Grade 10 learners' interpretations of graphical representations of functions

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Declaration

I declare that this research report is my own work, unless indicated as quotes, acknowledged in the text and the references. It is being submitted in fulfilment of the requirement for the degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other institution.

Signature of candidate

29th day of May 2019

Abstract

This study investigates the approaches and key features that Grade 10 learners used to interpret graphical representations of linear and quadratic functions. Data was collected with Grade 10 learners at a high school in the inner city of Johannesburg by means of two written tests and task-based interviews. The first test was used to select four learners to participate in the interviews and to guide the design of the interview tasks. A combination of approaches from Even (1998) and Thompson and Carlson (2017) was used as the lens to analyse the interview data. Findings revealed that learners reasoned visually, analytically or used combination of the two to interpret graphical representation tasks. Visual reasoning typically resulted in learners reading the graph as a picture labelled with numbers, with no attention to coordinating the *x*- and *y*-coordinates of points. However, as learners moved towards a pointwise approach, their analytical reasoning got stronger. Findings also revealed that intercepts was a dominant feature that learners used to interpret graphical representations of functions.

Key words: interpretation, reasoning, functions, approaches, graphical features

Dedication

I dedicate this study to my husband, Caswell, my soul mate, for supporting and encouraging me to endure till I finish this research report.

And

To my children Zakhele, Princess and Mpilwenhle. I know it was not a pleasant journey for all of you, especially my Princess getting burnt while trying to make food. I will restore the moments you missed being with me.

Acknowledgement

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A big thank you to my principal for affording me the opportunity to collect data from the school. I am indebted to the learners that participated in this study, especially the four learners who willingly contributed in the interviews.

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Chapter 1 : Introducing the study

1.1 Introduction

Mathematics is a crucial subject in the lives of many learners. According to the school curriculum, mathematics consists of a number of different topics including Algebra and Functions, Trigonometry, Geometry, Statistics and Probability (Department of Basic Education, 2011). Algebra and functions is seen as a major topic in mathematics and Usiskin (2004) states that it is the gatekeeper to and a prerequisite for all other mathematics. How and what learners learn in algebra and functions is therefore of great importance for the application of mathematical techniques in other mathematical topics and in higher education. According to the National Senior Certificate Diagnostic Report (Department of Basic Education, 2016) and the South African curriculum, functions is one of the topics that learners have difficulty in grasping. This is also foregrounded in previous research, which has found that learners experience difficulty in reasoning covariationally (Thompson & Carlson, 2017). This study sought to examine the way in which learners make sense of graphical representations of functions at Grade 10 level.

1.2 Background to the study

In my years of teaching mathematics, I have observed many learners struggling with the concept of functions. Previous research has shown that functions play a major role in mathematics as they represent relationships amongst phenomena (Leinhardt, Zaslavsky & Stein, 1990; Ponte, 1992). Functions consists of different representations such as tables, equations, graphs and verbal situations (Even, 1998; Kalchman & Koedinger, 2005; Leinhardt et al., 1990). The graphical representation of functions thus plays an important role in unifying concepts in mathematics (Denbel, 2015). However, many learners have difficulty in answering questions requiring them to interpret the graphical representation of functions in Grades 10, 11 and 12. This difficulty is one of the factors that contribute to the poor performance in Paper 1 of the Senior Certificate examination. Consequently, learners' struggles in this regard as well as their poor performance motivated me to conduct this research study, which aims at investigating Grade 10 learners' interpretation of graphical representations of linear and quadratic functions.

Research in the 1990s focused on the functional view of algebra (Ponte, 1992), in terms of which the curricula continually elaborated that "the Algebra Standards for Grades K-12 emphasized the relationships among quantities including ways of representing mathematical relationships" (Kieran, 2007, p. 709). Such research showed that most learners encounter difficulties in the interpretation of graphs because they fail to understand the meaning displayed by a graph (see for example Ayalon, Watson, & Lerman, 2015a, 2015b; Even, 1998; Oehrtman, Carlson, & Thompson, 2008). The bulk of this previous research focused on tertiary levels, while Ayalon et al. (2015a) focused on the curriculum in elementary and secondary grades in the context of Israel and the United Kingdom. However, there seems to be a literature gap in the secondary levels of South African schools concerning the interpretation of the graphical representation of functions. My research study seeks to bridge this gap.

My involvement in the professional development of teachers in the Wits Maths Connect Secondary project during the Transition Mathematics course sessions and the high school lesson studies prompted my interest in digging deeper into learners' learning processes, especially with regard to functions. It further motivated me to conduct this research study in order to gain a deeper understanding of the way learners interpret graphical representations of functions. The next section elaborates on the nature of graph interpretation.

1.3 What is graph interpretation?

According to Glazer (2011, p. 190), graph interpretation in general "refers to a graph reader's ability to obtain meaning from graphs created by others or by themselves". He further highlights that graph interpretation is an important skill that all learners require for their everyday lives in order to make sense of the information communicated using graphs, such as those found in newspapers and magazines. However, the general definition of graphical interpretation differs from the mathematical definition in terms of teachers' perspective and those provided in the literature. On the one hand, most teachers in South Africa view the interpretation of graphs as referring to answering questions on graphs such as finding the lengths of horizontal and vertical lines, the equation of a graph, the coordinates, domain and range of a function and reading the values of *x or y* for the inequality functions (Smith, 2010). This is usually done if one or two graphs are sketched on the same system of axes. Figure 1.1 provides an example demonstrating the interpretation of graphs from the teacher's perspective extracted from Smith (2010, p.111).

The example in Figure 1.1 (Smith, 2010, p. 110) illustrates a task representing the quadratic and linear graphs drawn on the same set of axes. This example is similar to the questions that usually appear in the National Senior Certificate examination papers (P1) where functions contribute ±30% to the content of Paper 1 of the Grade 10 mathematics in South Africa (Department of Basic Education, 2011). However, learners often struggle with these questions as stated in the 2016 National Diagnostic Report (Department of Basic Education, 2017).

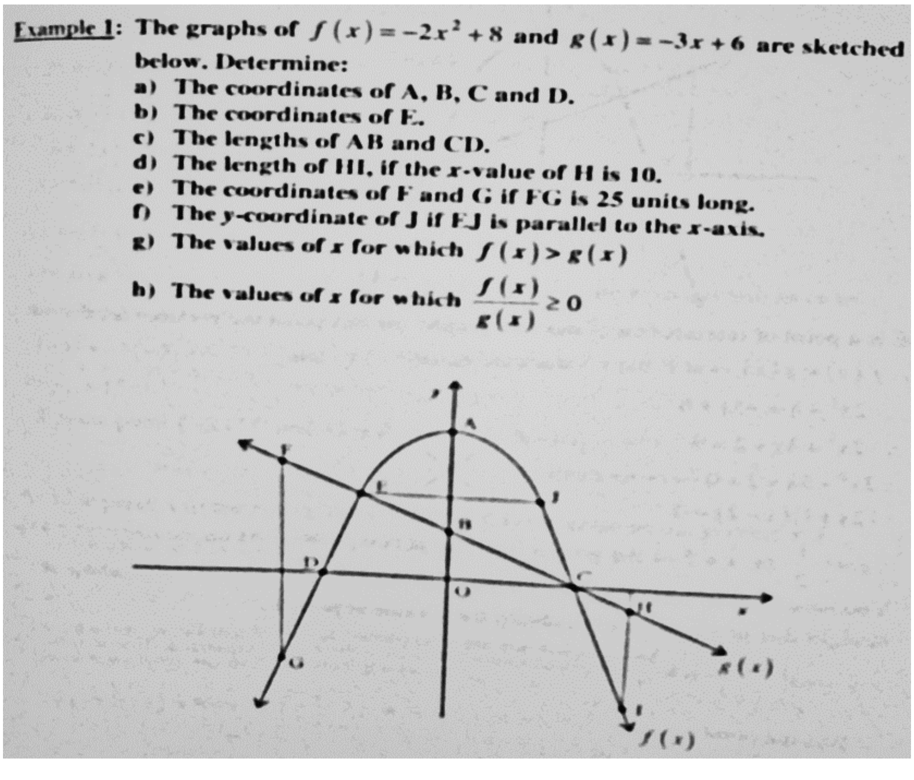


Figure 1.1: *Example of a question on the interpretation of graphs*

Nevertheless, the literature shows that the interpretation of graphs is about being able to write in words what one sees in a graphical representation in terms of how the axes are labelled, what the scale factor is, what the graph represents and what mathematical information can be extracted from the graph (Bell & Janvier, 1981; Leinhardt et al., 1990; Oehrtman et al., 2008). According to Leinhardt et al. (1990) the interpretation of graphs refers to learner's ability to look at the entire graph and make sense of the graph with regard to the relationships between the two variables. Most learners have difficulty in interpreting graphs because they fail to understand the meaning such graph convey (Even, 1998). Bell and Janvier (1981) mentioned that making sense of a graph can be done by translating the graph into an equation or a situation it represents. This means that the meaning gained from a graph "can either reside within the symbolic space of the graph or it can shift to a different space; the situation space or the algebraic space" (Leinhardt et al., 1990, p. 8). Therefore, spending time studying a graph prior to engaging with questions such as finding the values of x for which $f(x) > g(x)$, is a crucial step in interpreting graphs. According to my experience of teaching and the Grade 10 curriculum standard, the importance of studying a graph is not usually emphasised owing to the time allocation and the pressure of completing the curriculum.

1.4 Problem statement

The 2015 National Senior Certificate Diagnostic Report stated that questions on functions were poorly answered in the examination and that one of the areas of difficulty is that learners "did not know what the restricted domain meant and how it impacted on the straight line graph" (Department of Basic Education, 2016, p. 156). A similar performance was noted in the 2016 National Senior Certificate Diagnostic Report in that the transformation of graphs was poorly answered by most learners and as a result, only 5% of learners got the correct answer (Department of Basic Education, 2011). It was accordingly stated that "there is clearly a lack of understanding in obtaining a solution from a transformed graph" (Department of Basic Education, 2016, p. 157). This could be attributed to the fact that in Grade 10, learners are introduced to six new functions (parabola, exponential, hyperbola, sin, cos and tan graphs) within a short period of time, five weeks of teaching and learning to be precise - as stipulated in the Annual Teaching Plan (Department of Basic Education, 2011), which results in little time being spent on the interpretation of graphs.

In March 2017, I attended a cluster meeting where the 2016 matric performance in mathematics was discussed. At that meeting, matric teachers (who had marked the examination scripts) highlighted that learners' poor performance in the interpretation of functions was not improving. The district officials then made a plea to educators to come up with strategies that would improve learners' performance in functions. Accordingly, this study aimed at investigating learners' interpretation of graphical representations of functions at Grade 10 level. The results will be of great benefit for developing strategies to improve learner performance in mathematics, especially the section on functions.

1.5 Difficulties encountered with the initial analysis

Researchers often mention that doing a research study is not a linear process; I strongly agree with them because I experienced this. At the beginning of the study, I aimed at classifying the approaches that learners used to interpret graphs. using Even's notions of pointwise or global approach (Even, 1998) to develop a framework for analysing the interview data. However, as I used this framework to analyse the interview data, I experienced difficulties in getting it to fit the data and I felt that the framework was not really capturing the essence of what learners had written or talked about in the interview. This resulted in several adjustments being made to the framework which did not help as I still battled to fit the framework to the data. As the result, my supervisors advised me to abandon the framework and focus on the interpretation of data by looking at what each learner did to interpret graphs. This brought about the main contribution of this study which relates to the fact that some Grade 10 learners operated below the pointwise approach and interpreted the graph as a picture with numbers. Detailed information of my struggles with the initial analysis is given in Chapter 3 under section 3.9. This difficulty also gave an indication of the complexity of dealing with graphical representations of functions.

1.6 Purpose of the study

I started this study with an interest in looking at learners' mathematical thinking when interpreting graphs. However, as the study evolved, I described the purpose of mathematical thinking in the following ways:

- To explore the key features that learners attend to when making sense of the graphical representations of both linear and quadratic functions
- To investigate the approaches that Grade 10 learners use to interpret graphs

Knowing more about these issues will be of great benefit to mathematics educators, assisting them to comprehend the difficulties that learners face when dealing with graph interpretation.

1.7 Research questions

The objective of my study was to investigate learners' mathematical thinking when interpreting graphical representations of functions at Grade 10 level. My research study sought to answer the following critical questions:

1. *What key features of functions do learners pay attention to when responding to graphical representation tasks?*
2. *What approaches do learners use when working on graphical representation tasks?*

There is a close connection between the two research questions. The difference between them lies in the fact that the key features mentioned in research question 1 relate to the properties of the graph that learners use to make sense of the graph. However, the approach referred to in research question 2 addresses the ways in which learners apply these key features to draw conclusions on the behaviour of the function. The approach stems from the conception of the function as either process/object or correspondence/covariation or pointwise/global. These concepts are described in the next chapter.

1.8 Conclusion

In this chapter, I outlined the background to the study, the problem statement and the research questions. I also provided a definition of graph interpretation from teachers' perspective and from the literature, which guided the focus of the study. In addition, I provided my reflection on my initial experience while conducting this research, relating how I started the study with a broad understanding of mathematical thinking which contributed to a shift in the analysis of the interview data. Having done the research, I was able to break down mathematical thinking on functions into the key features and approaches that learners used to attempt graphical representation tasks.

This research report consists of six chapters. Chapter 2 outlines the learning theory, conceptual framework and reviews the literature on graphical representation of functions. To back up my study, social constructivism was the learning theory applied. I subsequently discuss the literature on the interpretation of functions as well as the mathematical language used when dealing with graphs. Chapter 3 provides the research design and methodology for collecting data. I used interviews (pre- and post-interviews) as a means of collecting the primary data, while the secondary data was collected using a pre- and post-test. I also conducted the intervention lessons prior to the post-test and post-interviews, however, the intervention lessons did not form part of the data analysis. Chapter 4 discusses the setup of an analytical framework and its use for analysing the interview data. It also provides the exemplar for applying the framework when analysing the interview data. Chapter 5 presents the analysis of the interview data, focusing on the key features and the approaches learners used to interpret graphs. Chapter 6 provides a discussion of the findings, the contribution of the study to the body of knowledge, my reflection while doing the research as well as the conclusion to the study.

Chapter 2 : Conceptual Framework and Literature Review

2.1 Introduction

In this chapter, I review the literature in regard to the conceptual framework for the interpretation of functions which guided my study. I begin the chapter by providing the learning theory on socio-constructivism, highlighting the way in which learners interact with other learners or the teacher to construct their own knowledge on functions. I then elaborate on the Curriculum and Assessment Policy Statement (CAPS) concerning the graphical representation of functions. This is followed by a mathematical analysis of the graphical representation of linear and quadratic functions in terms of the features of the two types of graph. I then provide the review of literature on the interpretation of functions. Lastly, I discuss the literature on how to approach and conceptualise graphical representations.

2.2 Learning theory: socio-cultural theory

Social constructivism is a theoretical perspective that acknowledges that knowledge is socially constructed through interaction with other people (Daniels, 2009; Vygotsky, 1978). Accordingly, my research study was informed by socio-cultural perspective. Such a perspectives focuses mainly on the fact that learning and the development of knowledge depends on society and the culture of the learning environment. In a school environment, learners interact with other individuals (teachers and other learners) based on the classroom culture, where learners are actively involved in constructing their own knowledge (English, 1995; Ernest, 1996). This means that the learning of mathematics is an active process that requires learners to construct their own meaning as they engage with mathematical activities (English, 1995).

Constructivism consists of two major levels of development: the actual development level (ADL) and the potential evelopment evel (PDL) (Vygotsky, 1978). The ADL refers to the mental functioning of learners that have been developed, whereas the PDL refers to those functions that a learner can potentially attain but which have not yet been developed. In the following sub sections I begin by elaborating on how the PDL could be transformed into ADL through the Zone of Proximal Development (ZPD). This is followed by a brief description of the mediators used during social interaction or the ZPD as well as how concepts are constructed. Lastly, I elaborate on the learning trajectories when constructing mathematical knowledge on functions.

2.2.1 Vygotsky's Zone of Proximal Development

Vygotsky (1978) claims that learning takes place in the ZPD. Donald, Lazarus and Lolwana (2007, p. 59) define the ZPD as "that critical space where a learner cannot quite understand something on her own but has the potential to do so through proximal interaction with another person". Such interaction may take place between a knowledgeable person and a learner to reach mutual agreement. The ZPD is the gap between what a learner has mastered (ADL) and what he/she would be able to master (PDL) with the assistance of a more knowledgeable person (Vygotsky, 1978). The way in which the learner interacts with the task, mediators and the reseacher at an adequate level of the ZPD may result in development. According to Vygotsky (1978), if an activity is pitched to learners' developmental level, there is a high possibility that they will grasp the activity, whereas if the same activity is pitched at a higher level, understanding of that activity is not guaranteed. This implies that if a learner's ZPD is large, the jump in content knowledge for that learner will also be large. Hence, a teacher would be expected to break up the content knowledge or scaffold the activity to suit the individual's current level of development. Accordingly, the PDL may be transformed to the ADL through the ZPD by means of suitable mediation.

2.2.2 Mediation

Social constructivism acknowledges that learning is socially mediated through the use of signs, tools and artefacts (Daniels, 2009; Vygotsky, 1978). These mediators bridge the gap in psychological meaning between the external social world and the internal mental processes or ADL and PDL. Daniels (2009) argues that tools and signs are means of strengthening and transforming conceptual understanding. He further highlights that tools, signs or artifacts are not the major focus, because the most important thing is the meaning encoded in them. When investigating how learners interpret graphical representation tasks, the most important thing is to examine the connection that learners engage in, in order to make sense of what the graph represents and to be able to respond to the task.

Signs, tools and artefacts are internally or externally oriented human responses such as the words, symbols and diagrams (Daniels, 2009) used to master a task. Functions, for example, are differentiated by means of words such as *linear, quadratic and cubic* and these functions are further represented by the symbols $y = x, y = x^2 and\ y = x^3$respectively. In addition, explicit words and symbols are used in each group of functions. For example, the word 'gradient' in reference to a straight line graph represented by symbol m is used to refer to the steepness of a linear function. The signifiers called *graphs* are the tools used to mediate learners' thinking on the nature and behaviour of a function when interpreting functions. These diagrams are used to differentiate the graphical representation of linear and quadratic functions, which are referred to as straight-line and parabola graphs respectively. Learners generally create artefacts in order to understand a task or a concept (Vygotsky, 1978). Caspi and Sfard (2012) refer to artefacts as metaphors, which are verbal or ideographic signifiers representing the same entities. When describing the shape of a parabola graph, one might use a 'happy smile' to refer to concave up or 'sad smile' to refer to concave down. These artefacts or metaphors are deeply embedded in the structural meaning or scenario. Accordingly, this study investigated how learners interpret graphs, which means how the signs, tools and artefacts for the graphical representation of functions illuminate the key features that enabled learners to respond to the task.

2.3 Graphical representation of functions in the curriculum and in textbooks

According to the Curriculum and Assessment Policy Statement (CAPS) for Grade 10 (Department of Basic Education, 2011), the concept of functions is covered in term 2 for a duration of five weeks. The work coverage in the CAPS document includes the straight line, parabola, hyperbola, exponential graphs as well as basic trigonometric graphs of sin θ, cos θ and tan θ in a form of $y = a. f(x) + c$. The syllabus covers the point by point plotting of graphs, investigating the effect of a and q, finding the equations of given graphs and interpreting graphs. This is a huge transition from Grade 9 to Grade 10, since learners are introduced to a straight line graph in Grade 8 and this is further investigated in Grade 9. Learning of functions is further complicated by trigonometric graphs that use angles (in degrees) as the domain as opposed to real numbers.

Research conducted by Leshota (2015) on the affordances of mathematics textbooks in South Africa found that the content relating to the teaching and learning of functions covers an interpretive approach, developing from pointwise to global action. She further indicates the sequence of teaching functions in terms of the approaches used:

> CAPS textbook reflects an approach to the teaching of Functions that begins with point by point strategies in content areas taught at the beginning of the topic; followed by a mixture of pointwise and global actions where properties of Functions and their transformations are determined, and then becoming completely global at the end of the topic where functional properties are being interpreted. This content-specific approach to the teaching of Functions constitutes yet another affordance of the textbook to the teacher's practice. (Leshota, 2015, p. 100)

The preceding quote indicates that South African textbooks express the interpretation of graphical properties in a strictly global manner. However, from my experience of teaching the interpretation of graphs, the actual learning process takes place from pointwise approach which gradually moves towards a global approach. Again, based on my teaching experience, the section in the curriculum on the interpretation of graphs is covered in one or two days in order mainly to consolidate the concepts of functions.

2.4 Mathematical analysis and key features for interpreting graphs

It is of great importance to unpack the mathematics involved when interpreting graphs at high school level. This was done by outlining the mathematical knowledge and skills required to interpret graphs, as well as the key graphical features. Underpinning the mathematical analysis involved in dealing with functions was a preliminary step for interpreting graphs. Even (1998) highlights three important factors necessary to make sense of the behaviour of the graph. These factors include the role of parameters, the connection between representations, as well as function notation. A brief description of these factors is provided in the next subsection where the focus is on linear and quadratic functions.

2.4.1 Role of parameters

The role of parameters in the general equation of a family of each functional group plays an important role. It is important for learners to differentiate a linear function from a quadratic function by recognising the degree of the function. Functions can be represented in different forms; in terms of graphs, equations, tables and verbal representations. The key features in each graphical representation of a functional group are discussed below.

a) Key features of the linear function

The key features such as intercepts, gradients, increasing, decreasing, range, domain and transformation are used to describe the linear function. However, in this study the key features that stood out were the intercepts, gradient, increasing, decreasing and transformation. Therefore, I focused in these features.

The graphical representation of a linear function is called a straight-line graph and its equation is written in the form of $y = mx + c$. It is of great importance for learners to know what *y, m, x and c* represents. It is therefore important to consider the effect of the parameter *m* and *c* in a straight-line graph. Figure 2.1 shows the role of parameter *c* as the *y*-intercept and also in transforming the graph by shifting it vertically. When contrasting the effect of adding *c*; positive *c* means that the graph shifts up and negative *c* implies that the graph shifts down.

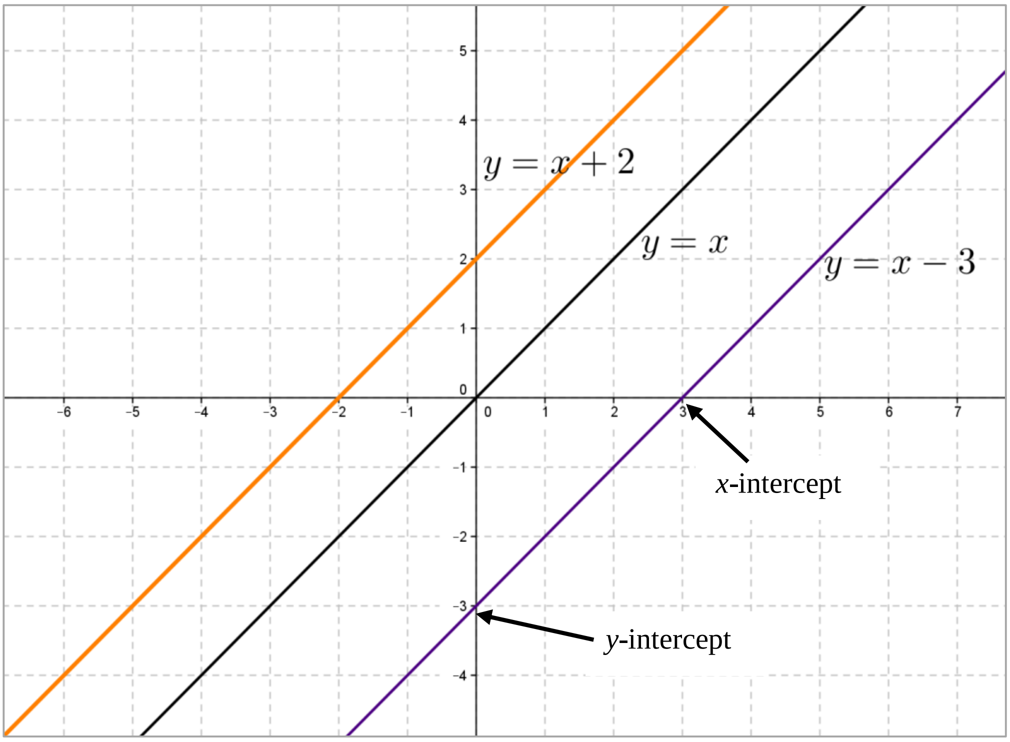


Figure 2.1: *Effect of c on a straight-line graph*

In school mathematics, the intercept of a graphical representation is classified into *x*- and *y*-intercepts, as illustrated in Figure 2.1. The *x*-intercept is the point on the Cartesian plane that crosses the *x*-axis, whereas the *y*-value in coordinate form is always zero, that is $(x; 0)$. Again, the *y*-intercept is the point on the Cartesian plane that cuts the *y*-axis and where the *x*-coordinate is always zero i.e. $(0; y)$.

<u>*Gradient*</u>

The coefficient of *x (m)* represents the gradient of a linear function, which refers to the slope or steepness of the line. The gradient is calculated using any two points from the graph by substituting into a formula: $m = \frac{y_2 - y_1}{x_2 - x_1}$; where $(x_1; y_1)$ refers to the coordinates of the first point and $(x_2; y_2)$ are the coordinates of the second point. Learners need to understand gradient beyond merely substituting into a formula; they should also be able to see it from a graphical representation whether it is increasing or decreasing (see Figures 2.2 and 2.3). In this study learners were expected to identify whether the graphical representation had a positive or negative gradient.

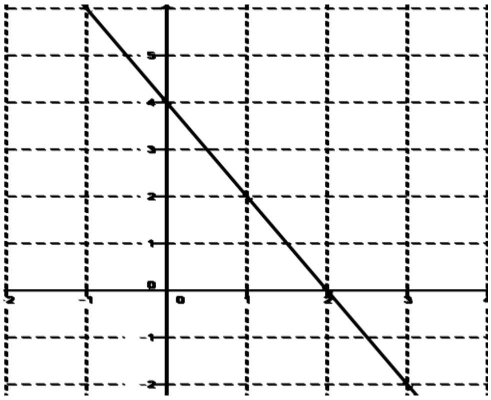


Figure 2.2: *Decreasing function with* $m < 0$

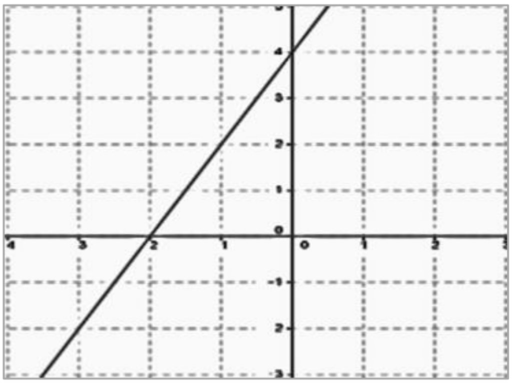


Figure 2.3: *Increasing function with* $m > 0$

Learners need to use their knowledge of gradient to identify whether a function is increasing or decreasing. Furthermore, they need to use covariation to extend the definition of increasing and decreasing. Increasing means that as the *x*-values increase, the *y*-values also increase whereas decreasing means that as the *x*-values increase, so the *y*-values decrease.

Again, learners must be able to relate and extend their understanding of gradient to compare gradients with the same sign by visualising the different steepness of graphs. This includes drawing conclusions such as the steeper the graph, the larger the value of the gradient or the less steep the graph, the smaller the gradient. See the different gradients in Figure 2.4.

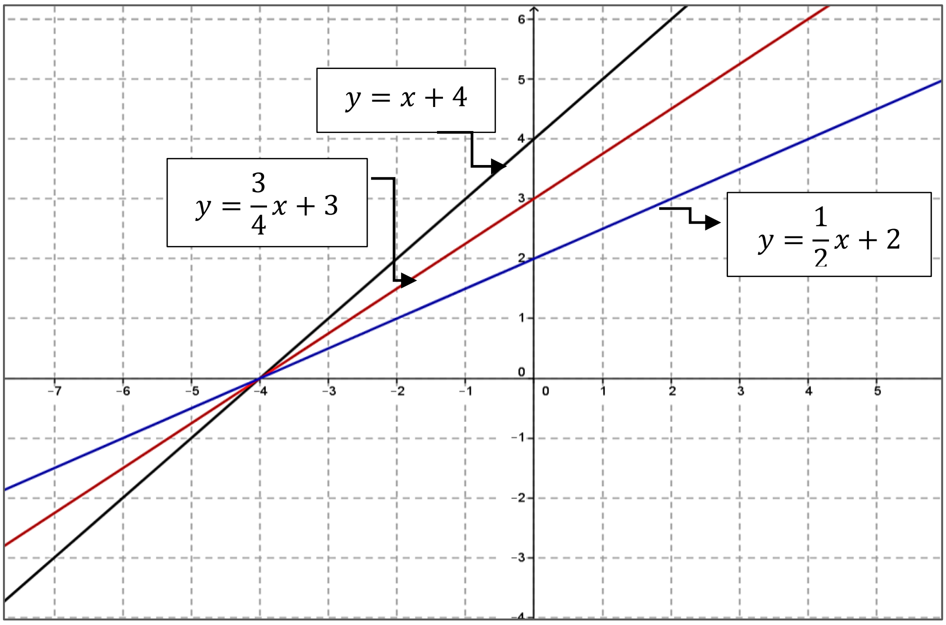


Figure 2.4: *The effect on the gradient when fixing the x-intercept*

b) Key features of the quadratic function

Key features such as intercepts, concavity, axis of symmetry, increasing, decreasing, range, domain, transformation, turning point, maximum and minimum are used to describe the quadratic function. However, in this study, the intercepts, axis of symmetry, increasing, decreasing, transformation, as well as turning point were the key features that stood out. I therefore focus in these features.

The graphical representation of a quadratic function is known as a parabola graph. Its general equation is written in the form of: $y = ax^2 + c$. The sign of the coefficient of x^2, a, denotes the concavity of the shape of the graph. If $a < 0$ it means the graph is concave down whereas if $a > 0$ the graph is concave up. The value of a irrespective of whether positive or negative, signifies the stretching of the graph. If $-1 < a < 1$, the graph stretches horizontally (becomes wider) whereas if $a > 1$ or $a < -1$, the graph stretches vertically (becomes narrower). See the effect of the value of a in Figure 2.5.

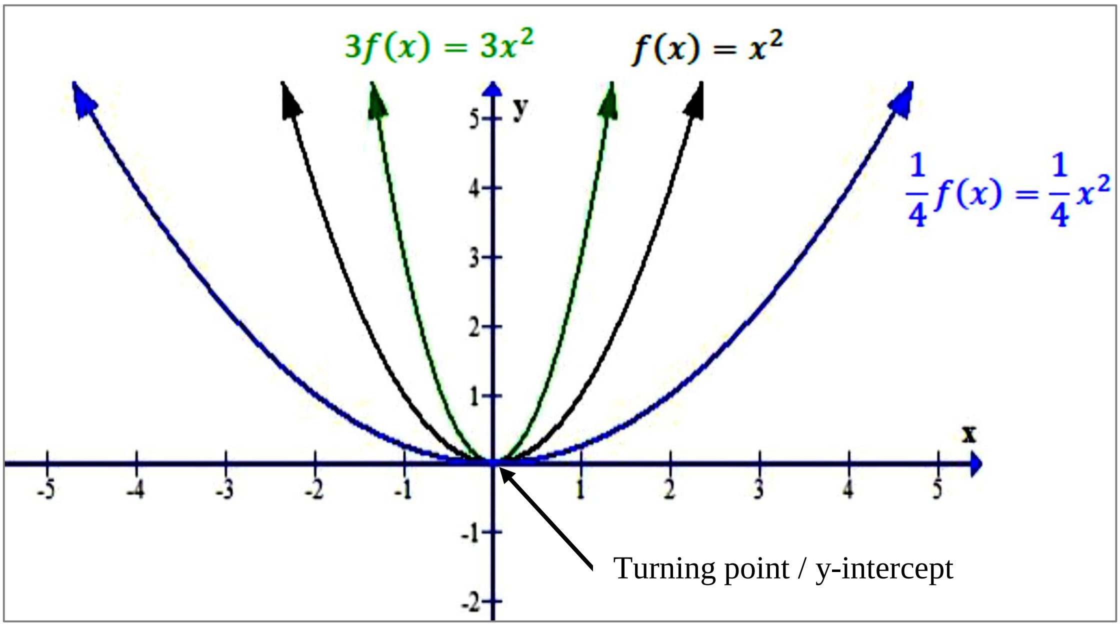


Figure 2.5: *The effect of changing the value of a in quadratic function*

Parameter *c* represents the *y*-intercept and the coordinate of the turning point. It also transforms the graph by shifting it up or down (see Figure 2.6). If $c < 0$, the graph of *f(x)* shifts vertically down and if $c > 0$, the graph shifts vertically up. The quadratic function has a maximum of two *x*-intercepts.

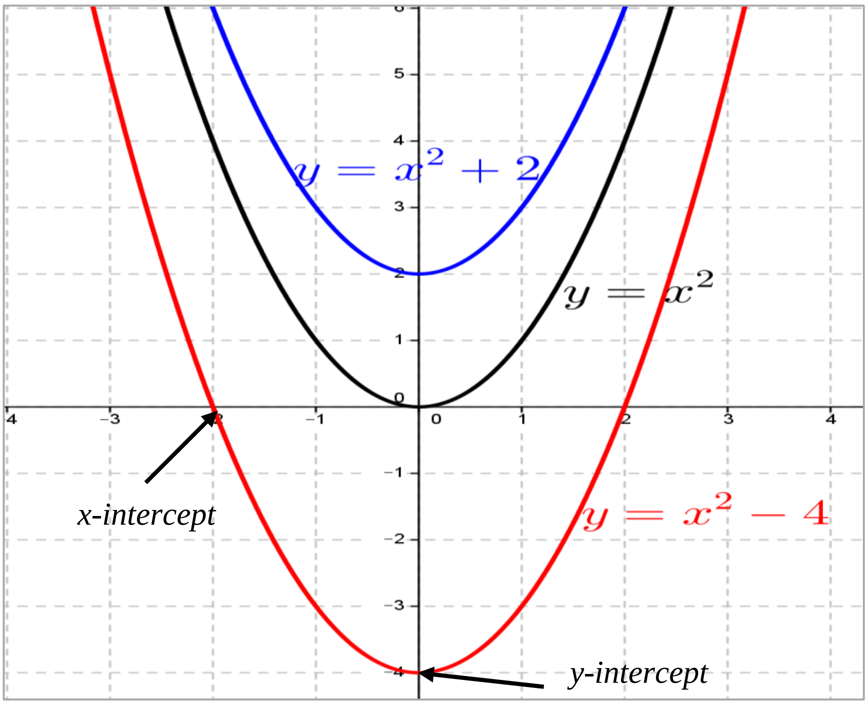


Figure 2.6: *Effect of c in a quadratic function*

Axis of symmetry

The x-value of the turning point represents the equation for the axis of symmetry. This is a line that divides the graph into two equal parts.

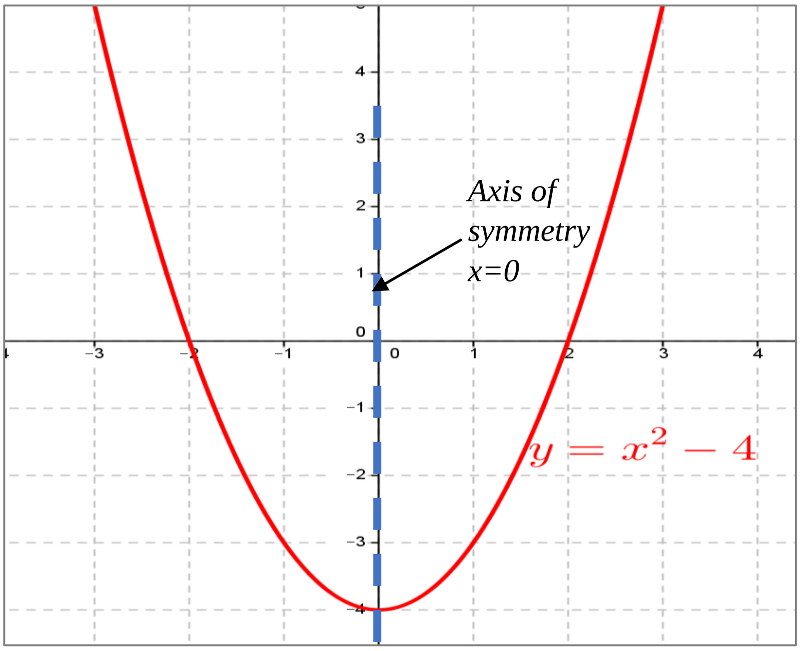


Figure 2.7: *Axis of symmetry for the quadratic function*

2.4.2 Connection between representations

This refers to the role of symbolic representation whereby the connection between the equation and the image of a graphical representation is emphasised (Even, 1998). Such mathematical analysis articulates whether seeing the straight-line graph brings to mind $y = mx + c$. Again, does $y = ax^2 + c$ bring to mind the parabola? If two straight-line graphs are 'joined' together to form a 'V' shape as illustrated in Figure 2.8, will learners be able to connect them with $y = mx + c$? According to Knuth (2000), understanding the connection between representations is fundamental to seeing functions as objects. This means that a learner might be able to view a function as an object if they are able to connect different representations such as the graph, table, and equation.

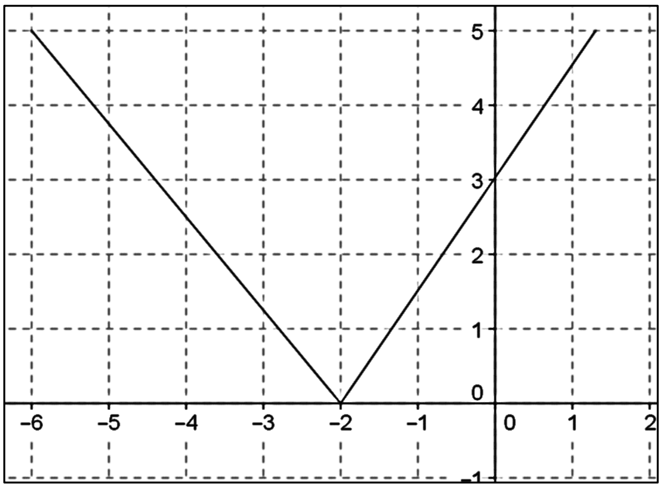


Figure 2.8: *Exemplar of two linear functions combined to form V*

2.4.3 Function notation

Learners often misunderstand the function notations such as *f(x), g(x) or f, g* (Even, 1998; Knuth, 2000; Oehrtman et al., 2008). Such poor understanding of function notation is attributed to the teaching and learning of functions, where *y =… is* usually referred to in the classroom and not *f(x) =…*. The underlying meaning of *f(x) or g(x)* is that the parentheses (*x*) represent the input values of *x* whereas *f(x) or g(x)* refers to the *y*-value, which are the output values (Oehrtman et al., 2008). Moreover, the label *f(x), g(x), f or g* on the graphical representation denotes the name of a function. This notation is further applied to include mathematical symbols such as $f(x) > 0$ to refer to all *y*-values that are positive.

2.5 Literature on the interpretation of functions

Bell and Janvier (1981) highlighted in their research on the interpretation of functions that most secondary school learners are weak in their ability of interpreting *global graphical features* to extract information from every day and scientific situations. They further indicated that learners viewed a graph as a picture of a situation as the result of having difficulty in coordinating information represented by two variables.

Leinhardt et al. (1990) highlighted in their study that the actions that relate to graphs and function tasks can be classified into interpretation and construction. They elaborated that a learner could interpret a graph from a level of algebraic construction in order to do the interpretation and from a more direct visual comparison. The interpretation of graphs can either be pointwise or global depending on the graphical features that learners are paying attention to (Even, 1998). According to Leinhardt et al. (1990, p. 9), the global features of a graph include “general shape of the graph, intervals of increase or decrease, and intervals of extreme increase or decrease”

The study conducted by Monk (1994) indicated that students had confidence in applying pointwise understanding of functions. However, it was not the case that a global understanding comes easily and automatically after a pointwise understanding has been established. He further highlighted that most educators and some researchers assume that there is a pathway that learners follow when they are taught functions by moving from pointwise to global understanding. This pathway involves reading a graph in terms of one point at a time using pointwise, then move to reading many points and then reading a graph at infinitely many points using global understanding. According to Monk (1994, p. 25), “pointwise understanding of graphs is prerequisite to global understanding, but the jump from the one to the other is a considerable one for students”.

Even (1998) concluded in her study that there were cases where the pointwise approach was more powerful especially in monitoring immature interpretations of graphs. Again, there were cases where the global approach was more powerful than pointwise approach. She highlighted that a learner could use global approach without understanding the meaning of the graph. Therefore, the combination of the two approaches was the most powerful.

Recently, Ayalon et al. (2017) highlighted that a process conception of functions affords a doorway into an object understanding of functions. However, learners have difficulty to abandon the pointwise approach of interpreting functions. This was evident when most learners used an input-output approach to interpret the statements given in the tasks. In addition, grade 12 learners showed a shift towards a global view while also retaining the pointwise view. The study indicated that very few learners demonstrated the understanding of covariation approach in terms of making sense of the graph in terms of the behavior of the changes in the input values with respect to the changes on the output values.

The review of the literature shows that most learners are acquainted with pointwise approach. This might be based on the fact that graphical representations of functions are introduced to learners using point-by-point sketching of graphs.

2.6 Approaches for interpreting graphs

Functions can be understood as a process or an object (Ayalon, Watson, & Lerman, 2017; Even, 1998; Monk, 1994). This is supported by Ronda (2009, p. 33), who states that "concepts are conceived as process before they are conceived as mathematical mental objects". According to Thompson (1994), a process conception refers to viewing a function as being for evaluating, whereas object conception has to do with the ability to view a function as using different representations such as tables, equations or graphs.

Thompson (1994) further states that functions can be perceived in two ways − correspondence and covariation. Correspondence extends from the definition of functions; that is, for every input value of *x* there is a corresponding output value of *y* (Ayalon et al., 2015b; Oehrtman et al., 2008). Learners in the study were able to read the corresponding *y*-value when given the *x*-value. Correspondence also deals with quantity where learners consider the operation on *x* to produce the *y*-value or plotting a graph using point by point method. Substituting a given *x*-value in an equation or formula to obtain a particular *y*-value or solution is an example of applying correspondence. Covariation involves dependant and independent variables and refers to perceiving a function in terms of the way one variable changes as the other variable changes (Ayalon et al., 2015b; Monk, 1994; Oehrtman et al., 2008; Thompson, 1994). This focuses on the ability to coordinate the movement of variables from y_m to y_{m+1} with the movement from x_m to x_{m+1} in order to describe the behaviour of a function. For example, asking learners to determine the values of x for which a function is greater than zero is one question that will require learners to coordinate variables.

Even (1998) also highlights two viewpoints that learners could use to interpret functions; namely a pointwise and a global approach. Even defines pointwise as the ability to "plot points on the graph, read individual points or deal with discrete points" (1998, p. 109) of a given graph. This also includes finding coordinates, the gradient and the lengths of horizontal and vertical lines. For example, if a graph of $y = x^2 - 4$ is given, a pointwise question would be to read the value of y if $x = 1$. A global approach refers to looking at the behaviour of a function in relation to interval reading of infinitely many points (Bell & Janvier, 1981; Even, 1998; Leinhardt et al., 1990). Leinhardt et al. (1990, p.9) further highlights that the global features comprise "the general shape of the graph, intervals of increase or decrease, and intervals of extreme increase or decrease". For example, if a graph of $y = x^2 - 4$ is given, the global question would be to find the values of x for which $y < 0$. A global approach would be evident if learners read infinitely many points at once and demonstrated the ability to read the whole graph.

According to Monk (1994), the ability to use a pointwise approach is a prerequisite for understanding the global approach. This means that learners can perceive a graph globally once they are able to make sense of it from pointwise perspective. He further states that it is not automatic for learners to move from a pointwise to a global approach.

Thompson and Carlson (2017) recently expanded on covariational reasoning, by coming up with a framework for learners' reasoning when coordinating changes in quantity values. They describe six levels of reasoning, namely; no coordination of values, pre-coordination of values, gross coordination of values, coordination of values, chunky continuous covariation and smooth continuous covariation. The chunky and smooth continuous covariation levels were not evident in my data. Hence, I draw only on the levels of no coordination and coordination of values in my analytical framework. All six levels are briefly described below.

a) No coordination of values

This is a level at which a learner has no image of variables varying together, instead s/he focuses on the variation one or another variable with no coordination of variables. An example of this is when learners are asked to provide the values of *x* for which $g(x) > 0$. Learners may subsequently respond by describing a graph using *x*-values only without coordinating them with the *y*-values.

b) Pre-coordination of values

A learner may imagine the values of two variables as varying but individually, that is, one variable changes and then the second variable changes. The learner anticipates creating pairs of values as multiplicative objects.

c) Gross coordination of values

The learner forms a gross image of the values of quantities as varying together, for example 'as the *x*-value increases, the y-value decreases'. The learner does not envision that individual values of quantities go together, instead s/he sees them as a loose, non-multiplicative link between the overall changes in values of two quantities.

d) Coordination of values

The learner coordinates the values of one variable (*x)* with the values of another variable *(y)* in the anticipation of creating a discrete collection of ordered pairs (*x; y).* An example of this is when learners are asked to provide the values of *x* for which $g(x) > 0$. Learners may subsequently respond by describing a graph using *x*-values together with its corresponding *y*-value and anticipating the collective pairs of coordinates.

e) Chunky continuous covariation

The learners envision changes in the value of one variable as happening simultaneously with changes in the value of another variable and they see both variables as varying with chunky continuous variation. For example, if learners are asked to provide values of x for which $g(x) > 0$. Learners may respond by saying as the x-values are going to negative infinity the graph is going up to infinity without specifying what happens to the y-values.

f) *Smooth continuous covariation*

Learners envision increases or decreases in the value of one variable as happening simultaneously with changes in the value of another variable and see both variables as varying smoothly and continuously. For example, learners might respond that as the x-values are increasing the y-values are also increasing when describing an increasing function.

The six terms (process, object, correspondence, covation, pointwise and global) for the conception of functions go hand in hand. The conception of a function in terms of process, correspondence and pointwise perspective all require a similar approach to evaluate a function. Again, object, covariation and global perspective could be grouped together because they deal with the behaviour of a function. This study focused on investigating the way in which Grade 10 learners describe the behaviour of graphical representations of functions. Therefore, correspondence and covariation were the more appropriate approaches to use as the lens for analysing learners' responses.

2.7 Conclusion

In this chapter, I discussed the socio-constructivism as the learning theory that informed this study. In particular, Vygotsky specifies that knowledge is mediated through signs, tools and artefacts. In my study, graphical representation is the tool used to represent linear and quadratic functions, which are generally written in terms of signs. The appearance of the graph will depend on the artefacts that learners have constructed while learning a particular function. Hence, learners are expected to operate on the signs, tools and artefacts in order to describe the behaviour of functions in terms of the approaches and the key features of graphs.

I also discussed Even's concept of approaching graphs in terms of using a pointwise or a global approach. The way learners approach a graph played a major role in my study because that determines the way in which learners make sense of graphical representations. For example, a learner might use calculations to describe the behaviour of a graph and, in so doing, they have used a pointwise approach because they used individual points from the graph to carry out their calculations. Even's concept goes hand in hand with the conception of functions in terms of correspondence or covariation. Correspondence and covariation look deeper into the behaviour of the graph in terms of coordinating variables. This conception was useful for my study because some learners operated below pointwise approach and looking deeper into the way in which they coordinated variables assisted me to understand their underlying thinking when interpreting graphs.

The mathematical analysis in terms of the role of parameters, function notation and connection between representations supported my study when considering the way in which learners translate graphs with regard to graphical language and the interpretation of symbols. This helped me to pay attention to how learners describe objects in graphical representations.

Chapter 3 : Research Design and Methodology

3.1 Introduction

This chapter focuses on the design of the study and explanations of the methods that I used to collect data from learners, which would later enable me to answer the research questions. As a reminder, this study seeks to investigate the approaches and key features that Grade 10 learners use to interpret the graphical representation of linear and quadratic functions. It is guided by the following research questions:

1. *What key features of functions do learners pay attention to when responding to graphical representation tasks?*
2. *What approaches do learners use when working on the graphical representation tasks?*

I begin this chapter by providing the approach adopted to the study followed by an in-depth explanation of the research design. I then elaborate on the intervention lessons, the rigour in the research study, as well as the ethical issues involved. Lastly, I discuss the way in which the data were analysed in detail.

3.2 Research approach and design

The interpretation of functions in mathematics, especially learners' underlying thinking, is very important yet would seem to be a challenging area for learners and education at large (Even, 1998; Friel, Curcio, & Bright, 2001). In order to gain insight into learners' thinking, semi-structured task interviews as well as the test-retest procedure were applied. Therefore a qualitative social research paradigm was deemed appropriate (Best & Kahn, 1998) because the interview data was collected by interacting with learners (Dawson, 2009). This resulted in the use of an interpretive analysis approach (Hatch, 2002) to make sense of learners' responses relating to linear and quadratic functions. This interpretive stance helped in constructing the analytical framework for describing the interview data. The results of the test analysis formed part of the secondary data and were not used for quantitative purposes, instead they were used to strengthen the qualitative results (from the interview data) based on learners' performance.

This qualitative research study did not require a large population, merely a sample that provided valuable information for answering the research questions (Creswell, 2012). The study focused on learners' mathematical thinking when interpreting graphical representation of linear and quadratic functions. The research design comprised five stages, namely, pre-test, pre-interviews, three intervention lessons, post-test and post interviews presented in the schematic diagram in Figure 3.1.

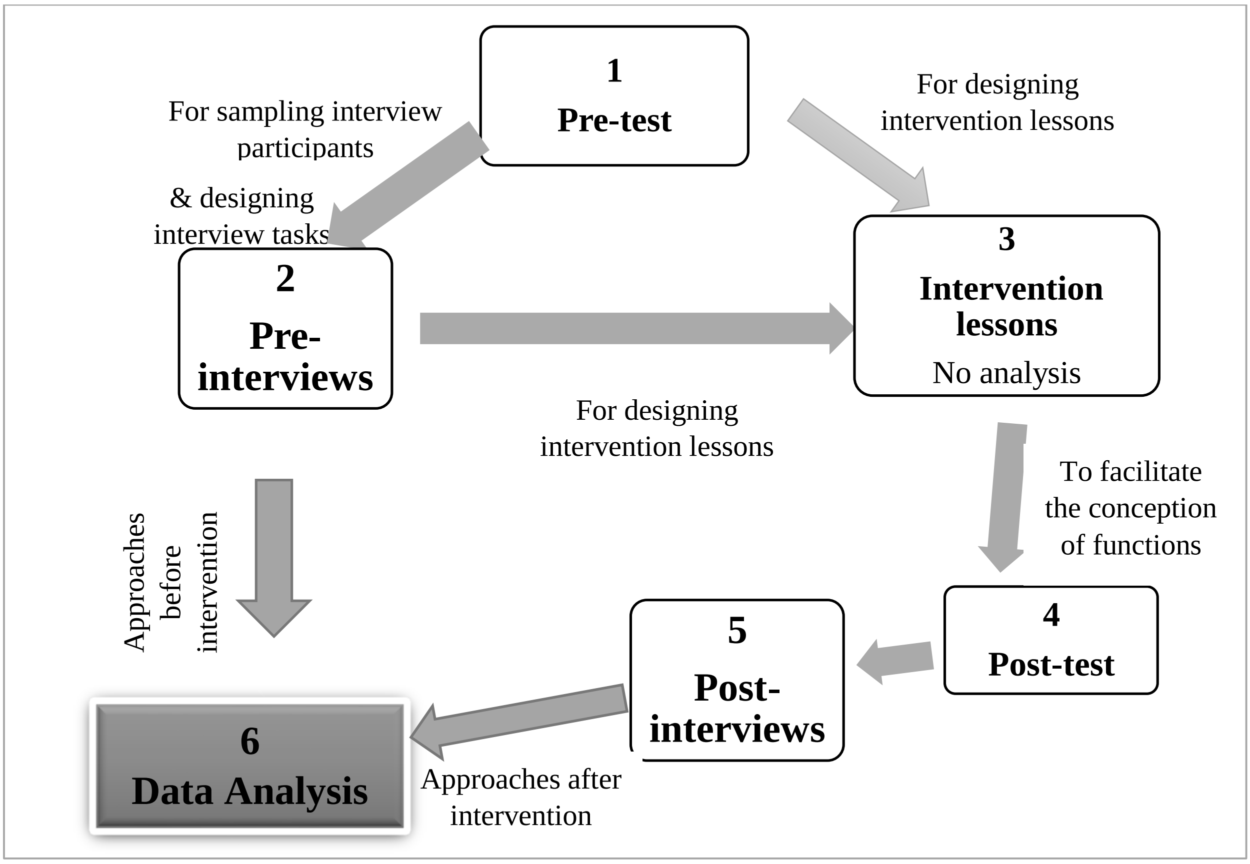


Figure 3.1: *Diagram illustrating various stages of the design*

Pre- and post-interviews formed the major stages of the study. These gave rise to the qualitative data which were analysed to gain insight into learners' underlying thinking when interpreting graphs. The pre- and post-test formed the minor stages and provided the information on the learners' performance while attempting graphical representation tasks. Please note that although the intervention lessons were conducted to facilitate learners' understanding of functions, they did not form part of data analysis. The five stages of research design are briefly discussed in the next subsection.

3.2.1. Setting and participants

This is a qualitative research study conducted in one Gauteng high school situated in inner city of Johannesburg. This school was identified because it was one of the schools participating in a bigger ongoing project conducted by Wits Maths Connect Secondary (WMCS). The school was easy to access because I work there (Creswell, 2012).

A group of 25 Grade 10 learners, with ages ranging from 14 to 17 years, participated in the research study. This was a convenience sample because I taught these learners. Moreover, these were the learners that gave consent (this is discussed under ethical consideration). All learners were given the pre-test. Thereafter, the five best performing learners in the pre-test were selected to take part in the pre-interviews. Accordingly, this was a purposeful sample (Creswell, 2012) which was chosen because graph interpretations require a higher level of thinking (Department of Basic Education, 2017). Moreover, these learners were anticipated to be able to communicate mathematically more effectively than low-achieving learners. A sample of five learners was chosen in order to provide enough data for the analysis even if one or two learners decided to discontinue the study. The same learners who participated in the pre-interviews also took part in the post-interviews in order to be able to contrast the pre- and post-interview data.

3.2.2. Data collection

Data were collected after school from 14h00 to 15h30, since the research was not meant to disturb the normal teaching programme. The data were collected in the third term after the concept of functions had been covered in the second term (Department of Basic Education, 2011). However, delays were experienced, starting from the piloting of the test instrument, to the intervention lessons, to the post-interviews, which made the data collection to be an uneven process. The delays resulted from uncontrollable factors such as sport activities in the school environment and the fact that learners were writing the control tests towards the end of term 3. In spite of all the delays, qualitative data were gathered throughout five stages (pre-test, pre-interviews, three intervention lessons, post-test and post interviews). A brief description of each stage is provided below.

3.2.2.1. Tests

The pre-test was administered in order to select a sample of five learners to participate in the interviews. It was also used as the guide to design the pre-interviews and the intervention tasks. It also helped to identify the difficulties that learners encountered when interpreting straight-line and parabolic graphs, which were later applied to design the intervention lessons. On the other hand, the post-test was administered to monitor and contrast learners' performance after participating in the intervention lessons. The results of the post-test were used as a guide for developing the post-interview tasks. The pre- and post-tests were administered using the same instrument. This was the test-retest procedure (Creswell, 2012) applied to examine the performance of learners when interpreting graphs. More details of the test instrument are given later.

3.2.2.2. Interviews

Interviews played a major role in this study as they allowed me to gain insight into learners' thinking while attempting to interpret graphs. Semi-structured, task-based interviews were conducted with each learner for a duration of 15 to 20 minutes using one-on-one verbal interaction (Creswell, 2012). A semi-structured interview approach (Maxwell, 2005) granted me the opportunity to probe questions and follow up on interesting responses. Moreover, probing further was done to allow learners to clarify their ideas and for the interviewer to gain deeper insight into learners' thinking on certain concepts. Individual interviews, as opposed to group interviews, allowed me to identify the thinking of each participant one at a time. Lastly, task-based interviews were chosen because learning is socially mediated through the use of signs, tools and artefacts (Daniels, 2009; Vygotsky, 1978); thus learner's thinking was triggered by the use of graphics.

Although I taught these learners, I started the interview by giving the interviewee a brief introduction to the study and encouraged them to be honest in their responses. Learners were allowed to ask or re-voice the questions as a means of gaining clarity prior to giving their responses. All interviews were video-recorded in order to capture learners' reactions and gestures while responding to the task. Gestures are essential traits of communication (Roth, 2001) and in this study were mainly used to point to the graph using verbal utterances such as "this side" or "this one" or "here". The video camera was projected in such a way that learners' identities were not revealed. The recordings enabled me to carry out a detailed analysis at a later stage.

The participants in the interviews were four boys and one girl. These learners were given pseudonyms: Sipho, Themba, Mpho, Zodwa and Bonga. The interview data pertaining to one of the five learners, Zodwa, did not form part of the analysis because she did not provide rich data (most of her responses were not justified). Hence, the analysis focused on four learners Sipho, Themba, Mpho and Bonga.

Pre-interviews were carried out after the pre-test but prior to the intervention lessons. The same learners who participated in the pre-interviews also took part in the post-interviews in order to monitor progress and contrast their performance. As already mentioned, delays were experienced in the post-interviews, with one learner being interviewed one week after interviewing the other four learners. However, this learner's responses gave no indication that he had obtained information about the task interview from the other learners.

3.2.3. Instruments

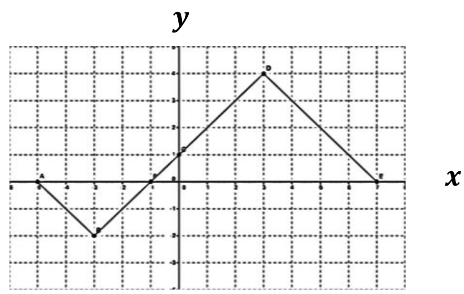
Instruments consisted of a pre- and a post-test, as well as the tasks for the pre- and post-interviews. These instruments are discussed further in the following subsections.

3.2.3.1. Pre- and post-test

The pre- and post-tests were designed based on the scope of the research to focus on linear and quadratic functions. I used the literature review as the basis for creating the test instrument, which was intended to identify the way in which learners interpret graphs in terms of the approaches and features of the graph (Even, 1998). Again, the knowledge and experience I gained from the WMCS project was of great benefit when designing the test instrument. The graphical representations were drawn in such a way that they were not familiar to the learners (not graphs usually done in class). This was not done to confuse learners but rather to explore how they applied their knowledge and understanding of functions in order to interpret unfamiliar graphs. Linear graphs were tested by combining three straight-line graphs to form a zig-zag shaped graph (see Figure 3.2). The graphical representation was then accompanied by five to six sub-questions.

The test instrument was discussed and approved by subject experts (my supervisors) following a discussion that was intended to improve the validity of the test. Questions items comprising this test instrument are included in Figures 3.2 and 3.3 for the reader's interest but a detailed discussion is not given because they are not the main focus of the study.

1. Study the graphical representation below and answer the questions that follow.



1.1 What is the general shape of the graph? ____________________________

1.2 For which values of x is $y>0$. ____

1.3 Give the intervals for increasing and decreasing function

Increasing __________________________

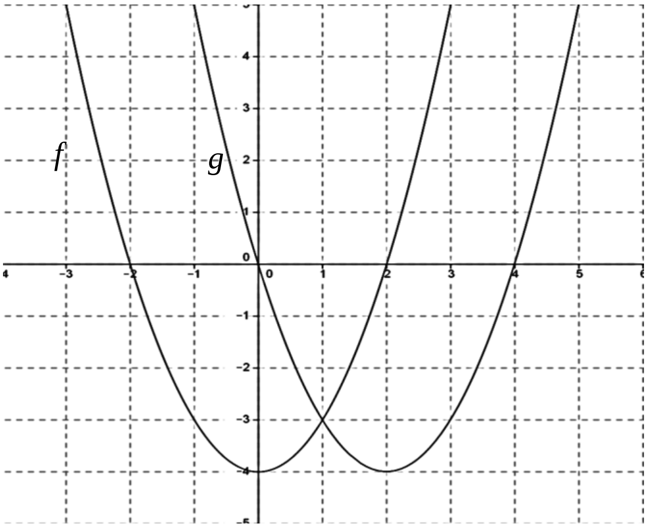
Decreasing _________________________

1.4 Find the gradient of line AB, BD and DE

Figure 3.2: *Example of the test items on linear functions*

A combination of two parabolic graphs drawn on the same system of axes was used to test learners' interpretation of quadratic functions. Graph *f* with the turning point on the y-axis was familiar to learners, whereas *g* was horizontally shifted to the right and its turning point was unfamiliar to Grade 10 learners. See an example of a graphical representation in Figure 3.3.

2. Study the graphical representation of *f* and *g* below and answer the questions that follows.



2.1 What is the general shape of the graph/s? ____________________________

2.2 Find the equation of *f*

____

____

____

2.3 Write down the coordinate where *f* intersects *g*. __________________________

2.4 For which values of *x* is the graphical representation of *f* and *g* less than zero?

____

2.5 What is the range and domain of *f* and *g*?

Range of *f:*	Range of *g:*
Domain of *f:*	Domain of *g:*

2.6 Find the equation of *g* using the equation of graph *f* or the turning point of *g*.

____

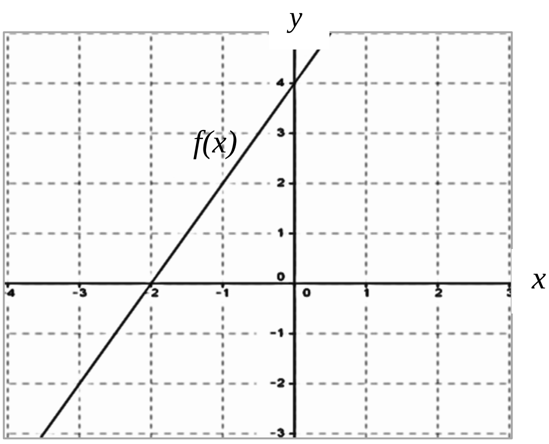
____

____

Figure 3.3: *Example of the test items on the quadratic function*

3.2.3.2. Task instrument for the pre- interviews

In the pre-interviews, the task instrument focused only on the interpretation of straight-line graphs. This was done based on the fact the foundation for linear graphs is laid in Grade 9 and therefore learners in Grade 10 should be able to respond mathematically. The task instrument was developed on the basis of literature review and the results of the pre-test. It consisted of two sessions, namely, a warm-up activity and the actual task interviews. The warm-up activity was given to learners as means of breaking the ice prior to responding to the actual task-based interviews. The actual task instrument was composed of three sub-questions, which were all centred on the straight-line graph presented in Figure 3.4. It should be noted that *the equation of the graph was not given to learners*. For the purpose of this research report, sub-questions 1 to 3 were treated as separate tasks which were numbered Task 1, Task 2 and Task 3.



Warm-up activity

a) What is the coordinates of the *x* and *y* intercept of graph *f*?

b) What is the gradient of graph *f*?

Task-based interviews

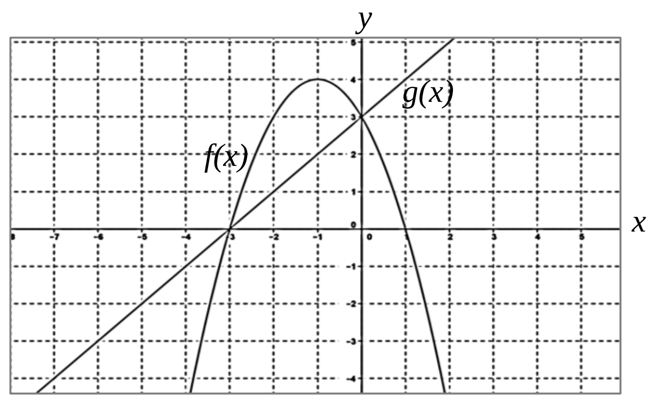
1. Construct a straight line graph *g(x)* with an opposite sign gradient as that of *f(x)* but keeping the same y-intercept. Explain how you constructed the line *g(x).*
2. For which values of *x* is the constructed graph *g(x)* greater than zero?
3. Construct *h(x)* using the same *x*-intercept as *f(x)* but with the *y*-intercept less than that of *f(x)* then compare the gradients of the two graphs whether the gradient of *h* is greater or less than that of *f*. Justify your answer.

Figure 3.4: *Task instrument for the pre-interviews*

The three tasks presented in Figure 3.4 had different purposes. Although Task 1 was not informed by the pre-test, it aimed at investigating whether learners were able to draw a straight-line graph using verbal instructions on graphical features without being given a specific method and thereafter verbalise the mathematical description of how they constructed the graph. Task 2 was included because the pre-test results indicated that none of the five learners was able to provide intervals for which the graph was greater/less than zero. This task aimed at examining learners' thinking regarding the intervals where the graph was greater than zero. Task 3 focused on exploring how learners would compare gradients of two lines with the same *x*-intercepts but different *y*-intercepts and whether they anticipated the effect of fixing the *x*-intercept and changing the *y*-intercept.

3.2.3.3. Task instrument for the post-interviews

The task instrument in the post-interviews concentrated on the combination of straight-line and parabolic graphs drawn on the same system of axes as illustrated in Figure 3.5. The pre-interviews and post-test informed the design of the task instrument and the three tasks formed part of the data analysis. These three sub-questions were treated as separate tasks for the purpose of this report. They were thus labelled as Task 4, Task 5 and Task 6 in the data analysis instead of Task 1, 2 and 3 in order to follow the sequencing of the pre-interviews tasks.



1. For which values of *x* is $g(x) > 0$?
2. For which values of *x* is *f(x)* increasing and decreasing
3. For which values of *x* is $f(x) > g(x)$?

Figure 3.5: *Task instrument for the post-interviews*

The question in Task 4 was similar to the question in the pre-interview except that the task was different as it was aimed at exploring the shift in the way learners approached this task. Tasks 5 and 6 were informed by the pre- and post-test, since the majority of learners had difficulty in answering such questions in the pre-test.

3.2.3.4. Piloting of tests instrument and interviews

Piloting the test instrument and interviews is the best way to perfect an instrument (Cohen, Manion, & Morrison, 2007). Moreover, it gives an indication of whether the instrument would be able to yield the intended results (Dawson, 2009). The pre-test instrument was piloted with a group of Grade 10 learners from a different school in the inner city of Johannesburg. About 15 learners participated in the pilot. The results from piloted group indicated that these learners misunderstood the word 'interval' but were able to answer similar questions where an inequality sign was used. Based on the pilot results, task interview instruments were adjusted by removing the word 'interval' and using the inequality sign.

Prior to actual interviews, interviews were also piloted in my school with learners who did not form part of the main sample. The interviews were piloted in order to gain an understanding of how learners would perform and to assist me to obtain a feel for the interviews especially with regard to whether the task interview instrument would produce the intended results (Creswell, 2012; Dawson, 2009).

3.2.3.5. Mathematical requirements of each task

It was important to underpin the mathematical demand of the task interviews as a preliminary step for learners to be able to deal with graphical representations. Table 3.1 provides a summary of the task requirements that are essential for learners to respond mathematically.

Table 3.1: *Summary of the mathematical analysis for the interview tasks*

Task 1	Task 2 & 4	Task 3	Task 5	Task 6
a) identify the graphical representation of a positive and negative gradient *b)* identify that opposite sign gradient means $m < 0$ *c)* be able to interpret the positive and negative gradient in terms of the changes in the *x* and *y* values *d)* be able to identify the *y*-intercept as the point on the Cartesian plane where the graph cuts the *y*-axis *e)* understand that the *x*-intercept changes to obtain a gradient with opposite sign *f)* if *x*-value changes and *y*-value stays the same then results to reflection g) the graph represent the relationship between *x* and *y*	a) know that the function notation *g(x)* refers to the *y*-values b) notation $g(x) > 0$ means that y-values are positive c) greater than zero refers to all positive real number where zero is not included d) notation $g(x) > 0$ is NOT the same as $x > 0$ e) *y* values are dependent on the *x*-values f) *x*-values for which *g(x)* is greater than zero refers to *x*-values where the graph has positive *y*-values g) the graph represents the relationship between *x* and *y* h) read interval of points	a) *x*-intercept is the point on the Cartesian plane where the graph cuts the *x*-axis b) same *x*-intercept means the point of intersection of both graphs and the *x*-axis is the same c) compare gradients using visual perception of the steepness of the graphical representations d) *y*-intercept is the point on the Cartesian plane where the graph cuts the *y*-axis e) *y*-values are dependent on *x*-values f) intercept less than that of *f(x)* refers to any value on the *y*-axis less than 4, such that if the *y*-intercept is less than 0, the gradient will change the sign g) for the *x* values greater than the *x*-intercept the *y*-values are reduced, whereas for the *x*-values less than the *x*-intercept the *y*-values are enlarged	a) function notation *f(x)* refers to the *y*-values b) know definition of increasing and decreasing c) coordinates of the turning point that the *x*-coordinate is on the *x*-axis and the y-coordinate on the y-axis d) *x*-coordinate of the turning point is the axis of symmetry e) *y* values are dependent on *x*-value f) increasing or decreasing refers to the behaviour of the graph with respect to the changes in the *y*-values as the *x*-values are changing g) read interval of points	a) know that function notation *f(x)* and *g(x)* refer to the *y*-values b) read individual points c) compare the *y*-values of the two graphs at the same *x*-value d) *y*-values are dependent to *x*-values e) read interval of points where $f(x) > g(x)$

3.4 Intervention lessons

Although the intervention lessons were not part of the data analysis, they were used to enhance learners' understanding when interpreting graphs. According to Creswell (2012), interventions are meant to make difference to learners' responses from pre- to post-interviews. This is clearly indicated in the analysis of the post-interviews, which is discussed in the subsequent chapters. Learners' responses in the pre-test and pre-interviews played a major role in designing the intervention lessons because they gave an indication of learners' conceptual knowledge of graphs and a guide to the difficulties learners experienced. This assisted me to focus the intervention lessons on the graphical features in interpreting linear and quadratic functions. My involvement to the WMCS project also assisted me to design the lessons especially the card matching activity.

Although only 25 learners returned the consent forms, the intervention lessons were offered to all Grade 10 learners (a group of 40) because they were treated as part of the extra classes. Three intervention lessons were conducted for a duration of 40 to 60 minutes each. All lessons took place in a classroom setting in which learners were working in groups in order to engage in group discussions. These lessons were taught by me after school from 14h00 to 15h00. There was a gap of one day between the first two lessons, which means they took place on Monday and Wednesday. The third lesson took place a week later due to the tests that were scheduled by the school. The lessons were not video recorded because the intervention lessons did not form part of the data analysis. The intervention lessons are described in detail in the following sub-sections.

3.4.1 Lesson 1

Lesson 1 concentrated on linear functions in which graphical features such as gradient and interval of increase/decrease were discussed. It consisted of two activities, namely, card matching and interpretation of graphs. The lesson began with a card matching activity, which was consisted of six functions with different representations; that is, graphs, equations, tables and rules written in words. Learners worked in groups to match each graph to its equation, rule, table and a comment on its behaviour, that is, whether it was increasing or decreasing. In cases where an equation, graph, rule or behaviour was missing, learners had to produce it. After matching the cards, each group was allocated a question number to which they had to write the answers on the board, which was then followed by class discussion.

The graphs used in the card matching activity were then further investigated. The focus was on interpreting graphs in terms of determining the gradient, behaviour of the graph in terms of why it was increasing or decreasing, and intervals of the graph where it was greater or less than zero. The lesson was conducted in such a way that learners gradually moved from a pointwise to a global approach to interpret the graphical representation of linear functions (Monk, 1994). However, there was no guarantee that the lesson would follow that sequence. Two graphs of the six graphs given were used as part of setting examples on how to interpret graphs. Learners were then given the other four graphs to interpret as part of their homework.

3.4.2 Lesson 2

Lesson 2 was similar to lesson 1 except that it focused on the quadratic functions. It also began with a card matching activity which consisted of six parabola graphs and equations. It was then followed by lesson on how to interpret the six parabolic graphs. The interpretation of graphs included finding intervals of increase or decrease, the domain and the range of a function as well as reading the values of x for which the graph is greater or less than zero. Again, two graphs were used as examples and learners were given the other four graphs as homework.

3.4.3 Lesson 3

In this lesson, the interpretation of both linear and quadratic functions drawn on the same system of axes was addressed. Delays were experienced when executing this lesson as the result of the annual control tests that took place towards the end of the term. The lesson was then taught during one of the Mathematics double periods, where graphs were interpreted based on the graphical features addressed in lessons 1 and 2. As part of the class activity, learners were asked to formulate their own questions based on the given graphs; these questions were then written on the board for the whole class to work on them. One of the questions that was formulated by learners was to provide the intervals for which the two graphs were either greater or less than each other.

3.4.4 Reflection on the intervention lessons

Although the intervention lessons were not the focal point for this study, I include a reflection for the reader's interest in order to give an indication of the way the intervention took place. The card matching activity played a major role in lessons 1 and 2 where learners were motivated to work with different representations of functions. This activity created more room for learners to gain an understanding of the behaviour of the graphs in terms of increasing and decreasing functions. It also stressed the importance of the role of m and c in the linear function, as well as a and c in the quadratic function. Moreover, it enabled learners to compete among themselves in groups.

The lessons turned out to be more interesting than usual as learners were interested in gaining deeper understanding of the interpretation of graphs especially the behaviour of the graph. I have to mention here that I have never used this teaching strategy to teach functions instead I participated in the group discussion in one of the session conducted by WMCS project. As the result of the card matching activity, learners concluded that the x-intercept plays a major role when finding the intervals where the graph is greater or less than zero for both linear and quadratic functions. This was one of the aspects that differed from my usual teaching, which is often hampered by the limited teaching timeframe. This resulted in looking at the deciding points or values. For example, the y-value of the turning point provides the range or maximum or minimum point, whereas the x-value of the turning point is the deciding point for the intervals of increase or decrease. In the usual way when teaching, I focus on the transformation of the graph in terms of the effect of a and c on the quadratic function and little attention is paid to the interval of increase and decrease owing to the time constraints.

The class activity in lesson 3, where learners were given an opportunity to create their own questions, motivated learners to challenge their peers. On the following day, learners asked if they could continue with this activity in which they took over to facilitation of the lesson, asking the whole class questions based on the interpretation of the given graphs. This challenged them to attempt to answer previous questions on the interpretation of functions.

3.5 Trustworthiness in qualitative research

In qualitative research, trustworthiness is seen as an appropriate concept to establish rigor (Opie 2004) because it relates to the need to ensure that the findings are closely conveying the meanings as expressed by the participants (Lietz, Langer, & Furman, 2006). According to Lietz et al. (2006) trustworthiness is not a natural occurrence. Instead it entails an attempt to ensure that the data is truthful and dependable according to the defined procedures and context in which the data was collected. However, there might be threats to trustworthiness such as the problem of reactivity and biases relating to the part of researcher and participants (Lietz et al., 2006; Opie, 2004).

Trustworthiness was ensured in this research study in order to embrace multiple realities and participant's meaning using data triangulation (Maxwell, 2005; Opie, 2004). Triangulation by different data sources is the strategy that uses more than one method to study the same research question, which also helped to reduce the threats to trustworthiness such as reactivity and biasness. The different data sources included a pre-test, pre-interviews, a post-test and post-interviews. These data sources sought to examine the learners' interpretation of graphical representations of linear and quadratic functions in terms of the key features of the functions and approaches that learners paid attention to. Moreover, data was collected by conducting interviews with four learners in order to see whether the same patterns recurred. The use of a video camera to capture the interviews enabled me to transcribe accurately the conversation between me and the interviewee. It also helped to clarify any ambiguous talk or gesture, for example if a learner said "here" or "this" to refer to the graph, I was able to elaborate and provide the exact communication. This allowed me to describe the data analysis accurately without including any ambiguous statements from learners' responses.

The trustworthiness of this research was ensured by applying appropriate methods of data collection to match the qualitative approach. Again, the use of the interpretive analysis driven by the analytical framework increased the trustworthiness of my findings. This was done by providing a thick description through the use of multiple methods when collecting data such as tests and interviews to ensure truthful and dependable data in the context under which the data collection was done. This was also an attempt that findings reflect the participant's meaning accumulated through the different data sources. The gleaned data analysis from the different sources and presentation of this analysis provided a coherent report.

3.6 Ethical considerations

Ethics Clearance Certificate with protocol number: 2017ECE022M was obtained from the Ethics Committee in Education prior to the collection of data. This enabled me to send consent forms to the principal, parents and learners in order to carry out the research study (see Appendices A to F). Consent forms were sent to parents because the participants are minors and, as such, needed permission from their parents to take part in this research (Maxwell, 2005; Punch, 2006). Participants were treated with respect and it was clearly stated in the information letters that participation in the study was voluntary and that participants could withdraw at any time. Documents generated by the participants were treated with confidentiality, with only the researcher and the supervisors having access to such information. Since anonymity was of great importance (Dawson, 2009; Maxwell, 2005; Punch, 2006), the real names of the learners and the school were not used in the report and instead pseudonyms were used. Videotaping during the interviews sessions was the prerequisite for the study to capture the graphical explanation of participants accurately. However, the video camera was not directed at learners' faces.

3.7 My different roles in the research

This is a qualitative study consisting of semi-structured interviews where there is a "provision for negotiation, discussion and expansion of the interviewee's response" (Opie, 2004, p. 118). However, some of the discussion or probing might portray some qualities of a researcher's or teacher's positioning. It is therefore important for me to clarify to the reader that while collecting the data during the interviews, I wore two different hats, namely, the researcher hat or the teacher. Below I provide an explanation of these two hats.

3.7.1 Researcher's hat

Although I worked at the school where I collected data and, as their teacher, had links with the participants, I maintained the researcher's hat by taking into consideration the ethical issues, in terms of which I had given the request letter to the principal and consent forms to learners and parents. My ultimate purpose was to collect data from the learners that would be sufficient for answering the research questions. This was done by introducing myself to the learners and giving them a brief introduction to the study. Learners were then given the tasks in the form of graphical representations and responses were followed up where necessary.

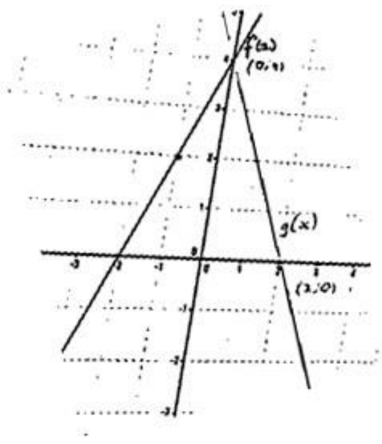
3.7.2 Teacher's hat

Having built a teacher-relationship with these learners, at one stage in the pre-interviews I tended to assume the role of teacher. The way I probed learners created a learning environment because I was persistent in questioning learners until they provided a correct response. My intention was not to teach learners instead I wanted to gain deeper understanding of what they were saying. I fell into a trap of guiding learners towards I wanted them to notice. This resulted to being able to transform learners' understanding of graphs from ADL to PDL (Daniels, 2009). However, this way of probing was noted and corrected in the post-interviews.

I struggled in terms of juggling the two hats and thus later on in the report (Chapter 6) I reflect on the dilemma of being both the researcher and teacher.

3.8 Preparing for data analysis

This is a qualitative study consisting of data collected by means of task-based interviews. Data from the interviews were first transcribed into a readable format (Maxwell, 2005). This was done by listening to the video tape and recording it verbatim. This process was repeated several times to ensure that the transcript was consistent with the video tape. There were instances where learners used gestures to indicate on the graph, together with words such as "this side" or "this one" or "here". *Gesture* referred to the movement of hands as part of learner's communication when interpreting graphic representation (Roth, 2001). Roth further identifies the *deictic* and *iconic* gestures where the deictic gesture refers to pointing whereas iconic gesture refers to the visual similarity of a trajectory. The gesture in the text was indicated in parenthesis together with a brief explanation to convey the flow of meaning and readability. The video was transcribed in conjunction with learners' written work. The extract in Excerpt 3.1 shows how the interview data were presented. In this extract learners were asked to provide the values of x for which $g(x) > 0$. The graphical representation shows the graph *g(x)* drawn by the learner.

Speaker	***Utterance***	***Graphical representation***
Researcher:	*For which values of x is graph g(x) greater than zero?*	
Sipho:	*When x is greater or equal to 2*	
Researcher:	*So which x values are greater than 2?*	
Sipho:	*Three, four…*	
Researcher:	*So which part of graph g is greater than zero?*	
Sipho:	*From here. (Pointing the part of the graph from the y-intercept all the way down to the fourth quadrant)*	
Researcher:	*So if I'm here at this point (referring to x=3 on the x-axis) what is the y value?*	
Sipho:	*It is -2*	
Researcher:	*Is -2 greater than zero?*	
Sipho:	*No*	
Researcher:	*May be let me be specific and say for which x values are the y values of g greater than zero?*	
Sipho:	*When x is less than 2 (pointing from x-intercepts of g(x) to negative infinity).*	

Excerpt 3.1: *Exemplar of the transcribed interview data*

Excerpt 3.1 shows an example of how the interview data were transcribed using brackets to represent the information supplied by the gesture.

The transcribed interview data were categorised into the themes that emerged from the data. Hatch (2002) refers to such an analysis as a typological analysis. This assisted me to create an analytical framework using Thompson and Carlson's (2017) framework of covariational reasoning and Even (1998). A detailed description of the way in which the framework was developed is provided in Chapter 4. The actual data analysis is discussed in Chapter 5.

3.9 My initial struggles to analyse the interview data

My study aimed at investigating how learners make sense of functions when working with tasks involving graphical representations. This was done by focusing on the graphical features that learners paid attention to as well as the approaches used to interpret graphs. I initially used the literature, which states that learners tend to either interpret graphs from a pointwise or global approach (Even, 1998). Researchers often mention that doing a research study is not a linear process; I strongly agree with them because I experienced a similar scenario. At the beginning of the study, I aimed at classifying the approaches that learners used to interpret graphs. I applied Even's concept of approaching graphs in terms of a pointwise or a global approach (Even, 1998) to develop a framework for analysing the interview data. I disaggregated Even's global approach to produce an analytical framework consisting of four categories for approaching graphs; namely, *pointwise, entry global, emerging global and global approach*. However, as I used this framework to analyse the interview data, I experienced difficulties in getting it to fit the data. One of the difficulties was to draw a clear distinction between an entry and an emerging global approach as well as an emerging and global approach. I also felt that the framework was not really capturing the essence of the data in terms of what learners had written or talked about in the interview. This resulted in several adjustments being made to the framework which I still battled to fit to the data. Given these difficulties, my supervisors and I agreed to abandon the framework and focus on the interpretation of data by looking at what each learner did to interpret graphs. I then realised that my battle with the framework stemmed from the fact that some of the learners were unable to coordinate the variables and hence operated "below" a pointwise approach. By "below" a pointwise approach, I mean that learners' attempts were less sophisticated than a pointwise approach as they were not able to coordinate variables. This came as a great surprise because I had thought that once learners were able to draw graphs they could also make sense of graph behaviour using a pointwise approach (based on the fact that they draw graphs by coordinating variables).

Subsequently, the focus of the data analysis changed, instead of looking at learners' approaches only, the underlying thinking behind learners' interpretation of graphs was incorporated. This means that the study focused on the aspects of functions that learners were paying attention to when responding to graphical representation tasks, as well as the approaches that learners used to interpret graphs. Therefore, Thompson and Carlson's (2017) covariation reasoning framework was found to be more appropriate to the way learners responded to tasks and subsequently resulted in the development of a new framework. Hence, this study contributes by looking at what learners were doing, including those that operated below the pointwise approach.

The main contribution of this study relates to the fact that some Grade 10 learners operated below the pointwise approach and interpreted the graph as a picture with numbers. In the next chapters, I will reveal why this discovery formed the major contribution.

Chapter 4 : Setting up and using analytical framework

4.1 Introduction

In this chapter, I present the setting up of the analytical framework as the result of the empirical data obtained and the theoretical lens used to analyse learners' responses. This framework has been used as the lens through which to examine the key features of the functions and the approaches that learners applied to respond to the graphical representation tasks. I begin by elaborating on how the analytical framework was created by working forward and backward between the interview data and the literature in order to generate the themes used to interpret the graphical representation. The description of themes is accompanied by the relevant examples. Lastly, I give an exemplar of the way in which the framework was applied to analyse the interview data.

4.2 The analytical framework for analysing interviews

4.2.1 Analytical framework for classifying learners' underlying thinking

In order to respond to the research questions, the approaches and the key features of the functions evident in learners' responses were classified using the covariational reasoning tool. In Chapter 2, I discussed the approaches used to interpret graphs provided by Even (1998), Thompson (1994), as well as Thompson and Carlson (2017). This included pointwise, global, correspondence, covariation, coordination and no coordinating of variables. Even (1998, p. 109) defines pointwise as the ability to "plot points on the graph, read individual points or deal with discrete points" of a given graph. She further defines a global approach as referring to the behaviour on a graph in relation to reading the interval of points or the whole graph (Even, 1998; Leinhardt et al., 1990). According to Thompson (1994), correspondence means for every input value there is a corresponding output value, whereas covariation refers to the change in one variable as the other variable changes in regard to dependant and independent variables (Ayalon et al., 2015b; Monk, 1994; Oehrtman et al., 2008; Thompson, 1994). Thompson and Carlson (2017) further describe covariational reasoning as the ability to envision two quantities changing *simultaneously*.

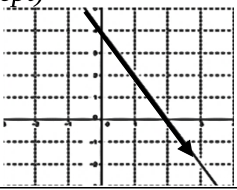
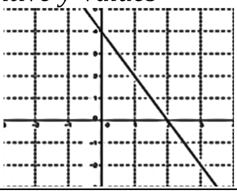
In Chapter 1, I elaborated on the struggles I encountered when I tried to analyse learners' responses using these approaches. When only pointwise and global approaches were used, the data would not fit into the framework. It was therefore in this regard that I built my analytical tool using a collection of approaches as informed by the data. I adapted the covariational reasoning on the coordination of the changes in variables and the ways in which learners conceive variables to vary in line with Thompson and Carlson (2017), as well as Even (see Table 4.1). Although Thompson and Carlson's framework for covariational reasoning consists of six approaches, I used only the two approaches that emerged from my data, that is, *coordination of values* and *no coordination of values*. In addition, I used Even's pointwise approach because global approach was not evident in my data. This means that my framework consisted of three approaches, namely, *no coordination of x and y values*, the *pointwise approach* and *the coordination of x and y values*. Below is the framework with the description of the approaches.

Table 4.1: *Analytical framework of the approaches emerged from the data*

<table>
<tr><th colspan="2">Approach</th><th>Description</th></tr>
<tr><td rowspan="3">Visual / Analytical</td><td>No coordination of x and y values</td><td>The learner does not have an image of variables either varying together or corresponding, instead s/he focuses on variation in one or other variable with no coordination of variables.</td></tr>
<tr><td>Pointwise</td><td>The learner coordinates variables by reading individual points. S/he anticipates creating discrete pairs of values without necessary seeing a trend or reads points with an aim of doing calculations.</td></tr>
<tr><td>Coordination of x and y values</td><td>The learner coordinates the values of one variable (x) with the values of another variable (y) in anticipation of creating a collection of ordered pairs (x; y) that are varying continuously and the learner is able to detect a trend.</td></tr>
</table>

In each of these approaches, learners used either an analytical or visual approach to make sense of the graph. Visual approach was predominant in the 'no coordination approach' where learners interpreted graphs as pictures labelled with numbers. However, as it becomes more sophisticated the analytical approach became dominant. Table 4.2 further clarify the description of the approaches by means of the indicators accompanied by examples in each approach.

Table 4.2: *Indicators and examples of each approach*

Approach	Indicators	Example
No coordination of *x* and *y* values	• Read axes separately without showing correspondence • Interpret graph as picture labelled with numbers	• *When x is* $-4;\ -3;\ -2$ *and* -1 • *when* $x < 4$ *(while pointing the y-intercept)* 
Pointwise	• Read correspondence of individual points (discrete point and no intervals) • Substitute points into a formula	*At* $x = 3$ *the corresponding y-value is* -2 *or let me do the calculations to verify that the gradient is negative.*
Coordination of values	• Read intervals of partial graph or given diagram • Showing traces of covariation and correspondence	*when x is less than 2, the graph will have positive y-values* 

The ideas of not coordinating and coordinating values from Thompson and Carlson (2017) were incorporated into the pointwise–global continuum (Even, 1998), as indicated in Figure 4.1. This was done because the pointwise–global continuum lens alone was not able to describe the data. However, the global approach was not evident in my data.

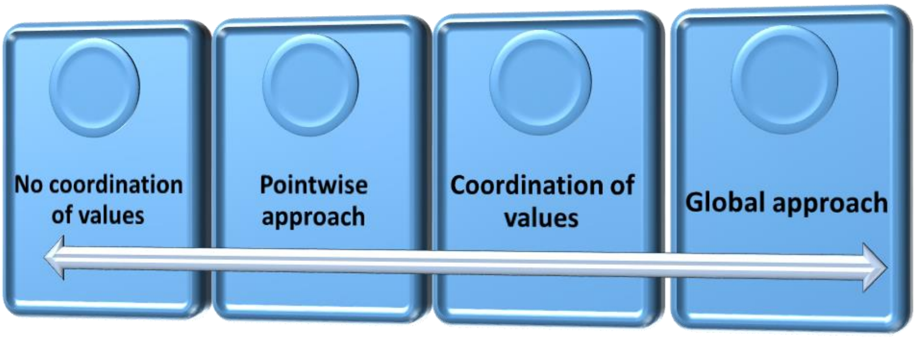


Figure 4.1: *Thompson and Carlson's ideas incorporated into Even's continuum*

The approach of "not coordinating values" is located on the left-hand side of the pointwise approach in terms of the pointwise–global continuum lens (Even, 1998), whereas the approach of "coordinating values" is located closer to the global approach. These approaches were taken from different literatures and combined to form one framework that I used as the lens to analyse my data.

4.2.2 Summary of graphical features

In Chapter 2, the key graphical features required at high school level were discussed. In this section, I summarise the features that the study focused on including intercepts, symmetry and transformation, and intervals of increase or decrease. These features of graphs played a major role in identifying the approaches used by learners when interpreting graphs. According to Carlson et al. (2002, p. 13) the ability "to represent and interpret important features in the shape of a graph of a dynamic function" and is a requirement for recognising the changes in the dependent and independent variables. Table 4.3 summarises the graphical features and their descriptions.

Table 4.3: *Summary of the description of the graphical features*

Graphical features	Description
Intercepts	*y-intercept:* the point on the Cartesian plane where the graph crosses the *y*-axis
	x-intercept: the point on the Cartesian plane where the graph crosses the *x*-axis
Gradient (*m*)	Positive if $m > 0 \Rightarrow$ increasing function
	Constant if $m = 0$
	Negative if $m < 0 \Rightarrow$ decreasing function
Transformation	*Reflection*: over the *y*-axis the $f(x) \to f(-x)$ over the *x*-axis $f(x) \to -f(x)$
	Horizontal shift by a units $f(x) \to f(x + a)$
	Vertical shift by c *units* $f(x) \to f(x) + c$
	Vertical compression refers to keeping the x-values constant while changing the y-values
Interval of change	*Increasing* means as the *x*-values increase, the *y*-values also increase
	Decreasing means as the *x*-values increase, the *y*-values decrease

4.3 Organising the data

All learners' responses to each task during the interviews were grouped using tables (see Table 4.4) in order to observe similarities and differences (Best & Kahn, 1998; Maxwell, 2005). The table is composed of learners' communication and the themes that emerged from the data. Hatch (2002) refers to such analysis as typological analysis; this type of analysis assisted me in creating an analytical framework. The themes used in Table 4.4 were graphical features, points/interval and justification. The theme for reading *points/interval* was taken from Even (1998) and Monk (1994) and it underpinned learners' way of reading the graph in terms of discrete points or interval. The category for *reasoning* was used to describe learners' justification in terms of analytical or visual.

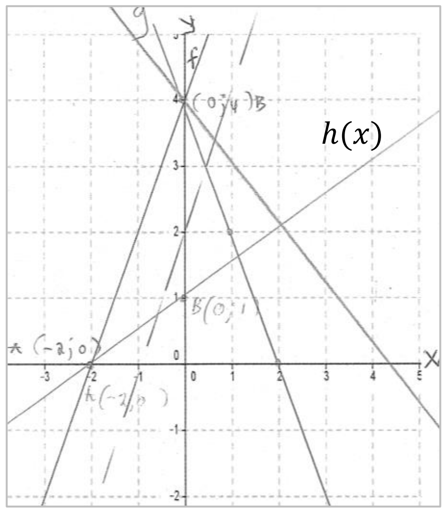
Table 4.4: *Grid for analysing learners' responses*

		Sipho	**Themba**	**Mpho**	**Bonga**
Learner's communication	**Responses (verbal & gesture)**				
Theme	**Graphical features**				
	Points/ interval				
	Reasoning				

4.3 Example of analysing interview data

In this section I provide an example to demonstrate how the interview data were analysed using the analytical framework. The basis of my study was to identify the key features of the functions and approaches learners used to respond to the graphical representation tasks. I used Mpho's response in Task 3 where he was asked to construct a line *h(x)* with the same *x*-intercept as graph *f(x)* but with the *y*-intercept being less than that of *f*. He drew graph *h* with the *x*-intercept of -2 and *y*-intercept of 1. He was then asked to compare the gradients of the two lines *f(x)* and *h(x)*.

Mpho responded by saying the gradient of *h(x)* was greater than that of *f* while pointing at graph *h* and *f*. He further justified this by saying that the difference (referring to the gap between *y*-intercepts of *h* and *f*) decreased as the gradient values increased. This statement was not clear and was never followed up because Mpho calculated the gradient of *h(x)* and compared it with that of *f(x)* as illustrated in the extract in Excerpt 4.1.

Speaker	Utterance	Graphical representation
Mpho:	*Greater than the gradient of f (pointing graph h and f)*	
Researcher:	*Why?*	
Mpho:	*The difference (referring to the gap between y-intercepts of h and f) … they decrease as the gradient values increase (pointing at the calculation of gradient f the change in y)*	
Researcher:	*Mmm….*	
Mpho:	*As I said previously, I label them (points) as B and A, where B (0;1) will be the second and A(-2;0) the first. It will be m=* $\frac{1-0}{0+2}=\frac{1}{2}$	
Researcher:	*So what can you decide, is the gradient greater or less (than that of f).*	
Mpho:	*It is less than that of the gradient of f*	
Researcher:	*What do you think makes it to be less?*	
Mpho:	*The y-intercept is less than…its lower than the first y-intercept*	

Excerpt 4.1: *Mpho comparing gradients of h and f using calculation*

Excerpt 4.1 shows that Mpho initially used visual reasoning to compare the gradients of *f(x)* and *h(x)* where he said gradient of *h* was less, based on the fact that the *y*-intercept was less than that of *f*. According to Costa, Manuel Matos and Carvalho Silva (2009, p. 2248), visual perception refers to the "information that comes through our eyes" which consist of visual processing followed by the interpretation of the shape using graphical features. Mpho's response in Excerpt 4.1 was organised and analysed using the themes as indicated in Table 4.5.

Table 4.5: *Analysis of Mpho's response*

Learner's communication	**Responses (verbal & gesture)**	*(drew h(x) passing (-2;0) and (0;1))* *It (gradient of h) is less. It's greater than the gradient of f, the difference (pointing at the calculation of gradient f the change in y)… they decrease as the gradient values increase.* *The line decreased… It will be* $m = \frac{1-0}{0+2} = \frac{1}{2}$ *It is less than that of the gradient of f.* *The y-intercept is less than…it is lower than the first y-intercept (which) does affect the gradient….by changing it*
Theme	**Graphical features**	Gradient, line decreased, *y*-intercept,
	Points/ interval	Many points (Δy), individual points
	Reasoning	Analytical

Table 4.5 indicates that Mpho used visual features such as the *y*-intercept to compare the gradients of the two lines. This was evident in his justification that the *y*-intercept of *h(x)* was lower than that of *f(x)*, which meant that the gradient of *h* was less than that of *f*. However, he did not pay explicit attention to the role of *x* in his sketch.

It is likely that when Mpho said the difference (*pointing and looking at the gap between y-intercepts of h and f)* decreased as the gradient values increased, he meant that $y_f - y_h$ would decrease if y_h increase but referred to y_h as gradient. This suggests that Mpho could not reason about the gradient outside of a formula, as he applied analytical reasoning to make meaningful comparison of the gradients of the two lines. This was evident when he literally used the formula to calculate the gradient of *h(x)* and then compared it with the gradients of *f(x)*. He demonstrated the ability to read and substitute individual points when he used the formula. Doing calculations was an indication of using analytical reasoning. But his response did not indicate the ability to value the effect of fixing the *x*-intercept. Therefore, his reasoning was based on pointwise approach.

4.3 Conclusion

This chapter focused on how the analytical framework was created as a result of the empirical data and the theoretical lens, which were refined by working forward and backward for several times. The discussion of the framework in this chapter was accompanied by a summary of the graphical features used by learners in this research as well as the sample table for analysing learners' responses. I also included an exemplar indicating how the framework was used to analyse the interview data in terms of the graphical features and approaches learners used to interpret the graphical representation tasks. In the next chapter, I provide a detailed analysis of the pre- and post-tests as well as the interview data. In particular, I use the themes that emerged from the data such as a visual approach, an analytical approach, no coordination and coordination of x and y values to provide a descriptive and interpretive analysis of learners' responses.

Chapter 5 : Analysis of the interview data

5.1 Introduction

This study investigated the way in which the Grade 10 learners used the key features and approaches to interpret the graphical representations of functions. In order to address the purpose of the study, the analysis of the interview data focused on underpinning learner thinking when responding to the graphical representation tasks. Learners' ways of interpreting graphs played a major role in my research, which sought to explore the key features and the kinds of complexities that learners encounter when dealing with graphical representations. In this chapter, I analysed the interview data using the framework developed in the previous chapter in terms of the themes that emerged from the empirical data and the theoretical lens. The framework resulted from combining different approaches in order to look through the interview data.

I begin this chapter by giving a summary of learners' performance in the pre- and post-tests which played a major role in designing the interview tasks. I then provide a summary of the interview tasks and the summary tables of learners' responses to each task. This is followed by an interpretive analysis which is guided by the themes that emerged from the data. Lastly, I discuss the summary of the analysis of learners' responses.

5.2 Summary of pre and post-tests

As noted earlier in Chapter 3, the same test instrument that was used to administer the pre-test was also used in the post-test. It should also be noted that the results of the pre and post-test formed part of the secondary data. These results played a major role in designing the interview tasks, which were further used to gain insights into the key features and approaches applied by learners while responding to the graphical representations. Chart 5.1 provides a summary of learners' scores for the pre- and post-tests. The vertical axis represents learners' test scores out of a total score of 16, while the horizontal axis is labelled with the names of learners who participated in the interviews and also formed part of the analysis. These results indicate that there was a clear improvement in learners' marks from the pre- to the post-test, however, this was not the focus of the study (see Chart 5.1).

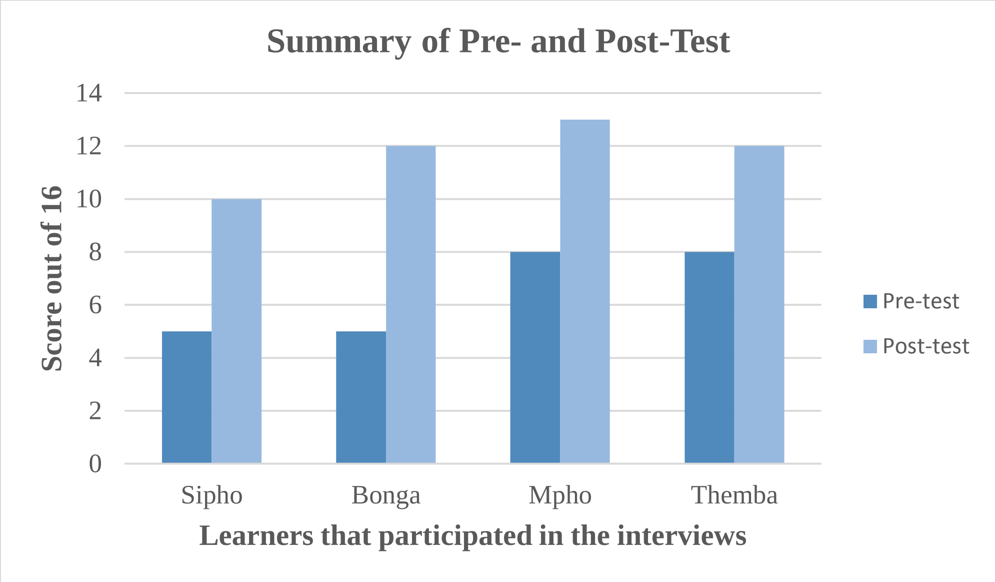


Chart 5.1: *Summary of learners' scores in the pre- and post-test*

Chart 5.1 shows that learners' scores ranged from 5 to 8 in the pre-test while in the post-test they ranged from 8 to 13. These results indicate that there was a substantial improvement in learners' competency in the post-test as compared to the pre-test. Bonga was the most improved learner with a difference of 7 marks, followed by Mpho with a difference of 5 marks. The next subsection outlines the interview tasks that learners engaged with.

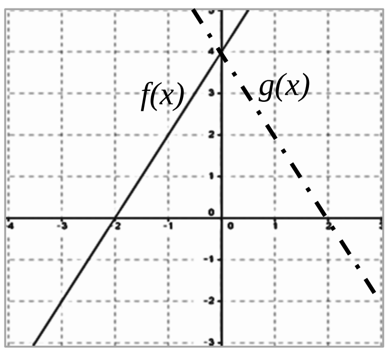
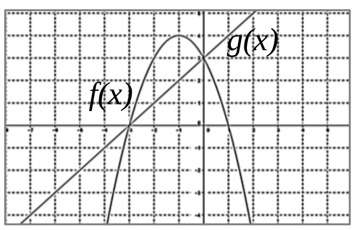
5.3 Summary of the interview tasks

In this section, I discuss the tasks that learners were asked to do during the interview. These tasks are summarised in terms of the task demand and task requirement. The reason for summarising the interview tasks prior to the analysis is to explain to the reader what each task required learners to do. Firstly, the task demand refers to the action that learners were required to perform in order to respond to the questions pertaining to the graphs. Such actions were classified into three ways of dealing with graphical representations, namely: constructing, interpreting or comparing. *Constructing* refers to the action of producing something new (Leinhardt et al., 1990) for example, *drawing a line with opposite sign gradient as f(x) while keeping the y-intercept the same*. Leinhardt et al. (1990)

further describe interpretation as an action that a learner uses to create or gain sense when attempting to respond to tasks dealing with graphical representations. An example of a task that required learners to interpret graphs was when they were asked *for which values of x is* $f(x) > 0$. The way learners made sense of the behaviour of *f(x)* in this example formed part of the interpretation. *Comparing* refers to creating a link between two graphs from a visual or analytical point of view and then formulating a conclusion. An example of such a task was when learners were asked *for which values of x is f(x) greater than g(x)*. This task required learners to compare the two graphs and then decide on the intervals. The terms *visual* and *analytical* were used in the analysis to categorise the interview data.

A summary of interview task questions and their demand is tabulated in Table 5.1. Column 1 indicates the numbering of tasks, Column 2 the task questions, Column 3 the task demand and Column 5 the graphical representations used in each task. Pre-interviews comprised Tasks 1 to 3 whereas post-interviews consisted of Tasks 4 to 6 as highlighted in the methodology chapter. Table 5.1 is included to help the reader to make sense of the data analysis discussed later in this chapter.

Table 5.1: *Summary of the interview task questions and its demand*

Task	Task question	Task demand	Graphical representation
1	Construct a line with an opposite sign gradient to *f(x)* while keeping the *y*-intercept the same. Explain how you constructed your line *g(x)*.	• Constructing • Interpreting	
2	For which values of *x* is *g(x)* greater than zero?	• Interpreting	
3	Draw another line with the same *x*-intercept and a *y*-intercept less than that of *f(x)* and label it as *h(x)*. Compare the gradients of *f(x)* and *h(x)* and decide which one is greater or less; then justify your answer.	• Constructing • Comparing • Interpreting	
4	For which values of *x* is *g(x)* greater than zero?	• Interpreting	
5	Give the values of *x* for which *f(x)* is increasing or decreasing.	• Interpreting	
6	For which values of *x* is *f(x)* greater than *g(x)?*	• Comparing • Interpreting	

Secondly, the task requirement refers to the basic knowledge required in each task in order for learners to be able to respond to the questions. For example, if learners were asked to find the gradient of the linear function, this task would require them to interpret the graph by:

- looking at the shape of the graph, that is, whether it has a negative or positive gradient
- read at least two points on the graph from the Cartesian plane
- identify and label the *x*-coordinate and *y*-coordinate
- recall the formula for calculating gradient
- do substitution into the formula and simplify calculations to obtain the answer

In this example, the key features such as the shape of graph, the definition of gradient, reading coordinates and the formula of the gradient are prerequisites for learners to attempt such a task.

In this study, the basic knowledge on the graphical features such as notation, intercepts, turning point, gradient and so forth, were the necessary requirements for learners to deal with linear and quadratic graphs. These features form part of the graphical language that comprises the mathematical talk for interpreting graphs.

5.4 Summary of learners' responses

Having summarised the task interviews, I now provide a summary of learners' responses to each task, which is tabulated in Tables 5.2 to 5.7. Row 1 shows learners' verbal and gesture communication. According to Roth (2001) gestures are essential traits for communication mainly focusing on mathematics about the graphic representation. Learners' responses were recorded as they appeared exactly in the transcript. Row 2 indicates the different themes, namely, graphical features, reading points/interval and reasoning. Reading of points/interval further highlights whether learners used individual/discrete points, intervals or whole graph to make sense of graphic representations. The theme for reasoning further elaborated the way in which learners perceived graphs analytically or visually.

Table 5.2: *Analysis of learners' responses in Task 1*

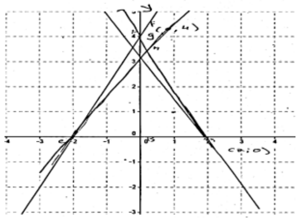
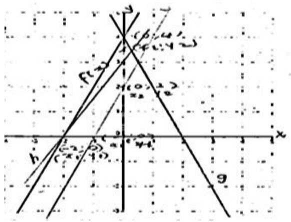
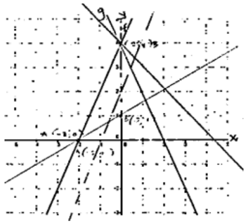
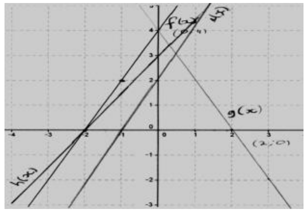
		Sipho	Themba	Mpho	Bonga
Learner's communication	**Response**	*Because it's (pointing the x-intercept of graph f and that of the constructed line) no longer on the negative side of the x-axis instead it shifted to the positive side while keeping the y-intercept the same. Now it is like the mirror, reflection of f.*	*Just looked at the axis of symmetry (indicating the y-axis and showing with both hands the parts from 0 to 2 and 0 to -2) on the x-axis, is it that if it is -2 this side, it is supposed to be 2 this side and so then keep the y-intercept the same as 4.*	*The gradient is negative because it reflect on the y-axis (pointing the y-axis as the reference point) for the line of f*	*Let me do the calculations (working out the gradient), the gradient is -2. Here (pointing the x-intercept) on f(x) we used -2 as the x-value so on the other one (referring to g(x)) I preferred to use positive 2*
	Picture				
Theme	**Graphical features**	*x*-axis, *x*-intercept, *y*-intercept, reflection	*x*-axis, axis of symmetry, *x*-intercept, *y*-intercept	Reflection, gradient, *y*-axis	Intercepts, *x*-value, gradient
	Points/ interval	Comparing individual points and whole graph reading	Comparing individual points and whole graph reading	Reading whole graph	Comparing individual points
	Reasoning	Visual	Visual	Visual	Visual and analytical

Table 5.3: *Analysis of learners' responses in Task 2*

		Sipho	Themba	Mpho	Bonga
Learner's communication	**Response**	*When x=4 (learner pointing the y-intercept), no when x is less than 4.Oh no, when x is less than 2, it's when the graph is decreasing, no increasing (pointing at the part of the line greater than zero that goes to infinity).When x is greater or equal to 2. When x is less than 2 (pointing from x-intercepts of g(x) to negative infinity).*	*When x is greater than 2. Oh, no, no, no….when x is less than 2. Because when x is greater than 2 it means here… (Indicating the part of g with negative y-values), therefore when x is less than 2.*	*When y is greater than zero. Y-values are greater than zero when the x values are greater than zero but less than 2. Let me choose these points here (looking at x-values far less than -1) From negative infinity but less than 2, the graph will be going up. Its values that are less than two. So x values less than 2, y will be greater than zero*	*Positive 3; 4 and 5. And 2 also, yah 2; 3; 4 and 5* *Yah going to the negative side (pointing the values on the x-axis) it will be increasing. It's going to be greater from positive 1 to infinity, negative infinity, it's going to be greater than 0 (referring to g(x)>0). (Negative infinity) to 1 point something*
Theme	**Graphical features**	*x*- and *y*-intercepts, decreasing, increasing,	*x*-intercept	*x*-intercept	Increasing, *x*-values
	Points/ interval	Infinitely many points	Infinitely many points	Many to infinitely many points	Individual to many points
	Reasoning	Visual	Traces of covariation, visual	visual	visual

***Table 5.4:** Analysis of learners' responses in task 3*

		Sipho	Themba	Mpho	Bonga
Learner's communication	**Response**	*(drew h(x) passing (-2;0) and (0;3))* *Graph f's gradient is greater than that of graph h's. The thing (pointing y-intercepts) that determines whether the gradient is negative or positive is the y-axis, the y-intercept. Since graph h's y-intercept is less than that of graph f meaning that graph h will have lesser gradient because if I calculate (mentally) the gradient for (line h) will give me three over two less than four over two for graph f*	*(drew h(x) passing (-2;0) and (0;3))* *It (gradient of h) is less than the gradient of graph f because the graph has shifted (indicating with hands downward movement in the y-axis), I'm not sure which shift but it is because the graph shifted. It has the vertical shift. The y-intercept has changed to 3*	*(drew h(x) passing (-2;0) and (0;1))* *It (gradient of h) is less. It's greater than the gradient of f, the difference (pointing at the calculation of gradient f the change in y)… they decrease as the gradient values increase. The line decreased…. It will be* $m = \frac{1-0}{0+2} = \frac{1}{2}$ *It is less than that of the gradient of f. The y-intercept is less than…it is lower than the first y-intercept (which) does affect the gradient….by changing it*	*(drew h(x) passing (-2;0) and (0;3))* *(learner quiet when asked to compare m) It (gradient of h) is less than (gradient of f) because the y-values are increasing* *I mean it (y-value) decreased to positive 3 (pointing at the y-intercept)*
Theme	**Graphical features**	Gradient, *y*-axis, *y*-intercept	Gradient, vertical shift, *y*-intercept	Gradient, line decreased, *y*-intercept,	Increasing, *y*-values
	Points/ interval	Whole graph, individual points	Whole graph, individual points	Many points (Δy), individual points	individual points
	Reasoning	Visual and analytical	Visual (Transformation)	Analytical	Visual comparison

Table 5.5: *Summary of learners' responses in Task 4*

		Sipho	Themba	Mpho	Bonga
Learners' communication	**Response**	*When x is greater than -3 (pointing on the graph and focused on the x-intercept. Also indicating upward movement of the graph)*	*(Re-voicing the question while pointing on the graph of g(x) and the y-axis then wrote down his solution). When x is greater than -3. I decided because it's the x-intercept (pointing x-intercept with a pencil).*	*(Writing down the solution) Ma'am x is the element of -3 to infinity. The deciding points are actually the x-intercepts, which was -3 and then from the values going um that are greater than -3* which is *the graph will never end because there's no stop. Anything indicating a no stop so it is to infinity*	*When x is greater than -3 cause mm eh (pointing the x-intercept) because when we start going on that side (pointing the direction of the x values greater than -3* and indicating that the graph is increasing *), the y-values are increasing. I think it's when x is greater than 0.*
Theme	**Graphical features**	*x*-intercept	*x*-intercept	*x*-intercept	*x*-intercept
	Points/ interval	Infinitely many points	Infinitely many points	Infinitely many points, whole graph	Infinitely many points
	Reasoning	Coordination of values; visual	Coordination of values; visual	Coordination of values, visual	Coordination of values, visual

Table 5.6: *Summary of learners' responses in Task 5*

		Sipho	Themba	Mpho	Bonga
Learners' communication	**Response**	*Increasing, it's increasing when x is less than, when x is greater than -3 and lesser than … 1 (pointing the part of f where f(x)>0). I think it's increasing when x is lesser than -3. It stops increasing when… yes it is increasing yes. The starting point isn't it the y-value of the turning point? It's increasing when x is greater than -3. the graph in general, I think it's increasing*	*(writing down the solution) increasing mm when x is less than -1….and decreasing when x … greater than -1*	*(writing down the solution and point the graph) Um it increases from negative infinity till -1 and decreases from -1 till infinity. Negative infinity it increases then it stops at -1. From -1 it decreases till negative infinity (R: To negative infinity or positive infinity?) To negative infinity ma'am*	*(Writing down the solution and point the graph). Increasing when x is -4; -3 and -2 and -1 sorry mam let me just see. Uh umm (writing the solution) when x is greater than 4, -4 , less than -1. (R: And decreasing?) When (writing the solution and pointing on the graph) x is greater than -1 and less than 2.*
Theme	**Graphical features**	Increasing, y-value of turning point, *x*-intercept	Turning point, *x-value*	*x*-values, turning point	*x*-values, turning point
	Points/ interval	Many points, infinitely many points	Whole graph	Infinitely many points	Individual, many points
	Reasoning	No coordination of values, visual	Coordination, visual	Coordination, visual	Pointwise, visual

Table 5.7: *Summary of learners' responses in Task 6*

		Sipho	Themba	Mpho	Bonga
Learner's Communication	**Response**	*(pointing the x-intercept of -3) When x is greater than -3 but less than 1. (R: if we look at this point (x=0.5). Which graph at 0.5 is greater? Is it g(x) or f(x)?) It's umm g(x). Meaning it's when x is eh… greater than -3 but less than 0, when x is less than 0. Less than 0 but greater than -3.*	*(Shading the part of the graph where f(x)>g(x) and writing the interval -3<x<0) when x is greater than -3 but less than 0. (Shaded) where f(x) is greater than g(x). (R: is nowhere else where f(x) greater than g(x)?) Yes there is nowhere else*	*-3 until 0, from -3 until 0 it is greater. It is this shaded area here (shading the part where f(x)>g(x)).*	*(Pointing on the graph while thinking) when x is greater than -3 and less than -1. Actually to zero. When x is greater than -3 and but less than 0*
Theme	**Graphical features**	*x*-intercept	*x*- and *y*-intercepts	*x*- and *y*-intercept	*x*- and *y*-intercept
	Points/ interval	Interval more than actual interval, adjusted to correct interval	Actual interval	Actual interval	Less than actual interval then adjusted to the correct interval
	Reasoning	visual	visual	visual	visual

Summary of learners' responses in Tables 5.2 to 5.7 was categorized by colour coding the key words in learner's talk, which helped me to identify important information. Learner's responses were coded as gesture using light blue; interval highlighted in green; justification highlighted in purple and was foregrounded by the combination of points/interval (green). The way in which learners explained their responses and the reasoning behind were informed by the points/interval and justification. These summary tables assisted me to structure the descriptive and interpretive analysis, which is discussed in the next section.

5.4 Data analysis

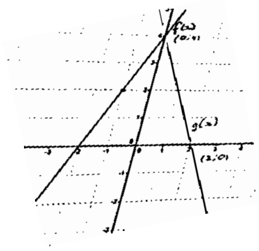
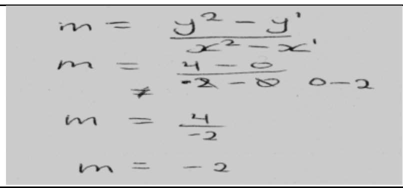
In this section, I provide an interpretive analysis of learners' responses according to the themes that emerged from the data. The analysis seek to provide answers to the following research questions:

1. *What key features of functions do learners pay attention to when responding to graphical representation tasks?*
2. *What approaches do learners use when working on the graphical representation tasks?*

In analysing the interview data, I categorised learners' responses according to the themes they used to make sense of the behaviour of the graphs. This was as a result of working forward and backward with the data and paying attention to the key features and approaches that learners applied. The way in which learners made sense of graphs in terms of points reading, coordination and variation played a major role when drawing conclusions on the approaches and graphical features used. The themes used are visual reasoning, analytical reasoning, coordinating and not coordinating *x*- and *y*-values. The following subsections provide a detailed analysis of the interviews in line with these themes.

5.5.1 Interpreting graphs using an analytical approach

Some learners were observed using analytical reasoning as a means of making sense of graphical representation tasks. *Analytical reasoning* in this section refers to responses where learners used calculations to make sense of the graph. This type of reasoning is associated with a pointwise approach because learners used individual points in order to carry out the calculations. According to Knuth (2000), learners have a strong tendency to approach graphs from an analytical point of view. Bonga's response to Task 1 is a good example of a pointwise approach. In Task 1, he was asked to construct a line with the same *y*-intercept as *f(x)* but with a gradient of an opposite sign. He subsequently drew line *g(x)* with an *x*-intercept of 2 and a *y*-intercept of 4 (see the diagram in Excerpt 5.1). Bonga's extract is provided in Excerpt 5.1.

Speaker	**What was said**	**What was done**
Bonga:	*(Drawing line g(x))*	
Researcher:	*Can you please take me through on how you decided that the line should be drawn like that?*	
Bonga:	*Ok, let me do the calculations. My gradient is -2*	
Researcher:	*So you got a gradient of -2. Is it of an opposite sign from that of 2 (referring to the gradient of f(x))*	
Bonga:	*Yes mam*	

Excerpt 5.1: *Bonga's explanatory talk for drawing g(x)*

Excerpt 5.1 shows that Bonga used calculations to convince the researcher that the gradient of the linear graph *g(x)* indeed had a negative sign. In so doing, he demonstrated the ability to read and substitute individual coordinates of the intercepts into a gradient formula, thus concluding that the gradient was -2. Ayalon et al. (2017), mention that learners often interpret graphs as input and output entities that can be manipulated using algebraic calculations. Bonga's interpretation of his constructed graph was driven by the visual features of the graph (*x*- and *y*-

intercepts) as well as the pointwise approach since he used individual points to do the calculations.

Sipho and Mpho also used analytical reasoning to respond to Task 3 when they were asked to draw another line with the same *x*-intercept as graph *f* but with the *y*-intercept less than that of *f*. Sipho drew line *h* with an *x*-intercept of -2 and a *y*-intercept of 3, while Mpho drew a line with the same *x*-intercept and *y*-intercept of 1. They were then asked in separate interviews to compare the gradients of the two lines *f(x)* and *g(x)*. The extract in Excerpt 5.2 provides learners' explanatory talk on how they compared the gradients of the two lines.

Speaker	What was said	What was done
Sipho:	*I will say that graph… f's gradient is greater than that of graph h's.*	*(pointing on the graph)*
Researcher:	*Why do you say that?*	
Sipho:	*Because the thing that determines whether the gradient is negative or positive is the y-axis, the y-intercept. Since graph h's y-intercept is less than that of graph f (indicating with hands by pointing the intercepts), meaning that graph h will have lesser gradient because if* ***I calculate the gradient for this (line h) will be (saying it verbally without writing) three minus zero which will give us three over two,*** *which is less than four over two of graph f.*	Verbally calculated gradient of line *h* to be $\frac{3}{2}$
Mpho:	*As I said previously, I label them (intercepts) as B(0;1) and A(-2;0), where B will be the second and A the first. It will be m=* $\frac{1-0}{0+2} = \frac{1}{2}$	$m = \frac{y_B - y_1}{x_B - x_1} = \frac{1-0}{0+2} = \frac{1}{2}$
Researcher:	*So what can you decide, is the gradient greater or less (than that of f)?*	
Mpho:	*It is less than that of the gradient of f*	
Researcher:	*What do you think makes it less?*	
Mpho:	*The y-intercept is less than…its lower than the first y-intercept*	

Excerpt 5.2: *Example in which calculations were used to interpret graphs*

Excerpt 5.2 indicates that both learners used the *y*-intercept and their calculations to substantiate that the gradient of line *h* was less than that of *f*. They accordingly demonstrated the ability to read and substitute coordinates into a gradient formula. However, they did not explicitly mention that the comparison of the two gradients also depends on the *x*-intercept being constant. According to these learners the smaller the y-intercept the smaller the gradient. This judgement is based on visual reasoning accompanied by analytical reasoning. Again, these learners used the intercepts as the main visual feature to calculate the gradient.

5.5.2 Interpreting graphs using a visual approach

The visual features of graphs such as intercepts and turning points usually trigger learners to reason from a visual point of view. This was the dominant approach applied by all four learners across the tasks that demanded them to either construct, interpret or compare graphical representations. Three out of four learners predominantly used visual reasoning in Tasks 1, 3 and 6. These tasks required learners to interpret graphs by comparing graphical representations. Again, visual reasoning was evident during probing when learners attempted to give definitions of certain concept or to justify their responses. Leinhardt et al. (1990) highlight that the way one interpret a graphs generally involves direct visual judgement since some definitions are embedded in visual representations (Thompson, 1994). The way learners applied visual reasoning either resulted in appropriate or inappropriate interpretation. These interpretations are discussed separately because interesting discoveries surfaced regarding what guided learners to reason visually either appropriately or inappropriately.

5.4.1.1 Appropriate use of the visual approach

Learners used visual graphical features to make sense of the behaviour of the graphs in Tasks 1, 3 and 6. It is generally expected that learners will make use of visual features to make sense of graphs because a graphs are visual representations of a function. In Excerpt 5.3, I provide a number of extracts to illustrate learners' use of visual reasoning to respond to Task 1. This task asked learners to construct a line $g(x)$ with an opposite sign gradient as $f(x) = 2x + 4$ (learners were given only the graph and not the equation) while keeping the y-intercept the same. All learners drew their graphs with an x-intercept of 2 and y-intercept of 4. The extracts in Excerpt 5.3 were captured from different interview slots.

Speaker	Utterances from different interviews
Sipho:	*Because it's no longer on the negative side of the x-axis (pointing the x-intercept of graph f and that of the constructed line) instead it shifted to the positive side while keeping the y-intercept the same. Now it's like the mirror, reflection of f.*
Themba:	*I just looked at the axis of symmetry (indicating the y-axis and showing with both hands the parts from 0 to 2 and 0 to -2) on the x-axis, is it that if it is -2 this side, it is supposed to be 2 this side and so then keep the y-intercept the same as 4*
Mpho: *Researcher:* *Mpho:*	*The gradient is negative* *Why do you say it's negative?* *Because it reflect on the y-axis (pointing the y-axis as the reference point) for the line of f*

Excerpt 5.3: *Exemplification of the way learners interpreted Task 1*

Excerpt 5.3 indicates that learners used the *x*-intercept, reflection and symmetry as the key features in order to attempt Task 1. Again, all three learners used a similar approach to draw line *g(x);* however, their justification was articulated differently. Themba mentioned that he looked at the axis of symmetry while pointing at the *y*-axis. It is likely that he was referring to the *y*-axis as the line of reflection but interpreted it as the axis of symmetry. All learners produced line *g(x)* with an opposite sign gradient by working visually using reflection. Costa et al. (2009) refers to such thinking as visual construction, where a learner modifies a shape to meet certain criteria by anticipating and organising the shape in a logical manner. This was evident when Sipho emphasised that line *g(x)* was a mirror image of *f(x)*, which means that *g(x)* was transformed from $f(x)\ to\ f(-x)$. Again, Sipho and Themba explicitly identified the changes in the *x*-intercept of *f(x)* and *g(x),* which allowed them to draw line *g(x)* with a negative gradient. This agrees with the task requirement which says for line *g(x)* to have the negative gradient, the *x*-intercept should be greater than zero. Another task requirement expected learners to interpret the negative gradient in term of the changes in *x*- and *y*-values; however, such interpretation was not evident in learners' responses; instead they opted to use visual graphical features.

Another instance that exemplified the correct use of visual interpretation was when Themba and Mpho were asked in Task 6 to provide the values of x for which $f(x) > g(x)$. See the extract in Excerpt 5.4.

Speaker	Utterances from different interviews
Themba:	*(Shading and writing down -3<x<0)*
Researcher:	*Mmm, so you are saying…*
Themba:	*When x is greater than -3 but less than 0.*
Mpho:	*(Writing down the answer in interval notation* $x \in [-3; 0]$*) Um…*
	Ehmm and your answer is?
Researcher:	*-3 until 0, from -3 until 0 it is greater. It is this shaded area here*
Mpho:	*(shading the part of graph where f(x)>g(x)*

Excerpt 5.4: *Exemplar of the appropriate use of visual reasoning on Task 6*

Excerpt 5.4 indicates that both Themba and Mpho responded to Task 6 by shading the region where $f(x) > g(x)$ prior to providing the interval of $-3 < x < 0$. It is likely that these two learners had a better understanding of comparing the two graphs because they applied the visual approach, which caused them to shade the appropriate interval as indicated in Figure 5.1.

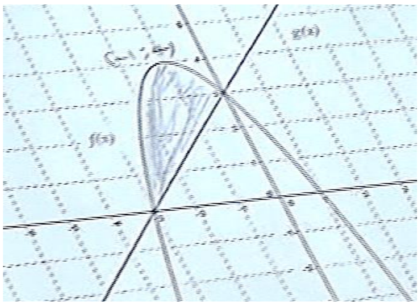
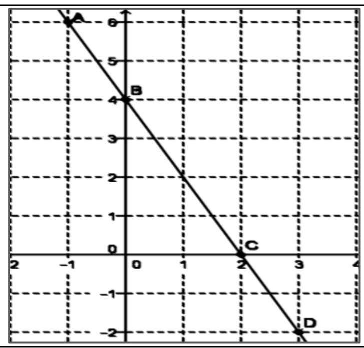


Figure 5.1: *Exemplar of how Themba shaded the region where* $f(x) > g(x)$

Although these learners were not explicit about the way they decided on their intervals (apart from visual perception), it is likely that they used the points where the two graphs intersect to choose the appropriate interval since their shaded region covered that area.

5.4.1.2 Inappropriate way of using a visual approach

The instances where learners used visual reasoning that produced partially incorrect interpretations indicated that learners were either reading the graph without making connection between the graph and the Cartesian plane or using graphical features incorrectly. Such responses were observed in Task 2 and 4. In Task 2 learners were asked to provide the values of *x* for which $g(x) > 0$. The extract indicating Sipho's response to this task is provided in Excerpt 5.5. It is clear that Sipho used the *y*-intercept to respond to the task but referred to it as the *x*-value. Learners often confuse the notation *f(x)* as referring to the *x*-values.

Speaker	**What was said**	**What was done / referred to**
Sipho:	*When x is equal to 4. No when x is less than 4*	*pointing to the y-intercept*
Researcher:	*Where is x equal to 4? I can see you pointing there (referring to y-intercept)*	
Sipho:	*Oh no, when x is less than 2 (pointing the x-intercept), it's when the graph is decreasing, no increasing (pointing at the part of the line greater than zero in the interval x∈(0;2)).*	
Researcher:	*Let me repeat, for which values of x is graph g greater than zero?*	
Sipho:	*When x is greater or equal to 2*	

Excerpt 5.5: *Sipho responding to Task 2*

Excerpt 5.5 indicates that Sipho initially responded to the task by looking and pointing at the *y*-intercept of 4, where he said $g(x) > 0$ when $x = 4$ and immediately adjusted it to $x < 4$. His deictic gesture of pointing at the *y*-intercept indicated that he was looking at the y-intercept but referred to it as $x = 4$. It appears that Sipho changed his response to $x < 4$ to describe the part of the graph below the *y*-intercept labelled as B (see the graph in Excerpt 5.5). This implies that he interpreted the graph as a picture labelled with the number 4. Asking him "where is $x = 4$", caused him to look at the *x*-axis, which prompted him to use the *x*-intercept and, thus to say *x* is less than 2. This suggests that he was still reading the part of the graph from C to B (see the graph in Excerpt 5.5), which he saw as being less than 2. This is also an indication that he used visual reasoning without coordinating the *x*- and *y*-values. Again, he made sense of the graph as a picture labelled with the number 2 (see Figure 5.2).

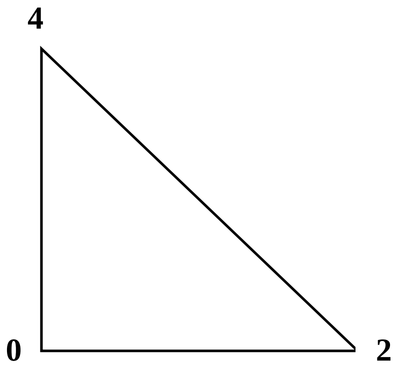


Figure 5.2: *Exemplar of reading graph as a picture or figure*

His justification that the graph was both decreasing and increasing suggests that he either read the graph from the *y*-intercept or the *x*-intercept. It is likely that he considered the graph as increasing when reading it from C to B, whereas reading from B to C he saw it as decreasing. It seems that Sipho described the same part of the graph (B to C) in terms of both the *y*-intercept and the *x*-intercept. Although Sipho used the graphical language of *increasing* and *decreasing,* he used it incorrectly because a linear function cannot increase and decrease. He reasoned visually whereas increasing and decreasing are analytically based on a definition.

It seemed that when the question was repeated to gain clarity on increasing and decreasing, Sipho was still reading the same part of the graph from C to B which he had referred to as being greater or equal to 2. This clearly shows that he overlooked the fact that the notation $g(x) > 0$ means that the *y*-values are positive; instead he assumed that the notation $g(x) > 0$ referred to $x > 0$. His interpretation was based mainly on the visual features of the graph which resulted in incorrect responses.

Another example that demonstrated the use of visual thinking which resulted in an incorrect response was Bonga's mathematical talk in Task 5. In this task, learners were asked to provide values of *x* for which the parabola graph *f (*see Figure 5.3) was increasing or decreasing.

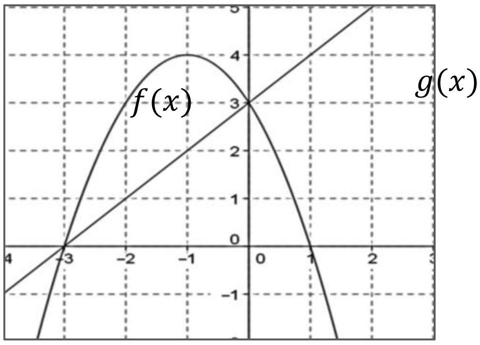


Figure 5.3: *Graphical representation of f(x) and g(x)*

Bonga started by interpreting this task using discrete points where he said *"increasing* when x *is* $-4;\ -3, -2$ *and* -1*"*. He then changed this to consider the interval notation, where he said that *f(x)* was increasing when *x* was greater than -4 and less than -1 and decreasing when *x* was greater -1 and less than 2. See his written response in Figure 5.4.

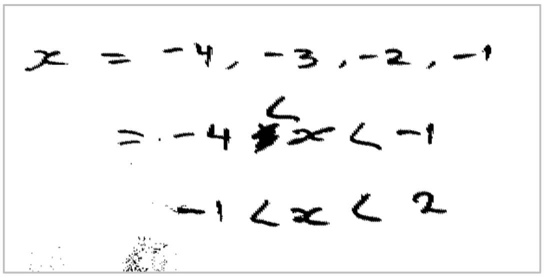
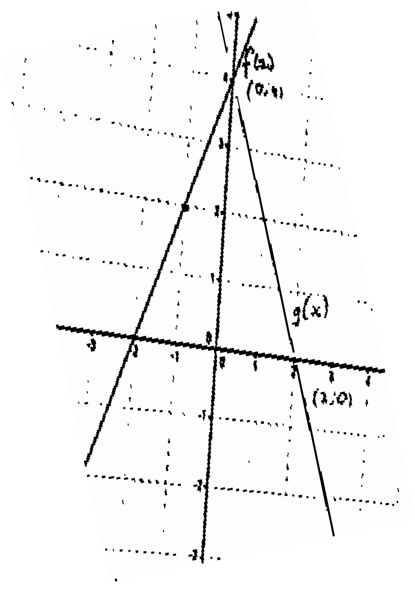


Figure 5.4: *Bonga's written response for the interval of increase and decrease*

Figure 5.2 shows that Bonga initially used individual points to talk about the intervals of increase because he had read the graph visually. He then moved from reading individual points to considering intervals. This agrees with the statement made by Even (1998) that thinking progresses from a pointwise to a global approach. Although Bonga gave the intervals of increase and decrease, his visual reasoning caused him to be limited by the given figure where he considered the domain to be $x \in [-4; 2]$ instead of $x \in (-\infty; \infty)$. This suggests that Bonga's partially incorrect interpretation of the intervals of increase and decrease of *f(x)* was embedded in visual approach.

5.5.3 Interpreting graphs without coordinating x- and y-values

How learners coordinated *x*- and *y*-values as they were making sense of the behaviour of graphical representations was of the utmost importance when analysing learners' responses. It was observed that most learners in the pre-interviews (Sipho, Mpho and Bonga) did not coordinate the changes in the *x*- and *y*-values when attempting to respond to tasks. They read either the axes independently of each other or interpreted the graph as a picture labelled with numbers. In the post-interview, only Sipho was seen to be not coordinating changes in the *x*-values with respect to the *y*-values. The extract in Excerpt 5.6 captures learners' responses from separate pre-interviews when they were asked to provide the values of *x* for which *g(x)* was greater than zero.

Speaker	***Utterance from separate interviews***	***Graphical representation***
Researcher: *Sipho:* *Researcher:* *Sipho:* *Researcher:* *Sipho:* *Researcher:* *Sipho:* *Researcher:* *Sipho:* *Researcher:* *Sipho:*	*For which values of x is graph g greater than* *zero?* *When x is greater or equal to 2* *So which x values are greater than 2?* *Three, four…* *So which part of graph g is greater than zero?* *From here. (Pointing the part of the graph from* *y-intercept along the line into the fourth* *quadrant)* *So if I'm here at this point (referring to x-value=3* *on the x-axis) what is the y value?* *It is -2* *Is -2 greater than zero?* *No* *May be let me be specific and say for which x* *values are the y values of g greater than zero?* *When x is less than 2 (pointing from x-intercepts* *of g(x) to negative infinity).*	
Mpho: *Researcher:* *Mpho:* *Researcher:* *Mpho:* *Researcher:* *Mpho:* *Researcher:* *Mpho:* *Researcher:* *Mpho:*	*When x is greater than zero.* *Can you choose one point when x is greater than* *zero* *It's two and also one* *Ok, let's go to one, the graph is…..* *It's (1; 2)* *Is the y-value positive or negative* *It's positive* *When x. is eh... 3, what is the y-value?* *It will be -2* *This means you need to refine your answer* *(referring to his response)* *The y-values are greater than zero when the x* *values are greater than zero but less than 2.*	
Bonga: *Researcher:*	*Its 3, ee… positive 3; 4 and 5. And 2 also, yah 2;* *3; 4 and 5 (pointing the values on the x-axis).*	

 Bonga: *Researcher:* *Bonga:* *Researcher:* *Bonga:* *Researcher:* *Bonga:*	*Thank you, so you are saying from 2. Now if I go* *to 3, you said also 3 even?* *Yes ma'am* *Or even 4, if I go to 3 what is my y value?* *The y-value will be 0 (pointing on the x-axis)* *I go down (from x-value=3 to the graph of g(x))* *if you go down (indicating with a pen) it will be -2* *-2. Is -2 greater than 0?* *No. So it's gonna be 1… only 1*	

Excerpt 5.6: *Exemplary of responses without coordination of variables*

The excerpt shows that all three learners gave positive *x*-values when defining the interval where $g(x) > 0$. This suggest that these learners were reading values on the *x*-axis only without coordinating them with the *y*-values. Bonga and Sipho's responses are good examples because they provided *x*-values greater and equal to 2, although Bonga gave individual points from 2 to 5, which is an indication of being limited by the given figure. In this interval $of\ x \geq 2$ the line $g(x)$ is less than zero.

From the above discussion, I argue that the three learners interpreted $g(x) > 0$ as referring to the *x*-values greater than zero; that is, to the right side of the *x*-axis. They were observed pointing to the graph and reading the *x*-values with respect to their positioning on the graph. For example, Sipho and Bonga provided *x*-values that were on the right side of the graph as they said $g(x) > 0$ when $x \geq 2$ and failed to relate these *x*-values to the *y*-values of the graph. I therefore conclude that they did not explicitly coordinate *x*- and *y*-values as a representation of the graph which represent the relationship between *x*- and *y*-values.

When these learners were asked to provide the corresponding *y*-value to $x = 3$, Sipho and Mpho (in separate interviews) gave a *y*-value of -2, whereas Bonga was still reading from the *x*-axis and gave 0, although he later gave the correct *y*-value of -2. These learners did not immediately make sense of the negative *y*-value until they were asked whether the graph was negative or positive at that point. It is likely that initially they did not make sense of the function notation *g(x)* as being the same as the *y*-values. This was evident when they all excluded the part of the graph in the third quadrant in their subsequent attempt. This probing appear to have enabled learners to shift from not being able to make sense of the notation *g(x)* to seeing it as the *y*-values. This is an indication of learners' ADL transformed into PDL, where they were able to exclude the part of the graph that was less than zero. See the extract taken from Excerpt 5.6.

Speaker	***Utterance***
Sipho:	*When x is less than 2 (pointing from x-intercepts of g(x) to negative infinity).*
Mpho:	*The y-values are greater than zero when the x values are greater than zero but less than 2.*
Bonga:	*So it's gonna be 1… only 1*

The above extract shows that there was an improvement in learners' interpretation of *g(x)* as a result of probing because they had excluded the part of the graph that consisted of negative y-values. Mpho and Bonga were still giving positive *x*-value/s; Bonga was still reading individual points, whereas Mpho demonstrated progress by referring to *g(x)* as the *y*-values. This suggests that Bonga interpreted the graph from a pointwise approach while Mpho showed improvement by interpreting the notation *g(x)* as being the same as the *y*-values.

Although Sipho gave the correct interval of *x* less than 2, there was no indication that he had coordinated these values with the *y*-values because he used an iconic gesture to refer to the *x*-axis only. A similar response by Sipho was observed in the post-interview, where he gave the definition of the increasing function. He said that increasing function means "*the x-values are going positively in the positive direction, it is increasing going towards the positive, eh positive side of the x-axis"*. This implies that Sipho defined increasing function in terms of the *x*-values going towards the right side of the *x*-axis. His definition focused on one variable *(x)* instead of coordinating the changes in the *x*-values with those of the *y*-values.

5.5.4 Interpreting graphs by coordinating x- and y-values

In the framework, it was highlighted that learners coordinate the values of one variable (*x)* with the values of another variable *(y)* in anticipation of creating a discrete collection of ordered pairs (*x; y).* Ayalon et al. (2015, p. 13) refers to the coordinating of variables as the covariation approach that deals with "coordinating two varying quantities while attending to the ways in which they change in relation to each other". Analysis showed that only Themba demonstrated the ability to coordinate variables in the pre-interview. However, Mpho later showed substantial improvement as the result of probing in the pre-interviews. Again, this is an example of probing that resulted in learning gains, where a learner was mediated from his ADL to PDL. The extract below indicates how Mpho began to recognise the coordination of x and y-values in the pre-interviews when he was asked to provide the values of x for which $g(x) > 0$.

Speaker	**Utterance**
Mpho:	*The y-values are greater than zero when the x values are greater than zero but less than 2.*
Researcher:	*Can you choose any value less than 0, let's see what the value of y is? May be choose -1.*
Mpho:	*Oh, it's still positive.*
Researcher:	*That means it is still certifying the requirement (values of x where the graph is greater than zero)*
Mpho:	*Soo…let me choose this point here (looking at x-values far less than -1)*
Researcher:	*Oh, so you now want to choose points further down (than -1)*
Mpho:	*Ja, ja, from negative infinity but less than 2*
Researcher:	*Ok, so negative infinity but less than 2*
Mpho:	*Yes, the graph will be going up (indicating with hands the direction of the graph g with y-values greater than zero)*

Excerpt 5.7: *Mpho's interpretation of Task 2 of the pre-interview*

Excerpt 5.7 shows that Mpho took the leading role when he said "*so…let me choose this point here (looking at the x-values far less than -1)".* This was an indication of a shift in the way he interpreted graphs (Vygotsky, 1978) because he moved from reading restricted intervals to reading open intervals where he said "*from negative infinity but less than 2"*. He further justified his choice of the interval by saying that *"the graph is going up"*. This implies that he considered the relationship between the *x*- and *y*-values because he spoke about the graph going up (meaning to positive infinity on the *y*-axis), as indicated by his iconic gesture to slid on the graph with respect to the *x*-values from 2 going to negative infinity. His iconic gesture suggests that he was no longer reading *x*-values only but had coordinated the *x*-values with the graph in terms of the *y*-values. I therefore argue that through probing Mpho began to approach the graph using the emerging coordination of values. This is because he demonstrated some traces of covariation when he said "*from negative* *infinity but less than 2 the graph is going up"* but did not envision that the individual values go together; instead he saw no multiplicative link between the overall changes in the values of the two quantities.

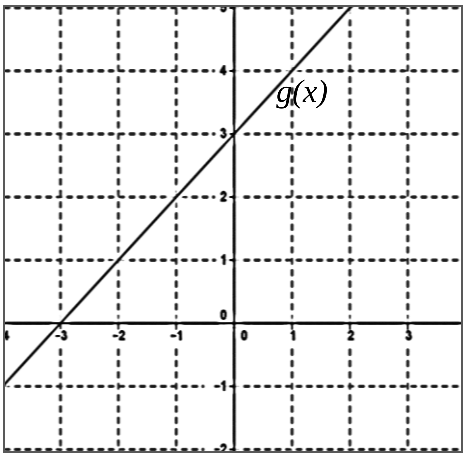
Themba also demonstrated the ability to coordinates *x*- and *y*-values in the pre-interviews. The extract below provides Themba's responses to Task 2 when he was asked to give the values of *x* for which $g(x) > 0$.

Speaker	Utterance
Themba:	*When x is greater than 2.*
Researcher:	*Eh, that was quick, when x is greater than 2. Can you show me, just pick one value where x is greater than 2?*
Themba:	*Oh, no, no, no…when x is less than 2. Yes mam, when x is less than 2 because when x is greater than 2 it means here … (used iconic gesture to indicate that the part of x>2 has negative y-values) therefore when x is less than 2.*

Excerpt 5.8: *Themba justifying the choice of his interval*

Excerpt 5.8 indicates that Themba applied coordination of variables to correct himself immediately without necessarily voicing the *x*-value with its corresponding *y*-value in that interval. When Themba was asked to pick a value within the interval of $x > 2$, it is likely that he realised that the *y*-values were negative and immediately corrected himself. His justification for comparing the two intervals of $x > 2$ and $x < 2$ using an iconic gesture to indicate whether the graph was less than or greater than zero, was an indication of the fact that he had coordinated *x*- and *y*-values. The comparison of the two intervals caused him to decide on the correct interval, that is, $g(x) > 0$ when $x < 2$. According to Carlson et al. (2002) the ability to compare input and output values and decide on the behaviour of the graph is a necessary characteristic for covariation reasoning. Therefore, Themba interpreted *g(x)* using the coordination approach because he coordinated the values of one variable *(x)* with the values of another variable *(y).*

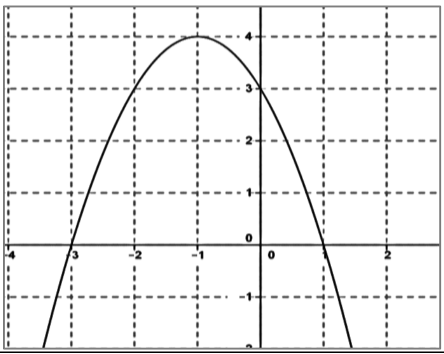
In the post-interviews, Themba, Mpho and Bonga approached the interpretation of graphs by coordinating *x*- and *y*-values. The analysis of Thembs and Mpho's responses revealed that they had moved from justifying their responses using iconic gesture to using graphical features together with iconic gesture. These two learners demonstrated an understanding of applying graphical features to make sense of graphs and provided constructive responses. This is evident in the extract in Excerpt 5.9. In Task 4, learners were asked to provide values of *x* for which $g(x) > 0$. This task was similar to Task 2 of the pre-interviews, except that the line *g(x)* had different intercepts and a positive gradient.

Speaker	***Utterances from separate interviews***	***Graphical representation***
Themba: *Researcher:* *Themba:*	*when x is greater than -3* *-3, ohk. So how did you decide on that one…. that was quick? I saw you pointing and pointing.* *I decided because it's the...... x-intercept*	
Mpho: *Researcher:* *Mpho:* *Researcher:* *Mpho:*	*Ma'am x is the element of -3 to infinity* *Ok how did you decide on your answer?* *The deciding points are actually the x-intercepts* *So you looked at the x-intercept?* *Yes, which was -3… and then from the values going um that are greater than -3 which is the graph will never end because there's no stop (while showing with hands that the upward movement of the graph in the first quadrant) anything indicating a no stop so it is to infinity*	

Excerpt 5.9*: Themba and Mpho justifying their reasoning in Task 4*

Excerpt 5.9 indicates that the way these learners approached this task was totally different from their approach in the pre-interview. Themba and Mpho justified that their use of the *x*-intercept to decide on the interval for which $g(x) > 0$. It is likely that these learners regarded at the behaviour of *g(x)* in the interval greater, as well as less than the *x*-intercept of -3 prior to choosing the correct interval. Using an iconic gesture, Mpho further illustrated that when $x > -3$ the graph continued into infinity in the first quadrant. Although Themba did not mention the coordination of variables, the fact that he used the *x*-intercept prior to deciding on the interval was an indication that he had considered the relationship between *x* and *y* in the part of the graph above and below the *x*-intercept. Therefore, Themba and Mpho coordinated the *x*- and *y*-values to interpret the behaviour of *g(x).* A similar approach was noticed in Tasks 5 and 6 when these learners decided on the intervals of increase and decrease for the parabola graph, as well as the interval where the parabola graph was greater than the straight-line graph ($f(x) > g(x)$).

Although Bonga interpreted graphs by coordinating *x*- and *y*-values in the post-interviews, his interpretation was limited by the given figure, which caused him to provide incorrect intervals of increase and decrease. See the extract below.

Speaker	**Utterance**	**Graphical representation**
Bonga:	*When x is -4; -3 and -2 and -1 sorry mam let me just see. Uh umm*	
Researcher:	*OK which one are you talking about increasing or decreasing, you need to be specific.*	
Bonga:	*Increasing…. when x is greater than -4 and less than -1*	
Researcher:	*And decreasing?*	
Bonga:	*(Paused, while pointing on the graph) when x is greater than -1 and less than 2.*	

Excerpt 5.10: *Bonga's explanatory talk for interval of increase and decrease*

The above excerpt indicates that Bonga provided the intervals of increase and decrease incorrectly because he considered the domain to be $x \in [-4; 2]$ instead of $x \in (-\infty; \infty)$. Leinhardt et al., (1990, p. 37) highlight that the incorrect reading and interpretation of graphs is "associated with students' learning to focus on the parts of the graph" instead of focusing more broadly on the overall shape. The excerpt also shows that Bonga moved from coordinating *x*- and *y*-values by reading individual points to reading restricted intervals. This means he demonstrated some emerging coordination characteristics when reading intervals but did not envisage the graph continuing into infinity. Therefore, his interpretation portrayed the coordination of *x*- and *y*-values.

5.5 Other interesting analyses

Other important responses were observed from all four learners which are not linked to the analytical framework. However, it is very important to discuss them as they highlight the level of difficulties learners encounter when interpreting graphical representations of functions.

5.5.1 Inappropriate use of graphical language

Analysis of the interview data revealed that Sipho, Themba and Bonga encountered difficulties with the use of new graphical terms[1]. The acquisition of new knowledge has a tendency to interfere with the existing knowledge structure (Wells, 1994) and the interference of new terms contributes to the complexity involved in interpreting graphical representations. This interference was mainly observed in learners' explanations in the pre-interviews, while in the post-interviews only Bonga showed traces of using graphical terms inappropriately. Learners misused graphical terms such as *shifted, increasing, decreasing, axis of symmetry and vertical shift;* instead of using *moved, positive or negative infinity, and reference line of reflection, decreased and positive values,* respectively. I will use Themba and Mpho to elaborate on the way the inappropriate use of graphical language interfered when interpreting graphical representations.

The extract below captures Themba's conversation as he was responding to Task 3. He was asked to construct another straight-line graph where the *x*-intercept was the same as graph *f* but the *y*-intercept was less than that of *f*. He was then probed to compare the gradients of *f* and *h.*

Speaker	**Utterance**
Themba:	*I think it (gradient of h) is less (than that of f(x))*
Researcher:	*Why?*
Themba:	*Because the graph has **shifted**, I'm not sure which shift but it is because the graph shifted*
Researcher:	*It shifted?*
Themba:	*Eh, it has the **vertical shift***
Researcher:	*Shifted how?*
Themba:	*The y-intercept has **changed to 3***

***Excerpt 5.11:** Themba's explanation talk in Task 3*

By new graphical terms, I mean the graphical language that learners become acquainted with in Grade 10, such as the axis of symmetry.

Excerpt 5.11 indicates that Themba used graphical language of "vertical shift" to justify why the gradient of *h(x)* was less than that of *f(x).* He said the gradient was less because "the graph shifted" and he later specified that "it has a vertical shift". The graphical language "graph shifted" and "vertical shift" interfered with the "*y*-intercept moved" and "it moved down". In this task, the *x*-intercept of *h(x)* was the same as that of *f(x),* therefore saying the graph shifted does not concur with the fact that the *x*-intercepts (or *x*-value/s) of both graphs were fixed. It is likely that Themba did not realise that only the *y*-values were changing, which resulted in the graph being vertically compressed. Learners have a tendency to apply scientific concepts that are not deeply rooted (Wells, 1994). However, when he was asked "how the graph shifted", he responded correctly by saying that *the y-intercept moved to 3.*

Similar interference was noted in Bonga's language in both the pre and post-interviews, where he referred to the positive y-values by saying that *the y-values are increasing*. In Excerpt 5.12, Bonga was asked in Task 4 to give the values of x for which $g(x) > 0$. See the extract in Excerpt 5.12 below.

Speaker	***Utterances***
Bonga:	*Mm… when x is… (pointing the x-intercept of -3) when x is greater than -3*
Researcher:	*-3, how did you decide on that answer?*
Bonga:	*Mm eh (pointing the x-intercept) because when we start going on that side (pointing from -3 towards positive side of the x-axis but stopped at 0),* ***the y-values are increasing*** *(sliding his finger on the positive y-axis)*

Excerpt 5.12: *Bonga's explanatory talk in Task 4*

It is likely that when Bonga mentioned *y-values are increasing,* he was doing so in line with the explanation that if the gradient of a line is positive, it means the line is increasing. So Bonga also applied a similar insight that positive y-values means that the *y*-values are increasing.

I claim that the inappropriate use of graphical language complicated the interpretation of graphical representations. This was evident from the two examples where graphical language was used partially incorrect. In the case of Themba, the use of the language "the graph vertically shifted" complicated the interpretation of fixing the *x*-intercept while changing the y-values. This caused him to compare the gradients by focusing on the y-intercept instead of comparing the steepness of *h(x)* and *f(x).* Therefore, the inappropriate use of graphical language was one of the difficulties learners encountered when making sense of the behaviour of the graphs.

5.6 Summary of the interview data analysis

In this section, I present a summary of the analysis of learners' responses given in the interviews, which are tabulated in Table 5.4. This table represents the approaches that learners used to interpret the tasks in the pre- and post-interviews. It should be noted that the types of task that learners were engaged with in the pre-interviews caused them to reason analytically, whereas in the post-interviews the tasks did not channel learners to use an analytical approach. Hence, the use of an analytical approach in Table 5.8 is not included in the post-interviews.

Table 5.8: *Summary of the analysis of learners' responses*

Learner	**Pre-interviews**			**Post-interviews**	
	No coordination of x and y	***Coordinating x and y***	***Analytical approach***	***No coordination of x and y***	***Coordinating x and y***
Sipho	✓		✓	✓	
Bonga	✓		✓		✓
Mpho	✓(initially)	✓(later)	✓		✓
Themba		✓			✓

Analysis of both the pre- and post-interviews showed that three learners did not coordinate *x*- and *y*-values when making sense of graphs. However, Bonga demonstrated great improvement in the post-test when his score moved from 5 to 12 out of 16. In the post-interviews, he also improved in the way he approached the tasks by coordinating variables. Mpho also showed great improvement in his test scores and the way in which he approached graphs. Mpho's shift in coordinating *x*- and *y*-values was prompted by further probing.

The summary table indicates that learners who made sense of graphs without coordinating the *x*- and *y*-values were observed using an analytical approach in other tasks. This argument was not foregrounded in the analysis since the themes for approaching graphs were considered separately and learners either applied an analytical approach or did not coordinate values. Learners who used an analytical approach or did not coordinate values were fluent in reading individual coordinates and substituting them into a gradient formula, but they did not make sense of these values. In their study, Adu-Gyamfi and Bossé (2014) found that although learners might be able to work successfully with a concept in graphical settings, this does not necessarily imply that they have sufficient understanding of the same concept to apply it in a different task for the same graph. One example of this statement was when Bonga and Sipho were able to read coordinates when calculating the gradient of the line but were unable to use the relationship between the x and y-values to describe the behaviour of the same graph.

Although three learners were able to coordinate variables in the post-interviews, this does not necessarily mean that they reasoned using covariation. The analytical framework indicated that the data had some traces of covariation. According to Monk (1994), it seems that global reasoning (covariation approach) does not come automatically after a learner has developed pointwise reasoning. This indicates the range of complexity that learners encounter when dealing with the interpretations of graphs.

5.7 Conclusion

This chapter aimed to identify the approaches that Grade 10 learners apply when interpreting graphs. This was done by analysing the interview data using the analytical framework as a lens. The analysis showed that learners had applied different ways of approaching graphs such as analytical reasoning, visual reasoning, coordinating and not coordinating variables on the Cartesian plane. Visual reasoning was a dominant approach across all four learners and it resulted in learners either approaching graphical representation tasks in an appropriate or inappropriate way. Learners who produced appropriate responses while reasoning visually were observed applying graphical features in the correct way, whereas those that provided inappropriate responses applied graphical features only partially correctly.

Coordinating variables is a necessary step for learners to interpret graphs both pointwise and globally. The analysis indicates that some learners interpreted graphs without coordinating x- and y-values. They were observed as either reading the x-axis only (without relating it to the y-axis) or reading the graph as a picture labelled with numbers. Analysis also revealed that learners encountered difficulties with the appropriate use of graphical language as new graphical language was used inappropriately.

Chapter 6 : Conclusion

6.1 Introduction

The purpose of this research study was to gain insight into Grade 10 learners' mathematical thinking when interpreting graphical representation tasks. In order to examine learners' mathematical thinking, I considered the key features and approaches that learners used to attempt graphical tasks. The research questions that guided this study were:

1. *What key features of functions do learners pay attention to when responding to graphical representation tasks?*
2. *What approaches do learners use when working on the graphical representation tasks?*

In order to explore these questions, I conducted task-based interviews with four high performing learners selected from the pre-test. The interviews were conducted in two stages: pre-interviews conducted prior to the intervention lessons and post-interviews done after the intervention lessons. The interview data collected were analysed using an analytical framework consisting of three approaches, namely, not coordinating values, a pointwise approach and coordinating values. The ideas of not coordinating and coordinating values from Thompson and Carlson (2017) were incorporated into the pointwise–global continuum (Even, 1998), as noted in Chapter 4 and indicated in Figure 6.1.

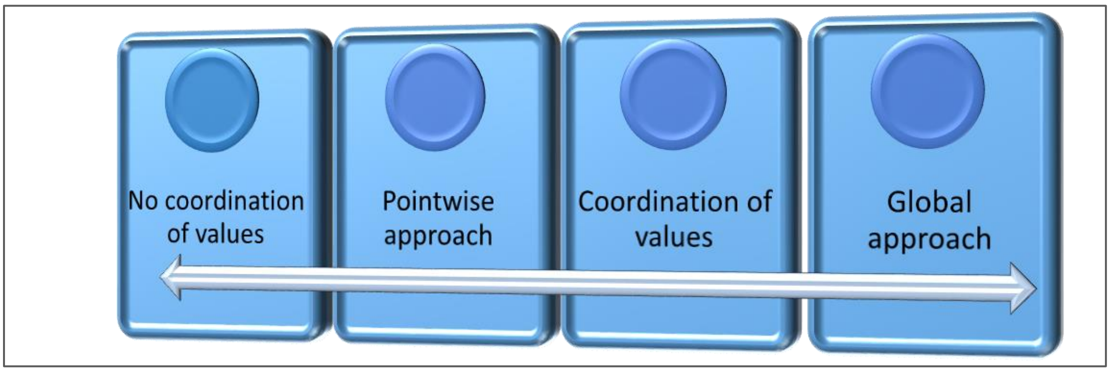


Figure 6.1: *Incorporating Thompson and Carlson's ideas into Even's continuum*

The different approaches were combined into one framework because the pointwise–global continuum lens alone was not able to describe the data. In terms of the pointwise–global continuum lens (Even, 1998), the approach of "not coordinating values" is located on the left-hand side of the pointwise approach, whereas the approach of "coordinating values" is located closer to the global approach. It should be noted that the global approach was not evident in my data.

In this chapter, I provide a summary of the findings which I relate to the research questions. I also present other interesting findings that emerged from the data. Moreover, I discuss the contribution made by the study, as well as the implications of the findings for teaching and learning the interpretation of functions. I then present my reflection on the research, as well as the limitations of the study and recommendations made based on the findings.

6.2 Findings

The Grade 10 curriculum specifies one of the learning outcomes of functions for learners as being able to read points from the graph by coordinating x- and y-values (Department of Basic Education, 2011). However, the analysis of the interview data revealed that although some learners could read coordinates from the graph, they were unable to interpret the graph by connecting the x- and y-axes. This was demonstrated in the discussion in Chapter 5.

This section is aimed at answering the research questions. The first subsection focuses on the findings obtained from the interview data and is aimed at answering research question 1. The second subsection addresses the findings related to research question 2. Lastly, I discuss unexpected findings.

6.2.1 What key features of the function do learners pay attention to when responding to graphical representation tasks?

Analysis of both the pre- and post-interviews indicated that learners paid more attention to the visual features of the graphs such as the intercepts and turning point when responding to graphical representation tasks. Leinhardt et al. (1990) refer to these visual features as the local features of the graph. All learners focused at the changes in the local features and overlooked the differences in the underlying behaviour of the graphs, such as the steepness, as they were comparing the two graphs. This finding concurs with Even's (1998) finding that some learners have a limited viewpoint because they interpret graphs without taking important components of the underlying characteristics of the graph such as steepness into consideration.

In all six tasks in both pre- and post-interviews, the intercepts were the dominant feature that learners used to make sense of graphs. This feature (the intercepts) triggered three learners in the pre-interviews to view the graph as a picture labelled with numbers. One such example was when Sipho attempted to give the values of x for which the graph was greater than zero, where he used both the y- and the x-intercept to express his interval. This perception may have arisen from the learners' experience of working with figures such as triangles, which are labelled with letters. Glazer (2011) highlights the challenges learners experienced in his study when interpreting graphs using visual features, which led to reading the graph as if it were a picture. He further explains that the learners in his study regarded the graph as a picture because they were unable to see it as an abstract representation of variables. In the post-interview in the current study, however, only one learner treated the graph as a picture.

Again, the visual features caused some of the learners in the pre-interviews to use calculations as a means of justifying their responses. Learners have a strong tendency to think algebraically (Knuth, 2000) especially when they talk about the gradient. This might be connected to the fact that gradient is introduced to learners through the use of a formula. Leshota (2015) states that although the textbook affords learners opportunities to compare the slope and the average gradients, learners nevertheless spend most of their time using a formula to calculate the gradient.

Learners also used transformation features such as the line of symmetry, reflections and vertical translation to respond to tasks that required them to construct graphs in the pre-interviews. These features were applied by learners as they attempted to justify how they drew the graph or when comparing the original graph with their constructed graph. Again, as part of their justification to translate or reflect a graph, the intercepts were used as the predominant feature.

The key feature of the intervals of increase or decrease was poorly attended to in the post-interviews. This poor performance was also observed in the pre- and post-test, with only two learners displaying a better understanding of the intervals of increase and decrease in the post-interviews. These learners were able to make sense of the graphs by providing constructive responses, as they specified the intervals where the graph was increasing and decreasing. The other two learners had difficulty perceiving that a quadratic function could be both increasing and decreasing. According to Leinhardt et al. (1990), intervals of increase or decrease form part of the global features and most learners have difficulty in answering questions in relation to these. They further elaborate that this difficulty is associated with viewing a function from the perspective of covariation, where one has to pay attention to the changes in one variable with respect to the other variable simultaneously. This links up with the second research question which is addressed in the next subsection.

6.2.2 What approaches do learners use when working with graphical representation tasks?

The analysis showed that the learners who participated in the interviews viewed a graphical representation either visually or analytically and used a combination of approaches depending on the graphical task they were attempting to interpret. These approaches included *not coordinating values, pointwise* and *coordination of values*. In the pre-interviews, the predominant approaches were 'not coordinating values' and a 'pointwise approach', whereas the approach 'coordination of values' was dominant in the post-interview. Three learners were observed not coordinating *x*- and *y*-values in the pre-interviews, thus, according to the extended pointwise–global continuum, these learners approached graphs from below a pointwise approach. Hence, they were reading values on separate axes without connecting quantities. For example, Sipho provided positive x-values when responding to the task *for which values of x is $g(x)>0$*; consequently saying "when x is greater than zero". His response showed that he did not connect the x- and y-axes because the graph was less than zero when x was greater than 2. This finding in relation to reading values on separate axes agrees with the findings of Bell and Janvier (1981) that learners viewed a graph as a picture of a situation because they had difficulty in coordinating information represented by two variables. Again, Lowrie and Diezmann (2007) discovered that learners examined the information on the graph in isolation by considering either the x- or the y-axis without looking at the relationship between the two axes. This suggests that even if learners can draw graphs using a pointwise approach, it is no guarantee that they will interpret the graph using the same approach. Nevertheless, a shift was experienced in the way learners approached graphs in the post-interviews because only one learner still interpreted graph without coordinating the x- and y-values. This shift may be associated either with the fact that learners participated in the intervention lessons or they were more alert to the questions after the pre-interviews as the result of probing. Ayalon et al. (2017), Even (1998) and Monk (1994) reported in their respective studies that learners interpreted graphs from a pointwise approach and beyond.

In the pre-interviews, three learners interpreted the behaviour of the graph using a pointwise approach. These learners either provided individual points or used calculations to justify their responses. It was observed that learners used calculations mainly in the pre-interviews where the graphical representation tasks required them to interpret graphs from a gradient point of view. Learners demonstrated a partial understanding of gradient as the measure of steepness because they interpreted it from a process point of view by constantly using calculations. As mentioned earlier, this may be attributed to the fact that gradient is introduced to learners through the use of the gradient formula. This finding agrees with Birgin's (2012) conclusion that learners' understanding of slope is dominated by the algebraic representations that trigger a process view of linear functions. He further highlights that the learners in his study had a limited understanding of slope representation. This agrees with the statement made by Leshota (2015) that in one of the teacher's classes in her study, learners remained at the pointwise level throughout the lesson because they spent most of their time using a formula to calculate gradients. Findings from other studies also support the finding that most learners interpret graphical representations from a pointwise view (Ayalon et al., 2017; Even, 1998; Monk, 1994).

Analysis revealed that two of the learners coordinated the values in the pre-interviews, with the other learner coordinating values at a later stage as the result of probing. This number increased to three learners in the post-interviews. These learners created a discrete collection of ordered pairs ($x; y$) to describe the behaviour of the graphs. Of the three learners, only one approached the graphical representation tasks by shifting to rough continuous covariation. This level required learners to envision changes in the value of one variable as happening simultaneously with changes in the value of another variable. This finding concurs with other research which has found that many learners have difficulty in interpreting graphical representations using a covariation approach (Even, 1998; Leinhardt et al., 1990; Oehrtman et al., 2008).

6.2.3 Other findings that emerged from the data

There were other interesting findings that arose from analysing learners' responses as to why they approached graphs in certain ways. These findings are briefly discussed below.

a) Inappropriate use of graphical language

Analysis revealed that there were instances where learners attempted to use the graphical language they had learnt in Grade 10 which interfered with the old knowledge. This resulted in the graphical representations of the functions being erroneously interpreted. One learner used the axis of symmetry to refer to the line of reflection as well as mentioning a vertical shift instead of vertical compression. This concurs with the findings of Mashazi (2014), which indicate that some errors in algebra are caused by the interference of new knowledge with the old knowledge. Similarly, in the current study the inappropriate use of graphical language interfered with the interpretation of graphical representation tasks.

b) Inappropriate definition of graphical features

Analysis revealed that learners used or defined graphical features inappropriately, which caused them to interpret the graphical representations without coordinating values or using a pointwise approach. The intervals of increase or decrease was one of the features where learners had difficulty using or defining. According to Bell and Janvier (1981) and Leinhardt et al. (1990), learners tend to focus a great deal on the individual points rather than global features. One example was when Sipho defined an increasing function by focusing on the *x*-values only by saying that increasing means *going positively in the positive side of the x-axis.* In other words, he did not coordinate the *x*- and *y*-values when attempting to define an increasing function.

Again, there were instances where two learners misinterpreted an increasing function as referring to "greater than zero". This resulted in them providing the interval where the graph was positive instead of giving the interval/s of increase.

6.3 Contribution to knowledge

This study has contributed to knowledge on mathematics education with regard to interpreting graphical representation tasks in three ways. Firstly, it revealed the level of complexity that Grade 10 learners face when working with graphical representations of functions. Previous research has shown that most learners interpret graphs by using a pointwise approach as their first approach; however, this study has shown that some learners interpret graphs below the pointwise approach because they focus on just one variable as they attempt to make sense of graph behaviour. This is new information and assists in clarifying why most learners have difficulty in relating two variables. Such difficulty was also evident when I struggled to locate my data within the pointwise–global continuum lens, which ultimately gave rise to the analytical framework briefly discussed in the second contribution below.

Secondly, the study makes a theoretical contribution by using a combination of approaches as a framework for the data that reveal that learners may reason visually or analytically or use a combination of the two reasoning methods to approach graphical representation tasks. In particular, learners reasoned visually when they operated in terms of a 'no coordination of values' approach, where a graph was interpreted as a picture labelled with numbers. However, as they moved toward a pointwise approach, their analytical reasoning became stronger and they used either one reasoning method or a combination of the two reasoning methods (visual and analytical) on a coordination of values approach. The way in which the two were linked together was not, however, perfectly formulated.

Thirdly, the study provided the information that Grade 10 learners interpret the graphs included in the South African curriculum by using various approaches. These include coordinating values, using a pointwise method and coordinating values to show traces of covariation. It also provided information on the way learners shifted in their approach from the pre- to the post-interviews. Again, it revealed that the visual features of the graph were dominant when the learners were making sense of graph behaviour.

6.4 Reflection

At the beginning of this research, I made an assumption that learners would interpret graphs using approaches ranging from a pointwise to a global approach. I initially used Even's (1998) concept of graph interpretation to set up a framework, which consisted of four levels, namely, pointwise, entry global, emerging global and global approaches. It would seem that I was not fully aware of what Grade 10 learners were able or unable to do. To my surprise, the data could not be fitted into the framework because it was difficult to make a clear distinction between these levels. Again, the framework was unable to clarify what each learner was able to do. I then used Thompson and Carlson's (2017) framework of covariation, which I adjusted to four levels, namely, not coordinating values, emerging coordination of values, coordination of values and roughly continuous covariation. This framework is more concerned with what learners are doing and what graphical features they pay attention to when interpreting graphs than is Even's framework, which is concerned with classifying how learners read the point/s from the graph. Through this exercise, I learnt that doing research is not about trying to push data into a framework but that data and theory should inform one another. However, Thompson and Carlson's (2017) framework was not adequate for explaining how learners read the points from the graph. Consequently, I incorporated both frameworks which resulted in a combination of approaches, namely, not coordinating values, pointwise and coordination of values.

It was in this way that I was able to discover that some of the learners who participated in the interviews operated below a pointwise approach and interpreted the graph as a picture labelled with numbers. This came as a great surprise because most of the learners who participated in the interview were able to draw graphs using a pointwise approach. I learnt that the fact that learners could draw graphs does not guarantee that they can coordinate *x*- and *y*-values to interpret the same graph. As a teacher, this finding played a major role in shaping my lessons and taught me not to make the assumption that learners operate from a pointwise approach. Instead, tasks should be provided that create awareness in learners of the necessity to coordinate *x*- and *y*-values.

I also reflect here on the fact that while conducting this research I wore two hats – a researcher hat and a teacher hat. As a researcher I concentrated on ethical issues and the gathering of data, whereas as a teacher I was concerned in the interviews about learners gaining graphical knowledge. The teacher hat resulted in persistent probing, an example of which is provided in Excerpt 6.1. In this extract, Mpho was asked to provide the values of *x* for which $f(x) > 0$.

Speaker	**Utterance**
Mpho:	*When x is greater than zero.*
Researcher:	*Can you choose one point when x is greater than zero?*
Mpho:	*It's two and also one.*
Researcher:	*Ok, let's go to one, the graph is …?*
Mpho:	*It's (1; 2).*
Researcher:	*Is the y-value positive or negative?*
Mpho:	*It's positive.*
Researcher:	*When x. is eh ... 3, what is the y-value?*
Mpho:	*(laughing and placing hands on his mouth) It will be -2.*
Researcher:	*This means you need to refine the answer (referring to his response).*
Mpho:	*The y-values are greater than zero when the x values are greater than zero but less than 2.*
Researcher:	*Oh, greater than zero but less than 2.*
Mpho:	*Yes.*
Researcher:	*Can you choose any value less than 0, let's see what the value of y. is May be let's go to 1.*
Mpho:	*It will greater than zero (reading x=1 instead of x=-1).*
Researcher:	*Mhmm …*
Mpho:	*Yes, oh … negative 1(referring to the x value).*
Researcher:	*Yes.*
Mpho:	*(Learner put the ruler on x=-1 and checked the y-value) It will also be negative.*
Researcher:	*Mmm … let's see, is the y value positive or negative?*
Mpho:	*Oh, it's still positive.*
Researcher:	*That means it is still satisfying (referring to the question).*
Mpho:	*Soo … let me choose this point here (looking at x-values on the left of -1).*
Researcher:	*Oh, so you now want to choose points further down (than -1)?*
Mpho:	*Ja, ja, from negative infinity but less than 2.*

Excerpt 6.1: *Example of transcript showing probing with teaching instances*

Excerpt 6.1 shows that most of the probing was teacher orientated because I was persistent in asking Mpho to read individual points in his choice of interval. It is clear that Mpho initially gave the wrong interval because he did not consider the relationship between *x*- and *y*-values. This was evident when he laughed prior to giving the corresponding *y*-value of -2 when $x = 3$. This probing channeled him to begin to consider the relationship between *x*- and *y*-values. It was not my intention to teach or guide learners but it happened that I fell into a trap of wanting to gain deeper understanding of whether learners were able to read corresponding values. This indicates that the teacher role elevated Mpho's mathematical thinking about how to make sense of graphs as the result of probing. The tension between the researcher and the teacher was dominant in the pre-interviews. However, later in the post-interviews I found it easier to assume the role of the researcher because

I had addressed most of the issues that had arisen in the pre-interviews in the intervention lessons.

6.5 Implications for teaching and learning

In Chapter 1, I mentioned that I had attended a cluster meeting in which the 2016 matric mathematics results were discussed. It transpired from the meeting that the poor performance of learners in functions was linked to the interpretation of graphical representations; however, the actual challenges could not be specified. Therefore, one of the outcomes of this study was to highlight the difficulties that learners encounter when interpreting graphs and, accordingly, offer strategies for improving the teaching and learning of functions in the FET phase and in education in general.

This study has revealed some of the difficulties that learners encounter when dealing with graphs. Such difficulties include mistakes such as interpreting graph as a picture labelled with numbers, not coordinating variables and the inappropriate use of graphical language. Some of these difficulties are not covered in the curriculum and, therefore, I suggest that educators develop items that would address these difficulties in their lessons so as to assist learners to interpret graphs using covariation categories.

When I developed the intervention lessons, I made the assumption that learners needed more clarity on the Grade 10 content on the graphical representation of functions. In most cases, educators are under the impression that learners have not understood the concept in that particular grade, while in reality they are operating far below the level of that grade. The findings of this study reveal that the interventions should rather have covered the content of the graphical work taught in the lower grades. This content should include reading a collection of discrete points and specifying whether the graph is positive or negative in certain intervals of these discrete points. I suggest that educators offer intervention lessons to learners who operate below a pointwise approach, using tasks that emphasise features such as the increasing/decreasing function to deal with the difficulties that learners encounter when interpreting graphical representations.

6.6 Limitations of the study

In this section, I elaborate on the limitations of this study. Firstly, one of the limitations lies in investigating participants whom I taught, since, as both researcher and teacher, I had to wear two hats. Despite the fact that having good relationships with the participants produced rich data, there were instances where I could not conclude whether a learner operated in terms of a certain approach or as a result of my probing.

Secondly, because I investigated learners from one school, the findings of this study cannot be generalised to the interpretation of graphs by all Grade 10 learners in South Africa. The performance of learners in this school is average compared to other schools in South Africa, because at the time of the study the matric pass rate in mathematics ranged from 55 to 75%. However, the findings in this study should not be ignored; instead they should be considered as a guide for the way in which Grade 10 learners interpret graphical representation tasks. One important discovery made by the study was that some Grade 10 learners operate below a pointwise approach by interpreting graphs as pictures labelled with numbers.

Thirdly, although the pre-test guided me on the content I should focus on, the use of task interviews that excluded certain content on graph interpretation such as the range and domain was a limitation. I would therefore suggest that the excluded content form part of future research.

6.7 Future research

There were a number of issues that the research was not able to resolve with regard to the framework to locate where visual versus analytical reasoning fits in the categories of covariation. The study established that visual reasoning was predominant in the 'not coordinating values approach' and the analytical approach was stronger for the pointwise approach. However, it was not clear where to draw the line between visual and analytical reasoning in the ‘coordination of values approach’. I recommend that future research should take the tools that I introduced and work with them to distinguish between visual and analytical reasoning within the approaches applied in the framework for covariation.

The study found that the Grade 10 learners in this study could not easily coordinate variables; hence, this finding opens up opportunities for future research to focus on a much bigger sample of learners who are operating below a pointwise approach by investigating whether they also interpret graphs as a picture labelled with numbers. Subsequently, interventions should be developed that would shift learners’ understanding so that they interpret graphs using covariational approaches.

6.8 Conclusion

This study gave a glimpse into the reasons why learners perform poorly when dealing with graphs, one of which is that they were observed reading a graph as a picture labelled with numbers. It also revealed that although learners could use a pointwise approach to produce graphs, this did not necessarily imply that they could use a pointwise approach to make sense of the same graph. This was evident when learners drew graphs but did not interpret the same graphs by coordinating x- and y-values. The findings of this study have influenced my teaching so that it now includes 1) lessons that address the importance of reading a collection of discrete points and making connections to the graph in terms of whether it is positive or negative in the particular interval; 2) specifying the graphical language of the increasing/decreasing function in terms of the changes in the x-values with respect to the changes to the y-values; and 3) addressing the importance of unpacking the task requirements such as *f(x) greater than zero* means that *positive y-values*. I conclude that this finding that learners interpret graphs as a picture should be investigated further with a bigger sample in the South African context.

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APPENDIX A: Ethics Clearance letter

WITS
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29 June 2017

Student Number: 0211524F

Protocol Number: 2017ECE022M

Dear Sbongile Mshazi

Application for Ethics Clearance: Master of Science

Thank you very much for your ethics application. The Ethics Committee in Education of the Faculty of Humanities, acting on behalf of the Senate has considered your application for ethics clearance for your proposal entitled:

Investigating Grade 10 learners' approaches and their underlying thinking when interpreting graphs

The committee recently met and I am pleased to inform you that **clearance was granted**. However, there were a few small issues which the committee would appreciate you attending to before embarking on your research.

The following comments were made:

- Section 6.2 should consider anonymising the school and also the possibility of other information that could identify learners.
- On the learner consent there are two categories for the test.
- The GDE form indicates that grade 1 learners (and not grade 10 learners) will be the participants.

Please use the above protocol number in all correspondence to the relevant research parties (schools, parents, learners etc.) and include it in your research report or project on the title page.

The Protocol Number above should be submitted to the Graduate Studies in Education Committee upon submission of your final research report.

All the best with your research project.

Yours sincerely,

MMabeta

Wits School of Education

011 717-3416

Cc Supervisor: Dr Craig Pournara and Dr Benita Nel

APPENDIX B: Letter and Consent forms

LETTER TO THE PRINCIPAL REQUESTING TO CONDUCT RESEARCH PROJECT
Protocol Number: 2017ECE022M

27 July 2017

Dear Mr ……………

My name is Mrs Sbongile Sibanda, one of your staff members. I am a Master's student in the School of Education at the University of the Witwatersrand.

I am doing research in Mathematics on the interpretation of graphs at Grade 10 level as part of my Master's degree in Maths Education.

My research involves collecting data by administering tests, conducting interviews and providing three intervention lessons of approximately 60 minutes each. The interviews will be video recorded to capture the graphics involved when dealing with functions. The research study will be done after school on Mondays and Wednesdays and one group of Grade 10 learners (taught by me) will participate.

The reason why I have chosen your school is because it has been involved in the bigger project conducted by Wits Maths Connect and I have been involved in this project for the past 6 years. I am inviting your school to participate in this research to improve learners' understanding in the field of my research study and hence, improve teaching and learning of mathematics.

The research participants will not be advantaged or disadvantaged in any way. They will be reassured that they can withdraw their permission at any time during this project without any penalty. There are no foreseeable risks in participating in this study. The participants will not be paid for this study.

The names of the research participants and identity of the school will be kept confidential at all times and in all academic writing about the study. Your individual privacy will be maintained in all published and written data resulting from the study. All research data will be destroyed between 3-5 years after completion of the project.

Please let me know if you require any further information. I look forward to your response as soon as is convenient.

Yours sincerely,
Mrs Sbongile Sibanda

____

7 Fay Street
Ridgeway
2091
smashazi@yahoo.com
072 816 9406

APPENDIX C: Information letter for learners

LEARNERS INFORMATION LETTER FOR RESEARCH PROJECT
Protocol Number: 2017ECE022M

1 August 2017

Dear Learner

My name is Mrs S Sibanda and I am a Master's student in the School of Education at the University of the Witwatersrand. I am doing research on the interpretation of graphs at Grade 10 level as part of my Master's degree in Maths Education.

My investigation involves collecting data by providing three intervention lessons of about 60 minutes each, which will help Grade 10 learners to understand functions. It will also include administering tests and conducting interviews. The interviews will be video recorded to capture the graphics involved when dealing with functions. It will be done after school on Mondays and Wednesdays. The intervention lessons will form part of the extra lessons.

Would you mind if I invite you to take part in the investigation. I need your help with the information which will be collected from you by means of the tests and interviews. The interviews and the lessons will be videotaped because of the graphical explanations.

Remember, this is not a test, it is not for marks and it is voluntary, which means that it is not compulsory for you to do it. Also, if you decide halfway through that you prefer to stop, this is completely your choice and will not affect you negatively in any way.

I will not be using your own name and the name of your school but I will make one up so no one can identify you. All information about you will be kept confidential in all my writing about the study. Also, all collected information will be stored safely and destroyed between 3-5 years after I have completed my project.

Your parents have also been given an information sheet and consent form, but at the end of the day it is your decision to join us in the study.
I look forward to working with you! Please feel free to contact me if you have any questions.
Thank you very much for understanding.

Yours sincerely,
Mrs S Sibanda

7 Fay Street
Ridgeway
2091
smashazi@yahoo.com
072 816 9406

APPENDIX D: Learner consent form

Learner Consent Form

Please fill in the reply slip below if you agree to participate in my study called:
Investigating Grade 10 learners' approaches and underlying thinking when interpreting graphs

My name is: ________________________

Permission to collect test **Circle one**
I agree that test scripts can be used for this study only.
YES/NO

Permission to be interviewed
I would like to be interviewed for this study.
YES/NO
I know that I can stop the interview at any time and don't have to answer all the questions asked.
YES/NO

Permission for test
I agree to write a test for this study.
YES/NO

Permission to be videotaped
I agree to be videotaped in interviews.
YES/NO
I know that the videotapes will be used for this project only.
YES/NO

Informed Consent
I understand that:

- My name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- I do not have to answer every question and can withdraw from the study at any time.
- I can ask not to be audiotaped, photographed and/or videotape
- All the data collected during this study will be destroyed within 3-5 years after completion of my project.

Sign______________________________ Date____________________________

APPENDIX E: Information letter for parents

PARENT INFORMATION LETTER FOR RESEARCH PROJECT
Protocol Number: 2017ECE022M

1 August 2017

Dear Parent

My name is Mrs S Sibanda and I am a Master's student in the School of Education at the University of the Witwatersrand. I am doing research on the interpretation of graphs at Grade 10 level, as part of my Master's degree in Maths Education.

My research involves collecting data by administering tests, conducting interviews and providing three intervention lessons of approximately 60 minutes each to Grade 10 learners. The interviews will be video recorded to capture the graphics involved when dealing with functions. The intervention lessons will be treated as part of the extra classes. The collection of data will take place after school on Mondays and Wednesdays.

The reason why I have chosen your child's class at Jules High School is because I am teaching them mathematics. Would you mind if I invite your child to participate in the research study. Your child's involvement will include taking part in the tests, intervention lesson (extra class) and interviews. The interviews and lessons will be videotaped to capture the graphical explanations.

Your child will not be advantaged or disadvantaged in any way. S/he will be reassured that s/he can withdraw her/his permission at any time during this project without any penalty. There are no foreseeable risks in participating and your child will not be paid for this study.

Your child's name and the name of my school will be kept confidential at all times and in all academic writing about the study. His/her individual privacy will be maintained in all published and written data resulting from the study.

All research data will be destroyed between 3-5 years after completion of the project.

Please let me know if you require any further information.

Thank you very much for your help.

Yours sincerely,
Mrs S Sibanda

____

7 Fay Street
Ridgeway
2091
smashazi@yahoo.com
072 816 9406

APPENDIX F: Parent consent form

Parent's Consent Form

Please fill in and return the reply slip below indicating your willingness to allow your child to participate in the research project called:
Investigating Grade 10 learners' approaches and underlying thinking when interpreting graphs

I, ____________________________ the parent of __________________________ give

Permission to collect test **Circle one**

I agree that my child's test script can be used for this study only.
YES/NO

Permission to be interviewed

I agree that my child may be interviewed for this study.
YES/NO
I know that he/she can stop the interview at any time and doesn't have to answer all the questions asked.
YES/NO

Permission for test

I agree that my child may write a test for this study.
YES/NO

Permission to be videotaped

I agree my child may be videotaped in interviews.
YES/NO
I know that the videotapes will be used for this project only.
YES/NO

Informed Consent

I understand that:

- My child's name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- He/she does not have to answer every question and can withdraw from the study at any time.
- He/she can ask not to be audiotaped, photographed and/or videotape
- All the data collected during this study will be destroyed within 3-5 years after completion of my project.

Sign_____________________________ Date___________________________