

Reviewing a framework to price a credit risky derivative post the credit crisis

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted as fulfillment of the Degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

A handwritten signature in black ink, consisting of a stylized 'C' and 'B' intertwined, with a horizontal line crossing through them.

C. B. Hunzinger
February 9, 2014

Abstract

The period between 2008 and 2009 was an interesting and dramatic time for financial markets. This period marked the beginning of the financial tsunami that would plague global markets for many years to come. This economic meltdown had massive effects on many everyday issues such as house prices, interest rates and inflation. Investment banks were also affected with numerous investment banks either defaulting or being taken over by the U.S. Federal Reserve to avoid default. This group of investment banks include names such as Lehman Brothers, Bear Sterns, Fannie Mae, Freddy Mac and many more. The myth of “*too big to fail*” was tested and failed because of the number of banks that were allowed to default during the crisis. Many things have changed because of the crisis. One area in finance that has changed is the pricing of financial derivatives. The realisation that huge investment banks can default has dried up the liquidity in capital markets. Therefore banks cannot borrow a shortfall of cash at a risk-free rate anymore but rather at a significant spread over the risk-free rate. The risk-free rate is a core concept of derivative pricing. If investment banks cannot borrow and lend at the risk-free rate then the Black-Sholes-Merton theory laid down in the 1970’s may not be applicable post the credit crisis. The aim of this dissertation is to review the framework of Piterbarg, Burgard and Kjaer to price a general derivative post the credit crisis. This review includes a variety of numerical methods to implement the framework.

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“No problem can be solved until it is reduced to some simple form. The changing of a vague difficulty into a specific, concrete form is a very essential element in thinking.”

- J. P. Morgan

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Chapter summary

The dissertation consists of six chapters. The aims and objectives of each chapter are listed below:

Chapter 1 - Introduction

- Describe how the pricing of a general derivative trade has changed because of the credit crisis.

Chapter 2 - Literature Review

- This chapter is a literature review of the topic at hand.
- This chapter reviews the current FVA debate. This includes the views of [Hull and White \(2012\)](#), [Laughton and Vaisbrot \(2012\)](#) and [Castagna \(2012a\)](#).
- The literature review also outlines the three frameworks that have so far been developed to price a general derivative trade post the credit crisis.

Chapter 3 - The Black-Scholes-Merton model which includes collateral

- This chapter reviews the paper by [Piterbarg \(2010\)](#).
- We develop a binomial model which is a discrete analog of the model presented in this paper.
- This model is derived using a replicating argument and is shown to hold for n number of steps, where n is a natural number.
- Next, we derive Piterbarg's PDE by making use of a replicating argument.
- The Feynman-Kac theorem is then applied to the PDE in order to represent the price of a collateralised derivative trade as an expectation.
- This shows that the discrete binomial model and the continuous PDE model are analogs of each other.
- The model is implemented numerically using *MATLAB*¹. The binomial tree is implemented using a standard algorithm for binomial trees. An *implicit-Euler* scheme is used to implement the PDE. A Quasi Monte Carlo simulation is used to evaluate the price of the collateralised derivative when written in expectation form.
- We further illustrate that the three approaches used to implement the model converge to a unique price.

¹MATLAB is a high-level technical computing language and interactive environment for algorithm development, data visualization, data analysis, and numerical computation.



Chapter 4 - A model used to price a derivative which includes counterparty credit risk and funding costs

- This chapter reviews the paper by [Burgard and Kjaer \(2011\)](#).
- We develop a tree model which is a discrete analog of the model presented in this paper.
- This model is derived using a hedging argument and is shown to hold for n number of steps, where n is a natural number.
- Next, the PDE is derived using a hedging argument.
- The Feynman-Kac theorem is then applied to the PDE in order to write the price of a credit risky derivative trade as an expectation.
- This shows that the discrete tree model and the continuous PDE model are analogs of each other.
- The Burgard and Kjaer PDE is solved when the derivative's value only takes on positive values or negative values throughout the lifetime of the trade. The solution to the PDE will be function of the risk-free price only.
- This model is implemented numerically via three approaches, namely a PDE, a tree lattice and a Quasi Monte Carlo approach. An implicit-Euler scheme is used to implement the PDE; however, the PDE is non-linear. In order to implement this PDE we use the implicit-Euler scheme in conjunction with a recursive algorithm. The tree lattice and the Quasi Monte Carlo approaches also require a recursive algorithm to compute the price of the credit risky derivative.
- We illustrate that the three approaches used to implement the model converge to a unique price.

Chapter 5 - A generalised framework which includes counterparty credit risk, funding costs and collateral

- This chapter reviews the paper by [Burgard and Kjaer \(2012b\)](#).
- This generalised framework is in the form of a PDE, which is derived using a hedging argument.
- The Feynman-Kac theorem is applied to the PDE in order to write the price of a credit risky derivative as an expectation.
- We show that this general framework reduces to three unique models.
- One of the three unique models is the model that is presented in Chapter 4.
- The three models derived in this chapter are implemented numerically via the recursive algorithm introduced in Chapter 4.
- The three unique models are used to price the same derivative. The prices are tabulated and compared.

Chapter 6 - Conclusion

- This chapter includes critique of the generalised framework reviewed in this dissertation.
- Finally, we relate this generalised framework to the banking industry.



List of Acronyms

Acronyms

BSDE Backward stochastic differential equation

BSM Black-Scholes-Merton

CDS Credit default swap

CollVA Collateral value adjustment

CSA Credit support annex

CVA Credit value adjustment

DVA Debit value adjustment

FASB Financial Accounting Standards Board

FBA Funding benefit adjustment

FCA Funding cost adjustment

FVA Funding value adjustment

IFRS International Financial Reporting Standards

ISDA International Swaps and Derivatives Association

JIBAR Johannesburg Interbank Agreed Rate

LIBOR London Interbank Offered Rate

MTM Mark-to-market

NACC Nominal annual continuous compounding

OIS Overnight index swap

OTC Over-the-counter

PDE Partial differential equation

SAFEX South African Futures Exchange

SDE Stochastic differential equation

TVA Total value adjustment



List of Notation

Notation

- Δ_t : amount of stock held at time t .
- C_{BVA} : bilateral CVA inclusive of collateral at time t_0 .
- B : bond amount used to invest excess cash and borrow a shortfall of cash for the various cash amounts.
- γ : cash in the replicating portfolio.
- γ_{P_B} : cash needed to purchase counterparty B 's bonds.
- γ_{P_C} : cash needed to purchase counterparty C 's bonds.
- $g(X, C)$: close-out function of the derivative trade.
- X : close-out value of the derivative trade.
- γ_C : collateral account in the replicating portfolio.
- C : collateral value.
- σ : constant volatility of the underlying.
- r_D : continuous dividend yield of the underlying stock.
- P_B : counterparty B 's bond price.
- P_C : counterparty C 's bond price.
- J_t : default indicator at time t .
- λ : default intensity.
- P_B^D : default state of counterparty B 's bonds.
- τ : default time.
- M : diagonal matrix used to encapsulate the discretisation of a PDE.
- d : downward factor of the underlying in a binomial asset pricing model.
- $\mathbb{E}$: expectation operator.
- $\mathcal{F}_t$: filtration generated by the stock price process at time t .
- θ : fractional amount of the risky derivative value held as collateral.
- γ_F : funding account in the replicating portfolio.
- F_j : funding and investing cash flow of a trade at time t_j .
- h_e : hedge error.
- T : maturity time of the trade.
- $\min(V, 0)$: negative part of the derivative.
- α_B : number of B bonds.
- α_C : number of C bonds.
- $\mathcal{A}_t$: parabolic operator.
- $\mathbb{P}$: real-world measure.
- $\mathbb{Q}^{r_R}$: pricing measure that makes the underlying drift at the rate r_R .

- q : probability that the stock moves upward in a binomial asset pricing model under the measure $\mathbb{Q}^{r_R}$.
- $\max(V, 0)$: positive part of the derivative.
- r_C : rate paid on collateral.
- r_F : rate paid on unsecured funding.
- γ_R : repo account in the replicating portfolio.
- R : recovery rate.
- r_R : repurchase rate of the underlying.
- V : risk-free derivative price.
- $\hat{V}$: risky derivative price.
- r : risk-free rate.
- $\mathbb{Q}$: risk-neutral measure.
- s_C : spread between the rate paid on collateral and the risk-free rate.
- s_F : spread between the rate paid on unsecured funding and the risk-free rate.
- W_t : standard Brownian motion.
- ξ_i : state of the world in a four state tree model.
- K : strike price of a general derivative.
- $F(0, T \wedge \tau, C)$: sum of the discounted expected discrete funding or investing cash flow of a trade at time t_0 .
- $R(0, T \wedge \tau, C)$: sum of the discounted expected discrete margining cash flows of a trade at time t_0 .
- U_t : total value adjustment at time t .
- S_t : underlying stock price at time t .
- u : upward factor of the underlying in a binomial asset pricing model.
- Π_t : value of the replicating portfolio at time t .
- $\bar{V}_j$: vector of risk-free derivative prices at time t_j .

Chapter 1

Introduction

The 2008 crisis has been a massive turning point for the world of finance and in particular for the pricing of financial derivatives. Before the crisis, pricing the value of a derivative was relatively straight forward. Universally, practitioners and academics agreed on the pricing method used to price a derivative. The method was simple: discount future expected cash flows under the risk-neutral measure to the present date using the risk-free rate. This method was derived from the fundamental theory laid down by *Black, Scholes* and *Merton* in the 1970's. The pricing of financial derivatives has become more complex because many more factors need to be considered now. These factors include counterparty credit risk and the inability to borrow and invest cash at the risk-free rate. The inclusion of these factors, into the pricing of derivatives, has not been universally accepted among the participants in the industry. The default of what we used to call “*too big to fail*” banks, such as *Lehman Brothers* and *Bear Stearns*, which defaulted in the crisis, disproved the myth that banks are risk-free. This realisation lead to many developments which include collateralisation of trades, a *funding value adjustment* (FVA) and a *debit value adjustment* (DVA). Next, we define and discuss these developments and show how they are related. We discuss what the *risk-free rate* is and how it affects the pricing of derivatives. This discussion leads to the topic of collateralised trades and funding value adjustment.

A crucial but complex question is - what is the “*risk-free rate*” in the market? We place the words risk-free rate in inverted commas to make the point that there is no rate that is 100% risk-free. Some may argue that government bond yields may be seen as risk-free. However, defaults in countries such as Russia in 1998 and the more recent austerity measures taken by countries such as Greece and Spain to avoid default, has halted this idea. The risk-free rate is an important quantity in the pricing of financial derivatives. It is the rate used to discount future expected risk-free cash flows to the present time. The *London Interbank Offered Rate* (LIBOR) and, in South Africa, the *Johannesburg Interbank Agreed Rate* (JIBAR) were seen as the risk-free rate before 2008. This is the rate that banks can freely borrow and lend at. The LIBOR rate includes a spread for the credit risk of the banks.

An *overnight indexed swap* (OIS) is an interest rate swap involving the overnight rate being exchanged for a fixed interest rate. An overnight index swap uses an overnight rate index, such as the Fed funds rate, as the underlying for its floating leg, while the fixed leg would be set at an assumed rate. The Fed funds rate is the interest rate earned on cash, when deposited in the Federal Reserve overnight. The OIS market is highly liquid for certain primary currencies as well as quoted for a broad range of maturities, an OIS-based curve can be readily bootstrapped¹ using traditional methods. This OIS rate is now seen as a proxy for the risk-free

¹ A procedure used to calculate the zero-coupon yield curve from market rates.



rate.

The LIBOR–OIS spread is the difference between the LIBOR and the OIS rate. Prior to the crisis, this spread was stable and not significant. The difference between the US 3-month LIBOR and OIS rate was only a few basis points prior to the crisis (see Gregory (2012) (pg. 286)). This spiked to hundreds of basis points in the aftermath of the Lehman default in September 2008 and has remained significant ever since. Figure 1.1 illustrates that the spread between the 3-month LIBOR and OIS rate, reached a maximum of 364.42 basis points in mid September 2008. These shifts made it apparent that LIBOR incorporates an adjustment for credit risk; therefore, LIBOR is an imperfect proxy for the risk-free rate. The relevant discounting rate for collateralised transactions is now typically the OIS rate.



Figure 1.1: The spread between 3-month LIBOR and OIS during the 2008 crisis. Source: [www.http://soberlook.com](http://soberlook.com), May 31, 2009.

In South Africa overnight index swaps do not trade actively. In Daniels *et al.* (2013) the authors propose two proxies, which could be used to approximate an OIS market in South Africa. The first proxy is to estimate a risk-free curve by shifting the JIBAR curve downward by 50 basis points in a parallel fashion. The second suggestion is to consider the *South African Futures Exchange* (SAFEX) rate paid on overnight margins.

Next, we discuss collateralised trades post the credit crisis. The International Swaps and Derivatives Association (ISDA) margin survey examines the state of collateral use and management among derivatives dealers and end-users. The 2010 ISDA margin survey suggests that 70% of net exposure arising from *over-the-counter* (OTC) derivative transactions are collateralised². Despite the increase in collateralised trades, many trades are not collateralised. This is mainly due to the counterparties in the trade. If a counterparty has liquidity issues and may find it difficult to meet Credit support annex (CSA)³ rules, then they will trade without a CSA agreement. The utility giant in South Africa, Eskom, would be a counterparty likely to do a trade without a CSA agreement because of a lack of liquidity in the parastatal.

²source: www2.isda.org.

³A Credit support annex (CSA) defines the terms or rules under which collateral is posted or transferred between trade counterparties to mitigate the credit risk arising from “in-the-money” derivative positions.

The Black-Scholes-Merton (BSM) theory states that one has to discount future expected cash flows under the risk-neutral measure using the risk-free rate. This theory may not apply anymore because post the credit crisis there exists a spread between LIBOR and the OIS rate. As a result this has stopped the banks from being able to lend and borrow funds at the risk-free rate. This is one of the assumptions of the BSM theory. The assumption allows a replicating portfolio to borrow (invest) the value of the stock position at the risk-free rate. This additional cost (benefit) of borrowing (lending) at a higher rate is exactly an FVA.

Piterbarg (2010) shows a fundamental result in the world of derivative pricing, which is when pricing a *zero-threshold CSA trade*⁴ where the collateral is cash and the same currency as the derivative, the cash flows should be discounted using the collateral rate. Collateral posted overnight will earn a rate similar to the index rate referenced in an OIS. Furthermore, he shows that when the trade is not collateralised, then the cash flows should be discounted using the funding rate of the bank. The author further shows that one may price a trade by always discounting the future expected cash flows using a collateral rate and making a funding value adjustment.

FVA is a correction made to the risk-free price of an OTC derivative to account for the funding cost in a financial institution. The funding value adjustment can be described as, when managing a trading position, one needs cash in order to do operations such as hedging or posting collateral. This shortfall of cash can be obtained from the treasury of the bank. The *funding cost adjustment* (FCA) is the cost of lending money at a funding rate which is higher than the risk-free rate. The firm may also receive cash in the form of collateral or a premium. The *funding benefit adjustment* (FBA) is the benefit earned when excess cash is invested at a higher rate than the risk-free rate. Therefore, the funding value adjustment has two components

$$FVA = FBA + FCA,$$

where the FBA and the FCA terms have opposite signs. From Piterbarg's paper, if the trade is not collateralised or traded with no CSA, then a FVA will appear in the price. If the trade is traded with a zero-threshold CSA, then the FVA term will reduce to zero. FVA can therefore be described as the difference between a collateralised trade and a non-collateralised trade.

We have not defined the FVA term mathematically. The quantity is a core subject of the dissertation and will be derived in the thesis. At this stage we propose the following formula to price a collateralised derivative trade

$$\text{risky price} = \text{risk-free price} + \text{FVA}.$$

The *risk-free price* is the price of a derivative that includes no credit risk and funding costs. This price is calculated by discounting all expected cash flows at the risk-free rate. The risky price is defined as the risk-free price plus any adjustments.

Next, we discuss the concept of credit value adjustment (CVA). The discussion will lead to the concept of debit value adjustment. Even before the crisis, banks realised that many corporate clients are not risk-free; therefore, banks charged their clients a credit value adjustment in an OTC derivative trade. CVA is the price of the hedging used to mitigate the credit risk in the trade. CVA is defined as the fair market value of the expected loss of an OTC derivative given that the opposite counterparty defaults. Many of the bank's clients believed that the bank was risk-free; therefore, the clients would accept the price that the bank offered them and in turn did not charge the bank a CVA on the trade. Reference papers on CVA include [Sorensen and](#)

⁴The threshold amount is the amount of unsecured credit risk that two counterparties are willing to accept before a collateral demand will be made.

Bollier (1994), Jarrow and Turnbull (1995) and Duffie and Huang (1996).

As a result of the crisis, banks are not seen as risk-free anymore. One implication of this is the inclusion of a *debit value adjustment* in the derivative's price. DVA is the price of the hedging used to mitigate own credit risk in the trade. DVA is defined as the fair market value of the expected gain of an OTC derivative given own default. The origins of DVA are found in Duffie and Huang (1996) who were the first to calculate DVA; however, their paper deals mostly with swaps. Gregory (2009) and Brigo and Capponi (2008) examine bilateral credit risk in general and derive DVA formally. In essence, DVA is the adjustment clients charge the bank for the bank's own credit risk. Therefore, from the client's point of view, the adjustment is known as CVA and from the bank's point of view the adjustment is known as DVA.

One aspect of pricing with CVA and DVA is that it allows two credit risky counterparties to still trade with each other. The DVA of the bank will be exactly the CVA charged by the client to the bank. If two counterparties charged each other a CVA and do not include the offsetting DVA term, then the two counterparties would not agree on the price of the derivative. The inclusion of DVA allows *symmetric* prices. The concept of symmetric prices means that two counterparties will price the derivative at the same price.

The approach used by banks to value a derivative, is to first calculate the price assuming no credit risk, then make adjustments for the expected gain or loss due to default by either counterparty. These adjustments are computed by the CVA desk in the bank. The derivative's price from the bank's point of view at time t_0 is calculated by

$$\text{risky price}_0 = \text{risk-free price}_0 + \text{CVA}_0 + \text{DVA}_0, \quad (1.1)$$

where the CVA and DVA terms have opposite signs.

DVA is a hotly debated and controversial quantity. The reason for the controversy is that the DVA amount can only be realised when the bank defaults. If the bank is *out-of-the-money* on a trade and defaults, then the bank only needs to pay a recovery of the *mark-to-market* (MTM); therefore, the bank benefits from own default. It can be compared to buying life insurance. The policy will only be realised after the death of the policy holder.

Another worrying property of DVA is that when a bank's credit rating deteriorates, it causes a paper profit on its books. For example, let us assume that the market factors affecting the value of the derivative and the credit quality of the client remains constant, whilst assuming that the bank's credit rating declines. In this example, we assume that the CVA is not dependent on the bank's credit quality. The value of the derivative's mark-to-market on the bank's book is

$$\text{risky price}_1 = \text{risk-free price}_1 + \text{CVA}_1 + \text{DVA}_1, \quad (1.2)$$

where $0 \leq \text{DVA}_0 \leq \text{DVA}_1$. Therefore, the derivative will have a favourable mark-to-market to the bank and will register a profit on its balance sheet. The Wall Street Journal reported on *October 18, 2011, 3:59 PM* that

“Goldmans DVA gains in the third quarter totaled \$450 million, about \$300 million of which was recorded under its fixed-income, currency and commodities trading segment and another \$150 million recorded under equities trading. That amount is comparatively smaller than the \$1.9 billion in DVA gains that J.P. Morgan Chase and Citigroup each recorded for the third quarter. Bank of America reported \$1.7 billion of DVA gains in its investment bank.”

The above extract illustrates the following phenomenon of DVA. A widening credit spread leads to a paper profit, whereas a decreasing credit spread leads to a paper loss. This is counterintuitive because one would



expect that when a firm's credit rating improves, the health of the firm improves. This is not the case with the inclusion of DVA.

One could argue that it is possible to hedge out this variability in *profit and loss* (P&L). To hedge out DVA, the bank would need to go short its own credit. An example of going short one's own credit is to sell a *credit default swap* (CDS) on itself. A credit default swap is a financial swap agreement that the seller of the CDS will compensate the buyer in the event of a loan default or a credit event. If the CDS is written by the party which the CDS references, then when the reference party (writer) defaults, the party long the CDS will likely only receive a fraction of the CDS's payoff. No rational buyer would buy a CDS from a firm who is the reference party in the CDS. [Hull and White \(2000\)](#) provide a more formal definition of a credit default swap.

Buying back their own bonds in order to hedge out DVA is another idea; however, this requires cash. If the firm is trying to hedge out their increasing DVA, then the firm would be in a situation where cash is scarce. They would have to fund the cash, defeating the point of buying back debt.

A more realistic option to hedge DVA is to sell protection on a basket of firms that are highly correlated with one's self. For example, Goldman Sachs could sell a CDS on a group of investment banks which includes names such as J.P. Morgan or Citigroup. The hedge effectiveness will depend on the correlation between Goldman Sachs and the basket of banks. The higher the correlation, the better the hedge. The problem that emerges is that if banks are highly correlated to default, then systematic risk is introduced into the hedge. The systematic risk is that if the banks in the basket begin to default, then the likelihood that Goldman Sachs defaults is high. Therefore, the chance that Goldman Sachs will deliver on their CDS diminishes, leading to financial contagion. [Kenyon and Stamm \(2012\)](#) (pg. 157) suggest that this is a highly inaccurate hedge. Some practitioners argue that DVA simply cannot be hedged effectively.

[Gregory and German \(2012\)](#) describe DVA as a double edged sword. On the one hand, it creates a symmetric world where counterparties can readily trade with one another, even when their underlying default probabilities are high. On the other hand, the nature of DVA and its implications and potential unintended consequences create some additional complexity and potential discomfort. From an accounting point of view, adding DVA to the price makes sense; however, the regulators are not so sure. Risk Magazine on 6th February 2012 reported that

“Accountants want banks to report as profits the impact of widening credit spreads on their liabilities, but regulators are moving in the other direction.”

(see [Carver \(2012\)](#)). The accounting rules IRFS 13 and Financial Accounting Standards Board (FASB) 157 require DVA. However, the Basel III committee has decided to ignore any DVA relief in capital calculations.

Up until now, we have not raised the issue of close-outs. A close-out is the value used to determine the mark-to-market value of the derivative when either counterparty defaults. The two cases, considered in this dissertation, are the risk-free value and the risky value. The risk-free value is the price obtained from the BSM model. The risky price will contain credit risk and funding costs. The impact of different close-out rules on CVA, DVA and FVA are discussed in [Brigo and Morini \(2010\)](#), [Brigo and Morini \(2011\)](#), [Burgard and Kjaer \(2012a\)](#) and [Gregory and German \(2012\)](#). We have discussed OIS discounting, collateralised trades, close-outs, FVA, CVA and DVA. This introduction will conclude by suggesting an expression to price a derivative trade post the credit crisis. This derivative price will include adjustments such as CVA, DVA and FVA.

The aim of this thesis is to review a framework to price a credit risky derivative with funding costs. At this stage, we suggest the following expression to price a credit risky derivative with funding costs

$$\text{risky price} = \text{risk-free price} + \text{CVA} + \text{DVA} + \text{FVA}. \quad (1.3)$$



The above expression is being debated furiously among academics and practitioners. There is currently quite an intense controversy in the financial quantitative industry as to whether we should include FVA at all when pricing financial instruments. This discussion reached its peak with the infamous article by [Hull and White \(2012\)](#) in Risk Magazine. Hull and White argue against FVA. They argue that the funding costs and benefits realised in a trade violate the idea of *risk-neutral pricing* and should not be included in the pricing of the derivative. The funding value adjustment violates the law of one price in the market because including FVA into the price of a derivative trade will mean that two counterparties will price a trade differently.

Many academics and practitioners have realised that if both FBA and DVA are included in the price, this will lead to double counting. We offer the following reason for this claim. FVA references the firms own funding spread ⁵ in both terms, FCA and FBA. The funding spread will be based on the credit rating of the firm. The counterparty's credit spread ⁶ is referenced in the CVA term and the firms own credit spread in the DVA term. A change in the credit rating of the bank will lead to a change in the price of the derivative. Since the DVA and FBA terms have the same sign, the change in the price will be reflected twice if both the DVA and FBA terms are included in the valuation.

⁵The funding spread is the difference between the funding rate of the bank and the risk-free rate.

⁶The credit spread of a firm is the difference between the yield on a firm's credit risky bond and the yield of a risk-free bond.

Chapter 2

Literature review

This chapter provides a literature review on pricing a derivative trade post the credit crisis. The literature review describes the current FVA debate and then reviews three frameworks that have been developed in order to price a derivative trade post the credit crisis. These frameworks encapsulate counterparty credit risk and funding or liquidity costs into an OTC derivative trade. The frameworks reviewed in this chapter are currently works in progress and no framework has been universally accepted by practitioners in the industry.

2.1 Funding value adjustment

Funding value adjustment is related to other factors such as collateral and debit value adjustment and thus cannot be reviewed without the inclusion of these related quantities. FVA is a consequence of the 2008 credit crisis and as a result all the literature on FVA is recent.

2.1.1 Hull and White

We now present the views of Hull and White on funding value adjustment. [Hull and White \(2012\)](#) are of the opinion that FVA should not be included in the valuation of financial derivatives. This statement sparked a huge debate among practitioners and academics. At the *Global Derivatives Trading & Risk Management conference 2013* John Hull said “We received an endless amount of emails about our article, roughly two thirds disagree with us”.

In the Black-Scholes-Merton theory, the value of an option can be calculated by solving the Black-Scholes partial differential equation (PDE) or equivalently taking the expectation of the future value of the derivative under the risk-neutral measure and discounting it back to time zero using the risk-free rate (see [Black and Scholes \(1973\)](#) and [Merton \(1973\)](#)). This is known as the economic value, because if this were not the price, one could form a portfolio to exploit an arbitrage.

[Hull and White \(2012\)](#) states that the reason why one discounts at the risk-free rate is because the risk-neutral valuation principle requires this, and not because the bank can fund at the risk-free rate. An immediate argument against this statement is that a price of a derivative is equal to its replicating portfolio. This will include funding costs and will not agree with the risk-neutral price. The funding of hedges is sometimes given as a reason for an FVA. [Hull and White \(2012\)](#) say that the decision to hedge does not affect valuations. They



further argue that the inclusion of FVA would violate a well established principle in corporate finance theory. This principle is known as the *Fisher-Hirshleifers Separation Principle*. This principle states that pricing should be kept separate from funding (see [Fisher \(1930\)](#) and [Hirshleifer \(1958\)](#)). Furthermore, when pricing an investment one must consider the risks of the investment and not how the firm will fund it.

FVA, in practice, causes a transfer pricing problem. The transfer pricing problem is: “Who carries the funding cost of the trade and who earns the funding benefit of the trade?”. For example, suppose that the treasury of the bank charges the trader on the derivatives desk a funding rate of LIBOR plus 20 basis points. The incentives of the trader will be assessed on the return on capital; therefore, the trader will want to pass this cost onto the counterparty in the trade in order to recover the funding costs. The trader will be incentivised not to pass on the funding benefit to the client. However, by not passing on the funding benefit to the client, the trader may lose the trade because the traders price may not be competitive in the market. This illustrates the difference between the economic price, which is the price the two counterparties agree on, and the private price, which is the price that the trader wants to trade at. The private price will include funding costs. Hull and White emphasise that the inclusion of FVA will violate the law of one price. They also say that banks are free to calculate the value of a derivative any way they wish. If they wish, they can include FVA; however, for the calculation of profit and loss from an accounting point of view, the bank has to use the best estimate of the fair market value of the contracts.

[Hull and White \(2012\)](#) illustrates the connection between FVA and collateral. If a trade is done with a two-way CSA¹ and the value of the derivative is posted as collateral, then the cost of funding is the rate paid on the collateral amount. No funding valuation adjustment term appears because the firm is effectively funding at the collateral rate which they assume to be a proxy for the risk-free rate. The above argument hinges on the fact that *rehypothecation* is possible. Rehypothecation is the practice by banks and brokers of using, for their own purposes, assets that have been posted as collateral by their clients (see [Brigo et al. \(2013\)](#) (pg. 259)). If a one-way CSA is in place, then one party will not be required to post collateral; therefore, the other counterparty will have to fund the trade at LIBOR plus a spread. The FVA of this counterparty would be set equal to the expected present value of the excess cost of funding over the risk-free rate. We conclude that, in their view, FVA should not be included in the price of the derivative.

2.1.2 Laughton and Vaisbrot

Next, the views of Laughton and Vaisbrot on funding value adjustment are presented. Many academics and practitioners disagree with Hull and White’s comments. [Laughton and Vaisbrot \(2012\)](#) responded to Hull and White’s comments about FVA in a follow-up article named “*In defense of FVA*” in Risk Magazine September 2012.

[Hull and White \(2012\)](#) use the Black-Scholes-Merton theory to argue that derivatives should be valued on the risk-neutral basis, independent of funding costs. However, [Laughton and Vaisbrot \(2012\)](#) argue that the theory hinges on the fact that all risks are fully hedged. Not all risk factors can be hedged because the market is incomplete. This is where theory and practice diverges. In theory, the Black-Scholes PDE is derived using a hedging portfolio that perfectly hedges the contingent claim; therefore, the market is complete (see [Shreve \(2000\)](#) (Definition 5.4.8, pg. 231)). Laughton and Vaisbrot suggest that in practice the market is not complete and the uniqueness of prices will fade and the law of one price (see [Harrison and Pliska \(1981\)](#)) will not hold anymore.

¹A two-way CSA enforces both counterparties to place collateral if they are out-of-the-money on a trade.

Laughton and Vaisbrot (2012) state that applying the so called FVA to the risk-neutral value, as widely as it happens nowadays, is justified. Banks with a lower funding rate will be more competitive on trades that require funding. This is fully consistent with the current situation in the markets, as theory should aim to be. In summary, they believe that the beautiful and elegant theory of BSM is not applicable in this day and age and needs to be rethought because of the theory's unrealistic assumptions, especially post the credit crisis.

2.1.3 Antonio Castagna

Next, the views of Antonio Castagna on funding value adjustment are presented. Within the same month of the article Hull and White (2012), Castagna released an article disagreeing with Hull and White (see Castagna (2012a)). The article confronts seven statements in Hull and White's article. The format of the paper is as follows: A statement of Hull and White (2012) is stated and then Castagna illustrates why the statement is flawed. We mentioned a few of Hull and White's comments and we will summarise why Castagna disagrees with them. The first statement is: *The inclusion of FVA would violate a well established principle in corporate finance theory. That is when pricing an investment one must consider the risks of the investment and not how the firm will fund it.* Castagna (2012a) comments that the Fisher-Hirshleifers Separation Principle only holds under specific conditions. He illustrates in his article that the principle does not apply to the valuation of financial derivatives.

The second statement is: *The reason why one discounts at the risk-free rate or OIS rate is because the risk-neutral valuation principle requires this and not because the bank can fund at the risk-free rate.* Castagna (2012a) makes the comment that we discount at the risk-free rate because it is required by the risk-neutral argument provided that we are pricing in an economy or world where there is no credit risk or funding costs. Castagna further argues that in the BSM model, to price a derivative, a replicating portfolio is constructed in such a way that the replicating portfolio perfectly replicates the derivative. In the BSM setting no funding costs will enter into the price. However, this only holds for a very specific economic environment which includes all the assumptions made in the BSM theory.

In the BSM model there is only one interest rate and that is the risk-free interest rate. Castagna (2012a) suggests if one considers a framework where more than one interest rate exists, such as a risk-free rate and a funding rate, then one could still produce a replicating portfolio which perfectly replicates the derivative. If a bank can only invest at the risk-free rate and fund at a higher funding rate, then it is well known that this will not impede the replication of the derivative (see Bergman (1995) and Rubenstein and Cox (1985)). This will lead to a different price for the buy side and the sell side; however, a closed form solution will still exist. Castagna thinks that models need to be amended in order to be more useful to traders. They should remove the assumption of the ability to borrow at the risk-free rate to finance trades and should be extended to include more unique interest rates.

2.2 General frameworks used to price a credit risky derivative with funding costs

In this section, we discuss the frameworks and models used to price a credit risky derivative with funding costs which have been proposed by various academics and practitioners. Currently, there are three unified frameworks which incorporate funding costs, collateral and credit risk into a derivative trade. The three frameworks are proposed by



- (i) V. Piterbarg, C. Burgard and M. Kjaer.
- (ii) D. Brigo, D. Perini and A. Pallavicini.
- (iii) S. Crépey.

Each framework takes a different approach to encapsulate funding costs, collateral and credit risk into the derivative's price. The approach by Piterbarg, Burgard and Kjaer prices a general derivative using a hedging argument to derive a PDE of the value of the credit risky derivative. The approach by Brigo, Perini and Pallavicini prices a general derivative by expectation. The third framework, which is developed by Stéphane Crépey, prices a general derivative using a *backward stochastic differential equation* (BSDE).

We now define some notation needed to present the frameworks. The notation presented here will be consistent with the entire dissertation. Consider two counterparties, counterparty B and counterparty C . Counterparty B can be seen as the bank and C as the bank's client. The frameworks will price a derivative trade from the point of view of counterparty B . The constant recovery rates of counterparty B and C are denoted by R_B and R_C , respectively. The deterministic hazard rates of the two counterparties are denoted by λ_B and λ_C . The default times of counterparty B and C are given by τ_B and τ_C , respectively, and $\tau = \min(\tau_B, \tau_C)$ denotes the first to default time.

The frameworks presented here will contain multiple interest rates. The *nominal annual compounded continuously* (NACC) deterministic interests rates in the frameworks include

- r the theoretical risk-free rate,
- r_C the rate paid on collateral,
- r_R the rate paid on a repurchase agreement on the underlying, and
- r_F the unsecured funding rate.

The price of the derivative that does not include any adjustment such as CVA, DVA and FVA is denoted by V and will be referred to as the risk-free price. Some practitioners refer to this price as the clean price. If the price of the derivative does include adjustments, then it is denoted by $\hat{V}$ and will be referred to as the risky price. We denote the underlying as S and the constant volatility of the underlying as σ . The positive part and the negative part of the derivative's price is denoted by

$$V^+ = \max(V, 0) \quad \text{and} \quad V^- = \min(V, 0),$$

respectively. We note that our notation for V^- is not standard. Furthermore, the value of collateral posted in the trade at some time t is denoted by C_t . The discount factor $D^{r_*}(s, t)$ represents the zero coupon bond with an appropriate interest rate r_* , hence

$$D^{r_*}(s, t) = e^{-\int_s^t r_{*,u} du},$$

where $0 \leq s \leq t$ are some finite time points.

Next, we discuss a model proposed by Hull and White to price a derivative post the credit crisis. The model is presented because each of the three frameworks will simplify to Hull and White's model if certain unrealistic assumptions are made in each framework. We then summarize the main results of the first two frameworks for comparison reasons. The chapter concludes with a summary of the framework developed by Stéphane Crépey.



2.2.1 Hull and White model

In [Hull and White \(2012\)](#) they extend the Black-Scholes-Merton model to incorporate credit risk and show that this value will be the best estimate of the fair market value. Their proposed price for a derivative trade post the credit crisis is given by the following theorem:

Theorem 1:

The price of a credit risky derivative trade satisfies the following PDE

$$\begin{cases} \frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = \lambda_C(1 - R_C)V^+ + \lambda_B(1 - R_B)V^- \\ \hat{V}_T = V_T \text{ is the final payoff of the derivative} \end{cases},$$

where the risk-free price satisfies the Black-Scholes PDE

$$\begin{cases} \frac{\partial V}{\partial t} + \mathcal{A}_t V - rV = 0 \\ V_T \text{ is the final payoff of the derivative} \end{cases}$$

and the parabolic differential operator is given by

$$\mathcal{A}_t = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + rS \frac{\partial}{\partial S}.$$

The terms $\lambda_C(1 - R_C)V^+$ and $\lambda_B(1 - R_B)V^-$ represent the CVA and DVA in the derivative trade. The CVA and DVA terms are adjustments made to the risk-free price in order to take into account counterparty default and own default into the derivative trade. The CVA term

$$\lambda_C(1 - R_C)V^+,$$

can be interpreted as the amount of the derivative's mark-to-market that counterparty B will lose if C had to default, scaled by the probability that C defaults. The DVA term

$$\lambda_B(1 - R_B)V^-,$$

can be interpreted as the amount of the derivative's mark-to-market that counterparty B will gain if B had to default, scaled by the probability that B defaults. This model does not include collateral or funding costs. We will show in this dissertation that this model is not adequate enough to price a derivative trade post the credit crisis. They only consider the risk-free close-out convention in this model.

2.2.2 Piterbarg, Burgard and Kjaer framework

The first framework is proposed by Piterbarg, Burgard and Kjaer. [Piterbarg \(2010\)](#) derives a model to price a collateralised derivative trade with the absence of credit risk. [Burgard and Kjaer \(2011\)](#) derives a model to price a credit risky derivative trade with the absence of collateral. These two papers are then extended in [Burgard and Kjaer \(2012b\)](#) to a generalised framework which includes counterparty credit risk, collateral and funding costs.

The approach by the authors is to derive the price of a derivative by constructing a hedging portfolio which perfectly hedges out all risks. If the hedging portfolio perfectly hedges out the derivative, then to avoid an arbitrage, the price of the derivative has to be equal to the short hedging portfolio.

Pricing a derivative trade with funding costs

We now outline the ideas of [Piterbarg \(2010\)](#). This model illustrates the difference between pricing a CSA derivative trade and a non-CSA derivative trade. The following theorem presents a PDE of the value of a collateralised derivative trade:

Theorem 2:

The value of a general collateralised derivative on a dividend paying stock satisfies the following PDE

$$\begin{cases} \frac{\partial V}{\partial t} + (r_R - r_D)S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = V r_F - C(r_F - r_C) \\ V_T \text{ is the final payoff of the derivative} \end{cases}.$$

The above PDE extends the one curve Black-Scholes model to a multi-curve model. The fundamental result illustrated in the paper is that if the derivative is traded with a zero-threshold CSA, then the collateral curve is used to discount future expected cash flows. If the derivative is traded with no CSA, then the funding curve is used to discount future expected cash flows. This paper will be reviewed in Chapter 3 of this dissertation. We provide a complete proof of Theorem 2 in that chapter and formally define funding value adjustment.

Pricing a derivative with counterparty credit risk and funding costs

Next, we outline the main result of [Burgard and Kjaer \(2011\)](#). The authors use a hedging portfolio to derive a PDE of the price of a general derivative which includes counterparty credit risk and funding costs. In the derivation of the PDE, Burgard and Kjaer make a few key assumptions:

- (i) The excess cash in the hedging portfolio is invested at the risk-free rate in order not to incur extra credit risk.
- (ii) The funding spread and the credit spread of the firm are implied from the firms bonds.
- (iii) The hedging strategy generates enough cash needed to repurchase B 's bonds in the secondary market in order to hedge out B 's own credit risk.

The third assumption is the most questionable. In practice, to repurchase one's own bonds on a trade basis to hedge own credit risk is unrealistic. A PDE of the value of a credit risky derivative is presented in the following theorem:

Theorem 3:

The value of a credit risky derivative on a non-dividend paying stock satisfies the following PDE

$$\begin{cases} \frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C) \hat{V} = s_F X^+ - \lambda_B (X^+ + R_B X^-) - \lambda_C (X^- + R_C X^+) \\ \hat{V}_T = V_T \text{ is the final payoff of the derivative} \end{cases}$$

and the parabolic differential operator is given by

$$\mathcal{A}_t = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} - r_R S \frac{\partial}{\partial S}.$$

The close-out value of the derivative trade is denoted by X .

If the close-out convention is the risky close-out, then the above PDE is non-linear because of the min and max functions applied to the close-out value on the right hand side of the PDE. In order to approximate the risky derivative's price a difference scheme in conjunction with an iterative algorithm is required. The term $s_F X^+$ represents the funding cost adjustment, which is the cost of borrowing the mark-to-market of the trade above the risk-free rate. The terms $\lambda_C (X^- + R_C X^+)$ and $\lambda_B (X^+ + R_B X^-)$ represent the CVA and DVA in the trade, respectively. When B defaults and B is in-the-money, then B will receive the full close-out value of the trade from C . When B is out-of-the-money on the trade and defaults, then B will only have to pay a recovery amount, R_B , of the close-out value of the trade. This holds for counterparty C too. This paper will be reviewed in Chapter 4 of this dissertation. We provide a proof of Theorem 3 in that chapter.

Pricing a collateralised derivative trade which includes counterparty credit risk and funding costs

Next, we briefly discuss [Burgard and Kjaer \(2012b\)](#). In this paper the authors provide a different approach to deriving the PDE as done in [Burgard and Kjaer \(2011\)](#). In this paper all risks are not fully hedged. They do not hedge in case of own default. The following theorem presents a PDE of the value of a credit risky collateralised derivative trade:

Theorem 4:

The value of a general credit risky collateralised derivative trade on a non-dividend paying stock satisfies the following PDE

$$\begin{cases} \frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C) \hat{V} = -\lambda_C g_C - \lambda_B g_B + \lambda_B h_e + s_c C \\ \hat{V}_T = V_T \text{ is the final payoff of the derivative} \end{cases}$$

and the parabolic differential operator is given by

$$\mathcal{A}_t = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} - r_R S \frac{\partial}{\partial S}.$$

The hedge error is denoted by h_e , and g_B and g_C are the close-out functions of counterparty B and C, respectively. The close-out functions and hedge error are defined as

$$\begin{aligned} g_B(X_t, C_t) &= C_t + (X_t - C_t)^+ + R_B(X_t - C_t)^-, \\ g_C(X_t, C_t) &= C_t + (X_t - C_t)^- + R_C(X_t - C_t)^+ \end{aligned}$$

and

$$h_e = g_B(X, C) + P_B^D - C,$$

respectively. The term P_B^D is defined as the default state of the positions taken in counterparty B's bonds.

The term $s_c C$ in Theorem 4 represents the *collateral value adjustment* (CollVA) term. The collateral value adjustment arises when the collateral rate is not equal to the theoretical risk-free rate. This adjustment will be discussed further in Chapter 5. The hedge error is the difference between the hedging portfolio and the short derivative value when counterparty B defaults. The expected discounted weighted hedge error is the funding cost adjustment of the trade. This becomes more clear when the Feynman-Kac theorem is applied to the above PDE. The terms $\lambda_C g_C$ and $\lambda_B g_B$ represent the CVA and DVA in the trade, respectively. The DVA and CVA are now a function of the collateral amount because when a default occurs, the surviving party has a claim on the collateral amount in order to recover an amount of its loss because of the opposite counterparty defaulting.

This framework is more general than Theorem 3 because the framework includes collateral and a hedge error. The size of this hedge error is dependent on the bond strategy employed by counterparty B. This general PDE reduces to three unique models when different assumptions are made about counterparty B's bonds used in the bond strategy. One of the three models is the model presented by Hull and White in Theorem 1. The second is the model presented in Theorem 3. This paper will be reviewed in Chapter 5 of this dissertation. We provide a complete proof of Theorem 4 in that chapter. The framework presented by Piterbarg, Burgard and Kjaer is the theme of this dissertation.

2.2.3 Brigo, Perini and Pallavicini framework

The second framework is proposed by Brigo, Perini and Pallavicini. Their approach differs from that of Piterbarg, Burgard and Kjaer which prices a derivative post the credit crisis by means of a hedging argument.



Brigo, Perini and Pallavicini approach the pricing of a derivative more generally by pricing by means of expectation. They argue that the markets are not complete and hence not all risks can be hedged out. In summary, this framework takes the expectation of all future cash flows under the risk-neutral measure and discounts them to the present time using the risk-free rate. Kenyon and Stamm (2012) (pg. 139) states that hedging is not the same as expectation in an incomplete market; therefore, this framework will differ to the framework of Piterbarg, Burgard and Kjaer.

Brigo and Capponi (2008) is the origin of this framework. The authors in that paper price a general derivative, which includes bilateral counterparty credit risk, by means of expectation. This model was adequate prior to the financial crisis. The framework is extended to include collateral, effects of close-outs and funding costs in the papers Brigo *et al.* (2011), Pallavicini *et al.* (2011a) and Pallavicini *et al.* (2011b), respectively.

Pricing a collateralised derivative trade which includes bilateral credit risk

We now outline the ideas of Brigo *et al.* (2011). The authors price a general derivative trade which includes bilateral counterparty credit risk and collateral by expectation. This framework is more general than the Piterbarg, Burgard and Kjaer framework because in this framework they consider the credit risk when rehypothecation is performed unlike, in previous framework. For comparison reasons we assume that one does not incur credit risk when performing rehypothecation. The following theorem presents a formula to value a credit risky derivative trade with collateral:

Theorem 5:

The value of a general collateralised derivative at time $t_0 = 0$ maturing at time T , is given by the following expression

$$\hat{V}_0 = V_0 + C_{BVA}(t_0, T \wedge \tau, C),$$

where $C_{BVA}(t_0, T \wedge \tau, C)$ denotes the bilateral credit value adjustment inclusive of collateral and is defined as

$$\begin{aligned} C_{BVA}(t_0, T \wedge \tau, C) = & - \mathbb{E}^{\mathbb{Q}} \left[\mathbb{I}_{\{\tau < T\}} e^{-\int_0^{\tau} r_u du} (V_{\tau} - \mathbb{I}_{\{\tau = \tau_C\}} X_B - \mathbb{I}_{\{\tau = \tau_B\}} X_C) \right] \\ & - \mathbb{E}^{\mathbb{Q}} \left[\mathbb{I}_{\{\tau = \tau_C < T\}} e^{-\int_0^{\tau} r_u du} (1 - R_C)(X_B - C_{\tau})^+ \right] \\ & - \mathbb{E}^{\mathbb{Q}} \left[\mathbb{I}_{\{\tau = \tau_B < T\}} e^{-\int_0^{\tau} r_u du} (1 - R_B)(X_C - C_{\tau})^- \right]. \end{aligned}$$

The expectation is under the risk-neutral measure $\mathbb{Q}$. The close-out value denoted by X_B and X_C indicates which counterparty survives. The surviving counterparty will determine the close-out value^a.

^aThis is inline with the recommendations set out by the International Swaps and Derivatives Association Master Agreement.

The bilateral credit value adjustment is a function of three adjustments. The first adjustment is the difference between the risk-free price and the sum of the close-out values, scaled by the default indicator function of the two counterparties. The second and third terms represent the DVA and CVA terms, respectively. When the close-out value is the risk-free value, the first term of the bilateral credit value adjustment reduces to zero and the CVA and DVA terms will reduce to the conventional bilateral CVA seen in most counterparty credit risk textbooks. However, when the close-out value is the risky price, then the first term does not reduce to zero. The intuition behind this term is that the derivative value at the time of the first default will be replaced

by some close-out value; therefore, the difference between these two values needs to be accounted for in the pricing of the derivative.

Pricing a collateralised derivative trade with bilateral credit risk and margining costs

Next, we outline the main results of Pallavicini *et al.* (2011a). In this paper the authors present a risk-neutral evaluation of a derivative trade inclusive of bilateral counterparty credit risk, collateral and margining. This is an extension to the model presented in Theorem 5.

Margining, in practice, happens at discrete time points often with lengthy periods between each time point. In this paper they extend the model by including discrete margining cash flows by summing all the discounted expected collateral cash flows between counterparty *B* and *C*. The following theorem presents an expression to price a general credit risky collateralised derivative which includes margining cash flows:

Theorem 6:

The value of a general collateralised derivative which includes discretely posted collateral at time $t_0 = 0$, maturing at time T , is given by the following expression

$$\hat{V}_0 = V_0 + C_{BVA}(t_0, T \wedge \tau, C) + R(t_0, T \wedge \tau, C),$$

where the sum of the expected discrete margining cash flows of the trade is denoted by $R(t_0, T \wedge \tau, C)$ and is defined as

$$R(t_0, T \wedge \tau, C) = \mathbb{E}^{\mathbb{Q}} \left[\sum_{k=1}^{n-1} \mathbb{I}_{\{t_k < T \wedge \tau\}} D^r(t_0, t_k) \left(C_k - C_k^+ \frac{D^r(t_k, t_{k+1})}{D^{r_C^+}(t_k, t_{k+1})} - C_k^- \frac{D^r(t_k, t_{k+1})}{D^{r_C^-}(t_k, t_{k+1})} \right) \right],$$

where $0 = t_0 \leq t_1 \leq \dots \leq t_k \leq \dots \leq t_n = T < \infty$. The collateral cash flows occurring at time t_k are denoted by C_k and the terms C_k^+ and C_k^- are the positive and negative parts of the cash flows, respectively. The rates r_C^+ and r_C^- denote the rates that are applied to negative and positive cash flows, respectively^a.

^aWe note this is an abuse of our notation.

We now describe the intuition behind the discrete margining cash flow expression in Theorem 6. We begin with the terms

$$C_k - C_k^+ \frac{D^r(t_k, t_{k+1})}{D^{r_C^+}(t_k, t_{k+1})} - C_k^- \frac{D^r(t_k, t_{k+1})}{D^{r_C^-}(t_k, t_{k+1})}. \quad (2.1)$$

At each discrete margin point the counterparty that is out-of-the-money will place collateral with the opposite counterparty. This is represented by the first cash flow in equation (2.1). The next two terms determine whether the cash flow is positive or negative. The two cash flows will grow at different rates depending on the sign. This means that the two counterparties may not pay the same rate on posted collateral. The collateral amount that is posted at time t_k is taken forward at an appropriate collateral rate and then discounted back at the risk-free rate. If

- $C_k > 0$, then counterparty *C* posts collateral with *B*.
- $C_k < 0$, then counterparty *B* posts collateral with *C*.

The cash flow represented by equation (2.1) is then discounted to time t_0 by the discount factor $D^r(t_0, t_k)$; however, this cash flow only happens if a default has not happened yet. This is taken care of by the indicator function $\mathbb{I}_{\{t_k < T \wedge \tau\}}$. All these margin cost cash flows are summed and the expectation operator under the risk-neutral measure is applied to the summation.

Pricing a collateralised derivative trade with bilateral credit risk, margining costs and funding costs

Next, we provide a summary of Pallavicini *et al.* (2011b). In this paper, the authors extend the framework further by including a funding value adjustment. Funding costs occur when a trading desk needs cash to perform operations such as hedging or paying a premium to enter into a trade.

In this paper they consider two ways to determine the funding rate of a trade. The single deal (micro) funding model where funding costs are deal specific and the homogeneous (macro) funding model where funding costs are firm specific. In the Piterbarg, Burgard and Kjaer framework they assume that the funding costs incurred by the trading desk are not deal specific. Pallavicini *et al.* (2011b) try to stay as general as possible. Funds can be sourced either directly from the market or from the treasury of the bank. This plays a role in the pricing of the derivative trade. For comparison reasons we assume that all funds are financed through the treasury of the bank at a specific rate. The following theorem presents a general framework to price a credit risky collateralised derivative trade which includes margining, funding and investing cash flows:

Theorem 7:

The value of a general collateralised derivative which includes margining, funding and investing costs at time $t_0 = 0$, maturing at time T , is given by the following expression

$$\hat{V}_0 = V_0 + C_{BVA}(t_0, T \wedge \tau, C) + R(t_0, T \wedge \tau, C) + F(t_0, T \wedge \tau, C),$$

where the funding or investing cost of the trade is denoted by $F(t_0, T \wedge \tau, C)$ and is defined as

$$F(t_0, T \wedge \tau, C) = \mathbb{E}^{\mathbb{Q}} \left[\sum_{j=1}^{m-1} \mathbb{I}_{\{t_j < T \wedge \tau\}} D^r(t_0, t_j) \left(F_j - F_j^- \frac{D^r(t_j, t_{j+1})}{D^{r_F^-}(t_j, t_{j+1})} - F_j^+ \frac{D^r(t_j, t_{j+1})}{D^{r_F^+}(t_j, t_{j+1})} \right) \right],$$

where $0 = t_0 \leq t_1 \leq \dots \leq t_j \leq \dots \leq t_m = T < \infty$. The funding or investing cash flows occurring at time t_j is denoted by F_j and the terms F_j^+ and F_j^- are the positive and negative parts of the cash flows, respectively. The rates r_F^+ and r_F^- denote the rates that are applied to negative and positive cash flows, respectively

The funding or investing cash flows and the collateral cash flows may not occur at the same discrete time points. This is illustrated by using a different index in the two summations. We now describe the intuition behind the funding or investing cost expression in Theorem 7. We begin with the terms

$$F_j - F_j^+ \frac{D^r(t_j, t_{j+1})}{D^{r_F^+}(t_j, t_{j+1})} - F_j^- \frac{D^r(t_j, t_{j+1})}{D^{r_F^-}(t_j, t_{j+1})}. \quad (2.2)$$

At each discrete time point counterparty B will need to finance a short fall of cash or invest a surplus of cash, which is represented by F_j in equation (2.2). If

- $F_j > 0$, then B has a surplus of cash in the trade.

- $F_j < 0$, then B will require cash to perform operations such as hedging.

The second and third terms in equation (2.2) represent the funding amount that is lent or invested at time t_j and taken forward at an appropriate funding rate, and then discounted back at the risk-free rate. The cash flow represented in equation (2.2) is then discounted back to time t_0 by the discount factor $D^r(t_0, t_j)$; however, this cash flow only happens if a default has not happened yet. This is taken care of by the indicator function $\mathbb{I}_{\{t_j < T \wedge \tau\}}$. All these funding and investing cash flows are summed and the expectation operator under the risk-neutral measure is applied to the summation.

The inclusion of a funding value adjustment makes the additive framework non-linear. The authors provide an explanation detailing why the inclusion of the FVA term makes the pricing formula recursive. This explanation would require the proof of the above framework which we have omitted; therefore, we leave it to the reader to verify this result. The authors recommend that the pricing equation can be solved by casting the equation into a set of iterative relationships which can be solved by means of standard least-square Monte Carlo techniques (see Longstaff and Schwartz (2001)).

The FVA term in this framework will reduce to zero if many unrealistic assumptions are placed on the model. These assumptions include:

- there exists no liquidity basis when funding, and
- that the risk-free rate is the OIS rate .

The authors make the comment that the FVA term only reduces to zero in a very specific case; therefore, they disagree with the comments made by Hull and White. Furthermore, if no collateral is posted in the trade, the FVA value is zero and markets are complete, then it can be shown that this framework reduces to Hull and White's model presented in Theorem 1. In the derivation of this framework the authors tried to keep many factors as general as possible. As a result, this framework is more general than the framework presented by Piterbarg, Burgard and Kjaer.

2.2.4 Stéphane Crépey framework

The framework developed by Stéphane Crépey prices a credit risky derivative with funding costs by means of a backward stochastic differential equation. The BSDE is a function of the price of the derivative and its relative hedging portfolio. This framework makes less assumptions than the Piterbarg, Burgard and Kjaer framework. For instance he assumes different rates when borrowing a shortfall of cash and when investing excess cash. In the Piterbarg, Burgard and Kjaer framework they assume that these cash flows grow at the same unique rate. The main ideas of this framework are presented in three papers, namely Crépey (2012a), Crépey (2012b) and Crépey *et al.* (2012c).

Crépey (2012a) presents the derivation of the “*price and hedge*” BSDE. The paper also shows that when one makes the same assumptions as in the Piterbarg, Burgard and Kjaer framework in this framework, then the results derived in this paper coincide with the ones derived in Burgard and Kjaer (2012b). Furthermore, if more unrealistic assumptions are made, such as if all funds can be borrowed and invested at the risk-free rate, then this BSDE will reduce to the Black-Scholes model.

In Crépey (2012b) an expression is derived for the total value adjustment of a credit risky derivative trade. The TVA term has four components

- (i) replacement cost or benefit,

- (ii) a debit value adjustment,
- (iii) a credit value adjustment, and
- (iv) a funding cost or benefit adjustment.

This differs to the total value adjustment presented in the Piterbarg, Burgard and Kjaer framework. The paper also shows the hedging portfolio needed to hedge out the total value adjustment.

Lastly, the paper [Crépey *et al.* \(2012c\)](#) is a numerical implementation of the previous two papers. The TVA presented in [Crépey \(2012b\)](#) is implemented numerically for an interest rate swap. The paper also shows how the four different parts of the TVA change when the amount of collateral posted in the trade and the recovery rates of the two counterparties change.

Chapter 3

The Black-Scholes-Merton model which includes collateral

In this chapter the paper [Piterbarg \(2010\)](#) is reviewed. Piterbarg was one of the first to show how the posting of collateral in a derivative trade affects the price. His paper extends the Black-Scholes-Merton framework to include collateral in the derivative trade. The paper won him the quant of the year award in 2011.

The chapter begins with a derivation of a binomial model, which includes dividends and collateral. This binomial model is our own original work. The derivation of the binomial model is followed by a derivation of Piterbarg's PDE. The Feynman-Kac theorem is then applied to Piterbarg's PDE in order to show that the two models are asymptotically equivalent. Furthermore, we show that the difference in price of a CSA trade and a non-CSA trade is a funding value adjustment. Then the model presented by Piterbarg is written in a different form in order to take expectations under the risk-neutral measure. The chapter concludes with a numerical implementation of the model.

3.1 Deriving a binomial asset pricing model which includes collateral

In this section, we develop a binomial model for a derivative trade which includes collateral on a dividend paying stock. We begin by defining the notation used in this chapter. Then the one period binomial model is derived using a replicating portfolio. Next, the n period model is derived using the *Tower Property*.

In a binomial setting a general derivative, V_t , can be priced by means of a replicating portfolio. The type of derivative that is being replicated is not specified and we assume that the trade is between two counterparties that cannot default during the life of the trade. All notation follows from [Piterbarg \(2010\)](#). The three different interest rates and the dividend yield used in this model are

- r_C the rate paid on collateral,
- r_R the rate paid on a repurchase agreement of non-dividend paying stock,
- r_F the rate paid on unsecured funding, and
- r_D the dividend yield on the stock.



The relationship between the rates is

$$r_C \leq r_R \leq r_F.$$

One assumption of this model is that the rates are NACC and deterministic.

The replicating portfolio, denoted by Π_t , is set up so that the portfolio perfectly replicates the derivative trade at all times and in all states of the world. The quantities making up the replicating portfolio are

- Δ amount of stock, and
- γ amount of cash.

The cash amount, γ , can be split into three different accounts, namely a funding account, a repo account and a collateral account

- (i) The amount in the funding account is denoted by γ_F . This amount is the difference between two cash amounts, namely the price of the hedging portfolio and the collateral amount. The cash price of the hedging portfolio is denoted by γ_P and the collateral amount is denoted by γ_C , hence $\gamma_F = \gamma_P - \gamma_C$. This amount is the amount needed to be borrowed or lent at the unsecured funding rate, r_F . The model makes the assumption that one can borrow or lend cash out at the funding rate without incurring counterparty credit risk. In the next chapter this assumption is dropped.
- (ii) The amount in the repo account is denoted by γ_R . This account will consist of the cash invested or borrowed in order to fund the stock position, ΔS_t , through a repurchase agreement.
- (iii) The amount in the collateral account is denoted by γ_C . This account will contain the collateral posted with the counterparty. This account will grow at the collateral rate r_C . The assumptions made about the posted collateral in this model are that no *hair cut* is applied to the collateral value and that the currency of the collateral and the settlement of the derivative is the same.

The dividend amount, γ_D , is the value of the dividend paid on the stock position. When the stock is placed in a repurchase agreement, then the dividend paid by the stock will be received by the counterparty holding the stock position in the repo. The counterparty that places the stock in the repo agreement will lose out on the dividend; however, the counterparty will be compensated by only paying the repo rate less the dividend yield on the repurchase agreement. Therefore, the repo rate of a dividend paying stock will be $r_R - r_D$.

The change in time between the nodes in the binomial lattice is denoted by dt . The short rate of a particular interest rate at time t is denoted by $r_*(t)$. This is the interest rate that occurs from time t until $t + dt$.

Remark 3.1. If the rate r_* is a time dependent interest rate compounded continuously, then the rate observed over the period $[t_j, t_{j+1}]$ is $\int_{t_j}^{t_{j+1}} r_*(s)ds$ where $j = 0, 1, \dots, n-1$. In order to ease notation and to avoid lengthy equations we define

$$r_*(t_j)(t_{j+1} - t_j) = \int_{t_j}^{t_{j+1}} r_*(s)ds, \quad (3.1)$$

where $dt = t_{j+1} - t_j$ for all $j = 0, 1, \dots, n-1$.

3.1.1 One period binomial model with collateral

A one period binomial model of a general derivative trade with collateral is now derived. The price of a derivative trade with collateral in a one period binomial is presented in the following theorem:



Theorem 8:

The price of a general collateralised derivative trade with the absence of counterparty credit risk in a one period binomial model is

$$V_0 = e^{-r_F(t_0)dt} \left[\mathbb{E}_0[V_1] + C_0 \left(e^{r_F(t_0)dt} - e^{r_C(t_0)dt} \right) \right]^a.$$

If we assume that the collateral rate is a proxy for the risk-free rate, then the price of the derivative trade is the sum of the risk-free price and a funding value adjustment,

$$V_0 = e^{-r_C(t_0)dt} \mathbb{E}_0[V_1] - e^{-r_C(t_0)dt} (V_0 - C_0) \left(e^{r_F(t_0)dt} - e^{r_C(t_0)dt} \right),$$

where V_1 is the payoff of the derivative at time t_1 , C_0 is the amount of collateral posted in the trade at time t_0 and $r_C(t_0)$ and $r_F(t_0)$ are the short collateral and funding rates at time t_0 , respectively. The expectation is taken under the $\mathbb{Q}^{r_R}$ -measure^b.

^aThe conditional expectation operator denoted by $\mathbb{E}_n$ is conditioned on the stock price at time t_n , where n is a natural number.

^bThe $\mathbb{Q}^{r_R}$ -measure is the pricing measure that makes a dividend paying stock drift at the repo rate less the dividend yield.

Proof. The derivation makes use of a replicating portfolio. If the replicating portfolio is equal to the derivative trade at all times, then to avoid an arbitrage the price of the derivative must be equal to the replicating portfolio. The initial value of the replicating portfolio consists of Δ_0 stock and γ cash, hence

$$\Pi_0 = \Delta_0 S_0 + \gamma.$$

The different accounts in the cash amount are invested or borrowed in different bonds, where each bond follows the following processes in the binomial model

$$\begin{aligned} dB_F(t, s) &= B_F(s + ds) - B_F(s) = \left(e^{r_F(s)ds} - 1 \right) B_F(s), \\ dB_R(t, s) &= B_R(s + ds) - B_R(s) = \left(e^{(r_R(s) - r_D)ds} - 1 \right) B_R(s), \\ dB_C(t, s) &= B_C(s + ds) - B_C(s) = \left(e^{r_C(s)ds} - 1 \right) B_C(s), \end{aligned}$$

where $t \leq s$ and at time t all the bonds are worth 1, i.e. $B(t, t) = 1$. In this discrete model $t = t_0$. The total growth of the cash accounts is the sum of the total growth of the individual bonds. At time t_1 the stock pays a dividend. The stock price will drop by exactly the dividend amount paid at time t_1 . The dividend amount of the stock price is denoted by $\delta = 1 - e^{-r_D(t_1 - t_0)}$. The value of the dividend is dependent on the outcome of the stock movement. The value of the dividend at time t_1 is

$$\gamma_D = \begin{cases} S_0 u \delta & \text{if the stock price moves up} \\ S_0 d \delta & \text{if the stock price moves down} \end{cases},$$

where u is the upward factor and d is the downward factor in the one period binomial tree. Figure 3.1 illustrates the drop in the stock price when paying out the dividend, where p is the real world probability that the stock moves upwards and $(1 - p)$ is the real world probability that the stock moves downward.

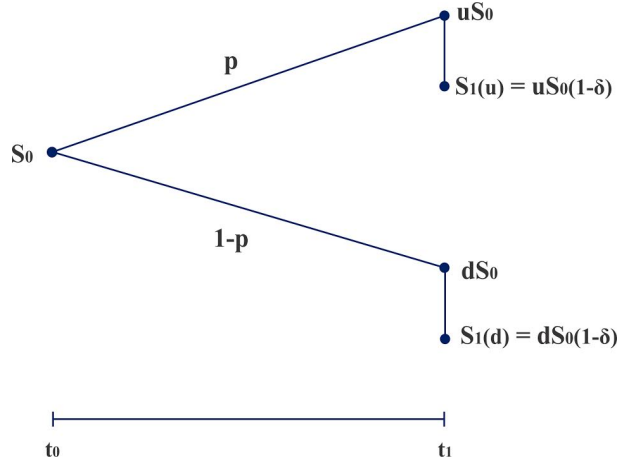


Figure 3.1: One period binomial asset pricing model with a discrete dividend.

At t_1 the replicating portfolio has two possible outcomes

$$\Pi_1(u) = \Delta_0 u S_0 (1 - \delta) + \gamma_F e^{r_F(t_0)dt} - \gamma_R e^{(r_R(t_0) - r_D)dt} + \gamma_C e^{r_C(t_0)dt},$$

$$\Pi_1(d) = \Delta_0 d S_0 (1 - \delta) + \gamma_F e^{r_F(t_0)dt} - \gamma_R e^{(r_R(t_0) - r_D)dt} + \gamma_C e^{r_C(t_0)dt}.$$

The parameters Δ_0 and γ_P are now chosen so that the replicating portfolio is equal to the derivative price at time t_1 ; therefore,

$$V_1(u) = \Delta_0 u S_0 (1 - \delta) + \gamma_F e^{r_F(t_0)dt} - \gamma_R e^{(r_R(t_0) - r_D)dt} + \gamma_C e^{r_C(t_0)dt}, \quad (3.2)$$

$$V_1(d) = \Delta_0 d S_0 (1 - \delta) + \gamma_F e^{r_F(t_0)dt} - \gamma_R e^{(r_R(t_0) - r_D)dt} + \gamma_C e^{r_C(t_0)dt}, \quad (3.3)$$

where $\gamma_F = \gamma_P - \gamma_C$. We subtract equation (3.3) from equation (3.2), and solve for Δ_0 , hence the delta of the stock position is

$$\Delta_0 = \frac{V_1(u) - V_1(d)}{S_0(u - d)(1 - \delta)}. \quad (3.4)$$

To solve for γ_P , the value of Δ_0 is substituted back into equation (3.2), to obtain the following

$$V_1(u) = \frac{(V_1(u) - V_1(d))u}{u - d} + \gamma_F e^{r_F(t_0)dt} - \gamma_R e^{(r_R(t_0) - r_D)dt} + \gamma_C e^{r_C(t_0)dt}.$$

Simplifying and rearranging terms and making the substitution $\gamma_F = \gamma_P - \gamma_C$ yields

$$\gamma_P = e^{-r_F(t_0)dt} \left[\frac{V_1(d)u - V_1(u)d}{u - d} + \gamma_C (e^{r_F(t_0)dt} - e^{r_C(t_0)dt}) + \gamma_R e^{(r_R(t_0) - r_D)dt} \right]. \quad (3.5)$$

The repo account can be written as

$$\gamma_R = S_0 \Delta_0 = \frac{V_1(u) - V_1(d)}{(u - d)(1 - \delta)}; \quad (3.6)$$

therefore, substituting equation (3.6) into equation (3.5) and simplifying yields

$$\begin{aligned} \gamma_P = & e^{-r_F(t_0)dt} \left[\frac{V_1(u) (e^{(r_R(t_0)-r_D)dt} - d(1-\delta)) + V_1(d) (u(1-\delta) - e^{(r_R(t_0)-r_D)dt})}{(u-d)(1-\delta)} \right] \\ & + e^{-r_F(t_0)dt} \left[\gamma_C (e^{r_F(t_0)dt} - e^{r_C(t_0)dt}) \right]. \end{aligned}$$

The parameters Δ_0 and γ_P have been chosen such that the replicating portfolio perfectly replicates the derivative payoff. If a contingent claim can be perfectly replicated, then to avoid an arbitrage, the price of the derivative has to be equal to the cost of the replicating portfolio; therefore, the price of the derivative is

$$\begin{aligned} V_0 = & e^{-r_F(t_0)dt} \left[\frac{V_1(u) (e^{(r_R(t_0)-r_D)dt} - d(1-\delta)) + V_1(d) (u(1-\delta) - e^{(r_R(t_0)-r_D)dt})}{(u-d)(1-\delta)} \right] \\ & + e^{-r_F(t_0)dt} \left[\gamma_C (e^{r_F(t_0)dt} - e^{r_C(t_0)dt}) \right]. \end{aligned} \quad (3.7)$$

If we define q as

$$q = \frac{e^{(r_R(t_0)-r_D)dt} - d(1-\delta)}{(u-d)(1-\delta)} = \frac{e^{r_R(t_0)dt} - d}{u-d}, \quad (3.8)$$

where $0 \leq q \leq 1$, then $d \leq e^{r_R(t_0)dt} \leq u$. Equation (3.7) can then be written as

$$V_0 = e^{-r_F(t_0)dt} \left[qV_1(u) + (1-q)V_1(d) + \gamma_C (e^{r_F(t_0)dt} - e^{r_C(t_0)dt}) \right].$$

If we interpret q as the probability of an upward movement in the stock price, then the expectation of the stock price at time t_1 can be written as

$$\mathbb{E}_0[S_1] = qS_0u + (1-q)S_0d = e^{(r_R(t_0)-r_D)dt}S_0; \quad (3.9)$$

therefore, under this $\mathbb{Q}^{r_R}$ -measure the stock price drifts at the repo rate less the dividend yield. The value of the derivative trade at time t_0 can be written as

$$V_0 = e^{-r_F(t_0)dt} \left[\mathbb{E}_0[V_1] + C_0 (e^{r_F(t_0)dt} - e^{r_C(t_0)dt}) \right], \quad (3.10)$$

where the expectation is taken under the $\mathbb{Q}^{r_R}$ -measure and C_0 is the amount of collateral posted in the trade at time t_0 . We can rearrange the terms in the above equation in order to write the price of the derivative trade as

$$V_0 = e^{-r_C(t_0)dt} \mathbb{E}_0[V_1] - e^{-r_C(t_0)dt} (V_0 - C_0) (e^{r_F(t_0)dt} - e^{r_C(t_0)dt}). \quad (3.11)$$

If we assume that the collateral rate is a proxy for the risk-free rate, then the price of the derivative trade is the sum of the risk-free price and a funding value adjustment. $\square$

The FVA is the adjustment made to the risk-free price in order to account for collateral in the trade. The FVA is defined as the difference between the price of the derivative and the collateral posted in the trade, multiplied by the difference between growing this amount at the funding rate and the collateral rate. This amount is then discounted back to the initial time using the collateral rate. If we were working in a one curve model, i.e. one interest rate, then no funding value adjustment would play a role in the price. However, a one curve model is not an adequate model to represent the market in this day and age.

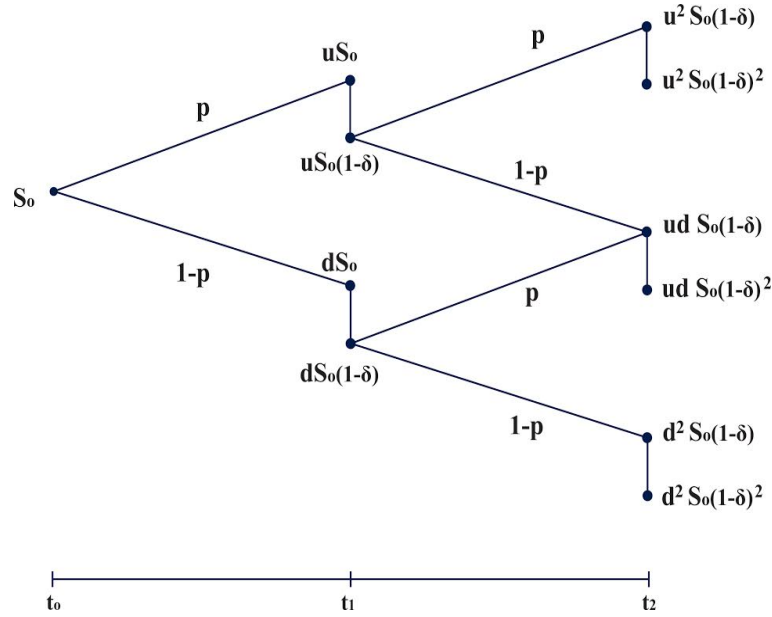


Figure 3.2: Two period binomial asset pricing model with discrete dividends.

3.1.2 n period binomial model with collateral

The one period model is now extended to an n period model. A two period binomial asset pricing model with discrete dividends is illustrated in Figure 3.2. The price of a collateralised derivative trade in an n period model is presented in the following theorem:

Theorem 9:

The price of a general collateralised derivative trade with the absence of counterparty credit risk in an n period binomial model is

$$V_0 = \mathbb{E}_0 \left[e^{-\sum_{j=0}^n r_F(t_j)dt} V_n \right] + \mathbb{E}_0 \left[\sum_{i=0}^{n-1} e^{-\sum_{j=0}^i r_F(t_j)dt} C_i \left(e^{r_F(t_i)dt} - e^{r_C(t_i)dt} \right) \right],$$

where the expectation is under the $\mathbb{Q}^{r_R}$ -measure. If the collateral rate is assumed to be a proxy for the risk-free rate, then the price of the derivative trade in an n period binomial model can be written as the sum of the risk-free price and a funding value adjustment,

$$V_0 = \mathbb{E}_0 \left[e^{-\sum_{j=0}^n r_C(t_j)dt} V_n \right] - \mathbb{E}_0 \left[\sum_{i=0}^{n-1} e^{-\sum_{j=0}^i r_C(t_j)dt} (V_i - C_i) \left(e^{r_F(t_i)dt} - e^{r_C(t_i)dt} \right) \right],$$

where V_i is the value of the derivative at time t_i , C_i is the amount of collateral posted in the trade at time t_i and $r_C(t_j)$ and $r_F(t_j)$ are the short collateral and funding rates at time t_j , respectively.

Proof. The proof uses the Tower Property to extend the one period model to n periods. The one period

binomial model in equation (3.10) holds for the i^{th} period; therefore,

$$V_i = e^{-r_F(t_i)dt} \left[\mathbb{E}_i[V_{i+1}] + C_i \left(e^{r_F(t_i)dt} - e^{r_C(t_i)dt} \right) \right], \quad (3.12)$$

where $i \in \{0, 1, \dots, n-1\}$. The price of the derivative at time t_{n-1} is

$$V_{n-1} = e^{-r_F(t_{n-1})dt} \left[\mathbb{E}_{n-1}[V_n] + C_{n-1} \left(e^{r_F(t_{n-1})dt} - e^{r_C(t_{n-1})dt} \right) \right] \quad (3.13)$$

and the price of the derivative at time t_{n-2} is

$$V_{n-2} = e^{-r_F(t_{n-2})dt} \left[\mathbb{E}_{n-2}[V_{n-1}] + C_{n-2} \left(e^{r_F(t_{n-2})dt} - e^{r_C(t_{n-2})dt} \right) \right]. \quad (3.14)$$

By substituting equation (3.13) into equation (3.14), gives us

$$\begin{aligned} V_{n-2} = & e^{-r_F(t_{n-2})dt} \mathbb{E}_{n-2} \left[e^{-r_F(t_{n-1})dt} \left[\mathbb{E}_{n-1}[V_n] + C_{n-1} \left(e^{r_F(t_{n-1})dt} - e^{r_C(t_{n-1})dt} \right) \right] \right. \\ & \left. + C_{n-2} \left(e^{r_F(t_{n-2})dt} - e^{r_C(t_{n-2})dt} \right) \right]. \end{aligned} \quad (3.15)$$

The Tower Property states that if $0 \leq n \leq m$, where n and m are whole numbers, then

$$\mathbb{E}_n[\mathbb{E}_m[V]] = \mathbb{E}_n[V]$$

(see [Shreve \(2004\)](#) (Theorem 2.3.2, pg. 33)). Applying the Tower Property to the conditional expectations and because the rates are deterministic, equation (3.15) becomes

$$\begin{aligned} V_{n-2} = & e^{-(r_F(t_{n-2})+r_F(t_{n-1}))dt} \mathbb{E}_{n-2}[V_n] + e^{-(r_F(t_{n-2})+r_F(t_{n-1}))dt} \mathbb{E}_{n-2} \left[C_{n-1} \left(e^{r_F(t_{n-1})dt} - e^{r_C(t_{n-1})dt} \right) \right] \\ & + e^{-r_F(t_{n-2})dt} C_{n-2} \left(e^{r_F(t_{n-2})dt} - e^{r_C(t_{n-2})dt} \right). \end{aligned}$$

If one repeats this methodology, then the initial price of the derivative trade is

$$V_0 = \mathbb{E}_0 \left[e^{-\sum_{j=0}^n r_F(t_j)dt} V_n \right] + \mathbb{E}_0 \left[\sum_{i=0}^{n-1} e^{-\sum_{j=0}^i r_F(t_j)dt} C_i \left(e^{r_F(t_i)dt} - e^{r_C(t_i)dt} \right) \right]. \quad (3.16)$$

If equation (3.12) is written as

$$V_i = e^{-r_C(t_i)dt} \mathbb{E}_i[V_{i+1}] - e^{-r_C(t_i)dt} (V_i - C_i) \left(e^{r_F(t_i)dt} - e^{r_C(t_i)dt} \right), \quad (3.17)$$

and the same methodology that was applied to equation (3.12) is applied to equation (3.17), and provided that the collateral rate is used as a proxy for the risk-free rate, then the price of the collateralised derivative trade can be written as the sum of the risk-free price and a funding value adjustment,

$$V_0 = \mathbb{E}_0 \left[e^{-\sum_{j=0}^n r_C(t_j)dt} V_n \right] - \mathbb{E}_0 \left[\sum_{i=0}^{n-1} e^{-\sum_{j=0}^i r_C(t_j)dt} (V_i - C_i) \left(e^{r_F(t_i)dt} - e^{r_C(t_i)dt} \right) \right]. \quad (3.18)$$

□

This is the discrete analog of the model presented in [Piterbarg \(2010\)](#). The continuous case is derived



in Section 3.2. The amount of collateral posted at each node will be contingent on the value of the derivative at each node. The collateral process in this binomial asset pricing model will be illustrated when it is implemented numerically.

Next, we show that the amount of collateral posted in the trade will determine which rate must be used to discount future expected cash flows. Consider two different values of collateral for equation (3.12). When $C_i = V_i$ for $i = 0, 1, 2, \dots, n-1$, which is the case when the seller has to place collateral to the value of the derivative at every node in the tree, then equation (3.18) becomes

$$V_0 = e^{-\sum_{j=0}^n r_C(t_j)dt} \mathbb{E}_0[V_n]. \quad (3.19)$$

On the other hand, if no collateral is placed, i.e. $C_i = 0$ for $i = 0, 1, 2, \dots, n-1$, then equation (3.16) becomes

$$V_0 = e^{-\sum_{j=0}^n r_F(t_j)dt} \mathbb{E}_0[V_n]. \quad (3.20)$$

The result that emerges is that if full collateral is posted at each node, then to value the derivative trade, future expected cash flows are discounted using a collateral curve. However, if no collateral is posted throughout the life of the trade, then to value the derivative trade, future expected cash flows are discounted using a funding curve of the bank. We want to stress that this model is a function of market rates only. The theoretical risk-free rate is not present in the pricing formula. Next, we consider the case when the amount of collateral posted in the trade is some fractional amount of the derivative's value.

3.1.3 Partial collateralisation

When entering into a CSA agreement, the two counterparties have to agree on what collateral will be posted, i.e. cash or bonds, and how much collateral needs to be posted. The above model can be generalised by letting the amount of collateral posted in a derivative trade become a fractional amount of the value of the derivative. Letting the collateral amount be some fractional amount of the derivative value in equation (3.12) yields

$$V_i = e^{-r_F(t_i)dt} \left[\mathbb{E}_i[V_{i+1}] + \theta V_i \left(e^{r_F(t_i)dt} - e^{r_C(t_i)dt} \right) \right]. \quad (3.21)$$

Solving for V_i , we get

$$V_i = \frac{\mathbb{E}_i[V_{i+1}]}{e^{r_F(t_i)dt} + \theta (e^{r_C(t_i)dt} - e^{r_F(t_i)dt})}. \quad (3.22)$$

The rate $r_\theta(t_i)$ is defined as

$$r_\theta(t_i) = r_F(t_i) + \frac{\log(1 + \theta (e^{r_C(t_i)dt} - e^{r_F(t_i)dt}))}{dt}, \quad (3.23)$$

so that

$$e^{-r_\theta(t_i)dt} = \frac{1}{e^{r_F(t_i)dt} + \theta (e^{r_C(t_i)dt} - e^{r_F(t_i)dt})}$$

and we can write

$$V_i = e^{-r_\theta dt} \mathbb{E}[V_{i+1}].$$

Repeating the previous arguments, we get that

$$V_0 = e^{-\sum_{i=0}^{n-1} r_\theta(t_i)dt} \mathbb{E}_0[V_n]. \quad (3.24)$$

The two special cases are when $C_i = V_i$ and $C_i = 0$, for all i ; this corresponds to values of $\theta = 1$ and $\theta = 0$, respectively. The discount factor rate will be $r_\theta(t_i) = r_F(t_i)$ if no collateral is posted, and $r_\theta(t_i) = r_C(t_i)$ if collateral to the value of the derivative is posted at each discrete time point throughout the tree. The following section derives the continuous analog of the model presented in this section.

3.2 Black-Scholes PDE with collateral and dividends

In this section, a PDE of the value of a derivative trade which includes collateral on a dividend paying stock is derived. We begin by stating the definition of a *self-financing* portfolio and Itô's lemma, which will be used in the derivation of the PDE.

The stock process is given by

$$dS_t = (r_R - r_D)S_t dt + \sigma S_t dW_t, \quad (3.25)$$

where S_0 is a constant, σ is the volatility of the stock and W_t is a standard Brownian motion. In the continuous case the stock pays a continuous dividend yield. The derivation of the PDE requires the following definition:

Definition 10:

*A portfolio is said to be **self-financing** if no money is injected or extracted from the portfolio.*

A self-financing replicating portfolio of the derivative is set up. The replicating portfolio consists of a stock position and cash,

$$\Pi_t = \Delta S_t + \gamma.$$

As in the discrete analog, the cash account is invested or borrowed in different bonds which satisfy the following bond processes:

$$dB_F(t, s) = r_F(s)B_F ds, \quad (3.26)$$

$$dB_R(t, s) = (r_R(s) - r_D)B_R ds, \quad (3.27)$$

$$dB_C(t, s) = r_C(s)B_C ds, \quad (3.28)$$

where $t \leq s$ and at time t the value of each of the bonds is worth 1. In order to compute the change in the derivative's price we make use of Itô's lemma (see Itô (1944) and Itô (1951)). Itô's lemma is stated in terms of the stock process in equation (3.25).

Lemma 11:

Let S_t , $t \geq 0$, be an Itô process determined by equation (3.25) and $V(t, S_t) \in C^{1,2}([0, T] \times \mathbb{R}^d)^a$. Then the process $V(t, S_t)$ follows the stochastic differential equation (SDE)

$$\begin{aligned} dV &= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt. \\ &= \left(\frac{\partial V}{\partial t} + (r_R - r_D) S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW_t, \end{aligned}$$

and is also an Itô process.

^a $C^{1,2}$ means that the function $V(t, S_t)$ is once continuously differentiable with respect to $t \in [0, T]$ and twice differentiable with respect to the space variable S_t .

The PDE is presented in the following theorem followed by its derivation:

Theorem 12:

The value of a general collateralised derivative trade with the absence of credit risk on a dividend paying stock satisfies the following PDE

$$\begin{cases} \frac{\partial V}{\partial t} + (r_R - r_D) S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = V r_F - C(r_F - r_C) \\ V_T \text{ is the final payoff of the derivative} \end{cases}.$$

Proof. In our derivation we are more precise than Piterbarg in the change in the cash account (see Piterbarg (2012)). The value of a derivative with collateral is derived by means of a replicating portfolio. The value of the replicating portfolio is

$$\Pi_t = \Delta S_t + \gamma.$$

From the self-financing condition on the replicating portfolio, the change in the replicating portfolio is

$$d\Pi_t = \Delta dS_t + (\gamma_F r_F - \gamma_R(r_R - r_D) + \gamma_C r_C) dt. \quad (3.29)$$

By applying Itô's lemma to the derivative, we have that

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS_t + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt. \quad (3.30)$$

The replicating portfolio will perfectly hedge out the short position of the derivative if and only if

$$d\Pi_t - dV_t = 0 \quad \text{and} \quad \Pi_T = V_T; \quad (3.31)$$

therefore, from equations (3.29) and (3.30), we have that

$$\Delta dS_t + \gamma_F r_F dt - \gamma_R(r_R - r_D) dt + \gamma_C r_C dt - \frac{\partial V}{\partial t} dt - \frac{\partial V}{\partial S} dS_t - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt = 0. \quad (3.32)$$

If we choose

$$\Delta = \frac{\partial V}{\partial S},$$

then equation (3.32) becomes

$$\frac{\partial V}{\partial t} dt + \frac{1}{2} \sigma_t^2 S_t^2 \frac{\partial^2 V}{\partial S^2} dt = \gamma_F r_F dt - \gamma_R (r_R - r_D) dt + \gamma_C r_C dt. \quad (3.33)$$

From equation (3.31), in order to avoid an arbitrage, we require that $\gamma_P = V$ and since $\gamma_R = \Delta S$, $\gamma_F = V - \gamma_C$ and $\gamma_C = C$, we have that

$$\frac{\partial V}{\partial t} + (r_R - r_D) S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = V r_F - C(r_F - r_C). \quad (3.34)$$

□

This PDE is a modification to the famous Black-Scholes-Merton option pricing PDE. If $r = r_C = r_F$, where r is the risk-free rate, hence secured lending and unsecured lending occurs at the same rate which is at the risk-free rate, then equation (3.34) reduces to the original Black-Scholes PDE on a dividend paying stock.

3.3 Showing that the PDE approach and the binomial lattice approach are equivalent

In this section, the Feynman-Kac theorem is applied to Piterbarg's PDE in order to write the PDE in the form of an expectation. This shows that the price of a derivative, in both the discrete and continuous case, can be written as an expectation. The amount of collateral posted will determine which discount factor is used when valuing a general derivative.

Karatzas and Shreve (1991) state and prove the Feynman-Kac theorem. We state the Feynman-Kac theorem in the following theorem without proof:

Theorem 13:

- (i) Suppose that $V(t, S_t) : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is continuous, is of class $C^{1,2}([0, T] \times \mathbb{R}^d)$, and satisfies the Cauchy problem

$$-\frac{\partial V}{\partial t} + kV = (r_R - r_D)S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + g; \text{ in } [0, T] \times \mathbb{R}^d,$$

$$V(T, S_T) = f(S_T); \quad S_t \in \mathbb{R}^d,$$

where the function $f(S_T)$ is the final payoff of the derivative contract and S_t is the stock process.

- (ii) Suppose $V(t, S_t)$ also satisfies the polynomial growth condition

$$\max_{0 \leq t \leq T} |V(t, S_t)| \leq M \left(1 + \|S_t\|^{2\mu}\right); \quad S_t \in \mathbb{R}^d,$$

for some $M > 0, \mu \geq 1$.

Then $V(t, S_t)$ admits the stochastic representation

$$V(t, S_t) = \mathbb{E}_t \left[f(S_T) e^{-\int_t^T k(u, S_u) du} + \int_t^T g(r, S_r) e^{-\int_t^r k(u, S_u) du} dr \middle| \mathcal{F}_t \right],$$

on $[0, T] \times \mathbb{R}^d$, where $\mathcal{F}_t$ is the filtration generated from the Brownian motion in the stock price process and the expectation is under the $\mathbb{Q}^{r_R}$ -measure. In particular, such a solution is unique.

Next, we write the PDE in equation (3.34) in the form of an expectation in the following corollary:

Corollary 14:

The price of a general collateralised derivative trade with the absence of counterparty credit risk is

$$V_0 = \mathbb{E}_0 \left[V_T e^{-\int_0^T r_F(u) du} + \int_0^T e^{-\int_0^s r_F(u) du} C_s (r_F(s) - r_C(s)) ds \right]^a,$$

where the expectation is under the $\mathbb{Q}^{r_R}$ -measure. If the collateral rate is assumed to be a proxy for the risk-free rate, then the price of the derivative trade can be written as the sum of the risk-free price and a funding value adjustment

$$V_0 = \mathbb{E}_0 \left[V_T e^{-\int_0^T r_C(u) du} - \int_0^T e^{-\int_0^s r_C(u) du} (V_s - C_s) (r_F(s) - r_C(s)) ds \right].$$

^aThe filtration is not included in the expectation operator for ease of notation. The subscript in the conditional expectation, i.e. $\mathbb{E}_t$, indicates that the expectation is conditioned on the filtration at time t .

Proof. The Feynman-Kac theorem is now applied to equation (3.34) by rearranging the terms in the PDE as

follows

$$-\frac{\partial V}{\partial t} + Vr_F = (r_R - r_D)S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + C(r_F - r_C), \quad (3.35)$$

The application of the Feynman-Kac theorem yields

$$V_0 = \mathbb{E}_0 \left[V_T e^{-\int_0^T r_F(u) du} + \int_0^T e^{-\int_0^s r_F(u) du} C_s(r_F(s) - r_C(s)) ds \right]. \quad (3.36)$$

We can also write equation (3.35) as

$$-\frac{\partial V}{\partial t} + Vr_C = (r_R - r_D)S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (V - C)(r_F - r_C),$$

and using the Feynman-Kac theorem, the price of the derivative can be written as

$$V_0 = \mathbb{E}_0 \left[V_T e^{-\int_0^T r_C(u) du} - \int_0^T e^{-\int_0^s r_C(u) du} (V_s - C_s)(r_F(s) - r_C(s)) ds \right]. \quad (3.37)$$

□

Equations (3.36) and (3.37) are the continuous analogs of equations (3.16) and (3.18). The following result emerges from equations (3.36) and (3.37). If the value of the collateral posted with the counterparty is equal to the value of the derivative during the life of the trade, i.e. $C_t = V_t$ for all $t \in [0, T]$, then in order to price the derivative, the future expected cash flows are discounted using the collateral curve, hence

$$V_0 = \mathbb{E}_0 \left[V_T e^{-\int_0^T r_C(u) du} \right]. \quad (3.38)$$

If no collateral is posted then the value of the derivative is obtained by discounting future expected cash flows using the funding curve of the bank, hence

$$V_0 = \mathbb{E}_0 \left[V_T e^{-\int_0^T r_F(u) du} \right]. \quad (3.39)$$

This was the same result we obtained in the binomial approach to pricing a derivative with collateral on a dividend paying stock. By examining equation (3.37), we see that the price of the derivative is the risk-free price plus a funding value adjustment. The funding value adjustment can be split into two terms, namely a funding benefit adjustment and a funding cost adjustment

$$\begin{aligned} \text{FVA} &= -\mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r_C(u) du} (V_s - C_s)^- (r_F(s) - r_C(s)) ds \right] \dots \text{FBA} \\ &\quad -\mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r_C(u) du} (V_s - C_s)^+ (r_F(s) - r_C(s)) ds \right] \dots \text{FCA}. \end{aligned} \quad (3.40)$$

Let us assume that counterparty B and C enter into a trade. The funding benefit (cost) adjustment term is the extra benefit (cost) gained (paid) in a trade from investing (borrowing) cash at a higher rate than the risk-free rate. Let us assume counterparty B enters into a trade that has an unfavourable mark-to-market, eg. an out-of-the-money interest rate swap. Counterparty B will receive a premium when entering into the trade. If B has to give back this premium by a CSA, then no funding benefit happens. However, if the trade is entered into

with no CSA, then B will place the premium of the trade on the money market desk and earn a rate better than the collateral rate earned if B had to place these funds as collateral to C . This discounted value of the spread earned on the value of the derivative is known as the FBA.

Let us assume now that the two counterparties enter into another trade. This time counterparty B enters into a trade that has a favourable mark-to-market to the bank, eg. an in-the-money interest rate swap. In this case B will pay a premium over to counterparty C . If full collateral is required by a CSA, then C will pay back the premium as collateral and no funding cost is incurred. In this case counterparty B will be effectively funding the premium at the collateral rate. However, if no collateral is posted by C , then the premium paid by B will have to be funded through the bank's treasury desk at a funding rate. This funding rate will be greater than the rate paid on collateral. This discounted value of the spread paid on the value of the derivative is known as the FCA. These arguments only hold if rehypothecation is possible. [Hull and White \(2012\)](#) argue that this excess cost must be absorbed by the trading desk and not passed onto the client as suggested in this example.

The model presented in Corollary 14 only contains market rates. The rates include the funding curve of the bank and a collateral curve implied from the OIS market. All the pricing is done under the $\mathbb{Q}^{rR}$ -measure. Furthermore, the theoretical risk-free rate does not enter into the pricing. We now rewrite the model presented in Corollary 14, so that the expectation is taken under the risk-neutral measure $\mathbb{Q}$. This idea is presented in [Castagna \(2012b\)](#). Castagna defines a different FVA to what is defined in Corollary 14 and defines a new term named liquidity value adjustment (LVA). We present his ideas in the following corollary:

Corollary 15:

The price of a general collateralised derivative trade with the absence of counterparty credit risk under the risk-neutral measure $\mathbb{Q}$ is

$$\begin{aligned} V_0 = & \mathbb{E}_0 \left[V_T e^{-\int_0^T r(u) du} \right] \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r(u) du} ((V_s - C_s)(r_F(s) - r(s)) - (r_R(s) - r(s))\Delta_s S_s) ds \right] \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r(u) du} C_s (r_C(s) - r(s)) ds \right]. \end{aligned}$$

Proof. The risk-free rate can be introduced into equation (3.35) by adding and subtracting the terms rV , $\frac{\partial V}{\partial S}rs$ and rC , and after rearranging terms we have that

$$-\frac{\partial V}{\partial t} + Vr = (r - r_D)S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \Delta S(r_R - r) - (V - C)(r_F - r) - C(r_C - r). \quad (3.41)$$

Applying the Feynman-Kac theorem to equation (3.41) yields

$$\begin{aligned} V_0 &= \mathbb{E}_0 \left[V_T e^{-\int_0^T r(u) du} \right] \\ &\quad - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r(u) du} ((V_s - C_s)(r_F(s) - r_C(s)) - (r_R - r)\Delta_s S_s) ds \right] \\ &\quad - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r(u) du} C_s(r_C(s) - r(s)) ds \right], \end{aligned} \quad (3.42)$$

where the expectation is under the risk-neutral measure $\mathbb{Q}$. $\square$

The discounting curve in this model is the theoretical risk-free curve unlike the collateral rate curve in Corollary 14. The funding value adjustment is defined as the first adjustment made to the risk-free price in equation (3.42), hence

$$\text{FVA} = -\mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r(u) du} ((V_s - C_s)(r_F(s) - r_C(s)) - (r_R - r)\Delta_s S_s) ds \right].$$

Castagna (2012b) defines FVA as the discounted value of the spread paid by the bank over the risk-free interest rate to finance the net amount of cash needed for the collateral amount and the underlying asset position in the dynamic replication strategy. The liquidity value adjustment is defined as the second adjustment made to the risk-free price, hence

$$\text{LVA} = -\mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r(u) du} C_s(r_C(s) - r(s)) ds \right]. \quad (3.43)$$

LVA is defined as the discounted value of the difference between the risk-free rate and the collateral rate paid (or received) on the collateral and it is the gain (or loss) produced by the liquidation of the *net present value* of the derivative contract due to the collateralisation agreement.

The LVA term is also included in the Piterbarg, Burgard and Kjaer framework; however, they refer to this term as a collateral value adjustment term (CollVA). The collateral value adjustment term will be introduced in Chapter 5 of this dissertation.

3.4 Numerical implementation

We now provide a numerical implementation of Piterbarg's model. The model will be implemented using a PDE approach, a binomial lattice approach and a Quasi Monte Carlo approach. An extensive breakdown on the procedures followed in order to code up these various methods are provided in this section. We show how the various methods converge to a unique price when pricing a European put option. The difference in price between a collateralised derivative and non-collateralised derivative is shown numerically. The section also illustrates how the different cash accounts and the stock positions change throughout a two period binomial model. Finally, a Quasi Monte Carlo simulation is used to price a derivative that has discrete recollateralisation time points.

3.4.1 PDE approach

Finite difference schemes are used to approximate the solutions to differential equations, in our case a parabolic partial differential equation. We use an *implicit-Euler* scheme to approximate the price surface of a general derivative trade (see Brandimarte (2006)) satisfying Piterbarg's PDE. The implementation of the implicit-Euler scheme is based on the work by Richardson (2009).

The asset value discretisation is given by $\{0, ds, 2ds, \dots, Mds = S_{max}\}$ and the time discretisation is given by $\{0, dt, 2dt, \dots, Ndt = T\}$. The value of the derivative at each grid point is denoted by

$$V_j^i = V(ids, jdt),$$

where $i = 1, 2, \dots, M$ and $j = 1, 2, \dots, N$. Figure 3.3 is an illustration of the time and space grid used to approximate the PDE.

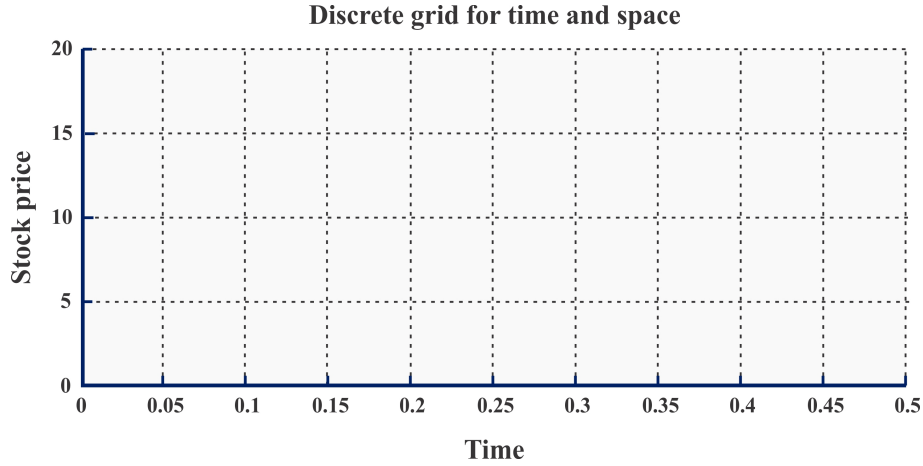


Figure 3.3: Discrete grid of time and space.

We are solving backwards in time from expiry of the option; therefore, a *backward difference* approximation is used for the first time derivative. A *central difference* is used for the first space derivative because it yields a more accurate approximation than a forward or backward difference and a *second-order difference* approximation is used for the second space derivative. Using these approximations, the discretisation of equation (3.34) is

$$\begin{aligned} \frac{V_j^i - V_{j-1}^i}{dt} + \frac{1}{2}\sigma^2(ids)^2 \frac{V_{j-1}^{i+1} - 2V_{j-1}^i + V_{j-1}^{i-1}}{(ds)^2} + (r_R - r_D) \frac{V_{j-1}^{i+1} - V_{j-1}^{i-1}}{2ds}(ids) \\ - r_F V_{j-1}^i + C_{j-1}^i(r_F - r_C) = 0, \end{aligned} \quad (3.44)$$

where C_j^i is the value of collateral posted in the trade at time jdt and stock price ids . In the implementation of the model, throughout the various methods, we assume that all rates are constant. We also assume that the collateral amount can be written as some fractional amount of the value of the derivative,

$$C_j^i = \theta V_j^i,$$

which yields

$$\begin{aligned} \frac{V_j^i - V_{j-1}^i}{dt} + \frac{1}{2}\sigma^2(ids)^2 \frac{V_{j-1}^{i+1} - 2V_{j-1}^i + V_{j-1}^{i-1}}{(ds)^2} + (r_R - r_D) \frac{V_{j-1}^{i+1} - V_{j-1}^{i-1}}{2ds} (ids) \\ - r_F V_{j-1}^i + \theta V_{j-1}^i (r_F - r_C) = 0. \end{aligned} \quad (3.45)$$

Rearranging equation (3.45), we arrive at

$$V_j^i = X_i V_{j-1}^{i-1} + Y_i V_{j-1}^i + Z_i V_{j-1}^{i+1},$$

where

$$\begin{aligned} X_i &= \frac{1}{2} ((r_R - r_D)i - \sigma^2 i^2) dt, \\ Y_i &= 1 + (\sigma^2 i^2 + r_F - \theta(r_F - r_C)) dt, \\ Z_i &= -\frac{1}{2} ((r_R - r_D)i + \sigma^2 i^2) dt. \end{aligned} \quad (3.46)$$

These equations can be written in matrix form

$$\bar{V}_j = M \bar{V}_{j-1}, \quad (3.47)$$

where the matrix M and the vector $\bar{V}_j$ is defined as

$$M = \begin{bmatrix} Y_1 & Z_1 & 0 & \dots & 0 \\ X_2 & Y_2 & Z_2 & \dots & 0 \\ 0 & X_3 & Y_3 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & Z_{M-1} \\ 0 & \dots & 0 & X_M & Y_M \end{bmatrix} \quad \text{and} \quad \bar{V}_j = \begin{bmatrix} V_j^1 \\ V_j^2 \\ V_j^3 \\ \vdots \\ V_j^M \end{bmatrix}.$$

The value of $\bar{V}_j$ is known and $\bar{V}_{j-1}$ is the quantity we require; therefore,

$$M^{-1} \bar{V}_j = \bar{V}_{j-1},$$

for $j = 1, 0, \dots, N$. The two extreme cases for the value of the collateral are considered. The first case is when the collateral amount is zero, i.e. $\theta = 0$. In this case equation (3.46) becomes

$$\begin{aligned} X_i &= \frac{1}{2} ((r_R - r_D)i - \sigma^2 i^2) dt, \\ Y_i &= 1 + (\sigma^2 i^2 + r_F) dt, \\ Z_i &= -\frac{1}{2} ((r_R - r_D)i + \sigma^2 i^2) dt. \end{aligned} \quad (3.48)$$

The second case is when collateral to the value of the derivative is posted continuously, i.e. $\theta = 1$. In this

case equation (3.46) becomes

$$\begin{aligned} X_i &= \frac{1}{2} ((r_R - r_D)i - \sigma^2 i^2) dt, \\ Y_i &= 1 + (\sigma^2 i^2 + r_C) dt, \\ Z_i &= -\frac{1}{2} ((r_R - r_D)i + \sigma^2 i^2) dt. \end{aligned} \quad (3.49)$$

For an implicit scheme the PDE will have a terminal condition and boundary conditions. The terminal and

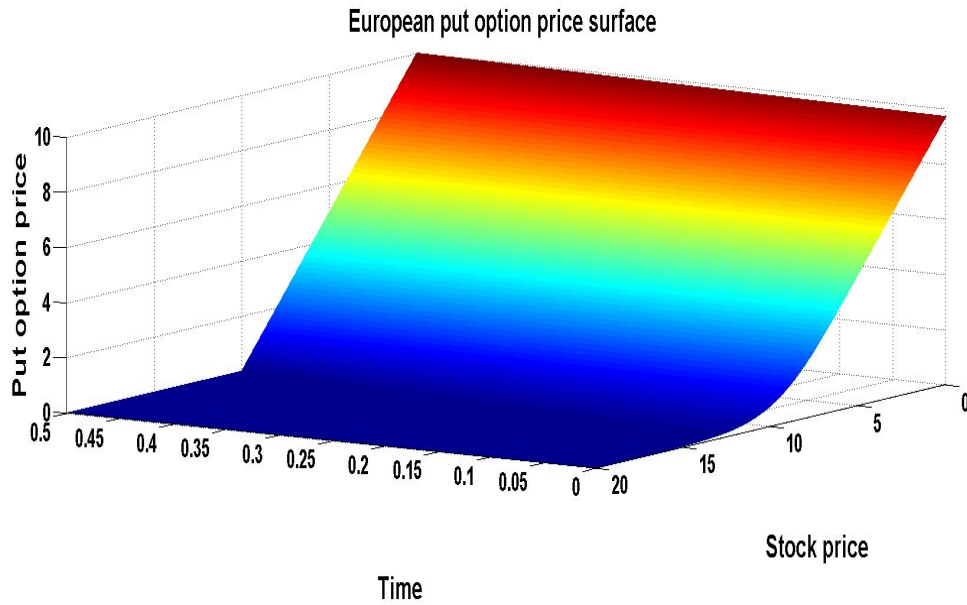


Figure 3.4: Collateralised European put option surface using the PDE approach with parameters: $S_0 \in [0, 20]$, $K = 10$, $dt = 0.0001$ (5 000 time increments), $ds = 0.02$ (1 000 space increments), $T = 0.5$, $t \in [0, T]$, $r_C = 0.04$, $r_R = 0.05$, $r_F = 0.06$, $r_D = 0.01$, $\sigma = 0.3$ and $\theta = 1$.

boundary conditions for a European put option are the following:

$$\begin{cases} V_N^i &= \max(K - ids, 0) \\ V_j^0 &= K e^{-r_\theta(N-j)dt} \\ V_j^{Mds} &= 0, \end{cases} \quad ,$$

where r_θ is defined in equation (3.23). Figure 3.4 illustrates the price surface for a European put option with parameters: $S_0 \in [0, 20]$, $K = 10$, $dt = 0.0001$ (5 000 time increments), $ds = 0.02$ (1 000 space increments), $T = 0.5$, $t \in [0, T]$, $r_C = 0.04$, $r_R = 0.05$, $r_F = 0.06$, $r_D = 0.01$, $\sigma = 0.3$ and $\theta = 1$.

The same European put option was priced; however, this time the trade was not collateralised. The difference in price of the two put options are plotted for varying initial stock prices in Figure 3.5. The difference is positive; therefore, the price of a collateralised European put option is higher than that of the non-collateralised put. The difference represents the funding value adjustment of a non-collateralised put option. Since

$$(V_s - C_s) \geq 0 \quad \text{for all } s, \quad (3.50)$$

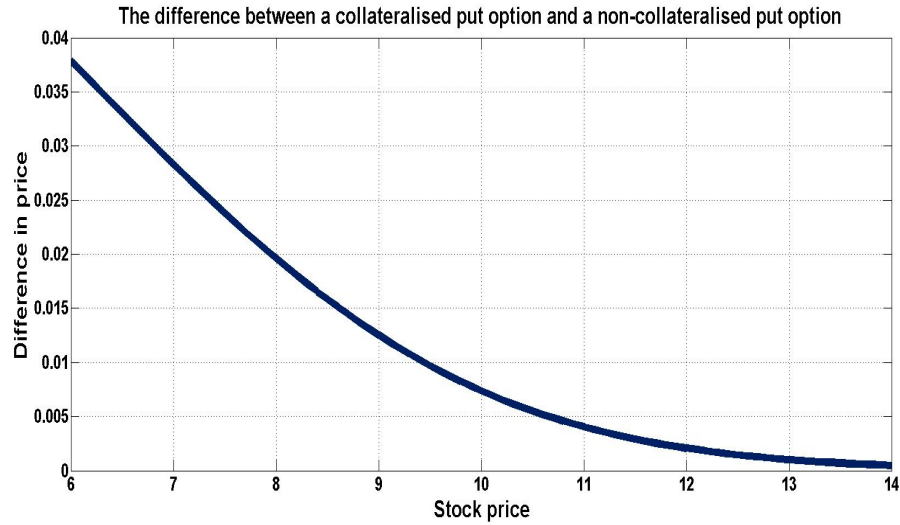


Figure 3.5: Difference in price between a collateralised European put option and a non-collateralised European put option for various initial stock prices with parameters: $S_0 = [8, 12]$, $K = 10$, $dt = 0.0001$ (5 000 time increments), $ds = 0.02$ (1 000 space increments), $T = 0.5$, $r_C = 0.04$, $r_R = 0.05$, $r_F = 0.06$, $r_D = 0.01$ and $\sigma = 0.3$.

for a European put option the funding benefit adjustment will reduce to zero and hence $FVA = FCA$. The shape of the FCA term is similar to the initial price of a European put option. As illustrated in Figure 3.5, the funding cost adjustment will increase when the underlying put option value increases. This happens when the initial stock price decreases.

3.4.2 Binomial lattice approach

The price of the same European put option is now computed using the binomial asset pricing model that was derived in Section 3.1. Let u be the upward factor and d the downward factor of the stock price. When constructing a binomial tree to represent the movements in a stock price, the parameters u and d are chosen to match the volatility and the drift of the stock price process. The parameters u and d are estimated using the *method of moments* in order to match the drift and volatility of the stock price. In order to give more insight into the workings of the method, the values of u and d are derived from first principles. The derivation of the parameters in the binomial asset pricing model are provided in Appendix A.1. The up and down factors of a dividend paying stock in the binomial tree are

$$u = e^{(r_R - \frac{1}{2}\sigma^2)dt + \sigma\sqrt{dt}}(1 - \delta)$$

and

$$d = e^{(r_R - \frac{1}{2}\sigma^2)dt - \sigma\sqrt{dt}}(1 - \delta).$$

The probability of an upward movement is

$$q = \frac{e^{r_R dt} - d}{(u - d)}.$$

The parameters for the stock price process are: $S_0 = 10$, $dt = 0.0001$, $r_R = 0.05$, $r_D = 0.01$ and $\sigma = 0.3$. Figure 3.6 illustrates a binomial asset price lattice.

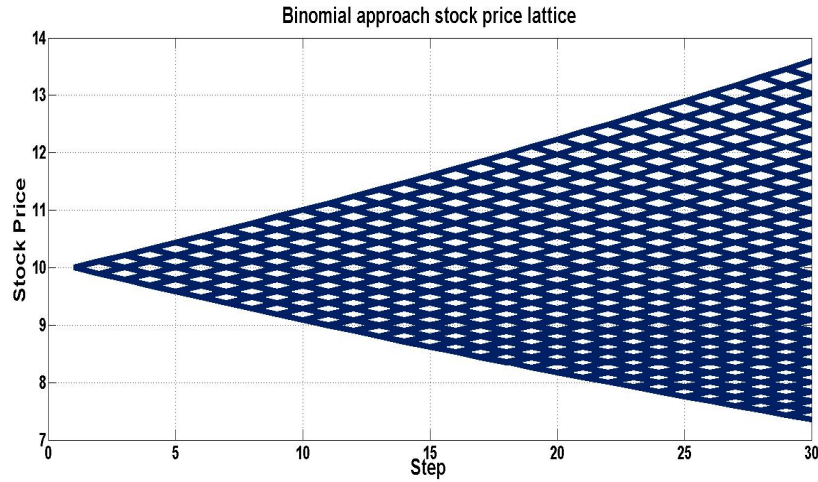


Figure 3.6: Stock price binomial lattice with parameters: $u = 1.039576$, $d = 0.962091$, $\mathbb{P}(\text{up}) = 0.5$ and $n = 30$.

The binomial asset pricing model is used to price the same European put option as in the PDE approach with the parameters: $S_0 \in [0, 20]$, $K = 10$, $dt = 0.0001$ (5 000 time increments), $T = 0.5$, $r_C = 0.04$, $r_R = 0.05$, $r_F = 0.06$, $r_D = 0.01$, $\sigma = 0.3$ and $\theta = 1$.

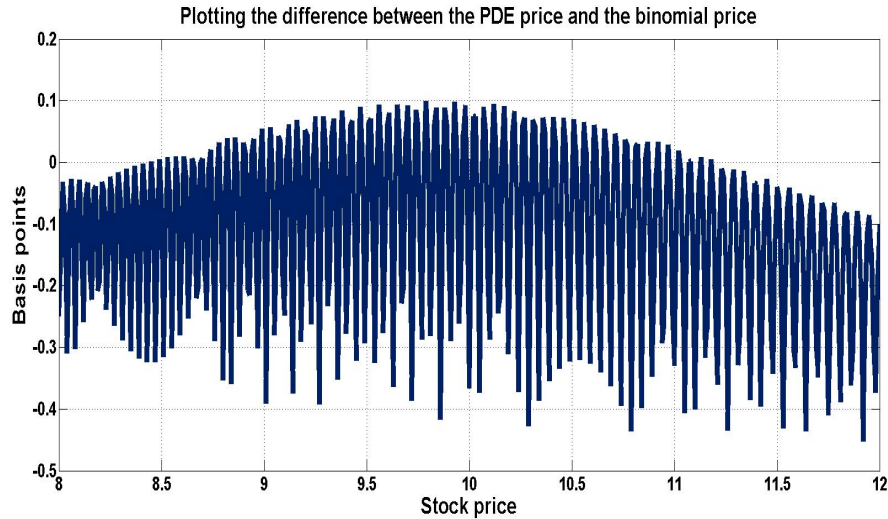


Figure 3.7: Difference between the PDE price and the binomial lattice price for a collateralised European put option with parameters: $S_0 \in [8, 12]$, $K = 10$, $dt = 0.0001$ (5 000 time increments), $ds = 0.02$ (1 000 space increments), $T = 0.5$, $r_C = 0.04$, $r_R = 0.05$, $r_F = 0.06$, $r_D = 0.01$, $\sigma = 0.3$ and $\theta = 1$.

In Figure 3.7, the difference between the PDE and binomial model approaches are graphed for varying initial stock prices. This shows that the two approaches converge to within 0.4 basis points of each other for

the given parameters. The case when no collateral is posted is shown in Appendix B.1. The convergence of the two approaches are also within 0.4 basis points when no collateral is posted.

European put option	PDE price	Binomial lattice price
Full collateral	0.741011	0.740992
No collateral	0.733637	0.733619
FCA	-0.007374	-0.007372

Table 3.1: Summary of collateralised and non-collateralised European put option prices for the PDE approach and the binomial lattice approach with parameters: $S_0 = 10$, $K = 10$, $T = 0.5$, $r_C = 0.04$, $r_R = 0.05$, $r_F = 0.06$, $r_D = 0.01$ and $\sigma = 0.3$.

Table 3.1 illustrates the at-the-money prices of the European put option for the PDE and binomial model approaches when collateral is posted and when no collateral is posted. This shows that when a European put option is traded with collateral, then the price of the derivative will be greater than the price of the same derivative which does not include collateral.

3.4.3 Analysing the replicating portfolio for a European put option in a two period binomial model

In this subsection, we examine what each account in the replicating portfolio does at each step in a two period binomial model. The same put option is priced as before; however, only a two step binomial model is now considered. At each node, we illustrate the replicating portfolio that perfectly replicates the European put option.

The case when the put option is fully collateralised at each step of the binomial lattice is considered. This is the case when the funding account is zero at each node in the tree. The European put option has parameters: $S_0 = 10$, $K = 10$, $dt = 0.25$, $T = 0.5$, $r_C = 0.04$, $r_R = 0.05$, $r_F = 0.06$, $r_D = 0.01$, $\sigma = 0.3$ and $\theta = 1$. The future expected cash flows are discounted using the collateral rate, as done in equation (3.19). Figure 3.8 illustrates the replicating portfolio process and also explicitly gives the value of the cash accounts and the stock position making up the replicating portfolio at each node in the tree. At time t_0 the value of the derivative is known. We set up a replicating portfolio to reproduce the value of the put at each node in the tree. The replicating portfolio at time t_0 is

$$\Pi_0 = \Delta_0 S_0 + \gamma_0.$$

The initial price of the derivative is 0.54721. We note that this price differs from that obtained by using the binomial lattice approach computed in the previous subsection and the PDE approach since in those prices $dt \rightarrow 0$, but in our two period tree lattice dt is still relatively large. The collateral posted in the trade is equal to the value of the derivative, hence the collateral account will be worth 0.54721. Since

$$\gamma_F = \gamma_P - \gamma_C = 0,$$

the cash account at time t_0 is worth

$$\gamma_0 = \gamma_{C,0} - \gamma_{R,0}.$$

The stock position will be financed in a repo agreement. The stock position at time t_0 will be worth -4.42363



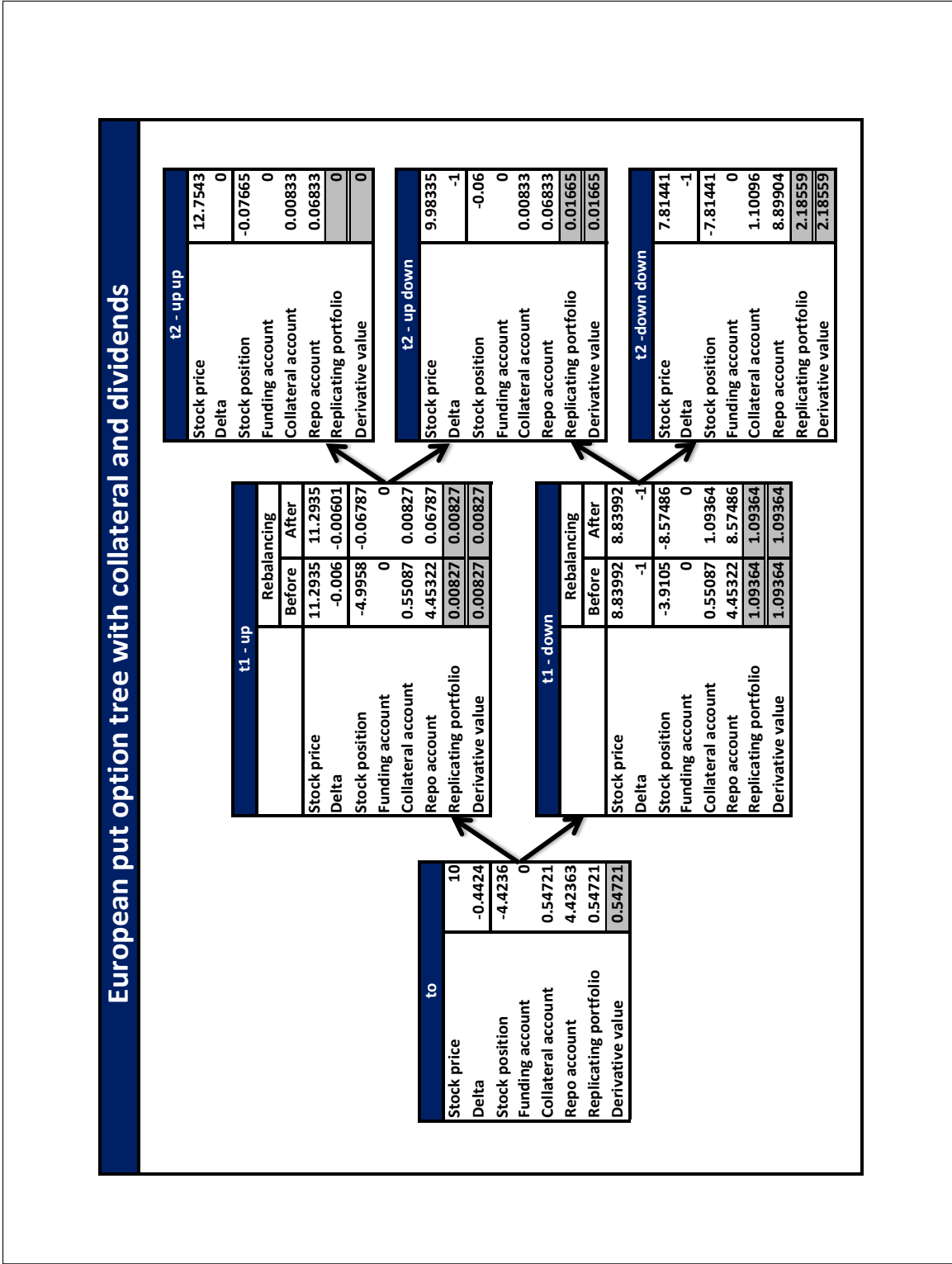


Figure 3.8: Replicating portfolio of a collateralised European put option in a two period binomial model and the accounts making up the replicating portfolio at each node in the tree.

and the repo account will be worth 4.42363. At time t_1 the value of the replicating portfolio is

$$\Pi_1 = \Delta_0 S_1(1 - \delta) + \gamma_{C,0}e^{r_C dt} - \gamma_{R,0}e^{(r_R - r_D)dt}.$$

The stock price at time t_1 denoted by S_1 is the price after it has paid a discrete dividend. Let us consider the case when the stock price moves up. The stock price will move from the initial stock price to 11.2935. The stock position at time t_1 will be the previous delta of the new stock price, hence $\Delta_0 S_1 = -4.995823$. The collateral account will earn the collateral rate over dt , amounting to 0.550872. The delta has been chosen so that the value of the replicating portfolio will be equal to the value of the derivative. The replicating portfolio is worth 0.008270, the exact value of the derivative value. The replicating portfolio is adjusted by liquidating the portfolio into cash, placing new collateral and entering into a new repo agreement. The replicating portfolio can now be written as

$$\Pi_1 = \Delta_1 S_1 + \gamma_{C,1} - \gamma_{R,1}.$$

The put option now moves to maturity. Let us assume that the next stock movement is downward, then the value of the replicating portfolio is

$$\Pi_2 = \Delta_1 S_2 + \gamma_{C,1}e^{r_C dt} - \gamma_{R,1}e^{(r_R - r_D)dt}.$$

The stock price will move from 11.293488 to 9.9833472. The stock position at time t_1 will be the previous delta of the new stock price; hence, $\Delta_1 S_2 = 0.06$. The collateral account will earn the collateral rate over dt , amounting to 0.008325. The value of the replicating portfolio will be equal to the final payoff of the put option, which is 0.016653.

3.4.4 Discrete recollateralisation

We now price a collateralised European put option assuming that the recollateralisation time points are five weeks apart from each other. A Quasi Monte Carlo simulation is used to price this European put option. The prices for the Quasi Monte Carlo simulations are plotted against the number of samples used in the simulations. The subsection concludes by extending Table 3.1 to include Quasi Monte Carlo estimates for a CSA European put option and non-CSA European put option.

Adjusting collateral continuously throughout the life time of the trade, as was assumed in the previous subsections, is unrealistic. In practice, collateral is adjusted at discrete time points. Basel III rules specifies that the time between adjusting collateral should be ten days for OTC trades.

Figure 3.9 illustrates the different ways we considered posting collateral in this chapter. The green line indicates that no collateral is ever posted during the life of the trade. The blue line indicates that collateral to the value of the derivative is posted continuously during the life of the trade. The more realistic case is the black lines. The black lines indicate that collateral is posted at discrete time points. We investigate how to price a derivative when collateral is posted at discrete time points only. The same European put option is priced that was used throughout this section. In this example, collateral is assumed to be adjusted every five weeks.

A different numerical pricing method, known as Monte Carlo simulations, is now used to price the derivative. The idea behind pricing via a Monte Carlo simulation is

- (i) Stock paths are generated until the maturity of the European put option. These stock paths are generated using the stock price process and pseudo random numbers.



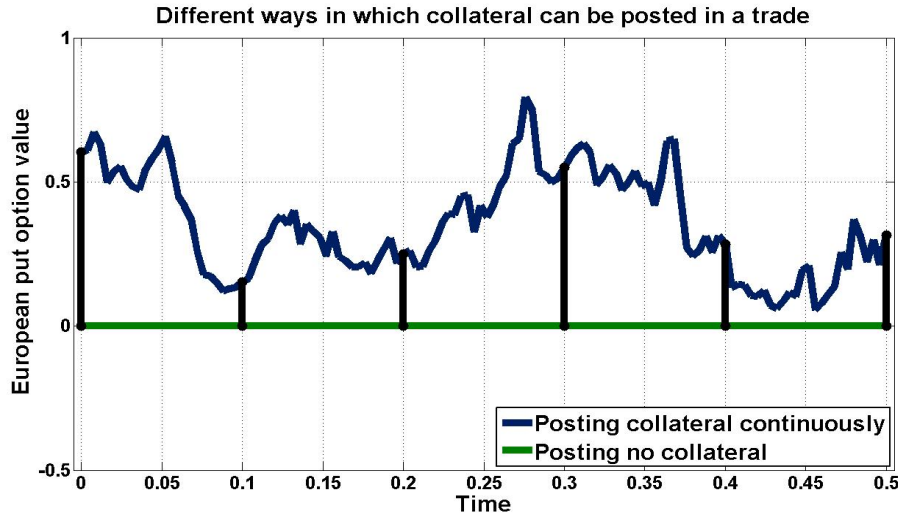


Figure 3.9: Recollateralisation of a European put option at discrete time points with parameters: $S_0 = 10$, $K = 10$, $T = 0.5$, $r_C = 0.04$, $r_R = 0.05$, $r_F = 0.06$, $r_D = 0.01$, $\sigma = 0.3$, and recollateralisation every 0.1 years.

- (ii) The payoff of each stock path in the payoff function is evaluated.
- (iii) The average payoff function is computed and discounted to the initial time.

Quasi Monte Carlo has better convergence than ordinary Monte Carlo because Quasi Monte Carlo uses low discrepancy sequences to generate the stock paths, rather than pseudo random numbers. In our implementation we use Quasi Monte Carlo simulations. A discrete version of equation (3.36) is coded up. The expression for the price of the derivative that is implemented is

$$V_0 = \mathbb{E}_0 \left[V_T e^{-r_F T} + \sum_{i=0}^5 e^{-r_F(i+1)dt} C_{idt} (r_F - r_C) dt \right], \quad (3.51)$$

where $T = ndt$ and $C_{idt} = \theta V_{idt}$. In order to compute equation (3.51), we require a recursive algorithm which is only presented in Chapter 4, Algorithm 23 (see Section 4.5.1). This section is presented here rather than in Chapter 4 because it falls in with the theme of the chapter. We urge the reader to return to this section again after reading Chapter 4.

For this example, the amount of collateral that is posted in a trade is equal to the value of the derivative, i.e. $\theta = 1$. The price of the put is 0.7395398 using 500 000 paths and parameters: $S_0 = 10$, $K = 10$, $T = 0.5$, $r_C = 0.04$, $r_R = 0.05$, $r_F = 0.06$, $r_D = 0.01$, $\sigma = 0.3$, $\theta = 1$ and recollateralisation every 0.1 years. The price of this European put option lies between the price of the put when collateral to the value of the put is posted continuously and the price of the put when no collateral is posted (see Table 3.1). This is a more reasonable price for a cash collateralised trade. This is closer to how trades happen in reality.

In Figure 3.10, the Quasi Monte Carlo prices for the put are plotted against the number of stock paths used. The green line represents the last Monte Carlo estimate. Figure 3.10 shows that the price estimates converge as the number of stock paths increase. The upper bound and lower bound illustrated in Figure 3.10 are the *Koksma-Hlawka bounds* for the Quasi Monte Carlo estimates. These bounds signify that no estimate

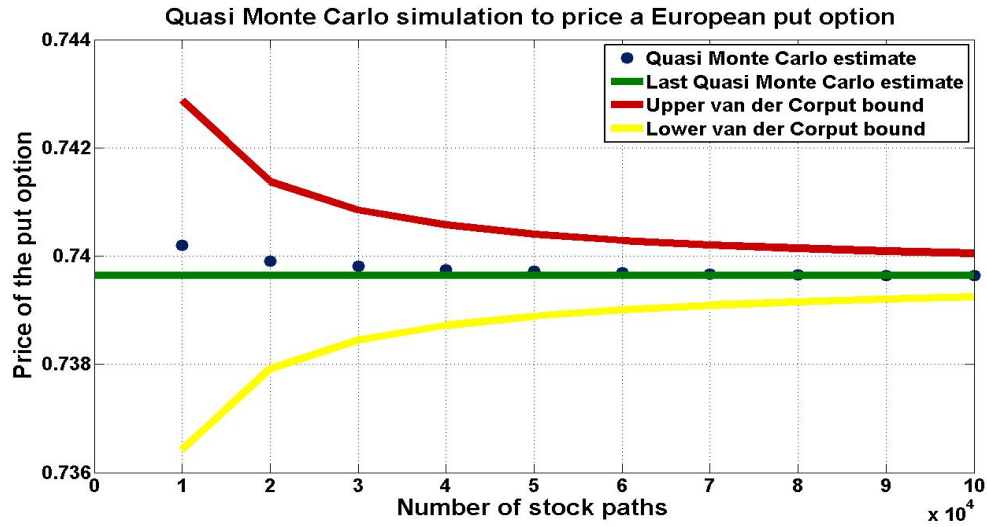


Figure 3.10: Quasi Monte Carlo estimates for a discrete collateralised European put option with parameters: $S_0 = 10$, $K = 10$, $T = 0.5$, $r_C = 0.04$, $r_R = 0.05$, $r_F = 0.06$, $r_D = 0.01$, $\sigma = 0.3$, $\theta = 1$, $b = 2$ for the van der Corput sequence and recollateralisation every 0.1 years.

will fall outside these boundaries. The bounds become tighter as the number of samples increase. We refer the reader to [Glasserman \(2003\)](#) (pg. 282) for an excellent reference on Quasi Monte Carlo simulations and the Koksma-Hlawka bounds.

We now test our implementation against what we have coded before. In Sections 3.4.1 and 3.4.2, the PDE and binomial lattice approaches were implemented to price a European put option that is fully collateralised continuously. In order to test our Quasi Monte Carlo implementation, we let the recollateralisation period tend to zero in this example. The values in Table 3.2 for the Monte Carlo estimates are for 500 000 paths and the time between recollateralisation time points is $dt = 0.0005$. The three approaches show consistent prices when no collateral is posted as well as when full collateral is posted, continuously. In Appendix C.1 we provide tables that include the prices for European calls and puts with varying strike prices.

European put option	PDE price	Binomial lattice price	Quasi Monte Carlo price
Full collateral	0.741011	0.740992	0.741020
No collateral	0.733637	0.733619	0.733676
FCA	-0.007374	-0.007344	-0.007314

Table 3.2: Summary of collateralised and non-collateralised European put option prices for the three different approaches with parameters: $S_0 = 10$, $K = 10$, $T = 0.5$, $r_C = 0.04$, $r_R = 0.05$, $r_F = 0.06$, $r_D = 0.01$ and $\sigma = 0.3$.

3.4.5 Default risk caused from readjusting collateral at discrete times

Next, we investigate the default risk of adjusting collateral at discrete time points in a derivative trade. If the market were to move a lot between two realigning (margining) dates, a significant loss could be faced if a counterparty defaults. The point of this exercise is to illustrate that even when a derivative trade is fully

collateralised and margined daily, there still exists significant counterparty credit risk. A specific realisation

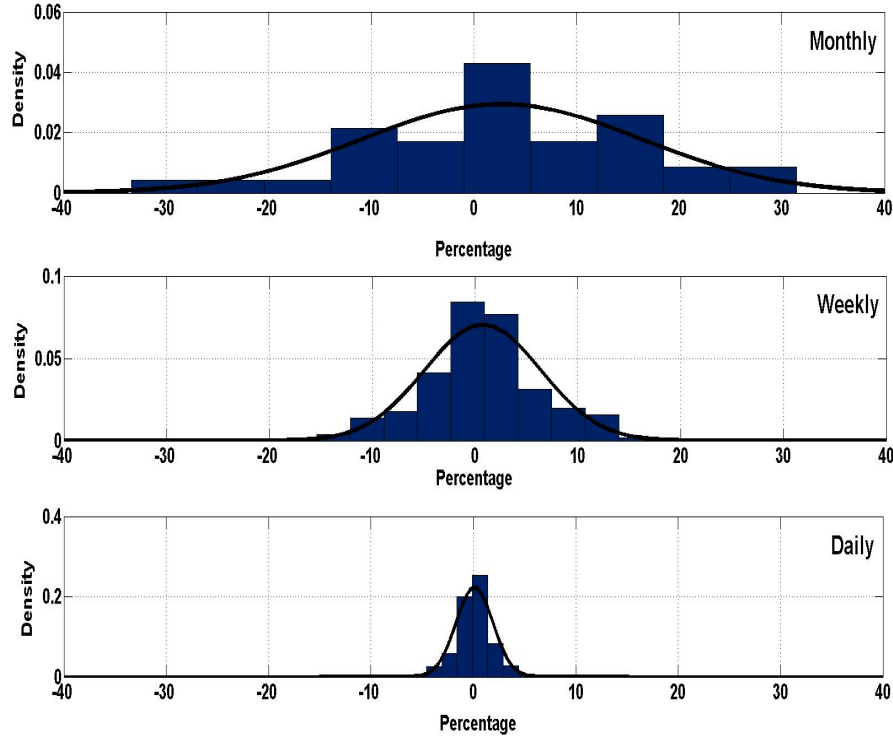


Figure 3.11: The distribution of the default risk of a collateralised European put option for three different recollateralisation periods with parameters: $S_0 = 10$, $K = 10$, $r_C = 0.04$, $r_R = 0.05$, $r_F = 0.06$, $r_D = 0.01$, $T = 3$ and $\sigma = 0.3$.

path for a European put option was simulated. The European put option used in this exercise is a three year put with the following parameters: $S_0 = 10$, $K = 10$, $r_C = 0.04$, $r_R = 0.05$, $r_F = 0.06$, $r_D = 0.01$ and $\sigma = 0.3$. Three different margin periods were considered, namely a one day period, a ten day period and a one month period. In this exercise we assumed that the seller of the option places the mark-to-market of the option as collateral at each discrete time point. The seller will earn the collateral rate on the posted margin. At the initial time t_0 , the seller receives the premium for the put option and at the same time returns the premium as collateral, hence $C_0 = V_0$. Then after dt time the value of the buyers position in the trade is

$$V_1 - V_0 e^{r_C dt},$$

where dt represents the length of the margin period. At time t_1 the collateral posted with the buyer will be readjusted to the value of V_1 . At each of the discrete margin time points, the difference between the exposure of the put option and the collateral amount in the margin account before readjustment is represented as a percentage of the exposure of the option, hence

$$\text{default risk}_i = \frac{V_i - V_{i-1} e^{r_C dt}}{V_i}, \quad (3.52)$$

where $i = 1, \dots, n$. The default risk at each of the i^{th} discrete margin time points are plotted in a histogram represented in Figure 3.11.

Figure 3.11 illustrates the distribution of the default risk for each of the dt values, namely one month, one week and one day. A normal distribution is superimposed over each default risk distribution. From Figure 3.11 one can see that as the margin period becomes shorter, the distribution of the default risk becomes less disperse and the distribution becomes more normal. Even for a margin period of one day, as in an exchange, there still exists significant default risk. If the seller of the European put option defaults on the trade between margin periods, then the buyer could lose up to 5% of the value of the European put option if the dt is one day and up to 30% if dt is one month. The margin period for an OTC derivative trade between a bank and its client may be up to one month, disproving the myth that full collateral completely mitigates credit risk exposure. In this exercise we have not taken into account thresholds; however, if we did consider thresholds in the marginings, then the dispersion of the default risk distribution would be even wider.

Brigo *et al.* (2013) (pg. 259-262) also look at default risk for a collateralised derivative. They investigate the bilateral counterparty credit risk exposure for an interest rate swap when the trade has varying margin periods.

Chapter 4

A model used to price a derivative which includes counterparty credit risk and funding costs

In this chapter, we review a model which is used to price a derivative which includes counterparty credit risk and funding costs. This model, which is presented in the paper [Burgard and Kjaer \(2011\)](#), extends the Black-Scholes model to include counterparty credit risk and funding costs into the value of the derivative. No collateral will be considered in this chapter.

The chapter begins by introducing theory on default probabilities and close-out conventions. Next, a derivation of a tree model, which includes counterparty credit risk, is illustrated. This tree model is our own original work. The derivation of the tree model is followed by a derivation of Burgard and Kjaer's continuous model to price a derivative with credit risk and funding costs. The Feynman-Kac theorem is then applied to the PDE to show that our tree model is a discrete analog of their continuous model. The chapter concludes with a numerical implementation of the model.

4.1 Probability of default

We now give a summary of default probabilities. In this chapter we will be dealing with counterparty credit risk, as a result we need to investigate how to model defaults. We assume throughout this dissertation that the default times of the two counterparties follow an exponential random variable:

- Counterparty B 's default time is denoted by τ_B with intensity λ_B .
- Counterparty C 's default time is denoted by τ_C with intensity λ_C .

The default indicator function of a counterparty is denoted by J_T . The indicator function and the time until default have the following relationship

$$J_T = \begin{cases} 0 & \text{if } \tau > T \\ 1 & \text{if } \tau \leq T \end{cases},$$



where T is some fixed time and τ is the default time of the counterparty. By definition of an exponential random variable, the probability that there is no default before some time T is equal to

$$\mathbb{P}(\tau > T) = \mathbb{P}(J_T = 0) = e^{-\lambda T}. \quad (4.1)$$

The probability of default before some time T is obtained by taking the compliment of $\mathbb{P}(\tau > T)$; therefore,

$$\mathbb{P}(\tau \leq T) = \mathbb{P}(J_T = 1) = 1 - e^{-\lambda T}. \quad (4.2)$$

The probability of default between t_n and t_{n+1} is

$$\begin{aligned} \mathbb{P}(\tau \in [t_n, t_{n+1}]) &= \mathbb{P}(\tau \leq t_{n+1}) - \mathbb{P}(\tau \leq t_n) \\ &= (1 - e^{-\lambda t_{n+1}}) - (1 - e^{-\lambda t_n}) \\ &= e^{-\lambda t_n} - e^{-\lambda t_{n+1}}. \end{aligned}$$

The probability that a counterparty defaults in some finite time $[t_n, t_{n+1}]$ is the difference between the survival functions of the end points of the finite interval. Next, close-out values are discussed.

4.2 Close-outs

Next, we discuss the two close-out conventions that will be used in this dissertation. Since we are working with a derivative trade where both counterparties can default, we need to address some technical issues around close-outs. A close-out amount is the value used to determine the mark-to-market of the derivative, in the case that either counterparty defaults. In this dissertation it is assumed that the close-out mark-to-market can take on only two values, namely a risk-free value or a risky value. The risky value will include adjustments such as CVA, DVA and FCA into the valuation, whereas the risk-free value will not include any counterparty specific adjustments. The risk-free value can be computed using the Black-Scholes model.

The *International Swaps and Derivatives Association* (ISDA) master agreement is the most commonly used master contract for OTC derivative trade transactions internationally. It is part of a framework of documents designed to enable OTC derivatives to be documented fully and flexibly. The master agreement is a document agreed upon between two counterparties that sets out standard terms that apply to all the transactions entered into between those counterparties.

When a counterparty defaults, the 2002 ISDA master agreement specifies that the derivative contract will return to the surviving party as the recovery value of its positive mark-to-market, whereas the full mark-to-market has to be paid to the defaulting party if the mark-to-market value is negative. In the case when the close-out convention is the risky close-out, the 2002 ISDA master agreement specifies that a dealer poll mechanism should be used to establish the mark-to-market of the derivative. One problem with the dealers determining the mark-to-market of the trade is that they may or may not include counterparty specific valuation adjustments such as a funding value adjustment into the price. As a result ISDA prefer to support a risk-free close-out (see Gregory (2012) (pg. 286)).

We let the close-out mark-to-market be denoted by $X_t(S)$. The close-out mark-to-market is

$$X_t = \begin{cases} V_t & \text{if Risk-free close-out} \\ \hat{V}_t & \text{if Risky close-out} \end{cases}.$$



4.3. TREE MODEL WHICH INCLUDES COUNTERPARTY CREDIT RISK AND FUNDING COSTS 49

The boundary conditions of the risky value of the derivative, $\hat{V}_t(S, J_B, J_C)$, upon default of the issuer or counterparty are:

$$\begin{aligned}\hat{V}_t(S, 1, 0) &= X_t^+ + R_B X_t^- \quad \text{if } B \text{ defaults first,} \\ \hat{V}_t(S, 0, 1) &= X_t^- + R_C X_t^+ \quad \text{if } C \text{ defaults first.}\end{aligned}\tag{4.3}$$

At this stage no collateral is introduced into the close-out value. In Chapter 5 these boundary conditions will be amended to include collateral.

4.3 Tree model which includes counterparty credit risk and funding costs

We now present our own tree model for a credit risky derivative trade that includes funding costs. The first valuation model that addresses the issue of counterparty credit risk is that of [Johnson and Stulz \(1987\)](#). They coined the term *vulnerable option*, which is an option that contains counterparty default risk. Their approach is an extension of the corporate bond model by [Merton \(1974\)](#) (see [Ammann \(2001\)](#) (pg. 66)). [Schönbucher \(2002\)](#) has developed a tree model in order to price a credit risky corporate bond. Our tree model differs from Schönbucher in the sense that we do not restrict ourselves to corporate bonds. In our tree model we can price a general credit risky derivative. All notation is consistent with the previous model; however, more notation needs to be introduced. In this model we assume that both counterparties have a liquid bond written on themselves:

- Counterparty B 's bond at time t will be denoted by $P_{B,t}$ with yield r_{P_B} .
- Counterparty C 's bond price at time t will be denoted by $P_{C,t}$ with yield r_{P_C} .

The constant recovery rates of the two counterparties on a derivative trade are R_B and R_C , respectively. The relationship between the default intensity per year, the spread of the corporate bond yield over the risk-free rate and the recovery rate R of the bond is

$$\lambda = \frac{s}{1 - R},\tag{4.4}$$

(see [Hull \(2012\)](#)). Because the bonds are recoveryless, i.e. $R = 0$, we have that

$$\lambda_B = r_{P_B} - r \quad \text{and} \quad \lambda_C = r_{P_C} - r.\tag{4.5}$$

4.3.1 One period tree model which includes counterparty credit risk and funding costs

The one step tree model of a credit risky derivative is now derived. The price of a credit risky derivative for each close-out is stated as a theorem followed by its derivation. The derivation makes use of a hedging portfolio that perfectly hedges the derivative to derive the value of the credit risky derivative. We first discuss the hedging portfolio and its various quantities before presenting the model in a theorem.

The hedging portfolio contains

- Δ number of stocks,



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- α_B number of counterparty B bonds,
- α_C number of counterparty C bonds, and
- γ amount of cash.

The time subscript for certain quantities will be omitted in order to make the equations easier to read. The value of the hedging portfolio at time t is

$$\Pi_t = \Delta S_t + \alpha_B P_B + \alpha_C P_C + \gamma. \quad (4.6)$$

The cash amount can be split into three different accounts, namely a funding account, a counterparty C bond cash account and a repo account:

- The cost of the hedging portfolio is denoted by γ_P , and the cash needed to buy a position in B 's bonds, or the cash obtained from selling B 's bonds is denoted by γ_{P_B} . The funding account is denoted by γ_F . The funding account is the difference between the cost of the hedging portfolio and the price of the position in counterparty B 's bonds, hence $\gamma_F = \gamma_P - \gamma_{P_B}$. The position in B 's bonds is used to hedge out counterparty B 's credit risk. The assumption that the funding account can invest and borrow cash at the same rate which we made in Chapter 3 is now dropped. The funding account will invest positive amounts of cash at the risk-free rate and will lend cash at an unsecured funding rate.
- The cash needed to buy a position in C 's bonds, or the cash obtained from selling C 's bonds is denoted by γ_{P_C} . The position in C 's bonds is used to hedge out counterparty C 's credit risk. The bond is placed in a repurchase agreement. [Burgard and Kjaer \(2011\)](#) assume that the repo rate is the risk-free rate for the bond. We make the same assumption in this dissertation.
- The repo account is denoted by γ_R . This account contains the amount of cash invested or borrowed in order to fund the stock position, ΔS_t , through a repurchase agreement.

Dividends paid by the stock have not been included in this model. The cash amount is invested or borrowed in different bonds where each bond follows the following processes in the tree model

$$dB_F(t, s) = \begin{cases} (e^{r_F(s)ds} - 1) B_F(s) & \text{if } \gamma_F \leq 0 \\ (e^{r(s)ds} - 1) B_F(s) & \text{if } \gamma_F > 0 \end{cases},$$

and

$$\begin{aligned} dB_{P_C}(t, s) &= (e^{r(s)ds} - 1) B_C(s), \\ dB_R(t, s) &= (e^{(r_R(s)ds)} - 1) B_R(s), \end{aligned}$$

where $t \leq s$ and at time t all the bonds are worth 1. In this discrete model $t = t_0$. The change in time between the nodes in the tree model is denoted by dt .

To incorporate defaults into the tree model, [Schönbucher \(2002\)](#) suggests adding an additional branching point to the tree. We make use of his idea; therefore, we first decide at each node of the tree whether either counterparty has defaulted. If a default has occurred, then the derivative value will move to the default branch. If no default has occurred, then the tree model will continue to the next period. Figure 4.1 illustrates a one period tree model with a default branch for each counterparty.



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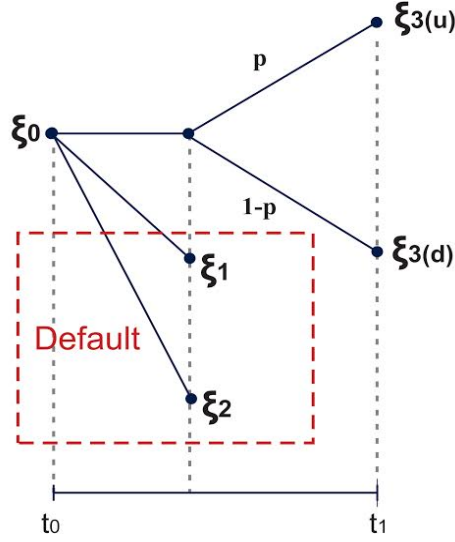


Figure 4.1: One period tree asset pricing model which includes counterparty credit risk and funding costs.

The price of the credit risky derivative that contains funding costs for both close-out conventions is provided by the following theorem:

Theorem 16:

The price of the credit risky derivative in a one period model when the close-out value is the risky price is

$$\begin{aligned} \hat{V}_0 &= e^{-rdt} \mathbb{E}_0 [\hat{V}_1] && \dots \text{Risk-free price} \\ &- e^{-rdt} \hat{V}_0^+ (1 - R_C) \lambda_C dt && \dots \text{CVA} \\ &- e^{-rdt} \hat{V}_0^- (1 - R_B) \lambda_B dt && \dots \text{DVA} \\ &- e^{-rdt} \hat{V}_0^+ s_F dt. && \dots \text{FCA}, \end{aligned}$$

where $\hat{V}_1$ is the payoff of the derivative at time t_1 . The price of the credit risky derivative in a one period model when the close-out value is the risk-free price is

$$\begin{aligned} \hat{V}_0 &= e^{-rdt} \mathbb{E}_0 [\hat{V}_1] && \dots \text{Risk-free price} \\ &- e^{-(r+\lambda_C+\lambda_B)dt} V_0^+ (1 - R_C) \lambda_C dt && \dots \text{CVA} \\ &- e^{-(r+\lambda_C+\lambda_B)dt} V_0^- (1 - R_B) \lambda_B dt && \dots \text{DVA} \\ &- e^{-(r+\lambda_C+\lambda_B)dt} V_0^+ s_F dt && \dots \text{FCA}. \end{aligned}$$

The price of the risky derivative is the sum of the risk-free value and three adjustments. These adjustments are a CVA, a DVA and a FCA. The expectation is taken under the $\mathbb{Q}^{TR}$ -measure.

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Proof. Our approach to derive the price of the credit risky derivative is to set up a hedging portfolio that perfectly hedges the derivative. If the hedging portfolio perfectly hedges the derivative, then in order to avoid an arbitrage the price of the hedging portfolio must be equal to the price of the short derivative. We take the view that counterparty B is long the derivative and is the counterparty setting up the hedging portfolio.

Consider the following four state model

$$\{\xi_i\}_{i=0}^3 = \{(t_0, 0, 0), (t_0, 1, 0), (t_0, 0, 1), (t_1, 0, 0)\},$$

where $\xi_i = (t, J_B, J_C)$. This is the default indicator which is defined in Section 4.1 and the subscript of the default indicator represents the counterparty. In state ξ_0 no counterparty defaults at time t_0 . In states ξ_1 and ξ_2 counterparty B and C default, respectively. State ξ_3 represents that in dt time no counterparty defaults and that the stock will be worth S_1 . The stock price can take on two possible values in state ξ_3 , namely an up value and a down value.

Next, we show what each quantity in the hedging portfolio value will be, given the state of the world. The stock price takes on the following values

$$S(\xi_0) = S(\xi_1) = S(\xi_2) = S_0$$

and

$$S(\xi_3) = \begin{cases} S_0 u & \text{if stock price moves up} \\ S_0 d & \text{if stock price moves down} \end{cases}.$$

Counterparty B 's zero recovery bond takes on the following values

$$P_B(\xi_0) = P_B(\xi_2) = P_B, \quad P_B(\xi_1) = 0 \quad \text{and} \quad P_B(\xi_3) = P_B e^{(r+\lambda_B)dt}.$$

Counterparty C 's zero recovery bond takes on the following values

$$P_C(\xi_0) = P_C(\xi_1) = P_C, \quad P_C(\xi_2) = 0 \quad \text{and} \quad P_C(\xi_3) = P_C e^{(r+\lambda_C)dt}.$$

The cash account takes on the following values

$$\gamma(\xi_0) = \gamma(\xi_1) = \gamma(\xi_2) = \gamma_F - \gamma_R - \gamma_{P_C} \quad (4.7)$$

and

$$\gamma(\xi_3) = (\gamma_P - \gamma_{P_B})^+ e^{r dt} + (\gamma_P - \gamma_{P_B})^- e^{r_F dt} - \gamma_{P_C} e^{r dt} - \gamma_R e^{r_R dt}. \quad (4.8)$$

The value of the derivative in the default states will be worth

$$\hat{V}(\xi_1) = X^+ + R_B X^- \quad \text{and} \quad \hat{V}(\xi_2) = R_C X^+ + X^-. \quad (4.9)$$

We now chose the parameters Δ , α_B , α_C and γ_P so that the hedging portfolio perfectly hedges the derivative in all states of the world; therefore,

$$\Pi(\xi_i) + \hat{V}(\xi_i) = 0 \quad \text{for } i = 0, 1, 2, 3.$$

We begin with state ξ_3 . In order to be hedged for the next time step, the hedging portfolio must satisfy the



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following equation

$$\Pi(\xi_3) + \hat{V}(\xi_3) = 0;$$

therefore, from equation (4.6) we have that

$$\Delta S(\xi_3) + \alpha_C P_C(\xi_3) + \alpha_B P_B(\xi_3) + \gamma(\xi_3) + \hat{V}(\xi_3) = 0. \quad (4.10)$$

Equation (4.10) contains the random variable $S(\xi_3)$ which can take on two values and results in the following two equations

$$\Delta S_0 u + \alpha_C P_C e^{(r+\lambda_C)dt} + \alpha_B P_B e^{(r+\lambda_B)dt} + \gamma(\xi_3) + \varphi(u) = 0, \quad (4.11)$$

$$\Delta S_0 d + \alpha_C P_C e^{(r+\lambda_C)dt} + \alpha_B P_B e^{(r+\lambda_B)dt} + \gamma(\xi_3) + \varphi(d) = 0, \quad (4.12)$$

where $\varphi(u)$ and $\varphi(d)$ indicates the derivative in the up state and the down state at time t_1 , respectively. Subtracting equation (4.12) from (4.11) and solving for Δ yields

$$\Delta = -\frac{\varphi(u) - \varphi(d)}{S_0(u - d)}. \quad (4.13)$$

Substituting equation (4.13) back into equation (4.11), gives us

$$-\frac{u\varphi(u) - u\varphi(d)}{u - d} + \alpha_C P_C e^{(r+\lambda_C)dt} + \alpha_B P_B e^{(r+\lambda_B)dt} + \gamma(\xi_3) + \varphi(u) = 0.$$

After rearranging and simplifying terms, we have that

$$-\frac{d\varphi(u) - u\varphi(d)}{u - d} + \alpha_C P_C e^{(r+\lambda_C)dt} + \alpha_B P_B e^{(r+\lambda_B)dt} + \gamma(\xi_3) = 0. \quad (4.14)$$

Substituting equation (4.8) into equation (4.14) and using the fact that $\gamma_R = \Delta_0 S_0$, we get

$$\begin{aligned} & \frac{(e^{r_R dt} - d)\varphi(u) + (u - e^{r_R dt})\varphi(d)}{u - d} + \alpha_C P_C (e^{(r+\lambda_C)dt} - e^{r dt}) \\ & + \alpha_B P_B e^{(r+\lambda_B)dt} + (\gamma_P - \gamma_{P_B})^+ e^{r dt} + (\gamma_P - \gamma_{P_B})^- e^{r_F dt} = 0. \end{aligned} \quad (4.15)$$

If we define q as

$$q = \frac{e^{r_R dt} - d}{u - d},$$

where $0 \leq q \leq 1$, then $d \leq e^{r_R dt} \leq u$. Equation (4.15) can then be written as

$$\begin{aligned} & q\varphi(u) + (1 - q)\varphi(d) + \alpha_C P_C (e^{(r+\lambda_C)dt} - e^{r dt}) + \alpha_B P_B e^{(r+\lambda_B)dt} \\ & + (\gamma_P - \gamma_{P_B})^+ e^{r dt} + (\gamma_P - \gamma_{P_B})^- e^{r_F dt} = 0, \end{aligned} \quad (4.16)$$

If we interpret q as the probability of an upward movement in the stock price, then

$$\mathbb{E}_0 [S_1] = S_0(qu + (1 - q)d) = S_0 e^{r_R dt},$$

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where the expectation is taken under the $\mathbb{Q}^{rR}$ -measure. Equation (4.16) then becomes

$$\begin{aligned} \mathbb{E}_0 [\varphi_1] + \alpha_C P_C \left(e^{(r+\lambda_C)dt} - e^{rdt} \right) + \alpha_B P_B e^{(r+\lambda_B)dt} \\ + (\gamma_P - \gamma_{P_B})^+ e^{rdt} + (\gamma_P - \gamma_{P_B})^- e^{r_F dt} = 0. \end{aligned} \quad (4.17)$$

The last two terms in equation (4.17) can be written as

$$(\gamma_P - \gamma_{P_B})^+ e^{rdt} + (\gamma_P - \gamma_{P_B})^- e^{r_F dt} = (\gamma_P - \gamma_{P_B}) e^{rdt} + (\gamma_P - \gamma_{P_B})^- (e^{r_F dt} - e^{rdt});$$

therefore, equation (4.17) becomes

$$\begin{aligned} \mathbb{E}_0 [\varphi_1] + \alpha_C P_C \left(e^{(r+\lambda_C)dt} - e^{rdt} \right) + \alpha_B P_B e^{(r+\lambda_B)dt} \\ + (\gamma_P - \gamma_{P_B}) e^{rdt} + (\gamma_P - \gamma_{P_B})^- (e^{r_F dt} - e^{rdt}) = 0. \end{aligned} \quad (4.18)$$

The above equation will be used to complete the derivation of the model.

Next, we consider the states of the world when a counterparty defaults. In order to be hedged for default, in states ξ_1 and ξ_2 the hedging portfolio must satisfy the following two equations

$$\Pi(\xi_1) + \hat{V}(\xi_1) = 0$$

and

$$\Pi(\xi_2) + \hat{V}(\xi_2) = 0;$$

therefore, equations (4.6) and (4.9) yield

$$\Delta_0 S_0 + \alpha_C P_C + \gamma + X^+ + R_B X^- = 0 \quad (4.19)$$

and

$$\Delta_0 S_0 + \alpha_B P_B + \gamma + R_C X^+ + X^- = 0. \quad (4.20)$$

The cash expression defined in equation (4.7) is substituted into equations (4.19) and (4.20), to get

$$-\alpha_B P_B + \gamma_P + X^+ + R_B X^- = 0, \quad (4.21)$$

$$-\alpha_C P_C + \gamma_P + R_C X^+ + X^- = 0. \quad (4.22)$$

If equations (4.13), (4.21) and (4.22) hold, then the hedging portfolio perfectly hedges the derivative in the states ξ_1 , ξ_2 and ξ_3 . In order to avoid an arbitrage the cost of the hedging portfolio must be equal to the short derivative; therefore,

$$\gamma_P = -\hat{V}(\xi_0). \quad (4.23)$$

To solve for the parameters α_B , α_C and γ_P of the hedging portfolio, we consider four different close-out values:

1. $X = \hat{V}(\xi_0) = -\gamma_P$
 - (a) $\gamma_P \leq 0$ and $\hat{V}(\xi_0) \geq 0$,
 - (b) $\gamma_P \geq 0$ and $\hat{V}(\xi_0) \leq 0$,

$$2. X = V(\xi_0)$$

$$(a) V(\xi_0) \geq 0,$$

$$(b) V(\xi_0) \leq 0.$$

Case 1.a: The risky close-out value and the value of the risky derivative is always positive

If $X = \hat{V}(\xi_0) = -\gamma_P \geq 0$, then equations (4.21) and (4.22) reduce to

$$\begin{aligned} -\alpha_B P_B + \gamma_P + \hat{V}(\xi_0) &= 0 \\ \alpha_B &= 0 \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} -\alpha_C P_C + \gamma_P + R_C \hat{V}(\xi_0) &= 0 \\ \alpha_C &= \frac{\gamma_P(1 - R_C)}{P_C}. \end{aligned}$$

These are the bond positions required to hedge out the derivative for this close-out value. Substituting the bond positions α_C and α_B into equation (4.18) yields

$$\mathbb{E}_0[\varphi_1] + \gamma_P(1 - R_C) \left(e^{(r+\lambda_C)dt} - e^{rdt} \right) + \gamma_P e^{rdt} + \gamma_P^- (e^{r_F dt} - e^{rdt}) = 0.$$

Since $\gamma_P = -\hat{V}(\xi_0) \leq 0$, and after rearranging the terms, the above equation becomes

$$\hat{V}(\xi_0) = e^{-rdt} \left[\mathbb{E}_0[\varphi_1] - \hat{V}(\xi_0)(1 - R_C) \left(e^{(r+\lambda_C)dt} - e^{rdt} \right) - \hat{V}(\xi_0) (e^{r_F dt} - e^{rdt}) \right]. \tag{4.25}$$

By only considering up to the first order of the Taylor series expansion of the exponential function, we can make the following simplifications

$$\left(e^{(r+\lambda_C)dt} - e^{rdt} \right) \approx \lambda_C dt$$

and

$$e^{r_F dt} - e^{rdt} \approx s_F dt.$$

Therefore, equation (4.25) can be written as

$$\hat{V}(\xi_0) \approx e^{-rdt} \mathbb{E}_0[\varphi_1] - e^{-rdt} \hat{V}(\xi_0)(1 - R_C) \lambda_C dt - e^{-rdt} \hat{V}(\xi_0) s_F dt. \tag{4.26}$$

The value of the risky derivative can be written as the sum of the risk-free price and a total value adjustment (TVA) term denoted by U_1 at time t_1 ; therefore, the final payoff of the derivative trade is equal to

$$\mathbb{E}_0[\varphi_1] = \mathbb{E}_0[\hat{V}_1] = \mathbb{E}_0[V_1 + U_1]. \tag{4.27}$$

From equation (4.27), and since $V_0 = e^{-rdt} \mathbb{E}_0[V_1]$, we can write equation (4.26) as

$$\hat{V}(\xi_0) = V_0 + e^{-rdt} \mathbb{E}_0[U_1] - e^{-rdt} \hat{V}(\xi_0)(1 - R_C) \lambda_C dt - e^{-rdt} \hat{V}(\xi_0) s_F dt.$$

For the one period model the total value adjustment at time t_1 is zero because at maturity of a derivative trade there is no credit risk or funding costs. There is no credit risk at maturity because the counterparties are assumed to only default at time t_0 , just after the two counterparties entered into the trade. At maturity there is no funding costs because counterparty B does not need to lend money at a higher rate for a non-zero amount of time. However, when this model is extended to n periods, then $U_1 \neq 0$. The price of the risky derivative can be written as $\hat{V}(\xi_0) = V(\xi_0) + U_0$, where U_0 is defined as

$$U_0 = e^{-r dt} \mathbb{E}_0 [U_1] - \overbrace{e^{-r dt} \hat{V}(\xi_0)(1 - R_C)\lambda_C dt}^{\text{CVA}} - \overbrace{e^{-r dt} \hat{V}(\xi_0)s_F dt}^{\text{FCA}}. \quad (4.28)$$

The first term represents the expected discounted cash flows arising from defaults and funding costs. The second term in the TVA represents a credit value adjustment and the third term represents a funding cost adjustment. Since $\hat{V}(\xi_0) \geq 0$ and $U_1 = 0$, the price of the credit risky derivative can be written as

$$\hat{V}^+(\xi_0) = V^+(\xi_0) - e^{-r dt} \hat{V}^+(\xi_0)(1 - R_C)\lambda_C dt - e^{-r dt} \hat{V}^+(\xi_0)s_F dt. \quad (4.29)$$

The price of the credit risky derivative is a function of itself; however, when the sign of the derivative is known, one can rewrite the above equation to explicitly make the price of the derivative the subject of the equation. The aim of this section is to price a credit risky derivative unconditioned on the sign of the derivative. We will show that the price of the derivative leads to a recursive formula.

Case 1.b: The risky close-out value and the value of the risky derivative is always negative

If $X = \hat{V}(\xi_0) = -\gamma_P \leq 0$, then equations (4.21) and (4.22) reduce to

$$\begin{aligned} -\alpha_B P_B + \gamma_P + R_B \hat{V}(\xi_0) &= 0, \\ \alpha_B &= \frac{\gamma_P(1 - R_B)}{P_B} \end{aligned} \quad (4.30)$$

and

$$\begin{aligned} -\alpha_C P_C + \gamma_P + \hat{V}(\xi_0) &= 0, \\ \alpha_C &= 0. \end{aligned}$$

These are the bond positions required to hedge out the derivative for this close-out value. Substituting the bond positions α_C and α_B into equation (4.18) yields

$$\begin{aligned} \mathbb{E}_0 [\varphi_1] + \gamma_P(1 - R_B)e^{(r+\lambda_B)dt} + (\gamma_P - \gamma_P(1 - R_B))e^{r dt} \\ + (\gamma_P - \gamma_P(1 - R_B))e^{r_F dt} - e^{r dt} &= 0. \end{aligned}$$

Since $\gamma_P = -\hat{V}(\xi_0) \geq 0$, and after simplifying the terms, the above equation becomes

$$\mathbb{E}_0 [\varphi_1] + \gamma_P(1 - R_B) \left(e^{(r+\lambda_B)dt} - e^{r dt} \right) + \gamma_P e^{r dt} = 0.$$

By rearranging the terms, the price of the risky derivative can be written as

$$\hat{V}(\xi_0) = e^{-r dt} \left[\mathbb{E}_0 [\varphi_1] - \hat{V}(\xi_0)(1 - R_B) \left(e^{(r+\lambda_B)dt} - e^{r dt} \right) \right]. \quad (4.31)$$

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We make the simplification

$$\left(e^{(r+\lambda_B)dt} - e^{rdt}\right) \approx \lambda_B dt;$$

therefore, equation (4.31) becomes

$$\hat{V}(\xi_0) \approx e^{-rdt} \mathbb{E}_0[\varphi_1] - e^{-rdt} \hat{V}(\xi_0)(1 - R_B)\lambda_B dt. \quad (4.32)$$

The value of the risky derivative can be written as $\hat{V}(\xi_0) = V(\xi_0) + U_0$, where

$$V(\xi_0) = e^{-rdt} \mathbb{E}_0[V_1]$$

and U_0 is defined as

$$U_0 = e^{-rdt} \mathbb{E}_0[U_1] - \overbrace{e^{-rdt} \hat{V}(\xi_0)(1 - R_B)\lambda_B dt}^{\text{DVA}}. \quad (4.33)$$

The second term in the TVA is a debit value adjustment. Since $\hat{V}(\xi_0) \leq 0$ and $U_1 = 0$, the price of the credit risky derivative can be written as

$$\hat{V}^-(\xi_0) = V^-(\xi_0) - e^{-rdt} \hat{V}^-(\xi_0)(1 - R_B)\lambda_B dt. \quad (4.34)$$

At this stage, we have the value of a risky derivative given a risky close-out and the sign of the derivative. We require the price of the derivative unconditioned on the sign of the derivative. The value of the risky derivative can be written as the sum of the negative part and the positive part of the price hence

$$\hat{V}_0 = \hat{V}_0^+ + \hat{V}_0^-. \quad (4.35)$$

The values $\hat{V}_0^+$ and $\hat{V}_0^-$ are given by equations (4.29) and (4.34), respectively. Considering only up to the first order of the Taylor series expansion, the price of the risky derivative is given by

$$\hat{V}_0 = V_0 + U_0,$$

where the total value adjustment is defined as

$$\begin{aligned} U_0 &= -e^{-rdt} \hat{V}_0^+(1 - R_C)\lambda_C dt \\ &\quad - e^{-rdt} \hat{V}_0^-(1 - R_B)\lambda_B dt \\ &\quad - e^{-rdt} \hat{V}_0^+ s_F dt. \end{aligned} \quad (4.36)$$

The price of the risky derivative is the sum of the risk-free value and three adjustments. These adjustments are a CVA, a DVA and a FCA.

Case 2.a: The risk-free close-out value and the value of the risk-free derivative is always positive

If $X = V(\xi_0) \geq 0$, then equations (4.21) and (4.22) reduce to

$$-\alpha_B P_B + \gamma_P + V(\xi_0) = 0,$$

$$\alpha_B = \frac{\gamma_P + V(\xi_0)}{P_B} \quad (4.37)$$

and

$$\begin{aligned} -\alpha_C P_C + \gamma_P + R_C V(\xi_0) &= 0, \\ \alpha_C &= \frac{\gamma_P + R_C V(\xi_0)}{P_C}. \end{aligned}$$

From equations (4.24), (4.30) and (4.37) we can observe that the position in B 's bonds is equal to the difference between the price of the risky derivative and the close-out value. Substituting the bond positions α_C and α_B into equation (4.18) yields

$$\begin{aligned} \mathbb{E}_0 [\varphi_1] + (\gamma_P + R_C V(\xi_0)) (e^{(r+\lambda_C)dt} - e^{rdt}) + (\gamma_P + V(\xi_0)) e^{(r+\lambda_B)dt} \\ + (\gamma_P - \gamma_P - V(\xi_0)) e^{rdt} + (\gamma_P - \gamma_P - V(\xi_0))^- (e^{r_F dt} - e^{rdt}) = 0. \end{aligned}$$

Since $\gamma_P = -\hat{V}(\xi_0)$ and $V(\xi_0) \geq 0$, the above equation becomes

$$\begin{aligned} \mathbb{E}_0 [\varphi_1] + (-\hat{V}(\xi_0) + R_C V(\xi_0)) (e^{(r+\lambda_C)dt} - e^{rdt}) + (-\hat{V}(\xi_0) + V(\xi_0)) e^{(r+\lambda_B)dt} \\ + (-\hat{V}(\xi_0) + \hat{V}(\xi_0) - V(\xi_0)) e^{rdt} + (-\hat{V}(\xi_0) + \hat{V}(\xi_0) - V(\xi_0))^- (e^{r_F dt} - e^{rdt}) = 0. \end{aligned}$$

If we only consider up to the first order in the Taylor series expansion, then the above equation can be written as

$$\mathbb{E}_0 [\varphi_1] + (-\hat{V}(\xi_0) + R_C V(\xi_0)) \lambda_C dt + (-\hat{V}(\xi_0) + V(\xi_0)) \lambda_B dt - e^{rdt} \hat{V}(\xi_0) - V(\xi_0) s_F dt = 0. \quad (4.38)$$

Using equation (4.27) and rearranging the terms in equation (4.38) yields

$$\hat{V}(\xi_0) (e^{rdt} + (\lambda_B + \lambda_C) dt) = e^{rdt} V_0 + \mathbb{E}_0 [U_1] + R_C V(\xi_0) \lambda_C dt + V(\xi_0) \lambda_B dt - V(\xi_0) s_F dt.$$

Considering only up to the first order of the Taylor series expansion in the above equation, and since $\hat{V}(\xi_0) \geq 0$ and $U_1 = 0$, the price of the credit risky derivative can be written as

$$\hat{V}^+(\xi_0) = e^{-(r+\lambda_B+\lambda_C)dt} [e^{rdt} V^+(\xi_0) + R_C V^+(\xi_0) \lambda_C dt + V^+(\xi_0) \lambda_B dt - V^+(\xi_0) s_F dt]. \quad (4.39)$$

Case 2.b: The risk-free close-out value and the value of the risk-free derivative is negative

If $X = V(\xi_0) \leq 0$, then equations (4.21) and (4.22) reduce to

$$\begin{aligned} -\alpha_B P_B + \gamma_P + R_B V(\xi_0) &= 0, \\ \alpha_B &= \frac{\gamma_P + R_B V(\xi_0)}{P_B} \end{aligned} \quad (4.40)$$

and

$$-\alpha_C P_C + \gamma_P + V(\xi_0) = 0,$$

$$\alpha_C = \frac{\gamma_P + V(\xi_0)}{P_C}.$$

Substituting the values of α_C and α_B into equation (4.18) yields

$$\begin{aligned} \mathbb{E}_0 [\varphi_1] + (\gamma_P + V(\xi_0)) \left(e^{(r+\lambda_C)dt} - e^{rdt} \right) + (\gamma_P + R_B V(\xi_0)) e^{(r+\lambda_B)dt} \\ + (\gamma_P - \gamma_P - R_B V(\xi_0)) e^{rdt} + (\gamma_P - \gamma_P - R_B V(\xi_0))^- (e^{r_F dt} - e^{rdt}) = 0. \end{aligned}$$

Since $\gamma_P = -\hat{V}(\xi_0)$ and $V(\xi_0) \leq 0$, the above equation becomes

$$\begin{aligned} \mathbb{E}_0 [\varphi_1] + \left(-\hat{V}(\xi_0) + V(\xi_0) \right) \left(e^{(r+\lambda_C)dt} - e^{rdt} \right) + \left(-\hat{V}(\xi_0) + R_B V(\xi_0) \right) e^{(r+\lambda_B)dt} \\ + \left(-\hat{V}(\xi_0) + \hat{V}(\xi_0) - R_B V(\xi_0) \right) e^{rdt} = 0. \end{aligned}$$

If we only consider up to the first order in the Taylor series expansion, we obtain

$$\mathbb{E}_0 [\varphi_1] + \left(-\hat{V}(\xi_0) + V(\xi_0) \right) \lambda_C dt + \left(-\hat{V}(\xi_0) + R_B V(\xi_0) \right) \lambda_B dt - \hat{V}(\xi_0) e^{rdt} = 0. \quad (4.41)$$

Using equation (4.27) and rearranging the terms in equation (4.41) yields

$$\hat{V}(\xi_0) (e^{rdt} + (\lambda_B + \lambda_C) dt) = e^{rdt} V(\xi_0) + \mathbb{E}_0 [U_1] + V(\xi_0) \lambda_C dt + R_B V(\xi_0) \lambda_B dt.$$

Considering only up to the first order of the Taylor series expansion in the above equation and since $V(\xi_0) \leq 0$ and $U_1 = 0$, the price of the risky derivative can be written as

$$\hat{V}^-(\xi_0) = e^{-(r+\lambda_B+\lambda_C)dt} \left[e^{rdt} V^-(\xi_0) + V^-(\xi_0) \lambda_C dt + R_B V^-(\xi_0) \lambda_B dt \right]. \quad (4.42)$$

The value of the risky derivative can be written as the sum of the negative part and the positive part of the price hence

$$\hat{V}_0 = \hat{V}_0^+ + \hat{V}_0^-.$$

The values $\hat{V}_0^+$ and $\hat{V}_0^-$ are given by equations (4.39) and (4.42), respectively. Considering only up to the first order of the Taylor series expansion and using the identity $V = V^+ + V^-$, the risky derivative can be written as

$$\hat{V}_0 = V_0 + U_0,$$

where the total value adjustment is defined as

$$\begin{aligned} U_0 = & - e^{-(r+\lambda_C+\lambda_B)dt} V^+(\xi_0) (1 - R_C) \lambda_C dt \\ & - e^{-(r+\lambda_C+\lambda_B)dt} V^-(\xi_0) (1 - R_B) \lambda_B dt \\ & - e^{-(r+\lambda_C+\lambda_B)dt} V^+(\xi_0) s_F dt. \end{aligned} \quad (4.43)$$

The price of the risky derivative in a one period model is the sum of the risk-free value and three adjustments, namely a CVA, a DVA and a FCA, as in the risky close-out case. $\square$

The CVA, DVA and FCA terms in Theorem 16 are adjustments made to the price in order to account for counterparty credit risk and funding costs. The difference between the two close-out prices is the derivative's

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price used in the TVA calculation and the discount factor. In the risky close-out convention, we discount future expected cash flows occurring from counterparty credit risk and funding costs with the risk-free rate, whereas in the risk-free close-out we discount future cash flows with the risk-free rate plus the sum of the two default intensities.

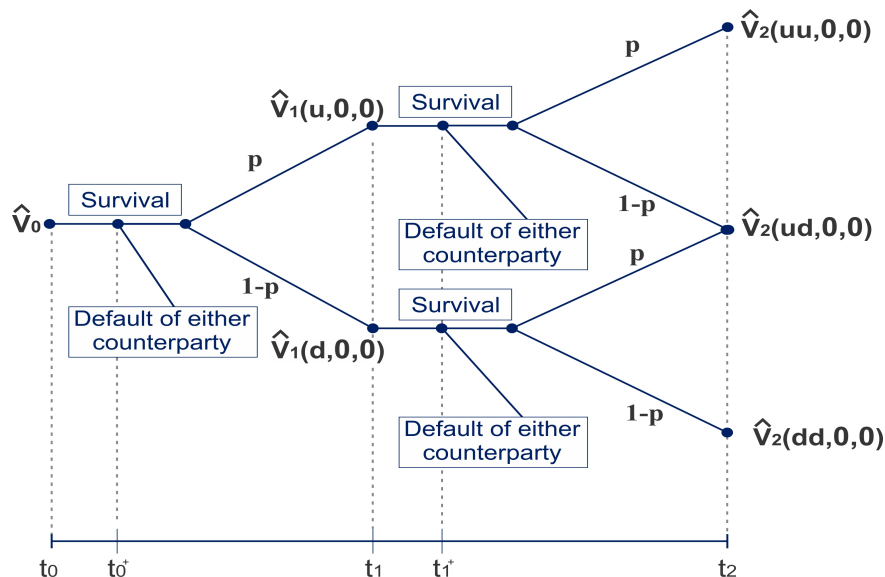


Figure 4.2: Two period tree asset pricing model which includes counterparty credit risk and funding costs.

4.3.2 n period tree model which includes counterparty credit risk and funding costs

The value of the derivative for the one period tree model is now extended for n periods using the Tower Property. The n period model is derived for both close-out conventions. Figure 4.2 illustrates the two period binomial lattice with defaults. The n period tree model for a risky derivative for both close-out conventions are presented in the following theorem:

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Theorem 17:

The price of the credit risky derivative when the close-out value is the risky price in an n period model is

$$\begin{aligned}\hat{V}_0 = V_0 &= \mathbb{E}_0 \left[\sum_{i=0}^{n-1} e^{-\sum_{j=0}^i r(t_j)dt} (1 - R_C) \lambda_C(t_i) \hat{V}_i^+ dt \right] \dots CVA \\ &- \mathbb{E}_0 \left[\sum_{i=0}^{n-1} e^{-\sum_{j=0}^i r(t_j)dt} (1 - R_B) \lambda_B(t_i) \hat{V}_i^- dt \right] \dots DVA \\ &- \mathbb{E}_0 \left[\sum_{i=0}^{n-1} e^{-\sum_{j=0}^i r(t_j)dt} s_F(t_i) \hat{V}_i^+ dt \right] \dots FCA. \quad (4.44)\end{aligned}$$

where $\hat{V}_n$ is the payoff of the derivative at time t_n . The price of the credit risky derivative when the close-out value is the risk-free price in an n period model is

$$\begin{aligned}\hat{V}_0 = V_0 &= \mathbb{E}_0 \left[\sum_{i=0}^{n-1} e^{-\sum_{j=0}^i (r(t_j) + \lambda_B(t_j) + \lambda_C(t_j))dt} (1 - R_C) \lambda_C(t_i) V_i^+ dt \right] \dots CVA \\ &- \mathbb{E}_0 \left[\sum_{i=0}^{n-1} e^{-\sum_{j=0}^i (r(t_j) + \lambda_B(t_j) + \lambda_C(t_j))dt} (1 - R_B) \lambda_B(t_i) V_i^- dt \right] \dots DVA \\ &- \mathbb{E}_0 \left[\sum_{i=0}^{n-1} e^{-\sum_{j=0}^i (r(t_j) + \lambda_B(t_j) + \lambda_C(t_j))dt} s_F(t_i) V_i^+ dt \right] \dots FCA. \quad (4.45)\end{aligned}$$

The price of the risky derivative is the sum of the risk-free value and three adjustments. These adjustments are a CVA, a DVA and a FCA. The expectation is taken under the $\mathbb{Q}^{rR}$ -measure.

Proof. In order to prove the above result, we apply the one period model to each node in an n period binomial lattice. In this n period model, we only consider up to the first order of the Taylor series expansion as we did in the one period model. The risky close-out price is derived first.

The value of a credit risky derivative at some time t_i can be written as the sum of the risk-free price and a total value adjustment hence

$$\hat{V}_i = V_i + U_i,$$

where $i = 0, 1, \dots, n-1$ and the total value adjustment at t_i is given by

$$U_i = e^{-r(t_i)dt} \left[\mathbb{E}_i [U_{i+1}] - \hat{V}_i^+ (1 - R_C) \lambda_C(t_i) dt - \hat{V}_i^- (1 - R_B) \lambda_B(t_i) dt - \hat{V}_i^+ s_F(t_i) dt \right]. \quad (4.46)$$

We start at the end of the binomial lattice and work backwards. At time t_{n-1} the value of the TVA is

$$U_{n-1} = e^{-r(t_{n-1})dt} \left[-\hat{V}_{n-1}^+ (1 - R_C) \lambda_C(t_{n-1}) dt - \hat{V}_{n-1}^- (1 - R_B) \lambda_B(t_{n-1}) dt - \hat{V}_{n-1}^+ s_F(t_{n-1}) dt \right], \quad (4.47)$$

where, by definition, $U_n = 0$. At time t_{n-2} the value of the TVA is

$$U_{n-2} = e^{-r(t_{n-2})dt} \mathbb{E}_{n-2} [U_{n-1}] + e^{-r(t_{n-2})dt} \left[-\hat{V}_{n-2}^+(1 - R_C)\lambda_C(t_{n-2})dt - \hat{V}_{n-2}^-(1 - R_B)\lambda_B(t_{n-1})dt - \hat{V}_{n-2}^+s_F(t_{n-1})dt \right]. \quad (4.48)$$

Substituting equation (4.47) into equation (4.48) yields

$$U_{n-2} = -e^{-\sum_{j=n-2}^{n-1} r(t_j)dt} \mathbb{E}_{n-2} [\hat{V}_{n-1}^+(1 - R_C)\lambda_C(t_{n-1})dt + \hat{V}_{n-1}^-(1 - R_B)\lambda_B(t_{n-1})dt + \hat{V}_{n-1}^+s_F(t_{n-1})dt] - e^{-r(t_{n-2})dt} [V_{n-2}^+(1 - R_C)\lambda_C(t_{n-2})dt + V_{n-2}^-(1 - R_B)\lambda_B(t_{n-2})dt + V_{n-2}^+s_F(t_{n-2})dt]. \quad (4.49)$$

If one repeats this methodology and applies the Tower Property, then the initial price of the derivative trade can be written as

$$\begin{aligned} \hat{V}_0 = V_0 &= \mathbb{E}_0 \left[\sum_{i=0}^{n-1} e^{-\sum_{j=0}^i r(t_j)dt} (1 - R_C)\lambda_C(t_i)\hat{V}_i^+dt \right] \\ &= \mathbb{E}_0 \left[\sum_{i=0}^{n-1} e^{-\sum_{j=0}^i r(t_j)dt} (1 - R_B)\lambda_B(t_i)\hat{V}_i^-dt \right] \\ &= \mathbb{E}_0 \left[\sum_{i=0}^{n-1} e^{-\sum_{j=0}^i r(t_j)dt} s_F(t_i)\hat{V}_i^+dt \right]. \end{aligned}$$

The risk-free close-out case follows similarly. □

Equations (4.44) and (4.45) are the discrete analog of the results that are derived in [Burgard and Kjaer \(2011\)](#). In equation (4.44), the value of the risky derivative at time t_0 is dependent on the TVA, which is in turn a function of the risky price. This will be implemented by means of a recursive algorithm.

4.4 Deriving the Black-Scholes PDE which includes counterparty credit risk and funding costs

In this subsection, we derive a PDE for the price of a credit risky derivative on a non-dividend paying stock. First we state Itô's lemma for jumps without proof, then we present the PDE in a theorem followed by its proof. Finally we show that the PDE has a non-recursive probabilistic solution under certain conditions.

The stock and bond price processes are defined as

$$dP_B = r_{P_B}P_Bdt - P_BdJ_B, \quad (4.50)$$

$$dP_C = r_{P_C}P_Cdt - P_CdJ_C \quad (4.51)$$

and the stock process is

$$dS_t = r_R S_t dt + \sigma S_t dW_t, \quad (4.52)$$

where W_t is a standard Brownian motion. The stock process is independent to the defaults of either counterparty. Itô's lemma for jumps is stated in terms of the stock price process in equation (4.52).

Lemma 18:

Let S_t , $t \geq 0$, be an Itô process determined by equation (4.52) and $V(t, S_t) \in C^{1,2}([0, T] \times \mathbb{R}^d)$. Then the process $V(t, S_t)$ follows the stochastic differential equation

$$\begin{aligned} d\hat{V} &= \frac{\partial \hat{V}}{\partial t} dt + \frac{\partial \hat{V}}{\partial S} dS + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \hat{V}}{\partial S^2} dt + \Delta \hat{V}_B dJ_B + \Delta \hat{V}_C dJ_C \\ &= \left(\frac{\partial \hat{V}}{\partial t} + r_R S \frac{\partial \hat{V}}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \hat{V}}{\partial S^2} \right) dt + \sigma S \frac{\partial \hat{V}}{\partial S} dW_t + \Delta \hat{V}_B dJ_B + \Delta \hat{V}_C dJ_C, \end{aligned}$$

where

$$\Delta \hat{V}_B = \hat{V}_t(S_t, 1, 0) - \hat{V}_t(S_t, 0, 0),$$

$$\Delta \hat{V}_C = \hat{V}_t(S_t, 0, 1) - \hat{V}_t(S_t, 0, 0),$$

and provided that the appropriate Lebesgue and Itô integrals exist.

The notation $\Delta \hat{V}_B$ is abuse of our notation. This does not mean the delta of the stock position multiplied by the value of the risky derivative, but it is defined as the difference between the derivative value when counterparty B defaults and the derivative value when both counterparties have not defaulted. This holds for both counterparties. Next, we state the price of a credit risky derivative as a PDE in the following theorem followed by its derivation:

Theorem 19:

The value of a credit risky derivative, $\hat{V}$, on a non-dividend paying stock satisfies the following PDE

$$\begin{cases} \frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C) \hat{V} = s_F X^+ - \lambda_B (X^+ + R_B X^-) - \lambda_C (X^- + R_C X^+) \\ \hat{V}_T = V_T \text{ is the final payoff of the derivative} \end{cases},$$

where the parabolic differential operator is given by

$$\mathcal{A}_t = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} - r_R S \frac{\partial}{\partial S}.$$

Proof. We derive the value of a credit risky derivative by means of a hedging portfolio. If the hedging portfolio hedges the derivative perfectly, then to avoid an arbitrage the cost of the hedging portfolio has to be equal to the short derivative. Firstly, a self-financing portfolio is set up. The self-financing hedging portfolio is given by

$$\Pi_t = \Delta S_t + \alpha_B P_B + \alpha_C P_C + \gamma.$$

As in the discrete analog, the cash account is invested or borrowed in different bonds and each bond satisfies

the following processes in the continuous model

$$dB_F(t, s) = \begin{cases} r_F(s)B_F(s)ds & \text{if } \gamma_F \leq 0 \\ r(s)B_F(s)ds & \text{if } \gamma_F > 0 \end{cases},$$

and

$$dB_{P_C}(t, s) = r(s)B_C(s)ds, \quad (4.53)$$

$$dB_R(t, s) = r_R(s)B_R(s)ds, \quad (4.54)$$

where $t \leq s$ and at time t all the bonds are worth 1. The self-financing portfolio condition on the hedging portfolio implies, that the change in the hedging portfolio is given by

$$d\Pi_t = \Delta dS + \alpha_B dP_B + \alpha_C dP_C + (r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R) dt. \quad (4.55)$$

In order to compute the change in the derivative's price with jumps dJ_B and dJ_C , we require Itô's lemma for jump processes. By applying Lemma 18 to the risky derivative value with jumps, we have that

$$d\hat{V} = \frac{\partial \hat{V}}{\partial t} dt + \frac{\partial \hat{V}}{\partial S} dS + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \hat{V}}{\partial S^2} dt + \Delta \hat{V}_B dJ_B + \Delta \hat{V}_C dJ_C. \quad (4.56)$$

Next, we choose the parameters Δ , α_C and α_B so that the hedging portfolio perfectly hedges the derivative. The hedging portfolio of the derivative will hedge out the derivative if and only if

$$d\Pi_t = -d\hat{V}_t \quad \text{and} \quad \Pi_T = -\hat{V}_T.$$

Substituting equations (4.55) and (4.56) into the above equation yields

$$\begin{aligned} & \Delta dS_t + \alpha_B dP_B + \alpha_C dP_C + (r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R) dt \\ &= - \left(\frac{\partial \hat{V}}{\partial t} dt + \frac{\partial \hat{V}}{\partial S} dS_t + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \hat{V}}{\partial S^2} dt + \Delta \hat{V}_B dJ_B + \Delta \hat{V}_C dJ_C \right). \end{aligned} \quad (4.57)$$

Next, the bond price processes in equations (4.50) and (4.51) are substituted into equation (4.57) hence

$$\begin{aligned} & \Delta dS_t + \alpha_B (r_{P_B} P_B dt - P_B dJ_B) + \alpha_C (r_{P_C} P_C dt - P_C dJ_C) + (r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R) dt \\ &= - \left(\frac{\partial \hat{V}}{\partial t} dt + \frac{\partial \hat{V}}{\partial S} dS_t + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \hat{V}}{\partial S^2} dt + \Delta \hat{V}_B dJ_B + \Delta \hat{V}_C dJ_C \right). \end{aligned} \quad (4.58)$$

If we choose

$$\Delta = -\frac{\partial \hat{V}}{\partial S}, \quad (4.59)$$

$$\alpha_B = \frac{\Delta \hat{V}_B}{P_B} = -\frac{\hat{V}_t - (X_t^+ + R_B X_t^-)}{P_B}, \quad (4.60)$$

$$\alpha_C = \frac{\Delta \hat{V}_C}{P_C} = -\frac{\hat{V}_t - (X_t^- + R_C X_t^+)}{P_C}, \quad (4.61)$$

then equation (4.58) becomes

$$\alpha_B r_{P_B} P_B + \alpha_C r_{P_C} P_C + r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R + \left(\frac{\partial \hat{V}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \hat{V}}{\partial S^2} \right) = 0. \quad (4.62)$$

We now simplify the following terms in equation (4.62),

$$\alpha_B r_{P_B} P_B + \alpha_C r_{P_C} P_C + r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R.$$

Since $\gamma_{P_B} = \alpha_B P_B$ and $\gamma_{P_C} = \alpha_C P_C$, and substituting $r_F = r + s_F$ and $\gamma_F = \gamma_P - \gamma_{P_B}$ into the above equation yields

$$\begin{aligned} & \alpha_B r_{P_B} P_B + \alpha_C r_{P_C} P_C + r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R = \\ & r\gamma_P + s_F\gamma_F^- - r_R\gamma_R + (r_{P_C} - r)\alpha_C P_C + (r_{P_B} - r)\alpha_B P_B. \end{aligned} \quad (4.63)$$

The repo account γ_R is defined as ΔS_t ; therefore, equation (4.63) becomes

$$\begin{aligned} & \alpha_B r_{P_B} P_B + \alpha_C r_{P_C} P_C + r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R = \\ & r\gamma_P + s_F\gamma_F^- - r_R\Delta_t S_t + (r_{P_C} - r)\alpha_C P_C + (r_{P_B} - r)\alpha_B P_B. \end{aligned}$$

The parameters Δ , α_C and α_B have been chosen so that the hedging portfolio perfectly hedges the derivative. The price of the hedging portfolio must be equal to the short derivative at time t in order to avoid an arbitrage, i.e. $\gamma_P = \hat{V}$, and from equation (4.5) we have that

$$\begin{aligned} & \alpha_B r_{P_B} P_B + \alpha_C r_{P_C} P_C + r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R = \\ & -r\hat{V}_t + s_F\gamma_F^- - r_R\Delta_t S_t + \lambda_C \alpha_C P_C + \lambda_B \alpha_B P_B. \end{aligned} \quad (4.64)$$

Substituting the counterparty bond positions given in equations (4.60) and (4.61) into the above equation, and rearranging the terms, gives us

$$\begin{aligned} & \alpha_B r_{P_B} P_B + \alpha_C r_{P_C} P_C + r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{P_C} - r_R\gamma_R = -(r + \lambda_B + \lambda_C)\hat{V} + \\ & s_F\gamma_F^- - r_R\Delta S_t + \lambda_B(X^+ + R_B X^-) + \lambda_C(X^- + R_C X^+). \end{aligned} \quad (4.65)$$

Substituting equation (4.65) back into equation (4.62) and rearranging the terms yields

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = (\lambda_B + \lambda_C)\hat{V} - s_F\gamma_F^- - \lambda_B(X^+ + R_B X^-) - \lambda_C(X^- + R_C X^+), \quad (4.66)$$

where the parabolic differential operator is defined as

$$\mathcal{A}_t = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} - r_R S \frac{\partial}{\partial S}.$$

The funding account term can be written as

$$\gamma_F^- = \left(-\hat{V}_t - \alpha_B P_B \right)^- = - \left(\hat{V}_t + \Delta \hat{V}_B \right)^+ = -(X^+ + R_B X^-)^+ = -X^+;$$

therefore, equation (4.66) becomes

$$\begin{cases} \frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C) \hat{V} = s_F X^+ - \lambda_B (X^+ + R_B X^-) - \lambda_C (X^- + R_C X^+) \\ \hat{V}_T = V_T \text{ is the final payoff of the derivative} \end{cases} \quad (4.67)$$

□

This is the PDE derived by Burgard and Kjaer (2011). The above PDE is a non-linear PDE due to the min and max functions applied to the derivative price on the right hand side of the PDE. However, when the close-out amount is the risk-free price, then the PDE becomes linear because we can compute the risk-free price by means of the Black-Scholes PDE. The PDE also becomes linear if the value of the derivative is either $V_t \geq 0$ or $V_t \leq 0$ for all t . This will be illustrated in Section 4.4.1.

The Feynman-Kac theorem is now applied to the above PDE in order to write the price as an expectation.

Corollary 20:

The price of a general credit risky collateralised derivative trade in expectation form is given by

$$\hat{V}_0 = V_0 + U_0,$$

where U_0 is the total value adjustment at time t_0 . The total value adjustment of the credit risky derivative when the close-out value is the risky price is

$$\begin{aligned} U_0 = & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r du} (1 - R_C) \lambda_C \hat{V}_s^+ ds \right] \dots CVA \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r du} (1 - R_B) \lambda_B \hat{V}_s^- ds \right] \dots DVA \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r du} s_F \hat{V}_s^+ ds \right] \dots FCA. \end{aligned}$$

The total value adjustment of the credit risky derivative when the close-out value is the risk-free price is

$$\begin{aligned} U_0 = & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r + \lambda_B + \lambda_C du} (1 - R_C) \lambda_C V_s^+ ds \right] \dots CVA \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r + \lambda_B + \lambda_C du} (1 - R_B) \lambda_B V_s^- ds \right] \dots DVA \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r + \lambda_B + \lambda_C du} s_F V_s^+ ds \right] \dots FCA. \end{aligned}$$

The total value adjustment for both close-out conventions is made up of a CVA, a DVA and a FCA. The expectation is taken under the $\mathbb{Q}^{rR}$ -measure.

Proof. As in the discrete case the price of a credit risky derivative can be written as the sum of the risk-free price and a sum of adjustments

$$\hat{V}_t = V_t + U_t. \quad (4.68)$$

Substituting equation (4.68) into equation (4.67) gives us

$$\begin{aligned} \frac{\partial(V+U)}{\partial t} + \mathcal{A}_t(V+U) - r(V+U) &= -\lambda_B(R_B X^- + X^+) - \lambda_C(R_C X^+ + X^-) \\ &\quad + (\lambda_B + \lambda_C)(V+U) + s_F X^+. \end{aligned}$$

The risk-free derivative value satisfies the following PDE

$$\begin{cases} \frac{\partial V}{\partial t} + \mathcal{A}_t V - rV = 0 \\ V_T \text{ is the final payoff of the derivative} \end{cases} ; \quad (4.69)$$

therefore, the total value adjustment satisfies the following non-linear PDE

$$\begin{cases} \frac{\partial U}{\partial t} + \mathcal{A}_t U - rU = -\lambda_B(R_B X^- + X^+) - \lambda_C(R_C X^+ + X^-) + (\lambda_B + \lambda_C)(V+U) + s_F X^+ \\ U_T = 0 \end{cases}. \quad (4.70)$$

The fact that there are no adjustments at maturity has been discussed in Section 4.3.1. Next, we apply the Feynman-Kac theorem to the PDE in equation (4.70). If the close-out value is the risky price, i.e. $X_t = \hat{V}_t$, then equation (4.70) becomes

$$\frac{\partial U}{\partial t} + \mathcal{A}_t U - rU = \lambda_B \hat{V}^-(1 - R_B) + \lambda_C \hat{V}^+(1 - R_C) + s_F \hat{V}^+. \quad (4.71)$$

By applying the Feynman-Kac theorem and using the fact that $U_T = 0$, the total value adjustment can be written as

$$\begin{aligned} U_0 = & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r du} (1 - R_C) \lambda_C \hat{V}_s^+ ds \right] \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r du} (1 - R_B) \lambda_B \hat{V}_s^- ds \right] \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r du} s_F \hat{V}_s^+ ds \right]. \end{aligned} \quad (4.72)$$

If the close-out value is the risk-free price, $X_t = V_t$, then equation (4.70) becomes

$$\frac{\partial U}{\partial t} + \mathcal{A}_t U - (r + \lambda_B + \lambda_C)U = \lambda_B V^-(1 - R_B) + \lambda_C V^+(1 - R_C) + s_F V^+. \quad (4.73)$$

By applying the Feynman-Kac theorem, the total value adjustment can be written as

$$\begin{aligned}
 U_0 = & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r + \lambda_B + \lambda_C du} (1 - R_C) \lambda_C V_s^+ ds \right] \\
 & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r + \lambda_B + \lambda_C du} (1 - R_B) \lambda_B V_s^- ds \right] \\
 & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r + \lambda_B + \lambda_C du} s_F V_s^+ ds \right].
 \end{aligned} \tag{4.74}$$

□

The three terms of U_0 represents a CVA, DVA and FCA, respectively. These adjustments are made to account for counterparty credit risk and funding costs. This is the continuous analog of the total value adjustment derived in the n period tree model in Section 4.3. For the risky close-out case the price of the derivative is dependent on the total value adjustment which in turn is dependent on the price of the derivative. This means that the price of the derivative has a recursive formula for the risky close-out.

4.4.1 Non-recursive probabilistic solution for the Burgard and Kjaer PDE

Next, we show that the Burgard and Kjaer PDE has a non-recursive probabilistic solution if certain conditions hold. It has been shown that the price of a general risky derivative can be written as the sum of the risk-free price and the total value adjustment. The derivative's price when the close-out is the risky price has a recursive nature. The value of the risky derivative depends on the value of U_0 , which in turn is a function of $\hat{V}_0$.

We show that if interest rates and hazard rates are constant and we are pricing a derivative where the value of derivative is either positive or negative for all time, i.e. a call option, where $V_t \geq 0$ or $V_t \leq 0$ for all t , then the risky derivative has a non-recursive probabilistic solution for both close-out conventions. This allows us to avoid a recursive algorithm in order to compute the price for this certain class of derivatives.

Theorem 21:

The value of a credit risky derivative with a risky close-out satisfies the following PDE

$$\begin{cases} \frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r \hat{V} = \lambda_C (1 - R_C) \hat{V}^+ + \lambda_B (1 - R_B) \hat{V}^- + s_F \hat{V}^+ \\ \hat{V}_T = V_T \text{ is the final payoff of the derivative} \end{cases},$$

where the maturity of the derivative contract is denoted by T and $t \in [0, T]$. If interest rates and hazard rates are constant and $V_t \geq 0$ or $V_t \leq 0$ for all t , then the PDE has the following non-recursive probabilistic solution

$$\begin{aligned}
 \hat{V}_t = & \mathbb{I}_{\{V_t \geq 0\}} e^{-(\lambda_C(1-R_C) + s_F)(T-t)} \mathbb{E}_t [e^{-r(T-t)} V_T] \\
 & + \mathbb{I}_{\{V_t < 0\}} e^{-\lambda_B(1-R_B)(T-t)} \mathbb{E}_t [e^{-r(T-t)} V_T],
 \end{aligned} \tag{4.75}$$

which is unique. The expectation is taken under the $\mathbb{Q}^{rR}$ -measure.

Proof. We now show that $\hat{V}_t$ given by equation (4.75) is a solution to the PDE in equation (4.67) when the close-out convention is the risky price. There are two cases to consider, when $V_t \geq 0$ and when $V_t < 0$ for all $t \in [0, T]$. First assume that $V_t \geq 0$ for all $t \in [0, T]$. Since the risk-free price forms a martingale under the $\mathbb{Q}^{r_R}$ -measure, equation (4.75) reduces to

$$\hat{V}_t = e^{-(\lambda_C(1-R_C)+s_F)(T-t)} V_t. \quad (4.76)$$

The first partial derivative of the risky price with respect to time and the parabolic operator acting on the risky price are given by

$$\frac{\partial \hat{V}_t}{\partial t} = e^{-(\lambda_C(1-R_C)+s_F)(T-t)} \left((\lambda_C(1-R_C) + s_F) V_t + \frac{\partial V_t}{\partial t} \right) \quad (4.77)$$

and

$$\mathcal{A}_t \hat{V}_t = e^{-(\lambda_C(1-R_C)+s_F)(T-t)} \mathcal{A}_t V_t, \quad (4.78)$$

respectively. Equations (4.77) and (4.78) yield

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r \hat{V} = e^{-(\lambda_C(1-R_C)+s_F)(T-t)} \left[\frac{\partial V}{\partial t} + \mathcal{A}_t V - r V \right] + \lambda_C(1-R_C) \hat{V} + s_F \hat{V}. \quad (4.79)$$

Since the risk-free price satisfies equation (4.69), equation (4.79) simplifies to

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r \hat{V} = \lambda_C(1-R_C) \hat{V} + s_F \hat{V}.$$

Therefore, it follows from equation (4.76) that $V_t \geq 0$ implies $\hat{V}_t \geq 0$ for all t ; hence, equation (4.76) satisfies the PDE.

The case when $V_t < 0$ for all $t \in [0, T]$ follows similarly.

Next, we show that the PDE has a unique solution. Define the function f as

$$f(\hat{V}(t, s)) = -r \hat{V} - \lambda_C(1-R_C) \hat{V}^+ - \lambda_B(1-R_B) \hat{V}^- - s_F \hat{V}^+.$$

Then $\hat{V}$ satisfies the following quasilinear parabolic partial differential equations

$$\begin{cases} \frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} + f(\hat{V}(t, s)) = 0, & \{s, t\} \in \mathbb{R} \times [0, T]. \\ \hat{V} = V, & \text{when } t = T. \end{cases} \quad (4.80)$$

Since f is globally *Lipschitz*, it follows from Peng (1992) (Theorem 3.1. pg. 176) (see also Pardoux and Peng (1990)) that (4.80) admits a unique solution which has the following representation

$$\hat{V}_t = V_t + \mathbb{E}_t \left[\int_t^T f(\hat{V}_r) dr \right].$$

This completes the proof that equation (4.75) is the only solution to the non-linear PDE in equation (4.67) when the derivative trade has a risky close-out. $\square$

When $r = r_R$, equation (4.69) will reduce to the original Black-Scholes PDE (see Black and Scholes

(1973)) which has an analytic solution; therefore, equation (4.67) will have an analytic solution.

Next, we solve the PDE when the close-out is the risk-free value.

Theorem 22:

The value of a credit risky derivative with a risk-free close-out satisfies the following PDE

$$\begin{cases} \frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C) \hat{V} = \lambda_C(1 - R_C)V^+ + \lambda_B(1 - R_B)V^- + s_F V^+ - (\lambda_B + \lambda_C)V \\ \hat{V}_T = V_T \text{ is the final payoff of the derivative} \end{cases}$$

where the maturity of the derivative contract is denoted by T and $t \in [0, T]$. If interest rates and hazard rates are constant and $V_t \geq 0$ or $V_t \leq 0$ for all t , then the PDE has the following non-recursive probabilistic solution

$$\begin{aligned} \hat{V}_t = & \mathbb{I}_{\{V_t \geq 0\}} \left(1 - ((1 - R_C)\lambda_C + s_F) \left(\frac{1 - e^{-(\lambda_B + \lambda_C)(T-t)}}{(\lambda_B + \lambda_C)} \right) \right) \mathbb{E}_t [e^{-r(T-t)} V_T] \\ & + \mathbb{I}_{\{V_t < 0\}} \left(1 - (1 - R_B)\lambda_B \left(\frac{1 - e^{-(\lambda_B + \lambda_C)(T-t)}}{(\lambda_B + \lambda_C)} \right) \right) \mathbb{E}_t [e^{-r(T-t)} V_T], \end{aligned} \quad (4.81)$$

which is unique. The expectation is taken under the $\mathbb{Q}^{rR}$ -measure.

Proof. We now show that $\hat{V}_t$ given by (4.81) is a solution to the PDE in (4.67) when the close-out convention is the risk-free price. There are two cases to consider, when $V_t \geq 0$ and when $V_t < 0$ for all $t \in [0, T]$. First assume that $V_t \geq 0$ for all $t \in [0, T]$. Since the risk-free price forms a martingale under the $\mathbb{Q}^{rR}$ -measure, equation (4.81) reduces to

$$\hat{V}_t = \left(1 - ((1 - R_C)\lambda_C + s_F) \left(\frac{1 - e^{-(\lambda_B + \lambda_C)(T-t)}}{(\lambda_B + \lambda_C)} \right) \right) V_t. \quad (4.82)$$

The first partial derivative of the risky price with respect to time and the parabolic operator acting on the risky price are given by

$$\begin{aligned} \frac{\partial \hat{V}_t}{\partial t} = & ((1 - R_C)\lambda_C + s_F) \left(e^{-(\lambda_B + \lambda_C)(T-t)} \right) V_t \\ & + \left(1 - ((1 - R_C)\lambda_C + s_F) \left(\frac{e^{-(\lambda_B + \lambda_C)(T-t)}}{(\lambda_B + \lambda_C)} \right) \right) \frac{\partial V_t}{\partial t} \end{aligned} \quad (4.83)$$

and

$$\mathcal{A}_t \hat{V}_t = \left(1 - ((1 - R_C)\lambda_C + s_F) \left(\frac{1 - e^{-(\lambda_B + \lambda_C)(T-t)}}{(\lambda_B + \lambda_C)} \right) \right) \mathcal{A}_t V_t, \quad (4.84)$$

respectively. From equations (4.69), (4.83) and (4.84) we get

$$\begin{aligned}
 \frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r \hat{V} &= ((1 - R_C)\lambda_C + s_F) \left(e^{-(\lambda_B + \lambda_C)(T-t)} \right) V \\
 &= \left(1 - ((1 - R_C)\lambda_C + s_F) \left(\frac{1 - e^{-(\lambda_B + \lambda_C)(T-t)}}{(\lambda_B + \lambda_C)} \right) \right) (\lambda_B + \lambda_C) V \\
 &\quad - (\lambda_B + \lambda_C) V + ((1 - R_C)\lambda_C + s_F) V \\
 &= (\lambda_B + \lambda_C) \hat{V} - (\lambda_B + \lambda_C) V + ((1 - R_C)\lambda_C + s_F) V.
 \end{aligned} \tag{4.85}$$

From equation (4.82), and since $V_t \geq 0$ for all t , we have that

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C) \hat{V} = -(\lambda_B + \lambda_C) V + ((1 - R_C)\lambda_C + s_F) V^+. \tag{4.86}$$

The case when $V_t < 0$ for all $t \in [0, T]$ follows similarly.

Next, we show that the PDE has a unique solution. If we define f as

$$f(\hat{V}(t, s)) = -(r + \lambda_B + \lambda_C) \hat{V} - \lambda_C(1 - R_C)V^+ - \lambda_B(1 - R_B)V^- - s_F V^+,$$

then $\hat{V}$ satisfies equation (4.80) and it follows that this solution is unique. This completes the proof that equation (4.81) is the only solution to the PDE in equation (4.67) when the derivative trade has a risk-free close-out. $\square$

Next, we interpret the TVA term for the risk-free close-out and when $s_F = 0$. The TVA term in equation (4.81) will reduce to the CVA term if $\hat{V}_t \geq 0$ for all t and the funding spread is zero. The CVA term at time t_0 can be written as the product of three terms

$$U_0 = - \overbrace{\left[1 - e^{-(\lambda_B + \lambda_C)T} \right]}^1 \overbrace{\left(\frac{\lambda_C}{\lambda_B + \lambda_C} \right)}^2 \overbrace{(1 - R_C)V_0}^3. \tag{4.87}$$

The following is a description of the three terms:

- (i) The first term represents the probability that either counterparty B or C defaults from the initial time until maturity T .
- (ii) The second term represents the probability that counterparty C will default first, given that a counterparty defaults. This is shown in [Ross \(2010\)](#) (Equation 5.5, pg. 302).
- (iii) The third term represents the amount that will be lost by B on the trade if C defaults.

The entire expression represents how much counterparty B will expect to lose given that counterparty C defaults during the specified time. The interesting fact about this result is that the CVA term of a derivative, which is $V_t \geq 0$ for all t , will depend on both counterparty's default intensities. The CVA term will only be realised if counterparty B survives longer than counterparty C and C defaults before the end of the trade.

4.5 Numerical implementation

The model discussed in this chapter is now implemented via three approaches as done in the previous chapter. The three approaches are a PDE approach, a tree lattice approach and a Quasi Monte Carlo approach. We show that the prices of the two close-out conventions are similar, given the parameters of the derivative trade used in this dissertation. In [Burgard and Kjaer \(2011\)](#) they only implement an asymmetric derivative using equation (4.67), i.e. a European call option. We implement the prices of European call and put options. Furthermore, we extend Burgard and Kjaer's work by implementing the price of a forward contract. When pricing a forward contract, Theorems 21 and 22 will not be useful because they require that the value of the derivative is either positive or negative for all time. A forward contract can take on alternating signs throughout the trade; therefore, we require a recursive algorithm in order to implement the non-linear PDE.

4.5.1 PDE approach

We begin with the PDE approach. As in Chapter 3, all interest and hazard rates are assumed to be constant in our implementation. We present a recursive algorithm used to solve the PDE in equation (4.67), numerically. The same finite difference approximations that were applied to equation (3.34) are now applied to equation (4.67) to obtain

$$\begin{aligned} & \frac{\hat{V}_j^i - \hat{V}_{j-1}^i}{dt} + \frac{1}{2}\sigma^2(ds)^2 \frac{\hat{V}_{j-1}^{i+1} - 2\hat{V}_{j-1}^i + \hat{V}_{j-1}^{i-1}}{(ds)^2} + r_R \frac{\hat{V}_{j-1}^{i+1} - \hat{V}_{j-1}^{i-1}}{2ds}(ids) = \\ & (\lambda_B + \lambda_C)\hat{V}_{j-1}^i + s_F X_{j-1}^{i+} - \lambda_B(X_{j-1}^{i+} + R_B X_{j-1}^{i-}) - \lambda_C(X_{j-1}^{i-} + R_C X_{j-1}^{i+}). \end{aligned}$$

Rearranging the terms and writing the above equation in matrix and vector form yields

$$\hat{V}_j = M\hat{V}_{j-1} + \left(s_F X_{j-1}^{i+} - \lambda_B(X_{j-1}^{i+} + R_B X_{j-1}^{i-}) - \lambda_C(X_{j-1}^{i-} + R_C X_{j-1}^{i+}) \right) dt, \quad (4.88)$$

where the matrix M and the vector $\hat{V}_j$ is defined as

$$M = \begin{bmatrix} Y_1 & Z_1 & 0 & \dots & 0 \\ X_2 & Y_2 & Z_2 & \dots & 0 \\ 0 & X_3 & Y_3 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & Z_{M-1} \\ 0 & \dots & 0 & X_M & Y_M \end{bmatrix} \quad \text{and} \quad \hat{V}_j = \begin{bmatrix} \hat{V}_j^1 \\ \hat{V}_j^2 \\ \hat{V}_j^3 \\ \vdots \\ \hat{V}_j^M \end{bmatrix},$$

where

$$\begin{aligned} X_i &= \frac{1}{2} (r_R i - \sigma^2 i^2) dt, \\ Y_i &= 1 + (\sigma^2 i^2 + (r + \lambda_B + \lambda_C)) dt, \\ Z_i &= -\frac{1}{2} (r_R i + \sigma^2 i^2) dt. \end{aligned}$$

When $X = V$, then the PDE becomes linear because the risk-free price is known. However, when $X = \hat{V}$, then the PDE cannot be rearranged in order to make the risky price $\hat{V}_{j-1}$ the subject of the equation, hence it becomes non-linear. The recursive algorithm used to approximate this PDE, is described in the

following extract of code used to implement the PDE. We warn the reader that this is only an extract of the full implementation. The extract only illustrates the iterative nature of the algorithm.

Algorithm 23:

```
% An outline of the recursive algorithm for the PDE approach

V_hat=BS; % Initial guess is the Black-Scholes price
for j=1:1:N

    for e=1:10 % After 10 iterations the price converges

        if CloseOut='Risk free' % Which close-out convention is used
            X(:,j)=BS(:,j);
        else
            X(:,j)=V_hat(:,j);
        end

        V_hat(:,j)=inv(M)((V_hat(:,j+1)-(s_F*max(X(:,j+1),0)
                        -Lamb_B*(Rb*min(X(:,j),0)+max(X(:,j),0))
                        -Lamb_C*(Rc*max(X(:,j),0)+min(X(:,j),0))*dt));

    end

    % The boundary conditions are handled in an appropriate way.
end
```

The idea behind this algorithm is that at each discrete time point the risky price is placed back into the formula to compute a new risky price. The initial risky price will be estimated with the risk-free price. After ten iterations the price of the risky derivative does not change significantly; therefore, it converges to a price.

A European put option is priced with the following parameters: $S_0 \in [0, 20]$, $K = 10$, $r = 0.04$, $r_R = 0.06$, $\sigma = 0.3$, $T = 0.5$, $r_{PB} = 0.08$, $r_{PC} = 0.08$, $R_B = 0.3$ and $R_C = 0.3$. In the implementation there are 5 000 time increments and 1 000 space increments. The at-the-money prices of the European put option for the risky and risk-free close-out calculating by using Algorithm 23 are 0.680335334 and 0.68044882, respectively. The difference in price using the two different close-out conventions is 0.00011495, which amounts to roughly 0.0168935% of the risk-free close-out price. The difference between the two prices is not significant. This result is true for the given parameters.

4.5.2 Tree lattice approach

Next, the tree model is implemented, which also requires a recursive algorithm to implement. Firstly, an extract of the recursive algorithm is presented, then the difference in price between the two close-out conventions is illustrated for a European put option. Lastly, the hedging portfolio used to perfectly hedge out the put option in a two period model is examined.

The value of the credit risky derivative is now computed using the tree model presented in Section 4.3. The following is an extract of the algorithm used to implement the tree model.

Algorithm 24:

```

% An outline of the recursive algorithm for the tree model approach

function [Price] = BinomialPriceCredit (Payoff,p,n,dt,CloseOut,Rb,Rc,rF,rB,rC,r,Tree)

Vbs=Payoff;
V=zeros(n,n);
V(:,end)=Payoff;
U=zeros(n,n);
Udis=zeros(n,n);

for i = n:-1:2
    if strcmp(CloseOut,'Risk free')

        Vbs=exp(-r*dt)*(p*Vbs(1:i-1)+(1-p)*Vbs(2:end)); % Risk-Free price

        for j=1:i-1
            if Vbs(j)>=0
                U(j,i-1)= -(1/(exp(rC*dt)-exp(r*dt)+exp(rB*dt)))*(Vbs(j)*(1-Rc)*(exp(rC*
                    dt)-exp(r*dt))+Vbs(j)*(exp(rF*dt)-exp(r*dt)));
                Udis(j,i-1)=(1/(exp(rC*dt)-exp(r*dt)+exp(rB*dt)))*(p*Udis(j,i)+(1-p)*Udis
                    (j+1,i))+U(j,i-1);
                V(j,i-1)=Vbs(j) + Udis(j,i-1);
            else
                % Similar to the previous case
            end
        end
    else
        Vbs=exp(-r*dt)*(p*Vbs(1:i-1)+(1-p)*Vbs(2:end));
        for j=1:i-1
            if Vbs(j)>=0
                V(j,i-1)=Vbs(j); % Intial guess
                for d=1:10 % After ten iterations the price converges
                    U(j,i-1)= -exp(-r*dt)*(V(j,i-1)*(1-Rc)*(exp(rC*dt)-exp(r*dt))+V(j,i-1)
                        *(exp(rF*dt)-exp(r*dt)));
                    Udis(j,i-1)=exp(-r*dt)*(p*Udis(j,i)+(1-p)*Udis(j+1,i))+U(j,i-1);
                    V(j,i-1)=Vbs(j) + Udis(j,i-1);
                end
            else
                % Similar to the previous case
            end
        end
    end
end
end

Price=V(1,1);

end

```

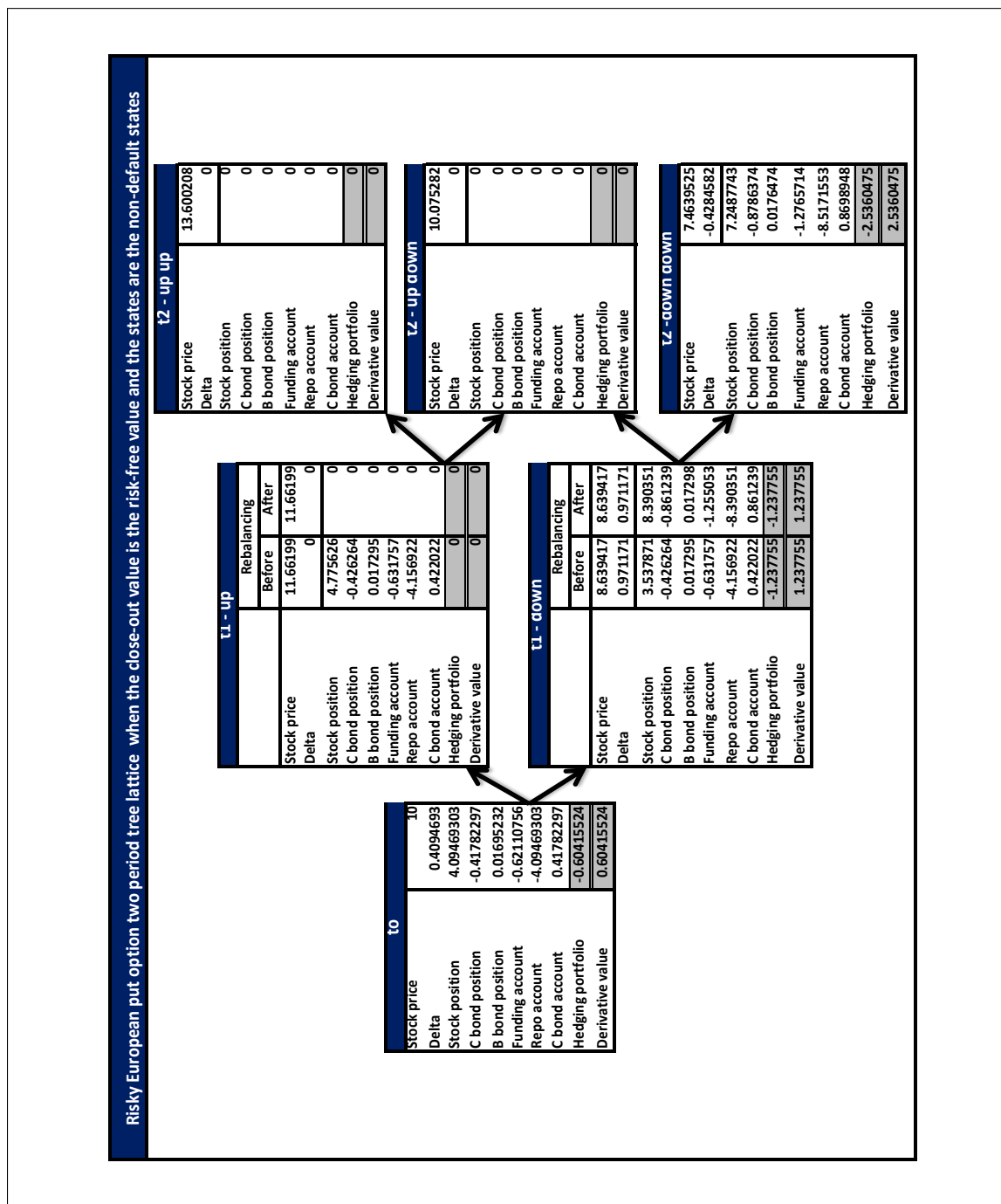


Figure 4.3: Risky European put option two period tree lattice when the close-out value is the risk-free value and the states are the non-default states.

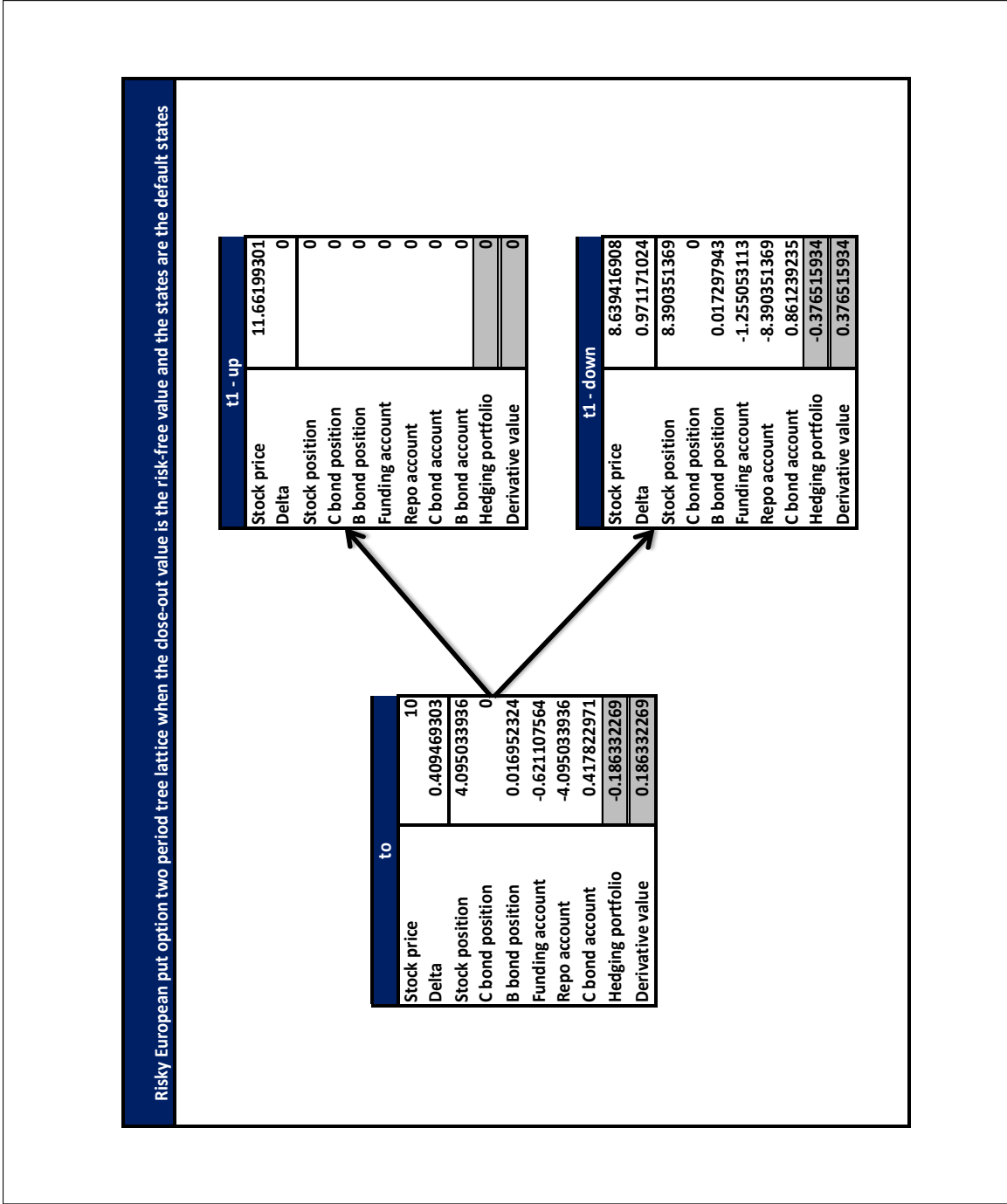


Figure 4.4: Risky European put option two period tree lattice when the close-out value is the risk-free value and the states are the default states.

The same idea that was used in Algorithm 23 is used in the tree model implementation. We would like to highlight to the reader that the variable called $Udis$ is the discounted total value adjustment from future nodes and the variable called U is the total value adjustment at time idt (see Section 4.3).

The same European put option that was priced in the PDE approach is priced using the tree model. The tree lattice used to price the put contains 5 000 steps. The price of the put with a risk-free close-out is 0.68049927, and the price of the put with a risky close-out is 0.68038432. The difference in prices using the two close-out conventions is 0.0001145, which amounts roughly to 0.016892% of the risk-free close-out price. Again, this shows that the difference in price between the two close-out conventions is not significant given the parameters of the put.

Next, we price the European put option in a two period model and examine the hedging portfolio used to perfectly hedge out the European put option in each state of the world. Suppose B buys a European put option from C and assume B sets up a portfolio in order to hedge out B 's own position in the put. The close-out value of the put option is the risk-free price. At each node, including the initial node, we first determine whether a default occurs. If a default has occurred, then the price of the derivative will move to a default branch. The value of the derivative in the default branch will depend on the party defaulting. If the counterparty defaulting is in-the-money, then the derivative value will be equal to the close-out value. If the counterparty defaulting is out-of-the-money, then the derivative value will be equal to the close-out value multiplied by the recovery rate of the defaulting counterparty. In our example, we assume only counterparty C defaults.

In Figure 4.3 the hedging portfolio and its various accounts throughout the two step model are illustrated. The tree in Figure 4.3 is a tree of non-default states. In these states counterparty C does not default. As one can see, the hedging portfolio perfectly hedges out the derivative at each node in the lattice. At time t_1 , counterparty B rebalances the hedging portfolio in order to be hedged for the next time step.

Figure 4.4 is a tree of default states. The default of counterparty C is determined before the stock moves. This means that C can default at the initial node just after the two counterparties enter into the trade and at time t_1 just before the put moves to maturity. At each node, when counterparty C does default, the value of the derivative will be equal to the risk-free price multiplied by the recovery rate R_C . As one can see, the hedging portfolio perfectly hedges the put option when C defaults and when C does not default. The risky close-out case is also considered. We provide the binomial lattices for the default states and the non-default states in Appendix B.2 and B.3.

4.5.3 Quasi Monte Carlo approach

Next, we price the same European put option that was priced in Sections 4.5.1 and 4.5.2; however this time using a Quasi Monte Carlo approach. The concept of Quasi Monte Carlo simulations was described in Section 3.4.4. The following extract illustrates the implementation of the Quasi Monte Carlo simulation in conjunction with a recursive algorithm.

Algorithm 25:

```

% An outline of the recursive algorithm for the Quasi Monte Carlo approach

U=zeros(length(Paths),N);
DummyV=zeros(Paths,1);
DummyU=zeros(Paths,1);

for i=1:length(Paths) % If we repeat experiment with different number of paths

    QuasiNumbers=vanderCorput(Paths(i)+1,b); % Generating quasi random numbers
    Z=norminv([QuasiNumbers(2:end)],0,1);

    for j=N-1:-1:1
        if strcmp(Closeout,'Risk free') % We know the close-out value in this case

            S=[So.*exp(repmat((rR-0.5*sigma.^2).*Time(j),Paths(i),1)+sigma.*sqrt(Time(j)).*Z)];
            RiskFreePrice=BSMC(S,T-Time(j),r,rR,sigma,K,Type);
            % The 'BSMC' function computes the risk-free price of the derivative

            DVA(i,j)=-mean(exp(-(r+Lambda_B+Lambda_C)*(Time(j)))*(1-Rb)*dt*Lambda_B*min(
                RiskFreePrice,0)); % DVA integral
            CVA(i,j)=-mean(exp(-(r+Lambda_B+Lambda_C)*(Time(j)))*(1-Rc)*dt*Lambda_C*max(
                RiskFreePrice,0)); % CVA integral
            FCA(i,j)=-mean(exp(-(r+Lambda_B+Lambda_C)*(Time(j)))*S_fund*dt*max(
                RiskFreePrice,0)); % FCA integral

        else

            S=[So.*exp(repmat((rR-0.5*sigma.^2).*Time(j),Paths(i),1)+sigma.*sqrt(Time(j)).*Z)];
            RiskFreePrice=BSMC(S,T-Time(j),r,rR,sigma,K,Type);
            DummyV= RiskFreePrice; % Initial guess of the risky price

            for e=1:10
                DummyU=(-(1-Rb)*Lambda_B*dt*min(DummyV,0)-(1-Rc)*Lambda_C*dt*max(DummyV,0)-
                    S_fund*dt*max(DummyV,0))+exp(-r*dt)*U(i,j+1); % TVA value include new
                    TVA and the discounted future TVA
                DummyV=RiskFreePrice+DummyU;
            end

            RiskyPrice(i,j)=mean(RiskFreePrice)+mean(DummyU);
            U(i,j)=mean(DummyU);
            DVA(i,j)=-mean(exp(-r*(Time(j)))*(1-Rb)*Lambda_B*dt*min(DummyV,0));
            CVA(i,j)=-mean(exp(-r*(Time(j)))*(1-Rc)*Lambda_C*dt*max(DummyV,0));
            FCA(i,j)=-mean(exp(-r*(Time(j)))*S_fund*dt*max(DummyV,0));

        end

        PriceMonteCarlo(i)=BSMC(So,T,r,rR,sigma,K,Type) + sum(DVA(i,:)) + sum(CVA(i,:))
        + sum(FCA(i,:));
    end
end

```

We used this implementation to price the same credit risky European put option. We used 800 000 stock paths and 5000 discrete time points. The price of the at-the-money put with a risk-free close-out is 0.68049204 and the at-the-money price of the put with a risky close-out is 0.68037749.

The prices of the three approaches are compared in Table 4.1. The table includes the risk-free price of the European put option. The difference between the risk-free price and the risky prices will be the TVA for the given close-out convention. Table 4.1 illustrates the convergence of the three approaches. All three methods converge within 0.5 basis points for at-the-money prices.

European put option	PDE	Tree lattice	Quasi Monte Carlo	Probabilistic solution
Risk-free price	0.699654	0.699704	0.699697	0.699697
Risk-free close-out	0.680449	0.680499	0.680492	0.680492
Risky close-out	0.680334	0.680384	0.680377	0.680377

Table 4.1: Summary of the prices of the three approaches and the probabilistic solution of a European put option with the following parameters: $S_0 = 10$, $K = 10$, $r = 0.04$, $r_R = 0.06$, $\sigma = 0.3$, $T = 0.5$, $r_{PB} = 0.08$, $r_{PC} = 0.08$, $R_B = 0.3$ and $R_C = 0.3$.

The probabilistic solution is computed using Theorems 21 and 22. The discounted expectation (risk-free price) in Theorems 21 and 22 is computed using a Quasi Monte Carlo simulation. Table 4.1 shows that the recursive algorithm converges to the probabilistic solution for a European put option.

The reader may argue that a European put option with constant interest rates and hazard rates can be priced using Theorems 21 and 22. In order to illustrate that our algorithms for the three approaches can price a derivative that can take on alternating signs, i.e. a forward contract, we provide the reader with the following table:

Strike	Contract	Risky close-out			Risk-free close-out		
		PDE	Monte Carlo	Tree	PDE	Monte Carlo	Tree
K=6	forward	4.102751	4.102826	4.102801	4.103444	4.103518	4.103495
	call	4.105528	4.105456	4.105523	4.106222	4.106170	4.106217
	put	0.002720	0.002720	0.002716	0.002721	0.002721	0.002716
K=8	forward	2.196010	2.196060	2.196061	2.196383	2.196449	2.196436
	call	2.300535	2.300499	2.300544	2.300924	2.300899	2.300933
	put	0.104008	0.104021	0.104004	0.104025	0.104039	0.104021
K=10	forward	0.284364	0.284355	0.284417	0.284434	0.284491	0.284486
	call	0.970577	0.970594	0.970681	0.970741	0.970763	0.970845
	put	0.680334	0.680374	0.680409	0.680449	0.680492	0.680524
K=12	forward	-1.640719	-1.640698	-1.640666	-1.640998	-1.640948	-1.640946
	call	0.310520	0.310506	0.310570	0.310573	0.310560	0.310623
	put	1.926531	1.926544	1.926565	1.926857	1.926880	1.926891
K=14	forward	-3.572297	-3.572260	-3.572245	-3.572940	-3.572897	-3.572888
	call	0.080325	0.080282	0.080312	0.080339	0.080286	0.080325
	put	3.602591	3.602577	3.602574	3.603199	3.603204	3.603183

Table 4.2: The prices of forward contracts, European put options and European call options for varying strikes and parameters: $S_0 = 10$, $r = 0.04$, $r_R = 0.06$, $\sigma = 0.3$, $T = 0.5$, $r_{PB} = 0.08$, $r_{PC} = 0.08$, $R_B = 0.3$ and $R_C = 0.3$.

Table 4.2 tabulates the prices for forward contracts, European calls and puts for varying strike prices. The prices in the table are generated using Algorithms 23, 24 and 25. This shows that our algorithm can price

any general derivative satisfying the PDE in equation (4.67). This is an extension to the work of Burgard and Kjaer (2011). This algorithm will be extended to include collateral in the derivative trade in Chapter 5.

Next, we simulate the price process of a European put option for one stock path realisation and illustrate the replicating portfolio used to replicate the put option. This realisation is one potential path that the put option value can follow. The put option being simulated has a half year maturity. The half year interval is partitioned into 25 equal parts. The close-out value of the put is the risky price; therefore, the bond position in counterparty B 's bonds in the replicating portfolio is zero (see equation (4.60)). At each of the 25 discrete time points the various accounts making up the replicating portfolio will be rebalanced in order to replicate the put until the next time point.

The put option path requires only one stock realisation. We generate this stock path from the initial time until maturity. At each of the 25 discrete time points the put option is revalued according to the simulated stock path. The put is priced using Quasi Monte Carlo simulations at each of the discrete time points. The number of stock paths used in the valuation of the put is 500 000 paths.

In Section 4.3 we priced a credit risky European put option and illustrated the replicating portfolio in a two period tree lattice. The delta value in that tree model is exact; however, in a Monte Carlo simulation the delta of the put option can only be estimated. Glasserman (2003) (pg. 378-401) suggests two ways to compute the delta using Monte Carlo estimates. Both methods are briefly described. The first method is the *Finite difference approximation*. The idea behind this method is that the delta value can be estimated by taking the forward difference of the derivative value with respect to the stock price; therefore

$$\Delta(S_t) \approx \frac{\hat{V}(S_t + \delta) - \hat{V}(S_t)}{\delta}, \quad (4.89)$$

where δ is a small change in the stock price and $\hat{V}(S_t + \delta)$ is the price of the risky derivative.

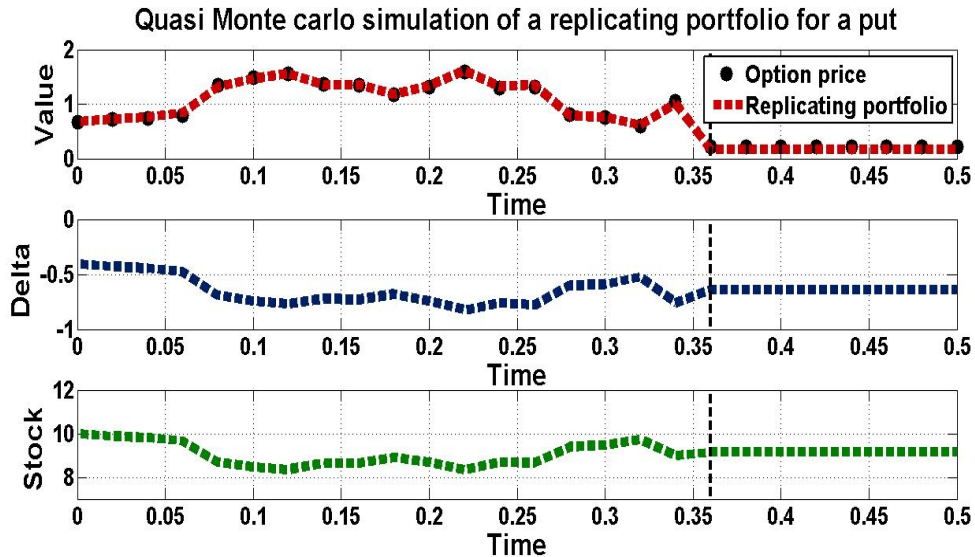


Figure 4.5: Quasi Monte Carlo simulation of a replicating portfolio for a risky European put option with parameters: $S_0 = 10$, $K = 10$, $r = 0.04$, $r_R = 0.06$, $\sigma = 0.3$, $T = 0.5$, $r_{P_B} = 0.08$, $r_{P_C} = 0.08$, $R_B = 0.3$ and $R_C = 0.3$.

The second method is the *Pathwise derivative estimates*. This approach lets the deviation, δ , tend to zero,

hence the estimate for the delta for a European put option is¹

$$\Delta(S_t) \approx e^{-rT} \frac{1}{n} \sum_{i=1}^n \frac{S_T(z_i)}{S_0} \mathbb{I}_{S_T(z_i) < K}, \quad (4.90)$$

where $z_i \sim \mathcal{N}(0, 1)$. This estimator reuses the price paths required for pricing, leading to a substantial computational improvement over the corresponding Finite difference approach. We compared the two approaches and found that they produce similar estimates; however, the Pathwise derivative estimates method is used to estimate the delta of the put because of its superior computational efficiency.

Since counterparty C is risky, a default during the life of the trade has a positive probability of occurring. To simulate a default event, we generated a uniform random variable and used the *inverse transform method* to transform the uniform random variable into an exponential random variable with intensity λ_C . This exponential random variable will generate the default time of counterparty C . We repeated the simulation a small number of times in order to have a simulation where counterparty C defaults during the life of the trade. In order to replicate the loss sustained in the event of a default, the replicating portfolio contains a long position in counterparty C 's bond. At the time of default, the value of C 's bonds will reduce to zero (zero recovery bond). The black line in Figure 4.5 shows the time when counterparty C defaulted in the simulation. In Figure 4.5 three quantities are illustrated:

- (i) The value of the European put option throughout the life of the trade.
- (ii) The estimated delta of the European put option throughout the life of the trade.
- (iii) The value of the underlying stock price throughout the life of the trade.

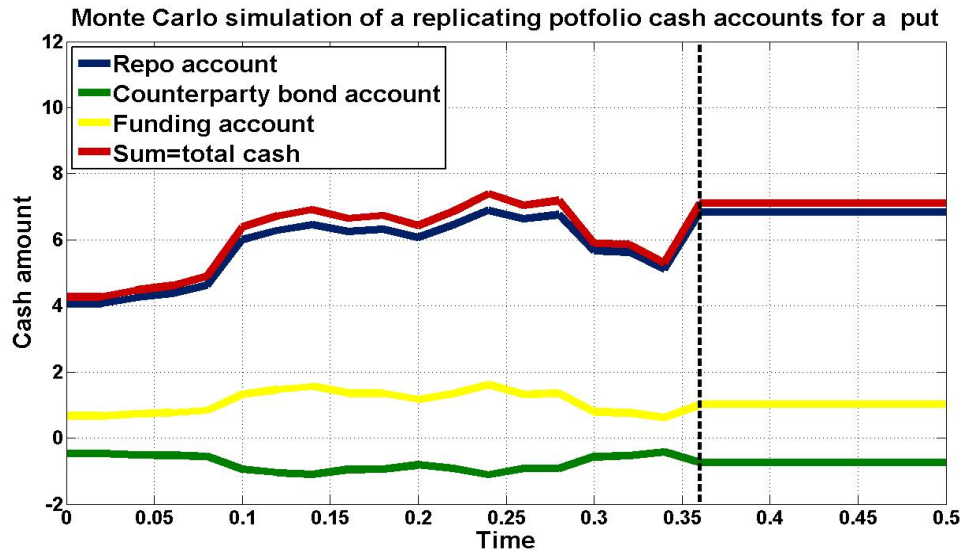


Figure 4.6: Quasi Monte Carlo simulation of the cash accounts of a replicating portfolio for a risky European put option with parameters: $S_0 = 10$, $K = 10$, $r = 0.04$, $r_R = 0.06$, $\sigma = 0.3$, $T = 0.5$, $r_B = 0.08$, $r_{P_C} = 0.08$, $R_{P_B} = 0.3$ and $R_C = 0.3$.

¹We omit the derivation and state only the formula.

One can see from Figure 4.5 that when the underlying stock price decreases, then the corresponding put option value increases. The magnitude of the delta value represents the probability that the put option will end in-the-money. Figure 4.6 illustrates the different cash accounts which make up the total cash account in the replicating portfolio. In Table B.4 the value of each of the accounts which make up the replicating portfolio at each of the discrete time points is tabulated. Since the close-out value is the risky price then as discussed the position in counterparty B 's bonds will be zero (see Section 4.4).

As illustrated in Table B.4, counterparty C defaults at time 0.34. At this time C 's bond is worth null. The difference between the “*Replicating portfolio*” column and the “*Put price*” column when no default occurs is relatively small. This difference is dependent on the delta estimate. In order to improve the delta estimates one needs to reduce the time between the discrete time points. An improvement in delta estimates will decrease the difference in price between the replicating portfolio and the put option.

The absolute difference between the replicating portfolio and the put option when C defaults is 0.0535924, which is roughly 24% of the value of the put option. The position in C 's bonds to replicate the loss when C defaults is determined on the previous price of the put option. The stock price moved from 8.9837 to 9.1330624 just before C defaulted. The stock price movement will result in a change in the put option's value. The larger the change in the put option's value, the more inaccurate the replicating portfolio becomes when a counterparty defaults.

We conclude this chapter with a numerical example highlighting the issues around funding costs. The problem with including a funding cost adjustment into the price of a derivative is that two counterparties will not agree on the price.

4.5.4 Practical example

Suppose counterparty B buys a put option from C . Counterparty B will have to pay over a premium to C for a long position in the put option. This premium needs to be borrowed at a funding rate. If the funding rate of the counterparty is the risk-free rate, then the funding cost adjustment will reduce to zero. Otherwise, the counterparty paying over a premium will incur a funding cost. In our example the counterparty incurring the funding costs is B .

The put option being traded is the same put option that was priced in this section; however, the maturity is now three years and the strike is equal to the stock price grown at the repo rate. We price the put option from the point of view of counterparty B . The price will contain a credit value adjustment and a funding cost adjustment. The PDE approach is used to price the European put option in this example.

Figure 4.7 shows the credit value adjustment surface for the put option. The surface is negative since this is the amount that B will charge C because of the exposure B faces when doing a trade with C . In essence, B will pay less for the put option because there is a positive probability that C will default on the derivative trade. As expected, the surface is zero at maturity because at maturity, B is not exposed to any credit risk from C . As the stock decreases, the magnitude of the CVA increases. This is due to the fact that when the stock price decreases, then the put option moves deeper into-the-money. If the derivative moves deeper into-the-money, then B will have more exposure to C . The more B is exposed to C , the more CVA will be charged.

Figure 4.8 illustrates the funding cost adjustment surface for a European put option. The surface is negative as this represents the funding cost that B passes on to C . In essence, B will pay less for the put option because it incurs a funding cost from funding the premium of the trade. As the derivative moves deeper into-the-money, we see a higher magnitude in funding costs. The reason for this is that the premium paid for a deep in-the-money put is higher than a put that is at-the-money; therefore, more funds are needed to finance

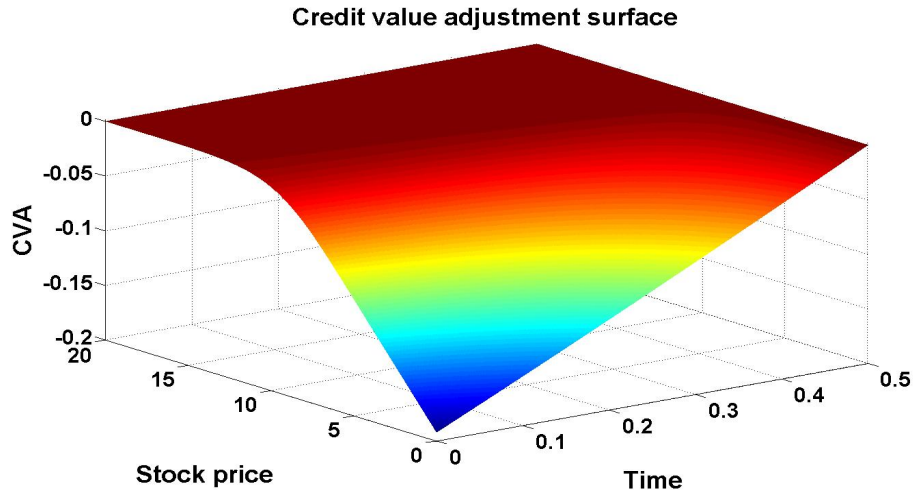


Figure 4.7: Credit value adjustment surface for a European put option with the following parameters: $S_0 \in [0, 20]$, $K = 10e^{r_R T}$, $r = 0.04$, $r_R = 0.06$, $\sigma = 0.3$, $T = 3$, $t \in [0, T]$, $r_{PB} = 0.08$, $r_{PC} = 0.08$, $R_B = 0.3$ and $R_C = 0.3$.

the premium at a funding rate.

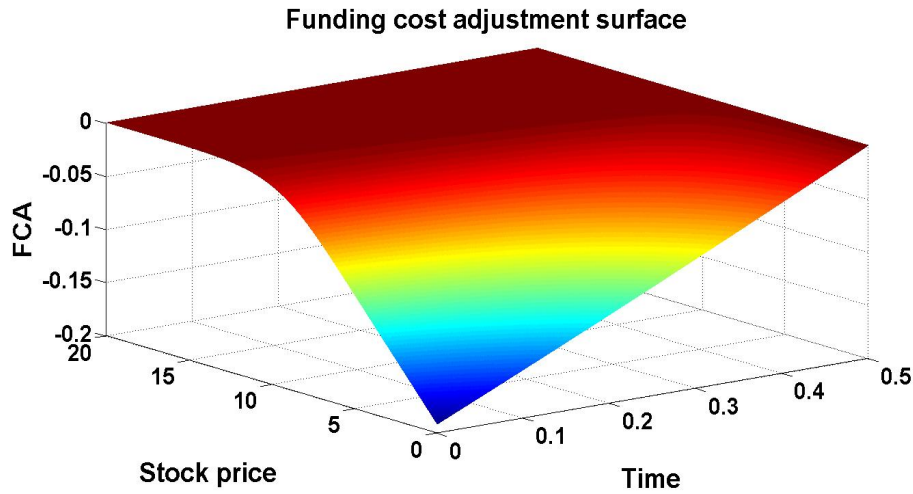


Figure 4.8: Funding cost adjustment surface for a risky European put option with parameters: $S_0 \in [0, 20]$, $K = 10e^{r_R T}$, $r = 0.04$, $r_R = 0.06$, $\sigma = 0.3$, $T = 3$, $t \in [0, T]$, $r_{PB} = 0.08$, $r_{PC} = 0.08$, $R_B = 0.3$ and $R_C = 0.3$.

Figure 4.9 illustrates the difference in price between a risk-free put option and a credit risky put option price with funding costs. Let us assume that both counterparties are banks. Prior to the 2008, crisis many banks were assumed to be risk-free; therefore, no CVA was charged to banks and the funding costs of the bank were insignificant. In this case counterparty B and C would agree on the risk-free price (blue line).

Now, let us assume B is a bank and C is a risky corporate client. Counterparty B would charge C a CVA on the trade. The price B charges C is the risk-free price plus a credit value adjustment. This price is represented by the green line in Figure 4.9. At this stage both counterparties still agree on the price of the trade.

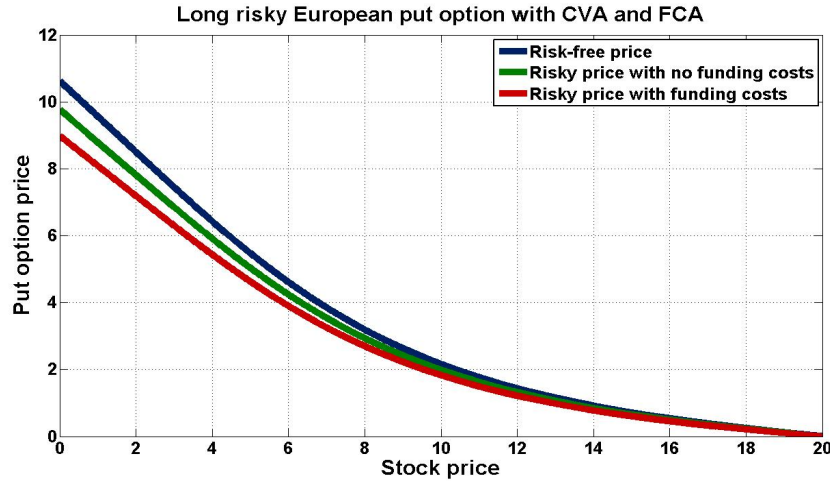


Figure 4.9: The risky price of a risky European put option with parameters: $S_0 \in [0, 20]$, $r = 0.04$, $r_R = 0.06$, $\sigma = 0.3$, $T = 3$, $r_{PB} = 0.08$, $r_{PC} = 0.08$, $K = S_0 e^{r_R T}$, $R_B = 0.3$ and $R_C = 0.3$.

Now, assume we are pricing this trade post the credit crisis. As a result of the crisis, banks are not seen as risk-free and therefore will be charged a liquidity premium for borrowing funds in the capital markets. Counterparty B will not be able to fund the premium of the trade at the risk-free rate anymore. This means B will incur a funding cost when doing the trade. This cost will be passed over to counterparty C . The price that counterparty B will offer counterparty C is the risk-free price plus a CVA and a FCA. This price is represented by the red line in Figure 4.9. The two counterparties will not agree on the price anymore. Counterparty C will not accept the funding costs that are being passed onto them. This shows that the law of one price will break down when pricing a trade with a funding cost adjustment. If a bank charges a client a funding cost adjustment, then they may lose the deal because an opposing bank may not include the funding costs into the price. In essence, if all the banks charge a funding cost to the client, then in theory *ceteris paribus* the bank with the best credit rating will snatch all the deals.

The same parameters that were used in the European put option example above are now used to price a forward contract. Figure 4.10 illustrates the price of a forward contract with a risky-close out from the point of view of counterparty B . The blue line is the risk-free price which does not need a model to compute. The green line is the risk-free price less a CVA charge. The green line is only present when the forward contract has a positive mark-to-market. This is due to the fact that positive exposure drives CVA. When the forward contract has a negative mark-to-market, then counterparty B is not exposed to C , but rather C is exposed to B . When the client is exposed to the bank, then the client will charge the bank a CVA but from the banks point of view a DVA. The red line between $S_0 \in [0, 10]$ represents the risk-free price plus a DVA charge. The red line between $S_0 \in [10, 20]$ represents the risk-free price plus CVA and FCA charges. This price is inline with current market conditions. In Figures 4.9 and 4.10, one can see that when the put and forward contract is around at-the-money, then the difference between the risk-free price and the risky price is not significant, given the parameters. We can conclude that exposure drives CVA, DVA and FCA charges.

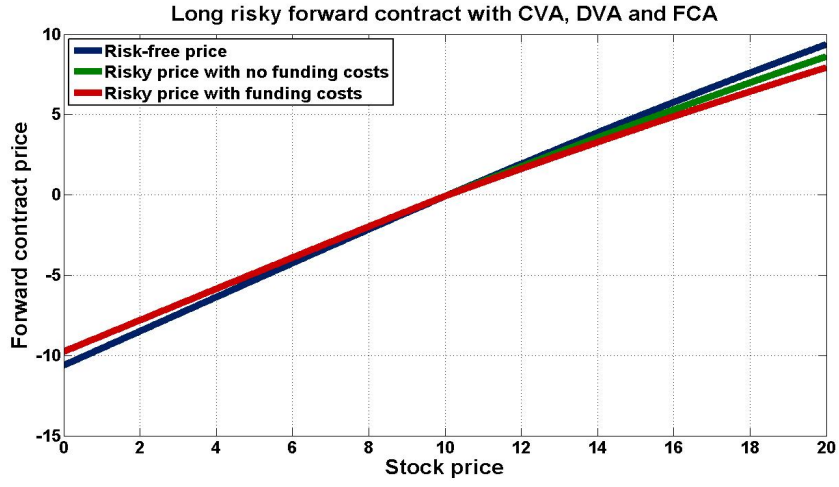


Figure 4.10: The risky price of a long forward contract with parameters: $S_0 \in [0, 20]$, $r = 0.04$, $r_R = 0.06$, $\sigma = 0.3$, $T = 3$, $r_{PB} = 0.08$, $r_{PC} = 0.08$, $K = S_0 e^{r_R T}$, $R_B = 0.3$ and $R_C = 0.3$.

Figures B.5, B.6, B.7 and B.8, which are included in the appendix, illustrate the prices for a short European put option, a short forward contract and a long and short European call option, respectively.

Chapter 5

A generalised framework which includes counterparty credit risk, funding costs and collateral

In Chapter 3 the model derived in [Piterbarg \(2010\)](#), which is used to price a collateralised derivative trade, was reviewed. In Chapter 4 the model derived in [Burgard and Kjaer \(2011\)](#), which is used to price a credit risky derivative trade with funding costs, was reviewed. The paper by [Burgard and Kjaer \(2012b\)](#), which is the focus of this chapter, extends these two models to create a general framework to price a credit risky derivative that is collateralised. This general framework, which is in the form of a PDE, reduces to the models presented in the previous two chapters if certain assumptions are made.

The PDE of the risky derivative is derived by means of a self-financing hedging portfolio. The Feynman-Kac theorem is applied to the derived PDE in order to obtain the value of the risky derivative as an expectation. The general framework is then shown to reduce to three unique models depending on the bonds available in the hedging portfolio. [Burgard and Kjaer \(2012b\)](#) only consider a risk-free close-out throughout their paper. We extend this paper by considering the risky close-out value for the general framework and each of the three subsequent models. Next, we provide a numerical implementation of the general framework, which is presented in a list of tables. The chapter concludes with a discussion on which model is the most appropriate model to use when pricing a credit risky collateralised derivative trade post the credit crisis.

5.1 Derivation of a general framework to price a collateralised derivative trade which includes counterparty credit risk

In this section, a PDE of the value of a credit risky collateralised derivative trade is derived. The trade is assumed to occur between two credit risky counterparties, namely B and C . In the derivation we assume that counterparty B is long the derivative. The PDE is derived by using a self-financing hedging portfolio. This derivation differs from that of Chapter 4. In that chapter the hedging portfolio perfectly hedges the derivative, whereas in this derivation the hedging portfolio only hedges out the derivative when counterparty B does not default. The difference between the hedging portfolio and the short derivative value when counterparty B

defaults is called the hedge error. We will show that the funding cost adjustment is exactly the discounted weighted expected value of the hedge error.

We now define further notation needed in the derivation of the PDE. In Chapter 4, counterparty B had a funding account in which excess funds were invested or a shortfall of funds were borrowed from. Positive amounts in this account would grow at the risk-free rate, whereas negative amounts in this account would grow at the funding rate. In this derivation, B does not have a funding account. All excess cash will be used to buy back B bonds and all shortfall cash is funded by issuing B bonds. In the previous chapter it was also assumed that B had one bond with yield r_{P_B} (see Section 4.4), and this bond was defined to be the difference between the risky price and the close-out value of the derivative (see equation (4.60)). We now assume that B has two bonds with different seniorities,

- P_1 is a junior issued bond with recovery $0 \leq R_1$ and yield r_1 , and
- P_2 is a senior issued bond with recovery $0 < R_2$ and yield r_2 ,

where $R_1 < R_2$. The bond price processes for these bonds are

$$dP_1 = r_1 P_1 dt - (1 - R_1) P_1 dJ_B \quad (5.1)$$

and

$$dP_2 = r_2 P_2 dt - (1 - R_2) P_2 dJ_B. \quad (5.2)$$

The relationship between the bond's yields are $r_1 > r_2$. The spread of the bond's yields over the risk-free rate is defined as

$$s_i = r_i - r, \quad \text{where } i \in \{1, 2\}.$$

From equation (4.4), the default intensity of counterparty B is

$$\lambda_B = \frac{s_i}{1 - R_i}, \quad \text{where } i \in \{1, 2\}. \quad (5.3)$$

The bond positions in the two bonds are:

- α_1 in P_1 , and
- α_2 in P_2 .

The total position in these bonds is defined as

$$P_B = \alpha_1 P_1 + \alpha_2 P_2, \quad (5.4)$$

and the default state of equation (5.4) is defined as

$$P_B^D = \alpha_1 R_1 P_1 + \alpha_2 R_2 P_2. \quad (5.5)$$

Furthermore, we assume that counterparty C only has one zero recovery bond. The bond price process for counterparty C 's zero recovery bond is given by equation (4.51). The stock price process is given by equation (4.52).

Next, the boundary conditions of the risky derivative which includes collateral are defined. The boundary conditions for a collateralised derivative trade upon default of either of the two counterparties is denoted by $g(X_t, C_t)$, where



- X_t is the close-out value, and
- C_t is value of the collateral at time of default.

The value of the risky derivative, $\hat{V}_t(S, J_B, J_C)$, is

$$\begin{aligned}\hat{V}_t(S, 1, 0) &= g_B(X_t, C_t) && \text{if } B \text{ defaults first,} \\ \hat{V}_t(S, 0, 1) &= g_C(X_t, C_t) && \text{if } C \text{ defaults first,}\end{aligned}$$

where the function, $g(X_t, C_t)$ is defined as

$$\begin{aligned}g_B(X_t, C_t) &= C_t + (X_t - C_t)^+ + R_B(X_t - C_t)^-, \\ g_C(X_t, C_t) &= C_t + (X_t - C_t)^- + R_C(X_t - C_t)^+.\end{aligned}\tag{5.6}$$

This boundary condition for a collateralised derivative trade is an extension to equation (4.3). If no collateral is posted in the trade, then equation (5.6) reduces to equation (4.3). In order to give more insight into the proposed boundary condition for a credit risky collateralised trade we discuss the various cases for equation (5.6).

- (i) Let us assume that B is in-the-money on the trade and receives collateral from C . If C defaults and the trade was

- (a) under-collateralised, i.e. $X_t - C_t > 0$, then B keeps the collateral and claims a recovery on the part that was not collateralised

$$C_t + R_C(X_t - C_t)^+.$$

- (b) over-collateralised, i.e. $X_t - C_t < 0$, then B has to give back the part that was over-collateralised

$$C_t + (X_t - C_t)^-.$$

- (ii) Let us assume that B is out-of-the-money on the trade and posts collateral to C . If C defaults and the trade was

- (a) under-collateralised, i.e. $X_t - C_t < 0$, then C can keep the collateral and B has to top up the difference

$$C_t + (X_t - C_t)^-.$$

The main idea here is that if B owes C money and C defaults, then B still owes them the money.

- (b) over-collateralised, i.e. $X_t - C_t > 0$, then B is down the collateral and B can only try to get back a recovery on the over-collateralisation part

$$C_t + R_C(X_t - C_t)^+.$$

The case when B defaults is identical to the case when C defaults.

Next, we state the generalised model as a theorem followed by its derivation.

Theorem 26:

The price of a general credit risky derivative trade satisfies the following PDE

$$\begin{cases} \frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C) \hat{V} = -\lambda_C g_C - \lambda_B g_B + \lambda_B h_e + s_C C \\ \hat{V}_T = V_T \text{ is the payoff of the derivative} \end{cases}, \quad (5.7)$$

where the parabolic differential operator is given by

$$\mathcal{A}_t = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} - r_R S \frac{\partial}{\partial S}.$$

The hedge error is defined as

$$h_e = g_B(X, C) + P_B^D - C.$$

Proof. If the hedging portfolio hedges out the derivative perfectly, then in order to avoid an arbitrage the price of the short derivative must be equal to the cost of the hedging portfolio. The hedging portfolio constructed in this proof only hedges out the derivative when counterparty B does not default.

If counterparty B does default, then B will gain a *windfall*¹. Counterparty B benefits from own default because if B is out-of-the-money on the trade and defaults, then B does not have to pay the full mark-to-market on a trade but rather only a recovery amount of the mark-to-market. We will show later that this windfall amount is equal to the hedge error of the hedging portfolio. The self-financing hedging portfolio consists of a stock position, a position in B 's bonds, a position in C 's bonds, collateral and cash,

$$\Pi_t = \Delta S_t + P_B + \alpha_C P_C - C + \gamma. \quad (5.8)$$

In Chapter 3, the collateral amount was included in the cash account, unlike the above equation. The cash account is given by

$$\gamma = -\gamma_R - \gamma_{P_C}, \quad (5.9)$$

where γ_R is the value of the stock position placed in a repurchase agreement and γ_{P_C} is the cash needed to purchase α_C number of C bonds, hence

$$\Delta S_t - \gamma_R = 0 \quad (5.10)$$

and

$$\alpha_C P_C - \gamma_{P_C} = 0. \quad (5.11)$$

Counterparty C 's bond position is financed through a repurchase agreement at the risk-free rate. The cash account is invested or borrowed in different bonds. The bond processes are given in equations (4.53) and (4.54).

Next, the parameters α_C and Δ are solved for so that the sum of the change in the derivative's price and the change in the hedging portfolio is equal to zero when counterparty B does not default. Then the hedge error is quantified.

Itô's lemma for jump diffusions is applied to the risky derivative to obtain the change in the derivative's

¹A sudden unexpected piece of good fortune or personal gain.

price, hence

$$d\hat{V} = \frac{\partial \hat{V}}{\partial t} dt + \frac{\partial \hat{V}}{\partial S} dS_t + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \hat{V}}{\partial S^2} dt + \Delta \hat{V}_B dJ_B + \Delta \hat{V}_C dJ_C, \quad (5.12)$$

where

$$\begin{aligned} \Delta \hat{V}_B &= V_t(S, 1, 0) - \hat{V}_t(S, 0, 0) = g_B(X_t, C_t) - \hat{V}_t(S, 0, 0), \\ \Delta \hat{V}_C &= V_t(S, 0, 1) - \hat{V}_t(S, 0, 0) = g_C(X_t, C_t) - \hat{V}_t(S, 0, 0). \end{aligned} \quad (5.13)$$

The self-financing portfolio condition on the hedging portfolio implies that the change in the hedging portfolio is

$$d\Pi_t = \Delta dS_t + dP_B + \alpha_C dP_C - r_C C dt - r \alpha_C P_C - \Delta S_t r_R dt.$$

Substituting equations (4.51), (5.1) and (5.2) into the above equation yields

$$\begin{aligned} d\Pi_t &= \Delta dS_t + \alpha_1(r_1 P_1 dt - (1 - R_1) P_1 dJ_B) + \alpha_2(r_2 P_2 dt - (1 - R_2) P_2 dJ_B) \\ &\quad + \alpha_C(r_{P_C} P_C dt - P_C dJ_C) - r_C C dt - r \alpha_C P_C - \Delta S_t r_R dt. \end{aligned}$$

After rearranging the terms and simplifying, the change in the hedging portfolio becomes

$$\begin{aligned} d\Pi_t &= \alpha_1 r_1 P_1 dt + \alpha_2 r_2 P_2 dt + \lambda_C \alpha_C P_C dt - r_R \Delta S_t dt - r_C C dt + \Delta dS_t \\ &\quad - \alpha_C P_C dJ_C + \alpha_1 R_1 P_1 dJ_B + \alpha_2 R_2 P_2 dJ_B - \alpha_1 P_1 dJ_B - \alpha_2 P_2 dJ_B. \end{aligned} \quad (5.14)$$

Definition of the default state of B 's bonds in equation (5.5) yields

$$\begin{aligned} d\Pi_t &= \alpha_1 r_1 P_1 dt + \alpha_2 r_2 P_2 dt + \lambda_C \alpha_C P_C dt - r_R \Delta S_t dt \\ &\quad - r_C C dt + \Delta dS_t - \alpha_C P_C dJ_C + (P_B^D - P_B) dJ_B. \end{aligned} \quad (5.15)$$

Summing the change in the hedging portfolio and the change in the derivative value and rearranging the terms gives us

$$\begin{aligned} d\Pi_t + d\hat{V}_t &= \left(\alpha_1 r_1 P_1 + \alpha_2 r_2 P_2 + \lambda_C \alpha_C P_C - r_R \Delta S_t + \frac{\partial \hat{V}}{\partial t} dt - r_C C + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \hat{V}}{\partial S^2} \right) dt \\ &\quad + \left(\Delta + \frac{\partial \hat{V}}{\partial S} \right) dS_t \\ &\quad + \left(\Delta \hat{V}_C - \alpha_C P_C \right) dJ_C \\ &\quad + \left(\Delta \hat{V}_B + P_B^D - P_B \right) dJ_B. \end{aligned} \quad (5.16)$$

The market risk and counterparty C 's credit risk can be eliminated in equation (5.16) by choosing

$$\alpha_C = \frac{\Delta \hat{V}_C}{P_C} \quad \text{and} \quad \Delta = -\frac{\partial \hat{V}}{\partial S}, \quad (5.17)$$

respectively. Then

$$\begin{aligned} d\Pi_t + d\hat{V}_t &= \left(\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} + r_1 \alpha_1 P_1 + r_2 \alpha_2 P_2 + \lambda_C \alpha_C P_C - r_C C_t \right) dt \\ &+ \left(\Delta \hat{V}_B + P_B^D - P_B \right) dJ_B, \end{aligned} \quad (5.18)$$

where the parabolic differential operator is given by

$$\mathcal{A}_t = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} - r_R S \frac{\partial}{\partial S}.$$

The hedge error is defined as the difference in value between the hedging portfolio and the short derivative when counterparty B defaults. From equations (5.8), (5.9), (5.10) and (5.11), the difference between the hedging portfolio and the short derivative when B does not default is given by

$$\Pi_t - (-\hat{V}_t) = \Pi_t + \hat{V}_t = \hat{V}_t + P_B - C_t. \quad (5.19)$$

The value of the derivative when counterparty B defaults is $g_B(X_t, C_t)$, and the value of the total B bond position when counterparty B defaults is given by equation (5.5); therefore, the hedge error is defined as

$$h_e = g_B(X_t, C_t) + P_B^D - C_t. \quad (5.20)$$

The hedge error will differ for the two close-out conventions.

Next, the PDE of the risky price of the derivative is derived. When B does not default during the life of the trade, then the change in the indicator function, dJ_B , in equation (5.18) is zero. Then equation (5.18) reduces to

$$\frac{\partial \hat{V}_t}{\partial t} + \mathcal{A}_t \hat{V} + r_1 \alpha_1 P_1 + r_2 \alpha_2 P_2 + \lambda_C \alpha_C P_C - r_C C_t = 0. \quad (5.21)$$

If the above equation holds, then the hedging portfolio and derivative will sum to zero, provided counterparty B does not default; therefore,

$$\Pi_t + \hat{V}_t = 0. \quad (5.22)$$

From equation (5.19) we have the following constraint

$$\hat{V}_t + P_B - C_t = 0. \quad (5.23)$$

This is known as the funding constraint. The intuition behind the funding constraint is that if the difference between the mark-to-market of the derivative and the posted collateral is negative, then the difference is funded by issuing B bonds. If the difference is positive, then the difference will be used to repurchase B issued bonds. If the collateral is equal to the value of the risky derivative, then the bond position will reduce to zero. If the collateral posted is zero, then the trade will be solely funded by B 's bonds. In this case, the funding rate of the hedging portfolio will be a combination of the junior and senior B bond yields.

The funding constraint is now used to simplify equation (5.21). From equations (5.3) and (5.17) and by the definition of the hedge error and the funding constraint, one can show that the last four terms of equation

(5.21) simplify to

$$r_1\alpha_1P_1 + r_2\alpha_2P_2 + \lambda_C\alpha_CP_C - r_C C_t = \lambda_C g_c + \lambda_B g_B - \lambda_B h_e - (r + \lambda_B + \lambda_C)\hat{V} - s_c C. \quad (5.24)$$

Substituting equation (5.24) into equation (5.21) and rearranging the terms, the proposed PDE for a credit risky collateralised derivative price is derived, hence

$$\begin{cases} \frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = -\lambda_C g_c - \lambda_B g_B + \lambda_B h_e + s_c C \\ \hat{V}_T = V_T \text{ is the payoff of the derivative} \end{cases}. \quad (5.25)$$

□

Comparing the PDE in equation (5.25) to the PDE in equation (4.67), we find that the PDE in equation (5.25) has two additional terms, namely $\lambda_B h_e$ and $s_c C$. The boundary values, g_B and g_C , are more general than those in the previous chapter. Burgard and Kjaer (2012b) make the comment that when counterparty B defaults, then there is a hedge error of h_e ; however, while counterparty B is still alive, B will incur a cost or gain of size $\lambda_B h_e$ per unit time.

We now show that the windfall amount is equal to the hedge error of the hedging portfolio. The windfall that counterparty B gains when it defaults is defined as the coefficient of the dJ_B term in equation (5.18). From equation (5.13) and by definition of the funding constraint, one can show that the windfall is equal to the hedge error

$$\begin{aligned} h_e &= g_B(X_t, C_t) + P_B^D - C_t \\ &= \Delta \hat{V}_B + P_B^D - P_B. \end{aligned} \quad (5.26)$$

A corollary of Theorem 26 now follows.

Corollary 27:

The price of a general credit risky collateralised derivative trade in expectation form is given by

$$\hat{V}_0 = V_0 + U_0.$$

The total value adjustment of a credit risky collateralised derivative with a risky close-out convention is

$$\begin{aligned} U_0 = & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r du} (1 - R_C) \lambda_C (\hat{V}_s - C_s)^+ ds \right] \dots CVA \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r du} (1 - R_B) \lambda_B (\hat{V}_s - C_s)^- ds \right] \dots DVA \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r du} \lambda_B h_e ds \right] \dots FCA \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r du} s_c C_s ds \right] \dots CollVA. \end{aligned} \quad (5.27)$$

The total value adjustment of a credit risky collateralised derivative with a risk-free close-out convention is

$$\begin{aligned} U_0 = & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s (r + \lambda_B + \lambda_C) du} (1 - R_C) \lambda_C (V_s - C_s)^+ ds \right] \dots CVA \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s (r + \lambda_B + \lambda_C) du} (1 - R_B) \lambda_B (V_s - C_s)^- ds \right] \dots DVA \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s (r + \lambda_B + \lambda_C) du} \lambda_B h_e ds \right] \dots FCA \\ & - \mathbb{E}_0 \left[\int_0^T s_c e^{-\int_0^s (r + \lambda_B + \lambda_C) du} C_s ds \right] \dots CollVA. \end{aligned} \quad (5.28)$$

The expectation is taken under the $\mathbb{Q}^{TR}$ -measure.

Proof. The above corollary is obtained by first deriving the PDE of the total value adjustment for both close-out conventions of a credit risk derivative with collateral and then applying the Feynman-Kac theorem to each PDE. The risky derivative can be written as the sum of the risk-free value and the total value adjustment term; therefore,

$$\hat{V}_t = V_t + U_t. \quad (5.29)$$

By making use of equation (5.29), we can write equation (5.25) as

$$\frac{\partial(V + U)}{\partial t} + \mathcal{A}_t(V + U) - r(V + U) = \lambda_C (\hat{V} - g_C) + \lambda_B (\hat{V} - g_B) + \lambda_B h_e + s_c C. \quad (5.30)$$

For the risky close-out convention, i.e. $X = \hat{V}$, the quantities $\hat{V}_s - g_C(\hat{V}_s, C_s)$ and $\hat{V}_s - g_B(\hat{V}_s, C_s)$

simplify to

$$\hat{V}_s - g_C(\hat{V}_s, C_s) = (1 - R_C)(\hat{V}_s - C_s)^+ \quad (5.31)$$

and

$$\hat{V}_s - g_B(\hat{V}_s, C_s) = (1 - R_B)(\hat{V}_s - C_s)^-, \quad (5.32)$$

respectively. Since the risk-free derivative satisfies the PDE in equation (4.69), equation (5.30) simplifies to

$$\frac{\partial U}{\partial t} + \mathcal{A}_t U - rU = \lambda_C(1 - R_C)(\hat{V} - C)^+ + \lambda_B(1 - R_B)(\hat{V} - C)^- + \lambda_B h_e + s_c C. \quad (5.33)$$

Applying the Feynman-Kac theorem to the above PDE, then the total value adjustment for a risky close-out is obtained

$$\begin{aligned} U_0 = & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r du} (1 - R_C) \lambda_C (\hat{V}_s - C_s)^+ ds \right] \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r du} (1 - R_B) \lambda_B (\hat{V}_s - C_s)^- ds \right] \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r du} \lambda_B h_e ds \right] \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r du} s_c C_s ds \right]. \end{aligned}$$

For the risk-free close-out convention, i.e. $X = \hat{V}$, the quantities $V_s - g_C(V_s, C_s)$ and $V_s - g_B(V_s, C_s)$ simplify to

$$V_s - g_C(V_s, C_s) = (1 - R_C)(V_s - C_s)^+ \quad (5.34)$$

and

$$V_s - g_B(V_s, C_s) = (1 - R_B)(V_s - C_s)^-, \quad (5.35)$$

respectively. The risk-free case follows similarly. □

The collateral value adjustment term is an adjustment made to the risk-free price when the collateral rate differs from the theoretical risk-free rate. The CollVA term is equivalent to the LVA defined in [Castagna \(2012b\)](#) and in equation (3.43). If no collateral is posted in the trade, then the CVA and DVA terms in the total value adjustment defined in equation (5.27) are equivalent to the CVA and DVA terms defined in equation (4.72). The funding cost adjustment in equations (5.27) and (5.28) is defined as the discounted weighted expected value of the hedge error. The hedge error is a function of the close-out value (see equation (5.26)); therefore, the funding cost adjustment for the two close-out values will differ. As in Chapter 4, for the risky close-out convention, the price of the derivative has a recursive formula.

5.2 Model I: Perfect hedging

The next three sections show that the general framework presented in Section 5.1 reduces to three unique models. Each model makes different assumptions about counterparty B 's bonds. In Hull and White (2012), they derive a model to price a credit risky derivative trade that contains no funding costs. If all risks are perfectly hedged, then a portfolio will grow at the risk-free rate. In the Burgard, Kjaer and Piterbarg framework, if all risks are perfectly hedged, then the windfall, h_e , gained by B when B defaults reduces to zero, and hence a funding cost adjustment will not appear in the price.

We show in this section that the Burgard, Kjaer and Piterbarg framework reduces to the model derived by Hull and White in Theorem 1 if certain unrealistic assumptions are made about counterparty B 's bonds.

Theorem 28:

The price of a credit risky collateralised derivative, when counterparty B has two liquid bonds with unique recoveries, satisfies the following PDE

$$\begin{cases} \frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C) \hat{V} = -\lambda_C g_C - \lambda_B g_B + s_c C \\ \hat{V}_T = V_T \text{ is the payoff of the derivative} \end{cases}, \quad (5.36)$$

where the parabolic differential operator is given by

$$\mathcal{A}_t = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} - r_R S \frac{\partial}{\partial S}.$$

The total value adjustment of the collateralised credit risky derivative with no funding costs and a risky close-out convention is given by

$$\begin{aligned} U_0 = & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r du} (1 - R_C) \lambda_C (\hat{V}_s - C_s)^+ ds \right] \dots \text{CVA} \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r du} (1 - R_B) \lambda_B (\hat{V}_s - C_s)^- ds \right] \dots \text{DVA} \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r du} s_c C_s ds \right] \dots \text{CollVA}. \end{aligned}$$

The total value adjustment of the collateralised credit risky derivative with no funding costs and a risk-free close-out convention is given by

$$\begin{aligned} U_0 = & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s (r + \lambda_B + \lambda_C) du} (1 - R_C) \lambda_C (V_s - C_s)^+ ds \right] \dots \text{CVA} \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s (r + \lambda_B + \lambda_C) du} (1 - R_B) \lambda_B (V_s - C_s)^- ds \right] \dots \text{DVA} \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s (r + \lambda_B + \lambda_C) du} s_c C_s ds \right] \dots \text{CollVA}. \end{aligned}$$

The expectation is taken under the $\mathbb{Q}^{TR}$ -measure.

Proof. The result is derived by solving the B bond positions that make the hedge error zero and satisfy the funding constraint, hence

$$h_e = g_B + \alpha_1 R_1 P_1 + \alpha_2 R_2 P_2 - C = 0 \quad (5.37)$$

and

$$\hat{V} + \alpha_1 P_1 + \alpha_2 P_2 - C = 0. \quad (5.38)$$

Solving for α_1 in equation (5.38) gives us

$$\alpha_1 = \frac{C - \alpha_2 P_2 - \hat{V}}{P_1}. \quad (5.39)$$

Substituting α_1 into equation (5.37) yields

$$g_B + C R_1 - \alpha_2 P_2 R_1 - \hat{V} R_1 + \alpha_2 R_2 P_2 - C = 0.$$

Solving for α_2 gives us

$$\alpha_2 = \frac{C(1 - R_1) - g_B + \hat{V} R_1}{P_2(R_2 - R_1)};$$

therefore, substituting α_2 into equation (5.39), we can solve for α_1

$$\alpha_1 = \frac{g_B - C(1 - R_2) - \hat{V} R_2}{P_1(R_2 - R_1)},$$

where $R_2 \neq R_1$, $P_1 \neq 0$ and $P_2 \neq 0$. These bond holdings reduce the hedge error to zero; therefore, all risks will be hedged. Since the hedge error reduces to zero, equation (5.25) simplifies to

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C) \hat{V} = -\lambda_C g_c - \lambda_B g_B + s_c C. \quad (5.40)$$

Because this model is equivalent to Theorem 26 when $h_e = 0$, from Corollary 27, the above theorem is proved. □

We now discuss the problems with the assumptions made on counterparty B 's bonds. If one dynamically manages their own balance sheet by trading in and out of junior and senior bonds, one could perfectly hedge out all risks. The problem is that dynamically trading ones own bonds is not realistic in practice. A firm cannot simply buy back senior issued bonds and issue more junior bonds and *vice versa* continuously in order to hedge out own credit risk. Burgard and Kjaer (2012c) discusses this idea further. In our opinion, this model is not realistic to price a collateralised credit risky derivative post the credit crisis.

5.3 Model II: Burgard and Kjaer two bond model

In this section, we show that this general framework reduces to the model derived in Chapter 4 of this dissertation when a specific B bond strategy is used. In this model, counterparty B is assumed to have two bonds,



namely a zero recovery bond and a bond with recovery R_2 , which are available in their bond strategy.

Let the zero recovery B bond be denoted by P_1 and let the bond with recovery R_2 be denoted by P_2 . The recovery of the bond P_2 is equivalent to the recovery rate of counterparty B on a derivative trade; therefore, $R_2 = R_B$. We now state the price of a collateralised credit risky derivative for this model in a theorem.

Theorem 29:

The price of a credit risky collateralised derivative, when counterparty B has one zero recovery bond and one bond with recovery R_B available to trade, satisfies the following PDE

$$\begin{cases} \frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C) \hat{V} = -\lambda_C g_C - \lambda_B g_B + \lambda_B (1 - R_B) (X - C)^+ + s_c C \\ \hat{V}_T = V_T \text{ is the payoff of the derivative} \end{cases}, \quad (5.41)$$

where the parabolic differential operator is given by

$$\mathcal{A}_t = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} - r_R S \frac{\partial}{\partial S}.$$

The total value adjustment for a credit risky derivative with a risky close-out is

$$\begin{aligned} U_0 = & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r du} (1 - R_C) \lambda_C (\hat{V}_s - C_s)^+ ds \right] \dots CVA \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r du} (1 - R_B) \lambda_B (\hat{V}_s - C_s)^- ds \right] \dots DVA \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r du} (1 - R_B) \lambda_B (\hat{V}_s - C_s)^+ ds \right] \dots FCA \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r du} s_c C_s ds \right] \dots CollVA. \end{aligned}$$

The total value adjustment for a credit risky derivative with a risk-free close-out is

$$\begin{aligned} U_0 = & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s (r + \lambda_B + \lambda_C) du} (1 - R_C) \lambda_C (V_s - C_s)^+ ds \right] \dots CVA \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s (r + \lambda_B + \lambda_C) du} (1 - R_B) \lambda_B (V_s - C_s)^- ds \right] \dots DVA \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s (r + \lambda_B + \lambda_C) du} (1 - R_B) \lambda_B (V_s - C_s)^+ ds \right] \dots FCA \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s (r + \lambda_B + \lambda_C) du} s_c C_s ds \right] \dots CollVA. \end{aligned}$$

The expectation is taken under the $\mathbb{Q}^{TR}$ -measure.

Proof. The derivation of the model first derives the value of the hedge error for the given bond strategy. The hedge error is then substituted into the the proposed PDE in equation (5.25) in order to obtain the price of the

risky derivative.

By the funding constraint in equation (5.23), we know that

$$P_B = \alpha_1 P_1 + \alpha_2 P_2 = -(\hat{V}_t - C_t). \quad (5.42)$$

In Chapter 4, the position in B 's zero recovery bond was used to hedge out B 's own credit risk. The zero recovery B bond position value was equal to the difference between the risky derivative price and the close-out value (see equation (4.60)). In this chapter the same bond strategy is employed; therefore,

$$\alpha_1 P_1 = -(\hat{V} - X). \quad (5.43)$$

From equation (5.42), the second B bond position is

$$\begin{aligned} \alpha_2 P_2 &= -(\hat{V} - C) + (\hat{V} - X) \\ &= -(X - C). \end{aligned} \quad (5.44)$$

From equations (5.5), (5.6), (5.43) and (5.44), the hedge error becomes

$$\begin{aligned} h_e &= g_B(\hat{V}, C) + P_B^D - C \\ &= C + (X - C)^+ + R_B(X - C)^- - R_B(X - C) - C \\ &= (1 - R_B)(X - C)^+. \end{aligned} \quad (5.45)$$

The hedge error is now substituted into the the proposed PDE for the general model in equation (5.25) to obtain

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C) \hat{V} = -\lambda_C g_C - \lambda_B g_B + \lambda_B (1 - R_B)(X - C)^+ + s_c C. \quad (5.46)$$

In order to obtain the total value adjustment in the form of an expectation for the two close-out values, we apply the result in Corollary 27 to equation (5.46) and substitute the hedge error into equations (5.27) and (5.28). This completes the proof. $\square$

When no collateral is posted in the derivative trade and

$$s_F = (1 - R_B)\lambda_B, \quad (5.47)$$

then the TVA expressions in Theorem 29 reduce to equations (4.72) and (4.74). This implies that the DVA term is equivalent to the FBA.

The funding cost adjustment term is the discounted weighted expected value of the hedge error; therefore, if counterparty B is always out-of-the-money, eg. a short option, then the funding cost adjustment reduces to zero, but if B is always in-the-money, eg. a long option, then the funding cost adjustment becomes the weighted expected value of the expression in equation (5.45).

The DVA term and the FCA term can be combined, for the two close-out conventions, in order to obtain

a funding value adjustment term. For the risky close-out convention the FVA term is

$$\text{FVA} = -\mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r \, du} (1 - R_B) \lambda_B (\hat{V}_s - C_s) \, ds \right],$$

and for the risk-free close-out convention the FVA term is

$$\text{FVA} = -\mathbb{E}_0 \left[\int_0^T e^{-\int_0^s (r + \lambda_B + \lambda_C) \, du} (1 - R_B) \lambda_B (V_s - C_s) \, ds \right].$$

The total value adjustment can either have a DVA and a FCA term which is known as the *asymmetric funding* approach, or it can have a FBA and a FCA term, which is known as the *symmetric funding* approach. If you include both the DVA term and the FBA term, then this would lead to double counting when pricing a derivative. The difference between the asymmetric funding approach and the symmetric funding approach will be discussed in the next chapter.

5.4 Model III: Burgard and Kjaer one bond model

In this section, we show that the general framework reduces to a third unique model. This model assumes that B only has one bond with recovery R_B . This bond will be seen as a funding account of a trader.

In our opinion, this model is the most realistic model out of the three presented in this chapter. The reason for this claim is that, when doing a trade, a trader cannot just issue bonds of the bank in order to raise cash for a trade. It is a function of the funding desk and senior management to decide whether or not to issue bonds into the market in order to raise cash to service the funding needs of the bank. The trader would request cash from the funding desk of the bank. The requested cash would be lent to the trader at a funding rate. If the trader has excess cash, then the trader will lend the cash back to the funding desk and earn a rate similar to the funding rate on the excess cash. We assume that the funding rate that the funding desk charges the trader is equal to the rate earned by the trader when the trader places funds on the funding desk. The third model is presented in the following theorem:

Theorem 30:

The price of a credit risky collateralised derivative, when counterparty B only has one bond with recovery R_B available to trade, satisfies the following PDE

$$\begin{cases} \frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C) \hat{V} = -\lambda_C g_C(X, C) - \lambda_B (1 - R_B) C - \lambda_B R_B \hat{V} + s_c C \\ \hat{V}_T = V_T \text{ is the payoff of the derivative} \end{cases},$$

where the parabolic differential operator is given by

$$\mathcal{A}_t = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} - r_R S \frac{\partial}{\partial S}.$$

The total value adjustment for a credit risky derivative with a risky close-out is

$$\begin{aligned} U_0 = & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r_F du} (1 - R_C) \lambda_C (\hat{V}_s - C_s)^+ ds \right] \dots \text{CVA} \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r_F du} (1 - R_B) \lambda_B (V_s - C_s) ds \right] \dots \text{FVA} \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r_F du} s_c C_s ds \right] \dots \text{CollVA}. \end{aligned}$$

The total value adjustment for a credit risky derivative with a risk-free close-out is

$$\begin{aligned} U_0 = & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s (r_F + \lambda_C) du} (1 - R_C) \lambda_C (V_s - C_s)^+ ds \right] \dots \text{CVA} \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s (r_F + \lambda_C) du} (1 - R_B) \lambda_B (V_s - C_s) ds \right] \dots \text{FVA} \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s (r_F + \lambda_C) du} s_c C_s ds \right] \dots \text{CollVA}. \end{aligned}$$

The expectation is taken under the $\mathbb{Q}^{r_R}$ -measure.

Proof. The hedge error for this model is derived first. Counterparty B only has one bond with recovery R_B . This bond can be seen as the funding account of a trader; however, we refer to it as a bond. The recovery rate R_B will be the recovery rate of the bank. Since counterparty B is assumed to only have one bond, the bond position in P_1 is zero, hence

$$\alpha_1 P_1 = 0. \quad (5.48)$$

From equation (5.4) and the funding constraint, the value of the B bond position is

$$\alpha_2 P_2 = -(\hat{V}_t - C_t). \quad (5.49)$$

Using equations (5.5), (5.29) (5.48) and (5.49), one can show that the hedge error is

$$h_e = g_B(X, C) - (1 - R_B)C_t - R_B V_t - R_B U_t. \quad (5.50)$$

Consider the risky close-out case first. After substituting the hedge error into equation (5.25) and rearranging the terms we arrive at

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C) \hat{V} = -\lambda_C g_C(\hat{V}, C) - \lambda_B(1 - R_B)C - \lambda_B R_B \hat{V} + s_c C. \quad (5.51)$$

From equation (4.69), the total value adjustment satisfies the following PDE

$$\frac{\partial U}{\partial t} + \mathcal{A}_t U - (r + (1 - R_B)\lambda_B)U = \lambda_C (\hat{V} - g_C(\hat{V}, C)) + (1 - R_B)\lambda_B (V - C) + s_c C. \quad (5.52)$$

By applying the Feynman-Kac theorem to equation (5.52) and by making use of the result in equation (5.31), the total value adjustment of the risky price for the risky close-out convention can be written as

$$\begin{aligned} U_0 = & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r_F du} (1 - R_C) \lambda_C (\hat{V}_s - C_s)^+ ds \right] \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r_F du} (1 - R_B) \lambda_B (V_s - C_s) ds \right] \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s r_F du} s_c C_s ds \right]. \end{aligned}$$

The total value adjustment for a risky close-out is a function of both the risky value and the risk-free value. The FVA term is a function of the risk-free price only.

We now consider the risk-free close-out case. Equation (5.52) can be written as

$$\frac{\partial U}{\partial t} + \mathcal{A}_t U - (r + (1 - R_B)\lambda_B + \lambda_C)U = \lambda_C (V - g_C(V, C)) + (1 - R_B)\lambda_B (V - C) + s_c C. \quad (5.53)$$

By applying the Feynman-Kac theorem to equation (5.53) and by making use of the result in equation (5.34), the total value adjustment of the risky price for the risk-free close-out convention can be written as

$$\begin{aligned} U_0 = & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s (r_F + \lambda_C) du} (1 - R_C) \lambda_C (V_s - C_s)^+ ds \right] \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s (r_F + \lambda_C) du} s_F (V_s - C_s) ds \right] \\ & - \mathbb{E}_0 \left[\int_0^T e^{-\int_0^s (r_F + \lambda_C) du} s_c C_s ds \right]. \end{aligned}$$

□

In this model, the funding cost adjustment does not correspond to the discounted expectation of the hedge error anymore. The discounting rate used to discount the adjustments is $r_F + \lambda_C$ rather than $r + \lambda_B + \lambda_C$, which was used in model II for the risk-free close-out. This means that the discount rate will be the funding

curve of the bank plus the default intensity of the counterparty.

We note that the FVA term for both close-out conventions is a function of the risk-free price only. Burgard and Kjaer (2012b) do not consider the risky close-out convention in their paper; therefore, the following corollary is an extension to their work.

Corollary 31:

Model II and model III are equivalent for the risky value close-out.

Proof. In model II, the zero recovery bond is equal to the difference between the risky price and the close-out value (see equation (5.43)). If the close-out value is the risky price, then the value of the zero recovery bond position is zero. In model III we assume that $\alpha_1 P_1$ is zero; therefore, the two bond strategies in models II and III for the risky close-out are equivalent. If the two bond strategies are equivalent, then the hedge error for the two models will be the same and they will result in the same model. $\square$

5.4.1 Non-recursive probabilistic solution for model II

Model II extends the model reviewed in Chapter 4 by including collateral. We now generalise the result obtained in Theorems 21 and 22 by including collateral into the PDE. The following results are not presented in Burgard and Kjaer (2012b) and is our own original idea.

Theorem 32:

The value of a credit risky derivative with a risky close-out satisfies the following PDE

$$\begin{cases} \frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r \hat{V} = \lambda_C(1 - R_C) (\hat{V} - C)^+ + \lambda_B(1 - R_B) (\hat{V} - C)^- + s_F (\hat{V} - C)^+ + s_C C, \\ V_T = \hat{V}_T \text{ is the payoff of the derivative} \end{cases}$$

where the maturity of the derivative contract is denoted by T and $t \in [0, T]$. If interest rates and hazard rates are constant and the amount of collateral posted for all time t is some fractional amount, $\theta \in [0, 1]$, of the risky price and $V_t \geq 0$ or $V_t \leq 0$ for all t , then the PDE has the following non-recursive probabilistic solution

$$\begin{aligned} \hat{V}_t = & \mathbb{I}_{\{V_t \geq 0\}} e^{-(\lambda_C(1-R_C)+s_F)(1-\theta)(T-t)} \mathbb{E}_t \left[e^{-(r+\theta s_C)(T-t)} V_T \right] \\ & + \mathbb{I}_{\{V_t < 0\}} e^{-\lambda_B(1-R_B)(1-\theta)(T-t)} \mathbb{E}_t \left[e^{-(r+\theta s_C)(T-t)} V_T \right], \end{aligned} \quad (5.54)$$

which is unique. The expectation is taken under the $\mathbb{Q}^{rR}$ -measure.

Proof. This corollary is a generalisation of the result shown in Theorem 21. The proof follows similarly to the proof presented in Theorem 21. $\square$

The above result shows that model II has a non-recursive probabilistic solution when the close-out value is the risky price and $V_t \geq 0$ or $V_t \leq 0$ for all $t \in [0, T]$. Model II reduces to model I when $s_F = 0$; hence, equation (5.54) satisfies equation (5.36) when $s_F = 0$. Furthermore, it following from Corollary 31

that model II is equivalent to model III when the close-out is the risky price; therefore, equation (5.54) also satisfies model III.

Next, we show that equation (5.54) reduces to Piterbarg's model, which was presented in Chapter 3. One of the key assumptions made in Chapter 3 was that counterparty C and B will not default during the life of the trade. If we assume that $\lambda_C = 0$ and that $s_F = (1 - R_B)\lambda_B$, then equation (5.54) reduces to

$$\begin{aligned}\hat{V}_t &= e^{-s_F(1-\theta)(T-t)} \mathbb{E}_t \left[e^{-(r+\theta s_c)(T-t)} V_T \right] (\mathbb{I}_{\{V_t \geq 0\}} + \mathbb{I}_{\{V_t < 0\}}) \\ &= e^{-(r+\theta s_c+s_F(1-\theta))(T-t)} \mathbb{E}_t [V_T].\end{aligned}\quad (5.55)$$

When $\theta = 1$, then $r + \theta s_c + s_F(1 - \theta) = r_C$, and equation (5.55) reduces to equation (3.38). When no collateral is posted in the trade, then $r + \theta s_c + s_F(1 - \theta) = r_F$, and equation (5.55) reduces to equation (3.39).

The case when the close-out value is the risk-free price is considered next.

Theorem 33:

The value of a credit risky derivative with a risk-free close-out satisfies the following PDE

$$\begin{cases} \frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C) \hat{V} = \lambda_C(1 - R_C)(V - C)^+ + \lambda_B(1 - R_B)(V - C)^- \\ \quad + s_F(V - C)^+ - (\lambda_B + \lambda_C)V \\ V_T = \hat{V}_T \text{ is the final payoff of the derivative} \end{cases},$$

where the maturity of the derivative contract is denoted by T and $t \in [0, T]$. If interest rates and hazard rates are constant and $V_t \geq 0$ or $V_t \leq 0$ for all t then the PDE has the following non-recursive probabilistic solution

$$\begin{aligned}\hat{V}_t &= \mathbb{I}_{\{V_t \geq 0\}} \left(1 - (1 - \theta)((1 - R_C)\lambda_C + s_F) \left(\frac{1 - e^{-(\lambda_B + \lambda_C)(T-t)}}{(\lambda_B + \lambda_C)} \right) \right) \mathbb{E}_t \left[e^{-(r+\theta s_c)(T-t)} V_T \right] \\ &\quad + \mathbb{I}_{\{V_t < 0\}} \left(1 - (1 - \theta)(1 - R_B)\lambda_B \left(\frac{1 - e^{-(\lambda_B + \lambda_C)(T-t)}}{(\lambda_B + \lambda_C)} \right) \right) \mathbb{E}_t \left[e^{-(r+\theta s_c)(T-t)} V_T \right],\end{aligned}\quad (5.56)$$

which is unique. The expectation is taken under the $\mathbb{Q}^{r_R}$ -measure.

Proof. This corollary is a generalisation of the result shown in Theorem 22. The proof follows similarly to the proof presented in Theorem 22. $\square$

When $s_F = 0$, then the solution in equation (5.56) will satisfy the PDE in model I. Corollary 31 does not hold for a risk-free close-out; therefore, equation (5.56) will not be a solution for the PDE in model III when the close-out is the risk-free price.

5.5 Numerical implementation

In this section, the three unique models derived in this dissertation are implemented. The models are compared by pricing a credit risky European put option. The parameters of the European put option are the same as in Chapter 4. The numerical results for the different models for the two close-out conventions are provided in a

list of tables. The section and chapter concludes with a comment on which model we recommend for pricing a credit risky derivative trade post the credit crisis.

In order to implement the models in this chapter we extend the PDE and Quasi Monte Carlo recursive algorithms (see Algorithms 23 and 25) to incorporate collateral. We now illustrate how to incorporate collateral into the PDE pricing algorithm. We only illustrate this extension for model II. The other models follow similarly. For model II, equation (5.46) can be represented as

$$\hat{V}_j = M\hat{V}_{j-1} - \lambda_C g_C \left(X_{j-1}^i, \theta \hat{V}_{j-1}^i \right) dt - \lambda_B g_B \left(X_{j-1}^i, \theta \hat{V}_{j-1}^i \right) dt + \lambda_B (1 - R_B) \left(X_{j-1}^i - \theta \hat{V}_{j-1}^i \right)^+ dt,$$

where the matrix M and the vector $\hat{V}_j$ is defined as

$$M = \begin{bmatrix} Y_1 & Z_1 & 0 & \dots & 0 \\ X_2 & Y_2 & Z_2 & \dots & 0 \\ 0 & X_3 & Y_3 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & Z_{M-1} \\ 0 & \dots & 0 & X_M & Y_M \end{bmatrix} \quad \text{and} \quad \hat{V}_j = \begin{bmatrix} \hat{V}_j^1 \\ \hat{V}_j^2 \\ \hat{V}_j^3 \\ \vdots \\ \hat{V}_j^M \end{bmatrix},$$

where

$$\begin{aligned} X_i &= \frac{1}{2} (r_R i - \sigma^2 i^2) dt, \\ Y_i &= 1 + (\sigma^2 i^2 + (r + \lambda_B + \lambda_C + \theta s_c)) dt, \\ Z_i &= -\frac{1}{2} (r_R i + \sigma^2 i^2) dt \end{aligned}$$

and the collateral amount is represented as some fractional amount, θ , of the risky price. The Monte Carlo recursive algorithm is also extended to include collateral.

We test our implementation to see whether the two algorithms still converge with the inclusion of collateral. We priced a risky derivative with a risky close-out using the two extended recursive algorithms. Figure 5.1 illustrates the price of an in-the-money forward contract with parameters: $S_0 = 10$, $r = 0.04$, $r_C = 0.05$, $r_R = 0.06$, $\sigma = 0.3$, $T = 0.5$, $r_{PB} = 0.08$, $r_{PC} = 0.08$, $K = 6$, $R_B = 0.3$, $R_C = 0.3$ and $\theta = 0.3$. The dots represent the Quasi Monte Carlo estimate when using a recursive algorithm for different sample sizes. The green line represents the most accurate Quasi Monte Carlo estimate. The red and yellow lines represent a one basis point upper and lower bound of the most accurate Quasi Monte Carlo estimate, respectively. The black line is the risky price estimate using the PDE approach with recursion. The two prices converge within a basis point for the given parameters.

We now use the PDE and Monte Carlo recursive algorithms to price derivatives that have a risky close-out and a risk-free close-out. We implement each model to price a European put option. We include Piterbarg's model from Chapter 3 for the analysis. The parameters of the European put option are: $K = 10$, $r = 0.04$, $r_C = 0.04$, $r_R = 0.06$, $r_{PC} = 0.08$, $r_F = 0.068$, $\sigma = 0.3$, $T = 0.5$, $R_B = 0.3$ and $R_C = 0.3$. The price of the European put option will be computed for three initial stock prices, namely $S_0 = \{8, 10, 12\}$. The PDE approach used 5 000 time steps and 1 000 space steps to approximate the price surface for each model. The Quasi Monte Carlo approach used 800 000 stock path simulations and 5 000 discrete time increments.

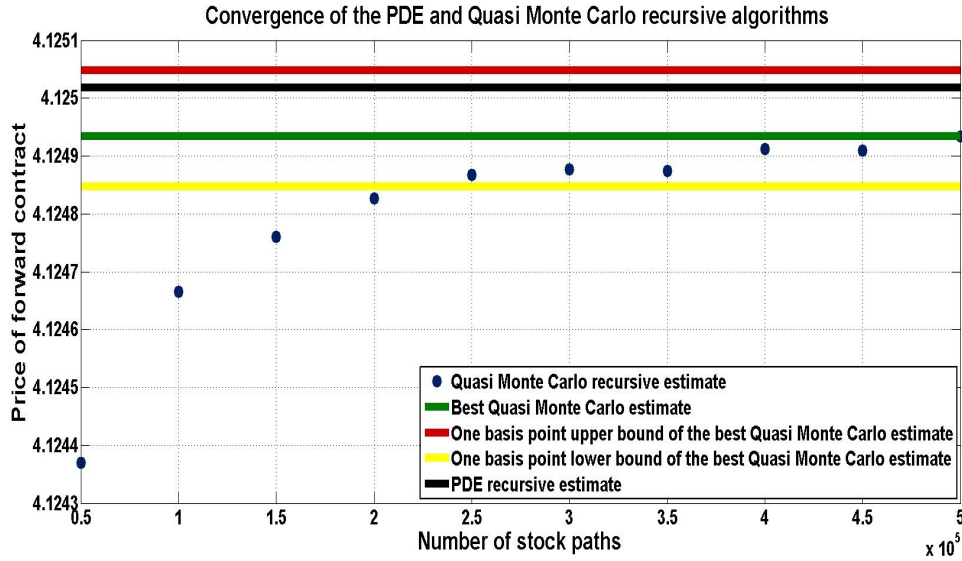


Figure 5.1: Convergence of the PDE and Quasi Monte Carlo recursive algorithms for parameters: $S_0 = 10$, $r = 0.04$, $r_C = 0.05$, $r_R = 0.06$, $\sigma = 0.3$, $T = 0.5$, $r_{PB} = 0.08$, $r_{PC} = 0.08$, $K = 6$, $R_B = 0.3$, $R_C = 0.3$ and $\theta = 0.3$.

Model	Risky close-out		Risk-free close-out	
	PDE	Monte Carlo	PDE	Monte Carlo
Piterbarg	-	-	0.689926	0.689903
Hull and White	0.689926	0.689970	0.690051	0.690095
Burgard and Kjaer two bond	0.680334	0.680374	0.680449	0.680492
Burgard and Kjaer one bond	0.680334	0.680374	0.680391	0.680435

Table 5.1: At-the-money European put option prices for the four unique models in the Burgard, Kjaer and Piterbarg framework when no collateral is posted in the trade with parameters: $S_0 = 10$, $K = 10$, $r = 0.04$, $r_C = 0.04$, $r_R = 0.06$, $r_{PC} = 0.08$, $r_F = 0.068$, $\sigma = 0.3$, $T = 0.5$, $R_B = 0.3$, $R_C = 0.3$ and $\theta = 0$.

We assume that the collateral curve is an approximation of the risk-free curve; therefore, the collateral value adjustment term in the models presented in this chapter will reduce to zero. Table 5.1 illustrates the at-the-money prices for the European put option for the four models when no collateral is posted. The price was computed two ways, using a PDE scheme and a Quasi Monte Carlo simulation. The table reflects these prices for each close-out value. Piterbarg's model does not have a close-out convention because in his model it is assumed that both counterparties do not default during the life of the trade. The price of the European put option, using Piterbarg's model, is quoted under the risk-free close-out convention.

It was shown in Corollary 31 that models II and III are equivalent when the close-out value is the risky price. Therefore, when pricing the put option using the *Burgard and Kjaer two bond* and *Burgard and Kjaer one bond* models for a risky close-out the prices are equivalent. The model presented by Hull and White does not include a funding value adjustment; however, does consider a CVA and a DVA. Both Burgard and Kjaer models include a CVA, a DVA and a FCA; therefore, the difference between the Burgard and Kjaer models and the Hull White model is a funding cost adjustment term. Hull and White's model is not realistic because

both counterparties will always pass their funding costs onto their opposite counterparty. This illustrates the difference between Hull and White's model and the models proposed by Burgard and Kjaer.

Model	Risky close-out		Risk-free close-out	
	PDE	Monte Carlo	PDE	Monte Carlo
Piterbarg	-	-	0.699654	0.699670
Hull and White	0.699654	0.699670	0.699654	0.699670
Burgard and Kjaer two bond	0.699654	0.699670	0.699654	0.699670
Burgard and Kjaer one bond	0.699654	0.699670	0.699654	0.699670

Table 5.2: At-the-money European put option prices for the four unique models in the Burgard, Kjaer and Piterbarg framework when collateral is posted in the trade with parameters: $S_0 = 10$, $K = 10$, $r = 0.04$, $r_C = 0.04$, $r_R = 0.06$, $r_{PC} = 0.08$, $r_F = 0.068$, $\sigma = 0.3$, $T = 0.5$, $R_B = 0.3$, $R_C = 0.3$ and $\theta = 1$.

Table 5.2 tabulates the prices for the European put option for each model when collateral is posted. The value of the collateral is equal to the value of the risky derivative. The prices of all the models reduce to the risk-free price. When collateral to the value of the derivative is posted continuously, then the credit risk in the trade is mitigated. We refer the reader to Section 3.4.5 to illustrate that even when collateral is adjusted at regular intervals credit risk will still play a role in the pricing of the derivative trade.

Model	Risky close-out		Risk-free close-out	
	PDE	Monte Carlo	PDE	Monte Carlo
Piterbarg	-	-	1.880422	1.880435
Hull and White	1.880422	1.880432	1.880763	1.880775
Burgard and Kjaer two bond	1.854280	1.854282	1.854593	1.854605
Burgard and Kjaer one bond	1.854280	1.854282	1.854436	1.854449

Table 5.3: In-the-money European put option prices for the four unique models in the Burgard, Kjaer and Piterbarg framework when no collateral is posted in the trade with parameters: $S_0 = 8$, $K = 10$, $r = 0.04$, $r_C = 0.04$, $r_R = 0.06$, $r_{PC} = 0.08$, $r_F = 0.068$, $\sigma = 0.3$, $T = 0.5$, $R_B = 0.3$, $R_C = 0.3$ and $\theta = 0$.

Tables 5.3 and 5.4 illustrate the in-the-money and the out-of-the-money prices for the European put option, respectively. The prices are for a put option when no collateral is posted in the trade. From the two tables one can see that the prices for the two methods converge even for in-the-money and out-of-the money prices. The prices for the PDE approach and the Quasi Monte Carlo approach converge within one basis point of each other for each model. The prices between model II and model III for the risk-free close-out are similar; however, the Hull and White model (model I) price is significantly higher than model II and III.

Model	Risky close-out		Risk-free close-out	
	PDE	Monte Carlo	PDE	Monte Carlo
Piterbarg	-	-	0.188959	0.188975
Hull and White	0.188959	0.188975	0.188993	0.189009
Burgard and Kjaer two bond	0.186331	0.186347	0.186363	0.186379
Burgard and Kjaer one bond	0.186331	0.186347	0.186347	0.186364

Table 5.4: Out-of-the-money European put option prices for the four unique models in the Burgard, Kjaer and Piterbarg framework when no collateral is posted in the trade with parameters: $S_0 = 12$, $K = 10$, $r = 0.04$, $r_C = 0.04$, $r_R = 0.06$, $r_{PC} = 0.08$, $r_F = 0.068$, $\sigma = 0.3$, $T = 0.5$, $R_B = 0.3$, $R_C = 0.3$ and $\theta = 0$.

In Appendix C.3, we provide three tables with the prices of European calls, European puts and forward contracts with varying strike prices.

We now recommend which model we believe is the most appropriate to price a general derivative post the credit crisis. From a bank's point of view, the trading desk will not absorb the funding costs associated with all their trades. This cost would negatively affect their *bottom line*. Traders are incentivised by making positive gains on trades; therefore, it is in the trader's best interests to pass the funding costs onto the bank's client. If the trader passes the entire funding cost onto the client and the other banks do not, then the trader may lose the trade to a more competitive bank. The model suggested by Hull and White is not appropriate in this day and age to price a credit risky derivative.

The Burgard and Kjaer two bond model makes the assumption that a counterparty can hedge out own default by trading their own bonds. This assumption is questionable because a firm cannot simply trade their own bonds on a daily basis in order to hedge out own default on a trade.

In our opinion, the most appropriate model to price a credit risky derivative trade post the credit crisis is the Burgard and Kjaer one bond model. In this model, no bonds are used to hedge own default. Only a funding account is used in which to place excess cash or borrow a shortfall of cash. This is more realistic. In practice, traders may not necessarily be allowed to issue a bank's bonds at will in order to raise cash. The trader raises cash through the treasury of the bank in order to fund their trading positions. Furthermore, this model only requires the funding curve of the bank and the default intensity of the counterparty. This model allows the bank to price a derivative trade with observable market interest and hazard rate curves. The total value adjustment for this model is written as the sum of the CVA term, the FVA term and the CollVA term. The DVA term does not appear in the price of the derivative; therefore, all the unusual properties of DVA do not enter into the pricing of the trade. By pricing the derivative with no DVA, but with a FBA term has accounting implications. This idea will be discussed further in the next chapter. In our view this model is the most appropriate method to price a general credit risky derivative trade with funding costs post the credit crisis, given the current literature.

Chapter 6

Conclusion

This dissertation reviewed one of three proposed frameworks in current literature to price a derivative trade post the credit crisis. This framework included quantities such as CVA, DVA, FVA and CollVA. The Piterbarg, Burgard and Kjaer framework used a hedging argument to derive the price of a collateralised credit risky derivative with funding costs. This differs to the framework derived by Brigo, Perini and Pallavicini, which prices a derivative post the credit crisis by means of discounting future expected cash flows under the risk-neutral measure. Two issues that arise with the Piterbarg, Burgard and Kjaer framework is firstly the unrealistic assumption that a firm can hedge out credit risk by buying and selling bonds. In practice this is unrealistic. However, Kjaer uses single name CDS's rather than bonds to hedge out counterparty credit risk (see [Kjaer \(2011\)](#)). The second critique is that the framework is unrealistically constrained to low dimensional problems.

The framework could be extended to be more general by considering stochastic interest rates and hazard rates. This would allow one to include *wrong way risk* and *right way risk* into the model. In this dissertation we only considered trying to model a general derivative trade; however, in practice, a bank and a counterparty may have a book of trades across multiple asset classes. On a portfolio level, the exposures between all the trades are netted and collateral is posted only against the netted exposure. This complexity is not realised on a trade level.

In our view the Brigo, Perini and Pallavicini framework is more general than the framework presented in this dissertation because, in their framework, the assumptions are less stringent. For instance, they do not assume that markets are complete, which is a more accurate state of the market in practice. One critique of their framework is when pricing a derivative trade, the price needs to be backed by a hedge; however, in their framework the composition of the hedging portfolio of the derivative trade is not apparent.

Further research work on this topic of quantitative finance may include

- Under what assumptions, if possible, will the three unique frameworks become equivalent.
- Introducing stochastic interest rates and hazard rates which are correlated with the underlying (see [Munõz \(2013\)](#)).
- Extend the Piterbarg, Burgard and Kjaer framework to price an interest rate derivative eg. cap or floor.

The focus of this dissertation was to derive a framework to price a derivative post the credit crisis from a quantitative finance point of view. Issues from an accounting and a regulatory point of view arise with the



inclusion of adjustments such as CVA, DVA and FVA. Hull and White's article in Risk Magazine placed FVA in the center stage of quantitative finance and banking. The inclusion of FVA leads to many issues, such as double counting when pricing a derivative with a DVA and a FVA. Some have gone as far as to say that DVA is being arbitrarily renamed as FBA. This is only true under very specific assumptions (see Section 5.3). At this stage there are, in our view, only two plausible approaches to pricing a derivative, namely a symmetric funding approach and an asymmetric funding approach. Gregory (2012) (pg. 296) summarises the two approaches:

- Symmetric funding and CVA implies that $TVA = CVA + FCA + FBA$. This would ignore the DVA benefit on the basis that monetisation of DVA is problematic (see Section 1). Model III takes this approach (see Section 5.4).
- Asymmetric funding and bilateral CVA implies that $TVA = CVA + DVA + FCA$. This includes DVA and considers a funding cost only. The model by Burgard and Kjaer (2011) takes an asymmetric approach (see Section 5.3).

We now consider which approach should be used from an accounting and regulatory point of view. Bilateral CVA arose from accounting practices and formally began in 2006, when the FAS¹ 157 determined that banks should record a DVA entry. FAS 157 states that

“Because non-performance risk includes the reporting entity's credit risk, the reporting entity should consider the effect of its credit risk on the fair value...”.

This means that the fair value of a trade must include both a CVA and DVA. This is in accordance with International Financial Reporting Standard (IFRS) 13 ‘Fair value measurement’, which has become effective from 1 January 2013. The asymmetric funding approach is consistent with accounting regulations; however, according to Basel III, DVA is not allowed to offset CVA when quantifying capital requirements.

From a pricing point of view the symmetric funding approach makes sense. On the one hand, one could see FBA as a way of monetising DVA. As mentioned before, DVA will only be realised once defaulted; however, with FBA, the interest on excess funds is received during the life of the trade. On the other hand, a problem with this approach is that the price is asymmetric since DVA is not included in the trade. This approach is in line with the Basel III rules for capital requirement, but is inconsistent with accounting practices.

Some banks question the use of DVA as it implies that they profit from declining credit quality and leads to hedges which may create wrong way risk and systematic risk. There are banks that see DVA more as a completely logical component, alongside CVA, which can be monetised with some difficulty. A counterparty risk and CVA survey compiled by Deloitte (2013) reports that one third of banks globally surveyed include full DVA into pricing, whilst only a quarter of the banks surveyed globally said that they give only a partial benefit. The rest of the banks either do not price with DVA, or the inclusion of DVA is planned for the future. A survey of the local South African banks compiled by KMPG (2013) showed that while local banks operating in South Africa have the capability to price the measure, it does not appear that they include it in pricing, nor do they hedge or monetise DVA.

The Deloitte (2013) bank survey also investigated which approach banks have adopted so far. The survey report presents a pie chart illustrating what percentage of banks use the following three approaches:

- CVA + DVA + symmetric funding $\approx 15\%$.

¹Financial Accounting Standards.

- CVA + symmetric funding $\approx 75\%$.
- CVA + DVA + partially asymmetric funding $\approx 10\%$.

The most popular approach is the CVA + symmetric funding approach. The banks that are using the CVA + DVA + symmetric funding approach are double counting when pricing a derivative trade. This is an extract from the bank survey

“Interestingly, it seems that banks are increasingly seeing DVA as a funding benefit and not as a benefit in the event they default. We note that whilst a bank may consider CVA + symmetric funding to be relevant, they may still refer to the funding benefit as DVA. An obvious reason for this is to remain consistent with accounting requirements.”.

In our opinion, the symmetric funding approach is the most appropriate approach to take for South African banks. According to the bank survey, if banks do not report DVA, then they should report a FBA on their balance sheet in order to reduce the minimum capital requirement set out by the Basel III committee.

The 2008 crisis has had a massive effect on derivative pricing and has plagued our markets with uncertainty. As this dissertation has highlighted, there is no general consensus about how to price a derivative trade in this day and age. This uncertainty has presented regulators, practitioners and academics with new challenges around financial markets. If all the market participants come together and share ideas, then these challenges could possibly be overcome.

Appendix A

Further derivations

A.1 Deriving the parameters for the binomial asset pricing model

Let q be the probability of an upward movement of the stock. Many different authors have slightly different views on how to approach this problem; however, all are based on the original binomial method paper by [Cox et al. \(1979\)](#). We take the approach by [Higham \(2008\)](#). The method of moment matching is used in order to solve for the parameters u and d .

Let R_i be an i.i.d *Bernoulli* random variable such that

$$R_i = \begin{cases} 1 & \text{with probability } q \\ 0 & \text{with probability } (1 - q) \end{cases},$$

and $\mathbb{E}[R_i] = q$ and $\mathbb{V}[R_i] = q(1 - q)$. If $R_i = 1$, then the stock moves up and if $R_i = 0$, then the stock moves down on the i^{th} step; therefore, the stock price at time $T = ndt$ is given by

$$S_{ndt} = S_0 u^{\sum_{i=1}^n R_i} d^{n - \sum_{i=1}^n R_i}, \quad (\text{A.1})$$

where n is the number of steps in the tree and dt is the time between steps. Rearranging equation (A.1), we have that

$$\frac{S_{ndt}}{S_0} = d^n \left(\frac{u}{d} \right)^{\sum_{i=1}^n R_i}.$$

Taking logs both sides of the equation yields

$$\log \left(\frac{S_{ndt}}{S_0} \right) = n \log(d) + \log \left(\frac{u}{d} \right) \sum_{i=1}^n R_i. \quad (\text{A.2})$$

Next, we solve for the expected value and variance of the log returns of equation (A.2), hence

$$\mathbb{E} \left[\log \left(\frac{S_{ndt}}{S_0} \right) \right] = n \log(d) + \log \left(\frac{u}{d} \right) nq, \quad (\text{A.3})$$

and

$$\mathbb{V} \left[\log \left(\frac{S_{ndt}}{S_0} \right) \right] = \log \left(\frac{u}{d} \right)^2 nq(1-q). \quad (\text{A.4})$$

Before applying the method of moment matching we first need to calculate the expected value and variance of the log returns of the stock process. The stock process is given by the following stochastic differential equation

$$\frac{dS_t}{S_t} = (r_R - r_D)dt + \sigma dW_t. \quad (\text{A.5})$$

From Itô's lemma we know that

$$d \log(S_t) = \left((r_R - r_D) - \frac{1}{2}\sigma^2 \right) dt + \sigma dW_t$$

and by integrating both sides

$$\log \left(\frac{S_t}{S_0} \right) = \left((r_R - r_D) - \frac{1}{2}\sigma^2 \right) t + \sigma W_t, \quad (\text{A.6})$$

where $W_t \sim \mathcal{N}(0, \sqrt{t})$. By taking the expected value and variance of equation (A.6) yields

$$\mathbb{E} \left[\log \left(\frac{S_t}{S_0} \right) \right] = \left((r_R - r_D) - \frac{1}{2}\sigma^2 \right) t \quad (\text{A.7})$$

and

$$\mathbb{V} \left[\log \left(\frac{S_t}{S_0} \right) \right] = \sigma^2 t. \quad (\text{A.8})$$

We now match the expected values in equations (A.3) and (A.7) and the variances in equations (A.4) and (A.8) to obtain the following two equations

$$\log(d) + \log \left(\frac{u}{d} \right) q = \left((r_R - r_D) - \frac{1}{2}\sigma^2 \right) dt, \quad (\text{A.9})$$

$$\log \left(\frac{u}{d} \right)^2 q(1-q) = \sigma^2 dt. \quad (\text{A.10})$$

We have two equations and three unknowns namely, q , u and d . We fix one of the three parameters and solve for the other two. To pick out a particular solution, the probability parameter is set to $q = \frac{1}{2}$ and then the other two parameters are solved using equations (A.9) and (A.10) hence

$$u = e^{(r_R - \frac{1}{2}\sigma^2)dt + \sigma\sqrt{dt}}(1 - \delta)$$

and

$$d = e^{(r_R - \frac{1}{2}\sigma^2)dt - \sigma\sqrt{dt}}(1 - \delta),$$

where $e^{-r_D dt} = 1 - \delta$.

Appendix B

Graphs

B.1 Convergence of the PDE approach and binomial approach when no collateral is posted

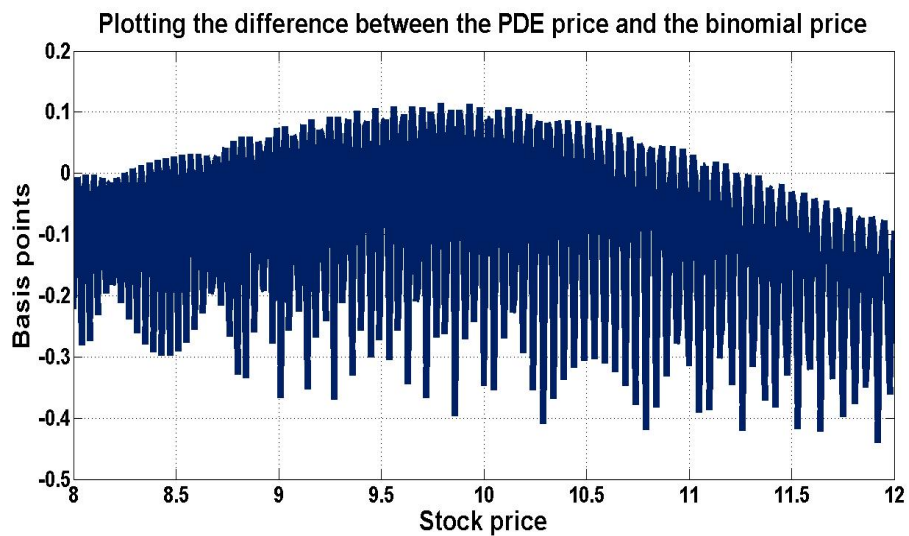


Figure B.1: Difference between the PDE price and binomial lattice price of a non-collateralised European put with parameters: $S_0 = 10$, $K = 10$, $dt = 0.0001$ (5 000 time increments), $ds = 0.02$ (1 000 space increments), $T = 0.5$, $r_C = 0.04$, $r_R = 0.05$, $r_F = 0.06$, $r_D = 0.01$, $\sigma = 0.3$ and $\theta = 0$.

B.2 Tree lattice implementation results

B.2.1 Non-default state tree

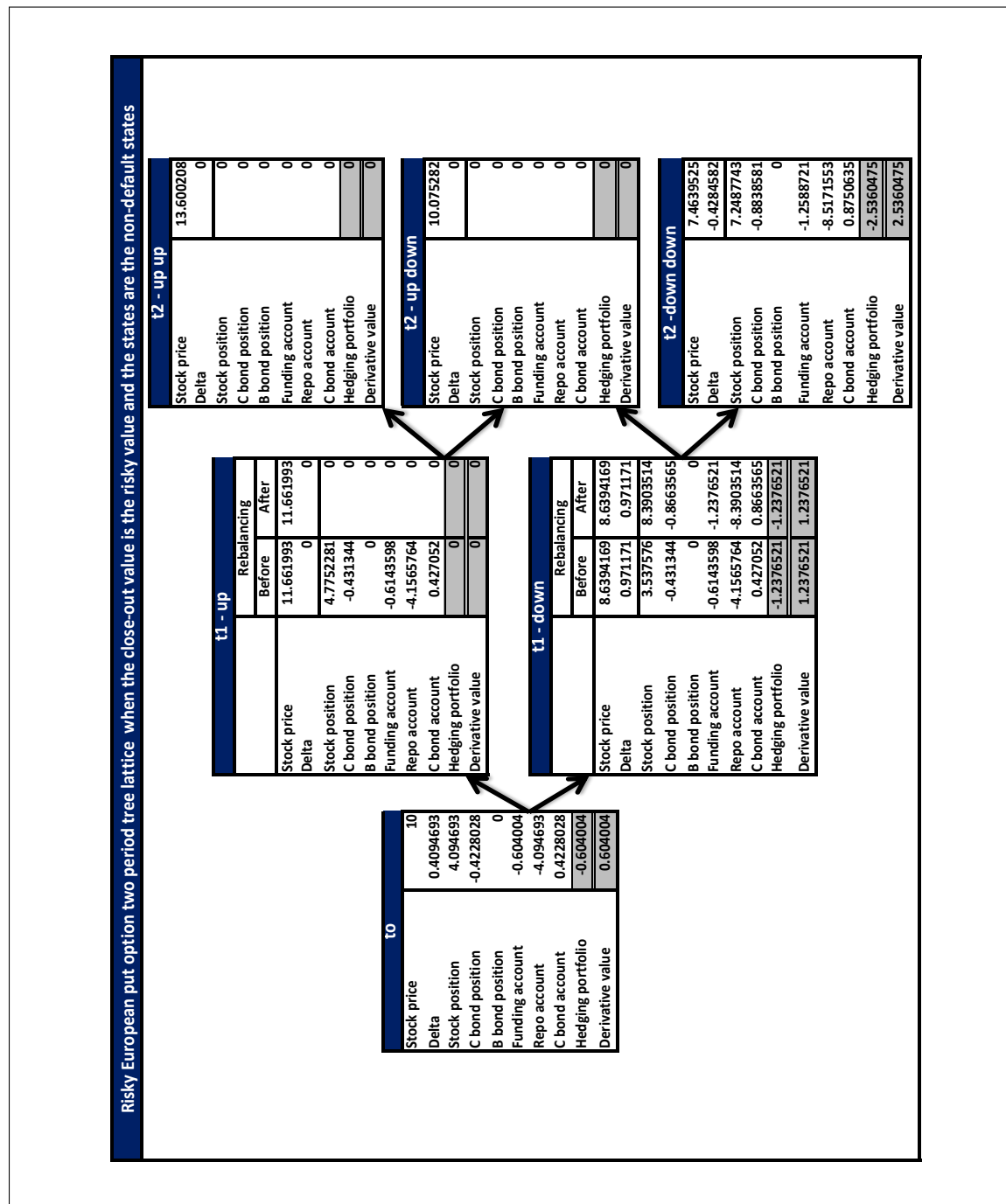


Figure B.2: Risky European put option two period tree lattice when the close-out value is the risky value and the states are the non-default states.

B.2.2 Default state tree

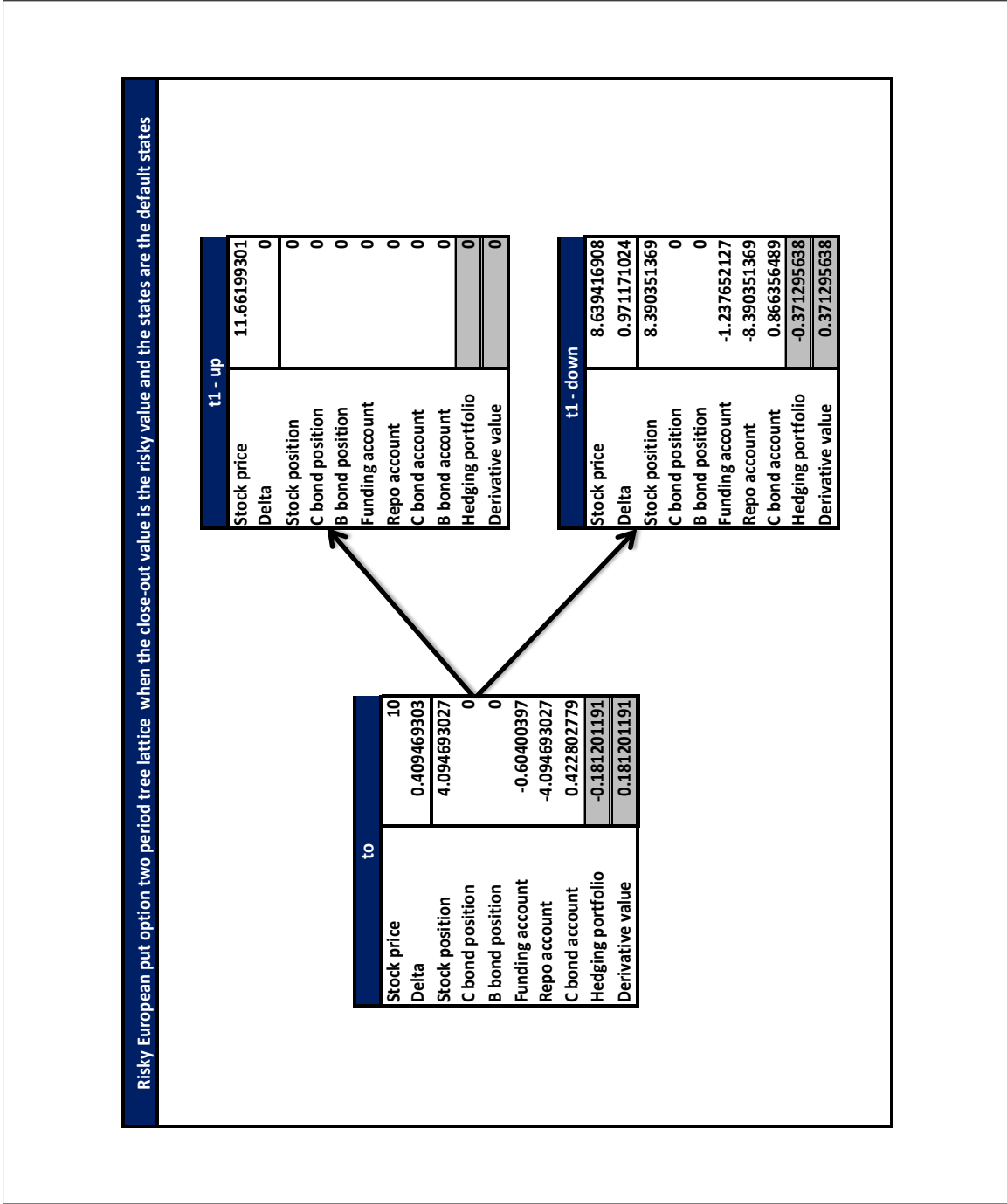


Figure B.3: Risky European put option two period tree lattice when the close-out value is the risky value and the states are the default states.

B.3 Quasi Monte Carlo simulation of a risky European put option when the close-out value is the risky price

Risky put option Quasi Monte Carlo simulation when the close-out value is the risky value											
Time	Stock price	Delta	Stock position	C bond position	Funding account	Bond account	Repo account	Total cash	Replicating portfolio	Put price	
0	10	-0.40631	-4.0631217	0.476268	0.68038293	-0.476268	4.0631217	4.2672366	0.680382926	0.680382926	
0.02	9.87205894	-0.4302	-4.0111377	0.4770307	0.7381716	-0.4770307	4.0111377	4.4715857	0.738171604	0.732410745	
0.04	9.81179286	-0.44421	-4.2209964	0.5143307	0.77019929	-0.5143307	4.2209964	4.6040866	0.770199287	0.748188203	
0.06	9.66081076	-0.47735	-4.2913905	0.5254104	0.8435363	-0.5254104	4.2913905	4.8899777	0.843536302	0.806049317	
0.08	8.67118998	-0.69199	-4.1391848	0.566043	1.32259587	-0.566043	4.1391848	6.3677704	1.322595865	1.362352363	
0.1	8.46430173	-0.7421	-5.8571843	0.9567032	1.47478808	-0.9567032	5.8571843	6.7017109	1.474788085	1.503949772	
0.12	8.35896483	-0.77142	-6.2032027	1.0561391	1.56252583	-1.0561391	6.2032027	6.9030889	1.562525831	1.579960497	
0.14	8.63981514	-0.72277	-6.6649611	1.1095171	1.35575155	-1.1095171	6.6649611	6.6385229	1.355751546	1.371835039	
0.16	8.64646114	-0.73031	-6.2493971	0.9633624	1.36030145	-0.9633624	6.2493971	6.7192926	1.360301448	1.362979403	
0.18	8.90846425	-0.68066	-6.5059482	0.9571435	1.17839389	-0.9571435	6.5059482	6.4174353	1.17839389	1.176100229	
0.2	8.68114544	-0.74216	-5.9089031	0.8259088	1.34200485	-0.8259088	5.9089031	6.8529538	1.342004854	1.329048119	
0.22	8.33190198	-0.82588	-6.1835789	0.9333155	1.61075467	-0.9333155	6.1835789	7.3730756	1.610754667	1.595726174	
0.24	8.6901002	-0.76206	-7.1769452	1.1205885	1.32537488	-1.1205885	7.1769452	7.0272568	1.325374881	1.312830627	
0.26	8.66819016	-0.77918	-6.6056382	0.9219769	1.35182108	-0.9219769	6.6056382	7.1771399	1.351821079	1.324698024	
0.28	9.39964648	-0.60151	-7.3240295	0.9302607	0.79181952	-0.9302607	7.3240295	5.8745364	0.791819517	0.81482267	
0.3	9.45660369	-0.59469	-5.6882665	0.572204	0.7654392	-0.572204	5.6882665	5.8516644	0.765439205	0.766719002	
0.32	9.71882516	-0.52445	-5.7797296	0.5384235	0.61729336	-0.5384235	5.7797296	5.2903573	0.617293357	0.604680301	
0.34	8.98365883	-0.75876	-4.7114615	0.4246329	1.00980353	-0.4246329	4.7114615	7.0848997	1.009803534	1.057314303	
0.36	9.13306241	-0.63794	-6.9297621	0	1.01036918	-0.7418984	6.8245859	7.0924629	0.163294537	0.219758309	

Figure B.4: Risky put option Quasi Monte Carlo simulation when the close-out is the risky value with parameters: $S_0 = 10$, $K = 10$, $r = 0.04$, $r_R = 0.06$, $\sigma = 0.3$, $T = 0.5$, $r_B = 0.08$, $r_{PC} = 0.08$, $R_{PB} = 0.3$ and $R_C = 0.3$.

B.4 PDE implementation for European call options, European put options and forward contracts which includes counterparty credit risk and funding costs

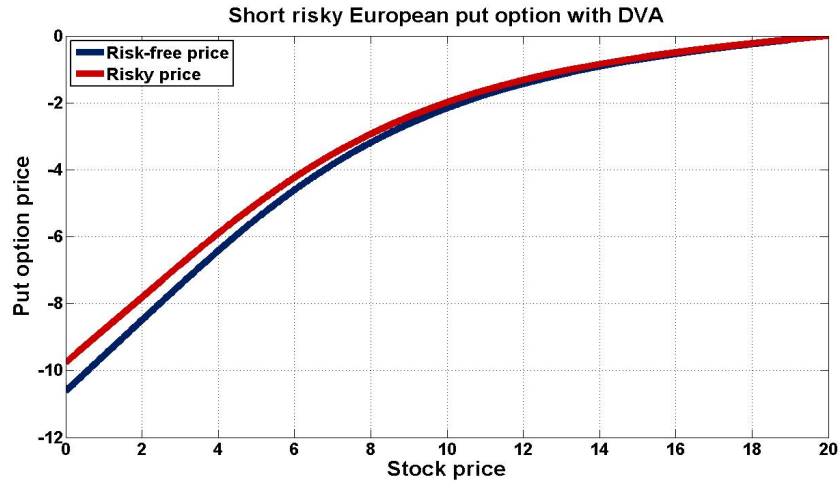


Figure B.5: The risky price of a short European put option with the following parameters: $S_0 = 10$, $r = 0.04$, $r_R = 0.04$, $\sigma = 0.3$, $T = 3$, $r_{PB} = 0.08$, $r_{PC} = 0.08$, $K = S_0 e^{r_R T}$, $R_B = 0.3$ and $R_C = 0.3$.

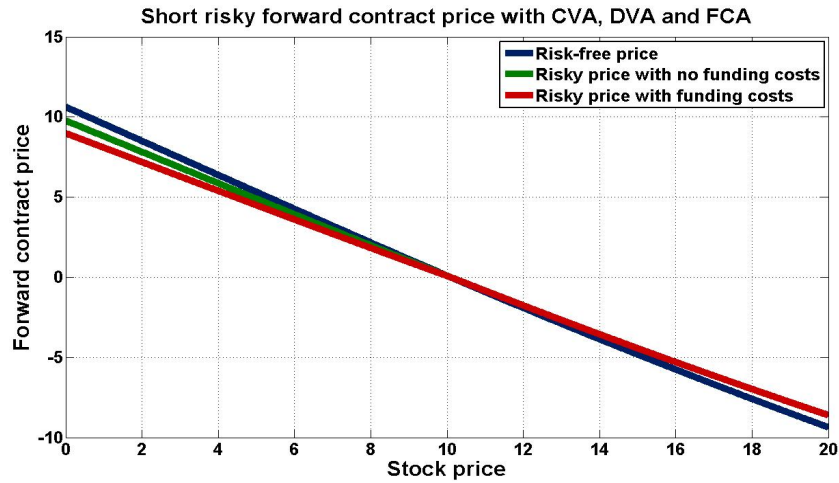


Figure B.6: The risky price of a short forward contract with parameters: $S_0 \in [0, 20]$, $r = 0.04$, $r_R = 0.06$, $\sigma = 0.3$, $T = 3$, $r_{PB} = 0.08$, $r_{PC} = 0.08$, $K = S_0 e^{r_R T}$, $R_B = 0.3$ and $R_C = 0.3$.

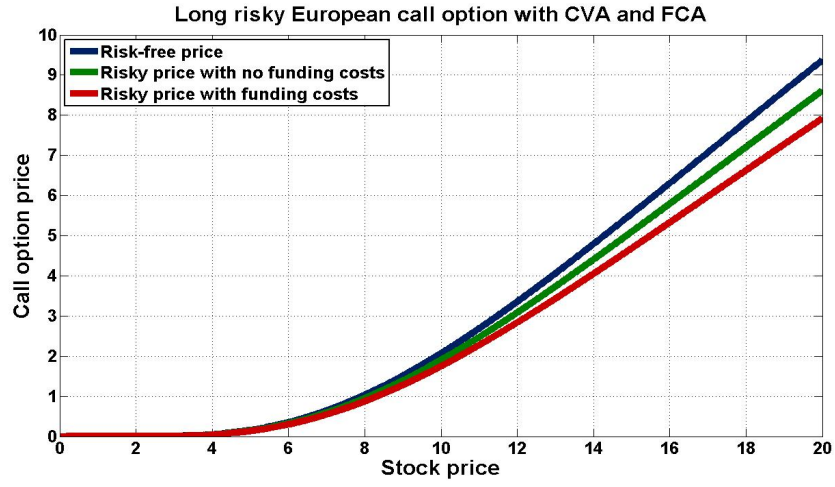


Figure B.7: The risky price of a long European call option with parameters: $S_0 \in [0, 20]$, $r = 0.04$, $r_R = 0.04$, $\sigma = 0.3$, $T = 3$, $r_{PB} = 0.08$, $r_{PC} = 0.08$, $K = S_0 e^{r_R T}$, $R_B = 0.3$ and $R_C = 0.3$.

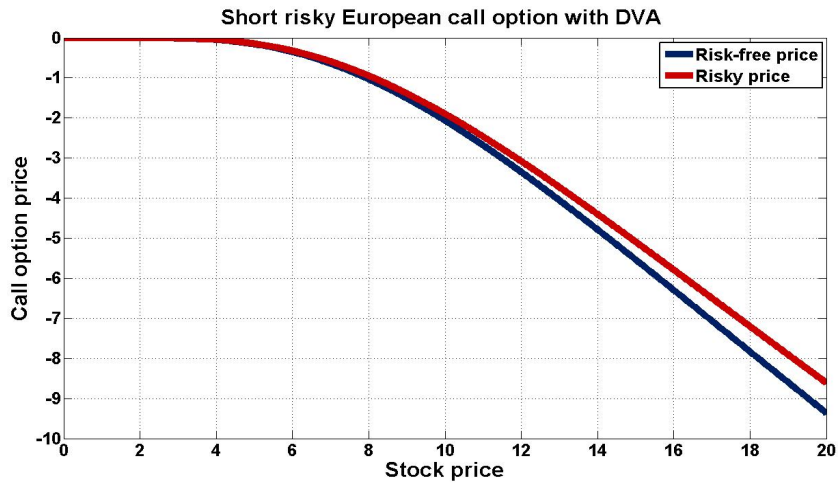


Figure B.8: The risky price of a short European call option with parameters: $S_0 \in [0, 20]$, $r = 0.04$, $r_R = 0.04$, $\sigma = 0.3$, $T = 3$, $r_{PB} = 0.08$, $r_{PC} = 0.08$, $K = S_0 e^{r_R T}$, $R_B = 0.3$ and $R_C = 0.3$.

Appendix C

Tabulated results

C.1 Prices of collateralised and non-collateralised European call options for varying strikes

European call option		PDE	Binomial model	Quasi Monte Carlo
K=2.5	full collateral	7.783695	7.783957	7.770085
	no collateral	7.331338	7.330710	7.330482
	funding cost adjustment	-0.452357	-0.453247	-0.439603
K=5	full collateral	5.650285	5.650496	5.640437
	no collateral	5.322162	5.321476	5.321323
	funding cost adjustment	-0.328123	-0.32902	-0.319114
K=7.5	full collateral	3.860938	3.861100	3.854281
	no collateral	3.637014	3.636274	3.636220
	funding cost adjustment	-0.223924	-0.224826	-0.218061
K=10	full collateral	2.549131	2.549334	2.544818
	no collateral	2.401606	2.400891	2.400841
	funding cost adjustment	-0.136248	-0.148443	-0.143978
K=12.5	full collateral	1.658999	1.659172	1.656321
	no collateral	1.563384	1.562561	1.562612
	funding cost adjustment	-0.095615	-0.096611	-0.093709
K=15	full collateral	1.076764	1.077363	1.075429
	no collateral	1.015384	1.014630	1.014585
	funding cost adjustment	-0.061380	-0.062733	-0.060844
K=17.5	full collateral	0.700794	0.702404	0.701200
	no collateral	0.662355	0.661504	0.661529
	funding cost adjustment	-0.038439	-0.040900	-0.039671

Table C.1: The prices of European call options with parameters: $S_0 = 10$, $dt = 0.0001$ (5 000 time increments), $ds = 0.02$ (1 000 space increments), $T = 0.5$, $r_C = 0.04$, $r_R = 0.05$, $r_F = 0.06$, $r_D = 0.01$ and $\sigma = 0.3$.

C.2 Prices of collateralised and non-collateralised European put options with varying strikes

European put option		PDE	Binomial model	Quasi Monte Carlo
K=2.5	full collateral	0.001293	0.001290	0.001291
	no collateral	0.001196	0.001215	0.001218
	funding cost adjustment	-0.000097	-0.000075	-0.000073
K=5	full collateral	0.085172	0.085168	0.085046
	no collateral	0.080211	0.080209	0.080234
	funding cost adjustment	-0.004961	-0.004959	-0.004812
K=7.5	full collateral	0.512630	0.513067	0.512291
	no collateral	0.482775	0.483194	0.483308
	funding cost adjustment	-0.029855	-0.029873	-0.028983
K=10	full collateral	0.741011	0.740992	0.741020
	no collateral	0.733637	0.733619	0.733676
	funding cost adjustment	-0.007374	-0.007344	-0.007314
K=12.5	full collateral	2.745815	2.745844	2.741134
	no collateral	2.585911	2.585958	2.586050
	funding cost adjustment	-0.159904	-0.159886	-0.155084
K=15	full collateral	4.380884	4.381370	4.373644
	no collateral	4.125765	4.126249	4.126700
	funding cost adjustment	-0.255119	-0.255121	-0.246944
K=17.5	full collateral	6.222219	6.223745	6.212816
	no collateral	5.859875	5.861346	5.861319
	funding cost adjustment	-0.362344	-0.362399	-0.351497

Table C.2: The prices of European put options with parameters: $S_0 = 10$, $dt = 0.0001$ (5 000 time increments), $ds = 0.02$ (1 000 space increments), $T = 0.5$, $r_C = 0.04$, $r_R = 0.05$, $r_F = 0.06$, $r_D = 0.01$ and $\sigma = 0.3$.

C.3 Prices of credit risky European call and put options and forward contracts for varying strike prices for both close-out conventions

C.3.1 Credit risky European call options with no collateral

Strike price	Model	Risky close-out		Risk-free close-out	
		PDE	Monte Carlo	PDE	Monte Carlo
K=4	Piterbarg	-	-	6.093795	6.093744
	Hull	6.093795	6.093736	6.094899	6.094848
	Burgard two bond	6.009076	6.008994	6.010091	6.010040
	Burgard one bond	6.009076	6.008994	6.009584	6.009533
K=6	Piterbarg	-	-	4.163410	4.163358
	Hull	4.163410	4.163353	4.164165	4.164113
	Burgard two bond	4.105528	4.105456	4.106222	4.106170
	Burgard one bond	4.105528	4.105456	4.105875	4.105824
K= 8	Piterbarg	-	-	2.332970	2.332945
	Hull	2.332970	2.332942	2.333393	2.333368
	Burgard two bond	2.300535	2.300499	2.300924	2.300899
	Burgard one bond	2.300535	2.300499	2.300730	2.300706
K = 10	Piterbarg	-	-	0.984263	0.984283
	Hull	0.984263	0.984282	0.984441	0.984461
	Burgard two bond	0.970577	0.970594	0.970741	0.970763
	Burgard one bond	0.970577	0.970594	0.970659	0.970681
K = 12	Piterbarg	-	-	0.314900	0.314886
	Hull	0.314900	0.314885	0.314957	0.314943
	Burgard two bond	0.310520	0.310506	0.310573	0.310560
	Burgard one bond	0.310520	0.310506	0.310547	0.310534
K = 14	Piterbarg	-	-	0.081458	0.081414
	Hull	0.081458	0.081414	0.081473	0.081429
	Burgard two bond	0.080325	0.080282	0.080339	0.080286
	Burgard one bond	0.080325	0.080282	0.080332	0.080289
K = 16	Piterbarg	-	-	0.018217	0.018178
	Hull	0.018217	0.018178	0.018221	0.018181
	Burgard two bond	0.017964	0.017925	0.017967	0.017928
	Burgard one bond	0.017964	0.017925	0.017965	0.017926
K = 18	Piterbarg	-	-	0.003711	0.003685
	Hull	0.003711	0.003685	0.003712	0.003685
	Burgard two bond	0.003660	0.003634	0.003660	0.003634
	Burgard one bond	0.003660	0.003634	0.003660	0.003634

Table C.3: European call option prices for the four unique models in the Burgard, Kjaer and Piterbarg framework with parameters: $S_0 = 10$, $r = 0.04$, $r_C = 0.04$, $r_R = 0.06$, $r_{P_C} = 0.08$, $r_F = 0.068$, $\sigma = 0.3$, $T = 0.5$, $R_B = 0.3$, $R_C = 0.3$ and $\theta = 0$.

C.3.2 Credit risky European put options with no collateral

Strike price	Model	Risky close-out		Risk-free close-out	
		PDE	Monte Carlo	PDE	Monte Carlo
K=4	Piterbarg	-	-	0.000001	0.000001
	Hull	0.000001	0.000001	0.000001	0.000001
	Burgard two bond	0.000001	0.000001	0.000001	0.000001
	Burgard one bond	0.000001	0.000001	0.000001	0.000001
K=6	Piterbarg	-	-	0.002759	0.002759
	Hull	0.002759	0.002759	0.002759	0.002759
	Burgard two bond	0.002720	0.002720	0.002721	0.002721
	Burgard one bond	0.002720	0.002720	0.002721	0.002721
K=8	Piterbarg	-	-	0.105475	0.105488
	Hull	0.105475	0.105488	0.105494	0.105507
	Burgard two bond	0.104008	0.104021	0.104025	0.104039
	Burgard one bond	0.104008	0.104021	0.104017	0.104031
K=10	Piterbarg	-	-	0.689926	0.689903
	Hull	0.689926	0.689903	0.690051	0.690095
	Burgard two bond	0.680334	0.680374	0.680449	0.680492
	Burgard one bond	0.680334	0.680374	0.680391	0.680435
K=12	Piterbarg	-	-	1.953693	1.953715
	Hull	1.953693	1.953715	1.954047	1.954070
	Burgard two bond	1.926531	1.926544	1.926857	1.926880
	Burgard one bond	1.926531	1.926544	1.926694	1.92672
K=14	Piterbarg	-	-	3.653380	3.653386
	Hull	3.653380	3.653386	3.654043	3.654049
	Burgard two bond	3.602591	3.602577	3.603199	3.603204
	Burgard one bond	3.602591	3.602577	3.602896	3.602900
K=16	Piterbarg	-	-	5.523276	5.523293
	Hull	5.523276	5.523293	5.524277	5.524294
	Burgard two bond	5.446490	5.446478	5.447410	5.447425
	Burgard one bond	5.446490	5.446478	5.446951	5.446966
K=18	Piterbarg	-	-	7.441846	7.441943
	Hull	7.441846	7.441943	7.443195	7.443292
	Burgard two bond	7.338388	7.338444	7.339627	7.339721
	Burgard one bond	7.338388	7.338444	7.339009	7.339102

Table C.4: European put option prices for the four unique models in the Burgard, Kjaer and Piterbarg framework with parameters: $S_0 = 10$, $r = 0.04$, $r_C = 0.04$, $r_R = 0.06$, $r_{P_C} = 0.08$, $r_F = 0.068$, $\sigma = 0.3$, $T = 0.5$, $R_B = 0.3$, $R_C = 0.3$ and $\theta = 0$.

C.3.3 Credit risky forward contract with no collateral

Strike price	Model	Risky close-out		Risk-free close-out	
		PDE	Monte Carlo	PDE	Monte Carlo
K=4	Piterbarg	-	-	6.093787	6.093718
	Hull	6.093743	6.093812	6.094848	6.094899
	Burgard two bond	6.009025	6.009111	6.010040	6.010124
	Burgard one bond	6.009025	6.009111	6.009533	6.009618
K=6	Piterbarg	-	-	4.160644	4.160575
	Hull	4.160600	4.160663	4.161354	4.161405
	Burgard two bond	4.102751	4.102826	4.103444	4.103518
	Burgard one bond	4.102751	4.102826	4.103098	4.103172
K=8	Piterbarg	-	-	2.227500	2.227432
	Hull	2.227457	2.227515	2.227861	2.227912
	Burgard two bond	2.196010	2.196010	2.196384	2.196449
	Burgard one bond	2.196060	2.196060	2.196197	2.196261
K=10	Piterbarg	-	-	0.294357	0.294289
	Hull	0.294314	0.294366	0.294367	0.294419
	Burgard two bond	0.284364	0.284355	0.284434	0.284491
	Burgard one bond	0.284364	0.284355	0.284388	0.284445
K=12	Piterbarg	-	-	-1.638786	-1.638854
	Hull	-1.638830	-1.638782	-1.639127	-1.639074
	Burgard two bond	-1.640719	-1.640698	-1.640998	-1.640948
	Burgard one bond	-1.640719	-1.640698	-1.640938	-1.640889
K=14	Piterbarg	-	-	-3.571929	-3.571997
	Hull	-3.571973	-3.571931	-3.572621	-3.572568
	Burgard two bond	-3.572297	-3.572260	-3.572940	-3.572897
	Burgard one bond	-3.572297	-3.572260	-3.572793	-3.572750
K=16	Piterbarg	-	-	-5.505073	-5.505140
	Hull	-5.505116	-5.505079	-5.506114	-5.506061
	Burgard two bond	-5.505168	-5.505132	-5.506166	-5.506127
	Burgard one bond	-5.505168	-5.505132	-5.505937	-5.505899
K=18	Piterbarg	-	-	-7.438216	-7.438283
	Hull	-7.438259	-7.438227	-7.439608	-7.439555
	Burgard two bond	-7.438267	-7.438236	-7.439616	-7.439583
	Burgard one bond	-7.438267	-7.438236	-7.439307	-7.439274

Table C.5: Forward contract prices for the four unique models in the Burgard, Kjaer and Piterbarg framework with parameters: $S_0 = 10$, $r = 0.04$, $r_C = 0.04$, $r_R = 0.06$, $r_{P_C} = 0.08$, $r_F = 0.068$, $\sigma = 0.3$, $T = 0.5$, $R_B = 0.3$, $R_C = 0.3$ and $\theta = 0$.

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