

UNIVERSITY OF THE WITWATERSRAND

MASTERS DISSERTATION

Skymions and vectorial wavefunctions

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Declaration of Authorship

I, Pedro ORNELAS, declare that this thesis titled, “Skyrmions and vectorial wave-functions” and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
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Abstract

Master of Science

Skyrmions and vectorial wavefunctions

by Pedro ORNELAS

The study and generation of robust structured light stand as compelling areas of focus in the exploration of future classical and quantum photonic technologies. While the appeal of structuring light in all its degrees of freedom (DOFs) is undeniable, achieving the generation of intricate light resilient to noise from multiple sources, such as faulty detectors, stray white light, and atmospheric turbulence, is imperative for its practical integration into forthcoming technologies.

Recently, there has been a lot of interest in generating states of light with identifiable topological features which are robust to local deformations thus providing such states with a possible mechanism for noise rejection. Topological structures known as optical skyrmions have garnered a lot of interest in the optics community of late as their magnetic counterparts have shown great promise as potential low-power information carriers. It has been shown that skyrmionic structures may be realised in classical free-space optical beams where their spatial and polarization DOFs are appropriately combined and manipulated to generate what are known as vector beams. Furthermore with the emergence of quantum structured light allowing for the manipulation of an individual photon's DOFs, such topological structures may also be utilized as a resource for photonic based quantum technologies.

In this dissertation we investigate the generation of classical optical skyrmions through the use of Bessel-Gaussian optical modes possessing interesting propagation dynamics which mimic magnetic systems under the application of a magnetic field. Furthermore, we extend the study of optical skyrmions to the quantum realm by generating and characterizing the topology of the quantum analogue to classical vector beams: hybrid entangled states where the spatial DOF of one photon is entangled with the polarization DOF of another. In this case the skyrmionic topology emerges as a shared property of both photons and can be identified through investigating their mutual correlations. We postulate a novel topological characterization of entangled states with the corollary that smooth deformations of these states do not change their topology and thus do not change how they are characterized. We show that the topology remains intact even when entanglement is fragile and further discuss how a typical mechanism for entanglement decay can be characterized as a smooth deformation. Lastly, we investigate the topological resilience of hybrid entangled states in the presence of isotropic noise usually attributed to external sources. We demonstrate the invariance of the topology of these states to varying levels of isotropic noise and discuss the associated mechanism for this invariance.

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List of Publications

Journal Papers

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Chapter 1

Introduction

Structured light being the manipulation of light in all its degrees of freedom (DOFs) [1], is an emerging field that promises many technological and scientific advancements such as providing higher speed optical communications [2–6], enabling the control and trapping of particles through the use of optical tweezers [7, 8], high resolution microscopy and imaging [9–11], topography measurements [12, 13], detection of motion for optical metrology [14, 15] and many more. Beyond classical sources of light, structured light has also been employed to advance the utility of single photons to produce complex, high dimensional quantum states [16–23] suitable for use in future quantum technologies such as quantum cryptography [24–30], quantum communication [31–34], quantum computing [35, 36] and quantum metrology [37, 38] to name just a few.

Beyond the significant utility offered by way of increased dimensionality and complexity, structured light also allows for the creation of robust states of light allowing for their feasible implementation in noisy environments. Approaches to develop these robust states range from increasing their tolerance to noise, which as an example can be done through an increase in dimensionality [39, 40], or by engineering and identifying state properties which are not altered by the presence of noise in the system [41–44]. The latter scheme is typically desired as it allows for reliable information encoding into invariant state properties. Topology is one such property which is invariant to local perturbations and has been investigated in optics by way of embedding topological structures such as loops, links and knots into different DOFs of free-space optical beams [45–48]. Recent observations of their resilience against local phase aberrations and system misalignments [49] have prompted further investigations into practical encoding schemes using their associated topological invariants [50–52]. Other more recent topological candidates known as skyrmions [53] have gathered much interest in the field of optics [54–58] since their magnetic counterparts have shown great promise as potential low-power information carriers [59, 60]. The combination of polarization and spatial DOFs of free-space optical beams used to generate vector beams, optical beams with a spatially varying polarization, have been shown to exhibit skyrmionic topology [57]. The plethora of tools available to generate and reliably characterize vector beams [61–66] has led to a flurry of research over the past few years into creating different types of optical skyrmions with arbitrary order [57, 67] and vectorial textures [68, 69] and investigating their resilience to perturbations [70, 71]. Furthermore, it is comparatively simpler to achieve higher-order optical skyrmionic topologies than their lower dimensional counterparts [50, 52] thus providing a possible advantage when using the skyrmionic topology as a high capacity information carrier. However, investigations into optical skyrmionic topology is still very much in its infancy with open questions around the robustness of the skyrmionic topology as well as how these

structures may manifest in quantum structured light.

In this chapter we consider the manipulation of light's spatial and polarization DOFs for the generation of arbitrary vectorial fields and study the accompanying skyrmionic topology embedded within. This is followed by an overview of entanglement between quantum structured photons and the realization of hybrid entangled states, where the polarization of one photon is entangled with the spatial DOF of another.

1.1 Structured light

1.1.1 Polarization and spin angular momentum of light

It is well-known that light is an electromagnetic wave, where electric and magnetic fields oscillate perpendicular to one another [72] (as presented in Fig. 1.1 (a)), leading to a property known as polarization - the direction in which the electric field (by convention) oscillates. In the paraxial regime the electric field oscillation in the plane perpendicular to propagation is considered. With this in mind, a general description for the polarization state of a light beam can be given by [73]

$$|P\rangle = \beta_1 |e_1\rangle + \beta_2 |e_2\rangle, \quad (1.1)$$

a 2-dimensional vector decomposed into the orthogonal basis vectors, $|e_1\rangle, |e_2\rangle$ and where $\beta_{1,2}$ are complex numbers, whose relative amplitudes and phases control the polarization in the transverse plane. It is important to note that we have adopted the quantum bra-ket notation [74] where a vector is denoted by a ket, $|\cdot\rangle$, and its complex conjugate by a bra, $\langle\cdot|$. If we assume that $|e_1\rangle, |e_2\rangle$ point along the x and y axes of the plane perpendicular to propagation then modifying the relative amplitude changes the orientation of the electric field oscillation, an example of which is shown in Fig. 1.1 (b) where the vertical oscillation has been modified to horizontal through a change in amplitude of the orthogonal components. Furthermore, relative phase between the polarization components induces a delay between them such that an elliptical polarization oscillation is achieved, an example of which is shown in Fig. 1.1 (b) where an induced phase shift of $\frac{\pi}{2}$ between the components yields a circular oscillation of the beams polarization. Importantly, it was discovered that photons whose polarization state is circular carry spin angular momentum (SAM) [75, 76] of $\pm\hbar$ per photon, where the sign is dependent on the handedness of the polarization. The states associated with SAM are the left-circular, $|L\rangle$, and right-circular, $|R\rangle$, polarization states. If we consider expressing the polarization in terms of these spin states, normalize the resultant state and ignore global phase terms, we can rewrite Eq. 1.1 as follows

$$|P\rangle = \cos\left(\frac{\theta}{2}\right) |R\rangle + e^{i\alpha} \sin\left(\frac{\theta}{2}\right) |L\rangle, \quad (1.2)$$

where $\theta \in [0, \pi]$ and $\alpha \in [0, 2\pi]$ are parameters controlling the relative amplitude and phase between the two orthogonal components, respectively. Therefore, a sphere is a natural choice for the space of all possible transverse polarization states. Similar to the Bloch sphere for spin states in quantum mechanics [74], polarization states are found on the surface of the Poincaré sphere [73], depicted by Fig. 1.1 (c).

It is convention to write Eq. 1.2 in the right-, $|R\rangle$, and left-, $|L\rangle$, circular basis as this follows the Bloch sphere convention of writing the spin states at the poles. However any choice of orthogonal bases is valid and in this work in addition to the circular basis, $\{|R\rangle, |L\rangle\}$, we will also use the horizontal and vertical basis, $\{|H\rangle, |V\rangle\}$, diagonal and anti-diagonal basis, $\{|D\rangle, |A\rangle\}$, and the general orthogonal arbitrary elliptical basis, $\{|P^+\rangle, |P^-\rangle\}$, described by Eq. 1.2.

Since polarization states are so readily described using simple vectors, it is convenient to describe them and operations on them using one of two formalisms, the Jones formalism [77], and the Stokes formalism [78]. The Jones formalism describes a polarization state as a 2-dimensional complex vector, $|P\rangle = [\cos(\frac{\theta}{2}) \ e^{i\alpha} \sin(\frac{\theta}{2})]^T$, which lives on a 2D Hilbert space [74]. Under this formalism unitary transformations can be written as simple 2×2 matrices acting on the state (discussed in more detail in section 2.1.2). Conventionally, the polarization basis states are written as the following Jones vectors $|H\rangle = [1 \ 0]^T$, $|V\rangle = [0 \ 1]^T$, $|D\rangle = \frac{1}{\sqrt{2}} [1 \ 1]^T$, $|A\rangle = \frac{1}{\sqrt{2}} [1 \ -1]^T$, $|R\rangle = \frac{1}{\sqrt{2}} [1 \ i]^T$ and $|L\rangle = \frac{1}{\sqrt{2}} [1 \ -i]^T$. Whilst compact, the Jones formalism does not yield outcomes which are measured directly as it contains imaginary components. This is where the Stokes formalism is useful as it builds a real-valued 3-dimensional vector living on S^2 (a 3D sphere of unit radius) describing the polarization state. The Hopf map [79] maps a state described by the Jones formalism to the Stokes formalism as follows $(\beta_1, \beta_2) \rightarrow \vec{S} = (2\text{Re}\{\beta_1\beta_2\}, 2\text{Im}\{\beta_1\beta_2\}, |\beta_1|^2 - |\beta_2|^2) = (S_1, S_2, S_3)$. Each component of the Stokes vector which describes a state can be extracted from a series of intensity measurements thus making the Stokes formalism useful when conducting experiments. It is important to note that by convention the Stokes vector has 4 components, with the first component, S_0 , being the total intensity of the field. If the field is considered to be normalized and fully polarized ($S_0^2 = S_1^2 + S_2^2 + S_3^2$) then this component can be ignored as its information can be inferred from the other components. Matrices which transform Stokes vectors are known as Mueller matrices [78] however are not of particular interest in this work.

Polarization states can be represented either by the Stokes vector in the paraxial plane, as indicated in Fig. 1.1 (d) on the left, or by its corresponding polarization ellipse where the ellipse parameters can be directly extracted from the angles which give a position on the surface of the Poincaré sphere.

1.1.2 Spatial distribution and orbital angular momentum of light

We will consider the spatial modes that are solutions to the Helmholtz equation in free-space [80, 81]

$$(\nabla^2 + k^2) U(x, y, z) = 0, \quad (1.3)$$

as appropriate smooth functions of the spatial coordinates $x, y, z \in \mathbb{R}$. Here, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian operator in cartesian coordinates, k is the magnitude of the wave vector and $U(x, y, z)$ is a complex scalar field. It should be noted that the Helmholtz equation arises from considering fields that satisfy the wave equation, $(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) F(x, y, z, t) = 0$, and assuming that $F(x, y, z, t)$ can be written in a separable manner, $F(x, y, z, t) = U(x, y, z)T(t)$ [80]. Assuming that the field can be written as $U(x, y, z) = u(x, y, z)e^{-ikz}$ where e^{-ikz} is a plane wave which points

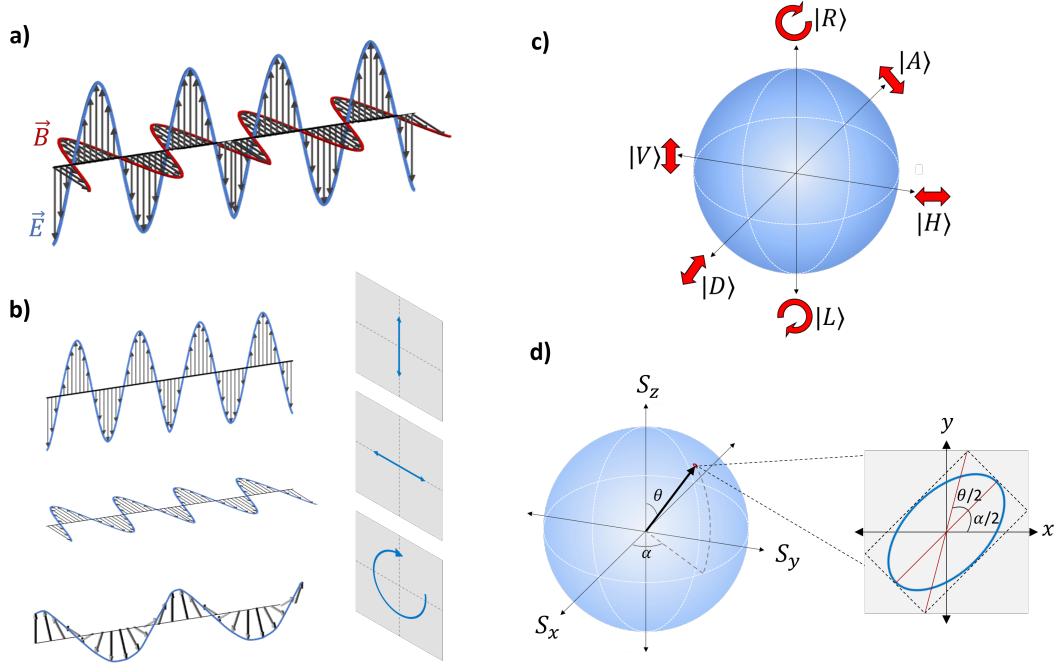


FIGURE 1.1: a) An electromagnetic wave consisting of an electric and magnetic field which are orthogonal to one another. The direction of the electric field oscillation gives the polarization of the field. b) Examples of (top) vertically, (middle) horizontally and circularly polarized light. c) The Poincaré sphere with right- and left-circular polarization states represented at the poles. d) Representations of polarization states, using a (left) Stokes vector in 3D and (right) a polarization ellipse.

along the direction of propagation of the field, we have that Eq. 1.3 after appropriate application of the product rule and simplification, becomes

$$\nabla_{\perp}^2 u(x, y, z) - 2ik \frac{\partial u(x, y, z)}{\partial z} + \frac{\partial^2 u(x, y, z)}{\partial z^2} = 0, \quad (1.4)$$

where $\nabla_{\perp}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Since we are interested in paraxial fields, we are free to perform the paraxial approximation which assumes that the divergence of the resultant spatial modes with propagation is minor [82], i.e., $u(x, y, z)$ does not vary significantly with change in z , which allows for the removal of the third term in Eq. 1.4 since $\left| \frac{\partial^2 u(x, y, z)}{\partial z^2} \right| \ll \left| \frac{\partial u(x, y, z)}{\partial z} \right|$. The result of this is the paraxial, helmholtz equation in free-space

$$\nabla_{\perp}^2 u(x, y, z) - 2ik \frac{\partial u(x, y, z)}{\partial z} = 0, \quad (1.5)$$

which when solved under different symmetries yields diverse sets of mode families that form complete orthonormal basis sets for complex scalar fields defined on $\mathbb{R}^2$. The completeness of these basis sets allow for a description of any arbitrary complex scalar field as linear combinations of basis elements [83]. Furthermore, the orthonormality of these basis sets allow for their unique detection, i.e., elements from these basis modes can be distinguished from one another [83, 84]. Another feature that all of these mode families share is that they are propagation invariant up to

a scaling factor which implies that they retain features such as orthornormality upon propagation [83, 85]. These features make such modes highly useful and practical to generate and detect in an experimental setting [84].

Free-space mode families

The first mode family that will be considered is the Hermite-Gaussian (HG) modes which have cartesian symmetry and are derived when solving the Helmholtz equation in cartesian coordinates. They are defined as follows [81, 86]

$$u(x, y, z) \equiv \text{HG}_n^m(x, y, z) = \frac{w_0}{w(z)} \sqrt{\frac{2^{(1-n-m)}}{\pi n! m!}} H_m \left(\sqrt{2} \frac{x}{w(z)} \right) H_n \left(\sqrt{2} \frac{y}{w(z)} \right) \exp \left(-\frac{r^2}{w^2(z)} \right) \times \exp \left(-i \left[\frac{kr^2}{2R(z)} - (n+m+1)\phi_G(z) + kz \right] \right), \quad (1.6)$$

where w_0 is the beam radius at $z = 0$, $k = \frac{2\pi}{\lambda}$, λ is the wavelength,

$$r = \sqrt{x^2 + y^2}, \quad (1.7)$$

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_R} \right)^2}, \quad (1.8)$$

is the beam radius after propagation,

$$R(z) = z \left[1 + \left(\frac{z_R}{z} \right)^2 \right], \quad (1.9)$$

is the radius of curvature of the beam,

$$\phi_G = \arctan \left(\frac{z}{z_R} \right), \quad (1.10)$$

is the mode dependent phase gathered by the beam on propagation, known as gouy phase [87, 88],

$$z_R = \frac{\pi w_0^2}{\lambda}, \quad (1.11)$$

is the Rayleigh range of the beam [89] and $H_{m,n}$ are the Hermite polynomials of the m -th, n -th order with $m, n \in \mathbb{Z}^+$ [90]. The Hermite polynomials are one-dimensional functions with m, n nodes along the x - and y -axes, respectively. In Fig. 1.2 the intensity and phase (placed as insets) of HG modes with varying indices are shown. HGs are identified with a clear disjointed intensity lobe structure where lobes of intensity are distributed along the x and y -axes. Clearly there are m, n null intensity "lines" across the x - and y -axes, respectively. The phase of each mode is a binary field, seen as a "checkerboard" pattern in Fig. 1.2, where the phase oscillates

between 0 and π with the switch between phase values occurring at the position of the intensity nulls.

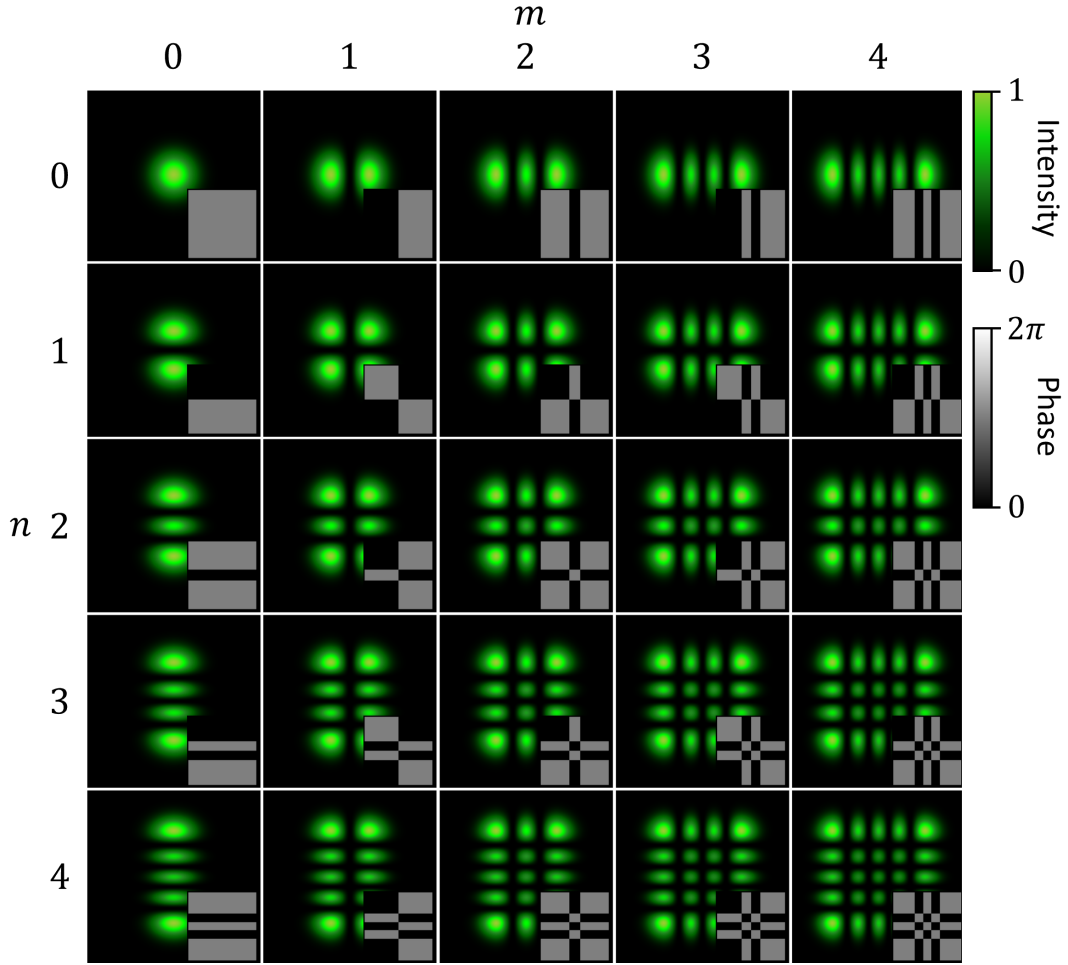


FIGURE 1.2: Intensity and phase (insets) of Hermite-Gaussian modes with varying m and n indices.

Another two mode families of interest are the Laguerre-Gaussian (LG) and Bessel-Gaussian (BG) modes which are cylindrically symmetric and are derived when solving the Helmholtz equation in polar coordinates [91]. Whilst LG modes are derived using the paraxial approximation of the Helmholtz equation, BG modes are the finite energy analogue to Bessel beams which are solutions to the full Helmholtz equation [92]. The LG modes are defined by [81]

$$u(x, y, z) \equiv \text{LG}_p^\ell(r, \phi) = \frac{w_0}{w(z)} \sqrt{\frac{2p!}{\pi(p+|\ell|)!}} \left(\frac{\sqrt{2}r}{w(z)} \right)^{|\ell|} L_p^{|\ell|} \left(\frac{2r^2}{w^2(z)} \right) \exp \left(-\frac{r^2}{w^2(z)} \right) \times \exp \left(-i \left[k \frac{r^2}{2R(z)} - (2p + |\ell| + 1)\phi_G(z) + kz + \ell\phi \right] \right), \quad (1.12)$$

where $\ell \in \mathbb{Z}$ and $p \in \mathbb{Z}^+$ are the azimuthal and radial indices, respectively and $L_p^{|\ell|}$ are the associated Laguerre polynomials [93]. In Fig. 1.3 the intensity and phase (placed as insets) of LG modes with different radial and azimuthal orders are

shown. The radial order, p , sets the number of radial nulls in the mode, or alternatively, the number of concentric intensity rings in the mode is set by $p + 1$. The effect of changing p is also evident in the inset phase of each mode where the phase structure presents some discontinuity lines at the radii where one finds the intensity nulls. Changing the azimuthal index results in a larger intensity null at the centre of the mode. More significant is that the phase of a mode possessing an azimuthal index of ℓ performs ℓ full 2π rotations about the azimuth. For $\ell < 0$, the rotation of the phase from 0 to 2π occurs in the clockwise direction while for $\ell > 0$ it occurs in the anti-clockwise direction. In such beams with non-zero azimuthal index one also observes regions of undefined phase at the centre of these beams, i.e., an optical singularity [94, 95]. These optical singularities are often defined by their topological charge, the number 2π rotations performed by the phase as well as the rotation direction.

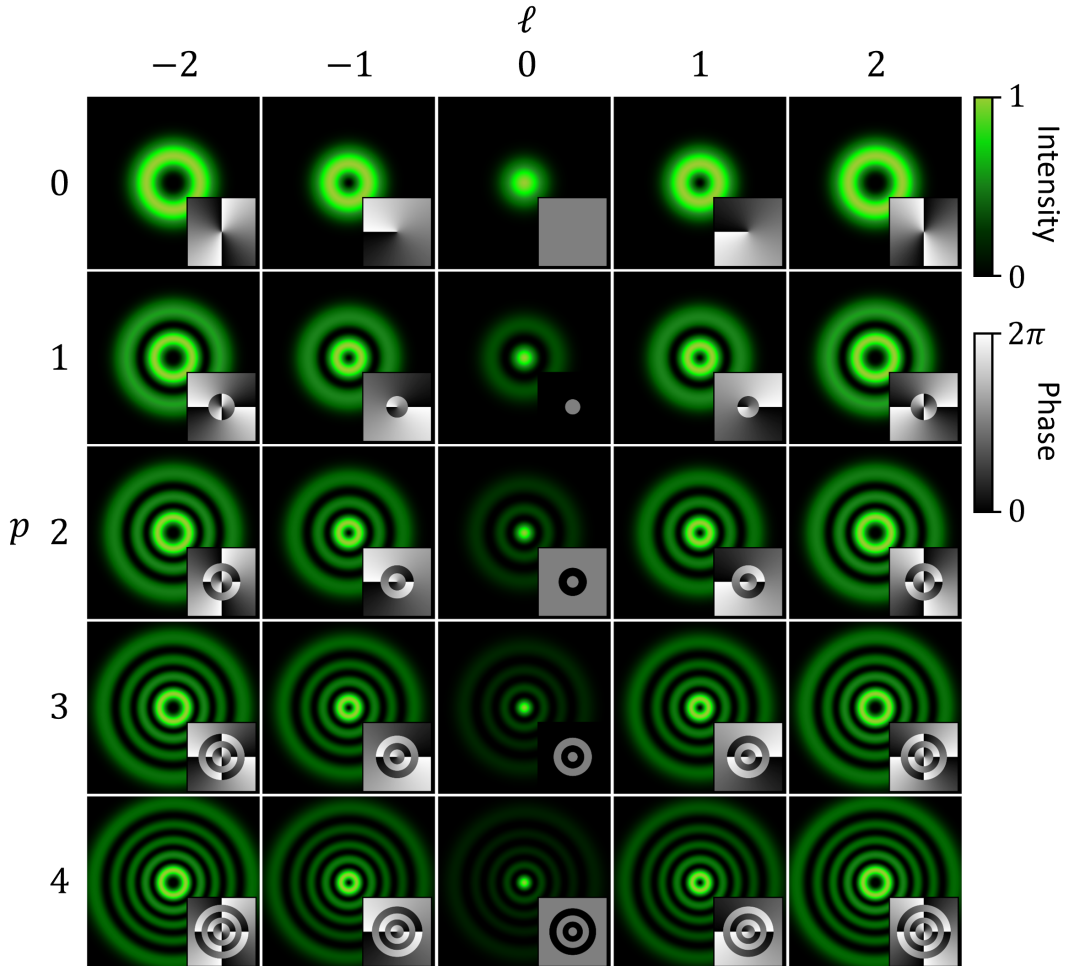


FIGURE 1.3: Intensity and phase (insets) of Laguerre-Gaussian modes with varying ℓ and p indices.

BG modes have a similar structure to LG modes, also possessing an azimuthal index, ℓ , except that their radial behaviour is set by a continuous parameter, $k_r = \sqrt{k^2 - k_x^2 - k_y^2}$, the radial wavevector. BG beams are defined by [81]

$$u(x, y, z) \equiv \text{BG}_{k_r}^\ell(r, \phi, z) = J_\ell(k_r r) \exp\left(-\frac{r^2}{w^2(z)}\right) \exp(i[\ell\phi - kz]), \quad (1.13)$$

where J_ℓ is the ℓ -th order bessel function of the first kind [96]. BG modes are of particular interest due to their non-diffracting nature [92], ability to "self-heal" around obstacles [97, 98], and interesting propagation dynamics [92, 99]. In Fig. 1.4 the intensity and phase (placed as insets) of BG modes with varying azimuthal index and radial wave vector are shown. It is clear that the structure of BG modes is almost identical to that of LG modes with a high radial order, except that the energy distribution spread between the central ring and concentric rings heavily favour the central ring, which is not the case for LG modes.

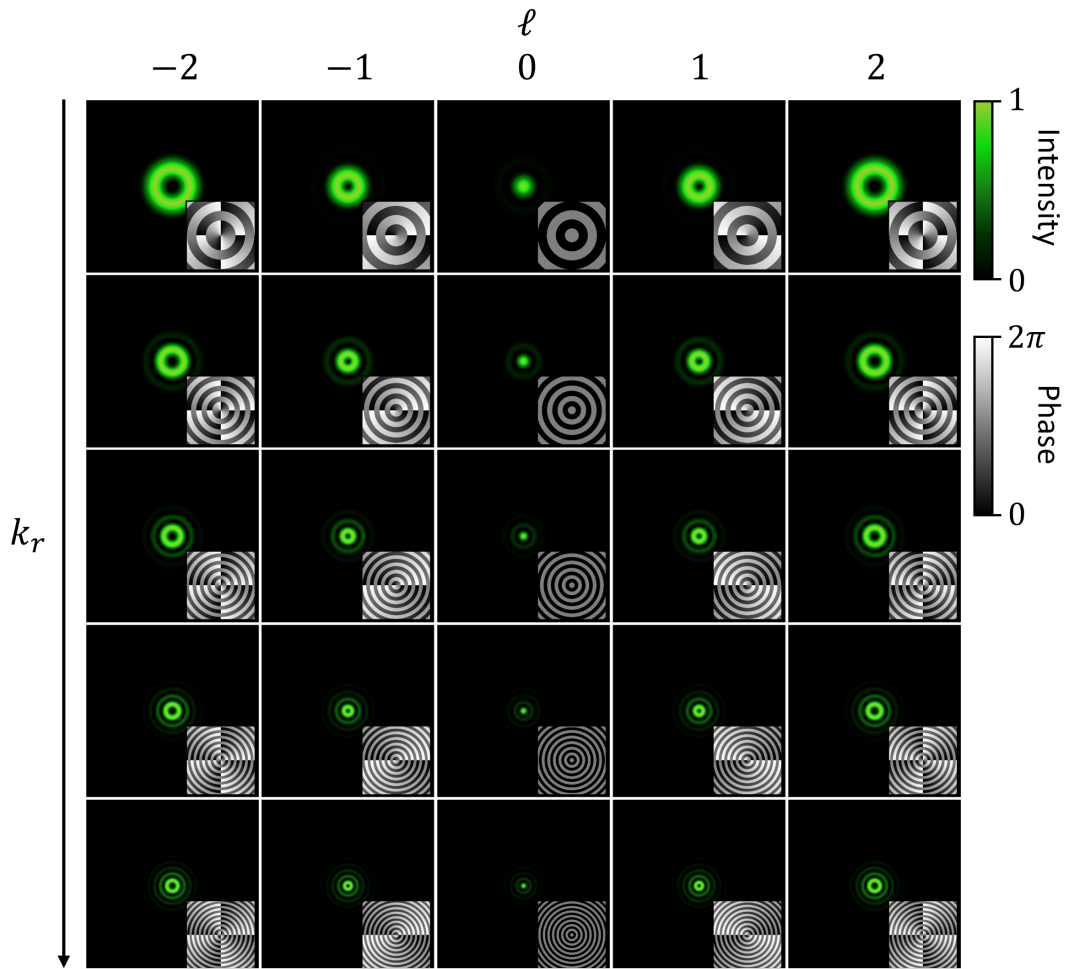


FIGURE 1.4: Intensity and phase (insets) of Bessel-Gaussian modes with varying ℓ index and magnitude of radial k_r vector.

An aspect of these cylindrically symmetric modes which have made them popular in the community, is their ability to carry orbital angular momentum (OAM), as a result of the number of "twists" in their phase structure [100]. The number of full 2π rotations performed by the phase function, gives the number of helical twists the wavefront of the light performs within a single wavelength. Much like polarization attributing single photons with SAM, the spatial distribution of an individual

photon's wavefunction can endow it with an OAM of $\ell\hbar$ per photon. This OAM property will be revisited in the context of individual photons described by quantum states in section 1.3.

In this work, we are particularly interested in the cylindrically symmetric BG and LG modes, with particular focus paid to LG modes in chapters 2, 4 and 5 and BG modes in chapter 3. Apart from their ability to carry OAM, the phase structure embedded in these modes allow for every possible relative phase between two orthogonal modes to be achieved in the transverse plane, a feature that is essential when constructing skyrmionic topologies, which will be discussed in section 1.2.

1.1.3 Scalar vs vector beams

In the previous sections the manipulation of polarization and spatial DOFs were discussed separately however these two DOFs can be manipulated independently. This allows for the generation of more complex beams, known as vector beams (VBs), where the polarization varies across the transverse plane of the beam [101, 102].

One can formulate a general description of such VBs through a straightforward generalization of Eq. 1.2

$$|\Psi\rangle = \cos\left(\frac{\theta(x,y)}{2}\right) |R\rangle + e^{i\alpha(x,y)} \sin\left(\frac{\theta(x,y)}{2}\right) |L\rangle, \quad (1.14)$$

where now the parameters which control the polarization state, θ and α are made to be functions of position. This concept is illustrated in Fig. 1.5 (a) where each block corresponds to a unique position in the plane perpendicular to the direction of propagation of a beam, and is associated its own polarization. In the case where the polarization is uniform across the beam profile, $\theta(x,y)$ is a constant, we have a scalar beam (SB). Mathematically these two types of beams can be distinguished by a property known as separability [103]. SBs are beams whose spatial and polarization DOFs are completely separable, while for VBs these DOFs are non-separable. As discussed previously, the choice of polarization basis is arbitrary, one is free to choose any pair of spatially dependent functions to control the spatial dependence of polarization across the beam profile. Due to their orthonormality it is convenient to express Eqn. 1.17 in terms of previously defined spatial modes

$$|\Psi\rangle = u_1(x,y) |R\rangle + u_2(x,y) |L\rangle. \quad (1.15)$$

As an example if we consider the case shown in Fig. 1.5 (b) (top row) where $u_1(x,y) = u_2(x,y) = LG_0^1$, the spatial components can be factorized out of the expression, thus making Eqn. 1.15 separable. However if we instead consider the case shown in Fig. 1.5 (b) (bottom row) where $u_1(x,y) = LG_0^1$ and $u_2(x,y) = LG_0^{-1}$, the spatial components cannot be factorized out of the expression thus making Eqn. 1.15 non-separable. The degree of non-separability of these beams also referred to as the vector quality factor (VQF) can be precisely quantified using a metric known as concurrence used to characterize the degree of entanglement between quantum subsystems [103–105]

$$\text{VQF} = 2\sqrt{\langle u_1|u_1\rangle \langle u_2|u_2\rangle - |\langle u_1|u_2\rangle|^2} \quad (1.16)$$

where $|u_i\rangle = \int |x, y\rangle u_i dx dy$ is the state description of the spatial modes with $|x, y\rangle$ identifying a particular position state and $\langle u_i | u_j \rangle = \int |x, y\rangle u_i^* u_j dx dy$ is the inner product of spatial mode states with the indices $i, j \in 1, 2$. It is important to note that the description given above is specific to pure/coherent states and a more general description used to quantify the degree of entanglement in quantum states is given in section 2.2.3. Given that we have chosen our spatial modes to be orthonormal, this implies that $\langle u_i | u_j \rangle = \frac{1}{2} \delta_{ij}$ where δ_{ij} is the Kronecker delta. For the case of the separable SB and non-separable VB discussed it is easy to verify that $VQF = 0$ and $VQF = 1$, respectively.

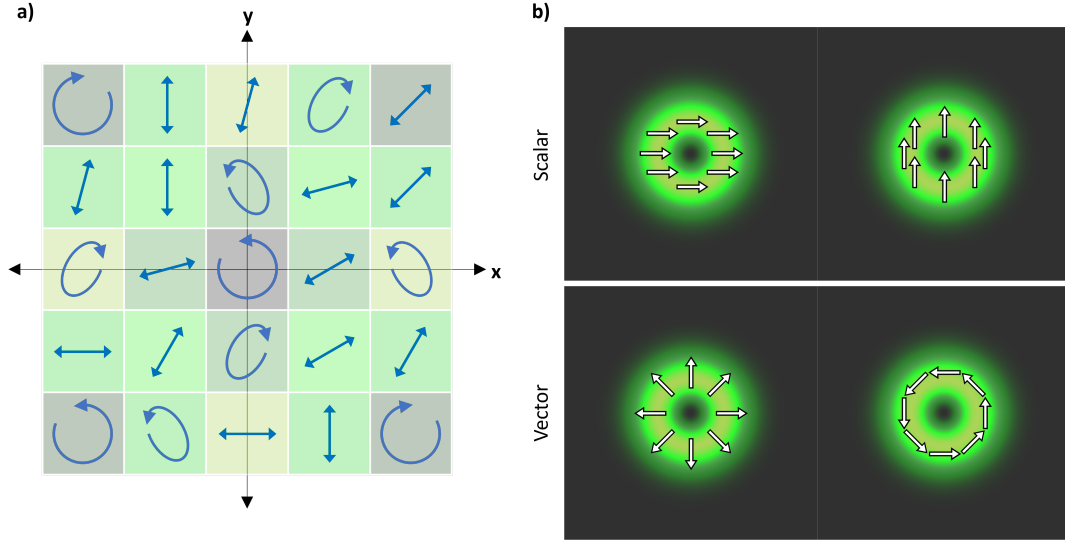


FIGURE 1.5: a) Inhomogeneous polarization structure achieved through combination of spatial and polarization DOFs. b) Difference between scalar and vector modes, where the polarization is homogeneous in the former and inhomogeneous in the latter.

To see how we can construct a plethora of diverse VBs, we may consider writing Eqn. 1.15 in terms of LG modes

$$|\Psi\rangle = \text{LG}_{p_1}^{\ell_1} |R\rangle + \text{LG}_{p_2}^{\ell_2} |L\rangle. \quad (1.17)$$

Since LG modes are cylindrically symmetric they can be rewritten as (for simplicity we set $p = 0$ and $z = 0$)

$$\text{LG}_0^\ell(r, \phi) = e^{-\frac{r^2}{w_0^2}} \sqrt{\frac{1}{\pi |\ell|!}} \left(\frac{\sqrt{2}r}{w_0} \right)^{|\ell|} \exp(i\ell\phi), \quad (1.18)$$

which indicates that the LG modes can be written in the separable form $\text{LG}_0^\ell(r, \phi) = A(r)e^{i\ell\phi}$, where the amplitude of the mode is dependent on the radial coordinate and the phase depends on the azimuthal coordinate. An example of this with $\ell_1 = 0$ and $\ell_2 = 1$ is depicted graphically in Fig. 1.6 (a). From this it is clear to see that the polarization changes in both the radial and azimuthal directions with the relative amplitudes and phases between orthogonal polarization components changing in the radial and azimuthal directions, respectively. This simple construction allows for an intuitive approach to create a plethora of diverse VBs, two examples of which are shown in Fig. 1.6 (b). Fig. 1.6 (b) depicts the polarization structure in two ways. The first representation being the Stokes vector plot which depicts a VB as a 3 dimensional vector field over the transverse plane. Here every position is associated

with a vector which points along the Poincaré sphere thus specifying a particular polarization state. From the previously discussed Hopf map, the Stokes vector field, $\vec{S}$, can be computed using Eq. 1.18 to give

$$\vec{S} = \begin{bmatrix} 2\text{Re}\{LG_0^{\ell_1} LG_0^{\ell_2*}\} \\ 2\text{Im}\{LG_0^{\ell_1} LG_0^{\ell_2*}\} \\ |LG_0^{\ell_1}|^2 - |LG_0^{\ell_2}|^2 \end{bmatrix}. \quad (1.19)$$

The second representation, the polarization ellipse plot, shows the actual oscillation of the electric field at each point in the transverse plane.

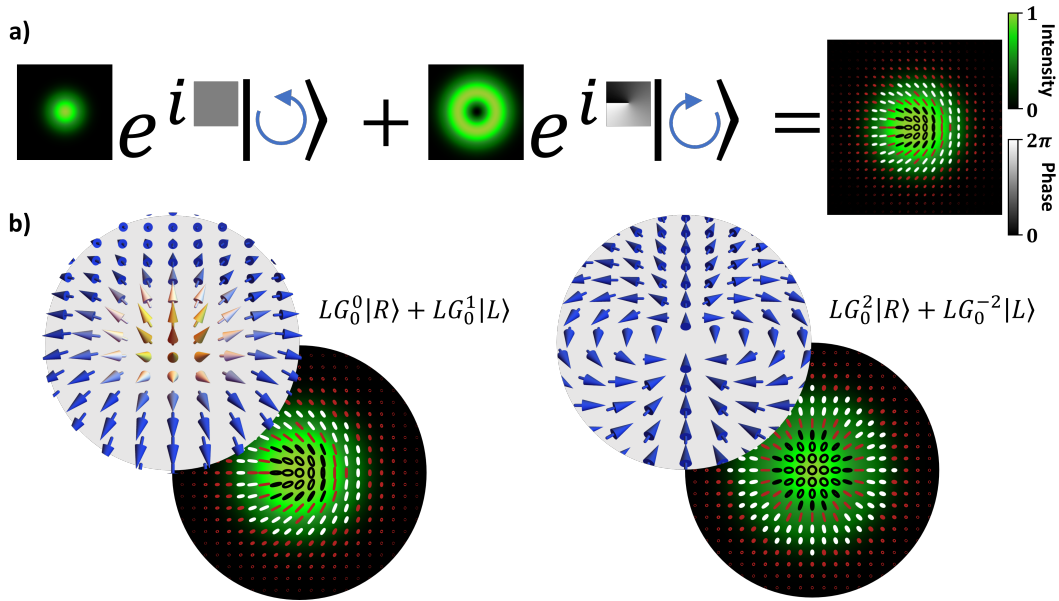


FIGURE 1.6: a) VB creation with LG modes, where their cylindrical symmetry allows for clear decomposition into amplitude and phase components. b) VBs formed from LGs with $\{\ell_1, p_1, \ell_2, p_2\} = \{0, 0, 1, 0\}$ (left) and $\{2, 0, -2, 0\}$ (right). The VBs are represented using a Stokes vector plot and polarization ellipse plot.

The study of VBs as well as the development of their generation and detection toolkit [101] has motivated several different applications such as the manipulation of the polarization and beam shape of tightly focused beams [106–108] allowing for optical control of nanoparticles [109], increased trapping efficiency in optical trapping experiments [7, 110–112], improved resolution for optical microscopy [113–115] and increased information capacity for optical communications [2, 116] to name just a few. Most recently it was shown that the degree of non-separability of VBs is invariant to unitary transformations induced by chiral liquids, optical fibres, atmospheric turbulence and complex optical aberrations even though the underlying spatial modes are heavily distorted [42], thus qualifying the degree of non-separability as a viable resource for high capacity optical communication through highly aberrated systems [43, 44]. Beyond these advances VBs have also recently been used for the generation of skyrmionic topologies [57] which will be discussed in the proceeding section.

1.2 Skyrmions

In the early 1960s Tony Skyrme proposed a non-linear meson field theory to describe sub-atomic particles as excitations of a single fundamental field, the pion [117–120]. To accomplish this he used the inherent mathematical structure of basic pion theory to postulate topologically non-trivial pion field configurations, now called skyrmions. A skyrmion is a topologically stable field configuration, characterized by an integer topological invariant, the skyrmion number. By definition, these skyrmions cannot be smoothly deformed by any means to attain a new field configuration with a different topology or equivalently, skyrmion number.

The skyrmion mapping was originally conceptualized as a map between 3-spheres (spheres that live in $\mathcal{R}^4$) where the skyrmion number indicates the number of times the first sphere wraps the second. However, this notion has since been extended to allow for mapping between spheres of arbitrary dimensions, i.e., maps from n -spheres to n -spheres. Furthermore the generality of this definition and its associated topological conserved quantity has allowed the notion to be extended beyond its initial intent [121, 122]. In particular, skyrmions have been instrumental in advances in magnetism and spintronics [123–130], nuclear physics [131, 132] and have recently featured as optical realizations [54, 57, 69, 133, 134].

1.2.1 Skyrmion mapping

In this work, of particular interest is the skyrmion topology formed by the mapping from the real sphere (a sphere living in real space), S^2 , to a parameter sphere, S^2 . Since one can stereographically project a sphere onto a plane (See appendix A), we can consider mappings from the real plane to the parameter sphere where the skyrmion number defines the number of times the parameter sphere is wrapped given a full traversal of the real plane. To realize such a mapping we may consider a simple vector field mapping the real plane onto a unit sphere,

$$\vec{n}(r, \phi) = \begin{pmatrix} \cos(\alpha(\phi)) \sin(\beta(r)) \\ \sin(\alpha(\phi)) \sin(\beta(r)) \\ \cos(\beta(r)) \end{pmatrix}, \quad (1.20)$$

where (r, ϕ) are coordinates on R^2 , $\alpha(\phi) = m\phi + \gamma$ with $\gamma \in [0, 2\pi)$ and $m \in \mathbb{Z}$ and $\beta(r)$ is an arbitrary continuous function of r . For simplicity, we choose $\beta(r)$ to be a monotonically increasing or decreasing function of r and $\beta(r) \in [0, \pi]$ (It must be noted that such a description rules out topological structures such as skyrmioniums [135] and target skyrmions [136] where the z-component of $\vec{n}(r, \phi)$ would complete multiple oscillations between 1 and -1 with distance from the origin, however these structures are not of interest in the current work). An example of such a mapping is shown in Fig 1.7 where the transverse plane is mapped onto a parameter sphere.

To quantify the skyrmion number, we consider that it computes the number of times R^2 wraps S^2 , therefore a computation over the solid angle of a unit sphere computes the skyrmion number [53, 121]

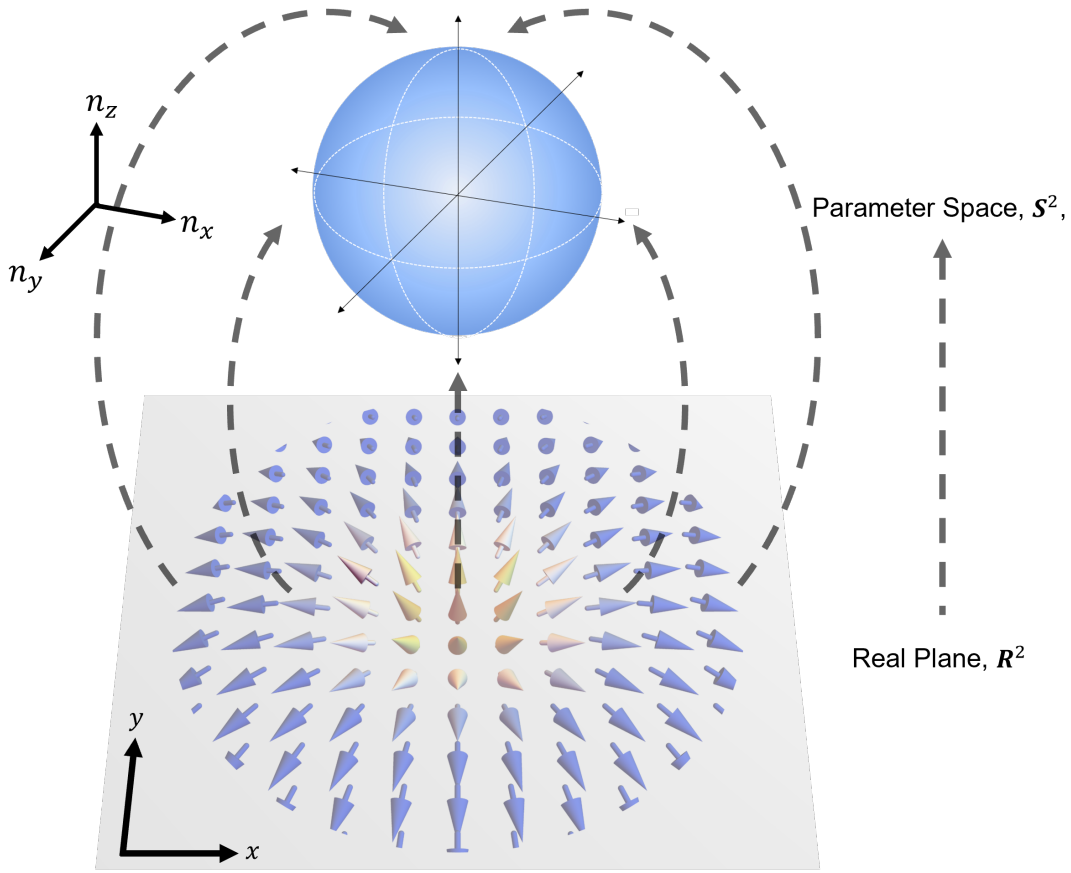


FIGURE 1.7: Mapping of transverse real plane, onto parameter space. Having a non-trivial skyrmion topology means that all possible states are achieved in the transverse plane at least once.

$$N = \frac{1}{4\pi} \iint_{R^2} \vec{n} \cdot \left[\frac{\partial \vec{n}}{\partial x} \times \frac{\partial \vec{n}}{\partial y} \right] dx dy = \frac{1}{4\pi} \iint_{R^2} \vec{n} \cdot \left[\frac{\partial \vec{n}}{\partial r} \times \frac{\partial \vec{n}}{\partial \phi} \right] dr d\phi \quad (1.21)$$

where the multiplication of a constant $\frac{1}{4\pi}$ ensures that the above computation computes only N wrappings and not $N \times$ surface Area of unit sphere.

To calculate the skyrmion number of the vector field defined in Eq. 1.20 we first calculate

$$\left[\frac{\partial \vec{n}}{\partial r} \times \frac{\partial \vec{n}}{\partial \phi} \right] = \alpha'(\phi) \beta'(r) \sin(\beta(r)) \begin{bmatrix} \cos(\alpha(\phi)) \sin(\beta(r)) \\ \sin(\alpha(\phi)) \sin(\beta(r)) \\ \cos^2(\alpha(\phi)) \cos(\beta(r)) + \sin^2(\alpha(\phi)) \cos(\beta(r)) \end{bmatrix} \quad (1.22)$$

where $f'(\cdot)$ indicates the derivative of the single-variable function, $f(\cdot)$, with respect to its variable. Then

$$\begin{aligned} \vec{n} \cdot \left[\frac{\partial \vec{n}}{\partial x} \times \frac{\partial \vec{n}}{\partial y} \right] &= \alpha'(\phi) \beta'(r) \sin(\beta(r)) [\cos^2(\alpha(\phi)) \sin^2(\beta(r)) \\ &\quad + \sin^2(\alpha(\phi)) \sin^2(\beta(r)) + \cos^2(\beta(r)) \cos^2(\alpha(\phi)) \\ &\quad + \sin^2(\alpha(\phi)) \cos^2(\beta(r))], \end{aligned} \quad (1.23)$$

which after using the simple trigonometric identity, $\sin^2 x + \cos^2 x = 1$, reduces to

$$\vec{n} \cdot \left[\frac{\partial \vec{n}}{\partial x} \times \frac{\partial \vec{n}}{\partial y} \right] = \alpha'(\phi) \beta'(r) \sin(\beta(r)). \quad (1.24)$$

Finally, substituting this result into Eq. 1.21 and integrating over all of space yields [69]

$$N = \frac{1}{4\pi} \alpha(\phi) \Big|_{\phi=0}^{\phi=2\pi} \cos \beta(r) \Big|_{r=0}^{r \rightarrow \infty} = qm \quad (1.25)$$

where $q = \frac{1}{2} [\cos \beta(r)]_{r=0}^{r \rightarrow \infty}$ and $m = \frac{1}{2\pi} [\alpha(\phi)]_{\phi=0}^{\phi=2\pi}$ [69]. Clearly, the skyrmion topology is set by the parameters m and q known as the vorticity and polarity, respectively. Furthermore, the parameter γ , known as the helicity, has no bearing on the topology of the vector field but simply controls its texture [69, 121]. Fig 1.8 depicts the different skyrmionic structures that are achievable through the tuning of these three parameters. Topologically equivalent Néel- (see Fig 1.8 (a)) and Bloch-type (See Fig 1.8 (b)) skyrmions are realised when $N = 1$ and $\gamma = 0, \frac{\pi}{2}$, respectively. Furthermore, $N = -1$ and $|N| > 1$ correspond to anti-skyrmions (See Fig 1.8 (c)) and higher-order skyrmions (See Fig 1.8 (d)), respectively.

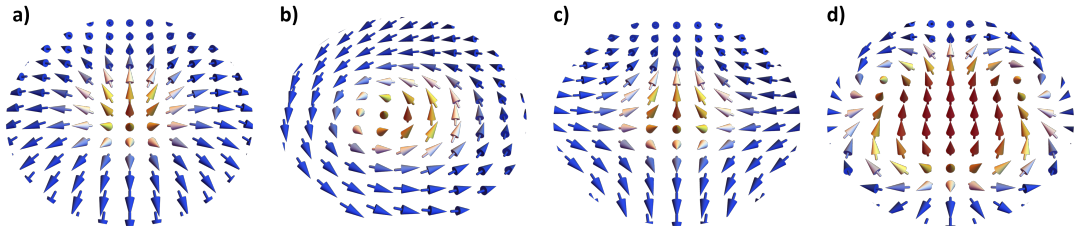


FIGURE 1.8: Skyrmions of different textures such as a) Néel- and b) Bloch-type as well as different topologies such as c) anti-skyrmion and d) higher-order skyrmions are all achievable through the manipulation of the vorticity, polarity and helicity DOFs.

1.2.2 Realizations of optical skyrmions

Optical skyrmions realized from linear optical fields was first observed by Tsesses. et. al. in 2018 in the evanescent electric field formed by the interference of surface plasmon polariton (SPP) waves within a hexagonal resonator [133]. Here, $\vec{n} = \vec{E}$ where $\vec{E}$ is the electric field formed by the superposition of standing SPP waves within the resonator. These optical configurations have been observed to form periodic lattices of Néel-type skyrmions with more general topological textures still yet

to be observed [121]. Beyond the electric field, it was observed that the excitation of a waveguided mode with a beam carrying OAM generates spin fields with individual isolated skyrmionic configurations [54] or lattice structures [55]. Optical skyrmions have also been realized in structured materials through the exploitation of multilayered chiral structures [56]. Further, by coupling space and time to form what are known as supertoroidal light pulses, skyrmionic topology was observed with varying textures at different cross-sections within these pulses [58]. Furthermore, whilst the construction suggested by Eq. 1.20 focused on mappings from real-space to a parameter sphere, it is also possible to replace real space with some alternative such as momentum space [137, 138].

In this work we are concerned with the generation of skyrmionic structures embedded within VBs [57] where the spatially varying polarization structure is described by the Stokes vector field, $\vec{S}(x, y)$ (now playing the role of $\vec{n}$), and can then be ascribed a skyrmion number based on the number of times the transverse plane of the beam wraps the Poincaré sphere (the chosen parameter space). The large structured light toolkit available for the manipulation of spatial and polarization DOFs has allowed for the realisation of a vast array of skyrmionic topologies with relative ease when compared to other optical realisations. Constructing VBs using propagation invariant modes such as LG modes ensures that the VB is topologically stable, i.e., if no external perturbations are present the topology of the VB remains intact during propagation [57, 121]. Furthermore, the polarity, vorticity and helicity can all be controlled via an appropriate choice of spatial modes and relative phases between the orthogonal polarization components thus allowing for the generation of Néel- and Bloch-type skyrmions as well as anti-skyrmions and arbitrary higher-order skyrmions [57, 69]. Skyrmionium and bimeron structures are achievable through a change in the choice of spatial mode basis [69] and polarization basis [68], respectively. Beyond 2D skyrmions, 3D skyrmionic structures called skyrmionic hopfions which realise mappings from the 3D real space to the 4D hypersphere have also been achieved by tracking lines of constant polarization in propagating VBs [139–141].

1.2.3 Constructing optical skyrmions with vector beams

Suppose we construct the following VB (where we have set $p = 0$)

$$|\Psi\rangle = \text{LG}^{\ell_1} |R\rangle + \text{LG}^{\ell_2} |L\rangle. \quad (1.26)$$

If we consider the simplified version of the LG modes given in Eq. 1.18 then we can identify certain boundary behaviours of the polarization vector field created by Eq. 1.27. We consider the vector field at three distinct regions, $r \rightarrow 0$, $r \rightarrow \infty$, and $r \rightarrow r_0 > 0$ with $\phi \in [0, 2\pi)$ depicted in Fig. 1.9 (a) as the red, green and blue regions, respectively. If $|\ell_1| = |\ell_2|$, then the amplitude structure of the LG modes are identical, thus the polarization state is independent of the radial coordinate since the relative amplitude between orthogonal polarization components is always 1. Suppose then that we select LG modes such that $|\ell_1| \neq |\ell_2|$ and without loss of generality $\ell_2 > \ell_1$. In the regions, $r \rightarrow 0$ and $r \rightarrow \infty$ the dominating behaviour is set by the function $\left(\frac{\sqrt{2}r}{w_0}\right)^\ell$ thus at $r \rightarrow 0$ we will only observe right circular polarization and at $r \rightarrow \infty$ we will only observe left circular polarization. In between these two regions we have that the relative amplitude of the two LG modes will change. A simple

example is shown in Fig. 1.9 (b) where $\ell_1 = 0$ and $\ell_2 = 1$. Clearly the beam with $\ell_1 = 0$ will dominate in the central region and the beam with $\ell_2 = 1$ will dominate "far-away" from the centre. Finally, observing the region $r \rightarrow r_0 > 0$ with $\phi \in [0, 2\pi)$ we see that the relative amplitudes between the orthogonal polarization components is constant, however the relative phase changes with azimuthal position. Fig. 1.9 (c) depicts the case where $\ell_1 = 0$ and $\ell_2 = 2$. The relative phase between the orthogonal components is given by $e^{i(\ell_2 - \ell_1)\phi}$, therefore one complete revolution in the azimuthal direction yields $\ell_2 - \ell_1$ complete relative phase cycles between the two orthogonal polarization components. To summarize, LG beams are an appropriate basis to construct skyrmionic topologies as they allow for all possible relative amplitudes and phases between orthogonal polarization states within the transverse plane.

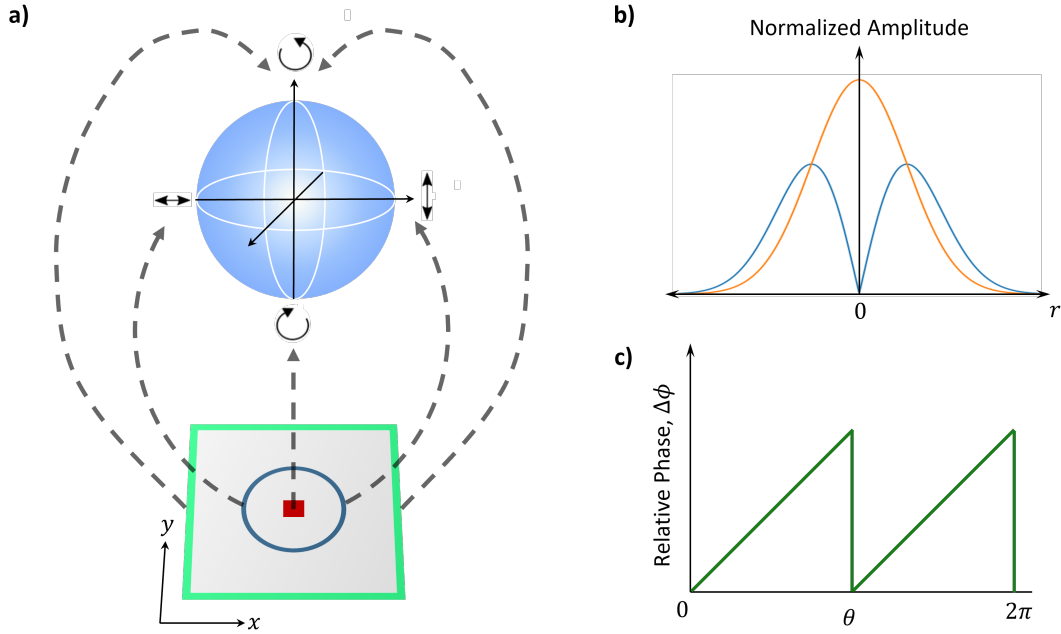


FIGURE 1.9: a) Simplest construction of a skyrmionic mapping, modelled after a stereographic map from an infinite real spatial plane, onto a spherical parameter space. b) Radial cross-section of amplitudes for two LG modes, with $\ell = 0, 1$. c) Azimuthal cross-section of relative phase function for two LG modes, with $\ell = 0, 2$.

We now consider modifying Eq. 1.19 so that it takes the form of Eq. 1.20. This allows for a simple extraction of an analytic expression for the skyrmion number in terms of parameters of the chosen LG modes. To do this, it is first necessary to locally normalize Eq. 1.27. Since only the relative amplitudes between the orthogonal polarization states matter (not the amplitudes of individual orthogonal polarization states) it is convenient to normalize the state as follows

$$|\Psi\rangle = \frac{|R\rangle + v(r)|L\rangle}{1 + v(r)}, \quad (1.27)$$

where $v(r) = f(r)e^{i\Delta\ell\phi}$, $f(r) = \left| \frac{LG^{\ell_2}}{LG^{\ell_1}} \right|$ and $\Delta\ell = \ell_2 - \ell_1$. Modifying the Stokes vector in Eq. 1.19 with Eq. 1.27 yields

$$\vec{S} = \begin{bmatrix} g(f(r)) \cos(\Delta\ell\phi) \\ g(f(r)) \sin(\Delta\ell\phi) \\ h(f(r)) \end{bmatrix}, \quad (1.28)$$

where $g(f(r)) = \frac{2f(r)}{1+f(r)^2}$ and $h(f(r)) = \frac{1-f^2(r)}{1+f^2(r)}$. We note that $g(f(r)) \in [0, 1]$ and $h(f(r)) \in [-1, 1]$ for $f(r), r \in [0, \infty)$. Furthermore, since we require a mapping from the real plane to the unit sphere, we require that $\vec{S} \cdot \vec{S} = 1 \implies g^2(f(r)) + h^2(f(r)) = 1$. Therefore, we can set $g(f(r)) = \sin(\Theta(r))$ and $h(f(r)) = \cos(\Theta(r))$ where $\Theta(r) \in [0, \pi]$. It must be noted that the explicit form of $\Theta(r)$ depends solely on the ratio of LG modes given by $f(r)$ which is set by their ℓ -indices. Since the Stokes vector now has the form of Eq. 1.20 we can extract the skyrmion number

$$N = \Delta\ell m, \quad (1.29)$$

where $m = \frac{1}{2}[\cos \Theta(r)]_{r=0}^{r \rightarrow \infty}$. Further, by evaluating $f(r)$ at $r = 0$ and $r \rightarrow \infty$ for different cases of relative ℓ -index between the two LG modes we find that [57, 142]

$$m = \begin{cases} 0, & |\ell_1| = |\ell_2| \\ 1, & |\ell_1| > |\ell_2| \\ -1, & |\ell_1| < |\ell_2|. \end{cases}$$

Here, $\Delta\ell$ plays the role of vorticity with $|\Delta\ell|$ setting the absolute order of the skyrmion topology and m plays the role of polarity as before. Furthermore, by including a relative phase of γ between the two orthogonal polarization components in Eq. 1.27 we can control the helicity of the skyrmion as well. By manipulating these parameters it is possible to generate a plethora of skyrmionic modes with varying skyrmion number (Fig. 1.10 (a)) and vectorial textures (Fig. 1.10 (b)).

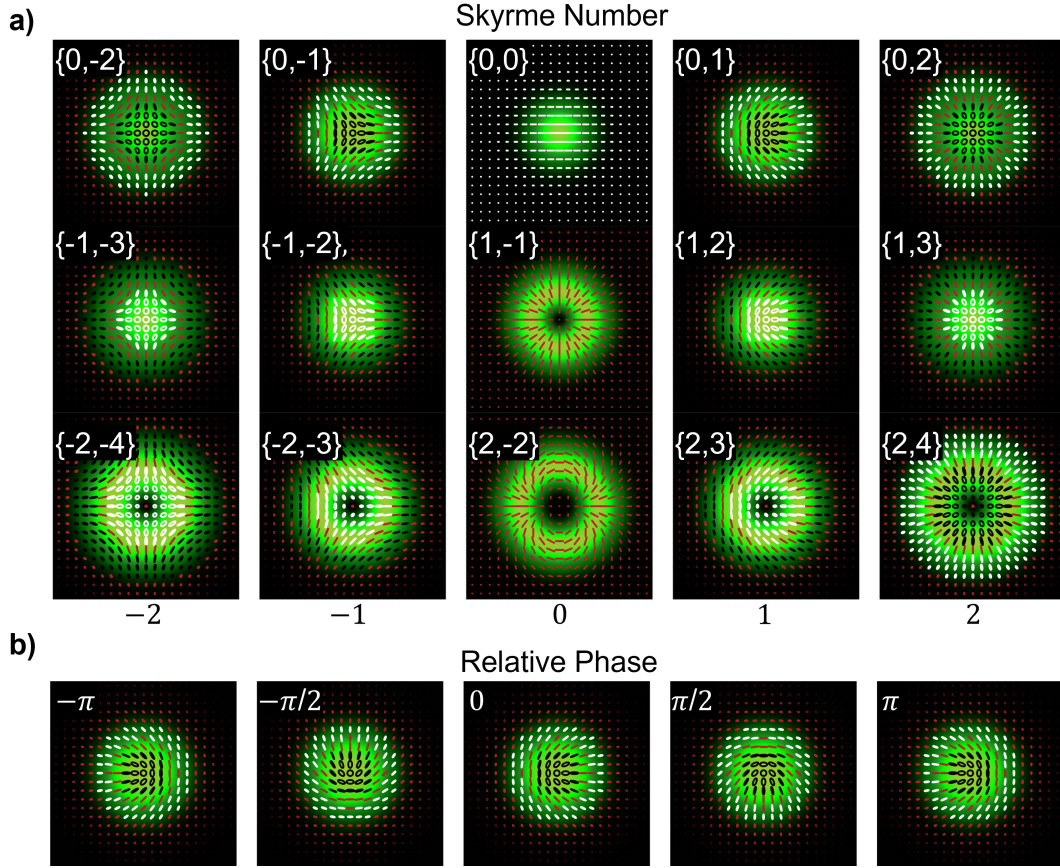


FIGURE 1.10: Skymion topologies generated by different LG modes. a) (column) Topologically equivalent skyrmions generated by different LG mode superpositions. Insets give the ℓ value for the right and left circular components respectively. (row) Different topological structures are also generated through the change in relative topological charges between the two LG modes. b) Topologically equivalent structures with different vectorial textures are generated through an induced relative phase between the orthogonal components.

1.3 Quantum structured light and vectorial wavefunctions

The particle-wave duality of light naturally suggests that while light can be quantized as photons, individual photons still possess wave-like features present if one studies their probability amplitude associated with properties of their wavefunction distributed in time and space [143]. Since we are particularly interested in the spatial and polarization DOFs we can consider the decomposition of a photon into said DOFs. Fig. 1.11 depicts the decomposition of a photon state into polarization (a) and spatial (b) DOFs which can be written as follows [143]

$$|\Psi\rangle = \iint \Phi_{\text{spatial}}(\mathbf{r}) |\mathbf{r}\rangle |P\rangle d^3r, \quad (1.30)$$

using the continuous spatial $|\mathbf{r} = (x, y, z)\rangle$ coordinate basis, with corresponding probability amplitudes, $\Phi_{\text{spatial}}(\mathbf{r})$, that constitute the photon wavepacket, while $|P\rangle$ denotes the polarization state of the photon. Here we have once again decided to use Dirac notation when referencing a state which lives in a d-dimensional Hilbert space, $\mathcal{H}_d$, where the dimensionality is dictated by the DOF in question and further the shorthand notation $|\cdot\rangle |\cdot\rangle \dots$ is used in place of writing down the tensor product between states, $|\cdot\rangle \otimes |\cdot\rangle \otimes \dots$. Written in the above form, it is clear that applying

the classical structured light toolbox to the quantum realm involves structuring the particle- (angular momentum) and wave- (spatially dependent complex probability amplitude) like properties of photons.

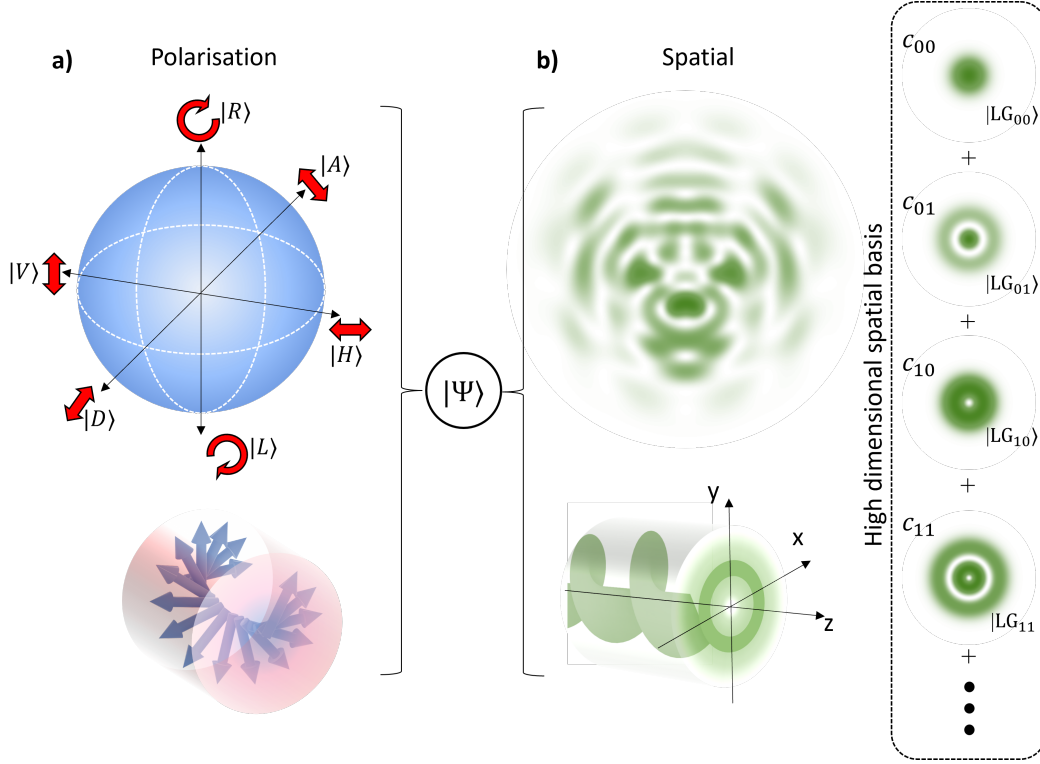


FIGURE 1.11: A decomposition of a photon into a) polarization and b) spatial DOFs. The polarization of a photon is connected to its inherent SAM and the spatial distribution of a photon's wavefunction can be associated to its OAM

1.3.1 Structuring single photons

The polarization degree of freedom was initially the best candidate for photon information processing due to its ease of control with conventional linear optical elements and was used to demonstrate numerous fundamental tests of quantum mechanics (Bell inequality violations [144] and quantum erasers [145]) and the initial demonstrations of quantum communication and cryptography (quantum key distribution [146], teleportation [147] and superdense coding [148]). However, since the basis for the polarization DOF only consists of two elements, a photon is restricted to a 2D state. Instead we may consider extending the dimensionality of the photon's state by manipulating its OAM as this can take on any integer value, thus is in principle infinite dimensional. We discussed earlier that the phase of spatially structured beams with non-zero OAM have an azimuthal dependence. Similarly, a photon possessing OAM can be described by a wavefunction whose distribution is characterized by the azimuthal phase dependent factor $\exp(il\phi)$ [100]. Optical modes with this characteristic are eigenstates of the OAM operator $\hat{L}_z = i\hbar \frac{\partial}{\partial \phi}$. In cylindrical coordinates $\mathbf{r} = (r, \phi, z)$, these modes take the form [143]

$$\Phi(\mathbf{r}) = R(\mathbf{r}) \exp(il\phi), \quad (1.31)$$

where $R(\mathbf{r})$ is an enveloping function that determines the radial amplitude profile of the photon. The form of Eq. 1.18 matches that of Eq. 1.31 thus the LG mode family forms a natural basis to describe OAM states, $|\ell\rangle$. Considering the case where the radial order, $p = 0$, we can write the OAM states in position basis

$$|\ell\rangle = \int_{\mathbf{R}_2} \text{LG}^\ell(\mathbf{r}) |\mathbf{r}\rangle d^2r. \quad (1.32)$$

It is clear that OAM states described in the position basis reveal the full content of the spatial wavefunction of the photon. The probability amplitude of the photon in the transverse plane is given by $|\text{LG}^\ell|$, while the characteristic azimuthal phase is given by $\arg(\text{LG}^\ell)$. These two pieces of information play an important role in describing the photon statistics as well as the evolution of the wavefunction due to propagation. A photon may be described in the OAM basis as follows

$$|\psi\rangle = \sum_{\ell} c_{\ell} |\ell\rangle, \quad (1.33)$$

where the complex coefficients c_{ℓ} can be tailored for a given application. For example, in quantum key distribution, such superpositions can be optimised to achieve unparalleled security [24, 29, 149, 150], transmit quantum states through fiber [151] or to tailor novel projective measurements for state characterisation [39, 152–154].

1.3.2 Entanglement of photons

In the classical regime, non-separability was engineered between internal degrees of freedom of a system, i.e, the non-separability between the spatial and polarization DOFs of a single beam. However, in the quantum regime, non-separability also known as entanglement is achieved through two or more subsystems or photons that are spatially separated [155]. These subsystems cannot be described independently from one another, for example, for two photons entangled in position, one cannot extract the information of a single photon's position without affecting the position of the other photon.

Suppose we consider a simple case of two spatially separated photons (photon A and photon B), each occupying a 2-level polarization system described by the basis $\{|R\rangle, |L\rangle\}$. Each photon's state can be described by the superposition given in Eq. 1.2. The composite system can be described as a tensor product between the two subsystems

$$\begin{aligned} |\psi\rangle_{AB}^{\text{sep}} &= |\psi\rangle_A |\psi\rangle_B \\ &= a_1 |R\rangle_A |R\rangle_B + a_2 |R\rangle_A |L\rangle_B + \\ &\quad a_3 |L\rangle_A |R\rangle_B + a_4 |L\rangle_A |L\rangle_B, \end{aligned} \quad (1.34)$$

where the complex coefficients a_i control the relative amplitudes and phases between each of the states. The state $|\psi\rangle_{AB}^{\text{sep}}$ is a separable tensor product ($|\psi\rangle_{AB}^{\text{sep}} = |\psi\rangle_A |\psi\rangle_B$) of the two subsystems. If $a_1 = a_2 = a_3 = a_4$ then the state is not entangled and said to be fully separable. Furthermore, this composite system is fully described through the use of four basis vectors thus the state lives on a 4 dimensional hilbert space, $\mathcal{H}_{AB}^4$. Whilst this composite system is not entangled, we are free to select a 2D state that lives on a subspace, $\mathcal{H}_{AB}^2$ of $\mathcal{H}_{AB}^4$ which is entangled. Suppose instead that we selected the state

Projection	Separable	Non-separable
$ R\rangle_A \langle R _A$	$ R\rangle_A (R\rangle_B + L\rangle_B)$	$ R\rangle_A L\rangle_B$
$ L\rangle_A \langle L _A$	$ L\rangle_A (R\rangle_B + L\rangle_B)$	$ L\rangle_A R\rangle_B$

TABLE 1.1: Projective measurement outcomes (normalization factors have been omitted) for the separable state described in Eq. 1.34 and non-separable state described in Eq. 1.35

$$|\psi\rangle_{AB}^{\text{ent}} = \frac{1}{\sqrt{2}} (|R\rangle_A |L\rangle_B + |L\rangle_A |R\rangle_B), \quad (1.35)$$

which is obtained from Eq. 1.34 by setting $a_1 = a_4 = 0$ and $a_2 = a_3 = \frac{1}{\sqrt{2}}$. It is clearly not possible to factor the state of either photon ($|\psi\rangle_{AB}^{\text{ent}} \neq |\psi\rangle_A |\psi\rangle_B$) therefore the state is non-separable, thus entangled. To illustrate this idea of separability and non-separability further, we consider the outcomes of both scenarios when we project photon A onto the basis states and observe the corresponding outcome on photon B. For simplicity we have assumed the case where $a_1 = a_2 = a_3 = a_4$ in Eq. 1.34. The measurement outcomes have been summarized in table 1.1. In the case of the separable state, regardless of the projection done on photon A, photon B always collapses into the same superposition of spin states. This demonstrates the fact that the state of photon B is independent of the state of photon A, and therefore they are not entangled. For the non-separable state, we observe the opposite behaviour. A projection of photon A onto the state $|R\rangle$ yields a collapse of photon B's state into $|L\rangle$ and vice versa. Therefore, photon A and B cannot be considered independent of one another, and are thus entangled. This relationship between the photons is always true despite any distance of separation between the photons. This prompted Einstein questioning the reality [156], calling this "spooky action at a distance" and has since been termed *quantum entanglement* [157].

The example given above serves to highlight the differences between entangled and non-entangled systems. In the proceeding section we will consider a more general definition for entanglement in the context of a 2-photon system.

High-dimensional entangled states

To generalise the description of entangled photons beyond $d = 2$, we need to add more linearly independent terms to Eq. (1.35) such that the state remains non-separable. However, as discussed before, the SAM DOF of photons can only form a basis containing 2 orthogonal components, thus this DOF is inherently limited to 2 dimensions. Instead, to create high dimensional entangled states we consider the infinite dimensional OAM DOF. It is important to note that any infinite dimensional DOF can be used in the discussion that follows and further that the DOFs used for each photon need not be the same. The composite system of two photons, whose states are each described by superpositions of OAM states can be written as

$$|\psi\rangle_{AB} = \sum_{\ell_A, \ell_B = -\infty, -\infty} c_{\ell_B}^{\ell_A} |\ell_A\rangle_A |\ell_B\rangle_B, \quad (1.36)$$

where $c_{\ell_B}^{\ell_A}$ are the complex coefficients which define state. As mentioned before, such a state is not entangled if it is factorizable as follows

$$|\psi\rangle_{AB} = |\psi_A\rangle_A |\psi_B\rangle_B, \quad (1.37)$$

where $|\psi_A\rangle_A = \sum_{\ell_A=-\infty}^{\infty} c_{\ell_A} |\ell_A\rangle_A$ and $|\psi_B\rangle_B = \sum_{\ell_B=-\infty}^{\infty} c_{\ell_B} |\ell_B\rangle_B$. If we consider the restriction applied on the coefficients of the state then we observe that for a separable state $c_{\ell_B}^{\ell_A} = c_{\ell_A} c_{\ell_B}$. However, if $c_{\ell_B}^{\ell_A} \neq c_{\ell_A} c_{\ell_B}$ then

$$|\psi\rangle_{AB} \neq |\psi_A\rangle_A |\psi_B\rangle_B, \quad (1.38)$$

and so by definition the state is non-separable and thus entangled. This provides a more general definition of entanglement for a 2-photon system.

Hybrid entanglement and vectorial wavefunctions

In section 1.1.3 we discussed free-space VBs where the polarization was spatially varying across the transverse profile of the beam. This behaviour originates from the non-separability between the spatial and polarization DOFs of the optical field. In fact these VBs are analogous to specific hybrid entangled states [22, 158] where the spatial DOF of one photon is entangled with the polarization DOF of another photon. Such a state can be written in the form

$$|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}} (|\ell_1\rangle_A |R\rangle_B + |\ell_2\rangle_A |L\rangle_B) \quad (1.39)$$

where the OAM of photon A is entangled with the SAM of photon B. Expressing the OAM DOF of photon A in position basis,

$$|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}} \int_{\mathcal{R}^2} |\mathbf{r}_A\rangle (LG_{\ell_1}(\mathbf{r}_A) |R\rangle_B + LG_{\ell_2}(\mathbf{r}_A) |L\rangle_B) d\mathbf{r}_A, \quad (1.40)$$

the correlations between position, $\mathbf{r}$, of photon A and polarization state of photon B become evident. Here the joint wavefunction of the entangled photons is not a simple complex scalar field, but is rather a complex vector field given by

$$\vec{\psi}_{AB}(\mathbf{r}_A) = LG_{\ell_1}(\mathbf{r}_A) |R\rangle_B + LG_{\ell_2}(\mathbf{r}_A) |L\rangle_B. \quad (1.41)$$

Clearly, Eq. 1.41 has the same form as the VB defined in Eq. 1.17. This implies that beyond engineering correlations between OAM and SAM, it is possible to precisely engineer vectorial wavefunctions describing the entanglement between two photons.

Exploiting the similarities between VBs and hybrid states has proven to be immensely fruitful. Several techniques used to generate and detect classical VBs have been deployed for the rapid realization and characterization of arbitrary hybrid states useful for higher dimensional quantum communication and cryptography [30, 116]. One study revealed that it is possible to characterize a quantum channel by studying its effects on a VB passed through it, so long as the channel only perturbs a single DOF

[158]. Further, a quantum inspired analysis of VBs revealed the invariance of their degree of non-separability to certain noisy channels [42]. This inspired the use of classical non-separability as a resource for communication over noisy channels [43, 44]. In this work, we further explore the connections between VBs and hybrid states, namely the skyrmionic properties they share and demonstrate the robustness of this new DOF against typical sources of noise.

1.3.3 Outline

Our work is structured as follows: In chapter 2 we will discuss classical and quantum structured light techniques. Classical techniques will be discussed to generate and control the spatial and polarization DOFs of light using liquid crystal technology and linear polarization optics. Following this, the generation of arbitrary vector modes using dynamic and geometric phase devices will be demonstrated. Finally, Stokes polarimetry will be employed as a technique for the characterization of vector beams. The classical techniques will then be extended into the quantum realm. The generation of high dimensional entangled states through spontaneous parametric down-conversion as well as the detection of OAM states, will be discussed. These high-dimensional OAM-OAM entanglement states form an ideal platform to construct arbitrary 2D hybrid entangled states. In the experimental setup required to generate these arbitrary hybrid states we will pay particular attention to the spatial-to-polarization conversion system allowing for the conversion of initial OAM entangled states to hybrid entangled states. To characterize such states we will discuss 2D quantum state tomography, a measurement technique capable of fully characterizing a 2D state as well as several other useful parameters (such as concurrence, fidelity and purity) that can be calculated once the state is reconstructed. The quantum analogue to Stokes polarimetry will also be developed.

In Chapter 3 we will demonstrate the generation of optical skyrmions through the use of Bessel Gaussian modes. Observing the transformations of their polarization structures on propagation reveals a beating between skyrmion types whilst leaving their topology unaltered. Furthermore, we observe the procession of the Stokes vectors with propagation, analogous to the procession of magnetic spins in the presence of a magnetic field. Lastly, through the use of a frozen wave construction, it will be shown that the evolution of the central on-axis polarization state can be precisely tuned during propagation.

In Chapter 4 we generate various 2D hybrid entangled states where the polarization of one photon is entangled to the OAM of another. We then present a novel result, that is the generation of non-local skyrmionic entangled states. To this end, we will discuss the characterization of quantum entangled states according to their topology, along with the implications that come along with it, such as characterizing any transformation that does not alter the topology as a smooth deformation or coordinate transformation. Lastly, resilience of the non-local topological features of the entangled states against entanglement decaying noise will be demonstrated and further the noise will be precisely associated to a coordinate transformation.

Lastly, in Chapter 5 we will build upon the ideas of chapter 4, where we then consider the effects of isotropic noise on the non-local topology of entangled states. We will demonstrate through theory and experiment that isotropic noise does not alter the topology of these states except when the maximally mixed state limit is reached.

Furthermore, we will describe the isotropic noise transformation, as a simple resizing of the state space, thus not altering the topology as expected.

Chapter 2

Experimental Techniques

In this chapter, we focus on the experimental techniques to control the polarization, amplitude and phase at every position within a classical optical field. We explore the use of static geometric optics as well as digital Spatial Light Modulators (SLMs) for the creation of vectorial beams. Furthermore we discuss Stokes polarimetry, an essential tool for the analysis of vectorial structures. We then consider the quantum analogue to the classical set of tools, by considering the generation and detection of entangled biphoton states. Complete control over 2 separate degrees of freedom (high dimensional spatial DOF and 2D polarization degree of freedom), allows for the digital creation and detection of a plethora of vectorial biphoton wavefunctions as well as different non-local topologies.

2.1 Structured light techniques

2.1.1 Amplitude and phase modulation

The goal of spatial amplitude and phase modulation of light, can be simply described as finding and performing the transformation $U_i(x, y) \xrightarrow{T} U_f(x, y) \implies U_f(x, y) = TU_i(x, y)$ where $U_{i,f}(x, y)$ are input (output) spatially dependent complex scalar fields, respectively with $T(x, y)$ being the transfer function required to engineer the desired final output field. Expressing the $U_i(x, y)$ and $U_f(x, y)$ as separable products of their amplitude and phase components we see that the transfer function is simply given by

$$T(x, y) = \frac{A_f(x, y)}{A_i(x, y)} e^{i(\phi_f(x, y) - \phi_i(x, y))}, \quad (2.1)$$

where $A_{i(f)}(x, y) = |U_{i(f)}(x, y)|$ and $e^{i(\phi_{i(f)}(x, y))} = \arg(U_{i(f)}(x, y))$. Clearly, to implement such transformations generally, we either require devices which can modulate the amplitude and phase simultaneously, or if we only have access to either amplitude or phase control, we can make use of Fourier modulation techniques to acquire the desired output fields [91]. In this chapter, we will focus on the use of phase-only devices to implement the desired modulation of an incoming optical field, $U_i(x, y)$. The first device we will discuss is the Liquid Crystal Spatial Light Modulator (which we will call an SLM), which can be digitally controlled to generate any desired mode. The second device is a static geometric phase device known as the q-plate, which will be discussed in more depth in a proceeding section under the context of generating vector modes.

The SLM functions as a Liquid Crystal (LC) display and allows the user to dynamically tune the orientation of the LCs within every pixel of the screen. The orientation of these LCs is set by an applied voltage across the LCs within every pixel. Due to the birefringence of the LCs, controlling their orientation equips the user with a tool to locally tune the refractive index of the SLM screen thus inducing a spatially dependent phase delay of the incoming light [159]. It is important to note that these devices are birefringent, therefore only light of a particular polarization will experience this phase delay. It is standard to orientate the SLM screen so that it modulates horizontally polarized light. A simplified structure of the SLM screen and function is depicted in Fig. 2.1.

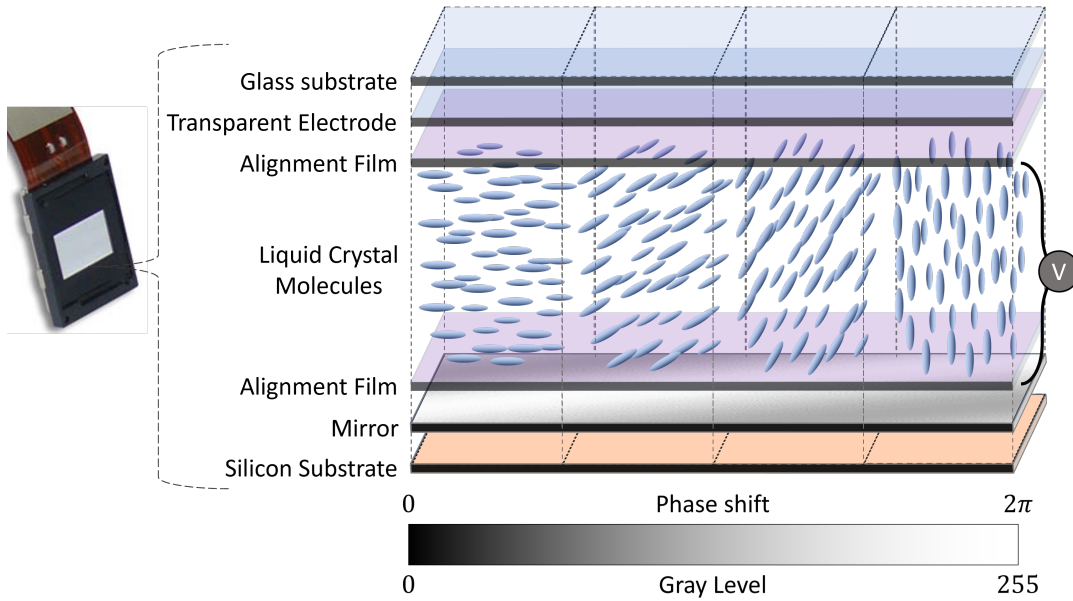


FIGURE 2.1: A working illustration of the SLM display. A zooming into the SLM screen, shows that the LC molecules are sandwiched between a glass substrate, transparent film and alignment film on the one side, and another alignment film, mirror and silicon substrate on the other side. Controlling the applied voltage across each pixel allows for dynamic control over the orientation of the LC molecules, and thus the local refractive index, which in turn induces a phase delay on incident light. The SLM can be controlled via the use of gray-scale images, where the local gray-scale value sets the applied voltage at that pixel, and thus the induced phase shift.

SLM technology has matured to a point where a wide range of devices with specific properties are currently available on the market. For this work, the Holoeye Pluto SLM series has been used, and is considered an all-rounded device with a pixel size of $8\ \mu\text{m}$ and a resolution of 1920×1080 pixels. These SLMs are capable of an 8-bit encoding depth, that is a 0 to 2π phase step modulation broken up into 256 levels. The voltage can be discretely controlled via the use of specially designed gray-scale images where the discrete phase shift levels from 0 to 2π are encoded into the colour gradient from black (0) to white (2π). These gray-scale images are known as digital holograms, and can be directly computed for any desired field modulation.

The two modulation techniques we will focus on are phase (with and without grating) and complex amplitude modulation. As mentioned before the SLM is a phase-only device, which means that any modulation of incoming light is achieved through the use of a phase holograms. In general, one can define the hologram that

encodes $\Phi^*(\mathbf{r})$ as [143, 160]

$$H(\Phi(\mathbf{r})) = f(\Phi^*(\mathbf{r})) \times h(\arg(\Phi^*(\mathbf{r}))), \quad (2.2)$$

where $f(\cdot)$ is the relation chosen to describe the grating depth capable of modulating for the correct amplitude in the image plane, while $h(\cdot)$ encodes the phase, $\arg(\Phi^*(\mathbf{r}))$.

By setting $f(\cdot) = 1$, and $h = \text{mod}[\arg(\Phi^*(\mathbf{r})), 2\pi]$, we perform a phase modulation that allows for the construction of the desired mode. This technique is known as phase modulation and works by constructing the desired phase of the final field at the SLM plane, and through propagation, the amplitude of the field is distributed to construct the desired field amplitude. An example of this modulation technique is shown in Fig. 2.2 a), where the phase hologram modulates the input field into the desired output field (Ideal) after propagation. The simplicity of this technique does not come without cost however, as there is typically some component of the incident light that is left unmodulated due to the inherent inefficiencies of the SLM. This unmodulated light follows the same optical path as the modulated light thus resulting in the final field being some superposition of the unmodulated (U_i) and the desired output field (U_f), shown as the superposition " $U_i + U_f$ " in Fig. 2.2 a) with an exaggerated contribution from the unmodulated light.

To avoid this issue, we can make use of a grating which will separate out the modulated and unmodulated light into spatially disjoint zeroth and first diffraction orders as shown Fig. 2.2 b). We achieve this by setting $h = \text{mod}[\arg(\Phi^*(\mathbf{r})) + \Lambda_{\text{grating}}, 2\pi]$, where the grating is controlled by the term Λ_{grating} . However, phase modulation still has shortcomings, the amplitude and phase attained after propagation is not yet ideal, rather there is some contribution from undesired modes albeit the phase can usually be corrected by a second phase modulation at the focus of a lens [161].

For a single-step modulation technique, which produces the desired output field at the plane of the SLM we can employ complex amplitude modulation, where both the phase and amplitude information of the output field is encoded into the phase hologram. The amplitude information can be directly encoded into the function, $f(\Phi(\mathbf{r}))$, which is proportional to the field amplitude. An example of this is shown in Fig. 2.2 c) where the SLM has been encoded with the desired field amplitude through control of the grating depth. Explained simply, this technique takes in an input field, usually a planar wave with flat phase and amplitude across the field, and the desired output field amplitude is "carved" out of the input field, whilst the desired phase is simultaneously embedded into the field profile. Due to the existing grating, there is no distortion of the output field due to unmodulated light and furthermore, the SLM plane can be imaged to any plane using a 4f system of lenses. This specific method of complex amplitude modulation is rather efficient, however it may come at the cost of some purity since the field amplitude does not map linearly with conversion efficiency. Several alternative methods of complex amplitude modulation exist [162–164] where efficiency and modal purity are some of the key parameters discussed when comparing them.[165] One scheme that can produce modes with high purity is when the amplitude function, $f(\cdot)$ is encoded as [163], $f(\Phi(\mathbf{r})) = J_1^{-1}(\Phi(\mathbf{r}))$. Here, $J_1^{-1}(\cdot)$ is the Bessel function of the first kind. The phase function is altered as $h = \sin(\arg(\Phi^*(\mathbf{r})) + \Lambda_{\text{grating}})$. Some experimentally generated spatial modes

using complex amplitude modulation with this mentioned scheme, are shown in Fig. 2.3, and are in excellent empirical agreement with the theoretical modes shown.

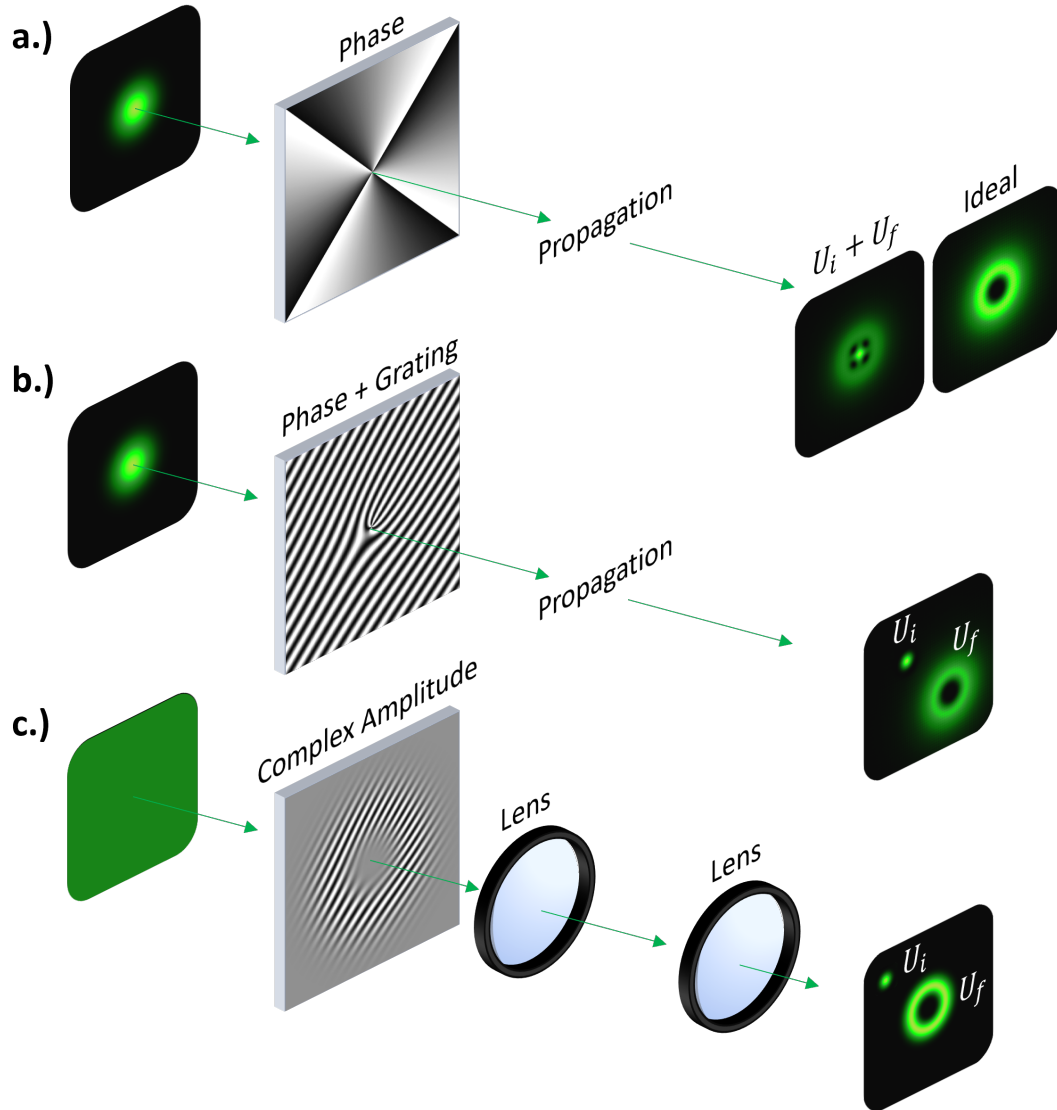


FIGURE 2.2: Illustration of the different modulation techniques capable of producing high fidelity spatial modes with a phase-only device. a) Phase modulation works by encoding the desired phase into the input field and observing the desired field after propagation. Ideally, all of the light is modulated by this phase, however, realistically, there is a portion of the light that is left unmodulated (U_i) and which interferes with the desired field (U_f). b) Adding a grating to the phase modulation technique, avoids interference issues between the modulated and unmodulated light by separating them out in space. c) Complex amplitude modulation accomplishes a single-step modulation where the desired field is generated at the plane of the SLM and can be imaged directly from the SLM plane Using a 4f imaging system

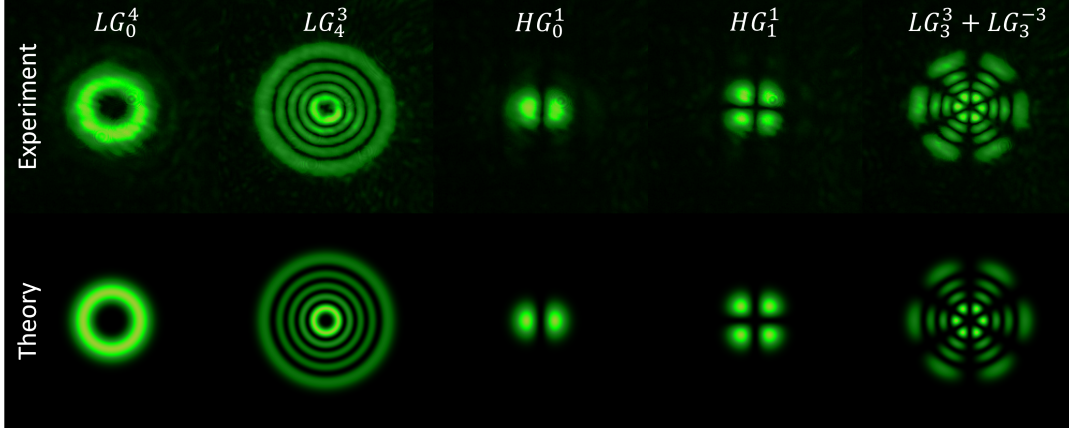


FIGURE 2.3: (Top) Experimental intensity images of scalar modes produced by displaying a digital hologram on a phase-only SLM using complex amplitude modulation. For empirical comparison, the (Bottom) simulated modes are also shown.

2.1.2 Polarization control

As discussed in section 1.1.1, we can describe a particular polarization state using Eq. 1.1 or equivalently using the Jones formalism

$$|P\rangle = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\alpha} \sin\left(\frac{\theta}{2}\right) \end{bmatrix}. \quad (2.3)$$

For the purposes of transforming and analysing polarization states, we only require three types of optics: linear polarizers (LPs), half-wave plates (HWP) quarter-wave plates (QWPs).

LPs are used to perform projective measurements onto a chosen linear state, and their action can be described using the Jones formalism as follows [77]

$$\text{LP}(\theta) = \begin{bmatrix} \cos^2(\phi) & \cos(\phi) \sin(\phi) \\ \cos(\phi) \sin(\phi) & \sin^2(\phi) \end{bmatrix}, \quad (2.4)$$

where ϕ is the angle between the optical axis and the horizontal. We can also describe the action of the LP as performing an overlap between an initial state $|P\rangle$ and a chosen linear state $|\phi\rangle = \cos(\phi)|H\rangle + \sin(\phi)|V\rangle$, such that the final state is given by $|P'\rangle = \langle\phi|P\rangle|\phi\rangle$.

HWPs and QWPs induce a π and $\frac{\pi}{2}$ phase delay, respectively, between orthogonal polarization components and in general their action can be described using the Jones formalism as follows

$$\text{HWP}(\theta) = e^{-i\frac{\pi}{2}} \begin{bmatrix} \cos^2(\phi) - \sin^2(\phi) & 2 \cos(\phi) \sin(\phi) \\ 2 \cos(\phi) \sin(\phi) & \sin^2(\phi) - \cos^2(\phi) \end{bmatrix}, \quad (2.5)$$

$$\text{QWP}(\theta) = e^{-i\frac{\pi}{4}} \begin{bmatrix} \cos^2(\phi) + i \sin^2(\phi) & (1-i) \cos(\phi) \sin(\phi) \\ (1-i) \cos(\phi) \sin(\phi) & \sin^2(\phi) + i \cos^2(\phi) \end{bmatrix}. \quad (2.6)$$

The phase delay incurred by a state passing through a HWP ensures that it only changes the states orientation, i.e., it maps linear states to linear states (example shown in Fig. 2.4 a) where the HWP rotates the state from $|H\rangle$ to $|V\rangle$) whilst the phase delay induced by a QWP changes the states ellipticity, i.e. it maps linear states to circular states and vice versa (example shown in Fig. 2.4 b) where the QWP maps the input state $|H\rangle$ to $|L\rangle$). The actions of these polarization optics are summarized in Fig. 2.4. With these two optics it is possible to attain every possible polarization state on the Poincaré sphere, in fact it has been shown that the use of two QWPs and one HWP (in any configuration), is sufficient to traverse the entirety of the Poincaré sphere[166]. The transformations between the six most common polarization states, $|H\rangle, |V\rangle, |D\rangle, |A\rangle, |R\rangle, |L\rangle$, is summarized in Table 2.1, along with the orientation angles of QWPs and HWPs required to achieve said states.

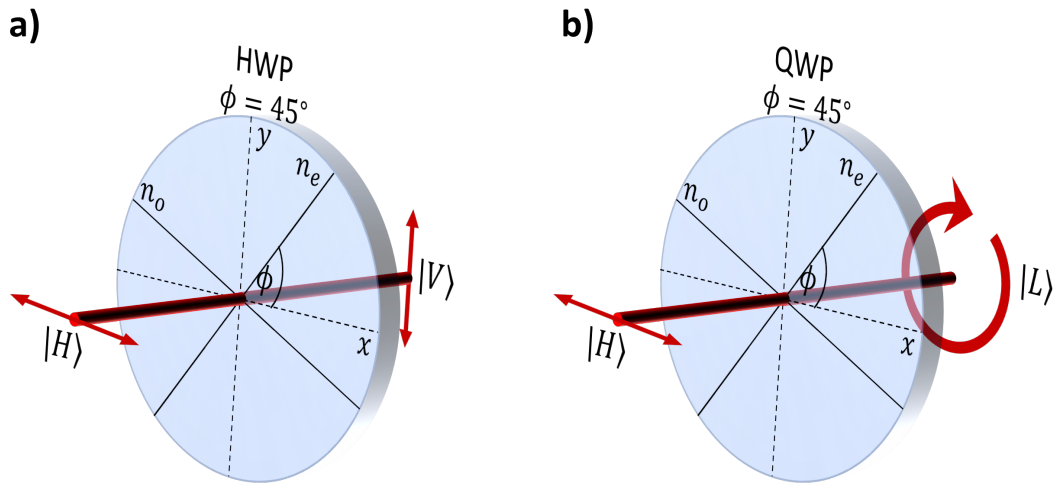


FIGURE 2.4: Polarization optics used for complete control over the polarization state of incident light. a) Half-wave plate (HWP) commonly used to rotate polarization states from one linear polarization state to another, and b) a Quarter-wave plate (QWP) commonly used to transform between linear and circular polarization states. For horizontally polarized incident light passing through a HWP and QWP orientated with their fast-axes 45° with respect to the x-axis, the output light has its polarization changed to vertical and left-circular, respectively.

wave plate	QWP			HWP			
	0°	45°	90°	0°	22.5°	45°	90°
$ H\rangle$	$ H\rangle$	$ L\rangle$	$ H\rangle$	$ H\rangle$	$ D\rangle$	$ V\rangle$	$ H\rangle$
$ V\rangle$	$ V\rangle$	$ R\rangle$	$ V\rangle$	$ V\rangle$	$ A\rangle$	$ H\rangle$	$ V\rangle$
$ D\rangle$	$ R\rangle$	$ D\rangle$	$ L\rangle$	$ A\rangle$	$ H\rangle$	$ D\rangle$	$ A\rangle$
$ A\rangle$	$ L\rangle$	$ A\rangle$	$ R\rangle$	$ D\rangle$	$ V\rangle$	$ A\rangle$	$ D\rangle$
$ R\rangle$	$ A\rangle$	$ H\rangle$	$ D\rangle$	$ L\rangle$	$ L\rangle$	$ L\rangle$	$ L\rangle$
$ L\rangle$	$ D\rangle$	$ V\rangle$	$ A\rangle$	$ R\rangle$	$ R\rangle$	$ R\rangle$	$ R\rangle$

TABLE 2.1: A summary of useful polarization transformations. Incident light of a particular polarization (far-left column) is transformed by a QWP and/or HWP, orientated with an angle ϕ with respect to the horizontal axis, into another polarization state.

2.1.3 Vector beam generation

Possessing the tools for full spatial and polarization control, we can now consider how to combine these tools in order to achieve the generation of vectorial beams, where the polarization structure of these beams is spatially dependent.

Digital devices

In section 2.1.1, the use of an SLM device to impart a spatially dependent dynamic phase on incident light was discussed. The dynamically controlled phase modulation was achieved through control over the orientation of birefringent LCs within each pixel of the device which would then grant the user control over the refractive index, thus inducing a phase shift at each pixel. This allows for complete control over the phase and amplitude at each point in space. However, due to the birefringent nature of this device, only incident light that is horizontally polarized (the exact polarization that gets modulated also depends on properties of the device itself as well as the orientation of the device with respect to the incident light, however, without loss of generality, we can configure the optical setup such that the SLM is setup to modulate horizontally polarized light) will be modulated. Therefore, in order to generate vector beams, we require an optical setup which is capable of modulating two orthogonally polarized beams, and then recombining them once again (assuming that the two orthogonally polarized beams are not spatially separated when they are modulated.) to form the desired superposition. Fig. 2.5 depicts the idea behind vector beam generation using SLMs. Two horizontally polarized beams are incident on SLMs. The fields U_1 and U_2 are encoded onto the SLMs with appropriately chosen holograms, and then are embedded onto the individual beams. After this modulation, one of the beams is sent through HWP, which rotates its polarization to vertical polarization before the two beams are superimposed using a beam splitter (BS). After this process, we attain the vector mode $|\Psi\rangle = U_1 |H\rangle + U_2 |V\rangle$. If one desires to change the polarization basis, to some arbitrary basis $\{|P^+\rangle, |P^-\rangle\}$, then we just need to pass the beam through an appropriate set of polarization optics (POs) in order to generate the final vector mode $|\Psi\rangle = U_1 |P^+\rangle + U_2 |P^-\rangle$.

Geometric optics

Another approach to generate vector beams is the use of geometric optics to impart different phases onto orthogonal polarization states, simultaneously.

Instead of imparting a spatially dependent dynamic phase, as is done with an SLM, geometric optics offer an alternative approach by using the ‘memory’ of locally applied transformations. Here, by altering parameters adiabatically in a closed loop fashion, a geometric phase [167] may be induced equal to half the solid area bounded by the transformations in the parameter space. Applied to light, one may use a change in polarization to induce such a relative phase in the electric field, as illustrated in Fig. 2.6 (a) and is otherwise known as Pancharatnam-Berry phase. [168–170]

To illustrate this phenomenon, we first consider the path that an incident beam’s polarization traces out on the surface of the Poincaré sphere as it passes through a series of polarization optics. In Fig. 2.6 a) diagonally polarized light passes through

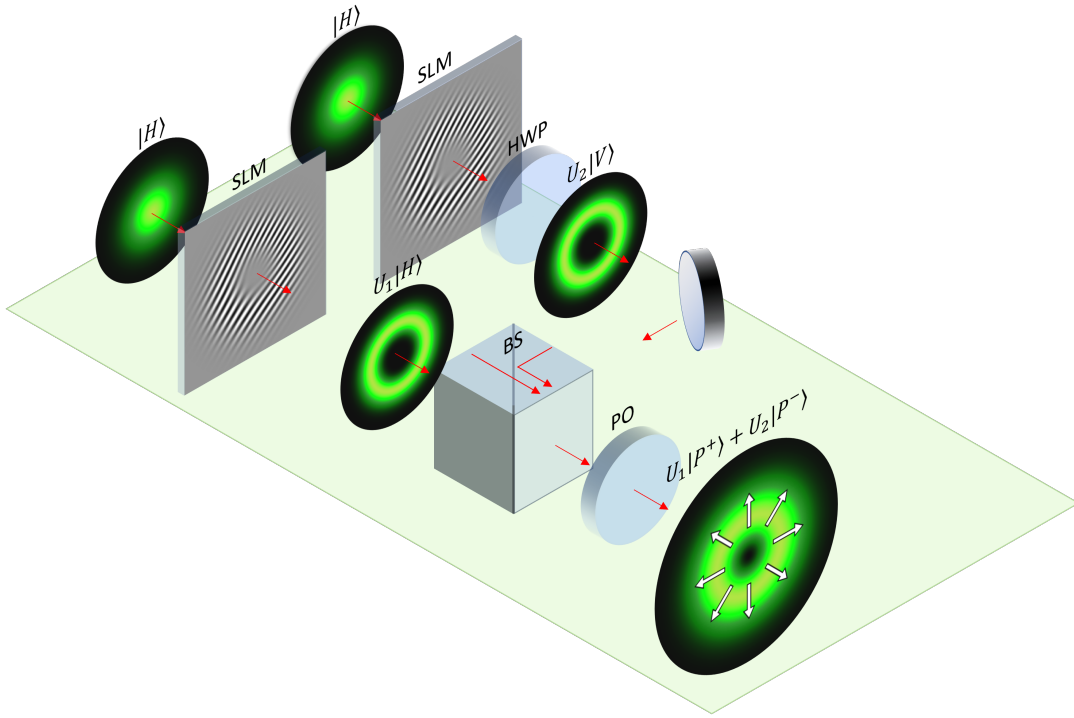


FIGURE 2.5: Concept for generation of arbitrary vector modes using SLMs. Two horizontally polarized beams are incident on SLMs which modulate them independently. Before recombining the beams with a beam splitter (BS), one is passed through a HWP to rotate it to the orthogonal polarization. A vector beam is then generated on the output port of the BS, and the final desired vector beam is generated after passing through an appropriate set of polarization optics (PO).

a HWP (A), QWP (B) and then one last QWP (C), such that its polarization transformation can be summarized as follows $|D\rangle \xrightarrow{A} |H\rangle \xrightarrow{B} |R\rangle \xrightarrow{C} e^{i\phi_G} |D\rangle$ or using shorthand $|D\rangle \xrightarrow{CBA} e^{i\phi_G} |D\rangle$, where ϕ_G is the geometric phase acquired due to the closed loop evolution of the state. To find this geometric phase, we consider the path traced out on the surface of the Poincaré sphere as depicted in Fig. 2.6 b). The transformations, A, B and C, form a closed path bounding a solid area Ω which means that the accrued geometric phase is given by $\phi_G = e^{i\Omega/2}$. Suppose then that instead of applying each transformation consecutively in time, we apply them spatially, that is that at every point across the transverse plane, the incident light sees a waveplate with a different orientation, $\tau(r, \theta)$, where r, θ are radial and azimuthal coordinates. As such a change in polarization occurs at points across the transverse spatial plane of the incident beam, an associated spatially-varying geometric phase is also generated. This is illustrated in Fig. 2.6 c), showing how geometric phase may be gathered in the transformation from right-circularly polarized light to left-circularly polarized light when passing through four arbitrary points (denoted $\tau_0(r, \phi)$ to $\tau_3(r, \phi)$) on an optic (c) which has effective HWP elements with different orientations at each transverse coordinate. With the polarization change occurring spatially, the ‘closed path’ is formed by the difference in path between the other elements, resulting in the geometric phase being both relative and spatially varying [171, 172]. One may therefore engineer the relative optical axis orientations to generate any number of variable geometric phase acquisitions that may be used to manipulate the spatial mode of the overall beam.

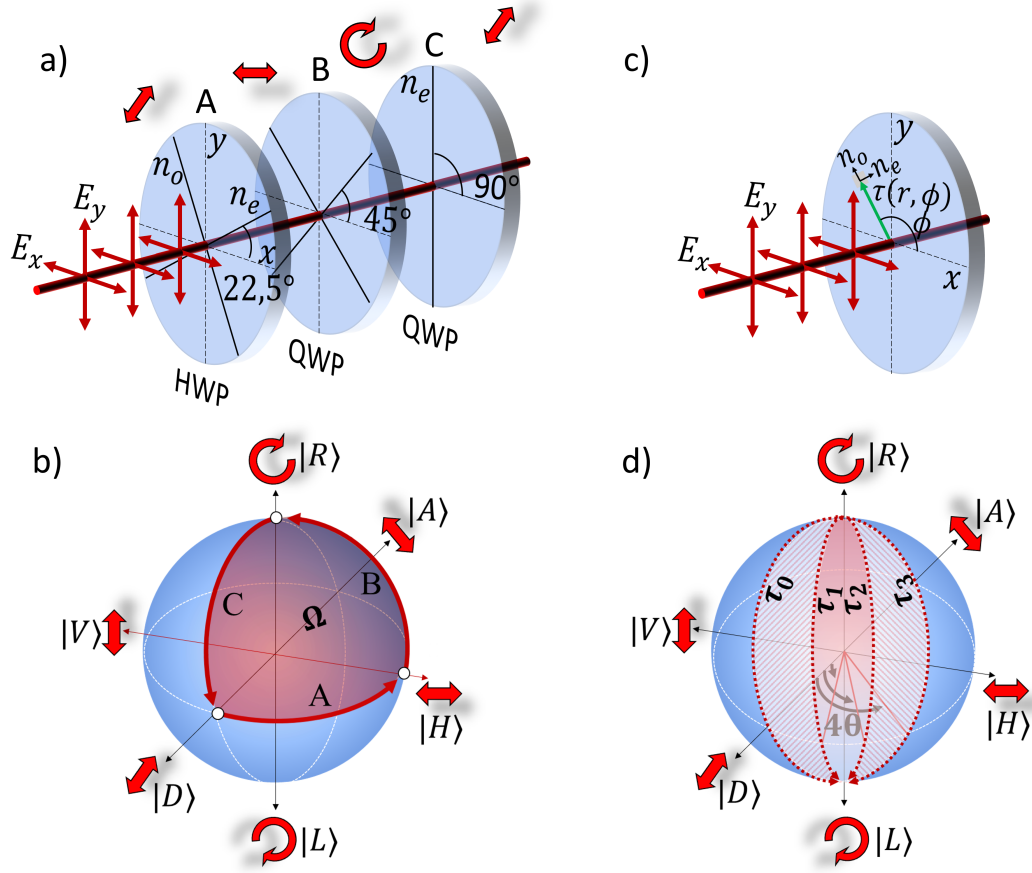


FIGURE 2.6: The concept of geometric phase. (a) Illustration of a diagonally polarized state traversing a series of waveplates that take the state through a closed loop transformation which is (b) depicted on the Poincaré sphere. (c) Depiction of an element with spatially varying optical indices of refraction which may be considered as a series of HWPs with varying optical axes. (d) Illustration of associated paths taken by different rays of a light beam traversing an element with four arbitrary HWPs with varying fast axis rotational angles.

A q-plate (QP) [173] is one such device which does this by patterning the optical axis orientation, such that the relative phase changes azimuthally and thus creates an azimuthal phase across the entire beam [173]. This patterning of the optical axis is described by

$$\alpha(r, \phi) = q\phi + \alpha_o. \quad (2.7)$$

Here the QP is taken to be in the xy -plane, α_o is the angle formed by the optical axis from the x -axis, q is a constant defining the number of times the optical axis rotates in a path as it traverses once around the plate center, also known as the QP charge and α_o is the permanent offset of the optical axis from the element's x -axis. A discontinuity occurring at $r = 0$ when $q \neq 0$ is evident due to the nature of the azimuthal coordinate, ϕ , being undefined at this point.

A simple description of an ideal QP (also referred to as being 'tuned') action can be given in terms of its Jones matrix[174]:

$$\text{QP} = i \begin{bmatrix} \cos(2\alpha(r, \phi)) & \sin(2\alpha(r, \phi)) \\ \sin(2\alpha(r, \phi)) & -\cos(2\alpha(r, \phi)) \end{bmatrix}. \quad (2.8)$$

It must be noted that Eq. 2.8 refers to an ideal QP, however QPs typically carry some inherent inefficiency even for their designed wavelength, which expectantly degrades the modal purity of the output mode, as the unmodulated light travels collinear to the modulated light, not unlike phase modulation using the SLM. Assuming that the inefficiency is minor, Eq. 2.8 perfectly captures the QP action.

One can further simplify the QP action by identifying the following set of simple selection rules [174]:

$$\text{QP} |\ell, R\rangle \rightarrow |l + 2q, L\rangle, \quad (2.9a)$$

$$\text{QP} |\ell, L\rangle \rightarrow |l - 2q, R\rangle, \quad (2.9b)$$

where we have used the shorthand notation, $|\ell, P^+\rangle$ to denote a state of light with OAM ℓ and polarization P^+ . While useful, the QP is limited by the physical principles governing geometric phase only. As shown in Eq. 2.9 the operation of the QP yields conjugate OAM contributions to orthogonal polarizations. An alternative to QPs uses the properties of meta-surfaces [175, 176] instead of liquid crystals. Here, the desired phases are achieved by spatially engineering the characteristics of sub-wavelength structures, known as meta-atoms. This is usually achieved through altering the orientation and dimensions of the meta-atom structures. Meta-surfaces can subsequently replicate the action of the QP [177–180], but can then be extended to have more control, enabling the controlled pairing of both geometric and propagation phase such that arbitrary polarization states can be mapped to arbitrary increments of OAM [181, 182], or even, superpositions thereof [179, 183]. For instance, one may engineer the metasurface such that a generalised conversion of OAM is possible for arbitrary linear polarization bases states,[181] $|P^+\rangle$ and $|P^-\rangle$, i.e., the action of such a device would yield the following selection rules

$$\text{JP} |\ell, P^+\rangle \rightarrow |\ell + m, (P^+)^*\rangle, \quad (2.10a)$$

$$\text{JP} |\ell, P^-\rangle \rightarrow |\ell + n, (P^-)^*\rangle, \quad (2.10b)$$

with $n, m \in \mathbb{Z}$.

These devices thus allow one to arbitrarily map between polarizations and orbital angular momentum, allowing one control of the full angular momentum (J) domain and, as such, have been denoted J-Plates.

The capabilities of these devices go beyond single-step spatial modulation. Rather, by selecting an appropriate polarization state for the incident light (a superposition of the polarization basis states on which the device acts independently), we attain that the orthogonal components of said state are modulated independently with different spatial modes, thus producing a vector mode in a single step. For example,

we consider the case shown in Fig. 2.7, where horizontally polarized light with $\ell = 0$ is incident on a QP with charge $\frac{1}{2}$. The QP transformation according to the selection rules given in equation 2.9 yields the following output state

$$\text{QP} |0, H\rangle = \frac{1}{\sqrt{2}} \text{QP} (|0, R\rangle + |0, L\rangle) \quad (2.11)$$

$$= \frac{1}{\sqrt{2}} (|1, L\rangle + |-1, R\rangle) \quad (2.12)$$

where $|1\rangle = e^{i\phi}$ and $|-1\rangle = e^{-i\phi}$. As before, a set of POs allows for the modulation of the basis polarization states of the vector beam. Finally, a lens is then used to observe the far-field of the QP (this is necessary as the QP performs phase modulation) where the desired vector beam is found $|\Psi\rangle = LG^1 |P^+\rangle + LG^{-1} |P^-\rangle$.

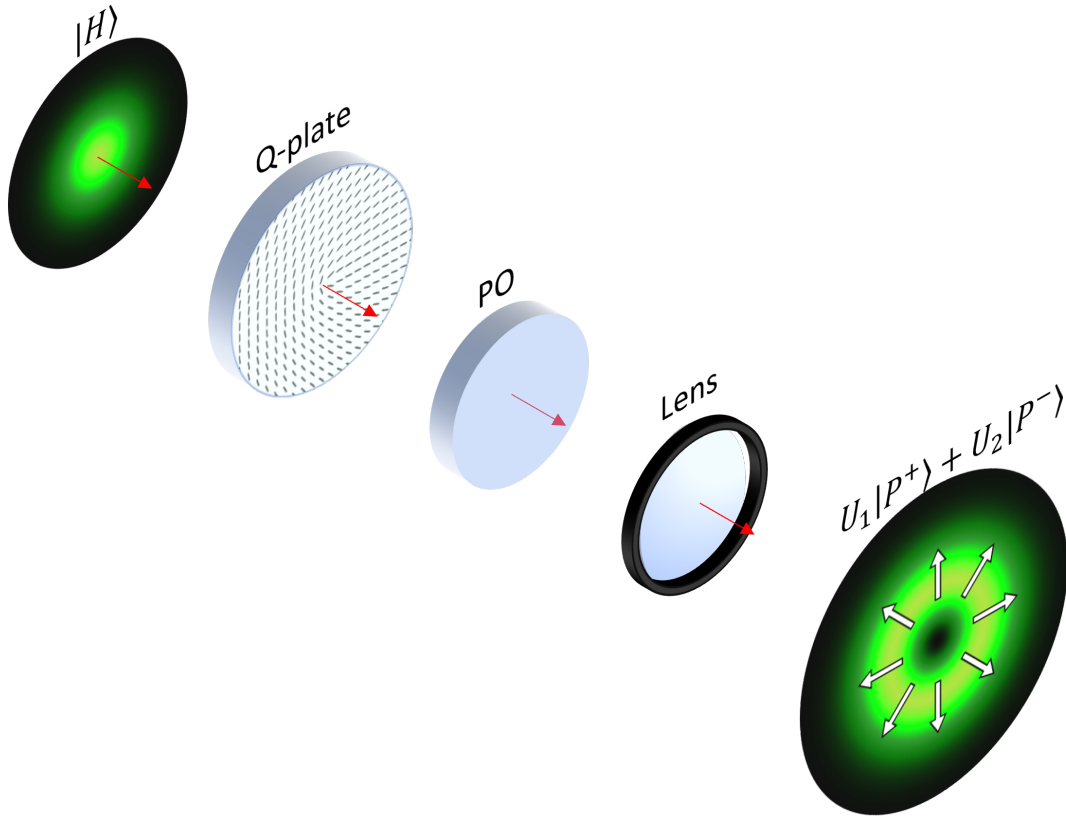


FIGURE 2.7: Vector beam (VB) generation using geometric optics. Horizontally polarized light is incident on a q-plate which independently modulates the right- and left- circular components of the field. Polarization optics (POs) are used to shift to the polarization basis of choice, $|P^+\rangle, |P^-\rangle$ desired polarization basis and a lens generates the desired final VB at its focal plane.

2.1.4 Vector beam characterization

Following the creation of VBs it is necessary to characterize their spatially dependent polarization and deduce its VQF. Since VBs naturally couple the polarization and

spatial degrees of freedom of light it becomes necessary to measure both degrees of freedom simultaneously. To do this we employ a technique known as Stokes polarimetry [63] which reconstructs the stokes field through a series of spatially-resolved intensity measurements by projecting different polarization states onto a CCD. The stokes field, $\vec{S}$, as previously discussed, is a vector field with 4 components, such that $\vec{S} = [S_0 \ S_1 \ S_2 \ S_3]^T$, where $S_i = S_i(x, y)$ are scalar fields known as the stokes parameters. With this in mind, characterizing a VB surmounts to measuring the Stokes parameters which enables the reconstruction of the stokes field of the VB. Each stokes parameter can be extracted from two intensity measurements according to

$$S_0 = I_H + I_V \quad (2.13a)$$

$$S_1 = I_H - I_V \quad (2.13b)$$

$$S_2 = I_D - I_A \quad (2.13c)$$

$$S_3 = I_R - I_L, \quad (2.13d)$$

where $I_H, I_V, I_D, I_A, I_R, I_L$ are spatially resolved intensity measurements obtained when projecting the VB into horizontal, vertical, diagonal, anti-diagonal, right-circular and left-circular states, respectively. If the degree of polarization (DOP) of the beam is unity (or close to unity), then the required number of measurements can be reduced to 4, a process typically known as reduced Stokes polarimetry [63]. In this process, we can still extract the desired stokes parameters by measuring I_H, I_V, I_D, I_R and computing the stokes parameters using

$$S_0 = I_H + I_V \quad (2.14a)$$

$$S_1 = 2I_H - S_0 \quad (2.14b)$$

$$S_2 = 2I_D - S_0 \quad (2.14c)$$

$$S_3 = 2I_R - S_0, \quad (2.14d)$$

Having extracted the Stokes parameters, we are then left with characterizing the quality of the VB. This is done by calculating its VQF from the extracted Stokes parameters. To do so we start by modifying the Stokes parameters above as follows [105, 184]

$$S_0 = I (|u_1|^2 + |u_2|^2), \quad (2.15a)$$

$$S_1 = I (|u_1|^2 - |u_2|^2) \quad (2.15b)$$

$$S_2 = 2I (|u_1||u_2| \cos(\arg(|u_1|) - \arg(|u_2|))) \quad (2.15c)$$

$$S_3 = 2I (|u_1||u_2| \sin(\arg(|u_1|) - \arg(|u_2|))), \quad (2.15d)$$

where $I = I_H + I_V$. We can then compute the terms within Eq. 1.16 as follows

$$\langle u_1 | u_2 \rangle = \int |u_1||u_2|^{i(\arg(|u_1|) - \arg(|u_2|))} dx dy = \frac{1}{2s_0} (s_2 - is_3), \quad (2.16)$$

$$\langle u_1 | u_1 \rangle \langle u_2 | u_2 \rangle = \int |u_1|^2 dx dy \int |u_2|^2 dx dy = \frac{1}{4} \left[1 - \left(\frac{s_1}{s_0} \right)^2 \right], \quad (2.17)$$

where $s_i = \int S_i dx dy$. Then substituting Eq. 2.16 and Eq. 2.17 into Eq. 1.16 we obtain

$$\text{VQF} = \sqrt{1 - \frac{s_1^2 + s_2^2 + s_3^2}{s_0^2}}. \quad (2.18)$$

We note that SBs have a $\text{VQF} = 0$ denoting the maximal separability between polarization and spatial DOFs and VBs $\text{VQF} = 1$ denoting the maximal non-separability between polarization and spatial DOFs.

2.1.5 Vector beam setup

Combining all the techniques discussed up to this point, allows for the full generation, detection and analysis of arbitrary vectorial beams.

The experimental setup is shown in Fig. 2.8 a). A HeNe $\lambda = 632.8$ nm wavelength beam is expanded and collimated onto an SLM using 2 lenses L_1 and L_2 . The SLM screen has been divided into two regions for simultaneous generation of two spatially separated, co-propagating beams with spatial mode profiles, as an example the complex amplitude holograms to encode the modes $U_1 = LG_{p=0}^{\ell=-1}$ and $U_2 = LG_{p=0}^{\ell=-1}$ have been displayed. The modulation technique chosen was complex amplitude modulation, and the holograms used are shown as insets alongside the SLM. Any unmodulated light including other orders besides the first order, are filtered using the combination of two lenses (L_3 and L_4) and an aperture (Ap). The co-propagating beams are then led into the b) Sagnac Interferometer (SI) system which spatially superimposes the two beams with orthogonal polarizations. In the SI system, two horizontally polarized fields, are rotated to diagonal polarization via a HWP before they are sent onto a polarizing beam splitter which splits each beam into its horizontal and vertical component beams. Through careful alignment of the two mirrors in the SI system, one can achieve the spatial overlap of the desired modes, such that we either attain the state $|\Psi_1\rangle = U_1 |H\rangle + U_2 |V\rangle$ or $|\Psi_2\rangle = U_1 |V\rangle + U_2 |H\rangle$, depending on which of the pairs of beams are overlapped. Lastly, the desired state $|\Psi\rangle = U_1 |P^+\rangle + U_2 |P^-\rangle$ (where $|P^+\rangle$ and $|P^-\rangle$ are arbitrary orthogonal polarization states) is achieved through the use of three polarization optics, QWP, HWP and QWP. The action, $\hat{S}_I$, of this interferometer can be modelled as: $\hat{S}_I : (U_1, U_2) \rightarrow \vec{U}_f = U_1 |P^+\rangle + U_2 |P^-\rangle$.

However, it must be noted that to achieve the generation of truly arbitrary vectorial modes, an approach is required which allows full, on-demand control over the selection of spatial modes. Whilst the previously discussed geometric devices allow for a compact, single-step VB generation, they are specially designed static optics, and would require transitioning between different optics to modify the output VB. For this reason the digital SLM approach presented above is the preferred as it offers dynamic approach to switching between different VBs through the control of the

holograms displayed on the SLM. In order to achieve a VB with geometric optics, it would be sufficient to replace the SLM and SI system, with the optics shown in Fig. 2.7.

For detection the following optics are required: two waveplates (HWP and QWP), and a linear polarizer in conjunction are used to perform the projective measurements on polarization states, and a camera allows for spatially resolved measurements. The measurement procedure is shown in Fig. 2.8 c) for a full stokes polarimetry procedure. To obtain the projections on the desired linear polarization states, I_H, I_V, I_D, I_A , we require the HWP at four different orientations with respect to the horizontal axis, $\theta_{HWP} = \{0, \frac{\pi}{4}, \frac{\pi}{8}, -\frac{\pi}{4}\}$, respectively, and the LP orientated to project onto horizontal polarization states, $\theta_{LP} = 0$. To obtain the remaining circular polarization, I_R, I_L , projections we simply need to place a QWP orientated at an angle $\theta_{QWP} = \frac{\pi}{4}$ before the HWP and rotate the HWP to two different angles, $\theta_{HWP} = \{0, \frac{\pi}{4}\}$, respectively (The LP is kept at $\theta_{LP} = 0$).

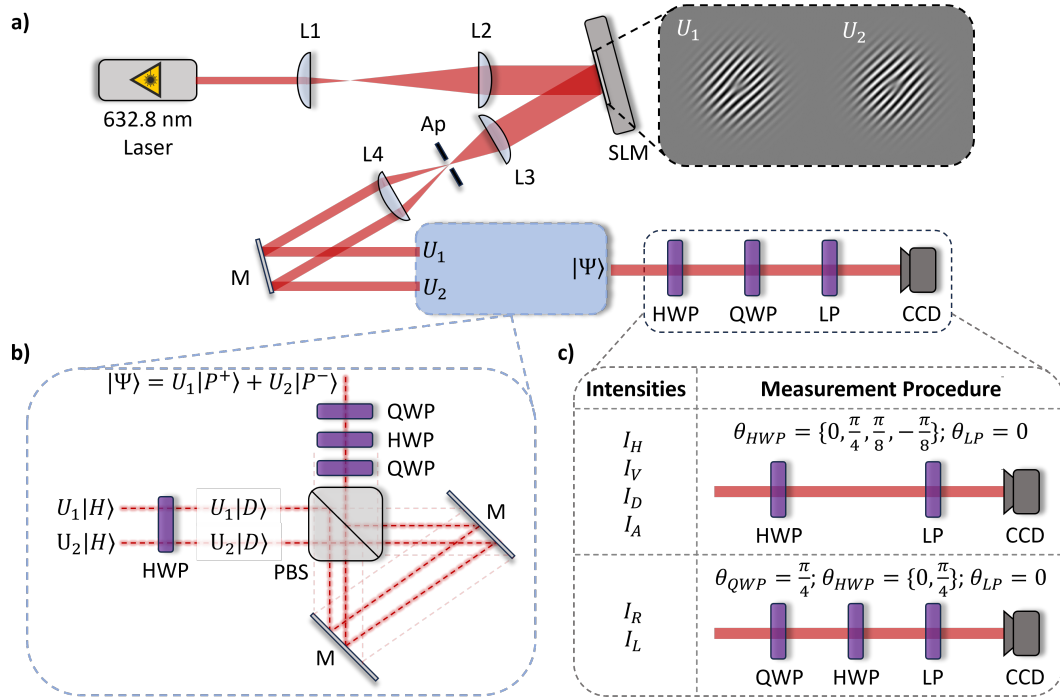


FIGURE 2.8: Setup for the generation and detection of vector beams. a) A $\lambda = 632.8$ nm wavelength laser is expanded and collimated onto an SLM screen that has been divided into two halves displaying the inset holgrams. The generated co-propagating beams are filtered from the unmodulated light and are sent into a b) Sagnac Interferometer system which spatially overlaps the two spatial modes with orthogonal polarizations generating the desired vector mode $|\Psi\rangle$. c.) The VB is then measured using a half-wave plate (HWP), quarter-wave plate, and linear polarizer (LP) to perform projective measurements on polarization along with a CCD camera used to perform the spatial projections.

Fig. 2.9 shows experimental data obtained for the digital approach using an SLM and the static geometric optic approach using a q-plate with charge $q = \frac{1}{2}$. The reduced set of intensity measurements for both cases are shown, empirically matching the theoretical insets. In the case of both projections into the right and left circular polarization states, we observe the expected intensity pattern produced by an $LG^{\pm 1}$. Furthermore the projections into horizontal and diagonal polarization states reveal the

expected HG intensity patterns produced from the superpositions $LG^1 + LG^{-1}$ and $LG^1 + e^{i\frac{\pi}{2}} LG^{-1}$ [185]. The associated stokes parameters were then calculated and also show excellent empirical agreement with the theoretical predictions shown as insets. In the case of both experimental approaches, S_0 reveals the expected total intensity pattern produced by $|LG^1|^2 + |LG^{-1}|^2$. Furthermore, S_1 (associated with basis states $\{|H\rangle, |V\rangle\}$) and S_2 (associated with basis states $\{|D\rangle, |A\rangle\}$) reveal regions of linear polarization. According to the colourmap used for the plot of $S_1(S_2)$, red regions are associated with predominantly $|V(D)\rangle$ polarized light whereas blue regions are associated with predominantly $|H(A)\rangle$ polarized light. As expected, both approaches yield a predominantly zero-valued scalar field for S_3 , since S_3 is computed from the difference in the LG mode intensities which in this case are identical. The polarization structure of each experimental VB is reconstructed, and shown to agree well with the expected theoretical radial vector mode [63]. Calculation of the VQF for both VBs produced using digital and geometric phase approaches shown in Fig. 2.9 yield values of $VQF = 0.99 \pm 0.01$ and $VQF = 0.98 \pm 0.01$, respectively, close to the desired value of 1. Therefore we can conclude beyond empirical arguments that both beams exhibit a high degree of non-separability.

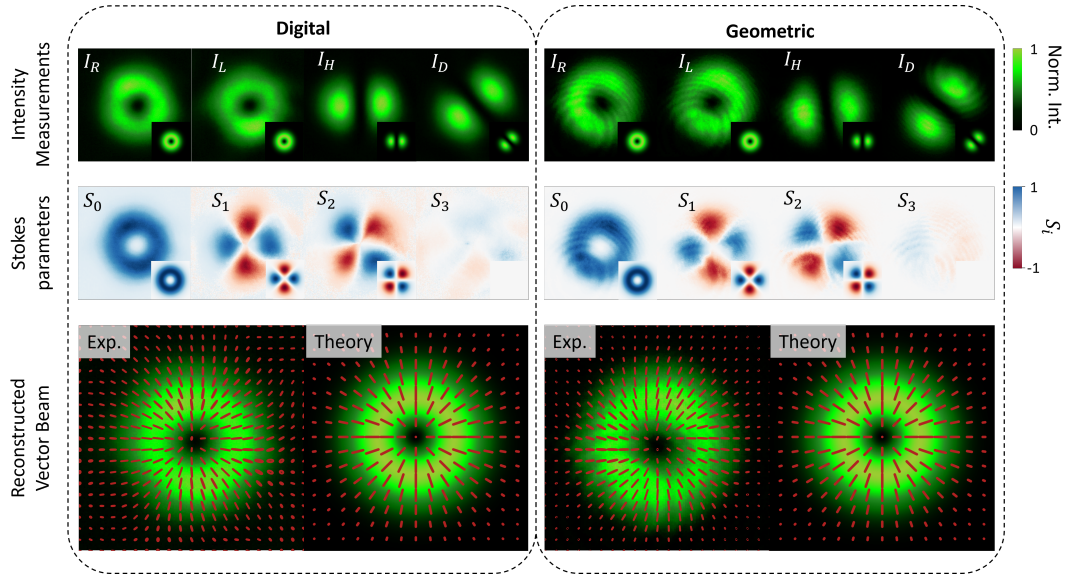


FIGURE 2.9: The data obtained from performing stokes measurements on a radially polarized vector mode generated using a digital SLM device as well as a static q-plate device. The reduced set of intensity measurements are obtained allowing for the calculation of the associated stokes parameters. Finally, using the stokes parameters, it is possible to reconstruct the polarization structure of the VB.

2.2 Quantum structured light techniques

Quantum structured light is a relatively new branch of quantum optics that borrows many tools from classical structured light techniques. The aim here is to engineer quantum states which utilize all the degrees of freedom available to a single photon in order to further enhance their utility for future quantum technologies.

Here we will discuss the engineering and detection of hybrid OAM-polarization entangled states. We start by discussing a technique to produce high dimensional OAM-OAM entangled states and characterize the dimensionality of the state. This is followed by a discussion on techniques to couple OAM to polarization thus allowing for the engineering of hybrid states out of high dimensional entangled states. Here, the high dimensionality of the initial OAM-OAM entangled state allows for the realization of a diverse landscape of hybrid states. Lastly, the characterization of the various features of these states will be discussed.

2.2.1 Entanglement through SPDC

One of the most commonly used approaches for generating quantum structured light uses crystals with a second-order susceptibility through a process called spontaneous parametric down-conversion (SPDC). Examples of crystals that enable this process are beta-barium borate (BBO), lithium niobate (LN), potassium dihydrogen phosphate (KDP) and potassium titanyl phosphate (KTP).

Figure 2.10 a) shows a cartoon description of the SPDC process where a pump (p) beam of angular frequency, ω_p , interacting with a non-linear crystal, results in the generation of two daughter photons of ω_s and ω_i , respectively. Due to the conservation of energy, the angular frequencies of said signal (s) and idler (i) are governed by the relation $\omega_p = \omega_s + \omega_i$. This relation is shown in the energy level diagram in Fig. 2.10. b). Accordingly, the entangled biphotons need not have the same energy, ($\omega_s = \omega_i$), termed *degenerate* SPDC, but can be *non-degenerate* ($\omega_s \neq \omega_i$).

Linear momentum is also conserved in the SPDC process; the momentum of the produced photons should add to that of the pump, resulting in $\mathbf{k}_p = \mathbf{k}_s + \mathbf{k}_i$, shown pictorially in Fig. 2.10 c), expressed using the wave-vectors, $\mathbf{k}$, of the respective pump, signal and idler photons. If energy and momentum conservation is met, one has perfect phase matching. On the contrary, phase mismatch occurs and is proportional to Δk_z , in the longitudinal momentum components as illustrated in Fig. 2.10 c). In order, to alter the phase-matching conditions, most bulk crystals require angle tuning as shown in Fig. 2.10 d), where the angle of incidence of the pump mode with respect to the crystal is varied. Alternatively, one can use periodically poled crystals (shown in Fig. 2.10 e), with engineered domain switching of a period of Λ [186, 187]. For these crystals, the phase matching conditions can be altered by changing the temperature of the crystal.

There are two types of phase-matching geometries that can be identified based on the matching of linear momentum. In Fig. 2.10 f), on the top panel, the output field propagates off-axis with respect to the pump, but with equal and opposite trajectories on either side, this is known as non-collinear SPDC. The emitted photons in this case are anti-correlated in transverse momentum and therefore the entangled

photons are located on any two opposite ends of the SPDC ring. For collinear SPDC, shown in the bottom panel of Fig. 2.10 f), the output field wave-vectors are also anti-correlated, except when $k_s = k_i = 0$. Experimental intensity images of different phase-matching geometries are shown in Fig. 2.10 g), where an angle tuned bulk crystal (BBO), was rotated to transition between collinear and non-collinear geometries.

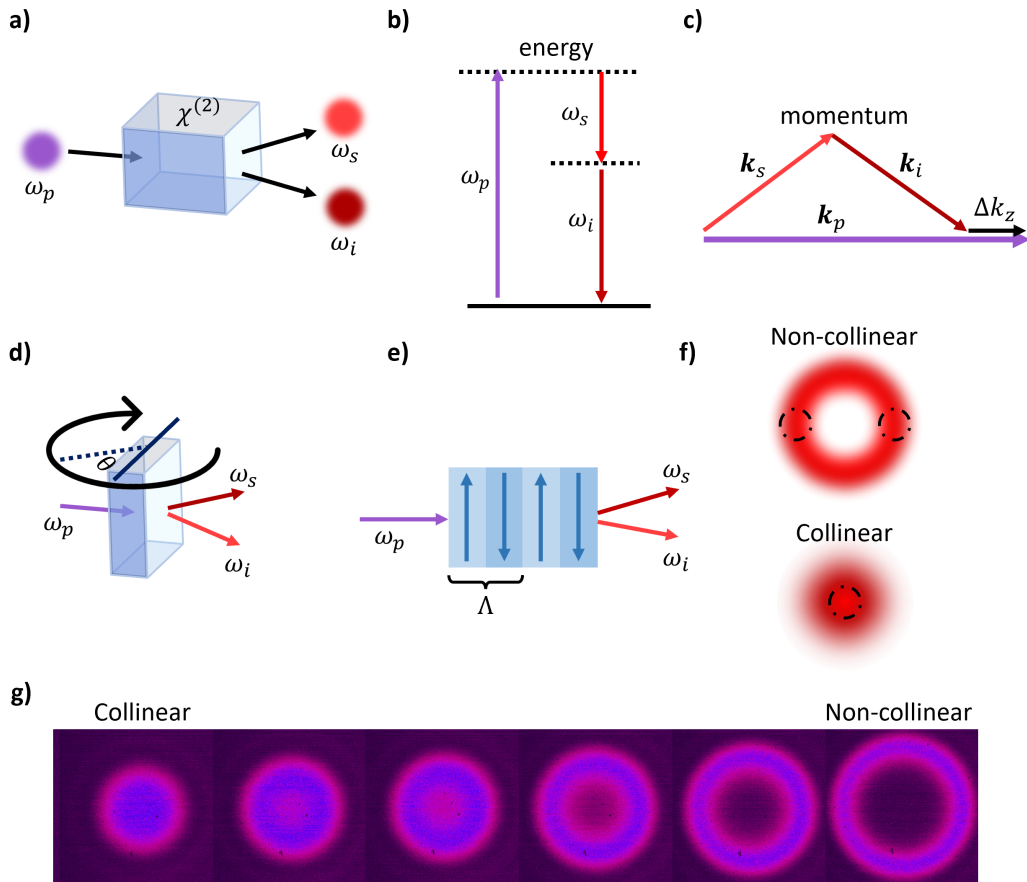


FIGURE 2.10: Spontaneous parametric down-conversion. a) Generation of SPDC photons from nonlinear crystals (NC) that are characterized by a second order nonlinear susceptibility tensor, $\chi^{(2)}$, 2). In this process a single photon with a frequency, ω_p , is absorbed by the crystal and twin photons with frequencies, ω_s and ω_i are produced. b) Energy diagram for the three photons involved in SPDC, showing energy conservation. c) Momentum conservation in the SPDC process. Momentum vectors of the emitted signal (s) and idler (i) photons must add up to that of the pump photon. The phase mismatch occurs in the longitudinal direction. The emission cones can be altered by changing the phase matching conditions. This can be achieved via d) angle tuning in bulk crystals and with e) temperature tuning in periodically poled (PP-) crystals. f) Far-field intensity profiles of SPDC photons showing non-collinear and collinear geometries where the entangled photons are found in the regions highlighted with dashed lines. g) Experimental far-field intensity images showing SPDC for an angle tuned bulk crystal (BBO) at different collinear and non-collinear regimes.

Due to the polarization sensitivity and phase-matching conditions, there are different phase-matching regimes which can be engineered. This results in different polarization pairings with active research considering exotic fabrications and crystal pairings, allowing one to engineer unique geometries and output states [188]. The

distinction between the different phase-matching conditions and geometries can be summarized into three types which rely on the relative polarizations with respect to the crystal ordinary (o) and extraordinary (e) axes. In general, the types can be summarized into the following, with examples shown in Fig. 2.11: a) Type-0 the signal and idler photons carry the same polarization as the pump, whereas in b) Type-I their polarization is orthogonal to that of the pump; for c) Type-II, however, the signal and idler photons are orthogonally polarized with respect to each other, leading to polarization entanglement. It must be noted that these characteristics are all determined by the nonlinear electric tensor of the crystals.

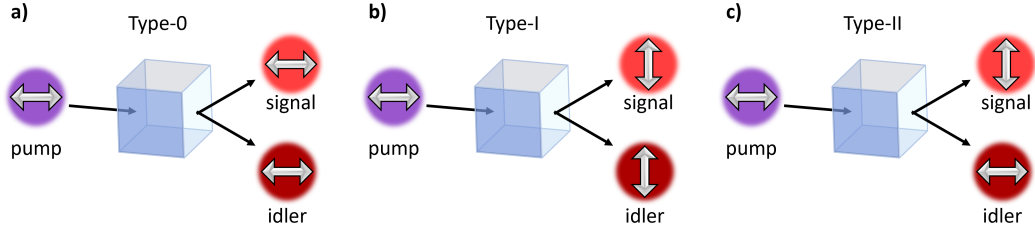


FIGURE 2.11: Types of nonlinear crystals with different phase-matching conditions. For a) Type-0 NCs, the pump photon is polarized in the same direction as the signal and idler photons, for b) Type-I NCs, the pump photon is orthogonally polarized to the signal and idler photons and for c) Type-II NCs, the signal and idler photons are orthogonally polarized to one another.

Spatial mode entanglement from SPDC

In the SPDC process, photon pairs can be entangled in their energy-time [189, 190], path [191] and transverse spatial degrees of freedom [17, 192–196] (see Ref. [197]). In this work, we are particularly interested in high-dimensional spatial mode entanglement, which we use as a resource to create a plethora of hybrid entangled states.

In order to sift out the presence of spatial mode entanglement, it is useful to consider the general expression of the SPDC state in terms of wavefunctions for the generated photons in the linear momentum basis. One method to derive this evolves the quantum state by using the non-linear phenomenological Hamiltonian [198–200], the main result, being the two photon state

$$|\psi_{s,i}\rangle = \int f(\mathbf{q}_s, \mathbf{q}_i) |\mathbf{q}_s\rangle |\mathbf{q}_i\rangle d^2q_s d^2q_i, \quad (2.19)$$

expressed in the transverse momentum coordinates, $\mathbf{q}_s$ ($\mathbf{q}_i$), where $f(\mathbf{q}_s, \mathbf{q}_i)$ is the two-photon joint probability amplitude of the pair of photons, s and i , respectively, i.e. the probability of observing the signal and idler photons with transverse momentum, $\mathbf{q}_s$ and $\mathbf{q}_i$, respectively. Assuming that the pump beam impinging on the crystal has a Gaussian profile in the far-field, the joint probability amplitude function is given by [188, 201]

$$f(\mathbf{q}_s, \mathbf{q}_i) = \underbrace{\mathcal{N} \exp\left(\frac{-w_p^2 |\mathbf{q}_p|^2}{4}\right)}_{\text{pump}} \underbrace{\text{sinc}\left(\frac{L\Delta k_z}{2}\right)}_{\text{phase-matching}}, \quad (2.20)$$

where w_p is the waist size of the pump beam in the near-field, $\mathbf{q}_p = \mathbf{q}_s + \mathbf{q}_i$, and the phase matching function, $\text{sinc}\left(\frac{L\Delta k_z}{2}\right)$, that is determined by the crystal length, L , in the longitudinal direction and the longitudinal wave-vector mismatch, Δk_z . The factor $\mathcal{N}$, is a normalisation constant. Furthermore, $\Delta k_z = k_{p,z} - k_{s,z} - k_{i,z}$, i.e., the longitudinal wave-vector mismatch depends mainly on the relative difference between the longitudinal components of the wave-vectors of the pump photon and emitted pairs. Therefore, if the emission angles of the SPDC photons (i.e., the angles between $k_{s,i}$ and k_p) are sufficiently small, then the longitudinal wave-vector mismatch can be approximated by [202]:

$$\Delta k_z = -\frac{|\mathbf{q}_s - \mathbf{q}_i|^2}{2k_p}. \quad (2.21)$$

Under these assumptions the phase matching function can be rewritten as $\text{sinc}\left(\frac{Lw_p^2}{8z_R}\right)$ and we can express the constants in the argument of the resulting phase matching function, as $\frac{Lw_p^2}{8z_R}$, where $z_R = \frac{1}{2}k_p w_p^2$ is the Rayleigh length of the pump beam. From here on, we assume that we can use the thin crystal limit, $L \ll z_R$, so that $f(\mathbf{q}_s, \mathbf{q}_i) \approx U(\mathbf{q}_s + \mathbf{q}_i)$. Now, our aim is to express the two photon states given any complete transverse spatial basis. Suppose we have a discrete Schmidt basis, $|j\rangle_s |k\rangle_i$, one can express our two photon states in Eq. (2.19) as an entangled two photon state

$$|\psi_{s,i}\rangle = \sum_{jk} c_{jk} |j\rangle_s |k\rangle_i, \quad (2.22)$$

where the coefficients are given by [203]:

$$c_{jk} = \int f(\mathbf{q}_s, \mathbf{q}_i) M_j^*(\mathbf{q}_s) M_k^*(\mathbf{q}_i) d^2 q_s d^2 q_i, \quad (2.23)$$

and the mode functions $M_{j(k)}(\mathbf{q})$ are eigen-functions that map onto the eigen-states $|j(k)\rangle$ expressed in the transverse momentum basis. Noting that the joint probability amplitude is approximated by the pump field (preferably a paraxial optical field), it is possible to map the integral to the spatial basis $|\mathbf{r}\rangle_{si}$ where it can be evaluated in the near field by Fourier transforming Eq. (2.23). Accordingly, the overlap integral becomes

$$c_{jk} = \mathcal{N} \int u(\mathbf{r}) m_j^*(\mathbf{r}) m_k^*(\mathbf{r}) d^2 r. \quad (2.24)$$

Here $u(\cdot)$ is the pump mode in the near field, while $m_{j,k}(\cdot)$ are the Fourier transforms of the modes $M_{j,k}(\cdot)$. Examples of suitable bases are the discrete pixel modes [196], Hermite-Gaussian [204], the Laguerre-Gaussian [16, 205] and Bessel-Gaussian [206] modes. As mentioned previously, of particular interest to this work is the LG modes, as such we will speak about those modes in particular. Using the LG basis formed from modes with the radial index set to $p = 0$, i.e., $|\ell\rangle \equiv |LG_{p=0}^\ell\rangle$. Suppose that the pump photon carries OAM, $u(\mathbf{r}) = LG_{p=0}^{\ell_p}(\mathbf{r}) \propto \exp(i\ell_p\phi)$. By evaluating the overlap integral in the cylindrical coordinates one finds that the azimuthal integral induces the restriction that the coefficients are nonzero when $\ell_p = \ell_s - \ell_i$ showing OAM conservation in the SPDC process. As such we arrive at the desired high-dimensional state

$$|\Psi_{si}\rangle = \sum_{\ell_s=-\infty}^{\infty} c_{\ell_s} |\ell_s\rangle_s |\ell_p - \ell_s\rangle_i. \quad (2.25)$$

It is important to note that whilst there are an infinite number of OAM states which satisfy the OAM conservation condition, practically Eq. 2.25 cannot be an infinite dimensional entangled state, as this would imply an ideal generation and OAM detection system, which is not practically possible. Instead, from experimental considerations, we can derive the actual OAM bandwidth, often termed, "spiral bandwidth", of the system, i.e., the entangled state with the highest dimension that the system can produce.

Optimising the spiral bandwidth

The limiting parameters affecting the extent of the spatial entanglement produced at the SPDC source can be reduced to two factors: the pump size and crystal length. This may be seen by decomposing the SPDC wavefunction (Eq. (2.19)) into the LG basis for $p = 0$ and the waist size is scaled with respect to that of the crystal and pump, $\beta = w_p \sqrt{\frac{L}{2z_R}}$ in order to determine the coefficients, c_ℓ in Eq. (2.25). Subsequent decomposition, yields the analytical expression [207],

$$c_\ell = \frac{4\alpha}{(1+\alpha)^2} \left| \frac{\alpha-1}{\alpha+1} \right|^{|\ell|}, \quad (2.26)$$

with $\alpha = w_p \sqrt{\frac{L}{k_p}}$ for pump wavenumber, k_p . The probability ($|c_\ell|^2$) of having the SPDC biphotons in the state $|\ell_s\rangle_s |-\ell_i\rangle_i$ can be calculated for the pump size and crystal length to give a generational bandwidth.

It is not difficult to deduce from geometric arguments that the generational spiral bandwidth must set an upper limit for the effective spiral bandwidth, which can be derived to give [208]

$$K_{gen} = 1 + 2\sqrt{\frac{\pi^2 w_p^2}{L\lambda}}, \quad (2.27)$$

given the pump waist, wavelength and crystal length. To incorporate the detection beam waists, further derivation results in a convolution of near and far-field bandwidths where [208]

$$K_{eff} = \left(\sqrt{\left(1 + 4\gamma_{s,i}^2\right)^{-2} + \left(1 + \frac{\pi^2 w_p^2}{\gamma_{s,i}^2 L\lambda}\right)^{-2}} \right)^{-1}, \quad (2.28)$$

which allows one to predict the effective dimensionality of the entangled states that one may detect, given a choice of the detection waist size $w_{s,i}$ backprojected to the crystal plane in the ratio $\gamma_{s,i} = \frac{w_p}{w_{s,i}}$.

Having discussed the generation of high-dimensional entangled states through SPDC, it is necessary to characterize the dimensionality of the state, which is typically done through a spiral bandwidth measurement achieved through the simultaneous detection of the OAM of entangled photons.

OAM detection of single photons

In section 2.1 the modulation of light (typically in the fundamental Gaussian mode, i.e., $|0\rangle \equiv |\text{LG}_{0,0}\rangle$) was discussed through the use of an SLM device. However, due to the reciprocal nature of light, the inverse action of detection is also possible, by "unmodulating" light that has some structure and then applying a filter which now only allows the unstructured light through. This process is done through a combination of an SLM (which modulates or unmodulates light) and an SMF (which acts as the filter only allowing through the fundamental Gaussian mode) [83, 209]. The concept is illustrated in Fig. 2.12 (a), where the fundamental mode is converted into a desired spatial pattern, $\Phi(\mathbf{r})$, and can be detected in reverse. Consequently, the fiber and the SLM form an optical correlation measurement that is analogous to performing an inner product measurement of the form $|\langle \Phi | \Psi \rangle|^2$ for some input field $\Psi(\mathbf{r})$ and some target (detection) mode $\Phi(\mathbf{r})$ [209]. Furthermore, to maximise the coupling efficiency, the creation and detection beam sizes need to be of a comparable size. As outlined, two elements are at play: (i) the detection mode that is encoded as a hologram ($H(\Phi(\mathbf{r}))$ (examples of appropriate phase and complex amplitude masks are given in Fig. 2.12 b) and c), respectively.), and (ii) the probability of the demodulated mode coupling into the single mode fiber.

Assuming that the detection system is optimized, the coefficients along with the dimension of the state can be changed through a change in the pump beam waist incident on the NC, as discussed in section 2.2.1, and in particular the dimension or Schmidt number can be extracted from a spiral bandwidth measurement, by calculating

$$K = \frac{1}{\sum_n |c_n|^4} \quad (2.29)$$

where $n \in \mathbb{Z}$ and c_n are the non-zero normalized coefficients of the state. Typically, to ensure correct characterization of the dimension of the state we perform OAM measurements beyond the limit set by K_{gen} .

To extract the dimensionality of the generated state, it is necessary to perform a spiral bandwidth measurement for our biphoton state. The measurement setup is summarized in Fig. 2.12 d). For this measurement, we perform a series of projective measurements over the chosen basis set of OAM states which span the range $[-\ell_{\text{max}}, \ell_{\text{max}}]$ where we ensure that $\ell_{\text{max}} < K_{\text{gen}}$. These projective measurements can be summarized in a matrix whose elements are given by $B_{ij} = \langle \ell_i | \ell_j \rangle$ where $i, j \in [0, 2\ell_{\text{max}}]$, an example of which is given in Fig. 2.13 a). From this matrix we can then extract the diagonal to obtain the coefficients of the Schmidt modes, seen as the bar plot in Fig. 2.13 b) and then we can calculate the Schmidt number through the calculation $K = \frac{[\sum_\ell |c_\ell|^2]^2}{\sum_\ell |c_\ell|^4}$. The data obtained for Fig. 2.13 yielded a Schmidt number $K = 11$. Since the high dimensional state is utilized as a resource, the Schmidt number sets the upper bounds of the highest OAM number chosen in the hybrid state by $\ell_{\text{max}} = \frac{K-1}{2}$, which in this case sets the highest OAM number as $\ell_{\text{max}} = 5$.

2.2.2 Experimental setup

The setup is summarized in Fig 2.14 (a), a pump laser with a Gaussian profile, is

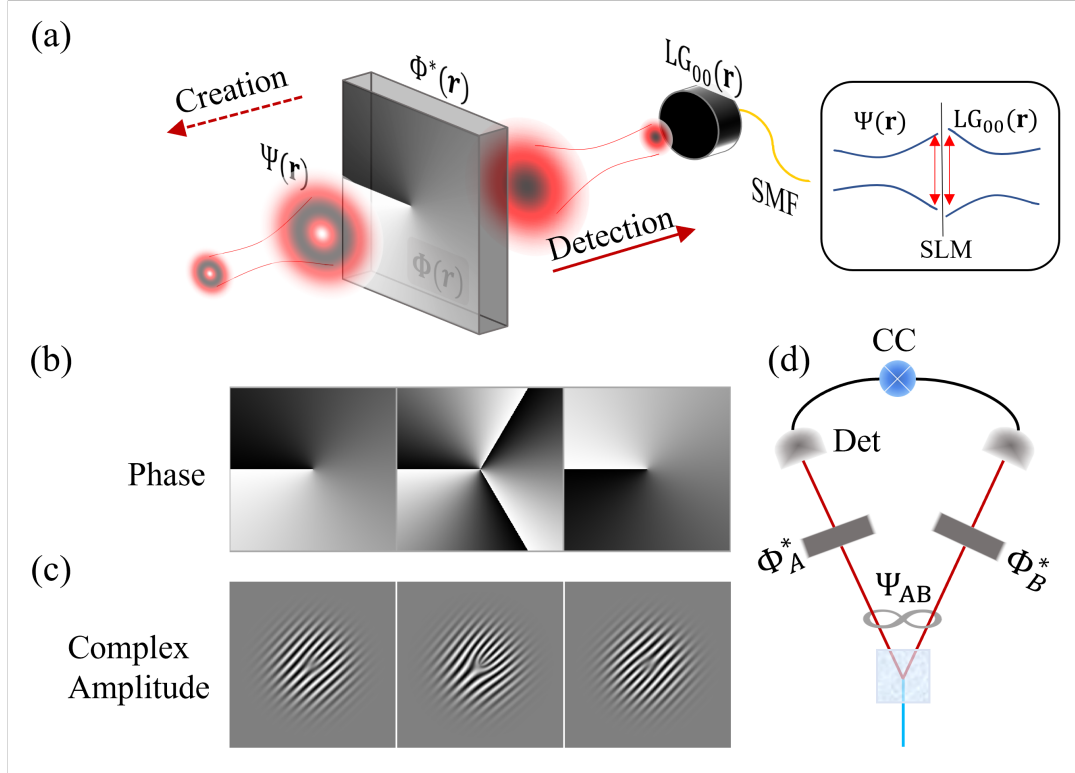


FIGURE 2.12: OAM detection scheme for high dimensional states (a) The detection scheme for an input field $\Psi(\mathbf{r})$ impinging it on an SLM encoded with a hologram $\Phi^*(\mathbf{r})$. The SMF then accepts only the LG_0^0 component of the output field. Reciprocally we see the usual SLM modulation scheme where the SMF now acts as our source producing the mode LG_0^0 which is incident on the SLM encoded with $\Phi(\mathbf{r})$, thereby producing the desired mode (or depending on the encoding scheme, a superposition of modes is created with the desired mode being among them). (b) Examples of encoded phase holograms (no grating) with encoded hologram function taking the form $H(\Phi(\mathbf{r})) = \exp(i\ell\phi)$ for $\ell = 1, 3, -1$. (c) Examples of complex-amplitude holograms (grating included) with encoded hologram function taking the form $H(\Phi(\mathbf{r})) = |LG_0^\ell| \sin(\arg(LG_0^\ell) + \Lambda_{grating})$ for $\ell = 1, 3, -1$. (d) Schematic of detection scheme for a biphoton state $|\Psi\rangle_{AB}$

incident on a non-linear crystal (NC) to produce entangled signal and idler photons with any residual pump photons being filtered by a band-pass filter (BPF). The entangled photons (photon A and photon B), occupy the joint state given in Eq. 2.25 with $\ell_p = 0$

$$|\Psi\rangle_{AB} = \sum_{\ell_s=-K}^K c_{\ell_s} |\ell_s\rangle_A |-\ell_s\rangle_B, \quad (2.30)$$

where $2K + 1$ is the Schmidt number of the state and determines the dimension of the entangled biphoton state, the c_{ℓ_s} 's are complex coefficients which control the weightings and relative phase of each of the individual states with respect to the entire entangled system. The photons are then separated into two distinct optical paths using a 50:50 beam-splitter (BS). Photon A is sent to a coupled detection system consisting of an SLM and Single Mode Fibre (SMF) which together are capable of measuring the spatial state of photon A, as discussed in the previous section. Photon B is sent through a spatial-polarization converter (SPC) (discussed in more detail in

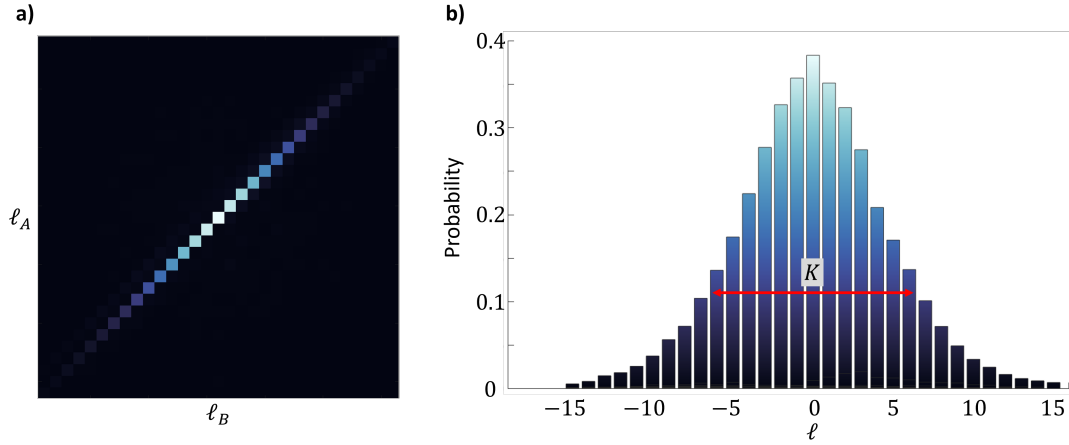


FIGURE 2.13: a) Measurement outcomes for simultaneous projective measurements done on both photons. b) Spiral bandwidth measurement, extracted from diagonal of measurement outcomes, can be used to determine the dimensionality of the system.

the proceeding section), which maps the initial spatial information of photon B onto polarization information, thus generating the desired hybrid state

$$|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}} (|\ell_1\rangle_A |H\rangle_B + |\ell_2\rangle_A |V\rangle_B), \quad (2.31)$$

and then photon B is sent through a series of polarization optics (POs) consisting of an LP and HWP, to perform projective measurements on polarization, similar to the required projective polarization measurements discussed in section 2.1.5

Spatial-to-Polarization Conversion

The aim of the SPC is to convert the high dimensional OAM-OAM entangled state to an arbitrary hybrid entangled state of the form

$$|\Psi\rangle = \alpha |\ell_1\rangle_A |H\rangle_B + \sqrt{1 - \alpha^2} e^{i\gamma} |\ell_2\rangle_A |V\rangle_B, \quad (2.32)$$

where $\alpha \in [0, 1]$ is a weighting parameter controlling the degree of entanglement between photons A and B and γ is a relative phase between the terms of the wavefunction. To engineer any desired state we require full control not only of the parameters α and γ but also of the desired OAM subspace. For this, typical methods used to generate hybrid entangled states using q-plates [210] and static meta-surfaces [211, 212] will not suffice, hence we make use of an all-digital SPC approach [23] which couples the spatial information of photon B to polarization. We must note that the limitations on the available OAM is set by the initial spiral bandwidth, as discussed in the previous section. In order to achieve the desired arbitrary state given in Eq. 2.32, the desired post-selected OAM states must be coupled to orthogonal polarization states, thus requiring separate modulations using a digital device. One approach of accomplishing digital SPC is shown in Fig. 2.14 b), where photon B is sent through a Half-Wave Plate (HWP) to convert it to diagonal polarization before having its OAM information coupled to polarization through a "double

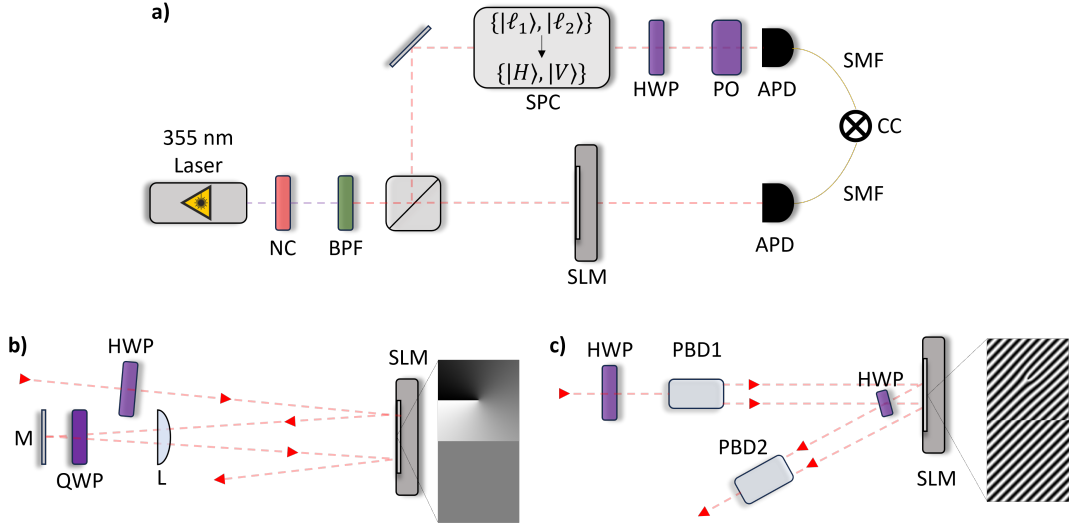


FIGURE 2.14: a) The summary of the experimental setup used to dynamically generate hybrid entangled states. Two photons entangled in OAM are sent down separate paths where one photon's OAM is measured via a joint SLM and SMF detection system and the other photon's polarization is measured using a set of polarization optics (PO) following the coupling of its OAM information to polarization through a spatial-to-polarization coupling (SPC) system. b) This SPC system utilizes a double bounce on an SLM where phase modulation (no grating) is used to modulate each orthogonal polarization separately. c) Utilizes polarization beam displacers (PBDs) to separate orthogonal polarizations into separate parallel optical paths before each polarization is modulated and then recombined using a second PBD.

bounce" on an SLM, thereby transferring the OAM-OAM correlations between photons A and B to OAM-Polarization correlations. To illustrate how this process functions, let's consider the case of an individual photon described by the state $|H, \ell\rangle$, i.e., it is horizontally polarized with OAM ℓ . Passing through the HWP, we have that the state becomes $|D, \ell\rangle$. On the first bounce off of the SLM, the horizontal component of the state is modulated by the displayed hologram, to $|H, \ell + \ell_1\rangle + |V, \ell\rangle$. It must be noted that the absence of a grating on the SLM ensures that the unmodulated and modulated remain collinear. The photon then passes through a QWP orientated at 45° with respect to the horizontal axis, reflects off a mirror and then passes through the same QWP once again. The transformation undergone by the state due to the QWP and mirror combination is given by $|H, \ell + \ell_1\rangle + |V, \ell\rangle \xrightarrow{QWP} |L, \ell + \ell_1\rangle + |R, \ell\rangle \xrightarrow{M} |R, \ell + \ell_1\rangle + |L, \ell\rangle \xrightarrow{QWP} |V, \ell + \ell_1\rangle + |H, \ell\rangle$. On the return bounce we have that we achieve the desired state $|V, \ell + \ell_1\rangle + |H, \ell + \ell_2\rangle$. This system can then be coupled to an SMF which will perform the projection $|0\rangle\langle 0|$, thereby erasing the OAM information from photon B. It is clear then that the desired OAM subspace, ℓ_1, ℓ_2 is selected by displaying those modes on the SLM. Mathematically, this SPC conversion can be summarized in a mapping of the form [23]

$$h_{SPC} : |\ell_2\rangle_B |V\rangle_B \rightarrow \alpha |0\rangle_B |H\rangle_B \quad (2.33)$$

$$: |\ell_1\rangle_B |H\rangle_B \rightarrow \sqrt{1 - \alpha^2} e^{i\gamma} |0\rangle_B |V\rangle_B. \quad (2.34)$$

We note further that the relative phase γ can be directly encoded on either SLM hologram and further the weighting α can be controlled via the first HWP or by

introducing a loss in the SPC conversion [142]. Since the double-bounce mechanism necessitates that the horizontally and vertically polarized photons follow the same path, we can induce a loss in the system by sending a precise portion of light away from the experimental path. This is done by adding a grating onto one of the halves and controlling the grating depth. In modifying the grating depth, we modify the efficiency of the SLM in the n th order, which is given by [213]

$$|c_n|^2 = \text{sinc}^2(\pi(n - M)), \quad (2.35)$$

where $|c_n|^2$ is the fraction of power in the n -th diffraction order and $M \in [0, 1]$ controls the grating depth from 0 to $2M\pi$. Since the SPC process used keeps the orthogonal polarizations collinear, α is dependent on the fraction of power sent to the zeroth order. Since the induced loss was to the vertically polarized term in equation 2.32, we have that $\alpha = 0.5(1)$ when $M = 0(1)$. As such we can show that α is dependent on M according to the expression

$$\alpha(M) \propto \frac{\sqrt{1 - |c_0|^2}}{2} + \frac{1}{2}. \quad (2.36)$$

This establishes a relation between α and the grating depth.

A second approach inspired by ref. [214] makes use of a HWP sandwiched between two polarization beam displacers (PBDs), as shown in Fig. 2.14 c), to separate the orthogonal polarizations into two separate, parallel paths, and recombine the two paths once both have been modulated. This approach is similar to that shown in Fig. 2.14 b) except that now one uses a grating to modulate in the first order, thus α is controlled instead by how much light is in the first-order instead of the zeroth-order. The advantage of this approach is the ability to use a grating to remove the possibility of unmodulated light affecting the purity of the final state.

2.2.3 Tools for quantum state characterization

In this section we will consider the necessary tools for quantum state characterization. The results shown from this point on, were obtained using the experiment shown in Fig. 2.14.

Before characterizing any quantum state, it is necessary to reconstruct said state through a series of projective measurements on the state. The process of quantum state tomography that will be discussed, allows a full reconstruction of the state, thus allowing us to build the density matrix of any generated state.

Quantum State Tomography

Features that identify a quantum state are not directly observable, and therefore a series of projective measurements need to be made in order to correctly characterize a given quantum state. The process of performing the full set of projective measurements is known as Quantum State Tomography (QST). Conceptually, QST simply tries to look at different aspects of a state, one at a time, and then having this information, we are able to algorithmically reconstruct the state which must have possessed those features. We can form an analogy between QST and typical orthographic projections done on 3D objects. Whilst we can observe an object in 3D, it is not possible to depict the entire object faithfully on a piece of paper if we wish to communicate

the object to someone else. Instead, one can produce projections of three distinct sides of the object, an example is shown in Fig. 2.15 a) where the projections are shown as the shadows formed by the object on a wall. Having these projections, someone can simply piece them together to reconstruct the original object. Whilst this analogy conveys the idea behind QST, it has its limitations. Quantum states live on Hilbert space, which means that for full state reconstruction it is necessary to extract both amplitude and phase information, the latter ensuring that the number of measurements required scales quadratically with the dimension of the state. For example, for a 2D biphoton state, like the ones that we are interested in, it would take 36 measurements to form a full QST of the state. In Fig. 2.15 b), a theoretical example is shown where each element of the matrix corresponds to the probability of observing the state in that particular configuration.

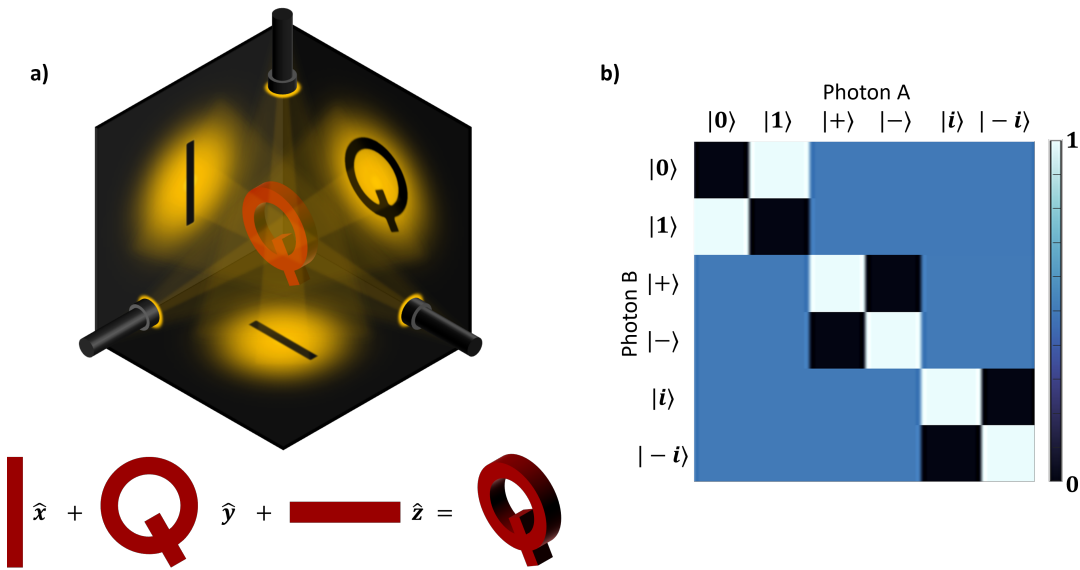


FIGURE 2.15: a) Conceptual idea of tomography with a 3D object. It is only necessary to perform 3 projections (shown as shadows cast by a torch on a wall for the 3D object) to obtain all of the information that would be necessary to reconstruct the object. In this way, we reconstruct the object merely by taking "snapshots" of its various features. b) For a 2D biphoton entangled state, we require 36 projective measurements to make up the full QST and which allows us to reconstruct the state without any ambiguity. The 36 measurements are made up of 6 spatially disjoint measurements made on each photon, where the photons are projected onto the computational basis states $\{|0\rangle, |1\rangle\}$ and superpositions $\{|+\rangle, |-\rangle, |i\rangle, |-i\rangle\}$ where the superpositions have the form $|j\rangle = |0\rangle + j|1\rangle$.

Since we are interested in hybrid OAM-polarization states, we will discuss performing QSTs for 2D entangled biphoton states. To reconstruct the biphoton hybrid state, we perform the tomography in the LG modal space in order to obtain the density matrix, ρ , of the system. An example of a generated QST and associated density matrix is shown in Fig. 2.16, derived from data captured for the state $|\Psi\rangle = |3\rangle_A |H\rangle_B + |0\rangle_A |V\rangle_B$. To achieve this, we performed spatially separated projective measurements, $M_{ij} = P_i^A \otimes P_j^B$, on each photon, where P_i^A and P_j^B are local projections of photon A and photon B in their independent spatial and polarization DoF, respectively.

In the computational basis, $\{|0\rangle, |1\rangle\}$, each of the photons are projected onto the eigenstates $|0\rangle$ and $|1\rangle$ as well as superposition states that form the mutually unbiased basis, $\{\frac{1}{\sqrt{2}}(|0\rangle \pm \exp(i\theta_{A(B)})|1\rangle)\}$, $\theta = (0, \pi/2)$, with respect to the standard basis. For photon A, this corresponds to the OAM eigenstates, $|\ell_{1,2}\rangle_A$ and the mutually unbiased superpositions $|\theta\rangle = \frac{1}{\sqrt{2}}(|\ell_1\rangle_A + \exp(i\theta_A)|\ell_2\rangle_A)$ which constitute our spatial basis.

As for photon B, we have the linear horizontal ($|H\rangle_B$) and vertical ($|V\rangle_B$) polarization states as our standard basis and the superposition states comprised of the diagonal ($|D\rangle_B$), anti-diagonal ($|A\rangle_B$), right circular ($|R\rangle_B$) and left circular ($|L\rangle_B$) polarization states. We performed these projections digitally using SLMs and linear optical elements only.

In Fig. 2.16 a), the detection probabilities for each possible projective measurement are shown with varying eigenstate superpositions for photon A and B shown across the rows and down the columns, respectively. The entire set of data is gathered digitally by varying the displayed holograms (see Fig. 2.16 b)) on an SLM and detecting the number of counts measured in coincidence for each configuration. A characteristic of a maximally entangled state is a set of 3 diagonal 2x2 matrices clearly evident in Fig. 2.15 b) as these correspond to projections on orthogonal states, thus we expect detection probabilities of either 0 or 1. This behaviour is also present in the QST taken of the experimental state in Fig. 2.16. The projections performed on the entangled state are shown in Fig. 2.16 b). The detection probabilities were used to determine the density matrix (see Fig. 2.16 c) via least squares fitting. In our model, we assumed that the density matrix follows the decomposition [155]

$$\rho = \frac{1}{4}(\mathbb{I}_4 + \sum_{m,n=1}^3 b_{mn}\sigma_{A,m} \otimes \sigma_{B,n}), \quad (2.37)$$

where $\mathbb{I}_4$ is the four dimensional identity matrix, b_{mn} are the coefficients that determine the state and $\sigma_{A,m}$ and $\sigma_{B,n}$ are the Pauli matrices that span the two-dimensional hybrid space for the spatial and polarization DoF of photon A and B, respectively. Here, the least squares fitting was used to minimise the relative errors between the measured and detection probabilities, given by

$$\chi = \sum_{mn} |p_{mn}(\mathbf{b}) - p_{mn}^M|^2, \quad (2.38)$$

where $p_{mn}(\mathbf{b})$ are the detection probabilities for a given density matrix that is determined by the coefficients b_{mn} which are elements of the matrix $\mathbf{b}$ while p_{mn}^M are the probabilities that were measured in the experiment.

Upon obtaining the density matrix ρ we can express it in our OAM and polarization basis

$$\rho = \sum_{pqst} \tau_{pqst} |\ell_p\rangle_A \langle \ell_q|_A \otimes |e_s\rangle_B \langle e_t|_B, \quad (2.39)$$

where τ_{pqst} are the coefficients that determine the state and the coefficients of b_{mn} in the bloch decomposition given in Eq. 2.37 can now be related to physical observables using Eq. 2.39 according to $b_{mn} = \sum_{pq} \langle \ell_p|_A \langle e_q|_B \sigma_{A,m} \otimes \sigma_{B,n} \rho |\ell_p\rangle_A |e_q\rangle_B$. The reconstructed density matrix shown in Fig. 2.16 empirically exhibits features of a maximally entangled hybrid state, as a separable state would have a singular peak

[215]. However, we can compute more concrete analytical properties from the reconstructed density matrix such as its fidelity, purity and concurrence (degree of entanglement).

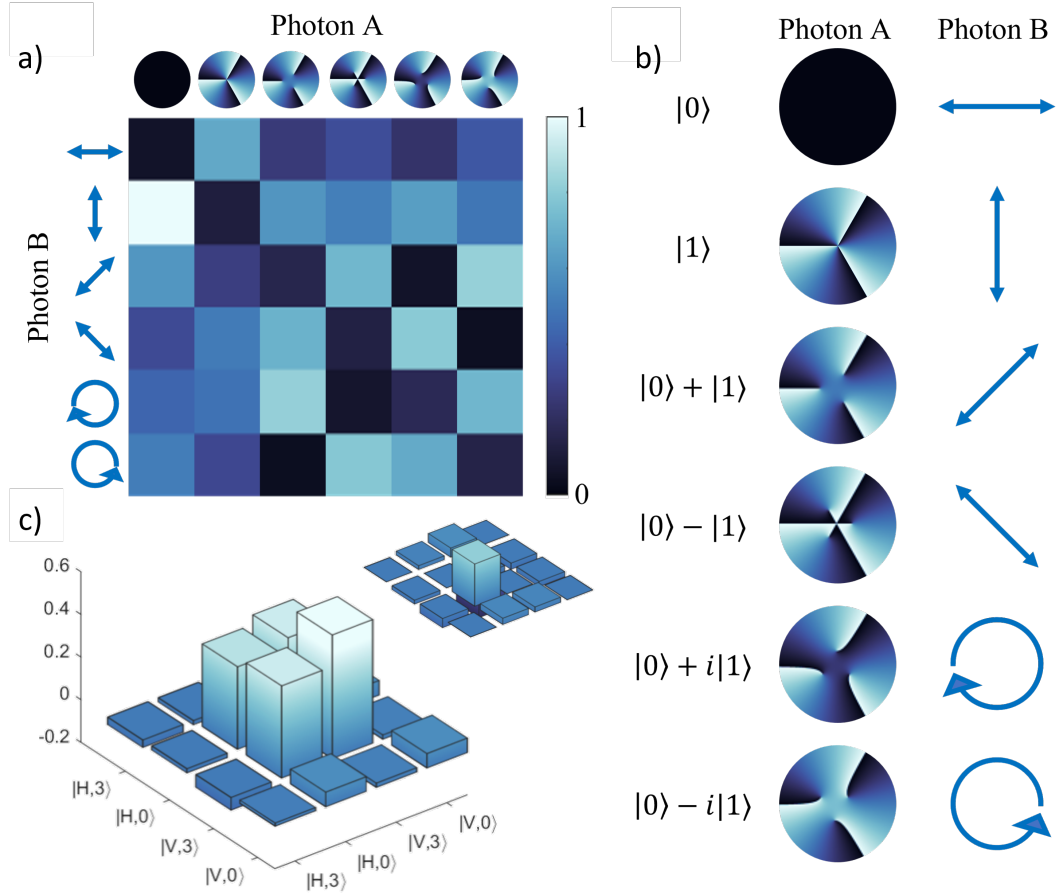


FIGURE 2.16: (a) Experimental example of QST with (b) spatial and polarization projections done on photon A and B, respectively, and (c) the reconstructed density matrix for the experimentally generated state, $|\Psi\rangle = |3\rangle_A |H\rangle_B + |0\rangle_A |V\rangle_B$. (a) Measured probabilities are shown for spatial (across each row) and polarization (down each column) projections on photon A and B, respectively. Top left 2x2 block shows approximate state where the probability Photon A is in the state $|3\rangle$ given that Photon B is measured to be in the state $|H\rangle$ ($|V\rangle$) is almost 100%(0%), whereas the probability of photon A being in the state $|0\rangle$ is 0%(100%), which indicates that Ψ is indeed at least an approximation of our final state. The remaining information provided by the QST gives insight into the purity of the generated state. This tomography is then used to generate (c) the real and imaginary parts (inset) of the density matrix which can then be mapped uniquely (up to some global phase) to the generated biphoton state. From the generated density matrix, one can see some small non-zero contributions appearing outside the central 2x2 matrix which contribute to impurity in the measured state, however this particular density matrix was identified to have a high fidelity of 91.61% when compared to the density matrix of the intended, ideal pure state. (c) The projective measurement states for photon A and B written in terms of the computational basis states and superpositions of said states.

Concurrence, fidelity and purity

To quantify the quality of the generated mode we will use the concurrence, fidelity and purity as our figures of merit.

The concurrence is used to measure the degree of entanglement between photons. For a 2D system it is measured by computing [104]

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \quad (2.40)$$

where λ_i are eigenvalues of the operator $R = \text{Tr}(\sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}})$ in descending order and $\tilde{\rho} = \sigma_y \otimes \sigma_y \rho^* \sigma_y \otimes \sigma_y$. The concurrence ranges from 0 for separable states to 1 for maximally entangled states. For the results shown in Fig. 2.16, we obtain $C(\rho) = 0.817$, indicating a high degree of entanglement.

The fidelity is calculated using [155]

$$F(\rho) = \left(\text{Tr} \left(\sqrt{\sqrt{\rho_T} \rho_M \sqrt{\rho_T}} \right) \right)^2, \quad (2.41)$$

where ρ_T is the target density matrix while ρ_M is the measured density matrix. The fidelity is 0 if the states are not identical or 1 when they are identical up to a global phase. For the results shown in Fig. 2.16, we obtain $F(\rho) = 0.9161$, when the state is compared to the desired output state.

To quantify the degree of purity of a state we compute [155]

$$\gamma(\rho) = \text{Tr}(\rho^2). \quad (2.42)$$

A purity of 0 indicates that the states are maximally mixed while a purity of 1 indicates a pure state. For the results shown in Fig. 2.16, we obtain $\gamma(\rho) = 0.805$, indicating a high degree of purity, well above the value of 0.5 indicating a maximally mixed state.

2.2.4 Quantum stokes

Motivated by the classical analysis of vector beams in the previous section, the aim here is to derive the nonlocal analogue to the Stokes parameters $\vec{S}(\vec{r}) = \langle S_x(\vec{r}), S_y(\vec{r}), S_z(\vec{r}) \rangle$. Traditionally, the spatially resolved Stokes parameters of a vectorial field are measured from the Pauli matrices in the polarization degree of freedom, i.e. $S_j(\vec{r}) = \langle \sigma_j \rangle(\vec{r})$ where $j \in \{x, y, z\}$ and σ_j are the usual 2×2 pauli matrices.

Here, we achieve this by extracting them from the reconstructed density matrix of the two photon state. We can therefore compute the Stokes parameters as

$$S_j = \langle |\vec{r}\rangle_A \langle \vec{r}|_A \otimes \sigma_{B,j} \rangle = \text{Tr}(|\vec{r}\rangle_A \langle \vec{r}|_A \otimes \sigma_{B,j} \rho), \quad (2.43)$$

where $\text{Tr}(\cdot)$ is the trace operator. Since the general decomposition of the density matrix is given as

$$\rho = \sum_{pqst=1}^2 \tau_{pqst} |\ell_p\rangle_A \langle \ell_q|_A \otimes |e_s\rangle_B \langle e_t|_B, \quad (2.44)$$

where τ_{pqst} are coefficients and $|\ell_{p(q)}\rangle_A$ and $|e_{s(t)}\rangle_B$ are the OAM and polarization basis states of photon A and B, respectively. It follows that we can now express

non-local stokes parameters as

$$S_j = \sum_{pqst=1}^2 \tau_{pqst} \text{Tr}(|\bar{r}\rangle_A \langle \bar{r}|_A |\ell_p\rangle_A \langle \ell_q|_A) \text{Tr}(\sigma_{B,j} |e_s\rangle_B \langle e_t|_B). \quad (2.45)$$

Next, we apply the trace of photon A in the position basis, $\{|\bar{r}\rangle_A \mid \bar{r} \in \mathcal{R}_A^2\}$, satisfying the orthogonality ($\langle \bar{r}_1 | \bar{r}_2 \rangle = \delta(\bar{r}_1 - \bar{r}_2)$) and the completeness relation ($\int |\bar{r}\rangle_A \langle \bar{r}|_A d^2r = \mathbb{I}_A$ with $\mathbb{I}_A$ being the identity matrix). By noting that the OAM eigenmodes can be projected onto the position basis, $\langle \bar{r} | \ell | \bar{r} \rangle = \text{LG}_\ell(\bar{r})$, we can perform the trace operation for photon A in Eq. (2.45) resulting in

$$S_j(\bar{r}) = \sum_{pqst=1}^2 \text{LG}_{\ell_p}(\bar{r}) \text{LG}_{\ell_q}^*(\bar{r}) \text{Tr}(\sigma_{B,j} |e_s\rangle_B \langle e_t|_B). \quad (2.46)$$

Furthermore the factor $\text{Tr}(\sigma_{B,j} |e_s\rangle_A \langle e_t|_A)$ can be written as two terms following the spectral decomposition of the Pauli matrices, i.e. $\sigma_{B,j} = \lambda_j^+ P_j^+ + \lambda_j^- P_j^-$ where $P_j^\pm = |\lambda_j^\pm\rangle \langle \lambda_j^\pm|$ for positive and negative eigenvalues $\lambda^\pm = \pm 1$, therefore

$$\begin{aligned} S_j(\bar{r}) &= \lambda_j^+ \sum_{pqst=1}^2 \tau_{pqst} \text{LG}_{\ell_p}(\bar{r}) \text{LG}_{\ell_q}^*(\bar{r}) \langle e_s|_B P_j^+ |e_t\rangle_B \\ &+ \lambda_j^- \sum_{pqst=1}^2 \tau_{pqst} \text{LG}_{\ell_p}(\bar{r}) \text{LG}_{\ell_q}^*(\bar{r}) \langle e_s|_B P_j^- |e_t\rangle_B. \end{aligned} \quad (2.47)$$

The expression can be further simplified by contracting the indices s and t , that iterate over the polarization eigenstates.

$$\begin{aligned} S_j(\bar{r}) &= \lambda_j^+ \sum_{pq} \tilde{\tau}_{pq}^{j+} \text{LG}_{\ell_p}(\bar{r}) \text{LG}_{\ell_q}^*(\bar{r}) \\ &+ \lambda_j^- \sum_{pq} \tilde{\tau}_{pq}^{j-} \text{LG}_{\ell_p}(\bar{r}) \text{LG}_{\ell_q}^*(\bar{r}), \\ &= \lambda_j^+ I_j^+(\bar{r}) + \lambda_j^- I_j^-(\bar{r}), \end{aligned} \quad (2.48)$$

Here the new coefficient, $\tilde{\tau}_{pq}^{j\pm}$, is obtained from contracting the indices, st , since the overlap probability $\langle e_s|_B P_j^\pm |e_t\rangle_B$ is a finite number.

In Fig. 2.17 the quantum stokes parameters for the state $|\Psi\rangle = |3\rangle_A |H\rangle_B + |0\rangle_A |V\rangle_B$ are shown. It is important to emphasize, that whilst these parameters seem practically identical to the classical stokes parameters, the underlying physics is completely different. Whilst classical stokes reveal a local vectorial polarization structure embedded in a classical beam, quantum stokes reveal the joint vectorial nature of a particular biphoton state, that is, if measured in coincidence, one would extract position information, (x, y) from photon A, and a polarization information, (s_0, s_1, s_2, s_3) from photon B, thus allowing for the reconstruction of the joint vectorial wavefunction.

Each quantum stokes parameter in Fig. 2.17 reveals certain features of the experimentally generated hybrid entangled state. S_0 reveals the probability density of photon A. The experimentally obtained S_0 is in agreement with the expected result

$|LG^3|^2 + |LG^0|^2$. S_1 reveals the regions where a measurement on photon A's position yields a likely outcome of photon B collapsing into the state $|H\rangle$ (dark red) or $|V\rangle$ (white), assuming that a projection measurement in the basis $\{|H\rangle, |V\rangle\}$ is performed on photon B. Likewise S_2 and S_3 reveal similar features except for the bases $\{|D\rangle, |A\rangle\}$ and $\{|R\rangle, |L\rangle\}$, respectively. Furthermore, unlike S_2 and S_3 , S_1 is independent of the azimuthal coordinate which is expected since theoretically $S_1 = |LG^0|^2 - |LG^3|^2$. Lastly, S_2 and S_3 exhibit a cyclic feature with changing azimuthal coordinate. In particular one observes the cyclic change of the parameters values between 1 and -1 three times within a complete revolution. This behaviour is expected since theoretically one can verify that $S_2 = |LG^0 + LG^3|^2$ and $S_3 = |LG^0 + iLG^3|^2$, i.e., S_2 and S_3 are produced from superpositions of LG^0 and LG^3 which locally exhibit a change in relative phase with azimuthal coordinate.

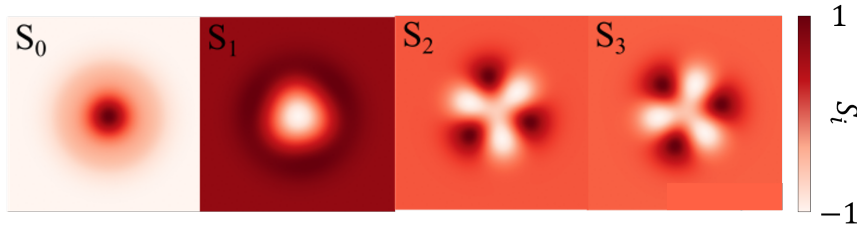


FIGURE 2.17: Experimental quantum Stokes parameters for the state $|\Psi\rangle = |3\rangle_A |H\rangle_B + |0\rangle_A |V\rangle_B$.

2.2.5 Bell measurements

It is common practice in most quantum entanglement experiments to demonstrate a violation of the Clauser-Horne-Shimony-Holt (CHSH) Bell inequality [216]. The violation of the inequality is a decisive indicator of entanglement or non-separability within the system. In principle, this measurement can be done on any quantum state regardless of the DOFs present as the projective measurements done on each photon are independent of each other. Furthermore, the measurement procedure is almost identical to that of QST, i.e., different projective measurements are made on the biphoton system and the probabilities are then of interest, the only difference being the chosen set of projective measurements.

We consider the general form of the hybrid entangled state discussed up until now, $|\Psi\rangle = |\ell_1\rangle_A |H\rangle_B + |\ell_2\rangle_A |V\rangle_B$. Photons A and B are projected onto superposition states of OAM and polarization, respectively, given by

$$|\theta_A\rangle = \frac{1}{\sqrt{2}} \left(|\ell\rangle + e^{i\ell\theta_A} |-\ell\rangle \right), \quad (2.49)$$

and

$$|\theta_B\rangle = \frac{1}{\sqrt{2}} \left(|H\rangle + e^{i\theta_B} |V\rangle \right), \quad (2.50)$$

where $\theta_{A,B}$ control the relative phases between the states in the superposition. Detection probabilities are shown in Fig. 2.18 for the states a) $|\Psi_1\rangle = |1\rangle_A |H\rangle_B + |-1\rangle_A |V\rangle_B$ and b) $|\Psi_2\rangle = |2\rangle_A |H\rangle_B + |-2\rangle_A |V\rangle_B$. In each plot the coincidence measurements between photon A and B are shown for $\theta_A = \{0, \frac{\pi}{2\ell}, \frac{\pi}{\ell}, \frac{3\pi}{2\ell}\}$, mapping the projections of photon A, while for photon B is varied between 0 and 2π , where the probabilities are proportional to $\cos^2(\ell(\theta_A - \theta_B))$. Subsequently, the CHSH-Bell

parameter S can be computed from [217]

$$S = |E(\theta_A, \theta_B) - E(\theta_A, \theta'_B) + E(\theta'_A, \theta_B) + E(\theta'_A, \theta'_B)|, \quad (2.51)$$

with $E(\theta_A, \theta_B)$ calculated from the coincidence events,

$$\begin{aligned} E(\theta_A, \theta_B) &= \frac{\zeta(\theta_A, \theta_B) - \zeta'(\theta_A, \theta_B)}{\zeta(\theta_A, \theta_B) + \zeta'(\theta_A, \theta_B)}, \\ \zeta(\theta_A, \theta_B) &= C(\theta_A, \theta_B) + C\left(\theta_A + \frac{\pi}{2}, \theta_B + \frac{\pi}{2}\right), \\ \zeta'(\theta_A, \theta_B) &= C\left(\theta_A + \frac{\pi}{2}, \theta_B\right) + C\left(\theta_A, \theta_B + \frac{\pi}{2}\right). \end{aligned} \quad (2.52)$$

Here $C(\theta_A, \theta_B)$ represents measured coincidence counts. The Bell parameter can be characterized as $S \leq 2$ for separable states and $2 < S \leq 2\sqrt{2}$ for entangled states. The upper bound of the Bell parameter, $S = 2\sqrt{2}$ is called the Tsirelson bound [218], which is achieved only for maximally entangled Bell states. For the two measurements in Fig. 2.18 a) and Fig. 2.18 b) we obtained Bell parameters of $S_1 = 2.51 \pm 0.07$ and $S_2 = 2.55 \pm 0.08$ violating the Bell inequality and confirming entanglement in the chosen subspaces of our generated SPDC photons.

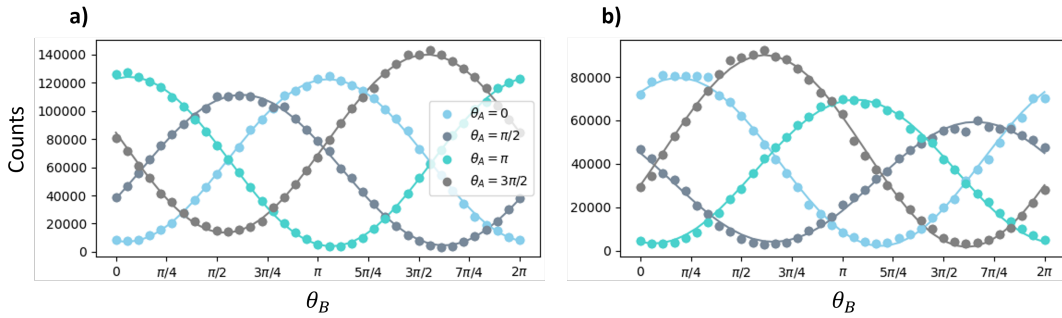


FIGURE 2.18: Experimental Bell measurement curves for hybrid entangled states a) $|\Psi_1\rangle = |1\rangle_A |H\rangle_B + |-1\rangle_A |V\rangle_B$ and b) $|\Psi_2\rangle = |2\rangle_A |H\rangle_B + |-2\rangle_A |V\rangle_B$.

2.3 Conclusion

In this section, we explored the application of classical structured light techniques to manipulate spatial and polarization DOFs. We emphasized the utilization of advanced technologies, such as SLMs and geometric optics, to encode spatial information onto beams. Additionally, we delved into phase and complex amplitude modulation techniques.

Our discussion extended to the essential polarization optics required for achieving comprehensive control over the polarization DOF. Subsequently, we combined these DOFs to create arbitrary VBs, thoroughly examining their characterization.

Transitioning into the quantum domain, we investigated the generation and characterization of entangled states through SPDC. Notably, we highlighted the tunability of the dimensionality of these states. Subsequently, we explored an experiment capable of realizing arbitrary hybrid entangled states, followed by an in-depth analysis of all the necessary tools for their characterization.

Chapter 3

Synthetic spin dynamics of Bessel Skyrmions

The work herein has been published in Ref. [219].

Skyrmions are topologically stable vector-field configurations which were initially proposed to explain strong force phenomena [117–120]. Observations of these structures in magnetic electron-spin systems have received increasing interest due to their superior stability, making them enticing candidates for spintronic information transfer [123–127, 220]. In parallel, optical realizations of skyrmion quasi-particles have garnered much interest in recent years as non-trivial topologically structured light in both classical and quantum regimes [142, 221–224], and have found applications in areas such as optical communication [225, 226] and particle trapping [227–229], with new advancements being made in tuning topological degrees of freedom [56, 69, 230, 231]. Observations of skyrmion-like structures in the optical-spin of evanescent waves [232] and focused propagating waves carrying orbital angular momentum revealed that the structures can exist at sub-wavelength scales [54]. In condensed matter systems skyrmions have been observed in plasmonic lattices and exciton-polaritons [233, 234]. It has been shown that full Poincaré beams [235, 236] contain transverse Stokes vector (i.e., psuedospin) distributions which satisfy the topological conditions for 2-dimensional skyrmions [53, 237, 238], a distribution formed by the map from the infinite plane, $\mathbb{R}^2 \cup \{\infty\}$, to the sphere, S^2 , and are consequently referred to as optical skyrmions [57, 134]. These optical skyrmions are commonly constructed using orthogonally polarized Laguerre-Gaussian (LG_p^ℓ) beams (having asymmetric azimuthal indices ℓ , with unequal absolute values). A mechanism for full parametric tuning between skyrmion types such as Néel, Bloch, and anti, which have respective diverging, spiraling and hyperbolic textures, has also been demonstrated [69]. Tracking the trajectories of individual psuedospin states with propagation also revealed mapping to the 4-dimensional hypersphere, which the full Skyrme field maps to, through a Hopf fibration [139, 239, 240].

Here, we demonstrate how Bessel-Gaussian (BG^ℓ) beams can be used to engineer optical skyrmions. This is achieved by superimposing two orthogonally polarized BG^ℓ beams of different order l , which ensures that all possible relative phases and amplitudes between the beams occur along their transverse profile. The relative phases are imposed along the azimuthal coordinate, φ , by the differing helical phases $e^{i\ell\varphi}$ while the relative amplitudes are achieved in the radial direction by the differing distributions of Bessel functions $J^\ell()$ having different orders. The resulting distribution of Stokes (i.e., psuedospin) vectors have global topologies which are conveniently confined to a finite on-axis region. We then reveal how the engineered difference between continuously variable radial k_r and axial k_z wavevector components can be

used to control the periodic precessional beating of local Stokes vectors. This linear beating with propagation is the result of the Gouy phase difference between the two beams, which is proportional to the difference in k_z . The global effect of the local precession of pseudospins is a change in skyrmion texture with propagation, while preserving the Skyrme number. We then show analytically and by numerical simulation that superpositions of BG^ℓ beams forming optical frozen-waves can be used to engineer pseudospin vectors which, in addition to precession about the polar axis of the Poincaré sphere, also trace a trajectory between the poles. This level of control over propagation dynamic pseudospin trajectory forms a powerful analogy to temporal dynamics of magnetic spin systems in the presence of effective magnetic fields, which can be understood using the Maxwell-Schrödinger formalism. A strong similarity can be drawn to temporal dynamics of pseudospin states in polariton condensates, where the physical mechanism inducing the precession is transverse-electric and transverse-magnetic mode splitting which is also modelled using effective magnetic fields [241]. Another phenomenon, also found in polariton condensates, which shares similarities with our free-space work is the Rabi trajectories of quantum state vectors on the Bloch-Sphere [242]. This highlights how the analogy can extend beyond spin/polarization states to spatial/Bloch states. This connection between the optical and solid-state regimes allows for the powerful free-space optical toolkit to be applied in understanding dynamics of more complex systems, treating the optical skyrmions as quasi-particles with stability imbued courtesy of their topological soliton nature.

3.1 Theory

Consider the superposition of two BG^ℓ beams in the orthogonal right-circular (left-circular) $R(L)$ polarization basis

$$|\Psi\rangle = BG^{\ell_R}(r, \varphi, z, k_z^R)|R\rangle + BG^{\ell_L}(r, \varphi, z, k_z^L)|L\rangle, \quad (3.1)$$

$$BG^\ell(r, \varphi, z, k_z) = e^{-\frac{r^2}{w_0^2}} J^\ell(rk_r) e^{i\ell\varphi} e^{ik_z z}, \quad (3.2)$$

where r, φ and z are the respective radial, azimuthal and axial cylindrical coordinates and $J^\ell()$ denotes the ℓ^{th} order Bessel function of the first kind. $|R\rangle$ and $|L\rangle$ represent the appropriate polarization Jones vectors, k_z and $k_r = \sqrt{k^2 - k_z^2}$, are the respective axial and radial wavevector components. We use l to label the topological charge and w_0 is the width of the Gaussian amplitude envelope which will be subsequently omitted for brevity. Now let us recall that the Skyrme field components Σ_i are given by [57]

$$\Sigma_i(x, y, z) = \epsilon_{ijk} \epsilon_{pqr} S_p \partial_j S_q \partial_k S_r, \quad (3.3)$$

where $(i, j, k) = (x, y, z)$ and $(p, q, r) = (1, 2, 3)$, $S_p(x, y, z)$ are the (locally normalized) Stokes parameters (where we have omitted the spatial dependence for brevity) and $\epsilon_{ijk(pqr)}$ are the Levi-Cevita permutation operators. If we confine ourselves to a single transverse plane (e.g., $z = 0$), we can calculate the field's Skyrme number, $\mathcal{N}$, as

$$\mathcal{N} = \frac{1}{4\pi} \int_0^\infty \int_0^{2\pi} \Sigma_z(r \cos \varphi, r \sin \varphi) d\varphi dr, \quad (3.4)$$

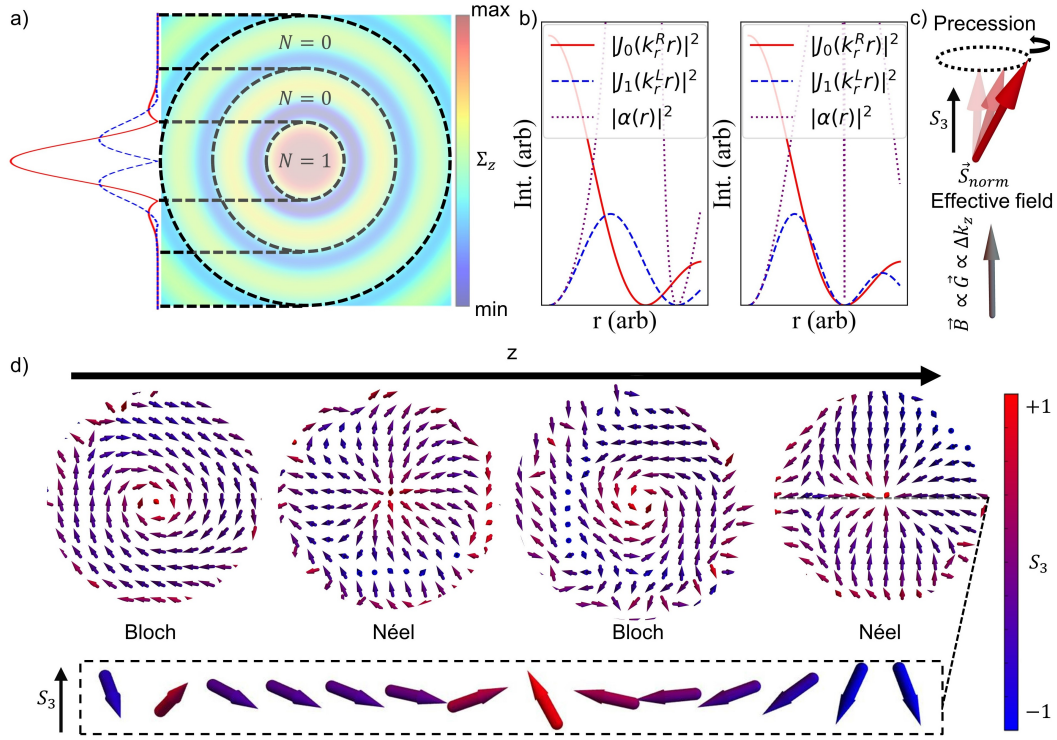


FIGURE 3.1: a) Left: Overlaid theoretical intensity plot of constituent orthogonally polarized Bessel modes as shown by the dashed rings. Right: Skyrme field with Skyrme number $\mathcal{N}$ associated to the concentric annular regions placed as insets. b) Theoretical plots of the relative intensity of the constituent BG^ℓ components ($\ell_R = 0, \ell_L = 1$) for valid ($k_L^L = 1.2k_R^R$) and invalid ($k_L^L = 1.6k_R^R$) choice of radial wavevectors, where $|\alpha(r)|^2 = |J^{\ell_L}(k_L^L r)|^{2q} |J^{\ell_R}(k_R^R r)|^{-2q}$ with q defined in the text. c) Conceptual illustration of local Stokes vector precession about the S_3 axis due to the effective inhomogeneity $\vec{G}$ caused by the Gouy-phase difference between orthogonally polarized constituents. d) Experimentally measured vector plots of BG^ℓ optical skyrmions beating between continuously deformable Bloch- and Néel-types, as well as a section indicating the radial variation of S_3 .

where the Σ_z component of the Skyrme field is given by

$$\begin{aligned} \Sigma_z(r \cos \varphi, r \sin \varphi) = \Sigma_z(x, y) = & S_1 \partial_x S_2 \partial_y S_3 + S_2 \partial_x S_3 \partial_y S_1 + S_3 \partial_x S_1 \partial_y S_2 \\ & - (S_1 \partial_x S_3 \partial_y S_2 + S_2 \partial_x S_1 \partial_y S_3 + S_3 \partial_x S_2 \partial_y S_1). \end{aligned} \quad (3.5)$$

Note that we choose to define Σ_z in Cartesian coordinates (as this correlates better with experimentally acquired CCD images), while the integral is performed in polar coordinates to exploit the cylindrical symmetry of the fields. It can be shown that by exploiting the cylindrical symmetry of the Bessel-Gaussian modes, the above calculation yields the following simplified integral [69, 243]

$$\mathcal{N} = q \Delta \ell \int_0^\infty g(\alpha) dr \quad (3.6)$$

where $g(\alpha) = \frac{\partial \alpha^2}{\partial r} \frac{1}{(1+\alpha^2)^2}$, $\alpha(r) = |J^{\ell_L}(rk_r)|^q |J^{\ell_R}(rk_r)|^{-q}$, $\Delta \ell = \ell_R - \ell_L$ with $|\ell_R| \neq |\ell_L|$ and $q = \frac{|\ell_L| - |\ell_R|}{||\ell_L| - |\ell_R||}$. It is clear then for the roots of the lower order Bessel function, we have that $\alpha \rightarrow \infty$. This then means that α is only continuous in the domains

$(0, r_0) \bigcup_{i=0}^{\infty} (r_i, r_{i+1})$, where $r_i \in \mathbb{R}$ are the roots of the lower order Bessel function. As such, to evaluate Eqn. 3.6 such that we obtain an integer Skyrme number, we choose to break up the integral as follows

$$\mathcal{N} = q\Delta\ell \left[\int_0^{r_0} g(\alpha)dr + \sum_{i=0}^{\infty} \int_{r_i}^{r_{i+1}} g(\alpha)dr \right], \quad (3.7)$$

which can then be evaluated to give

$$\mathcal{N} = q\Delta\ell \left[\left(\frac{1}{1+\alpha(0)^2} - \frac{1}{1+\alpha(r_0)^2} \right) + \sum_{i=0}^{\infty} \left(\frac{1}{1+\alpha(r_i)^2} - \frac{1}{1+\alpha(r_{i+1})^2} \right) \right]. \quad (3.8)$$

Clearly the second term in Eqn. 3.8 will always evaluate to zero unless the two Bessel functions share roots. However, this is not the case, as according to Bourgat's Hypothesis [244], Bessel functions of varying positive orders do not have any roots in common. And so we are left with

$$\mathcal{N} = q\Delta\ell \left(\frac{1}{1+\alpha(0)^2} - \frac{1}{1+\alpha(r_0)^2} \right), \quad (3.9)$$

which analytically indicates that the generated state given in Eqn. 3.1 has a skyrmionic topology in the localized region $(0, r_0)$. From the definition given to α , it is clear that the term within the brackets will always evaluate to an integer as $\alpha(r)$ can only tend to either 0 or ∞ at both $r = 0$ and $r = r_0$ which implies that our field is in fact skyrmionic. Furthermore, since the ratio given in the definition of α inverts depending on the sign of q , we have that the first term in the brackets always evaluates to 1 while the second always evaluates to 0, as such the Skyrme number in this central region is easily controlled by modifying q and $\Delta\ell$, which independently control the polarity [69] (affected by which component in Eqn. 3.1 dominates at $r = 0, r_0$.) and the order/vorticity [69] of the Skyrme number. A truncated segment of the Skyrme field, Σ_z , is shown in Fig. 3.1.a, where we can see that the central region's Skyrme number evaluates to 1, whereas the Skyrme number in concentric annulus regions around the origin evaluate to 0. By subdividing each of these outer annulus regions into regions where the function $\zeta = \alpha + \alpha^{-1}$ is continuous, we obtain Skyrme number contributions of $\mathcal{N}$ and $-\mathcal{N}$ which compensate for one another thereby yielding an effective skyrmion contribution of 0. This is evident in Fig. 3.1.a where each concentric annulus ring can clearly be divided into two annulus rings which contribute only positive or negative values to the Skyrme field, Σ_z . Physically this phenomenon can be seen as wrapping around the Poincaré sphere and then unwrapping again within the same region. This, along with the approximately non-diffracting nature of BG^ℓ beams, allows for a more practical implementation than the infinite bounds required by LG_p^ℓ skyrmions. We note that the above analysis has been done for $k_z^R = k_z^L$, however as long as for $k_z^R \neq k_z^L$, the first zeroes do not overlap, then Eqn. 3.9 will still evaluate to an integer, thereby forming a skyrmionic topology which wraps the Poincaré sphere $\mathcal{N}$ times. For practical purposes, we do require that the zeroes of the Bessel functions be sufficiently far apart.

In Fig. 3.1.b, we show theoretical plots of relative intensities of the orthogonally polarized components for valid and invalid choices of $k_r^{R(L)}$. An invalid choice is characterized by values of $k_r^{R(L)}$ which result in $J_{\ell_R(\ell_L)}(k_r^{R(L)}r)$ distributions with first

minima (excluding those at $r = 0$) which overlap. An overlap will result in a pseudospin field which does not map to the complete Poincaré sphere, and therefore does not constitute a skyrmion.

The Stokes parameters of our vector beam are calculated using Pauli matrices σ_p - and $\sigma_0 = \mathcal{I}$ as $S_p = \langle \Psi | \sigma_p | \Psi \rangle$, resulting in

$$S_0 = |J^{\ell_R}(rk_r^R)|^2 + |J^{\ell_L}(rk_r^L)|^2, \quad (3.10)$$

$$S_1 = 2J^{\ell_R}(rk_r^R)J^{\ell_L}(rk_r^L)\cos(\Delta l\varphi + \Delta k_z z), \quad (3.11)$$

$$S_2 = -2J^{\ell_R}(rk_r^R)J^{\ell_L}(rk_r^L)\sin(\Delta l\varphi + \Delta k_z z), \quad (3.12)$$

$$S_3 = |J^{\ell_R}(rk_r^R)|^2 - |J^{\ell_L}(rk_r^L)|^2, \quad (3.13)$$

where $\Delta\ell = \ell_R - \ell_L$ and $\Delta k_z = k_z^R - k_z^L$. From Eqns. 3.10 to 3.13 we can see that each local transverse Stokes vector [i.e., fixed $\vec{r}_T = (r, \varphi)$] will experience a periodic precession about the S_3 axis proportional to Δk_z as the vector field propagates. Important cases are where $r = 0$ and $r = r_0$ where we get $S_3 = \pm 1$, which is constant with z . It is also notable that due to the relationship $k_r = \sqrt{k^2 - k_z^2}$, either $\Delta k_r = k_r^R - k_r^L$ or Δk_z could be used to tune the precession rate. The linear Gouy phase of our BG^ℓ beams allow for true periodic beating between (continuously deformable) skyrmion types over a well defined distance, which differs from the asymptotic arctan behaviour of conventional free-space LG_p^ℓ optical skyrmion dynamics. It can be noted that in polariton condensates periodic trajectories of state/pseudospin vectors has been realized through various mechanisms [241, 242]. It should also be noted that Eqns. 3.10 and 3.13 experience symmetry under the interchange of z and ϕ which means that complete precession leads to skyrmion textures in planes orthogonal to the propagation direction. This can be seen by taking a plane at a fixed azimuthal angle and projecting it along r and z until the radial bound for the skyrmion and the axial bound of one complete pseudospin precession respectively. This results in a skyrmion texture in rectangular planes orthogonal to the beam axis for all azimuthal angles. To draw a comparison to anisotropic optical systems we can adopt the Maxwell-Schrödinger formalism, which describes pseudospin transformations of light propagating in anisotropic media and reveals the equation of motion [245]

$$\lambda \frac{\partial \vec{S}}{\partial z} = \vec{S} \times \vec{G}. \quad (3.14)$$

Here λ is the wavelength, $\vec{S} = S_0^{-1}(S_1, S_2, S_3)^T$ is the locally normalized Stokes vector and $\vec{G} = (2\alpha, 2\beta, 2\gamma)^T$ is the vector describing the anisotropic medium. The parameters α , β and γ specify the axes of linear birefringence, the retardance between this axes and the retardance between pure circular polarizations respectively. In the case we have been examining, we observe a pseudospin evolution in free-space when $\alpha = \beta = 0$ and $\gamma = \frac{1}{2}\lambda\Delta k_z$. A resemblance between Eqn. 3.14 and the temporal equation of motion of a magnetic spin system in a magnetic field is clear, where λ plays the role of Planck's constant and $\vec{G}$ resembles an applied magnetic field [245]. A schematic illustrating the Larmor-like precession of the normalized Stokes vector about the $\vec{G}$ axis is included in Fig. 3.1.c. In order to further highlight the value of the free-space optical analogy (beyond static homogeneous $\vec{G}$ fields), we can utilize so-called optical frozen waves (FWs) which are constructed using superpositions of BG^ℓ beams given by [246, 247]

$$\text{FW}(r, z, f(z), Q) = e^{iQz} \sum_{n=-N}^N A_n(f(z)) J^0(k_{r,n} r) e^{i \frac{2\pi}{L_{FW}} n z}, \quad (3.15)$$

$$A_n = \frac{1}{L_{FW}} \int_0^{L_{FW}} f(z) e^{-i \frac{2\pi}{L_{FW}} n z} dz. \quad (3.16)$$

Here Q is the 'central' k_z about which the superposition is constructed, $2N + 1$ is the number of BG^ℓ modes in the superposition and $f(z)$ is an arbitrary function which the central intensity of the FW will approximate (i.e., $|\text{FW}(0, z, f(z))|^2 \approx |f(z)|^2$). L_{FW} is the propagation distance over which the FW is defined. We can use these FWs as the constituent components of our vector field

$$|\Psi\rangle = \text{FW}(r, z, f^R(z), Q_R)|R\rangle + \text{FW}(r, z, f^L(z), Q_L)|L\rangle. \quad (3.17)$$

By choosing $Q_R \neq Q_L$ with an appropriate choice of $f^{R(L)}(z)$, we can have an on-axis (i.e., $r = 0$) Stokes vector which, in addition to the precession about S_3 , can trace paths between the poles of the Poincaré sphere. As an example we can consider $f^R(z) = \sqrt{z/L_{FW}}$ and $f^L(z) = \sqrt{1 - z/L_{FW}}$ resulting in Stokes parameters approximated by

$$S_0 \approx 1, \quad (3.18)$$

$$S_1 \approx 2 \sqrt{\frac{z}{L_{FW}}} \sqrt{1 - \frac{z}{L_{FW}}} \cos(\Delta k_z z), \quad (3.19)$$

$$S_2 \approx -2 \sqrt{\frac{z}{L_{FW}}} \sqrt{1 - \frac{z}{L_{FW}}} \sin(\Delta k_z z), \quad (3.20)$$

$$S_3 \approx \frac{2z}{L_{FW}} - 1. \quad (3.21)$$

We then model axially dependent $\vec{G}$, analogous to a time-varying $\vec{B}$ field, in free-space

$$\vec{G}(z) \approx \begin{pmatrix} \frac{\sin(\Delta k_z z)}{L_{FW} \sqrt{\frac{z}{L_{FW}}} \sqrt{1 - \frac{z}{L_{FW}}}} \\ \frac{\cos(\Delta k_z z)}{L_{FW} \sqrt{\frac{z}{L_{FW}}} \sqrt{1 - \frac{z}{L_{FW}}}} \\ \frac{(L_{FW} - 2\Delta k_z(z-1)z - 2z) \cot(\Delta k_z) + L_{FW} - 2z}{2z(L_{FW} - z)} \end{pmatrix}. \quad (3.22)$$

The case of FWs is distinct from the case of optical skyrmions as the control is over a single on-axis pseudospin vector as opposed to a transverse distribution of pseudospins, our implementation of FWs serves as a first step toward the use of spatially varying FWs which could exhibit changes of not only skyrmion textures but also Skyrme number during propagation [248, 249]. It should be noted that while FWs have been generated experimentally, the large range of k_z values required to accurately approximate $f(z)$ creates some requirements for large system apertures to acquire this. Due to these restrictions our demonstration of pseudospin control using FWs is based on simulations involving propagating the field using numerical angular spectrum propagation [250].

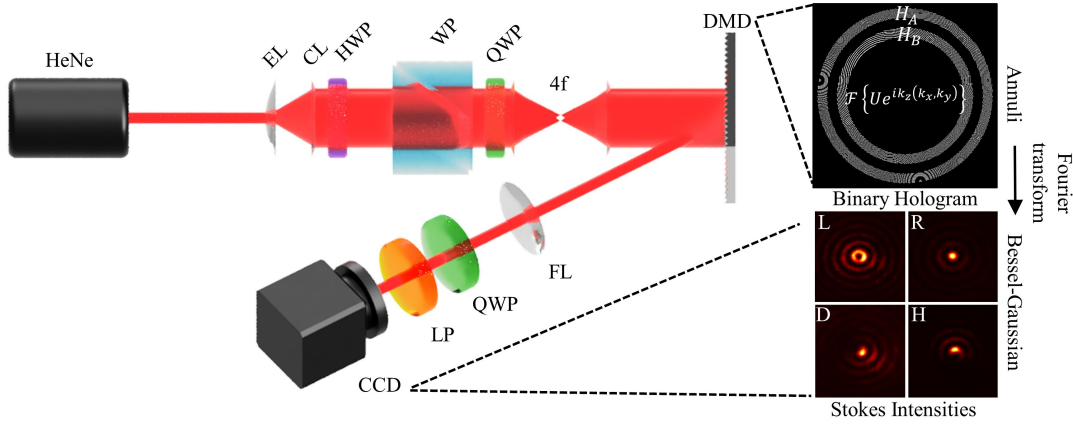


FIGURE 3.2: Diagram showing the experimental arrangement used to generate and analyse BG^l based optical skyrmions [WP - Wollaston prism, EL - expansion lens, CL - collimation lens, HWP - half-wave plate, QWP - quarter-wave plate, DMD - digital micro-mirror device, FL - Fourier lens, LP - linear polarizer]. The top right inset shows an example of binary amplitude holograms featuring individual rings for each of the two multiplexed holograms $H_{A(B)}$ (far-field). The bottom right inset shows examples of Stokes intensities measured by the CCD (near-field)

3.2 Experiment

To investigate the predicted behaviour of the vector fields described in Sec.3.1, we used the arrangement shown in Fig.3.2. A horizontally polarized Gaussian beam from a HeNe laser (operating at a wavelength of $\lambda = 632.5$ nm) was expanded and collimated using lenses EL ($f = 2$ mm) and CL ($f = 250$ mm) respectively, such that the central transverse region of the beam approximated a plane wave of constant amplitude. The expanded beam then had its polarization axis rotated by 45° with a half-wave plate (HWP) before it was passed through a Wollaston prism (WP) and quarter-wave plate (QWP) which separated the right- (R) and left-circularly (L) polarized components of the expanded beam by an angle of $\approx 1^\circ$ (in the horizontal $x - z$ plane). A $4f$ system was then used to image the plane at the WP onto the screen of a digital micro-mirror device (DMD, TI-DLP6500). To facilitate complex amplitude modulation of the incident light [251], we make use of two binary amplitude holograms of the form

$$H_{A(B)}(\vec{r}) = \frac{1}{2} - \frac{1}{2} \left(\cos \left(\pi \Phi(\vec{r}) + 2\pi \vec{g}^{A(B)} \cdot \vec{r} \right) + \cos(\pi A(\vec{r})) \right), \quad (3.23)$$

which were multiplexed and displayed by the DMD. We define $\vec{r} = (x, y)$ as the Cartesian pixel coordinates, $\Phi(\vec{r}) = \frac{\arg(T^{A(B)}(\vec{r}))}{\pi}$ and $A(\vec{r}) = \frac{\arcsin(|T^{A(B)}(\vec{r})|)}{\pi}$ are the re-normalized phases and amplitudes of the complex transmission functions $T^{A(B)}(\vec{r})$. With the 0^{th} diffraction orders of the R(L) components leaving the DMD screen at the same 1° separation angle, the spatial carrier frequencies $\vec{g}^{A(B)} = (g_x^{A(B)}, g_y^{A(B)})$ were chosen such that the 1^{st} diffraction order containing the complex field distributions $T^A(\vec{r})$ in the R polarization spatially overlapped (co-linearly) with the 1^{st} order of the $T^B(\vec{r})$ distribution in the L polarization [252]. It should be noted that the resulting gratings have spatially constant frequencies (for each hologram

$H_{A(B)}$), with local shifts in the fringes associated with the desired phase modulation [253]. Finally a Fourier lens (F: $f = 0.2$ m) was used to project the far-field intensity onto a CCD camera. The transmission functions were set according to $T^{A(B)}(\vec{r}) = \mathcal{F}\{\langle R(L)|\Psi\rangle_{z=0}\}e^{iq_z(q_x, q_y)z}$, where $\mathcal{F}\{\}$ represents the Fourier transform, $q_z(q_x, q_y)$ is the spatially varying transverse angular spectrum used to digitally propagate the field captured at the CCD by the distance z [254]. A QWP and linear polarizer (LP) were used to project Stokes intensities of the generated fields onto the CCD [63]. It is notable that the digital propagation scheme should result in similar results to those which could be obtained by physically translating the CCD along z . The digital scheme was preferable as it allowed for automated measurements and for the Stokes intensities to be more accurately aligned since there was no risk of the CCD shifting.

3.3 Results and Discussion

Using the arrangement in Sec.3.2, we generated vector fields of the form in Eqn. 3.1. We set the parameters $\ell_R = 0$, $\ell_L = [1, -2, 3]$ and $k_z^{R(L)} = 0.999995(0.999986)k$. Using Stokes intensity measurements of the R , L , D (diagonal) and H (horizontal) polarization projections, we extracted the transverse Stokes vector distribution at 26 evenly spaced planes over the digitally propagated range $z \in [-0.9, 0.9]$ m. In Fig. 3.1.d we show the experimentally measured Stokes vector fields of the case with $\ell_L = 1$ at various planes where the transition between Bloch- and Néel-type skyrmions, characterized by the respective ‘hedgehog’ and ‘spiral’ textures, is clear. Inspection of the local behaviour of individual vectors reveals the precessional dynamics akin to that of anisotropic and magnetic systems. In Fig. 3.3.a we show how the topological index of $\mathcal{N}$ is preserved over the propagation range, here $\mathcal{N}$ was calculated using Eqn. 3.4 where the transverse gradients were determined using a numerical second-order finite difference approximation, while the error-bands were calculated by propagating the observed 2 % shot noise through Eqn. 3.4. Notwithstanding the visible fluctuations in the approximated $\mathcal{N}$, the invariance of the distinct values with propagation is evident. The cause of the excessive fluctuations in the $\ell_L = 1$ case is due to the effect of rapidly changing relative amplitudes for lower order skyrmions, which induce errors in the computation method used to calculate gradients. The invariance of the Skyrme number highlights the stability of the structures to the effect of the $\vec{G}$ fields in the Maxwell-Schrödinger formalism. Plots of the locally normalized Stokes vector fields (sampled at every 3 pixels of the measured data) for various $\Delta\ell$ are shown in Fig. 3.3.b where the azimuthally varying vector orientations give the characteristic textures associated with each Skyrme number. In Fig. 3.3.c we show plots of the paths, on the Poincaré sphere, traced by sample Stokes vectors (at the positions indicated in Fig. 3.3.b). The predicted complete rotation of the local vectors about the S_3 axis over a finite distance is clear, it is notable that to observe a complete rotation utilizing LG_p^ℓ modes one would need to observe the field propagation to infinity. In Fig. 3.4.a we show the numerically simulated spiraling path traced out by the central Stokes vector of a field taking the form in Eqn. 3.17, modeling the dynamics of an electron spin/Stokes vector and a magnetic field/anisotropic material of the form in Eqn. 3.22. The dynamics were simulated over a 200 mm distance with $Q_R(L) = (0.99997)0.99985$. We can note that the path deviates from an ideal spiral at the poles, this is due to the approximate nature of the FW which is highlighted when comparing the simulated relative phase and amplitude to the ideal case shown in Fig. 3.4.b. This system emulates a dynamic applied magnetic

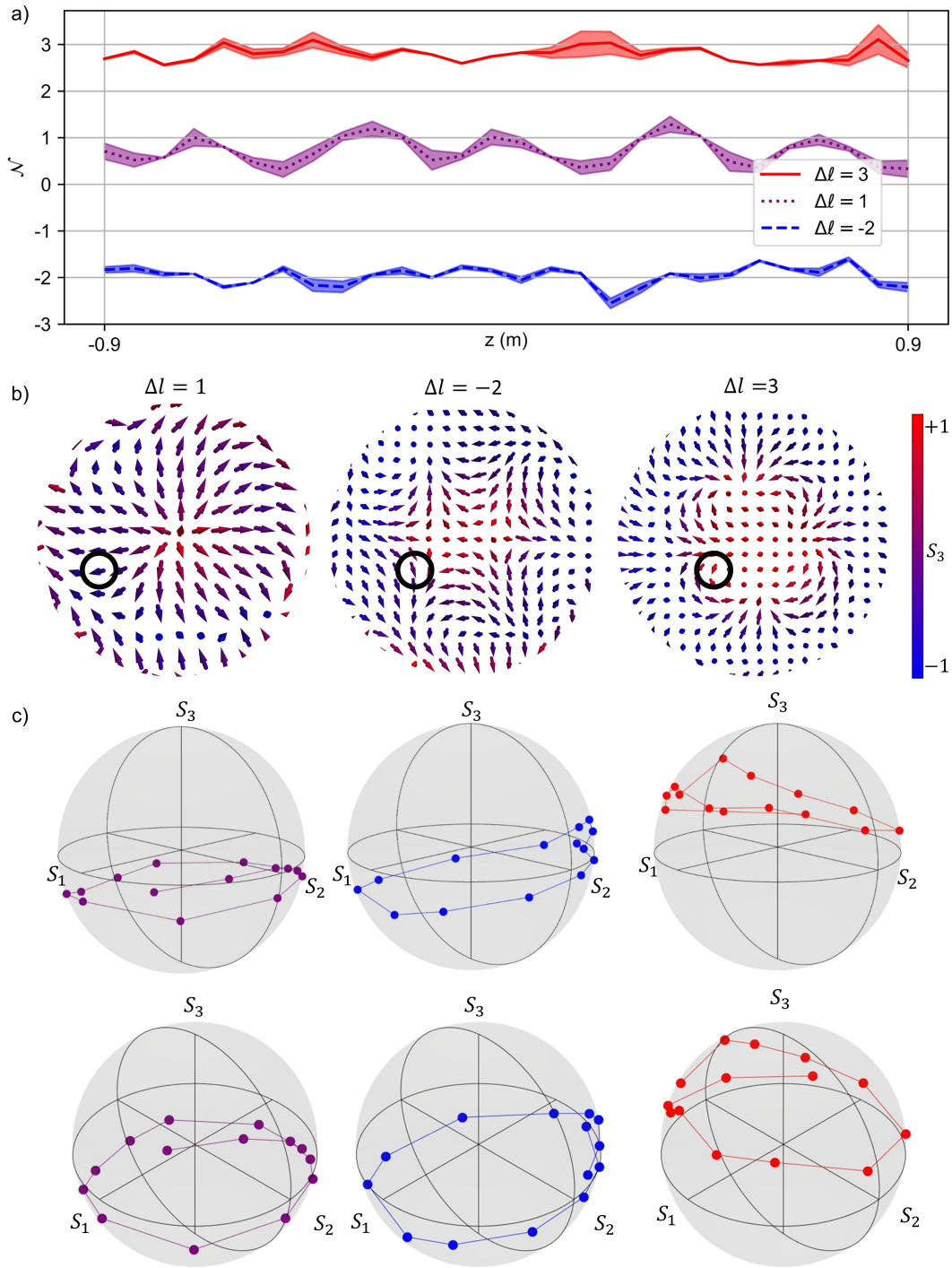


FIGURE 3.3: a) Plot showing the experimentally measured Skyrme number of various BG^ℓ optical (anti-)skyrmions propagated over a well defined distance. b) Experimentally measured Stokes vector fields of the corresponding BG^ℓ skyrmions in a). c) Plots showing the experimentally measured precession of a local Stokes vectors (from the corresponding beams of a) about the S_3 axis with propagation for various $\mathcal{N}$.

field or, equivalently an axially dependent anisotropic material which results in a (pseudo)spin state which accesses a large portion of possible orientations over a well defined propagation length. It is also notable that this propagation dynamic trajectory is similar to that observed in time dynamic Bloch vectors which evolve in

polariton condensates [242]. This extension of the Maxwell-Schrödinger formalism, which is generally restricted to anisotropic media, to the realm of free-space optics provides a more accessible form of the analogy [245].

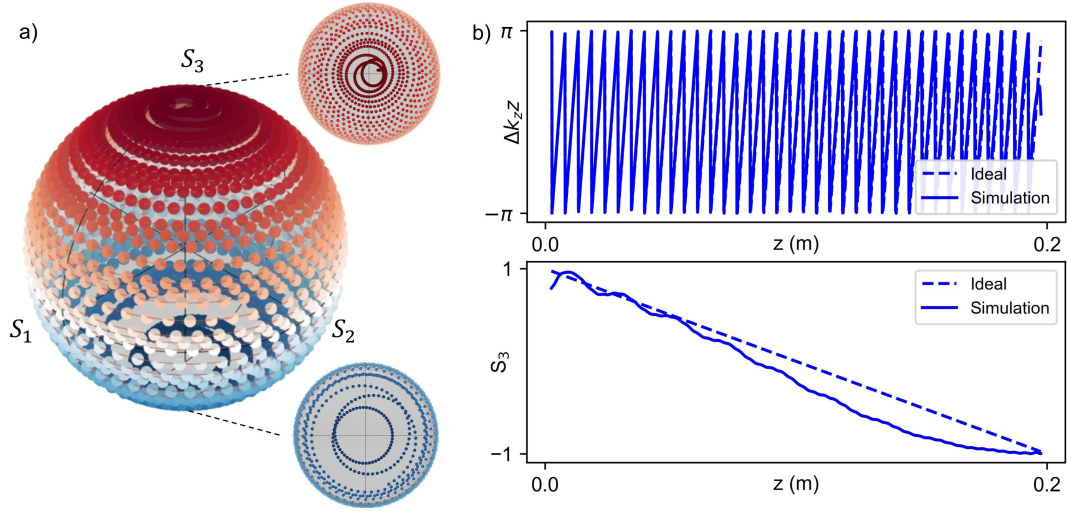


FIGURE 3.4: a) Plot showing the path traced during propagation of a numerically simulated central ($r = 0$) Stokes vector generated using optical 'frozen-waves', insets show the views from each of the poles. b) Plots showing the ideal (theoretical) and simulated (via the angular spectrum method) relative phase $\Delta k_z z$ and amplitude S_3 of the constituent components of the vector field used to produce the Stokes vector rotation in (a).

3.4 Conclusions

We have generated and analysed a variation of skyrmions in the Stokes field engineered using BG^ℓ beams, with different absolute values in their ℓ indices. We have shown how the global topological invariant can be extracted from a finite region and how unequal axial wavevector components in the constituent beams lead to precessional dynamics of pseudospins which cause a global transformation between Bloch and Néel type skyrmions with different initial angles. We experimentally verify the invariance of the topological index of the generated fields with propagation. We also propose a technique to extend the optical control of pseudospin dynamics by exploiting optical frozen-waves constructed by superpositions of BG^ℓ beams. We show how these propagation dynamics of pseudospin states are analogous to time dynamics of magnetic spin states, where the skyrmion beating is characteristic of behaviour in a static magnetic field and the frozen-wave case allows for dynamic magnetic fields to be emulated. The analogy can be described using the Maxwell-Schrödinger formalism, which often relates optical propagation through anisotropic media to magnetic systems. Further work exploiting spatially structured frozen-waves, could see Stokes vector fields with Skyrme numbers which change with propagation - analogous to temporal dynamics of spin systems in applied magnetic fields which vary spatially and temporally [248, 249].

Chapter 4

Non-local skyrmions as topologically resilient quantum entangled states of light

The work herein has been published in Ref. [142].

In the early 1960s Tony Skyrme proposed a non-linear meson field theory to describe sub-atomic particles as excitations of a single fundamental field, the pion [117–120]. To accomplish this he used the inherent mathematical structure of basic pion theory to postulate topologically non-trivial pion field configurations, now called skyrmions. A skyrmion is a topologically stable field configuration, characterized by an integer topological invariant, the skyrme number. A skyrmion cannot be smoothly deformed into another field configuration with a different skyrme number. The generality of this definition and its associated topological conserved quantity allowed the notion to be extended beyond its initial intent [121, 122]. In particular, skyrmions have been instrumental in advances in magnetism and spintronics, where their topological stability makes them ideal candidates for information storage and transfer [123–128, 220]. Here the quantum properties of these localised magnetic textures have been theoretically studied [255–260] and suggested as a basis for quantum information processing [261], where their macroscopic quantum tunneling and energy-level quantization are indicative of quantum behavior, with some quantum dynamics already observed in magnetic systems [262].

Beyond magnetism, skyrmions have been revisited in the context of nuclear physics to resolve long standing debates on the nucleon-nucleon spin-orbit potential [131] and have been observed in atomic matter [132], chiral liquid crystals [263] and in acoustics [264]. Optical realisations have only recently been explored, including observations in evanescent waves [133], in focused orbital angular momentum (OAM) beams at sub-wavelength scales [54], in certain classes of full Poincaré beams [57, 69, 134] and even as toroidal pulses [58]. Tracking the trajectories of individual pseudospin states with propagation exposes a mapping to the 4-dimensional hypersphere, realizing the skyrme field as a Hopf fibration [139, 140].

These impressive advances are all local realisations of skyrmions as particles and fields, without any essential role for entanglement, as illustrated in Fig. 4.1a. Creating skyrmions as quantum entangled states with non-local quantum correlations would allow potent applications of spatially structured topologies to quantum topological photonics [265]. Despite being very much in its infancy, the merging of topological and quantum structure holds great promise for information robustness even in non-ideal quantum systems, including stable quantum emitters [266–268], robust transport of quantum states through quantum circuits [269, 270] and entanglement

storage through light matter interaction [271].

Here we report the first non-local quantum entangled state with a non-trivial topology that is skyrmionic in nature. Intriguingly, the non-trivial topological structure does not exist in the properties of the individual (local) photons in the two-photon entangled state, but rather it emerges from the (non-local) entanglement between them. We demonstrate how the topology of the quantum wavefunction remains intact under a smooth deformation (with entanglement decay as an example), only vanishing once the entanglement itself vanishes, and here we report the first demonstration of this invariance. Our approach allows the topology of these quantum wavefunctions to be fully controlled, which we outline theoretically and demonstrate experimentally across a wide variety of skyrmion types, and allowing us to distinguish entanglement classes according to their quantum topology. Our work leverages on topological photonics and quantum state engineering, offering a promising avenue for preservation of quantum information by topologically engineered quantum states that persist even when entanglement is fragile.

4.1 Concept

An optical skyrmion can be represented as a topologically protected spin textured field, where every point on the Poincaré Sphere (S^2) is in correspondence with a point in the 2D transverse spatial plane ($\mathcal{R}^2$). The optical skyrmion is a mapping from the 2D transverse spatial plane to the space of polarization states, i.e., a mapping $\mathcal{R}^2 \rightarrow S^2$, with the skyrmion number characterizing the number of times $\mathcal{R}^2$ wraps S^2 . We create this by entanglement, through engineering two entangled photons (A and B) to occupy the quantum state $|\Psi\rangle = \frac{1}{\sqrt{2}} (|\ell_1\rangle_A |H\rangle_B + e^{i\gamma} |\ell_2\rangle_A |V\rangle_B)$, where ℓ_1 and ℓ_2 denote OAM of $\ell_1\hbar$ and $\ell_2\hbar$ per photon, respectively, H, V are the orthogonal horizontally and vertically polarized states, while γ allows for a rotation of the state vector, and can be extended to include the affects of quantum gouy phase [272] on the state, which merely alters γ in propagation (full details given in appendix B). The reduced state of photon A is an incoherent mixture of OAM, while that of photon B is unpolarized, so that *individually* neither has any salient topological structure. Quantum correlations between the two photons imply a rather different picture: the collapsed state of one photon is determined by the measurement choice on the other, so that a joint measurement on both reveals *non-local* quantum topological structure. Intriguingly, each localised position of photon A in real space $\mathcal{R}^2$ is associated with a polarization state, occupied by its entangled twin (photon B) and parameterised on a sphere, S^2 . We expand on this mathematically later in the context of a wider meaning. Consequently photon A holds the possibility for every spatial position, shown in Fig. 4.1b as both a spatial mode Bloch Sphere (top)[217] and the corresponding spatial probability distribution (bottom). Photon B meanwhile holds the possibility for every polarization state, shown in Fig. 4.1c as both a polarization Bloch Sphere (top) and the corresponding Poincaré Sphere (bottom). A spatial measurement on photon A collapses photon B into a particular polarization state, producing a mapping from space to the Poincaré Sphere, revealed by joint measurements in coincidence, so that we have indeed obtained the mapping $\mathcal{R}^2 \rightarrow S^2$ as desired. While the topological structure of each individual photon is always trivial and with zero skyrme number, the non-local skyrme number (we call it the quantum skyrme number) denoted by N , can be tailored as desired, with

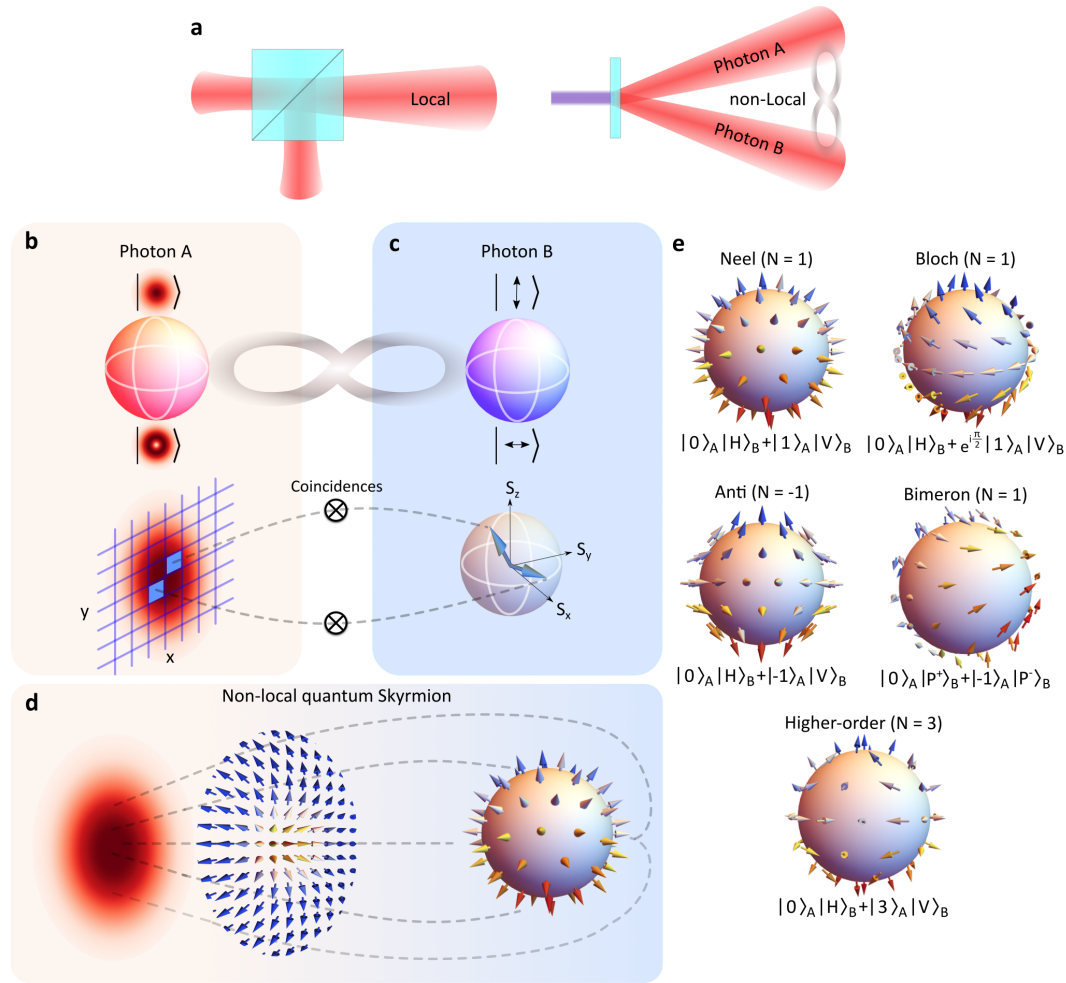


FIGURE 4.1: Non-local quantum skyrmions. **a**, A typical local field configuration giving rise to a skyrmion can be produced by superimposed, orthogonally polarized spatial modes. The non-local quantum skyrmion exists as a shared property of spatially separated photons A and B. The non-local skyrmionic configuration is formed in the entanglement between the two photons, with **b**, photon A carrying the spatial degree of freedom (DoF) expressed on an OAM Bloch Sphere, and **c**, photon B carrying the polarization DoF expressed on the polarization Bloch Sphere. A spatial measurement on photon A collapses photon B into some definite polarization state, linking space to polarization non-locally by a coincidence measurement (bottom panel). A complete spatial mapping of photon A reveals all polarization possibilities in photon B. **d**, The entanglement between the photons is now skyrmionic, as denoted by the vectorial arrows in space, which can be mapped stereographically to an abstract sphere that holds spatial information (coverage) and the skyrmionic state (arrows and location). **e**, Full control over the quantum wavefunction allows for the manipulation of the texture and precise topology of the non-local skyrmion thereby giving access to a plethora of topological structures.

the skyrmion existing in the entanglement between the photons themselves. This non-local quantum entangled skyrmion is depicted graphically in Fig. 4.1d. The skyrmion topology is shown as vectorial arrows in space, the position and direction of which are derived from the joint state of photons A and B. Photon A also contributes the probability of detection, spatially varying, shown as a false colour density plot. This information can be visualised holistically by a stereographic mapping to an abstract sphere whose spatial coverage is given by photon A and the skyrmion

topology (position and directions of the arrows) by the joint state of photons A and B. Furthermore, with complete control over the quantum wavefunction the topology and texture can be modified at will, examples of which are given in Fig. 4.1e. Here the topology (skyrmion, anti- and higher-order skyrmions, etc.) is largely dictated by the OAM values selected while the texture (Néel, Bloch and Bimeron, etc.) is determined by the relative phase between the states and polarization basis of the states, i.e., a change from $\{|H\rangle, |V\rangle\} \rightarrow \{|P^+\rangle, |P^-\rangle\}$, where $|P^+\rangle$ and $|P^-\rangle$ are any orthogonal polarization states, also changes the texture [68].

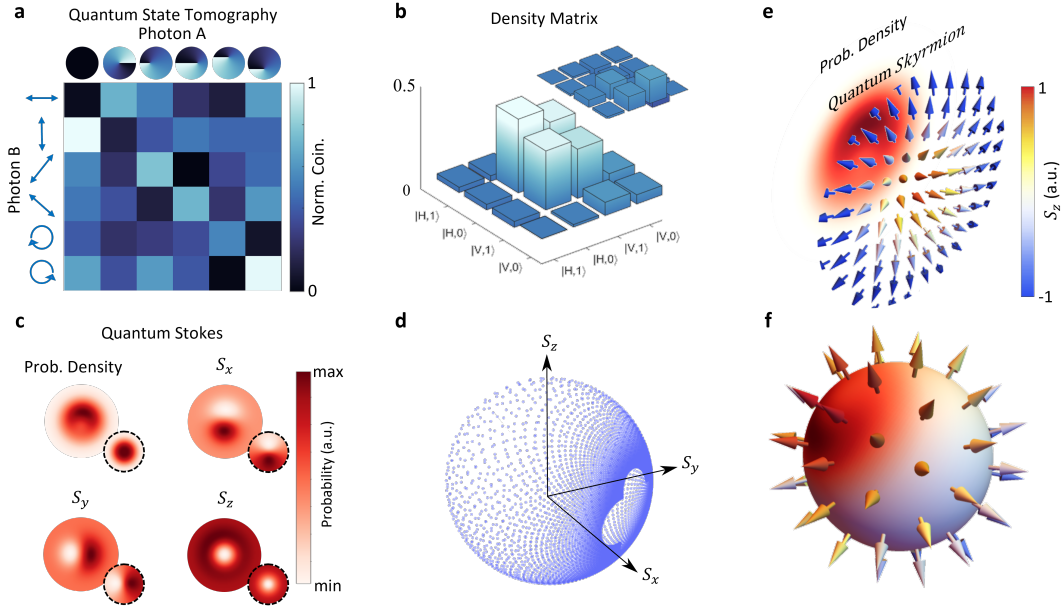


FIGURE 4.2: Experimental quantum skyrmion. **a**, A quantum state tomography was performed on the selected entangled state, $|\Psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle_A|H\rangle_B + |0\rangle_A|V\rangle_B)$ by spatial measurements on photon A (columns) and polarization measurements on photon B (rows), with coincidences collected for all outcomes shown as false colour from low (black) to high (white). **b**, The real and imaginary (inset) parts of the density matrix for the reconstructed state, indicative of the desired state. **c**, Measured quantum Stokes projections directly from the tomography. **d**, Reconstruction of the coverage on the Poincaré Sphere, with experimental data shown as blue dots. **e**, The experimental probability density of finding a photon in space, given by the spatial structure of photon A, and the experimentally reconstructed skyrmion as a spin textured field with a defined topology. **f**, A stereographic mapping of the experimental data showing spatial information (coverage) and probability of detection (colour), with the skyrmionic state depicted by the direction of the arrows and their location, a direct outcome from the joint state of photons A and B.

4.2 Creating and detecting quantum skyrmions

To verify this concept, we prepared our entangled state from an initial spontaneous parametric downconversion (SPDC) source and performed a unitary operation on the OAM of Photon A while performing a spatial-to-polarization (SPC) conversion, exchanging the spatial information of photon B for polarization information, with full experimental details given in section 2.2.1. This altered the typical SPDC OAM

spiral spectrum to the desired asymmetric spiral bandwidth for a skyrmionic two-photon quantum wavefunction. The non-local two-photon wavefunction was then analysed by projective measurements with a spatial light modulator (SLM) on photon A and polarization projections on photon B, measured in coincidence. The results from this QST are shown in Fig. 4.2a for an example state with $\ell_1 = 1$ and $\ell_2 = 0$ and $\gamma = 0$, where the detected coincidences for each projective measurement has been normalized against the maximum coincidences detected. The top-left 2×2 matrix partially depicts the behaviour of the generated state, that is when projecting onto $|\ell\rangle = |0\rangle$ for photon A, we only detect coincidences for vertically polarized photon B and similarly when projecting onto $|\ell\rangle = |1\rangle$ we only detect coincidences for horizontally polarized photon B. This is consistent with our expected state behaviour, a measurement of photon A revealing $\ell = 0(1)$ collapses photon B into the state $|V\rangle$ ($|H\rangle$). Further projections are performed to build an over-complete QST which enabled the reconstruction of the density matrix for our skyrmionic quantum state, shown in Fig. 4.2b. The real and imaginary (inset) parts shown contain all the information of our quantum state. When compared with the expected pure theoretical state, the experimentally generated state revealed a fidelity of $F = 95.0\%$. Additionally, the QST contains the information necessary to reconstruct the non-local Quantum Stokes parameters for the state, shown in Fig. 4.2c after local normalization, with the theoretical Quantum Stokes parameters given as insets. The probability density is given by $\langle I_A \otimes I_B \rangle$ and depicts the probability of finding photon A in a particular position and photon B in the corresponding polarization state. Furthermore, the remaining Quantum Stokes parameters are calculated as the expectation values of the Pauli matrices. The Quantum Stokes parameters reveal a mapping of each point in space to a position on the Poincaré sphere such that our complete set of spatial measurements on photon A reveals a collapse of photon B into 97.3% of all possible polarization states parametrized by the Poincaré sphere. This coverage is shown in Fig. 4.2d. In Fig. 4.2e the true skyrmionic nature of the generated biphoton state is depicted. The probability density and spatial information is extracted by measurement of photon A and in coincidence the polarization information is extracted from measurement of photon B, revealing a Néel-type configuration. A stereographic projection of the configuration shown in Fig. 4.2e yields the hedgehog-like polarization texture shown in Fig. 4.2f of the entangled state on the surface of a sphere. An experimental quantum skyrme number of $N_{\text{exp}} = 0.972$ which is very close to the theoretical value of $N_{\text{th}} = 1$, for the first realisation of a non-local quantum entangled skyrmion.

Next, we traverse the skyrmionic quantum landscape, altering the texture and topology by control of γ and N , respectively. We illustrate this in Fig. 4.3a, moving from anti-skyrmions ($N < 0$) through non-skyrmionic states ($N = 0$) to $N = 1$ examples of Bloch-type ($\gamma = \pi/2$) and Néel-type ($\gamma = 0$) and lastly to higher-order skyrmions ($N > 1$), all faithfully produced with high fidelity when compared to maximally entangled states, as shown in Fig. 4.3b and measured skyrme numbers (N_{exp}), as shown in Fig. 4.3c in excellent agreement with theoretically predicted integer values. See appendix C.1 for fidelity and skyrme number values for the generated states referenced in Fig. 4.3. In Fig. 4.3d we show the experimentally inferred topology of the generated states as stereographic projections onto the surface of S^2 , while in 4.3e we show selected experimental and theoretical topologies (with the experimental and theoretical probability densities of each state given as insets), in excellent agreement. Notice that when $\ell_2 = -\ell_1$ i.e., opposite twists to the OAM components, as would be created in typical spin-orbit hybrid entanglement experiments [205], no quantum skyrmion is produced and $N = 0$. For $N = -1$, we see the

characteristic hyperbolic texture embedded into the wavefunction, and for the Néel and Bloch type skyrmions we see the characteristic hedgehog and spiral textures. Full experimental data sets are given in appendix C.

4.3 Topology and entanglement

A convenient basis of states in which to expand the wavefunction of photon A is provided by the position space states $|\mathbf{r}_A\rangle$. Using the explicit expression $|\ell\rangle = \int_{\mathcal{R}^2} |\text{LG}^\ell(\mathbf{r}_A)| e^{i\ell\phi_A} |\mathbf{r}_A\rangle d^2r_A$ for the OAM eigenstates, we write our entangled photon state as

$$|\Psi\rangle = \int_{\mathcal{R}^2} |\mathbf{r}_A\rangle \left(a(\mathbf{r}_A) |H\rangle_B + b(\mathbf{r}_A) e^{i\Theta(\phi_A)} |V\rangle_B \right) d^2r_A, \quad (4.1)$$

where $a(\mathbf{r}_A) \equiv |\text{LG}^{\ell_1}(\mathbf{r}_A)|$, $b(\mathbf{r}_A) \equiv |\text{LG}^{\ell_2}(\mathbf{r}_A)|$, $\Theta(\phi_A) = \Delta\ell\phi_A + \gamma$ and $\Delta\ell = \ell_2 - \ell_1$. The coefficient of $|\mathbf{r}_A\rangle$ above defines a state for photon B

$$|\psi_{B|A}\rangle = \cos(\theta(\mathbf{r}_A)) |H\rangle_B + \sin(\theta(\mathbf{r}_A)) e^{i\Theta(\phi_A)} |V\rangle_B, \quad (4.2)$$

where the state $|\psi_{B|A}\rangle$ describes photon B's state having measured photon A at position $\mathbf{r}_A$. The $\theta(\mathbf{r}_A)$ dependence arises since we have normalized the state for every position $\mathbf{r}_A$, i.e., $a(\mathbf{r}_A)^2 + b(\mathbf{r}_A)^2 = 1$ for $\mathbf{r}_A \in \mathcal{R}^2$ which implies that we may set $a(\mathbf{r}_A) = \cos(\theta(\mathbf{r}_A))$ and $b(\mathbf{r}_A) = \sin(\theta(\mathbf{r}_A))$ for $\theta(\mathbf{r}_A) \in [0, \pi/2]$. The polarization state space for photon B is a two dimensional sphere $\mathcal{S}^2$. Using a stereographic projection, the original position state space $\mathcal{R}^2$ can also be identified as an $\mathcal{S}^2$, that is, we can map it to a sphere. In this way, by associating $|\mathbf{r}_A\rangle$ and $|\psi_{B|A}\rangle$, the entangled photon wavefunction defines a map from a sphere to a sphere: from the position state space $\mathcal{S}^2$ (associated to photon A) to the polarization state space $\mathcal{S}^2$ (associated to photon B).

Every map from $\mathcal{S}^2$ to $\mathcal{S}^2$ has an integer degree, N , measuring how many times the first sphere wraps the second. This degree is a topological invariant: it is unchanged by continuous deformation of the map. This rich topological structure of the maps from spheres to spheres is the source of the topological structure in our quantum two-photon wavefunctions. In the current context, the degree N is the quantum skyrme number. An important corollary of the existence of a topological invariant is that the space of continuous maps is partitioned into equivalence classes [273]. Any two maps in the same class can be continuously deformed into each other, while this is not possible for maps in distinct classes [273]. The connection we have established between entangled photon wavefunctions and maps from $\mathcal{S}^2$ to $\mathcal{S}^2$ implies that the space of all possible wavefunctions is itself partitioned into equivalence classes with each class labeled by its quantum skyrme number.

To illustrate this, we show examples of experimentally measured mappings with various winding numbers in Fig. 4.4. All entangled states with $N = 0$ map to rings on $\mathcal{S}^2$ and hence are topologically equivalent, with the results in Fig. 4.4a confirming the mapping. This topology groups conventional hybrid entangled states, as is usually created by geometric phase approaches [274], into a single class. Note the deviation between the perturbation-free prediction (solid line in the inset) and the experimental points, which we return to shortly in the context of robustness. States with $N \neq 0$ wrap the entire $\mathcal{S}^2$ space N times, as shown in Fig. 4.4b and Fig. 4.4c for

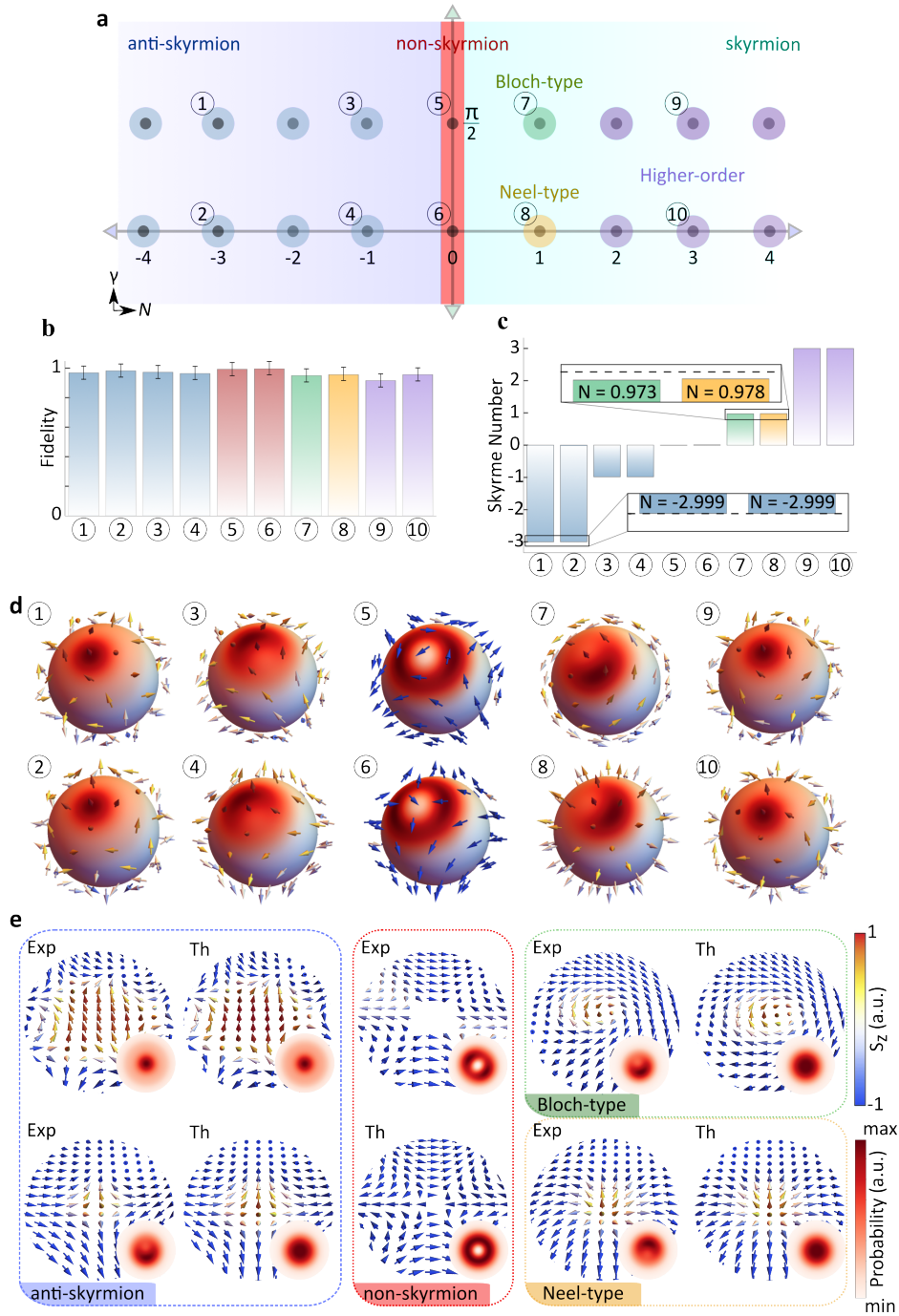


FIGURE 4.3: Traversing the quantum skyrmionic landscape. **a**, Varying textures (down a column) and skyrmion numbers (across a row) were created by controlling γ and the OAM difference ($\Delta\ell$) in the quantum state to indirectly control N . Example states are shown graphically and labelled with numbers 1 through 10, traversing the skyrmionic landscape from anti-skyrmions ($N < 0$) through non-skyrmionic states ($N = 0$) to Bloch-type ($N = 1, \gamma = \pi/2$), Néel-type ($N = 1, \gamma = 0$) and higher-order skyrmions $N > 1$. **b**, Experimental fidelities for the example states when compared to theoretical maximally entangled pure states. **c**, Experimental skyrmion numbers for example states. **d**, Examples of experimental stereographic projections. **e**, Example skyrmion topologies with varying textures shown as experimental (Exp) and theoretical (Th) reconstructions. The insets show the probability of detection as measured (Exp) and calculated (Th) from photon A.

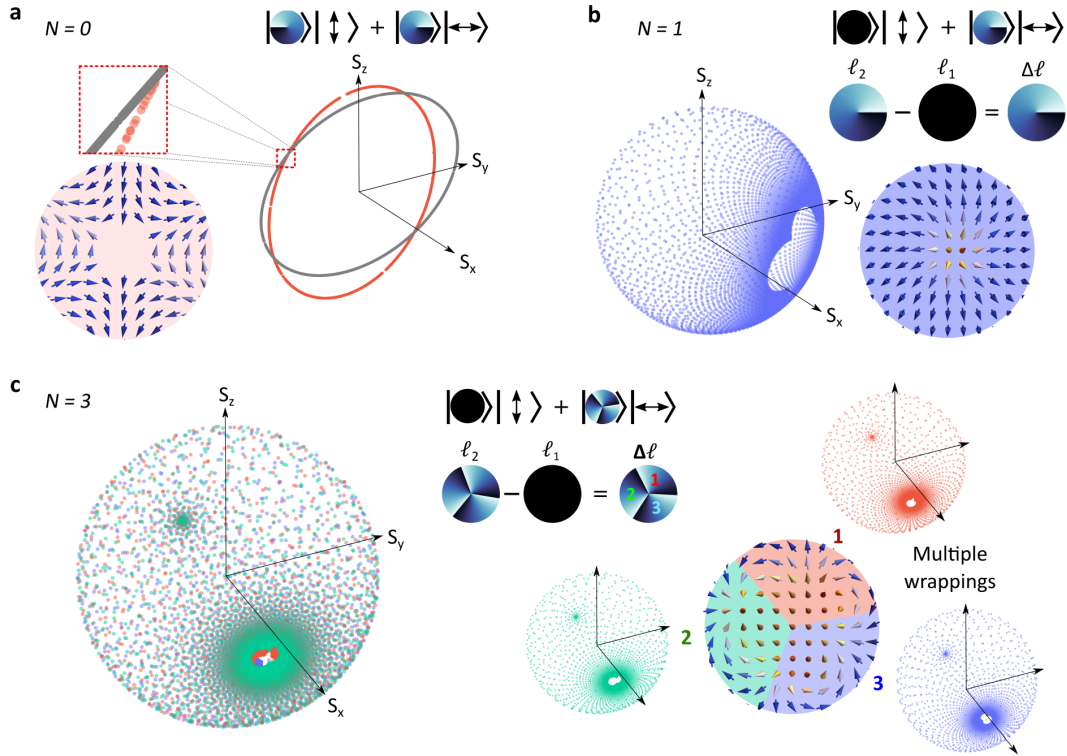


FIGURE 4.4: Topology of quantum entangled states. Bi-photon entangled states can be classified according to their topology. **a**, Entangled States where $|\ell_1| = |\ell_2|$ have $N = 0$ and map to rings on S^2 with close to zero coverage and are topologically trivial. Experimental results (dots) together with the perturbation-free prediction (solid line) are shown in the inset. States with $|\ell_1| \neq |\ell_2|$ necessarily have $N \neq 0$ and map to the entirety of S^2 , through N wrappings. This is shown for **(b, $N = 1$ and c, $N = 3$** , with experimental data shown as dots on the spheres. In **b** the coverage is close to 100% for $N = 1$. **c**, For $N = 3$, the non-local field has three segments, each with a full coverage of S^2 , shown as red, green and blue experimental dots. The entire state maps to S^2 with three windings for a coverage of 299%, close to 300%, illustrated in the composite sphere (right) with all three experimental data sets.

$N = 1$ and $N = 3$, respectively. The experimentally measured coverage of approximately 97% ($N = 1$) and 299% ($N = 3$) come close to the theoretical values of 100% and 300%, respectively. The small deviation can be seen at the pole of the mapped sphere, a natural artefact of mapping from $\mathcal{R}^2$ to S^2 , where data at spatially infinite distances cannot be recovered. Furthermore we note that for any entanglement class, the spatial state space may be partitioned into N segments, each of which will be found to wrap the entire S^2 state space, as shown in Fig. 4.4c.

Having identified the origin of the topological structure in the two-photon quantum state, we now offer a connection between topology and entanglement. A non-trivial topology is not a property of either of the S^2 spaces on their own, but rather it is a shared emergent property originating in the non-trivial global structure of the map between the two spaces. Entanglement itself is not a property of the state of either particle on its own, but rather it too is a shared emergent property. There is a quantitative relationship between the two: it is only the topologically trivial $N = 0$ class of two photon wavefunctions that can have vanishing entanglement. Wavefunctions belonging to any other class are always entangled. This argument is supported by noting that since for the trivial $N = 0$ topology, the entangled states

map onto rings on S^2 , and so it is possible to continuously deform the map so that all points on the first sphere map to a single point on the second sphere. This map corresponds to a wavefunction for which every position state $|\mathbf{r}_A\rangle$ in Equation 4.1 multiplies the same state from the polarization state space. This product state is a separable state, therefore not entangled. For *any* other winding number $N \neq 0$, the coefficient of different position space states range N times over the complete polarization state space, so the corresponding state is entangled. The inescapable conclusion is that non-trivial topology $N \neq 0$ implies non-zero entanglement. On the other hand, trivial topology does *not* imply that the corresponding wavefunction has vanishing entanglement. Indeed, wavefunctions corresponding to maps that partially cover the polarization S^2 are entangled, but still topologically trivial with $N = 0$. The point however, is that these maps can be continuously deformed to a map which corresponds to the zero entanglement wavefunction. For any other $N \neq 0$ class, it is not possible to reach the map of a zero entanglement wavefunction by continuous deformations. In this sense the topology of the wavefunction partitions the space of all possible wavefunction into entanglement classes.

4.4 Topological resilience

Our connection between topology and entanglement suggests that (1) it is not possible to have no entanglement and a non-trivial topology ($N \neq 0$), (2) it is possible to have a trivial ($N = 0$) topology and no entanglement, while (3) the local deformation preserving topology suggests that so long as the entanglement persists the topology is robust - a quantum form of topological resilience. The results reported in Fig. 4.5a support these statements. We introduced an amplitude damping operation [275] applied to the wavefunction that resulted in entanglement decay from a maximally entangled state to no entanglement, all the while monitoring the topology through the quantum skyrme number, N . The results for $N = 1$ and $N = 3$ reveal a new form of topological resilience, with the quantum skyrme number intact in the presence of decaying entanglement, satisfying (1) and (3), falling off to zero only when the entanglement itself vanishes, satisfying (2). To the best of our knowledge, this is the first observation of an invariance (N) to entanglement decay derived from the topology of the entanglement between particles, which holds exciting promise for topologically protected quantum information processing.

We now sketch a simple argument explaining how this occurs for quantum wavefunctions which we believe is not to be found elsewhere. As a first step, can we understand what we gain by coding information into topology? Topology is a systematic approach to characterizing those quantities that are insensitive to smooth deformation. Our observed topological resilience of the skyrmionic wavefunction can be explained by considering the entanglement decay operation as a smooth deformation of the state which in our problem is the statement that the quantum skyrmion number is unchanged by a change of coordinates, $(x, y) \rightarrow (x', y')$, for photon A. To see this, start from the formula for the skyrmion number

$$N = \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2} \epsilon_{pqr} S_p \frac{\partial S_q}{\partial x} \frac{\partial S_r}{\partial y} dx dy, \quad (4.3)$$

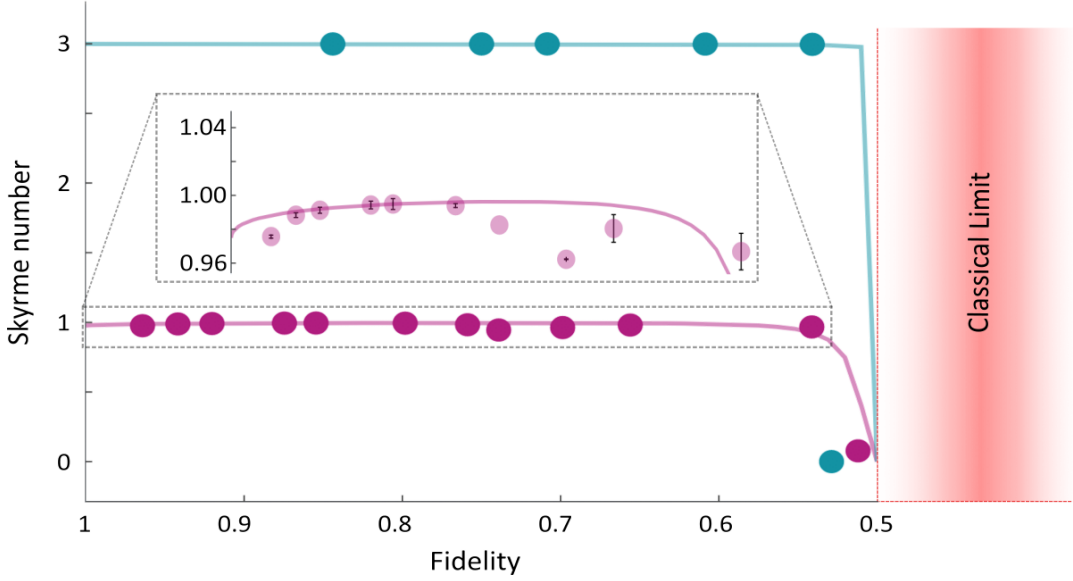


FIGURE 4.5: Quantum topological invariance to entanglement decay. Measured quantum Skyrme number (N) as a function of state fidelity (F) against a maximally entangled state. Here F decreased by a deliberately induced decay in entanglement, from fully entangled ($F = 1$) to no entanglement ($F = 0.5$). Results are shown for initial $N = 1$ and $N = 3$, with the invariance of N clearly revealing that the topology remains intact in the presence of decaying entanglement. Experimental data is shown as dots and theory as solid curves, in excellent agreement with one another.

where $p, q, r \in \{x, y, z\}$. Consider a change of coordinates from x, y to x', y' where both are arbitrary functions of x, y . First consider the transformation of the integrand under this change of coordinates. A simple application of the chain rule gives

$$\begin{aligned}
 & \frac{1}{2} \epsilon_{pqr} S_p \frac{\partial S_q}{\partial x} \frac{\partial S_r}{\partial y} \\
 &= \frac{1}{2} \epsilon_{pqr} S_p \left(\frac{\partial x'}{\partial x} \frac{\partial S_q}{\partial x'} + \frac{\partial y'}{\partial x} \frac{\partial S_q}{\partial y'} \right) \left(\frac{\partial x'}{\partial y} \frac{\partial S_r}{\partial x'} + \frac{\partial y'}{\partial y} \frac{\partial S_r}{\partial y'} \right) \\
 &= \frac{1}{2} \epsilon_{pqr} S_p \left(\frac{\partial x'}{\partial x} \frac{\partial x'}{\partial y} \frac{\partial S_q}{\partial x'} \frac{\partial S_r}{\partial x'} + \frac{\partial x'}{\partial x} \frac{\partial y'}{\partial y} \frac{\partial S_q}{\partial x'} \frac{\partial S_r}{\partial y'} \right. \\
 & \quad \left. + \frac{\partial y'}{\partial x} \frac{\partial x'}{\partial y} \frac{\partial S_q}{\partial y'} \frac{\partial S_r}{\partial x'} + \frac{\partial y'}{\partial x} \frac{\partial y'}{\partial y} \frac{\partial S_q}{\partial y'} \frac{\partial S_r}{\partial y'} \right). \tag{4.4}
 \end{aligned}$$

There are four terms in the final expression above. The indices q and r are summed. The first and fourth terms above are sums of an expression that is symmetric under interchange of q and r times ϵ_{pqr} which is antisymmetric under the interchange of q and r . Consequently the first and fourth terms vanish after the sums over q and r are performed (for example, the first term under the sum over q and r becomes $\frac{1}{2} S_p \frac{\partial x'}{\partial x} \frac{\partial x'}{\partial y} \frac{\partial S_q}{\partial x'} \frac{\partial S_r}{\partial x'} - \frac{1}{2} S_p \frac{\partial x'}{\partial x} \frac{\partial x'}{\partial y} \frac{\partial S_r}{\partial x'} \frac{\partial S_q}{\partial x'} = 0$). In the third term we relabel the indices $q \leftrightarrow r$ to obtain

$$\frac{1}{2} \epsilon_{pqr} S_p \frac{\partial S_q}{\partial x} \frac{\partial S_r}{\partial y}$$

$$= \left(\frac{\partial x'}{\partial x} \frac{\partial y'}{\partial y} - \frac{\partial y'}{\partial x} \frac{\partial x'}{\partial y} \right) \frac{1}{2} \epsilon_{pqr} S_p \frac{\partial S_q}{\partial x'} \frac{\partial S_r}{\partial y'}. \quad (4.5)$$

The integration measure transforms as usual

$$dxdy = Jdx'dy', \quad (4.6)$$

where the Jacobian is given by

$$\begin{aligned} J &= \left(\frac{\partial x}{\partial x'} \frac{\partial y}{\partial y'} - \frac{\partial y}{\partial x'} \frac{\partial x}{\partial y'} \right) \\ &= \left(\frac{\partial x'}{\partial x} \frac{\partial y'}{\partial y} - \frac{\partial y'}{\partial x} \frac{\partial x'}{\partial y} \right)^{-1}. \end{aligned} \quad (4.7)$$

Putting these transformation rules together we learn that

$$\begin{aligned} N &= \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2} \epsilon_{pqr} S_p \frac{\partial S_q}{\partial x} \frac{\partial S_r}{\partial y} dxdy \\ &= \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2} \epsilon_{pqr} S_p \frac{\partial S_q}{\partial x'} \frac{\partial S_r}{\partial y'} dx'dy', \end{aligned} \quad (4.8)$$

which tells us that the skyrmion number is independent of the choice of coordinates used for the calculation. This is the complete content of the statement that the skyrmion number is invariant under smooth deformations, since any smooth deformation is equivalent to a judiciously chosen change of coordinate.

To complete the argument, we need to explain why the freedom to change coordinates leads to resilience against local perturbations. The argument is most simply developed in a single spatial dimension. In the presence of a perturbation, the wave function $\psi(x)$ is changed into $\psi'(x)$, as illustrated in Fig. 4.6 a) (top panel). If the perturbation is not too severe, the equality $\psi'(x) = \psi(x')$ defines a change of coordinates from x to a new coordinate x' . See Fig. 4.6 a) (middle panel) which shows an example of this coordinate transformation. We are interested in computing a topological invariant associated with the distorted wave function. Since the topological invariant is unchanged by coordinate transformations, we are free to perform the transformation $x \rightarrow x'$ that we have just defined. The transformation replaces $\psi'(x) \rightarrow \psi(x')$, so that we return to our undistorted wave function in the new coordinates as shown in Fig. 4.6 a) (bottom panel), demonstrating the topological perturbation invariance mechanism. In Fig. 4.6 b) a perturbation with a greater amplitude is applied, leading to a new local maximum and a new local minimum. The equation $\psi'(x) = \psi(x')$ continues to define a change of coordinates from x to the new coordinate x' , but now the path must backtrack to produce the maxima and minima, so that the transformation between x and x' zig-zags as shown. This is still a perfectly smooth change of coordinate and the topological invariant remains unchanged. Finally, as shown in Fig. 4.6 c), we might induce such a large perturbation, i.e severely distort the wave function such that there is no coordinate transformation that manages to completely restore the wave function. In this case the best we can do is to require that $\psi'(x) = A\psi(x')$, where both the relation between x and x' as well

as the amplitude A are parameters we can vary. Recall that physical states are rays in Hilbert space. Two wavefunctions that differ only in magnitude and/or global phase, correspond to the same ray and consequently they define the same physical state. Thus the condition $\psi'(x) = A\psi(x')$ continues to identify the original and perturbed states correctly. As the amplitude of the perturbation is increased, we will not manage to satisfy the condition $\psi'(x) = A\psi(x')$ exactly, and the perturbation invariance is no longer perfect. Ultimately, if the perturbation is too severe, there is no coordinate transformation that even approximately obeys $\psi'(x) = \psi(x')$. At this point we can no longer smoothly deform ψ' into ψ , i.e. the two wave functions have different topologies and topological perturbation invariance will fail. At these very high perturbation levels even basic properties, like the number of nodes of the wave function, are changed.

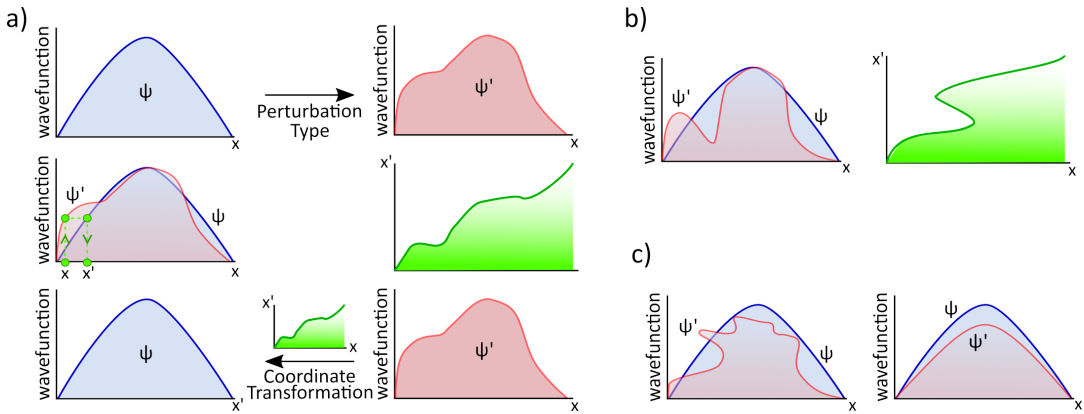


FIGURE 4.6: The origin of topologically protected wavefunctions. This figure describes the coordinate transformation defined by the original and noisy wavefunctions. To obtain an x' starting from a given x , as in (a) we follow point x vertically up to ψ' , then traverse horizontally from ψ' to ψ and finally project down from ψ to x' . If the noise adds extra local maxima and minima, as shown in (b), the map between x and x' will zig-zag as shown. Finally, the noise can introduce a discrepancy between the amplitude of ψ and the amplitude of ψ' , as shown in (c). In this case we solve the relation $\psi'(x) = A\psi(x')$, with the constant A magnifying ψ to match ψ' . Since wavefunctions are rays in Hilbert space, two wavefunctions that differ only in magnitude and/or global phase, are the same state. Consequently $\psi'(x) = A\psi(x')$ continues to identify the original and noisy states correctly.

Now that the basic idea is clear, we will describe how this argument applies to the actual problem of interest. Recall that the wave function, after expanding in the basis of position space states $|\mathbf{r}_A\rangle$, is given by Eq. 4.1. As we have remarked, the coefficient of $|\mathbf{r}_A\rangle$ in Eq. 4.1 defines a state for photon B

$$|\psi_{B|A}\rangle = \cos(\theta(\mathbf{r}_A)) |H\rangle_B + \sin(\theta(\mathbf{r}_A)) e^{i\Theta(\phi_A)} |V\rangle_B, \quad (4.9)$$

where the state $|\psi_{B|A}\rangle$ describes photon B 's state having measured photon A at position $\mathbf{r}_A$. Giving up normalization, we can write this state as

$$|\psi_{B|A}\rangle \propto |H\rangle_B + \tan(\theta(\mathbf{r}_A)) e^{i\Theta(\phi_A)} |V\rangle_B. \quad (4.10)$$

Both the entanglement structure of this state as well as its topology i.e. its skyrmion number, are completely specified by the coefficient $\tan(\theta(\mathbf{r}_A)) e^{i\Theta(\phi_A)}$. In the presence of some perturbation this coefficient is modified to $\tan(\theta'(\mathbf{r}_A)) e^{i\Theta'(\phi_A)}$. The coordinate transformation which establishes that both the entanglement of the wave

function and it's topology are robust against local perturbations is defined by

$$\tan(\theta'(\mathbf{r}_A))e^{i\Theta'(\phi_A)} = \tan(\theta(\mathbf{r}'_A))e^{i\Theta(\phi'_A)}. \quad (4.11)$$

This coordinate transformation protects the relative weighting of the two components of the wave function, ensuring that it does indeed protect both the entanglement and topology of the wave function.

Returning to the experimental demonstration of the topological resilience to an amplitude damping channel resulting in entanglement decay Fig. 4.5, we can now discuss this in terms of the arguments provided above. First we note that our robustness argument will hold for any quantum channel that can be shown to be a smooth deformation of the wavefunction and thus map preserving. The isomorphism between quantum channels and quantum states [276] suggests that our notion of topological entanglement classes can be used to characterise quantum channels too: those that are smooth deformations and those that are not. In the present case we consider the topology preserving amplitude damping operation, $\hat{T}_d$, and describe it topologically as a simple coordinate transformation as it preserves the topology of a given wavefunction so long as the new wavefunction describes an entangled state. Suppose we start with the entangled state, $|\Psi(a(r_A), b(r_A), \Theta(\phi_A))\rangle$ given in Eq. 4.10, and we apply an entanglement decay operation on this state such that it is changed to $|\Psi'(a'(r_A), b'(r_A), \Theta(\phi_A))\rangle$ with $a'(r_A) = \alpha a(r_A)$, $b'(r_A) = \sqrt{1 - \alpha^2}b(r_A)$ and $\alpha \in (0, 1)$, as depicted in Fig. 4.7.

To equate this transformation to a coordinate transformation, we consider what happens to its structure locally. In other words, suppose that under the state $|\Psi\rangle$, a spatial measurement on photon A, at $\mathbf{r}_A = (r_A, \phi_A)$ results in a collapse of photon B into the polarization state $|P\rangle$, and under the state $|\Psi'\rangle$, a spatial measurement on photon A, at $\mathbf{r}'_A = (r'_A, \phi'_A)$ results in a collapse of photon B into the polarization state $|P\rangle$. If a coordinate mapping, $\mathbf{r}_A \rightarrow \mathbf{r}'_A$ exists such that the scenario above is satisfied for all $\mathbf{r}_A, \mathbf{r}'_A \in \mathbb{R}$, then we can define a smooth deformation describing the action of $\hat{T}_d$ on our wavefunction.

Lending from the discussion above and using a modified version of equation 4.11, we have that a coordinate transformation must exist such that

$$\frac{b(r_A)}{a(r_A)} \xrightarrow{\hat{T}_d} \frac{b'(r_A)}{a'(r_A)} = \frac{b(r'_A)}{a(r'_A)} \quad (4.12)$$

where the terms dependent on the azimuthal coordinate ϕ_A were factored out of the expression as the transformation did not modify them. Substituting the appropriate Laguerre Gaussian functions into Eqn. 4.12 yields

$$\begin{aligned} \frac{b'(r_A)}{a'(r_A)} &= \sqrt{\frac{|\ell_2|!}{|\ell_1|!}} \left[\frac{\sqrt{2}}{\omega_0} r_A \left(\frac{\sqrt{1 - \alpha^2}}{\alpha} \right)^{\frac{1}{|\ell_2| - |\ell_1|}} \right]^{| \ell_2 | - | \ell_1 |} \\ &= \sqrt{\frac{|\ell_2|!}{|\ell_1|!}} \left[\frac{\sqrt{2}}{\omega_0} r'_A \right]^{| \ell_2 | - | \ell_1 |} = \frac{b(r'_A)}{a(r'_A)}, \end{aligned} \quad (4.13)$$

from which we extract the coordinate transformation $(r_A, \phi_A) \rightarrow (r'_A, \phi'_A)$ where $r'_A = r_A \left(\frac{\sqrt{1-\alpha^2}}{\alpha} \right)^{\frac{1}{|l_2|-|l_1|}}$ and $\phi'_A = \phi_A$.

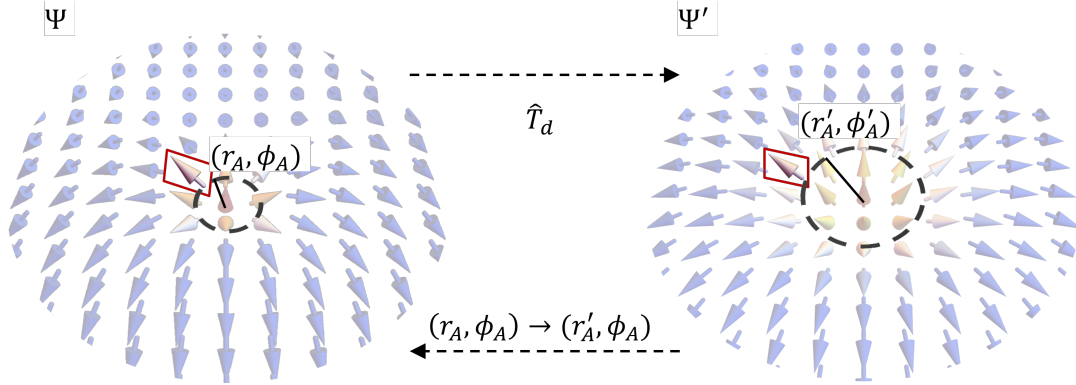


FIGURE 4.7: Entanglement Decay as smooth deformation The entanglement decay operation, $\hat{T}_d$, can be described as a smooth deformation as it does not affect the topology of the quantum state. Physically, if we consider any biphoton state with $|N| > 1$ (As an example we depict $|N| = 1$), the action of the channel on the state works to shift all polarization states towards or away from the centre such that one can perform a simple coordinate change on the radial position coordinate, i.e $|\Psi(r_A, \phi_A)\rangle \xrightarrow{\hat{T}_d} |\Psi'(r_A, \phi_A)\rangle = |\Psi(r'_A, \phi_A)\rangle$.

Therefore, the amplitude damping operation manifests as a radial position coordinate scaling for polarization state vectors which lie at some r , as shown in Fig. 4.8, so that a simple linear coordinate transformation of the form $r \rightarrow r'$ where $r' = \eta(\alpha)r$ yields the original entangled state. We see from Fig. 4.5 that as we asymptotically approach vanishing entanglement, so the topology abruptly changes from non-trivial ($N \neq 0$) to trivial ($N = 0$), emphasising that the skyrmionic topology is a *shared* property of the entangled photons, which necessarily vanishes once there is no entanglement. As shown in Fig. 4.8 this corresponds to the mapping of the radial coordinate of all the polarization state vectors to infinity, thus a simple coordinate transformation cannot retrieve the state.

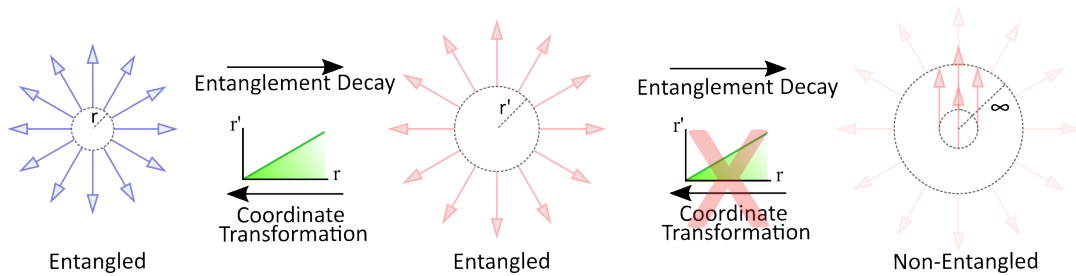


FIGURE 4.8: Entanglement Decay as smooth deformation The entanglement decay operation, $\hat{T}_d$, can be described as a smooth deformation as it does not affect the topology of the quantum state. Physically, if we consider any biphoton state with $|N| > 1$ (As an example we depict $|N| = 1$), the action of the channel on the state works to shift all polarization states towards or away from the centre such that one can perform a simple coordinate change on the radial position coordinate, i.e $|\Psi(r_A, \phi_A)\rangle \xrightarrow{\hat{T}_d} |\Psi'(r_A, \phi_A)\rangle = |\Psi(r'_A, \phi_A)\rangle$.

Another topical example of this topological invariance is that against quantum Gouy phase [272]. We have shown that quantum Gouy phase originates from the propagation of the two entangled photons and results in the coordinate transformation (Full

details are given in appendix B)

$$r_A \rightarrow r'_A = \frac{r_A}{1 + \left(\frac{z}{z_R}\right)^2}, \quad (4.14)$$

$$\tilde{\theta}(\phi_A, z) \rightarrow \tilde{\theta} = \Delta\ell\phi_A + \Delta\phi \quad (4.15)$$

where z is the propagation distance, z_R is the Rayleigh range and $\Delta\phi(z) = (|\ell_2| - |\ell_1|) \arctan\left(\frac{z}{z_R}\right)$.

Because perturbing the wavefunction corresponds to deforming the map, information about the entanglement class of the wavefunction is naturally resilient to such a perturbation. Further, any topologically non-trivial ($N \neq 0$) class enjoys this resilience, while the topologically trivial class does not. An experimental demonstration of this can be visualised by the mappings in Fig. 4.4: since the topologically trivial class is mapped to a ring, as shown in Fig. 4.4a, its orientation can easily be discerned. Indeed, in Fig. 4.4a we can see the deviation from the ideal prediction of the ring (solid line) and the experimental reconstruction (data points), the latter always with some inherent perturbation. Not only can the ring alter in orientation but it can be deformed to a point since states with trivial topology ($N = 0$) could be entangled or not. Conversely, one cannot discern an orientation change due to perturbation when the entire sphere is mapped ($N > 0$), as is the case for all non-trivial classes with examples shown in Fig. 4.4b and Fig. 4.4c: full (or more) coverage always looks the same regardless of your perspective. As we have discussed, the topology of such states remains intact so long as there is *some* entanglement in the system, and hence can be considered robust even when entanglement is fragile.

4.5 Discussion and Conclusion

We have reported the first realisation of skyrmionic fields with non-local quantum correlations, which have hitherto been elusive. The topologically non-trivial two-photon entangled state has a skyrme number that can be any non-zero integer, and this topological structure is not derived from either local photon state, but rather, from the entanglement between them. We demonstrated experimental control over the topology of our two-photon entangled states showing exotic topologies ($N \in \{-3, -1, 0, 1, 3\}$) and textures (anti-, Néel- and Bloch types). We have outlined a new tool for the classification of these entangled states according to their topology, from trivial classes that map to rings on S^2 , to non-trivial classes with N wrappings of S^2 , shown up to $N = 3$. We have shown that such quantum textures imbue the state with topological resilience, preserving topology until the entanglement itself vanishes and establishing that one may identify an entangled state's topology even when entanglement is fragile. This invariance of the topology of the state could be combined with topological approaches to preserve entanglement [265] which would see robustness of both the information and the entanglement by topology. The non-local nature of our topology lends itself to exciting advances, for instance, distributing topology across distant parties, topological teleportation, topology exchange between light and matter, and topology transport down non-map preserving channels by judicious choice of the transported photon (e.g., the polarization superposition photon through single mode fibre).

Evidently, an important outcome of this work is the emerging understanding of the link between quantum entangled states and their topology. Topology quantifies the failure of these spaces to factorise into a product, as entanglement does for states in Hilbert space. Our connection reveals a hitherto hidden commonality in seemingly disparate states that allows them to be grouped into entanglement classes by their topology, exposing the potential for topological prediction of entanglement properties. The isomorphism between quantum channels and quantum states [276] suggests that our notion of topological entanglement classes can be used to characterise quantum channels too, grouping them into classes and negating the need to study each channel as a separate entity. We anticipate that the topology will remain intact in all channels that can be shown to be smooth deformations of the state, an open challenge to be addressed.

Our results should inspire new directions in engineering topological quantum states, for instance, for a quantum version of vectorial holography [277], and could benefit from resonant metasurfaces for compact sources of such states [278]. The fact that the skyrmion number can take any quantised integer value should also inspire the use of non-local topology as a high dimensional encoding alphabet which can be useful for quantum communication and information processing. While fuelling new research avenues, we believe that our work holds exciting prospects for information robustness by topology even in non-ideal quantum systems that use entanglement as a resource.

Chapter 5

Topological rejection of noise by non-local quantum skyrmions

The work herein is based on a future publication that is currently under preparation.

In this chapter we demonstrate that the skyrmionic topology of non-local quantum photon states is invariant to noise. Whilst several advancements have been made in improving the robustness of quantum states to environmental contributions of noise, common entanglement witnesses still remain susceptible to high noise levels. We provide an alternative property of these states which is invariant to environmental noise, its non-local skyrmionic topology. Furthermore, we discuss how noise can be considered a smooth deformation of the state, and using a white light source, we experimentally demonstrate the invariance of its topological invariant, the skyrmion number, only changing once the state is maximally mixed.

5.1 Introduction

Quantum entanglement is an essential and viable resource for future quantum technologies. It has been used in contexts involving secure quantum communication such as the use of quantum key distribution protocols [279–281] to create secure satellite based networks [282, 283], quantum computing [284–286], quantum imaging of complex [287] and 3D [288] objects, quantum metrology with entangled probes [289] and lithography as it can be leveraged to beat the diffraction limit [290, 291].

However, in such applications, decoherence arising from noisy detectors (shot noise)[292], stray white light or other irreversible environmental effects play a prevalent role in diminishing our ability to certify, identify or exploit the full benefits of quantum entanglement for information processing. This is because entanglement measures are sensitive to the ratio of signal-to-noise; increasing the noise results in the gradual decay of the state and so the measures themselves decay, as expected [275, 293–295].

On the other hand, high dimensional entangled states are able to push the threshold to which the usual entanglement witnesses certify entanglement and confirm the integrity of the state [39, 296–300]. However, such systems remain susceptible to high noise levels and there is no general consensus about whether they are truly resilient to decoherence or not [40]. Another practical challenge is that these states are difficult to characterize as the number of measurements required scales polynomially with dimensionality and exponentially with the number of particles involved, therefore placing high limitations on the utilisation of quantum entanglement.

In this chapter, we propose an alternative and new approach that instead harnesses a nontrivial topology that emerges from quantum entanglement. To this end, we consider our recently discovered nonlocal skyrmionic topology, a shared property of two entangled photons (depicted in Fig. 5.1), and its associated topological invariant, the skyrmion number, N [142]. We show that the skyrmion number, N , characterising the topological features of our two photon states, is invariant to external noise that can be modelled as depolarization channels and map pure states to noisy Werner states (i.e., isotropic states). Furthermore we provide an intuitive explanation of why isotropic noise can be considered a smooth deformation of the state, i.e., an operation which does not modify its topology. We then present experimental results which verify the invariance of the topology for non local biphoton states with different topologies. Our findings show that the topological invariant can withstand noise levels of up to 99% relative to the signal. We propose these non-local two photon topological states as a candidate for information encoding through noisy quantum channels.

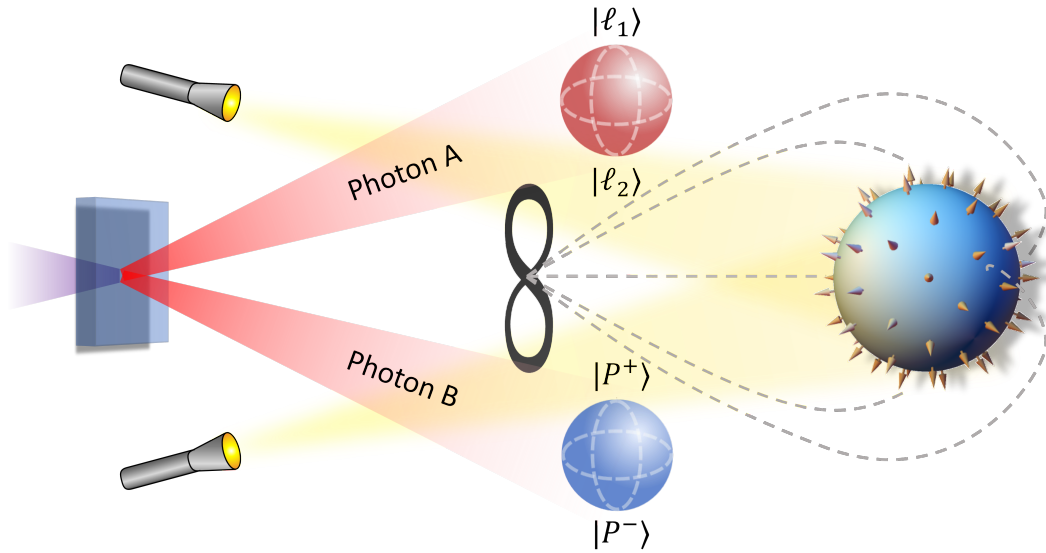


FIGURE 5.1: A biphoton entangled state, exhibits non-local skyrmionic topology which is a shared emergent property of the two photons. Here photon A carries the spatial degree of freedom and photon B carries the polarization degree of freedom. The shared topology is invariant to isotropic noise induced through the presence of an incoherent, white noise source (torches).

5.2 Theory

We will first theoretically show the resilience in our skyrmionic biphoton entangled states emerges and subsequently provide experimental evidence for it. To this end, we first start by constructing a pure non-local biphoton hybrid entangled state which in general can be written as

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|\ell_1\rangle_A |P^+\rangle_B + e^{i\delta} |\ell_2\rangle_A |P^-\rangle_B \right), \quad (5.1)$$

where ℓ_1 and ℓ_2 denote OAM of $\ell_1\hbar$ and $\ell_2\hbar$ per photon, respectively, and $|P^+\rangle, |P^-\rangle$ are orthogonal polarization states, while δ allows for a rotation of the state vector. In

Fig. 5.1 this is depicted graphically with photon A and B's states depicted as living on the Hilbert spaces spanned by the states $\{|\ell_1\rangle, |\ell_2\rangle\}$ and $\{|P^+\rangle, |P^-\rangle\}$, respectively. For states with $|\ell_1| \neq |\ell_2|$ it has been shown that the correlations between these photons form the desired skyrmionic mapping from $\mathcal{R}^2 \rightarrow \mathcal{S}^2$ [142] as shown in Fig. 5.2 (a). In this context, the skyrmion number characterizes the number of times $\mathcal{R}^2$ (stereographically projected Hilbert space of photon A) wraps $\mathcal{S}^2$ (Hilbert space of photon B, the Poincaré sphere in this case). In other words, "should one complete a full set of spatial position measurements on photon A, how many times would the vector characterizing photon B's collapsed states, wrap the entirety of its own Hilbert space. This mapping is evident when expressing the OAM states of photon A in the position basis, using $|\ell\rangle = \int_{\mathcal{R}^2} |\text{LG}^\ell(\mathbf{r})| e^{i\ell\phi} |\mathbf{r}\rangle$ where $\text{LG}^\ell(\mathbf{r})$ are the Laguerre-Gaussian fields and $|\mathbf{r}\rangle$ are position states, thus altering Eqn. 5.1 to

$$|\Psi\rangle = \int_{\mathcal{R}^2} |\mathbf{r}_A\rangle \left(a(\mathbf{r}_A) |P^+\rangle_B + b(\mathbf{r}_A) e^{i\Theta(\phi_A)} |P^-\rangle_B \right) d^2r_A, \quad (5.2)$$

where $a(\mathbf{r}_A) \equiv |\text{LG}^{\ell_1}(\mathbf{r}_A)|$, $b(\mathbf{r}_A) \equiv |\text{LG}^{\ell_2}(\mathbf{r}_A)|$, $\Theta(\phi_A) = \Delta\ell\phi_A + \gamma$ and $\Delta\ell = \ell_2 - \ell_1$. The Skyrmion number of such a state is then calculated using the equation $N = \frac{1}{4\pi} \int_{\mathcal{R}^2} \Sigma_z(x, y) dx dy$ [57] where $\Sigma_z(x, y) = \frac{1}{2} \epsilon_{pqr} S_p \frac{\partial S_q}{\partial x} \frac{\partial S_r}{\partial y}$ is the skyrmion field, with ϵ_{pqr} being the anti-symmetric tensor.

Importantly, the quantum stokes parameters, S_i , are given by the expectation values of the pauli matrices which are calculated by taking the partial trace over photon A such that $S_i = \langle \bar{r} \rangle_A \langle \bar{r} |_A \otimes \sigma_{B,i} \rangle = \text{Tr}(|\bar{r}\rangle_A \langle \bar{r}|_A \otimes \sigma_{B,i} |\Psi\rangle \langle \Psi|)$. From this it can be shown that the skyrmion number depends on the difference between ℓ_1 and ℓ_2 , according to $N = m\Delta\ell$ with $m = \text{sign}(|\ell_1| - |\ell_2|)$ [57, 68, 69, 134]. We note further that the integral in the skyrmion number calculation computes the covered surface area of photon B's parameter space according to the wavefunction mapping given in Eqn. 5.2. Thus by assuming that the parameter space has unit radius, dividing by the surface area of the sphere (4π), we attain the wrapping number N of the non-local topology.

Suppose that our entangled state is not pure, and contains a degree of mixture induced by isotropic noise. Its density matrix can then be written as

$$\rho = p |\Psi\rangle \langle \Psi| + \frac{1-p}{d^2} \mathbb{1}_{d^2}, \quad (5.3)$$

where $p \in [0, 1]$ is a parameter controlling the degree of purity of the state with $p = 1$ giving a pure state ($|\Psi\rangle \langle \Psi|$) and $p = 0$ giving a maximally mixed state ($\frac{1}{d^2} \mathbb{1}_{d^2}$) and $\mathbb{1}_{d^2}$ is the $d^2 \times d^2$ identity matrix. Since the purity, γ , of a state is given by $\text{Tr}(\rho^2)$, it follows that γ is related to p according to

$$\begin{aligned}
\gamma &= \text{Tr} \left(\left(p |\Psi\rangle \langle \Psi| + \frac{1-p}{d^2} \mathbb{1}_{d^2} \right)^2 \right) \\
&= \text{Tr} \left(p^2 |\Psi\rangle \langle \Psi| + 2p \frac{(1-p)}{d^2} |\Psi\rangle \langle \Psi| + \frac{(1-p)^2}{d^4} \mathbb{1}_{d^2} \right) \\
&= p^2 + 2p \frac{(1-p)}{d^2} + \frac{(1-p)^2}{d^2} \\
&= \gamma = p^2 + \frac{1-p^2}{d^2}.
\end{aligned} \tag{5.4}$$

To calculate the skyrmion number we must consider how the quantum stokes parameters transform under the isotropic model given by Eqn. 5.3,

$$\begin{aligned}
S'_i &= \text{Tr} (|\bar{r}\rangle_A \langle \bar{r}|_A \otimes \sigma_{B,i} \rho) \\
&= \text{Tr} (|\bar{r}\rangle_A \langle \bar{r}|_A \otimes \sigma_{B,i} (p |\Psi\rangle \langle \Psi| + \frac{1-p}{d^2} \mathbb{1}_{d^2}))
\end{aligned} \tag{5.5}$$

Using the linearity of the trace operation, the property $\text{Tr}(A \otimes B) = \text{Tr}(A)\text{Tr}(B)$ and the fact that the pauli matrices are traceless, we have

$$\begin{aligned}
S'_i &= \text{Tr} (|\bar{r}\rangle_A \langle \bar{r}|_A \otimes \sigma_{B,i} (p |\Psi\rangle \langle \Psi| + \frac{1-p}{d^2} \mathbb{1}_{d^2})) \\
&= p \text{Tr} (|\bar{r}\rangle_A \langle \bar{r}|_A \otimes \sigma_{B,i} |\Psi\rangle \langle \Psi|) + \\
&\quad \frac{1-p}{d^2} \text{Tr} (|\bar{r}\rangle_A \langle \bar{r}|_A \otimes \sigma_{B,i}) \\
&= p \text{Tr} (|\bar{r}\rangle_A \langle \bar{r}|_A \otimes \sigma_{B,i} |\Psi\rangle \langle \Psi|) \\
&= p S_i
\end{aligned} \tag{5.6}$$

here S_i is the quantum stokes parameters for a pure state. Therefore, isotropic noise acts on the state by multiplying these Stokes parameters, S_i , by a constant factor p . We note however that the skyrmion number assumes normalization such that $\sum_i^3 S_i^2 = 1$. And so we renormalize the state such that this condition is met, therefore this constant p factor does not contribute to the calculation of the skyrmion number. Thus we can conclude that the skyrmionic topology of the state is invariant to isotropic noise so long as a portion of the pure state survives (i.e., $p > 0$ or $\gamma > \frac{1}{d^2}$).

This noise rejection mechanism can be understood intuitively if we consider the spectral decomposition of the pauli observables, given by $\sigma_{B,i} = \lambda_i^+ P_i^+ + \lambda_i^- P_i^-$ where $P_i^\pm = |\lambda_i^\pm\rangle \langle \lambda_i^\pm|$ for positive and negative eigenvalues $\lambda^\pm = \pm 1$. Substituting this into Eqn. 5.6 we get

$$\begin{aligned}
S'_i &= \text{Tr} \left(\mathbb{1}_A \otimes P_i^+ \left[p |\Psi\rangle \langle \Psi| + \frac{1-p}{d^2} \mathbb{1} \right] \right) - \\
&\quad \text{Tr} \left(\mathbb{1}_A \otimes P_i^- \left[p |\Psi\rangle \langle \Psi| + \frac{1-p}{d^2} \mathbb{1} \right] \right) \\
&= (I_{i,\text{pure}}^+ + I_{i,\text{noise}}^+) - (I_{i,\text{pure}}^- + I_{i,\text{noise}}^-),
\end{aligned} \tag{5.7}$$

where $I_{i,\text{pure}}^\pm = \text{Tr}(\mathbb{1}_A \otimes P_i^\pm [p|\Psi\rangle\langle\Psi|])$ and $I_{i,\text{noise}}^\pm = \text{Tr}(\mathbb{1}_A \otimes P_i^\pm [\frac{1-p}{d^2}\mathbb{1}])$. Experimentally, we measure the quantities $I_{i,\text{exp}}^+ = I_{i,\text{pure}}^+ + I_{i,\text{noise}}^+$ and $I_{i,\text{exp}}^- = I_{i,\text{pure}}^- + I_{i,\text{noise}}^-$. Due to the fact that the pauli matrices are traceless, we have that $\text{Tr}(P_i^+) = \text{Tr}(P_i^-)$ and so $I_{i,\text{noise}}^+ = I_{i,\text{noise}}^-$, therefore $S_i' = I_{i,\text{exp}}^+ - I_{i,\text{exp}}^- = I_{i,\text{pure}}^+ - I_{i,\text{pure}}^-$. This shows that each pair of projective measurements required to calculate each Pauli observable carry equivalent noise contributions and therefore noise rejection is inherent in the measurement of the Pauli observables.

Since we have shown that the isotropic noise does not modify the topology, we can consider the action of the isotropic noise channel as a smooth deformation of the state. Normalizing each of the S_i' parameters by $S_0'^2 = (\text{Tr}(|\bar{r}\rangle_A \langle\bar{r}|_A \otimes \mathbb{1}_B \rho))^2$ reveals the smooth deformation that takes place under the addition of isotropic noise to the system. Solving for $S_0'^2$ reveals

$$S_0'^2 = \frac{4p^2}{(p+1)^2}, \quad (5.8)$$

which under the substitution of the purity equation for a 2-dimensional system, $\gamma = p^2 + \frac{1-p^2}{4}$ becomes

$$S_i^2 = \frac{4(\gamma-1)}{4\gamma-1+2\sqrt{3(4\gamma-1)}+\sqrt{3}}, \quad (5.9)$$

which reveals that under the addition of isotropic noise, photon B's state is mapped to a state which lives on a shell found in the interior of its Hilbert space. The radius of this shell is then dictated by the purity of the state according to Eqn. 5.9. This has been depicted in Fig. 5.2 where it can be seen that the noise performs an operation on photon B which maps its state onto a shell within its Hilbert space and thus does not change the topology, " $\mathcal{R}^2$ still maps to a sphere, $\mathcal{S}^2$, albeit one of a smaller radius, thus the topology is not affected." For varying values of purity, $\frac{1}{4} < \gamma \leq 1$, the radius of this shell decreases whilst the skyrmion number remains invariant (depicted in Fig. 5.2 (a,b)). However, when the state is maximally mixed, $\gamma = \frac{1}{4}$, then the calculated skyrmion number is zero as the radius of the shell has reduced to zero, and so $\mathcal{R}^2$ maps to a point (depicted in Fig. 5.1 (c)) therefore the topology has changed. This completes the argument of the invariance of non-local skyrmionic topology to isotropic noise.

5.3 Experimental results and discussion

Now that we have provided theoretical proof for the invariance, we now verify the findings experimentally. To create the non-local skyrmionic states, firstly OAM-OAM entangled pairs of photons are generated through SPDC. The generated entangled pair of photons, photons A and B, are then spatially separated. To generate the desired arbitrary hybrid states, Photon B's spatial information is coupled to polarization through an all-digital SPC process. Photon A and Photon B are then detected with a coupled SLM and SMF for photon A and a set of polarization optics for photons B. The photons are measured in coincidence allowing for a full QST to be performed in order to reconstruct the state. For full experimental details, see section 2.2.1. To induce the isotropic noise in our model, we use an incoherent, white light

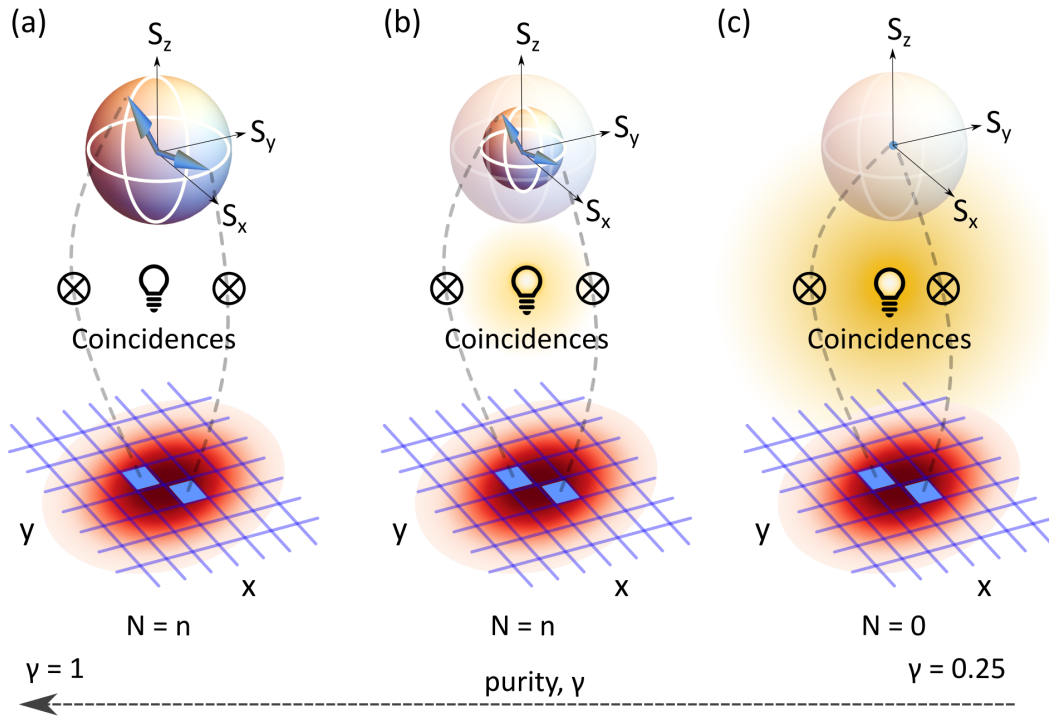


FIGURE 5.2: (a) In the presence of no isotropic noise, i.e the state describing the bi-photon system is pure, we witness a full mapping from real space R^2 to the poincaré sphere, S^2 , of unit radius, with an integer skyrmie number $N = n$. (b) By increasing the isotropic noise contribution to the state, the state becomes partially mixed, thus yielding a full mapping from real space R^2 to a shell within the volume of the poincaré sphere, S^2 , of radius < 1 , whilst still maintaining an integer skyrmie number $N = n$. (c) Further increasing the isotropic noise contribution to the state saturates the state such that it becomes a completely mixed state which results in a mapping from real space R^2 to a point at the centre of S^2 thereby yielding a skyrmie number $N = 0$.

source with variable intensity. This allowed us to controllably scale the p parameter for our isotropic state. The contribution of the white light source was also estimated from the quantum contrast, QC, which estimates the signal to noise (SNR) ratio in our system, reaching values as high as 24.5 and as low as 1.01 for low and high levels of noise, respectively. Furthermore, the impact of the external white light source on the hybrid entangled state was measured using the purity of the reconstructed density matrix. Finally, for each noise level we measured our topological invariant quantity using the reconstructed density matrix.

The skyrmion number was then computed for 7 different states with varying topology ($N \in -3, -2, -1, 0, 1, 2, 3$) for various isotropic noise levels. The data is shown in Fig. 5.3 with insets showing the density matrices (top panel) and stokes parameters (middle panel) for three selected data points for a state with topology $N = 3$ (Fig. 5.3 (a,b,c) show data for states with purity $\gamma = 0.80, 0.61$ and 0.33 and skyrmion number 2.99 ± 0.12 and $2.98 \pm 0.13, 2.98 \pm 0.15$). It is clear from the inset data that the density matrix describing the states tends towards that of the isotropic state, i.e an identity matrix, as the purity is decreased, as expected from theory. Furthermore, the Stokes parameters clearly do not change in structure as the purity decreases but rather the entire distribution is multiplied by some constant factor as expected from

theory, effectively only disappearing once the purity drops to the classical limit of $\gamma = 0.25$. The data shown in Fig. 5.3 (d) (theory shown with a dashed-line and experiment shown with dots) for all 7 topological states, demonstrates that the skyrmion number is invariant to a decrease in purity induced by an increase in isotropic noise. The zoomed-in inset shows experimental data for isotropic noise near the classical limit, clearly the topology changes only once the state is maximally mixed.

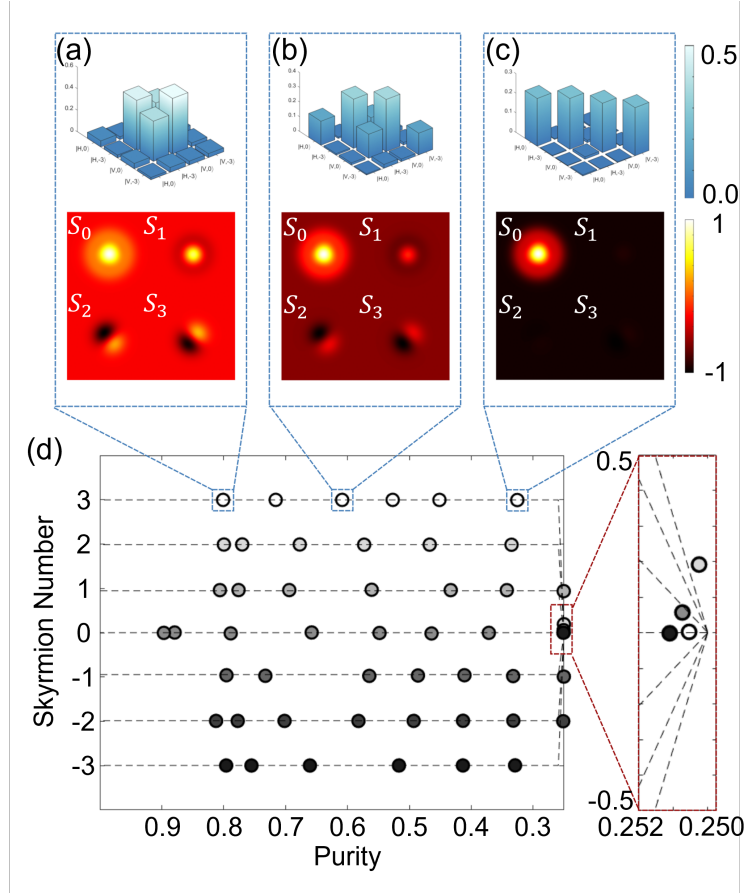


FIGURE 5.3: Experimental data for the characterization of the skyrmion number of non-local quantum states. In the top panel experimental density matrices for purities of (a) 0.80 (b) 0.61 and (c) 0.33 are shown with their corresponding calculated Pauli observables. (d) Skyrmion number (theory plotted as dashed lines and experimental data plotted as points) plotted against purity. Decreasing the purity of the state results in no change of the skyrmion number/topology of the state until the maximally mixed state limit is reached, shown as a zoomed-in inset.

5.4 Conclusion

In conclusion we have shown that the skyrmionic topology of non-local states is invariant under transformations due to isotropic noise and therefore isotropic noise can be thought of as an operation which yields a smooth deformation of the state. In particular, isotropic noise reduces the radius of the hilbert space of the photon which does not carry the spatial degree of freedom. Lastly, we have demonstrated this topological robustness experimentally, showing that even under high isotropic

noise conditions, a topology can still be identified, until the state becomes maximally mixed.

Chapter 6

Conclusion

In this dissertation we investigated the use of structured light techniques in classical and quantum regimes in order to create non-separable vectorial wavefunctions. Furthermore, we realised topological features in classical and quantum light with the focus of extending the already existing classical toolbox as well as presenting a novel notion of non-local topology. We were able to demonstrate the utility of these non-local topologies through the postulation of their use as a new mechanism for state characterization. Lastly, we investigated the robustness of the topological features of the states to noise and found them to be invariant, thus opening future avenues for information encoding using non-local topological properties.

In chapter 2 we discussed classical and quantum structured light techniques. Classical techniques were discussed to generate and control the spatial and polarization DOFs of light using liquid crystal technology and linear polarization optics. Following this, the generation of arbitrary vector modes using dynamic and geometric phase devices was demonstrated. Finally, Stokes polarimetry was employed as a technique for the characterization of vector beams. The classical techniques were then extended into the quantum realm. The generation of high dimensional entangled states through spontaneous parametric down-conversion as well as the detection of OAM states, were discussed. These high-dimensional OAM-OAM entanglement states formed an ideal platform to construct arbitrary 2D hybrid entangled states. In the experimental setup required to generate these arbitrary hybrid states we paid particular attention to the spatial-to-polarization conversion system allowing for the conversion of initial OAM entangled states to hybrid entangled states. To characterize such states we discussed 2D quantum state tomography, a measurement technique capable of fully characterizing 2D quantum states thus allowing for the extraction of useful parameters from the state such as concurrence, fidelity and purity. The quantum analogue to Stokes polarimetry was also developed.

In Chapter 3 the generation of optical skyrmions through the use of Bessel Gaussian modes was demonstrated. Observing the transformations of their polarization structures on propagation revealed a beating between skyrmion types whilst leaving their topology unaltered. Furthermore, we observed the precession of the Stokes vectors with propagation, analogous to the precession of magnetic spins in the presence of a magnetic field. Lastly, through the use of a frozen wave construction, it was shown that the evolution of the central on-axis polarization state can be precisely tuned during propagation.

In Chapter 4 various 2D hybrid entangled states were generated. We then presented a novel result, that is the generation of non-local skyrmionic entangled states. We then discussed the characterization of quantum entangled states according to their

topology along with the implications that come along with it, such as the characterization of any transformation that does not alter the topology as a smooth deformation or coordinate transformation. Lastly, we investigated the resilience of these non-local topological features against entanglement decaying noise and demonstrated that such noise can be considered to be a simple coordinate transformation.

Lastly, in Chapter 5 we built upon the ideas of chapter 4, where we then considered the effects of isotropic noise on the non-local topology of entangled states. We demonstrated through theory and experiment that isotropic noise does not alter the topology of these states except when the maximally mixed state limit is reached. Furthermore, we described the isotropic noise transformation, as a simple resizing of the state space, thus not altering the topology as expected.

Appendix A

Stereographic Projection

To compactly represent the topology of an entangled state we may stereographically project the polarization and spatial information onto a single sphere. The spatial projection of $\mathcal{R}^2$ to $\mathcal{S}^2$ is done via the mapping

$$(x', y', z') = \left(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{-1 + x^2 + y^2}{1 + x^2 + y^2} \right) \quad (\text{A.1})$$

where $(x, y) \in \mathcal{R}^2$ and $(x', y', z') \in \mathcal{S}^2$ with the usual relation $x'^2 + y'^2 + z'^2 = 1$ holding for all x', y', z' . An example of this mapping is shown in Figure A.1a, with additional probability density information also being carried over to the surface of the sphere. Expressed in polar coordinates the mapping can be written as

$$(\phi, \theta) = \left(2 \arctan \left(\frac{1}{r} \right), \Theta \right) \quad (\text{A.2})$$

where $r, \Theta \in \mathcal{R}^2$ and $\phi, \theta \in \mathcal{S}^2$ with . Beyond the inherent compactness of this representation, one is free to scale r and Θ by arbitrary real constants $a, b^2 > 0$ which allow for a simple scheme to restructure the representation so as to emphasize regions of interest. The validity of such a restructuring of the representation follows from the fact that the topology is invariant to smooth deformations caused by coordinate changes, and in fact, any scaling can be reversed through a second inverse scaling.

Next, for a clearer depiction of the vectorially textured field formed in the entanglement, we represent the arbitrary elliptical states found in coincidence with spatial measurements on the partner photon, with Poincaré vector states, as shown in Figure A.1b. Extracting the Quantum Stokes Parameters from the entangled state, allows for the description of a spatially dependent vector $\mathbf{S} = (S_x(r, \Theta), S_y(r, \Theta), S_z(r, \Theta))$ which carries the local normalization condition $\mathbf{S} \cdot \mathbf{S} = 1$, thus describing a vector field with vectors pointing from the centre of the Poincaré sphere to its surface and therefore denoting particular polarization states.

Combining these two representation schemes, we can compactly represent the entirety of the topological information on the surface of a sphere, an example of which is shown in Figure A.1c.

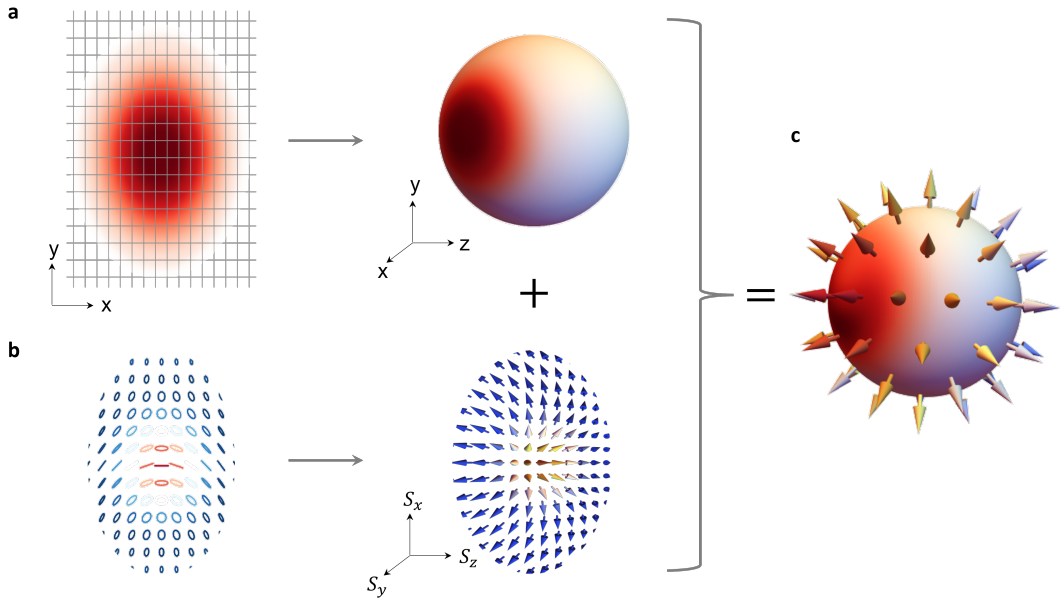


FIGURE A.1: Stereographic projection of entangled state onto state space **a**, Through a stereographic projection, the transverse plane, $\mathcal{R}^2$, is mapped onto the surface of a sphere $\mathcal{S}^2$. **b**, Arbitrary elliptical polarization states are represented as state vectors pointing from the centre of the Poincaré sphere to its surface. **c**, Combining the stereographic projection of $\mathcal{R}^2$ to $\mathcal{S}^2$ with the exchange of polarization ellipses for Poincaré state vectors, we get the condensed representation of the topology of our entangled state.

Appendix B

Topological resilience against quantum Gouy phase

Another topical example that has recently been explored is that of quantum gouy phase [272]. In applying this approach to two photons, we consider the propagation of photon A from an initial plane at $z = 0$ to an arbitrary final plane at $z = z$. The spatial LG modes of photon A have the general structure (for a radial quantum number of $p = 0$) following

$$LG_\ell(\bar{r}, z) = |LG_\ell(\bar{r}, z)| e^{i\ell\phi} e^{i\eta(r,z)} e^{i\Phi_\ell(z)} \quad (\text{B.1})$$

where $|LG_\ell(\bar{r}, z)| \propto \left(\frac{r}{w(z)}\right)^{|\ell|} e^{-\left(\frac{r}{w(z)}\right)^2}$ is the radially dependent amplitude with $\bar{r} \in \mathcal{R}^2$ representing the transverse coordinate of length $r = |\bar{r}|$. The scaling parameter $w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2}$ controls the radial extent of the field, while z_R is the rayleigh range and $e^{i\ell\phi}$ is the characteristic field profile of the vortex mode with the corresponding topological charge ℓ . Here, $\eta(r, z)$, is the phase gained due to curvature, and $\Phi_\ell(z)$ is the propagation dependent Gouy phase that is given by (for $p = 0$)

$$\Phi_\ell(z) = (|\ell| + 1) \arctan\left(\frac{z}{z_R}\right). \quad (\text{B.2})$$

Accordingly, we can modify our two-photon state from Eq. (1) in the main text to read as

$$|\tilde{\Psi}\rangle = \int d^2r_A \left(a(r_A, z) |H\rangle_B + b(r_A, z) e^{i\tilde{\Theta}(\phi_A, z)} |V\rangle_B \right) |\bar{r}_A\rangle, \quad (\text{B.3})$$

where $a(r_A, z) \equiv |LG_{\ell_1}(r_A, z)|$ and $b(r_A, z) \equiv |LG_{\ell_2}(r_A, z)|$ while $\tilde{\Theta}(\phi_A, z) = \arg(LG_{\ell_2}(r, z)) - \arg(LG_{\ell_1}(r, z))$. The radial amplitude change in the field is somewhat trivial as it is not mode dependent, which means that both LG functions scale by the same z -dependent scaling factor. To this end, the radial scaling can be seen as resulting from a change in radial coordinates via a propagation dependent scaling factor, thus mapping the radial coordinate as

$$r_A \rightarrow r'_A = \frac{r_A}{1 + \left(\frac{z}{z_R}\right)^2}. \quad (\text{B.4})$$

Now, for the propagation dependent phases, we have

$$\tilde{\theta}(\phi_A, z) = \arg(LG_{\ell_2}(r, z)) - \arg(LG_{\ell_1}(r, z)) = \Delta\ell\phi_A + \Delta\phi \quad (\text{B.5})$$

having the same form as $\theta(\phi_A) = \Delta\ell\phi_A + \gamma$, from Eq. (1) in the main text, where we have the constant phase γ replaced by a z -dependent phase, $\Delta\phi(z) = (|\ell_2| - |\ell_1|) \arctan\left(\frac{z}{z_R}\right)$ (See Figure B.1 (A) which shows the simulated behaviour of (z) for different topologies/skyrmion numbers, N , with propagation). Thus the propagation can be considered a transformation of the form

$$|\Psi\rangle \rightarrow |\tilde{\Psi}\rangle = d^2 r_A \left[a(r'_A) |L\rangle_B + b(r_A)' e^{i\tilde{\theta}(\phi_A)} |R\rangle_B \right] |r_A\rangle, \quad (\text{B.6})$$

which yields the following coordinate transformations

$$r_A \rightarrow r'_A = \frac{r_A}{1 + \left(\frac{z}{z_R}\right)^2}, \quad (\text{B.7})$$

$$\tilde{\theta}(\phi_A, z) \rightarrow \tilde{\theta} = \Delta\ell\phi_A + \Delta\phi \quad (\text{B.8})$$

This confirms that propagation is a smooth transformation which does not alter the topology, specifically, it scales the radial coordinate by a z -dependent factor, and introduces a z -dependent relative phase by way of the Gouy phase and simply yields a change in the texture (not topology) of the skyrmionic state, as shown in Figure B.1 (B) where simulations were performed for skyrmionic states with $N \in \{-3, -2, -1, 1, 2, 3\}$ and various relative phase values, $\gamma \in [0, 2\pi)$. In Figure B.1 (B), the general case for relative phase was considered, which by default covers any possible phase that may be induced by quantum gouy phase. The simulations agree with theory and demonstrate that as expected, the non-local topology is invariant to addition of relative phase. Furthermore, Figure B.1 (C) depicts the effect of relative phase on the topological texture of the state, simply yielding a rotation of the state vectors in place, whilst not altering the global topological feature of the state. For $N = 1$, B.1 (C) shows a transition between Néel ($\gamma = 0, \pi$) and bloch ($\gamma = \frac{\pi}{2}, \frac{3\pi}{2}$) texture types.

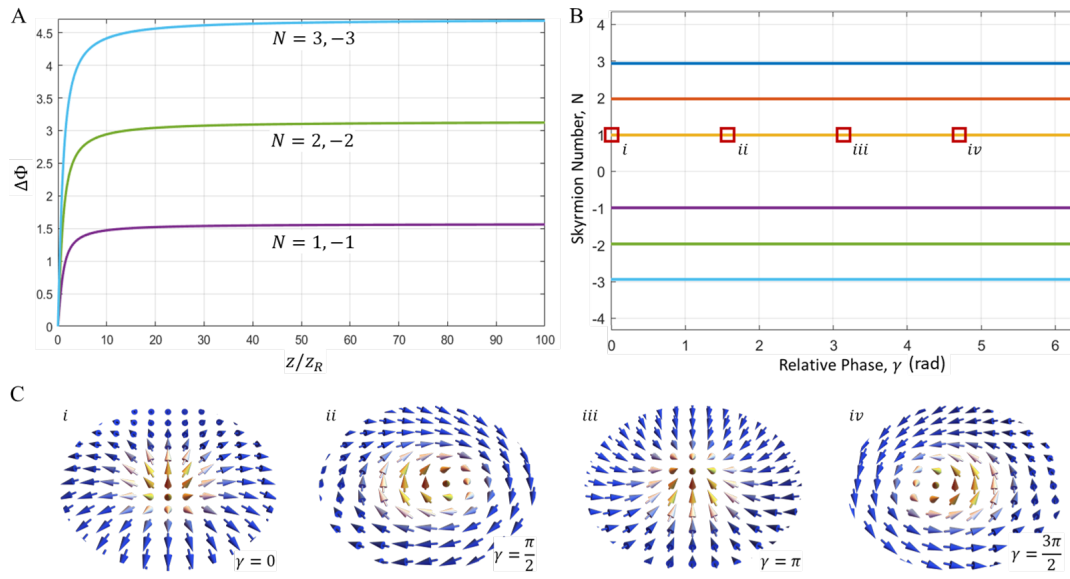


FIGURE B.1: Relative phase as a smooth deformation. (A) Simulation results showing the behaviour of relative quantum gouy phase, $\Delta\Phi$ for different topologies/skyrmion numbers. $\Delta\Phi(\frac{z}{z_R})$ asymptotically approaches $|N|\frac{\pi}{2}$ as $z \rightarrow \infty$. (B) The skyrmion number computed for different relative phase values, clearly demonstrating that the non-local topology is invariant to a change in relative phase and thus can be considered a smooth deformation. (C) If we consider the textures of the skyrmionic states, we see that they are modified by a change in relative phase. Some examples are shown for relative phases $\gamma = 0, \frac{\pi}{2}, \pi$ and $\gamma = \frac{3\pi}{2}$, which correspond to the textures Néel, Bloch, Néel and Bloch, respectively.

Appendix C

Additional Data Sets for "Non-local skyrmions as topologically resilient quantum entangled states of light"

C.1 Experimental Data

Skyrmion	N_{exp}	F	C
$\{-3, \frac{\pi}{2}\}$	$-2.999 \pm 3.292 \times 10^{-5}$	0.9724	0.8472
$\{-3, 0\}$	$-2.999 \pm 4.654 \times 10^{-5}$	0.9683	0.8655
$\{-1, \frac{\pi}{2}\}$	$-0.981 \pm 5.458 \times 10^{-4}$	0.9637	0.7003
$\{-1, 0\}$	$-0.982 \pm 5.087 \times 10^{-4}$	0.9818	0.7351
$\{0, \frac{\pi}{2}\}$	$-0.004 \pm 1.231 \times 10^{-2}$	0.9926	0.9147
$\{0, 0\}$	$-0.008 \pm 8.315 \times 10^{-3}$	0.9955	0.8921
$\{1, \frac{\pi}{2}\}$	$0.978 \pm 7.518 \times 10^{-4}$	0.9559	0.8438
$\{1, 0\}$	$0.973 \pm 5.993 \times 10^{-4}$	0.9498	0.7381
$\{3, \frac{\pi}{2}\}$	$2.998 \pm 3.368 \times 10^{-5}$	0.9189	0.8346
$\{3, 0\}$	$2.998 \pm 4.558 \times 10^{-5}$	0.9161	0.8165

TABLE C.1: Table of experimental data. The Skyrmion topology, N , and type ϕ is given in the form $\{N, \phi\}$ along with the associated measured values for the experimental skyrme number, N_{exp} , the fidelities of each reconstructed density matrix, F , and their concurrences.

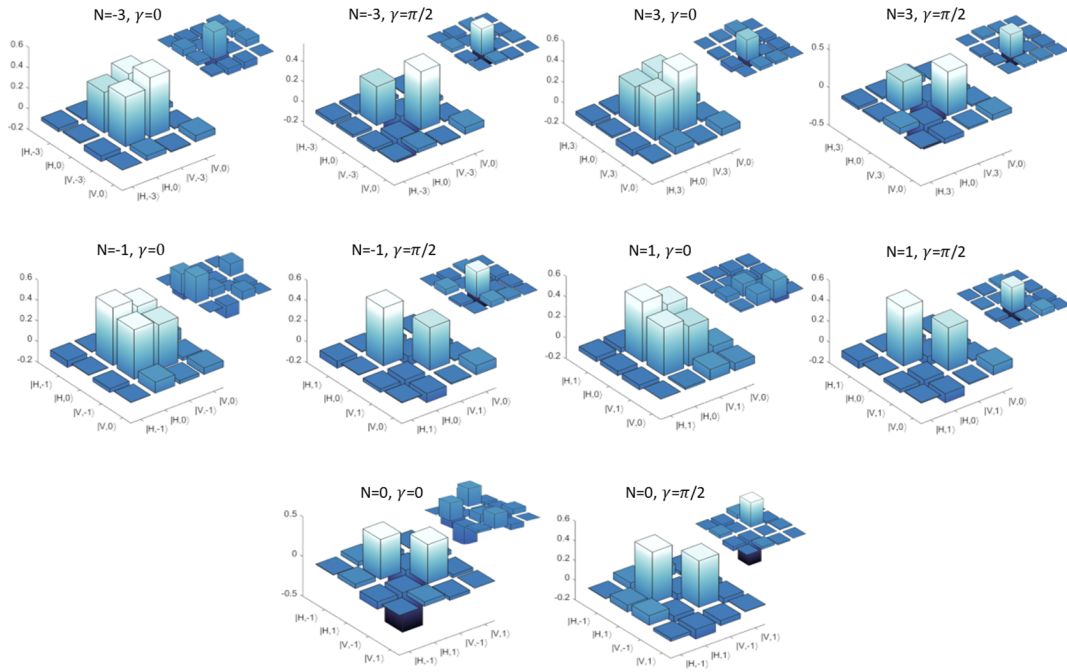


FIGURE C.1: Reconstructed Density Matrices. The real part of the reconstructed density matrices (imaginary parts are shown as insets) for all the data discussed in the paper is shown above. The skyrmion number and γ values for each Density Matrix representing our created states are also shown. The corresponding obtained fidelities when compared with theoretically maximally entangled states, are shown in table C.1.

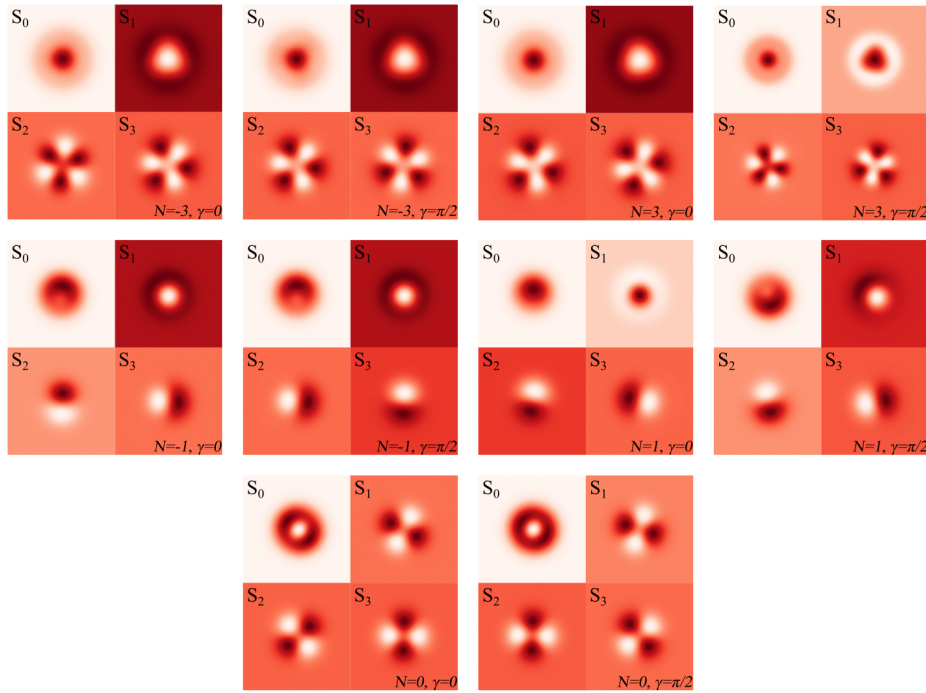


FIGURE C.2: Reconstructed Quantum Stokes. The reconstructed Stokes parameters for all the data discussed in the paper is shown above, with skyrme number and γ given as insets. The data shown above is globally normalised against S_0 , however when calculating the skyrme number, each field is locally normalized such that $S_1(x, y)^2 + S_2(x, y)^2 + S_3(x, y)^2 = 1 \forall x, y \in R^2$ in order for every point of the Stokes vector field to map onto the surface of the Poincaré sphere.

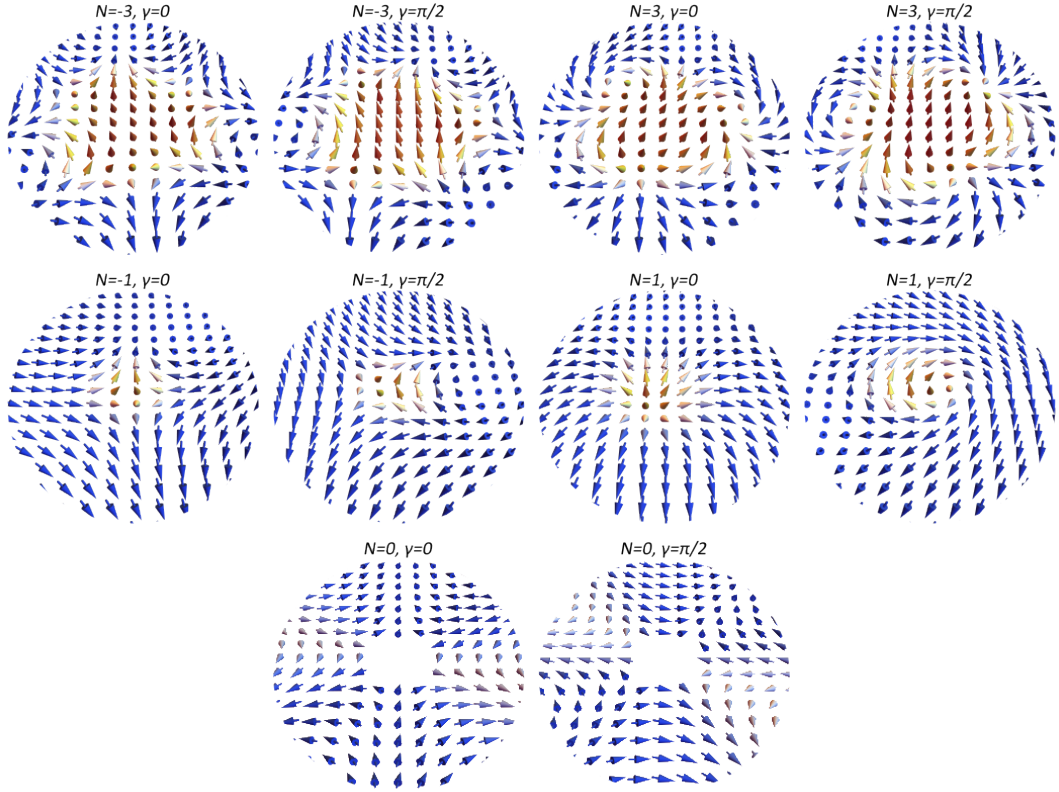


FIGURE C.3: Reconstructed state topologies. The measured topological structure for all the data discussed in the paper is shown above. Vector fields for states with varying topologies (i.e varying skyrme number) and γ are given as insets. The above vector fields are formed by sub-sampling the Stokes vector, $\vec{S}$. As expected, since the entangled states were created in the $|H\rangle, |V\rangle$ basis, which corresponds to the stokes parameter $S_1 = S_x$, we have that at the centre of the vector field, the vectors point to the left, corresponding to vertical polarization and as one moves away from the centre we attain elliptical polarization states (indicated with arrows pointing in and out of the plane) till at the periphery we obtain vectors pointing to the right indicating horizontal polarization.

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