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Research Article

Bargraphs of combinations with repetition

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Abstract

Generating functions that track some geometrical features of combinations with repetition are developed; namely, the semi-perimeter, inner site-perimeter, and outer site-perimeter, each of whose meanings depends on the interpretation of the combination as a bargraph. The paper has three theorems, which respectively give the total number for each of these statistics based on separate generating functions tracking these statistics.

Keywords: combinations with repetition; generating functions; semi-perimeter; site-perimeter.

2020 Mathematics Subject Classification: 05A10, 05A15.

1. General introduction

In elementary mathematics, choosing (allowing replacement of) p objects from n different objects, regardless of the order of choice, is called a combination with repetition. The number of possible such combinations is a basic and elementary statistic. Indeed, choosing a combination of j objects with repetition from a set of n objects can be done in $\binom{n+j-1}{j}$ ways. By definition, such a combination is of the form $\{c_1, c_2, \cdots, c_j\}$ where

$$1 \le c_1 \le c_2 \le \cdots \le c_j \le n.$$

In this paper, we track the number of ways of choosing j integers from $[n]$, where we allow repetitions but ignore the order in which we choose the j integers. What is different here is that we track certain geometrical statistics of the combinations, namely the semi-perimeter, inner site-perimeter, and outer site-perimeter, which are defined below. The idea for this paper follows that of Mansour and Shabani [8], where these same statistics are studied in combinations that do not allow repetition. The geometric parameters above are based on the interpretation of such a combination as a bargraph. For a survey of topics related to bargraphs, see [7]. A bargraph representation of a combination $c_1, c_2, \ldots c_j$ of j integers from $[n]$ is a left-to-right sequence of columns whose lower boundaries are at the same level and where the ith column has c_i vertical squares, each of which is called a cell. We think of the cells of a bargraph as lying within a larger grid of such cells, some of which are inside the bargraph and the rest being outside.

In this paper, we study three statistics. Firstly, the semi-perimeter of a bargraph is half the number of edges on the boundary of the bargraph. Secondly, the inner site-perimeter is the number of cells inside the bargraph that have at least one common edge with an outside cell. Thirdly, the outer site-perimeter is the number of cells outside the bargraph that have at least one common edge with a cell inside the bargraph.

The statistic that is fixed over all three subsequent sections is the number of elements or the cardinality of combinations. We develop a different generating function in each of these sections; in the next section, the semi-perimeter, then the outer site-perimeter, and finally in the last section, the inner site-perimeter (see [6]).

As an example of the inner site-perimeter and outer site-perimeter, we illustrate in Figure 1.1, the combination $\{2, 4, 4, 5\}$. The outer site-perimeter is the sum of the cells marked by “o”, i.e., 16, whereas the inner site-perimeter is the sum of the cells marked by “x”, i.e., 12. The semi-perimeter of the bargraph is 9. The number of parts is clearly 4 and this is a combination whose parts come from $[n]$, where n is any integer greater than or equal to the largest part of the combination, say $n = 6$.

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Figure 1.1: Outer and inner site-perimeters of the combination $\{2, 4, 4, 5\}$ of $[6]$ with four parts.

2. Semi-perimeter

We denote by $card(\pi)$ the cardinality of the combination π of $[n]$. Let $P_n(p, q)$ be the generating function for the number of combinations π of $[n]$ with repetition according to the statistics $card$ tracked by p and semi-perimeter (sp) tracked by q.

$$P_n(p, q) := \sum_{\pi \in C_n} p^{card(\pi)} q^{sp(\pi)}.$$

Since each combination of $[n]$ either contains n or not, we have

$$P_n(p, q) = P_{n-1}(p, q) + P_n(p, q|n), \tag{1}$$

where $P_n(p, q|i)$ is defined to be the generating function for such combinations ending in i. We consider the two cases of cardinality 1 or larger (where the last two columns are jn) and therefore obtain:

$$P_n(p, q|n) = pq^{n+1} + p \sum_{j=1}^{n} q^{n+1-j} P_j(p, q|j) \tag{2}$$

which implies

$$qP_{n-1}(p, q|n-1) = pq^{n+1} + p \sum_{j=1}^{n-1} q^{n+1-j} P_j(p, q|j) \tag{3}$$

with $P_1(p, q|1) = P_1(p, q) = \frac{pq^2}{1-pq}$. Subtracting Equation (3) from (2), we obtain

$$P_n(p, q|n) = \frac{qP_{n-1}(p, q|n-1)}{1 - pq},$$

which we iterate $n - 1$ times to obtain

$$P_n(p, q|n) = \frac{pqq^n}{(1 - pq)^n}.$$

So, from Equation (1), we have

$$P_n(p, q) = P_{n-1}(p, q) + \frac{pqq^n}{(1 - pq)^n}.$$

Again, we iterate the latter equation to obtain

$$P_n(p, q) = pq \sum_{j=1}^{n} \frac{q^j}{(1 - pq)^j} = \frac{pq^2 \left(1 - q^n (1 - pq)^{-n}\right)}{1 - (1 + p)q}.$$

To find the generating function for the total semi-perimeter we differentiate the last equation with respect to q and set $q = 1$ to obtain

$$\frac{\partial}{\partial q} \frac{pq^2 \left(1 - q^n (1 - pq)^{-n}\right)}{1 - (1 + p)q}\bigg|_{q=1} = -\frac{(1 - p)^{-1-n} \left(1 - (1 - p)^n + (-2 - n + 2(1 - p)^n)\, p - (-1 + (1 - p)^n)\, p^2\right)}{p}. \tag{4}$$

Now, we extract the coefficient of p^j and obtain

$$\frac{(1 + j(1 + j + n))\binom{j+n}{j+1}}{j + n}.$$

Finally, dividing by $\binom{j+n-1}{j}$ (which is the number of combinations of $[n]$ with j parts allowing repetition), we obtain the next result.

Theorem 2.1. *For fixed n and j, the average semi-perimeter for the class of size $\binom{j+n-1}{j}$ of all combinations of $[n]$ with repetition and with j parts, is*

$$\frac{1+j(1+j+n)}{1+j}.$$

Example 2.1. *When $n=3$ and $j=4$, we obtain the following list of the 15 combinations with repetition*

$$\begin{gathered}\{\{1,1,1,1\},\{1,1,1,2\},\{1,1,1,3\},\{1,1,2,2\},\{1,1,2,3\},\\ \{1,1,3,3\},\{1,2,2,2\},\{1,2,2,3\},\{1,2,3,3\},\{1,3,3,3\},\\ \{2,2,2,2\},\{2,2,2,3\},\{2,2,3,3\},\{2,3,3,3\},\{3,3,3,3\}\}\end{gathered} \tag{5}$$

and average semi-perimeter of $33/5$ *as asserted by Theorem 2.1.*

Next, we briefly consider the following problem. Suppose we are given a combination from $[n]$ with $j \leq n$ parts and semi-perimeter k, where $k \geq j$. Can we find the generating function for the area of the different combinations having these properties?

The bargraph representation of the combination $c_1, c_2, \cdots, c_j$ has area $c_1 + c_2 + \cdots + c_j$ and fits within a rectangle of width j and height n. The number of combination bargraphs that have exactly j columns and fit within such a rectangle is given by

$$q^j \binom{n+j-1}{j}_q$$

where

$$\binom{n+j-1}{j}_q$$

is the q-binomial coefficient. This is in accordance with Equation (7.1) in [1]. Also $q^j\binom{n+j-1}{j}_q$ is the generating function tracked by area for all combination bargraphs with j parts and semi-perimeter $\leq n+j$.

Hence the generating function for all combinations with j parts and semi-perimeter k is given by

$$q^j\binom{k-1}{j}_q - q^j\binom{k-2}{j}_q.$$

Thus, by letting $q \to 1$, the number of bargraphs with semi-perimeter k is

$$\binom{k-1}{j} - \binom{k-2}{j}.$$

This gives the formula for the total semi-perimeter:

$$\sum_{k=j+1}^{j+n} k\left(\binom{k-1}{j} - \binom{k-2}{j}\right) = \frac{n(j(j+n+1)+1)\left(\binom{j+n}{j} - \binom{j+n-1}{j}\right)}{j(j+1)}.$$

Dividing this by $\binom{n+j-1}{j}$ we re-derive the average given in Theorem 2.1.

3. Outer site-perimeter

Outer site-perimeter is defined as the number of nearest neighbour cells outside the bargraph. See for example, [4, 5], where the statistic is simply called the site-perimeter. Outer site-perimeter is illustrated in the left portion of Figure 1.1.

Let $P_n(p,q)$ be the generating function for the number of combinations π of $[n]$ according to the statistics $card$ tracked by p and op (outer site-perimeter) tracked by q.

Also, we define $P_0(p,q) = 1$. First, let us write a recurrence relation for $P_n(p,q)$. Since each combination of $[n]$ either contains n or not, we have

$$P_n(p,q) = P_{n-1}(p,q) + P_n(p,q|n), \tag{6}$$

where $P_n(p,q|n)$ is the generating function for the number of combinations of $[n]$ that contain n, again according to the statistics $card$ and op. Since each combination π of $[n]$ that contains n either has only one element, or the second maximal element in π is either j, $1 \leq j \leq n-1$ or n, we obtain

$$P_n(p,q|n) = pq^{2n+2} + p\sum_{j=1}^{n-1} q^{2n+1-2j} P_j(p,q|j) + pq^2 P_n(p,q|n).$$

Hence

$$P_n(p,q|n) = \frac{pq^{2n+2}}{1-pq^2} + \frac{p}{1-pq^2}\sum_{j=1}^{n-1} q^{2n+1-2j} P_j(p,q|j),$$

from which we obtain

$$q^2 P_{n-1}(p,q|n-1) = \frac{pq^{2n+2}}{1-pq^2} + \frac{p}{1-pq^2} - \sum_{j=1}^{n-2} q^{2n+1-2j} P_j(p,q|j).$$

Subtracting the latter equation from the previous one, we obtain the recursion

$$P_n(p,q|n) = \left(q^2 + \frac{pq^3}{1-pq^2}\right) P_{n-1}(p,q|n-1)$$

where

$$P_1(p,q|1) = \frac{pq^4}{1-pq^2},$$

and iterating this yields

$$P_n(p,q|n) = \frac{pq^2}{1+pq-pq^2}\left(\frac{q^2(-1+p(-1+q)q)}{-1+pq^2}\right)^n.$$

Now, substituting this into Equation (6) we obtain

$$\begin{aligned} P_n(p,q) &= 1 + \sum_{j=1}^{n} P_n(p,q|j) \\ &= \frac{1-(1+p)q^2 - pq^3 - pq^4\left(-2 + \left(\frac{q^2(-1+p(-1+q)q)}{-1+pq^2}\right)^n\right)}{1-(1+p)q^2-pq^3+pq^4}. \end{aligned} \tag{7}$$

Differentiating with respect to q and putting $q=1$, we find the generating function for the total outer site-perimeter to be

$$\left.\frac{\partial P_n(p,q)}{\partial q}\right|_{q=1} = -3 + \frac{2}{p} + \left(5 + 2n - \frac{2}{p} + (-3-n)p + np^2\right)\left(\frac{1}{1-p}\right)^{n+1}.$$

Next, we extract the coefficient of this partial derivative to obtain

$$\begin{aligned} [p^j]\left.\frac{\partial P_n(p,q)}{\partial q}\right|_{q=1} &= (5+2n)\binom{n+j}{j} - 2\binom{n+j+1}{j+1} - (3+n)\binom{n+j-1}{j-1} + n\binom{n+j-2}{j-2} \\ &= \frac{-3+j^2+2j^3+3\left(1+j^2\right)n+2jn^2}{(n+j)(n+j-1)}\binom{j+n}{j+1}. \end{aligned} \tag{8}$$

Now, dividing by $\binom{j+n-1}{j}$, we obtain the next result.

Theorem 3.1. *For fixed n and j, the average outer site-perimeter for the class of size $\binom{j+n-1}{j}$ of all combinations of $[n]$ with repetition and with j parts, is*

$$\frac{3-j^2-2j^3-3\left(1+j^2\right)n-2jn^2}{(1+j)(1-j-n)}.$$

Example 3.1. *When $n=3$ and $j=4$, the list of the 15 combinations with repetition given in (5) has average outer site-perimeter of* $61/5$ *as asserted by Theorem 3.1.*

4. Inner site-perimeter

The inner site-perimeter is defined as the number of cells inside the bargraph that have at least one-edge lying on the bargraph perimeter; for example, see [2,3]. The inner site-perimeter is illustrated in the right-hand portion of Figure 1.1.

Let $P_n(p,q)$ be the generating function for the number of combinations π of $[n]$ according to the statistics $card$ tracked by p and ip (i.e., inner site-perimeter) tracked by q.

Notationally, we use $P_n(p,q|j_1,j_2,\ldots,j_s)$ to mean the generating function for combinations of $[n]$ that end with the parts $j_1, j_2, \ldots j_s$.

The strategy for obtaining the generating function in this section is to express $P_n(p,q|jn)$ in terms of $P_{j+1}(p,q|j(j+1))$. This will enable us to solve for the generating function recursively as specified later, in Equation (17).

Also, we define $P_0(p,q)=1$. First, let us write a recurrence relation for $P_n(p,q)$. Since each combination of $[n]$ either contains n or not, we have

$$P_n(p,q)=P_{n-1}(p,q)+P_n(p,q|n) \tag{9}$$

where $P_n(p,q|n)$ is the generating function for the number of combinations of $[n]$ that contain n, again according to the statistics $card$ and ip. Since each combination π of $[n]$ that contains n either has only one element, or the second maximal element in π is either j ($1\le j\le n-1$) or n, we obtain

$$P_n(p,q|n)=pq^n+\sum_{j=1}^{n-1}P_n(p,q|jn)+P_n(p,q|nn) \tag{10}$$

$$=pq^n+\sum_{j=1}^{n-1}q^{n-1-j}P_{j+1}(p,q|j(j+1))+P_n(p,q|nn), \tag{11}$$

where the summand $q^{n-1-j}P_{j+1}(p,q|j(j+1))$ is a replacement for the summand of the previous line $P_n(p,q|jn)$ by exchanging n with $j+1$ and compensating with q^{n-1-j}. Now, focusing on $P_n(p,q|nn)$, we obtain

$$P_n(p,q|nn)=p^2q^{2n}+\sum_{j=1}^{n-1}P_n(p,q|jnn)+P_n(p,q|nnn)$$

$$=p^2q^{2n}+\sum_{j=1}^{n-1}pq^{n+1-j}P_n(p,q|jn)+pq^2P_n(p,q|nn). \tag{12}$$

Hence

$$P_n(p,q|nn)=\frac{p^2q^{2n}}{1-pq^2}+\frac{p}{1-pq^2}\sum_{j=1}^{n-1}q^{n+1-j}P_n(p,q|jn). \tag{13}$$

Now, we substitute the latter equation into Equation (10) to obtain

$$P_n(p,q|n)=pq^n+\sum_{j=1}^{n-1}P_n(p,q|jn)+\frac{p^2q^{2n}}{1-pq^2}+\frac{p}{1-pq^2}\sum_{j=1}^{n-1}q^{n+1-j}P_n(p,q|jn)$$

$$=pq^n+\frac{p^2q^{2n}}{1-pq^2}+\sum_{j=1}^{n-1}\left(1+\frac{pq^{n+1-j}}{1-pq^2}\right)P_n(p,q|jn)$$

$$=pq^n+\frac{p^2q^{2n}}{1-pq^2}+\sum_{j=1}^{n-1}\left(1+\frac{pq^{n+1-j}}{1-pq^2}\right)q^{n-1-j}P_{j+1}(p,q|j(j+1)), \tag{14}$$

where the last line follows using (11).

Next, for $j>1$, we have

$$P_{j+1}(p,q|j(j+1))=p^2q^{2j+1}+\sum_{i=1}^{j}P_{j+1}(p,q|ij(j+1))$$

$$=p^2q^{2j+1}+\sum_{i=1}^{j-1}q^{2(j-1-i)}P_{i+2}(p,q|i(i+1)(i+2))+P_{j+1}(p,q|jj(j+1))$$

$$=p^2q^{2j+1}+p\sum_{i=1}^{j-1}q^{2j+1-2i}P_{i+1}(p,q|i(i+1))+pq^2P_{j+1}(p,q|j(j+1)).$$

Hence for $j>1$, we have

$$P_{j+1}(p,q|j(j+1))=\frac{p^2q^{2j+1}}{1-pq^2}+\frac{p}{1-pq^2}\sum_{i=1}^{j-1}q^{2j+1-2i}P_{i+1}(p,q|i(i+1)). \tag{15}$$

Thus, for $j>2$, we obtain

$$P_{j+1}(p,q|j(j+1))-q^2P_j(p,q|(j-1)j)=\frac{p}{1-pq^2}q^3P_j(p,q|(j-1)j). \tag{16}$$

$$P_{j+1}(p,q|j(j+1))=q^2\left(\frac{pq}{1-pq^2}+1\right)P_j(p,q|(j-1)j). \tag{17}$$

Since $P_3(p,q|23) = \frac{1}{1-pq}\frac{p^2q^5}{1-pq^2}$, by iteration for $j \geq 2$, we have

$$P_{j+1}(p,q|j(j+1)) = P_3(p,q|23)q^{2j-4}\left(\frac{pq}{1-pq^2}+1\right)^{j-2} \tag{18}$$

Substitute this into Equation (14) to obtain

$$\begin{aligned}
P_n(p,q|n) &= pq^n + \frac{p^2q^{2n}}{1-pq^2} + \left(1+\frac{pq^n}{1-pq^2}\right)q^{n-2}P_2(p,q|12)\\
&+ \sum_{j=2}^{n-1}\left(1+\frac{pq^{n+1-j}}{1-pq^2}\right)q^{n-1-j}P_3(p,q|23)q^{2j-4}\left(1+\frac{pq}{1-pq^2}\right)^{j-2}\\
&= pq^n + \frac{p^2q^{2n}}{1-pq^2} + \left(1+\frac{pq^n}{1-pq^2}\right)q^{n-2}P_2(p,q|12)\\
&+ \frac{1}{1-pq^2}\sum_{j=2}^{n-1}q^{n-1-j}\frac{p^2q^5}{1-pq}q^{2j-4}\left(1+\frac{pq}{1-pq^2}\right)^{j-2}\left(1+\frac{pq^{n+1-j}}{1-pq^2}\right)\\
&= pq^n + \frac{p^2q^{2n}}{1-pq^2} + \left(1+\frac{pq^n}{1-pq^2}\right)q^{n-2}P_2(p,q|12)\\
&+ \frac{p^2q^n}{(1-pq)(1-pq^2)}\sum_{j=2}^{n-1}q^j\left(1+\frac{pq}{1-pq^2}\right)^{j-2}\left(1+\frac{pq^{n+1-j}}{1-pq^2}\right).\\
&= \frac{pq^n\left(1-pq^2\right)\left(1-q+pq-2pq^2+pq^3-pq^{1+n}\left(1+\frac{pq}{1-pq^2}\right)^n\right)}{(1-pq)(1+p(1-q)q)\left(1-q-2pq^2+pq^3\right)}.
\end{aligned} \tag{19}$$

Now, by substituting this into Equation (9), we obtain

$$\begin{aligned}
P_n(p,q) &= 1+\sum_{j=1}^{n}P_n(p,q|j)\\
&= 1+pq\left(1-pq^2\right)\times\frac{\left(1-(1+2p)q^2-pq^3+pq^4-q^n(1-(1+p)q^2-pq^3+pq^4)+pq^{2+2n}\left(1+\frac{pq}{1-pq^2}\right)^n\right)}{(1-pq)\left(1-q-2pq^2+pq^3\right)\left(1-(1+p)q^2-pq^3+pq^4\right)},
\end{aligned} \tag{20}$$

where the simplification is a result of summing a finite geometric series. Differentiating with respect to q and putting $q=1$, we find the generating function for the total inner site-perimeter to be

$$\left.\frac{\partial P_n(p,q)}{\partial q}\right|_{q=1} = \frac{3-4p+np+2p^2-np^2}{(1-p)p} - \frac{\left(3-4p-2np+2p^2+np^2-np^3\right)}{(1-p)p}\left(\frac{1}{1-p}\right)^n. \tag{21}$$

Next, we extract the coefficient of this partial derivative to obtain

$$\begin{aligned}
[p^j]\left.\frac{\partial P_n(p,q)}{\partial q}\right|_{q=1} &= -3\binom{n+j}{j+1}+(1+2n)\binom{n+j-1}{j}+(n-1)\binom{n+j-2}{j-1}+(2n-1)\sum_{i=2}^{j}\binom{n+j-1-i}{j-i}+1\\
&= (2n-1)\binom{j+n-2}{j-2}+(n-1)\binom{j+n-2}{j-1}+(2n+1)\binom{j+n-1}{j}-3\binom{j+n}{j+1}+1.
\end{aligned} \tag{22}$$

Dividing by $\binom{j+n-1}{j}$, we obtain the next theorem.

Theorem 4.1. *For fixed* n *and* j*, the average inner site-perimeter for the class of size* $\binom{j+n-1}{j}$ *of all combinations of* $[n]$ *with repetition and with j parts, is*

$$\frac{(2j-1)n}{1+j}+j-2+\frac{3}{1+j}-\frac{j}{n}+\frac{(j-1)j}{j+n-1}+\frac{1}{\binom{j+n-1}{j}}.$$

Example 4.1. *When* $n=3$ *and* $j=4$*, the list of the 15 combinations with repetition given in (5) has average inner site-perimeter of* $113/15$ *as asserted by Theorem 4.1.*

References

[1] G. Andrews, K. Eriksson, *Integer Partitions*, Cambridge University, Cambridge, 2004.
[2] A. Blecher, C. Brennan, A. Knopfmacher, The inner site-perimeter of compositions, *Quaest. Math.* **43** (2020) 55–66.
[3] A. Blecher, C. Brennan, A. Knopfmacher, The inner site-perimeter of bargraphs, *Online J. Anal. Combin.* **16** (2021) #2.
[4] A. Blecher, C. Brennan, A. Knopfmacher, The site-perimeter of compositions, *Discrete Math. Appl.* **32** (2022) 75–89.
[5] M. Bousquet-Mélou, A. Rechnitzer, The site-perimeter of bargraphs, *Adv. Appl. Math.* **31** (2003) 86–112.
[6] T. Mansour, Semi-perimeter and inner site-perimeter of k-ary words and bargraphs, *Art Discrete Appl. Math.* **4** (2021) #P1.06.
[7] T. Mansour, A. Shabani, Enumerations on bargraphs, *Discrete Math. Lett.* **2** (2019) 65–94.
[8] T. Mansour, A. Shabani, Combinations as bargraphs, *Discrete Math. Lett.* **8** (2022) 106–110.