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ORIGINAL RESEARCH

Asymptotics of the Eigenvalues of Self-adjoint Fourth Order Boundary Value Problems

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Abstract
A regular fourth order differential equation which depends quadratically on the eigenvalue parameter λ is considered with classes of separable boundary conditions, where exactly one of the boundary conditions depends on λ linearly. This problem is described by a quadratic operator polynomial with self-adjoint operators. The location of the eigenvalues is investigated and the first four terms of the eigenvalues are provided.

Keywords Self-adjoint · Fourth order · Boundary value problem · Birkhoff regularity · Asymptotic expansions of eigenvalues

Mathematics Subject Classification 34B08 · 34B09 · 34L20

Introduction

Sturm–Liouville problems are well known and understood. In addition, they have applications in physics and engineering. Classical Sturm–Liouville problems have been extended to higher order differential equations which occur in applications with or without the eigenvalue parameter in the boundary conditions. Some recent developments of higher order problems whose boundary conditions may depend on the eigenvalue parameter can be found in [1, 3–6, 8–11, 16, 17].

The generalized Regge problem and the small transversal vibration of a homogeneous beam compressed or stretched problems, investigated in [6, 15], have boundary conditions with partial first derivatives with respect to the time variable. The fourth order problems [8, 14], the self-adjoint sixth order problem [11] and the higher order problem [9] have the same type of boundary conditions described above. The mathematical model of these problems has an operator representation of the form

$$L(\lambda)y = \lambda^2 My - i\lambda Ky - Ay \tag{1}$$

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in the Hilbert space $H = L_2(I) \oplus \mathbb{C}^k$, where I is an interval, k is the number of eigenvalue dependent boundary conditions, while M, K and A are coefficient operators.

The generalized Regge problem is realized by a second order differential operator, whereas the problem considered in [6] leads to a boundary eigenvalue problem of the form

$$y^{(4)} - (gy')' + hy = \lambda^2 y, \tag{2}$$

$$B_j(\lambda)y = 0,\ j = 1, 2, 3, 4, \tag{3}$$

where $a > 0$, $g \in C^1[0, a]$ and $h \in C[0, a]$ are real valued functions, while (3) are separated boundary conditions, taken at the endpoint 0 for $j = 1, 2$ and at the endpoint a for $j = 3, 4$. The B_j are constant or depend on λ linearly. Note that the operator realization of the problems (2), (3) is

$$L(\lambda)y = \lambda^2 My - i\alpha\lambda Ky - Ay, \quad \lambda \in \mathbb{C}, \tag{4}$$

where α is a positive constant, $M \geq 0$, $K \geq 0$ are bounded, $M + K \gg 0$, and A is bounded below with compact resolvent.

In [6], one particular set of boundary conditions has been considered, whose operator realization has self-adjoint operators. In [8], necessary and sufficient conditions have been obtained for a class of boundary conditions for which the problem (2), (3) has the representation (4) as a pencil with self-adjoint operators. For a subclass of such boundary conditions which lead to self-adjoint operators in the operator pencil (4), we have derived asymptotic expansions of the eigenvalues in [10].

In this paper, we extend our previous work to a class of boundary conditions where the boundary conditions at the endpoints a differ from those in [10]. In "The quadratic operator pencil L", we construct the operator pencil and we introduce the class of boundary conditions to be considered while in "Asymptotics of eigenvalues for $g = 0$ and $h = 0$", we investigate the eigenvalues for the case $g = h = 0$. In "Birkhoff regularity", we prove that the boundary value problems under investigation are Birkhoff regular. Finally, in "Asymptotic expansions of eigenvalues", we use techniques from [2] to derive the first four terms of the eigenvalue asymptotics that we compare the first four terms of the eigenvalue asymptotics to those obtained in [10]. Note that similar techniques can be found in [12] and [13].

The Quadratic Operator Pencil *L*

We recall that the quasi-derivatives associated with (2) are given by

$$y^{[0]} = y,\ y^{[1]} = y',\ y^{[2]} = y'',\ y^{[3]} = y^{(3)} - gy',\ y^{[4]} = y^{(4)} - (gy')' + hy,$$

see [7, page 272]. In this paper, we consider a class of boundary conditions (3) determined by:

$$B_j(\lambda)y = y^{[p_j]}(a_j)$$

for $j \in \{1, 2, 3\}$, while

$$B_4(\lambda)y = y^{[p_4]}(a_4) - i\alpha\lambda y^{[q_4]}(a_4),$$

where $a_j = 0$ for $j = 1, 2$ and $a_j = a$ for $j = 3, 4$, $\alpha > 0$. We will assume that the numbers p_1, p_2 as well as the numbers p_3, p_4, q_4 are mutually disjoint and $\{p_4, q_4\} = \{0, 3\}$.

In applications using separation of variables, the parameter λ emanates from derivatives with respect to the time variable in the partial differential equation and its boundary conditions. Hence, it is appropriate that the highest space derivative occurs in a term without time derivative. Therefore, the most relevant λ-dependent boundary condition would have $q_4 < p_4$, so that $q_4 = 0$ and $p_4 = 3$. Further assumptions on p_1, p_2, p_3, p_4 and q_4 will be made later according to the classification given in [8].

We denote by U the collection of the boundary conditions (3) and consider the linear operators $A(U)$, K and M in the space $L_2(0, a) \oplus \mathbb{C}$ with domains

$$\mathscr{D}(A(U)) = \left\{ \widetilde{y} = \begin{pmatrix} y \\ -y(a) \end{pmatrix} : \ y \in W_4^2(0, a),\ y^{[p_j]}(a_j) = 0 \text{ for } j = 1, 2, 3 \right\}$$
$$\mathscr{D}(K) = \mathscr{D}(M) = L_2(0, a) \oplus \mathbb{C},$$

given by

$$(A(U))\widetilde{y} = \begin{pmatrix} y^{[4]} \\ y^{[3]}(a) \end{pmatrix} \text{ for } \widetilde{y} \in \mathscr{D}(A(U)),\ K = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } M = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

It is easy to check that $K \geq 0$, $M \geq 0$, $M + K = I$ and $M|_{\mathscr{D}(A(U))} > 0$. We associate the quadratic operator pencil

$$L(\lambda, \alpha) = \lambda^2 M - i\alpha\lambda K - A(U), \quad \lambda \in \mathbb{C}, \tag{5}$$

in the space $L_2(0, a) \oplus \mathbb{C}$ with the problem (2), (3). We observe that (5) is an operator representation of the eigenvalue problem (2), (3) in the sense that a function y satisfies (2), (3) if and only if $L(\lambda, \alpha)\widetilde{y} = 0$ with $\widetilde{y}^{\top} = (y, -y(a))$.

The conditions under which the differential operator $A(U)$ is self-adjoint are given in

Theorem 2.1 ([8], Theorem 1.2) *Denote by P_0 the set of p in $y^{[p]}(0) = 0$ for the λ-independent boundary conditions and by P_a the corresponding set for $y^{[p]}(a) = 0$. Then the differential operator $A(U)$ associated with this boundary value problem is self-adjoint if and only if $p + q = 3$ for all boundary conditions of the form $y^{[p]}(a_j) + i\alpha\varepsilon_j\lambda y^{[q]}(a_j) = 0$ and $\varepsilon_j = 1$ if q is even in case $a_j = 0$ or odd in case $a_j = a$, $\varepsilon_j = -1$ otherwise, $\{0, 3\} \not\subset P_0$, $\{1, 2\} \not\subset P_0$, $\{0, 3\} \not\subset P_a$ and $\{1, 2\} \not\subset P_a$.*

We have to observe that $h = 0$ in [8, Theorem 1.2]. However, it is easy to see that [8, Theorem 1.2] also holds in case $h \neq 0$.

Proposition 2.2 *The operator pencil $L(\cdot, \alpha)$ is a Fredholm valued operator function with index 0. The spectrum of the Fredholm operator $L(\cdot, \alpha)$ consists of discrete eigenvalues of finite multiplicities and all eigenvalues of $L(\cdot, \alpha)$, $\alpha \geq 0$, lie in the closed upper half-plane and on the imaginary axis and are symmetric with respect to the imaginary axis.*

Proof As in [6, Section 3], we can argue that for all $\lambda \in \mathbb{C}$, $L(\lambda, \alpha)$ is a relatively compact perturbation of $L(0, 0)$, where $L(0, 0)$ is well known to be a Fredholm operator. The statement on the location of the spectrum now follows as in [6, Lemma 3.1]. $\square$

It follows from Theorem 2.1 that we have eight different cases of boundary conditions $B_j(\lambda)y = 0$. These boundary conditions are determined by the values of p_1, p_2 and p_3. Hence, we will consider

Case 1: $p_1 = 0$, $p_2 = 1$, $p_3 = 1$; **Case 2**: $p_1 = 0$, $p_2 = 1$, $p_3 = 2$;
Case 3: $p_1 = 0$, $p_2 = 2$, $p_3 = 1$; **Case 4**: $p_1 = 0$, $p_2 = 2$, $p_3 = 2$;
Case 5: $p_1 = 1$, $p_2 = 3$, $p_3 = 1$; **Case 6**: $p_1 = 1$, $p_2 = 3$, $p_3 = 2$;
Case 7: $p_1 = 2$, $p_2 = 3$, $p_3 = 1$; **Case 8**: $p_1 = 2$, $p_2 = 3$, $p_3 = 2$.

The corresponding boundary operators are then

$$B_1y = y(0) \text{ and } B_2y = y'(0) \qquad \text{(Cases 1 and 2)}, \tag{6}$$

$$B_1y = y(0) \text{ and } B_2y = y''(0) \qquad \text{(Cases 3 and 4)}, \tag{7}$$

$$B_1y = y'(0) \text{ and } B_2y = y^{[3]}(0) \qquad \text{(Cases 5 and 6)}, \tag{8}$$

$$B_1y = y''(0) \text{ and } B_2y = y^{[3]}(0) \qquad \text{(Cases 7 and 8)}, \tag{9}$$

$$B_3y = y'(a) \qquad \text{(Cases 1,3,5,7)}, \tag{10}$$

$$B_3y = y''(a) \qquad \text{(Cases 2,4,6,8)}, \tag{11}$$

$$B_4(\lambda)y = y^{[3]}(a) - i\alpha\lambda y(a). \tag{12}$$

Asymptotics of Eigenvalues for $g = 0$ and $h = 0$

In this section, we consider the boundary value problem (2), (3) with $g = h = 0$. We count all eigenvalues with their algebraic multiplicities and develop a formula for the asymptotic distribution of the eigenvalues, which is used to obtain the corresponding formula for general g and h. Observe that for $g = h = 0$, the quasi-derivatives $y^{[j]}$ coincide with the standard derivatives $y^{(j)}$. We take the canonical fundamental system $y_j(\cdot, \lambda)$, $j = 1, \ldots, 4$, of (2) with $y_j^{(m)}(0, \lambda) = \delta_{j,m+1}$ for $m = 0, \ldots, 3$. It is well known that the functions $y_j(\cdot, \lambda)$ are analytic on $\mathbb{C}$ with respect to λ. Putting

$$M(\lambda) = (B_i(\lambda)y_j(\cdot, \lambda))_{i,j=1}^4,$$

the eigenvalues of the boundary value problem (2), (3) are the eigenvalues of the analytic matrix function M, where the corresponding geometric and algebraic multiplicities coincide, see [2, Theorem 6.3.2].

Setting $\lambda = \mu^2$ and

$$y(x, \mu) = \frac{1}{2\mu^3} \sinh(\mu x) - \frac{1}{2\mu^3} \sin(\mu x)$$

it is easy to see that

$$y_j(x,\lambda) = y^{(4-j)}(x,\mu), \quad j = 1,\dots,4.$$

The first two rows of $M(\lambda)$ have exactly one entry 1 and all the other entries zero. Hence, an expansion of $M(\lambda)$ shows that $\det M(\lambda) = \pm\phi(\mu)$, where

$$\phi(\mu) = \det\begin{pmatrix} B_3(\mu^2)y_{\sigma(1)}(\cdot,\mu) & B_3(\mu^2)y_{\sigma(2)}(\cdot,\mu) \\ B_4(\mu^2)y_{\sigma(1)}(\cdot,\mu) & B_4(\mu^2)y_{\sigma(2)}(\cdot,\mu) \end{pmatrix},$$

with

$$(\sigma(1),\sigma(2)) = \begin{cases} (3,4) & \text{in Cases 1 and 2,} \\ (2,4) & \text{in Cases 3 and 4,} \\ (1,3) & \text{in Cases 5 and 6,} \\ (1,2) & \text{in Cases 7 and 8.} \end{cases}$$

In view of (10), (11) and (12) this gives

$$\begin{aligned}\phi(\mu) = i\alpha\mu^2\Big(&y_{\sigma(1)}(a)B_3y_{\sigma(2)}(a,\mu) - y_{\sigma(2)}(a)B_3y_{\sigma(1)}(a,\mu)\Big)\\ &+ y^{(3)}_{\sigma(2)}(a)B_3y_{\sigma(1)}(a,\mu) - y^{(3)}_{\sigma(1)}(a)B_3y_{\sigma(2)}(a,\mu).\end{aligned}$$

Each of the summands in ϕ is a product of a power in μ and a product of two sums of a trigonometric and a hyperbolic functions. The term with the highest μ-power in $\phi(\mu)$ occurs with

$$y^{(3)}_{\sigma(2)}(a)B_3y_{\sigma(1)}(a,\mu) - y^{(3)}_{\sigma(1)}(a)B_3y_{\sigma(2)}(a,\mu).$$

Hence, we are going to investigate the zeros of

$$\phi_0(\mu) = 2\Big[y^{(3)}_{\sigma(2)}(a)B_3y_{\sigma(1)}(a,\mu) - y^{(3)}_{\sigma(1)}(a)B_3y_{\sigma(2)}(a,\mu)\Big].$$

In the above eight cases we obtain:

Case 1: $p_1 = 0$, $p_2 = 1$, $p_3 = 1$:

$$\begin{aligned}\phi_0(\mu) &= \frac{1}{2\mu}[(\cosh(\mu a) + \cos(\mu a))\\ &\qquad\times(\sinh(\mu a) + \sin(\mu a)) - (\sinh(\mu a) - \sin(\mu a))(\sinh(\mu a) + \sin(\mu a))]\\ &= \frac{1}{\mu}(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)).\end{aligned}$$

Case 2: $p_1 = 0$, $p_2 = 1$, $p_3 = 2$:

$$\begin{aligned}\phi_0(\mu) &= \frac{1}{2}[(\cosh(\mu a) + \cos(\mu a))^2 - (\sinh(\mu a) - \sin(\mu a))(\sinh(\mu a) + \sin(\mu a))]\\ &= 1 + \cos(\mu a)\cosh(\mu a).\end{aligned}$$

Case 3: $p_1 = 0$, $p_2 = 2$, $p_3 = 1$:

$$\begin{aligned}\phi_0(\mu) &= \frac{1}{2}[(\cosh(\mu a) + \cos(\mu a))^2 - (\cosh(\mu a) - \cos(\mu a))^2]\\ &= 2\cos(\mu a)\cosh(\mu a).\end{aligned}$$

Case 4: $p_1 = 0, p_2 = 2, p_3 = 2$:

$$\begin{aligned}\phi_0(\mu) &= \frac{\mu}{2}[(\cosh(\mu a) + \cos(\mu a))(\sinh(\mu a) - \sin(\mu a)) \\ &\quad - (\cosh(\mu a) - \cos(\mu a))(\sinh(\mu a) + \sin(\mu a))] \\ &= \mu(\cos(\mu a)\sinh(\mu a) - \sin(\mu a)\cosh(\mu a)).\end{aligned}$$

Case 5: $p_1 = 1, p_2 = 3, p_3 = 1$:

$$\begin{aligned}\phi_0(\mu) &= \frac{\mu^2}{2}[(\sinh(\mu a) - \sin(\mu a))^2 - (\sinh(\mu a) + \sin(\mu a))^2] \\ &= -2\mu^2 \sin(\mu a)\sinh(\mu a).\end{aligned}$$

Case 6: $p_1 = 1, p_2 = 3, p_3 = 2$:

$$\begin{aligned}\phi_0(\mu) &= \frac{\mu^3}{2}(\sinh(\mu a) - \sin(\mu a))(\cosh(\mu a) - \cos(\mu a)) \\ &\quad - (\sinh(\mu a) + \sin(\mu a)(\cosh(\mu a) + \cos(\mu a)) \\ &= -\mu^3(\cos(\mu a)\sinh(\mu a) + \sin(\mu a)\cosh(\mu a)).\end{aligned}$$

Case 7: $p_1 = 2, p_2 = 3, p_3 = 1$:

$$\begin{aligned}\phi_0(\mu) &= \frac{\mu^3}{2}[(\cosh(\mu a) - \cos(\mu a))(\sinh(\mu a) - \sin(\mu a)) \\ &\quad - (\sinh(\mu a) + \sin(\mu a))(\cosh(\mu a) + \cos(\mu a))] \\ &= -\mu^3(\cos(\mu a)\sinh(\mu a) + \sin(\mu a)\cosh(\mu a)).\end{aligned}$$

Case 8: $p_1 = 2, p_2 = 3, p_3 = 2$:

$$\begin{aligned}\phi_0(\mu) &= \mu^4[(\cosh(\mu a) - \cos(\mu a))^2 - (\sinh(\mu a) + \sin(\mu a))(\sinh(\mu a) - \sin(\mu a))] \\ &= \mu^4(1 - \cos(\mu a)\cosh(\mu a)).\end{aligned}$$

Next, we give the asymptotic distributions of the zeros of $\phi_0(\mu)$ with their proper counting.

Lemma 3.1 **Case 1** : $p_1 = 0$, $p_2 = 1$, $p_3 = 1$: ϕ_0 *has no zero at* 0, *exactly one simple zero in each interval* $((k - \frac{1}{2})\frac{\pi}{a}, (k + \frac{1}{2})\frac{\pi}{a})$ *for positive integers k with asymptotics*

$$\tilde{\mu}_k = (4k-1)\frac{\pi}{4a} + o(1), \quad k = 1, 2, \dots,$$

simple zeros at $-\tilde{\mu}_k$, $\tilde{\mu}_{-k} = i\tilde{\mu}_k$, -$i\tilde{\mu}_k$, *for* $k = 1, 2, \dots$, *and no other zeros.*

Case 2 : $p_1 = 0$, $p_2 = 1$, $p_3 = 2$: ϕ_0 *has no zero at* 0, *exactly one simple zero in each interval* $\left[2m\frac{\pi}{a}, \left(2m + \frac{1}{2}\right)\frac{\pi}{a}\right]$, $\left[\left(2m + \frac{3}{2}\right)\frac{\pi}{a}, (2m+2)\frac{\pi}{a}\right]$, *respectively, for nonnegative integers m with asymptotics*

$$\tilde{\mu}_k = (2k-3)\frac{\pi}{2a} + o(1), \quad k = 2, 3, \dots,$$

simple zeros at $-\tilde{\mu}_k$, $\tilde{\mu}_{-k} = i\tilde{\mu}_k$, -$i\tilde{\mu}_k$, *for* $k = 1, 2, \dots$, *and no other zeros.*

Case 3 : $p_1 = 0$, $p_2 = 2$, $p_3 = 1$: ϕ_0 *has no zero at* 0, *simple zeros*

$$\tilde{\mu}_k = (2k-1)\frac{\pi}{2a}, \quad k = 1, 2, \dots,$$

simple zeros at $-\tilde{\mu}_k$, $\tilde{\mu}_{-k} = i\tilde{\mu}_k$, -$i\tilde{\mu}_k$, *for* $k = 1, 2, \dots$, *and no other zeros.*

Case 4 : $p_1 = 0$, $p_2 = 2$, $p_3 = 2$: ϕ_0 *has a zero of multiplicity four at* 0, *exactly one simple zero in each interval* $((k-\frac{1}{2})\frac{\pi}{a}, (k+\frac{1}{2})\frac{\pi}{a})$ *for positive integers k with asymptotics*

$$\tilde{\mu}_k = (4k-1)\frac{\pi}{4a} + o(1), \quad k = 1, 2, \dots,$$

simple zeros at $-\tilde{\mu}_k$, $\tilde{\mu}_{-k} = i\tilde{\mu}_k$, -$i\tilde{\mu}_k$, *for* $k = 1, 2, \dots$, *and no other zeros.*

Case 5 : $p_1 = 1$, $p_2 = 3$, $p_3 = 1$: ϕ_0 *has a zero of multiplicity four at* 0, *simple zeros*

$$\tilde{\mu}_k = (k-1)\frac{\pi}{a}, \quad k = 2, 3, \dots,$$

simple zeros at $-\tilde{\mu}_k$, $\tilde{\mu}_{-k} = i\tilde{\mu}_k$, -$i\tilde{\mu}_k$, *for* $k = 2, 3, \dots$, *and no other zeros.*

Case 6 : $p_1 = 1$, $p_2 = 3$, $p_3 = 2$: ϕ_0 *has a zero of multiplicity four at* 0, *exactly one simple zero in each interval* $\left(\left(k-\frac{1}{2}\right)\frac{\pi}{a}, \left(k+\frac{1}{2}\right)\frac{\pi}{a}\right)$ *for positive integers k with asymptotics*

$$\tilde{\mu}_k = (4k-5)\frac{\pi}{4a} + o(1), \quad k = 2, 3, \dots,$$

simple zeros at $-\tilde{\mu}_k$, $\tilde{\mu}_{-k} = i\tilde{\mu}_k$, $-i\tilde{\mu}_k$, *for* $k = 2, 3, \dots$, *and no other zeros.*

Case 7 : $p_1 = 2$, $p_2 = 3$, $p_3 = 1$: ϕ_0 *has a zero of multiplicity four at* 0, *exactly one simple zero in each interval* $\left(\left(k-\frac{1}{2}\right)\frac{\pi}{a}, \left(k+\frac{1}{2}\right)\frac{\pi}{a}\right)$ *for positive integers k with asymptotics*

$$\tilde{\mu}_k = (4k-5)\frac{\pi}{4a} + o(1), \quad k = 2, 3, \dots,$$

simple zeros at $-\tilde{\mu}_k$, $\tilde{\mu}_{-k} = i\tilde{\mu}_k$, -$i\tilde{\mu}_k$, *for* $k = 2, 3, \dots$, *and no other zeros.*

Case 8 : $p_1 = 2$, $p_2 = 3$, $p_3 = 2$: ϕ_0 *has a zero of multiplicity eight at* 0, *exactly one simple zero in each interval* $\left[2m\frac{\pi}{a}, \left(2m+\frac{1}{2}\right)\frac{\pi}{a}\right]$, $\left[\left(2m+\frac{3}{2}\right)\frac{\pi}{a}, (2m+2)\frac{\pi}{a}\right]$, *respectively, for nonnegative integers m with asymptotics*

$$\tilde{\mu}_k = (2k-5)\frac{\pi}{2a} + o(1), \quad k = 2, 3, \dots,$$

simple zeros at $-\tilde{\mu}_k$, $\tilde{\mu}_{-k} = i\tilde{\mu}_k$, -$i\tilde{\mu}_k$, *for* $k = 2, 3, \dots$, *and no other zeros.*

Proof The result is obvious in Case 3 and Case 5.

For Case 6, it is easy to see that ϕ_0 has a zero of multiplicity 4 at 0. Next, we are going to find the zeros of ϕ_0 on the positive real axis. One can observe that for $\mu \neq 0$, $\phi_0(\mu) = 0$ implies $\cosh(\mu a) \neq 0$ and $\cos(\mu a) \neq 0$, whence the positive zeros of ϕ_0 are those $\mu > 0$ for which $\tan(\mu a) + \tanh(\mu a) = 0$. Since $\tan'(\mu a) \geq 1$ and $\tanh'(\mu a) \geq 0$ for $\mu \in \mathbb{R}$, the function $\mu \mapsto \tan(\mu a) + \tanh(\mu a)$ is increasing with positive derivative on each interval $\left(\left(k-\frac{1}{2}\right)\frac{\pi}{a}, \left(k+\frac{1}{2}\right)\frac{\pi}{a}\right)$, $k \in \mathbb{Z}$. On each of these intervals, the function moves from $-\infty$ to ∞, thus we have exactly one simple zero $\tilde{\mu}_k$ of $\tan(\mu a) + \tanh(\mu a)$ in each interval

$\left(\left(k-\frac{1}{2}\right)\frac{\pi}{a}, \left(k+\frac{1}{2}\right)\frac{\pi}{a}\right)$, where k is a positive integer, and no zero in $\left(0, \frac{\pi}{2a}\right)$. Since $\tanh(\mu a) \to 1$ as $\mu \to \infty$, we have

$$\tilde{\mu}_k = (4k-5)\frac{\pi}{4a} + o(1),\ k = 2, 3, \ldots .$$

The location of the zeros on the other three half-axes follows by repeated application of $\phi_0(i\mu) = -\phi_0(\mu)$.

The proof will be complete if we show that all zeros of ϕ_0 lie on the real or the imaginary axis. To this end we observe that the product-to-sum formula for trigonometric functions gives

$$\begin{aligned} \phi_0(\mu) &= -\frac{1}{2}\mu^3\left[\cosh(\mu a)\sin(\mu a) + \sinh(\mu a)\cos(\mu a)\right] \\ &= -\frac{1}{4}\mu^3\Big[\sin((1+i)\mu a) + \sin((1-i)\mu a) - i\sin((1+i)\mu a) \\ &\qquad + i\sin((1-i)\mu a)\Big] \\ &= -\frac{1}{4}\mu^3\left[(1-i)\sin((1+i)\mu a) + (1+i)\sin((1-i)\mu a))\right]. \end{aligned} \tag{13}$$

Putting $(1+i)\mu a = x + iy$, $x, y \in \mathbb{R}$, it follows for $\mu \neq 0$ that

$$\begin{aligned} \phi_0(\mu) = 0 &\Rightarrow |\sin((1+i)\mu a)| = |\sin((1-i)\mu a)| \\ &\Leftrightarrow |\sin(x+iy)| = |\sin(y-ix)| \\ &\Leftrightarrow \cosh^2 y - \cos^2 x = \cosh^2 x - \cos^2 y \\ &\Leftrightarrow \cosh^2(|y|) + \cos^2(|y|) = \cosh^2(|x|) + \cos^2(|x|). \end{aligned} \tag{14}$$

Since $\cosh^2 x + \cos^2 x = \frac{1}{2}\cosh(2x) + \frac{1}{2}\cos(2x) + 1$ has a positive derivative on $(0, \infty)$, this function is strictly increasing, and $\phi_0(\mu) = 0$ therefore implies by (14) that $|y| = |x|$ and thus $y = \pm x$. Then

$$\mu = \frac{x+iy}{(1+i)a} = \frac{1 \pm i}{1+i}\,\frac{x}{a}$$

is either real or pure imaginary.

Cases 1 and 7 easily follow from the result for Case 6. For Case 4, a power series expansion shows that ϕ_0 has a zero of multiplicity 4 at 0. For the zeros on the positive real axis we just need to replace the function $\mu \mapsto \tan(\mu a) + \tanh(\mu a)$ in the proof of Case 6 by $\mu \mapsto \tan(\mu a) - \tanh(\mu a)$ and observe that $\tanh'(\mu a) < 1$. Furthermore, in this case we have a representation of ϕ_0 similar to (13), except that on the right hand side, the factors $1 - i$ and $1 + i$ in front of the sine functions are interchanged. Hence, (14) also holds in this case, and all zeros must be real or pure imaginary.

In Case 2, it is easy to see that ϕ_0 has no zero at 0. Next, we are going to find the zeros of ϕ_0 on the positive real axis. Let $f(\mu) = \cos(\mu a)\cosh(\mu a) + 1$ and $I_{m,j} = \left[(2m + \frac{j}{2})\frac{\pi}{a}, (2m + \frac{j+1}{2})\frac{\pi}{a}\right]$, $m = 0, 1, \ldots$, $j = 0, 1, 2, 3$. The zeros of ϕ_0 are the zeros of f. It is obvious that for all m and $x \in I_{m,0} \cup I_{m,3}$, $f(\mu) \geq 1$. On $I_{m,1}$, $\mu \mapsto \cos(\mu a)$ is decreasing and negative, while $\mu \mapsto \cosh(\mu a)$ is increasing and positive, so that f is decreasing. At the endpoints of this interval, f has the values $f((2m + \frac{1}{2})\frac{\pi}{a}) = 1$ and

$f((2m+1)\frac{\pi}{a}) = -\cosh((2m+1)\pi)+1<0$. Hence, f has exactly one simple zero on $I_{m,1}$. From $f''(\mu) = -2a^2\sin(\mu a)\sinh(\mu a)$ we see that f is strictly convex on $I_{m,2}$ with

$$f\left((2m+1)\frac{\pi}{a}\right) = -\cosh((2m+1)\pi)+1<0 \text{ and } f\left(\left(2m+\frac{3}{2}\right)\frac{\pi}{a}\right) = 1.$$

Hence, f has exactly one simple zero on $I_{m,2}$. Since $-\dfrac{1}{\cosh(\mu a)} \to 0$ as $\mu \to \infty$, we have

$$\tilde{\mu}_m^1 = \left(2m+\frac{1}{2}\right)\frac{\pi}{a}+o(1) \text{ and } \tilde{\mu}_m^2 = \left(2m+\frac{3}{2}\right)\frac{\pi}{a}+o(1), \quad m=0,1,\dots.$$

The location of the zeros on the other three half-axes follows by repeated application of $\phi_0(i\mu) = \phi_0(\mu)$.

The proof for Case 2 will be complete if we show that all zeros of ϕ_0 lie on the real or the imaginary axis. To this end we observe that the operator associated with the eigenvalue problem

$$y^{(4)} = \tau y,\ y(0)=0,\ y'(0)=0,\ y''(a)=0,\ y^{(3)}(a)=0, \tag{15}$$

is self-adjoint, see Theorem 2.1. It is easy to see that this operator is non-negative. The substitution $\tau = \mu^4$ shows that f as a function of μ is the characteristic function of the problem (15). Hence, the zeros of f are fourth roots of nonnegative real numbers, which means that all zeros of f are real or pure imaginary.

For Case 8, it is easy to see that 0 is a zero of ϕ_0 of multiplicity 8. Let $k(\mu) = \cos(\mu a)\cosh(\mu a)-1$. The zeros of ϕ_0 are the zeros of k. An obvious modification of the proof of Case 7 shows that k has no zeros in the interval $I_{m,1}$ and $I_{m,2}$, whereas it has simple zeros in the intervals $I_{m,0}$ and $I_{m,3}$. The reasonings for the asymptotics and the zeros on the other three semiaxes is the same as in the proof of Case 2. To complete the proof, observe that as for (15) the eigenvalues of

$$y^{(4)} = \tau y,\ y''(0)=0,\ y^{(3)}(0)=0,\ y''(a)=0,\ y^{(3)}(a)=0, \tag{16}$$

are nonnegative real numbers. Hence, the eigenvalues of the problem (16) are real and nonnegative. The substitution $\tau = \mu^4$ shows that $\mu \mapsto \mu^{-4}g(\mu)$ is the characteristic function of the problem (16), so that all zeros of h are real or pure imaginary. □

Proposition 3.2 *For $h=0$ and $g=0$, there exists a positive integer k_0 such that the eigenvalues $\hat{\lambda}_k$, $k\in\mathbb{Z}$, counted with multiplicity, of the problem (2)–(3), where $B_1(\lambda)y = y^{[p_1]}(0)$, $B_2(\lambda)y = y^{[p_2]}(0)$, $B_3(\lambda)y = y^{[p_3]}(a)$, $B_4(\lambda)y = y^{[3]}(a) - i\alpha\lambda y(a)$, $\alpha>0$, can be indexed in such a way that the eigenvalues $\hat{\lambda}_k$ are pure imaginary for $|k|<k_0$, and satisfy $\hat{\lambda}_{-k} = -\hat{\lambda}_k$ and $\Im\lambda_k \ge 0$ for $k\ge k_0$. For $k>0$, we can write $\hat{\lambda}_k = \hat{\mu}_k^2$, where the $\hat{\mu}_k$ have the following asymptotic representation as $k\to\infty$:*

Case 1 : $p_1=0,\ p_2=1,\ p_3=1$:

$$\hat{\mu}_k = (4k-1)\frac{\pi}{4a}+o(1);$$

Case 2 : $p_1=0,\ p_2=1,\ p_3=2$:

$$\hat{\mu}_k = (2k-3)\frac{\pi}{2a}+o(1);$$

Case 3 : $p_1 = 0, p_2 = 2, p_3 = 1$:

$$\hat{\mu}_k = (2k - 1)\frac{\pi}{2a} + o(1);$$

Case 4 : $p_1 = 0, p_2 = 2, p_3 = 2$:

$$\hat{\mu}_k = (4k - 1)\frac{\pi}{4a} + o(1);$$

Case 5 : $p_1 = 1, p_2 = 3, p_3 = 1$:

$$\hat{\mu}_k = (k - 1)\frac{\pi}{a} + o(1);$$

Case 6 : $p_1 = 1, p_2 = 3, p_3 = 2$:

$$\hat{\mu}_k = (4k - 5)\frac{\pi}{4a} + o(1);$$

Case 7 : $p_1 = 2, p_2 = 3, p_3 = 1$:

$$\hat{\mu}_k = (4k - 5)\frac{\pi}{4a} + o(1);$$

Case 8 : $p_1 = 2, p_2 = 3, p_3 = 2$:

$$\hat{\mu}_k = (2k - 5)\frac{\pi}{2a} + o(1).$$

In particular, the number of pure imaginary eigenvalues is even in each case.

Proof In each case, we will show that the zeros of ϕ are asymptotically close to the zeros of ϕ_0. We will start with Case 7. Since the remaining cases are very similar, we will just indicate the changed needed in the other 7 cases.

Case 7: A straightforward calculation gives

$$\begin{aligned}\phi(\mu) = &-\frac{1}{2}\mu^3(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a))\\ &+ \frac{1}{2}i\alpha\mu^2(1 + \cos(\mu a)\cosh(\mu a)).\end{aligned} \tag{17}$$

Up to the constant factor $\frac{1}{2}$, the first term equals $\phi_0(\mu)$. It follows that for μ with $\phi_0(\mu) \neq 0$, $\sin(\mu a) \neq 0$ and $\sinh(\mu a) \neq 0$, we have

$$\begin{aligned}\phi_1(\mu) &= \frac{2\phi(\mu) - \phi_0(\mu)}{\phi_0(\mu)}\\ &= -\frac{i\alpha}{\mu}\frac{1}{\tan(\mu a) + \tanh(\mu a)} - \frac{i\alpha}{\mu}\frac{1}{\tan(\mu a) + \tanh(\mu a)}\frac{1}{\cos(\mu a)\cosh(\mu a)}.\end{aligned} \tag{18}$$

Fix $\varepsilon \in (0, \frac{\pi}{4a})$ and for $k = 2, 3, \ldots$. Let $R_{k,\varepsilon}$ be the squares determined by the vertices $(4k - 5)\frac{\pi}{4a} \pm \varepsilon \pm i\varepsilon$, $k \in \mathbb{Z}$. These squares do not intersect due to $\varepsilon < \frac{\pi}{2a}$. Since $\tan z = -1$ if and only if $z = j\pi - \frac{\pi}{4}$ and $j \in \mathbb{Z}$, it follows from the periodicity of tan that the number

$$C_1(\varepsilon) = 2\min\{|\tan(\mu a) + 1| : \mu \in R_{k,\varepsilon}\}$$

is positive and independent of ε. Since $\tanh(\mu a) \to 1$ uniformly in the strip $\{\mu \in \mathbb{C} :\ \mathrm{Re}\ \mu \ge 1, |\mathrm{Im}\ \mu| \le \frac{\pi}{4a}\}$ as $|\mu| \to \infty$, there is and integer $k_1(\varepsilon)$ such that

$$|\tan(\mu a) + \tanh(\mu a)| \ge C_1(\varepsilon) \text{ for all } \mu \in R_{k,\varepsilon} \text{ with } k > k_1(\varepsilon).$$

By periodicity, there is a number $C_2(\varepsilon) > 0$ such that $|\cos(\mu a)| > C_2(\varepsilon)$ for all $\mu \in R_{k,\varepsilon}$ and all k. Observing $|\cosh(\mu a)| \ge |\sinh(\Re \mu a)|$, it follows that there exists $k_2(\varepsilon) \ge k_1(\varepsilon)$ such that for all μ on the squares $R_{k,\varepsilon}$ with $k \ge k_2(\varepsilon)$ the estimate $|\phi_1(\mu)| < \alpha$ holds.

We can assume from Lemma 3.1 that $\tilde{\mu}_k$ is inside of $R_{k,\varepsilon}$ and no other zero of ϕ_0 has this property. By definition of ϕ_1 in (18) we have

$$\phi(\mu) = \frac{1}{2}(\phi_1(\mu) + 1)\phi_0(\mu) \text{ for } \mu \in R_{k,\varepsilon}. \tag{19}$$

Hence, it follows by Rouché's theorem that there is exactly one (simple) zero $\hat{\mu}_k$ of ϕ in each $R_{k,\varepsilon}$ for $k \ge k_2(\varepsilon)$. In view of $\phi_0(i\mu) = \phi_0(\mu)$ and $\phi_1(i\mu) = -\phi_1(\mu)$ for all $\mu \in \mathbb{C}$, the same reasoning applies to the corresponding squares along the positive imaginary semiaxis. Observing that ϕ is an even function, it follows that the same estimate applies to the corresponding squares along the other remaining two semiaxes. Therefore, ϕ has zeros $\pm\hat{\mu}_k, \pm\hat{\mu}_{-k}$ for $k > k_2(\varepsilon)$ with the same asymptotic behaviour as the zeros $\pm\tilde{\mu}_k, \pm i\tilde{\mu}_k$ of ϕ_0 as stated in Lemma 3.1.

Next, we are going to estimate ϕ_1 on the squares S_k, $k \in \mathbb{N}$, whose vertices are $\pm k\frac{\pi}{a} \pm ik\frac{\pi}{a}$. For $k \in \mathbb{Z}$ and $\gamma \in \mathbb{R}$,

$$\tan\left(\left(\frac{k\pi}{a} + i\gamma\right)a\right) = \tan(i\gamma a) = i\tanh(\gamma a) \in i\mathbb{R}. \tag{20}$$

Therefore, we have for $\mu = \frac{k\pi}{a} + i\gamma$, where $k \in \mathbb{Z}$ and $\gamma \in \mathbb{R}$, that

$$|\tan(\mu a)| < 1 \text{ and } |\tan(\mu a) \pm 1| \ge 1. \tag{21}$$

For $\mu = x + iy$, $x, y \in \mathbb{R}$ and $x \ne 0$, we have

$$\tanh(\mu a) = \frac{e^{(ax+iay)} - e^{-(ax+iay)}}{e^{(ax+iay)} + e^{-(ax+iay)}} \to \pm 1 \tag{22}$$

uniformly in y as $x \to \pm\infty$. Hence, there is $\widetilde{k}_1 > 0$ such that for all $k \in \mathbb{Z}$, $|k| \ge \widetilde{k}_1$, and $\gamma \in \mathbb{R}$,

$$\left|\tanh\left(\left(\frac{k\pi}{a} + i\gamma\right)a\right) - \mathrm{sgn}(k)\right| < \frac{1}{2}. \tag{23}$$

It follows from (21) and (23) for $\mu = \frac{k\pi}{a} + i\gamma$, $k \in \mathbb{Z}$, $|k| \ge \widetilde{k}_1$, and $\gamma \in \mathbb{R}$ that

$$\left|\tan(\mu a) + \tanh(\mu a)\right| \ge \frac{1}{2}. \tag{24}$$

Furthermore, we will make use of the estimates

$$\left|\cosh\left(\left(\frac{k\pi}{a} + i\gamma\right)a\right)\right| \ge |\sinh(k\pi)|, \tag{25}$$

$$\left|\cos\left(\left(\frac{k\pi}{a} + i\gamma\right)a\right)\right| = \cosh(\gamma a) \ge 1, \tag{26}$$

which hold for all $k \in \mathbb{Z}$ and all $\gamma \in \mathbb{R}$. Therefore, it follows from (24) to (26) and the corresponding estimates with μ replaced by $i\mu$ that there is $\hat{k}_1 \geq \tilde{k}_1$ such that $|\phi_1(\mu)| < \alpha$ for all $\mu \in S_k$ with $k > \hat{k}_1$. Again from (19) and Rouché's theorem we conclude that the functions ϕ_0 and ϕ have the same number of zeros in the square S_k, for $k \in \mathbb{N}$ with $k \geq \hat{k}_1$.

Since ϕ_0 has $4k+4$ zeros inside S_k and thus $4k+4+4$ zeros inside S_{k+1}, it follows that ϕ has no large zeros other than the zeros $\pm\hat{\mu}_k$ found above for $|k|$ sufficiently large, and that there are $\hat{\mu}_k$ for small $|k|$ such that the $\hat{\lambda}_k = \hat{\mu}_k^2$ account for all eigenvalues of the problem (2)–(3) since each of these eigenvalues gives rise to two zeros of ϕ, counted with multiplicity. By Proposition 2.2, all eigenvalues with nonzero real part occur in pairs $\hat{\lambda}_k, -\hat{\lambda}_k$ with $\Re\hat{\lambda}_k \geq 0$, which shows that we can index all such eigenvalues as $\hat{\lambda}_{-k} = -\hat{\lambda}_k$. Since there is an even number of remaining indices, the number of pure imaginary eigenvalues must be even.

The functions ϕ is cases 1, 4 and 6 are respectively the following:

Case 1:

$$\begin{aligned}\phi(\mu) = {} & \frac{1}{2\mu}\Big(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)\Big)\\ & + \frac{i\alpha}{2\mu^2}\Big(1 - \cos(\mu a)\cosh(\mu a)\Big).\end{aligned} \tag{27}$$

Case 4:

$$\phi(\mu) = \frac{1}{2}\mu\Big(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)\Big) + i\alpha\sin(\mu a)\sinh(\mu a). \tag{28}$$

Case 6:

$$\phi(\mu) = -\frac{1}{2}\mu^3\Big(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)\Big) + i\alpha\mu^2\cos(\mu a)\cosh(\mu a). \tag{29}$$

Then all the estimates are as in Case 7, and the results respectively in Case 1, Case 4 and Case 6 immediately follow from that in Case 7.

Case 5: A straightforward calculation gives

$$\phi(\mu) = -\mu^2\sin(\mu a)\sinh(\mu a) + \frac{1}{2}i\alpha\mu\Big(\sin(\mu a)\cosh(\mu a) + \cos(\mu a)\sinh(\mu a)\Big). \tag{30}$$

Then

$$\phi_1(\mu) = \frac{2\phi(\mu) - \phi_0(\mu)}{\phi_0(\mu)} = \frac{1}{2\mu}i\alpha\Big(\coth(\mu a) + \cot(\mu a)\Big).$$

The result follows with reasonings and estimates as in the proof of Case 7, replacing μ by $\mu \pm \frac{\pi}{2}$ and $\mu \pm i\frac{\pi}{2}$ respectively.

Case 3: The function ϕ in this case is

$$\phi(\mu) = \cos(\mu a)\cosh(\mu a) + \frac{1}{2\mu}i\alpha\Big(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)\Big). \tag{31}$$

The result is similar to Case 5 with each trigonometric and hyperbolic function replaced by its derivative.

Case 2: A straightforward calculation gives

$$\begin{aligned}\phi(\mu) = &\frac{1}{2}\Big(\cos(\mu a)\cosh(\mu a) + 1\Big)\\ &- \frac{1}{2\mu} i\alpha\Big(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)\Big).\end{aligned} \tag{32}$$

Then

$$\begin{aligned}\phi_1(\mu) &= \frac{2\phi(\mu) - \phi_0(\mu)}{\phi_0(\mu)}\\ &= \frac{i\alpha}{\mu}\Big(\tanh(\mu a) - \tan(\mu a)\Big) - \frac{i\alpha}{\mu}\,\frac{\tanh(\mu a) - \tan(\mu a)}{\cos(\mu a)\cosh(\mu a) + 1}\end{aligned}$$

Observing that in the strip $\{\mu = x + iy : x \geq 1, |y| \leq \frac{\pi}{2a}\}$ we have $|\tan(\mu a)| < C_1$ and $|\tanh(\mu a)| < C_2$ with suitable constants C_j, the result follows with reasonings and estimates similar to those for Case 7.

Case 8: The function ϕ in this case is

$$\begin{aligned}\phi(\mu) = &\frac{1}{2}\mu^4\Big(1 - \cos(\mu a)\cosh(\mu a)\Big)\\ &+ \frac{1}{2} i\alpha\mu^3\Big(\sin(\mu a)\cosh(\mu a) - \cos(\mu a)\sinh(\mu a)\Big),\end{aligned} \tag{33}$$

and a reasoning as in Case 2 completes the proof. □

Birkhoff Regularity

We refer to [2, Definition 7.3.1] for the definition of Birkhoff regularity.

Proposition 4.1 *The boundary value problems (2), (6)–(12) are Birkhoff regular for $\alpha > 0$ with respect to the eigenvalue parameter μ given by $\lambda = \mu^2$.*

Proof The characteristic function of (2) as defined in [2, (7.1.4)] is $\pi(\rho) = \rho^4 - 1$, and its zeros are $i^{k-1},\ k = 1, \dots, 4$. We can choose

$$C(x, \mu) = \operatorname{diag}\Big(1, \mu, \mu^2, \mu^3\Big)\left(i^{(k-1)(j-1)}\right)_{k,j=1}^{4}$$

according to [2, Theorem 7.2.4.A]. The boundary conditions (6)–(12) can be written in the form

$$B_j(\lambda)y = \hat{B}_j(\mu)(y(a_j), y'(a_j), y''(a_j), y^{(3)}(a_j)),\ j = 1, 2, 3, 4,$$

where

$$\begin{aligned}\hat{B}_j(\mu) &= \varepsilon_{p_j+1}^{\top} \text{ for } j = 1, 2, 3,\\ \hat{B}_4(\mu) &= (-i\alpha\mu^2, 0, 0, 1).\end{aligned}$$

Thus, the boundary matrices defined in [2, (7.3.1)] are given by

$$W^{(0)}(\mu) = \begin{pmatrix} \hat{B}_1(\mu) \\ \hat{B}_2(\mu) \\ 0 \\ 0 \end{pmatrix} C(0,\mu) = \begin{pmatrix} \mu^{p_1} & \mu^{p_1} i^{p_1} & \mu^{p_1} i^{2p_1} & \mu^{p_1} i^{3p_1} \\ \mu^{p_2} & \mu^{p_2} i^{p_2} & \mu^{p_2} i^{2p_2} & \mu^{p_2} i^{3p_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$W^{(1)}(\mu) = \begin{pmatrix} 0 \\ 0 \\ \hat{B}_3(\mu) \\ \hat{B}_4(\mu) \end{pmatrix} C(a,\mu) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mu^{p_3} & \mu^{p_3} i^{p_3} & \mu^{p_3} i^{2p_3} & \mu^{p_2} i^{3p_3} \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix},$$

where $\beta_j = (-1)^{j-1}\mu^3 - i\alpha\mu^2$. Choosing $C_2(\mu) = \operatorname{diag}(\mu^{p_1}, \mu^{p_2}, \mu^{p_3}, \mu^3)$, it follows that $C_2(\mu)^{-1} W^{(j)}(\mu) = W_0^{(j)} + O(\mu^{-1})$, where

$$W_0^{(0)} = \begin{pmatrix} 1 & i^{p_1} & i^{2p_1} & i^{3p_1} \\ 1 & i^{p_2} & i^{2p_2} & i^{3p_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad W_0^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & i^{p_3} & i^{2p_3} & i^{3p_3} \\ 1 & -i & -1 & i \end{pmatrix}.$$

The Birkhoff matrices are

$$W_0^{(0)} \Delta_j + W_0^{(1)} (I - \Delta_j), \tag{34}$$

where Δ_j, $j = 1, 2, 3, 4$ are the 4×4 diagonal matrices with two consecutive ones and two consecutive zeros in the diagonal in a cyclic arrangement, see [2, Definition 7.3.1 and Proposition 4.1.7]. It is easy to see that after a permutation of columns, the matrices (34) are block diagonal matrices consisting of 2×2 blocks taken from two consecutive columns (in the sense of cyclic arrangement) of the first two rows of $W_0^{(0)}$ and the last two rows of $W_0^{(1)}$, respectively. Hence, the determinants of the Birkhoff matrices (34) are

$$\begin{vmatrix} i^{(j-1)p_1} & i^{jp_1} \\ i^{(j-1)p_2} & i^{jp_2} \end{vmatrix} \begin{vmatrix} i^{(j-1)p_3} & i^{jp_3} \\ (-i)^{j-1} & (-i)^j \end{vmatrix} = -i^{(j-1)(p_1+p_2+p_3+1)} (i + i^{p_3})(i^{p_2} - i^{p_1}) \neq 0.$$

Therefore, the problems (2), (6)–(12) are Birkhoff regular. □

Asymptotic Expansions of Eigenvalues

Let D, as function of μ with $\lambda = \mu^2$, be the characteristic function of any the problems (2), (6)–(12) with respect to the fundamental system y_j, $j = 1, 2, 3, 4$, with $y_j^{[m]}(0) = \delta_{j,m+1}$ for $m = 0, 1, 2, 3$, where δ is the Kronecker delta. Denote by D_0 the corresponding characteristic function for $g = 0$ and $h = 0$. Note that the characteristic functions D_0 and ϕ_0 of any of the problems, considered in "Asymptotics of eigenvalues for $g = 0$ and $h = 0$", have the same zeros, counted with multiplicity. Due to the Birkhoff regularity, g and h only influence lower order terms in D, see [2, Sections 4.3 and 7.3]. Therefore, it can be inferred that outside the interior of the small squares R_k, $-R_k$, iR_k, $-iR_{-k}$ around the zeros of D_0, $|D(\mu) - D_0(\mu)| < |D_0(\mu)|$ if $|\mu|$ is sufficiently large. Since the fundamental system y_j, $j = 1, 2, 3, 4$, depends analytically on μ, also D and D_0 are analytic functions. Hence, applying Rouché's theorem both to the large squares S_k and to the small squares which are sufficiently far away from the origin, it follows that the eigenvalues of the boundary value problems for general g and h have the same asymptotic distribution as for $g = 0$ and $h = 0$. Whence, Proposition 3.2 leads to

Proposition 5.1 *For $h \in C[0,a]$ and $g \in C^1[0,a]$, there exists a positive integer k_0 such that the eigenvalues λ_k, $k \in \mathbb{Z}$, counted with multiplicity, of the problems (2), (6)–(12), can be enumerated in such a way that the eigenvalues λ_k are pure imaginary for $|k| < k_0$, and $\lambda_{-k} = -\overline{\lambda_k}$ for $k \geq k_0$. For $k > 0$, we can write $\lambda_k = \mu_k^2$, where the μ_k have the following asymptotic representations as $k \to \infty$:*

Case 1 : $p_1 = 0, p_2 = 1, p_3 = 1$:

$$\hat{\mu}_k = (4k-1)\frac{\pi}{4a} + o(1);$$

Case 2 : $p_1 = 0, p_2 = 1, p_3 = 2$:

$$\hat{\mu}_k = (2k-3)\frac{\pi}{2a} + o(1);$$

Case 3 : $p_1 = 0, p_2 = 2, p_3 = 1$:

$$\hat{\mu}_k = (2k-1)\frac{\pi}{2a} + o(1);$$

Case 4 : $p_1 = 0, p_2 = 2, p_3 = 2$:

$$\hat{\mu}_k = (4k-1)\frac{\pi}{4a} + o(1);$$

Case 5 : $p_1 = 1, p_2 = 3, p_3 = 1$:

$$\hat{\mu}_k = (k-1)\frac{\pi}{a} + o(1);$$

Case 6 : $p_1 = 1, p_2 = 3, p_3 = 2$:

$$\hat{\mu}_k = (4k-5)\frac{\pi}{4a} + o(1);$$

Case 7 : $p_1 = 2, p_2 = 3, p_3 = 1$:

$$\hat{\mu}_k = (4k-5)\frac{\pi}{4a} + o(1);$$

Case 8 : $p_1 = 2, p_2 = 3, p_3 = 2$:

$$\hat{\mu}_k = (2k-5)\frac{\pi}{2a} + o(1).$$

In particular, the number of pure imaginary eigenvalues is even in each case.

In the remainder of the section, we are going to establish more precise asymptotic expansions of the eigenvalues. According to [2, Theorem 8.2.1], (2) has an asymptotic fundamental system $\{\eta_1, \eta_2, \eta_3, \eta_4\}$ of the form

$$\eta_\nu^{(j)}(x,\mu) = \delta_{\nu,j}(x,\mu)e^{i^{\nu-1}\mu x},\ \nu = 1, \dots, 4,\ j = 0, \dots, 3, \tag{35}$$

where

$$\delta_{\nu,j}(x,\mu) = \left[\frac{d^j}{dx^j}\right]\left\{\sum_{r=0}^{2}(\mu i^{\nu-1})^{-r}\varphi_r(x)e^{i^{\nu-1}\mu x}\right\}e^{-i^{\nu-1}\mu x} + \{o(\mu^{-2+j})\}_\infty, \tag{36}$$

$[\frac{d^j}{dx^j}]$ means that we omit those terms of the Leibniz expansion which contain a function $\varphi_r^{(k)}$ with $k > 4 - r$, and where $\{o(\cdot)\}_\infty$ means that the estimate is uniform in x.

Since the coefficient of $y^{(3)}$ in (2) is zero, we have $\varphi_0(x) = 1$, see [2, (8.2.3)]. We will now determine the functions φ_1 and φ_2. In this regard, observe that $n_0 = 0$ and $l = 4$ in the notation of [2, (8.1.2) and (8.1.3)], see [2, Theorem 8.1.2]. From [2, (8.2.45)] we know that

$$\varphi_r = \varphi_{1,r} = \varepsilon_1^\top V Q^{[r]}\varepsilon_1, \tag{37}$$

where ε_ν is the ν-th unit vector in $\mathbb{C}^4$, $V = (i^{(j-1)(k-1)})_{j,k=1}^4$, and $Q^{[r]}$ are 4×4 matrices given by [2, (8.2.28), (8.2.33) and (8.2.34)], that is, $Q^{[0]} = I_4$,

$$\Omega_4 Q^{[1]} - Q^{[1]}\Omega_4 = Q^{[0]\prime} = 0, \tag{38}$$

$$\Omega_4 Q^{[2]} - Q^{[2]}\Omega_4 = Q^{[1]\prime} - \frac{1}{4}g\Omega_4\varepsilon\varepsilon^\top\Omega_4^{-2}Q^{[0]}, \tag{39}$$

$$0 = \varepsilon_\nu^\top\Big(Q^{[2]\prime} + \frac{1}{4}\sum_{j=1}^{2}k_{3-j}\Omega_4\varepsilon\varepsilon^\top\Omega_4^{-1-j}Q^{[2-j]}\Big)\varepsilon_\nu \quad (\nu = 1,2,3,4), \tag{40}$$

where $k_2 = -g$, $k_1 = -g'$, $\Omega_4 = \operatorname{diag}(1, i, -1, -i)$ and $\varepsilon^\top = (1,1,1,1)$. Let $G(x) = \int_0^x g(t)dt$. A lengthy but straightforward calculation gives

$$\varphi_1 = \frac{1}{4}G, \quad \varphi_2 = \frac{1}{32}G^2 - \frac{1}{8}g \tag{41}$$

and thus

$$\begin{aligned}\eta_\nu(x,\mu) = \Big(1 + \frac{1}{4}i^{-\nu+1}G(x)\mu^{-1} + (-1)^{\nu-1}\Big(\frac{1}{32}G(x)^2 - \frac{1}{8}g(x)\Big)\mu^{-2}\Big)e^{i^{\nu-1}\mu x}\\ + \{o(\mu^{-2})\}_\infty e^{i^{\nu-1}\mu x}\end{aligned} \tag{42}$$

for $\nu = 1,2,3,4$.

The characteristic function of (2), (6)–(12) is

$$D(\mu) = \det(\gamma_{jk}\exp(\varepsilon_{jk}))_{j,k=1}^4,$$

where

$$\begin{aligned}&\varepsilon_{1k} = \varepsilon_{2k} = 0,\ \varepsilon_{3k} = \varepsilon_{4k} = i^{k-1}\mu a,\ \gamma_{1k} = \delta_{k,p_1}(0,\mu),\\ &\gamma_{2k} = \delta_{k,p_2}(0,\mu) \text{ if } p_2 \le 2, \quad \gamma_{2k} = \delta_{k,3}(0,\mu) - g(0)\delta_{k,1}(0,\mu) \text{ if } p_2 = 3,\\ &\gamma_{3k} = \delta_{k,p_3}(a,\mu) \text{ if } p_3 = 1,2,\ \gamma_{4k} = \delta_{k,3}(a,\mu) - g(a)\delta_{k,1}(a,\mu) + i\alpha\mu^2\delta_{k,0}(a,\mu).\end{aligned}$$

Note that

$$D(\mu) = \sum_{m=1}^{5} \psi_m(\mu) e^{\omega_m \mu a}, \tag{43}$$

where $\omega_1 = 1 + i$, $\omega_2 = -1 + i$, $\omega_3 = -1 - i$, $\omega_4 = 1 - i$, $\omega_5 = 0$, and each of the functions $\psi_1, \dots, \psi_5$ has an asymptotic representation of the form $c_k \mu^k + c_{k-1}\mu^{k-1} + \cdots + c_{k_0}\mu^{k_0} + o(\mu^{k_0})$.

It follows from (43) that

$$D_1(\mu) := D(\mu) e^{-\omega_1 \mu a} = \psi_1(\mu) + \sum_{m=2}^{5} \psi_m(\mu) e^{(\omega_m - \omega_1)\mu a}, \tag{44}$$

where $\omega_2 - \omega_1 = -2$, $\omega_3 - \omega_1 = -2 - 2i$, $\omega_4 - \omega_1 = -2i$, $\omega_5 - \omega_1 = -1 - i$. If $\arg \mu \in [-\frac{3\pi}{8}, \frac{\pi}{8}]$, we have $|e^{(\omega_m - \omega_1)\mu a}| \le e^{-\sin \frac{\pi}{8} |\mu| a}$ for $m = 2, 3, 5$ and the terms $\psi_m(\mu) e^{(\omega_m - \omega_1)\mu a}$ for $m = 2, 3, 5$ can be absorbed by $\psi_1(\mu)$ as they are of the form $o(\mu^{-s})$ for any integer s. Hence, for $\arg \mu \in [-\frac{3\pi}{8}, \frac{\pi}{8}]$,

$$D_1(\mu) = \psi_1(\mu) + \psi_4(\mu) e^{(\omega_4 - \omega_3)\mu a} = \psi_1(\mu) + \psi_4(\mu) e^{-2i\mu a}, \tag{45}$$

where

$$\psi_1(\mu) = \left[\gamma_{13}\gamma_{24} - \gamma_{14}\gamma_{23}\right] \left[\gamma_{31}\gamma_{42} - \gamma_{32}\gamma_{41}\right], \tag{46}$$

$$\psi_4(\mu) = \left[\gamma_{12}\gamma_{23} - \gamma_{13}\gamma_{22}\right] \left[\gamma_{31}\gamma_{44} - \gamma_{34}\gamma_{41}\right]. \tag{47}$$

A straightforward calculation gives for $p_3 = 1$ that

$$\begin{aligned} \gamma_{31}\gamma_{42} - \gamma_{32}\gamma_{41} &= -2i\mu^4 + ((1-i)\alpha - 2(1+i)\varphi_1(a))\mu^3 \\ &\quad - 2(i\alpha\varphi_1(a) + \varphi_1^2(a))\mu^2 + o(\mu^2), \end{aligned} \tag{48}$$

$$\begin{aligned} \gamma_{31}\gamma_{44} - \gamma_{34}\gamma_{41} &= 2i\mu^4 + ((1+i)\alpha - 2(1-i)\varphi_1^2(a))\mu^3 \\ &\quad + 2(i\alpha\varphi_1(a) - \varphi_1^2(a))\mu^2 + o(\mu^2), \end{aligned} \tag{49}$$

while for $p_3 = 2$ we have

$$\begin{aligned} \gamma_{31}\gamma_{42} - \gamma_{32}\gamma_{41} &= (1-i)\mu^5 + (2\alpha - 2i\varphi_1(a))\mu^4 \\ &\quad + (2(1-i)\alpha\varphi_1(a) - (1+i)\varphi_1^2(a) - \frac{3}{4}(1+i)g(a))\mu^3 + o(\mu^3), \end{aligned} \tag{50}$$

$$\begin{aligned} \gamma_{31}\gamma_{44} - \gamma_{34}\gamma_{41} &= (1+i)\mu^5 + (2\alpha + 2i\varphi_1(a))\mu^4 \\ &\quad + (2(1+i)\alpha\varphi_1(a) - (1-i)\varphi_1^2(a) - \frac{3}{4}(1-i)g(a))\mu^3 + o(\mu^3). \end{aligned} \tag{51}$$

For the the other two factors in (46) and (47) we have to consider four different cases.

Cases 1 and 2: $p_1 = 0$, $p_2 = 1$. Here we have

$$\gamma_{13}\gamma_{24} - \gamma_{14}\gamma_{23} = (1-i)\mu + \frac{1}{4}(1+i)\mu^{-1} g(0) + o(\mu^{-1}), \tag{52}$$

$$\gamma_{12}\gamma_{23}-\gamma_{13}\gamma_{22}=-(1+i)\mu-\frac{1}{4}(1-i)\mu^{-1}g(0)+o(\mu^{-1}). \tag{53}$$

Cases 3 and 4: $p_1=0$, $p_2=2$. Here we have

$$\gamma_{13}\gamma_{24}-\gamma_{14}\gamma_{23}=-2\mu^2+o(1), \tag{54}$$

$$\gamma_{12}\gamma_{23}-\gamma_{13}\gamma_{22}=2\mu^2+o(1). \tag{55}$$

Cases 5 and 6: $p_1=1$, $p_2=3$. Here we have

$$\gamma_{13}\gamma_{24}-\gamma_{14}\gamma_{23}=-2i\mu^4+o(\mu^2), \tag{56}$$

$$\gamma_{12}\gamma_{23}-\gamma_{13}\gamma_{22}=-2i\mu^4+o(\mu^2). \tag{57}$$

Cases 7 and 8: $p_1=2$, $p_2=3$. Here we have

$$\gamma_{13}\gamma_{24}-\gamma_{14}\gamma_{23}=-(1-i)\mu^5+\frac{3}{4}(1+i)\mu^3g(0)+o(\mu^3), \tag{58}$$

$$\gamma_{12}\gamma_{23}-\gamma_{13}\gamma_{22}=(1+i)\mu^5-\frac{3}{4}(1-i)\mu^3g(0)+o(\mu^3). \tag{59}$$

Case 1: We get from (48), (49) (52) and (53)

$$\begin{aligned}\psi_1(\mu)=&-2(1+i)\mu^5+(G(a)-2i\alpha)\mu^4\\&-\left[\frac{1}{8}(1-i)G^2(a)+\frac{1}{4}(1+i)\alpha G(a)-\frac{1}{2}(1-i)g(0)\right]\mu^3+o(\mu^3),\end{aligned} \tag{60}$$

$$\begin{aligned}\psi_4(\mu)=&\,2(1-i)\mu^5+(G(a)-2i\alpha)\mu^4\\&+\left[\frac{1}{4}(1+i)G^2(a)+(1-i)\alpha G(a)-\frac{1}{2}(1+i)g(0)\right]\mu^3+o(\mu^3).\end{aligned} \tag{61}$$

Case 2: We have from (50), (51) (52) and (53)

$$\begin{aligned}\psi_1(\mu)=&-2i\mu^6-\left[\frac{1}{2}(1+i)G(a)-2(1-i)\alpha\right]\mu^5\\&-\left[\frac{1}{8}G^2(a)+i\alpha G(a)+\frac{3}{2}g(a)-\frac{1}{2}g(0)\right]\mu^4+o(\mu^4),\end{aligned} \tag{62}$$

$$\begin{aligned}\psi_4(\mu)=&\,2i\mu^6+\left[\frac{1}{2}(1-i)G(a)-2(1+i)\right]\mu^5\\&+\left[\frac{1}{8}G^2(a)-i\alpha G(a)+\frac{3}{2}g(a)-\frac{1}{2}g(0)\right]\mu^4+o(\mu^4).\end{aligned} \tag{63}$$

Case 3: We have from (48), (49) (54) and (55)

$$\begin{aligned}\psi_1(\mu)=&\,4i\mu^6+((1+i)G(a)-2(1-i)\alpha)\mu^5\\&+\left[\frac{1}{4}G^2(a)+i\alpha G(a)\right]\mu^4+o(\mu^4),\end{aligned} \tag{64}$$

$$\begin{aligned}\psi_4(\mu) &= 4i\mu^6 - ((1-i)G(a) - 2(1+i)\alpha G(a))\mu^5 \\ &\quad - \left[\frac{1}{4}G^2(a) - i\alpha G(a)\right]\mu^4 + o(\mu^4).\end{aligned} \tag{65}$$

Case 4: We have from (50), (51) (54) and (55)

$$\begin{aligned}\psi_1(\mu) &= 2(1+i)\mu^7 + (iG(a) + 4\alpha)\mu^6 \\ &\quad - \left[\frac{1}{8}(1-i)G^2(a) - (1+i)\alpha G(a) + \frac{3}{2}(1-i)g(a)\right]\mu^5 + o(\mu^5),\end{aligned} \tag{66}$$

$$\begin{aligned}\psi_4(\mu) &= 2(1+i)\mu^7 + (iG(a) + 4\alpha)\mu^6 \\ &\quad - i\left[\frac{1}{8}(1-i)G^2(a) - (1+i)\alpha G(a) = \frac{3}{2}(1-i)g(a)\right]\mu^5 + o(\mu^5).\end{aligned} \tag{67}$$

Case 5: We have from (48), (49) (56) and (57)

$$\begin{aligned}\psi_1(\mu) &= -4\mu^8 - ((1+i)\alpha + (1-i)G(a))\mu^7 \\ &\quad + \left[\frac{1}{4}iG^2(a) - \alpha G(a)\right]\mu^6 + o(\mu^6),\end{aligned} \tag{68}$$

$$\begin{aligned}\psi_4(\mu) &= 4\mu^8 + (2(1+i)G(a) + 2(1-i)\alpha)\mu^7 \\ &\quad - \left[\frac{1}{4}iG^2(a) + \alpha G(a)\right]\mu^6 + o(\mu^6).\end{aligned} \tag{69}$$

Case 6: We have from (50), (51) (56) and (57)

$$\begin{aligned}\psi_1(\mu) &= -2(1+i)\mu^9 - (G(a) + 4i\alpha)\mu^8 \\ &\quad - \left[\frac{1}{8}(1-i)G^2(a) + (1+i)\alpha G(a) + \frac{3}{2}(1-i)g(a)\right]\mu^7 + o(\mu^7),\end{aligned} \tag{70}$$

$$\begin{aligned}\psi_4(\mu) &= 2(1-i)\mu^9 + (G(a - 4i\alpha)\mu^8 \\ &\quad + \left[\frac{1}{8}(1+i)G^2(a) + (1-i)\alpha G(a) + \frac{3}{2}(1+i)g(a)\right]\mu^7 + o(\mu^7).\end{aligned} \tag{71}$$

Case 7: We have from (48), (49) (58) and (59)

$$\begin{aligned}\psi_1(\mu) &= 2(1+i)\mu^9 + (2i\alpha + G(a))\mu^8 \\ &\quad + i\left[\frac{1}{8}(1-i)G^2(a) + \frac{1}{2}(1+i)\alpha G(a) + \frac{3}{2}(1-i)g(0)\right]\mu^7 + o(\mu^7),\end{aligned} \tag{72}$$

$$\begin{aligned}\psi_4(\mu) &= -2(1-i)\mu^9 - (G(a) - 2i\alpha)\mu^8 \\ &\quad - \left[\frac{1}{8}(1+i)G^2(a) + \frac{1}{2}(1-i)\alpha G(a) + \frac{3}{2}(1+i)g(0)\right]\mu^7 + o(\mu^7).\end{aligned} \tag{73}$$

Case 8: It follows from (50), (51) (58) and (59)

$$\psi_1(\mu) = 2i\mu^{10} + \left[\frac{1}{2}(1+i)G(a) - 2(1-i)\right]\mu^9 + \left[\frac{1}{8}G^2(a) + i\alpha G(a) + \frac{3}{2}(g(a)+g(0))\right]\mu^8 + o(\mu^8), \tag{74}$$

$$\psi_4(\mu) = 2i\mu^{10} - \left[\frac{1}{2}(1-i)G(a) - 2(1+i)\alpha\right]\mu^9 - \left[\frac{1}{8}G^2(a) - i\alpha G(a) + \frac{3}{2}(g(a)+g(0))\right]\mu^8 + o(\mu^8). \tag{75}$$

We already know by Proposition 5.1 that the zeros μ_k of D satisfy the asymptotic representations $\mu_k = k\frac{\pi}{a} + \tau_0 + o(1)$ as $k \to \infty$. In order to improve on these asymptotic representations, write

$$\mu_k = k\frac{\pi}{a} + \tau(k),\ \tau(k) = \sum_{m=0}^{n} \tau_m k^{-m} + o(k^{-n}), \quad k = 1, 2, \dots. \tag{76}$$

Because of the symmetry of the eigenvalues, we will only need to find the asymptotic expansions as $k \to \infty$. We know τ_0 from Proposition 5.1, and it is our aim to find τ_1 and τ_2. To this end we will substitute (76) into $D_1(\mu_k) = 0$ and we will then compare the coefficients of k^0, k^{-1} and k^{-2}.

Observe that

$$e^{-2i\mu_k a} = e^{-2i\tau(k)a} = e^{-2i\tau_0 a}\exp\left(-2ia\left(\frac{\tau_1}{k} + \frac{\tau_2}{k^2} + o(k^{-2})\right)\right) = e^{-2i\tau_0 a}\left(1 - 2ia\tau_1\frac{1}{k} - \left(2a^2\tau_1^2 + 2ia\tau_2\right)\frac{1}{k^2} + o(k^{-2})\right), \tag{77}$$

while

$$\frac{1}{\mu_k} = \frac{a}{\pi k}\left(1 + \frac{a\tau(k)}{k\pi}\right)^{-1} = \frac{a}{k\pi} - \frac{a^2\tau_0}{k^2\pi^2} + o(k^{-2}). \tag{78}$$

Using (45), $D_1(\mu_k) = 0$ can be written as

$$\mu_k^{-\gamma}\psi_1(\mu_k) + \mu_k^{-\gamma}\psi_4(\mu_k)e^{-2i\tau_k a} = 0, \tag{79}$$

where γ is the highest μ-power in $\psi_1(\mu)$ and $\psi_4(\mu)$. Substituting (62), (63), (77) and (78) into (79) and comparing the coefficients of k^0, k^{-1} and k^{-2} we get

Theorem 5.2 *For $g \in C^1[0,a]$ and $h \in C[a,b]$, there exists a positive integer k_0 such that the eigenvalues λ_k, $k \in \mathbb{Z}$, counted with multiplicity, of the problems (2), (6)–(12), where $k \in \mathbb{Z} \setminus \{0\}$ can be enumerated in such a way that the eigenvalues λ_k are pure imaginary for $|k| < k_0$, and $\lambda_{-k} = -\overline{\lambda_k}$ for $k \geq k_0$, where $\lambda_k = \mu_k^2$ and the μ_k have the asymptotic representations*

$$\mu_k = k\frac{\pi}{a} + \tau_0 + \frac{\tau_1}{k} + \frac{\tau_2}{k^2} + o(k^{-2})$$

and the numbers τ_0, τ_1, τ_2 are as follows:

$$\text{Case 1: } \tau_0 = -\frac{\pi}{4a},\ \tau_1 = -\frac{\alpha}{2\pi} + \frac{1}{4}\frac{G(a)}{\pi},$$
$$\tau_2 = \frac{a\alpha}{4\pi^2} - \frac{\alpha}{8\pi} + \frac{1}{16}\frac{G(a)}{\pi} - \frac{1}{4}\frac{a}{\pi^2}g(0).$$
$$\text{Case 2: } \tau_0 = -\frac{3\pi}{2a},\ \tau_1 = -\frac{\alpha}{\pi} + \frac{1}{4}\frac{G(a)}{\pi},$$
$$\tau_2 = \frac{3}{8\pi}G(a) + \frac{3a}{4}\frac{g(a)}{\pi^2} - \frac{a}{4}\frac{g(0)}{\pi^2} - \frac{a\alpha^2}{\pi^2} - \frac{3\alpha}{2\pi}.$$
$$\text{Case 3: } \tau_0 = -\frac{\pi}{2a},\ \tau_1 = -\frac{\alpha}{2\pi} + \frac{1}{4}\frac{G(a)}{\pi},$$
$$\tau_2 = \frac{a\alpha^2}{4\pi^2} - \frac{\alpha}{4\pi} + \frac{1}{8}\frac{G(a)}{\pi}.$$
$$\text{Case 4: } \tau_0 = -\frac{\pi}{4a},\ \tau_1 = -\frac{i\alpha}{\pi} - \frac{i}{4}\frac{G(a)}{\pi},$$
$$\tau_2 = \frac{a\alpha^2}{\pi^2} - \frac{i\alpha}{4\pi} - \frac{i}{16}\frac{G(a)}{\pi} - \frac{i\alpha a}{2}\frac{G(a)}{\pi^2} - \frac{a}{16}\frac{G^2(a)}{\pi^2}.$$
$$\text{Case 5: } \tau_0 = -\frac{\pi}{a},\ \tau_1 = -\frac{\alpha}{2\pi} + \frac{1}{4}\frac{G(a)}{\pi},$$
$$\tau_2 = \frac{a\alpha^2}{4\pi^2} - \frac{\alpha}{2\pi} + \frac{1}{4}\frac{G(a)}{\pi}.$$
$$\text{Case 6: } \tau_0 = -\frac{5\pi}{4a},\ \tau_1 = -\frac{\alpha}{\pi} + \frac{1}{4}\frac{G(a)}{\pi},$$
$$\tau_2 = \frac{a\alpha^2}{\pi^2} - \frac{5}{4}\frac{\alpha}{\pi} + \frac{3a}{4}\frac{g(a)}{\pi^2} + \frac{5}{16}\frac{G(a)}{\pi}.$$
$$\text{Case 7: } \tau_0 = -\frac{5\pi}{4a},\ \tau_1 = -\frac{\alpha}{2\pi} + \frac{1}{4}\frac{G(a)}{\pi},$$
$$\tau_2 = \frac{a\alpha^2}{4\pi^2} - \frac{5\alpha}{8\pi} + \frac{5}{16}\frac{G(a)}{\pi} + \frac{3}{4}\frac{a}{\pi^2}g(0).$$
$$\text{Case 8: } \tau_0 = -\frac{5\pi}{2a},\ \tau_1 = -\frac{\alpha}{\pi} + \frac{1}{4}\frac{G(a)}{\pi},$$
$$\tau_2 = \frac{a\alpha^2}{\pi^2} - \frac{5\alpha}{2\pi} + \frac{3}{4}\frac{a}{\pi^2}g(0) + \frac{3}{4}\frac{a}{\pi^2}g(a) + \frac{5}{8}\frac{G(a)}{\pi}.$$

In particular, the number of pure imaginary eigenvalues is even in each case.

Remark 5.3 The number of pure imaginary eigenvalues is even in each of the cases in this paper while they are odd in [10]. In addition, the first four terms of the asymptotics of the eigenvalues in Case 2 and in Case 8, and the functions ϕ_0 in this paper differ from those in [10]. However, it is only the last two terms of the asymptotics of the eigenvalues in the remaining cases which differ from those in [10].

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