AN INVERSE PROBLEM FOR AN INHOMOGENEOUS STRING WITH AN INTERVAL OF ZERO DENSITY AND A CONCENTRATED MASS AT THE END POINT

By

DANIEL SIPHO MDHLULI

SUBMITTED IN FULFILLMENT OF THE REQUIREMENT
FOR THE DEGREE
OF
DOCTOR OF PHILOSOPHY

AT

THE UNVERSITY OF THE
WITWATERSRAND
PRIVATE BAG
WITS
2050

THE UNIVERSITY OF THE WITWATERSRAND
SCHOOL OF MATHEMATICS

DECLARATION

I Daniel Sipho Mdhluli with Student number 9713739J and ID number 7811045341081 hereby declare that this research is my own unaided work. It is being submitted for the degree of Doctor of Philosophy at the University Witwatersrand, Johannesburg. I further state that this thesis is the final submission to the university and is a representation of my original work that has not been submitted at any other university.

Signature

Date
27January 2016

AN INVERSE PROBLEM FOR AN INHOMOGENEOUS STRING WITH AN INTERVAL OF ZERO DENSITY AND A CONCENTRATED MASS AT THE END POINT

DANIEL SIPHO MDHLULI

ABSTRACT. The direct and inverse spectral problems for an inhomogeneous string with an interval of zero density and a concentrated mass at the endpoint moving with damping are investigated. The partial differential equation is mapped into an ordinary differential equation using separation of variables which in turn is transformed into a Sturm-Liouville differential equation with boundary conditions depending on the separation variable λ. The Marchenko approach is employed in the inverse problem to recover the potential, density and other parameters from the knowledge of the two spectra and length of the string.

CONTENTS

1. INTRODUCTION

A typical problem that involves investigating certain outcomes in the light of certain forces which may be expressed as a set of differential equations subject to initial conditions is known as the Direct Problem. This problem has been extensively solved not only by scholars but also by normal civilians in their pursuit to understanding the problem "What would happen if I do this, or change it like this?" The opposite of the Direct Problem involves beginning with a set of observable (i.e. frequencies of vibrations/eigenvalues for scientist) and one is concerned with understanding the underlying forces that led to those outcomes, and is known as the Inverse Problem.

Isaac Newton in his day solved an Inverse Problem when he developed his theory of gravitational force between two separate masses. The history of the Direct and Inverse problem can be found in a book by Chadan and Sabatier in [2]. The success of this branch of mathematics has led to many solutions to other inverse problems some of which have found application in the field of medicine, for example, in the detection of leukemia amongst other applications.

The generalised approach to inverse problems was given by Mark Kac (1966) in a lecture entitled *Can one hear the shape of a drum?* in [5]. We all know that one does not hear the shape but rather hears the sound of the drum which vibrates at certain frequencies which are called normal modes. Mark Kac was able to show in his manuscript that by carefully identifying frequencies of vibration of the drum that one can be able to deduce geometric information about the drum.

Krein, M.G and Nudelman, A.A gave impetus to research activities in the problem of small transversal vibrations of inhomogeneous strings with damping. My interest into this research was triggered by a manuscript written by Reinhard Mennicken and Vyacheslav Pivovarchick entitled *An inverse problem for an inhomogeneous string with an interval of zero density* which was published in the journal *Mathematische Nachrichten*, see [12], which also investigated transversal vibrations of strings belonging to a certain class.

The problem under consideration in this thesis involves a string with density

$$B(s) = \begin{cases} b(s) & \text{if } s \in [0, l_1] \\ 0 & \text{if } s \in (l_1, l], \end{cases}$$

where $0 < l_1 < l < \infty$, $b(s) \in W_2^2(0, l_1)$ and give the description of the spectrum of the problem:

$$\frac{\partial^2 w}{\partial s^2} - B(s)\frac{\partial^2 w}{\partial t^2} = 0, \tag{1.1}$$

$$w(0,t) = 0, \tag{1.2}$$

$$\left(\frac{\partial w}{\partial s} + \nu\frac{\partial w}{\partial t} + M\frac{\partial^2 w}{\partial t^2}\right)\bigg|_{s=l} = 0 \tag{1.3}$$

where $b(s) \geq \varepsilon > 0$, $b(s) \in W_2^2(0, l_1)$, $\nu > 0$ is the coefficient of damping and M the mass of the concentrated ring at the endpoint.

We begin our investigation of the Direct and Inverse problem associated with the above partial differential equation by going through some of the preliminary material (Riemann's Formular and Transformation Operators) that form the basis of the route undertaken here and can be found in the book by Marchenko.V.A. in [11] with varying degree of details. The preliminary material is included in this thesis so as make the entire thesis self contained. We know that the Sturm-Liouville differential equation can be rewritten as some integral equation, however the problem with some of them is that the solution function appears on both sides of the equation. Transformation operators are therefore employed in this thesis since they allow for an explicit representation of the solution function in terms of another solution function to the same differential equation but satisfying different boundary conditions. The explicit solution function obtained using transformation operators has a form of a Fredholm integral equation whose kernel does not depend on the eigenvalue parameter. As we will see later on, this route will become important when we pursue the inverse problem since the kernel of that integral equation bear some relation with the potential of the Sturm Liouvlle differential equation.

The Direct problem in this thesis is investigated for strings whose density belongs to three different classes namely, $b(s) \in W_2^2(0, l_1)$, $b(s) \in W_2^4(0, l_1)$ and lastly $b(s) \in W_2^6(0, l_1)$. The above space-time partial differential equation shown by equation (1.1) is reduced to a second order ordinary differential equation that depends on the space and an eigenvalue parameter λ (energy parameter). As will be seen, the differential equation that results does not conform to the Sturm-Liouville form for which we know so much about, however it is transformed into one using the Sturm-Liouville transformation.

A further application of the Sturm-Liouville transformation on all boundary conditions leads to an entire function whose zeroes corresponds to the eigenvalues of the Sturm-Liouville operator. The asymptotic expansions of the eigenvalues are then obtained for string with density $b(s)$ belonging to $W_2^2(0,l_1)$, $W_2^4(0,l_1)$, and $W_2^6(0,l_1)$. In comparison to the Pivovarchik and Mennicken case in [12], it is found in this instance of the problem that the general form of the asymptotic expansion remains unaltered when a concentrated mass is introduced at the endpoint, but the term Q_2 is equal zero in the eigenvalues expansion for all strings classes of strings considered here, and this contrasts sharply with known results from Pivovarchik and Mennicken which showed that $Q_2 > 0$. It is only when we consider the case $b(s) \in W_2^6(0,l_1)$ that it becomes clear from the asymptotic expansion that the majority of the eigenvalues are located in the upper half, infact in this thesis we prove that all the eigenvalues are located in the upper half plane.

The second half of this thesis focuses on the Inverse Problem for strings belonging to the class $W_2^6(0,l_1)$, and the approach we follow is the Marchenko method which is outlined in his book [11]. We begin this thesis by considering some preliminary material shown in Section 2, 3 and 4 that we will need to pursue both the Direct and Inverse Problem. In Section 2 (Rieman's Formular) we consider a twice continuously differentiable solution of the Cauchy problem

$$u_{tt}(t,x) - q_1(x)u(t,x) = u_{xx}(t,x) - q_2(x)u(t,x),$$

satisfying the initial data conditions

$$u(t,0) = \varphi(t);\ u_x(t,0) = \psi(t),$$

and we seek to obtain an explicit expression for the u at the point (t_0,x_0) using Riemann's functions. In section 3 (Transformation Operators) we will obtain an explicit solution $e_0(\lambda,x)$ to the Sturm-Liouville differential equation

$$y'' - q(x)y + \lambda^2 y = 0$$

where $q(x)$ satisfies some condition and the function $e_0(\lambda,x)$ also satisfies

$$e_0(\lambda,0) = 1,\ e_0'(\lambda,0) = i\lambda.$$

Then using the results from Section 2 we show how the Riemann's function are used to construct Kernels which are in an integral equation used to obtain an explicit expression for the solution to the Sturm-Liouville differential equation. In Section 4 (Zeroes of Entire functions) we consider three distinct different classes of entire functions E_1, E_2 and E_3 and show that the zeroes of these functions are properly enumerated and then obtain the asymptotic expansions of their zeroes. Section 5 of the thesis proves the main results of the The Direct Problem by giving the description of the spectrum to the problem (1.1) - (1.3) for classes of strings whose density $b(s)$ belongs to $W_2^2(0,l_1)$, $W_2^4(0,l_1)$ and lastly to $W_2^6(0,l_1)$. In Section 6 of this thesis we follow the Marchenko method to prove that we can find the S-matrix which satisfies special conditions such that the Marchenko integral equation

$$G(x,t) + F(x+t) + \int_x^\infty G(x,s)F(s+t)ds = 0.$$

has a unique solution $G(x,t)$, and we further obtain that

$$q(x) = -2\frac{dG(x,x)}{dx}$$

In essence we prove that given the spectrum of problem (5.17) - (5.19) that we can find the set $\{a, q(x), M_1, M_2, M_3, M_4\}$ where $a > 0, M_1 > 0, M_2 > 0, M_3 - M_4 > 0$ and $q(x) \in L_2(0,a)$.

Lastly, in section 7 (The Inversion) we begin with the obtained set $\{a, q(x), M_1, M_2, M_3, M_4\}$ and our task there is to find $b(s), \nu, l_1, M$ such that $\nu > 0$, $l_1 > 0$, $M > 0$, $b \in W_2^2(0, l_1)$ and $b(s) > 0$ for $s \in [0, l_1]$ and generates the Sturm-Liouville problem having the spectrum $\{\lambda_k : k \in \mathbb{Z} \cup \{0^+\}\}$ of the Direct problem.

2. Riemann's Formula

Let $u(t,x)$ $(-\infty < t < \infty,\ -\infty < x < \infty)$ be a twice continuously differentiable solution of the Cauchy problem:

$$u_{tt}(t,x) - q_1(x)u(t,x) = u_{xx}(t,x) - q_2(x)u(t,x), \tag{2.1}$$

$$u(t,0) = \varphi(t);\ u_x(t,0) = \psi(t), \tag{2.2}$$

where φ, ψ, q_1, and q_2 are continuous complex-valued functions (see pg. 1 of [11]). Let the value of the function u at the point (t_0, x_0) be thought of as a value of a linear functional $T_{t_0}^{x_0}$ on the vector function (φ, ψ)

$$u(t_0, x_0) = T_{t_0}^{x_0}[\varphi, \psi]. \tag{2.3}$$

An expression for this functional was first found by B. Riemann using the following argument. For fixed t_0 and $x_0 \geq 0$ let $R(t,x;t_0,x_0) = R(t,x)$ denote the twice differentiable solution of the equation

$$R_{tt} - q_1(x)R = R_{xx} - q_2(x)R, \tag{2.4}$$

in the domain shown in Figure 1

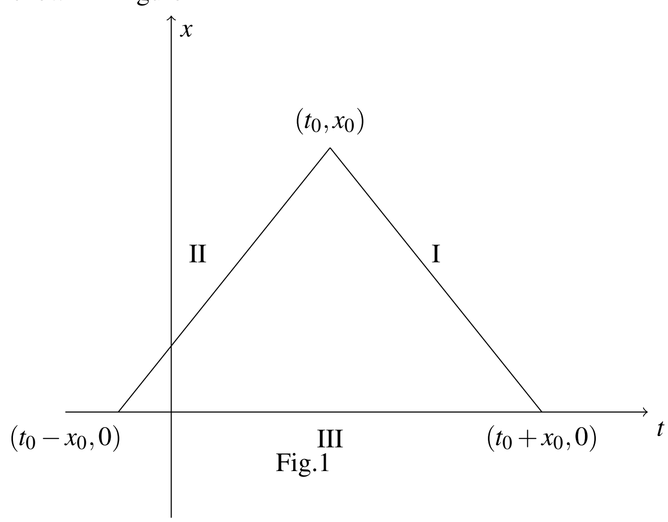

Fig.1

which takes the value 1 on the characteristics $x_0 - x = |t - t_0|$ of the equation. The existence of this solution is proved on page 3 of [11]. Now multiplying equation (2.1) and (2.4) by R and u respectively and then subtracting the second equation from the first equation we obtain

$$u_{tt}R - uR_{tt} = u_{xx}R - uR_{xx},$$

in D, where D is the region enclosed by lines I, II and III. The above expression can equivalently be written as

$$\frac{\partial(u_tR - uR_t)}{\partial t} - \frac{\partial(u_xR - uR_x)}{\partial x} = 0.$$

Therefore integrating over the region D on both sides we get

$$\int\int_D \Big[\frac{\partial(u_tR-uR_t)}{\partial t}-\frac{\partial(u_xR-uR_x}{\partial x}\Big]dtdx=0,$$

and recalling that Green's formula states that

$$\int_\Gamma Ldx+Mdy=\int\int_D\Big(\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y}\Big)dxdy$$

whenever L and M have continuous partial derivatives. Therefore an application of Green's formula gives

$$\int_\Gamma(u_tR-uR_t)dx+(u_xR-uR_x)dt=0, \tag{2.5}$$

where Γ denotes the anticlockwise oriented boundary of D which consists of three line segments we call Γ_1, Γ_2 and Γ_3. So we may also write

$$\int_\Gamma(u_tR-uR_t)dx+(u_xR-uR_x)dt=I_1+I_2+I_3, \tag{2.6}$$

where

$$I_j=\int_{\Gamma_j}(u_tR-uR_t)dx+(u_xR-uR_x)dt,\ j=1,2,3.$$

On the line segment Γ_1 we have that

$$x=x_0-(t-t_0)\text{ and }dx=-dt,$$

so then evaluating the integral on Γ_1 gives that

$$\begin{aligned}I_1&=\int_{t_0+x_0}^{t_0}\Big(-u_tR+uR_t+u_xR-uR_x\Big)(t,x_0-(t-t_0))dt\\&=-\int_{t_0+x_0}^{t_0}\Big((u_t-u_x)R+(R_t-R_x)u-2u(R_t-R_x)\Big)(t,x-(t-t_0))dt.\end{aligned} \tag{2.7}$$

We know that for any differentiable function F we can write

$$\frac{d}{dt}F(t,x_0-(t-t_0))=(F_t-F_x)(t,x_0-(t-t_0)),$$

therefore using the above we simplify the following:

$$(u_t-u_x)R+(R_t-R_x)u\Big|_{x=x_0-(t-t_0)}=\frac{d}{dt}(uR\Big|_{x=x_0-(t-t_0)})$$

and

$$(R_t-R_x)\Big|_{x=x_0-(t-t_0)}=\frac{d}{dt}(R(t,x_0-(t-t_0))=0,$$

since R is identically 1 on the line $x=x_0-(t-t_0)$ for $t_0\le t\le t_0+x_0$. Combining these results with equation (2.7) we obtain

$$\begin{aligned}I_1&=-\int_{t_0+x_0}^{t_0}\frac{d}{dt}(uR\Big|_{x=x_0-(t-t_0)})dt\\&=-u(t,x_0-(t-t_0))\Big|_{t_0+x_0}^{t_0}.\end{aligned}$$

Hence

$$\begin{aligned}I_1&=-u(t_0,x_0)+u(t_0+x_0,0)\\&=-u(t_0,x_0)+\varphi(t_0+x_0),\end{aligned}$$

where we have used condition (2.2). Similar computation on segment Γ_2 where $x = x_0 + (t - t_0)$ and $dx = dt$, yields

$$\begin{aligned} I_2 &= \int_{t_0}^{t_0 - x_0} \Big(u_t R - u R_t + u_x R - u R_x \Big)(t, x_0 + (t - t_0))dt \\ &= \varphi(t_0 - x_0) - u(t_0, x_0). \end{aligned}$$

Finally on segment Γ_3, we have that $dx = 0$ and that

$$u(t,0) = \varphi(t), \;\; u_x(t,0) = \psi(t),$$

so we get that

$$\begin{aligned} I_3 &= \int_{t_0 - x_0}^{t_0 + x_0} \Big(u_x R - u R_x \Big)(t,0)dt \\ &= \int_{t_0 - x_0}^{t_0 + x_0} (\psi(t)R(t,0) - \varphi(t)R_x(t,0))dt. \end{aligned}$$

Substituting I_1, I_2 and I_3 into (2.6) we get that

$$\begin{aligned} 0 = &-2u(t_0,x_0) + \varphi(t_0 + x_0) + \varphi(t_0 - x_0) \\ &+ \int_{t_0 - x_0}^{t_0 + x_0} (\psi(t)R(t,0) - \varphi(t)R_x(t,0))dt, \end{aligned}$$

or equivalently

$$u(t_0,x_0) = \frac{\varphi(t_0 + x_0) + \varphi(t_0 - x_0)}{2} + \frac{1}{2} \int_{t_0 - x_0}^{t_0 + x_0} \Big(\psi(t)R(t,0) - \varphi(t)R_x(t,0) \Big) dt. \tag{2.8}$$

The transformation $x \to -x$ leaves the general form of equation (2.1) unchanged when the variable x is replaced by $-x$ and $q_{1,2}(x)$ replaced by $q_{1,2}(-x)$, whereas $\psi(t) = u_x(t,0)$ and $R_x(t,0)$ change sign. From the above transformation and equation (2.8) it is quite clear to see that the form of equation (2.8) is preserved for $x_0 < 0$. The function $R(t,x;t_0,x_0)$ is called the Riemann function and the expression (2.8) for the functional $T_{t_0}^{x_0}$ is known as the Riemann formula. Note that a special case occurs when $\varphi'(t) = \psi(t)$, in which case, integrating the first term in the integral by parts we obtain

$$\begin{aligned} u(t_0,x_0) = &\frac{\varphi(t_0 + x_0) + \varphi(t_0 - x_0)}{2} + \frac{1}{2} \varphi(t)R(t,0) \Big|_{t_0 - x_0}^{t_0 + x_0} \\ &- \frac{1}{2} \int_{t_0 - x_0}^{t_0 + x_0} (R_t(t,0;t_0,x_0) + R_x(t,0;t_0,x_0))\varphi(t)dt, \end{aligned}$$

which upon simplification gives

$$u(t_0,x_0) = \varphi(t_0 + x_0) - \frac{1}{2} \int_{t_0 - x_0}^{t_0 + x_0} (R_t(t,0;t_0,x_0) + R_x(t,0;t_0,x_0))\varphi(t)dt. \tag{2.9}$$

In the next section we show how Riemann functions and Riemann's formula are used to construct kernels which are encountered when writing explicit solutions to the Sturm-Liouville equation using transformation operators.

3. TRANSFORMATION OPERATORS

Consider the Sturm-Liouville differential equation

$$y'' - q(x)y + \lambda^2 y = 0 \tag{3.1}$$

on the interval $(-a,a)$, $0<a<\infty$, where $q(x)$ is a continuous complex-valued function on this interval and λ a complex parameter. Let $e_0(\lambda,x)$ denote the solution to equation (3.1) with initial data

$$e_0(\lambda,0)=1,\ e_0'(\lambda,0)=i\lambda. \tag{3.2}$$

The function

$$u(t,x)=e^{i\lambda t}e_0(\lambda,x)$$

is obviously twice differentiable for $-\infty<t<\infty$, $-a<x<a$ and solves the Cauchy problem

$$u_{tt}=u_{xx}-q(x)u, \tag{3.3}$$

with initial conditions

$$u(t,0)=e^{i\lambda t},\ u_x(t,0)=i\lambda e^{i\lambda t}.$$

Equation (2.9) gives that the value of the function u at (t_0,x_0) is given by

$$e^{i\lambda t_0}e_0(\lambda,x_0)=e^{i\lambda(t_0+x_0)} - \frac{1}{2}\int_{t_0-x_0}^{t_0+x_0}(R_t(t,0;t_0,x_0)+R_x(t,0;t_0,x_0))e^{i\lambda t}dt.$$

Letting $t_0=0$ and denoting

$$K(x_0,t)=-\frac{1}{2}(R_t(t,0;0,x_0)+R_x(t,0;0,x_0)),\quad |t|\le|x_0|, \tag{3.4}$$

we get

$$e_0(\lambda,x_0)=e^{i\lambda x_0}+\int_{-x_0}^{x_0}K(x_0,t)e^{i\lambda t}dt.$$

Thus we have proved the following

Theorem 3.1. *The solution $e_0(\lambda,x)$ of equation (3.1) with the initial data (3.2) admits the representation*

$$e_0(\lambda,x)=e^{i\lambda x}+\int_{-x}^{x}K(x,t)e^{i\lambda t}dt, \tag{3.5}$$

where $K(x,t)$ is a continuous function expressed in terms of the Riemann function by equation (3.4).

The integral operator X defined by

$$(Xf)(x)=f(x)+\int_{-x}^{x}K(x,t)f(t)dt$$

is called the transformation operator preserving the initial data at the point $x=0$ where f represents the solution to some simpler differential equation $y''+\lambda^2y=0$ with the same initial data as the function $(Xf)(x)$. The operator maps the function $e^{i\lambda x}$ (i.e. the solution of the simplest form of equation of (3.1) with initial conditions (3.2) into the solution of equation (3.1) with condition (3.2). Since $e^{i\lambda x}$ and $e^{-i\lambda x}$ constitute a fundamental system of solutions to equation $y''+\lambda^2y=0$, then the operator X transforms any solutions of this equation into a solution of equation (3.1) with the same initial conditions at $x=0$. Suppose that we impose initial conditions $w(\lambda,0,h)=1,\ w'(\lambda,0,h)=h$ on the differential equation $y''+\lambda^2y=0$, then its solution is given by

$$\tilde{y}(x)=\cos\lambda x+h\frac{\sin\lambda x}{\lambda},$$

so then by an application of Theorem 3.1 we have that the solution to equation (3.1) with the initial data

$$w(\lambda,0,h) = 1,\ w'(\lambda,0,h) = h,$$

can be expressed as

$$\begin{aligned} w(\lambda,x,h) &= \cos\lambda x + h\frac{\sin\lambda x}{\lambda} + \int_{-x}^{x} K(x,t)\Big(\cos\lambda t + h\frac{\sin\lambda t}{\lambda}\Big)dt \\ &= \cos\lambda x + \int_0^x (h + K(x,t) + K(x,-t))\cos\lambda t \\ &\quad + h\int_0^x \Big[(K(x,t) - K(x,-t))\int_0^t \cos\lambda\xi d\xi\Big]dt \\ &= \cos\lambda x + \int_0^x K(x,t,h)\cos\lambda t dt, \end{aligned}$$

where

$$K(x,t,h) = h + K(x,t) + K(x,-t) + h\int_t^x (K(x,\xi) - K(x,-\xi))d\xi. \tag{3.6}$$

Similarly, the solution $w(\lambda,x,\infty)$ of (3.1) with initial data

$$w(\lambda,0,\infty) = 0,\ w'(\lambda,0,\infty) = 1,$$

admits the representation

$$\begin{aligned} w(\lambda,x,\infty) &= \frac{\sin\lambda x}{\lambda} + \int_{-x}^{x} K(x,t)\frac{\sin\lambda t}{\lambda}dt \\ &= \frac{\sin\lambda x}{\lambda} + \int_0^x (K(x,t) - K(x,-t))\frac{\sin\lambda t}{\lambda}dt \\ &= \frac{\sin\lambda x}{\lambda} + \int_0^x K(x,t,\infty)\frac{\sin\lambda t}{\lambda}dt, \end{aligned}$$

where

$$K(x,t,\infty) = K(x,t) - K(x,-t).$$

Note that if $h = 0$ we have that $K(x,t,0) = K(x,t) + K(x,-t)$ which is not the same as $K(x,t,\infty) = K(x,t) - K(x,-t)$. Now rewriting equation (3.1) in the form

$$y'' + \lambda^2 y = q(x)y, \tag{3.7}$$

and regarding the right hand side of equation (3.7) as known, we seek the solution $e_0(\lambda,x)$ of this equation by method of variation of constants. Let $q(x) = 0$ in (3.7), then the system $\{e^{-i\lambda x}, e^{i\lambda x}\}$ is a fundamental system of solutions to the differential equation $y'' = -\lambda^2 y$ and has Wronskian

$$W(x) = \begin{vmatrix} e^{-i\lambda x} & e^{i\lambda x} \\ -i\lambda e^{-i\lambda x} & i\lambda e^{i\lambda x} \end{vmatrix} = 2i\lambda.$$

Now, the variation of parameter states that if y_1, y_2 are linearly independent solutions of

$$a(t)y'' + b(t)y' + c(t)y(t) = 0,$$

then the solution y_p to

$$a(t)y'' + b(t)y' + c(t)y(t) = f(t)$$

is given by

$$y_p = -y_1\int \frac{y_2 f(t)}{a(t)W(t)}dt + y_2\int \frac{y_1 f(t))}{a(t)W(t)}dt,$$

where $W(t)$ is again the Wronskian of a fundamental system. Now, replacing f above with qy, we get that the solution $y(x)$ of

$$y'' + \lambda^2 y = qy$$

is given by

$$\begin{aligned} y(x) &= -e^{-i\lambda x}\int_0^x \frac{e^{-i\lambda t}q(t)y(t)}{2i\lambda}dt + e^{i\lambda x}\int_0^x \frac{e^{-i\lambda t}q(t)y(t)}{2i\lambda}dt \\ &= \int_0^x \frac{(e^{i\lambda(x-t)} - e^{-i\lambda(x-t)})q(t)y(t)}{2i\lambda}dt \\ &= \int_0^x \frac{\sin\lambda(x-t)}{\lambda}q(t)y(t)dt. \end{aligned}$$

Suppose that y_3 is the solution to the homogeneous equation $y'' + \lambda^2 y = 0$ and y_z a solution to the inhomogeneous equation $y'' + \lambda^2 y = g$, then $y_3 + y_z$ is a solution to the inhomogeneous equation $y'' + \lambda^2 y = g$ and implies that the solution

$$e_0(\lambda,x) = e^{i\lambda x} + \int_0^x \frac{\sin\lambda(x-t)}{\lambda}q(t)e_0(\lambda,t)dt, \tag{3.8}$$

which is the sum of solutions to the homogeneous and inhomogeneous equations is equivalent to problem (3.1) with initial condition (3.2). This is an integral equation for the function $e_0(\lambda,x)$ (which is known as the Sturm-Liouville integral equation). We seek its solution in the form of (3.5). In order for the function of the form (3.5) to satisfy (3.8), it is necessary that the equality

$$\begin{aligned} \int_{-x}^x K(x,t)e^{i\lambda t}dt = \int_0^x &\frac{\sin\lambda(x-t)}{\lambda}q(t)e^{i\lambda t}dt \\ &+ \int_0^x \frac{\sin\lambda(x-t)}{\lambda}q(t)\int_{-t}^t K(t,\xi)e^{i\lambda\xi}d\xi dt \end{aligned}$$

hold (where we have used Theorem 3.1). Next we express the right hand side of the above equation as a Fourier transform. Since

$$\frac{\sin\lambda(x-t)}{\lambda}e^{i\lambda\xi} = \frac{1}{2}\int_{\xi-(x-t)}^{\xi+(x-t)} e^{i\lambda u}du, \tag{3.9}$$

then it follows that

$$\begin{aligned} \int_0^x \frac{\sin\lambda(x-t)}{\lambda}q(t)e^{i\lambda t}dt &= \frac{1}{2}\int_0^x q(t)\int_{2t-x}^x e^{i\lambda u}dudt \\ &= \frac{1}{2}\int_{-x}^x e^{i\lambda u}\int_0^{\frac{x+u}{2}} q(t)dtdu. \end{aligned}$$

Changing the variables of integration we get

$$\int_0^x \frac{\sin\lambda(x-t)}{\lambda}q(t)e^{i\lambda t}dt = \int_{-x}^x \frac{1}{2}\Big(\int_0^{\frac{x+t}{2}} q(\xi)d\xi\Big)e^{i\lambda t}dt. \tag{3.10}$$

Using equation (3.9) once more, we obtain the equality

$$\begin{aligned} &\int_0^x \frac{\sin\lambda(x-t)}{\lambda}q(t)\int_{-t}^t K(t,\xi)e^{i\lambda\xi}d\xi dt \\ &= \frac{1}{2}\int_0^x q(t)\Big(\int_{-t}^t K(t,\xi)\int_{\xi-(x-t)}^{\xi+(x-t)} e^{i\lambda u}dud\xi\Big)dt. \end{aligned}$$

Next, upon extending the function $K(t,\xi)$ by zero for $|\xi| > |t|$, we can write

$$\int_{-t}^{t} K(t,\xi)\int_{\xi-(x-t)}^{\xi+(x-t)} e^{i\lambda u}dud\xi = \int_{-\infty}^{\infty} K(t,\xi)\int_{\xi-(x-t)}^{\xi+(x-t)} e^{i\lambda u}dud\xi,$$

and changing the order of integration

$$= \int_{-\infty}^{\infty} e^{i\lambda u}\int_{u-(x-t)}^{u+(x-t)} K(t,\xi)d\xi du.$$

Now since $K(t,\xi) = 0$ for $|\xi| > |t|$, so if $|u-(x-t)| = |t|$, then this equality implies that $u-(x-t) = t$ which in turn implies that $u = x$ so that if $u > x$ then we have that $|u-(x-t)| > |t|$, in which case $K(t,\xi) = 0$ on that interval. Similarly $|u+(x-t)| = |t|$ implies that $u = -x$ so that if $u < -x$ then we have that $|u+(x-t)| > |t|$ in which case we have again that $K(t,\xi) = 0$ on that interval. Altogether we get that $|u| \le x$, so we can write

$$\int_{-t}^{t} K(t,\xi)\int_{\xi-(x-t)}^{\xi+(x-t)} e^{i\lambda u}dud\xi = \int_{-x}^{x} e^{i\lambda u}\int_{u-(x-t)}^{u+(x-t)} K(t,\xi)d\xi du,$$

for any choice of $t \in (0,x)$. Consequently,

$$\int_0^x q(t)\int_{-t}^{t} K(t,\xi)\int_{\xi-(x-t)}^{\xi+(x-t)} e^{i\lambda u}dud\xi dt = \int_{-x}^{x} e^{i\lambda u}\int_0^x q(t)\int_{u-(x-t)}^{u+(x-t)} K(t,\xi)d\xi dt du.$$

Renaming the variables of integration,

$$\begin{aligned}(3.11)\qquad &\int_0^x \frac{\sin\lambda(x-t)}{\lambda}q(t)\int_{-t}^{t} K(t,\xi)e^{i\lambda\xi}d\xi dt\\ &= \int_{-x}^{x} e^{i\lambda t}\Big(\frac{1}{2}\int_0^x q(u)\int_{t-(x-u)}^{t+(x-u)} K(u,\xi)d\xi du\Big)dt.\end{aligned}$$

Equation (3.10) and (3.11) show that the equality (3.8) leads to

$$\begin{aligned}&\int_{-x}^{x} K(x,t)e^{i\lambda t}dt\\ &= \frac{1}{2}\int_{-x}^{x}\Big(\int_0^{\frac{x+t}{2}} q(u)du + \int_0^x q(u)\int_{t-(x-u)}^{t+(x-u)} K(u,\xi)d\xi du\Big)e^{i\lambda t}dt,\end{aligned}$$

and satisfies the equation

$$(3.12)\qquad K(x,t) = \frac{1}{2}\int_0^{\frac{x+t}{2}} q(u)du + \frac{1}{2}\int_0^x q(u)\int_{t-(x-u)}^{t+(x-u)} K(u,\xi)d\xi du.$$

The domain of integration of the double integral on the right hand side of formula (3.12) is shown in Figure 2.

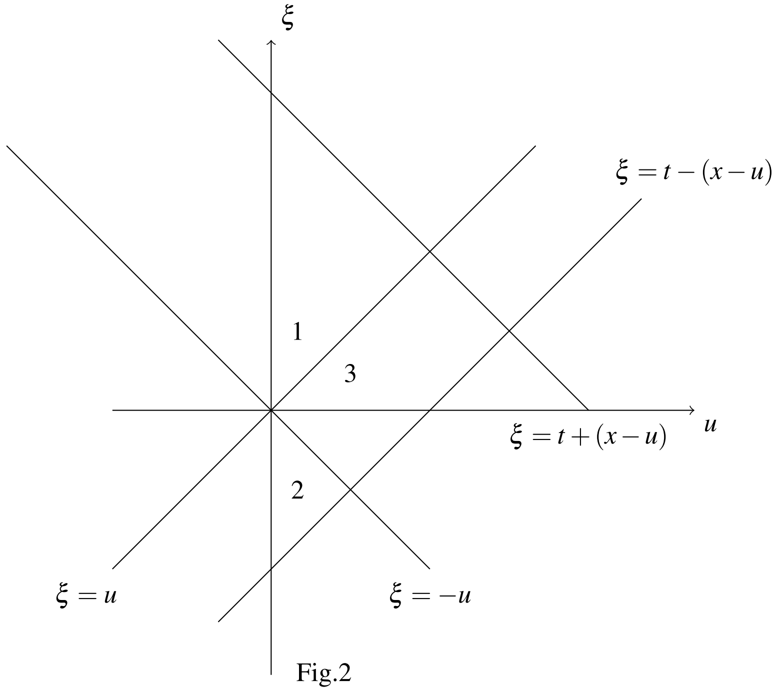

Fig.2

It consist of three regions, 1, 2, 3. In region 1 and 2, $|\xi| > |u|$ hence $K(u,\xi) = 0$ there. So the integral is actually taken over region 3. Performing a change of variables

$$u+\xi = 2\alpha \text{ and } u-\xi = 2\beta$$

in this integral, we get

$$K(x,t) = \frac{1}{2}\int_0^{\frac{x+t}{2}} q(y)dy + \int_0^{\frac{x+t}{2}}\int_0^{\frac{x-t}{2}} q(\alpha+\beta)K(\alpha+\beta,\alpha-\beta)d\beta d\alpha,$$

in which it is already taken into account that $K(x,t) = 0$ for $|t| > |x|$.

Theorem 3.2 (Rouche's theorem)**.** *Let f and g be holomorphic inside and on γ, where γ is a simple closed path. Suppose that $|f(z)| > |g(z)|$ on γ. Then f and $f+g$ have the same number of zeroes in the interior of γ.*

Theorem 3.3 (The Identity theorem)**.** *Let G be a region and suppose that f is holomorphic in G. If the set of zeroes of f in G has a limit point in G, then f is identically zero in G.*

4. Zeroes Of Entire Functions

In this section we will investigate the behavior of the zeroes of the entire function E_1 of the form:

(4.1)

$$\begin{aligned} E_1(\lambda) = &-K_1\lambda^2\cos\lambda a + i\lambda K_2\cos\lambda a + \lambda K_3\sin\lambda a + K_4\cos\lambda a + iK_5\sin\lambda a \\ &+ iK_6\frac{\cos\lambda a}{\lambda} + K_7\frac{\sin\lambda a}{\lambda} + K_8\frac{\cos\lambda a}{\lambda^2} + \lambda.\int_0^a \zeta_1(t)\sin\lambda t dt + \int_0^a \zeta_2(t)\cos\lambda t dt \\ &+ i\int_0^a \zeta_3(t)\sin\lambda t dt + i\int_0^a \zeta_4(t)\frac{\cos\lambda t}{\lambda}dt + \int_0^a \zeta_5(t)\frac{\sin\lambda t}{\lambda}dt + \int_0^a \zeta_6(t)\frac{\cos\lambda t}{\lambda^2}dt, \end{aligned}$$

where $\zeta_1, \ldots, \zeta_6$ are real valued functions belonging to $L_2[0,a]$ and $K_j \in \mathbb{R}$ for all $j = 1,2,\ldots,8$ and $K_1 \neq 0$. We will also investigate the behavior of another function of the form

(4.2)

$$E_2(\lambda) = -\lambda^2 K_1 \cos\lambda a + i\lambda K_2 \cos\lambda a + \lambda K_3 \sin\lambda a + K_4 \cos\lambda a + iK_5 \sin\lambda a + iK_6 \frac{\cos\lambda a}{\lambda} + K_7 \frac{\sin\lambda a}{\lambda} + \int_0^a \xi_1(t) \frac{\sin\lambda t}{\lambda} dt + i\int_0^a \xi_2(t) \frac{\cos\lambda t}{\lambda} dt,$$

where ξ_1 and ξ_2 are real valued functions belonging to $L_2[0,a]$ and $K_j \in \mathbb{R}$ for all $j = 1,2,\ldots,7$ and $K_1 \neq 0$. Lastly, we will investigate the zeroes of the function of the form

(4.3)

$$\begin{aligned} E_3(\lambda) = &-\lambda^2 K_1 \cos\lambda a + i\lambda K_2 \cos\lambda a + \lambda K_3 \sin\lambda a + K_4 \cos\lambda a + iK_5 \sin\lambda a + iK_6 \frac{\cos\lambda a}{\lambda} \\ &+ K_7 \frac{\sin\lambda a}{\lambda} + K_8 \frac{\cos\lambda a}{\lambda^2} + iK_9 \frac{\sin\lambda a}{\lambda^2} + iK_{10} \frac{\cos\lambda a}{\lambda^3} + K_{11} \frac{\sin\lambda a}{\lambda^3} \\ &+ \int_0^a \xi_1(t) \frac{\sin\lambda t}{\lambda^3} dt + i\int_0^a \xi_2(t) \frac{\cos\lambda t}{\lambda^3} dt, \end{aligned}$$

where again ξ_1 and ξ_2 are real valued functions belonging to $L_2[0,a]$ and $K_j \in \mathbb{R}$ for all $j = 1,2,\ldots,11$ and $K_1 \neq 0$. As a first step in our investigation let us consider the following theorem

Theorem 4.1. *Let $E_1(\lambda)$ be an entire function of the form*

$$\begin{aligned} E_1(\lambda) = &-K_1\lambda^2 \cos\lambda a + i\lambda K_2 \cos\lambda a + \lambda K_3 \sin\lambda a + K_4 \cos\lambda a + iK_5 \sin\lambda a \\ &+ iK_6 \frac{\cos\lambda a}{\lambda} + K_7 \frac{\sin\lambda a}{\lambda} + K_8 \frac{\cos\lambda a}{\lambda^2} + \lambda . \int_0^a \zeta_1(t) \sin\lambda t dt + \int_0^a \zeta_2(t) \cos\lambda t dt \\ &+ i\int_0^a \zeta_3(t) \sin\lambda t dt + i\int_0^a \zeta_4(t) \frac{\cos\lambda t}{\lambda} dt + \int_0^a \zeta_5(t) \frac{\sin\lambda t}{\lambda} dt + \int_0^a \zeta_6(t) \frac{\cos\lambda t}{\lambda^2} dt, \end{aligned}$$

where $\zeta_1, \ldots, \zeta_6$ are real valued and belong to $L_2[0,a]$ and $K_j \in \mathbb{R}$ for all $j = 1,2,\ldots,8$ and $K_1 \neq 0$. Then under proper enumeration, the zeroes λ_k of E_1 satisfy the following properties:

(1) The sequence $\{\lambda_k : k \in H\}$ is symmetric with respect to the imaginary axis, where H is an index set.
(2) The number of pure imaginary λ_k is even

The enumeration is said to be proper if

(1) the multiplicities are taken into account.
(2) $\mathrm{Re}\lambda_{k+1} \geq \mathrm{Re}\lambda_k$.

This enumeration is arbitrary in other respects. It should be mentioned that such enumeration is possible for all λ_k considered in this thesis and this will be shown later.

Proof. From equation (4.1) we can see that

$$\begin{aligned}
E_1(-\bar\lambda) &= -K_1\bar\lambda^2\cos\bar\lambda a - i\bar\lambda K_2\cos\bar\lambda a + \bar\lambda K_3\sin\bar\lambda a + K_4\cos\bar\lambda a - iK_5\sin\bar\lambda a \\
&\quad - iK_6\frac{\cos\bar\lambda a}{\bar\lambda} + K_7\frac{\sin\bar\lambda a}{\bar\lambda} + K_8\frac{\cos\bar\lambda a}{\bar\lambda^2} + \bar\lambda.\int_0^a \zeta_1(t)\sin\bar\lambda t dt + \int_0^a \zeta_2(t)\cos\bar\lambda t dt \\
&\quad - i\int_0^a \zeta_3(t)\sin\bar\lambda t dt - i\int_0^a \zeta_4(t)\frac{\cos\bar\lambda t}{\bar\lambda}dt + \int_0^a \zeta_5(t)\frac{\sin\bar\lambda t}{\bar\lambda}dt + \int_0^a \zeta_6(t)\frac{\cos\bar\lambda t}{\bar\lambda^2}dt \\
&= \overline{-K_1\lambda^2\cos\lambda a + i\lambda K_2\cos\lambda a + \lambda K_3\sin\lambda a + K_4\cos\lambda a + iK_5\sin\lambda a} \\
&\quad \overline{+iK_6\frac{\cos\lambda a}{\lambda} + K_7\frac{\sin\lambda a}{\lambda} + K_8\frac{\cos\lambda a}{\lambda^2} + \lambda.\int_0^a \zeta_1(t)\sin\lambda t dt + \int_0^a \zeta_2(t)\cos\lambda t dt} \\
&\quad \overline{+i\int_0^a \zeta_3(t)\sin\lambda t dt + i\int_0^a \zeta_4(t)\frac{\cos\lambda t}{\lambda}dt + \int_0^a \zeta_5(t)\frac{\sin\lambda t}{\lambda}dt + \int_0^a \zeta_6(t)\frac{\cos\lambda t}{\lambda^2}dt} \\
&= \overline{E_1(\lambda)},
\end{aligned}$$

from which we obtain that if λ_k is a zero of E_1, then so is $-\bar\lambda_k$ a zero of E_1 with matching multiplicity. We therefore have that the sequence $\{\lambda_k : k \in H\}$ is symmetric with respect to the imaginary axis and this establishes property (1) of the theorem. It will be shown in the next theorem that the indexing set H is countably infinite subset of $\mathbb{Z}$. Now since the above sequence is symmetric with respect to the imaginary axis we can write $\lambda_{-k} = -\bar\lambda_k$ for all not pure imaginary λ_k where we assume for now that $k \in H$ is a integer. It is evident that if we consider the same function on the imaginary axis ($\lambda = i\tau$, $\tau \in \mathbb{R}$) together with the identity $E_1(-\bar\lambda) = \overline{E_1(\lambda)}$ we can deduce that the function is real valued. We replace sines and cosines in E_1 with their exponential forms as follows:

$$\begin{aligned}
E_1(\lambda) &= -K_1\lambda^2\Big(\frac{e^{i\lambda a}+e^{-i\lambda a}}{2}\Big) + i\lambda K_2\Big(\frac{e^{i\lambda a}+e^{-i\lambda a}}{2}\Big) + \lambda K_3\Big(\frac{e^{i\lambda a}-e^{-i\lambda a}}{2i}\Big) \\
&\quad + K_4\Big(\frac{e^{i\lambda a}+e^{-i\lambda a}}{2}\Big) + iK_5\Big(\frac{e^{i\lambda a}-e^{-i\lambda a}}{2i}\Big) + iK_6\Big(\frac{e^{i\lambda a}+e^{-i\lambda a}}{2\lambda}\Big) \\
&\quad + K_7\Big(\frac{e^{i\lambda a}-e^{-i\lambda a}}{2i\lambda}\Big) + K_8\Big(\frac{e^{i\lambda a}+e^{-i\lambda a}}{2\lambda^2}\Big) + \lambda.\int_0^a \zeta_1(t)\Big(\frac{e^{i\lambda t}-e^{-i\lambda t}}{2i}\Big)dt \\
&\quad + \int_0^a \zeta_2(t)\Big(\frac{e^{i\lambda t}+e^{-i\lambda t}}{2}\Big)dt + i\int_0^a \zeta_3(t)\Big(\frac{e^{i\lambda t}-e^{-i\lambda t}}{2i}\Big)dt \\
&\quad + i\int_0^a \zeta_4(t)\Big(\frac{e^{i\lambda t}+e^{-i\lambda t}}{2\lambda}\Big)dt + \int_0^a \zeta_5(t)\Big(\frac{e^{i\lambda t}-e^{-i\lambda t}}{2i\lambda}\Big)dt \\
&\quad + \int_0^a \zeta_6(t)\Big(\frac{e^{i\lambda t}+e^{-i\lambda t}}{2\lambda^2}\Big)dt,
\end{aligned}$$

and then evaluating E_1 on the imaginary axis ($\lambda = i\tau$, $\tau \in \mathbb{R}$) we get

$$E_1(\pm i\tau) \underset{\tau\to\infty}{=} \frac{\tau^2}{2}K_1 e^{\tau a} + O(\tau e^{\tau a})$$

from which we deduce that the number of pure imaginary λ_k (including multiplicity) is even. Hence property (2) of the theorem is proved. $\square$

We now show that the zeroes of E_1 can be enumerated in a proper manner.

Lemma 4.2. *The zeroes λ_k of the function E_1 are countably infinite and have a proper enumeration (in a sense defined in Theorem 4.1)*

Proof. We know that the zeroes $\lambda_k^{(0)}$ of the function $-K_1\cos\lambda a$ which satisfy

$$\lambda_k^{(0)} = \frac{\pi}{a}(k-\frac{1}{2}), k\in\mathbb{Z}$$

are properly enumerated according to our sense of proper enumeration. We are going to show that the zeroes λ_k of E_1 satisfy

$$\lambda_k = \lambda_k^{(0)} + O(1),$$

from which it will becomes evident that the remaining zeroes can be arranged such that proper enumeration holds. Suppose that $\lambda\neq 0$, and let us define the function E as

(4.4)

$$\begin{aligned}E(\lambda) &= \frac{E_1}{\lambda^2}\\ &= -K_1\cos\lambda a + iK_2\frac{\cos\lambda a}{\lambda} + K_3\frac{\sin\lambda a}{\lambda} + K_4\frac{\cos\lambda a}{\lambda^2} + iK_5\frac{\sin\lambda a}{\lambda^2} + K_6\frac{\cos\lambda a}{\lambda^3}\\ &+ K_7\frac{\sin\lambda a}{\lambda^3} + K_8\frac{\cos\lambda a}{\lambda^4} + \int_0^a \zeta_1(t)\frac{\sin\lambda t}{\lambda}dt + \int_0^a \zeta_2(t)\frac{\cos\lambda t}{\lambda^2}dt + \int_0^a \zeta_3\frac{\sin\lambda t}{\lambda^2}dt\\ &+ i\int_0^a \zeta_4(t)\frac{\cos\lambda t}{\lambda^3}dt + \int_0^a \zeta_5(t)\frac{\sin\lambda t}{\lambda^3}dt + \int_0^a \zeta_6(t)\frac{\cos\lambda t}{\lambda^4}dt,\end{aligned}$$

which need not necessarily be entire. Suppose that there there exists a subsequence $\{\lambda_{k_m}\}_1^\infty$ (which without loss of generality we assume is countable for now) of $\{\lambda_k : k\in H\}$ such that $\mathrm{Im}\lambda_{k_m}\to\infty$ as $m\to\infty$, then from the definition of the function E we can see that

$$E(\lambda_{k_m}) + K_1\cos\lambda_{k_m}a = O(\frac{e^{|\mathrm{Im}\lambda_{k_m}a|}}{|\lambda_{k_m}|}) \tag{4.5}$$

which contradicts that $E(\lambda_{k_m}) = 0$. So there exists a number $A>0$ such that for all $k\in H$ we have that $\mathrm{Im}\lambda_k\leq A$. Similarly, it can be shown that $\mathrm{Im}\lambda_k$ is bounded below. Altogether, we can find a number $A_1>0$ such that for all $k\in H$ we have that $|\mathrm{Im}\lambda_k|\leq A_1$ (say $A_1\geq\frac{\pi}{2a}$). Now from equation (4.4) it can be seen that we can find an integer $k_0>0$ such that

$$\left|E(\lambda) + K_1\cos\lambda a\right| \leq \frac{C}{|\lambda|}e^{|\mathrm{Im}\lambda|a} \tag{4.6}$$

for all $\lambda\in\{\mathbb{C} : Re\lambda\geq k_0, |\mathrm{Im}\lambda|\leq A_1\}$ where $C>0$. Since $\lambda\neq 0$ then multiplying both sides by $|\lambda^2|$ we get

$$|E_1(\lambda) + K_1\lambda^2\cos\lambda a| \leq C|\lambda|e^{|\mathrm{Im}\lambda|a} \tag{4.7}$$

for all $\lambda\in\{\mathbb{C} : Re\lambda\geq k_0, |\mathrm{Im}\lambda|\leq A_1\}$. Let us denote by $C_{\rho k}$ the closed disc of radius $\rho\in(0,\frac{\pi}{2a})$ with centers at the point $\frac{\pi}{a}(k-\frac{1}{2})$. Immediately we can see that $\bigcup_{-\infty}^{\infty}C_{\rho k}$ is bounded by the strip $|\mathrm{Im}\lambda|\leq A_1$. Furthermore, from the above remark we can see that there exists a $C>0$ such that

$$\left|E(\lambda) + K_1\cos\lambda a\right| \leq \frac{C}{|\lambda|}e^{|\mathrm{Im}\lambda|a} \text{ for all } \lambda\in\bigcup_{|k|\geq k_0} C_{\rho k}.$$

Let $\Gamma_{\rho k}$ denote the boundary of $C_{\rho k}$,which is closed and bounded (compact). Since the zeroes of $\cos\lambda a$ are located on the center of $C_{\rho k}$, then $|\cos\lambda a|\neq 0$ on every point in the boundary $\Gamma_{\rho k}$. Due to periodicity of the cosine function, we can restrict ourselves to one particular disc $C_{\rho k}$. We know that the cosine function is continuous in the entire complex plane, certainly it is continuous on $\Gamma_{\rho k}$, hence $|\cos\lambda a|$ is continuous on $\Gamma_{\rho k}$ which is closed and bounded. Thus, the function attains a maximum and a minimum on $\Gamma_{\rho k}$, and that minimum is not zero since $|\cos\lambda a| \neq 0$ for all $\lambda \in \bigcup_{-\infty,k\neq 0}^{\infty}\Gamma_{\rho k}$, say $d > 0$. Then

$$|K_1\lambda^2\cos\lambda a| > d'|\lambda|^2 \;\; \forall\lambda \in \Gamma_{\rho k},$$

where $d' = K_1 d$. So for $\lambda \in \bigcup_{-\infty,k\neq 0}^{\infty}\Gamma_{\rho k} \cap \{\lambda \in \mathbb{C} : \; |\lambda| > Cd'^{-1}\}$, we can find another k_0', ($k_0' > k_0$) such that $C_{\rho k} \subset \{\lambda \in \mathbb{C} : |\lambda| > Cd'^{-1}\}$ for $|k| > k_0'$ and

$$|K_1\lambda^2\cos\lambda a| > d'|\lambda|^2 > d'|\lambda|\frac{C}{d'} = C|\lambda| > |E_1(\lambda) + K_1\lambda^2\cos\lambda a|.$$

Then applying Rouche's theorem we get that the number of zeroes of $K_1\lambda^2\cos\lambda a$ coincide with the number of zeroes of E_1 in the domain $C_{\rho k} \subset \{\lambda \in \mathbb{C} : |\lambda| > Cd'^{-1}\}$. So we can conclude each of the disc $C_{\rho k}$, $|k| > k_0'$ which lie in the domain $\{\lambda \in \mathbb{C} : |\lambda| > Cd'^{-1}\}$ contains exactly one simple zero of E_1. Hence

$$\lambda_k = \lambda_k^{(0)} + O(1).$$

Now let us choose a positively oriented boundary Γ of a rectangle enclosing the region $\{\lambda \in \mathbb{C} : |\mathrm{Re}\lambda| \leq k_0', |\mathrm{Im}\lambda| = B\}$, where $B = max\{A_1, Cd'^{-1}\}$. So then if $\lambda \in \Gamma = \{\mathbb{C} : |\mathrm{Re}\lambda| = k_0', |\mathrm{Im}\lambda| = B\}$ then we have that $|\lambda| > |\mathrm{Im}\lambda| > B > Cd'^{-1}$ and consequently that

$$|K_1\lambda^2\cos\lambda a| > d'|\lambda|^2 > d'|\lambda|\frac{C}{d'} = C|\lambda| > |E_1(\lambda) + K_1\lambda^2\cos\lambda a|,$$

which gives that the number of zeroes of the function E_1 coincide with the number of zeroes of the functions $-K_1\lambda^2\cos\lambda a$ inside Γ. Altogether we have that outside the rectangle Γ the circles $C_{\rho k}$, $k > k_0'$ contain exactly one zero written as λ_k of the function E_1, and inside the rectangle the function E_1 has exactly the same number of zeroes as $-K_1\lambda^2\cos\lambda a$ which are finite, hence the indexing set is countable. For $|k| \leq k_0'$ we have $2k_0' + 2$ zeroes which are distinct from those found when $k > k_0'$. Therefore the indexing set $H = \mathbb{Z} \cup \{0^+\}$ is enough to index the zeroes of the function E_1 completely. □

Theorem 4.3. *Under proper enumeration, the zeroes λ_k of E_1 satisfy the following*

$$\lambda_k \underset{k\to\infty}{=} \frac{\pi}{a}(k - \frac{1}{2}) + \frac{P_1}{(k - \frac{1}{2})} + \frac{b_k}{k}$$

where $\{b_k\}_{-\infty}^{\infty} \in l_2$ and $P_1 = -\frac{K_3}{\pi K_1}$.

Proof. We have already established in Lemma 4.2 that λ_k satisfy the following asymptotic $\lambda_k = \frac{\pi}{a}(k - \frac{1}{2}) + O(1)$, then substituting λ_k for k sufficiently large into equation (4.1) to obtain

$$\begin{aligned}0 = &-K_1\lambda_k^2\cos\lambda_k a + i\lambda_k K_2\cos\lambda_k a + \lambda_k K_3\sin\lambda_k a + K_4\cos\lambda_k a + iK_5\sin\lambda_k a\\ &+ iK_6\frac{\cos\lambda_k a}{\lambda_k} + K_7\frac{\sin\lambda_k a}{\lambda_k} + K_8\frac{\cos\lambda_k a}{\lambda_k^2} + \lambda_k\int_0^a \zeta_1(t)\sin\lambda_k t dt + \int_0^a \zeta_2(t)\cos\lambda_k t dt\\ &+ i\int_0^a \zeta_3(t)\sin\lambda_k t dt + i\int_0^a \zeta_4(t)\frac{\cos\lambda_k t}{\lambda_k}dt + \int_0^a \zeta_5(t)\frac{\sin\lambda_k t}{\lambda_k}dt + \int_0^a \zeta_6(t)\frac{\cos\lambda_k t}{\lambda_k^2}dt.\end{aligned}$$

Dividing both sides by $\lambda_k^2 K_1$, (possible since $\lambda_k \neq 0$ and $K_1 \neq 0$) to get

$$\begin{aligned}
0 &= -\cos\lambda_k a + i\frac{K_2}{K_1}\frac{\cos\lambda_k a}{\lambda_k} + \frac{K_3}{K_1}\frac{\sin\lambda_k a}{\lambda_k} + \frac{K_4}{K_1}\frac{\cos\lambda_k a}{\lambda_k^2} + i\frac{K_5}{K_1}\frac{\sin\lambda_k a}{\lambda_k^2} \\
&+ i\frac{K_6}{K_1}\frac{\cos\lambda_k a}{\lambda_k^3} + \frac{K_7}{K_1}\frac{\sin\lambda_k a}{\lambda_k^3} + \frac{K_8}{K_1}\frac{\cos\lambda_k a}{\lambda_k^4} + \frac{1}{K_1}\int_0^a \zeta_1(t)\frac{\sin\lambda_k t}{\lambda_k}dt \\
&+ \frac{1}{K_1}\int_0^a \zeta_2(t)\frac{\cos\lambda_k t}{\lambda_k^2}dt + i\frac{1}{K_1}\int_0^a \zeta_3(t)\frac{\sin\lambda_k t}{\lambda_k^2}dt + i\frac{1}{K_1}\int_0^a \zeta_4(t)\frac{\cos\lambda_k t}{\lambda_k^3}dt \\
&+ \frac{1}{K_1}\int_0^a \zeta_5(t)\frac{\sin\lambda_k t}{\lambda_k^3}dt + \frac{1}{K_1}\int_0^a \zeta_6(t)\frac{\cos\lambda_k t}{\lambda_k^4}dt, \\
&= -\cos\lambda_k a + iL_1\frac{\cos\lambda_k a}{\lambda_k} + L_2\frac{\sin\lambda_k a}{\lambda_k} + \frac{\Psi(\lambda_k)}{\lambda_k}
\end{aligned}$$

where $L_1 = \frac{K_2}{K_1}$, $L_2 = \frac{K_3}{K_1}$ and

$$\begin{aligned}
\Psi(\lambda_k) &= \frac{K_4}{K_1}\frac{\cos\lambda_k a}{\lambda_k} + i\frac{K_5}{K_1}\frac{\sin\lambda_k a}{\lambda_k} + i\frac{K_6}{K_1}\frac{\cos\lambda_k a}{\lambda_k^2} + \frac{K_7}{K_1}\frac{\sin\lambda_k a}{\lambda_k^2} + \frac{K_8}{K_1}\frac{\cos\lambda_k a}{\lambda_k^3} \\
&+ \frac{1}{K_1}\int_0^a \zeta_1(t)\sin\lambda_k t dt + \frac{1}{K_1}\int_0^a \zeta_2(t)\frac{\cos\lambda_k t}{\lambda_k}dt + i\frac{1}{K_1}\int_0^a \zeta_3(t)\frac{\sin\lambda_k t}{\lambda_k}dt \\
&+ \frac{i}{K_1}\int_0^a \zeta_4(t)\frac{\cos\lambda_k t}{\lambda_k^2}dt + \frac{1}{K_1}\int_0^a \zeta_5(t)\frac{\sin\lambda_k(t)}{\lambda_k^2}dt + \frac{1}{K_1}\int_0^a \zeta_6(t)\frac{\cos\lambda_k(t)}{\lambda_k^3}dt.
\end{aligned}$$

Let $\lambda_k = x + y$ where $x = \frac{\pi}{a}(k - \frac{1}{2})$ and $y = \Delta_k$, (as per Lemma 4.2 we have that $\Delta_k = O(1)$) then Ψ can be further reduced as follows

$$\Psi(\lambda_k) = L_3\frac{\cos\lambda_k a}{\lambda_k} + iL_4\frac{\sin\lambda_k a}{\lambda_k} + O(\frac{1}{k^2}),$$

where $L_3 = \frac{K_4}{K_1}$ and $L_4 = \frac{K_5}{K_1}$. Now substituting λ_k back to find the zeroes using $\cos\lambda_k a = (-1)^k \sin ay$ and $\sin\lambda_k a = (-1)^{k+1}\cos ay$ we obtain

$$-(-1)^k \sin ay + iL_1(-1)^k\frac{\sin ay}{\lambda_k} + L_2(-1)^{k+1}\frac{\cos ay}{\lambda_k} + \frac{\Psi(\lambda_k)}{\lambda_k} = 0.$$

Multiplying both sides by $(-1)^k$ and then dividing by i we get

$$i\sin ay + L_1\frac{\sin ay}{\lambda_k} + iL_2\frac{\cos ay}{\lambda_k} + i(-1)^{k+1}\frac{\Psi(\lambda_k)}{\lambda_k} = 0.$$

Now replacing sine and cosine with their exponential form in the above

$$\begin{aligned}
0 &= i\Big(\frac{e^{ayi} - e^{-ayi}}{2i}\Big) + L_1\Big(\frac{e^{ayi} - e^{-ayi}}{2i\lambda_k}\Big) + iL_2\Big(\frac{e^{ayi} + e^{-ayi}}{2\lambda_k}\Big) + (-1)^k\frac{\Psi(\lambda_k)}{\lambda_k} \\
&= \Big(\frac{e^{ayi} - e^{-ayi}}{2}\Big) - iL_1\Big(\frac{e^{ayi} - e^{-ayi}}{2\lambda_k}\Big) + iL_2\Big(\frac{e^{ayi} + e^{-ayi}}{2\lambda_k}\Big) + (-1)^k\frac{\Psi(\lambda_k)}{\lambda_k},
\end{aligned}$$

and then multiplying both sides again by e^{ayi}

$$\begin{aligned}
0 &= \Big(\frac{e^{2ayi} - 1}{2}\Big) - iL_1\Big(\frac{e^{2ayi} - 1}{2\lambda_k}\Big) + iL_2\Big(\frac{e^{2ayi} + 1}{2\lambda_k}\Big) + (-1)^k e^{ayi}\frac{\Psi(\lambda_k)}{\lambda_k} \\
&= e^{2ayi}\Big(\frac{1}{2} + i\frac{(-L_1 + L_2)}{2\lambda_k}\Big) + (-1)^k e^{ayi}\frac{\Psi(\lambda_k)}{\lambda_k} + \Big(-\frac{1}{2} + i\frac{(L_1 + L_2)}{2\lambda_k}\Big)
\end{aligned}$$

which is a quadratic equation in e^{ayi}. Observe that

$$\Big(\frac{1}{2}+i\frac{(-L_1+L_2)}{2\lambda_k}\Big)\Big(-\frac{1}{2}+i\frac{(L_1+L_2)}{2\lambda_k}\Big)=\Big(-\frac{1}{4}+i\frac{(L_1+L_2+L_1-L_2)}{4\lambda_k}+\frac{(L_1^2-L_2^2)}{4\lambda_k^2}\Big)$$
$$\Big(-\frac{1}{4}+i\frac{L_1}{2\lambda_k}+\frac{(L_1^2-L_2^2)}{4\lambda_k^2}\Big),$$

then solving the quadratic equation for its zeroes we see that

$$e^{ayi}=\frac{(-1)^{k+1}\frac{\Psi(\lambda_k)}{\lambda_k}\pm\Big[\frac{\Psi^2(\lambda_k)}{\lambda_k^2}+\Big(1-i\frac{2L_1}{\lambda_k}+\frac{(L_1^2-L_2^2)}{\lambda_k^2}\Big)\Big]^{\frac{1}{2}}}{2\Big(\frac{1}{2}+i\frac{(-L_1+L_2)}{2\lambda_k}\Big)}.$$

As $k\to\pm\infty$, y converges to zero, then taking the principal branch of the logarithmic function with center 1 and radius 1, we obtain

(4.8)

$$ai\Delta_k=\log\Big((-1)^{k+1}\frac{\Psi(\lambda_k)}{\lambda_k}\pm\Big[1-i\frac{2L_1}{\lambda_k}+\frac{(L_1^2-L_2^2)}{\lambda_k^2}+\frac{\Psi^2(\lambda_k)}{\lambda_k^2}\Big]^{\frac{1}{2}}\Big)-\log\Big(1+i\frac{(-L_1+L_2)}{\lambda_k}\Big).$$

Since we are looking for Δ_k and we know that $\Delta_k=o(1)$, the question then is, given the two square roots

$$\pm\Big[1-i\frac{2L_1}{\lambda_k}+\frac{(L_1^2-L_2^2)}{\lambda_k^2}+O(\frac{1}{k^2})\Big]^{\frac{1}{2}}$$

which one describes our Δ_k? We note that the function

$$\Big(-i\frac{2L_1}{\lambda_k}+\frac{(L_1^2-L_2^2)}{\lambda_k^2}+O(\frac{1}{k^2})\Big)+(-1)^{k+1}\frac{\Psi(\lambda_k)}{\lambda_k}$$

converges to zero as $k\to\infty$. So in the limit as k tends to infinity we have that the overall expression

$$\pm\Big[1-i\frac{2L_1}{\lambda_k}+\frac{(L_1^2-L_2^2)}{\lambda_k^2}+O(\frac{1}{k^2})\Big]^{\frac{1}{2}}+(-1)^{k+1}\frac{\Psi(\lambda_k)}{\lambda_k}$$

tends to ±1. Now thinking of our coefficients as coming from $\mathbb{C}$ and having that for $z\in\mathbb{C}$, $z=|r|e^{i\theta}$,

$$\log z=\log|r|+i\theta,$$

we see that choosing the minus sign makes an argument to be $-\pi+2\pi k$, k an integer. And in the limit as k tends to infinity we have that the logarithm of

$$(-1)^{k+1}\frac{\Psi(\lambda_k)}{\lambda_k}-\Big[1-i\frac{2L_1}{\lambda_k}+\frac{(L_1^2-L_2^2)}{\lambda_k^2}+O(\frac{1}{k^2})\Big]^{\frac{1}{2}}$$

does not tend to zero. Therefore by process of elimination, the positive square root describes our Δ_k. So

$$\log\Big((-1)^{k+1}\frac{\Psi(\lambda_k)}{\lambda_k}+\Big[1-i\frac{2L_1}{\lambda_k}+\frac{(L_1^2-L_2^2)}{\lambda_k^2}+\frac{\Psi^2(\lambda_k)}{\lambda_k^2}\Big]^{\frac{1}{2}}\Big)-\log\Big(1+i\frac{(-L_1+L_2)}{\lambda_k}\Big)$$
$$=\log\Big((-1)^{k+1}\frac{\Psi(\lambda_k)}{\lambda_k}+1+\frac{1}{2}\Big(-i\frac{2L_1}{\lambda_k}+\frac{(L_2^2-L_1^2)}{\lambda_k^2}+\frac{\Psi^2(\lambda_k)}{\lambda_k^2}\Big)+O(\frac{1}{k^2})\Big)$$
$$-\log\Big(1+i\frac{(-L_1+L_2)}{\lambda_k}\Big)$$
$$=\log\Big(1-i\frac{L_1}{\lambda_k}+(-1)^{k+1}\frac{\Psi(\lambda_k)}{\lambda_k}+O(\frac{1}{k^2})\Big)-\log\Big(1+i\frac{(-L_1+L_2)}{\lambda_k}\Big)$$
$$=-i\frac{L_1}{\lambda_k}+(-1)^{k+1}\frac{\Psi(\lambda_k)}{\lambda_k}-i\frac{(-L_1+L_2)}{\lambda_k}+O(\frac{1}{k^2}).$$

Now substituting back into equation (4.8) gives

$$\begin{aligned}ai\Delta_k&=-i\frac{L_2}{\lambda_k}+i\frac{(L_1+L_2)}{\lambda_k}+(-1)^{k+1}\frac{\Psi(\lambda_k)}{\lambda_k}+O(\frac{1}{k^2})\\&=\frac{1}{\lambda_k}\Big(-iL_2\Big)+(-1)^{k+1}\frac{\Psi(\lambda_k)}{\lambda_k}+O(\frac{1}{k^2}).\end{aligned}$$

Note that the above expression still has a λ_k occurring in the denominator and needs to be eliminated in order to arrive at the asymptotic expansion, and so to do that we use

$$\begin{aligned}\frac{1}{\lambda_k}&=\frac{1}{x(1+\frac{y}{x})}\\&=\frac{1}{x}\Big(1-\frac{y}{x}+(\frac{y}{x})^2+\dots\Big)\\&=\frac{1}{x}+O(\frac{1}{k^2}),\end{aligned}$$

for large enough x. So then

$$\begin{aligned}ai\Delta_k&=\Big(-iL_2+(-1)^{k+1}\Psi(\lambda_k)\Big)\Big(\frac{1}{\frac{\pi}{a}(k-\frac{1}{2})}+O(\frac{1}{k^2})\Big)+O(\frac{1}{k^2})\\&=(-1)^{k+1}\frac{\Psi(\lambda_k)}{\frac{\pi}{a}(k-\frac{1}{2})}+\frac{-iaL_2}{\pi(k-\frac{1}{2})}+O(\frac{1}{k^2}),\end{aligned}$$

hence

$$\begin{aligned}\Delta_k&=\frac{-L_2}{\pi(k-\frac{1}{2})}+i(-1)^k\frac{\Psi(\lambda_k)}{\pi(k-\frac{1}{2})}+O(\frac{1}{k^2})\\&=\frac{-L_2}{\pi(k-\frac{1}{2})}+\frac{b_k}{k},\end{aligned}$$

where

$$b_k=O(\frac{1}{k})+i(-1)^k\frac{\Psi(\lambda_k)}{\pi}\Big(\frac{k}{k-\frac{1}{2}}\Big).$$

Note that to complete the proof we need to show that the sequence b_k is square summable. For purpose of length of proof we will end it here and continue from the next proposition onwards to show that the sequence b_k is square summable. □

Proposition 4.4. *$h_k = O(\frac{1}{k})$ is square summable.*

Proof. $h_k = O(\frac{1}{k})$ by definition means that there exist constants n_0 and $c > 0$ such that

$$|h_k| < c\frac{1}{|k|}, \text{ for } |k| > n_0.$$

So for $|k| > n_0$ we have

$$\sum_{|k|>n_0} |h_k|^2 < c^2 \sum_{|k|>n_0} \frac{1}{|k|^2} < \infty,$$

and adding a finite number of terms to a convergent series produce another convergent series. Thus

$$\sum_{k=-\infty}^{\infty} |h_k|^2 < \infty.$$

□

Lemma 4.5. *Suppose that for a function $f \in L_2(0,\pi)$ we define a function $\tilde{f}$ as follows*

$$\tilde{f}(\lambda) = \int_0^{\pi} f(x)e^{-2i\lambda x}dx,$$

and further suppose that we have a numeric sequence $a_k = k + b + h_k$ such that

$$h_k = O(\frac{1}{k}) \textit{ for } k \to \pm\infty$$

then it follows that we must have that

$$\sum_{-\infty}^{\infty} |\tilde{f}(a_k)|^2 < \infty.$$

Proof. We will prove the results by showing that

$$\tilde{f}(a_k) = \tilde{f}(k+b) + k^{-1}g(k), \text{ and}$$

$$\sum_{k=-\infty}^{\infty} |\tilde{f}(k+b)|^2 < \infty, \quad \sum_{k=-\infty}^{\infty} |g(k)|^2 < \infty,$$

so that $\{\tilde{f}(a_k)\}_{-\infty}^{\infty} \in l_2$ will follow as a consequence of adding two square summable sequences. From the equality

$$\begin{aligned}\tilde{f}(a_k) &= \int_0^{\pi} f(x)e^{-2i(k+b)x}.e^{-2ih_kx}dx \\ &= \int_0^{\pi} f(x)e^{-2i(k+b)x}[1 - 2ih_kx + O(h_k^2)]dx,\end{aligned}$$

it follows that

$$\begin{aligned}\tilde{f}(a_k) &= \tilde{f}(k+b) + h_k\tilde{f}'(k+b) + O(h_k^2) \\ &= \tilde{f}(k+b) + k^{-1}g(k),\end{aligned}$$

where

$$g(k) = kh_k\tilde{f}'(k+b) + k^{-1}O(k^2h_k^2).$$

Now since

$$\tilde{f}(k+b) = \int_0^{\pi} f(x)e^{-2ibx}.e^{-2ikx}dx, \text{ and}$$

$$\tilde{f}'(k+b) = -2i\int_0^{\pi} f(x)xe^{-2ibx}e^{-2ikx}dx,$$

are Fourier coefficients of the functions $f(x)e^{-2ibx}$ and $2if(x)xe^{-2ibx}$ which belong to $L_2(0,\pi)$, Bessel's inequality implies that

$$\sum_{k=-\infty}^{\infty}|\tilde{f}(k+b)|^2<\infty,\quad \sum_{-\infty}^{\infty}|\tilde{f}'(k+b)|^2<\infty,$$

hence $\sum_{k=-\infty}^{\infty}|g(k)|^2<\infty$ because by assumption $\sup|kh_k|<\infty$, $kh_k=O(1)$ and by Proposition 4.4 is square summable. □

Now that all the machinery is in place, we therefore return to complete the proof of Theorem 4.3 by showing square summability.

Corollary 4.6. *Let $f\in L_2(0,a)$, and if $\tilde{h}$ is defined by*

$$\tilde{h}(\lambda)=\int_0^a f(x)\cos\lambda x dx,$$

and further suppose that we have a numeric sequence $b_k=\frac{\pi}{a}k+c+h_k$ such that $h_k=O(\frac{1}{k})$ for $k\to\pm\infty$ then $\tilde{h}(b_k)$ is square summable.

Proof. From the definition of $\tilde{h}$ above we replace the cosine function with its exponential form as follows

$$\begin{aligned}\tilde{h}(\lambda)&=\int_0^a f(x)\Big(\frac{e^{i\lambda x}+e^{-i\lambda x}}{2}\Big)dx,\\&=\frac{1}{2}\int_0^a(f(x)e^{i\lambda x}+f(x)e^{-i\lambda x})dx,\\&=\frac{1}{2}(f_1(\lambda)+f_2(\lambda)),\end{aligned}$$

where

$$f_1(\lambda)=\int_0^a f(x)e^{i\lambda x}dx \text{ and } f_2(\lambda)=\int_0^a f(x)e^{-i\lambda x}dx.$$

Let us consider the function

$$f_2(\lambda)=\int_0^a f(x)e^{-i\lambda x}dx.$$

Now performing a change of variables given by $t=\frac{\pi}{a}x$, then $\frac{a}{\pi}dt=dx$ and

$$f_2(\lambda)=\int_0^\pi\frac{a}{\pi}f(\frac{a}{\pi}t)e^{-i(\frac{a}{\pi}\lambda)t}dt.$$

Certainly $\frac{a}{\pi}f(\frac{a}{\pi}t)\in L_2(0,\pi)$. Now to apply Lemma 4.5 we need the following correspondence

$$\lambda\leftrightarrow\frac{a}{2\pi}\lambda,\;\; f(x)\leftrightarrow\frac{a}{\pi}f(\frac{a}{\pi}t)\in L_2(0,\pi).$$

Since we had obtained the results of Lemma 4.5 strictly for the numeric sequence $a_k=k+b+h_k$, then we must show that with $\frac{a}{2\pi}\lambda$ in place of λ that we do obtain the same numeric sequence. Given that $b_k=\frac{\pi}{a}k+b+h_k$, then

$$\begin{aligned}c_k&=\frac{a}{2\pi}(\frac{\pi}{a}k+b+h_k),\\&=\frac{k}{2}+c+h_k',\; h_k'=O(\frac{1}{k})\end{aligned}$$

So for k even, $k=2n$, n an integer, we obtain

$$c_k=n+c+h_k',$$

which gives a numeric sequence satisfying the asumptions. And if $k = 2n+1$, n an integer, we get

$$\begin{aligned} c_k &= n + \frac{1}{2} + c + h'_k, \\ &= n + d + h'_k, \end{aligned}$$

which also gives the same numeric sequence as in Lemma 4.6. So applying Lemma 4.5 we obtain that $f_2(b_k) \in l_2$. Similarly for $f_1(b_k) \in l_2$. Therefore, the function defined by

$$f(\lambda) = \int_0^a f(x) \cos \lambda x dx,$$

where $f(x) \in L_2(0,a)$ satisfies

$$\sum_{-\infty}^{\infty} |f(b_k)|^2 < \infty.$$

$\square$

This then completes the proof of Theorem 4.3.

Theorem 4.7. *Under proper enumeration the zeroes λ_k of E_2 as defined in equation (4.2) satisfy the following asymptotic*

$$\lambda_k = \frac{\pi}{a}(k - \frac{1}{2}) + \frac{Q_1}{(k-\frac{1}{2})} + i\frac{Q_2}{(k-\frac{1}{2})^2} + \frac{Q_3}{(k-\frac{1}{2})^3} + \frac{\beta_k}{k^3}$$

where

$$\begin{aligned} Q_1 &= -\frac{P_2}{\pi} \\ Q_2 &= -\frac{a(P_1P_2 + P_4)}{\pi^2} \\ Q_3 &= \frac{(-P_6 - P_2P_3 + P_1P_4 + P_1^2P_2 + \frac{1}{3}P_2^3)a^2}{\pi^3} - a\frac{P_2^2}{\pi^3} \end{aligned}$$

and

$$\beta_k = \frac{(-1)^k a^2 i\chi(\lambda_k)}{\pi^3}\Big(\frac{k}{k-\frac{1}{2}}\Big)^3 + \kappa_k + O(\frac{1}{k})$$

which is square summable. The function χ is defined by

$$\chi(\lambda_k) = \frac{1}{K_1}\Big(-i\int_0^a \xi_1(t)\sin\lambda_k t dt + \int_0^a \xi_2(t)\cos\lambda_k t dt\Big),$$

where the constants P_j are determined by $P_j = \frac{K_{j+1}}{K_1}$, $j = 1,2,\ldots,6$.

Proof. Given

$$\begin{aligned} E_2(\lambda) = &-\lambda^2 K_1 \cos\lambda a + i\lambda K_2 \cos\lambda a + \lambda K_3 \sin\lambda a + K_4 \cos\lambda a + iK_5 \sin\lambda a \\ &+ iK_6\frac{\cos\lambda a}{\lambda} + K_7\frac{\sin\lambda a}{\lambda} + \int_0^a \xi_1(t)\frac{\sin\lambda t}{\lambda}dt + i\int_0^a \xi_2(t)\frac{\cos\lambda t}{\lambda}dt, \end{aligned}$$

where $K_j \in \mathbb{R}$ for $j = 1,2,\ldots,7$ and $\xi_{1,2} \in L_2[0,a]$ are real valued functions. Suppose that λ_k is a zero of E_2, so dividing by $i\lambda_k^2 K_1$ in $E_2(\lambda_k) = 0$ to obtain

$$\begin{aligned} 0 = &\, i\cos\lambda_k a + P_1\frac{\cos\lambda_k a}{\lambda_k} - iP_2\frac{\sin\lambda_k a}{\lambda_k} - iP_3\frac{\cos\lambda_k a}{\lambda_k^2} \\ &+ P_4\frac{\sin\lambda_k a}{\lambda_k^2} + P_5\frac{\cos\lambda_k a}{\lambda_k^3} - iP_6\frac{\sin\lambda_k a}{\lambda_k^3} + \frac{\chi(\lambda_k)}{\lambda_k^3} \end{aligned}$$

and

$$\chi(\lambda_k) = \frac{1}{K_1}\Big(-i\int_0^a \xi_1(t)\sin\lambda_k t dt + \int_0^a \xi_2(t)\cos\lambda_k t dt\Big).$$

Let $\lambda_k = x+y$ where $x = \frac{\pi}{a}(k-\frac{1}{2}), y = \Delta_k$ and using $\cos\lambda_k a = (-1)^k \sin ay,\ \sin\lambda_k a = (-1)^{k+1}\cos ay$ to get that

$$\begin{aligned}
0 &= i(-1)^k \sin ay + P_1(-1)^k \frac{\sin ay}{\lambda_k} - iP_2(-1)^{k+1}\frac{\cos ay}{\lambda_k} \\
&\quad - iP_3(-1)^k \frac{\sin ay}{\lambda_k^2} + P_4(-1)^{k+1}\frac{\cos ay}{\lambda_k^2} + P_5(-1)^k\frac{\sin ay}{\lambda_k^3} \\
&\quad - iP_6(-1)^{k+1}\frac{\cos ay}{\lambda_k^3} + \frac{\chi(\lambda_k)}{\lambda_k^3}.
\end{aligned}$$

Multiplying by $(-1)^k$ on both sides, we then get

$$\begin{aligned}
0 &= i\sin ay + P_1\frac{\sin ay}{\lambda_k} + iP_2\frac{\cos ay}{\lambda_k} - iP_3\frac{\sin ay}{\lambda_k^2} \\
&\quad - P_4\frac{\cos ay}{\lambda_k^2} + P_5\frac{\sin ay}{\lambda_k^3} + iP_6\frac{\cos ay}{\lambda_k^3} + (-1)^k\frac{\chi(\lambda_k)}{\lambda_k^3}.
\end{aligned}$$

Now writing sines and cosines in their exponential forms to get

$$\begin{aligned}
0 &= i\Big(\frac{e^{ayi}-e^{-ayi}}{2i}\Big) + P_1\Big(\frac{e^{ayi}-e^{-ayi}}{2i\lambda_k}\Big) + iP_2\Big(\frac{e^{ayi}+e^{-ayi}}{2\lambda_k}\Big) \\
&\quad - iP_3\Big(\frac{e^{ayi}-e^{-ayi}}{2i\lambda_k^2}\Big) - P_4\Big(\frac{e^{ayi}+e^{-ayi}}{2\lambda_k^2}\Big) + P_5\Big(\frac{e^{ayi}-e^{-ayi}}{2i\lambda_k^3}\Big) \\
&\quad + iP_6\Big(\frac{e^{ayi}+e^{-ayi}}{2\lambda_k^3}\Big) + (-1)^k\frac{\chi(\lambda_k)}{\lambda_k^3}
\end{aligned}$$

which can also be written as

$$\begin{aligned}
0 &= \Big(\frac{e^{ayi}-e^{-ayi}}{2}\Big) - iP_1\Big(\frac{e^{ayi}-e^{-ayi}}{2\lambda_k}\Big) + iP_2\Big(\frac{e^{ayi}+e^{-ayi}}{2\lambda_k}\Big) \\
&\quad - P_3\Big(\frac{e^{ayi}-e^{-ayi}}{2\lambda_k^2}\Big) - P_4\Big(\frac{e^{ayi}+e^{-ayi}}{2\lambda_k^2}\Big) - iP_5\Big(\frac{e^{ayi}-e^{-ayi}}{2\lambda_k^3}\Big) \\
&\quad + iP_6\Big(\frac{e^{ayi}+e^{-ayi}}{2\lambda_k^3}\Big) + i(-1)^{k+1}\frac{\chi(\lambda_k)}{\lambda_k^3}.
\end{aligned}$$

Now multiplying both sides by e^{iay} we obtain

$$
\begin{aligned}
0 &= \Big(\frac{e^{2ayi}-1}{2}\Big) - iP_1\Big(\frac{e^{2ayi}-1}{2\lambda_k}\Big) + iP_2\Big(\frac{e^{2ayi}+1}{2\lambda_k}\Big) \\
&- P_3\Big(\frac{e^{2ayi}-1}{2\lambda_k^2}\Big) - P_4\Big(\frac{e^{2ayi}+1}{2\lambda_k^2}\Big) - iP_5\Big(\frac{e^{2ayi}-1}{2\lambda_k^3}\Big) \\
&+ iP_6\Big(\frac{e^{2ayi}+1}{2\lambda_k^3}\Big) + e^{ayi}i(-1)^{k+1}\frac{\chi(\lambda_k)}{\lambda_k^3} \\
&= e^{2ayi}\Big(\frac{1}{2} + i\frac{(-P_1+P_2)}{2\lambda_k} - \frac{(P_3+P_4)}{2\lambda_k^2} + i\frac{(-P_5+P_6)}{2\lambda_k^3}\Big) + e^{ayi}(-1)^k\frac{\chi(\lambda_k)}{\lambda_k^3} \\
&+ \Big(-\frac{1}{2} + i\frac{(P_1+P_2)}{2\lambda_k} + \frac{(P_3-P_4)}{2\lambda_k^2} + i\frac{(P_5+P_6)}{2\lambda_k^3}\Big)
\end{aligned}
$$

a quadratic equation in e^{ayi} of form $az^2+bz+c=0$. Now before solving the quadratic equation we first multiply the following

$$
\begin{aligned}
&\Big(\frac{1}{2} + i\frac{(-P_1+P_2)}{2\lambda_k} - \frac{(P_3+P_4)}{2\lambda_k^2} + i\frac{(-P_5+P_6)}{2\lambda_k^3}\Big)\Big(-\frac{1}{2} + i\frac{(P_1+P_2)}{2\lambda_k} + \frac{(P_3-P_4)}{2\lambda_k^2} + \frac{(P_5+P_6)}{2\lambda_k^3}\Big) \\
&= \Big(\frac{1}{2} + i\frac{(-P_1+P_2)}{2\lambda_k}\Big)\Big(-\frac{1}{2} + i\frac{(P_1+P_2)}{2\lambda_k}\Big) + \Big(\frac{1}{2} + i\frac{(-P_1+P_2)}{2\lambda_k}\Big)\Big(\frac{(P_3-P_4)}{2\lambda_k^2} + \frac{(P_5+P_6)}{2\lambda_k^3}) \\
&+ \Big(-\frac{(P_3+P_4)}{2\lambda_k^2} + i\frac{(-P_5+P_6)}{2\lambda_k^3}\Big)\Big(-\frac{1}{2} + i\frac{(P_1+P_2)}{2\lambda_k}\Big) + O(\frac{1}{k^4}) \\
&= \Big(-\frac{1}{4} + i\frac{P_1}{2\lambda_k} + \frac{(P_1^2-P_2^2+2P_3)}{4\lambda_k^2} + \frac{2i(P_5-P_1P_3-P_2P_4)}{4\lambda_k^3} + O(\frac{1}{k^4})\Big).
\end{aligned}
$$

Then solving the quadratic equation by taking the principal branch of the logarithm on the disc of radius 1 and center 1, we obtain

$$
\begin{aligned}
ai\Delta_k &= \log\Big((-1)^{k+1}\frac{\chi(\lambda_k)}{\lambda_k^3} \pm \Big[\frac{\chi^2(\lambda_k)}{\lambda_k^6} - \Big(-1 + i\frac{2P_1}{\lambda_k} + \frac{(P_1^2-P_2^2-2P_3)}{\lambda_k^2} \\
&+ \frac{2i(P_5-P_1P_3-P_2P_4)}{\lambda_k^3} + O(\frac{1}{k^4})\Big)\Big]^{\frac{1}{2}}\Big) \\
&- \log\Big(1 + i\frac{(-P_1+P_2)}{\lambda_k} - \frac{(P_3+P_4)}{\lambda_k^2} + i\frac{(-P_5+P_6)}{\lambda_k^3}\Big) \\
&= \log\Big((-1)^{k+1}\frac{\chi(\lambda_k)}{\lambda_k} + \Big[1 - i\frac{2P_1}{\lambda_k} - \frac{(P_1^2-P_2^2-2P_3)}{\lambda_k^2} - \frac{2i(P_5+P_1P_3+P_2P_4)}{\lambda_k^3} \\
&+ O(\frac{1}{k^4})\Big]^{\frac{1}{2}}\Big) \\
&- \log\Big(1 + i\frac{(-P_1+P_2)}{\lambda_k} - \frac{(P_3+P_4)}{\lambda_k^2} + i\frac{(-P_5+P_6)}{\lambda_k^3}\Big).
\end{aligned}
$$

Using the arguments of Theorem 4.3 we therefore take the positive square root and evaluate the term:

$$
\begin{aligned}
&\Big[1-i\frac{2P_1}{\lambda_k}-\frac{(P_1^2-P_2^2-2P_3)}{\lambda_k^2}-\frac{2i(P_5-P_1P_3-P_2P_4)}{\lambda_k^3}\\
&+O(\frac{1}{k^4})\Big]^{\frac{1}{2}}\\
&=1+\frac{1}{2}\Big(-i\frac{2P_1}{\lambda_k}-\frac{(P_1^2-P_2^2-2P_3)}{\lambda_k^2}-\frac{2i(P_5-P_1P_3-P_2P_4)}{\lambda_k^3}\\
&+O(\frac{1}{k^4})\Big)\\
&+\frac{1}{2}(-\frac{1}{2})\frac{\Big(-i\frac{2P_1}{\lambda_k}-\frac{(P_1^2-P_2^2-2P_3)}{\lambda_k^2}-\frac{2i(P_5-P_1P_3-P_2P_4)}{\lambda_k^3}+O(\frac{1}{k^4})\Big)^2}{2}\\
&+\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})\frac{\Big(-i\frac{2P_1}{\lambda_k}-\frac{(P_1^2-P_2^2-2P_3)}{\lambda_k^2}-\frac{2i(P_5-P_1P_3-P_2P_4)}{\lambda_k^3}+O(\frac{1}{k^4})\Big)^3}{6}\\
&=1-i\frac{P_1}{\lambda_k}+\frac{(-P_3-\frac{1}{2}(P_1^2-P_2^2)+\frac{1}{2}P_1^2)}{\lambda_k^2}\\
&+i\frac{(-P_5-P_1P_3+P_2P_4+P_1P_3-\frac{1}{2}P_1^3+\frac{1}{2}P_1P_2^2+\frac{1}{2}P_1^3)}{\lambda_k^3}+O(\frac{1}{k^4})\\
&=1-i\frac{P_1}{\lambda_k}+\frac{(-P_3+\frac{1}{2}P_2^2)}{\lambda_k^2}+i\frac{(-P_5+P_2P_4+\frac{1}{2}P_1P_2^2)}{\lambda_k^3}+O(\frac{1}{k^4})
\end{aligned}
$$

and since $\chi^2(\lambda_k)$ is bounded then we can replace $\frac{\chi^2(\lambda_k)}{\lambda_k^6}$ by $O(\frac{1}{k^4})$ to get

$$
\begin{aligned}
&\log\Big(1-i\frac{P_1}{\lambda_k}+\frac{(-P_3+\frac{1}{2}P_2^2)}{\lambda_k^2}+i\frac{(-P_5+P_2P_4+\frac{1}{2}P_1P_2^2)}{\lambda_k^3}+(-1)^{k+1}\frac{\chi(\lambda_k)}{\lambda_k^3}+O(\frac{1}{k^4})\Big)\\
&=-i\frac{P_1}{\lambda_k}+\frac{(-P_3+\frac{1}{2}P_2^2)}{\lambda_k^2}+i\frac{(-P_5+P_2P_4+\frac{1}{2}P_1P_2^2)}{\lambda_k^3}+(-1)^{k+1}\frac{\chi(\lambda_k)}{\lambda_k^3}+O(\frac{1}{k^4})\\
&-\frac{1}{2}\Big(-i\frac{P_1}{\lambda_k}+\frac{(-P_3+\frac{1}{2}P_2^2)}{\lambda_k^2}+i\frac{(-P_5+P_2P_4+\frac{1}{2}P_1P_2^2)}{\lambda_k^3}+(-1)^{k+1}\frac{\chi(\lambda_k)}{\lambda_k^3}+O(\frac{1}{k^4})\Big)^2\\
&+\frac{1}{3}\Big(-i\frac{P_1}{\lambda_k}+\frac{(-P_3+\frac{1}{2}P_2^2)}{\lambda_k^2}+i\frac{(-P_5+P_2P_4+\frac{1}{2}P_1P_2^2)}{\lambda_k^3}+(-1)^{k+1}\frac{\chi(\lambda_k)}{\lambda_k^3}+O(\frac{1}{k^4})\Big)^3\\
&=-i\frac{P_1}{\lambda_k}+\frac{(-P_3+\frac{1}{2}P_2^2+\frac{1}{2}P_1^2)}{\lambda_k^2}\\
&+i\frac{(-P_5-P_2P_4+\frac{1}{2}P_1P_2^2-P_1P_3+\frac{1}{2}P_1P_2^2+\frac{1}{3}P_1^3)}{\lambda_k^3}+(-1)^{k+1}\frac{\chi(\lambda_k)}{\lambda_k^3}+O(\frac{1}{k^4})
\end{aligned}
$$

whereas

$$
\begin{aligned}
&\log\Big(1+i\frac{(-P_1+P_2)}{\lambda_k}-\frac{(P_3+P_4)}{\lambda_k^2}+i\frac{(-P_5+P_6)}{\lambda_k^3}\Big)\\
&=i\frac{(-P_1+P_2)}{\lambda_k}-\frac{(P_3+P_4)}{\lambda_k^2}+i\frac{(-P_5+P_6)}{\lambda_k^3}\\
&-\frac{1}{2}\Big(i\frac{(-P_1+P_2)}{\lambda_k}-\frac{(P_3+P_4)}{\lambda_k^2}+i\frac{(-P_5+P_6)}{\lambda_k^3}\Big)^2\\
&+\frac{1}{3}\Big(i\frac{(-P_1+P_2)}{\lambda_k}-\frac{(P_3+P_4)}{\lambda_k^2}+i\frac{(-P_5+P_6)}{\lambda_k^3}\Big)^3+O(\frac{1}{k^4})\\
&=i\frac{(-P_1+P_2)}{\lambda_k}-\frac{(P_3+P_4)}{\lambda_k^2}+\frac{1}{2}\frac{(P_1^2-2P_1P_2+P_2^2)}{\lambda_k^2}\\
&+i\frac{(-P_5+P_6)}{\lambda_k^3}+i\frac{(-P_1+P_2)(P_3-P_4)-\frac{1}{3}(-P_1+P_2)^3}{\lambda_k^3}+O(\frac{1}{k^4})\\
&=i\frac{(-P_1+P_2)}{\lambda_k}+\frac{(-P_3-P_4+\frac{1}{2}P_1^2-P_1P_2+\frac{1}{2}P_2^2)}{\lambda_k^2}\\
&+i\frac{(-P_5+P_6-P_1P_3-P_1P_4+P_2P_3+P_2P_4+\frac{1}{3}(P_1^3-3P_1^2P_2+3P_1P_2^2-P_2^3))}{\lambda_k^3}+O(\frac{1}{k^4}).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
ai\Delta_k=&-i\frac{P_1}{\lambda_k}+\frac{(-P_3+\frac{1}{2}P_2^2+\frac{1}{2}P_1^2)}{\lambda_k^2}\\
&+i\frac{(-P_5-P_2P_4+\frac{1}{2}P_1P_2^2-P_1P_3+\frac{1}{2}P_1P_2^2+\frac{1}{3}P_1^3)}{\lambda_k^3}+(-1)^{k+1}\frac{\chi(\lambda_k)}{\lambda_k^3}+O(\frac{1}{k^4})\\
&-i\frac{(-P_1+P_2)}{\lambda_k}-\frac{(-P_3-P_4+\frac{1}{2}P_1^2-P_1P_2+\frac{1}{2}P_2^2)}{\lambda_k^2}\\
&-i\frac{(-P_5+P_6-P_1P_3-P_1P_4+P_2P_3+P_2P_4+\frac{1}{3}(P_1^3-3P_1^2P_2+3P_1P_2^2-P_2^3))}{\lambda_k^3}\\
=&-i\frac{P_2}{\lambda_k}+\frac{(P_4+P_1P_2)}{\lambda_k^2}+i\frac{(-P_6-P_2P_3+P_1P_4+P_1^2P_2+\frac{1}{3}P_2^3)}{\lambda_k^3}+(-1)^{k+1}\frac{\chi(\lambda_k)}{\lambda_k^3}+O(\frac{1}{k^4}).
\end{aligned}
$$

So we have that
(4.9)

$$
ai\Delta_k=-i\frac{P_2}{\lambda_k}+\frac{(P_4+P_1P_2)}{\lambda_k^2}+i\frac{(-P_6-P_2P_3+P_1P_4+P_1^2P_2+\frac{1}{3}P_2^3)}{\lambda_k^3}+(-1)^{k+1}\frac{\chi(\lambda_k)}{\lambda_k^3}+O(\frac{1}{k^4})
$$

Again here we have that our Δ_k is expressed in terms of λ_k, but we know from Theorem 4.3 that

$$
\lambda_k=\frac{\pi}{a}(k-\frac{1}{2})-\frac{P_2}{\pi(k-\frac{1}{2})}+\frac{b_k}{k},
$$

where $\{b_k\}_{-\infty}^{\infty} \in l_2$. Then

$$\begin{aligned}
\frac{1}{\lambda_k} &= \frac{1}{\frac{\pi}{a}(k-\frac{1}{2}) - \frac{P_2}{\pi(k-\frac{1}{2})} + \frac{b_k}{k}} \\
&= \frac{1}{\frac{\pi}{a}(k-\frac{1}{2})\Big[1+\Big(-\frac{P_2}{\pi(k-\frac{1}{2})}+\frac{b_k}{k}\Big)(\frac{\pi}{a}(k-\frac{1}{2}))^{-1}\Big]} \\
&= \frac{a}{\pi(k-\frac{1}{2})}\Big[1-\Big(-\frac{P_2}{\pi(k-\frac{1}{2})}+\frac{b_k}{k}\Big)(\frac{\pi}{a}(k-\frac{1}{2}))^{-1}+O(\frac{1}{k^4})\Big] \text{ for } k \text{ large enough} \\
&= \frac{a}{\pi(k-\frac{1}{2})} - \frac{-a^2P_2}{\pi^3(k-\frac{1}{2})^3} + \frac{b_k a^2}{\pi^2 k(k-\frac{1}{2})^2} + O(\frac{1}{k^5}) \\
&= \frac{a}{\pi(k-\frac{1}{2})} - \frac{-a^2P_2}{\pi^3(k-\frac{1}{2})^3} + \frac{b_k a^2}{\pi^2 k^3(1-\frac{1}{2k})^2} + O(\frac{1}{k^5}) \\
&= \frac{a}{\pi(k-\frac{1}{2})} - \frac{-a^2P_2}{\pi^3(k-\frac{1}{2})^3} + \frac{b_k a^2}{\pi^2 k^3}\Big(1+O(\frac{1}{k})\Big) + O(\frac{1}{k^5}) \\
&= \frac{a}{\pi(k-\frac{1}{2})} - \frac{-a^2P_2}{\pi^3(k-\frac{1}{2})^3} + \frac{b_k a^2}{\pi^2 k^3} + O(\frac{1}{k^4}) + O(\frac{1}{k^5}) \\
&= \frac{a}{\pi(k-\frac{1}{2})} + \frac{a^2P_2}{\pi^3(k-\frac{1}{2})^3} + \frac{\kappa_k}{k^3},
\end{aligned}$$

where $\kappa_k = \frac{b_k a^2}{\pi^2} + O(\frac{1}{k})$, which is also square summable. So we have shown that

$$\lambda_k^{-1} = \frac{a}{\pi(k-\frac{1}{2})} + \frac{a^2P_2}{\pi^3(k-\frac{1}{2})^3} + \frac{\kappa_k}{k^3},$$

then

$$\lambda_k^{-2} = \frac{a^2}{\pi^2(k-\frac{1}{2})^2} + O(\frac{1}{k^4})$$

and

$$\lambda_k^{-3} = \frac{a^3}{\pi^3(k-\frac{1}{2})^3} + O(\frac{1}{k^4}).$$

Now substituting λ_k into

$$ai\Delta_k = -i\frac{P_2}{\lambda_k} + \frac{(P_4+P_1P_2)}{\lambda_k^2} + i\frac{(-P_6-P_2P_3+P_1P_4+P_1^2P_2+\frac{1}{3}P_2^3)}{\lambda_k^3} + (-1)^{k+1}\frac{\chi(\lambda_k)}{\lambda_k^3} + O(\frac{1}{k^4})$$

to get

$$\begin{aligned}
ai\Delta_k = &-iP_2\Big(\frac{a}{\pi(k-\frac{1}{2})} - \frac{a^2P_2}{\pi^3(k-\frac{1}{2})^3} + \frac{\kappa_k}{k^3}\Big) + (P_4+P_1P_2)\Big(\frac{a^2}{\pi^2(k-\frac{1}{2})^2} + O(\frac{1}{k^4})\Big) \\
&+ i(-P_6-P_2P_3+P_1P_4+P_1^2P_2+\frac{1}{3}P_2^3)\Big(\frac{a^3}{\pi^3(k-\frac{1}{2})^3} + O(\frac{1}{k^4})\Big) \\
&+ (-1)^{k+1}\chi(\lambda_k)\Big(\frac{a^3}{\pi^3(k-\frac{1}{2})^3} + O(\frac{1}{k^4})\Big) + O(\frac{1}{k^4})
\end{aligned}$$

which upon simplification gives that

$$\Delta_k = \frac{Q_1}{(k-\frac{1}{2})} + i\frac{Q_2}{(k-\frac{1}{2})^2} + \frac{Q_3}{(k-\frac{1}{2})^3} + \frac{\beta_k}{k^3}$$

where

$$Q_1 = -\frac{P_2}{\pi}$$

$$Q_2 = -\frac{a(P_1P_2+P_4)}{\pi^2}$$

$$Q_3 = \frac{(-P_6 - P_2P_3 + P_1P_4 + P_1^2P_2 + \frac{1}{3}P_2^3)a^2}{\pi^3} - a\frac{P_2^2}{\pi^3}$$

and

$$\beta_k = \frac{(-1)^k a^2 i\chi(\lambda_k)}{\pi^3}\Big(\frac{k}{k-\frac{1}{2}}\Big)^3 + \kappa_k.a + O(\frac{1}{k})$$

which is square summable. □

Theorem 4.8. *Under proper enumeration the zeroes λ_k of E_3 as defined in equation (4.3) satisfy the following asymptotic*

$$\lambda_k = \frac{\pi}{a}(k-\frac{1}{2}) + \frac{Q_1}{(k-\frac{1}{2})} + i\frac{Q_2}{(k-\frac{1}{2})^2} + \frac{Q_3}{(k-\frac{1}{2})^3} + i\frac{Q_4}{(k-\frac{1}{2})^4} + \frac{Q_5}{(k-\frac{1}{2})^5} + \frac{\beta_k}{k^5}$$

where $\{\beta_k\} \in l_2$, Q_1, Q_2, Q_3 are as in Theorem 4.7 while Q_4 and Q_5 are defined as follows

$$Q_4 = \frac{aP_2Q_2}{\pi^2} + \frac{(P_4+P_1P_2)2a^2Q_1}{\pi^3} - \frac{a^3(P_8+P_2P_5+P_1P_6+P_3P_4+2P_1P_2P_3-P_2^2P_4-P_1^2P_4-P_2^3P_1-P_1^3P_2)}{\pi^4}$$

$$\begin{aligned} Q_5 = {} & \frac{aP_2Q_3}{\pi^2} - \frac{2a^2Q_2(P_4+P_1P_2)}{\pi^3} - \frac{3a^3Q_1(-P_6-P_2P_3+P_1P_4+\frac{1}{3}P_2^2+P_2P_1^2)}{\pi^4} \\ & + \frac{(-P_{10}-P_2P_7+P_1P_8-P_3P_6+P_4P_5+P_2^2P_6+2P_1P_2P_5-P_2P_3^2)a^4}{\pi^5} \\ & + \frac{(-P_2P_4^2+P_1^2P_6+2P_1P_3P_4-3P_1P_2^2P_4+3P_1^2P_2P_3+P_2^3P_3-P_1^3P_4-2P_1^2P_2^3-P_1^4P_2-\frac{1}{5}P_2^5)a^4}{\pi^5} - \frac{P_2^3a^2}{\pi^5} \end{aligned}$$

The constants P_j are again defined as $P_j = \frac{K_{j+1}}{K_1}$, $j = 1, 2, \ldots, 10$

Proof. We proceed as before by dividing $E_3(\lambda) = 0$ by $i\lambda^2K_1$ to obtain

$$\begin{aligned} 0 = {} & i\cos\lambda a + P_1\frac{\cos\lambda a}{\lambda} - iP_2\frac{\sin\lambda a}{\lambda} - iP_3\frac{\cos\lambda a}{\lambda^2} + P_4\frac{\sin\lambda a}{\lambda^2} + P_5\frac{\cos\lambda a}{\lambda^3} \\ & - iP_6\frac{\sin\lambda a}{\lambda^3} - iP_7\frac{\cos\lambda a}{\lambda^4} + P_8\frac{\sin\lambda a}{\lambda^4} + P_9\frac{\cos\lambda a}{\lambda^5} - iP_{10}\frac{\sin\lambda a}{\lambda^5} - i\frac{\psi(\lambda)}{K_1\lambda^5}. \end{aligned}$$

Substituting $\lambda_k = x+y$ where again $y=\Delta_k$ and $y=\frac{\pi}{a}(k-\frac{1}{2})$ an then using the following $\cos\lambda a = (-1)^k \sin ay$ and $\sin\lambda a = (-1)^{k+1}\cos ay$ we get

$$
\begin{aligned}
0 &= i(-1)^k \sin ay + P_1(-1)^k \frac{\sin ay}{\lambda_k} - iP_2(-1)^{k+1}\frac{\cos ay}{\lambda_k} - iP_3(-1)^k \frac{\sin ay}{\lambda_k^2} + P_4(-1)^{k+1}\frac{\cos ay}{\lambda_k^2} \\
&+ P_5(-1)^k \frac{\sin ay}{\lambda_k^3} - iP_6(-1)^{k+1}\frac{\cos ay}{\lambda_k^3} - iP_7(-1)^k\frac{\sin ay}{\lambda_k^4} + P_8(-1)^{k+1}\frac{\cos ay}{\lambda_k^4} \\
&+ P_9(-1)^k\frac{\sin ay}{\lambda_k^5} - iP_{10}(-1)^{k+1}\frac{\cos ay}{\lambda_k^5} - i\frac{\psi(\lambda_k)}{K_1\lambda_k^5} \\
&= i\sin ay + P_1\frac{\sin ay}{\lambda_k} + iP_2\frac{\cos ay}{\lambda_k} - iP_3\frac{\sin ay}{\lambda_k^2} - P_4\frac{\cos ay}{\lambda_k^2} + P_5\frac{\sin ay}{\lambda_k^3} + iP_6\frac{\cos ay}{\lambda_k^3} - iP_7\frac{\sin ay}{\lambda_k^4} \\
&- P_8\frac{\cos ay}{\lambda_k^4} + P_9\frac{\sin ay}{\lambda_k^5} + iP_{10}\frac{\cos ay}{\lambda_k^5} - i(-1)^{k+1}\frac{\psi(\lambda_k)}{K_1\lambda_k^5}.
\end{aligned}
$$

Now we replace the sine and cosine with their exponential form and then grouping like terms we get

$$
\begin{aligned}
0 &= i\Big(\frac{e^{ayi}-e^{-ayi}}{2i}\Big) + P_1\Big(\frac{e^{ayi}-e^{-ayi}}{2i\lambda_k}\Big) + iP_2\Big(\frac{e^{ayi}+e^{-ayi}}{2\lambda_k}\Big) - iP_3\Big(\frac{e^{ayi}-e^{-ayi}}{2i\lambda_k^2}\Big) \\
&- P_4\Big(\frac{e^{ayi}+e^{-ayi}}{2\lambda_k^2}\Big) + P_5\Big(\frac{e^{ayi}-e^{-ayi}}{2i\lambda_k^3}\Big) + iP_6\Big(\frac{e^{ayi}+e^{-ayi}}{2\lambda_k^3}\Big) - iP_7\Big(\frac{e^{ayi}-e^{-ayi}}{2i\lambda_k^4}\Big) \\
&- P_8\Big(\frac{e^{ayi}+e^{-ayi}}{2\lambda_k^4}\Big) + P_9\Big(\frac{e^{ayi}-e^{-ayi}}{2i\lambda_k^5}\Big) + iP_{10}\Big(\frac{e^{ayi}+e^{-ayi}}{2\lambda_k^5}\Big) - i(-1)^{k+1}\frac{\psi(\lambda_k)}{K_1\lambda_k^5} \\
&= \Big(\frac{e^{ayi}-e^{-ayi}}{2i}\Big) - iP_1\Big(\frac{e^{ayi}-e^{-ayi}}{2\lambda_k}\Big) + iP_2\Big(\frac{e^{ayi}+e^{-ayi}}{2\lambda_k}\Big) - P_3\Big(\frac{e^{ayi}-e^{-ayi}}{2\lambda_k^2}\Big) \\
&- P_4\Big(\frac{e^{ayi}+e^{-ayi}}{2\lambda_k^2}\Big) - iP_5\Big(\frac{e^{ayi}-e^{-ayi}}{2\lambda_k^3}\Big) + iP_6\Big(\frac{e^{ayi}+e^{-ayi}}{2\lambda_k^3}\Big) - P_7\Big(\frac{e^{ayi}-e^{-ayi}}{2\lambda_k^4}\Big) \\
&- P_8\Big(\frac{e^{ayi}+e^{-ayi}}{2\lambda_k^4}\Big) - iP_9\Big(\frac{e^{ayi}-e^{-ayi}}{2\lambda_k^5}\Big) + iP_{10}\Big(\frac{e^{ayi}+e^{-ayi}}{2\lambda_k^5}\Big) - i(-1)^{k+1}\frac{\psi(\lambda_k)}{K_1\lambda_k^5}.
\end{aligned}
$$

Now multiplying both sides by e^{ayi} we get

$$
\begin{aligned}
&= \Big(\frac{e^{2ayi}-1}{2i}\Big) - iP_1\Big(\frac{e^{2ayi}-1}{2\lambda_k}\Big) + iP_2\Big(\frac{e^{2ayi}+1}{2\lambda_k}\Big) - P_3\Big(\frac{e^{2ayi}-1}{2\lambda_k^2}\Big) \\
&- P_4\Big(\frac{e^{2ayi}+1}{2\lambda_k^2}\Big) - iP_5\Big(\frac{e^{2ayi}-1}{2\lambda_k^3}\Big) + iP_6\Big(\frac{e^{2ayi}+1}{2\lambda_k^3}\Big) - P_7\Big(\frac{e^{2ayi}-1}{2\lambda_k^4}\Big) \\
&- P_8\Big(\frac{e^{2ayi}+1}{2\lambda_k^4}\Big) - iP_9\Big(\frac{e^{2ayi}-1}{2\lambda_k^5}\Big) + iP_{10}\Big(\frac{e^{2ayi}+1}{2\lambda_k^5}\Big) - i(-1)^{k+1}e^{ayi}\frac{\psi(\lambda_k)}{K_1\lambda_k^5} \\
&= e^{2yai}\Big(\frac{1}{2} + i\frac{(P_2-P_1)}{2\lambda_k} - \frac{(P_3+P_4)}{2\lambda_k^2} + i\frac{(P_6-P_5)}{2\lambda_k^3} - \frac{(P_7+P_8)}{2\lambda_k^4} + i\frac{(P_{10}-P_9)}{2\lambda_k^5}\Big) \\
&+ i(-1)^{k+1}e^{ayi}\frac{\psi(\lambda_k)}{K_1\lambda_k^5} \\
&+ \Big(-\frac{1}{2} + i\frac{(P_2+P_1)}{2\lambda_k} + \frac{(P_3-P_4)}{2\lambda_k^2} + i\frac{(P_6+P_5)}{2\lambda_k^3} + \frac{(P_7-P_8)}{2\lambda_k^4} + i\frac{(P_{10}+P_9)}{2\lambda_k^5}\Big).
\end{aligned}
$$

which again is a quadratic equation in e^{ayi}. We now compute the following product

$$
\begin{aligned}
&\Big(\frac{1}{2}+i\frac{(P_2-P_1)}{2\lambda_k}-\frac{(P_3+P_4)}{2\lambda_k^2}+i\frac{(P_6-P_5)}{2\lambda_k^3}-\frac{(P_7+P_8)}{2\lambda_k^4}+i\frac{(P_{10}-P_9)}{2\lambda_k^5}\Big)\cdot\Big(-\frac{1}{2}\\
&+i\frac{(P_2+P_1)}{2\lambda_k}+\frac{(P_3-P_4)}{2\lambda_k^2}+i\frac{(P_6+P_5)}{2\lambda_k^3}+\frac{(P_7-P_8)}{2\lambda_k^4}+i\frac{(P_{10}+P_9)}{2\lambda_k^5}\Big)\\
&=\Big(\frac{1}{2}+i\frac{(P_2-P_1)}{2\lambda_k}-\frac{(P_3+P_4)}{2\lambda_k^2}+i\frac{(P_6-P_5)}{2\lambda_k^3}-\frac{(P_7+P_8)}{2\lambda_k^4}\Big)\cdot\Big(-\frac{1}{2}\\
&+i\frac{(P_2+P_1)}{2\lambda_k}+\frac{(P_3-P_4)}{2\lambda_k^2}+i\frac{(P_6+P_5)}{2\lambda_k^3}+\frac{(P_7-P_8)}{2\lambda_k^4}\Big)\\
&+\Big(\frac{1}{2}+i\frac{(P_2-P_1)}{2\lambda_k}-\frac{(P_3+P_4)}{2\lambda_k^2}+i\frac{(P_6-P_5)}{2\lambda_k^3}-\frac{(P_7+P_8)}{2\lambda_k^4}\Big)\cdot i\frac{(P_9+P_{10})}{2\lambda_k^5}\\
&+\Big(-\frac{1}{2}+i\frac{(P_2+P_1)}{2\lambda_k}+\frac{(P_3-P_4)}{2\lambda_k^2}+i\frac{(P_6+P_5)}{2\lambda_k^3}+\frac{(P_7-P_8)}{2\lambda_k^4}\Big)\cdot i\frac{(P_{10}-P_9)}{2\lambda_k^5}+O(\frac{1}{k^{10}})\\
&=\Big(\frac{1}{2}+i\frac{(P_2-P_1)}{2\lambda_k}\Big)\cdot\Big(-\frac{1}{2}+i\frac{(P_2+P_1)}{2\lambda_k}\Big)+\Big(\frac{1}{2}+i\frac{(P_2-P_1)}{2\lambda_k}\Big)\cdot\Big(\frac{(P_3-P_4)}{2\lambda_k^2}+i\frac{(P_6+P_5)}{2\lambda_k^3}\Big)\\
&+\Big(\frac{-(P_3+P_4)}{2\lambda_k^2}+i\frac{(P_6-P_5)}{2\lambda_k^3}\Big)\cdot\Big(-\frac{1}{2}+i\frac{(P_2+P_1)}{2\lambda_k}\Big)\\
&+\Big(\frac{-(P_3+P_4)}{2\lambda_k^2}+i\frac{(P_6-P_5)}{2\lambda_k^3}\Big)\cdot\Big(\frac{(P_3-P_4)}{2\lambda_k^2}+i\frac{(P_6+P_5)}{2\lambda_k^3}\Big)\\
&+\Big(\frac{1}{2}+i\frac{(P_2-P_1)}{2\lambda_k}\Big)\cdot\frac{(P_7-P_8)}{2\lambda_k^4}+O(\frac{1}{k^6})+\frac{-(P_7+P_8)}{2\lambda_k^4}\Big)\cdot\Big(-\frac{1}{2}+i\frac{(P_2+P_1)}{2\lambda_k}\Big)+O(\frac{1}{k^6})\\
&i\frac{(P_9+P_{10})}{4\lambda_k^5}+O(\frac{1}{k^6})-i\frac{(P_{10}-P_9)}{4\lambda_k^5}++O(\frac{1}{k^6}).
\end{aligned}
$$

Multiplying all terms and grouping like term we get

$$
\begin{aligned}
&=-\frac{1}{4}+i\frac{(P_1+P_2)}{4\lambda_k}-i\frac{(-P_1+P_2)}{4\lambda_k}-\frac{-(P_2-P_1)(P_1+P_2)}{4\lambda_k^2}+\frac{(P_3-P_4)}{4\lambda_k^2}+i\frac{(P_5+P_6)}{4\lambda_k^3}\\
&+i\frac{(P_2-P_1)(P_3-P_4)}{4\lambda_k^3}+\frac{(-P_2-P_1)(P_5+P_6)}{4\lambda_k^4}+\frac{(P_3+P_4)}{4\lambda_k^2}-i\frac{(P_3+P_4)(P_1+P_2)}{4\lambda_k^3}\\
&-i\frac{(P_6-P_5)}{4\lambda_k^3}+\frac{(-P_6+P_5)(P_1+P_2)}{4\lambda_k^4}+\frac{(P_3+P_4)(-P_3+P_4)}{4\lambda_k^4}-i\frac{(P_3+P_4)(P_5+P_6)}{4\lambda_k^5}\\
&+i\frac{(P_6-P_5)(P_3-P_4)}{4\lambda_k^5}+O(\frac{1}{k^6})+\frac{(P_7-P_8)}{4\lambda_k^4}+i\frac{(P_2-P_1)(P_7-P_8)}{4\lambda_k^5}+O(\frac{1}{k^6})\\
&+\frac{(P_7+P_8)}{4\lambda_k^4}-i\frac{(P_7+P_8)(P_1+P_2)}{4\lambda_k^5}+O(\frac{1}{k^6})+i\frac{(P_9+P_{10})}{4\lambda_k^5}+O(\frac{1}{k^6})-i\frac{(P_{10}-P_9)}{4\lambda_k^5}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{4} + i\frac{(P_1+P_2-P_2+P_1)}{4\lambda_k} - \frac{(P_2^2+P_1^2)}{4\lambda_k^2} + \frac{(P_3-P_4)}{4\lambda_k^2} + i\frac{(P_5+P_6)}{4\lambda_k^3} \\
&\quad + i\frac{(P_2P_3-P_2P_4-P_1P_3+P_1P_4)}{4\lambda_k^3} + \frac{(-P_2P_5-P_2P_6+P_1P_5+P_1P_6)}{4\lambda_k^4} + \frac{(P_3+P_4)}{4\lambda_k^2} \\
&\quad - i\frac{(P_3P_1+P_3P_2+P_1P_4+P_4P_2)}{4\lambda_k^3} - i\frac{(P_6-P_5)}{4\lambda_k^3} + \frac{(-P_1P_6-P_2P_6+P_1P_5+P_2P_5)}{4\lambda_k^4} \\
&\quad + \frac{(P_4^2-P_3^2)}{4\lambda_k^4} - i\frac{(P_3P_5+P_3P_6+P_4P_5+P_4P_6)}{4\lambda_k^5} + i\frac{(P_3P_6-P_6P_4-P_5P_3+P_4P_5)}{4\lambda_k^5} \\
&\quad + O(\frac{1}{k^6}) + \frac{(P_7-P_8)}{4\lambda_k^4} + i\frac{(P_2P_7-P_2P_8-P_1P_7+P_1P_8)}{4\lambda_k^5} + O(\frac{1}{k^6}) + \frac{(P_7+P_8)}{4\lambda_k^4} \\
&\quad - i\frac{(P_1P_7+P_2P_7+P_1P_8+P_8P_2)}{4\lambda_k^5} + O(\frac{1}{k^6}) + i\frac{(P_9+P_{10})}{4\lambda_k^5} + O(\frac{1}{k^6}) - i\frac{(P_{10}-P_9)}{4\lambda_k^5} + O(\frac{1}{k^6}) \\
&= -\frac{1}{4} + i\frac{2P_1}{4\lambda_k} + \frac{(P_1^2-P_2^2+2P_3)}{4\lambda_k^2} + i\frac{(2P_5-2P_2P_4-2P_1P_3)}{4\lambda_k^3} + \frac{-2P_2P_6+2P_1P_5+P_4^2-P_3^2+2P_7)}{4\lambda_k^4} \\
&\quad + i\frac{(-2P_3P_5-2P_4P_6-2P_2P_8-2P_1P_7+2P_9)}{4\lambda_k^5}.
\end{aligned}
$$

Then solving the quadratic equation by taking the principle branch of the logarithm on the disc of radius 1 center 1, we obtain

$$
\begin{aligned}
ai\Delta_k &= \log\Big(i\frac{(-1)^k\psi(\lambda_k)}{K_1\lambda_k^5} \pm \Big[\frac{\psi^2(\lambda_k)}{K_1^2\lambda_k^{10}} - \Big(-1 + i\frac{2P_1}{\lambda_k} + \frac{(P_1^2-P_2^2+2P_3)}{\lambda_k^2} + i\frac{(2P_5-2P_2P_4-2P_1P_3)}{\lambda_k^3} \\
&\quad + \frac{(-2P_2P_6+2P_1P_5+P_4^2-P_3^2+2P_7)}{\lambda_k^4} + i\frac{(-2P_3P_5-2P_4P_6-2P_2P_8-2P_1P_7+2P_9)}{\lambda_k^5}\Big)\Big) \\
&\quad - \log\Big(1 + i\frac{(P_2-P_1)}{\lambda_k} - \frac{(P_3+P_4)}{\lambda_k^2} + i\frac{(P_6-P_5)}{\lambda_k^3} - \frac{(P_7+P_8)}{\lambda_k^4} + i\frac{(P_{10}-P_9)}{\lambda_k^5}\Big).
\end{aligned}
$$

We must evaluate the following square root

$$
\begin{aligned}
&\Big[1 - i\frac{2P_1}{\lambda_k} - \frac{(P_1^2-P_2^2+2P_3)}{\lambda_k^2} - i\frac{(2P_5-2P_2P_4-2P_1P_3)}{\lambda_k^3} \\
&- \frac{(-2P_2P_6+2P_1P_5+P_4^2-P_3^2+2P_7)}{\lambda_k^4} - i\frac{(-2P_3P_5-2P_4P_6-2P_2P_8-2P_1P_7+2P_9)}{\lambda_k^5}\Big]^{\frac{1}{2}},
\end{aligned}
$$

but before evaluating the above square root we let

$$A = -2P_2P_6 + 2P_1P_5 + P_4^2 - P_3^2 + 2P_7,$$

$$B = -2P_3P_5 - 2P_4P_6 - 2P_2P_8 - 2P_1P_7 + 2P_9.$$

Then using the Taylor series expansion of a square root function we evaluate its first few terms which gives

$$\begin{aligned}
\frac{1}{2}\Big[\ \Big] &= \frac{1}{2}\Big[-i\frac{2P_1}{\lambda_k} - \frac{(P_1^2-P_2^2+2P_3)}{\lambda_k^2} - i\frac{(2P_5-2P_2P_4-2P_1P_3)}{\lambda_k^3} \\
&\quad - \frac{(-2P_2P_6+2P_1P_5+P_4^2-P_3^2+2P_7)}{\lambda_k^4} - i\frac{(-2P_3P_5-2P_4P_6-2P_2P_8-2P_1P_7+2P_9)}{\lambda_k^5}\Big] \\
&= -i\frac{P_1}{\lambda_k} + \frac{(-\frac{1}{2}P_1^2+\frac{1}{2}P_2^2-P_3)}{\lambda_k^2} + i\frac{(-P_5+P_2P_4+P_1P_3)}{\lambda_k^3} - \frac{(-\frac{1}{2}A)}{\lambda_k^4} - i\frac{(\frac{1}{2}B)}{\lambda_k^5}.
\end{aligned}$$

The second term of the series is then given by

$$\begin{aligned}
-\frac{1}{8}\Big[\ \Big]^2 &= -\frac{1}{8}\Big[-i\frac{2P_1}{\lambda_k} - \frac{(P_1^2-P_2^2+2P_3)}{\lambda_k^2} - i\frac{(2P_5-2P_2P_4-2P_1P_3)}{\lambda_k^3} \\
&\quad - \frac{A}{\lambda_k^4} - i\frac{B}{\lambda_k^5}\Big]^2 \\
&= -\frac{1}{8}\Big[\Big(\frac{-2iP_1}{\lambda_k}\Big)^2 + \Big(\frac{-2iP_1}{\lambda_k}\Big)\Big(\frac{P_1^2-P_2^2+2P_3}{\lambda_k^2}\Big) - \Big(\frac{2P_1}{\lambda_k}\Big)\Big(\frac{2P_5-2P_2P_4-2P_1P_3}{\lambda_k^3}\Big) \\
&\quad + \Big(\frac{2iP_1}{\lambda_k}\Big)\Big(\frac{-2P_2P_6+2P_1P_5+P_4-P_3^2+2P_7}{\lambda_k^4}\Big) + \Big(\frac{2iP_1}{\lambda_k}\Big)\Big(\frac{P_1^2-P_2^2+2P_3}{\lambda_k^2}\Big) \\
&\quad + \frac{(P_1^2-P_2^2+2P_3)^2}{\lambda_k^4} + i\frac{(P_1^2-P_2^2+2P_3)(2P_5-2P_2P_4-P_1P_3)}{\lambda_k^5} \\
&\quad - \Big(\frac{2iP_1}{\lambda_k}\Big)\Big(\frac{2P_5-2P_2P_4-2P_1P_3}{\lambda_k^3}\Big) + i\frac{(P_1^2-P_2^2+2P_3)(2P_5-2P_2P_4-P_1P_3)}{\lambda_k^5} \\
&\quad + \Big(\frac{2iP_1}{\lambda_k}\Big)\Big(\frac{-2P_2P_6+2P_1P_5+P_4-P_3^2+2P_7}{\lambda_k^4}\Big)\Big] \\
&= -\frac{1}{8}\Big[-4\frac{P_1^2}{\lambda_k^2} + i\frac{(4P_1^3-4P_1P_2^2+8P_1P_3)}{\lambda_k^3} \\
&\quad + \frac{(-8P_1P_5+8P_1P_2P_4+12P_1^2P_3+P_1^4-2P_1^2P_2^2-4P_2^2P_3+4P_3^2+P_2^4)}{\lambda_k^4} \\
&\quad + i\frac{(12P_1^2P_5-4P_1^2P_2P_4-4P_1^3P_3-4P_2^2P_5+4P_2^3P_4+4P_1P_2^2P_3+8P_3P_5)}{\lambda_k^5} \\
&\quad + i\frac{(-8P_2P_3P_4-12P_1P_3^2-8P_1P_2P_6+4P_1P_4^2+8P_1P_7)}{\lambda_k^5}.
\end{aligned}$$

Therefore

$$
\begin{aligned}
-\frac{1}{8}\Big[\ \Big]^2 = \frac{P_1^2}{2\lambda_k} &+ i\frac{(-\frac{1}{2}P_1^3+\frac{1}{2}P_1P_2^2-P_1P_3)}{\lambda_k^3}\\
&+\frac{(P_1P_5-P_1P_2P_4-\frac{3}{2}P_1^2P_3-\frac{1}{8}P_1^4+\frac{1}{4}P_1^2P_2^2+\frac{1}{2}P_2^2P_3-\frac{4}{8}P_3^2-\frac{1}{8}P_2^4)}{\lambda_k^4}\\
&+i\frac{(-\frac{12}{8}P_1^2P_5+\frac{1}{2}P_1^2P_2P_4+\frac{1}{2}P_1^3P_3+\frac{1}{2}P_2^2P_5-\frac{1}{2}P_2^3P_4-\frac{1}{2}P_1P_2^2P_3-P_3P_5)}{\lambda_k^5}\\
&+i\frac{(P_2P_3P_4+\frac{3}{2}P_1P_3^2+P_1P_2P_6-\frac{1}{2}P_1P_4^2-P_1P_7)}{\lambda_k^5}
\end{aligned}
$$

And now the third term of the series gives

$$
\begin{aligned}
\frac{1}{16}\Big[\ \Big]^3 = {}& \frac{1}{16}\Big[\Big(-i\frac{2P_1}{\lambda_k}-\frac{(P_1^2-P_2^2+2P_3)}{\lambda_k^2}-i\frac{(2P_5-2P_2P_4-2P_1P_3)}{\lambda_k^3}\\
&-\frac{A}{\lambda_k^4}-i\frac{B}{\lambda_k^5}\Big)\\
&\cdot\Big(-4\frac{P_1^2}{\lambda_k^2}+i\frac{(4P_1^3-4P_1P_2^2+8P_1P_3)}{\lambda_k^3}\\
&+\frac{(-8P_1P_5+8P_1P_2P_4+12P_1^2P_3+P_1^4-2P_1^2P_2^2-4P_2^2P_3+4P_3^2+P_2^4)}{\lambda_k^4}\Big)\Big]\\
={}& \frac{1}{16}\Big[\Big(-i\frac{2P_1}{\lambda_k}\Big)\Big(-4\frac{P_1^2}{\lambda_k^2}\Big)+\Big(-i\frac{2P_1}{\lambda_k}\Big)\Big(\frac{(4P_1^3-4P_1P_2^2+8P_1P_3)}{\lambda_k^3}\Big)\\
&+\Big(-i\frac{2P_1}{\lambda_k}\Big)\Big(+\frac{(-8P_1P_5+8P_1P_2P_4+12P_1^2P_3+P_1^4-2P_1^2P_2^2-4P_2^2P_3+4P_3^2+P_2^4)}{\lambda_k^4}\Big)\\
&+\frac{(P_1^2-P_2^2+2P_3)(4P_1^2)}{\lambda_k^4}-i\frac{(P_1^2-P_2^2+2P_3)(4P_1^3-4P_1P_2^2+8P_1P_3)}{\lambda_k^5}\\
&+i\frac{4P_1^2(2P_5-2P_2P_4-2P_1P_3)}{\lambda_k^5}\Big]\\
={}& \frac{1}{16}\Big[i\frac{8P_1^3}{\lambda_k^3}+\frac{(12P_1^4-12P_1^2P_2^2+24P_1^2P_3)}{\lambda_k^4}\\
&+i\frac{(24P_1^2P_5-24P_1^2P_2P_4-48P_1^3P_3-6P_1^5+12P_1^3P_2^2+24P_1P_2^2P_3-24P_1P_3^2-6P_1P_2^4)}{\lambda_k^5}\Big]\\
={}& i\frac{P_1^3}{2\lambda_k^3}+\frac{(\frac{3}{4}P_1^4-\frac{3}{4}P_1^2P_2^2+\frac{3}{2}P_1^2P_3)}{\lambda_k^4}\\
&+i\frac{(\frac{3}{2}P_1^2P_5-\frac{3}{2}P_1^2P_2P_4-3P_1^3P_3-\frac{6}{16}P_1^5+\frac{3}{4}P_1^3P_2^2+\frac{3}{2}P_1P_2^2P_3-\frac{3}{2}P_1P_3^2-\frac{3}{8}P_1P_2^4)}{\lambda_k^5},
\end{aligned}
$$

while the fourth term

$$\begin{aligned}
\frac{-5}{128}\Big[\ \Big]^4 &= \frac{-5}{2^7}\Big[\Big(-i\frac{2P_1}{\lambda_k}-\frac{(P_1^2-P_2^2+2P_3)}{\lambda_k^2}\Big)\cdot\Big(i\frac{8P_1^3}{\lambda_k^3}+\frac{(12P_1^4-12P_1^2P_2^2+24P_1^2P_3)}{\lambda_k^4}\Big)\\
&= \frac{-5}{128}\Big[\frac{16P_1^4}{\lambda_k^4}+i\frac{(-32P_1^5+32P_1^3P_2^2-64P_1^3P_3)}{\lambda_k^5}\Big]\\
&= \frac{-5}{8}\frac{P_1^4}{\lambda_k^4}+i\frac{(\frac{5}{4}P_1^5-\frac{5}{4}P_1^3P_2^2+\frac{5}{4}P_1^3P_3)}{\lambda_k^5}
\end{aligned}$$

and finally the last term gives

$$\begin{aligned}
\frac{7}{256}\Big[\ \Big]^5 &= \frac{7}{2^7}\Big[-i\frac{2P_1}{\lambda_k}\Big]^5\\
&= -i\frac{7}{8}\frac{P_1^5}{\lambda_k^5}.
\end{aligned}$$

Altogether we have that

$$\begin{aligned}
&\Big[1-i\frac{2P_1}{\lambda_k}-\frac{(P_1^2-P_2^2+2P_3)}{\lambda_k^2}-i\frac{(2P_5-2P_2P_4-2P_1P_3)}{\lambda_k^3}\\
&-\frac{(-2P_2P_6+2P_1P_5+P_4^2-P_3^2+2P_7)}{\lambda_k^4}-i\frac{(-2P_3P_5-2P_4P_6-2P_2P_8-2P_1P_7+2P_9)}{\lambda_k^5}\Big]^{\frac{1}{2}}.\\
&=1-i\frac{P_1}{\lambda_k}+\frac{(\frac{1}{2}P_2^2-P_3)}{\lambda_k^2}+i\frac{(-P_5+P_2P_4+\frac{1}{2}P_2^2P_1)}{\lambda_k^3}\\
&+\frac{(P_2P_6-\frac{1}{2}P_4^2-P_7-P_1P_2P_4-\frac{1}{8}P_2^4-\frac{1}{2}P_1^2P_2^2+\frac{1}{2}P_2^2P_3)}{\lambda_k^4}\\
&+i\frac{(P_4P_6+P_2P_8-P_9-P_1^2P_2P_4+\frac{1}{2}P_2^2P_5-\frac{1}{2}P_2^3P_4+P_1P_2^2P_3)}{\lambda_k^5}\\
&+i\frac{(P_2P_3P_4+P_1P_2P_6-\frac{1}{2}P_1P_4^2-\frac{3}{8}P_1P_2^4-\frac{1}{2}P_1^3P_2^2)}{\lambda_k^5}.
\end{aligned}$$

Since the function $\psi(\lambda_k)$ is bounded above we can replace $\frac{\psi(\lambda_k)}{\lambda_k^{10}}$ by $O(\frac{1}{k^{10}})$, first we evaluate

$$\begin{aligned}
&\log\Big(1-i\frac{P_1}{\lambda_k}+\frac{(\frac{1}{2}P_2^2-P_3)}{\lambda_k^2}+i\frac{(-P_5+P_2P_4+\frac{1}{2}P_2^2P_1)}{\lambda_k^3}+\frac{A'}{\lambda_k^4}+i\frac{B'}{\lambda_k^5}\Big)\\
&=\Big[\ \Big]-\frac{1}{2}\Big[\ \Big]^2+\frac{1}{3}\Big[\ \Big]^3-\frac{1}{4}\Big[\ \Big]^4+\frac{1}{5}\Big[\ \Big]^5+O(\frac{1}{k^6})
\end{aligned}$$

where

$$A' = P_2P_6-\frac{1}{2}P_4^2-P_7-P_1P_2P_4-\frac{1}{8}P_2^4-\frac{1}{2}P_1^2P_2^2+\frac{1}{2}P_2^2P_3)$$

and

$$\begin{aligned}
B' =&P_4P_6+P_2P_8-P_9-P_1^2P_2P_4+\frac{1}{2}P_2^2P_5-\frac{1}{2}P_2^3P_4+P_1P_2^2P_3+P_2P_3P_4+P_1P_2P_6-\frac{1}{2}P_1P_4^2\\
&-\frac{3}{8}P_1P_2^4-\frac{1}{2}P_1^3P_2^2.
\end{aligned}$$

Now using the above expansion we get that the second term is given by

$$
\begin{aligned}
-\frac{1}{2}\Big[\;\Big]^2 &= -\frac{1}{2}\Big[-i\frac{P_1}{\lambda_k}+\frac{(\frac{1}{2}P_2^2-P_3)}{\lambda_k^2}+i\frac{(-P_5+P_2P_4+\frac{1}{2}P_2^2P_1)}{\lambda_k^3}+\frac{A'}{\lambda_k^4}+i\frac{B'}{\lambda_k^5}\Big]^2\\
&= -\frac{1}{2}\Big[\Big(-i\frac{P_1}{\lambda_k}\Big)^2+\Big(-i\frac{P_1}{\lambda_k}\Big)\Big(\frac{\frac{1}{2}P_2^2-P_3}{\lambda_k^2}\Big)+\Big(-i\frac{P_1}{\lambda_k}\Big)\Big(\frac{(-P_5+P_2P_4+\frac{1}{2}P_2^2P_1)}{\lambda_k^3}\Big)\\
&+\Big(-i\frac{P_1}{\lambda_k}\Big)\Big(\frac{A'}{\lambda_k^4}\Big)+\Big(-i\frac{P_1}{\lambda_k}\Big)\Big(\frac{\frac{1}{2}P_2^2-P_3}{\lambda_k^2}\Big)+\frac{(\frac{1}{2}P_2^2-P_3)^2}{\lambda_k^4}\\
&+i\frac{(\frac{1}{2}P_2^2-P_3)(P_2P_4-P_5+\frac{1}{2}P_1P_2^2)}{\lambda_k^5}\\
&+\Big(-i\frac{P_1}{\lambda_k}\Big)\Big(\frac{i(-P_5+P_2P_4+\frac{1}{2}P_2^2P_1)}{\lambda_k^3}\Big)+i\frac{(\frac{1}{2}P_2^2-P_3)(P_2P_4-P_5+\frac{1}{2}P_1P_2^2)}{\lambda_k^5}\\
&+-i\frac{A'P_1}{\lambda_k}\Big]\\
&= -\frac{1}{2}\Big[-\frac{P_1^2}{\lambda_k^2}+i\frac{(2P_1P_3-P_1P_2^2)}{\lambda_k^3}+\frac{(2P_1P_2P_4-2P_1P_5+P_1^2P_2^2+\frac{1}{4}P_2^4-P_2^2P_3+P_3^2)}{\lambda_k^4}\\
&+i\frac{(P_2^3P_4-P_2^2P_5+\frac{1}{2}P_1P_2^4-2P_2P_3P_4+2P_3P_5-P_1P_2^2P_3-2A'P_1)}{\lambda_k^5}\Big]\\
&= \frac{P_1^2}{2\lambda_k^2}+i\frac{(\frac{1}{2}P_1P_2^2-P_1P_3)}{\lambda_k^3}+\frac{(-P_1P_2P_4+P_1P_5-\frac{1}{2}P_1^2P_2^2-\frac{1}{8}P_2^4+\frac{1}{2}P_2^2P_3-\frac{1}{2}P_3^2)}{\lambda_k^4}\\
&+i\frac{(-\frac{1}{2}P_2^3P_4+\frac{1}{2}P_2^2P_5-\frac{1}{4}P_1P_2^4+P_2P_3P_4-P_3P_5+\frac{1}{2}P_1P_2^2P_3+A'P_1)}{\lambda_k^5}\Big]
\end{aligned}
$$

We follow the certain procedure and evaluate the remaining terms

$$
\frac{1}{3}\Big[\;\Big]^3 = i\frac{\frac{1}{3}P_1^3}{\lambda_k^3}+\frac{(P_1^2P_3-\frac{1}{2}P_1^2P_2^2)}{\lambda_k^4}+i\frac{(P_1^2P_5-P_1^2P_2P_4-\frac{1}{2}P_1^3P_2^2+P_1P_2^2P_3-P_1P_3^2-\frac{1}{4}P_1P_2^4)}{\lambda_k^5},
$$

$$
\begin{aligned}
-\frac{1}{4}\Big[\;\Big]^4 &= \Big[i\frac{P_1}{\lambda_k}+\frac{(\frac{1}{2}P_2^2-P_3)}{\lambda_k^2}+i\frac{(-P_5+P_2P_4+\frac{1}{2}P_2^2P_1)}{\lambda_k^3}+O(\frac{1}{k^4})\Big]^4\\
&= -\frac{\frac{1}{4}P_1^4}{\lambda_k^4}+i\frac{(-\frac{1}{2}P_1^3P_2^2+P_1^3P_3)}{\lambda_k^5},
\end{aligned}
$$

and lastly

$$
\frac{1}{5}\Big[\;\Big]^5 = -i\frac{\frac{1}{5}P_1^5}{\lambda_k^5}.
$$

Altogether we have that

$$\begin{aligned}
&\log\Big(1-i\frac{P_1}{\lambda_k}+\frac{(\frac{1}{2}P_2^2-P_3)}{\lambda_k^2}+i\frac{(-P_5+P_2P_4+\frac{1}{2}P_2^2P_1)}{\lambda_k^3}+\frac{A'}{\lambda_k^4}+i\frac{B'}{\lambda_k^5}\Big)\\
&=-i\frac{P_1}{\lambda_k}+\frac{(\frac{1}{2}P_1^2+\frac{1}{2}P_2^2-P_3)}{\lambda_k^2}+i\frac{(-P_5+P_2P_4+P_2^2P_1-P_1P_3+\frac{1}{3}P_1^3)}{\lambda_k^3}\\
&+\frac{(P_2P_6-\frac{1}{2}P_4^2-P_7-2P_1P_2P_4-\frac{1}{4}P_2^4-\frac{3}{2}P_1^2P_2^2+P_2^2P_3+P_1P_5-\frac{1}{2}P_3^2+P_1^2P_3-\frac{1}{4}P_1^4)}{\lambda_k^4}\\
&+\frac{(P_4P_6+P_2P_8-P_9-3P_1^2P_2P_4-P_2^3P_4+3P_1P_2^2P_3+2P_2P_3P_4+2P_1P_2P_6-P_1P_4^2)}{\lambda_k^5}\\
&+i\frac{(-P_1P_2^4-2P_1^3P_2^2-P_1P_7-P_3P_5+P_1^2P_5-P_1P_3^2+P_1^3P_3+\frac{1}{5}P_1^5)}{\lambda_k^5}.
\end{aligned}$$

We repeat to process again

$$\begin{aligned}
&\log\Big(1+i\frac{(P_2-P_1)}{\lambda_k}-\frac{(P_3+P_4)}{\lambda_k^2}+i\frac{(P_6-P_5)}{\lambda_k^3}-\frac{(P_7+P_8)}{\lambda_k^4}+i\frac{(P_{10}-P_9)}{\lambda_k^5}\Big)\\
&=\Big[\ \Big]-\frac{1}{2}\Big[\ \Big]^2+\frac{1}{3}\Big[\ \Big]^3-\frac{1}{4}\Big[\ \Big]^4+\frac{1}{5}\Big[\ \Big]^5+O(\frac{1}{k^6}),
\end{aligned}$$

we compute the second term

$$\begin{aligned}
-\frac{1}{2}\Big[\ \Big]^2&=-\frac{1}{2}\Big[i\frac{(P_2-P_1)}{\lambda_k}-\frac{(P_3+P_4)}{\lambda_k^2}+i\frac{(P_6-P_5)}{\lambda_k^3}-\frac{(P_7+P_8)}{\lambda_k^4}+i\frac{(P_{10}-P_9)}{\lambda_k^5}\Big]^2\\
&=\frac{(\frac{1}{2}P_2^2-P_1P_2+\frac{1}{2}P_1^2)}{\lambda_k^2}+i\frac{(P_2P_3+P_2P_4-P_1P_3-P_1P_4)}{\lambda_k^3}\\
&+\frac{(P_2P_6-P_2P_5-P_1P_6+P_1P_5-\frac{1}{2}P_3^2-P_3P_4-\frac{1}{2}P_4^2)}{\lambda_k^4}\\
&+i\frac{(P_2P_7+P_2P_8-P_1P_7-P_1P_8+P_3P_6-P_3P_5+P_4P_6-P_4P_5)}{\lambda_k^5}.
\end{aligned}$$

Again using the expansion the third term is given by

$$\begin{aligned}
&\frac{1}{3}\Big[\ \Big]^3=\frac{1}{3}\Big[i\frac{(P_2-P_1)}{\lambda_k}-\frac{(P_3+P_4)}{\lambda_k^2}+i\frac{(P_6-P_5)}{\lambda_k^3}-\frac{(P_7+P_8)}{\lambda_k^4}+i\frac{(P_{10}-P_9)}{\lambda_k^5}\Big]^3\\
&=i\frac{(-\frac{1}{3}P_2^3+P_2^2P_1-P_2P_1^2+\frac{1}{3}P_1^3)}{\lambda_k^3}+\frac{(P_2^2P_3-2P_1P_2P_3+P_1^2P_3+P_2^2P_4-2P_1P_2P_4+P_1^2P_4)}{\lambda_k^4}\\
&+i\frac{(-P_2^2P_6+P_2^2P_5+2P_1P_2P_6-2P_1P_2P_5+P_2P_3^2+2P_2P_3P_4+P_2P_4^2-P_1^2P_6)}{\lambda_k^5}\\
&+i\frac{(P_1^2P_5-P_1P_3^2-2P_1P_3P_4-P_1P_4^2)}{\lambda_k^5},
\end{aligned}$$

while the fourth term is given by

$$
\begin{aligned}
-\frac{1}{4}\Big[\ \Big]^4 &= -\frac{1}{4}\Big[i\frac{(P_2-P_1)}{\lambda_k}-\frac{(P_3+P_4)}{\lambda_k^2}+i\frac{(P_6-P_5)}{\lambda_k^3}-\frac{(P_7+P_8)}{\lambda_k^4}+i\frac{(P_{10}-P_9)}{\lambda_k^5}\Big]^4\\
&= \frac{(-\frac{1}{4}P_2^4+P_2^3P_1-\frac{3}{2}P_2^2P_1^2+P_2P_1^3-\frac{1}{4}P_1^4)}{\lambda_k^4}\\
&+i\frac{(3P_1P_2^2P_3+3P_1P_2^2P_4-3P_1^2P_2P_3-3P_1^2P_2P_4-P_2^3P_3-P_2^3P_4+P_1^3P_3+P_1^3P_4)}{\lambda_k^5}
\end{aligned}
$$

and finally

$$
\begin{aligned}
\frac{1}{5}\Big[\ \Big]^5 &= \frac{1}{5}\Big[i\frac{(P_2-P_1)}{\lambda_k}-\frac{(P_3+P_4)}{\lambda_k^2}+i\frac{(P_6-P_5)}{\lambda_k^3}-\frac{(P_7+P_8)}{\lambda_k^4}+i\frac{(P_{10}-P_9)}{\lambda_k^5}\Big]^5\\
&= i\frac{(\frac{1}{5}P_2^5-P_2^4P_1+2P_2^3P_1^2-2P_2^2P_1^3+P_2P_1^4-\frac{1}{5}P_1^5)}{\lambda_k^5}.
\end{aligned}
$$

We then substitute everything we get that

$$
\begin{aligned}
ai\Delta_k &= -i\frac{P_2}{\lambda_k}+\frac{(P_4+P_1P_2)}{\lambda_k^2}+i\frac{(-P_6-P_2P_3+P_1P_4+\frac{1}{3}P_2^2+P_2P_1^2)}{\lambda_k^3}\\
&+\frac{(P_8+P_2P_5+P_1P_6+P_3P_4+2P_1P_2P_3-P_2^2P_4-P_1^2P_4-P_2^3P_1-P_1^3P_2)}{\lambda_k^4}\\
&+i\frac{(-P_{10}-P_2P_7+P_1P_8-P_3P_6+P_4P_5+P_2^2P_6+2P_1P_2P_5-P_2P_3^2)}{\lambda_k^5}\\
&+i\frac{(-P_2P_4^2+P_1^2P_6+2P_1P_3P_4-3P_1P_2^2P_4+3P_1^2P_2P_3+P_2^3P_3-P_1^3P_4-2P_1^2P_2^3-P_1^4P_2-\frac{1}{5}P_2^5)}{\lambda_k^5}\\
&= -i\frac{P_2}{\lambda_k}+\frac{(P_4+P_1P_2)}{\lambda_k^2}+i\frac{A''}{\lambda_k^3}+\frac{B''}{\lambda_k^4}+i\frac{C''}{\lambda_k^5}+O(\frac{1}{k^6}),
\end{aligned}
$$

where A'', B'' and C'' correspond to higher order terms We also know from Theorem 4.8 that

$$
\lambda_k=\frac{\pi}{a}(k-\frac{1}{2})+\frac{Q_1}{(k-\frac{1}{2})}+i\frac{Q_2}{(k-\frac{1}{2})^2}+\frac{Q_3}{(k-\frac{1}{2})^3}+\frac{\beta_k}{k^3},
$$

then

$$
\begin{aligned}
\frac{1}{\lambda_k} &= \frac{1}{\left(\frac{\pi}{a}(k-\frac{1}{2}) + \frac{Q_1}{(k-\frac{1}{2})} + i\frac{Q_2}{(k-\frac{1}{2})^2} + \frac{Q_3}{(k-\frac{1}{2})^3} + \frac{\beta_k}{k^3}\right)} \\
&= \frac{1}{\frac{\pi}{a}(k-\frac{1}{2})\left[1 + \left(\frac{Q_1}{(k-\frac{1}{2})} + i\frac{Q_2}{(k-\frac{1}{2})^2} + \frac{Q_3}{(k-\frac{1}{2})^3} + \frac{\beta_k}{k^3}\right)\left(\frac{\pi}{a}(k-\frac{1}{2})\right)^{-1}\right]} \\
&= \left(\frac{\pi}{a}(k-\frac{1}{2})\right)^{-1}\left[1 - \left(\frac{Q_1}{(k-\frac{1}{2})} + i\frac{Q_2}{(k-\frac{1}{2})^2} + \frac{Q_3}{(k-\frac{1}{2})^3} + \frac{\beta_k}{k^3}\right)\left(\frac{\pi}{a}(k-\frac{1}{2})\right)^{-1}\right. \\
&\quad \left. + \left(\frac{Q_1}{(k-\frac{1}{2})} + i\frac{Q_2}{(k-\frac{1}{2})^2} + \frac{Q_3}{(k-\frac{1}{2})^3}\right)^2\left(\frac{\pi}{a}(k-\frac{1}{2})\right)^2 + O(\frac{1}{k^5})\right] \\
&= \frac{a}{\pi(k-\frac{1}{2})} - \frac{a^2 Q_1}{\pi^2(k-\frac{1}{2})^3} - i\frac{a^2 Q_2}{\pi^2(k-\frac{1}{2})^4} - \frac{a^2 Q_3}{\pi^2(k-\frac{1}{2})^5} \\
&\quad + \left(\frac{Q_1}{(k-\frac{1}{2})} + i\frac{Q_2}{(k-\frac{1}{2})^2} + \frac{Q_3}{(k-\frac{1}{2})^3}\right)^2\left(\frac{\pi}{a}(k-\frac{1}{2})\right)^2 + \frac{a^2\beta_k}{\pi^2 k^5(1-\frac{1}{2k})^2} + O(\frac{1}{k^6}) \\
&= \frac{a}{\pi(k-\frac{1}{2})} - \frac{a^2 Q_1}{\pi^2(k-\frac{1}{2})^3} - i\frac{a^2 Q_2}{\pi^2(k-\frac{1}{2})^4} - \frac{a^2 Q_3}{\pi^2(k-\frac{1}{2})^5} + \frac{Q_1^2 a^3}{\pi^3(k-\frac{1}{2})^5} \\
&\quad + \frac{a^2\beta_k}{\pi^2 k^5}\left(1 + O(\frac{1}{k})\right) + O(\frac{1}{k^5}) \\
&= \frac{a}{\pi(k-\frac{1}{2})} - \frac{a^2 Q_1}{\pi^2(k-\frac{1}{2})^3} - i\frac{a^2 Q_2}{\pi^2(k-\frac{1}{2})^4} - \frac{a^2 Q_3}{\pi^2(k-\frac{1}{2})^5} + \frac{Q_1^2 a^3}{\pi^3(k-\frac{1}{2})^5} + \frac{\kappa_k}{k^5}.
\end{aligned}
$$

Now let $\kappa_k = a^2\frac{\beta_k}{\pi^2} + O(\frac{1}{k})$, which is square summable. Then

$$
\lambda_k^{-1} = \frac{a}{\pi(k-\frac{1}{2})} - \frac{a^2 Q_1}{\pi^2(k-\frac{1}{2})^3} - i\frac{a^2 Q_2}{\pi^2(k-\frac{1}{2})^4} - \frac{a^2 Q_3}{\pi^2(k-\frac{1}{2})^5} + \frac{Q_1^2 a^3}{\pi^3(k-\frac{1}{2})^5} + \frac{\kappa_k}{k^5},
$$

from which we deduce that

$$
\lambda_k^{-2} = \frac{a^2}{\pi^2(k-\frac{1}{2})^2} - \frac{2a^3 Q_1}{\pi^3(k-\frac{1}{2})^4} - i\frac{2a^3 Q_2}{\pi^3(k-\frac{1}{2})^5} + O(\frac{1}{k^6}),
$$

$$
\lambda_k^{-3} = \frac{a^3}{\pi^3(k-\frac{1}{2})^3} - \frac{3a^4 Q_1}{\pi^4(k-\frac{1}{2})^5} + O(\frac{1}{k^7})
$$

$$
\lambda_k^{-4} = \frac{a^4}{\pi^4(k-\frac{1}{2})^4} + O(\frac{1}{k^6})
$$

$$
\lambda_k^{-5} = \frac{a^5}{\pi^5(k-\frac{1}{2})^5} + O(\frac{1}{k^6}),
$$

then substituting all the above we get that

$$
\begin{aligned}
ai\Delta_k &= -iP_2\Big(\frac{a}{\pi(k-\frac{1}{2})} - \frac{a^2Q_1}{\pi^2(k-\frac{1}{2})^3} - i\frac{a^2Q_2}{\pi^2(k-\frac{1}{2})^4} - \frac{a^2Q_3}{\pi^2(k-\frac{1}{2})^5} + \frac{Q_1^2a^3}{\pi^3(k-\frac{1}{2})^5} + \frac{\kappa_k}{k^5}\Big)\\
&+ (P_4+P_1P_2)\Big(\frac{a^2}{\pi^2(k-\frac{1}{2})^2} - \frac{2a^3Q_1}{\pi^3(k-\frac{1}{2})^4} - i\frac{2a^3Q_2}{\pi^3(k-\frac{1}{2})^5} + O(\frac{1}{k^6})\Big)\\
&+ iA''\Big(\frac{a^3}{\pi^3(k-\frac{1}{2})^3} - \frac{3a^4Q_1}{\pi^4(k-\frac{1}{2})^5} + O(\frac{1}{k^7})\Big) + B''\Big(\frac{a^4}{\pi^4(k-\frac{1}{2})^4} + O(\frac{1}{k^6})\Big)\\
&+ iC\Big(\frac{a^5}{\pi^5(k-\frac{1}{2})^5} + O(\frac{1}{k^6})\Big)\\
&= \frac{i}{(k-\frac{1}{2})}\Big[-\frac{aP_2}{\pi}\Big] + \frac{1}{(k-\frac{1}{2})^2}\Big[\frac{(P_4+P_1P_2)a^2}{\pi^2}\Big] + \frac{i}{(k-\frac{1}{2})^3}\Big[\frac{a^2P_2Q_1}{\pi^2} + \frac{A''a^3}{\pi^3}\Big]\\
&+ \frac{1}{(k-\frac{1}{2})^4}\Big[-\frac{P_2a^2Q_2}{\pi^2} - \frac{(P_4+P_1P_2)2a^3Q_1}{\pi^3} + \frac{B''a^4}{\pi^4}\Big]\\
&+ \frac{i}{(k-\frac{1}{2})^5}\Big[-\frac{P_2Q_1^2a^3}{\pi^3} + \frac{a^2P_2Q_3}{\pi^2} - \frac{2a^3Q_2(P_4+P_1P_2)}{\pi^3} - \frac{3a^4Q_1A''}{\pi^4} + \frac{C''a^5}{\pi^5}\Big] + \frac{\kappa_k}{k^5}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\Delta_k &= \frac{1}{(k-\frac{1}{2})}\Big[-\frac{P_2}{\pi}\Big] - i\frac{1}{(k-\frac{1}{2})^2}\Big[\frac{(P_4+P_1P_2)a}{\pi^2}\Big] + \frac{1}{(k-\frac{1}{2})^3}\Big[\frac{aP_2Q_1}{\pi^2} + \frac{A''a^2}{\pi^3}\Big]\\
&+ \frac{i}{(k-\frac{1}{2})^4}\Big[\frac{aP_2Q_2}{\pi^2} + \frac{(P_4+P_1P_2)2a^2Q_1}{\pi^3} - \frac{B''a^3}{\pi^4}\Big]\\
&+ \frac{1}{(k-\frac{1}{2})^5}\Big[-\frac{P_2^3a^2}{\pi^5} + \frac{aP_2Q_3}{\pi^2} - \frac{2a^2Q_2(P_4+P_1P_2)}{\pi^3} - \frac{3a^3Q_1A''}{\pi^4} + \frac{C''a^4}{\pi^5}\Big] - i\frac{\kappa_k}{ak^5},
\end{aligned}
$$

hence results. □

We finally conclude this section by considering necessary conditions that will enable the function Ψ to be an entire function, where the function Ψ is defined by

$$
\begin{aligned}
\Psi(\lambda) = &-K_1\cos\lambda a + iK_2\frac{\cos\lambda a}{\lambda} + K_3\frac{\sin\lambda a}{\lambda} + K_4\frac{\cos\lambda a}{\lambda^2} + iK_5\frac{\sin\lambda a}{\lambda^2} + iK_6\frac{\cos\lambda a}{\lambda^3}\\
&+ K_7\frac{\sin\lambda a}{\lambda^3} + K_8\frac{\cos\lambda a}{\lambda^4} + +iK_9\frac{\sin\lambda a}{\lambda^4} + iK_{10}\frac{\cos\lambda a}{\lambda^5} + K_{11}\frac{\sin\lambda a}{\lambda^5}\\
&+ \frac{\chi(\lambda)}{\lambda^5},
\end{aligned}
$$

and $\chi(\lambda)$ is a function that belongs to $L_2(-\infty,\infty)$. This means that the principal part of the Laurent series about the point $\lambda = 0$ must vanish, and in order for the principal part of the Laurent series to vanish it is necessary that the following coefficients a_i, $i = -5,-4,\ldots,-1$ of the Laurent series about the point $\lambda = 0$ be equal to zero. Therefore eliminating the coefficient a_{-5} we see that it leads to

$$-\chi(0) = iK_{10} \tag{4.10}$$

and eliminating the coefficient a_{-4} we get

$$-\chi'(0) = aK_{11} + K_8, \tag{4.11}$$

while eliminating the coefficient a_{-3} gives that

$$\frac{-\chi''(0)}{2} = i(-\frac{a^2}{2}K_{10} + aK_9 + K_6) \tag{4.12}$$

and $a_{-2} = 0$ implies

$$-\frac{\chi'''(0)}{3!} = -\frac{a^3}{3!}K_{11} - \frac{a^2}{2}K_8 + aK_7 + K_4 \tag{4.13}$$

and finally $a_{-1} = 0$ yields

$$-\frac{\chi^{(iv)}(0)}{4!} = i\Big(\frac{a^4}{4!}K_{10} - \frac{a^3}{3!}K_9 - \frac{a^2}{2}K_6 + aK_5 + K_2\Big). \tag{4.14}$$

Now if we write the function χ in the form

$$\chi(\lambda) = \int_0^a \zeta(t)\sin\lambda t\,dt + i\int_0^a \eta(t)\cos\lambda t\,dt,$$

then we must determine the necessary conditions that ζ and η must satisfy given the above initial conditions on χ. Beginning with ζ, the function must satisfy the following conditions

$$-\chi'(0) = -\int_0^a \zeta(t)t\,dt = A_1$$

and

$$-\frac{\chi'''(0)}{3!} = -\int_0^a \zeta(t)t^3\,dt = A_2.$$

We choose $\zeta(t) = At + B$ and then substitute back into the above to get

$$-A_1 = \int_0^a (At^2 + Bt)\,dt = \frac{1}{3}Aa^3 + \frac{1}{2}Ba^2 \tag{4.15}$$

Again setting $-\frac{\chi'''(0)}{3!} = A_2$ and then substituting ζ we get

$$-3!A_2 = \int_0^a (At^4 + Bt^3)\,dt = \frac{1}{5}Aa^5 + \frac{1}{4}Ba^4 \tag{4.16}$$

By solving the simultaneous equation for A and B we get an expression for $\zeta(t)$. Similarly, setting $-\chi(0) = B_1$, $-\frac{\chi''(0)}{2} = iB_2$ and $-\frac{\chi^{(iv)}(0)}{4!} = iB_3$ suggest choosing $\eta(t) = Ct^2 + Dt + E$ and then proceeding as before to solve the simultaneous equation gives an explicit expression for $\eta(t)$.

5. The Direct Problem

In this section we are going to investigate the behaviour of eigenvalues associated with the differential equation and initial conditions. The boundary value problem considered in this thesis is as follows:

Given the density

$$B(s) = \begin{cases} b(s) & \text{if } s \in [0, l_1] \\ 0 & \text{if } s \in (l_1, l], \end{cases}$$

we consider the problem

$$\frac{\partial^2 w}{\partial s^2} - B(s)\frac{\partial^2 w}{\partial t^2} = 0, \tag{5.1}$$

$$w(0,t) = 0, \tag{5.2}$$

$$\Big(\frac{\partial w}{\partial s} + \nu\frac{\partial w}{\partial t} + M\frac{\partial^2 w}{\partial t^2}\Big)\Big|_{s=l} = 0 \tag{5.3}$$

where $b(s) \geq \varepsilon > 0$, $b(s) \in W_2^2(0,l_1)$. The problem describes small transversal vibrations of a string at the point s. The term $w(s,t)$ represent the transversal displacement of the string of length l with a thread on the interval (l_1,l) hence the density there is set to zero. The coefficient of damping is $\nu > 0$ while the mass of the concentrated ring at the right endpoint is M and is free to move while the left end is fixed. In what follows we are going to show that the partial differential equation can be reduced into a Sturm Liouville equation, and then deduce different asymptotic expansions for eigenvalues of strings with density $b(s)$ belonging to different Sobolev spaces.

Proposition 5.1. *Problem (5.1)-(5.3) can be transformed into an ordinary differential equation*

$$u_{ss}(\lambda,s) + b(s)\lambda^2 u(\lambda,s) = 0, \quad s \in [0,l_1] \tag{5.4}$$

satisfying boundary conditions

$$u(\lambda,0) = 0 \tag{5.5}$$

$$u_s(\lambda,l_1) + i\lambda\nu[u_s(\lambda,l_1)(l-l_1) + u(\lambda,l_1)] - M\lambda^2[u_s(\lambda,l_1)(l-l_1) + u(\lambda,l_1)] = 0. \tag{5.6}$$

Proof. We write the solution w of (5.1)-(5.3) using separation of variables as follows

$$w(s,t) = u(\lambda,s)e^{i\lambda t},$$

then substituting w into (5.1) we get

$$u_{ss}(\lambda,s)e^{i\lambda t} + B(s)\lambda^2 e^{i\lambda t}u(\lambda,s) = 0, \quad s \in [0,l]$$

which holds for all values of t. Now multiplying both sides by $e^{-i\lambda t}$ reduces the above equation into

$$u_{ss}(\lambda,s) + \lambda^2 B(s)u = 0, \quad s \in [0,l]$$

and using the piecewise definition of $B(s)$ on the above equation further simplifies it to

$$u_{ss}(\lambda,s) + \lambda^2 b(s)u = 0, \quad s \in (0,l_1). \tag{5.7}$$

Furthermore, substituting w into equation (5.2) gives the following initial condition

$$u(\lambda,0) = 0. \tag{5.8}$$

Smoothness conditions at l_1 gives that

$$u(\lambda,l_1-0) = u(\lambda,l_1+0) = u(\lambda,l_1) \tag{5.9}$$

$$u_s(\lambda,l_1-0) = u_s(\lambda,l_1+0) = u_s(\lambda,l_1). \tag{5.10}$$

We now revisit equation (5.1) where our density B is restricted to the interval $(l_1,l]$ which gives that $u_{ss}(\lambda,s) = 0$ and implies that for $s \in [l_1,l]$ we have the following

$$u(\lambda,s) = ds + b, \quad s \in [l_1,l] \tag{5.11}$$

where b and d are integration constants. Now using (5.9) on (5.11) we obtain that

$$u(\lambda,l_1) = dl_1 + b$$

and (5.10) also gives that $u_s(\lambda,l_1) = d$, so then solving for b we get that

$$b = u(\lambda,l_1) - u_s(\lambda,l_1)l_1.$$

Substituting b and d into (5.11) gives that the function u is given by

$$u(\lambda,s) = u_s(\lambda,l_1)(s-l_1) + u(\lambda,l_1), \quad s \in [l_1,l]. \tag{5.12}$$

Lastly, substituting w into (5.3) we obtain

$$u_s(\lambda,l)+i\lambda\nu u(\lambda,l)-M\lambda^2u(\lambda,l)=0 \tag{5.13}$$

while using u from (5.12) into equation (5.13) we deduce that

$$u_s(\lambda,l_1)+i\lambda\nu[u_s(\lambda,l_1)(l-l_1)+u(\lambda,l_1)]-M\lambda^2[u_s(\lambda,l_1)(l-l_1)+u(\lambda,l_1)]=0$$

which proves our proposition. $\square$

So far we have shown that our problem can be reduced to a differential equation (5.4) with initial condition (5.5) and (5.6) by seeking its solution in the form of separation of variables. We also note that the above mentioned second order differential equation has a weight factor $b(s)$ in front of the eigenparameter λ. We will show that the above second order differential equation can be further simplified using the Sturm-Liouville transformation. For $b(s)\in W_2^2(0,l_1)$ and $b(s)\geq\varepsilon>0$, let us define the Sturm Liouville transformation as follows

$$x(s)=\int_0^s b^{\frac{1}{2}}(s')ds' \tag{5.14}$$

which transforms the independent variable and maps the interval $[0,l_1]$ in the variable s into the the interval $[0,a]$ in the variable x where a is defined as

$$a=\int_0^{l_1} b^{\frac{1}{2}}(s')ds'. \tag{5.15}$$

The dependent variable is transformed by the Sturm Liouville transformation using the equation

$$y(\lambda,x)=b^{\frac{1}{4}}(s(x))u(\lambda,s(x)). \tag{5.16}$$

Proposition 5.2. *The differential equation (5.4) satisfying (5.5) and (5.6) can be transformed into the following differential equation*

$$-y''(\lambda,x)+q(x)y(\lambda,x)=\lambda^2y(\lambda,x), \tag{5.17}$$

where $q(x)$ is a real valued potential in $L_2(0,a)$ and the function y satisfies

$$y(\lambda,0)=0 \tag{5.18}$$

and

$$(1+i\lambda M_1^{-1}-\lambda^2M_2^{-1})y_x(\lambda,a)+\Big[M_4+i\lambda M_1^{-1}M_3-\lambda^2M_2^{-1}M_3\Big]y(\lambda,a)=0 \tag{5.19}$$

where the constants M_j for $j=1,2,3,4$ are real valued constants given by

$$M_1=\nu^{-1}(l-l_1^{-1}) \tag{5.20}$$

and

$$M_2=M^{-1}(l-l_1)^{-1} \tag{5.21}$$

$$M_3=-\frac{1}{4}b^{-1}(l_1)\frac{d}{dx}(b(s(x)))\Big|_{x=a}+(l-l_1)^{-1}b^{-\frac{1}{2}}(l_1) \tag{5.22}$$

$$M_4=-\frac{1}{4}b^{-1}(l_1)\frac{d}{dx}(b(s(x)))\Big|_{x=a}. \tag{5.23}$$

Proof. Differentiating equation (5.16) with respect to x we obtain

$$y'(\lambda,x) = \frac{d}{dx} b^{\frac{1}{4}}(s(x))u(\lambda,s) + b^{\frac{1}{4}}(s)u_s(\lambda,s)\frac{ds}{dx}.$$

From the definition of $x(s)$ it is obvious that $\frac{ds}{dx} = b^{-\frac{1}{2}}(s)$, so then substituting further we get

$$\begin{aligned} y'(\lambda,x) &= \frac{d}{dx} b^{\frac{1}{4}}(s(x))u(\lambda,s(x)) + b^{\frac{1}{4}}(s(x))b^{-\frac{1}{2}}(s(x))u_s(\lambda,s(x)) \\ &= \frac{d}{dx} b^{\frac{1}{4}}(s(x))u(\lambda,s(x)) + b^{-\frac{1}{4}}(s(x)) \cdot u_s(\lambda,s(x)). \end{aligned} \tag{5.24}$$

Since $b(s(x)) > 0$ we can multiply both sides by $b^{\frac{1}{4}}(s(x))$ to get

$$b^{\frac{1}{4}}(s(x))y'(\lambda,x) = \frac{d}{dx} b^{\frac{1}{4}}(s(x))y(\lambda,x) + u_s(\lambda,s).$$

Now differentiating once more with respect to x

$$\begin{aligned} \frac{d}{dx} b^{\frac{1}{4}}(s(x)).y'(\lambda,x) + b^{\frac{1}{4}}(s(x))y''(\lambda,x) = \frac{d^2}{dx^2} b^{\frac{1}{4}}(s(x))y(\lambda,x) + \frac{d}{dx} b^{\frac{1}{4}}(s(x))y'(\lambda,x) \\ + u_{ss}(\lambda,s)b^{-\frac{1}{2}}(s), \end{aligned}$$

and simplifies to

$$b^{\frac{1}{4}}(s(x))y''(\lambda,x) = \frac{d^2}{dx^2} b^{\frac{1}{4}}(s(x))y(\lambda,x) + u_{ss}(\lambda,s)b^{-\frac{1}{2}}(s).$$

But $u_{ss}(\lambda,s) = -\lambda^2 b(s)u(\lambda,s)$, then substituting into the above we obtain that

$$b^{\frac{1}{4}}(s(x))y''(\lambda,x) = \frac{d^2}{dx^2} b^{\frac{1}{4}}(s(x))y(\lambda,x) - b(s(x))\lambda^2 u(\lambda,s)b^{-\frac{1}{2}}(s(x)),$$

therefore

$$\begin{aligned} y''(\lambda,x) &= b^{-\frac{1}{4}}(s(x))\frac{d^2}{dx^2} b^{\frac{1}{4}}(s(x))y(\lambda,x) - \lambda^2 b^{\frac{1}{4}}(s(x)) \cdot u(\lambda,s) \\ &= b^{-\frac{1}{4}}(s(x))\frac{d^2}{dx^2} b^{\frac{1}{4}}(s(x))y(\lambda,x) - \lambda^2 y(\lambda,x). \end{aligned}$$

Let us define the function $q(x)$ as follows

$$q(x) = b^{\frac{1}{4}}(s(x))\frac{d^2}{dx^2} b^{\frac{1}{4}}(s(x)), \tag{5.25}$$

then since $b \in W_2^2(0,l_1)$ we have by definition that $\frac{d^2 b(s)}{dx^2} \in L_2(0,l_1)$. Furthermore, since b is also differentiable then it is continuous on $[0,l_1]$ and therefore attains attains max and min, so we have that $b \in L_\infty(0,l_1)$. Now $\frac{d}{dx}(b(s(x)))^{\frac{1}{4}} = \frac{1}{4}(b(s(x)))^{-\frac{3}{4}}.\frac{d}{dx}b(s(x))$ also belongs to $L_2(0,a)$ by Hölder's theorem on L_∞ and L_2-functions. Differentiating once more we get that

$$\frac{d^2}{dx^2}(b(s(x)))^{\frac{1}{4}} = -\frac{3}{16}(b(s(x)))^{\frac{-7}{4}} \cdot \frac{d}{dx}(b(s(x))) + \frac{1}{4}(b(s(x)))\frac{d^2}{dx^2}(b(s(x)))$$

which is real valued and belongs in $L_2(0,l_1)$ by repeated application of Hölder's theorem and the fact that the sum of two square integrable functions is another square integrable function. Altogether we have that $q(x) \in L_2(0,a)$, and using the potential $q(x)$ defined as in equation (5.25) it is clear that

$$-y''(\lambda,x) + q(x)y(\lambda,x) = \lambda^2 y(\lambda,x),$$

which establishes equation (5.17). The condition $u(\lambda,0)=0$ together with equation (5.16) gives that

$$y(\lambda,0)=0,$$

which is equation (5.18). We also know that

$$[1+i\lambda\nu(l-l_1)-\lambda^2M(l-l_1)]u_s(\lambda,l_1)=(\lambda^2M-i\lambda\nu)u(\lambda,l_1),$$

therefore

$$(1+i\lambda M_1^{-1}-\lambda^2M_2^{-1})u_s(\lambda,l_1)=(\lambda^2M-i\lambda\nu)u(\lambda,l_1), \tag{5.26}$$

where M_1 and M_2 are as defined in equation (5.20) and (5.21) respectively. We now replace $y_x(\lambda,x)$ in the following expression by (5.24)

$$\begin{aligned}(1+i\lambda M_1^{-1}-\lambda^2M_2^{-1})y_x(\lambda,a) =&(1+i\lambda M_1^{-1}-\lambda^2M_2^{-1})\frac{d}{dx}b^{\frac{1}{4}}(s(x))\Big|_{x=a}\cdot u(\lambda,l_1)\\
&+b^{-\frac{1}{4}}(l_1)(1+i\lambda M_1^{-1}-\lambda^2M_2^{-1})\cdot u_s(\lambda,l_1)\\
&=(1+i\lambda M_1^{-1}-\lambda^2M_2^{-1})\frac{d}{dx}b^{\frac{1}{4}}(s(x))\Big|_{x=a}\cdot u(\lambda,l_1)\\
&+b^{-\frac{1}{4}}(l_1)(\lambda^2M-i\lambda\nu)u(\lambda,l_1), \quad \text{and now using (5.26)}\\
&=\frac{1}{4}(1+i\lambda M_1^{-1}-\lambda^2M_2^{-1})b^{-1}(l_1)\frac{d}{dx}(b(s(x))\Big|_{x=a}\\
&\cdot u(\lambda,l_1)b^{\frac{1}{4}}(l_1)+(\lambda^2M-i\lambda\nu)u(\lambda,l_1)b^{\frac{1}{4}}(l_1)b^{-\frac{1}{2}}(l_1)\\
&=\frac{1}{4}(1+i\lambda M_1^{-1}-\lambda^2M_2^{-1})b^{-1}(l_1)\frac{d}{dx}(b(s(x)))\Big|_{x=a}y(\lambda.a)\\
&+(\lambda^2M-i\lambda\nu)y(\lambda,a)b^{-\frac{1}{2}}(l_1).\end{aligned}$$

Hence

$$\begin{aligned}(1+i\lambda M_1^{-1}-\lambda^2M_2^{-1})y_x(\lambda,a)=&\Big[\frac{1}{4}b^{-1}(l_1)\frac{d}{dx}(b(s(x)))\Big|_{x=a}\\
&+i\lambda\Big(\frac{1}{4}M_1^{-1}b^{-1}(l_1)\frac{d}{dx}(b(s(x)))\Big|_{x=a}-\nu b^{-\frac{1}{2}}(l_1)\Big)\\
&+\lambda^2\Big(Mb^{-\frac{1}{2}}(l_1)-\frac{1}{4}M_2^{-1}b^{-1}(l_1)\frac{d}{dx}(b(s(x)))\Big|_{x=a}\Big)\Big]y(\lambda,a),\end{aligned}$$

which with the use of equation (5.22) and (5.23) can be rewritten as

$$(1+i\lambda M_1^{-1}-\lambda^2M_2^{-1})y_x(\lambda,a)+\Big[M_4+i\lambda M_1^{-1}M_3-\lambda^2M_2^{-1}M_3\Big]y(\lambda,a)=0$$

which establishes (8). □

From the definitions of M_1, M_2 it is clear that $M_1>0$, $M_2>0$ and that

$$M_3-M_4=(l-l_1)^{-1}b^{-\frac{1}{2}}(l_1)>0.$$

Denote by $s(\lambda,x)$ the solution of (5.17) satisfying the conditions

$$s(\lambda,0)=0,\ \ s'(\lambda,0)=1 \tag{5.27}$$

which is called a regular solution. As discussed in the Transformation of Operators section (see page 6 of this thesis), the solution to the Sturm Liouville equation can be written in the form

$$s(\lambda,x)=\frac{\sin\lambda x}{\lambda}+\int_0^x K(x,t)\frac{\sin\lambda t}{\lambda}dt \tag{5.28}$$

where $K(x,t) = \tilde{K}(x,t) - \tilde{K}(x,-t)$ and $\tilde{K}(x,t)$ is the solution of the integral equation

$$\tilde{K}(x,t) = \frac{1}{2}\int_0^{\frac{x+t}{2}} q(s)ds + \int_0^{\frac{x+t}{2}}\int_0^{\frac{x-t}{2}} q(s+p)\tilde{K}(s+p,s-p)dpds$$

and $\tilde{K}(x,t) = 0$ for $|t| > |x|$ where evidently $K(x,0) = 0$. Integrating the second term on the right by parts gives

$$s(\lambda,x) = \frac{\sin\lambda x}{\lambda} - K(x,x)\frac{\cos\lambda x}{\lambda^2} + \int_0^x K_t(x,t)\frac{\cos\lambda t}{\lambda^2}dt. \tag{5.29}$$

We differentiate (5.29) with respect to x to obtain

$$s'(\lambda,x) = \cos\lambda x + K(x,x)\frac{\sin\lambda x}{\lambda} + \int_0^x K_x(x,t)\frac{\sin\lambda t}{\lambda}dt, \tag{5.30}$$

then the function defined by

$$\varphi(\lambda) = (1 + i\lambda M_1^{-1} - \lambda^2 M_2^{-1})s'(\lambda,a) + \left[M_4 + i\lambda M_1^{-1}M_3 - \lambda^2 M_2^{-1}M_3\right]s(\lambda,a) \tag{5.31}$$

is an entire functions in the λ-plane since $s(\lambda,a)$ and $s'(\lambda,a)$ are entire functions. Furthermore, we identify equation (5.31) as the characteristic equation since the spectrum of problem (5.17), (5.18) and (5.19) coincide with the set of zeroes of the entire function $\varphi(\lambda)$ defined above.

Proposition 5.3. *If $b \in W_2^2(0,l_1)$ then the characteristic equation (5.31) can be written in the form*

$$\begin{aligned}\varphi(\lambda) = &-\lambda^2 K_1\cos\lambda a + i\lambda K_2\cos\lambda a + \lambda K_3\sin\lambda a + K_4\cos\lambda a + iK_5\sin\lambda a\\ &+ iK_6\frac{\cos\lambda a}{\lambda} + K_7\frac{\sin\lambda a}{\lambda} + K_8\frac{\cos\lambda a}{\lambda^2} + \lambda\int_0^a \zeta_1(t)\sin\lambda t dt\\ &+ \int_0^a \zeta_2(t)\cos\lambda t dt + i\int_0^a \zeta_3(t)\sin\lambda t dt + i\int_0^a \zeta_4(t)\frac{\cos\lambda t}{\lambda}dt\\ &+ \int_0^a \zeta_5(t)\frac{\sin\lambda t}{\lambda}dt + \int_0^a \zeta_6(t)\frac{\cos\lambda t}{\lambda^2}dt.\end{aligned}$$

where

$$\begin{aligned}&K_1 = M_2^{-1} && K_2 = M_1^{-1}\\ &K_3 = -M_2^{-1}K(a,a) - M_2^{-1}M_3 && K_4 = M_2^{-1}M_3K(a,a) + 1\\ &K_5 = M_1^{-1}K(a,a) + M_1^{-1}M_3 && K_6 = -M_1^{-1}M_3K(a,a)\\ &K_7 = K(a,a) + M_4 && K_8 = -M_4K(a,a)\end{aligned}$$

are all real valued constants and

$$\begin{aligned}&\zeta_1(t) = -M_2^{-1}K_x(a,t) && \zeta_2(t) = -M_2^{-1}M_3K_t(a,t)\\ &\zeta_3(t) = M_1^{-1}K_x(a,t) && \zeta_4(t) = M_1^{-1}M_3K_t(a,t)\\ &\zeta_5(t) = K_x(a,t) && \zeta_6(t) = M_4K_t(a,t)\end{aligned}$$

are also real valued functions belonging to $L_2(0,a)$.

Proof. Let us substitute equation (5.29) and equation (5.30) into equation (5.31) to get

$$
\begin{aligned}
\varphi(\lambda) &= (1+i\lambda M_1^{-1} - \lambda^2 M_2^{-1})\Big(\cos\lambda a + K(a,a)\frac{\sin\lambda a}{\lambda} + \int_0^a K_x(a,t)\frac{\sin\lambda t}{\lambda}dt\Big) \\
&+ [M_4 + i\lambda M_1^{-1}M_3 - \lambda^2 M_2^{-1}M_3]\Big(\frac{\sin\lambda a}{\lambda} - K(a,a)\frac{\cos\lambda a}{\lambda^2} + \int_0^a K_t(a,t)\frac{\cos\lambda t}{\lambda^2}dt\Big) \\
&= \cos\lambda a + K(a,a)\frac{\sin\lambda a}{\lambda} + \int_0^a K_x(a,t)\frac{\sin\lambda t}{\lambda}dt + i\lambda M_1^{-1}\cos\lambda a + iM_1^{-1}K(a,a)\sin\lambda a \\
&+ iM_1^{-1}\int_0^a K_x(a,t)\sin\lambda t dt - \lambda^2 M_2^{-1}\cos\lambda a - \lambda M_2^{-1}K(a,a)\sin\lambda a \\
&- M_2^{-1}\lambda\int_0^a K_x(a,t)\sin\lambda t dt + M_4\frac{\sin\lambda a}{\lambda} - M_4K(a,a)\frac{\cos\lambda a}{\lambda^2} \\
&+ M_4\int_0^a K_t(a,t)\frac{\cos\lambda t}{\lambda^2}dt + iM_1^{-1}M_3\sin\lambda a - iM_1^{-1}M_3K(a,a)\frac{\cos\lambda a}{\lambda} \\
&+ iM_1^{-1}M_3\int_0^a K_t(a,t)\frac{\cos\lambda t}{\lambda}dt - M_2^{-1}M_3\lambda\sin\lambda a + M_2^{-1}M_3K(a,a)\cos\lambda a \\
&- M_2^{-1}M_3\int_0^a K_t(a,t)\cos\lambda t dt \\
&= -\lambda^2 M_2^{-1}\cos\lambda a + iM_1^{-1}\lambda\cos\lambda a + \lambda(-M_2^{-1}K(a,a) - M_2^{-1}M_3)\sin\lambda a \\
&+ (M_2^{-1}M_3K(a,a)+1)\cos\lambda a + i(M_1^{-1}K(a,a) + M_1^{-1}M_3)\sin\lambda a - iM_1^{-1}M_3K(a,a)\frac{\cos\lambda a}{\lambda} \\
&+ (K(a,a)+M_4)\frac{\sin\lambda a}{\lambda} - M_4K(a,a)\frac{\cos\lambda a}{\lambda^2} - \lambda\int_0^a M_2^{-1}K_x(a,t)\sin\lambda t dt \\
&- \int_0^a M_2^{-1}M_3K_t(a,t)\cos\lambda t dt + i\int_0^a M_1^{-1}K_x(a,t)\sin\lambda t dt + i\int_0^a M_1^{-1}M_3K_t(a,t)\frac{\cos\lambda t}{\lambda}dt \\
&+ \int_0^a K_x(a,t)\frac{\sin\lambda t}{\lambda}dt + \int_0^a M_4K_t(a,t)\frac{\cos\lambda t}{\lambda^2}dt.
\end{aligned}
$$

Then setting $K_i,\ i=1,2,\ldots,8$ and $\zeta_j,\ j=1,2,\ldots,6$ as in the proposition we obtain the results. □

Proposition 5.4. *If $b\in W_2^4(0,l_1)$ then the characteristic equation (5.31) can be written in the form*

(5.32)

$$
\begin{aligned}
\varphi(\lambda) &= -\lambda^2 N_1\cos\lambda a + i\lambda N_2\cos\lambda a + \lambda N_3\sin\lambda a + N_4\cos\lambda a + iN_5\sin\lambda a + iN_6\frac{\cos\lambda a}{\lambda} \\
&\quad + N_7\frac{\sin\lambda a}{\lambda} + \int_0^a \xi_1(t)\frac{\sin\lambda t}{\lambda}dt + i\int_0^a \xi_2(t)\frac{\cos\lambda t}{\lambda}dt,
\end{aligned}
$$

where

$$N_1 = M_2^{-1} \qquad N_2 = M_1^{-1}$$
$$N_3 = -M_2^{-1}M_3 - M_2^{-1}K(a,a) \qquad N_4 = M_2^{-1}K_x(a,a) + 1 + M_2^{-1}M_3K(a,a)$$
$$N_5 = M_1^{-1}K(a,a) + M_1^{-1}M_3 \qquad N_6 = -M_1^{-1}M_3K(a,a) - M_1^{-1}K_x(a,a)$$
$$N_7 = -M_2^{-1}M_3K_t(a,a) - M_2^{-1}K_{xt}(a,a) + K(a,a) + M_4$$

and

$$\xi_1(t) = K_x(a,t) + M_2^{-1}K_{xtt}(a,t) + M_4K(a,t) + M_2^{-1}M_3K_{tt}(a,t),\ \xi_2(t) = M_1^{-1}M_3K_t(a,t) + M_1^{-1}K_{xt}(a,t)$$

which are also real valued functions belonging to $L_2(0,a)$.

Proof. We begin by substituting equation (5.28) and equation (5.30) into equation (5.31) to get

$$\begin{aligned}\varphi(\lambda) =& \cos\lambda a + K(a,a)\frac{\sin\lambda a}{\lambda} + \int_0^a K_x(a,t)\frac{\sin\lambda t}{\lambda}dt + i\lambda M_1^{-1}\cos\lambda a + iM_1^{-1}K(a,a)\sin\lambda a\\ &+ i\int_0^a M_1^{-1}K_x(a,t)\sin\lambda t dt - \lambda^2 M_2^{-1}\cos\lambda a - \lambda M_2^{-1}K(a,a)\sin\lambda a\\ &- \int_0^a M_2^{-1}K_x(a,t)\lambda\sin\lambda t dt + M_4\frac{\sin\lambda a}{\lambda} + \int_0^a M_4 K(a,t)\frac{\sin\lambda t}{\lambda}dt + iM_1^{-1}M_3\sin\lambda a\\ &+ i\int_0^a M_1^{-1}M_3K(a,t)\sin\lambda t dt - \lambda M_2^{-1}M_3\sin\lambda a - \int_0^a M_2^{-1}M_3K(a,t)\lambda\sin\lambda t dt,\end{aligned}$$

and then evaluate the following integrals

$$i\int_0^a M_1^{-1}K_x(a,t)\sin\lambda t dt = -iM_1^{-1}K(a,a)\frac{\cos\lambda a}{\lambda} + i\int_0^a M_1^{-1}K_x(a,t)\frac{\cos\lambda t}{\lambda}dt,$$

$$\begin{aligned}-\int_0^a M_2^{-1}K_x(a,t)\lambda\sin\lambda t dt =& M_2^{-1}K_x(a,a)\cos\lambda a - M_2^{-1}K_{xt}(a,a)\frac{\sin\lambda a}{\lambda}\\ &+ \int_0^a M_2^{-1}K_{xt}(a,t)\frac{\sin\lambda t}{\lambda}dt,\end{aligned}$$

$$i\int_0^a M_1^{-1}M_3K(a,t)\sin\lambda t = -iM_1^{-1}M_3K(a,a)\frac{\cos\lambda a}{\lambda} + i\int_0^a M_1^{-1}M_3K_t(a,t)\frac{\cos\lambda t}{\lambda}dt$$

and

$$\begin{aligned}-\int_0^a M_2^{-1}M_3K(a,t)\lambda\sin\lambda t dt =& +M_2^{-1}M_3K(a,a)\cos\lambda a - M_2^{-1}M_3K_t(a,a)\frac{\sin\lambda a}{\lambda}\\ &+ \int_0^a M_2^{-1}M_3K_{tt}(a,t)\frac{\sin\lambda t}{\lambda}dt.\end{aligned}$$

A direct substitution of the above integrals and grouping like terms reduces the characteristic equation to

$$\begin{aligned}\varphi(\lambda) =& -\lambda^2 M_2^{-1}\cos\lambda a + iM_1^{-1}\lambda\cos\lambda a + \lambda(-M_2^{-1}M_3 - M_2^{-1}K(a,a))\sin\lambda a\\ &+ (M_2^{-1}K_x(a,a) + 1 + M_2^{-1}M_3K(a,a))\cos\lambda a + i(M_1^{-1}K(a,a) + M_1^{-1}M_3)\sin\lambda a\\ &- i(M_1^{-1}M_3K(a,a) + M_1^{-1}K_x(a,a))\frac{\cos\lambda a}{\lambda}\\ &+ (-M_2^{-1}M_3K_t(a,a) - M_2^{-1}K_{xt}(a,a) + K(a,a) + M_4)\frac{\sin\lambda a}{\lambda}\\ &+ \int_0^a \frac{\sin\lambda t}{\lambda}(K_x(a,t) + M_2^{-1}K_{xtt}(a,t) + M_4K(a,t) + M_2^{-1}M_3K_{tt}(a,t))dt\\ &+ i\int_0^a \frac{\cos\lambda t}{\lambda}(M_1^{-1}M_3K_t(a,t) + M_1^{-1}K_{xt}(a,t))dt.\end{aligned}$$

Then using the above definitions of $N_j, j = 1,2,\ldots,7$ and of the functions ξ_1 and ξ_2 we can see that

$$\varphi(\lambda) = -\lambda^2 N_1 \cos\lambda a + i\lambda N_2 \cos\lambda a + \lambda N_3 \sin\lambda a + N_4 \cos\lambda a + iN_5 \sin\lambda a + iN_6 \frac{\cos\lambda a}{\lambda} + N_7 \frac{\sin\lambda a}{\lambda} + \int_0^a \xi_1(t) \frac{\sin\lambda t}{\lambda} dt + i\int_0^a \xi_2(t) \frac{\cos\lambda t}{\lambda} dt.$$

□

Note that the relationship between the constants $N_j,\ j = 1,2,\ldots,7$ and M_1, M_2, M_3 and M_4 after equation (5.32) does not allow one to express M_1, M_2, M_3 and M_4 in terms of constants $N_j,\ j = 1,2,\ldots,7$ because there are so many unknowns and fewer equations to eliminate unknowns.

Proposition 5.5. *If $b \in W_2^6(0,l_1)$ then the characteristic equation (5.31) can be written as*

(5.33)

$$\varphi(\lambda) = -\lambda^2 K_1 \cos\lambda a + i\lambda K_2 \cos\lambda a + \lambda K_3 \sin\lambda a + K_4 \cos\lambda a + iK_5 \sin\lambda a + iK_6 \frac{\cos\lambda a}{\lambda} + K_7 \frac{\sin\lambda a}{\lambda} + K_8 \frac{\cos\lambda a}{\lambda^2} + iK_9 \frac{\sin\lambda a}{\lambda^2} + iK_{10} \frac{\cos\lambda a}{\lambda^3} + K_{11} \frac{\sin\lambda a}{\lambda^3} + \int_0^a \xi_1(t) \frac{\sin\lambda t}{\lambda^3} dt + i\int_0^a \xi_2(t) \frac{\cos\lambda t}{\lambda^3} dt$$

where

$$\begin{aligned}
K_1 &= M_2^{-1} \\
K_2 &= M_1^{-1} \\
K_3 &= -(K(a,a)M_2^{-1} + M_2^{-1}M_3) \\
K_4 &= 1 + M_2^{-1}K_x(a,a) + M_2^{-1}M_3K(a,a) \\
K_5 &= M_1^{-1}K(a,a) + M_1^{-1}M_3 \\
K_6 &= -(M_1^{-1}K_x(a,a) + M_1^{-1}M_3K(a,a)) \\
K_7 &= K(a,a) - M_2^{-1}K_{xt}(a,a) + M_4 - M_2^{-1}M_3K_t(a,a) \\
K_8 &= -K_x(a,a) - M_2^{-1}K_{xtt}(a,a) - M_4K(a,a) - M_2^{-1}M_3K_{tt}(a,a) \\
K_9 &= M_1^{-1}K_{xt}(a,a) + M_1^{-1}M_3K_t(a,a) \\
K_{10} &= M_1^{-1}K_{xtt}(a,a) + M_1^{-1}M_3K_{tt}(a,a) \\
K_{11} &= K_{xt}(a,a) + M_2^{-1}K_{xttt}(a,a) + M_4K_t(a,a) + M_2^{-1}M_3K_{ttt}(a,a)
\end{aligned}$$

and

$$\begin{aligned}
\xi_1(t) &= -K_{xtt}(a,t) - M_2^{-1}K_{xtttt}(a,t) - M_4K_{tt}(a,t) - M_2^{-1}M_3K_{tttt}(a,t) \\
\xi_2(t) &= M_1^{-1}K_{xttt}(a,t) - M_1^{-1}M_3K_{ttt}(a,t)
\end{aligned}$$

are real valued functions which belong to $L_2(0,a)$.

Proof. We begin by substituting into the following equation

$$\varphi(\lambda) = (1 + i\lambda M_1^{-1} - \lambda^2 M_2^{-1})y_x(\lambda,a) + \Big[M_4 + i\lambda M_1^{-1}M_3 - \lambda^2 M_2^{-1}M_3\Big]y(\lambda,a),$$

to get

$$\begin{aligned}
\varphi(\lambda) &= (1+i\lambda M_1^{-1}-\lambda^2 M_2^{-1})\Big(\cos\lambda a + K(a,a)\frac{\sin\lambda a}{\lambda}+\int_0^a K_x(a,t)\frac{\sin\lambda t}{\lambda}dt\Big)\\
&+\Big[M_4+i\lambda M_1^{-1}M_3-\lambda^2 M_2^{-1}M_3\Big]\Big(\frac{\sin\lambda a}{\lambda}-K(a,a)\frac{\cos\lambda a}{\lambda^2}+\int_0^a K_x(a,t)\frac{\cos\lambda t}{\lambda^2}dt\Big)\\
&=\cos\lambda a+K(a,a)\frac{\sin\lambda a}{\lambda}+\int_0^a K_x(a,t)\frac{\sin\lambda t}{\lambda}dt+i\lambda M_1^{-1}\cos\lambda a+iM_1^{-1}K(a,a)\sin\lambda a\\
&+i\int_0^a M_1^{-1}K_x(a,t)\sin\lambda t dt-\lambda^2 M_2^{-1}\cos\lambda a-\lambda M_2^{-1}K(a,a)\sin\lambda a\\
&-\int_0^a M_2^{-1}K_x(a,t)\lambda\sin\lambda t dt+M_4\frac{\sin\lambda a}{\lambda}-M_4K(a,a)\frac{\cos\lambda a}{\lambda^2}+\int_0^a M_4K_t(a,t)\frac{\cos\lambda t}{\lambda^2}dt\\
&+iM_1^{-1}M_3\sin\lambda a-iM_1^{-1}M_3K(a,a)\frac{\cos\lambda a}{\lambda}+i\int_0^a M_1^{-1}M_3K_t(a,t)\frac{\cos\lambda t}{\lambda}dt-\lambda M_2^{-1}M_3\sin\lambda a\\
&+iM_2^{-1}M_3K(a,a)\cos\lambda a-\int_0^a M_2^{-1}M_3K_t(a,t)\cos\lambda t dt.
\end{aligned}$$

Integrating a few terms of the integrals by parts and grouping like terms we finally obtain that

$$\begin{aligned}
\varphi(\lambda) &= -\lambda^2 M_2^{-1}\cos\lambda a+i\lambda\cos\lambda a-\lambda(K(a,a)M_2^{-1}+M_2^{-1}M_3\sin\lambda a\\
&+(1+M_2^{-1}K_x(a,a)+M_2^{-1}M_3K(a,a))\cos\lambda a+i(M_1^{-1}K(a,a)+M_1^{-1}M_3)\sin\lambda a\\
&-i(M_1^{-1}K_x(a,a)+M_1^{-1}M_3K(a,a))\frac{\cos\lambda a}{\lambda}\\
&+(K(a,a)-M_2^{-1}K_{xt}(a,a)+M_4-M_2^{-1}M_3K_t(a,a))\frac{\sin\lambda a}{\lambda}\\
&+(-K_x(a,a)-M_2^{-1}K_{xtt}(a,a)-M_4K(a,a)-M_2^{-1}M_3K_{tt}(a,a))\frac{\cos\lambda a}{\lambda^2}\\
&+i(M_1^{-1}K_{xt}(a,a)+M_1^{-1}M_3K_t(a,a))\frac{\sin\lambda a}{\lambda^2}+i(M_1^{-1}K_{xtt}(a,a)+M_1^{-1}M_3K_{tt}(a,a))\frac{\cos\lambda a}{\lambda^3}\\
&+(K_{xt}(a,a)+M_2^{-1}K_{xttt}(a,a)+M_4K_t(a,a)+M_2^{-1}M_3K_{ttt}(a,a))\frac{\sin\lambda a}{\lambda^3}\\
&+i\int_0^a(M_1^{-1}K_{xttt}(a,t)-M_1^{-1}M_3K_{ttt}(a,t))\frac{\cos\lambda t}{\lambda^3}dt\\
&+\int_0^a(-K_{xtt}(a,t)-M_2^{-1}K_{xtttt}(a,t)-M_4K_{tt}(a,t)-M_2^{-1}M_3K_{tttt}(a,t))\frac{\sin\lambda t}{\lambda^3}dt
\end{aligned}$$

hence results. □

Theorem 5.6. *If $b\in W_2^4(0,l_1)$, $b(s)>\varepsilon>0$ for $s\in[0,l_1]$, $0<l_1<l<\infty$, then the spectrum $\{\lambda_k : k\in\mathbb{Z}\cup\{0^+\}\}$ of problem (5.17), (5.18) and (5.19) has the following properties*

1. *(1) The sequence $\{\lambda_k : k\in\mathbb{Z}\cup\{0^+\}\}$ is symmetric, with respect to the imaginary axis*
2. *(2) For all $k\in\mathbb{Z}\cup\{0^+\}$ we have that $Im\lambda_k>0$,*
3. *(3) The number of pure imaginary λ_k is even.*

Proof. (1): We already know that if $b\in W_2^4(0,l_1)$ that the characteristic equation takes the form described in Proposition 5.4 and using Theorem 4.1 we get that property 1 of Theorem 5.6 is satisfied.
(2): If λ_k is an eigenvalue and $u(\lambda_k,s)$ an eigenfunction, then integrating the equation just

before equation (5.7) gives

$$\begin{aligned} 0 &= \int_0^l u_{ss}\bar{u}ds + \lambda_k^2 \int_0^l B(s)|u|^2 ds \\ &= u_s\bar{u}\Big|_0^l - \int_0^l |u_s|^2 ds + \lambda_k^2 \int_0^l B(s)|u|^2 ds. \end{aligned}$$

Taking (5.8) and (5.13) into account gives that

$$\bar{u}(l)(M\lambda_k^2 - i\lambda_k\nu)u(l) - \int_0^l |u_s|^2 ds + \lambda_k^2 \int_0^{l_1} b(s)|u|^2 ds = 0$$

$$(M\lambda_k^2 - i\lambda_k\nu)|u(l)|^2 - \int_0^l |u_s|^2 ds + \lambda_k^2 \int_0^{l_1} b(s)|u|^2 ds = 0. \tag{5.34}$$

The imaginary part of (5.34) is given by

$$\begin{aligned} 0 &= (2M\mathrm{Re}\lambda_k\mathrm{Im}\lambda_k - \mathrm{Re}\lambda_k\nu)|u(l)|^2 + 2\mathrm{Re}\lambda_k\mathrm{Im}\lambda_k \int_0^{l_1} b(s)|u|^2 ds \\ &= -\mathrm{Re}\lambda_k\nu|u(l)|^2 + 2\mathrm{Re}\lambda_k\mathrm{Im}\lambda_k\Big(\int_0^{l_1} b(s)|u|^2 ds + M|u(l)|^2\Big). \end{aligned}$$

If $u(l) = 0$, then the identity $u_s(l) = (M\lambda_k^2 - i\lambda_k\nu)u(l)$ would imply that u is identically zero, which is impossible since u is an eigenfunction. If $\mathrm{Re}\lambda_k \neq 0$, then diving both sides by $\mathrm{Re}\lambda_k$ we obtain

$$\mathrm{Im}\lambda_k = \frac{\nu.|u(l)|^2}{\Big(\int_0^{l_1} b(s)|u|^2 ds + M|u(l)|^2\Big)},$$

hence $\mathrm{Im}\lambda_k > 0$ since M, $|u(l)|$, ν and $\int_0^{l_1} b(s)|u|^2 ds$ are all positive. Now if $\mathrm{Re}\lambda_k = 0$, then taking the real part of (5.34) we get

$$\begin{aligned} 0 &= \Big(-M(\mathrm{Im}\lambda_k)^2 + \mathrm{Im}\lambda_k\nu\Big)|u(l)|^2 - \int_0^l |u_s|^2 ds - (\mathrm{Im}\lambda_k)^2\Big(\int_0^{l_1} b(s)|u|^2 ds\Big) \\ &= \mathrm{Im}\lambda_k\nu|u(l)|^2 - \int_0^l |u_s|^2 ds - (\mathrm{Im}\lambda_k)^2\Big(\int_0^{l_1} b(s)|u|^2 ds + M|u(l)|^2\Big), \end{aligned}$$

from which we obtain that $\mathrm{Im}\lambda_k > 0$. Hence property (2) of Theorem 5.6 is obtained. Now to prove property 3 of the theorem we note that $\varphi(-\bar{\lambda}) = \overline{\varphi(\lambda)}$ which implies that the function is real valued on the imaginary axis. Furthermore, we know that

$$\begin{aligned} \varphi(\lambda) = &-\lambda^2 N_1 \cos\lambda a + i\lambda N_2 \cos\lambda a + \lambda N_3 \sin\lambda a + N_4 \cos\lambda a \\ &+ iN_5 \sin\lambda a + iN_6 \frac{\cos\lambda a}{\lambda} + N_7 \frac{\sin\lambda a}{\lambda} + \int_0^a \xi_1(t)\frac{\sin\lambda t}{\lambda} dt \\ &+ i\int_0^a \xi_2(t)\frac{\cos\lambda t}{\lambda} dt, \end{aligned}$$

and replacing sine and cosine functions with their exponential form,

$$\begin{aligned}\varphi(\lambda) =& -\lambda^2 N_1\Big(\frac{e^{i\lambda a}+e^{-i\lambda a}}{2}\Big)+i\lambda N_2\Big(\frac{e^{i\lambda a}+e^{-i\lambda a}}{2}\Big)\\&+\lambda N_3\Big(\frac{e^{i\lambda a}-e^{-i\lambda a}}{2}\Big)+N_4\Big(\frac{e^{i\lambda a}+e^{-i\lambda a}}{2}\Big)+iN_5\Big(\frac{e^{i\lambda a}-e^{-i\lambda a}}{2i}\Big)\\&+iN_6\Big(\frac{e^{i\lambda a}+e^{-i\lambda a}}{2\lambda}\Big)+N_7\Big(\frac{e^{i\lambda a}-e^{-i\lambda a}}{2i\lambda}\Big)\\&+\int_0^a \xi_1(t)\Big(\frac{e^{i\lambda t}-e^{-i\lambda t}}{2i\lambda}\Big)dt+i\int_0^a \xi_2(t)\Big(\frac{e^{i\lambda t}+e^{-i\lambda t}}{2\lambda}\Big)dt,\end{aligned}$$

from which we obtain that

$$\varphi(\pm i\tau) \underset{\tau\to\infty}{=} \frac{\tau^2}{2}N_1 e^{\tau a}+O(\tau e^{\tau a})$$

and we can deduce that $\pm\infty$ are not zeroes of φ, and that the number of pure imaginary zeroes (counting multiplicity) is even. Hence property (3) of the Theorem 5.6 is proved. □

Theorem 5.7. *If $b\in W_2^2(0,l_1)$, $b(s)>\varepsilon>0$ for $s\in[0,l_1]$, $0<l_1<l<\infty$, $\nu>0$, then the spectrum $\{\lambda_k : k\in\mathbb{Z}\cup\{0^+\}\}$ of (5.17), (5.18) and (5.19) satisfy the following asymptotics*

$$\lambda_k \underset{k\to\infty}{=} \frac{\pi}{a}(k-\frac{1}{2})+\frac{P_1}{(k-\frac{1}{2})}+\frac{b_k}{k},$$

where $\{b_k\}_{-\infty}^{\infty}\in l_2$ and $P_1=-\frac{K_2}{\pi K_1}$.

Proof. We know from Proposition 5.3 that if $b\in W_2^2(0,l_1)$ then characteristic equation (5.31) takes the form

$$\begin{aligned}\varphi(\lambda) =& -\lambda^2 K_1\cos\lambda a+i\lambda K_2\cos\lambda a+\lambda K_3\sin\lambda a+K_4\cos\lambda a+iK_5\sin\lambda a\\&+iK_6\frac{\cos\lambda a}{\lambda}+K_7\frac{\sin\lambda a}{\lambda}+K_8\frac{\cos\lambda a}{\lambda^2}+\lambda\int_0^a\zeta_1(t)\sin\lambda t dt\\&+\int_0^a\zeta_2(t)\cos\lambda t dt+\int_0^a\zeta_3(t)\sin\lambda t dt+i\int_0^a\zeta_4(t)\frac{\cos\lambda t}{\lambda}dt\\&+\int_0^a\zeta_5(t)\frac{\sin\lambda t}{\lambda}dt+\int_0^a\zeta_6(t)\frac{\cos\lambda t}{\lambda^2}dt.\end{aligned}$$

where the constants K_j, $j=1,2,\ldots,8$ and the functions ζ_i, $i=1,2,\ldots,6$ are as defined in Proposition 5.3, and the $M_j, j=1,2,\ldots,4$ are defined as before. Now suppose that λ_k is a zero of φ, then since $\mathrm{Im}\lambda_k>0$ for all k we can divide $\varphi(\lambda_k)=0$ through by $i\lambda_k^2 K_1$ to get

$$i\cos\lambda_k a+L_1\frac{\cos\lambda_k a}{\lambda_k}+iL_2\frac{\sin\lambda_k a}{\lambda_k}+\frac{\Psi_1(\lambda_k)}{\lambda_k}=0$$

where

$$\Psi_1(\lambda_k) = \frac{1}{i\lambda_k K_1}\Big(K_4\cos\lambda_k a + iK_5\sin\lambda_k a + iK_6\frac{\cos\lambda_k a}{\lambda_k} + K_7\frac{\sin\lambda_k a}{\lambda_k}$$
$$+\lambda_k\int_0^a \zeta_1(t)\sin\lambda_k t dt + K_8\frac{\cos\lambda_k a}{\lambda_k^2} + \int_0^a \zeta_2(t)\cos\lambda_k t dt + \int_0^a \zeta_3(t)\sin\lambda_k t dt$$
$$+ i\int_0^a \zeta_4(t)\frac{\cos\lambda_k t}{\lambda_k}dt + \int_0^a \zeta_5(t)\frac{\sin\lambda_k t}{\lambda_k}dt$$
$$+\int_0^a \zeta_6(t)\frac{\cos\lambda_k t}{\lambda_k^2}dt\Big)$$

Then a direct application of Theorem 4.4 gives the results. □

Theorem 5.8. *If $b \in W_2^4(0,l_1)$, $b(s) > \varepsilon > 0$ for $s \in [0,l_1]$, $0 < l_1 < l < \infty$, $\nu > 0$, then the spectrum $\{\lambda_k : k \in \mathbb{Z}\cup\{0^+\}\}$ of (5.17), (5.18) and (5.19) satisfy the following asymptotics:*

$$\lambda_k = \frac{\pi}{a}(k-\frac{1}{2}) + \frac{Q_1}{(k-\frac{1}{2})} + i\frac{Q_2}{(k-\frac{1}{2})^2} + \frac{Q_3}{(k-\frac{1}{2})^3} + \frac{\beta_k}{k^3}$$

where

$$Q_1 = \frac{P_2}{\pi}$$

$$Q_2 = -\frac{a(P_1P_2+P_4)}{\pi^2}$$

$$Q_3 = \frac{(-P_6 - P_2P_3 + P_1P_4 + P_1^2P_2 + \frac{1}{3}P_2^3)a^2}{\pi^3} - a\frac{P_2^2}{\pi^3},$$

$P_j = \frac{K_{j+1}}{K_1}$, $j = 1,2,\ldots,6$, *and* $\{\beta_k\}_{-\infty}^{\infty} \in l_2$.

Proof. Recall that if $b \in W_2^4(0,l_1)$ that Proposition 5.4 gives that the characteristic equation has the form

$$\varphi(\lambda) = -\lambda^2 N_1\cos\lambda a + i\lambda N_2\cos\lambda a + \lambda N_3\sin\lambda a + N_4\cos\lambda a$$
$$+ iN_5\sin\lambda a + iN_6\frac{\cos\lambda a}{\lambda} + N_7\frac{\sin\lambda a}{\lambda} + \int_0^a \xi_1(t)\frac{\sin\lambda t}{\lambda}dt + i\int_0^a \xi_2(t)\frac{\cos\lambda t}{\lambda}dt.$$

Therefore applying Theorem 4.8 we obtain results. □

When comparing these results to Lemma 2.3 of the case considered by Pivovarchik and Mennicken in [12] we see that the asymptotic expansion of eigenvalues still retains the form it had in [12] when a concentrated mass was introduced at the endpoint with the exception that $Q_2 > 0$ in [12] whereas in this case we have that $Q_2 = 0$ as will be shown later.

Theorem 5.9. *If $b \in W_2^6(0,l_1)$, $b(s) > \varepsilon > 0$ for $s \in [0,l_1]$, $0 < l_1 < l < \infty$, $\nu > 0$, then the spectrum $\{\lambda_k : k \in \mathbb{Z}\cup\{0^+\}\}$ of (5.17), (5.18) and (5.19) satisfy the following asymptotics:*

$$\lambda_k = \frac{\pi}{a}(k-\frac{1}{2}) + \frac{Q_1}{(k-\frac{1}{2})} + i\frac{Q_2}{(k-\frac{1}{2})^2} + \frac{Q_3}{(k-\frac{1}{2})^3} + i\frac{Q_4}{(k-\frac{1}{2})^4} + \frac{Q_5}{(k-\frac{1}{2})^5} + \frac{\beta_k}{k^5}$$

where $\{\beta_k\}_{-\infty}^{\infty} \in l_2$*,* Q_1, Q_2, Q_3 *are as in Theorem 5.8 while* Q_4 *and* Q_5 *are defined as follows*

$$\begin{aligned}Q_4 = &\frac{aP_2Q_2}{\pi^2} + \frac{(P_4+P_1P_2)2a^2Q_1}{\pi^3}\\ &- \frac{a^3(P_8+P_2P_5+P_1P_6+P_3P_4+2P_1P_2P_3-P_2^2P_4-P_1^2P_4-P_2^3P_1-P_1^3P_2)}{\pi^4}\end{aligned}$$

$$\begin{aligned}Q_5 = &\frac{aP_2Q_3}{\pi^2} - \frac{2a^2Q_2(P_4+P_1P_2)}{\pi^3} - \frac{3a^3Q_1(-P_6-P_2P_3+P_1P_4+\frac{1}{3}P_2^2+P_2P_1^2)}{\pi^4}\\ &+ \frac{(-P_{10}-P_2P_7+P_1P_8-P_3P_6+P_4P_5+P_2^2P_6+2P_1P_2P_5-P_2P_3^2)a^4}{\pi^5}\\ &+ \frac{(-P_2P_4^2+P_1^2P_6+2P_1P_3P_4-3P_1P_2^2P_4+3P_1^2P_2P_3+P_2^3P_3-P_1^3P_4-2P_1^2P_2^3-P_1^4P_2-\frac{1}{5}P_2^5)a^4}{\pi^5}\\ &- \frac{P_2^3a^2}{\pi^5}\end{aligned}$$

Proof. According to Proposition 5.5 if $b \in W_2^6(0,a)$ then the characteristic function takes the form

$$\begin{aligned}\varphi(\lambda) = &-\lambda^2K_1\cos\lambda a + i\lambda K_2\cos\lambda a + \lambda K_3\sin\lambda a + K_4\cos\lambda a + iK_5\sin\lambda a + iK_6\frac{\cos\lambda a}{\lambda}\\ &+ K_7\frac{\sin\lambda a}{\lambda} + K_8\frac{\cos\lambda a}{\lambda^2} + iK_9\frac{\sin\lambda a}{\lambda^2} + iK_{10}\frac{\cos\lambda a}{\lambda^3} + K_{11}\frac{\sin\lambda a}{\lambda^3}\\ &+ \int_0^a \xi_1(t)\frac{\sin\lambda t}{\lambda^3}dt + i\int_0^a \xi_2(t)\frac{\cos\lambda t}{\lambda^3}dt\end{aligned}$$

where the constants K_j, $j = 1, 2, \ldots, 11$ and the functions ξ_1 and ξ_2 are as defined in Proposition 5.5. Then a direct application of Theorem 4.8 gives the desired asymptotic expansion. □

We conclude this section by doing some simple algebra which will help us to draw important conclusions which affect the direct and inverse problem. Using the definition of K_j, $j = 1, 2, \ldots, 11$ for the case where $b \in W_2^6(0,l_1)$ given in the previous theorem we can see that

$$K_1K_5 = M_1^{-1}M_2^{-1}K(a,a) + M_1^{-1}M_2^{-1}M_3$$

and

$$K_2K_3 = -K(a,a)M_1^{-1}M_2^{-1} - M_1^{-1}M_2^{-1}M_3$$

so then

$$K_1K_5 + K_2K_3 = 0. \tag{5.35}$$

We also know that

$$K_4 = 1 + M_2^{-1}K_x(a,a) + M_2^{-1}M_3K(a,a),$$

and

$$K_6 = -K_2K_x(a,a) - K_2M_3K(a,a)$$

then it follows that

$$K_2K_4 - K_2 + K_1K_6 = 0.$$

It is clear from Theorem 5.5 how M_1 and M_2 are related to the constants K_1 and K_2, we now show the relationship between M_3 and M_4 and the coefficient K_i, $i = 1, 2, \ldots, 11$. From the definition of K_5 we can see that

$$\frac{K_5}{K_2} = K(a,a) + M_3. \tag{5.36}$$

Now using the definition of K_7 we obtain that

$$K_2K_7 = M_1^{-1}K(a,a) - M_1^{-1}M_2^{-1}K_{xt}(a,a) + M_1^{-1}M_4 - M_1^{-1}M_2^{-1}M_3K_t(a,a),$$

furthermore, the definition of K_9 also gives that

$$K_1K_9 = M_1^{-1}M_2^{-1}K_{xt}(a,a) + M_1^{-1}M_2^{-1}M_3K_t(a,a)$$

then combining both equations we deduce that

$$-(K_2K_7 + K_1K_9) = -(M_1^{-1}K(a,a) + M_1^{-1}M_4)$$

therefore

$$K_5 - (K_2K_7 + K_1K_9) = M_1^{-1}M_3 - M_1^{-1}M_4. \tag{5.37}$$

Now dividing by M_1 we get that

$$M_3 - M_4 = \frac{K_5 - (K_2K_7 + K_1K_9)}{M_1^{-1}}. \tag{5.38}$$

Similarly, using the definition of K_{10} and K_8 we also see that

$$K_1K_{10} = M_1^{-1}M_2^{-1}K_{xtt}(a,a) + M_1^{-1}M_2^{-1}M_3K_{tt}(a,a)$$

while

$$K_2K_8 = -M_1^{-1}K_x(a,a) - M_1^{-1}M_2^{-1}K_{xtt}(a,a) - M_1^{-1}M_4K(a,a) - M_1^{-1}M_2^{-1}M_3K_{tt}(a,a)$$

from which we get that

$$K_1K_{10} + K_2K_8 = -M_1^{-1}K_x(a,a) - M_1^{-1}M_4K(a,a).$$

So then combining this with K_6 we see that

$$K_6 - (K_1K_{10} + K_2K_8) = -(M_1^{-1}M_3 - M_1^{-1}M_4)K(a,a),$$

therefore using equation (5.37) we obtain that

$$K(a,a) = -\Big(\frac{K_6 - (K_1K_{10} + K_2K_8)}{K_5 - (K_2K_7 + K_1K_9)}\Big).$$

Altogether, from the above deductions we observe that M_3 and M_4 satisfy the following relationship

$$M_3 = \frac{K_5}{K_2} + \Big(\frac{K_6 - (K_1K_{10} + K_2K_8)}{K_5 - (K_2K_7 + K_1K_9)}\Big)$$

while

$$M_4 = \Big(\frac{K_6 - (K_1K_{10} + K_2K_8)}{K_5 - (K_2K_7 + K_1K_9)}\Big) + \frac{K_2K_7 + K_1K_9}{K_2}$$

Corollary 5.10. *The term Q_2 in all the asymptotics expansion is equal to zero.*

Proof. We know from Theorem 5.8 that

$$Q_2 = -\frac{a(P_1P_2 + P_4)}{\pi^2}$$

now using the definition $P_i = \frac{K_{i+1}}{K_1}$ we can see that

$$Q_2 = -\frac{a}{\pi^2}\frac{1}{K_1^2}(K_1K_5 + K_2K_3)$$

and using equation (5.35) we deduce that $Q_2 = 0$. □

Corollary 5.11.

$$P_8 + P_2P_5 + P_1P_6 + P_3P_4 + 2P_1P_2P_3 - P_2^2P_4 - P_1^2P_4 - P_2^3P_1^3 - P_1^3P_2 = \frac{M_1^{-1}(M_4 - M_3)}{M_2^{-2}}$$

Proof. We will use the identity $P_4 + P_1P_2 = 0$ which was established in Corollary 5.10 to simplify the following

$$\begin{aligned}
&P_8 + P_2P_5 + P_1P_6 + P_3P_4 + 2P_1P_2P_3 - P_2^2P_4 - P_1^2P_4 - P_2^3P_1^3 - P_1^3P_2 \\
&= P_8 + P_2P_5 + P_1P_6 + P_3P_4 + 2P_1P_2P_3 - P_2^2(P_4 + P_1P_2) - P_1^2(P_4 + P_1P_2) \\
&= P_8 + P_2P_5 + P_1P_6 + P_3(P_4 + P_1P_2) + P_1P_2P_3 \\
&= P_8 + P_2P_5 + P_1P_6 + P_1P_2P_3 \\
&= \frac{K_9}{K_1} + \frac{(K_3K_6 + K_2K_7)}{K_1^2} + \frac{K_2K_3K_4}{K_1^3}.
\end{aligned}$$

Now using the definition of K_7 we can see that

$$K_2K_7 = M_1^{-1}K(a,a) - M_1^{-1}M_2^{-1}K_{xt}(a,a) + M_1^{-1}M_4 - M_1^{-1}M_2^{-1}M_3K_t(a,a)$$

while using the definition of K_9 we can deduce that

$$K_1K_9 = M_1^{-1}M_2^{-1}K_{xt}(a,a) + M_1^{-1}M_2^{-1}M_3K_t(a,a)$$

therefore

$$K_2K_7 + K_1K_9 = M_1^{-1}K(a,a) + M_1^{-1}M_4. \tag{5.39}$$

Furthermore, using the definitions of K_3, K_4 and K_6 we can see that

$$K_1K_3K_6 = M_2^{-1}(K(a,a)M_2^{-1} + M_2^{-1}M_3)(M_1^{-1}K_x(a,a) + M_1^{-1}M_3K(a,a))$$

while

$$K_2K_3K_4 = -M_1^{-1}(K(a,a)M_2^{-1} + M_2^{-1}M_3)(1 + M_2^{-1}K_x(a,a) + M_2^{-1}M_3K(a,a))$$

so then summing the two terms we get that

$$K_1K_3K_6 + K_2K_3K_4 = -M_1^{-1}(K(a,a)M_2^{-1} + M_2^{-1}M_3). \tag{5.40}$$

Lastly, using (5.39) and (5.40) we can obtain that

$$K_1(K_2K_7 + K_1K_9) + (K_1K_3K_6 + K_2K_3K_4) = M_1^{-1}M_2^{-1}(M_4 - M_3), \tag{5.41}$$

therefore

$$P_8 + P_2P_5 + P_1P_6 + P_3P_4 + 2P_1P_2P_3 - P_2^2P_4 - P_1^2P_4 - P_2^3P_1^3 - P_1^3P_2 = \frac{M_1^{-1}(M_4 - M_3)}{M_2^{-2}}$$

□

This result can also be seen in the light of Theorem 5.6 as further reinforcement to the postulate that for all k we have that $\mathrm{Im}\lambda_k > 0$.

Corollary 5.12.

$$K_1K_5 + K_1K_3K_6 + K_2K_3K_4 = 0$$

Proof. The results can be deduced by using the two equations for $M_3 - M_4$ in (5.38) and (5.41) or by using the definition K_i, $i = 1, \dots, 11$. □

It is clear from the above equations that there is more than one way of defining M_3 and M_4 due to the multitude of identities that exist in this problem, therefore no preference will be given to either one of the identities and these identities will be assumed to hold from this point onwards.

6. Inverse Problem

In this section we will recover the following parameters $\{a, q(x), M_1, M_2, M_3, M_4\}$ from the knowledge of the spectrum and the length of the string. It will be shown that from the asymptotic expansion of the eigenvalues and all the properties of a spectrum that we can construct an entire function whose zeroes correspond to our eigenvalues and has coefficients that contains some information about the constants M_1, M_2, M_3, and M_4 which in turn are related to the parameters $b(s)$, $\nu > 0$, l_1 of the string and the mass of the ring M. We will also give a construction of the S-matrix which will be used in the Marchenko integral equation to recover the potential uniquely. Lastly, once the potential has been recovered we will then give an algorithm on how to recover the parameters $b(s)$, $\nu > 0$, l_1 and the mass of the ring M.

Denote by SL the class $\{a, q(x), M_1, M_2, M_3, M_4\}$ satisfying the following conditions

(1) $a > 0$, $M_j \in \mathbb{R}$ for $j = 1, 2 \cdots 4$, $M_1 > 0$, $M_2 > 0$, $M_3, M_4 \in \mathbb{R}$.
(2) The function $q(x)$ is real valued and belongs to $L_2(0,a)$
(3) The operator A acting on $L_2(0,a)$ according to

$$D(A) = \{f : \ \ f \in W_2^2(0,a),\ f(0) = 0,\ \ f'(a) + M_4 f(a) = 0\}$$

$$Af = -f'' + q(x)f$$

is strictly positive.

Theorem 6.1. *Let $\{\lambda_k : k \in \mathbb{Z} \cup \{0^+\}\}$ be a properly enumerated sequence of complex numbers satisfying the following conditions*

(1) For all $k \in \mathbb{Z} \cup \{0^+\}$ we have that $\mathrm{Im}\lambda_k > 0$,
(2) The number of pure imaginary λ_k is even,
(3) The sequence $\{\lambda_k : k \in \mathbb{Z} \cup \{0^+\}\}$ is symmetric with respect to the imaginary axis and the corresponding multiplicities of symmetric points coincide,
(4) Being enumerated in a proper way, the sequence $\{\lambda_k : k \in \mathbb{Z} \cup \{0^+\}\}$ admits the following asymptotic

$$\lambda_k \underset{k\to\infty}{=} \frac{\pi}{a}(k - \frac{1}{2}) + \frac{Q_1}{(k-\frac{1}{2})} + i\frac{Q_2}{(k-\frac{1}{2})^2} + \frac{Q_3}{(k-\frac{1}{2})^3} + i\frac{Q_4}{(k-\frac{1}{2})^4} + \frac{Q_5}{(k-\frac{1}{2})^5} + \frac{\beta_k}{k^5} \tag{6.1}$$

where $\{\beta_k\}_{-\infty}^{\infty} \in l_2$, $a > 0$, $Q_j \in \mathbb{R}$, for $j = 1, 2, 3, 4$, furthermore $Q_2 = 0$, and $Q_4 > 0$,

then there exist a unique set $\{a, q(x), M_1, M_2, M_3, M_4\}$ ∈SL such that $\{\lambda_k : k \in \mathbb{Z} \cup \{0^+\}\}$ is the spectrum of (5.17)-(5.19).

The proof of the above theorem will be broken down into a number of theorems, lemmas, proposition and corollaries, and that quest to prove Theorem 6.1 begins from this point onwards. We begin by recovering the set $\{a, q(x), M_1, M_2, M_3, M_4\}$ from all the conditions of Theorem 6.1. We know from condition 4 of Theorem 6.1 that

$$\lambda_k \underset{k\to\infty}{=} \frac{\pi}{a}(k-\frac{1}{2}) + \frac{Q_1}{(k-\frac{1}{2})} + i\frac{Q_2}{(k-\frac{1}{2})^2} + \frac{Q_3}{(k-\frac{1}{2})^3} + i\frac{Q_4}{(k-\frac{1}{2})^4} + \frac{Q_5}{(k-\frac{1}{2})^5} + \frac{\beta_k}{k^5}$$

then it follows that

$$0 < a = \lim_{k\to\infty} \frac{\pi k}{\lambda_k} \tag{6.2}$$

hence a is recovered. In order to characterize the growth of an entire function for large values of λ we introduce the function

$$M_f(r) = \max_{|z|=r} |f(z)|.$$

Let f be an arbitrary entire function of order 1. A function f is said to be of exponential type σ if and only if there exist a greatest lower bound σ of positive numbers A for which asymptotically

$$M_f(r) < e^{Ar} \text{ for } r \geq r_0(\sigma).$$

Let us consider an auxiliary function χ^0 defined by

$$\begin{aligned}\chi^0(\lambda) = \cos\lambda a - iP_1\frac{\cos\lambda a}{\lambda} - P_2\frac{\sin\lambda a}{\lambda} - P_3\frac{\cos\lambda a}{\lambda^2} - iP_4\frac{\sin\lambda a}{\lambda^2} - iP_5\frac{\cos\lambda a}{\lambda^3}\\ - P_6\frac{\sin\lambda a}{\lambda^3} - P_7\frac{\cos\lambda a}{\lambda^4} - iP_8\frac{\sin\lambda a}{\lambda^4} - iP_9\frac{\cos\lambda a}{\lambda^5} - P_{10}\frac{\sin\lambda a}{\lambda^5} + \frac{\phi(\lambda)}{\lambda^5},\end{aligned}$$

where $P_i = \frac{K_{1+1}}{K_1}$, $i = 1, 2, \ldots, 10$ are real valued constants, $K_1 \neq 0$ and ϕ is an entire function of exponential type $\leq a$. To show that the auxiliary function χ^0 is of exponential type $a > 0$ we note that for $\lambda = \sigma + i\rho$ where σ and ρ are arbitrary numbers

$$\begin{aligned}|\cos\lambda a| &= \left|\frac{e^{i\lambda a} + e^{-i\lambda a}}{2}\right|\\ &\leq e^{|\rho|a},\end{aligned}$$

and similarly $|\sin\lambda a| \leq e^{|\rho|a}$. Therefore, using these inequalities we can now write that

$$\chi^0(\lambda) = \cos\lambda a + O(\frac{e^{|\rho|a}}{\lambda}) \tag{6.3}$$

for sufficiently large λ, from which we deduce that χ^0 has type $a > 0$.

Definition 6.2. A function $S(\lambda)$ is said to be a function of sine type if it is an entire function of exponential type $\sigma > 0$ satisfying the following conditions:

(1) All its zeroes lie in horizontal strip $|\text{Im}\lambda| < h$; for some $h > 0$.
(2) The inequalities
$$0 < m \leq |S(\lambda)| \leq M < \infty$$
holds on some line $\text{Im}\lambda = const$
(3) The types of $S(\lambda)$ in the upper and lower half planes are equal.

It is sometimes convenient to use the following definition of a function of a sine type, equivalent to the one above (see equation (1.2) on p.80 of [7]). An entire function $S(\lambda)$ of exponential type $\sigma > 0$ is said to be a function of sine type if for some $H > 0$

$$0 < m < |S(x+iy)|e^{-\sigma|y|} \leq M < \infty \ \ (|y| \geq H).$$

We will state without proof the proposition found in lemma 5 of [7].

Proposition 6.3. *Let $\{\lambda_k\}_{-\infty}^{\infty}$ be the sequence of zeroes of a function S of sine type $\leq \sigma$ belonging to $L_p(-\infty,\infty)$, and let the sequence $\{\psi_k\}_{-\infty}^{\infty}$ be of the form*

$$\psi_k = b\lambda_k^{-m} + b_k\lambda_k^{-m},$$

where m is a positive integer and $\{b_k\}_{-\infty}^{\infty} \in l_p, p > 1$. Then the function defined by

$$\tilde{S}(z) = \lim_{p\to\infty}\prod_{k=-p}^{p}\Big(1 - \frac{z}{(\lambda_k+\psi_k)}\Big)$$

where $\lambda_k + \psi_k \neq 0$ for all k, has a representation of the form

$$\tilde{S}(z) = CS(z)\{1 + B_1z^{-1} + B_2z^{-2} + \cdots + B_mz^{-m}\} - aCS'(z)z^{-m} + f_m(z)z^{-m},$$

where

$$C = \prod_{-\infty}^{\infty}\Big(1+\psi_k\lambda_k^{-1}\Big)^{-1},$$

and $B_1, B_2, \cdots B_m$ are constants, $f_m(z)$ is an entire function of exponential type σ belonging to $L_2(-\infty,\infty)$.

Now, let us introduce the function

$$\chi(\lambda) = \lim_{n\to\infty}\prod_{k=+0}^{n}\Big(1-\frac{\lambda}{\lambda_k}\Big)\prod_{k=-n}^{-0}\Big(1-\frac{\lambda}{\lambda_k}\Big), \tag{6.4}$$

whose limit need not exist yet where λ_k satisfy the following asymptotic

$$\lambda_k = \frac{\pi}{a}(k-\frac{1}{2}) + \frac{Q_1}{(k-\frac{1}{2})} + i\frac{Q_2}{(k-\frac{1}{2})^2} + \frac{Q_3}{(k-\frac{1}{2})^3} + i\frac{Q_4}{(k-\frac{1}{2})^4} + \frac{Q_5}{(k-\frac{1}{2})^5} + O(1).$$

The fact that the set $\{\lambda_k : k \in \mathbb{Z}\cup\{0^+\}\}$ is symmetric with respect to the imaginary axis will be expressed by stating that $\lambda_{-k} = -\bar{\lambda}_k$ for all λ_k not purely imaginary. Then pairing λ_k not pure imaginary we compute the following

$$\begin{aligned}
&\Big(1-\frac{\lambda}{\lambda_k}\Big)\Big(1-\frac{\lambda}{\lambda_{-k}}\Big)\\
&=\Big(1-\lambda\frac{(\lambda_k+\lambda_{-k})}{\lambda_k\lambda_{-k}} - \frac{\lambda^2}{|\lambda_k|^2}\Big)\\
&=\Big(1+i\lambda\frac{(2\mathrm{Im}\lambda_k)}{|\lambda_k|^2} - \frac{\lambda^2}{|\lambda_k|^2}\Big) \quad (\text{using }\ \lambda_{-k} = -\bar{\lambda}_k)\\
&=\Big(1+\lambda.O((k-\frac{1}{2})^{-2}) - \frac{\lambda^2}{|\lambda_k|^2}\Big),
\end{aligned}$$

from which we obtain that over a compact subset of $\mathbb{C}$ the function χ is an entire function of λ that is uniformly convergent. Let us defined the following entire function

(6.5)

$$\begin{aligned}
\Gamma(\lambda) = &-\lambda^2K_1\cos\lambda a + i\lambda K_2\cos\lambda a + \lambda K_3\sin\lambda a + K_4\cos\lambda a + iK_5\sin\lambda a + iK_6\frac{\cos\lambda a}{\lambda}\\
&+K_7\frac{\sin\lambda a}{\lambda} + K_8\frac{\cos\lambda a}{\lambda^2} + iK_9\frac{\sin\lambda a}{\lambda^2} + iK_{10}\frac{\cos\lambda a}{\lambda^3} + K_{11}\frac{\sin\lambda a}{\lambda^3}\\
&+\frac{\psi(\lambda)}{\lambda^3},
\end{aligned}$$

where K_j, $j = 1,2,\ldots,11$ are constants and ψ is an entire function of exponential type $\leq a$ which belongs to $L_2(-\infty,\infty)$. We further identify the function χ^0 as $\chi^0(\lambda) = -\frac{\Psi(\lambda)}{K_1}$ where the function Ψ is as defined on page 38 and satisfies the following equations

$$Q_1 = -\frac{P_2}{\pi} \tag{6.6}$$

$$Q_2 = -\frac{(P_4 + P_1P_2)a}{\pi^2} \tag{6.7}$$

$$Q_3 = \frac{(-P_6 - P_2P_3 + P_1P_4 + P_1^2P_2 + \frac{1}{3}P_2^3)a^2}{\pi^3} - a\frac{P_2^2}{\pi^3} \tag{6.8}$$

and

$$\begin{aligned} Q_4 = {} & \frac{aP_2Q_2}{\pi^2} + \frac{(P_4 + P_1P_2)2a^2Q_1}{\pi^3} \\ & - \frac{a^3(P_8 + P_2P_5 + P_1P_6 + P_3P_4 + 2P_1P_2P_3 - P_2^2P_4 - P_1^2P_4 - P_2^3P_1 - P_1^3P_2)}{\pi^4} \end{aligned} \tag{6.9}$$

(6.10)

$$\begin{aligned} Q_5 = {} & \frac{aP_2Q_3}{\pi^2} - \frac{2a^2Q_2(P_4 + P_1P_2)}{\pi^3} - \frac{3a^3Q_1(-P_6 - P_2P_3 + P_1P_4 + \frac{1}{3}P_2^2 + P_2P_1^2)}{\pi^4} \\ & + \frac{(-P_{10} - P_2P_7 + P_1P_8 - P_3P_6 + P_4P_5 + P_2^2P_6 + 2P_1P_2P_5 - P_2P_3^2)a^4}{\pi^5} \\ & + \frac{(-P_2P_4^2 + P_1^2P_6 + 2P_1P_3P_4 - 3P_1P_2^2P_4 + 3P_1^2P_2P_3 + P_2^3P_3 - P_1^3P_4 - 2P_1^2P_2^3 - P_1^4P_2 - \frac{1}{5}P_2^5)a^4}{\pi^5} \\ & - \frac{P_2^3a^2}{\pi^5}, \end{aligned}$$

where $P_j = \frac{K_{j+1}}{K_j}$, $j = 1,2,\ldots,10$.

Lemma 6.4. *The auxiliary function $\chi^0(\lambda)$ is an entire function of sine type where ϕ is an entire function of exponential type $\leq a$ which belongs to $L_2(-\infty,\infty)$ and satisfy $\phi(-\bar{\lambda}) = -\overline{\phi}(\lambda)$.*

Proof. We recall from section 4 (see pg 38) that the function $\phi(\lambda)$ can be written as

$$\phi(\lambda) = \int_0^a \zeta(t)\sin\lambda t dt + i\int_0^a \xi(t)\cos\lambda t dt \tag{6.11}$$

where ζ and ξ are real valued and belong to $L_2(0,a)$. We can observe that

$$\begin{aligned} \phi(-\bar{\lambda}) &= \int_0^a \zeta(t)\sin(-\bar{\lambda})t dt + i\int_0^a \xi(t)\cos(-\bar{\lambda})t dt, \\ &= -\Big(\int_0^a \zeta(t)\sin\bar{\lambda}t dt + \bar{i}\int_0^a \xi(t)\cos\bar{\lambda}t dt\Big), \\ &= -\overline{\phi(\lambda)}. \end{aligned}$$

To show that the function χ^0 is a function of sine type we note that $\lambda^2\chi^0$ has the same form as the function E_3 of section 4 (shown in equation (4.3)), and by Theorem 4.8 the zeroes of the function χ^0 behave asymptotically as

$$\lambda_k^0 \underset{k\to\infty}{=} \frac{\pi}{a}(k - \frac{1}{2}) + \frac{\tilde{Q}_1}{(k-\frac{1}{2})} + i\frac{\tilde{Q}_2}{(k-\frac{1}{2})^2} + \frac{\tilde{Q}_3}{(k-\frac{1}{2})^3} + i\frac{\tilde{Q}_4}{(k-\frac{1}{2})^4} + \frac{\tilde{Q}_5}{(k-\frac{1}{2})^5} + \frac{\beta_k^0}{k^5} \tag{6.12}$$

where $\{\beta_k^0\}_{-\infty}^{\infty} \in l_2$, from which we deduce that

$$\lim_{|k|\to\infty} \beta_k^0 = 0.$$

We similarly observe that

$$\frac{\beta_k^0}{k^5} \to 0, \;\; i\frac{\tilde{Q}_2}{(k-\frac{1}{2})^2} \to 0 \;\text{ and} i\frac{\tilde{Q}_4}{(k-\frac{1}{2})^4} \to 0 \text{ as } \;|k| \to \infty,$$

which means there exists an $h > 0$ such that the zeroes λ_k^0 of χ^0 lie in the strip $|\mathrm{Im}\lambda| < h$, hence condition 1 of Definition 6.2 holds. Now let $\lambda = \sigma + i\rho$ with $\rho > h > \sup_k |\lambda_k^0|$. Since the sine and the cosine functions are bounded on the line $\mathrm{Im}\lambda = \rho$, we have that for any $\varepsilon > 0$ given, there exists an $\nu_0 > 0$ such that for $|\lambda| > \nu_0$ on $\mathrm{Im}\lambda = \rho$ we have that

$$\begin{aligned}&\Big| -iP_1\frac{\cos\lambda a}{\lambda} - P_2\frac{\sin\lambda a}{\lambda} - P_3\frac{\cos\lambda a}{\lambda^2} - iP_4\frac{\sin\lambda a}{\lambda^2} - iP_5\frac{\cos\lambda a}{\lambda^3}\\ &- P_6\frac{\sin\lambda a}{\lambda^3} - P_7\frac{\cos\lambda a}{\lambda^4} - iP_8\frac{\sin\lambda a}{\lambda^4} - iP_9\frac{\cos\lambda a}{\lambda^5} - P_{10}\frac{\sin\lambda a}{\lambda^5} + \frac{\phi(\lambda)}{\lambda^5}\Big| < \varepsilon\end{aligned}$$

Let us set

$$\begin{aligned}R_1(\lambda) = &-iP_1\frac{\cos\lambda a}{\lambda} - P_2\frac{\sin\lambda a}{\lambda} - P_3\frac{\cos\lambda a}{\lambda^2} - iP_4\frac{\sin\lambda a}{\lambda^2} - iP_5\frac{\cos\lambda a}{\lambda^3}\\ &- P_6\frac{\sin\lambda a}{\lambda^3} - P_7\frac{\cos\lambda a}{\lambda^4} - iP_8\frac{\sin\lambda a}{\lambda^4} - iP_9\frac{\cos\lambda a}{\lambda^5} - P_{10}\frac{\sin\lambda a}{\lambda^5} + \frac{\phi(\lambda)}{\lambda^5}\end{aligned}$$

then the above gives that for all $\varepsilon > 0$ there exist an ν_0 such that

$$|\lambda| > \nu_0, \;\; \mathrm{Im}\lambda = \rho \Rightarrow |R_1(\lambda) - 0| < \varepsilon.$$

So for $|\lambda| > \nu_0$ also on the line $\mathrm{Im}\lambda = \rho$ we have that

$$\begin{aligned}|\cos\lambda a + R_1(\lambda)| &\le |\cos\lambda a| + |R_1(\lambda)|\\ &< |\cos\lambda a| + \varepsilon.\end{aligned}$$

We also know that the cosine is bounded above as follows

$$\begin{aligned}&|\cos\lambda a| = \Big|\frac{e^{i\lambda a} + e^{-i\lambda a}}{2}\Big|,\\ &\le |\frac{e^{-i\lambda a}}{2}| + |\frac{e^{i\lambda a}}{2}|\\ &= \frac{e^{\rho a}}{2} + \frac{e^{-\rho a}}{2}\\ &= N,\end{aligned}$$

so using the upper bound we obtain that for λ sufficiently large on the line $\mathrm{Im}\lambda = \rho$, we have that

$$\begin{aligned}|\chi^0(\lambda)| &\le |\cos\lambda a| + \varepsilon\\ &\le N + \varepsilon = M < \infty.\end{aligned}$$

Similarly we have

$$\begin{aligned}|\chi^0(\lambda)| &\ge |\cos\lambda a| - |R_1(\lambda)|\\ &\ge |\cos\lambda a| - \varepsilon\end{aligned}$$

where

$$\begin{aligned}|\cos\lambda a| &\geq |\frac{e^{-i\lambda a}}{2}| - |\frac{e^{i\lambda a}}{2}|\\ &= \frac{e^{\rho a}}{2} - \frac{e^{-\rho a}}{2}\\ &= m > 0.\end{aligned}$$

Therefore

$$\begin{aligned}|\chi^0(\lambda)| &\geq |\cos\lambda a| - \varepsilon\\ &\geq m - \varepsilon > 0,\end{aligned}$$

where ε is arbitrary small. Altogether we obtain that for$|\lambda| > \nu_0$ on the line $\text{Im}\lambda = \rho = const$, the following is satisfied

$$0 < m - \varepsilon \leq |\chi^0(\lambda)| \leq M + \varepsilon < \infty.$$

Now for $|\lambda| \leq \nu_0$ (compact) still on the line $\text{Im}\lambda = \rho$ we have that χ^0 attains a minimum and a maximum m' and M' respectively with $m' \neq 0$ (since $\rho > h$), so we can write

$$0 < m' \leq |\chi^0(\lambda)| \leq M' < \infty,$$

which satisfies condition 2 of Definition 6.2 holds. Condition 3 of Definition 6.2 follows from equation (6.3). Therefore the function χ^0 is of sine type. □

Lemma 6.5. *The function $\chi(\lambda)$ can be written as the function $\Gamma(\lambda)$ defined in (6.5)*

Proof. To prove this lemma we consider the zeroes λ_k as perturbation of zeroes λ_k^0. Comparing (6.1) with (6.12) we see that

$$\begin{aligned}\lambda_k - \lambda_k^0 &= \frac{(\beta_k - \beta_k^0)}{k^5}\\ &= \frac{\gamma_k}{k^5},\end{aligned}$$

where $\{\gamma_k\}_{-\infty}^{\infty} \in l_2$. We also know that $\lambda_k^0 = O(k)$, then this implies that $\exists c \neq 0$ such that

$$|\lambda_k^0| \leq c|k|$$

which is the same as

$$|k|^{-5} \leq c|\lambda_k^0|^{-5}$$

so then we can write $k^{-5} = O((\lambda_k^0)^{-5})$, and substituting back

$$\begin{aligned}\lambda_k - \lambda_k^0 &= \frac{\gamma_k}{k^5}\\ &= \gamma_k O((\lambda_k^0)^{-5})\\ &= \delta_k(\lambda_k^0)^{-5},\end{aligned}$$

$$\lambda_k = \lambda_k^0 + \delta_k(\lambda_k^0)^{-5}, \tag{6.13}$$

where $\{\delta_k\}_{-\infty}^{\infty} \in l_2$. Before applying Proposition 6.3 we note the following correspondence

$$\psi_k \leftrightarrow \delta_k(\lambda_k^0)^{-5},\ \tilde{S}(z) \leftrightarrow \chi(\lambda)\Big(1 - \frac{\lambda}{\lambda_{-0}}\Big)^{-1} \cdot \Big(1 - \frac{\lambda}{\lambda_{+0}}\Big)^{-1},\ S(z) \leftrightarrow \chi^0(\lambda),\ b \leftrightarrow 0,$$

where $\lambda_{\pm 0}$ are not purely imaginary. Now applying Proposition 6.3 to the function χ^0 we get
(6.14)
$$\chi(\lambda)\Big(1-\frac{\lambda}{\lambda_{-0}}\Big)^{-1}\Big(1-\frac{\lambda}{\lambda_{+0}}\Big)^{-1}=C_0\chi^0(\lambda)\Big(1+\frac{T_1}{\lambda}+\frac{T_2}{\lambda^2}+\frac{T_3}{\lambda^3}+\frac{T_4}{\lambda^4}+\frac{T_5}{\lambda^5}\Big)+\frac{\varphi(\lambda)}{\lambda^5},$$
where φ is an entire function belonging to $L_2(-\infty,+\infty)$ and

$$C_0=\prod_{-\infty,k\neq 0}^{\infty}\Big(1+\delta_k(\lambda_k^0)^{-6}\Big)^{-1}. \tag{6.15}$$

Now since we have that $\lambda_{-k}=-\bar{\lambda}_k$ for all not pure imaginary, then we can write
$$\chi(\lambda)=\prod_{-n}^{n}\Big(1-\frac{\lambda}{\lambda_k}\Big)\prod_{|k|>n}\Big(1-\frac{\lambda}{\lambda_k}\Big)$$
where the first product is over the pure imaginary λ_k. From the product representation it follows that
$$\begin{aligned}\chi(-\bar{\lambda})&=\prod_{-n}^{n}\Big(1-\frac{\bar{\lambda}}{-\lambda_k}\Big)\prod_{|k|>n}\Big(1-\frac{\bar{\lambda}}{-\lambda_k}\Big)\\&=\prod_{-n}^{n}\Big(1-\frac{\bar{\lambda}}{\bar{\lambda}_{-k}}\Big)\prod_{|k|>n}\Big(1-\frac{\bar{\lambda}}{\bar{\lambda}_{-k}}\Big)\\&=\overline{\chi(\lambda)}.\end{aligned}$$
Now since $\lambda_{\pm 0}$ are not pure imaginary we compute the following product
$$\begin{aligned}\Big(1-\frac{-\bar{\lambda}}{\lambda_{-0}}\Big)\Big(1-\frac{-\bar{\lambda}}{\lambda_{+0}}\Big)&=\Big(1-\frac{\bar{\lambda}}{\bar{\lambda}_{+0}}\Big)\Big(1-\frac{\bar{\lambda}}{\bar{\lambda}_{-0}}\Big)\\&=\overline{\Big(1-\frac{\lambda}{\lambda_{-0}}\Big)\Big(1-\frac{\lambda}{\lambda_{+0}}\Big)}.\end{aligned}$$
Furthermore, if for any function g we have that $g(\lambda)=K\cdot f(\lambda)$ where g satisfy $g(-\bar{\lambda})=\overline{g(\lambda)}$ and $K\in\mathbb{C}$, then we must have that $K\in\mathbb{R}$ which in turn implies that $C_0\in\mathbb{R}$ since both χ,χ^0 satisfy $g(-\bar{\lambda})=\overline{g(\lambda)}$. Now substituting $-\bar{\lambda}$ into equation (6.14) we get

$$\chi(-\bar{\lambda})\Big(1-\frac{-\bar{\lambda}}{\lambda_{-0}}\Big)^{-1}\Big(1-\frac{-\bar{\lambda}}{\lambda_{+0}}\Big)^{-1}=C_0\chi^0(-\bar{\lambda})\Big(1-\frac{T_1}{\bar{\lambda}}+\frac{T_2}{\bar{\lambda}^2}-\frac{T_3}{\bar{\lambda}^3}+\frac{T_4}{\bar{\lambda}^4}-\frac{T_5}{\bar{\lambda}^5}\Big)-\frac{\varphi(-\bar{\lambda})}{\bar{\lambda}^5},$$
while the conjugate of (6.14) gives
$$\overline{\chi(\lambda)\Big(1-\frac{\lambda}{\lambda_{-0}}\Big)^{-1}\Big(1-\frac{\lambda}{\lambda_{+0}}\Big)^{-1}}=C_0\overline{\chi^0(\lambda)}\Big(1+\frac{\bar{T}_1}{\bar{\lambda}}+\frac{\bar{T}_2}{\bar{\lambda}^2}+\frac{\bar{T}_3}{\bar{\lambda}^3}+\frac{\bar{T}_4}{\bar{\lambda}^4}+\frac{\bar{T}_5}{\bar{\lambda}^5}\Big)+\frac{\bar{\varphi(\lambda)}}{\bar{\lambda}^5},$$
then taking the difference between the above two equations and using $\chi(-\bar{\lambda})=\overline{\chi(\lambda)}$ and $\chi^0(-\bar{\lambda})=\overline{\chi^0(\lambda)}$ we get
$$0=C_0\overline{\chi(\lambda)}\Big[\Big(\frac{\bar{T}_1+T_1}{\bar{\lambda}}\Big)+\Big(\frac{\bar{T}_2-T_2}{\bar{\lambda}^2}\Big)+\Big(\frac{\bar{T}_3+T_3}{\bar{\lambda}^3}\Big)+\Big(\frac{\bar{T}_4-T_4}{\bar{\lambda}^4}\Big)+\Big(\frac{\bar{T}_5+T_5+\overline{\varphi(\lambda)}+\varphi(-\bar{\lambda})}{\bar{\lambda}^5}\Big)\Big].$$
Since $\frac{1}{\lambda},\frac{1}{\lambda^2},\frac{1}{\lambda^3},\frac{1}{\lambda^4},\frac{1}{\lambda^5}$ are linearly independent we can deduce that $\bar{T}_1=-T_1$, $\bar{T}_2=T_2$, $\bar{T}_3=-T_3$, $\bar{T}_4=T_4$ and $-\varphi(\bar{\lambda})=-2\mathrm{Re}T_5-\varphi(-\bar{\lambda})$, which implies that T_1,T_3 are pure imaginary,

while $T_2, T_4 \in \mathbb{R}$. Now if $\mathrm{Re}T_5 \neq 0$ then this contradicts the fact that $\varphi \in L_2(-\infty,\infty)$, so we have that $\mathrm{Re}T_5 = 0$ and $\varphi(-\bar{\lambda}) = -\overline{\varphi(\lambda)}$. Altogether we have that

$$\chi(\lambda) = \Big(1 - \frac{\lambda}{\lambda_{+0}} - \frac{\lambda}{\lambda_{-0}} + \frac{\lambda^2}{\lambda_{+0}\lambda_{-0}}\Big) C_0 \chi^0(\lambda) \Big(1 + \frac{T_1}{\lambda} + \frac{T_2}{\lambda^2} + \frac{T_3}{\lambda^3} + \frac{T_4}{\lambda^4} + \frac{T_5}{\lambda^5}\Big)$$
$$+ \frac{\varphi(\lambda)}{\lambda^5}\Big(1 - \frac{\lambda}{\lambda_{+0}} - \frac{\lambda}{\lambda_{-0}} - \frac{\lambda^2}{|\lambda_{+0}|^2}\Big),$$

which upon simplification gives

$$\chi(\lambda) = -\lambda^2 K_1 \cos\lambda a + i\lambda K_2 \cos\lambda a + \lambda K_3 \sin\lambda a + K_4 \cos\lambda a + iK_5 \sin\lambda a + iK_6 \frac{\cos\lambda a}{\lambda}$$
$$+ K_7 \frac{\sin\lambda a}{\lambda} + K_8 \frac{\cos\lambda a}{\lambda^2} + + iK_9 \frac{\sin\lambda a}{\lambda^2} + iK_{10} \frac{\cos\lambda a}{\lambda^3} + K_{11} \frac{\sin\lambda a}{\lambda^3} + \frac{\psi(\lambda)}{\lambda^3}$$

where $K_1 = \frac{C_0}{|\lambda_{+0}|^2} \neq 0$. □

Proposition 6.6. $K_1 > 0$

Proof. We are going to prove the proposition by proving that $C_0 > 0$ using $\lambda^0_{-k} = -\bar{\lambda}^0_{-k}$ and that $\lambda_{-k} = -\bar{\lambda}_k$, for all not pure imaginary. Recall that

$$C_0 = \prod_{-\infty, k\neq 0}^{\infty} \Big(1 + \delta_k (\lambda^0_k)^{-6}\Big)^{-1},$$

and suppose that λ^0_k is not pure imaginary, then

$$\begin{aligned} \lambda_{-k} &= \lambda^0_{-k} + \delta_{-k}(\lambda^0_{-k})^{-5} \\ &= -\bar{\lambda}^0_k + \delta_{-k}(-\bar{\lambda}^0_k)^{-5}, \end{aligned}$$

while

$$-\bar{\lambda}_k = -\bar{\lambda}^0_k - \bar{\delta}_k(\bar{\lambda}^0_k)^{-5}$$

then taking the difference between the two equation we obtain

$$0 = -\frac{1}{(\bar{\lambda}^0_k)^5}(-\delta_{-k} + \bar{\delta}_k), \quad \forall k, \;\; |k| > n$$

which implies that $\delta_{-k} = \bar{\delta}_k$. So for $|k| > n$, we note that

$$\begin{aligned} &(1 + \delta_{-k}(\lambda^0_{-k})^{-6})^{-1}(1 + \delta_k(\lambda^0_k)^{-6})^{-1} \\ &= (1 + \bar{\delta}_k(\bar{\lambda}^0_k)^{-6})^{-1}(1 + \delta_k(\lambda^0_k)^{-6})^{-1} \\ &= \overline{(1 + \delta_k(\lambda^0_k)^{-6})^{-1}}(1 + \delta_k(\lambda^0_k)^{-6})^{-1} \\ &= \frac{1}{\Big|1 + \delta_k(\lambda^0_k)^{-6}\Big|^2} > 0. \end{aligned}$$

We can also write (6.13) as

$$\frac{\lambda_k}{\lambda^0_k} = 1 + \delta_k(\lambda^0_k)^{-6},$$

so then

$$C_0 = \prod_{k=-n,k\neq 0}^{n} \Big(1+\delta_k(\lambda_k^0)^{-6}\Big)^{-1} \prod_{|k|>n} \Big(1+\delta_k(\lambda_k^0)^{-6}\Big)^{-1}$$
$$= \prod_{k=-n,k\neq 0}^{n} \Big(\frac{\lambda_k}{\lambda_k^0}\Big)^{-1} \prod_{|k|>n} \Big(1+\delta_k(\lambda_k^0)^{-6}\Big)^{-1} > 0,$$

since $\mathrm{Im}\lambda_k$ and $\mathrm{Im}\lambda_k^0 > 0$ for all k (where the first product is over the pure imaginary λ_k and λ_k^0). Note that the above computation assumed that the non pure imaginary eigenvalue λ_k^0 is perturbed into λ_k which is also not pure imaginary. Suppose now that the pure imaginary eigenvalue λ_k^0 is perturbed into λ_k which is not purely imaginary (we have only a finite number of such λ_k^0 to consider since there are only a finite number of λ_k^0 which are pure imaginary). Therefore, the product

$$C_0 = \prod_{k=-n,k\neq 0}^{n} \Big(\frac{\lambda_k}{\lambda_k^0}\Big)^{-1} \prod_{|k|>n} \Big(1+\delta_k(\lambda_k^0)^{-6})^{-1} > 0$$

since for any pair λ_k and λ_k^0 from the first product we have that

$$\frac{\lambda_k}{\lambda_k^0}\cdot\frac{\lambda_{-k}}{\lambda_{-k}^0} = \frac{-\lambda_k\cdot\bar{\lambda}_k}{-\mathrm{Im}\lambda_k^0\cdot\mathrm{Im}\lambda_{-k}^0} > 0.$$

□

Now that we know from Lemma 6.5 that the function $\chi(\lambda)$ takes the form

$$\chi(\lambda) = -\lambda^2 K_1\cos\lambda a + i\lambda K_2\cos\lambda a + \lambda K_3\sin\lambda a + K_4\cos\lambda a + iK_5\sin\lambda a + iK_6\frac{\cos\lambda a}{\lambda}$$
$$+K_7\frac{\sin\lambda a}{\lambda} + K_8\frac{\cos\lambda a}{\lambda^2} + +iK_9\frac{\sin\lambda a}{\lambda^2} + iK_{10}\frac{\cos\lambda a}{\lambda^3} + K_{11}\frac{\sin\lambda a}{\lambda^3} + \frac{\psi(\lambda)}{\lambda^3}$$

we now write the following relationships

$$\chi(\frac{2\pi k}{a}) = -(\frac{2\pi k}{a})^2 K_1 + i\frac{2\pi k}{a}K_2 + K_4 + i\frac{K_6}{(\frac{2\pi k}{a})} + K_8\Big(\frac{a}{2\pi k}\Big)^2 + iK_{10}\Big(\frac{a}{2\pi k}\Big)^3 + \frac{\psi(\frac{2\pi k}{a})}{(\frac{2\pi k}{a})^3}$$

and

$$\chi\Big(\frac{2\pi}{a}(k+\frac{1}{4})\Big) = \frac{2\pi}{a}(k+\frac{1}{4})K_3 + iK_5 + \frac{K_7}{\frac{2\pi}{a}(k+\frac{1}{4})} + iK_9\Big(\frac{a}{2\pi(k+\frac{1}{4})}\Big)^2 + K_{11}\Big(\frac{a}{2\pi(k+\frac{1}{4})}\Big)^3$$
$$+\frac{\psi(\frac{2\pi}{a}(k+\frac{1}{4}))}{(\frac{2\pi}{a}(k+\frac{1}{4}))^3}.$$

It now follows that the constants K_i, $i = 1,\ldots,11$ can be obtained as follows

$$K_1 = -\Big(\frac{a}{2\pi}\Big)^2 \lim_{k\to\infty} k^{-2}\chi(\frac{2\pi k}{a}),$$
$$K_2 = -i\Big(\frac{a}{2\pi}\Big)\lim_{k\to\infty} k^{-1}\Big(\chi(\frac{2\pi k}{a}) + (\frac{2\pi k}{a})^2 K_1\Big),$$
$$K_3 = (\frac{a}{2\pi})\lim_{k\to\infty} k^{-1}\Big(\chi(\frac{2\pi}{a}(k+\frac{1}{4}))\Big),$$
$$K_4 = \lim_{k\to\infty}\Big(\chi(\frac{2\pi k}{a}) + (\frac{2\pi k}{a})^2 K_1 - (\frac{2\pi k}{a})iK_2\Big),$$
$$K_5 = -i\lim_{k\to\infty}\Big(\chi(\frac{2\pi}{a}(k+\frac{1}{4})) - \frac{2\pi}{a}(k+\frac{1}{4})K_3\Big),$$

$$K_6=\frac{2\pi}{a}\lim_{k\to\infty}k\Big(\chi(\frac{2\pi k}{a})+(\frac{2\pi k}{a})^2K_1-(\frac{2\pi k}{a})iK_2-K_4\Big),$$
$$K_7=\frac{2\pi}{a}\lim_{k\to\infty}k\Big(\chi(\frac{2\pi}{a}(k+\frac{1}{4}))-\frac{2\pi}{a}(k+\frac{1}{4})K_3-iK_5\Big),$$
$$K_8=\Big(\frac{2\pi}{a}\Big)^2\lim_{k\to\infty}k^2\Big(\chi(\frac{2\pi k}{a})+(\frac{2\pi k}{a})^2K_1-i\Big(\frac{2\pi k}{a}\Big)K_2-K_4-iK_6\Big(\frac{a}{2\pi k}\Big)\Big)$$
$$K_9=-i\Big(\frac{2\pi}{a}\Big)^2\lim_{k\to\infty}k^2\Big(\chi(\frac{2\pi}{a}(k+\frac{1}{4}))-\frac{2\pi}{a}(k+\frac{1}{4})K_3-iK_5-K_7\Big(\frac{a}{2\pi(k+\frac{1}{4})}\Big)\Big)$$
$$K_{10}=-i\Big(\frac{2\pi}{a}\Big)^3\lim_{k\to\infty}k^3\Big(\chi(\frac{2\pi k}{a})+(\frac{2\pi k}{a})^2K_1-i\Big(\frac{2\pi k}{a}\Big)K_2-K_4-iK_6\Big(\frac{a}{2\pi k}\Big)-K_8\Big(\frac{a}{2\pi k}\Big)^2\Big)$$
$$K_{11}=\Big(\frac{2\pi}{a}\Big)^3\lim_{k\to\infty}k^3\Big(\chi(\frac{2\pi}{a}(k+\frac{1}{4}))-\frac{2\pi}{a}(k+\frac{1}{4})K_3-iK_5-K_7\Big(\frac{a}{2\pi(k+\frac{1}{4})}\Big)-iK_9\Big(\frac{a}{2\pi(k+\frac{1}{4})}\Big)^2\Big)$$

In order to recover M_1, M_2, M_3 and M_4 we define

$$M_2=K_1 \tag{6.16}$$

$$M_1=K_2 \tag{6.17}$$

$$M_3=\frac{K_5}{K_2}+\Big(\frac{K_6-(K_1K_{10}+K_2K_8)}{K_5-(K_2K_7+K_1K_9)}\Big) \tag{6.18}$$

$$M_4=\Big(\frac{K_6-(K_1K_{10}+K_2K_8)}{K_5-(K_2K_7+K_1K_9)}\Big)+\frac{K_2K_7+K_1K_9}{K_2} \tag{6.19}$$

We have to remark here that M_3 and M_4 can not be uniquely expressed in terms of K_i given a number of identities that exists in this problem, so from now onwards we will use interchangeably the following identity

$$\frac{K_2(K_5-(K_2K_7+K_1K_9))}{K_1^2K_2}=-\frac{(K_1(K_2K_7+K_1K_9)+(K_1K_3K_6+K_2K_3K_6))}{K_1^3}$$

Definition 6.7. The entire function $w(\lambda)$ is said to be a function of Hermite-Biehler class (HB) if it has no zeroes in the closed lower half plane and if

$$\Big|\frac{w(\lambda)}{\bar{w}(\lambda)}\Big|<1 \text{ for } \operatorname{Im}\lambda>0.$$

We now give an equivalent definition of a function of HB type, the proof of which can be found in Theorem 6 of [8].

If a function $w(z)$ has a representation

$$w(z)=z^m e^{u(z)+i(\gamma z+\delta)}\prod_1^\infty\Big(1-\frac{z}{a_k}\Big)e^{RP_m(\frac{z}{a_k})},$$

and

$$\sum_1^\infty\Big|\operatorname{Im}(\frac{1}{a_k})\Big|<\infty,\ \ \operatorname{Im}(a_k)>0\ \ \forall k,$$

where $u(z)$ is a real entire function, $\gamma\geq 0$, δ a real constant and RP_m the real part of P_m where

$$P_m(\frac{z}{a_k})=\Big(\frac{z}{a_k}+\frac{z^2}{2a_k^2}+\cdots+\frac{z^m}{ma_k^m}\Big) \text{ for } m\geq 0$$

(called the Weierstrass primary factors) then $w(z)\in HB$.

Lemma 6.8. $\chi(\lambda) \in HB$

Proof. Using the equivalent definition of HB functions to prove the lemma and the asymptotic expansion of λ_k

$$\lambda_k \underset{k\to\infty}{=} \frac{\pi}{a}(k-\frac{1}{2}) + \frac{Q_1}{(k-\frac{1}{2})} + i\frac{Q_2}{(k-\frac{1}{2})^2} + \frac{Q_3}{(k-\frac{1}{2})^3} + i\frac{Q_4}{(k-\frac{1}{2})^4} + \frac{Q_5}{(k-\frac{1}{2})^5} + \frac{\beta_k}{k^5}$$

we observe that

$$\mathrm{Im}(\frac{1}{\lambda_k}) \underset{k\to\infty}{=} \frac{-\frac{Q_2}{(k-\frac{1}{2})^2} - \frac{Q_4}{(k-\frac{1}{2})^4} - \frac{\mathrm{Im}(\beta_k)}{k^5}}{\left|\frac{\pi}{a}(k-\frac{1}{2}) + \frac{Q_1}{(k-\frac{1}{2})} + i\frac{Q_2}{(k-\frac{1}{2})^2} + \frac{Q_3}{(k-\frac{1}{2})^3} + i\frac{Q_4}{(k-\frac{1}{2})^4} + \frac{Q_5}{(k-\frac{1}{2})^5} + \frac{\beta_k}{k^5}\right|^2}.$$

Since $\{\beta_k\}_{-\infty}^{\infty} \in l_2$ then we must have that $\lim_{|k|\to\infty}\beta_k = 0$, so then

$$\frac{\mathrm{Im}(\beta_k)}{k^5} \to 0, \quad \frac{Q_4}{(k-\frac{1}{2})^4} \to 0 \text{ and} -\frac{Q_2}{(k-\frac{1}{2})^2} \to 0 \text{ as } |k| \to \infty,$$

which allows us to simplify the following

$$-\frac{Q_2}{(k-\frac{1}{2})^2} - \frac{Q_4}{(k-\frac{1}{2})^4} - \frac{\mathrm{Im}(\beta_k)}{k^5} = O(1),$$

and

$$\left|\frac{\pi}{a}(k-\frac{1}{2}) + \frac{Q_1}{(k-\frac{1}{2})} + i\frac{Q_2}{(k-\frac{1}{2})^2} + \frac{Q_3}{(k-\frac{1}{2})^3} + i\frac{Q_4}{(k-\frac{1}{2})^4} + \frac{Q_5}{(k-\frac{1}{2})^5} + \frac{\beta_k}{k^5},\right|^2 = O((k-\frac{1}{2})^2),$$

from which we ultimately deduce that

$$\left|\mathrm{Im}(\frac{1}{\lambda_k})\right| = \left|O((k-\frac{1}{2})^{-2})\right|,$$

therefore

$$\sum_{|k|>n_0} \left|\mathrm{Im}(\frac{1}{\lambda_k})\right| < \infty.$$

Altogether we have that

$$\sum_{-\infty,k\neq 0}^{\infty} \left|\mathrm{Im}(\frac{1}{\lambda_k})\right| < \infty,$$

and we know that for all k that $\mathrm{Im}\lambda_k > 0$, and χ as defined by equation (6.4) admits the product representation depicted in Theorem 6 of [8], therefore $\chi(\lambda) \in HB$. □

Lemma 6.9.

$$K_2K_3 + K_1K_5 = 0 \tag{6.20}$$

Proof. We know from Lemma 6.5 that the function χ can be represented as in equation (6.5). Now applying Theorem 4.8 we get that

$$\begin{aligned}
\frac{\pi^2}{a}Q_2 &= -P_1P_2 - P_4 \\
&= \frac{K_2}{K_1}\Big(-\frac{K_3}{K_1}\Big) - \frac{K_5}{K_1} \\
&= -\frac{K_2K_3}{K_1^2} - \frac{K_5}{K_1} \\
&= -\Big(\frac{K_2K_3 + K_1K_5}{K_1^2}\Big).
\end{aligned}$$

Since we know that $a, K_1^2 > 0$ and $Q_2 = 0$ it then follows that $K_2K_3 + K_1K_5 = 0$. $\square$

Lemma 6.10. *The derivative of the function χ at zero satisfy $Re\chi'(0) = 0$ and $Im\chi'(0) > 0$*

Proof. Recall that

$$\chi(\lambda) = \lim_{n\to\infty} \prod_{k=+0}^{n} \Big(1 - \frac{\lambda}{\lambda_k}\Big) \prod_{k=-n}^{-0} \Big(1 - \frac{\lambda}{\lambda_k}\Big),$$

then differentiating the function logarithmically we obtain

$$\chi'(\lambda) = \sum_{-\infty}^{\infty} \frac{f'(\lambda)}{f(\lambda)}.\chi(\lambda),$$

where $f(\lambda) = (1 - \frac{\lambda}{\lambda_k})$. Therefore

$$\begin{aligned}\chi'(0) =& \sum_{0}^{\infty} \Big(-\frac{1}{\lambda_k} - \frac{1}{\lambda_{-k}}\Big)\chi(0)\\ =& -\sum_{0}^{\infty} \Big(\frac{\lambda_{+k} + \lambda_{-k}}{\lambda_{+k}\lambda_{-k}}\Big)\\ =& \sum_{0}^{\infty} \Big(\frac{i(\mathrm{Im}\lambda_{+k} + \mathrm{Im}\lambda_{-k})}{-\lambda_{+k}\lambda_{-k}}\Big),\end{aligned}$$

from which the results follow. $\square$

Corollary 6.11. $K_2 > 0$

Proof. Let us consider the function $\chi(\lambda)$ as a perturbation of the function $\varphi_1(\lambda)$ defined by

$$\varphi_1(\lambda) = -\lambda^2 K_1 \cos\lambda a = -\lambda^2 K_1 \prod_{1}^{\infty} \Big(1 - \frac{4a^2\lambda^2}{(2k-1)^2\pi^2}\Big).$$

We now rewrite our function $\chi(\lambda)$ as follows

$$\begin{aligned}\chi(\lambda) =& \varphi_1(\lambda). \lim_{n\to\infty} \prod_{+0}^{n} \Big(1 - \frac{\lambda}{\lambda_k}\Big) \prod_{-n}^{-0} \Big(1 - \frac{\lambda}{\lambda_k}\Big).(-\lambda^{-2}K_1^{-1}) \prod_{1}^{\infty} \Big(1 - \frac{4a^2\lambda^2}{(2k-1)^2\pi^2}\Big)^{-1}\\ =& \varphi_1(\lambda)\Big(1 - \frac{\lambda}{\lambda_{+0}}\Big)\Big(1 - \frac{\lambda}{\lambda_{-0}}\Big) \lim_{n\to\infty} \prod_{+1}^{n} \Big(1 - \frac{\lambda}{\lambda_k}\Big) \prod_{-n}^{-1} \Big(1 - \frac{\lambda}{\lambda_k}\Big)\\ &\cdot \prod_{1}^{\infty} \Big(1 - \frac{4a^2\lambda^2}{(2k-1)^2\pi^2}\Big)^{-1} (-\lambda^{-2}K_1^{-1})\\ =& -\lambda^{-2}K_1^{-1}\varphi_1(\lambda)\Big(1 - \frac{\lambda}{\lambda_{+0}}\Big)\Big(1 - \frac{\lambda}{\lambda_{-0}}\Big) \lim_{n\to\infty} \prod_{1}^{n} \Big(1 - \frac{\lambda}{\lambda_{+k}}\Big)\Big(1 - \frac{\lambda}{\lambda_{-k}}\Big)\\ &\cdot \prod_{1}^{\infty} \Big(1 - \frac{4a^2\lambda^2}{(2k-1)^2\pi^2}\Big)^{-1}\end{aligned}$$

$$\begin{aligned}\chi(\lambda) &= -\lambda^{-2}K_1^{-1}\varphi_1(\lambda)\Big(1-i\lambda\frac{\mathrm{Im}\lambda_{+0}+\mathrm{Im}\lambda_{-0}}{\lambda_{+0}\lambda_{-0}}+\frac{\lambda^2}{\lambda_{+0}\lambda_{-0}}\Big)\\ &\cdot\prod_1^{\infty}\Big(1-i\lambda\frac{\mathrm{Im}\lambda_{+k}+\mathrm{Im}\lambda_{-k}}{\lambda_{+k}\lambda_{-k}}+\frac{\lambda^2}{\lambda_{+k}\lambda_{-k}}\Big).\Big(1-\frac{4a^2\lambda^2}{(2k-1)^2\pi^2}\Big)^{-1}\\ &= -K_1^{-1}\varphi_1(\lambda)\Big(\lambda^{-2}-i\frac{\mathrm{Im}\lambda_{+0}+\mathrm{Im}\lambda_{-0}}{\lambda_{+0}\lambda_{-0}\lambda}+\frac{1}{\lambda_{+0}\lambda_{-0}}\Big)\\ &\cdot\prod_1^{\infty}\Big(\lambda^{-2}-i\frac{\mathrm{Im}\lambda_{+k}+\mathrm{Im}\lambda_{-k}}{\lambda_{+k}\lambda_{-k}\lambda}+\frac{1}{\lambda_{+k}\lambda_{-k}}\Big)\\ &.\Big(\lambda^{-2}-\frac{4a^2}{(2k-1)^2\pi^2}\Big)^{-1}.\end{aligned}$$

Now substituting $\lambda = \frac{2\pi n}{a}$, $n \in \mathbb{N}$ and evaluating the function as $n \to \infty$

$$\begin{aligned}\chi(\lambda) &= -K_1^{-1}(-\lambda^2K_1)\Big(-i\frac{\mathrm{Im}\lambda_{+0}+\mathrm{Im}\lambda_{-0}}{\lambda_{+0}\lambda_{-0}\lambda}+\frac{1}{\lambda_{+0}\lambda_{-0}}\Big)\\ &\cdot\lim_{p\to\infty}\prod_1^{p}\Big(-i\frac{\mathrm{Im}\lambda_{+k}+\mathrm{Im}\lambda_{-k}}{\lambda_{+k}\lambda_{-k}\lambda}+\frac{1}{\lambda_{+k}\lambda_{-k}}\Big).\Big(\frac{-(2k-1)^2\pi^2}{4a^2}\Big)+O(1)\\ &= \frac{\lambda^2}{\lambda_{+0}\lambda_{-0}}\Big(-i\frac{\mathrm{Im}\lambda_{+0}+\mathrm{Im}\lambda_{-0}}{\lambda}+1\Big)\\ &\cdot\lim_{p\to\infty}\prod_1^{p}\frac{1}{\lambda_{+k}\lambda_{-k}}\Big(-i\frac{\mathrm{Im}\lambda_{+k}+\mathrm{Im}\lambda_{-k}}{\lambda}+1\Big).\Big(-\frac{(2k-1)^2\pi^2}{4a^2}\Big)+O(1).\\ &= \lim_{p\to\infty}\prod_1^{p}\frac{\lambda^2}{\lambda_{+0}.\lambda_{-0}.\lambda_{+k}.\lambda_{-k}}\Big[\frac{-(\mathrm{Im}\lambda_{-0}+\mathrm{Im}\lambda_{+0})(\mathrm{Im}\lambda_{-k}+\mathrm{Im}\lambda_{+k})}{\lambda^2}\\ &-i\frac{(\mathrm{Im}\lambda_{-0}+\mathrm{Im}\lambda_{+0}+\mathrm{Im}\lambda_{-k}+\mathrm{Im}\lambda_{+k})}{\lambda}+1\Big].\prod_1^{\infty}\Big(-\frac{(2k-1)^2\pi^2}{4a^2}\Big)+O(1)\end{aligned}$$

Then taking the imaginary part of the above equation we get

$$\mathrm{Im}\chi(\lambda) = -\sum_{k=0}^{\infty}\lambda.(\mathrm{Im}\lambda_{+k}+\mathrm{Im}\lambda_{-k})\prod_1^{\infty}\frac{1}{|\lambda_{+0}|^2|\lambda_k|^2}.\prod_1^{\infty}\Big(-\frac{(2k-1)^2}{4a^2}\Big),$$

which is positive since $\mathrm{Im}\lambda_k > 0$, for all $k \in \mathbb{Z}^+$ and comparing this equation with equation (6.5) imply that $K_2 > 0$. □

Lemma 6.12. *Assuming the identity*

$$\frac{K_2(K_5-(K_2K_7+K_1K_9))}{K_1^2K_2} = -\frac{(K_1(K_2K_7+K_1K_9)+(K_1K_3K_6+K_2K_3K_6))}{K_1^3}$$

then

$$M_3 - M_4 > 0$$

Proof. We know from equation 6.1 that

$$\lambda_k \underset{k\to\infty}{=} \frac{\pi}{a}(k-\frac{1}{2})+\frac{Q_1}{(k-\frac{1}{2})}+i\frac{Q_2}{(k-\frac{1}{2})^2}+\frac{Q_3}{(k-\frac{1}{2})^3}+i\frac{Q_4}{(k-\frac{1}{2})^4}+\frac{Q_5}{(k-\frac{1}{2})^5}+\frac{\beta_k}{k^5}$$

where $\{\beta_k\}_{-\infty}^{\infty} \in l_2$, $a>0$, $Q_i \in \mathbb{R}$, $j=1,2,\dots,5$ and that the function χ can be written as

$$\begin{aligned}\chi(\lambda) =& -\lambda^2 K_1 \cos\lambda a + i\lambda K_2 \cos\lambda a + \lambda K_3 \sin\lambda a + K_4 \cos\lambda a + iK_5 \sin\lambda a + iK_6 \frac{\cos\lambda a}{\lambda} \\ & + K_7 \frac{\sin\lambda a}{\lambda} + K_8 \frac{\cos\lambda a}{\lambda^2} + iK_9 \frac{\sin\lambda a}{\lambda^2} + iK_{10} \frac{\cos\lambda a}{\lambda^3} + K_{11} \frac{\sin\lambda a}{\lambda^3} \\ & + \frac{\psi(\lambda)}{\lambda^3},\end{aligned}$$

where K_j, $j=1,2,\dots,11$ are constants and ψ is an entire function of exponential type $\leq a$ which belongs to $L_2(-\infty,\infty)$,. Then applying Theorem 4.8 we obtain that

$$\begin{aligned}Q_4 =& \frac{aP_2Q_2}{\pi^2} + \frac{(P_4+P_1P_2)2a^2Q_1}{\pi^3} \\ & - \frac{a^3(P_8+P_2P_5+P_1P_6+P_3P_4+2P_1P_2P_3-P_2^2P_4-P_1^2P_4-P_2^3P_1-P_1^3P_2)}{\pi^4}.\end{aligned}$$

We also know from Theorem 4.8 that

$$Q_2 = -\frac{a(P_1P_2+P_4)}{\pi^2}$$

and Theorem 6.1 states that $Q_2=0$ which leads to

$$P_1P_2+P_4=0.$$

So then using this identity we deduce that

$$\begin{aligned}& P_8+P_2P_5+P_1P_6+P_3P_4+2P_1P_2P_3-P_2^2P_4-P_1^2P_4-P_2^3P_1-P_1^3P_2 \\ &= P_8+P_2P_5+P_1P_6+P_1P_2P_3 \\ &= \frac{(K_1(K_2K_7+K_1K_9)+(K_1K_3K_6+K_2K_3K_4))}{K_1^3} \\ &= -\frac{K_2(K_5-(K_2K_7+K_1K_9))}{K_2K_1^2} \\ &= -\frac{K_2}{K_1^2}(M_3-M_4).\end{aligned}$$

Since $Q_4>0$ and $K_2>0$ it follows that $M_3-M_4>0$ □

We now write the function χ as follows

$$\chi(\lambda) = P(\lambda) - \lambda^2 K_1 Q(\lambda) + i\lambda K_2 Q(\lambda) \tag{6.21}$$

where

$$\begin{aligned}i\lambda K_2 Q(\lambda) =& i\lambda K_2 \cos\lambda a + iK_5 \sin\lambda a + iK_6 \frac{\sin\lambda a}{\lambda} + iK_9 \frac{\sin\lambda a}{\lambda^2} + iK_{10} \frac{\cos\lambda a}{\lambda^3} \\ & + \Big(\frac{\psi(\lambda)+\psi(-\lambda)}{2\lambda^3}\Big),\end{aligned}$$

from which we get that the function Q is given by

$$Q(\lambda) = \cos\lambda a + \frac{K_5}{K_2}\frac{\sin\lambda a}{\lambda} + \frac{K_6}{K_2}\frac{\cos\lambda a}{\lambda^2} + \frac{K_9}{K_2}\frac{\sin\lambda a}{\lambda^3} + \frac{K_{10}}{K_2}\frac{\cos\lambda a}{\lambda^4} - i\Big(\frac{\psi(\lambda)+\psi(-\lambda)}{K_2\lambda^4}\Big). \tag{6.22}$$

Similarly, equating

$$P(\lambda)-\lambda^2K_1Q(\lambda)=-\lambda^2K_1\cos\lambda a+\lambda K_3\sin\lambda a+K_4\cos\lambda a+K_7\frac{\sin\lambda a}{\lambda}+K_8\frac{\cos\lambda a}{\lambda^2}+K_{11}\frac{\sin\lambda a}{\lambda^3}+\frac{(\psi(\lambda)-\psi(-\lambda))}{\lambda^3},$$

we get that the function P is given by

$$P(\lambda)=\cos\lambda a+\Big(\frac{K_2K_7+K_1K_9}{K_2}\Big)\frac{\sin\lambda a}{\lambda}+\Big(\frac{K_1K_{10}+K_2K_8}{K_2}\Big)\frac{\cos\lambda a}{\lambda^2}+K_{11}\frac{\sin\lambda a}{\lambda^3}-i\Big(\frac{K_1\psi(\lambda)+K_1\psi(-\lambda)}{K_2\lambda^2}\Big)+\Big(\frac{\psi(\lambda)-\psi(-\lambda)}{\lambda^3}\Big). \tag{6.23}$$

Now let us denote by $\{\tau_k\}_{-\infty,k\neq 0}^{\infty}$ the zeroes of the function $Q(\lambda)$ and by $\{\theta_k\}_{-\infty}^{\infty}$ the zeroes of the function $P(\lambda)$.

Definition 6.13. A function $w(\lambda)$ of complex variable λ is said to be an R function (Nevanlinna function) if

(1) It is holomorphic in each of the half planes $\mathrm{Im}\lambda>0$ and $\mathrm{Im}\lambda<0$.
(2) $w(\bar{\lambda})=\bar{w}(\lambda)$.
(3) $\mathrm{Im}\lambda\cdot\mathrm{Im}w(\lambda)\geq 0$, $\mathrm{Im}\lambda\neq 0$.

Now since our function $\chi(\lambda)$ can be written as

$$\chi(\lambda)=P(\lambda)-\lambda^2K_1Q(\lambda)+i\lambda K_2Q(\lambda),$$

where $P(\lambda)$ and $Q(\lambda)$ are as defined before, and we consider the following function $\eta(\lambda,\nu)=\chi(\lambda)+\lambda^2\nu Q(\lambda)$ as a perturbation of the function $\chi(\lambda)$. The zeroes of the function $\eta(\lambda,\nu)$ are continuous in ν as ν approaches zero and do not cross the real axis as ν change from zero to a positive number. Suppose that the zeroes of $\eta(\lambda,\nu)$ crossed the real axis as ν changed from zero to $K_1>0$, then we have that $\eta(\lambda,K_1)=0$ for some $\lambda\in\mathbb{R}$ which implies that $P(\lambda)=\lambda Q(\lambda)=0$ since P and Q are real entire functions. This in turn implies that $\chi(\lambda)=0$ for some $\lambda\in\mathbb{R}$, which contradicts that $\mathrm{Im}\lambda_k>0$ for all zeroes λ_k of χ. So then we have that the zeroes of function $\eta(\lambda,\nu)$ remains in the upper half planes when ν changes from zero to a positive number. We also know that $\eta(\lambda,K_1)=P(\lambda)+i\lambda Q(\lambda)$ is precisely the case depicted by equation (3.15) in [12] and Corollarry 3.9 also of [12] gives that $\frac{Q}{P}(\lambda)$ is a Nevanlinna function. We note that the function $P(\lambda)$ and $P(\lambda)-\lambda^2K_1Q(\lambda)$ are even functions, they can be thought as function of variables $\zeta=\lambda^2$.

Corollary 6.14. *The function* $\frac{Q(\sqrt{\zeta})}{P(\sqrt{\zeta})-K_1\zeta Q(\sqrt{\zeta})}$ *is a Nevanlinna function*

Proof.

$$\begin{aligned}\frac{Q(\sqrt{\zeta})}{P(\sqrt{\zeta})-K_1\zeta Q(\sqrt{\zeta})}&=\Big(\frac{1}{\frac{P(\sqrt{\zeta})-K_1\zeta Q(\sqrt{\zeta})}{Q(\sqrt{\zeta})}}\Big)\\&=\Big(\frac{1}{-K_1\zeta+\frac{P}{Q}(\sqrt{\zeta})}\Big)\\&=\Big(-K_1\zeta+\Big(\frac{Q}{P}(\sqrt{\zeta})\Big)^{-1}\Big)^{-1}.\end{aligned}$$

Then taking the imaginary part of the above equation gives that

$$\begin{aligned}
\operatorname{Im}\Big(\frac{Q(\sqrt{\zeta})}{P(\sqrt{\zeta})-K_1\zeta Q(\sqrt{\zeta})}\Big) &= \operatorname{Im}\Big(-K_1\zeta+\Big(\frac{Q}{P}(\sqrt{\zeta})\Big)^{-1}\Big)^{-1}\\
&= \frac{\operatorname{Im}\Big(\overline{-K_1\zeta}+\overline{\Big(\frac{Q}{P}(\sqrt{\zeta})\Big)^{-1}}\Big)}{\Big|\Big(-K_1\zeta+\Big(\frac{Q}{P}(\sqrt{\zeta})\Big)^{-1}\Big)\Big|^2}\\
&= \frac{\operatorname{Im}\Big(\overline{-K_1\zeta}+\Big(\frac{\frac{Q}{P}(\sqrt{\zeta})}{|\frac{Q}{P}(\sqrt{\zeta})|^2}\Big)^{-1}\Big)}{\Big|\Big(-K_1\zeta+\Big(\frac{Q}{P}(\sqrt{\zeta})\Big)^{-1}\Big)\Big|^2}>0, \quad \text{for } \operatorname{Im}\zeta>0
\end{aligned}$$

since $\frac{Q}{P}$ is a Nevanlinna function. □

Let us define the following functions

$$u_1(\lambda)=\frac{Q(\lambda)-P(\lambda)}{M_3-M_4} \tag{6.24}$$

$$u_2(\lambda)=\frac{M_3P(\lambda)-M_4Q(\lambda)}{M_3-M_4}. \tag{6.25}$$

Since we noted that our functions $P(\lambda)$ and $Q(\lambda)$ are even entire functions, then u_1, u_2 are also even entire functions which can be considered functions of variables $\zeta=\lambda^2$.

Lemma 6.15. *$\frac{u_1}{u_2}(\zeta)$ is a Nevanlinna function.*

Proof. We first note that

$$\frac{u_1}{u_2}(\bar{\zeta})=\frac{Q(\bar{\zeta})-P(\bar{\zeta})}{M_3P(\bar{\zeta})-M_4Q(\bar{\zeta})}=\overline{\frac{u_1}{u_2}}(\zeta),$$

and we also know that $M_3-M_4>0$ then evaluating the following

$$\begin{aligned}
\frac{u_1(\zeta)}{u_2(\zeta)} &= (Q(\zeta)-P(\zeta))(M_3P(\zeta)-M_4Q(\zeta))^{-1}\\
&= \Big(\frac{Q}{P}(\zeta)-1\Big)\Big(M_3-M_4\frac{Q}{P}(\zeta)\Big)^{-1}\\
&= \frac{\Big(\frac{Q}{P}(\zeta)-1\Big)\Big(M_3-M_4\overline{\frac{Q}{P}(\zeta)}\Big)}{\Big|M_3-M_4\frac{Q}{P}(\zeta)\Big|^2}\\
&= \frac{\Big(M_3\frac{Q}{P}(\zeta)-M_4|\frac{Q}{P}(\zeta)|^2-M_3+M_4\overline{\frac{Q}{P}(\zeta)}\Big)}{\Big|M_3-M_4\frac{Q}{P}(\zeta)\Big|^2}.
\end{aligned}$$

so then

$$\operatorname{Im}\frac{u_1(\zeta)}{u_2(\zeta)}=\frac{\Big(M_3-M_4\Big)\operatorname{Im}\frac{Q}{P}(\zeta)}{\Big|M_3-M_4\frac{Q}{P}(\zeta)\Big|^2}>0 \quad \text{for } \operatorname{Im}\zeta\neq 0$$

Since $\frac{Q}{P}$ is a Nevanlinna function we have that for any nonzero complex number with $\operatorname{Im}\zeta\neq 0$ and $M_3-M_4>0$ then we have that $M_3-M_4\frac{Q}{P}(\zeta)$ takes on non zero values in the upper and lower half plane, therefore the function $\frac{u_1}{u_2}$ is holomorphic in each half plane. Hence $\frac{u_1}{u_2}$ is a Nevanlinna function. □

Corollary 6.16. *The set $\{\mu_k^2\}_1^\infty$ of zeroes of $u_1(\lambda)$ interlace with the set $\{\nu_k^2\}_1^\infty$ of zeroes of the function $u_2(\lambda)$*

Proof. Since $\frac{u_1}{u_2}$ is a Nevanlinna function, then according to the Theorem 1 of [8] (see pg 308) we can deduce that

$$\nu_k < \mu_k < \nu_{k+1}, \quad \mu_{-1} < 0 < \nu_1$$

therefore

$$\nu_1^2 < \mu_1^2 < \nu_2^2 < \mu_2^2 < \nu_3^2 \cdots$$

□

Now let us set $v(\lambda) = u_1(\lambda)$ and $u(\lambda) = u_2(\lambda)$.

Lemma 6.17. *The set $\{\mu_k^2\}_1^\infty$ of zeroes of $u(\lambda)$ and the set $\{\nu_k^2\}_1^\infty$ of zeroes of the function $v(\lambda)$ behave asymptotically as follows:*

$$\nu_k = \frac{\pi k}{a} - \frac{a}{\pi k}\Big(\frac{K_6 - K_1K_{10} - K_7K_8}{K_5 - K_2K_7 - K_1K_9)}\Big) + \frac{b_k^-}{k}, \tag{6.26}$$

$$\mu_k = \frac{\pi}{a}(k - \frac{1}{2}) - \frac{a}{\pi k}\Big(\frac{K_6 - K_1K_{10} - K_7K_8}{K_5 - K_2K_7 - K_1K_9)}\Big) + \frac{b_k^+}{k} \tag{6.27}$$

where $\{b_k^\mp\}_{-\infty}^\infty \in l_2$.

Proof. Recall that

$$P(\lambda) = \cos\lambda a + \Big(\frac{K_2K_7 + K_1K_9}{K_2}\Big)\frac{\sin\lambda a}{\lambda} + \Big(\frac{K_1K_{10} + K_2K_8}{K_2}\Big)\frac{\cos\lambda a}{\lambda^2} + K_{11}\frac{\sin\lambda a}{\lambda^3}$$
$$- i\Big(\frac{K_1\psi(\lambda) + K_1\psi(-\lambda)}{K_2\lambda^2}\Big) + \Big(\frac{\psi(\lambda) - \psi(-\lambda)}{\lambda^3}\Big),$$

and that

$$Q(\lambda) = \cos\lambda a + \frac{K_5}{K_2}\frac{\sin\lambda a}{\lambda} + \frac{K_6}{K_2}\frac{\cos\lambda a}{\lambda^2} + \frac{K_9}{K_2}\frac{\sin\lambda a}{\lambda^3} + \frac{K_{10}}{K_2}\frac{\cos\lambda a}{\lambda^4} - i\Big(\frac{\psi(\lambda) + \psi(-\lambda)}{K_2\lambda^4}\Big)$$

then substituting $P(\lambda)$ and $Q(\lambda)$ into the following

$$v(\lambda) = u_1(\lambda) = \frac{Q(\lambda) - P(\lambda)}{(M_3 - M_4)}$$
$$= \Big[\Big(\cos\lambda a + \frac{K_5}{K_2}\frac{\sin\lambda a}{\lambda} + \frac{K_6}{K_2}\frac{\cos\lambda a}{\lambda^2} + \frac{K_9}{K_2}\frac{\sin\lambda a}{\lambda^3} + \frac{K_{10}}{K_2}\frac{\cos\lambda a}{\lambda^4} - i\Big(\frac{\psi(\lambda) + \psi(-\lambda)}{K_2\lambda^4}\Big)\Big)$$
$$- \Big(\cos\lambda a + \Big(\frac{K_2K_7 + K_1K_9}{K_2}\Big)\frac{\sin\lambda a}{\lambda} + \Big(\frac{K_1K_{10} + K_2K_8}{K_2}\Big)\frac{\cos\lambda a}{\lambda^2} + K_{11}\frac{\sin\lambda a}{\lambda^3}$$
$$- i\Big(\frac{K_1\psi(\lambda) + K_1\psi(-\lambda)}{K_2\lambda^2}\Big) + \Big(\frac{\psi(\lambda) - \psi(-\lambda)}{\lambda^3}\Big)\Big)\Big](M_3 - M_4)^{-1}$$
$$= \Big[\Big(\frac{K_5}{K_2} - \frac{(K_2K_7 + K_1K_9)}{K_2}\Big)\frac{\sin\lambda a}{\lambda} + \Big(\frac{K_6}{K_2} - \frac{(K_1K_{10} + K_2K_8)}{K_2}\Big)\frac{\cos\lambda a}{\lambda^2} + \Big(\frac{K_9}{K_2} - K_{11}\Big)\frac{\sin\lambda a}{\lambda^3}$$
$$+ \frac{K_{10}}{K_2}\frac{\cos\lambda a}{\lambda^4} - i\Big(\frac{\psi(\lambda) + \psi(-\lambda)}{K_2\lambda^4}\Big) + i\Big(\frac{K_1\psi(\lambda) + K_1\psi(-\lambda)}{K_2\lambda^2}\Big) - \Big(\frac{\psi(\lambda) - \psi(-\lambda)}{\lambda^3}\Big)\Big](M_3 - M_4)^{-1}$$
$$= \Big[\Big(\frac{K_5}{K_2} - \frac{(K_2K_7 + K_1K_9)}{K_2}\Big)\frac{\sin\lambda a}{\lambda} + \Big(\frac{K_6}{K_2} - \frac{(K_1K_{10} + K_2K_8)}{K_2}\Big)\frac{\cos\lambda a}{\lambda^2} + \frac{\psi_2(\lambda)}{\lambda^2}\Big](M_3 - M_4)^{-1}$$

where

$$\psi_2(\lambda) = \Big(\frac{K_9}{K_2} - K_{11}\Big)\frac{\sin\lambda a}{\lambda} + \frac{K_{10}}{K_2}\frac{\cos\lambda a}{\lambda^2} - i\Big(\frac{\psi(\lambda)+\psi(-\lambda)}{2K_2\lambda^2}\Big)$$
$$+ i\Big(\frac{K_1\psi(\lambda)+K_1\psi(-\lambda)}{2K_2}\Big) - \Big(\frac{\psi(\lambda)-\psi(-\lambda)}{2\lambda}\Big)\Big](M_3-M_4)^{-1}.$$

From the definitions of M_3 and M_4 we can see that

$$M_3 - M_4 = \frac{K_5 - K_2K_7 - K_1K_9}{K_2},$$

so then our equation reduces to

$$v(\lambda) = \frac{\sin\lambda a}{\lambda} + \frac{(K_6 - (K_1K_{10} + K_2K_8))}{(K_5 - (K_2K_7 + K_1K_9))}\frac{\cos\lambda a}{\lambda^2} + \frac{\psi_2(\lambda)}{\lambda^2}.$$

We recall that our entire function $\chi(\lambda)$ has the form

$$\chi(\lambda) = -\lambda^2K_1\cos\lambda a + i\lambda K_2\cos\lambda a + \lambda K_3\sin\lambda a + K_4\cos\lambda a + iK_5\sin\lambda a + iK_6\frac{\cos\lambda a}{\lambda}$$
$$+ K_7\frac{\sin\lambda a}{\lambda} + K_8\frac{\cos\lambda a}{\lambda^2} + iK_9\frac{\sin\lambda a}{\lambda^2} + iK_{10}\frac{\cos\lambda a}{\lambda^3} + K_{11}\frac{\sin\lambda a}{\lambda^3} + \frac{\psi(\lambda)}{\lambda^3},$$

and implies that the following relationship holds

$$\psi(0) = -iK_{10}$$
$$\psi'(0) = -aK_{11} - K_8$$
$$\psi''(0) = i(a^2K_{10} - 2aK_9 - 2K_6).$$

So then evaluating the function ψ_2 at zero gives that

$$\begin{aligned}
\psi_2(0) &= \Big[\Big(\frac{K_9}{K_2} - K_{11}\Big)a + \frac{K_{10}}{K_2}\frac{(1-\frac{a^2\lambda^2}{2})}{\lambda^2} - \frac{i}{K_2}\frac{(2\psi(0)+\psi''(0))}{2\lambda^2} \\
&\quad + i\frac{K_1}{K_2}\psi(0) - \psi'(0)\Big](M_3-M_4)^{-1} \\
&= \Big[\Big(\frac{K_9}{K_2} - K_{11}\Big)a + \frac{K_{10}}{\lambda^2K_2} - \frac{K_{10}a^2}{2K_2} - \frac{K_{10}}{\lambda^2K_2} \\
&\quad - i\frac{(a^2K_{10} - 2aK_9 - 2K_6)}{2K_2} + i\frac{K_1}{K_2}(-iK_{10}) - (aK_{11} - K_8)\Big](M_3-M_4)^{-1} \\
&= -\Big(\frac{K_6 - (K_2K_8 + K_1K_{10})}{K_2}\Big)(M_3-M_4)^{-1} \\
&= -\frac{(K_6 - (K_2K_8 + K_1K_{10}))}{(K_5 - (K_2K_7 + K_1K_9))}.
\end{aligned}$$

To prove the assymptotic distribution of zeroes we are going to use Lemma 3.4.2 of [11] which states that:

> In order that the function $y(z)$ admits the representation
>
> $$y(z) = \sin\pi z + A\pi\frac{4z}{4z^2-1}\cos\pi z + \frac{f(z)}{z},$$
>
> where
>
> $$f(z) = \int_0^\pi \tilde{f}(t)\cos zt\,dt, \quad \tilde{f}(t) \in L_2(0,\pi), \quad f(0) = 0,$$

it is necessary and sufficient that

$$y(z) = \pi z \prod_{1}^{\infty} k^{-2}(\mu_k^2 - z^2),$$

where $\mu_k = k - \frac{A}{k} + \frac{\alpha_k}{k}$ with $\{\alpha_k\}_1^\infty \in l_2$.

We make a change of variables given by $\lambda = \frac{\pi}{a} z$ on the function

$$\lambda v(\lambda) = \sin \lambda a + \frac{(K_6 - (K_2K_8 + K_1K_{10}))}{(K_5 - (K_2K_7 + K_1K_9))} \frac{\cos \lambda a}{\lambda} + \frac{\psi_2(\lambda)}{\lambda}$$

to get

$$(\frac{\pi}{a} z) v(\frac{\pi}{a} z) = \sin \pi z + L \frac{\cos \pi z}{z} + \frac{\psi_2(\frac{\pi}{a} z)}{(\frac{\pi}{a} z)},$$

where $L = \frac{a}{\pi} \frac{(K_6 - (K_2K_8 + K_1K_{10}))}{(K_5 - (K_2K_7 + K_1K_9))}$. We rewrite the above function as follows

$$\begin{aligned}
(\frac{\pi}{a} z) v(\frac{\pi}{a} z) &= \sin \pi z + L \frac{\cos \pi z}{z} + \frac{a}{\pi} \frac{\psi_2(\frac{\pi}{a} z)}{z} \\
&= \sin \pi z + L \frac{4z^2 - 1}{4z^2 - 1} \frac{\cos \pi z}{z} + \frac{a}{\pi} \frac{\psi_2(\frac{\pi}{a} z)}{z} \\
&= \sin \pi z + L \frac{4z}{4z^2 - 1} \frac{\cos \pi z}{z} + \frac{\tilde{\varphi}(z)}{z},
\end{aligned}$$

where

$$\tilde{\varphi}(z) = \frac{a}{\pi} \psi_2(\frac{\pi}{a} z) - L \frac{\cos \pi z}{4z^2 - 1},$$

and it is evident that $\tilde{\varphi}(0) = 0$ and that $\tilde{\varphi}(z) \in L_2(-\infty, \infty)$ since $\psi_2(z) \in L_2(-\infty, \infty)$ and is an entire function. In order to apply Lemma 3.4.2 of [11] we must first show that the function $\tilde{\varphi}(z)$ can be wtitten in the form

$$\tilde{\varphi}(z) = \int_0^\pi \tilde{g}(t) \cos zt \, dt$$

where $\tilde{g} \in L_2(0, \pi)$. Without positing the existence of the function $\tilde{g}$ we continue and expand the cosine of the above equation as follows

$$\begin{aligned}
\tilde{\varphi}(z) &= \int_0^\pi \tilde{g}(t) \Big(\frac{e^{izt} + e^{-izt}}{2} \Big) dt \\
&= \frac{1}{2} \Big(\int_0^\pi \tilde{g}(t) e^{izt} dt + \int_0^\pi \tilde{g}(t) e^{-izt} dt \Big) \\
&= \frac{1}{2} \Big(\int_0^\pi \tilde{g}(t) e^{izt} dt - \int_{-\pi}^0 \tilde{g}(-t) e^{-izt} dt \Big) \\
&= \frac{1}{2} \Big(\int_{-\pi}^\pi \tilde{h}(t) e^{izt} dt \Big)
\end{aligned}$$

where

$$\tilde{h}(t) = \begin{cases} \tilde{g}(t) & \text{if } t \in [0, \pi] \\ -\tilde{g}(-t) & \text{if } t \in [-\pi, 0), \end{cases}$$

Now since our function $\tilde{\varphi}$ is an entire function of exponential type $\leq a$ and belongs to $L_2(-\infty, \infty)$ then according to the Paley Weiner theorem which states that:

An entire function $f(z)$ is of exponential type and belongs to $L_2(-\infty,\infty)$ on the real axis if and only if

$$f(z)=\int_{-\tau}^{\tau}e^{izt}\phi(t)dt$$

where $\phi\in L_2(-\tau,\tau)$.

Therefore, an application of the Paley Wiener theorem on our function $\tilde{\varphi}(z)$ gives that for our function $\tilde{\varphi}(z)$ there exist a function $\tilde{g}$ such that $\tilde{\varphi}$ can be expressed $\tilde{\varphi}(\lambda)=\frac{1}{2}\int_{-\pi}^{\pi}\tilde{g}(t)e^{izt}dt$. The function $\tilde{g}$ is an even function since $\tilde{\varphi}(z)$ is also so, furthermore applying the inverse Fourier transform we can then set $\tilde{h}=\tilde{g}\Big|_{[0,\pi]}$ from which it is evident that the reversal of the above computation gives that the function can indeed be written as $\tilde{\varphi}(z)=\int_0^{\pi}\tilde{g}(t)\cos zt dt$ and we have also shown that $\tilde{\varphi}(0)=0$. Now an application of Lemma 3.4.2 of [11] gives that the zeroes satisfy the following asymptotic

$$\nu_k=\frac{\pi k}{a}-\frac{a}{\pi k}\Big(\frac{K_6-K_1K_{10}-K_7K_8}{K_5-K_2K_7-K_1K_9)}\Big)+\frac{b_k^-}{k},$$

where $\{b_k^-\}_{-\infty}^{\infty}\in l_2$. We substitute the function P and Q into the function u

$$\begin{aligned}
u(\lambda)=u_2(\lambda)&=\frac{M_3P(\lambda)-M_4Q(\lambda)}{(M_3-M_4)}\\
&\Big[M_3\Big(\cos\lambda a+\Big(\frac{K_2K_7+K_1K_9}{K_2}\Big)\frac{\sin\lambda a}{\lambda}+\Big(\frac{K_1K_{10}+K_2K_8}{K_2}\Big)\frac{\cos\lambda a}{\lambda^2}+K_{11}\frac{\sin\lambda a}{\lambda^3}\\
&-i\Big(\frac{K_1\psi(\lambda)+K_1\psi(-\lambda)}{K_2\lambda^2}\Big)+\Big(\frac{\psi(\lambda)-\psi(-\lambda)}{\lambda^3}\Big)\Big)\\
&-M_4\Big(\cos\lambda a+\frac{K_5}{K_2}\frac{\sin\lambda a}{\lambda}+\frac{K_6}{K_2}\frac{\cos\lambda a}{\lambda^2}+\frac{K_9}{K_2}\frac{\sin\lambda a}{\lambda^3}\\
&+\frac{K_{10}}{K_2}\frac{\cos\lambda a}{\lambda^4}-i\Big(\frac{\psi(\lambda)+\psi(-\lambda)}{K_2\lambda^4}\Big)\Big)\Big](M_3-M_4)^{-1}\\
&=\Big[(M_3-M_4)\cos\lambda a+\Big(\frac{M_3(K_2K_7+K_1K_9)}{K_2}-M_4\frac{K_5}{K_2}\Big)\frac{\sin\lambda a}{\lambda}\\
&+\Big(M_3\frac{(K_1K_{10}+K_2K_8)}{K_2}-M_4\frac{K_6}{K_2}\Big)\frac{\cos\lambda a}{\lambda^2}+\Big(M_3K_{11}-M_4\frac{K_9}{K_2}\Big)\frac{\sin\lambda a}{\lambda^3}\\
&-M_4\frac{K_{10}}{K_2}\frac{\cos\lambda a}{\lambda^4}-iM_3\frac{K_1}{K_2}\frac{(\psi(\lambda)+\psi(-\lambda))}{\lambda^2}+M_3\frac{(\psi(\lambda)+\psi(-\lambda))}{\lambda^3}\\
&+iM_4\frac{(\psi(\lambda)+\psi(-\lambda))}{K_2\lambda^4}\Big](M_3-M_4)^{-1}\\
&=\cos\lambda a+\frac{(M_3(K_2K_7+K_1K_9)-M_4K_5)}{K_2(M_3-M_4)}\frac{\sin\lambda a}{\lambda}\\
&+\frac{(M_3(K_1K_{10}+K_2K_8)-M_4K_6)}{K_2(M_3-M_4)}\frac{\cos\lambda a}{\lambda^2}+\frac{(M_3K_{11}K_2-M_4K_9)}{K_2(M_3-M_4)}\frac{\sin\lambda a}{\lambda^3}\\
&-\frac{M_4K_{10}}{K_2(M_3-M_4)}\frac{\cos\lambda a}{\lambda^4}-i\frac{M_3K_1}{K_2(M_3-M_4)}\frac{(\psi(\lambda)+\psi(-\lambda))}{\lambda^2}\\
&+\frac{M_3}{M_3-M_4}\frac{(\psi(\lambda)+\psi(-\lambda))}{\lambda^3}+i\frac{M_4}{M_3-M_4}\frac{(\psi(\lambda)+\psi(-\lambda))}{K_2\lambda^4}.
\end{aligned}$$

We recall again that $M_3 - M_4 = \frac{K_5 - K_2K_7 - K_1K_9}{K_2}$, then substituting M_4 into the above equation we get

$$\begin{aligned}
u(\lambda) &= \cos\lambda a + \frac{(M_3(K_2K_7 + K_1K_9) - (M_3 - (K_5 - K_2K_7 - K_1K_9))K_5)}{K_2(M_3 - M_4)}\frac{\sin\lambda a}{\lambda} + \frac{\psi_1(\lambda)}{\lambda} \\
&= \cos\lambda a + \frac{(\frac{K_5 - K_2K_7 - K_1K_9}{K_2})(\frac{K_5}{K_2} - M_3)}{M_3 - M_4}\frac{\sin\lambda a}{\lambda} + \frac{\psi_1(\lambda)}{\lambda} \\
&= \cos\lambda a + \frac{(M_3 - M_4)(\frac{K_5}{K_2} - M_3)}{M_3 - M_4}\frac{\sin\lambda a}{\lambda} + \frac{\psi_1(\lambda)}{\lambda} \\
&= \cos\lambda a - \Big(\frac{K_6 - K_1K_{10} - K_7K_8}{K_5 - K_2K_7 - K_1K_9)}\Big)\frac{\sin\lambda a}{\lambda} + \frac{\psi_1(\lambda)}{\lambda},
\end{aligned}$$

where we have used the definition of M_3 and $\psi_1(\lambda)$ is given by

$$\begin{aligned}
\psi_1(\lambda) = & \frac{(M_3(K_1K_{10} + K_2K_8) - M_4K_6)}{K_2(M_3 - M_4)}\frac{\cos\lambda a}{\lambda} + \frac{(M_3K_{11}K_2 - M_4K_9)}{K_2(M_3 - M_4)}\frac{\sin\lambda a}{\lambda^2} \\
& - \frac{M_4K_{10}}{K_2(M_3 - M_4)}\frac{\cos\lambda a}{\lambda^3} - i\frac{M_3K_1}{K_2(M_3 - M_4)}\frac{(\psi(\lambda) + \psi(-\lambda))}{\lambda} \\
& + \frac{M_3}{M_3 - M_4}\frac{(\psi(\lambda) + \psi(-\lambda))}{\lambda^2} - i\frac{M_4}{M_3 - M_4}\frac{(\psi(\lambda) + \psi(-\lambda))}{K_2\lambda^3}
\end{aligned}$$

From here onwards we proceed as before to obtain the results. □

We observe that both functions

$$u(\lambda) = \cos\lambda a - \Big(\frac{K_6 - K_1K_{10} - K_7K_8}{K_5 - K_2K_7 - K_1K_9)}\Big)\frac{\sin\lambda a}{\lambda} + \frac{\psi_1(\lambda)}{\lambda}$$

and

$$v(\lambda) = \frac{\sin\lambda a}{\lambda} + \frac{(K_6 - (K_1K_{10} + K_2K_8))}{(K_5 - (K_2K_7 + K_1K_9))}\frac{\cos\lambda a}{\lambda^2} + \frac{\psi_2(\lambda)}{\lambda^2}$$

satisfy the condition that

$$\begin{aligned}
u(-\lambda) &= \Big(\frac{K_6 - K_1K_{10} - K_7K_8}{K_5 - K_2K_7 - K_1K_9)}\Big)\frac{\sin(-\lambda a)}{(-\lambda)} + \frac{\psi_1(-\lambda)}{(-\lambda)} \\
&= u(\lambda),
\end{aligned}$$

$$\begin{aligned}
v(-\lambda) &= \frac{\sin(-\lambda a)}{(-\lambda)} + \frac{(K_6 - (K_1K_{10} + K_2K_8))}{(K_5 - (K_2K_7 + K_1K_9))}\frac{\cos(-\lambda a)}{\lambda^2} + \frac{\psi_2(-\lambda)}{\lambda^2} \\
&= v(\lambda).
\end{aligned}$$

We will now construct the potential using two sequences $\{\mu_k\}_{-\infty}^{\infty}$ and $\{\nu_k\}_{-\infty}^{\infty}$ as the spectra of the Dirichlet-Dirichlet and Dirichlet-Neumann problem. We follow the approach described in [11] and define the function

$$S(\lambda) = \Big(\frac{u(\lambda) + i\lambda v(\lambda)}{u(\lambda) - i\lambda v(\lambda)}\Big)e^{-2i\lambda a}, \quad -\infty < \lambda < \infty$$

which we further note that it has the following symmetry

$$\begin{aligned} S(-\lambda) &= \frac{u(-\lambda) - i\lambda v(-\lambda)}{u(-\lambda) + i\lambda v(-\lambda)} e^{2i\lambda a} \\ &= \Big(\frac{u(\lambda) + i\lambda v(\lambda)}{u(\lambda) - i\lambda v(\lambda)}\Big)^{-1} e^{-2i\lambda a} \\ &= (S(\lambda))^{-1}. \end{aligned}$$

In addition, it is evident that for $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} S(-\lambda) &= \frac{u(-\lambda) - i\lambda v(-\lambda)}{u(-\lambda) + i\lambda v(-\lambda)} e^{2i\lambda a} \\ &= \overline{\Big(\frac{u(\lambda) + i\lambda v(\lambda)}{u(\lambda) - i\lambda v(\lambda)}\Big) e^{-2i\lambda a}} \\ &= \overline{S(\lambda)}. \end{aligned}$$

Altogether we have that S satisfies

$$S(\lambda) = \overline{S(-\lambda)} = (S(-\lambda))^{-1} \text{ for } \lambda \in \mathbb{R}.$$

Proposition 6.18. *The function S is continuous on the real line, $S(\lambda) = \overline{S(-\lambda)} = (S(-\lambda))^{-1}$, also $1 - S(\lambda)$ tends to zero as $|\lambda| \to \infty$ and is the Fourier transform of the function*

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 - S(\lambda) e^{i\lambda x}) d\lambda, \tag{6.28}$$

which in turn can be written as a sum of two functions, one which belongs to $L_1(-\infty,\infty)$ while the second is bounded and belongs to $L_2(-\infty,\infty)$. On the positive half line, $F(x)$ has a derivative $F'(x)$ and $F'(x) \in L_2(0,\infty)$.

Proof. Recalling that

$$u(\lambda) = \cos\lambda a - A\frac{\sin\lambda a}{\lambda} + \frac{\psi_1(\lambda)}{\lambda}$$

and

$$v(\lambda) = \frac{\sin\lambda a}{\lambda} + A\frac{\cos\lambda a}{\lambda^2} + \frac{\psi_2(\lambda)}{\lambda^2},$$

where $A = \frac{K_6 - K_1K_{10} - K_7K_8}{K_5 - K_2K_7 - K_1K_9}$ and ψ_1, ψ_2 are bounded and belong to $L_2(0,a)$, then defining

$$e(\lambda) = (u(\lambda) + i\lambda v(\lambda))e^{-i\lambda a}$$

we get that

$$u(\lambda) + i\lambda v(\lambda) = (\cos\lambda a + i\sin\lambda a) + A\frac{(i\cos\lambda a - \sin\lambda a)}{\lambda} + \frac{(\psi_1(\lambda) + i\psi_2(\lambda))}{\lambda}.$$

But $\cos\lambda a + i\sin\lambda a = e^{i\lambda a}$, so

$$u(\lambda) + i\lambda v(\lambda) = e^{i\lambda a} + Ai\frac{e^{i\lambda a}}{\lambda} + \frac{(\psi_1(\lambda) + i\psi_2(\lambda))}{\lambda}.$$

Therefore

$$e(\lambda) = 1 + i\frac{A}{\lambda} + \frac{e^{-i\lambda a}}{\lambda}[\psi_1(\lambda) + i\psi_2(\lambda)], \tag{6.29}$$

Since the function u and v are holomorphic in the lower half plane it follows that the function $e(\lambda)$ is holomorphic in the lower half. We restrict ourself to the lower half plane just to ensure convergence of the limit which will be evaluated. We recall that Hölder's inequality gives that an L_2-function on a compact interval $[0,a]$ is also L_1-function on the same interval, so we have that $\psi_1,\ \psi_2 \in L_1(0,a)$. Now Lemma 1.3.1 of [11] states that:

If $f(x) \in L_1(0,\pi)$ then

$$\lim_{|\lambda|\to\infty} e^{-|\mathrm{Im}\lambda\pi|}\int_0^\pi f(x)\cos\lambda x dx = \lim_{|\lambda|\to\infty} e^{-|\mathrm{Im}\lambda\pi|}\int_0^\pi f(x)\sin\lambda x dx = 0,$$

so then applying Lemma 1.3.1 of [11] on equation (6.29) we obtain

$$e(\lambda) \underset{\mathrm{Im}\lambda\le 0,\ |\lambda|\to\infty}{=} 1+O(\lambda^{-1}). \tag{6.30}$$

We will assume for now that the function $e(\lambda)$ does not have any zeroes in the closed lower half plane (this will be proved in the next theorem). We define the function

$$S(\lambda) = \frac{e(\lambda)}{e(-\lambda)}, \qquad (-\infty < \lambda < \infty) \tag{6.31}$$

Since we assumed that the function $e(-\lambda)$ has no zeroes in the closed upper half plane, the function S is continuous on the whole real line. From equation (6.31) we can immediately see that $S(0) = 1$. We can also see that by using equation (6.29) we can rewrite the function $e(\lambda)$ as follows

$$e(\lambda) = 1+\lambda^{-1}g(\lambda)$$

where $g(\lambda) = Ai + e^{-i\lambda a}[\psi_1(\lambda)+i\psi_2(\lambda)]$ is bounded. Then using the above representation of $e(\lambda)$ we proceed to write

$$\begin{aligned} S(\lambda) &= \frac{e(\lambda)}{e(-\lambda)} \\ &= \frac{1+\lambda^{-1}g(\lambda)}{1-\lambda^{-1}g(-\lambda)} \\ &= (1+\lambda^{-1}g(\lambda))(1+\lambda^{-1}g(-\lambda)+O(\lambda^{-2})), \end{aligned}$$

for sufficiently large λ, where

$$\begin{aligned} g(-\lambda) &= Ai + e^{i\lambda a}[\psi_1(-\lambda)+i\psi_2(-\lambda)] \\ &= Ai + e^{i\lambda a}[-\psi_1(\lambda)+i\psi_2(\lambda)] \\ &= Ai - e^{i\lambda a}[\psi_1(\lambda)-i\psi_2(\lambda)]. \end{aligned}$$

For real values of λ that are sufficiently large we can write

$$\begin{aligned} S(\lambda) &= (1+\lambda^{-1}g(\lambda))(1+\lambda^{-1}g(-\lambda)+O(\lambda^{-2})) \\ &= 1+\lambda^{-1}\Big(2Ai + e^{-i\lambda a}[\psi_1(\lambda)+i\psi_2(\lambda)] - \overline{e^{-i\lambda a}[\psi_1(\lambda)+i\lambda\,\psi_2(\lambda)]}\Big) + O(\lambda^{-2}) \\ &= 1+\frac{2iA}{\lambda} + 2i\mathrm{Im}\Big(\frac{e^{-i\lambda a}}{\lambda}[\psi_1(\lambda)+i\psi_2(\lambda)]\Big) + O(\lambda^{-2}). \end{aligned}$$

Therefore

$$1-S(\lambda) = -\frac{2iA}{\lambda}2i\mathrm{Im}\Big(\frac{e^{-i\lambda a}}{\lambda}[\psi_1(\lambda)+i\psi_2(\lambda)]\Big) + O(\lambda^{-2}). \tag{6.32}$$

from which we see that $1-S(\lambda)$ tends to zero as $|\lambda|\to\infty$. To show the last part of the proposition we note that since $\psi_1,\ \psi_2$ are bounded and belong to $L_2(-\infty,\infty)$, so then the function

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(1-S(\lambda))e^{i\lambda x}d\lambda$$

as a Fourier transform of $(1-S(\lambda))$ is bounded and belongs to $L_2(-\infty,\infty)$. Furthermore, from equation (6.32) we can write $(1-S(\lambda))$ as

$$1-S(\lambda)=\frac{-2iA}{(\lambda+i)}\frac{(\lambda+i)}{\lambda}+\frac{h(\lambda)}{\lambda},$$

where $h(\lambda)=-2i\mathrm{Im}e^{-i\lambda a}[\psi_1(\lambda)+i\psi_2(\lambda)]+O(\lambda^{-1})$ and belongs to $L_2(-\infty,\infty)$ since $\psi_1,\psi_2\in L_2(-\infty,\infty)$. We further simplify the above equation as follows

$$\begin{aligned}1-S(\lambda)&=-\frac{2iA}{\lambda+i}\Big(1+\frac{i}{\lambda}\Big)+\frac{h(\lambda)}{(|\lambda|+1)}\Big(\frac{(|\lambda|+1)}{\lambda}\Big)\\&=-\frac{2iA}{\lambda+i}+\frac{\psi(\lambda)}{(|\lambda|+1)},\end{aligned}$$

where

$$\psi(\lambda)=\frac{2A}{\lambda^2+i\lambda}+h(\lambda)\Big(\frac{(|\lambda|+1)}{\lambda}\Big)$$

which also belongs to $L_2(-\infty,\infty)$. Let x be positive and Γ be the positively oriented boundary of a semi circle in the upper half plane with radius R (passing through the origin and symmetric with respect to the imaginary axis). Then

$$\int_\Gamma\frac{-2iAe^{i\lambda x}}{\lambda+i}d\lambda=\int_{-R}^{R}\frac{-2iAe^{i\lambda x}}{\lambda+i}d\lambda+\int_0^\pi\frac{-2iAe^{(iRe^{i\theta})x}}{Re^{i\theta}+i}Rie^{i\theta}d\theta.$$

Since $\frac{-2iAe^{i\lambda x}}{\lambda+i}$ has no poles in the upper half plane, then

$$\int_\Gamma\frac{-2iAe^{i\lambda x}}{\lambda+i}d\lambda=0$$

which implies

$$0=\lim_{R\to\infty}\Big(\int_{-R}^{R}\frac{-2iAe^{i\lambda x}}{\lambda+i}d\lambda+\int_0^\pi\frac{-2iAe^{(iRe^{i\theta})x}}{Re^{i\theta}+i}Rie^{i\theta}d\theta\Big),$$

so then we obtain

$$\int_{-\infty}^{\infty}\frac{2iAe^{i\lambda x}}{\lambda+i}d\lambda=\lim_{R\to\infty}\Big(\int_0^\pi\frac{-2iAe^{(iRe^{i\theta})x}}{Re^{i\theta}+i}Rie^{i\theta}d\theta\Big).$$

Let

$$f_R(\theta)=\frac{-2iAe^{(iRe^{i\theta})x}}{Re^{i\theta}+i}Rie^{i\theta},$$

and since we know that $e^{i\theta}=\cos\theta+i\sin\theta$ and that $0\le\sin\theta\le1$ for $\theta\in[0,\pi]$, then

$$\begin{aligned}|f_R(\theta)|&=2|A|e^{-Rx\sin\theta}\Big|\frac{Re^{i\theta}}{Re^{i\theta}+i}\Big|\\&\le2|A|\Big|\frac{Re^{i\theta}}{Re^{i\theta}+i}\Big|.\end{aligned}$$

Evidently we also have that $\lim_{R\to\infty}\Big(\frac{Re^{i\theta}}{Re^{i\theta}+i}\Big)=1$, which means there exist a $c>0$ such that

$$\Big|\frac{Re^{i\theta}}{Re^{i\theta}+i}\Big|\le c,\ \forall R\ge1.$$

Altogether we obtain that

$$|f_R(\theta)|\le2|A|c,$$

and we know that

$$\lim_{R\to\infty} f_R(\theta) = \lim_{R\to\infty} -2iAe^{(iRe^{i\theta})x}\Big(\frac{Rie^{i\theta}}{Re^{i\theta}+i}\Big)$$
$$= 0 \text{ on } (0,\pi).$$

Lebesgue's dominated convergence theorem applied gives that

$$\lim_{R\to\infty}\int_0^{\pi} f_R(\theta)d\theta = \int_0^{\pi}\lim_{R\to\infty} f_R(\theta)d\theta$$
$$= 0.$$

Therefore

$$\int_{-\infty}^{\infty}\frac{2iAe^{i\lambda x}}{\lambda+i}d\lambda = \lim_{R\to\infty}\Big(\int_0^{\pi}\frac{-2iAe^{(iRxe^{i\theta})}}{Re^{i\theta}+i}Rie^{i\theta}d\theta\Big)$$
$$= 0.$$

So for the function $1-S$ given by

$$1-S(\lambda) = \frac{-2iA}{\lambda+i}+\frac{\psi(\lambda)}{(|\lambda|+1)},$$

we have that

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\frac{-2iA}{\lambda+i}+\frac{\psi(\lambda)}{(|\lambda|+1)}\Big)e^{i\lambda x}d\lambda$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\psi(\lambda)}{(|\lambda|+1)}e^{i\lambda x}d\lambda,$$

which also shows that $F(x)$ is absolutely continuous on the positive half line, and

$$F'(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(\lambda)\Big(\frac{i\lambda}{|\lambda|+1}\Big)e^{i\lambda x}d\lambda$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{h}(\lambda)e^{i\lambda x}d\lambda,$$

where $\tilde{h}(\lambda) = \psi(\lambda)\Big(\frac{i\lambda}{|\lambda|+1}\Big) \in L_2(-\infty,\infty)$, then $F'(x)\in L_2(0,\infty)$. □

Proposition 6.19. *The following identity holds*

$$\frac{\ln S(0^+)-\ln S(+\infty)}{2\pi i} - \frac{1-S(0)}{4} = 0$$

Proof. We know from Corollary 6.16 that the sets of zeroes $\{\mu_j^2\}_1^\infty$ of the function $u(\lambda)$ interlace with the set of zeroes $\{\nu_j^2\}_1^\infty$ of the function $v(\lambda)$

$$\nu_k < \mu_k < \nu_{k+1} < \cdots,$$

so

$$\nu_k^2 < \mu_k^2 < \nu_{k+1}^2 < \mu_{k+1}^2.$$

We want to determine the increment of the argument of the function $u(\lambda)+i\lambda v(\lambda)$ on a segment $[\nu_{k-1},\nu_k]$ where $k-1$ is an even nonnegative index. From the asymptotic expansion of the zeroes $\{\mu_j^2\}$ and $\{\nu_j^2\}$ combined with an application of Lemma 3.4.2 of [11] we can write the functions $u(\lambda)$ and $v(\lambda)$ as follows

$$u(\lambda) = \prod_{j=1}^{\infty}\Big(\frac{\pi}{a}(j-\frac{1}{2})\Big)^{-2}(\mu_j^2-\lambda^2) \tag{6.33}$$

$$v(\lambda)=\prod_{j=1}^{\infty}\left(\frac{\pi}{a}j\right)^{-2}(\nu_j^2-\lambda^2) \tag{6.34}$$

So then equation (6.33) gives that the function λv is always nonnegative (since we have that there are $k-1$ factors $(\nu_j^2-\lambda^2)$ which are non-positive and the other factors are non-negative, where $k-1$ is even and $0<\nu_1<\nu_2,\cdots$, which gives that the product in (6.34) is nonnegative). The interlacing property

$$\nu_{k-1}<\mu_k<\nu_k$$

and equation (6.33) gives that the function u is positive on the interval $[\nu_{k-1},\mu_k)$ (by a similar argument) where $\mu_k<\nu_k$. The simplicity of the zeroes of u gives that u is negative on $(\mu_k,\nu_k]$. So we obtain that the increment of the argument of the function $u+i\lambda v$ on the segment $[\nu_{k-1},\nu_k]$ is π. Since the zeroes of u and λv are simple and interlace, it follows that the increment of the argument of the function $u+i\lambda v$ on any segment $[\nu_{j-1},\nu_j]$, $j\in\mathbb{N}$ is π. So when λ runs through $[-\nu_k,\nu_k]$ the increment of the argument of the function $u+i\lambda v$ is $2k\pi$, whereas the increment of the argument of the function $e^{-i\lambda a}$ on the same segment is $-2\nu_k a$. Thus the increment of the argument of the function $e(\lambda)$ on the segment $[-\nu_k,\nu_k]$ is $-2\nu_k a+2k\pi=2(k\pi-\nu_k a)$, which tends to zeros as k tends to infinity. Now using equation (6.29) we easily see that

$$\lim_{\lambda\to\pm\infty}e(\lambda)=1,$$

and so we can conclude that the increment of the argument of the function $e(\lambda)$ when λ changes from $-\infty$ to ∞ is equal to zero,

$$\arg e(+\infty)-\arg e(-\infty)=0. \tag{6.35}$$

By the principle of the argument, this equality and equation (6.29) imply that $e(\lambda)$ has no zeroes in the closed lower half plane, hence the assumption used in previous theorem about the zeroes is now validated. Now equation (6.35) gives that

$$\begin{aligned}\arg S(0)-\arg S(\infty)&=\arg e(0)-\arg e(0)-\arg e(\infty)+\arg e(-\infty)\\&=-(\arg e(\infty)-\arg e(-\infty))\\&=0 \text{ by } (6.35).\end{aligned}$$

Since $S(0)=1$, we get that the following identity holds

$$\frac{\ln S(0^+)-\ln S(+\infty)}{2\pi i}-\frac{1-S(0)}{4}=0.$$

$\square$

Proposition 6.20. *The Fourier transform F(x) of the function $1-S(\lambda)$, where the function $S(\lambda)$ is as defined in equation (6.37) has a derivative that vanishes for $x>2a$, and satisfies*

$$\int_0^\infty x|F'(x)|dx<\infty.$$

Proof. Now since the functions $e(\lambda)$ and $e(-\lambda)$ are entire functions and we know that $e(-\lambda)$ has no roots in the closed upper half plane (since $e(\lambda)$ has none in the closed lower half plane), then the function S is holomorphic in the closed upper half plane. We know that

$$e(\lambda)=1+\frac{iA}{\lambda}+\frac{e^{-i\lambda a}}{\lambda}[\psi_1(\lambda)+i\psi_2(\lambda)],$$

therefore

$$e(-\lambda) = 1 - \frac{Ai}{\lambda} - \frac{e^{i\lambda a}}{\lambda}[\psi_1(-\lambda) + i\psi_2(-\lambda)].$$

Now using Lemma 1.3.1 of [11] stated on page 74 we get that there exists an $R > 0$ such that for $|\lambda| > R$ in the upper half plane

$$e(-\lambda) = 1 + O(\lambda^{-1}).$$

We also know that $\psi_1(\lambda) = \int_0^a \tilde{f}(t)\cos\lambda t\,dt$, where $\tilde{f}(t) \in L_2(-\infty,\infty)$, so we can write

$$\begin{aligned}
|\psi_1(\lambda)| &= |\int_0^a \tilde{f}(t)\cos\lambda t\,dt| \\
&\le \int_0^a |\tilde{f}(t)||\cos\lambda t|dt \\
&\le \max_{t\in[0,a]} |\cos\lambda t| \int_0^a |\tilde{f}(t)|dt \\
&\le \max_{t\in[0,a]} |\cos\lambda t| \sqrt{a}|\tilde{f}|_{L_2[0,a]} \\
&= C_1 \max_{t\in[0,a]} |\cos\lambda t|,
\end{aligned}$$

and

$$\begin{aligned}
|\cos\lambda t| &= \Big|\frac{e^{i\lambda t} + e^{-i\lambda t}}{2}\Big| \\
&\le \frac{e^{\mathrm{Im}\lambda t} + e^{-\mathrm{Im}\lambda t}}{2} \\
&\le e^{|\mathrm{Im}\lambda|t}.
\end{aligned}$$

Now using $\max_{t\in[0,a]} |\cos\lambda t| \le e^{|\mathrm{Im}\lambda|a}$, we further obtain that

$$|\psi_1(\lambda)| \le C_1 e^{|\mathrm{Im}\lambda|a},$$

similarly

$$|\psi_2(\lambda)| \le C_2 e^{|\mathrm{Im}\lambda|a}.$$

So then for $|\lambda| > R$ in the upper half plane, we have that

$$\frac{1}{1 + O(\lambda^{-1})} = O(1).$$

Altogether then if $|\lambda| > R$ in the upper half plane we have that

$$\begin{aligned}
|1 - S(\lambda)| &= \Big|1 - \frac{(1 + \frac{Ai}{\lambda} + \frac{e^{-i\lambda a}}{\lambda}[\psi_1(\lambda) + i\psi_2(\lambda)])}{1 + O(\lambda^{-1})}\Big| \\
&= \Big|\frac{\frac{e^{-i\lambda a}}{\lambda}[\psi_1(\lambda) + i\psi_2(\lambda)] + O(\lambda^{-1})}{1 + O(\lambda^{-1})}\Big| \\
&= \frac{1}{|\lambda|}\Big|\frac{e^{-i\lambda a}[\psi_1(\lambda) + i\psi_2(\lambda)] + O(1)}{1 + O(\lambda^{-1})}\Big| \\
&\le \frac{1}{|\lambda|} F' e^{2a\mathrm{Im}\lambda a}
\end{aligned}$$

for some constant F'. Since S is continuous in the closed upper half plane, we get that the above inequality also holds on the real axis. Altogether we have obtained for $|\lambda| > R$ in the closed upper half plane that there exists $F' \neq 0$ such that the following holds:

$$|1-S(\lambda)| \leq \frac{F'}{|\lambda|}e^{2a\mathrm{Im}\lambda}.$$

Since the function $e(-\lambda)$ has no zeroes in the closed upper half plane it follows from equation (6.31) that $1-S$ is also holomorphic in the closed upper half plane since S is holomorphic in the closed upper half plane. So for $|\lambda| \leq R$, $\mathrm{Im}\lambda \geq 0$ (which is compact), $1-S$ attains a minimum and a maximum, i.e. there exists $c_1 \neq 0$ such that for $|\lambda| \leq R$ in the closed upper half plane

$$|1-S(\lambda)| \leq \frac{c_1}{R}.$$

But for $|\lambda| \leq R$ in the closed upper half plane we have that

$$\begin{aligned}|1-S(\lambda)| &\leq \frac{c_1}{R} \leq \frac{c_1}{|\lambda|}\\ &\leq \frac{c_1}{|\lambda|}e^{2a\mathrm{Im}\lambda}.\end{aligned}$$

Therefore the following holds on the closed upper half plane:

$$|1-S(\lambda)| \leq \frac{F}{|\lambda|}e^{2a\mathrm{Im}\lambda}, \tag{6.36}$$

for some constant F. Let Γ be a positively oriented boundary of the semi circle (as before). Then

$$\int_\Gamma \Big(1-S(\lambda)\Big)e^{i\lambda x}d\lambda = \int_{-R}^{R}(1-S(\lambda))e^{i\lambda x}d\lambda + \int_0^\pi (1-S(Re^{i\theta}))e^{i(Re^{i\theta})x}Rie^{i\theta}d\theta.$$

Since $1-S$ is holomorphic in the closed upper half plane, then

$$\int_\Gamma (1-S(\lambda))e^{i\lambda x}d\lambda = 0$$

gives that

$$\lim_{R\to\infty}\Big(-\int_{-R}^{R}(1-S(\lambda))e^{i\lambda x}d\lambda\Big) = \lim_{R\to\infty}\Big(\int_0^\pi (1-S(Re^{i\theta}))e^{iRe^{i\theta}x}Rie^{i\theta}d\theta\Big),$$

i.e

$$\int_{-\infty}^{\infty}(1-S(\lambda))e^{i\lambda x}d\lambda = -\lim_{R\to\infty}\Big(\int_0^\pi (1-S(Re^{i\theta}))e^{i(Re^{i\theta}x)}Rie^{i\theta}d\theta\Big).$$

Let

$$g_R(\theta) = (1-S(Re^{i\theta}))e^{iR(e^{i\theta}x)}Rie^{i\theta}.$$

Then

$$\begin{aligned}|g_R(\theta)| &= \Big|(1-S(Re^{i\theta}))e^{iR(e^{i\theta}x)}Rie^{i\theta}\Big|\\ &\leq \frac{F}{|Re^{i\theta}|}e^{2a\mathrm{Im}Re^{i\theta}}|e^{iR(e^{i\theta}x)}Rie^{i\theta}| \text{ by (6.36)}\\ &= \frac{F}{|Re^{i\theta}|}e^{2aR\sin\theta}e^{-Rx\sin\theta}|Rie^{i\theta}|\\ &= Fe^{(2a-x)R\sin\theta}.\end{aligned}$$

If $x > 2a$, then

$$|g_R(\theta)| \leq F = s(\theta) \text{ holds on } [0,\pi]$$

and

$$\lim_{R\to\infty}|g_R(\theta)| = 0 \text{ for } \theta\in(0,\pi).$$

Applying Lebesgue's dominated convergence theorem we get

$$\begin{aligned}\lim_{R\to\infty}\int_0^{\pi} g_R(\theta)d\theta &= \int_0^{\pi}\lim_{R\to\infty} g_R(\theta)d\theta\\ &= 0.\end{aligned}$$

Therefore for $x > 2a$, we obtain that

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(1-S(\lambda))e^{i\lambda x}d\lambda = 0,$$

and

$$\int_0^{\infty} x|F'(x)|dx = \int_0^{2a} x|F'(x)|dx < \infty.$$

□

It is worth reminding the reader that we are still in hot pursuit of the proof of Theorem 6.1. So far we have managed to obtain a, M_1, M_2, M_3, M_4 and showed that $M_1, M_2 > 0$ and that $M_3 - M_4 > 0$, and we are still yet to obtain the potential $q(x)$. We will now give motivation for the Marchenko integral equation in a less rigorous manner and merely refer the reader to the book by Marchenko [11] for further details. The Jost solution is the solution to the Sturm Liouville equation which satisfy the following conditions at infinity:

$$\lim_{r\to\infty} e^{-ikr}f(k,r) = 1$$

and

$$\lim_{r\to\infty} e^{-ikr}f'(k,r) = ik.$$

It can shown that the Jost solution can be written as

$$f(k,r) = e^{ikr} - \int_r^{\infty}\frac{\sin k(r-t)}{k}q(t)f(k,t)dt \tag{6.37}$$

which is a Volterra integral equation, and

$$f'(k,r) = ike^{ikr} - \int_r^{\infty}\cos k(r-t)q(t)f(k,t)dt. \tag{6.38}$$

Equation (6.37) is solved using an iterative method which starts with the zeroth order solution (q(t)=0)

$$f^{(0)}(k,r) = e^{ikr}$$

and then define the n'th order solution by

$$f^{(n)}(k,r) = e^{ikr} - \int_r^{\infty}\frac{\sin k(r-t)}{k}q(t)f^{(n-1)}(k,t)dt.$$

The sequence $\{f^{(n)}\}$ converges uniformly since we assume that $\int_0^{\infty} t|q(t)|dt < \infty$. Furthermore, it can be shown that

$$\left|f(k,r) - e^{ikr}\right| \le \frac{Ke^{-\mathrm{Im}\,\mathrm{k}r}}{|k|}\int_r^{\infty}|q(t)|dt \tag{6.39}$$

where Im k > 0 and r is fixed and $r > 0$. It follows from equation (6.39) that the function $g(k,r) = f(k,r) - e^{ikr}$ is an L_2 function of k on any line parallel to the real axis and that

$$\int_{-\infty}^{\infty}|g(\sigma+i\tau)|^2 d\sigma = O(e^{-2\tau r}).$$

Theorem 4.3.4 of [1] gives that the Jost solution $f(k,r)$ and $g(k,r)$ are continuous on the real line.

Theorem 6.21 (Titchmarsh)**.** *A necessary and sufficient condition that the function $F(x) \in L_2(\infty,\infty)$ should be the limit as $y \to 0$ of an analytic function $F(z = x+iy)$ in $y > 0$ such that*

$$\int_{-\infty}^{\infty} |F(x+iy)|^2 dx = O(e^{2\lambda y})$$

is that

$$\int_{-\infty}^{\infty} F(x)e^{-itx} dx = 0$$

for all values of $x < -\lambda$.

Now applying the Titchmarsch theorem to $g(k,r)$ we get that

$$A(r,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [f(k,r) - e^{ikr}]e^{-ikt} dk = 0, \quad t < r$$

and a Fourier inversion formula gives that

$$f(k,r) = e^{ikr} + \int_{r}^{\infty} A(r,t)e^{ikt} dt,$$

where Im k ≥ 0 and $A(r,t) \in L_2$ in the variable t. Since we already know in our case that there are no negative eigenvalues (no bound states) then the completeness relation takes the form

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(k,r)[f(-k,t) - S(k)f(k,t)]dk = \delta(r-t),$$

so then replacing the Jost solutions $f(-k,t)$ and $f(k,t)$ we obtain that

$$\int_{-\infty}^{\infty} f(k,r)[e^{-ikt} - S(k)e^{ikt}]dk + \int_{t}^{\infty} A(r,t)\Big[\int_{-\infty}^{\infty} f(k,r)[e^{-iku} - S(k)e^{iku}dk\Big]du = 0.$$

Now letting

$$G_r(t) = \int_{-\infty}^{\infty} f(k,r)[e^{-ikt} - S(k)e^{ikt}]dk$$

we can see that the above equation is a homogeneous Volterra integral equation of the form

$$G_r(t) + \int_{t}^{\infty} A(t,u)G_r(u)du = 0,$$

where the kernel $A(t,u) \in L_2(t,\infty)$ in the variable u. According to Theorem 5.38 of [14] we have that $G_r(t) = 0$ which means that

$$\int_{-\infty}^{\infty} f(k,r)[e^{-ikt} - S(k)e^{ikt}]dk = 0, \quad \text{for all } \ t > r.$$

Lastly, substituting the Jost solution $f(k,r)$ into the above equation gives that

$$A(r,t) = A_0(r+t) + \int_{r}^{\infty} A(r,s)A_0(s+t)ds$$

which is known as the Marchenko integral equation where

$$A_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [S(k)-1]e^{ikt} dk.$$

Note that the function $A_0(t)$ is none other than what we have been calling $F(x)$. Now given the kernel

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (1 - S(\lambda))e^{i\lambda x} d\lambda,$$

of the integral equation

$$G(x,t)+F(x+t)+\int_x^\infty G(x,s)F(s+t)ds=0 \tag{6.40}$$

which we will call the fundamental equation, Theorem 3.3.1 of [11] states that if Proposition 6.18 and Proposition 6.19 (of this thesis) are satisfied then the fundamental equation has a unique solution $G(x,t)$, and

$$q(x)=-2\frac{dG(x,x)}{dx} \tag{6.41}$$

The corollary to Theorem 3.3.3 of [11] gives that since Proposition 6.18 and Proposition 6.19 are satisfied that there exists a real valued potential $q(x)$ for which $q(x)=0$ for $x>a$ and that $q\in L_2(0,a)$ such that the function $S(\lambda)$ is the scattering function. Therefore the above defined potential corresponds to the potential we are looking for. We note that with the above obtained potential that the increment of the argument of the function S and the number n of negative eigenvalues of the boundary value problem

$$-y''+q(x)y=\lambda^2y,\ \ y(0)=0$$

are related by the formula (see pg 210 of [11])

$$n=\frac{\ln S(0)-\ln S(\infty)}{2\pi i}-\frac{1-S(0)}{4}$$

and $n=0$ in our case since there are no such eigenvalues. We will prove that the Sturm Liouville equation with the potential as defined above and the M_j $j=1,2,3,4$ possess the spectrum $\{\lambda_k\}_{-\infty}^{\infty}$.

So far we have recovered the following parameters a, M_1, M_2, M_3,M_4 and the potential $q(x)$, but we have not shown that the recovered potential is indeed the potential that is considered in our problem. The next theorem will use a number of theorems and propositions to show that the recovered potential is exactly the potential we considered.

Theorem 6.22. *The differential equation*

$$y''(\lambda,x)+\lambda^2y(\lambda,x)-q(x)y(\lambda,x)=0$$

with conditions

$$y(\lambda,0)=0$$

$$(1+i\lambda M_1^{-1}-\lambda^2M_2^{-1})y_x(\lambda,a)+\left[M_4+i\lambda M_1^{-1}M_3-\lambda^2M_2^{-1}M_3\right]y(\lambda,a)=0$$

has the spectrum $\{\lambda_k : k\in\mathbb{Z}\cup\{0^+\}\}$, where the potential $q(x)$ is as defined by equation (6.41)

Proof. Theorem 3.4.3 of [11] states that:

> In order that any two sequences of numbers $\{\lambda_k\}$ and $\{\nu_k\}$ be the spectra of the boundary value problems generated by one and same Sturm-Liouville equation $-y''+q(x)y=\lambda^2y$, $0\le x\le\pi$ with real valued potential $q(x)\in L_2(0,\pi)$, and boundary conditions $y(0)=y(\pi)=0$ and $y(0)=y'(\pi)=0$ respectively, it is necessary and sufficient that they interlace
>
> $$-\infty<\nu_1<\lambda_1<\nu_2<\lambda_2<\cdots$$
>
> and that they satisfy the asymptotic formulas
>
> $$\lambda_k=k^2-2A+\alpha_k, \tag{6.42}$$

$$\gamma_k = (k - \frac{1}{2})^2 - 2A + \beta_k, \tag{6.43}$$

where A is arbitrary and $\{\alpha_k\}_1^\infty, \{\beta_k\}_1^\infty \in l_2$.

We observe that the asymptotic formulas (6.42) and (6.43) are equivalent to the formulas

$$\sqrt{\lambda_k} = k - \frac{A}{k} + \frac{\tilde{\alpha}_k}{k},$$

$$\sqrt{\gamma_k} = k - \frac{1}{2} - \frac{A}{k} + \frac{\tilde{\beta}_k}{k},$$

where $\{\tilde{\alpha}_k\}_1^\infty$, $\{\tilde{\beta}_k\}_1^\infty \in l_2$. Since the sets $\{\mu_k\}_{-\infty,k\neq 0}^\infty$, $\{\nu_k\}_{-\infty,k\neq 0}^\infty$ of the zeroes of the functions $u(\lambda)$ and $v(\lambda)$ respectively interlace and satisfy the following asymptotic:

$$\nu_k = \frac{\pi k}{a} - \frac{a}{\pi}\Big(\frac{K_6 - K_1K_{10} - K_7K_8}{K_5 - K_2K_7 - K_1K_9}\Big) + \frac{b_k^-}{k},$$

$$\mu_k = \frac{\pi}{a}(k - \frac{1}{2}) - -\frac{a}{\pi}\Big(\frac{K_6 - K_1K_{10} - K_7K_8}{K_5 - K_2K_7 - K_1K_9}\Big) + \frac{b_k^+}{k},$$

where $\{b_k^\mp\}_{-\infty}^\infty \in l_2$, then from the above theorem we get that the set $\{\mu_k\}_{-\infty,k\neq 0}^\infty$ is the spectrum of the Dirichlet-Neumann problem generated by the set $\{a, q(x)\}$

$$y'' + \lambda^2 y - q(x)y = 0$$

and

$$y(\lambda, 0) = y'(\lambda, a) = 0 \tag{6.44}$$

and the set $\{\nu_k\}_{-\infty,k\neq 0}^\infty$ is the spectrum of the Dirichlet-Dirichlet problem generated by the set $\{a, q(x)\}$

$$y'' + \lambda^2 y - q(x)y = 0$$

and

$$y(\lambda, 0) = y(\lambda, a) = 0 \tag{6.45}$$

Since Proposition 6.18, Proposition 6.19 and Proposition 6.20 are satisfied then Theorem 3.3.3 below gives that there exists a real valued potential $q(x) \in L_2(0, a)$, $q(x) = 0$ for $x > a$, such that $S(\lambda)$ is the scattering function of the boundary value problem

$$-y'' + q(x)y = \lambda^2 y, \; y(0) = 0$$

for $0 \leq x < \infty$. Theorem 3.3.3 of [11] states that:

> For the function S to be the scattering function of the above mentioned boundary value problem also with the above mentioned potential, it is necessary and sufficient that Proposition 6.18 and Proposition 6.19 (of this thesis) be satisfied.

But the function

$$S(\lambda) = \frac{e(-\lambda, 0)}{e(\lambda, 0)},$$

where $e(\lambda, x)$ is the solution (Jost solution) of

$$-y'' + q(x)y = \lambda^2 y$$

satisfies Proposition 6.18 and Proposition 6.19, then combining the definition of the scattering function with (6.31), we get

$$\frac{e(-\lambda, 0)}{e(\lambda, 0)} = \frac{e(\lambda)}{e(-\lambda)},$$

or equivalently

$$\frac{e(-\lambda,0)}{e(\lambda)}=\frac{e(\lambda,0)}{e(-\lambda)}.$$

The function on the right (left) hand side of the last equality is holomorphic in the upper (respectively lower) half plane and tends uniformly in it to one as $|\lambda|\to\infty$. Liouville's theorem applied gives that $\frac{e(\lambda,0)}{e(-\lambda)}=1$, which implies

$$e(\lambda,0)=e(-\lambda),$$

and we identify the function $e(-\lambda)$ as the Jost function. Since

$$e(-\lambda)=e^{i\lambda a}[u(\lambda)-i\lambda v(\lambda)],$$

and Lemma 3.4.1 of [11] states that:

> If in
>
> $$-y''+q(x)y=\lambda^2 y,$$
>
> the potential is square summable on the interval $(0,\pi)$ and vanishes for $x>\pi$ then
>
> $$e(\lambda,0)=e^{i\lambda\pi}[s'(\lambda,\pi)-i\lambda s(\lambda,\pi)]$$
>
> where s is the solution to the Sturm-Liouville equation.

In our case we do have such a potential, and the identity $e(\lambda,0)=e(-\lambda)$ implies that

$$(u(\lambda)-i\lambda v(\lambda))e^{i\lambda a}=e^{i\lambda a}(s'(\lambda,a)-i\lambda s(\lambda,a)),$$

i.e.

$$(u(\lambda)-s'(\lambda,a))-i\lambda(v(\lambda)-s(\lambda,a))=0\ \forall\lambda\in\mathbb{C}.$$

So then the function $u(\lambda)-s'(\lambda,a)-i\lambda(v(\lambda)-s(\lambda,a))$ is holomorphic in $\mathbb{C}$ and identically zeroes on the real axis (i.e. $u(\lambda)-s'(\lambda,a)=0$ and $v(\lambda)-s(\lambda,a)=0$ since s is real valued for real λ, then the identity theorem gives that

$$u(\lambda)=s'(\lambda,a)\text{ and }\ v(\lambda)=s(\lambda,a).$$

Therefore, the function

$$u(\lambda)=\prod_1^\infty\Big((k-\frac{1}{2})\frac{\pi}{a}\Big)^{-2}(\mu_k^2-\lambda^2)$$

is nothing other than $s'(\lambda,a)$, where $s(\lambda,a)$ is the solution of the Sturm-Liouville equation (5.17) with the obtained potential satisfying condition (5.24). Similarly the function

$$v(\lambda)=a\prod_1^\infty\Big(\frac{\pi k}{a}\Big)^{-2}(\nu_k^2-\lambda^2)$$

is nothing else than $s(\lambda,a)$. Due to (6.21), (6.24), (6.25), (6.16)-(6.19) and (5.33) we identify $\chi(\lambda)$ as $\varphi(\lambda)$. That means that $\{\lambda_k : k\in\mathbb{Z}\cup\{0^+\}\}$ is the spectrum of problem (5.17), (5.18) and (5.19). □

Corollary 6.23. *The operator A acting on $L_2(0,a)$ according to the formula*

$$D(A)=\{f:\ \ f\in W_2^2(0,a),\ f(0)=0,\ \ f'(a)+M_4 f(a)=0\}$$
$$Af=-f''+q(x)f$$

is strictly positive

Proof. The results follow since the definition of our operator coincide with the operator which was defined in section 3 of [12] which is strictly positive. □

Recall that we started this section with Theorem 6.1 and we stated that the the proof of Theorem 6.1 will be made of a number of theorems, lemmas, propositions, and corollaries, then it is at this stage that all the work that has been covered so far constitute a full proof for Theorem 6.1

7. The Inversion

Since we have obtained the set $\{a, q(x), M_1, M_2, M_3, M_4\}$ our task now is to find $b(s), \nu, l_1, M$ such that $\nu > 0$, $l_1 > 0$, $M > 0$, $b \in W_2^2(0, l_1)$ and $b(s) > 0$ for $s \in [0, l_1]$ which generates the Sturm Liouville problem having the spectrum $\{\lambda_k : k \in \mathbb{Z} \cup \{0^+\}\}$.

Theorem 7.1. *Let $\{\lambda_k : k \in \mathbb{Z} \cup \{0^+\}\}$ be a sequence described in equation (6.1). Then for any positive l there exist $l_1 \in (0, l)$, $\nu > 0$, $M > 0$ and $b(s) \in W_2^2(0, l_1)$, $b(s) > 0$ such that the spectrum of the corresponding problem (5.6)-(5.11) coincide with $\{\lambda_k : k \in \mathbb{Z} \cup \{0^+\}\}$*

Proof. From the definition of the potential we write

$$\frac{d^2}{dx^2} b^{\frac{1}{4}}(s(x)) - q(x) b^{\frac{1}{4}}(s(x)) = 0.$$

We also know that M_1, M_2, M_4 are defined as follows

$$M_1 = \nu^{-1}(l - l_1)^{-1},$$
$$M_2 = M^{-1}(l - l_1)^{-1},$$

and

$$M_4 = -\frac{1}{4} b^{-1}(l_1) \frac{d}{dx}(b(s(x)))\Big|_{x=a}.$$

We now consider the following Cauchy problem

$$\frac{d^2}{dx^2} f - q(x) f = 0 \tag{7.1}$$

$$f(a) = 1 \tag{7.2}$$

$$\frac{d}{dx} f(x)\Big|_{x=a} = -M_4. \tag{7.3}$$

Since $q(x) \in L_2(0, a)$ then from Theorem 3.2 of [4] we obtain that the initial value problem has a unique solution $f(x) \in W_2^2(0, a)$. Theorem 13.2 of [15] states that:

> Let τ be a regular Sturm-Liouville expression on (a, b) with $p(x) > 0$ and A a self adjoint realization of τ with separated boundary conditions (α, β), $\alpha \in [0, \pi)$, $\beta \in (0, \pi]$. Then the eigenvalues of A are bounded from below. If the eigenvalues are arranged such that
>
> $$\lambda_1 < \lambda_2 < \lambda_3 < \cdots \lambda_n \to \infty,$$
>
> then the eigenfunction u_n corresponding to λ_n has $n - 1$ zeroes in (a, b).

Since we have that the operator defined by

$$D(A) = \{f : f \in W_2^2(0, a), f(0) = 0, f'(a) + M_4 f(a) = 0\},$$
$$Af = -f'' + q(x) f,$$

is self adjoint with separated boundary conditions and its eigenvalues can be arranged as above, we can conclude that the eigenfunction y_1 corresponding to $\lambda_1 > 0$ has no zeroes in $(0, a)$. We rewrite the following equation

$$-y_1'' + q(x) y_1 = \lambda_1 y_1$$

as

$$-y_1'' + q_1(x)y_1 = 0 \tag{7.4}$$

where $q_1(x) = q(x) - \lambda_1 < q(x)$. Now Theorem 13.1 of [15] states that:

> Let $p_2(x) \leq p_1(x)$ and $g_1(x) \leq g_2(x)$ and assume that u_1 and u_2 are solutions to
>
> $$-(p_1u_1')' + g_1u_1 = 0, \text{ and } -(p_2u_2')' + g_2u_2 = 0,$$
>
> respectively. If $g_2(x) > g_1(x)$ holds on the interval (d,c), then $\theta_2(c) \geq \theta_1(c)$ implies $\theta_2(x) > \theta_1(x)$ for $x \in (a,c)$.

Now applying the above theorem to equation (7.1) and (7.4) after noting that our $p_1(x) = p_2(x)$, $q_1(x) < q(x)$ for all $x \in (0,a)$ and that the Prüffer angle $\theta(a) = \tan^{-1}\left(\frac{y(a)}{y'(a)}\right)$ corresponding to the solutions to equation (7.1) and (7.4) are equal (or can be made equal because of the periodicity of the tan function) then Theorem 13.1 of [15] gives that $\theta_2(x) < \theta_1(x)$ for all $x \in (0,a)$. This and the fact that $y_1(x)$ has no zeroes in $(0,a)$ give that $f(x)$ has no zeroes in $(0,a)$. Since $f(a) = 1$ we obtain that f is strictly positive. We must now find the solution to the differential equation

$$\frac{d^2}{dx^2}b^{\frac{1}{4}}(s(x)) - q(x)b^{\frac{1}{4}}(s(x)) = 0$$

and satisfying

$$M_4 = -\frac{1}{4}b^{-1}(s(a))\frac{d}{dx}b(s(x))\Big|_{x=a}$$

using the above Cauchy problem. Firstly, we note that

$$b^{-\frac{1}{4}}(s(a))\frac{d}{dx}b^{\frac{1}{4}}(s(x))\Big|_{x=a} = \frac{1}{4}b^{-1}(s(a))\frac{d}{dx}(b(s(x)))\Big|_{x=a},$$

therefore the above differential equation and condition take the form

$$\frac{d^2}{dx^2}b^{\frac{1}{4}}(s(x)) - q(x)b^{\frac{1}{4}}(s(x)) = 0 \tag{7.5}$$

and

$$M_4 = -b^{-\frac{1}{4}}(s(a))\frac{d}{dx}b^{\frac{1}{4}}(s(x))\Big|_{x=a} \tag{7.6}$$

from which we get that any solution to equation (7.5) satisfying equation (7.6) is a multiple of the solution to the Cauchy problem (7.1)-(7.3). It then follows that

$$b(s(x)) = Cf^4(x) \tag{7.7}$$

for some constant C. Recall from equation (5.14) that

$$x(s) = \int_0^s b^{\frac{1}{2}}(s')ds',$$

and that this mapping is one to one. The inverse Sturm-Liouville transformation $s(x)$ satisfies

$$x(s(y)) = y.$$

Differentiating the above equation gives

$$\begin{aligned} s'(y) &= \frac{1}{x'(s(y))} \\ &= \frac{1}{b^{\frac{1}{2}}(s(y))}, \end{aligned}$$

and integrating from zero to x we obtain that the inverse Sturm-Liouville transformation is given by

$$s(x) = \int_0^x b^{-\frac{1}{2}}(s(x'))dx',$$

from which we obtain that

$$l_1 = \int_0^a b^{-\frac{1}{2}}(s(x'))dx'. \tag{7.8}$$

Substituting (7.7) into (7.8) gives that

$$l_1 = C^{-\frac{1}{2}} \int_0^a f^{-2}(x')dx', \tag{7.9}$$

and now using the definitions of M_3 and M_4 we obtain

$$\begin{aligned} M_3 - M_4 &= (l - l_1)^{-1} b^{-\frac{1}{2}}(l_1) \\ &= (l - l_1)^{-1} C^{-\frac{1}{2}} f^{-2}(a) \\ &= (l - l_1)^{-1} C^{-\frac{1}{2}}. \end{aligned}$$

We also know that $l_1 = C^{\frac{1}{2}} A$ where $A = \int_0^a f^{-2}(x)dx$, so then

$$(M_3 - M_4)(l - l_1) = C^{-\frac{1}{2}},$$

and using $l_1 = C^{-\frac{1}{2}} A$ we get that

$$\begin{aligned} (M_3 - M_4)l &= (M_3 - M_4)l_1 + \frac{l_1}{A} \\ &= l_1(A^{-1} + M_3 - M_4), \end{aligned}$$

which implies that

$$l_1 = (M_3 - M_4)lA(1 + (M_3 - M_4)A)^{-1}$$

and substituting A we can then write it as

$$l_1 = (M_3 - M_4)l \int_0^a f^{-2}(x)dx \Big(1 + (M_3 - M_4) \int_0^a f^{-2}(x)dx\Big)^{-1} \tag{7.10}$$

from which it follows that $l_1 > 0$ and that C is positive since we have $l_1 = C^{-\frac{1}{2}} A$, hence $b(s(x)) = Cf^4(x) > 0 \; \forall x \in [0, a]$. Now $M_1 = \nu^{-1}(l - l_1)^{-1}$ gives that

$$\nu = M_1^{-1} l^{-1} (1 - \frac{l_1}{l})^{-1},$$

but (7.10) gives that

$$\begin{aligned} \Big(1 - \frac{l_1}{l}\Big) &= 1 - (M_3 - M_4)A(1 + (M_3 - M_4)A)^{-1} \\ &= \frac{1 + (M_3 - M_4)A - (M_3 - M_4)A}{(1 + (M_3 - M_4)A)} \\ &= (1 + (M_3 - M_4)A))^{-1}. \end{aligned}$$

Since $(M_3 - M_4)A) > 0$, this gives that $l_1 < l$, therefore

$$\nu = M_1^{-1} l^{-1} (1 - \frac{l_1}{l})^{-1}$$

yields

$$\nu = M_1^{-1} l^{-1} \Big(1 + (M_3 - M_4) \int_0^a f^{-2}(x) dx \Big) \tag{7.11}$$

which is positive since $M_1 > 0$. Similarly, from $M_2 = M^{-1}(l - l_1)^{-1}$ we obtain that

$$M = M_2^{-1} l^{-1} (1 - \frac{l_1}{l})^{-1}$$

which when expanded gives that

$$M = M_2^{-1} l^{-1} \Big(1 + (M_3 - M_4) \int_0^a f^{-2}(x) dx \Big) \tag{7.12}$$

which is positive since $M_2 > 0$. □

8. Conclusion

We conclude by summarizing our entire journey. We started with the following boundary value problem: Given the density

$$B(s) = \begin{cases} b(s) & \text{if } s \in [0, l_1] \\ 0 & \text{if } s \in (l_1, l], \end{cases}$$

where $0 < l_1 < l < \infty$, $b(s) \in W_2^2(0, l_1)$ and our task was to give the description of the spectrum of the problem:

$$\frac{\partial^2 w}{\partial s^2} - B(s) \frac{\partial^2 w}{\partial t^2} = 0,$$
$$w(0,t) = 0,$$
$$\Big(\frac{\partial w}{\partial s} + \nu \frac{\partial w}{\partial t} + M \frac{\partial^2 w}{\partial t^2} \Big) \Big|_{s=l} = 0$$

where $b(s) \geq \varepsilon > 0$, $b(s) \in W_2^2(0, l_1)$, $\nu > 0$ is the coefficient of damping and M the mass of the concentrated ring at the endpoint. Using separation of variables we transformed the above partial differential equation into the following equation

$$u_{ss}(\lambda, s) + b(s) \lambda^2 u(\lambda, s) = 0, \quad s \in [0, l_1]$$

satisfying boundary conditions

$$u(\lambda, 0) = 0$$
$$u_s(\lambda, l_1) + i\lambda\nu[u_s(\lambda, l_1)(l - l_1) + u(\lambda, l_1)] - M\lambda^2[u_s(\lambda, l_1)(l - l_1) + u(\lambda, l_1)] = 0.$$

We further observed that the above second order differential equation had the weight factor $b(s)$ in front of the parameter λ which we removed using the Sturm-Liouville transformation which is a one to one map and transformed the above equations into the Sturm-Liouville differential equation

$$-y''(\lambda, x) + q(x) y(\lambda, x) = \lambda^2 y(\lambda, x),$$

where $q(x)$ is a real valued potential in $L_2(0, a)$ and the function y satisfied

$$y(\lambda, 0) = 0$$

and

$$(1 + i\lambda M_1^{-1} - \lambda^2 M_2^{-1}) y_x(\lambda, a) + \Big[M_4 + i\lambda M_1^{-1} M_3 - \lambda^2 M_2^{-1} M_3 \Big] y(\lambda, a) = 0$$

where the constants M_j for $j = 1, 2, 3, 4$ are real valued constants given by

$$M_1 = \nu^{-1}(l - l_1^{-1})$$

and

$$M_2 = M^{-1}(l-l_1)^{-1}$$
$$M_3 = -\frac{1}{4}b^{-1}(l_1)\frac{d}{dx}(b(s(x)))\Big|_{x=a} + (l-l_1)^{-1}b^{-\frac{1}{2}}(l_1)$$
$$M_4 = -\frac{1}{4}b^{-1}(l_1)\frac{d}{dx}(b(s(x)))\Big|_{x=a}.$$

We further identified the entire function φ

$$\varphi(\lambda) = (1 + i\lambda M_1 - \lambda^2 M_2)s'(\lambda,a) + \Big[M_4 + i\lambda M_1 M_3 - \lambda^2 M_2 M_3\Big]s(\lambda,a)$$

as the characteristic equation and showed that for strings with density $b(s) \in W_2^6(0,l_1)$ the characteristic equation had the form

$$\begin{aligned}\varphi(\lambda) = &-\lambda^2 K_1 \cos\lambda a + i\lambda K_2 \cos\lambda a + \lambda K_3 \sin\lambda a + K_4 \cos\lambda a + iK_5 \sin\lambda a + iK_6 \frac{\cos\lambda a}{\lambda} \\ &+ K_7 \frac{\sin\lambda a}{\lambda} + K_8 \frac{\cos\lambda a}{\lambda^2} + iK_9 \frac{\sin\lambda a}{\lambda^2} + iK_{10}\frac{\cos\lambda a}{\lambda^3} + K_{11}\frac{\sin\lambda a}{\lambda^3} \\ &+ \int_0^a \xi_1(t)\frac{\sin\lambda t}{\lambda^3}dt + i\int_0^a \xi_2(t)\frac{\cos\lambda t}{\lambda^3}dt.\end{aligned}$$

Furthermore, we showed that under proper enumeration that the spectrum $\{\lambda_k : k \in \mathbb{Z}\cup\{0^+\}\}$ satisfied the following:

(1) For all $k \in \mathbb{Z}\cup\{0^+\}$ we have that $\mathrm{Im}\lambda_k > 0$,
(2) The number of pure imaginary λ_k is even,
(3) The sequence $\{\lambda_k : k \in \mathbb{Z}\cup\{0^+\}\}$ is symmetric with respect to imaginary axis and the corresponding multiplicities of symmetric points coincide
(4) Being enumerated in a proper way, the sequence $\{\lambda_k : k \in \mathbb{Z}\cup\{0^+\}\}$ admits the following asymptotic

$$\lambda_k \underset{k\to\infty}{=} \frac{\pi}{a}(k-\frac{1}{2}) + \frac{Q_1}{(k-\frac{1}{2})} + i\frac{Q_2}{(k-\frac{1}{2})^2} + \frac{Q_3}{(k-\frac{1}{2})^3} + i\frac{Q_4}{(k-\frac{1}{2})^4} + \frac{Q_5}{(k-\frac{1}{2})^5} + \frac{\beta_k}{k^5}$$

where $\{\beta_k\}_{-\infty}^{\infty} \in l_2$, $a > 0$, Q_1,$Q_2 = 0$, $Q_3, Q_4 > 0, Q_5 \in \mathbb{R}$,

The Direct Problem for smooth strings with damping at one endpoint was considered by Pivovarchick in [13] and he showed there that the spectrum satisfied the same conditions as above except for the asymptotic expansion which had the form

$$\lambda_n \underset{n\to\infty}{=} \frac{\pi n}{a} + \frac{P_1}{n} + i\frac{P_2}{n^2} + \frac{P_3}{n^3} + \frac{\tilde{\beta}_k}{k^3},$$

from which it is evident how the introduction of a string with an inhomogeneous density affected the spectrum.

The Inverse Problem on the other hand began with all the conditions satisfied by our spectrum as listed above and we proved that there exist a unique set $\{a,q(x),M_1,M_2,M_3,M_4\}$ such that $\{\lambda_k\}_{-\infty}^{\infty}$ is the spectrum of our problem. We also gave a brief discussion regarding the difficulties encountered in the inverse problem when classes of strings whose density belonged to $W_2^2(0,l_1)$ and $W_2^4(0,l_1)$ are considered. Therefore the inverse problem was considered for a much narrower class of string, namely $b(s) \in W_2^6(0,l_1)$. We followed the Marchenko approach in the inverse problem and constructed the S-matrix as

$$S(\lambda) = \Big(\frac{u(\lambda) + i\lambda v(\lambda)}{u(\lambda) - i\lambda v(\lambda)}\Big)e^{-2i\lambda a}, \quad -\infty < \lambda < \infty$$

and further identified the sets $\{\mu_k\}_{-\infty,k\neq 0}^{\infty}$, $\{\nu_k\}_{-\infty,k\neq 0}^{\infty}$ of the zeroes of the functions $u(\lambda)$ and $v(\lambda)$ as the spectrum of Dirichlet-Neumann problem generated by the set $\{a,q(x)\}$

$$y'' + \lambda^2 y - q(x)y = 0$$

and

$$y(\lambda,0) = y'(\lambda,a) = 0$$

and the set $\{\nu_k\}_{-\infty,k\neq 0}^{\infty}$ as the spectrum of the Dirichlet-Dirichlet problem generated by the set $\{a,q(x)\}$

$$y'' + \lambda^2 y - q(x)y = 0$$

and

$$y(\lambda,0) = y(\lambda,a) = 0$$

Armed with the above information we then showed that Marchenko fundamental equation

$$G(x,t) + F(x+t) + \int_x^{\infty} G(x,s)F(s+t)ds = 0$$

had a unique solution $G(x,t)$ since the S-matrix satisfied Proposition 6.18 and Proposition 6.19, and we also obtained that the potential given by

$$q(x) = -2\frac{dG(x,x)}{dx}$$

matched the potential from our Direct problem.

Finally, in the last section we showed how solutions to the Cauchy problem may be used to unlock parameters of the string which are imbedded in the set $\{a,q(x),M_1,M_2,M_3,M_4\}$ which we defined at the begin of the Inverse problem.

REFERENCES

[1] Chadan, K. An introduction to inverse scattering and inverse spectral problems, SIAM Monographs on Mathematical Modeling and Computation. Society for Industrial and Applied Mathematics (SIAM) Philadelphia, Philadelphia, PA, 1997. x+198 pp.

[2] Chadan, K. and Sabatier, P.C. Inverse problems in quantum scattering theory, Texts and Monographs in Physics. Springer-Verlag, New York-Berlin, 1977. xxii+344 pp.

[3] Dennemeyer, R. Introduction to partial differential equations and boundary value problems. McGraw-Hill Book Co., New York-Toronto, Ont.-Lond 1968 viii+376 pp.

[4] Hinton, D and Schaefer, P. Spectral theory and computational methods of Sturm-Liouville problems, Lecture Notes in Pure and Applied Mathematics, 191. Marcel Dekker, Inc., New York, 1997. viii+399 pp.

[5] Kac, M. Can one hear the shape of a drum? Amer. Math. Monthly 73 1966 no. 4, part II, 1-23.

[6] Krein, M.G. and Nudelman, I. V. Some spectral properties of a nonhomogeneous string with a dissipative boundary condition. J. Operator Theory 22 (1989), no. 2, 369-395.

[7] Levin, B.Ja. and Ostrovskii, I.V. Small perturbations of the set of roots of sine-type functions. Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), no. 1, 87-110, 238.

[8] Levin, B.Ja. Distribution of zeros of entire functions. Translations of Mathematical Monographs, 5. American Mathematical Society, Providence, R.I., 1980. xii+523 pp.

[9] Marchenko, V.A. Concerning the theory of a differential operator of the second order. Doklady Akad. Nauk SSSR. (N.S.) 72, (1950). 457-460.

[10] Marchenko, V.A. Some questions of the theory of one-dimensional linear differential operators of the second order. I. Trudy Moskov. Mat. Ob. 1, (1952), 327-420.

[11] Marchenko, V.A. Sturm-Liouville operators and applications. Revised edition. AMS Chelsea Publishing, Providence, RI, 2011. xiv+396 pp.

[12] Mennicken, R. and Pivovarchick, V. An inverse problem for an inhomogeneous string with an interval of zero density. Math. Nachr. 259 (2003), 51-65.

[13] Pivovarchick, V. Inverse problem for a smooth string with damping at one end. J. Operator Theory 38 (1997), no. 2, 243-263.

[14] Roach, G.F. Wave Scattering by time-dependent pertubations, Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ, 2007. xii+287 pp.

[15] Weidmann, J. Spectral theory of ordinary differential operators. Lecture Notes in Mathematics, 1258. Springer-Verlag, Berlin, 1987. vi+303 pp.

[16] Zygmund, A. Trigonometric series. 2nd ed. Vols. I, II. Cambridge University Press, New York 1959 Vol. I. xii+383 pp.; Vol. II. vii+354 pp.