$. One of the partition sets, say $X(u)$, contains $x_{n+1}$. Since $X(u) = x(u') + X(u^*)$, one has $x_{n+1} = x(u') + y_{n+1}$ for some $y_{n+1} \in X(u^*)$.

We choose a clopen symmetric neighborhood of $1$ as $X(\emptyset_{n+1})$ such that after $(4)_{n+1}$ has been defined, the partial operation $X(w^\frown 0) = x(w) + X(\emptyset_{n+1}) \subset X(w)$, $\sigma(X(w) \setminus X(w^\frown 0)) = \sigma(X(w))$, $w \in B_n$, $X(\emptyset_n)$ is contained in the centralizer of each element of order $2$ from the set $\{x(w) : w \in B_n\}$ and $y_{n+1} \notin X(u^{*\frown} 0)$, if, of course, $y_{n+1} \neq x(u^*)$. Put $x(w^\frown 0) = x(w)$, $w \in B_n$.

Let $w \in B_n$, and $w^{-1} = w$. Pick the elements $x(w^\frown i)$ for $i = 1, 2, 3$ from the set $U = X(w) \setminus X(w^\frown 0)$ such that $(x(w^\frown 1))^{-1} = x(w^\frown 3)$, $(x(w^\frown 2))^{-1} = x(w^\frown 2)$ and $y_{n+1} \in \{x(w^\frown i) : i \in \mathbb{Z}(4)\}$, if $w = u^*$. Then we choose a clopen invariant partition $\{X(w^\frown i) : i = 1, 2, 3\}$ of $U$ such that $X(w^\frown i)$ is spectrally minimal neighborhood of $x(w^\frown i)$.

Let $w \in B_n$, and $w^{-1} \neq w$. Put $U_1 = X(w) \setminus X(w^\frown 0)$, $U_2 = X(w^{-1}) \setminus X(w^{-1\frown} 0)$, and $U = U_1 \cup U_2$. Since $X(\emptyset_{n+1})$ is symmetric, $U^{-1} = U_2$. Pick the distinct elements $x(w^\frown i) \in U_1$, $x(w^{-1\frown} i) \in U_2$, for $i = 1, 2, 3$, such that $(x(w^\frown i))^{-1} = x(w^{-1\frown} i)$ and $y_{n+1} \in \{x(w^\frown i), x(w^{-1\frown} i) : i \in \mathbb{Z}(4)\}$, if $u^* \in \{w, w^{-1}\}$. Then we inscribe the clopen invariant partition $\{X(v^\frown i) : v = w, w^{-1},$ for $i = 1, 2, 3\}$ into the partition $\{U_1, U_2\}$ of $U$ such that $X(v^\frown i)$ is a spectrally minimal neighborhood of $x(v^{-1})$.

When $X(w), x(w)$ and $x(w) + y$ have been defined for all $w \in B_{n+1}$, we shall define them for $w \in W_{n+1} \backslash B_{n+1}$ by conditions $(5)_{n+1}, (6)_{n+1}$.

After we are done with this process, we will get the bijection $F' \ni w \mapsto x(w) \in X$ and the partial operation $X(w) + y$, where $w \in F', y \in X(\emptyset_{r(w)+1})$. Trivially, the operation $x(w) + y$, for $y \in X(\emptyset_{r(w)+1})$ maps homeomorphically the neighborhood $X(\emptyset_{r(w)+1})$ of $1$ onto the neighborhood $X(w^\frown 0)$ of $x(w)$. Furthermore, this holds for any topology $\rho \supseteq \tau$ with continuous shifts. To see that $x(u + v) = x(u) + x(v)$, if $r(u) + 1 < l(v)$, use the induction in the length of the canonical representation of $v$. That is,

$$\begin{aligned}
x(u+v) &= x(u + v_1 + \cdots + v_q) \\
&= x(u + v_1 + \cdots + v_{q-1}) + x(v_q) \\
&= x(u + v_1 + \cdots + v_{q-2}) + (x(v_{q-1}) + x(v_q)) \\
&= x(u + v_1 + \cdots + v_{q-2}) + x(v_{q-1} + v_q) \\
&= x(u) + x(v).
\end{aligned}$$

It follows that the partial operation $+$ is associative. If $r(u) + 1 < l(v)$ and $r(v) + 1 < l(w)$, then

$$\begin{aligned}(x(u) + x(v)) + x(w) &= x(u + v) + x(w)\\ &= x(u + v + w)\\ &= x(u) + x(v + w)\\ &= x(u) + (x(v) + x(w)).\end{aligned}$$

Thus, $X_\rho$ is a local left topological group. Now we need to check that inversion on $X_\rho$ is a local automorphism. Let $w \in F', y \in X(\emptyset_{r(w)+1})$. If $w \in B, w = w^{-1}$ then

$$\begin{aligned}(x(w) + y)^{-1} &= (x(w) \cdot y)^{-1}\\ &= (y \cdot x(w))^{-1} = (x(w))^{-1} \cdot y^{-1}\\ &= (x(w))^{-1} + y^{-1}.\end{aligned}$$

If $w \in B, w \neq w^{-1}$ and the last letter of $w$ is 3 from the set $\{1, 3\}$, then the last letter of $w^{-1}$ is $1$ from the same set and one has ;

$$\begin{aligned}(x(w)+y)^{-1} = (y \cdot x(w))^{-1} &= x(w))^{-1} \cdot y^{-1} \\ &= x(w)^{-1} + y^{-1} \\ &= (x(w))^{-1} + y^{-1}.\end{aligned}$$

We use induction in the length of the canonical representation of $w$ for a more general case. In particular;

$$\begin{aligned}(x(w)+y)^{-1} &= (x(w') + (x(w^*)+y))^{-1} \\ &= (x(w'))^{-1} + (x(w^*)+y))^{-1} \\ &= \vdots \\ &= (x(w'))^{-1} + (x(w^*)^{-1} + (y)^{-1}) \\ &= x(((w)^{-1})') + (x(((w)^{-1})^*) + y^{-1}) \\ &= x(w^{-1}) + y^{-1} \\ &= (x(w))^{-1} + y^{-1}.\end{aligned}$$

Lastly, Case (1) can be considered as similar to the above proved case (Case (2)) and even easier. To prove it we take a semigroup of words on the alphabet $\mathbb{Z}(3)$ as a semigroup $F$ and apply similar methods. $\square$

As our first results we have the following.

**Theorem 5.3.4.** *In a countable group $G$ endowed with a regular $\omega$-irresolvable topology with continuous shifts and inversion, $B(G)$ is a neighborhood of the identity.*

*Proof.* See ( [51], Theorem 6.1). □

**Corollary 5.3.2.** *Every countable $\omega$- irresolvable topological group contains an open Boolean subgroup.*

**Theorem 5.3.5.** *Every countable group with a finite Boolean part, that can be embedded in a compact topological group, can be partitioned into countably many subsets dense in any nondiscrete topology with continuous shifts and inversion.*

*Proof.* Let $G$ be a countable group with Boolean part $B(G)$ that can be embedded in a compact topological group. Let $\tau$ be a totally bounded topological group on a countable group $G$ and let $X$ be a local left topological group obtained from $(G, \tau)$ by removing elements of order $2$, let $X$ have the partial operarion $+$ guaranteed by Theorem 5.3.3 and let $\{X_n : n < \omega\}$ be a partition of a local left topological group $(X, +)$ by inversion guaranteed by Theorem 5.3.1. We check that each $X_n$ is dense in any nondiscrete topology $\rho$ on $G$ with continuous shifts and inversion. Let $\tau_1 = \tau \vee \rho$ be the least upper bound of $\tau$ and $\rho$. Neighborhoods of the identity in the topology $\tau_1$ are subsets of the form $U \cap V$, where $U, V$ are the neighborhoods of the identity in topologies $\tau, \rho$, respectively. It is easy to see that shifts and inversion in topology $\tau_1$ are continuous. Also, $\tau_1$ is nondiscrete since $\tau$ is totally bounded. We have that $\tau_1 \supseteq \tau$, and $(X, \tau_1|_X, +)$ is a nondiscrete local left topological group and the inversion is a nontrivial local automorphism on it. Thus, $X_n$ is dense in $\tau_1$ and, therefore $X_n$ is dense in $\rho$. $\square$

**Corollary 5.3.3.** *Every countable group with finitely many elements of order* $2$*, that can be embedded in a compact topological group, is absolutely* $\omega$*-resolvable.*

# Chapter 6

# Absolute Resolvability

In this chapter we will deal mostly with Abelian groups and we shall use additive notation even if the group is not supposed to be Abelian.

## 6.1 Preliminaries

**Definition 6.1.1.** Let $G$ be a group. A subset $A \subseteq G$ is *absolutely dense* if $A$ is dense in every nondiscrete group topology on $G$. Given a cardinal $\kappa \geq 2$, $G$ is *absolutely $\kappa$-resolvable* (*absolutely resolvable if* $\kappa = 2$) if $G$ can be partitioned into $\kappa$ absolutely dense subsets.

Note that an Abelian group can be absolutely $\kappa$-resolvable only for $\kappa \leq \omega$.

**Lemma 6.1.1.** *Let $G$ be an Abelian group and let $A \subseteq G$. If $A$ is absolutely dense, then $G \backslash A$ contains no coset modulo infinite subgroup.*

*Proof.* Suppose that there exist an infinite subgroup $H$ of $G$ and $g \in G$ such that $g + H \subseteq G \backslash A$. Pick any nondiscrete group topology $\mathcal{T}_H$ on $H$ and extend it to the group topology $\mathcal{T}$ on $G$ by declaring $H$ to be an open subgroup. Then $g + H$ is an open subset of $(G, \mathcal{T})$ disjoint from $A$, so $A$ is not dense. Hence, $A$ is not absolutely dense. □

**Proposition 6.1.1.** *Whenever an infinite Boolean group is partitioned into finitely many subsets, there is a coset modulo infinite subgroup contained in one subset of the partition.*

*Proof.* Let $B$ be an infinite Boolean group and let $\{A_i : i < r\}$ be finite partition of $B$. By Hindmans Theorem, there exist $i < r$ and a one-to-one sequence $(x_n)_{n<\omega}$ in $B$ such that $\mathrm{FS}((x_n)_{n<\omega}) \subseteq A_i$. Let $H = \mathrm{FS}((x_n)_{n<\omega}) \cup \{0\}$. Then $H$ is a subgroup of $B$, since $B$ is Boolean, and $x_0 + H \subseteq \mathrm{FS}((x_n)_{n<\omega}) \subseteq A_i$.

□

We obtain the following Corollary from Proposition 6.1.1 and Lemma 6.1.1.

**Corollary 6.1.1.** *If an Abelian group contains an infinite Boolean subgroup, then it is not absolutely resolvable.*

**Definition 6.1.2.** Let $G$ be a group written additively. Given a sequence $(x_n)_{n<\omega}$ in $G$, the set of finite sums with inverses of the sequence is defined by

$$\mathrm{FSI}((x_n)_{n<\omega}) = \left\{ \sum_{n \in F} \epsilon_n^F x_n : F \in \mathcal{P}_f(w) \text{ and } \epsilon_n^F \in \{1, 1\} \text{ for all } n \in F \right\}.$$

# CHAPTER 6. ABSOLUTE RESOLVABILITY

The aim of this chapter is to prove the following theorem.

**Theorem 6.1.1.** *Let $G$ be an Abelian group and let $A = G \backslash B(G)$ the complement of Boolean subgroup of $G$, be infinite. Then there exists a partition $\{A_m : m < \omega\}$ of $A$ such that whenever the sequence $(x_n)_{n<\omega}$ is one-to-one in $A$, and $m < \omega$, we get*

$$(g + \mathrm{FSI}((x_n)_{n<\omega})) \cap A_m \neq \emptyset$$

*where $g$ is an element in $G$.*

Theorem 6.1.1 gives two important corollaries. One of the corollaries is related to the results from [16] which states as follows; If we let $G$ be an Abelian group containing an infinite subgroup of *bounded* order, then whenever the group is partitioned into finitely many segments, there exists cosets modulo randomly large finite subgroups contained in one segment. Furthermore, if $G$ has Boolean subgroup that is infinite, then Finite Sums Theorem gives the required cosets modulo infinite subgroups contained specifically in one segment.

The construction of a 2-partition of an arbitrary infinite Abelian group not containing an infinite Boolean subgroup such that every coset modulo infinite subgroup meets each cell of the partition is contained in [50]. However, the question whether there is such a 3-partition, even for the group $\bigoplus_{\omega_1} \mathbb{Z}(3)$, remained open. Now Theorem 6.1.1 gives us the complete answer to this question.

**Corollary 6.1.2.** *Let $G$ be an infinite Abelian group not containing an infinite Boolean subgroup. Then there is a partition $\{A_m : m < \omega\}$ of $G$ such that for every infinite subgroup $H$ of $G$, $g \in G$ and $m < \omega$, one has $(g + H) \cap A_m \neq \emptyset$.*

*Proof.* Let $(x_n)_{n<\omega}$ be a one-to-one sequence in $H \backslash B(G)$. Then $\mathrm{FSI}((x_n)_{n<\omega}) \subseteq H$. By Theorem 6.1.1, $(g + \mathrm{FSI}((x_n)_{n<\omega})) \cap A_m \neq \emptyset$, so $(g + H) \cap A_m \neq \emptyset$. □

The second corollary is concerned with the problem of characterizing the absolutely resolvable groups raised in [11]. For countable Abelian groups the absolute resolvability problem was solved in [48] and for all Abelian groups and $\kappa = 2$ in [50]. Theorem 6.1.1 gives us the final solution of the problem for all Abelian groups.

**Corollary 6.1.3.** *Let $G$ be an infinite Abelian group. Then for every $G$, the following statements are equivalent;*

1. *$G$ is absolutely resolvable,*
2. *$G$ is absolutely $\omega$-resolvable,*
3. *$G$ contains no infinite Boolean subgroup.*

*Proof.* $1) \implies 2)$

is trivial.

$1) \implies 3)$

Let $\{A_0, A_1\}$ be a partition of $G$ into absolutely dense subsets and suppose the contrary that $G$ contains an infinite Boolean subgroup $B$. For each $i < 2$, let

$C_i = A_i \cap B$. By the Finite Sums Theorem, there is one-to-one sequence $(x_n)_{n<\omega}$ in $B$ with $\mathrm{FS}((x_n)_{n<\omega})$ contained in one cell, say $C_0$. Let $B_0 = \mathrm{FS}((x_n)_{1\leq n<\omega}) \cup \{0\}$. Then $B_0$ is an infinite subgroup, as $B$ is Boolean, and

$$x_0 + B_0 \subseteq \mathrm{FS}((x_n)_{n<\omega}) \subseteq C_0.$$

Let $\mathcal{T}$ be any nondiscrete group topology on $G$ in which $B$ is open. Then $x_0 + B$ is open and, since $x_0 + B \subseteq C_0$, $(x_0 + B) \cap A_1 = \emptyset$. Hence, $A_1$ is not dense in $\mathcal{T}$, which is a contradiction.

3) $\implies$ 2)

Let $\{A_m : m < \omega\}$ be a partition of $A = G \backslash B(G)$ guaranteed by Theorem 6.1.1. We claim that each $A_m$ is absolutely dense. To see this, let $\mathcal{T}$ be a nondiscrete group topology on $G$, let $U$ be an open neighborhood of $0 \in G$, and let $g \in G$. Pick $x_0 \in U \backslash B(G)$ with $-x_0 \in U$. Fix $k < \omega$ and suppose that we have constructed a one-to-one sequence $(x_n)_{n\leq k}$ in $U \backslash B(G)$ with $\mathrm{FSI}((x_n)_{n\leq k}) \subseteq U$. Let $H_k = \mathrm{FSI}((x_n)_{n\leq k}) \cup \{0\}$. Pick $x_{k+1} \in U \backslash (B(G)) \cup \{x_n : n \leq k\})$ such that $H_k + \epsilon x_{k+1} \subseteq U$ for each $\epsilon \in \{1, -1\}$. As a result, we obtain a one-to-one sequence $(x_n)_{n<\omega}$ in $U \backslash B(G)$ with $\mathrm{FSI}((x_n)_{n<\omega}) \subseteq U$. Since $(g\mathrm{FSI}((x_n)_{n<\omega})) \cap A_m \neq \emptyset$, $(g + U) \cap A_m \neq \emptyset$ as well. $\square$

Now, the application of Corollary 6.1.3 gives rise to more exposition about infinite Abelian groups not containing infinite Boolean subgroups. In particular, the following Corollary is a consequence of Corollary 6.1.3 and Lemma 6.1.1.

**Corollary 6.1.4.** *Every infinite Abelian group not containing an infinite Boolean subgroup can be partitioned into $\omega$ subsets such that every coset modulo infinite subgroup meets each subset of the partition.*

Note that the cardinal number $\omega$ in Theorem 6.1.1, Corollary 6.1.3 and Corollary 6.1.4 is maximally possible.

An element $p$ of a semigroup is an idempotent if $pp = p$. The following proposition tells us that the relationship between idempotents and finite products is intimate.

**Proposition 6.1.2.** *Let $S$ be a semigroup and let $(x_n)_{n=1}^{\infty}$ be a sequence in $S$. Then $\bigcap_{m=1}^{\infty} \overline{\mathrm{FP}((x_n)_{n=m}^{\infty})}$ is a closed subsemigroup of $\beta S$. Consequently, there is an idempotent $p \in \beta S$ such that $\mathrm{FP}((x_n)_{n=m}^{\infty}) \in p$ for every $m \in \mathbb{N}$.*

**Lemma 6.1.2.** *Let $G$ be a totally bounded topological group and let $(x_n)_{n<\omega}$ be a sequence in $G$. Then whenever $U$ and $U_x$, $x \in U$, are neighborhoods of $0 \in G$, there exists a sum subsystem $(y_n)_{n<\omega}$ of $(x_n)_{n<\omega}$ such that $y_0 \in U$ and for every $n < \omega$, $y_{n+1} \in \bigcap_{i \leq n} U_{y_i}$.*

*Proof.* We first show that for every $m < \omega$ and for every neighborhood $V$ of $0 \in G$,

$$\mathrm{FS}((x_n)_{m\leq n<\omega} \cap V \neq \emptyset.$$

To this end, we define a closed subsemigroup $T \subseteq \beta G_d$ by

$$T = \bigcap_{m<\omega} \overline{\mathrm{FS}((x_n)_{m\leq n<\omega})}$$

Using Proposition 6.1.2, and pick idempotent $p \in T$. We then have that $\mathrm{FS}((x_n)_{m\leq n<\omega}) \in p$ for every $m < \omega$. But since $G$ is totally bounded, we also have that $V \in p$ for every neighborhood $V$ of $0 \in G$. It follows that $\mathrm{FS}((x_n)_{m\leq n<\omega}) \cap V \neq \emptyset$ for every $m < \omega$ and for every neighborhood $V$ of $0 \in G$.

Now pick $y_0 \in \mathrm{FS}((x_n)_{n<\omega}) \cap U$. Fix $k < \omega$ and suppose that we have constructed a sum subsystem $(y_n)_{n\leq k}$ of $(x_n)_{n<\omega}$ such that $y_0 \in U$ and for every $n < k$, $y_{n+1} \in \bigcap_{i\leq n} U_{y_i}$. Let $y_k = \sum_{n\in H_k} x_n$ and let $n_{k+1} = \max H_k + 1$. Pick

$$y_{k+1} \in \mathrm{FS}((x_n)_{n_{k+1}\leq n<\omega}) \cap \bigcap_{i\leq k} U_{y_i}.$$

The sequence $(y_n)_{n<\omega}$ is constructed as required. $\square$

## 6.2 Direct sums of Finite groups

In this section we prove Theorem 6.1.1 for the group

$$G = \bigoplus_{\alpha<\kappa} G_\alpha$$

where $\kappa$ is an infinite cardinal and for every $\alpha < \kappa$, we have $G_\alpha$ as a finite group, not necessarily Abelian.

The following Lemma will give us the crucial idea of the construction needed to prove first partial part of Theorem 6.1.1.

**Lemma 6.2.1.** *Let $(c_n)_{n<\kappa}$ be an increasing sequence in $\mathbb{N}$ such that $c_0 = 1$ and $c_{n+1} - c_n \to \infty$. Then there is a function $\varphi : \mathbb{N} \to [\mathbb{N}]^{<\omega}$ with the following properties:*

*1) if $a \in [c_n, c_{n+1})$, then $\varphi(a) = \{a_0, ..., a_{n-1}\}$, where $a_k \in [c_k, c_{k+1})$ for all $k \leq n-1$ (in particular, if $a \in [c_0, c_1)$, then $\varphi(a) = \emptyset$), and*

*2) for every $d \in \mathbb{N}$, there exist $c \in \mathbb{N}$ such that whenever $a, b \in \mathbb{N}$, $0 < |a-b| \leq d$, $u \in \varphi(a)$, $v \in \varphi(b)$ and $u, v \geq c$, one has $|u - v| > d$.*

*Proof.* Without loss of generality one may suppose that $c_{n+1} - c_n \geq 5$ for all $n < \omega$. For each $n < \omega$, pick an integer $d_n \geq 2$ such that $d_n^2 + d_n - 1 \leq c_{n+1} - c_n$ and $d_n \to \infty$. To define our $\varphi$, let $k < \omega$ and let $a \geq c_{k+1}$. Choose the largest integer $l \geq 0$ for which $c_{k+1} + l d_k^2 \leq a$, then choose the lagest integer $i \geq 0$ for which $c_{k+1} + ld_k^2 + id_k \leq a$, and then put $j = a - c_{k+1} - ld_k^2 - id_k$. Thereby we write $a$ in the form

$$a = c_{k+1} + ld_k^2 + id_k + j,$$

where $l$ is a nonnegative integer and $i, j \in \{0, ..., d_k - 1\}$. Put

$$a_k = c_k + jd_k + i.$$

Since $a_k \in [c_k, c_k + d_k^2)$, the function $\varphi$ so defined satisfies the condition 1). Now to check 2), let $d \in \mathbb{N}$ be given. Choose $n_0 < \omega$ such that $d_n \geq d + 2$ for all $n \geq n_0$ and put $c = c_{n_0}$. Now, let $a, d \in \mathbb{N}$, with $0 < |a - b| \leq d$, $u \in \varphi(a)$, $v \in \varphi(b)$ and $u, v \geq c$. Since

$$\begin{aligned}
c_{k+1} - a_k &= c_{k+1} - (c_k + jd_k + i) \\
&= (c_{k+1} - c_k) - jd_k - i \\
&\geq d_n^2 + d_n - 1 - jd_k - i \\
&\geq d_k,
\end{aligned}$$

one may suppose that $u = a_k$ an $v = b_k$ for some $k \geq n_0$. Let

$$a = c_{k+1} + ld_k^2 + id_k + j,$$

$$b = c_{k+1} + l'd_k^2 + i'd_k + j'$$

Then

$$a_k = jd_k + i,$$

$$b_k = j'd_k + i'.$$

Thus, we have that

$$\begin{aligned} a - b &= (c_{k+1} + ld_k^2 + id_k + j) - (c_{k+1} + l'd_k^2 + i'd_k + j) \\ &= [(l - l')d_k + (i - i')]d_k + (j - j'), \\ a_k - b_k &= (jd_k + i) - (j'd_k + i') \\ &= (j - j')d_k + (i - i'). \end{aligned}$$

Notice that $|i - i'| < d_k$ and $|j - j'| < d_k$. Then it follows from the second equality that if $|j - j'| > 1$, $|a_k - b_k| > d_k$. Therefore one may suppose that $|j - j'| \leq 1$. But then it follows from the first equality that $a - b$ differs from a multiple of $d_k$ by $0, 1$ or $-1$. Since $|a - b| \leq d \leq d_k - 2$, this multiple of $d_k$ must be zero, so

$$(l - l')d_k + (i - i') = 0.$$

This gives us $l = l'$ and $i = i'$. Consequently, $|j - j'| = 1$, and we obtain that $|a_k - b_k| = d_k$. $\square$

**Theorem 6.2.1.** *Let $\kappa \geq \omega$. For each $\alpha < \kappa$, let $G_\alpha$ be a finite group written additively, let $G = \bigoplus_{\alpha<\kappa} G_\alpha$, and let $A = G \backslash B(G)$ be finite. Then there is a disjoint family $\{A_m : m < \omega\}$ of subsets of $A$ such that whenever a sequence $(x_n)_{n<\omega}$ is one-to-one in $A$, $g \in G$ and $m < \omega$, one has*

$$g + \mathrm{FSI}((x_n)_{n<\omega})) \cap A_m \neq \emptyset.$$

*Proof.* For each $x \in G$, let

$$\mathrm{supp}(x) = \{\alpha < \kappa : x(\alpha) \neq 0_\alpha\} \text{ and } \mathrm{supp}_0(x) = \{\alpha < \kappa : x(\alpha) \notin B(G_\alpha)\}.$$

Since $B(G) = \bigoplus_{\alpha<\kappa} B(G_\alpha)$, we have that $x \in A$ if and only if $\mathrm{supp}_0(x) \neq \emptyset$. For every $\alpha < \kappa$, $G_\alpha \backslash B(G_\alpha)$ is a disjoint union of 2-element subsets of the form $\{y, -y\}$ and we put $\mathrm{sgn}(y) = 1$ and $\mathrm{sgn}(-y) = -1$. Thus, there is a nonempty finite sequence of $\mathrm{sgn}(x(\alpha))$ assigned to every $x \in A$, where $\alpha \in \mathrm{supp}_0(x)$.

Let $\varphi : \mathbb{N} \to [\mathbb{N}]^{<\omega}$ be a function guaranteed by Lemma 6.2.1. For every $x \in A$, define the subset $\mathrm{supp}_\varphi(x) \subseteq \mathrm{supp}_0(x)$ and the nonnegative integer $\nu(x)$ as follows. Let $|\mathrm{supp}_0(x)| = l$ and let let $l \in [c_n, c_{n+1})$. Then $\mathrm{supp}_0(x) = \{\alpha_0, ..., \alpha_{l-1}\}$ for some $\alpha_0 < \cdots < \alpha_{l-1} < \kappa$ and $\varphi(l) = \{l_0, ..., l_{n-1}\}$ for some $l_k \in [c_k, c_{k+1})$, $k \leq n-1$. Put

$$\mathrm{supp}_\varphi(x) = \{\alpha_{l_0}, \ldots, \alpha_{l_{n-1}}\}$$

and define $\nu(x)$ to be the number of pairs of distinct neighbouring elements in the sequence

$$\mathrm{sgn}(x(\alpha_{l_0})), ..., \mathrm{sgn}(x(\alpha_{l_{n-1}}))$$

(if $l \in [c_0, c_1)$, then $\varphi(l) = \emptyset$, and then $\mathrm{supp}_\varphi(x) = \emptyset$ and $\nu(x) = 0$). We define the disjoint family $\{A_m : m < \omega\}$ of subsets of $A$ by

$$A_m = \{x \in A : \nu(x) \equiv 2^m (\mathrm{mod} 2^{m+1})\}.$$

Now let $(x_n)_{n<\omega}$ be a one-to-one sequence in $A$, $g \in G$ and $m < \omega$. We shall show that

$$(g + \mathrm{FSI}((x_n)_{n<\omega})) \cap A_m \neq \emptyset.$$

Note that if $(y_n)_{n<w}$ is a sum subsystem of $(x_n)_{n<\omega}$, then

$$\mathrm{FSI}((y_n)_{n<w}) \subseteq \mathrm{FSI}((x_n)_{n<\omega}).$$

By Lemma 6.1.2, one may suppose that

$$\mathrm{supp}(g) \cap \mathrm{supp}(x_i) = \emptyset \quad \textbf{and} \quad \mathrm{supp}(x_i) \cap \mathrm{supp}(x_j) = \emptyset$$

for all $i < j < \omega$. Also one may suppose that the sequence $(\min \mathrm{supp}_0(x_n))_{n<\omega}$ is increasing. Let

$$\gamma = \sup\{\mathrm{minsupp}_0(x_n) : n < \omega\}.$$

Every $x \in G$ can be uniquely written in the form $x = x' + x''$, where $x', x'' \in G$,

$$\mathrm{supp}(x') = \mathrm{supp}(x) \cap \gamma \quad \textbf{and} \quad \mathrm{supp}(x'') = \mathrm{supp}(x) \backslash \mathrm{supp}(x').$$

Put $k = 2^{m+1} - 1$. Choose inductively a sum subsystem $(y_n)_{n<k}$ of $(x_n)_{n<\omega}$ such that

a) $\max\, \mathrm{supp}_0(g') < \min\, \mathrm{supp}_0(y_0')$ and $\max\, \mathrm{supp}_0(y_i') < \min\, \mathrm{supp}_0(y_{i+1}')$ for all $i < k - 1$, and

b) each of the intervals $(|\mathrm{supp}_0(g')|, |\mathrm{supp}_0(g' + y_0')|]$ and

$$(|\mathrm{supp}_0(g' + y_0' + \cdots + y_i')|, |\mathrm{supp}_0(g' + y_0' + \cdots + y_i' + y_{i+1}')|],$$

where $i < k - 1$, contains some interval $[c_n, c_{n+1})$.

Let $h = g + y_0 + \cdots + y_{k-1}, y_{k-1} = \sum_{n \in H} x_n$ and $n_0 = \max H + 1$. We claim that there is $y_k \in \mathrm{FS}((x_n)_{n_0 \le n < \omega})$ such that

i) $\max \mathrm{supp}_0(h') < \min \mathrm{supp}_0(y_k)$,

ii) the interval $(|\mathrm{supp}_0(h')|, |\mathrm{supp}_0(h + y_k|]$ contains some $[c_n, c_{n+1}]$, and

iii) $\mathrm{supp}_\varphi(h + y_k) \cap \mathrm{supp}_0(h'') = \emptyset$.

Assume the contrary. Then by Corollary 6.1.1, $\exists\, \delta \in \mathrm{supp}_0(h'')$ and a sum subsystem $(z_n)_{n<\omega}$ of $(x_n)_{n_0 \le n < \omega}$ such that

$$\max \mathrm{supp}_0(h') < \min \mathrm{supp}_0(z_0), \quad \text{and}$$
$$\delta \in \mathrm{supp}_\varphi(h + z) \text{ for all } z \in \mathrm{FS}((z_n)_{n<\omega}).$$

Put $d = |\mathrm{supp}_0(h + z_0)|$ and let $c$ be a constant guaranteed by Lemma 6.1.1. Pick $z \in \mathrm{FS}((z_n)_{1 \le n < \omega})$ such that

$$\max \mathrm{supp}_0(h' + z_0') < \min \mathrm{supp}_0(z), \quad \text{and}$$
$$|\mathrm{supp}_0(h' + z_0')| \ge c.$$

Let $a = |\mathrm{supp}_0(h' + z_0')| \ge c$ and $b = |\mathrm{supp}_0(h + z)|$. Then

$$0 < a - b = |\mathrm{supp}_0(z_0)| \le |\mathrm{supp}_0(h + z_0)| = d.$$

Let $u = |\mathrm{supp}_0(h + z_0 + z)| \cap (\delta + 1)|$, $v = |\mathrm{supp}_0(h + z) \cap (\delta + 1)|$, $w = |\mathrm{supp}_0(h + z_0) \cap (\delta + 1)|$ and $t = |\mathrm{supp}_0(h + z_0) \cap (\delta + 1)|$. Then $u = v + w - t$. and so

$$u - v = w - t,$$

which is a contradition, since $u \in \varphi(a)$, $v \in \varphi(b)$, $u$, $v \geq c$ and $w - t \leq w \leq d$.

Now consider the element

$$g_o = g + y_0 + y_1 + \cdots + y_k.$$

By the construction

$$\operatorname{supp}_\varphi(g_0) \cap \operatorname{supp}_0(y_k) \neq \emptyset,$$

and for each $i \leq k-1$,

$$\emptyset \neq \operatorname{supp}_\varphi(g_o) \cap \operatorname{supp}_0(y_i) \subseteq \operatorname{supp}_0(y_i').$$

Therefore we have that

$$\beta_0 < \alpha_1 \leq \beta_1 < \alpha_2 \leq \beta_2 < \cdots < \alpha_{k-1} \leq \beta_{k-1} < \alpha_k,$$

where

$$\alpha_i = \min(\operatorname{supp}_\varphi(g_0) \cap \operatorname{supp}_0(y_i)),$$

$$\beta_i = \max(\operatorname{supp}_\varphi(g_0) \cap \operatorname{supp}_0(y_i)).$$

Put $\epsilon_0 = 1$ and, by induction on $i = 1, ..., k$, choose $\epsilon_i \in \{1, -1\}$ so that

$$\epsilon_{i-1} y_{i-1}(\beta_{i-1}) = \epsilon_i y_i(\alpha_i).$$

Without loss of generality one may suppose that all $\epsilon_1 = 1$. For each $i \leq k$, define $g_i \in g + \mathrm{FSI}((y_n)_{n \leq k})$ by

$$g_i = g + y_0 - y_1 + \cdots + (-1)^i y_i + (-1)^i y_{i+1} + \cdots + (-1)^i y_k.$$

Then

$$\nu(g_i) = \nu(g_0) + i.$$

Hence, there is $j \leq k \equiv 2^{m+1} - 1$ such that

$$\nu(g_j) \equiv 2^m (\mathrm{mod} 2^{m+1}),$$

and so $g_i \in A_m$.

$\square$

## 6.3 General case

In this section we prove Theorem 6.1.1 in the general case.

We need the following additional results before attempting the theorem.

**Proposition 6.3.1.** *Let $G$ be a countable Abelian topological group. Suppose that $B(G)$ is neither discrete nor open. Then there is a continuous bijection $f : G \to \bigoplus_\kappa \mathbb{Z}_4$ such that.*

*i)* $f(-x) = -f(x)$ *for all* $x \in G$,

*ii)* $f(x + y) = f(x) + f(y)$ *for all* $x, y \in G\backslash\{0\}$ *with* $\max \operatorname{supp}(f(x)) + 2 \leq \min \operatorname{supp}(f(y))$.

*Proof.* This proposition is an immediate consequence of Theorem 5.2.1. □

However, will only show how condition ii) of this proposition can replaced by condition 2) of the next theorem (Theorem 6.3.1).

Indeed, each nonzero element $a \in \bigoplus_\omega \mathbb{Z}_4$ has a unique canonical decomposition

$$a = a_1 + \cdots + a_n,$$

where for each $i = 1, \ldots, n$, $\operatorname{supp}(a_i)$ is a nonempty interval in $\omega$ and for each $i = 1, \ldots, n-1$, $\max \operatorname{supp}(a_i) + 2 \leq \min \operatorname{supp}(a_{+1})$. Further from ii), if $x_i = f^{-1}(a_i)$ for each $i = 1, \ldots, n$, we have that

$$f(x_1 + \cdots + x_n) = a = a_1 + \cdots + a_n.$$

Now, let $a = f(x)$, $b = f(y)$ and $S(a) \cap S(b) = \emptyset$. Let $a = a_1 + \cdots + a_n$ and $b = b_1 + \cdots + b_m$ be the canonical decompositions. Since $S(a) \cap S(b) = \emptyset$, there is a permutation $c_1, \ldots, c_{n+m}$ of $a_1, \ldots, a_n, b_1, \ldots, b_m$ such that

$$a + b = c_1 + \cdots + c_{n+m}$$

is a canonical decomposition. Let $x_i = f^{-1}(a_i), y_ij = f^{-1}(b_j)$ and $z_k = f^{-1}(c_k)$. Then $x = x_1 + \cdots + x_n$, $y = y_1 + \cdots + y_m$ and since $G$ is Abelian, we have that $x + y = z_1 + \cdots + z_{n+m}$. Lastly, we get that

$$\begin{aligned} f(x+y) &= f(z_1 + \cdots + z_{n+m}) \\ &= c_1 + \cdots + c_{n+m} \\ &= (a_1 + \cdots + a_n) + (b_1 + \cdots + b_m) \\ &= f(x) + f(y). \end{aligned}$$

**Theorem 6.3.1.** *Let $G$ be an infinite Abelian group endowed with the largest totally bounded group topology and let $|G| = \kappa$. Then there is a continuous injection $f : G \to \bigoplus_\kappa \mathbb{Z}_4$ such that*

1. $f(-x) = -f(x)$ *for all* $x \in G$.

2. $f(x+y) = f(x) + f(y)$ *for all* $x, y \in G$ *with* $S(f(Y)) = \emptyset$, *where for each* $a \in \bigoplus_\kappa \mathbb{Z}_4$. $S(a) = \mathrm{supp}(a) \cup (\mathrm{supp}(a) + 1)$.

*Here,* $\bigoplus_\kappa \mathbb{Z}_4$ *is endowed with the topology induced by the product topology on* $\prod_\kappa \mathbb{Z}_4$

Theorem 6.3.1 allows us to identify an Abelian group $G$ of cardinality $\kappa$ with a subset of $\bigoplus_\kappa \mathbb{Z}_4$

a) for every $x \in G$, $-x \in \bigoplus_\kappa \mathbb{Z}_4$ is the inverse of $x$ in $G$.

b) for every $x, y \in G$ with $S(x) \cap S(y) = \emptyset$, $x + y \in \bigoplus_\kappa \mathbb{Z}_4$ is the sum of $x$ and $y$ in $G$, and

c) the largest totally bounded group topology on $G$ is stronger than that induced from $\bigoplus_\kappa \mathbb{Z}_4$.

And that is all that we need to extend the proof of Theorem 6.2.1 to Theorem 6.1.1.

*Proof.* To prove Theorem 6.3.1, we start by assumption. Without loss of generality one may suppose that

$$G = \bigcup_{\alpha<\kappa} G_\alpha,$$

where for each $\alpha < \kappa$,

$$|G_\alpha| = |B(G_\alpha)| = |G_\alpha : B(G_\alpha)| = \omega.$$

For each $\alpha < \kappa$, endow $G_\alpha$ with a totally bounded group topology and let

$$f_\alpha : G_\alpha \to \bigcup_\alpha \mathbb{Z}_4$$

be a continuous bijection guaranteed by Proposition 6.3.1. We define

$$f : G \to \bigcup_{\alpha<\kappa} \bigcup_{[\alpha,\alpha+\omega)} \mathbb{Z}_4$$

by

$$f = \bigcup_{\alpha<\kappa} f_\alpha.$$

$\square$

# Chapter 7

# Conclusion

Sufficient unfolding and consolidating of the required set theory driving the final results was properly unpacked. As a result, we showed a base theorem of a wide group of homeomorphisms of finite order on countable regular spaces. The fundamental steps were to give a theorem describing the structure of a local automorphism of finite order. In particular, by defining $X$ as a local left topological group, we showed that a countably nondiscrete regular local left topological group $(X, \tau)$ can be partitioned into countably many subsets dense in any nondiscrete topology $\rho$ on $X$ such that; i) $(X, \rho)$ is a local left topological group, ii) for $f$ a nontrivial stable local automorphism on $(X, \tau)$ of finite order, $f$ is a homeomorphism on $(X, \rho)$ and iii) for a defined least number $s$ of $\mathrm{spec}(f)\backslash\{1\}$, we have that $s \in \mathrm{spec}(f, U \cap V)$ for any neighborhoods $U, V$ of the identity in topologies $\rho$ and $\tau$.

This led us to more unfoldings as we obtained that i) if a countable regular space declares a nontrivial regular homeomorphism of finite order, then it is $\omega$-resolvable. ii) In a countable group endowed with a regular $\omega$-irresolvable topology with continuous shifts, the centralizer of any element of finite order is open. It was also shown that every countable group with finitely many elements of order 2, that can be embedded in a compact topological group, is absolutely $\omega$-resolvable. As a result of showing this, we were able to show that every countable nondiscete topological group not containing an open Boolean subgroup is $\omega$-resolvable. In addition, we showed how this results were again proved using a theorem describing the structure of a larger family of homeomorphism of finite order on countably infinite regular spaces. We thus say with certainty that every countable infinite nondiscrete $\omega$-irresolvable topological group contains a countably infinite open Boolean subgroup.

Also, we showed that every infinite Abelian group not containing an infinite Boolean subgroup is absolutely $\omega$-resolvable. As a consequence, we obtained that every infinite Abelian group not containing infinite Boolean subgroup can be partitioned into $\omega$ subset such that every coset modulo infinite subgroup meets each subset of the partition. Furthermore, we got that for a totally bounded topological group $G$ and a sequence $(x_n)_{n<\omega}$ in $G$, whenever we have some neighborhoods of $0 \in G$, say $U$ and $U_x$, for $x \in U$, there is a sum subsystem $(y_n)_{n<\omega}$ of $(x_n)_{n<\omega}$ such that $y_0 \in U$ and for each $n < \omega$, we have $y_{n+1} \in \bigcap_{i\leq n} U_{y_i}$.

It was finally shown that there is a disjoint family $\{A_m : m < \omega\}$ of subsets of $A$ such that whenever a sequence $(x_n)_{n<\omega}$ is one-to-one in $A$, $g \in G$ and $m < \omega$ one has $g + \mathrm{FSI}((x_n)_{n<\omega}) \bigcap A_m \neq \emptyset$. In particular, we showed that every infinite Abelian group not containing an infinite Boolean subgroup is absolutely resolvable.

# Bibliography


[1] S. I. ADIAN, *Classifications of periodic words and their application in group theory*, in Burnside groups, Springer, 1980, pp. 1–40.

[2] R. ARENS, *The adjoint of a bilinear operation*, Proceedings of the American Mathematical Society, 2 (1951), pp. 839–848.

[3] A. ARHANGELSKII, *Groupes topologiques extremalement discontinus*, CR Acad. Sci. Paris, 265 (1967), pp. A822–A825.

[4] A. ARHANGELSKII AND M. TKACHENKO, *Topological Groups and Related Structures, An Introduction to Topological Algebra.*, vol. 1, Springer Science & Business Media, 2008.

[5] V. I. ARNAUTOV, *Nondiscrete topologizability of countable rings*, in Doklady Akademii Nauk, vol. 191, Russian Academy of Sciences, 1970, pp. 747–750.

[6] N. BOURBAKI, *Elements of mathematics: General topology*, Springer, 1995.

[7] H. CARTAN, *Théorie des filtres*, CR Acad. Sci. Paris, 205 (1937), pp. 595–598.

[8] E. CECH, *On bicompact spaces*, Annals of Mathematics, (1937), pp. 823–844.

[9] V. CHVÁTAL AND F. HARARY, *Generalized ramsey theory for graphs. iii. small off-diagonal numbers*, Pacific Journal of mathematics, 41 (1972), pp. 335–345.

[10] W. COMFORT, *Topological groups*, in Handbook of set-theoretic topology, Elsevier, 1984, pp. 1143–1263.

[11] W. COMFORT AND J. VAN MILL, *Groups with only resolvable group topologies*, Proceedings of the American Mathematical Society, 120 (1994), pp. 687–696.

[12] M. M. DAY, *Lumpy subsets in left-amenable locally compact semigroups*, Pacific Journal of Mathematics, 62 (1976), pp. 87–92.

[13] D. DIKRANJAN, *Topological groups*, in Encyclopedia of General Topology, Elsevier, 2003, pp. 359–364.

[14] A. EL'KIN, *Regular maximal spaces*, Matematicheskie Zametki, 27 (1980), pp. 301–305.

[15] R. GRAHAM AND S. BUTLER, *Rudiments of Ramsey theory*, vol. 123, American Mathematical Soc., 2015.

[16] R. L. GRAHAM AND B. L. ROTHSCHILD, *Ramseys theorem for $n$-parameter sets*, Transactions of the American Mathematical Society, 159 (1971), pp. 257–292.

[17] R. L. GRAHAM, B. L. ROTHSCHILD, AND J. H. SPENCER, *Ramsey theory*, vol. 20, John Wiley & Sons, 1990.

[18] E. HEWITT, *A problem of set-theoretic topology*, Duke Mathematical Journal, 10 (1943), pp. 309–333.

[19] E. HEWITT AND K. A. ROSS, *Abstract Harmonic Analysis: Volume I Structure of Topological Groups Integration Theory Group Representations*, vol. 115, Springer Science & Business Media, 2012.

[20] N. HINDMAN AND D. STRAUSS, *Algebra in the stone-čech compactification*, de Gruyter, Berlin, 1998.

[21] N. HINDMAN, D. STRAUSS, AND Y. ZELENYUK, *Large rectangular semigroups in stone-cech compactifications*, Transactions of the American Mathematical Society, 355 (2003), pp. 2795–2812.

[22] A. ILLANES, *Finite and $\omega$-resolvability*, Proceedings of the American Mathematical Society, 124 (1996), pp. 1243–1246.

[23] T. J. JECH, *The axiom of choice*, Courier Corporation, 2008.

[24] M. KATÉTOV, *On topological spaces containing no disjoint dense sets*, Matematiceskij sbornik, 63 (1947), pp. 3–12.

[25] V. I. MALYKHIN, *Extremally disconnected and similar groups*, in Doklady Akademii Nauk, vol. 220, Russian Academy of Sciences, 1975, pp. 27–30.

[26] A. MARKOV, *On unconditionally closed sets*, Mat. Sbornik, 18 (1946), pp. 3–28.

[27] P. S. NOVIKOV AND S. I. ADJAN, *Infinite periodic groups. i*, Mathematics of the USSR-Izvestiya, 2 (1968), p. 209.

[28] A. OLŠANSKII, *A note on a countable nontopologizable group*, Vestnik Moskov. Univ. Ser. I Mat. Mekh, 103 (1980).

[29] L. S. PONTRYAGIN, *Topological groups*, (1946).

[30] I. PROTASOV, *Decompositions of direct products of groups*, Ukrainian Mathematical Journal, 49 (1997), pp. 1559–1569.

[31] I. PROTASOV, *Irresolvable topologies on groups*, Ukrainian Mathematical Journal, 50 (1998), pp. 1879–1887.

[32] I. PROTASOV, *Maximal topologies on groups*, Siberian Mathematical Journal, 39 (1998), pp. 1184–1194.

[33] F. P. RAMSEY, *On a problem of formal logic*, in Classic Papers in Combinatorics, Springer, 2009, pp. 1–24.

[34] F. RIESZ, *Stetigkeitsbegriff und abstrakte Mengenlehre*, Tip. della R. Accademia dei Lincei, 1909.

[35] W. RUDIN, *Homogeneity problems in the theory of čech compactifications*, Duke Mathematical Journal, 23 (1956), pp. 409–419.

[36] S. SHELAH, *On a problem of kurosh, jónsson groups, and applications*, Word problems. II. Amsterdam, (1980), pp. 373–394.

[37] S. SHELAH, *Proper forcing*, Springer-Verlag. Berlin, 1982.

[38] S. SIROTA, *The product of topological groups and extremal disconnectedness*, Mathematics of the USSR-Sbornik, 8 (1969), p. 169.

[39] M. H. STONE, *Applications of the theory of boolean rings to general topology*, Transactions of the American Mathematical Society, 41 (1937), pp. 375–481.

[40] S. ULAM, *Concerning functions of sets*, Fundamenta Mathematicae, 14 (1929), pp. 231–233.

[41] D. VAN DANTZIG, *Ueber topologisch homogene kontinua*, Fundamenta Mathematicae, 15 (1930), pp. 102–125.

[42] D. VAN DANTZIG, *Zur topologischen algebra*, Mathematische Annalen, 107 (1933), pp. 587–626.

[43] D. VAN DANTZIG, *Zur topologischen algebra. ii. abstrakte bv-adische ringe*, Compositio Mathematica, 2 (1935), pp. 201–223.

[44] D. VAN DANTZIG, *Zur topologischen algebra. iii. brouwersche und cantorsche gruppen*, Compositio Mathematica, 3 (1936), pp. 408–426.

[45] E. K. VAN DOUWEN, *Applications of maximal topologies*, Topology and its Applications, 51 (1993), pp. 125–139.

[46] H. WALLMAN, *Lattices and topological spaces*, Annals of Mathematics, (1938), pp. 112–126.

[47] Y. ZELENYUK, *Topological groups with finite semigroups of ultrafilters*, Matematychni Studii, 6 (1996), pp. 41–52.

[48] Y. ZELENYUK, *Decomposability of topological groups*, Ukrainian Mathematical Journal, 51 (1999), pp. 44–50.

[49] Y. ZELENYUK, *On topologies on groups with continuous shifts and inversion, visn*, Kyiv. Univ. Ser. Fiz.-Mat. Nauki, (2000), pp. 252–256.

[50] Y. ZELENYUK, *Partitions of groups into absolutely dense subsets*, Mathematics Zametki, 67 (2000), pp. 706–711.

[51] Y. ZELENYUK, *On partitions of groups into dense subsets*, Topology and its Applications, 126 (2002), pp. 327–339.

[52] Y. ZELENYUK, *On extremally disconnected topological groups*, Topology and its Applications, 153 (2006), pp. 2382–2385.

[53] Y. ZELENYUK, *On the ultrafilter semigroup of a topological group*, Semigroup Forum 73, (2006), pp. 301–307.

[54] Y. ZELENYUK, *Partitions and sums with inverses in abelian groups*, Journal of Combinatorial Theory, Series A, 115 (2008), pp. 331–339.

[55] Y. ZELENYUK, *Topologies on groups determined by discrete subsets*, Topology and its Applications, 155 (2008), pp. 1332–1339.

[56] Y. ZELENYUK, *Ultrafilters and topologies on groups*, vol. 50, Walter de Gruyter, 2011.