Chapter 1

Introduction

> *Teachers face a double challenge: in the short term, students still must learn the numeracy/mathematical content set out in the curriculum and demonstrate skills in an understanding of this content. But given the unpredictability of future numeracy demands, the second challenge is how to teach in ways that encourage students to see themselves as successful learners of mathematics and develop the confidence and dispositions to go on and learn, if need be, more mathematics later.*
>
> *- Mike Askew (Keynote address titled: New curriculum, new futures: learning mathematics for life, 2013)*

It has been shown that South African primary school learners have a poor grasp of elementary foundational mathematical concepts (Taylor, Fleisch and Schindler, 2008; Fleisch, 2008; Reddy, 2006; Howie, 2001). Low levels of opportunity-to-learn (OTL), with teachers spending a complete lesson period focusing on two to three low cognitive demand mathematical problems (Taylor and Vinjevold, 1999; Taylor, 2008), continue to plague South African classrooms. Despite the national government allocating billions of rands to improve education in the country through providing training for teachers and also improving the availability of resources, the problems remain (Scholar, 2008; Spaull and Simkins, 2013). Taylor, Fleisch and Schindler (2008) maintain that poor progress of students in high school and higher education is due to insufficient preparation of primary learners in literacy and numeracy. In South Africa it was found that one third of teachers of mathematics are either under- or unqualified (Reddy, 2006) with South African Grade 6 teachers placed at the bottom end of the spectrum, in terms of skills, when compared to their colleagues in Eastern and Southern African countries (McCarthy and Oliphant, 2013). Spaull (2013) and Simkins (2013) both ascribe poor learner performance to the inadequate teaching of mathematics. A positive correlation has previously been established between teacher subject knowledge and learners' levels of achievement (Howie, 2001; Taylor, 2011; Carnoy et.al, 2011). Taylor and Vinjevold (1999) argue that teachers with a poor grasp of fundamental mathematical concepts provide their learners with low level concepts, unchallenging tasks and poor learning objectives. Such classrooms show a dearth of more challenging, higher order tasks and a scarcity of OTL. Teachers with weak conceptual mathematical knowledge are limited for they 'cannot teach what they do not know' (Taylor, 2008, p.24) and are, therefore, 'constrained by this in the classroom' (Howie, 2001, p.155).

More recently, Venkat and Spaull (2015) revealed that 79% of a sample of 401 Grade 6 teachers demonstrated content knowledge below the Grade 6/7 band. Venkat and Naidoo (2012) also focused on Foundation Phase (FP) classrooms, where they observed poor coherence across pedagogic communication and activities, as well as the random selection and sequencing of exercises that inhibit meaning-making. They argued that the random selection and sequencing of tasks and the weak coherence gave rise to what they term 'extreme localization[1]'. It is, therefore, essential to develop both the pedagogical knowledge, as well as the conceptual and content knowledge of teachers in order to enable them to activate learners' cognition of both elementary and more advanced mathematical ideas.

Patterns, Algebra and Functions: The South African Curriculum Specifications

This study will focus on Content Area 2 (CA2), which consists of Patterns, Functions and Algebra and will investigate the conceptual and relational knowledge that is intended in this section of the Curriculum Assessment Policy Statement (CAPS) (2010). Firstly, the *general content focus* is described as follows:

> Algebra is the language for investigating and communicating most of Mathematics and can be extended to the *study of functions* and other *relationships between variables*. A central part of this content area is for the learners to achieve efficient manipulative skills in the use of algebra. It also focuses on the:
> - description of patterns and relationships through the use of symbolic expressions, graphs and tables; and the
> - identification and analysis of regularities and change in patterns, and relationships that enable learners to make predictions and solve problems.
>
> (CAPS, 2010, p.14).

The *skills* that are understood as important are description, identification, analysis, prediction and problem solving. The *pedagogical resources* that are suggested are symbolic expressions, graphs and tables. The *pedagogic practice* is shaped by a search for regularity and change within a set of mathematical items where patterns may emerge. All these notions are in line with what has been foregrounded by various researchers as being key to the development of

[1] By the time a new example is introduced in a working sequence, the processes and facts associated with previous examples seemed to have vanished.

algebraic reasoning skills. It is thus reasonable to expect the supporting and resource texts to contain elements that enact these skills through activities that are designed to enhance learners' ability to develop competence in CA 2.

Secondly, the CAPS (2010) outlines the *specific content focus* of the FP as follows:

> In this phase, learners work with both
>
> - number patterns (e.g. skip counting); and
> - geometric patterns (e.g. pictures).
>
> Learners should use physical objects, drawings and symbolic forms to copy, extend, describe and create patterns.
>
> Copying the pattern helps learners to see the *logic* of how the pattern is made.
>
> *Extending* the pattern helps learners to check that they have properly understood the logic of how the pattern is made.
>
> *Describing* the pattern helps learners to develop their language skills.
>
> Focusing on the logic of patterns lays the basis for developing *algebraic thinking skills.*
>
> Number patterns support the number concept development and operational sense built in Numbers, Operations and Relations[2].
>
> Geometric patterns include sequences of lines, shapes and objects but also patterns in the world. In geometric patterns learners apply their knowledge of Space and Shape[3].
>
> (Italics added by author) (CAPS, 2010, p.14).

The term 'logic' is used to describe the relational aspects embedded in the sequenced set of items, and thus its structure. The terms 'copy', 'extend' and 'describe' are given short descriptors that flag the intended skill expansion achieved through each of these actions. Specific reference is made to a *focus on the structure,* which is intended to *enhance algebraic reasoning skills* in the FP classroom. The notion of structure, however, is not described at a micro-level to include spatial, cyclic or growth structures, but merely mentioned vaguely as noticing (identifying) and using (applying) the *logic* of the pattern. Structural exploration in CA 2 is thus left unexamined and invisible by these descriptions. Based on these extended descriptors, however, it is reasonable to expect that activities in workbooks and assessment items will deal with structure (logic) and elements that involve the application of structural aspects to develop algebraic reasoning skills.

[2] Numbers, Operations and Relations make up CA 1 in the CAPS (2010) which has the most time assigned to it.
[3] Shape and Space form CA 3 in the CAPS (2010).

The Triad of Pattern, Structure and Algebraic Reasoning

It is evident from the extract of the CAPS document above that the triad[4] of pattern, structure and algebraic reasoning is underscored by the description provided in the specific content focus of the curriculum documents. The triad of pattern, structure and algebraic reasoning has been the focus of research since the early eighties, and has provided a body of research that calls for recognition of the inter-connectedness of pattern, structure and algebraic reasoning and their ability to enhance basic numeracy in young learners (eg. Wilkie and Clarke, 2016; Mulligan, Cavanagh and Keanan-Brown, 2012; Papic, Mulligan and Mitchelmore, 2011; Mason, Stephens and Watson, 2009; Mulligan and Mitchelmore, 2009; Yeap and Kaur, 2008; Kaput, 2008; Warren and Cooper, 2008; Carraher and Schliemann, 2007; Carraher, Schliemann, Brizuela and Earnest, 2006; Radford, 2006; Mason, Graham and Johnstone-Wilder, 2006; Blanton and Kaput, 2005; Warren, 2005). In order to improve basic numeracy in the FP it is necessary to investigate ways in which the OTL and the levels of cognitive demand can be improved in mathematical learning, particularly through a focus on the triad of pattern, structure and algebraic reasoning. In the sections that follow, the components of the triad are explored to provide a background and a rationale for the study of structure in the FP mathematics class in the context of CA 2.

Pattern and Sequencing: A Subsumptive Relationship

The term 'pattern' (as used in the CAPS documents) is really a nebulous description which exists without clear definition and is often used to describe (i) a broad category of 'recognition skills', (ii) an observed regularity, (iii) the regularity itself, (iv) a domain of study and also (v) a sequence of (ordered) items. Each one of these elicits its own unique concept image. The Merriam-Webster dictionary (2003 Edition) defines 'pattern' as 'a discernible coherent system based on an intended interrelationship of component parts'. This definition focuses on two very important components, namely: the relational aspects between items in a sequence, as well as pattern as a coherent system determined by the relational. Not all patterns involve mathematics or even recurrence. Patterns can also be found in geometric applications such as the particular iteration of a shape (in wallpaper, beadwork or textiles). Papic, Mulligan and Mitchelmore (2011) assert that pattern means 'any replicable regularity' and they identify three contexts where patterns have significance in mathematics:

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[4] The notion of a triad is used here to signify the bond or the union of the three concepts of pattern, structure and algebraic reasoning working together to enhance conceptual mathematical understanding. This will be elaborated upon in the sections that follow.

- Within a single object, in which some of its components are consistently related;
- Within an ordered set of objects, in which there is a consistent relation between each component and the next; and
- Between two ordered set of objects, in which the corresponding elements are paired in some way. (p.239)

They maintain that children up to Grade 2 level, experience all three kinds of patterns.

The term 'sequence' or 'sequencing' involves a specific, clearly defined object or action which is based on the application of a particular regularity. A sequence also has clearly defined members which are referred to as *terms* of the sequence. These can extend indefinitely, or they can be limited to a certain number of terms or items only. The word 'sequencing' is used in the English language to describe the idea of succession, that is the ordering of one thing after another. In mathematics a sequence is an ordered set of quantities, elements or objects (Webster New World College Dictionary, 2015 Edition; Wolfram World of Mathematics, 2015). In a *sequence structure*, one action leads to the next ordered action which explicates the pre-determined fixed regularity in a set. This fixed regularity can represent sequenced items that repeat themselves, or the set of items can develop through a growth element in a precise routine. In this sense, a 'pattern', as used in CA 2, is a component of a sequenced set of items which can consist of numbers, objects and shapes.

In this study, the use of the words 'sequence' (noun) and 'sequencing' (verb) is intended to describe a set of items (number, objects or shapes) that are arranged in a precise manner by applying a pre-determined regularity to form an array of items from which a pattern can be explicated. Sequencing is meant to encompass various forms of mathematical order / regularity (growth or repeat) that are applied to the items. Simply put: ordering objects or numbers through applying and retaining some form of regularity (an observed pattern) is what is understood to describe the notion of sequencing. This regularity can be, but is not limited to, a spatial or numerical regularity. The notion of a pattern exists within the object called a sequence.

Sequencing: The Heart of Mathematical Activity

Sequencing, and thus pattern, is studied in CA 2 and designates the mathematical study of repeating patterns and growth patterns. The notion of sequencing can range from simple patterns, where there is a constant growth quantified by a numerical regularity, or the addition of a geometric shape to an existing array of shapes, to more complex patterns. Such complex patterns involve compounded events where, for example, the shape, orientation and colour of a repeating pattern display variance. Complex number sequences (not to be confused with complex number e.g. $\sqrt{-4} = 2i$) make use of co-variance and structure that characterises functional thinking. Complex sequences can also move away from the exclusive use of additive reasoning to the building of a sequence of numbers that rely on a multiplicative regularity. Using the term 'sequencing' is intentional and includes any form of mathematical activity that explores and uses regularity and change in its various forms and levels of complexity. Sequencing, and thus pattern, is one of the primary objectives in mathematics learning as it is the tool that, through structural exploration, leads to the use of observed regularity and change to advance generalised expression and understanding of mathematics.

Mathematics has been described by Devlin (1994) and Steen (1988) as the science of pattern. Liljedahl (2004) signposts number pattern as the cornerstone of mathematics – 'the genesis, the motivation, and the foundation of mathematical knowledge' (Ibid, p.1). Clements and Sarama (2009) present a persuasive argument to view pattern as a habit of mind which is more than a content area: 'it is a process and a domain of study' (Ibid, p.2). Threlfall (1999) sees patterns as crucial in developing mathematical reasoning skills that are essential for the later conceptual learning of algebra. Pattern is thus positioned as the foundation of all mathematical learning, as it enriches the conceptual ground that enhances the development of algebraic reasoning skills at an early age and prefigures later learning of functions. The afore-mentioned authors use the notion of pattern as the 'study of regularity and variance'. It is thus essential to revisit the notion of sequencing through exploring patterns as they are presented in the curriculum and the supporting materials that enact the curriculum in the classroom, to establish a cohesive understanding of what is presented as the 'object of learning' when dealing with sequencing at FP level. The three levels of enactment – classroom teaching, curriculum and supporting texts – work collaboratively to reach the optimal mathematical thinking intended in CA 2 at FP, and thus coherently work towards developing numeracy in FP mathematics classrooms.

Structure, Structural Sense, Structural Knowledge, Structural Thinking and Structural Extension.

Structuring is a fundamental process which assists people to organize the world (Freudenthal, 1987). The development, understanding and generalisation of mathematical concepts rely on the ability to extract structural relationships in and between quantities or objects (Sfard, 1991). It is necessary to describe what is meant by 'structure', 'structure sense', 'structural thinking' and 'structural extension' as there are a number of ways in which researchers and mathematicians formulate and use these concepts.

Structure is the way in which the various elements in a sequenced set are organised and related (Mulligan and Mitchelmore, 2009, p.1). Structure refers to the relationship between mathematical entities, rather than the entities themselves (Resnick, Shapiro and Colyvan, 1997, p.1). Structure is also taken to mean the identification of general properties which are instantiated in particular situations as relationships between elements or subsets of elements of a set (Mason, Stephens and Watson, 2009).

Knowing structure is possessing knowledge about the relationships between mathematical objects and the properties of these objects (Morris, 1999). *Structural knowledge* means being able to identify 'all the equivalent forms of the expression' (Kieran, 1998). Hoch (2003) suggested that *structure sense* is a collection of abilities, separate from manipulative ability, which enables students to make better use of previously learned algebraic techniques. This idea was refined over the years and Hoch and Dreyfus (2004, 2005 and 2006) arrived at the following definition:

> A student is said to display *structure sense* for high school algebra if s/he can:
> - recognise a familiar structure in its simplest form.
> - deal with a compound term as a single entity, and through an appropriate substitution, recognise a familiar structure in a more complex form.
> - choose appropriate manipulations to make best use of a structure.
>
> (Hoch and Dreyfus, 2007, p.1).

This definition describes the notion of structure sense at a high school level, but is equally appropriate for young learners who work with both simple and complex sequenced items at a primary level. The power of mathematics 'lies in the relations and transformations which give rise to patterns and generalisations. Abstracting patterns is the basis of structural knowledge, the goal of mathematics learning' (Warren, 2005, p.305).

Structural thinking can emerge from, or underlie, mathematical concepts, procedures and relationships (Mulligan, et. al., 2010). Mason, Stephens and Watson (2009) argue that structural thinking is more than merely identifying properties of a relationship; it is about using these properties in such a way that it demonstrates a deeper understanding of how these properties are connected. They provide a detailed description of the notion of a *relation*, describing it as a 'connection or association between elements or subsets which have themselves been discerned. When a relationship is seen as instantiation of a property, the relation becomes (part of) a structure' (Ibid, p.1). They argue that recognising a relationship between two or more items is not necessarily structural or relational thinking. Kieran (1989) draws a distinction between two aspects of structure: *surface* structure, for example $2(x-1)$ meaning that 1 is subtracted from x and the difference is multiplied by 2, and *systemic* structure - equivalent forms according to properties of operations where, for example, $2(x-1)$ is the same as $2x-2$ or $(x-1)\times 2$.

The notions described above are what inform the idea of *structural extension* in arithmetic. Structural knowledge, in the domain of sequencing can be understood to work with both the surface structure and systemic structure through extending arithmetic in order to enhance relational aspects that exist between sequenced objects or numbers. I understand structural extension to operate deliberately to decompose sequenced items according to the relational elements that shape the sequencing principles of regularity and change. Structural extension is relational in its approach to sequenced items and functions to express underlying generalised characteristics of sequenced items and use these to advance to a generalised description of these relations. The products of structural extension range from being able to give several other instances of the same property to providing fully developed generalisations (Mason, Stephens and Watson, 2009, p.20). Structural extension is not only an extension of arithmetic, but also a decomposition of the elements that give arithmetic relations their form. Structural extension provides arithmetic with a tool that allows the form of mathematical ideas to become visible and, thus made available, as a cognitive tool.

Structure and Sequencing in Foundation Phase Mathematics

Working structurally with elementary mathematical objects enhances algebraic reasoning habits and skills and should start when learning arithmetic in the primary grades (Carraher, et al., 2006). Working structurally suggests the development of reasoning skills where the objects of reasoning are the relationships between objects rather than the objects themselves (Davydov, 1982; Sfard, 1991; Mulligan, Mitchelmore, English and Robertson, 2010). Learners who recognise and work with structural processes and representations of mathematical ideas develop a sound conceptual understanding of mathematics (Pitta-Pantazi, Gray and Christou, 2004). More recently the focus of research has shifted to the enhancement of structural thinking at an early age (Carraher, Schliemann, Brizuela and Earnest, 2006; Kaput, 2008; Mason, Stephens, and Watson, 2009; Clements and Sarama, 2009; Mulligan, Vale and Stephens, 2009; Mulligan, Mitchelmore, English and Robertson, 2010; Stephens and Allesandro, 2014; Stephens, 2015).

Structural exploration in the domain of sequencing engages the various processes proposed by Kieran (2004, p.19): 'analysing relationships amongst quantities, foregrounding structure, studying variance, generalising and problem solving'. Research with young learners has shown that pattern and structure can be generalised across various content domains in mathematics (Papic, 2007; Vermeulen, 2007). As Stephens (2015) asserts:

> 'Structural thinking is in this sense productive – pointing to the fact that the products of structural thinking can extend from being able to give several other instances of the same property to giving fully developed generalisations across domains in mathematics' (p.1).

Across the mathematical landscape, there are opportunities to work explicitly with structure in its various conceptual embodiments. Learners working with the underlying additive structure of three can realise that the sequences 1; 4; 7; 10; 13; ... and 2; 5; 8; 11; 14; ... and 3; 6; 9; 12; 15; ... all signpost a growth unit of three between consecutive terms, which can then be utilised to move towards a generalised expression for each of the sequences. Merely recognising (or seeing) the growth unit of three does not lead to structural reasoning, according to Mason, Stephens and Watson (2009). They maintain that it is the *use* of this growth component to elicit more generalised ideas which explicate relational aspects as 'instantiations of properties' (Ibid, p.1). Thus, embedded in each of the sequences above is a growth of three (a surface feature of the sequence) and *using* this growth in a process that enhances the structural aspects of the sequence is what Mason et al (2009) define as structural thinking. By making use of the growth element of three, learners can be guided towards more generalised embodiments of the sequences mentioned above. The sequence 1; 4; 7; 10; 13; ... can thus be expressed structurally in various forms as illustrated below in Figure 1.1. Both generalised items do, however, lead to the same general form, even though the initial relational thinking is different.

Term 1	Term 2	Term 3	Term 4	Term 5	General term
1	4	7	10	13	term t
Using the additive / multiplicative reasoning					
1	1+3	1+3+3	1+3+3+3	1+3+3+3+3	$1+(t-1)\times 3$
$1+0\times 3$	$1+1\times 3$	$1+2\times 3$	$1+3\times 3$	$1+4\times 3$	
Using multiples of three to express the generalized term					
3 - 2	6 - 2	9 - 2	12 - 2	15 - 2	$t\times 3-2$
$1\times 3-2$	$2\times 3-2$	$3\times 3-2$	$4\times 3-2$	$5\times 3-2$	

Figure 1.1: Using the additive/multiplicative structure in the number sequence to work towards a generalized expression which can be used to find any term in the sequence.

Patterns, and thus the general notion of sequencing, offer a way of engaging with mathematical quantities that goes beyond the procedural and answer-driven nature of current teaching in arithmetic. The Ontario Ministry of Education states that by 'eliciting multiple solution strategies from learners, teachers should allow learners to think about mathematical structure and engage learners in offering conjectures and justify their thinking' (Ontario Ministry of Education, 2015, p.11). Sequencing should move beyond predicting the next number in the sequence towards developing generalised notions of the sequenced items, and using the surface features in a sequence (growth of three) to give rise to generalised ways of expressing relationships and thinking structurally.

Structural Thinking Enhances Algebraic Reasoning

The research conducted by Hoch and Dreyfuss (2004, 2005, 2006) on structural thinking and also by Mulligan, Mitchelmore, Marston, Highfield and Kemp (2008) and Mulligan, Vale and Stephens (2009), advanced patterns as the means for developing algebraic reasoning skills. It has been shown by numerous researchers that a focus on the structural aspects of arithmetic (Freudenthal, 1987; Davis, 1985; Mason, 1996; Mason, Graham and Johnson-Wilder, 2005) enhances the relational understanding of mathematical concepts in early mathematics learning, which has a positive effect on the numeracy outcomes for younger learners (Carraher, Schliemann, Brizuela and Earnest, 2006; Kaput, 2008; Mason, Stephens, and Watson, 2009; Clements and Sarama, 2009; Mulligan, Vale and Stephens, 2009).

Davis (1985) claimed that learners can build a conceptual awareness prior to the introduction of formal notation, and thus advocated the introduction of algebra to FP learners as young as Grade 3. He maintained that the idea of function can be built intuitively, as learners explore situations that require identification of increasingly more challenging patterns. Furthermore, researchers argued that young learners who were engaged in problem solving activities, which required justifications for their solution strategies, developed a deeper understanding of sophisticated algebraic ideas (Kieran, 1996; Yerushalmy, 2000; Kaput, Carraher and Blanton, 2008; Maher, Powell and Uptegrove, 2010). Kieran (2004, p.141) cautions that considerable adjustments are, however, required in developing an algebraic way of thinking and suggests that these include a focus on:

- relations, and not merely calculations of a numerical answer
- operations, as well as their inverses, and on the related idea of doing/undoing
- both representing and solving a problem, rather than merely solving it

- both numbers and letters, rather than on numbers alone, to include
  - working with letters as unknowns, variables and as parameters
  - accepting unclosed literal expressions as responses
  - comparing expressions for equivalence based on relational properties, rather than numerical evaluation.

Kieran (2004) provides us with a clear roadmap with which early algebra (EA) can build towards the inclusion of greater conceptual depth in the domain of arithmetic, and CA 2 in the CAPS (2010) provides the platform where the necessary skills of working structurally are focused and developed[5].

Working Towards the General

Constructing meaningful relational understanding of algebra and algebraic objects occurs 'within the generational activity of algebra' (Kieran, 2004, p.142). For a mathematical problem to become a meaningful activity that enhances learners' algebraic reasoning skills, relationships are studied by identifying and analysing variance. This will require some form or level of generalisation as mathematical structure is 'expressed in some form of a generalisation' (Mulligan and Mitchelmore, 2009, p4). Working relationally leads to activity that augments and uses the structure of mathematical objects. Relational thinking helps to develop the ability to identify and communicate attributes of objects, as well as the similarities and differences between these objects. Learners who work structurally with arithmetic acquire a deeper conceptual understanding of mathematics, as they are encouraged to move from an operational view[6] of mathematics to a structural view[7] (Sfard, 1991). A structural approach enhances algebraic reasoning skills in which the focus is on general mathematical relationships and on expressing them in an increasingly more sophisticated manner. The shift towards a relational understanding of arithmetic is made possible by *identifying* and *using* the structural aspects embedded in a set of sequenced items, describing structure in words or diagrams, and then finally using symbols to express the generality concisely in order to carry meaning beyond the activity and the context. Young learners are able to make generalisations and develop abstract mathematical skills that make use of mathematical structure (Carraher, Schliemann, Brizuela and Ernest, 2006).

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[5] Early Algebra will be defined and discussed in detail in Chapter 2, pp. 24 to 26.
[6] Focusing on mathematical processes
[7] Focusing on the properties of, and relationships between, mathematical objects

Background to the Problem: Theoretical Perspectives

Structural Extension and Generalisation: An Essential Ingredient for Developing Algebraic Reasoning Skills

The purpose of working with structure in mathematical sequencing activities at an early age is to build towards the notion of algebraic thinking that will support the subsequent learning of more complex, abstract mathematical concepts, which are expressed as generalisations (Kaput, 1999; Lannin, 2005). As Kaput (1999, p.4) argues:

> Generalisation involves *deliberately extending* the range of reasoning or communication *beyond the case* or cases considered, explicitly *identifying* and *exposing commonality across cases*, or lifting the reasoning or communication to a level where the focus is no longer on the cases or situations themselves, but rather on the patterns, procedures, structures, and the relations across and among them (which, in turn, become new, *higher level objects of reasoning* or communication).
>
> (Italics added by author)

Algebraic activity is a deliberate action that moves the discourse and activity beyond the surface and the perceptual to create brand new objects of learning which are made available to be operated upon as general representations of the specific.

> These distinctions, together with theories accounting for the transformation of processes into concepts, have helped to shift attention from doing mathematics to conceptualizing mathematics.
>
> (Gray, Pitta and Tall, 2000, p.2)

The manner in which procedures become conceptual objects has been described by Sfard (1991) as the process of reification. Reification of the primary processes (of arithmetic) is a warranty of relational understanding (Sfard and Linchevski, 1994) and it involves (re)constructing something that is operational, and based mainly on a set of process-orientated underpinnings, into a more sophisticated object which is an encapsulation of the process into a concrete object that is grounded in structural conceptions. This implies a shift towards the more structural and relational view of learning mathematics which pursues regularity to explicate mathematical processes to a generalised encapsulation of primary learning objects.

Arithmetic – Algebra Dichotomy: Developing Mathematics Education Theories

For years algebra and algebraic activity have been believed to be beyond the level of ability of younger learners. This belief was grounded in cognitive and developmental psychology. Jean Piaget (1896-1980) suggested that algebra requires a skill-set that develops over time and that cognitive ability advances progressively as the learner matures. The ability of learners to deal with abstract notions was linked to their development, and it was widely held that young learners were not capable or cognitively ready to deal with the abstract. This belief shaped the conviction of 'arithmetic first, and algebra later' and resulted in worldwide curricula that reinforced this dichotomous view of arithmetic and algebra. The mathematical activity that resulted as a consequence of this, namely 'transition mathematics', was viewed as inherently transitory in nature, as it was meant to serve as a bridge between arithmetic and algebra (Herscovics and Kieran, 1980; Kieran, 1985; Vergnaud, 1988; Vergnaud, Cortes and Favre-Artigue, 1988; Filloy and Rojano, 1989; Bernarz, 2001; Fujii and Stephens, 2001). The aim of transition mathematics was to narrow the cognitive gap[8] (Herscovics and Linchevski, 1994; Sfard and Linchevski, 1994) between arithmetic and algebra, and to provide a stepping stone to transition from arithmetical thinking to algebraic thinking. This view of primary mathematics held that there was a discernible point where learners crossed over from arithmetic to algebra (Filloy and Rojano, 1989). The curriculum in South Africa, prior to 1994, was founded on the same sequence and philosophical underpinnings. The National Council of Teachers of Mathematics (NCTM) in the United States of America until 1988 advocated a transitional approach to the introduction of algebra by introducing algebra as *generalised arithmetic* in Grades 5 to 8, and then studying the *logic and structures* of algebra from Grades 9 to 12.

Towards Early Algebra: A Symbiosis

In 2000, the NCTM's *Principles and Standards for School Mathematics* endorsed the view that algebraic reasoning should be nurtured from as early as kindergarten and that algebraic notation form part of mathematics from Grade 3 onwards. Researchers are increasingly advancing the view that algebra has an important place in early mathematics curricula and underscores a symbiotic approach to arithmetic and algebra. Schliemann et. al. (2002) cautions that EA[9] promotes 'teaching arithmetic and other topics of early mathematics more deeply, and not teaching algebra earlier' (p.101). They raise the following questions in their book entitled 'Bringing out the Algebraic Character of Arithmetic: From Children's Ideas to Classroom Practice':

[8] Van Amerom (2002) also refers to the *didactical cut* which characterises the troublesome transition from arithmetic to algebra (p.3).

[9] Early Algebra will be defined and discussed in depth in Chapter 2, p.24 to 26.

> People generally believe that arithmetic should precede algebra in the curriculum. And they can find ample evidence to support their view: Arithmetic is easy; algebra is difficult. Arithmetic is about operations involving particular numbers; algebra is about generalised numbers. Arithmetic appears in all cultures; algebra appears only in some, and, even in those, it made its appearance only recently. But what if there are good reasons for thinking otherwise? What if there were compelling research showing that young students can learn algebra? What if there were good mathematical justifications for teaching algebra early? What if history is not always a trustworthy guide for ordering topics in the mathematics curriculum?
>
> (p.6)

Schliemann et. al. (Ibid) propose a view that arithmetic is part of algebra and that the categories of arithmetic and algebra are not mutually exclusive' (Ibid, p.10). Kaput (1995) posits that algebraic thinking throughout the primary school curriculum will lend coherence, depth and power to school mathematics. This is supported by other researchers (Vergnaud, 1988; Davydof, 1991; Bodanskii, 1991; Schoenveld, 1995), who advocate that the 'full range of algebraic activity be present in all the processes of teaching and learning of mathematics' (Schliemann et al, 2002, p.26). Subramaniam and Banerjee (2011) suggests that the role of algebra be viewed explicitly as a foundation rather than a mere generalisation of arithmetic. These views convey a symbiotic view of algebra and arithmetic, and suggest that arithmetic is merely a subset within algebra. It is becoming clear that algebraic activity in the FP classroom will strengthen arithmetic and general mathematical cognition.

Research with Young Learners: Algebra in Elementary School

Research conducted with young students, who were engaged in activities that involve algebraic reasoning and algebraic representation, investigated how young learners deal with algebraic concepts and notation. The results all point to the feasibility of introducing algebraic reasoning in elementary school (Schliemann et al, 2002, p.27). In one study, conducted by Davydov and his team of researchers in 1991, the results showed that six-year-olds, who received instruction on using algebraic notation when working on verbal problems, outperformed their peers throughout the school year and achieved better results in algebra problem solving than their peers in Grades 6 and 7 who followed the traditional approach to instruction that places algebra after arithmetic. Booth (1988) maintains that the 'difficulties that students experience in algebra are not so much difficulties in algebra itself as problems in arithmetic that remain uncorrected' (p.29). Booth explains that the traditional primary mathematics curriculum focuses mainly on the computational aspects of arithmetic and that this makes it difficult for learners to work algebraically later on. Booth explains that learners' difficulties in beginner algebra result from an inconsistent use of letters in arithmetic and algebra: the letters *m* and *c* are used as labels that represent metres and cents in arithmetic and not the *number of* metres or the *number of* cents, as they would in algebra. Research also shows that learners' difficulties with understanding letters as variables may be due to shortcomings in instruction (Carpenter and Levi, 2000; Carpenter and Franke, 2001). These studies indicate clearly that learning and understanding algebra and its structures is possible at a young age, and that learners' difficulties are due to constrained instruction in arithmetic. Researchers are increasingly calling for the inclusion of EA in the elementary mathematics experience, with some proposing that algebra be introduced as early as pre-school.

The Need to Generalise in Arithmetic

Generalisation forms part of the process of developing algebraic reasoning (Kaput, 2004; Mason, Stephens and Watson, 2009). Mason (1996) argues that elementary mathematics must focus on developing generalisation habits. Schifter (1999) reported implicit algebraic reasoning and generalisations in a class where young learners were reasoning about mathematical relations. Third graders make robust generalisations and provide intuitive supporting arguments when discussing operations on odd and even numbers, considering these as placeholders or variables (Blanton and Kaput, 2000). Seven-year-old learners were found to display fluency at handling the basic logic that underpins additive transformations in equations (Carraher, Schliemann and Brizuela, 2000; Carraher, Brizuela and Earnest, 2001). Blanton and Kaput (2004) provide compelling evidence that young learners can understand equivalence and also form generalisations, and that they can develop functional thinking and engage in co-variation thinking as early as kindergarten. They propose that primary grade mathematics should extend beyond patterns that focus on uni-variate relations to include functional thinking as well. Mason (1996) and Lins and Kaput (2004) argue that learners have a natural inclination to generalise and express generalities and that tapping into this naturally occurring capacity, for pedagogical purposes, can foster the development of algebraic reasoning skills.

It is clear that learners, at an early age, can engage in practices that are inherently algebraic, and can create and use algebraic objects coherently and with flexibility. Learners are also able to reason algebraically and generalise across items using variables and placeholders to argue, and work with, both uni-variate and co-variate relationships.

Research on Teachers Using Structure in Arithmetic

Much of the research in the 1980s and 1990s on early numeracy, which focused on the development of arithmetic strategies, influenced further research on the relationship between arithmetic structure and algebraic thinking (Carpenter, Franke and Levi, 2003; Irwin, 1996; Warren, 2003). There is a considerable body of research showing that low-achieving students, of all ages, have a poor grasp of mathematical patterns and structure (Mulligan, Mitchelmore and Prescott, 2006). Young learners can be taught to seek and recognise mathematical patterns and structure, and the effect of this structural approach on their overall mathematics achievement can be substantial (Ibid, p.4). These findings arose over a series of research studies conducted since 2001 in Australia by the Pattern and Structure Mathematics Awareness Project (PASMAP), which focused on the development of patterning and early algebraic reasoning among four to eight-year-old learners. The PASMAP aimed at developing teachers' pedagogical knowledge and their awareness of learners' use of pattern and structure across key mathematical concepts. The PASMAP also aimed to advance a strong foundation for mathematical development through a focus on critical underlying features of mathematics learning at a much earlier stage than was previously thought possible (Mulligan, Mitchelmore, English and Crevensten, 2010, p.1). The effects of the PASMAP intervention on young, low-achieving learners, aged between four and six years, suggest that explicit assessment and teaching of mathematical pattern and structure, delivered in a manner suitable for low-achieving learners, has the potential to improve learners' acquisition of key mathematical concepts and processes effectively within a relatively short time-frame (Mulligan, Prescott, Papic and Mitchelmore, 2006). The PASMAP has also highlighted the critical importance of the type, level and consistency of teacher support required during academic assistance, and the need for additionally trained support staff and provision of practical resources (Mulligan et al., 2006).

South African Research on Structure and Early Algebra

In South Africa, Liebenberg, Linchevski, Olivier and Sasman (1998) worked with structure in Grades 6 and 7 using a transition view of arithmetic to algebra, and worked on developing a sense of structure in a numerical context. They emphasised the structure of expressions while still doing calculations to verify the equivalence of expressions. Liebenberg et.al. (1998) conclude that focusing on the structural aspects of numerical expressions will improve learners' ability to calculate. Other research includes the work of Vermeulen (2007) who, in an analysis of the curriculum documents, investigated the extent to which the Revisited National Curriculum Statement (RNCS) encourages and supports the development of algebraic reasoning skills in the early grades in South Africa. He concludes by raising concerns with regard to the elusive reference in the RNCS to algebra at FP level, and cautions that algebra must be treated as building from FP through to Grade 12 level.

> Do they (*teachers and teacher educators*) read into the curriculum that algebra is not only a high school topic, but that its roots lie in and must be developed right from the Foundation Phase, through the Intermediate and Senior Phases, with increasing levels of generalisation and symbolisation?
>
> (Vermeulen, 2007, p31)

A study was also conducted by Roberts (2010), which compared the RNCS of South Africa with the British curriculum, and concluded that the RNCS, in contrast to the British curriculum, pays only 'lip service' to the notion of EA.

The Statement of the Problem

While the importance of the conceptual links between the triad of pattern, structure and algebraic reasoning has been established in the literature, methods of working with teachers to promote this awareness still require substantial research. There are three areas of interest related to the teaching of sequencing at FP level in South Africa: (i) the support given to teachers through resource texts that can enhance an EA approach to teaching CA 2 at FP, (ii) the prominence given to developing the algebraic reasoning skills of FP learners and (iii) a structural approach to teaching sequencing in FP CA 2.

In South Africa there is no research available on teachers' classroom work with structure at FP level, and these waters remain uncharted despite the reference in the curriculum documents to learners 'seeing' and *using* the 'logic' of the pattern. No research is available in the South African context with regard to FP teachers' teaching of sequencing using structural extension in CA 2, particularly addressing pedagogical shifts which enhance the development of algebraic reasoning skills in CA 2 and thus advancing the notion of EA at FP. International research has shown that an approach which enhances structure has a positive effect on the numeracy outcomes of young learners (Mulligan et. al., 2011), and this suggests it will be useful to explore the use of structure in the South African FP classroom.

The CAPS (2010) emphasises CA 1 with the aim being to 'secure operational fluency' (DbE, 2011, p.8). As a result, FP teachers focus on an operational approach to arithmetic without enhancing learners' relational thinking. Particularly in CA 2, the need to guide learners successfully through the development of their algebraic reasoning abilities means that teachers themselves need to have an in-depth knowledge of the mathematics and the processes involved (Ma, 1999; Dobrynina and Tsankova, 2005). The facilitation of structural extension involves a level of understanding of mathematical concepts and processes that lie embedded beneath the surface, and the engagement of specialist and deliberate interventions by the teacher is essential to bringing these relations to the surface for the learner. To understand and develop current practice in FP mathematics, it is necessary to investigate what is made available for the learning of sequencing in FP CA 2 in order to establish the degree to which EA is included in the pedagogical approached taken by FP teachers.

To investigate these matters, this study focused on answering three research questions.

Research Questions

1. To what extent and in what ways do the curriculum documents, the supporting materials and the materials used for teaching encourage and support teachers in their development of structural thinking and algebraic reasoning when working with sequencing?
2. How do teachers work with sequencing at a Grade 2 level?
3. How are teachers able to change or supplement their teaching after a series of workshops on algebraic reasoning and structural thinking?

Significance of the Study

For teachers to instil an understanding of structure in arithmetic in their learners, requires the effective use of the essential pedagogical resources across various situations and contexts. Making sense of the relationships between the components of an expression relies heavily on the understanding of structural relationships (Liebenberg, Linchevski, Olivier and Sasman, 1998). Research in the area of the teaching and learning of algebra has indicated the importance of understanding the structure of arithmetical expressions in order to make sense of algebraic expressions and to facilitate their manipulation (Subramaniam and Bannerjee, 2004; Banerjee and Subramaniam, 2005; Warren, 2004 and 2005). In the mathematical domain of sequencing, it has become increasingly important to use pattern as a pedagogical and cognitive tool (Zazkis and Liljedahl, 2004; Mason, 1996; Threlfall, 1999). This positions *pattern* beyond being merely content that must be covered in the curriculum in CA 2, to being a conceptual tool that deals with sequencing items, and structurally extending those items in order to move beyond the realm of perception into the territory of algebraic activity and generalisation. The pedagogical choices that teachers make, and the discourse and materials they use to enact and activate highly specialised reasoning processes, are key in shaping learners' ability to deal with patterns in a sophisticated manner. Collectively they will ultimately advance numeracy in the FP classroom.

This study recognises CA 2 as focusing on developing the specialised knowledge of working with an unknown, advancing the understanding and skills of sequencing and generalisation, and thus supporting and enhancing EA at FP level. The unique contribution this study will make is based on its focus on the use of structure and structural extension by FP teachers when working in CA 2. The research design enables a rigorous study of a series of intervention workshops, designed to introduce FP teachers to the notions of structure and structural extension, to advance the broader pedagogical and didactical choices that teachers make when teaching sequencing in the FP classroom.

Defining Key Mathematical Concepts

This study works with the following key mathematical concepts:

- A *sequence* of items is any set of shapes, objects or numbers that are presented in an ordered form, through the application of a distinct regularity.
- *Growth patterns* are patterns that are formed by sequencing items using the principle of numbers growing in magnitude, or objects growing in size or quantity. This growth can be positive (increasing) or negative (decreasing). Examples are the sequence of numbers 3; 6; 9; ... which possesses constant additive growth, or the sequence 2; 4; 8; ... which possesses multiplicative growth.
- A *number pattern* is a sequenced set of numbers with an underlying growth structure.
- A *repeated pattern* is a pattern that shows cyclic structure where the *core* of the pattern is identified as unique to the pattern, and the component of the pattern that repeats. The *form* of the pattern is recognised across patterns that are similar. The core of the pattern is always uninterrupted. The form of the pattern is expressed using symbol notation. A simple example will be the pattern 123123123, with core elements 123 and the form ABC.

- *Structure* is the meta-level description of the relational features that exist between sequenced items. It is the statement of the principle that underlies the production of the next item in a sequence.
- *Structural extension* is the process by which arithmetical and spatial features are decomposed into the key relational aspects that allow for the emergence of a generalised understanding across items.
- *Spatial structure* is understood to be the invariant relational attributes that exists in geometric shapes, in arrays and in grids.

Organisation of the Thesis

This thesis is organised into eight chapters. The first chapter provides an introduction to the study by considering the numeracy demands and shortfalls in the South African mathematics classroom. The triadic notion of pattern, structure and algebraic reasoning is discussed as a tool that can help in the attainment of basic levels of numeracy in the FP. The broader notions of structure are discussed and defined. Chapter 2 provides a review of the literature pertaining to the triad of pattern, structure and algebraic reasoning. Chapter 3 provides a description of the research methodology and design. The document analysis is discussed in Chapter 4, and Chapter 5 contains a description and analysis of the series of three intervention workshops. Chapter 6 provides an analysis of the broader notion of sequencing in the FP classrooms studied, by combining observations of both pre- and post-intervention lessons and teaching. Chapter 7 provides insight into the use of structure in the teaching of sequencing in FP, from a pre- and post-intervention workshop perspective. Chapter 8 summarises the findings, recommendations and conclusions of the study, and comments on its limitations and the implications for teaching, learning and future research.