COMPLETE CLASSIFICATION FOR THE (1+1)D NON-LINEAR BOUSSINESQ EQUATIONS

by

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Declaration

I declare that the contents of this thesis are original except where due references have been made. It has not been submitted previously for any degree from any other institution.

C.Carvalho
C. Carvalho

In loving memory of Victor Eugenio Rodrigues

(12 April 1971 - 8 November 2019)

“If an answer does not give rise to a new question from itself, it falls out of the dialogue” -

M.M. Bakhtin

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Abstract

This thesis contributes to the known analytical solutions of the (1+1)D non-linear Boussinesq (NB) equations, which effectively describe solitary wave motion in shallow water (typically used to describe tsunami wave motion) using a systematic symmetry approach. These results can be used to enrich the analytical validation of numerical models for tsunami detection systems.

Tsunami detection systems based on numerical models provide real-time forecasting and inundation mapping, playing a critical role in evacuation plans. These models have become more sophisticated over the past three decades through precise and rigorous analytical and experimental validation. Although analytical and experimental benchmarking cannot ensure that numerical models that pass benchmark tests will always produce realistic inundation predictions, validation largely reduces the uncertainty in the results to initial geophysical conditions [1].

The true benefit of analytical solutions in tsunami model predictions is the ability to identify the dependence of the desired results (such as run-up height, wave height and wave speed) on the independent variables of the model (such as water depth and beach slope). Problem scaling is also made simpler through analytical calculations, as opposed to numerical techniques, which usually require repeated numerical computations. Lastly, by comparing the numerical solutions to exact analytical solutions, systematic errors can be identified, making analytical solutions useful in validating complex numerical models with realistic applications [1]. The use of conserved quantities as an analytical validation mechanism is a good example of how analytical solutions can assist with numerical tsunami model validation. Upon substitution of these quantities into the numerical models, these conserved quantities should remain constant throughout the calculation. Although analytical solutions are powerful for the reasons previously stated, these solutions are complex to obtain, and therefore, not as widely used in comparison to experimental validation for tsunami detection system development.

The (1+1)D Boussinesq equations investigated in this work are considered for the equa-

tions' ability to model tsunamis via solitary wave motion. Three models are investigated, namely the (1+1)D Boussinesq equations for a moving bathymetry, the (1+1)D linear Boussinesq (LB) equations at a constant water depth and the NB equations at a constant water depth.

We obtain three cases of solutions for the NB equations at a constant water depth dependent on the wave speed c^2, namely (i) $0 < c^2 < \infty$, $c^2 \neq \frac{1}{15}$ and $c^2 \neq \frac{1}{3}$; (ii) $c^2 = \frac{1}{15}$ and (iii) $c^2 = \frac{1}{3}$. To our knowledge, the invariant exact travelling wave solutions for the NB equations for cases (i) and (iii) are new and can be used in analytical validation of numerical models for tsunami detection systems, whilst the remaining case, case (ii), is identical to that displayed in the work of Dutykh and Dias [2].

We provide physically realistic explanations of the results relating to the horizontal component of the fluid velocity and surface elevation of the solitary wave for the invariant exact solutions. This is done by considering seven cases dependent on the wave speed c^2, namely $0 < c^2 < \frac{1}{15}$, $c^2 = \frac{1}{15}$, $\frac{1}{15} < c^2 < \frac{1}{3}$, $c^2 = \frac{1}{3}$, $\frac{1}{3} < c^2 < 1$, $c^2 = 1$ and $1 < c^2 < \infty$. To our knowledge, these are newly discovered solutions and results.

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Nomenclature

Symbol	SI Units	Description
t	s	Time
x, z	m	Spatial components (or lengths)
η	m	Surface elevation
h	m	Water depth
l	m	Wave length
h_0	m	Characteristic water depth
a_0	m	Wave amplitude
ζ	m	Dynamic water depth
u, w	$m.s^{-1}$	Fluid velocities in the x and z directions respectively
g	$m.s^{-2}$	Gravitational acceleration
m	kg	Mass
ρ_a	$kg.m^{-2}$	Area density
$\mathcal{E}_{fd}$	$kg.s^{-3}$	Energy flux density
$\dot{m}$	$kg.s^{-1}$	Mass flow rate
p	$kg.m.s^{-1}$	Momentum
$\mathcal{E}$	$kg.m^2.s^{-2}$	Energy

Chapter 1

Introduction

As tsunamis, a series of long oceanic waves generated by impulsive geophysical events such as earthquakes, propagate shoreward, their height increases significantly due to shoaling. Upon reaching the shoreline, the effects of this powerful phenomenon are observed as the waves travel inland for relatively long distances destroying anything in their path. Since 1998, there have been more than 250 000 deaths due to tsunamis worldwide, including more than 227 000 deaths due to the Indian Ocean tsunami in 2004 [3]. The Indian Ocean tsunami in 2004 displaced 1.8 million people, spread over a dozen countries, and is by far the worst natural disaster of its kind in recorded human history. In financial terms, the humanitarian aid mobilization (over USD 14 billion) is the largest ever international response to a natural disaster on record [4]. Given the potentially deadly and far reaching consequences of tsunami events, their ongoing investigation is of prime importance and forms the basis of this research. It is crucial to note that most of the damage associated with tsunamis is related to their run-up at the coastline. Therefore, the inundation modelling of these waves is critical in order to gain insight into the maximum run-up height. By accurately modelling this phenomenon, town planners may construct evacuation routes and manage the placement of infrastructure in coastal areas more efficiently.

The aim of this thesis is to contribute to the known analytical solutions of the NB equations, which effectively describe solitary wave motion in shallow water (typically used to describe tsunami wave motion), using a systematic symmetry approach. We attempt to determine the effect that linearizing these equations has on the solution. We also endeavour to provide new conserved quantities with physical significance for both the linear and non-linear $(1+1)D$ Boussinesq equations. These results enrich the analytical validation of numerical models for tsunami detection systems. Tsunami detection systems based on numerical models provide real-time forecasting and inundation mapping, playing a critical role in evacuation plans. Inaccuracies in these models can have serious implications, such as lives lost or unnecessary evacuations, potentially leading to a loss of credibility in these systems [1].

An initial attempt at a theory of water waves was made by Newton in 1687, with his work being used extensively throughout the mid-17^{th} century [5]. The 18^{th} and early 19^{th} centuries gave rise to theoretical advances in linear wave theory, made by French mathematicians Laplace, Lagrange, Poisson and Cauchy. During this time, neighbouring Germany saw advances in non-linear wave theory where Gerstner considered non-linear waves, while the Weber brothers performed experimental work in the same field. It was only in 1837 that Britain started making contributions to wave theory. Over the next decade, Russell, Green, Kelland, Airy and Earnshaw made advances in the field, with their findings used in subsequent work by Stokes, Boussinesq, Rayleigh and others. A concise summary of many of these equations is presented in a book written by Johnson [6].

A system of equations that is of particular interest within this context is the system of non-linear, dispersive Boussinesq equations, a generalization of the Korteweg-de Vries (KdV) equation when considering a higher-dimensional co-ordinate system. Boussinesq equations are widely used in coastal and oceanic engineering. One example of their application is in tsunami wave modelling via a solitary wave form. Solitary wave motion in shallow water is effectively described by these equations, where the shape and propagation velocity remain constant as a consequence of the balancing of the non-linear and dispersive effects.

Solitary waves or combinations of negative and positive solitary-like waves (N-waves) are often used to simulate the run-up and shoreward inundation of waves [7, 8, 9, 10, 11, 12, 13, 14, 15]. Such waves are easily reproduced in an experimental set-up and can be described entirely by their amplitude, making the modelling of tsunamis via solitary waves advantageous. In fact, important characteristics of the run-up can be obtained analytically, numerically and experimentally by modelling tsunamis in this fashion. Thus, solitary waves or N-waves will be the focus of this thesis, specifically focusing on tsunamis.

Significant developments of the Boussinesq equations occurred in 1871 and 1872. Of particular importance was the paper presented by Boussinesq on solitary wave motion in 1871 [16]. Boussinesq's derivation is restricted to (1+1)D Boussinesq equations with a flat bottom. These equations have a hyperbolic structure (identical to the non-linear shallow-water equations) and contain high-order derivatives, which model the wave dispersion. Further development of these equations is seen in work done by Peregrine [17], Nwogu [18], Wei *et al.* [19] and Madsen and Schaffer [20]. Various other forms of the Boussinesq equations exist, depending on the choice of the velocity variable. In the majority of cases, the velocity is chosen at an arbitrary water level or the depth-averaged velocity vector. The right choice of the velocity variable can significantly improve the propagation of moderately long waves, as the model performance is highly dependent on the linear dispersion properties [21].

There has been renewed interest in the study of the Boussinesq equations in recent years. Such a study regarding the dynamics of shallow water waves has been presented by Krishnan *et al.* [22]. Chen [23] presents several systematic ways to determine exact travelling wave solutions of the systems describing small amplitude long waves in a water channel, formally equivalent to the classical first-order Boussinesq system. Yang *et al.* [24] makes use of the modified Jacobi elliptic function expansion method and the pseudo spectral method to obtain solutions of homogeneous and inhomogeneous dissipative Boussinesq equations. Dutykh and Dias [2] present two models in which the addition of a dissipative term to the Boussinesq equations is considered. The dissipative Boussinesq equations are solved numerically to determine whether the solutions remain close to the non-dissipative Boussinesq equations. By following work done by Chen [23], Dutykh and Dias [2] obtain analytical solutions for the NB equations in order to validate numerical results found. The results they obtain are not physically realistic, as the fluid velocity of the solitary wave in the horizontal direction is negative where it is expected to be positive for a solitary wave propagating from right to left.

Early investigations of long wave run-up have mainly considered an analytical approach. Carrier and Greenspan [25] pioneered the investigation of non-linear run-up analysis and obtained a periodic solution using a hodograph transformation of the non-linear shallow water equations. A solution for run-up of non-breaking solitary waves has been obtained by Synolakis [7]. In his work, he simplifies the transformation used by Carrier and Greenspan [25] and derives a significant wave breaking criterion, comparing favourably to experiments. Zhang [26] obtains a linear solution for the 3D run-up of a solitary wave using Fourier analysis. By considering varying linear topographies, Kanoglu and Synolakis [27] obtain solutions using the linear shallow water equations. Aydin and Kanoglu [28] solve the non-linear shallow water wave equations over a linearly sloping beach as an initial boundary value problem under general initial conditions using an eigenfunction expansion method. The travelling wave solutions of the foam drainage equation and non-linear evolution equations of fourth-order are found by Rani *et al.* [29] using the (G'/G)-expansion method.

Since 1970, numerical approaches have become the preferred method of solution due to the complex beach geometries and wide range of wave parameters that exist. An analysis of wave run-up on rough inclines, considering long shallow water equations using a Lax-Wendroff scheme, has been presented by Kobayashi *et al.* [30]. Liu *et al.* [31] uses an explicit leap-frog finite difference model to simulate tsunami propagation based on linear shallow water equations. A Lagrangian finite-element Boussinesq model is used by Zelt [8] to describe breaking and non-breaking solitary wave run-up on impermeable beaches. The results compare favourably with Synolakis' [32] maximum run-up laboratory data, where the run-up of non-breaking long waves on plane beaches is investigated. Titov and Synolakis [33] present a variable grid finite difference approximation to an identical physical situation, which Zelt [8] considers without taking into account the artificial viscosity. The scheme effectively predicts the solitary wave run-up without making any assumptions about viscosity and friction. In turn, non-breaking and breaking solitary wave run-up has been investigated by Li and Raichlen [14], where the post breaking condition is considered via a bore structure. A weighted non-oscillatory shock capturing scheme is used to solve the linear shallow water equations and an accurate estimate for the energy dissipation associated with wave breaking is given. A moving boundary technique has been developed by Lynett *et al.* [34] to investigate wave run-up and run-down with depth-integrated equations. They solve highly non-linear and weakly dispersive equations via a high-order finite difference scheme. Amini-Afshar [35] uses the finite-volume method for the spatial discretization of odd equations, modelling the propagation of fifth-order Stokes waves in deep water conditions. Fan [36] presents an Euler solver for non-linear water waves using a modified staggered grid and Gaussian quadrature approach. A meshless method, based on exponential basis functions, has been developed by Zandi *et al.* [37]. This method is used to simulate the propagation of solitary waves and run-up on the slope. It has been verified, through experiments, that this new formula is more accurate than the preceding formulae in predicting the maximum run-up of non-breaking solitary waves. Smith *et al.* [38] have investigated non-breaking solitary waves and measurements of swash zone dynamics on an inclined beach, using a boundary integral technique, combined with a boundary layer model. The velocity profiles at the lower regions of the beach compare favourably to experiments, whilst the maximum run-up is over predicted by this method.

Tsunami numerical models have become more sophisticated over the past three decades through precise and rigorous analytical and experimental validation. Although analytical and experimental benchmarking cannot ensure that numerical models that pass benchmark tests will always produce realistic inundation predictions, validation largely reduces the uncertainty in the results to initial geophysical conditions [1]. The true benefit of analytical solutions in tsunami model predictions is the ability to identify the dependence of the desired results (such as run-up height, wave height and wave speed) on the independent variables of the model (such as water depth and beach slope). Problem scaling is also made simpler through analytical calculations, as opposed to numerical techniques usually requiring repeated numerical computations. Lastly, by comparing the numerical solutions to exact analytical solutions, systematic errors can be identified, making analytical solutions useful in validating complex numerical models with realistic applications [1]. The use of conserved quantities as an analytical validation mechanism is a good example of how analytical solutions can assist with numerical tsunami model validation. Upon substitution of these quantities into the numerical models, these conserved quantities should remain constant throughout the calculation.

A common analytical approach used to solve wave equations requires the use of conservation laws. Conservation laws play an important role in studying the integrability of equations. The family of KdV equations emit infinitely many conservation laws, and hence, the equations may be fully integrable and written as a Hamiltonian system [6]. The conservation laws and conserved quantities obtained for a system of equations may also assist in the solution process. Upon consideration of the family of KdV equations, the integral relations, acting as existence conditions, are properties of the wave and its motions; in fact they represent the conserved quantities of mass, momentum and energy. These integral relations (i.e. conservation laws) define how the system behaves and must be conserved at the initial state. For a solitary wave unbounded in the horizontal directions, these initial quantities must be conserved and, as the wave evolves, must still be maintained. Given that initial quantities are difficult to obtain in general, conservation laws provide us with properties at the initial state of the system. Solitary wave and soliton solutions satisfy these conservation laws. However, periodic solutions do not, and therefore, the latter are not considered as possible solutions for the solitary wave [6]. Thus, obtaining conservation laws is an important part of the process of obtaining physically meaningful solutions.

There are many methods used to obtain conservation laws and conserved quantities of a system of Partial Differential Equations (PDEs) [39]. By considering a method heavily reliant on physical insight of the problem, Johnson [6] established the conservation of mass, momentum and energy relating to the LB equations. A systematic method of obtaining conservation laws and conserved quantities is the Multiplier method. This method was first introduced by Steudel [40] and has been used extensively by Naz [39] in obtaining conservation laws and conserved quantities for the various KdV equations. We will employ this method for the purpose of isolating such conserved quantities.

The approach of obtaining symmetries and using these symmetries to obtain invariant solutions for a given PDE was developed by Sophus Lie (1842-1899). He developed the theory of Lie group analysis, providing an algorithmic method for obtaining invariant solutions to a given PDE via the use of Lie point symmetries [41, 42]. Using a symmetry approach to obtain analytical solutions to water wave equations is a common method, as the equations describing these waves are usually non-linear and dispersive in nature, making them difficult to solve. Rui *et al.* [43] uses a Lie symmetry approach to determine the invariant solutions of the (2+1)D Boussinesq equation. Conservation laws for the underlying equation are derived by utilizing the new conservation theorem and the partial Lagrange approach. Chen and Jiang [44] make use of the Multiplier method and Ibragimov theorem to determine conservation laws and invariant solutions of the generalized Kaup Boussinesq equation.

This thesis comprises of eleven chapters. Chapter 2 provides a brief overview of the theory of Lie group analysis. In Chapter 3, we consider the non-linear, dispersive Boussinesq equations, derived by Dutykh and Dias [2] from first principles. We derive the NB equations representing a solitary wave at a constant water depth h, propagating from left to right. We further derive the LB equations representing a solitary wave at a constant water depth h, propagating from left to right, by considering a linear approximation of the NB equations on condition that the wave amplitude is small [6].

In Chapter 4, we obtain the conserved quantities for the LB equations in dimensional form by employing the systematic Multiplier method. The full set of newly discovered Lie point symmetries for the LB equations is obtained in Chapter 5.

The Multiplier method is again used in Chapter 6 to obtain the conserved quantities for the NB equations in dimensionless form. We obtain the full set of newly discovered Lie point symmetries for the NB equations in Chapter 7. In Chapter 8, a linear combination of the time and spatial symmetries leads to the invariant exact travelling wave solutions. Results relating to the horizontal component of the fluid velocity and the surface elevation for the NB equations are discussed in Chapters 9 and 10. The main results and conclusions of this thesis are summarised in the final chapter.

We have included five appendices. Appendix A contains detailed calculations of the multipliers and conserved vectors for the LB equations. Appendix B contains expressions for the extended infinitesimal generators and for the determining equations for the LB equations. Detailed calculations relating to the multipliers for the NB equations are presented in Appendix C. In Appendix D, we present expressions for the extended infinitesimal generators and for the determining equations for the NB equations. In Appendix E, we present tables of results for the NB equations.

Chapter 2

Methodology

In this chapter, we outline the definitions and concepts relating to Lie group analysis used throughout this thesis. In particular, we discuss the concept of conservation laws and conserved vectors and its relation to conserved quantities. We discuss the Multiplier method, the method chosen to obtain conservation laws in this work.

2.1 Conservation laws and conserved vectors

Ibragimov [42] discusses the concept of a conservation law for a differential equation

$$F(x, u, u_1, \ldots, u_k) = 0, \tag{2.1}$$

where the partial derivatives are denoted by a subscript. This concept is motivated by the conservation of such quantities as energy, linear and angular momentum, etc. for equations that arise in classical particle mechanics. These quantities (here denoted by T) are conserved in the sense that they are constant on each trajectory, i.e. if t denotes time, $q(t)$ a trajectory, and $\dot{q}(t)$ the corresponding velocity, then $T(t) = T(t, q(t), \dot{q}(t))$ satisfies the equation

$$\frac{dT}{dt} = 0. \tag{2.2}$$

The conservation law given by (2.2) generalizes to continuum mechanics in the form

$$\frac{\partial \tau}{\partial t} + div\rho = 0, \tag{2.3}$$

where t is again time and div is the divergence operator with respect to spatial variables $x = (x^1, \ldots, x^p)$. The function τ is called a conserved density, called such because of its integral form

$$T = \int_{\mathbb{R}^p} \tau \, dx, \tag{2.4}$$

which also satisfies

$$\frac{dT}{dt} = 0, \tag{2.5}$$

under the usual assumption that ρ vanishes at infinity. The conserved density τ is a function of the dependent variables, t and x, and their partial derivatives up to some order, whereas $\rho = (\rho^1, \ldots, \rho^p)$ is a vector with the same arguments as τ. Since the differentiation with respect to t and x must be understood as a total differentiation, (2.3) can be written as

$$D_t(\tau) + \sum_{i=1}^{p} D_i(p^i) = 0. \tag{2.6}$$

For an arbitrary differential equation we consider

$$F(x^i, u^\alpha, u_1^\alpha, \ldots, u_k^\alpha) = 0, \tag{2.7}$$

where we do not distinguish between space and time variables, instead let x^i, $i = 1, 2, \ldots, n$ be n independent variables and u^α, $\alpha = 1, 2, \ldots, N$ be N dependent variables [45]. In this case a conservation equation is written in the form

$$D_i(T^i) = 0, \tag{2.8}$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \ldots, \quad i = 1, 2, \ldots, n. \tag{2.9}$$

Note that $T^i \in \mathscr{A}$ (i.e. each T^i is a differential function of finite-order, and $\mathscr{A}$ denotes the space of all differential functions of all finite-orders). Thus (2.8) is called a conservation law of (2.7), a more general form of (2.1), if the equation

$$D_i \left[T^i \left(x^i, u^\alpha(x^i), u_1^\alpha(x^i), \ldots, u_k^\alpha(x^i) \right) \right] = 0, \tag{2.10}$$

is satisfied for all solutions $u^\alpha(x^i)$ of (2.7). The vector T^i, $i = 1, 2, \ldots n$, is called a conserved vector.

2.1.1 The Multiplier method

The Multiplier method is one of the methods used to obtain conservation laws. The advantage of this method is that it may be used to obtain conservation laws for PDEs that do not admit a Lagrangian or a partial Lagrangian. Naz *et al.* [46] consider multipliers to be functions of the position variables as well as the function itself.

For a given PDE,

$$F(x^i, u^\alpha, u_1^\alpha, \ldots, u_k^\alpha) = 0, \tag{2.11}$$

we consider a multiplier of the form $\Lambda = \Lambda(x^i, u^\alpha, u_1^\alpha, \ldots, u_k^\alpha)$. By the property of multipliers as presented by Bluman and Kumei [41], the following equation is obtained

$$\Lambda F(x^i, u^\alpha, u_1^\alpha, \ldots, u_k^\alpha) = D_i\left[T^i\left(x^i, u^\alpha(x^i), u_1^\alpha(x^i), \ldots, u_k^\alpha(x^i)\right)\right] = 0, \tag{2.12}$$

where $T^i, i = 1, 2, \ldots n$, are the components of the conserved vector and the derivative operators are defined as

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \ldots, \quad i = 1, 2, \ldots, n. \tag{2.13}$$

The determining equation is given by

$$E_{u^\alpha}(\Lambda F(x^i, u^\alpha, u_1^\alpha, \ldots, u_k^\alpha)) = 0, \tag{2.14}$$

where the Euler operator, used for its divergence annihilating property, is given by [45]

$$E_{u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \ldots D_{i_s} \frac{\partial}{\partial u_{i_1 \ldots i_s}^\alpha}, \quad \alpha = 1, 2, \ldots N. \tag{2.15}$$

Expanding (2.14) by using operators (2.15) and (2.13) successively, a resulting determining equation is obtained. This equation can now be separated accordingly in order to reduce the multipliers to functions of the dependent and independent variables only, i.e. $\Lambda = \Lambda(x^i, u^\alpha)$. Depending on the nature of the equation, different separation approaches may be considered; certain resulting equations are separated according to products and powers of the partial derivatives of the dependent variables $u^\alpha, \alpha = 1, 2, \ldots, N$. By substituting the multipliers into (2.12), we may write (2.12) in conserved form by considering elementary manipulations. Under the condition that the dependent variables $u^\alpha, \alpha = 1, 2, \ldots, N$, are solutions to (2.11), (2.12) simplifies to

$$D_i\left[T^i\left(x^i, u^\alpha(x^i), u_1^\alpha(x^i), \ldots, u_k^\alpha(x^i)\right)\right] = 0, \tag{2.16}$$

where $T^i, i = 1, 2, \ldots n$. Equation (2.16) takes the form of (2.10), thus, it may be called a conservation law for the PDE given by (2.11) and the resulting vector $T^i, i = 1, 2, \ldots n$, is known as a conserved vector.

2.2 Lie group analysis

While the equations considered in this research are difficult to solve due to their non-linear, dispersive nature, we still aim to obtain analytical solutions if possible. We will therefore consider a symmetry approach, which was initially investigated by Sophus Lie (1842-1899). He developed the theory of Lie group analysis, providing an algorithmic method for obtaining invariant solutions to a given PDE via the use of Lie point symmetries [41]. Since the development of this theory, many advances have been made in this field. Ibragimov [42] and Bluman and Kumei [41] have provided detailed explanations regarding Lie group analysis, and hence, we refer the reader to these for further clarification.

Consider the PDE

$$F(x^i, u^\alpha, u_1^\alpha, \ldots, u_k^\alpha) = 0, \tag{2.17}$$

where $x^i, i = 1, \ldots, n$ denote the dependent variables and $u^\alpha, \alpha = 1, \ldots, N$, denote the independent variables. Let

$$X = \xi^i(x^i, u^\alpha)\frac{\partial}{\partial x^i} + \eta^\alpha(x^i, u^\alpha)\frac{\partial}{\partial u^\alpha}, \tag{2.18}$$

be the infinitesimal generator of the one-parameter Lie group of transformations

$$\begin{aligned} \bar{x}^i &= X\left(x^i, u^\alpha, \epsilon\right), \\ \bar{u}^\alpha &= U\left(x^i, u^\alpha, \epsilon\right), \end{aligned} \tag{2.19}$$

then (2.19) is admitted by PDE (2.17) if and only if

$$X^{(k)}F(x^i, u^\alpha, u_1^\alpha, \ldots, u_k^\alpha) = 0, \tag{2.20}$$

when (2.17) holds, where $X^{(k)}$ is defined to be the k^{th} extended infinitesimal generator of (2.18). Bluman and Kumei [41] provide an elegant definition for the invariant solution of the k^{th} order PDE (2.17), admitting a one-parameter Lie group of transformations with infinitesimal generator (2.18). We consider the definition as follows.

Definition 1 (Invariant Solution) *Assuming that $\xi^i(x^i, u^\alpha) \neq 0$, $u = \Theta(x)$ is an invariant solution of PDE (2.17) corresponding to (2.18) admitted by PDE (2.17) if and only if*

1. *$u = \Theta(x)$ is an invariant surface of (2.18),*

2. *$u = \Theta(x)$ solves (2.17).*

It follows that $u = \Theta(x)$ is an invariant solution of (2.17) resulting from its invariance under (2.18) if and only if $u = \Theta(x)$ satisfies

1. *$X(u - \Theta(x)) = 0$ when $u = \Theta(x)$, i.e.*

$$\xi_i(x, \Theta(x))\frac{\partial \Theta}{\partial x_i} = \eta(x, \Theta(x)); \tag{2.21}$$

2. *$F(x^i, u^\alpha, u_1^\alpha, \ldots, u_k^\alpha) = 0$, where $u_{i_1 i_2 \ldots i_s} = \frac{\partial^s \Theta(x)}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_s}}$, for $i_s = 1, 2, \ldots, n$ for $s = 1, 2, \ldots, k$.*

Equation (2.21) is called the *invariant surface condition* for invariant solutions corresponding to (2.18). The components of (2.20) constitute the determining equations where the prolongation of the generator X is dependent on the derivatives in (2.17). The invariant solution is deduced from the determining equations.

Chapter 3

Derivation of the (1+1)D Boussinesq equations

Some of the work in this chapter has appeared in:

C. Carvalho and C. Harley, *Conservation laws and conserved quantities for (1+1)D linearized Boussinesq equations*, Communications in Nonlinear Science and Numerical Simulation, 46, 37-48, 2017.

3.1 NB equations for a constant water depth

In this chapter, we are guided by the work of Dutykh and Dias [2]. Consider a (x,z) cartesian co-ordinate system with the x-axis along the undisturbed water level, extending infinitely to the left and right. Consider a z-axis perpendicular to the undisturbed water level. We investigate the inviscid region where $u(t,x,z)$ and $w(t,x,z)$ represent the x and z-components of the fluid velocity of the solitary wave, respectively - refer to Figure 3.1. Let Ω_t represent the fluid domain, changing with respect to time in $\mathbb{R}^2$.

The domain is vertically bounded below by the moving seabed $z=-h(t,x)$ and above by the free surface $z=\eta(t,x)$. We assume that the fluid extends infinitely in the horizontal direction [6]. Since the fluid is irrotational, no wave breaking will occur and we can intro-

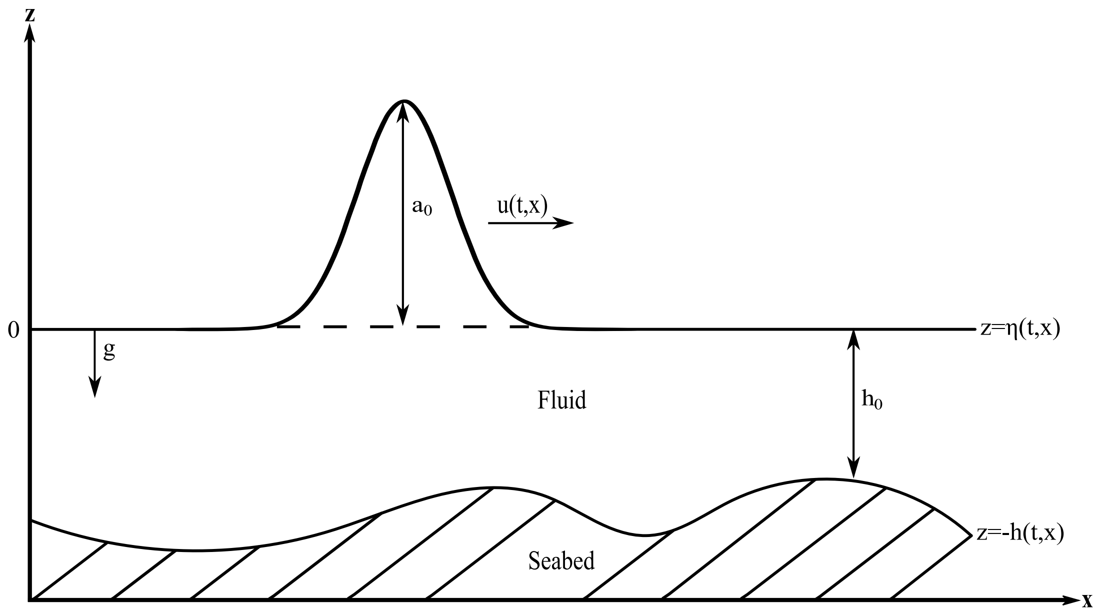

Figure 3.1: Visual representation of the fluid domain.

duce a scalar potential $\phi(t,x,z)$ such that

$$\underline{v} = \underline{\nabla}\phi, \tag{3.1}$$

where $\underline{v} = (u, w)$, $\underline{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right)^T$ and, hence, $u = \frac{\partial \phi}{\partial x}$ and $w = \frac{\partial \phi}{\partial z}$. Since the flow is incompressible the continuity equation applies

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{3.2}$$

Upon substitution of (3.1) into (3.2) we obtain

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0, \ (x,z) \in \Omega_t = \mathbb{R} \times [-h, \eta], \tag{3.3}$$

therefore, $\phi(t,x,z)$ satisfies Laplace's equation.

We now consider two types of boundary conditions:

1. A dynamic boundary condition specifying the forces and pressures at the free surface $z = \eta(t,x)$,

2. A kinematic boundary condition specifying the flow velocity $\underline{v}(t,x)$ at the moving seabed $z = -h(t,x)$ and the free surface $z = \eta(t,x)$.

The dynamical boundary condition on the free surface can be derived by considering Bernoulli's equation for unsteady flow under the assumptions that surface tension effects are neglected and the pressure remains constant on the free surface. Thus the dynamic boundary condition at the free surface is given by

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] + g\eta = 0, \quad z = \eta(t,x). \tag{3.4}$$

Given that a fluid particle on the moving seabed remains on the seabed our kinematic boundary condition, $z + h(t,x) = 0$, traces the motion of a particle on the seabed. In this instance, the material operator is used to trace the path of a particle in time, which follows the fluid flow, providing a condition at the seabed $z = -h(t,x)$ given by

$$\frac{\partial h}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial \phi}{\partial z} = 0, \tag{3.5}$$

where $\frac{Dy}{Dt} = \frac{\partial y}{\partial t} + \underline{v} \cdot \underline{\nabla} y$. In turn, the kinematic boundary condition at the free surface $z = \eta(t,x)$ is as follows

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} = \frac{\partial \phi}{\partial z}. \tag{3.6}$$

Thus equations (3.3)-(3.6) create the system of full potential flow equations given by

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} &= 0, \quad (x,z) \in \Omega_t = \mathbb{R} \times [-h, \eta(t,x)], \\
\frac{\partial \phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] + g\eta &= 0, \qquad z = \eta(t,x), \\
\frac{\partial h}{\partial t} + \frac{\partial \phi}{\partial z} + \frac{\partial \phi}{\partial x}\frac{\partial h}{\partial x} &= 0, \qquad z = -h(t,x), \\
\frac{\partial \phi}{\partial z} - \frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} &= 0, \qquad z = \eta(t,x).
\end{aligned} \tag{3.7}$$

We now scale the full potential flow equations using the classic Boussinesq scalings

$$\begin{aligned}
x^* &= \frac{x}{l}, \quad z^* = \frac{z}{h_0}, \quad t^* = \frac{\sqrt{gh_0}\,t}{l}, \\
h^* &= \frac{h}{h_0}, \quad \eta^* = \frac{\eta}{a_0}, \quad \phi^* = \frac{\sqrt{gh_0}\,\phi}{ga_0 l}.
\end{aligned} \tag{3.8}$$

where we define the following

h_0	-	characteristic water depth,
l	-	wavelength, measured as the distance between the maximum points of any two consecutive wave crests,
a_0	-	amplitude of the wave,
$\sqrt{gh_0}$	-	constant celerity or phase velocity, used as the characteristic velocity.

Employing (3.8) in order to scale the system of equations (3.7), we obtain the following dimensionless full potential flow equations in which the * notation is removed to avoid cumbersome representation

$$\mu^2\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}=0, \qquad (x,z)\in\Omega_t=\mathbb{R}\times[-h,\eta(t,x)], \tag{3.9}$$

$$\mu^2\frac{\partial\phi}{\partial t}+\frac{\epsilon\mu^2}{2}\left(\frac{\partial\phi}{\partial x}\right)^2+\frac{\epsilon}{2}\left(\frac{\partial\phi}{\partial z}\right)^2+\mu^2\eta=0, \qquad z=\epsilon\eta, \tag{3.10}$$

$$\frac{\partial\phi}{\partial z}+\frac{\mu^2}{\epsilon}\frac{\partial h}{\partial t}+\mu^2\frac{\partial\phi}{\partial x}\frac{\partial h}{\partial x}=0, \qquad z=-h, \tag{3.11}$$

$$\frac{\partial\phi}{\partial z}-\mu^2\frac{\partial\eta}{\partial t}-\epsilon\mu^2\frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x}=0, \qquad z=\epsilon\eta, \tag{3.12}$$

where $\epsilon=\frac{a_0}{h_0}$ and $\mu=\frac{h_0}{l}$ are the non-linearity and frequency dispersion parameters respectively. It is important to note that upon inspection of (3.9)-(3.12), under the assumptions that $\mu,\epsilon\ll 1$, we may claim that ϕ is not dependent on z and h is steady i.e. $\frac{\partial\phi}{\partial z}=0$ and $\frac{\partial h}{\partial t}=0$. These conclusions will in fact come about from the calculations which are to follow.

We consider an expansion of the velocity potential ϕ in powers of μ^2

$$\phi=\phi_0+\mu^2\phi_1+\mu^4\phi_2+\mu^6\phi_3+\ldots. \tag{3.13}$$

In order to solve for ϕ_0,ϕ_1,ϕ_2, we substitute (3.13) into (3.9) and obtain

$$\mu^2\left(\frac{\partial^2\phi_0}{\partial x^2}+\mu^2\frac{\partial^2\phi_1}{\partial x^2}+\mu^4\frac{\partial^2\phi_2}{\partial x^2}+\ldots\right)+\frac{\partial^2\phi_0}{\partial z^2}+\mu^2\frac{\partial^2\phi_1}{\partial z^2}+\mu^4\frac{\partial^2\phi_2}{\partial z^2}+\ldots=0. \tag{3.14}$$

Separating coefficients according to the powers of the frequency dispersive parameter μ, we obtain the following relations

$$\mu^0:\frac{\partial^2\phi_0}{\partial z^2}=0, \tag{3.15}$$

$$\mu^2:\frac{\partial^2\phi_0}{\partial x^2}+\frac{\partial^2\phi_1}{\partial z^2}=0, \tag{3.16}$$

$$\mu^4 : \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial z^2} = 0. \tag{3.17}$$

Similarly, we consider the kinematic boundary condition at the seabed and substitute (3.13) into (3.11) to obtain the following relations

$$\mu^0 : \frac{\partial \phi_0}{\partial z} = 0, \tag{3.18}$$

$$\mu^2 : \frac{\partial \phi_1}{\partial z} + \frac{1}{\epsilon}\frac{\partial h}{\partial t} + \frac{\partial \phi_0}{\partial x}\frac{\partial h}{\partial x} = 0, \tag{3.19}$$

$$\mu^4 : \frac{\partial \phi_2}{\partial z} + \frac{\partial \phi_1}{\partial x}\frac{\partial h}{\partial x} = 0. \tag{3.20}$$

The above relations make it evident that ϕ_0 is independent of z, such that $\phi_0 = \phi_0(t,x)$. Considering a zeroth-order approximation, eliminating the vertical component of the velocity from the equations, we define the velocity in the x-direction as

$$u(t,x) = \frac{\partial \phi_0}{\partial x}. \tag{3.21}$$

Given (3.16), we know that $\frac{\partial^2 \phi_1}{\partial z^2} = -\frac{\partial u}{\partial x}$. Upon double integration of (3.16) with respect to z and making use of (3.19) to solve for the constant of integration, we obtain an expression for ϕ_1 given by

$$\phi_1 = -\frac{1}{2}(z+h)^2\frac{\partial u}{\partial x} - z\left(\frac{1}{\epsilon}\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}\right). \tag{3.22}$$

We find ϕ_2 by firstly substituting the double derivative of expression (3.22) with respect to x into (3.17). We then integrate this result with respect to z and combine it with (3.18). Thus the expression for ϕ_2 is given by

$$\begin{aligned}\phi_2 = {} & \frac{1}{2}z^2\left(\frac{\partial h}{\partial x}\right)^2\frac{\partial u}{\partial x} + \frac{(z+h)^3}{6}\frac{\partial^2 h}{\partial x^2}\frac{\partial u}{\partial x} + \frac{(z+h)^3}{3}\frac{\partial h}{\partial x}\frac{\partial^2 u}{\partial x^2} + \frac{(z+h)^4}{24}\frac{\partial^3 u}{\partial x^3} + \frac{z^3}{6\epsilon}\frac{\partial^3 h}{\partial x^2 \partial t} \\ & + \frac{z^3}{6}\frac{\partial^2 u}{\partial x^2}\frac{\partial h}{\partial x} + \frac{z^3}{3}\frac{\partial u}{\partial x}\frac{\partial^2 h}{\partial x^2} + \frac{z^3}{6}u\frac{\partial^3 h}{\partial x^3} - \frac{zh}{\epsilon}\frac{\partial h}{\partial x}\frac{\partial^2 h}{\partial t \partial x} - zhu\frac{\partial h}{\partial x}\frac{\partial^2 h}{\partial x^2} - \frac{zh^2}{2\epsilon}\frac{\partial^3 h}{\partial t \partial x^2} \\ & - \frac{zh^2}{2}\frac{\partial^2 u}{\partial x^2}\frac{\partial h}{\partial x} - zh^2\frac{\partial u}{\partial x}\frac{\partial^2 h}{\partial x^2} - \frac{zh^2}{2}u\frac{\partial^3 h}{\partial x^3}.\end{aligned} \tag{3.23}$$

The flow is incompressible and inviscid, and therefore, the wave amplitude is identical to the amplitude of the moving bottom, i.e. $\frac{1}{\epsilon}\frac{\partial h}{\partial t}$ may be expressed in this form due to the fact that the moving seabed, given in (3.23) may be written in scale form as $h(t,x) = h_0(x) - \epsilon\zeta(t,x)$ where $h_0(x)$ is the static seabed and $\zeta(t,x)$ is the dynamic component.

$$\frac{1}{\epsilon}\frac{\partial h}{\partial t} = -\frac{\partial \zeta}{\partial t} = \mathcal{O}(1). \tag{3.24}$$

We consider dispersion terms of up to order $\mathcal{O}\left(\mu^2\right)$ since the Ursell number is also assumed to be $\mathcal{O}(1)$, i.e.

$$U = \frac{\epsilon}{\mu^2} = \mathcal{O}(1), \tag{3.25}$$

termed the non-linearity dispersion relation. Therefore we neglect $\mathcal{O}\left(\epsilon^2\right)$ and $\mathcal{O}\left(\epsilon\mu^2\right)$ terms.

We now have all the information needed to derive the non-linear, dispersive Boussinesq equations. The substitution of the asymptotic expansion (3.13) into the free surface kinematic boundary condition (3.12) leads to the free surface elevation equation, the first of the NB equations, represented in asymptotic expansion form as

$$\frac{\partial \phi_0}{\partial z} + \mu^2 \frac{\partial \phi_1}{\partial z} + \mu^4 \frac{\partial \phi_2}{\partial z} = \mu^2 \frac{\partial \eta}{\partial t} + \epsilon\mu^2 \frac{\partial \phi_0}{\partial x}\frac{\partial \eta}{\partial x} + \mathcal{O}\left(\epsilon^2 + \epsilon\mu^4 + \mu^6\right). \tag{3.26}$$

It is evident that the first term is equal to zero and by making use of the expressions found for ϕ_1 and ϕ_2, we can evaluate $\frac{\partial \phi_1}{\partial z}$ and $\frac{\partial \phi_2}{\partial z}$ on the free surface, where the former is given as

$$\frac{\partial \phi_1}{\partial z} = -\left(\epsilon\eta + h\right)\frac{\partial u}{\partial x} - \left(\frac{1}{\epsilon}\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}\right). \tag{3.27}$$

From this point onwards, our derivation differs from the work done by Dutykh and Dias [2] in terms of the expression we obtain for $\frac{\partial \phi_2}{\partial z}$ and the subsequent effects thereafter. The expression we obtain for $\frac{\partial \phi_2}{\partial z}$ is given by

$$\begin{aligned}\frac{\partial \phi_2}{\partial z} = {} & \epsilon\eta\left(\frac{\partial h}{\partial x}\right)^2 \frac{\partial u}{\partial x} + \frac{(\epsilon\eta + h)^2}{2}\frac{\partial^2 h}{\partial x^2}\frac{\partial u}{\partial x} + (\epsilon\eta + h)^2 \frac{\partial h}{\partial x}\frac{\partial^2 u}{\partial x^2} + \frac{1}{6}(\epsilon\eta + h)^3 \frac{\partial^3 u}{\partial x^3} \\ & + \frac{1}{2}\epsilon\eta^2\left(\frac{1}{\epsilon}\frac{\partial^3 h}{\partial x^2 \partial t} + \frac{\partial^2 u}{\partial x^2}\frac{\partial h}{\partial x} + 2\frac{\partial u}{\partial x}\frac{\partial^2 h}{\partial x^2} + u\frac{\partial^3 h}{\partial x^3}\right) - \frac{h}{\epsilon}\frac{\partial h}{\partial x}\frac{\partial^2 h}{\partial t \partial x} \\ & - hu\frac{\partial^2 h}{\partial x^2}\frac{\partial h}{\partial x} - \frac{h^2}{2\epsilon}\frac{\partial^3 h}{\partial x^2 \partial t} - \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}\frac{\partial h}{\partial x} - h^2 \frac{\partial u}{\partial x}\frac{\partial^2 h}{\partial x^2} - \frac{h^2}{2}u\frac{\partial^3 h}{\partial x^3}.\end{aligned} \tag{3.28}$$

It seems that the expression obtained by Dutykh and Dias [2] considers the term $u\frac{\partial h}{\partial x}$ to be equivalent to zero, which is peculiar as the expression they obtained for $\frac{\partial \phi_1}{\partial z}$ contains this term. By substituting the relevant expressions into (3.26), dividing by μ^2 and retaining terms of order $\mathcal{O}\left(\epsilon + \mu^2\right)$, we obtain the free surface elevation equation for a moving seabed expressed in terms of the fluid velocity of the solitary wave in the horizontal direction and the surface elevation of the solitary wave given by

$$\begin{aligned}\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left((h + \epsilon\eta)u\right) = {} & -\left(\frac{1}{\epsilon}\frac{\partial h}{\partial t} + \frac{h^2}{2}\mu^2 \frac{\partial^2}{\partial x^2}\left(\frac{1}{\epsilon}\frac{\partial h}{\partial t}\right) + h\mu^2 \frac{\partial h}{\partial x}\frac{\partial}{\partial x}\left(\frac{1}{\epsilon}\frac{\partial h}{\partial t}\right)\right) + \mu^2 \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3} \\ & + \mu^2 h\left(-\frac{h}{2}\frac{\partial u}{\partial x}\frac{\partial^2 h}{\partial x^2} + \frac{h}{2}\frac{\partial^2 u}{\partial x^2}\frac{\partial h}{\partial x} - \frac{h}{2}u\frac{\partial^3 h}{\partial x^3} - u\frac{\partial^2 h}{\partial x^2}\frac{\partial h}{\partial x}\right).\end{aligned} \tag{3.29}$$

The second NB equation representing the evolution of the fluid velocity of the solitary wave in the horizontal direction is now derived. Upon evaluating ϕ_0 and $\frac{\partial \phi_1}{\partial t}$ on the free surface, we substitute these expressions into the asymptotic form of the dynamic boundary condition, (3.10) on the free surface. We further divide by μ^2 and by retaining terms of order $\mathcal{O}\left(\epsilon+\mu^2\right)$ we obtain

$$\frac{\partial \phi_0}{\partial t} - \mu^2\left(h\frac{\partial h}{\partial t}\frac{\partial u}{\partial x} + \eta\frac{\partial^2 h}{\partial t^2} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x \partial t}\right) + \frac{1}{2}\epsilon u^2 + \eta = 0. \tag{3.30}$$

We differentiate (3.30) with respect to x in order to obtain the evolutionary equation of the horizontal fluid velocity for the NB equations

$$\begin{aligned} 0 = &\frac{\partial u}{\partial t} + \epsilon\frac{\partial u}{\partial x}u + \frac{\partial \eta}{\partial x} - \mu^2\left(\frac{\partial h}{\partial x}\frac{\partial h}{\partial t}\frac{\partial u}{\partial x} + h\frac{\partial^2 h}{\partial t \partial x}\frac{\partial u}{\partial x} + h\frac{\partial h}{\partial t}\frac{\partial^2 u}{\partial x^2} + \frac{\partial \eta}{\partial x}\frac{\partial^2 h}{\partial t^2} + \eta\frac{\partial^3 h}{\partial t^2 \partial x}\right.\\ &\left. + h\frac{\partial h}{\partial x}\frac{\partial^2 u}{\partial x \partial t} + \frac{h^2}{2}\frac{\partial^3 u}{\partial x^2 \partial t}\right). \end{aligned} \tag{3.31}$$

Finally we make a simplifying assumption by allowing the water depth to be constant ($h =$ constant). We apply this assumption to (3.29) and (3.31). Thus the system of dimensionless NB equations for a constant water depth of order accuracy $\frac{\epsilon}{\mu^2} = \mathcal{O}(1)$ are given by

$$\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial x} + \epsilon\frac{\partial}{\partial x}(u\eta) = \frac{\mu^2}{6}\frac{\partial^3 u}{\partial x^3}, \tag{3.32}$$

$$\frac{\partial u}{\partial t} + \epsilon u\frac{\partial u}{\partial x} + \frac{\partial \eta}{\partial x} = \frac{\mu^2}{2}\frac{\partial^3 u}{\partial x^2 \partial t}, \tag{3.33}$$

with the following lateral boundary conditions

$$u(t,\pm\infty) = \frac{\partial u(t,\pm\infty)}{\partial t} = \frac{\partial u(t,\pm\infty)}{\partial x} = \eta(t,\pm\infty) = \frac{\partial \eta(t,\pm\infty)}{\partial t} = \frac{\partial \eta(t,\pm\infty)}{\partial x} = 0. \tag{3.34}$$

Boundary conditions (3.34) describe the fluid velocity of the solitary wave in the x-direction, its derivatives and the surface elevation of the solitary wave and its derivatives where $|x| \to \infty$. As the fluid extends to infinity in the x-direction, there is no fluid motion, hence, the fluid velocity of the solitary wave in the x-direction, its derivatives, and the surface elevation of the solitary wave are zero [6]. Using (3.8) to scale the system of equations (3.32) and (3.33), we obtain the following dimensional NB equations for a constant water depth

$$\frac{\partial \eta}{\partial t} + h\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}(u\eta) = \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}, \tag{3.35}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial \eta}{\partial x} = \frac{h^2}{2}\frac{\partial^3 u}{\partial x^2 \partial t}, \tag{3.36}$$

with the following lateral boundary conditions

$$u(t,\pm\infty) = \frac{\partial u(t,\pm\infty)}{\partial t} = \frac{\partial u(t,\pm\infty)}{\partial x} = \eta(t,\pm\infty) = \frac{\partial \eta(t,\pm\infty)}{\partial t} = \frac{\partial \eta(t,\pm\infty)}{\partial x} = 0. \quad (3.37)$$

3.2 LB equations for a constant water depth

In this section, we consider a linear approximation of the governing equations and boundary conditions presented in the previous section. We assume $\epsilon \to 0$ on condition that the amplitude a_0 is small, thus, (3.32), (3.33), (3.35) and (3.36) become linearized [6].

Considering (3.32) and (3.33) under the assumption proposed, we obtain the following dimensionless LB equations for a constant water depth

$$\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial x} = \frac{\mu^2}{6}\frac{\partial^3 u}{\partial x^3}, \quad (3.38)$$

$$\frac{\partial u}{\partial t} + \frac{\partial \eta}{\partial x} = \frac{\mu^2}{2}\frac{\partial^3 u}{\partial x^2 \partial t}, \quad (3.39)$$

with the following lateral boundary conditions

$$u(t,\pm\infty) = \frac{\partial u(t,\pm\infty)}{\partial t} = \frac{\partial u(t,\pm\infty)}{\partial x} = \eta(t,\pm\infty) = \frac{\partial \eta(t,\pm\infty)}{\partial t} = \frac{\partial \eta(t,\pm\infty)}{\partial x} = 0. \quad (3.40)$$

Using (3.8) to scale the system of equations (3.38) and (3.39), we obtain the following dimensional LB equations for a constant water depth

$$\frac{\partial \eta}{\partial t} + h\frac{\partial u}{\partial x} = \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}, \quad (3.41)$$

$$\frac{\partial u}{\partial t} + g\frac{\partial \eta}{\partial x} = \frac{h^2}{2}\frac{\partial^3 u}{\partial x^2 \partial t}, \quad (3.42)$$

with the following lateral boundary conditions

$$u(t,\pm\infty) = \frac{\partial u(t,\pm\infty)}{\partial t} = \frac{\partial u(t,\pm\infty)}{\partial x} = \eta(t,\pm\infty) = \frac{\partial \eta(t,\pm\infty)}{\partial t} = \frac{\partial \eta(t,\pm\infty)}{\partial x} = 0. \quad (3.43)$$

Chapter 4

Conserved Quantities for the LB Equations

Some of the work in this chapter has appeared in:

C. Carvalho and C. Harley, *Conservation laws and conserved quantities for (1+1)D linearized Boussinesq equations*, Communications in Nonlinear Science and Numerical Simulation, 46, 37-48, 2017.

4.1 Conservation laws and conserved vectors for the LB equations

The Multiplier method is a very systematic approach to finding conservation laws that do not require the system of equations to emit a Lagrangian. A high number of conservation laws for a PDE makes it more likely that the PDE is strongly integrable and can be linearized or explicitly solved [47]. In this section, we make use of the Multiplier method to determine the conservation laws and conserved vectors for the dimensional LB equations. The conservation laws and conserved quantities obtained for a system of equations assists in the solution process. Upon consideration of the family of KdV equations, the integral relations, acting as existence conditions are properties of the wave and its motions; in fact they represent the conserved quantities of mass, momentum and energy. These integral relations (i.e. conservation laws) define how the system behaves and must be conserved at the initial state. In our case, for a solitary wave unbounded in the horizontal directions, these initial quantities must be conserved and, as the wave evolves, must still be maintained. Given that initial quantities are difficult to obtain in general, conservation laws provide us with properties at the initial state of the system. Solitary wave and soliton solutions satisfy these conservation laws. However, periodic solutions do not, and thus, the latter are not considered as possible solutions for the solitary wave [6]. Obtaining conservation laws is, therefore, an important part of the process of obtaining physically meaningful solutions. The interested reader is referred to the works [39, 40, 48, 49] for further clarification on the method employed here.

We consider the system of LB equations in dimensional form, (3.41) and (3.42), given by

$$\frac{\partial \eta}{\partial t}+h\frac{\partial u}{\partial x}=\frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}, \tag{4.1}$$

$$\frac{\partial u}{\partial t}+g\frac{\partial \eta}{\partial x}=\frac{h^2}{2}\frac{\partial^3 u}{\partial t\partial x^2}, \tag{4.2}$$

with lateral boundary conditions

$$u(t,\pm\infty)=\frac{\partial u(t,\pm\infty)}{\partial t}=\frac{\partial u(t,\pm\infty)}{\partial x}=\eta(t,\pm\infty)=\frac{\partial \eta(t,\pm\infty)}{\partial t}=\frac{\partial \eta(t,\pm\infty)}{\partial x}=0, \tag{4.3}$$

where t is time, x is the horizontal spatial component, h is constant water depth, g is gravitational acceleration, $u(t,x)$ is the fluid velocity of the solitary wave in the x-direction and $\eta(t,x)$ is the surface elevation of the solitary wave.

In order to derive the conservation laws for the system given by (4.1) and (4.2) using the Multiplier method, we need to first obtain the multipliers of the system. The latter will be obtained via a consideration of (4.4)

$$\Lambda_1(\eta_t+hu_x-\frac{h^3}{6}u_{xxx})+\Lambda_2(u_t+g\eta_x-\frac{h^2}{2}u_{xxt})=D_tT^1+D_xT^2, \tag{4.4}$$

which derives from properties of the multipliers themselves. Given the dimensional LB equations, given by (4.1) and (4.2), we consider multipliers of the form $\Lambda_{1,2}(t,x,u,\eta,u_t,\eta_t,$

$u_x, \eta_x, u_{xx}, u_{xt}, u_{tt}, \eta_{xx}, \eta_{xt}, \eta_{tt}$). We have defined $T = (T^1, T^2)$, present in (4.4), as the components of the conserved vector, where the total derivative operators D_t and D_x are defined by

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + \eta_t \frac{\partial}{\partial \eta} + u_{tt} \frac{\partial}{\partial u_t} + \eta_{tt} \frac{\partial}{\partial \eta_t} + u_{tx} \frac{\partial}{\partial u_x} + \eta_{tx} \frac{\partial}{\partial \eta_x} + \ldots, \tag{4.5}$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + \eta_x \frac{\partial}{\partial \eta} + u_{xx} \frac{\partial}{\partial u_x} + \eta_{xx} \frac{\partial}{\partial \eta_x} + u_{xt} \frac{\partial}{\partial u_t} + \eta_{xt} \frac{\partial}{\partial \eta_t} + \ldots. \tag{4.6}$$

We make use of existing Maple Software in order to obtain these multipliers. Some of these results were corroborated manually [50].

We make use of the Euler operator to derive the determining equations. This allows us to set the divergence of our conservation laws to zero. Thus, by employing (4.4) as shown above, along with this property, we structure our determining equations by expanding

$$E_u[\Lambda_1(\eta_t + h u_x - \frac{h^3}{6} u_{xxx}) + \Lambda_2(u_t + g\eta_x - \frac{h^2}{2} u_{xxt})] = 0, \tag{4.7}$$

$$E_\eta[\Lambda_1(\eta_t + h u_x - \frac{h^3}{6} u_{xxx}) + \Lambda_2(u_t + g\eta_x - \frac{h^2}{2} u_{xxt})] = 0, \tag{4.8}$$

where the Euler operators E_u and E_η are defined by

$$E_u = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + D_t^2 \frac{\partial}{\partial u_{tt}} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_t D_x \frac{\partial}{\partial u_{tx}} + \ldots, \tag{4.9}$$

$$E_\eta = \frac{\partial}{\partial \eta} - D_t \frac{\partial}{\partial \eta_t} - D_x \frac{\partial}{\partial \eta_x} + D_t^2 \frac{\partial}{\partial \eta_{tt}} + D_x^2 \frac{\partial}{\partial \eta_{xx}} + D_t D_x \frac{\partial}{\partial \eta_{tx}} + \ldots. \tag{4.10}$$

The determining equations are then separated with respect to products and powers of the derivatives of u and η. This allows us to obtain the sets of multipliers as given in Tables 4.1-4.3. The reader can refer to Appendix A for calculations of the multipliers. The conserved vectors $T = (T^1, T^2)$ can now be obtained using the Multiplier method. Please refer to Appendix A for detailed calculations of the conserved vectors using the Multiplier method.

When $u(t,x)$ and $\eta(t,x)$ are solutions of the system of equations (4.1) and (4.2), we may obtain a set of conserved vectors by satisfying (4.4) such that $D_t[T_1] + D_x[T_2] = 0$. If we were to consider Case 2 presented in Table 4.1 we may write such an instance as follows

$$\Lambda_1(\eta_t + h u_x - \frac{h^3}{6} u_{xxx}) + \Lambda_2(u_t + g\eta_x - \frac{h^2}{2} u_{xxt}) = D_t[\eta] + D_x\left[h u - \frac{h^3}{6} u_{xx}\right] = 0, \tag{4.11}$$

where we have employed one set of vectors (Case 2), as displayed in Table 4.1. Hence, for each set of multipliers we can obtain the corresponding conserved vectors as given in Tables 4.1- 4.3, where $c_4 = \frac{\sqrt{g(3h+6g+\alpha\sqrt{36g^2+12gh+9h^2})}}{\sqrt{2}gh}$ and $\alpha = \pm 1$. Cases 5 to 8 are the special cases where we obtain non-local conserved vectors. In these cases $gh = 1$ and $c_4 = \alpha\frac{\sqrt{6}}{h}$.

Table 4.1: Multipliers and conserved vectors for the LB equations

Case	Multiplier	Conserved vector
1	$\Lambda_1 = 0$ $\Lambda_2 = 1$	$T^1 = u$ $T^2 = g\eta - \frac{h^2}{2}u_{xt}$
2	$\Lambda_1 = 1$ $\Lambda_2 = 0$	$T^1 = \eta$ $T^2 = hu - \frac{h^3}{6}u_{xx}$

4.2 Conserved quantities for the LB equations

We consider a system with a collection of constraints. Some of these constraints can be interpreted as physical observables such as energy, mass and momentum, whilst there are additional constraints that assist in reducing the solution space. This particular problem is identical to a wide range of other wave problems both physically and mathematically. However, we note that for the problem under consideration, the additional conservation laws obtained give us a solution space that distinguishes this problem from apparently equivalent problems.

A systematic Multiplier approach is used to obtain these conservation laws in dimensional form - refer to Section 4.1. Conserved quantities are then derived by integrating the conservation laws in the direction of wave propagation and imposing the boundary conditions. In this section, while Tables 4.1- 4.3 provide the conserved vectors, we will endeavour to investigate only those providing physical insight into the problem.

Table 4.2: Multipliers and conserved vectors for the LB equations continued

Case	Multiplier	Conserved vector
3	$\Lambda_1 = -\frac{x}{h}$ $\Lambda_2 = t$	$T^1 = t\,u - \frac{1}{h}x\eta$ $T^2 = -x\,u + g\,t\eta - \frac{h^2}{2}t\,u_{xt} + \frac{h^2}{6}[x\,u_{xx} - u_x]$
4	$\Lambda_1 = e^{\alpha\frac{\sqrt{6}}{h}x}e^{c_4x+ght}$ $\Lambda_2 = 0$	$T^1 = e^{\epsilon\frac{\sqrt{6}}{h}x}\eta$ $T^2 = \frac{h^3}{6}e^{\epsilon\frac{\sqrt{6}}{h}x}\left[\epsilon\frac{\sqrt{6}}{h}u_x - u_{xx}\right]$
5	$\Lambda_1 = c_4 g e^{-c_4x+ght}$ $\Lambda_2 = e^{-c_4x+t}$	$T^1 = e^{-c_4x+t}\left[g\eta_x - 2u\right]$ $T^2 = e^{-c_4x+t}\eta_t\left[-g - \frac{1}{c_4}u_{xx} - u_x - \frac{3}{c_4{}^2}u_{xt} - \frac{3}{c_4}u_t + \int 2u\,dx\right]$
6	$\Lambda_1 = -c_4 g e^{-c_4x-ght}$ $\Lambda_2 = e^{-c_4x-t}$	$T^1 = e^{-c_4x-t}\left[-g\eta_x - 2u\right]$ $T^2 = e^{-c_4x-t}\left[g\eta_t + \frac{1}{c_4}u_{xx} + u_x - \frac{3}{c_4{}^2}u_{xt} - \frac{3}{c_4}u_t - \int 2u\,dx\right]$
7	$\Lambda_1 = -c_4 g\ e^{c_4x+ght}$ $\Lambda_2 = e^{c_4x+t}$	$T^1 = e^{c_4x+t}\left[g\eta_x - 2u\right]$ $T^2 = e^{c_4x+t}\left[-g\eta_t + \frac{1}{c_4}u_{xx} - u_x - \frac{3}{c_4{}^2}u_{xt} + \frac{3}{c_4}u_t + \int 2u\,dx\right]$
8	$\Lambda_1 = c_4 g e^{c_4x-ght}$ $\Lambda_2 = e^{c_4x-t}$	$T^1 = e^{c_4x-t}\left[g\eta_x - 2u\right]$ $T^2 = e^{c_4x-t}\left[g\eta_t - \frac{1}{c_4}u_{xx} - u_x - \frac{3}{c_4{}^2}u_{xt} + \frac{3}{c_4}u_t + \int 2u\,dx\right]$
9	$\Lambda_1 = -g\,u_{xx}$ $\Lambda_2 = u_{xt}$	$T^1 = -g\,u_{xx}\eta$ $T^2 = \frac{1}{2}\left[\frac{h^3}{6}g(u_{xx})^2 - gh(u_x)^2 - \frac{h^2}{2}(u_{xt})^2 + (u_t)^2\right] + g\,u_{xt}\eta$
10	$\Lambda_1 = -g\,u_{xt}$ $\Lambda_2 = u_{tt}$	$T^1 = \frac{1}{2}\left[-\frac{gh^3}{6}(u_{xx})^2 + (u_t)^2 - gh(u_x)^2 + \frac{h^2}{2}(u_{xt})^2\right] + g\,u_t\eta_x$ $T^2 = \frac{1}{2}\left[\frac{gh^3}{3}u_{xt}u_{xx} - h^2u_{tt}u_{xt}\right] - g\,u_t\eta_t$
11	$\Lambda_1 = -2u_{xx} - \dfrac{3}{h}\eta_{xt}$ $\Lambda_2 = u_{xx}$	$T^1 = \frac{1}{2}\left[-\frac{h^2}{2}(u_{xx})^2 - (u_x)^2\right]$ $T^2 = \frac{h^3}{6}(u_{xx})^2 - \frac{3}{2h}(\eta_t)^2 - h(u_x)^2 + \frac{h^2}{2}\left[\eta_{xt}u_{xx} - \eta_{xxt}u_x + \eta_{xxxt}u\right]$ $+u_xu_t + g\,u_{xx}\eta - u\eta_{xt} - 2u_x\eta_t + \int\left(-\frac{h^2}{2}\eta_{xxxt}u - g\,u_{xxx}\eta + u\eta_{xxt}\right)dx$
12	$\Lambda_1 = -2u_{xt} - \dfrac{3}{h^2}\eta_{tt}$ $\Lambda_2 = \eta_{xt}$	$T^1 = \frac{g}{2}(\eta_x)^2 - h(u_x)^2 - \frac{3}{2h^2}(\eta_t)^2 - 3u_{xt}\eta - \frac{h^2}{2}\eta_{xxxt}u - \frac{h^3}{3}u\,u_{xxxx}$ $+\dfrac{h}{2}\eta_t u_{xxx} - \frac{h}{2}\eta u_{xxxt}$ $T^2 = \eta_t u_t - \frac{h^2}{2}\eta_{xt}u_{xt} + \frac{h^2}{2}\eta_{xxt}u_t + \frac{h^3}{3}u_tu_{xxx} - \frac{3}{h}u\eta_{tt}$ $+\int\left(\frac{3}{h}u\eta_{ttx} + 3u_{xtt}\eta + \frac{h^2}{2}\eta_{xxxtt}u + \frac{h^3}{3}u\,u_{xxxxt} + \frac{h}{2}\eta u_{xxxtt}\right)dx$

Table 4.3: Multipliers and conserved vectors for the LB equations continued

Case	Multiplier	Conserved vector
13	$\Lambda_1 = \dfrac{g}{h}\eta$ $\Lambda_2 = u - \dfrac{1}{6}h^2 u_{xx}$	$T^1 = \frac{g}{2h}\eta^2 + \frac{1}{2}u^2 + \frac{h^4}{24}(u_{xx})^2 + \frac{h}{4}(u_x)^2 + \frac{h^2}{12}(u_x)^2$ $T^2 = g\eta u - \frac{gh^2}{6}u_{xx}\eta + \frac{h}{2}u_x u_t + \frac{h^2}{6}u_x u_t$
14	$\Lambda_1 = u - \dfrac{1}{2}h^2 u_{xx}$ $\Lambda_2 = \eta$	$T^1 = \eta u - \frac{h^2}{2}u_{xx}\eta - \frac{h^3}{4}(u_x)^2$ $T^2 = \frac{h}{2}u^2 - \frac{h^3}{6}u u_{xx} + \frac{h^3}{12}(u_{xx})^2 + \frac{h^5}{24}(u_{xx})^2 + \frac{g}{2}\eta^2$

4.2.1 Case 1: Conservation of the mass flow rate

Consider the following conserved vector

$$\begin{aligned} T^1 &= u, \\ T^2 &= g\eta - \frac{h^2}{2}u_{xt}. \end{aligned} \tag{4.12}$$

The conservation law for the system given by (4.1) and (4.2) is given by

$$\frac{\partial T^1}{\partial t} + \frac{\partial T^2}{\partial x} = D_t T^1 + D_x T^2 = 0. \tag{4.13}$$

The first conserved quantity for the dimensional LB equations is obtained by substituting (4.12) into (4.13) and subsequently integrating (4.13) with respect to x on the interval $(-\infty, \infty)$, having fixed t. Using Leibniz's rule of integration, we obtain

$$\frac{d}{dt}\int_{-\infty}^{\infty} u\, dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} u\, dx. \tag{4.14}$$

Applying boundary conditions (4.3), (4.14) reduces to

$$\int_{-\infty}^{\infty} u\, dx = J_1, \tag{4.15}$$

where J_1 is a constant independent of time. Thus, the mass flow rate, $\rho_a J_1$, where ρ_a represents the area density, is constant along the wave. The mass flow rate is simply the amount of mass passing by a single point over a duration of time [51].

4.2.2 Case 2: Conservation of mass

We now turn to the following conserved vector

$$
\begin{aligned}
T^1 &= \eta, \\
T^2 &= hu - \frac{h^3}{6} u_{xx}.
\end{aligned}
\tag{4.16}
$$

The second conserved quantity for the dimensional LB equations is obtained by substituting (4.16) into (4.13) and subsequently integrating (4.13) with respect to x on the interval $(-\infty, \infty)$, having fixed t. Using Leibniz's rule of integration, we obtain

$$
\frac{d}{dt} \int_{-\infty}^{\infty} \eta \, dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} \eta \, dx. \tag{4.17}
$$

Using boundary conditions (4.3), (4.17) reduces to

$$
\int_{-\infty}^{\infty} \eta \, dx = J_2, \tag{4.18}
$$

where J_2 is a constant independent of time. Thus, the mass density, $\rho_a J_2$, where ρ_a represents the area density, is constant along the wave. Hence, for all time and for all surface waves represented by $\eta(t,x)$, the mass of the fluid associated with the wave is conserved. Johnson [6] established the conserved mass (4.18) directly from the dimensionless equations for $1D$ gravity wave propagation at a constant water depth. In this instance, decay conditions at infinity are invoked by integrating the continuity equation over the fluid depth.

4.2.3 Case 9: Conservation of the energy flux density

Consider the following conserved vector

$$
\begin{aligned}
T^1 &= -g u_{xx} \eta, \\
T^2 &= \frac{1}{2} \left[\frac{h^3}{6} g (u_{xx})^2 - g h (u_x)^2 - \frac{h^2}{2} (u_{xt})^2 + (u_t)^2 \right] + g u_{xt} \eta.
\end{aligned}
\tag{4.19}
$$

Similary for this case, we substitute (4.19) into (4.13) and subsequently integrate (4.13) with respect to x on the interval $(-\infty, \infty)$, while fixing t. Employing Leibniz's rule of integration and making use of boundary conditions (4.3), we obtain

$$
\int_{-\infty}^{\infty} \left(-g u_{xx} \eta \right) \, dx = J_3, \tag{4.20}
$$

where J_3 is a constant independent of time. Thus, the energy flux density, $\rho_a J_3$, where ρ_a represents the area density, is constant along the wave. The energy flux can be defined as the rate of transfer of energy through a unit area [52]. In terms of the conservation of energy flux in water wave theory, the physical significance of a conserved energy flux density can be interpreted as the magnitude of the wave current being conserved. Simply put, the amount of water that flows through a cross-section of the wave at each second is conserved [52].

4.2.4 Case 13: Conservation of energy

Consider the following conserved vector

$$\begin{aligned} T^1 &= \frac{g}{2h}\eta^2 + \frac{1}{2}u^2 + \frac{h^4}{24}(u_{xx})^2 + \frac{h}{4}(u_x)^2 + \frac{h^2}{12}(u_x)^2, \\ T^2 &= g\eta u - \frac{gh^2}{6}u_{xx}\eta + \frac{h}{2}u_x u_t + \frac{h^2}{6}u_x u_t. \end{aligned} \tag{4.21}$$

The conserved quantity is obtained by substituting (4.21) into (4.13) and subsequently integrating (4.13) with respect to x on the interval $(-\infty, \infty)$, while fixing t. Employing Leibniz's rule of integration and making use of boundary conditions (4.3), we obtain

$$\int_{-\infty}^{\infty} \left(\frac{g}{2h}\eta^2 + \frac{1}{2}u^2 + \frac{h^4}{24}(u_{xx})^2 + \frac{h}{4}(u_x)^2 + \frac{h^2}{12}(u_x)^2 \right) dx = J_5, \tag{4.22}$$

where J_5 is a constant independent of time. Thus, the energy density, $\rho_a J_5$, where ρ_a represents the area density, is constant along the wave. Johnson [6] established the same result directly from the dimensionless equations for $1D$ gravity wave propagation. The differences observed are a result of the way we define the free surface and the seabed. We also eliminate the z-component of the fluid velocity of the solitary wave, w, by making use of an asymptotic expansion on the potential flow equations and assuming the zeroth Boussinesq approximation. Johnson [6], in turn, writes the conserved energy in terms of w. Lastly, the conserved energy we obtain is in dimensional form whilst it takes on a dimensionless form in the work by Johnson [6].

4.2.5 Case 14: Conservation of momentum

Consider the conserved vector

$$T^1 = \eta u - \frac{h^2}{2} u_{xx}\eta - \frac{h^3}{4}(u_x)^2,$$
$$T^2 = \frac{h}{2}u^2 - \frac{h^3}{6}u u_{xx} + \frac{h^3}{12}(u_{xx})^2 + \frac{h^5}{24}(u_{xx})^2 + \frac{g}{2}\eta^2. \tag{4.23}$$

Once again the conserved quantity is obtained by substituting (4.23) into (4.13) and subsequently integrating (4.13) with respect to x on the interval $(-\infty, \infty)$ while fixing t. Employing Leibniz's rule of integration and making use of boundary conditions (4.3), we obtain

$$\int_{-\infty}^{\infty} \left(\eta u - \frac{h^2}{2} u_{xx}\eta - \frac{h^3}{4}(u_x)^2 \right) dx = J_6, \tag{4.24}$$

where J_6 is a constant independent of time. Thus, the momentum density, $\rho_a J_6$, where ρ_a represents the area density, is constant along the wave. Johnson [6] established the conserved energy directly from the dimensionless equations for $1D$ gravity wave propagation. Once again, the conserved momentum (4.24) looks slightly different from that found in Johnson [6] due to the differences discussed above when we considered Case 13.

4.3 Concluding remarks

We considered the conservation laws for the dimensional LB equations that were obtained using the Multiplier method. The equations produced eight local and six non-local conservation laws [53]. From these conservation laws we were able to obtain six conserved quantities by integrating across the wave in the horizontal direction. Conserved mass, energy and momentum quantities were found in a more systematic way than that employed in [6]. In addition to these conserved quantities, we were able to obtain two, as yet, unobserved conserved quantities, namely a conserved mass flow rate and a conserved energy flux density.

Of additional interest is Case 10, where we were able to observe a conserved quantity that we, as yet, cannot relate to any known physical quantities. However, for the sake of completion, we will include our discussion thereon. The following conserved vector is obtained via

$$
\begin{aligned}
T^1 &= \frac{1}{2}\left[-\frac{gh^3}{6}(u_{xx})^2+(u_t)^2-gh(u_x)^2+\frac{h^2}{2}(u_{xt})^2\right]+gu_t\eta_x,\\
T^2 &= \frac{1}{2}\left[\frac{gh^3}{3}u_{xt}u_{xx}-h^2u_{tt}u_{xt}\right]-gu_t\eta_t.
\end{aligned}
\tag{4.25}
$$

The conserved quantity is obtained by substituting (4.25) into (4.13) and subsequently integrating (4.13) with respect to x on the interval $(-\infty,\infty)$, fixing t. Employing Leibniz's rule of integration and making use of boundary conditions (4.3), we obtain

$$
\int_{-\infty}^{\infty}\left(\frac{1}{2}\left[-\frac{gh^3}{6}(u_{xx})^2+(u_t)^2-gh(u_x)^2+\frac{h^2}{2}(u_{xt})^2\right]+gu_t\eta_x\right)dx = J_4,
\tag{4.26}
$$

where J_4 is a constant independent of time. The conserved density, J_4, is a new result for which we have no physical interpretation as yet.

Chapter 5

Lie point symmetries and invariant solutions for the LB equations

In this chapter, we attempt to find the invariant solutions for the LB equations by making use of the full Lie point symmetry approach. This method is preferred over the associated Lie point symmetry approach for this problem due to the large number of conserved vectors obtained. We make use of a linear combination of the time and spatial symmetry in an attempt to obtain a travelling wave solution. We conclude this chapter by presenting the associated Lie point symmetry to determine if there are any invariant travelling wave solutions associated with the conserved quantities of the problem. The associated Lie point symmetry approach was introduced by Kara and Mahomed [54, 55].

5.1 Lie point symmetries for the LB equations

Although all calculations in this section could be computed using software, the symmetries obtained have all been calculated manually. The advantage of not using software is that the expressions are neater and we avoid unnecessary repetitive terms, usually occurring when software is used to obtain symmetries.

From (3.41) and (3.42) the dimensional LB equations are given by

$$\frac{\partial \eta}{\partial t} + h\frac{\partial u}{\partial x} = \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}, \tag{5.1}$$

$$\frac{\partial u}{\partial t}+g\frac{\partial \eta}{\partial x}=\frac{h^2}{2}\frac{\partial^3 u}{\partial t\partial x^2}. \tag{5.2}$$

where t is time, x is the horizontal spatial component, h is the constant water depth, g is the gravitational acceleration, $u(t,x)$ is the fluid velocity of the solitary wave in the x-direction and $\eta(t,x)$ is the surface elevation of the solitary wave.

We proceed to obtain the first determining equation by considering (5.1) with the Lie point operator given by

$$X=\xi^1(t,x,u,\eta)\frac{\partial}{\partial t}+\xi^2(t,x,u,\eta)\frac{\partial}{\partial x}+\psi^1(t,x,u,\eta)\frac{\partial}{\partial u}+\psi^2(t,x,u,\eta)\frac{\partial}{\partial \eta}, \tag{5.3}$$

where ξ^1, ξ^2, ψ^1 and ψ^2 are the infinitesimals. This operator is a Lie point symmetry generator of the governing equation (5.1) if and only if the infinitesimal criterion for invariance holds, i.e.

$$X^{[3]}\left(\frac{\partial \eta}{\partial t}+h\frac{\partial u}{\partial x}-\frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}\right)\bigg|_{(5.1),(5.2)}=0. \tag{5.4}$$

Since (5.1) depends on third derivatives, we take the third prolongation. That is

$$\begin{aligned}X^{[3]}=&\,\xi^1\frac{\partial}{\partial t}+\xi^2\frac{\partial}{\partial x}+\psi^1\frac{\partial}{\partial u}+\psi^2\frac{\partial}{\partial \eta}+\varphi^1_t\frac{\partial}{\partial u_t}+\varphi^1_x\frac{\partial}{\partial u_x}+\varphi^2_t\frac{\partial}{\partial \eta_t}+\varphi^2_x\frac{\partial}{\partial \eta_x}+\varphi^1_{tt}\frac{\partial}{\partial u_{tt}}\\&+\varphi^1_{tx}\frac{\partial}{\partial u_{tx}}+\varphi^1_{xx}\frac{\partial}{\partial u_{xx}}+\varphi^2_{tt}\frac{\partial}{\partial \eta_{tt}}+\varphi^2_{tx}\frac{\partial}{\partial \eta_{tx}}+\varphi^2_{xx}\frac{\partial}{\partial \eta_{xx}}+\varphi^1_{ttt}\frac{\partial}{\partial u_{ttt}}+\varphi^1_{ttx}\frac{\partial}{\partial u_{ttx}}\\&+\varphi^1_{txx}\frac{\partial}{\partial u_{txx}}+\varphi^1_{xxx}\frac{\partial}{\partial u_{xxx}}+\varphi^2_{ttt}\frac{\partial}{\partial \eta_{ttt}}+\varphi^2_{ttx}\frac{\partial}{\partial \eta_{ttx}}+\varphi^2_{txx}\frac{\partial}{\partial \eta_{txx}}+\varphi^2_{xxx}\frac{\partial}{\partial \eta_{xxx}},\end{aligned} \tag{5.5}$$

where we can define explicitly the extended infinitesimals as

$$\varphi^\alpha_i=D_i\left(\psi^\alpha\right)-u^\alpha_j D_i\left(\xi^j\right), \tag{5.6}$$

$$\varphi^\alpha_{ij}=D_j\left(\varphi^\alpha_i\right)-u^\alpha_{il} D_j\left(\xi^l\right), \tag{5.7}$$

where $u^1=u$, $u^2=\eta$, $D_i=D_1=D_t$ and $D_j=D_2=D_x$. Thus by applying the third prologation (5.5) onto (5.1) we obtain

$$\left(\varphi^2_t+h\varphi^1_x-\frac{h^3}{6}\varphi^1_{xxx}\right)\bigg|_{(5.1),(5.2)}=0. \tag{5.8}$$

The extended infinitesimals for (5.8) are given by (B.1), (B.2), (B.3) in Appendix B. We substitute the extended infinitesimals (B.1), (B.2), (B.3) into (5.8) to give the first determining equation given by (B.4) in Appendix B, subject to (5.1) and (5.2). The determining equation (B.4) can be split into the following derivative products and powers of u and η

$$(\eta_t)^2 : -\frac{\partial \xi^1}{\partial \eta} = 0, \tag{5.9}$$

$$u_{xt}u_{xx} : \frac{h^3}{6}\left(3\frac{\partial \xi^1}{\partial u}\right) = 0, \tag{5.10}$$

$$u_{xt} : -\frac{h^3}{6}\left(-3\frac{\partial^2 \xi^1}{\partial x^2}\right) = 0. \tag{5.11}$$

From (5.9)-(5.11) we obtain

$$\xi^1 = \xi^1(t,x) = c_1(t)x + c_2(t). \tag{5.12}$$

We separate the first determining equation (B.4) by the first-order derivative products of u and η, given by (5.13) and (5.14), and note that ξ^2 is independent of u and η

$$\eta_t\eta_x : 2\frac{\partial \xi^2}{\partial \eta} = 0, \tag{5.13}$$

$$u_t\eta_x : -\frac{\partial \xi^2}{\partial u} + \frac{h^3}{6}\left(3\frac{\partial^3 \xi^2}{\partial x^2 \partial \eta}\right) = 0. \tag{5.14}$$

Next, we split (B.4) by η_{xxx}, given by (5.15), and see that ξ^2 is independent of η

$$\eta_{xxx} : -\frac{h^3}{6}\left(\frac{\partial \psi^1}{\partial \eta}\right) = 0. \tag{5.15}$$

By separating (B.4) by u_t we obtain

$$u_t : \frac{\partial \psi^2}{\partial u} - \frac{h^3}{6}\left(-\frac{\partial^3 \xi^1}{\partial x^3}\right) = 0. \tag{5.16}$$

Upon substituting (5.11) into (5.16), we deduce that ψ^2 is independent of u. We now separate (B.4) by u_x and u_{xx} as follows

$$u_x : -\frac{h^3}{6}\left(3\frac{\partial^3 \psi^1}{\partial x^2 \partial u} - \frac{\partial^3 \xi^2}{\partial x^3}\right) + 2h\frac{\partial \xi^2}{\partial x} = 0, \tag{5.17}$$

$$u_{xx} : \frac{\partial^2 \psi^1}{\partial u \partial x} - \frac{\partial^2 \xi^2}{\partial x^2} = 0. \tag{5.18}$$

Differentiating (5.18) with respect to x and substituting this result into (5.17), we obtain the following third-order PDE

$$h^2\frac{\partial^3 \xi^2}{\partial x^3} = 6\frac{\partial \xi^2}{\partial x}. \tag{5.19}$$

Solving (5.19) gives

$$\xi^2(t,x)=\frac{e^{\frac{\sqrt{6}x}{h}}hc_3(t)-e^{\frac{-\sqrt{6}x}{h}}hc_4(t)}{\sqrt{6}}+c_5(t), \tag{5.20}$$

where c_3, c_4 and c_5 are arbitrary functions of t.

In summary we have the following expressions

$$\xi^1=\xi^1(t,x)=c_1(t)x+c_2(t), \tag{5.21}$$

$$\xi^2=\xi^2(t,x)=\frac{e^{\frac{\sqrt{6}x}{h}}hc_3(t)-e^{\frac{-\sqrt{6}x}{h}}hc_4(t)}{\sqrt{6}}+c_5(t), \tag{5.22}$$

$$\psi^1=\psi^1(t,x,u), \tag{5.23}$$

$$\psi^2=\psi^2(t,x,\eta), \tag{5.24}$$

which upon subsitution into (B.4) reduce the determining equation to

$$\begin{aligned}0=&\left(\frac{\partial\psi^2}{\partial t}+\frac{\partial\psi^2}{\partial\eta}\eta_t-\frac{\partial\xi^1}{\partial t}\eta_t-\frac{\partial\xi^2}{\partial t}\eta_x\right)+h\left(\frac{\partial\psi^1}{\partial x}+\frac{\partial\psi^1}{\partial u}u_x-\frac{\partial\xi^1}{\partial x}u_t-\frac{\partial\xi^2}{\partial x}u_x\right)\\&-\frac{h^3}{6}\left[\left(\frac{\partial^3\psi^1}{\partial x^3}+\frac{\partial^3\psi^1}{\partial x^2\partial u}u_x\right)+\left(2\frac{\partial^3\psi^1}{\partial x^2\partial u}u_x+2\frac{\partial^3\psi^1}{\partial u^2\partial x}(u_x)^2+2\frac{\partial^2\psi^1}{\partial u\partial x}u_{xx}\right)\right.\\&+\left(\frac{\partial^3\psi^1}{\partial u^2\partial x}(u_x)^2+\frac{\partial^3\psi^1}{\partial u^3}(u_x)^3+2\frac{\partial^2\psi^1}{\partial u^2}u_xu_{xx}\right)+\left(\frac{\partial^2\psi^1}{\partial u\partial x}u_{xx}+\frac{\partial^2\psi^1}{\partial u^2}u_xu_{xx}\right)\\&\left.-\left(\frac{\partial^3\xi^2}{\partial x^3}u_x+\frac{\partial^2\xi^2}{\partial x^2}u_{xx}\right)-\left(3\frac{\partial\xi^1}{\partial x}\frac{2}{h^2}(u_t+g\eta_x)\right)-\left(2\frac{\partial^2\xi^2}{\partial x^2}u_{xx}+3\frac{\partial\xi^2}{\partial x}\frac{6}{h^3}(\eta_t+hu_x)\right)\right].\end{aligned} \tag{5.25}$$

Finally after applying the third prolongation onto (5.25) we are left with the following remaining equations

$$(u_x)^3:-\frac{h^3}{6}\left(\frac{\partial^3\psi^1}{\partial u^3}\right)=0, \tag{5.26}$$

$$(u_x)^2:-\frac{h^3}{6}\left(3\frac{\partial^3\psi^1}{\partial u^2\partial x}\right)=0, \tag{5.27}$$

$$u_xu_{xx}:-\frac{h^3}{6}\left(3\frac{\partial^2\psi^1}{\partial u^2}\right)=0, \tag{5.28}$$

$$\eta_t:\frac{\partial\psi^2}{\partial\eta}-\frac{\partial\xi^1}{\partial t}-\frac{\partial\psi^1}{\partial u}+3\frac{\partial\xi^2}{\partial x}=0, \tag{5.29}$$

$$\eta_x:-\frac{\partial\xi^2}{\partial t}+hg\frac{\partial\xi^1}{\partial x}=0, \tag{5.30}$$

$$u_x:-\frac{h^3}{6}\left(3\frac{\partial^3\psi^1}{\partial x^2\partial u}-\frac{\partial^3\xi^2}{\partial x^3}\right)+2h\frac{\partial\xi^2}{\partial x}=0, \tag{5.31}$$

$$u_{xx}: -\frac{h^3}{6}\left(3\frac{\partial^2\psi^1}{\partial u\partial x} - 3\frac{\partial^2\xi^2}{\partial x^2}\right) = 0, \tag{5.32}$$

$$1: \frac{\partial\psi^2}{\partial t} + h\frac{\partial\psi^1}{\partial x} - \frac{h^3}{6}\frac{\partial^3\psi^1}{\partial x^3} = 0, \tag{5.33}$$

which will be of use when we come back to it later in this calculation.

We now proceed to ascertain the second determining equation. A similar procedure is followed so that we once again consider the Lie point operator given by (5.3)

$$X = \xi^1(t,x,u,\eta)\frac{\partial}{\partial t} + \xi^2(t,x,u,\eta)\frac{\partial}{\partial x} + \psi^1(t,x,u,\eta)\frac{\partial}{\partial u} + \psi^2(t,x,u,\eta)\frac{\partial}{\partial \eta},$$

where ξ^1, ξ^2, ψ^1 and ψ^2 are the infinitesimals. This operator is a Lie point symmetry generator of the governing equation (5.2) if and only if the infinitesimal criterion for invariance holds, i.e.

$$X^{[3]}\left(u_t + g\eta_x - \frac{h^2}{2}u_{xxt}\right)\bigg|_{(5.1),(5.2)} = 0. \tag{5.34}$$

Since (5.2) also depends on third-order derivatives, we take the third prolongation given by (5.5) where the extended infinitesimals and the derivative operators have been explictly defined. Thus by applying the third prologation (5.5) onto (5.2) we obtain

$$\left(\varphi^1_t + g\varphi^2_x - \frac{h^2}{2}\varphi^1_{xxt}\right)\bigg|_{(5.1),(5.2)} = 0. \tag{5.35}$$

The extended infinitesimals for (5.35) are given by the following (subject to results obtained from the first determining equation, namely equations (5.21)-(5.24))

$$\begin{aligned}\varphi^1_t &= D_t\left(\psi^1\right) - u_t D_t\left(\xi^1\right) - u_x D_t\left(\xi^2\right)\\ &= \frac{\partial\psi^1}{\partial t} + u_t\frac{\partial\psi^1}{\partial u} - u_t\frac{\partial\xi^1}{\partial t} - u_x\frac{\partial\xi^2}{\partial t},\end{aligned} \tag{5.36}$$

and similarly

$$\begin{aligned}\varphi^2_x &= D_x\left(\psi^2\right) - u_t D_x\left(\xi^1\right) - u_x D_x\left(\xi^2\right)\\ &= \frac{\partial\psi^2}{\partial x} + \eta_x\frac{\partial\psi^2}{\partial \eta} - u_t\frac{\partial\xi^1}{\partial x} - u_x\frac{\partial\xi^2}{\partial x},\end{aligned} \tag{5.37}$$

and finally

$$
\begin{aligned}
\varphi^1_{xxt} &= D_t\left(\varphi^1_{xx}\right) - u_{xxt} D_t\left(\xi^1\right) - u_{xxx} D_t\left(\xi^2\right) \\
&= \left(\frac{\partial^3\psi^1}{\partial x^2\partial t} + u_t \frac{\partial^3\psi^1}{\partial x^2\partial u}\right) + \left(2u_x \frac{\partial^3\psi^1}{\partial u\partial x\partial t} + 2u_x u_t \frac{\partial^3\psi^1}{\partial u^2\partial x} + 2u_{xt}\frac{\partial^2\psi^1}{\partial u\partial x}\right) \\
&+ \left((u_x)^2 \frac{\partial^3\psi^1}{\partial u^2\partial t} + (u_x)^2 u_t \frac{\partial^3\psi^1}{\partial u^3} + 2u_x u_{xt}\frac{\partial^2\psi^1}{\partial u^2}\right) + \left(u_{xx}\frac{\partial^2\psi^1}{\partial u\partial t} + u_t u_{xx}\frac{\partial^2\psi^1}{\partial u^2}\right) \\
&- \left(u_x \frac{\partial^3\xi^2}{\partial x^2\partial t} + u_{xt}\frac{\partial^2\xi^2}{\partial x^2}\right) - \left(2u_{xt}\frac{\partial^2\xi^1}{\partial x\partial t} + 2u_{xtt}\frac{\partial\xi^1}{\partial x}\right) - u_{xxx}\frac{\partial\xi^2}{\partial t} \\
&- \left(2u_{xx}\frac{\partial^2\xi^2}{\partial x\partial t} + 2u_{xxt}\frac{\partial\xi^2}{\partial x}\right) + u_{xxt}\frac{\partial\psi^1}{\partial u} - u_{xxt}\frac{\partial\xi^1}{\partial t}.
\end{aligned}
\tag{5.38}
$$

We substitute the extended infinitesimals (5.36)-(5.38) into (5.35) to give the determining equation

$$
\begin{aligned}
0 &= \frac{\partial\psi^1}{\partial t} + u_t\frac{\partial\psi^1}{\partial u} - u_t\frac{\partial\xi^1}{\partial t} - u_x\frac{\partial\xi^2}{\partial t} + g\left(\frac{\partial\psi^2}{\partial x} + \eta_x\frac{\partial\psi^2}{\partial\eta} - u_t\frac{\partial\xi^1}{\partial x} - u_x\frac{\partial\xi^2}{\partial x}\right) \\
&- \frac{h^2}{2}\left[\left(\frac{\partial^3\psi^1}{\partial x^2\partial t} + u_t\frac{\partial^3\psi^1}{\partial x^2\partial u}\right) + \left(2u_x\frac{\partial^3\psi^1}{\partial u\partial x\partial t} + 2u_x u_t\frac{\partial^3\psi^1}{\partial u^2\partial x} + 2u_{xt}\frac{\partial^2\psi^1}{\partial u\partial x}\right)\right. \\
&+ \left((u_x)^2\frac{\partial^3\psi^1}{\partial u^2\partial t} + (u_x)^2 u_t\frac{\partial^3\psi^1}{\partial u^3} + 2u_x u_{xt}\frac{\partial^2\psi^1}{\partial u^2}\right) + \left(u_{xx}\frac{\partial^2\psi^1}{\partial u\partial t} + u_t u_{xx}\frac{\partial^2\psi^1}{\partial u^2}\right) \\
&- \left(u_x\frac{\partial^3\xi^2}{\partial x^2\partial t} + u_{xt}\frac{\partial^2\xi^2}{\partial x^2}\right) - \left(2u_{xt}\frac{\partial^2\xi^1}{\partial x\partial t} + 2u_{xtt}\frac{\partial\xi^1}{\partial x}\right) - u_{xxx}\frac{\partial\xi^2}{\partial t} + u_{xxt}\frac{\partial\psi^1}{\partial u} \\
&\left.- \left(2u_{xx}\frac{\partial^2\xi^2}{\partial x\partial t} + 2u_{xxt}\frac{\partial\xi^2}{\partial x}\right) - u_{xxt}\frac{\partial\xi^1}{\partial t}\right],
\end{aligned}
\tag{5.39}
$$

subject to (5.1) and (5.2).

The determining equation (5.39) can be separated by u_{xtt} as follows

$$
u_{xtt} : \frac{\partial\xi^1}{\partial x} = 0. \tag{5.40}
$$

From (5.40) we see that ξ^1 is independent of x, therefore, (5.21) becomes

$$
\xi^1 = \xi^1(t) = c_2(t), \tag{5.41}
$$

where c_2 is an arbitrary function of t.

Next, we separate (5.39) by η_t as follows

$$
\eta_t : \frac{\partial\xi^2}{\partial t} = 0. \tag{5.42}
$$

From equation (5.42) we see that ξ^2 is independent of t, hence, (5.22) becomes

$$
\xi^2 = \xi^2(x) = \frac{e^{\frac{\sqrt{6}x}{h}} h c_3 - e^{\frac{-\sqrt{6}x}{h}} h c_4}{\sqrt{6}} + c_5, \tag{5.43}
$$

where c_3, c_4 and c_5 are arbitrary constants.

Next, we separate (5.39) by u_{xt} as follows

$$u_{xt}: \quad 2\frac{\partial^2 \psi^1}{\partial u \partial x} - \frac{\partial^2 \xi^2}{\partial x^2} = 0. \tag{5.44}$$

By substituting (5.32) into (5.44), we get the following second-order PDE

$$\frac{\partial^2 \psi^1}{\partial u \partial x} = 2\frac{\partial^2 \psi^1}{\partial u \partial x}. \tag{5.45}$$

Equation (5.45) is satisfied if and only if

$$\frac{\partial^2 \psi^1}{\partial u \partial x} = 0. \tag{5.46}$$

By integrating (5.46) once with respect to u and then once with respect to x we obtain

$$\psi^1 = c_6 u + c_7(t,x). \tag{5.47}$$

Next, we separate (5.39) by u_t as follows

$$u_t: \frac{h^2}{2}\frac{\partial^3 \psi^1}{\partial x^2 \partial u} - 2\frac{\partial \xi^2}{\partial x} = 0, \tag{5.48}$$

and using the result obtained in (5.46), (5.48) simplifies to $\xi^2 = c_5$ where c_5 is an arbitrary constant.

We now separate equation (5.39) by η_x as follows

$$\eta_x: \quad \frac{\partial \psi^2}{\partial \eta} - \frac{\partial \psi^1}{\partial u} + \frac{\partial \xi^1}{\partial t} = 0. \tag{5.49}$$

Combining (5.49) and (5.29), we conclude that $c_2(t) = c_2$ and obtain

$$\psi^2 = c_6 \eta + c_8(t,x). \tag{5.50}$$

The remaining determining equations cannot be simplified further, and hence, we are left with the following expressions for ξ^1, ξ^2, ψ^1 and ψ^2

$$\xi^1 = c_2, \quad \xi^2 = c_5, \tag{5.51}$$

$$\psi^1 = c_6 u + c_7(t,x), \quad \psi^2 = c_6 \eta + c_8(t,x), \tag{5.52}$$

subject to the remaining determining equations

$$\frac{\partial c_8(t,x)}{\partial t} + h\frac{\partial c_7(t,x)}{\partial x} = \frac{h^3}{6}\frac{\partial^3 c_7(t,x)}{\partial x^3}, \tag{5.53}$$
$$\frac{\partial c_7(t,x)}{\partial t} + g\frac{\partial c_8(t,x)}{\partial x} = \frac{h^2}{2}\frac{\partial^3 c_7(t,x)}{\partial x^2 \partial t}, \tag{5.54}$$

where c_2, c_5 and c_6 are arbitrary constants and where c_7 and c_8 are arbitrary functions of t and x. Interestingly, (5.53) and (5.54) resemble our original system of equations. Substituting (5.51) and (5.52) into (5.3), we obtain the infinitesimal generator

$$X = c_2\frac{\partial}{\partial t} + c_5\frac{\partial}{\partial x} + (c_6 u + c_7(t,x))\frac{\partial}{\partial u} + \left(c_6\eta + c_8(t,x)\right)\frac{\partial}{\partial \eta}, \tag{5.55}$$

producing the symmetries

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \tag{5.56}$$
$$X_3 = u\frac{\partial}{\partial u} + \eta\frac{\partial}{\partial \eta}, \quad X_4 = c_7(t,x)\frac{\partial}{\partial u} + c_8(t,x)\frac{\partial}{\partial \eta}, \tag{5.57}$$

subject to the following

$$\frac{\partial c_8(t,x)}{\partial t} + h\frac{\partial c_7(t,x)}{\partial x} = \frac{h^3}{6}\frac{\partial^3 c_7(t,x)}{\partial x^3}, \tag{5.58}$$
$$\frac{\partial c_7(t,x)}{\partial t} + g\frac{\partial c_8(t,x)}{\partial x} = \frac{h^2}{2}\frac{\partial^3 c_7(t,x)}{\partial x^2 \partial t}, \tag{5.59}$$
$$gh\frac{\partial^2 c_8(t,x)}{\partial x^2} - 2h\frac{\partial^2 c_7(t,x)}{\partial x \partial t} = 3\frac{\partial^2 c_8(t,x)}{\partial t^2}, \tag{5.60}$$

where (5.60) is obtained from combining (5.58) and (5.59).

5.2 Invariant solutions for the LB equations

In this section, we attempt to obtain invariant travelling wave solutions relating to the fluid velocity in the x-direction $u(t,x)$ and the surface elevation $\eta(t,x)$ for the LB equations representing a solitary wave propogating rightward. We refer to the generator obtained in the previous section, namely

$$X = c_2\frac{\partial}{\partial t} + c_5\frac{\partial}{\partial x} + (c_6 u + c_7(t,x))\frac{\partial}{\partial u} + \left(c_6\eta + c_8(t,x)\right)\frac{\partial}{\partial \eta}. \tag{5.61}$$

Now,

$$u = f(t,x), \tag{5.62}$$

$$\eta = k(t,x), \tag{5.63}$$

are invariant solutions for the system, (5.1) and (5.2), provided

$$X\left(u - f(t,x)\right)\Big|_{u=f} = 0, \tag{5.64}$$

$$X\left(\eta - k(t,x)\right)\Big|_{\eta=k} = 0. \tag{5.65}$$

Expanding (5.64) using (5.61), we obtain the following PDE in terms of $f(t,x)$

$$c_2 \frac{\partial f}{\partial t} + c_5 \frac{\partial f}{\partial x} = c_6 f + c_7(t,x). \tag{5.66}$$

Similarly, expanding (5.65) using (5.61) gives the following PDE in terms of $k(t,x)$

$$c_2 \frac{\partial k}{\partial t} + c_5 \frac{\partial k}{\partial x} = c_6 k + c_8(t,x). \tag{5.67}$$

We are looking for solutions in the form of travelling waves, thus, we consider $c_7(t,x) = 0$ and $c_8(t,x) = 0$ where $c_2 \neq 0$. We obtain the following differential equations of the characteristic curves of (5.66)

$$\frac{dt}{c_2} = \frac{dx}{c_5} = \frac{df}{c_6 f}. \tag{5.68}$$

Considering the first two terms of (5.68) we obtain

$$x - \frac{c_5}{c_2} t = b_1. \tag{5.69}$$

where $c_2 > 0$, $c_5 > 0$ and b_1 is a constant. Through a consideration of the first and last terms of (5.68) we find that

$$\frac{df}{dt} - \frac{c_6}{c_2} f = 0. \tag{5.70}$$

We let the integrating factor be

$$e^{-\frac{c_6}{c_2}\int dt} = e^{-\frac{c_6}{c_2} t}, \tag{5.71}$$

and then substitute (5.71) into (5.70) to get the following expression for $f(t,x)$

$$f = e^{\frac{c_6}{c_2} t} b_2, \tag{5.72}$$

where b_2 is a constant. From (5.69) we know that the general solution takes the form of $b_2 = F(b_1)$, therefore,

$$f(t,x) = e^{\frac{c_6}{c_2}t} F\left(x - \frac{c_5}{c_2}t\right), \tag{5.73}$$

where $c_2 > 0$ and $c_5 > 0$. Similarly, we use the differential equations of the characteristic curves of (5.67) and by following the same procedure we obtain the following solution for $k(t,x)$

$$k(t,x) = e^{\frac{c_6}{c_2}t} G\left(x - \frac{c_5}{c_2}t\right). \tag{5.74}$$

Substituting the results from (5.73) and (5.74) into (5.62) and (5.63) provides the following expressions for the invariant solutions $u(t,x)$ and $\eta(t,x)$

$$u(t,x) = e^{\frac{c_6}{c_2}t} F\left(x - \frac{c_5}{c_2}t\right) = e^{\frac{c_6}{c_2}t} F(\xi), \tag{5.75}$$

$$\eta(t,x) = e^{\frac{c_6}{c_2}t} G\left(x - \frac{c_5}{c_2}t\right) = e^{\frac{c_6}{c_2}t} G(\xi), \tag{5.76}$$

where $\xi = x - ct$, $c = \frac{c_5}{c_2}$ and $c > 0$.

Next, we substitute (5.75) and (5.76) into the system, (5.1) and (5.2), and simplify to obtain the following system of ODEs

$$\frac{c_6}{c_2}G(\xi) - \frac{c_5}{c_2}\frac{dG}{d\xi} + h\frac{dF}{d\xi} = \frac{h^3}{6}\frac{d^3F}{d\xi^3}, \tag{5.77}$$

$$\frac{c_6}{c_2}F(\xi) - \frac{c_5}{c_2}\frac{dF}{d\xi} + g\frac{dG}{d\xi} = \frac{h^2}{2}\left(\frac{c_6}{c_2}\frac{d^2F}{d\xi^2} - \frac{c_5}{c_2}\frac{d^3F}{d\xi^3}\right). \tag{5.78}$$

subject to the following boundary conditions

$$F(\pm\infty) = \frac{dF(\pm\infty)}{d\xi} = \frac{d^2F(\pm\infty)}{d\xi^2} = \frac{d^3F(\pm\infty)}{d\xi^3} = G(\pm\infty) = \frac{dG(\pm\infty)}{d\xi} = \frac{d^2G(\pm\infty)}{d\xi^2} = 0. \tag{5.79}$$

Now, we solve (5.77) and (5.78) simultaneously. Firstly, we multiply (5.77) by g to get

$$g\frac{c_5}{c_2}\frac{dG}{d\xi} = g\frac{c_6}{c_2}G(\xi) + gh\frac{dF}{d\xi} - g\frac{h^3}{6}\frac{d^3F}{d\xi^3}, \tag{5.80}$$

and secondly multiply (5.78) by c_5 to obtain

$$gc_5\frac{dG}{d\xi} = -c_5c_6F(\xi) + c_5^2\frac{dF}{d\xi} + \frac{h^2}{2}\left(c_5c_6\frac{d^2F}{d\xi^2} - c_5^2\frac{d^3F}{d\xi^3}\right). \tag{5.81}$$

Equating (5.80) and (5.81) gives

$$gG(\xi) = -c_5F(\xi) + \frac{(c_5^2 - ghc_2)}{c_6}\frac{dF}{d\xi} + \frac{h^2}{2}\left(c_5\frac{d^2F}{1d\xi^2}\right) + \frac{h^2}{6c_6}(ghc_2 - 3c_5^2)\frac{d^3F}{d\xi^3}. \tag{5.82}$$

Solving for $g\frac{dG}{d\xi}$ by differentiating (5.82) with respect to ξ gives

$$g\frac{dG}{d\xi}=-c_5\frac{dF}{d\xi}+\frac{\left(c_5^2-ghc_2\right)}{c_6}\frac{d^2F}{d\xi^2}+\frac{h^2}{2}\left(c_5\frac{d^3F}{d\xi^3}\right)+\frac{h^2}{6c_6}\left(ghc_2-3c_5^2\right)\frac{d^4F}{d\xi^4}. \tag{5.83}$$

Upon substituting (5.83) into (5.78) we obtain

$$\frac{h^2}{6c_6}\left(ghc_2-3c_5^2\right)\frac{d^4F}{d\xi^4}+h^2\frac{c_5}{c_2}\frac{d^3F}{d\xi^3}+\frac{1}{c_6}\left(c_5^2-ghc_2-\frac{h^2}{2}\frac{c_6^2}{c_2}\right)\frac{d^2F}{d\xi^2}-c_5\left(1+\frac{1}{c_2}\right)\frac{dF}{d\xi}+c_6F=0. \tag{5.84}$$

Multiplying (5.84) by c_6, where c_6 is an arbitrary constant thus allowing for this operation, and substituting $c_6=0$ into the result gives the following fourth-order ODE in terms of $F(\xi)$

$$\frac{h^2}{6}\left(ghc_2-3c_5^2\right)\frac{d^4F}{d\xi^4}+\left(c_5^2-ghc_2\right)\frac{d^2F}{d\xi^2}=0. \tag{5.85}$$

As commonly observed in classical water wave theory we assume a solution of the form

$$F(\xi)=Ae^{\lambda\xi}. \tag{5.86}$$

Substituting (5.86) into (5.85) gives

$$\lambda^2\left[\lambda^2\frac{h^2}{6}\left(ghc_2-3c_5^2\right)F+\left(c_5^2-ghc_2\right)F\right]=0. \tag{5.87}$$

Therefore

$$\lambda=0 \qquad \text{or} \qquad \lambda^2=-\frac{2(c_5^2-ghc_2)}{h^2(c_5^2-\frac{1}{3}ghc_2)}, \tag{5.88}$$

and

$$F(\xi)=A_1+A_2\xi+A_3e^{\alpha x}+A_4e^{-\alpha x}=A_1+A_2\xi+B_1\cosh(\alpha x)+B_2\sinh(\alpha x), \tag{5.89}$$

where

$$\alpha^2=-\frac{2(c_5^2-ghc_2)}{h^2(c_5^2-\frac{1}{3}ghc_2)} \qquad \text{and} \qquad \lambda=\pm\alpha. \tag{5.90}$$

The expression (5.89), relating to the fluid velocity of the solitary wave in the horizontal direction, takes the form of an oscillatory solution, which does not satisfy the decaying lateral boundary conditions (5.79). As such, the solution presented is not a possible solution for the problem posed. $G(\xi)$ will have a similar form to that of (5.89) and, therefore, there is no solitary wave that satisfies the boundary conditions (5.79). We conclude that linearizing the NB equations does not lead to physically realistic solutions as the lateral boundary conditions are not satisfied.

5.3 Associated Lie point symmetry for the LB equations

In this section, we present the associated Lie point symmetry, which we use to determine if there are any invariant travelling wave solutions associated with the conserved quantities of the problem. To avoid repetition only calculations relating to the elementary conserved vectors are presented. We refer to the infinitesimal generator obtained in the previous section, namely

$$X = c_2\frac{\partial}{\partial t} + c_5\frac{\partial}{\partial x} + (c_6 u + c_7(t,x))\frac{\partial}{\partial u} + \big(c_6\eta + c_8(t,x)\big)\frac{\partial}{\partial \eta}. \tag{5.91}$$

We are looking for solutions in the form of travelling waves, therefore, we consider $c_7(t,x) = 0$ and $c_8(t,x) = 0$. Thus, (5.91) becomes

$$X = c_2\frac{\partial}{\partial t} + c_5\frac{\partial}{\partial x} + c_6 u\frac{\partial}{\partial u} + c_6\eta\frac{\partial}{\partial \eta}. \tag{5.92}$$

In general, we have the infinitesimal generator

$$X = \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \phi^1\frac{\partial}{\partial u} + \phi^2\frac{\partial}{\partial \eta}, \tag{5.93}$$

where

$$\begin{aligned} \xi^1 = \xi^1(t,x,u,\eta) = 1, \quad & \xi^2 = \xi^2(t,x,u,\eta) = c_5, \\ \phi^1 = \phi^1(t,x,u,\eta) = c_6 u, \quad & \phi^2 = \phi^2(t,x,u,\eta) = c_6\eta. \end{aligned} \tag{5.94}$$

We introduce the association formula

$$X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0. \tag{5.95}$$

For $i = 1$ and $k = 1,2$ we obtain

$$X(T^1) + T^1\big(D_1(\xi^1) + D_2(\xi^2)\big) - T^1 D_1(\xi^1) - T^2 D_2(\xi^1) = 0 \tag{5.96}$$

$$\Rightarrow X(T^1) + T^1 D_x(\xi^2) - T^2 D_x(\xi^1) = 0, \tag{5.97}$$

and for $i = 2$ and $k = 1,2$ we get

$$X(T^2) + T^2\big(D_1(\xi^1) + D_2(\xi^2)\big) - T^1 D_1(\xi^2) - T^2 D_2(\xi^2) = 0$$

$$\Rightarrow X(T^2) + T^2 D_t(\xi^1) - T^1 D_t(\xi^2) = 0. \tag{5.98}$$

In addition, we use the prolongation formulae

$$\begin{aligned}\zeta^\alpha_i = D_i(\phi^\alpha) - u^\alpha_s D_i(\xi^s), \quad & \zeta^\alpha_{ij} = D_j(\zeta^\alpha_i) - u^\alpha_{is} D_j(\xi^s),\\ \zeta^\alpha_{ijk} = D_k(\zeta^\alpha_{ij}) - u^\alpha_{ijs} D_k(\xi^s), \quad & \zeta^\alpha_{ijkl} = D_l(\zeta^\alpha_{ijk}) - u^\alpha_{ijks} D_l(\xi^s),\end{aligned} \tag{5.99}$$

where

$$t = x^1, \qquad x = x^2, \qquad u = u^1, \qquad \eta = u^2. \tag{5.100}$$

5.3.1 Case 1: Conservation of the mass flow rate

Consider the conserved vector

$$\begin{aligned}T^1 &= u,\\ T^2 &= g\eta - \frac{h^2}{2} u_{xt}.\end{aligned} \tag{5.101}$$

Substituting (5.101) into (5.97), using the infinitesimal generator

$$X = c_2 \frac{\partial}{\partial t} + c_5 \frac{\partial}{\partial x} + c_6 u \frac{\partial}{\partial u} + c_6 \eta \frac{\partial}{\partial \eta} + \zeta^1_{21} \frac{\partial}{\partial u_{xt}}, \tag{5.102}$$

gives

$$\begin{aligned}& X(u) + u D_x(c_5) - (g\eta - \frac{h^2}{2} u_{xt}) D_x(1) = 0\\ \Rightarrow & c_6 = 0.\end{aligned} \tag{5.103}$$

Similarly, substituting (5.101) into (5.98), using the infinitesimal generator

$$X = c_2 \frac{\partial}{\partial t} + c_5 \frac{\partial}{\partial x} + c_6 u \frac{\partial}{\partial u} + c_6 \eta \frac{\partial}{\partial \eta} + \zeta^1_{21} \frac{\partial}{\partial u_{xt}}, \tag{5.104}$$

gives

$$\begin{aligned}& X(g\eta - \frac{h^2}{2} u_{xt}) + (g\eta - \frac{h^2}{2} u_{xt}) D_t(1) - u D_t(c_5) = 0\\ \Rightarrow & c_6 \left(g\eta - \frac{h^2}{2} u_{tx} \right) = 0\\ \Rightarrow & c_6 = 0.\end{aligned} \tag{5.105}$$

The results of (5.103) and (5.105) simplify (5.102) to give the associated Lie point symmetry

$$X = c_2 \frac{\partial}{\partial t} + c_5 \frac{\partial}{\partial x}, \tag{5.106}$$

where c_2 and c_5 are arbitrary.

The associated Lie point symmetry (5.106) takes the form of a symmetry that produces travelling wave solutions. As such, the conserved mass flow rate is associated with invariant travelling wave solutions that do not satisfy the decaying boundary conditions, as the solutions are ossilatory functions.

5.3.2 Case 2: Conservation of mass

Consider the conserved vector

$$\begin{aligned} T^1 &= \eta, \\ T^2 &= hu - \frac{h^3}{6} u_{xx}. \end{aligned} \tag{5.107}$$

Following the same approach as adopted in Case 1, we again obtain the result that $c_6 = 0$. The result simplifies to give the associated Lie point symmetry (5.106) obtained in Case 1.

5.4 Concluding remarks

The invariant solutions discovered in this chapter are physically unrealistic. We observe that the effect of linearizing the NB equations leads to ossilatory type solutions that do not satisfy the decaying lateral boundary conditions. We conclude that applying a linear approximation to the NB equations leads to physically unrealistic results and so one should not linearize these types of equations.

We discovered that all the conserved quantities presented in the previous chapter are associated, via the associated Lie point symmetry, with invariant travelling wave solutions that do not satisfy the decaying boundary conditions, as the solutions are ossilatory functions. Finally, we note that the conserved quantities are not needed to solve for the constants in an attempt to obtain linear solutions.

Chapter 6

Conserved Quantities for the NB Equations

6.1 Conservation laws and conserved vectors for the NB equations

In this section, we again make use of the Multiplier method to determine the conservation laws and conserved vectors for the system of NB equations. We reiterate that, for a solitary wave unbounded in the horizontal directions, the initial quantities must be conserved and, as the wave evolves, must still be maintained. Given that initial quantities are difficult to obtain in general, conservation laws provide us with properties at the initial state of the system. Solitary wave and soliton solutions satisfy these conservation laws. However, periodic solutions do not, and thus, the latter are not considered as possible solutions for the solitary wave [6]. Obtaining conservation laws is, therefore, an important part of the process of obtaining physically realistic solutions.

We consider the system of NB equations in dimensionless form, (3.32) and (3.33), given by

$$\frac{\partial \eta}{\partial t}+\frac{\partial u}{\partial x}+\epsilon\left(u\frac{\partial \eta}{\partial x}+\eta\frac{\partial u}{\partial x}\right)=\frac{\mu^2}{6}\frac{\partial^3 u}{\partial x^3}, \tag{6.1}$$

$$\frac{\partial u}{\partial t}+\epsilon u\frac{\partial u}{\partial x}+\frac{\partial \eta}{\partial x}=\frac{\mu^2}{2}\frac{\partial^3 u}{\partial x^2\partial t}, \tag{6.2}$$

with lateral boundary conditions

$$u(t,\pm\infty)=\frac{\partial u(t,\pm\infty)}{\partial t}=\frac{\partial u(t,\pm\infty)}{\partial x}=\eta(t,\pm\infty)=\frac{\partial \eta(t,\pm\infty)}{\partial t}=\frac{\partial \eta(t,\pm\infty)}{\partial x}=0, \quad (6.3)$$

where $u(t,x)$ is the fluid velocity of the solitary wave in the x-direction, $\eta(t,x)$ is the surface elevation of the solitary wave, ϵ and μ are the non-linearity and frequency dispersion parameters respectively.

In order to derive the conservation laws for the system given by (6.1) and (6.2) using the Multiplier method, we consider

$$\Lambda_1(\eta_t+u_x+\epsilon u\eta_x+\epsilon\eta u_x-\frac{\mu^2}{6}u_{xxx})+\Lambda_2(u_t+\epsilon uu_x+\eta_x-\frac{\mu^2}{2}u_{xxt})=D_tT^1+D_xT^2, \quad (6.4)$$

where the multipliers are given by $\Lambda_{1,2}(t,x,u,\eta,u_t,\eta_t,u_x,\eta_x)$ and $T=(T^1,T^2)$ are the components of the conserved vector.

We make use of the Euler operator to derive the determining equations. This allows us to set the divergence of our conservation laws to zero. Thus by employing (6.4) as shown above, together with this property, we structure our determining equations by expanding

$$E_u[\Lambda_1(\eta_t+u_x+\epsilon u\eta_x+\epsilon\eta u_x-\frac{\mu^2}{6}u_{xxx})+\Lambda_2(u_t+\epsilon uu_x+\eta_x-\frac{\mu^2}{2}u_{xxt})]=0, \quad (6.5)$$

$$E_\eta[\Lambda_1(\eta_t+u_x+\epsilon u\eta_x+\epsilon\eta u_x-\frac{\mu^2}{6}u_{xxx})+\Lambda_2(u_t+\epsilon uu_x+\eta_x-\frac{\mu^2}{2}u_{xxt})]=0. \quad (6.6)$$

The determining equations are then separated with respect to products and powers of the derivatives of u and η. This allows us to obtain the following set of multipliers

$$\Lambda_1=k_1 \text{ and } \Lambda_2=k_2, \quad (6.7)$$

where k_1 and k_2 are arbitrary. Refer to Appendix C for calculations of the multipliers. The conserved vector $T=(T^1,T^2)$ can now be obtained using the Multiplier method.

When $u(t,x)$ and $\eta(t,x)$ are solutions for the system of equations (6.1) and (6.2), we may obtain a set of conserved vectors by satisfying (6.4) such that $D_t[T_1]+D_x[T_2]=0$. Given that the multipliers (6.7) are arbitrary constants, we will now consider various possible values thereof in order to obtain the conserved vectors for the NB equations.

6.1.1 Case 1: $\Lambda_1 = k_1 = 0, \Lambda_2 = k_2 = 1$

Consider the case where $\Lambda_1 = k_1 = 0, \Lambda_2 = k_2 = 1$. We obtain the following conservation law

$$\begin{aligned} & u_t + \epsilon u u_x + \eta_x - \frac{\mu^2}{2} u_{xxt} = D_t T^1 + D_x T^2 = 0, \\ \Rightarrow & D_t(u) + D_x\left(\frac{\epsilon}{2} u^2 + \eta - \frac{\mu^2}{2} u_{xt}\right) = 0, \end{aligned} \tag{6.8}$$

where the conserved vector is given by

$$\begin{aligned} T^1 &= u, \\ T^2 &= \frac{\epsilon}{2} u^2 + \eta - \frac{\mu^2}{2} u_{xt}. \end{aligned} \tag{6.9}$$

6.1.2 Case 2: $\Lambda_1 = k_1 = 1, \Lambda_2 = k_2 = 0$

Considering the case where $\Lambda_1 = k_1 = 1, \Lambda_2 = k_2 = 0$, we obtain the second conservation law given by

$$\begin{aligned} & \eta_t + u_x + \epsilon u \eta_x + \epsilon \eta u_x - \frac{\mu^2}{6} u_{xxx} = D_t T^1 + D_x T^2 = 0, \\ \Rightarrow & D_t(\eta) + D_x\left(u + \epsilon u \eta - \frac{\mu^2}{6} u_{xx}\right) = 0, \end{aligned} \tag{6.10}$$

where the conserved vector is given by

$$\begin{aligned} T^1 &= \eta, \\ T^2 &= u + \epsilon u \eta - \frac{\mu^2}{6} u_{xx}. \end{aligned} \tag{6.11}$$

6.2 Conserved quantities for the NB equations

In this section, we determine the conserved quantities through the use of the conserved vectors obtained in the previous section. These are derived by integrating the conservation laws, obtained via the conserved vectors from Section 6.1, in the direction of wave propagation and imposing the boundary conditions.

6.2.1 Case 1: Conservation of the mass flow rate

Consider the following conserved vector

$$\begin{aligned} T^1 &= u, \\ T^2 &= \frac{\epsilon}{2}u^2 + \eta - \frac{\mu^2}{2}u_{xt}. \end{aligned} \tag{6.12}$$

The conservation law for the system given by (6.1) and (6.2) is given by

$$\frac{\partial T^1}{\partial t} + \frac{\partial T^2}{\partial x} = D_t T^1 + D_x T^2 = 0. \tag{6.13}$$

The first conserved quantity for the dimensionless NB equations is obtained by substituting (6.12) into (6.13) and subsequently integrating (6.13) with respect to x on the interval $(-\infty, \infty)$, having fixed t. Using Leibniz's rule of integration, we obtain

$$\frac{d}{dt}\int_{-\infty}^{\infty} u\, dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} u\, dx. \tag{6.14}$$

Applying boundary conditions (6.3), (6.14) reduces to

$$\rho_a \int_{-\infty}^{\infty} u\, dx = J_1, \tag{6.15}$$

where J_1 is a constant independent of time. Thus, the mass flow rate, $\rho_a J_1$, where ρ_a represents the area density, is constant along the wave. The mass flow rate is simply the amount of mass passing by a single point over a duration of time [51].

6.2.2 Case 2: Conservation of mass

We now turn to the following conserved vector

$$\begin{aligned} T^1 &= \eta, \\ T^2 &= u + \epsilon u\eta - \frac{\mu^2}{6}u_{xx}. \end{aligned} \tag{6.16}$$

The second conserved quantity for the dimensionless NB equations is obtained by substituting (6.16) into (6.13) and subsequently integrating (6.13) with respect to x on the interval $(-\infty, \infty)$, having fixed t. Using Leibniz's rule of integration, we obtain

$$\frac{d}{dt}\int_{-\infty}^{\infty} \eta\, dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} \eta\, dx. \tag{6.17}$$

Applying boundary conditions (6.3), (6.17) reduces to

$$\rho_a \int_{-\infty}^{\infty} \eta \, dx = J_2, \tag{6.18}$$

where J_2 is a constant independent of time. Thus, the mass density, $\rho_a J_2$, where ρ_a represents the area density, is constant along the wave. Hence, for all time and for all surface waves represented by $\eta(t, x)$, the mass of the fluid associated with the wave is conserved.

6.3 Concluding remarks

We considered the conservation laws for the dimensionless NB equations, obtained via a Multiplier approach. The equations produced two local conservation laws. From these conservation laws we were able to obtain the conserved mass quantity and a newly discovered mass flow rate by integrating across the wave in the horizontal direction.

Chapter 7

Lie point symmetries for the NB equations

In this chapter, we obtain the full set of Lie point symmetries for the NB equations (7.1) and (7.2) representing a solitary wave at a constant water depth h propagating rightward. The Lie point symmetries will assist in determining the invariant solutions for the NB equations. This approach is advantageous, as it is systematic in nature and does not rely on extensive physical insights to obtain the solutions. We also present the associated Lie point symmetry, which we use to determine the conserved vectors that are associated with the linear combination of the time and spatial Lie point symmetries for the NB equations.

7.1 Lie point symmetries for the NB equations

As outlined in Section 5.1, the symmetries obtained in this section have also been calculated manually.

From (3.32) and (3.33)

$$\frac{\partial \eta}{\partial t}+\frac{\partial u}{\partial x}+\epsilon\left(u\frac{\partial \eta}{\partial x}+\eta\frac{\partial u}{\partial x}\right)=\frac{\mu^2}{6}\frac{\partial^3 u}{\partial x^3}, \tag{7.1}$$

$$\frac{\partial u}{\partial t}+\epsilon u\frac{\partial u}{\partial x}+\frac{\partial \eta}{\partial x}=\frac{\mu^2}{2}\frac{\partial^3 u}{\partial x^2\partial t}, \tag{7.2}$$

where t is time, x is the horizontal spatial component, μ is the dispersive parameter, ϵ is the non-linear parameter, u is the fluid velocity of the solitary wave in the x-direction and η is the surface elevation of the solitary wave.

We proceed to obtain the first determining equation by considering (7.1) with the Lie point operator given by

$$X = \xi^1(t,x,u,\eta)\frac{\partial}{\partial t} + \xi^2(t,x,u,\eta)\frac{\partial}{\partial x} + \psi^1(t,x,u,\eta)\frac{\partial}{\partial u} + \psi^2(t,x,u,\eta)\frac{\partial}{\partial \eta}, \tag{7.3}$$

where ξ^1, ξ^2, ψ^1 and ψ^2 are the infinitesimals. This operator is a Lie point symmetry generator of the governing equation (7.1) if and only if the infinitesimal criterion for invariance holds, i.e.

$$X^{[3]}\left(\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial x} + \epsilon u \frac{\partial \eta}{\partial x} + \epsilon \eta \frac{\partial u}{\partial x} - \frac{\mu^2}{6}\frac{\partial^3 u}{\partial x^3}\right)\Bigg|_{(7.1),(7.2)} = 0. \tag{7.4}$$

Since (7.1) depends on third derivatives, we take the third prolongation and obtain

$$\begin{aligned} X^{[3]} = {} & \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \psi^1\frac{\partial}{\partial u} + \psi^2\frac{\partial}{\partial \eta} + \varphi^1_t\frac{\partial}{\partial u_t} + \varphi^1_x\frac{\partial}{\partial u_x} + \varphi^2_t\frac{\partial}{\partial \eta_t} + \varphi^2_x\frac{\partial}{\partial \eta_x} + \varphi^2_{ttx}\frac{\partial}{\partial \eta_{ttx}} \\ & + \varphi^1_{tt}\frac{\partial}{\partial u_{tt}} + \varphi^1_{tx}\frac{\partial}{\partial u_{tx}} + \varphi^1_{xx}\frac{\partial}{\partial u_{xx}} + \varphi^2_{tt}\frac{\partial}{\partial \eta_{tt}} + \varphi^2_{tx}\frac{\partial}{\partial \eta_{tx}} + \varphi^2_{xx}\frac{\partial}{\partial \eta_{xx}} + \varphi^2_{txx}\frac{\partial}{\partial \eta_{txx}} \\ & + \varphi^1_{ttt}\frac{\partial}{\partial u_{ttt}} + \varphi^1_{ttx}\frac{\partial}{\partial u_{ttx}} + \varphi^1_{txx}\frac{\partial}{\partial u_{txx}} + \varphi^1_{xxx}\frac{\partial}{\partial u_{xxx}} + \varphi^2_{ttt}\frac{\partial}{\partial \eta_{ttt}} + \varphi^2_{xxx}\frac{\partial}{\partial \eta_{xxx}}, \end{aligned} \tag{7.5}$$

where we can define explicitly the extended infinitesimals as

$$\varphi^\alpha_i = D_i\left(\psi^\alpha\right) - u^\alpha_j D_i\left(\xi^j\right), \tag{7.6}$$

$$\varphi^\alpha_{ij} = D_j\left(\varphi^\alpha_i\right) - u^\alpha_{il} D_j\left(\xi^l\right), \tag{7.7}$$

where $u^1 = u$, $u^2 = \eta$, $D_i = D_1 = D_t$ and $D_j = D_2 = D_x$ for $i, j, l = 1, 2$.

Thus by applying the third prologation (7.5) onto (7.1) we obtain

$$\left(\epsilon\psi^1\eta_x + \epsilon\psi^2 u_x + \left(1+\epsilon\eta\right)\varphi^1_x + \varphi^2_t + \epsilon u\varphi^2_x - \frac{\mu^2}{6}\varphi^1_{xxx}\right)\Bigg|_{(7.1),(7.2)} = 0. \tag{7.8}$$

Similarly for (7.2) we obtain

$$\left(\epsilon\psi^1 u_x + \varphi^1_t + \epsilon u\varphi^1_x + \varphi^2_x - \frac{1}{2}\mu^2\varphi^1_{xxt}\right)\Bigg|_{(7.1),(7.2)} = 0. \tag{7.9}$$

The extended infinitesimals for (7.9) are given by (D.1), (D.2), (D.3) and (D.4) in Appendix D. We substitute the extended infinitesimals (D.1), (D.2), (D.3), (D.4) into (7.9) to give the first determining equation given by (D.5) in Appendix D, subject to (7.1) and (7.2). The determining equation (D.5) can be split into the following derivative products of u and η, containing third-order derivatives of u

$$u_{ttx}: \frac{\partial \xi^1}{\partial x} = 0, \tag{7.10}$$
$$u_x u_{ttx}: \frac{\partial \xi^1}{\partial u} = 0, \tag{7.11}$$
$$\eta_x u_{ttx}: \frac{\partial \xi^1}{\partial \eta} = 0. \tag{7.12}$$

Considering (7.10)-(7.12), we note that ξ^1 is only dependent on t.

Next, we separate (D.5) by the following derivative products of u and η containing third-order derivatives of η

$$\eta_{txx}: \frac{\partial \psi^1}{\partial \eta} = 0, \tag{7.13}$$
$$u_t \eta_{txx}: \frac{\partial \xi^1}{\partial \eta} = 0, \tag{7.14}$$
$$u_x \eta_{txx}: \frac{\partial \xi^2}{\partial \eta} = 0. \tag{7.15}$$

Considering (7.13)-(7.15), it can be seen that ξ^2 and ψ^1 are independent of η.

We further separate (D.5) by $u_{xt} u_{xx}$ as follows

$$u_{xt} u_{xx}: \frac{\partial \xi^1}{\partial u} + 3\frac{\partial \xi^2}{\partial u} = 0. \tag{7.16}$$

From (7.16) we note that ξ^2 is independent of u.

Next, we separate (D.5) by η_t as follows

$$\eta_t: \frac{\partial \xi^2}{\partial t} = 0. \tag{7.17}$$

From (7.17) we see that ξ^2 is independent of t.

Finally, we separate (D.5) by the following derivative products and powers of u and η

$$u_{xx}: \frac{\partial^2 \psi^1}{\partial t \partial u} = 0, \tag{7.18}$$
$$u_x u_{xt}: \frac{\partial^2 \psi^1}{\partial u^2} = 0, \tag{7.19}$$
$$u_{xt}: \frac{d^2 \xi^2}{d x^2} = 2\frac{\partial^2 \psi^1}{\partial x \partial u}, \tag{7.20}$$
$$\eta_x: \frac{d\xi^2}{dx} + \frac{d\xi^1}{dt} = \frac{\partial \psi^1}{\partial u} - \frac{\partial \psi^2}{\partial \eta}, \tag{7.21}$$

$$u_x : \epsilon u\left(\frac{d\xi^2}{dx}+\frac{d\xi^1}{dt}\right)=\mu^2\frac{\partial^3\psi^1}{\partial t\partial x\partial u}-\epsilon\psi^1-\frac{\partial\psi^2}{\partial u}, \tag{7.22}$$

$$u_t : \frac{d\xi^2}{dx}=\frac{\mu^2}{4}\frac{\partial^3\psi^1}{\partial u\partial x^2}, \tag{7.23}$$

$$1 : \frac{\partial\psi^1}{\partial t}+\epsilon u\frac{\partial\psi^1}{\partial x}+\frac{\partial\psi^2}{\partial x}-\frac{\mu^2}{2}\frac{\partial^3\psi^1}{\partial t\partial x^2}=0, \tag{7.24}$$

where

$$\xi^1=\xi^1(t), \qquad \xi^2=\xi^2(x), \tag{7.25}$$

$$\psi^1=\psi^1(t,x,u), \qquad \psi^2=\psi^2(t,x,u,\eta). \tag{7.26}$$

We now proceed to ascertain the next determining equation; that is we consider (7.8). The extended infinitesimals for (7.8) are given by the following (subject to results obtained from the first determining equation, namely (7.25) and (7.26)),

$$\begin{aligned}\varphi_t^2&=D_t\left(\psi^2\right)-\eta_t D_t\left(\xi^1\right)-\eta_x D_t\left(\xi^2\right)\\&=\frac{\partial\psi^2}{\partial t}+\frac{\partial\psi^2}{\partial u}u_t+\frac{\partial\psi^2}{\partial\eta}\eta_t-\frac{d\xi^1}{dt}\eta_t.\end{aligned} \tag{7.27}$$

Similarly we obtain

$$\begin{aligned}\varphi_x^1&=D_x\left(\psi^1\right)-u_t D_x\left(\xi^1\right)-u_x D_x\left(\xi^2\right)\\&=\frac{\partial\psi^1}{\partial x}+\frac{\partial\psi^1}{\partial u}u_x-\frac{d\xi^2}{dx}u_x,\end{aligned} \tag{7.28}$$

and

$$\begin{aligned}\varphi_x^2&=D_x\left(\psi^2\right)-\eta_t D_x\left(\xi^1\right)-\eta_x D_x\left(\xi^2\right)\\&=\frac{\partial\psi^2}{\partial x}+\frac{\partial\psi^2}{\partial u}u_x+\frac{\partial\psi^2}{\partial\eta}\eta_x-\frac{d\xi^2}{dx}\eta_x,\end{aligned} \tag{7.29}$$

and finally

$$\begin{aligned}\varphi_{xxx}^1&=D_x\left(\varphi_{xx}^1\right)-u_{xxt}D_x\left(\xi^1\right)-u_{xxx}D_x\left(\xi^2\right)\\&=\frac{\partial^3\psi^1}{\partial x^3}+3\frac{\partial^3\psi^1}{\partial x^2\partial u}u_x+3\frac{\partial^3\psi^1}{\partial x\partial u^2}(u_x)^2+3\frac{\partial^2\psi^1}{\partial x\partial u}u_{xx}+\frac{\partial^3\psi^1}{\partial u^3}(u_x)^3\\&\quad+3\frac{\partial^2\psi^1}{\partial u^2}u_x u_{xx}+\frac{\partial\psi^1}{\partial u}u_{xxx}-\frac{d^3\xi^2}{dx^3}u_x-3\frac{d^2\xi^2}{dx^2}u_{xx}-3\frac{d\xi^2}{dx^2}u_{xxx}.\end{aligned} \tag{7.30}$$

We substitute the extended infinitesimals (7.27)-(7.30) into (7.8) to give the determining equation subject to (7.1) and (7.2)

$$
\begin{aligned}
0=&\epsilon\psi^1\eta_x+\epsilon\psi^2u_x+(1+\epsilon\eta)\left[\frac{\partial\psi^1}{\partial x}+\frac{\partial\psi^1}{\partial u}u_x-\frac{d\xi^2}{dx}u_x\right]+\frac{\partial\psi^2}{\partial t}+\frac{\partial\psi^2}{\partial u}u_t-\frac{d\xi^1}{dt}\eta_t\\
&-\frac{\mu^2}{6}\left[\frac{\partial^3\psi^1}{\partial x^3}+3\frac{\partial^3\psi^1}{\partial x^2\partial u}u_x+3\frac{\partial^3\psi^1}{\partial x\partial u^2}(u_x)^2+3\frac{\partial^2\psi^1}{\partial x\partial u}u_{xx}+\frac{\partial^3\psi^1}{\partial u^3}(u_x)^3+3\frac{\partial^2\psi^1}{\partial u^2}u_xu_{xx}\right.\\
&\left.+\frac{\partial\psi^1}{\partial u}u_{xxx}-\frac{d^3\xi^2}{dx^3}u_x-3\frac{d^2\xi^2}{dx^2}u_{xx}-3\frac{d\xi^2}{dx^2}u_{xxx}\right]+\epsilon u\left[\frac{\partial\psi^2}{\partial x}+\frac{\partial\psi^2}{\partial u}u_x+\frac{\partial\psi^2}{\partial\eta}\eta_x-\frac{d\xi^2}{dx}\eta_x\right].
\end{aligned}
\tag{7.31}
$$

Equation (7.31) can be separated by u_{xxt} as follows

$$u_{xxt}:\frac{\partial\psi^2}{\partial u}=0. \tag{7.32}$$

From (7.32) we note that ψ^2 is independent of u.

Next, we separate (7.31) by the following derivative products and powers of u and η

$$u_xu_{xx}:\frac{\partial^2\psi^1}{\partial u^2}=0, \tag{7.33}$$

$$u_{xx}:\frac{\partial^2\psi^1}{\partial x\partial u}=\frac{d^2\xi^2}{dx^2}, \tag{7.34}$$

$$(u_x)^2:\frac{\partial^3\psi^1}{\partial x\partial u^2}=0, \tag{7.35}$$

$$u_x:\epsilon\psi^2+2(1+\epsilon\eta)\frac{d\xi^2}{dx}-\frac{\mu^2}{6}\left[3\frac{\partial^3\psi^1}{\partial x^2\partial u}-\frac{d^3\xi^2}{dx^3}\right]=0, \tag{7.36}$$

$$\eta_x:\epsilon\psi^1-\epsilon u\frac{\partial\psi^1}{\partial u}+\epsilon u\frac{\partial\psi^2}{\partial\eta}+2\epsilon u\frac{d\xi^2}{dx}=0, \tag{7.37}$$

$$\eta_t:-\frac{\partial\psi^1}{\partial u}+\frac{\partial\psi^2}{\partial\eta}-\frac{d\xi^1}{dt}+3\frac{d\xi^2}{dx}=0, \tag{7.38}$$

$$1:(1+\epsilon\eta)\frac{\partial\psi^1}{\partial x}+\frac{\partial\psi^2}{\partial t}+\epsilon u\frac{\partial\psi^2}{\partial x}-\frac{\mu^2}{6}\left[\frac{\partial^3\psi^1}{\partial x^3}\right]=0, \tag{7.39}$$

where

$$\xi^1=\xi^1(t),\qquad \xi^2=\xi^2(x), \tag{7.40}$$

$$\psi^1=\psi^1(t,x,u),\qquad \psi^2=\psi^2(t,x,\eta). \tag{7.41}$$

We now consider (7.18)-(7.24) and (7.33)-(7.39); we integrate (7.20) twice with respect to x and obtain

$$\xi^2(x)=c_1x+c_2. \tag{7.42}$$

Using (7.45) and (7.42), we can simplify (7.36) to obtain

$$\psi^2(t,x,\eta) = -2\frac{(1+\epsilon\eta)}{\epsilon}c_1. \tag{7.43}$$

We now integrate (7.19) twice with respect to u and substitute the result into (7.18) giving

$$\psi^1(t,x,u) = uA(x) + B(t,x). \tag{7.44}$$

By (7.20) and (7.34),

$$\frac{\partial^2\psi^1}{\partial x\partial u} = 0. \tag{7.45}$$

We now substitute (7.44) into (7.45) to get $A(x) = c_3$, and hence,

$$\psi^1(t,x,u) = c_3 u + B(t,x). \tag{7.46}$$

By substituting (7.46) into (7.23) we obtain $c_1 = 0$, therefore, $\psi^2 = 0$.

In summary

$$\xi^1 = \xi^1(t), \quad \xi^2 = c_2, \tag{7.47}$$

$$\psi^1 = c_3 u + B(t,x), \quad \psi^2 = 0. \tag{7.48}$$

Substituting (7.47) and (7.48) into (7.21) and integrating once with respect to t gives

$$\xi^1(t) = c_3 t + c_4. \tag{7.49}$$

Similarly substituting (7.47)-(7.49) into (7.22) and separating the result by u we deduce that $c_3 = 0$, thus, $B(t,x) = 0$.

In summary

$$\xi^1 = c_4, \quad \xi^2 = c_2, \tag{7.50}$$

$$\psi^1 = 0, \quad \psi^2 = 0. \tag{7.51}$$

Note that (7.18)-(7.24) and (7.33)-(7.39) are now identically satisfied. Substituting (7.50) and (7.51) into (7.3), we obtain the infinitesimal generator

$$X = c_4\frac{\partial}{\partial t} + c_2\frac{\partial}{\partial x}, \tag{7.52}$$

with symmetries given by $X_1 = \dfrac{\partial}{\partial t}$ and $X_2 = \dfrac{\partial}{\partial x}$.

7.2 Associated Lie point symmetry for the NB equations

In this section, we present the associated Lie point symmetry, which we use to determine the conserved vectors that are associated with the linear combination of the time and spatial Lie point symmetries derived in Section 7.1 for all values of c_2 and c_4 for the NB equations.

7.2.1 Case 1: Conservation of the mass flow rate

Consider the conserved vector

$$\begin{aligned} T^1 &= u, \\ T^2 &= \frac{\epsilon}{2}u^2 + \eta - \frac{\mu^2}{2}u_{xt}. \end{aligned} \tag{7.53}$$

Consider the first invariant condition

$$X(T^1) + T^1 D_x(\xi^2) - T^2 D_x(\xi^1) = 0, \tag{7.54}$$

where $\xi^1 = c_4$ and $\xi^2 = c_2$.

Equation (7.54) simplifies to

$$\left(c_4\frac{\partial}{\partial t} + c_2\frac{\partial}{\partial x}\right)u + uD_x(c_2) - \left(\frac{\epsilon}{2}u^2 + \eta - \frac{\mu^2}{2}u_{xt}\right)D_x(c_4) = 0, \tag{7.55}$$

upon substitution of the conserved vector and infinitesimal generator. Thus, equation (7.55) is identically satisfied and the conserved vector is associated with the linear combination of the time and spatial Lie point symmetries for all values of c_2 and c_4.

7.2.2 Case 2: Conservation of mass

Consider the conserved vector

$$\begin{aligned} T^1 &= \eta, \\ T^2 &= u + \epsilon u\eta - \frac{\mu^2}{6}u_{xx}. \end{aligned} \tag{7.56}$$

Again consider the first invariant condition (7.54), which upon substitution of (7.56) gives

$$\left(c_4\frac{\partial}{\partial t} + c_2\frac{\partial}{\partial x}\right)\eta + \eta D_x(c_2) - \left(u + \epsilon u\eta - \frac{\mu^2}{6}u_{xx}\right)D_x(c_4) = 0. \tag{7.57}$$

Thus, (7.57) is identically satisfied and the conserved vector is associated with the linear combination of the time and spatial Lie point symmetries for all values of c_2 and c_4.

7.3 Concluding remarks

We have derived the full set of Lie point symmetries for the NB equations. We conclude that all conserved vectors are associated with the linear combination of the time and spatial Lie point symmetries for all values of c_2 and c_4 for the NB equations.

Chapter 8

Invariant exact solutions for the NB equations

In this chapter, we make use of a linear combination of the time and spatial symmetry to obtain invariant travelling wave solutions relating to the fluid velocity in the x-direction $u(t,x)$ and the surface elevation $\eta(t,x)$ for the NB equations representing a solitary wave propogating rightward. These solutions can be used to enrich the analytical validation of numerical models for tsunami detection systems. Through the process of obtaining these solutions, we observe that there are three cases of solutions dependent on the wave speed c^2, namely (i) $0 < c^2 < \infty$ where $c^2 \neq \frac{1}{15}$ and $c^2 \neq \frac{1}{3}$, (ii) $c^2 = \frac{1}{15}$ and (iii) $c^2 = \frac{1}{3}$. It is important to note that, although $c^2 = \frac{1}{15}$ can form part of the general case $0 < c^2 < \infty$, we have instead considered it separately as a special case since it is the only c^2 value for which the solution takes the form of an explicit hyperbolic function, as will be shown below. In fact, the case $c^2 = \frac{1}{15}$ is most commonly considered when studying the NB equations due to its explicit functional form, making analysis simpler [2, 23, 56].

We now consider the system (3.32) and (3.33) given by

$$\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial x} + \epsilon\left(u\frac{\partial \eta}{\partial x} + \eta\frac{\partial u}{\partial x}\right) = \frac{\mu^2}{6}\frac{\partial^3 u}{\partial x^3}, \tag{8.1}$$

$$\frac{\partial u}{\partial t} + \epsilon u\frac{\partial u}{\partial x} + \frac{\partial \eta}{\partial x} = \frac{\mu^2}{2}\frac{\partial^3 u}{\partial x^2 \partial t}, \tag{8.2}$$

describing the fluid velocity in the x-direction $u(t,x)$ and the surface elevation $\eta(t,x)$ for the NB equations representing a solitary wave propogating rightward along with the lateral boundary conditions

$$u(t,\pm\infty) = \frac{\partial u(t,\pm\infty)}{\partial t} = \frac{\partial u(t,\pm\infty)}{\partial x} = \eta(t,\pm\infty) = \frac{\partial \eta(t,\pm\infty)}{\partial t} = \frac{\partial \eta(t,\pm\infty)}{\partial x} = 0, \quad (8.3)$$

where t is time, x is the horizontal spatial component, ϵ is the non-linear parameter, μ is the dispersive parameter.

8.1 Case (i): $0 < c^2 < \infty$, $c^2 \neq \frac{1}{15}$ and $c^2 \neq \frac{1}{3}$

We recall the infinitesimal generator derived in the previous chapter, namely

$$X = c_4 \frac{\partial}{\partial t} + c_2 \frac{\partial}{\partial x}. \quad (8.4)$$

Now,

$$u = f(t,x) \text{ and } \eta = k(t,x), \quad (8.5)$$

are invariant solutions for the system, (8.1) and (8.2), provided

$$X(u - F(t,x))|_{u=F(t,x)} = 0, \quad (8.6)$$

$$X\left(\eta - G(t,x)\right)\Big|_{\eta=G(t,x)} = 0. \quad (8.7)$$

Expanding (8.6) using (8.4), we obtain the following PDE in terms of $F(t,x)$

$$c_4 \frac{\partial F}{\partial t} + c_2 \frac{\partial F}{\partial x} = 0. \quad (8.8)$$

Similarly, expanding equation (8.7) using (8.4) gives the following PDE in terms of $G(t,x)$

$$c_4 \frac{\partial G}{\partial t} + c_2 \frac{\partial G}{\partial x} = 0. \quad (8.9)$$

The differential equations of the characteristic curves of the system (8.8) and (8.9) are

$$\frac{dt}{c_4} = \frac{dx}{c_2}, \quad (8.10)$$

where $c_4 \neq 0$. This differential equation is variable separable and integrating (8.10) once gives

$$x - \frac{c_2}{c_4} t = k_1, \quad (8.11)$$

where k_1 is a constant and $c_4 \neq 0$. The general solutions for $F(k_1)$ and $G(k_1)$ are

$$k_2 = F(k_1), \tag{8.12}$$

$$k_3 = G(k_1), \tag{8.13}$$

where k_2 and k_3 are arbitrary. Since $u = F(t,x)$ and $\eta = G(t,x)$, the general solutions for $F(\xi)$ and $G(\xi)$ are

$$u(t,x) = F(\xi), \text{ and } \eta(t,x) = G(\xi), \tag{8.14}$$

where $\xi = x - ct$, and $c = \frac{c_2}{c_4} > 0$.

We now substitute (8.14) into (8.1) and (8.2) and simplify to obtain

$$-c\frac{dG}{d\xi} + \frac{dF}{d\xi} + \epsilon G\frac{dF}{d\xi} + \epsilon F\frac{dG}{d\xi} = \frac{\mu^2}{6}\frac{d^3F}{d\xi^3}, \tag{8.15}$$

$$-c\frac{dF}{d\xi} + \epsilon F\frac{dF}{d\xi} + \frac{dG}{d\xi} = -\frac{\mu^2}{2}c\frac{d^3F}{d\xi^3}. \tag{8.16}$$

Using (8.14), we then express the boundary conditions (8.3) in terms of $F(\xi)$ and $G(\xi)$ given by

$$F(\pm\infty) = \frac{\partial F(\pm\infty)}{\partial\xi} = \frac{\partial F(\pm\infty)}{\partial\xi} = G(\pm\infty) = \frac{\partial G(\pm\infty)}{\partial\xi} = \frac{\partial G(\pm\infty)}{\partial\xi} = 0. \tag{8.17}$$

Integrating (8.15) and (8.16) with respect to ξ we obtain

$$-cG + F + \epsilon GF = \frac{\mu^2}{6}\frac{d^2F}{d\xi^2} + C_1, \tag{8.18}$$

$$-cF + \frac{\epsilon}{2}F^2 + G = -\frac{1}{2}c\mu^2\frac{d^2F}{d\xi^2} + C_2. \tag{8.19}$$

From boundary conditions (8.17) we deduce that $C_1 = C_2 = 0$, thus, (8.18) and (8.19) give

$$-cG(\xi) + F(\xi) + \epsilon G(\xi)F(\xi) = \frac{\mu^2}{6}\frac{d^2F}{d\xi^2}, \tag{8.20}$$

$$-cF(\xi) + \frac{\epsilon}{2}F^2(\xi) + G(\xi) = -\frac{\mu^2}{2}c\frac{d^2F}{d\xi^2}. \tag{8.21}$$

We now determine the relation between $G(\xi)$ and $F(\xi)$. From (8.20) and (8.21), we obtain

$$\frac{d^2F}{d\xi^2} = \frac{6}{\mu^2}(-cG + F + \epsilon FG), \tag{8.22}$$

$$\frac{d^2F}{d\xi^2} = -\frac{2}{\mu^2}\left(-cF + \frac{\epsilon}{2}F^2 + G\right). \tag{8.23}$$

Equating (8.22) and (8.23), we obtain the following expression for $G(\xi)$ in terms of $F(\xi)$

$$G(F) = \frac{-F(4c+\epsilon F)}{2(1-3c^2+3c\epsilon F)}. \tag{8.24}$$

We now rewrite the system of equations (8.20) and (8.21) in terms of $F(\xi)$. From (8.21),

$$G(F) = cF - \frac{\epsilon}{2}F^2 - \frac{\mu^2}{2}c\frac{d^2F}{d\xi^2}. \tag{8.25}$$

Substituting (8.25) into (8.20), we obtain the following second-order ordinary differential equation in terms of $F(\xi)$

$$\frac{d^2F}{d\xi^2} = \frac{2}{\epsilon c\mu^2}\frac{F\left(-\frac{\epsilon}{2}F^2 + \frac{3}{2}\epsilon cF + \left(1-c^2\right)\right)}{F+A}, \tag{8.26}$$

where

$$A = \frac{1}{\epsilon c}\left(\frac{1}{3} - c^2\right). \tag{8.27}$$

We now introduce a dummy variable $H(\xi) = F(\xi) + A$, and express (8.26) in terms of $H(\xi)$ and A to get

$$\frac{d^2H}{d\xi^2} = \frac{\epsilon}{c\mu^2}\left(-H^2 + BH + D + \frac{E}{H}\right), \tag{8.28}$$

where

$$B = \frac{1}{\epsilon c},\ D = \frac{1}{3\epsilon^2c^2}\left(3c^4+6c^2-1\right) \text{ and } E = \frac{5}{3\epsilon^3c^3}\left(\frac{1}{3}-c^2\right)\left(\frac{1}{15}-c^2\right). \tag{8.29}$$

Equation (8.28) is a second-order autonomous ODE, which can be reduced and expressed as a first-order ODE using the chain rule

$$\frac{\partial^2H}{\partial\xi^2} = \frac{d}{d\xi}\left(\frac{dH}{d\xi}\right) = \frac{d}{dH}\left(\frac{dH}{d\xi}\right)\frac{dH}{d\xi} = \frac{1}{2}\frac{d}{dH}\left(\frac{dH}{d\xi}\right)^2. \tag{8.30}$$

Thus (8.28) can be expressed as

$$\frac{\partial^2H}{\partial\xi^2} = \frac{1}{2}\frac{d}{dH}\left(\frac{dH}{d\xi}\right)^2 = \frac{\epsilon}{c\mu^2}\left(-H^2 + BH + D + \frac{E}{H}\right). \tag{8.31}$$

Integrating (8.31) once with respect to $H(\xi)$ gives

$$\left(\frac{dH}{d\xi}\right)^2 = \frac{2\epsilon}{c\mu^2}\left(-\frac{1}{3}H^3 + \frac{B}{2}H^2 + DH + E\ln(H) + k_4\right), \tag{8.32}$$

where k_4 is a constant of integration. We now consider the boundary conditions on $H(\xi)$, recalling that $H(\xi) = F(\xi) + A$ and $\frac{dH}{d\xi} = \frac{dF}{d\xi}$, thus,

$$H(\pm\infty) = A \qquad \text{as} \qquad F(\pm\infty) = 0, \tag{8.33}$$

$$\frac{dH(\pm\infty)}{d\xi} = 0 \qquad \text{as} \qquad \frac{dF(\pm\infty)}{d\xi} = 0. \tag{8.34}$$

Transforming (8.32) back in terms of $F(\xi)$, we obtain

$$\left(\frac{dF}{d\xi}\right)^2 = \frac{2\epsilon}{c\mu^2}\left(-\frac{1}{3}(F+A)^3 + \frac{B}{2}(F+A)^2 + D(F+A) + E\ln(F+A) + k_4\right). \tag{8.35}$$

Imposing boundary conditions (8.17) on (8.35), provides

$$k_4 = \frac{1}{3}A^3 - \frac{B}{2}A^2 - DA - E\ln(A). \tag{8.36}$$

Substituting (8.29) and (8.36) into (8.35) and simplifying gives

$$\left(\frac{dF}{d\xi}\right)^2 = \frac{2\epsilon}{3c\mu^2}\left[-F^3 + \frac{3}{\epsilon c}\left(c^2 + \frac{1}{6}\right)F^2 - \frac{5}{\epsilon^2 c^2}\left(\frac{1}{15} - c^2\right)F + \frac{5}{\epsilon^3 c^3}\left(\frac{1}{3} - c^2\right)\left(\frac{1}{15} - c^2\right)\ln\left(\frac{F+A}{A}\right)\right], \tag{8.37}$$

Now we let

$$P(F) = -F^3 + \frac{3}{\epsilon c}\left(c^2 + \frac{1}{6}\right)F^2 - \frac{5}{\epsilon^2 c^2}\left(\frac{1}{15} - c^2\right)F + \frac{5}{\epsilon^3 c^3}\left(\frac{1}{3} - c^2\right)\left(\frac{1}{15} - c^2\right)\ln\left(\frac{F+A}{A}\right), \tag{8.38}$$

giving

$$\left(\frac{dF}{d\xi}\right)^2 = \frac{2\epsilon}{3c\mu^2}P(F), \tag{8.39}$$

and allows us to express (8.39) as

$$\int^F \frac{dF}{\sqrt{P(F)}} = \pm\sqrt{\frac{2\epsilon}{3c\mu^2}}\,\xi + k_5, \tag{8.40}$$

where k_5 is a constant of integration.

Through the general analysis, shown later in Chapter 9, we observe that there are two distinct regions within which two possible distinct solitary waves may occur. In each case, we are able to isolate the region in which $F(\xi)$ exists so that we may distinguish between these two events. The two regions representing these two distinct events are defined by our choice of the origin of the co-ordinate ξ, since, through this choice, the minimum fluid velocity of the solitary wave and the maximum fluid velocity of the solitary wave, both travelling in the horizontal direction, will correspond to this choice of $\xi = \xi_{origin}$. The reason for this is that these velocities are both defined as $F(\xi) = F(\xi_{origin})$ within their respective regions.

We choose the origin such that $\xi = 0$. As such, the minimum fluid velocity of the solitary wave in the horizontal direction is defined as $F(0) = F_2$ for $F_2 \leq F \leq 0$, and similarly the maximum fluid velocity of the solitary wave in the horizontal direction is defined as $F(0) = F_1$ for $0 \leq F \leq F_1$.

The choice of the origin allows for eloquently written solutions and visually pleasing graphs since the maximum or minimum fluid velocity coincides with the vertical axis for the fluid velocity in the horizontal direction $F(\xi)$ versus ξ. As a consequence of the dependency of the surface elevation of the solitary wave $G(\xi)$ on $F(\xi)$, shown later in this section, the maximum or minimum surface elevation of the solitary wave, namely G_2 for $F_2 \leq F \leq 0$ and G_1 for $0 \leq F \leq F_1$, coincides with the vertical axis for the surface elevation of the solitary wave $G(\xi)$ versus ξ. We emphasize that there are two distinct solitary wave solutions, one in each region. It is important to note that the two solitary waves cannot be generated by the same event.

Consider the region $0 \leq F \leq F_1$ where, given the choice of origin $F(0) = F_1$ with F_1 the maximum fluid velocity of the solitary wave in the horizontal direction, so that

$$\int_F^{F_1} \frac{dF}{\sqrt{P(F)}} = k_5, \tag{8.41}$$

where k_5 is a constant of integration. Equating (8.40) and (8.41) and solving for ξ we obtain

$$\xi = \pm\sqrt{\frac{3c\mu^2}{2\epsilon}} \int_F^{F_1} \frac{dF}{\sqrt{P(F)}}. \tag{8.42}$$

Similarly, when we consider $F(\xi)$ such that we are in the region $F_2 \leq F \leq 0$ where, given the choice of origin $F(0) = F_2$ with F_2 the minimum fluid velocity of the solitary wave in the horizontal direction, we obtain

$$\xi = \pm\sqrt{\frac{3c\mu^2}{2\epsilon}} \int_{F_2}^{F} \frac{dF}{\sqrt{P(F)}}. \tag{8.43}$$

In summary, the invariant exact solution for the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$, where $\xi = x - ct$, is given by

$$\xi = \pm\sqrt{\frac{3c\mu^2}{2\epsilon}} \int_F^{F_1} \frac{dF}{\sqrt{P(F)}}, \tag{8.44}$$

when $0 \leq F \leq F_1$ and

$$\xi = \pm\sqrt{\frac{3c\mu^2}{2\epsilon}} \int_{F_2}^{F} \frac{dF}{\sqrt{P(F)}}, \tag{8.45}$$

when $F_2 \leq F \leq 0$ where

$$P(F) = -F^3 + \frac{3}{\epsilon c}\left(c^2 + \frac{1}{6}\right)F^2 - \frac{5}{\epsilon^2 c^2}\left(\frac{1}{15} - c^2\right)F + \frac{5}{\epsilon^3 c^3}\left(\frac{1}{3} - c^2\right)\left(\frac{1}{15} - c^2\right)\ln\left(\frac{F+A}{A}\right), \tag{8.46}$$

$$A = \frac{1}{\epsilon c}\left(\frac{1}{3} - c^2\right), \tag{8.47}$$

for $c^2 \neq \frac{1}{3}$, and $c > 0$. We note that the solution $F(\xi)$ takes the form of an implicit function.

The invariant exact solution for the surface elevation of the solitary wave $G(\xi)$, where $\xi = x - ct$, is expressed in terms of $F(\xi)$. Thus, the invariant exact solution for the surface elevation of the solitary wave $G(F)$ is given by

$$G(F) = -\frac{F\left(\frac{4c}{\epsilon} + F\right)}{6c\,(A+F)} \text{ where } A = \frac{1}{\epsilon c}\left(\frac{1}{3} - c^2\right). \tag{8.48}$$

8.2 Case (ii): $c^2 = \frac{1}{15}$

We now obtain the invariant exact solution for the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$, where $\xi = x - ct$, for this special case. Substituting $c^2 = \frac{1}{15}$ into (8.44) and (8.46) gives

$$\xi = \pm\sqrt{\frac{3c\mu^2}{2\epsilon}} \int_{F}^{F_1} \frac{dF}{\sqrt{P(F)}}, \tag{8.49}$$

where

$$P(F) = F^2(F_1 - F). \tag{8.50}$$

Upon simplifying (8.50), we notice that we have two roots: 0 and $\frac{7}{10\epsilon c}$. These roots are of importance because they indicate the values which $F(\xi)$ cannot take without causing ξ to be indeterminate. As such, we can easily see that the region within which the associated solution must live in this case is $0 \leq F \leq F_1 = \frac{7}{10\epsilon c}$. We have now defined $F_1 = \frac{7}{10\epsilon c}$ as the first root of (8.50) and the maximum fluid velocity of the solitary wave in the horizontal direction and $F = 0$ as the second root of (8.50). We note that

$$\left(\frac{dF}{d\xi}\right)^2 = \frac{2\epsilon}{3c\mu^2}P(F). \tag{8.51}$$

We determine the invariant exact solution for the fluid velocity of the solitary wave $F(\xi)$, where $\xi = x - c\,t$, by assuming a solution of the form

$$F(\xi) = F_1 \operatorname{sech}^2(\theta) \tag{8.52}$$

$$\Rightarrow dF(\xi) = \frac{-2F_1 \sinh(\theta)}{\cosh^3(\theta)} d\theta. \tag{8.53}$$

Rewriting (8.49) and (8.50) in terms of θ by making use of (8.52), (8.53) and the trigonometric identity $\operatorname{sech}^2(\theta) + \tanh^2(\theta) = 1$, we obtain

$$\xi = \pm\sqrt{\frac{3c\mu^2}{2\epsilon}} \frac{2}{\sqrt{F_1}} \theta + k, \tag{8.54}$$

where k is a constant. Solving for θ gives

$$\theta = \pm\xi\left(\frac{\sqrt{7}}{2\mu}\right) + k. \tag{8.55}$$

Substituting (8.55) into (8.52) gives

$$F(\xi) = F_1 \operatorname{sech}^2\left(\pm\xi\left(\frac{\sqrt{7}}{2\mu}\right) + k\right). \tag{8.56}$$

We choose the origin of the ξ coordinate so that the maximum corresponds to $\xi = 0$. Since sech^2 is an even function, we can take the positive ξ value. Thus, the invariant exact solution for the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$, where $\xi = x - c\,t$, is given by

$$F(\xi) = \frac{21}{2\sqrt{15\epsilon}} \operatorname{sech}^2\left(\frac{\sqrt{7}}{2\mu}\xi\right). \tag{8.57}$$

We now determine the invariant exact solution for the surface elevation of the solitary wave $G(\xi)$, where $\xi = x - c\,t$. Substituting (8.57) into (8.48) gives the following invariant exact solution for $G(\xi)$

$$G(\xi) = -\frac{7}{4\epsilon} \operatorname{sech}^2\left(\frac{\sqrt{7}}{2\mu}\xi\right). \tag{8.58}$$

The solutions take the form of explicit hyperbolic functions, as there is no logarithmic term in $P(F)$ in this case.

8.3 Case (iii): $c^2 = \frac{1}{3}$

We obtain the invariant exact solution for the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$, where $\xi = x - ct$, for this special case by substituting $c^2 = \frac{1}{3}$ into (8.44) and (8.46) to obtain

$$\xi = \pm\sqrt{\frac{3c\mu^2}{2\epsilon}} \int_F^{F_1} \frac{dF}{\sqrt{P(F)}}, \tag{8.59}$$

where

$$P(F) = F(F_1 - F)(F - F_2). \tag{8.60}$$

Via a similar agrument to that given in Section 8.3, by simplifying (8.60) we notice that we have three roots: 0, $\frac{1}{4\epsilon}\left(3\sqrt{3}+\sqrt{91}\right)$ and $\frac{1}{4\epsilon}\left(3\sqrt{3}-\sqrt{91}\right)$. The regions within which the associated solution must live in this case is $0 \le F \le F_1 = \frac{1}{4\epsilon}\left(3\sqrt{3}+\sqrt{91}\right)$ and $F_2 = \frac{1}{4\epsilon}\left(3\sqrt{3}-\sqrt{91}\right) \le F \le 0$. We have now defined $F_1 = \frac{1}{4\epsilon}\left(3\sqrt{3}+\sqrt{91}\right)$ as the first root of (8.60) and the maximum fluid velocity of the solitary wave in the horizontal direction, $F_2 = \frac{1}{4\epsilon}\left(3\sqrt{3}-\sqrt{91}\right)$ as the second root of (8.60) and the minimum fluid velocity of the solitary wave in the horizontal direction and $F = 0$ as the third root of (8.60). We note that

$$\left(\frac{dF}{d\xi}\right)^2 = \frac{2\epsilon}{3c\mu^2} P(F). \tag{8.61}$$

In summary, the invariant exact solution for the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$, where $\xi = x - ct$, is given by

$$\xi = \pm\sqrt{\frac{\sqrt{3}\mu^2}{2\epsilon}} \int_F^{F_1} \frac{dF}{\sqrt{F^2(F_1 - F)(F - F_2)}}, \tag{8.62}$$

where $F_1 = \frac{1}{4\epsilon}\left(3\sqrt{3}+\sqrt{91}\right) = \frac{3.684}{\epsilon}$ and $F_2 = \frac{1}{4\epsilon}\left(3\sqrt{3}-\sqrt{91}\right) = -\frac{1.086}{\epsilon}$. We note that the solution $F(\xi)$ takes the form of an implicit function.

The invariant exact solution for the surface elevation of the solitary wave $G(\xi)$, where $\xi = x - ct$, is expressed in terms of $F(\xi)$. Thus, the invariant exact solution for the surface elevation of the solitary wave $G(F)$ is given by

$$G(F) = -\frac{1}{6c}\left(\frac{4c}{\epsilon} + F\right). \tag{8.63}$$

8.4 Concluding remarks

Considering the general case (i) where $0 < c^2 < \infty$, $c^2 \neq \frac{1}{15}$ and $c^2 \neq \frac{1}{3}$ and the special case (iii) $c^2 = \frac{1}{3}$, we find solutions in implicit form for the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ and the surface elevation of the solitary wave $G(\xi)$. To our knowledge, these are newly discovered solutions, which can be used to enrich the analytical validation of numerical models for tsunami detection systems.

For the special case (ii) $c^2 = \frac{1}{15}$, the solutions take the form of explicit hyperbolic functions, as there is no logarithmic term in $P(F)$ in this case. The reader can refer to the work done by Dutykh and Dias [2] who obtain the same set of solutions for this specific case.

Finally, we note that the conserved quantities are not needed to solve for the constants so as to obtain the non-linear solutions.

Chapter 9

General analysis for the NB equations

We recall that there are two distinct regions to consider, namely $F_2 \le F \le 0$ and $0 \le F \le F_1$. The two regions are defined by our choice of the origin of the ξ co-ordinate, as, through this choice the minimum fluid velocity of the solitary wave in the horizontal direction can be defined as $F(0) = F_2$ such that $F_2 \le F \le 0$, and the maximum fluid velocity of the solitary wave in the horizontal direction can be defined as $F(0) = F_1$ such that $0 \le F \le F_1$ corresponds to $\xi = 0$. As shown in Chapter 8, the maximum or minimum surface elevation of the solitary wave is dependent on $F(\xi)$, namely G_2 for $F_2 \le F \le 0$ and G_1 for $0 \le F \le F_1$.

In this chapter, we analyse the properties of the fluid velocity of the solitary wave in the horizontal direction and the surface elevation of the solitary wave in the above mentioned regions for the three cases of solutions dependent on the wave speed c^2, namely (i) $0 < c^2 < \infty$ where $c^2 \neq \frac{1}{15}$ and $c^2 \neq \frac{1}{3}$, (ii) $c^2 = \frac{1}{15}$ and (iii) $c^2 = \frac{1}{3}$. Although these are the three main cases, certain sections will consider different cases dependent on the property under investigation. The study of these properties is important in our pursuit of attaining physically realistic results, used to enrich the analytical validation of numerical models for tsunami detection systems.

9.1 Properties of the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$

There are three existance conditions on $\left(\frac{dF}{d\xi}\right)^2$ for the fluid velocity of the solitary wave $F(\xi)$ that need to be satisfied for a physically realistic result to occur. There is no physical interpretation for $\left(\frac{dF}{d\xi}\right)^2$ and it is merely a mathematical artifact used to study the properties of the integral in the solutions. This investigation is important, as $P(F)$ contains a logarithmic term that will produce both real and imaginary results. However, we are considering physically realistic measurements such as velocity and surface elevation, thus, we are only interested in the real results. Consider the relation between $\left(\frac{dF}{d\xi}\right)^2$ and $F(\xi)$. The regions of physical significance satisfy the conditions

1. $\left(\dfrac{dF}{d\xi}\right)^2$ is real and $\left(\dfrac{dF}{d\xi}\right)^2 \geq 0$,
2. $\left(\dfrac{dF}{d\xi}\right)^2 = 0$ at $F = 0$,
3. $\left(\dfrac{dF}{d\xi}\right) = 0$ is bounded.

Condition (1) ensures that the physical region is real whilst condition (2) ensures that the boundary conditions are satisfied. Finally condition (3) ensures that the amplitude of the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ is finite.

Using the existance condition framework detailed above, we analyse the properties of the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$. The results of this investigation will assist us in determining whether the solutions for $F(\xi)$ produce physically realistic results. This section is broken down into the following two subsections:

- Properties of $\frac{dw}{dF}\Big|_{F=0}$ to determine the turning points of $w = \left(\frac{dF}{d\xi}\right)^2$, where we investigate existance condition (1),
- Behaviour of $\left(\frac{dF}{d\xi}\right)^2$ versus $F(\xi)$ as $F(\xi)$ tends to infinity, where we investigate existance conditions (2) and (3).

9.1.1 Properties of $\left.\frac{dw}{dF}\right|_{F=0}$ to determine the turning points of w

9.1.1.1 Case (i): $0 < c^2 < \infty$, $c^2 \neq \frac{1}{15}$ and $c^2 \neq \frac{1}{3}$

We consider the exact invariant solution for the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$. We investigate the nature of the turning points of $w = \left(\frac{dF}{d\xi}\right)^2$ at $F = 0$. The results of this investigation will assist us in determining whether $\left(\frac{dF}{d\xi}\right)^2$ is real and $\left(\frac{dF}{d\xi}\right)^2 \geq 0$ i.e. whether condition (1) is satisfied.

Let

$$w = \left(\frac{dF}{d\xi}\right)^2 = \frac{2\epsilon}{3c\mu^2} P(F), \tag{9.1}$$

where $P(F)$ is given by (8.46) and (8.47). Differentiating (8.46) with respect to $F(\xi)$ gives

$$\frac{dP}{dF} = -3F^2 + \frac{6}{\epsilon c}\left(c^2 + \frac{1}{6}\right)F - \frac{5}{\epsilon^2 c^2}\left(\frac{1}{15} - c^2\right) + \frac{5}{\epsilon^3 c^3}\left(\frac{1}{3} - c^2\right)\left(\frac{1}{15} - c^2\right)\frac{1}{F+A}, \tag{9.2}$$

and differentiating (9.2) with respect to $F(\xi)$ results in

$$\frac{d^2P}{dF^2} = -6F + \frac{6}{\epsilon c}\left(c^2 + \frac{1}{6}\right) - \frac{5}{\epsilon^3 c^3}\left(\frac{1}{3} - c^2\right)\left(\frac{1}{15} - c^2\right)\frac{1}{(F+A)^2}, \tag{9.3}$$

where $A = \frac{1}{\epsilon c}\left(\frac{1}{3} - c^2\right)$ per equation (8.47). Equations (9.2) and (9.3) are evaluated at $F = 0$ and both give zero such that

$$\left.\frac{dw}{dF}\right|_{F=0} = 0, \tag{9.4}$$

is a turning point of w at $F = 0$. The nature of the turning point is given by

$$\left.\frac{d^2w}{dF^2}\right|_{F=0} = \frac{4}{\mu^2}\frac{\left(1 - c^2\right)}{\left(\frac{1}{3} - c^2\right)}. \tag{9.5}$$

If $\left.\frac{d^2w}{dF^2}\right|_{F=0} > 0$, a minimum turning point occurs at $F = 0$ else if $\left.\frac{d^2w}{dF^2}\right|_{F=0} < 0$, a maximum turning point occurs at $F = 0$. We now analyse the nature of the turning points by specific cases.

Case: $0 < c^2 < \frac{1}{3}$, $c^2 \neq \frac{1}{15}$ **and** $1 < c^2 < \infty$

From (9.5), for the cases $0 < c^2 < \frac{1}{3}$, $c^2 \neq \frac{1}{15}$ and $1 < c^2 < \infty$,

$$\frac{dw}{dF}(0) = 0, \tag{9.6}$$

and

$$\frac{d^2w}{dF^2}(0) = \frac{4}{\mu^2}\frac{\left(1 - c^2\right)}{\left(\frac{1}{3} - c^2\right)} > 0, \tag{9.7}$$

thus, the turning point at $F=0$ is a minimum. Therefore, in the neighbourhood of $F=0$, $\left(\frac{dF}{d\xi}\right)^2 \geq 0$ for both $F<0$ and $F>0$.

Case: $c^2 = 1$

From (9.5), for the case $c^2=1$,

$$\frac{dw}{dF}(0)=0, \tag{9.8}$$

and

$$\frac{d^2w}{dF^2}(0)=\frac{4}{\mu^2}\frac{\left(1-c^2\right)}{\left(\frac{1}{3}-c^2\right)}=0, \tag{9.9}$$

thus, a point of inflection occurs at $F=0$.

Case: $\frac{1}{3}<c^2<1$

From (9.5), for the case $\frac{1}{3}<c^2<1$,

$$\frac{dw}{dF}(0)=0, \tag{9.10}$$

and

$$\frac{d^2w}{dF^2}(0)=\frac{4}{\mu^2}\frac{\left(1-c^2\right)}{\left(\frac{1}{3}-c^2\right)}<0, \tag{9.11}$$

thus, the turning point at $F=0$ is a maximum. Therefore, in the neighbourhood of $F=0$, $\left(\frac{dF}{d\xi}\right)^2 \geq 0$ for both $F<0$ and $F>0$.

9.1.1.2 Case (ii): $c^2=\frac{1}{15}$

Once more we let

$$w=\left(\frac{dF}{d\xi}\right)^2=\frac{2\epsilon}{3c\mu^2}P(F). \tag{9.12}$$

where $P(F)=F^2(F_1-F)$ and $F_1=\frac{7}{10\epsilon c}$. We can express $P(F)$ as follows

$$P(F)=-F^3+\frac{3}{\epsilon c}\left(c^2+\frac{1}{6}\right)F^2-\frac{5}{\epsilon^2c^2}\left(\frac{1}{15}-c^2\right)F. \tag{9.13}$$

Employing the same approach as has been done in Section 9.1.1.1, we obtain the following turning point of w, occurring at $F=0$

$$\left.\frac{dw}{dF}\right|_{F=0}=0, \tag{9.14}$$

and

$$\left.\frac{d^2w}{dF^2}\right|_{F=0}=\frac{14}{\mu^2}>0. \tag{9.15}$$

Thus, the turning point at $F = 0$ is a minimum. Therefore, in the neighbourhood of $F = 0$, $\left(\frac{dF}{d\xi}\right)^2 \geq 0$ for both $F < 0$ and $F > 0$.

9.1.1.3 Case (iii): $c^2 = \frac{1}{3}$

Again we let

$$w = \left(\frac{dF}{d\xi}\right)^2 = \frac{2\epsilon}{3c\mu^2} P(F), \tag{9.16}$$

where $P(F) = F(F_1 - F)(F - F_2)$, $F_1 = \frac{1}{4\epsilon}\left(3\sqrt{3} + \sqrt{91}\right) = \frac{3.684}{\epsilon}$ and $F_2 = \frac{1}{4\epsilon}\left(3\sqrt{3} - \sqrt{91}\right) = -\frac{1.086}{\epsilon}$. We can express $P(F)$ as follows

$$P(F) = -F^3 + \frac{3}{\epsilon c}\left(c^2 + \frac{1}{6}\right)F^2 - \frac{5}{\epsilon^2 c^2}\left(\frac{1}{15} - c^2\right)F. \tag{9.17}$$

Employing the same approach as has been done in Section 9.1.1.1, we obtain the following turning point of w, which is the only non-zero turning point

$$\left.\frac{dw}{dF}\right|_{F=0} = \frac{8\sqrt{3}}{3\mu^2\epsilon} > 0, \tag{9.18}$$

and

$$\left.\frac{d^2w}{dF^2}\right|_{F=0} = \frac{6}{\mu^2} > 0, \tag{9.19}$$

thus, the turning point at $F = 0$ is a minimum. Therefore, in the neighbourhood of $F = 0$, $\left(\frac{dF}{d\xi}\right)^2 \geq 0$ for both $F < 0$ and $F > 0$.

9.1.2 Behaviour of $\left(\frac{dF}{d\xi}\right)^2$ versus $F(\xi)$ as $F(\xi)$ tends to infinity

We now consider the behaviour of $w = \left(\frac{dF}{d\xi}\right)^2$ as $F(\xi)$ tends to infinity in order to understand the interaction between the third-order algebraic term and the logarithmic term in $P(F)$. The study of this interaction is important due to the nature of the decaying boundary conditions at infinty in both the negative and positive directions. The results of this investigation will assist us in determining whether $\left(\frac{dF}{d\xi}\right)^2 = 0$ at $F = 0$ and $\left(\frac{dF}{d\xi}\right) = 0$ is bounded i.e. whether conditions (2) and (3) are satisfied.

9.1.2.1 Case (i): $0 < c^2 < \infty$, $c^2 \neq \frac{1}{15}$ and $c^2 \neq \frac{1}{3}$

Considering equations (8.46) and (8.47), we obtain five cases, namely $0 < c^2 < \frac{1}{15}$, $\frac{1}{15} < c^2 < \frac{1}{3}$, $\frac{1}{3} < c^2 < 1$, $c^2 = 1$ and $1 < c^2 < \infty$.

Case: $0 < c^2 < \frac{1}{15}$

In the region $F < 0$, consider the logarithmic term of $P(F)$ in (8.46), for $F > -A$ as $F \to -$A from the right,

$$\frac{5}{\epsilon^3 c^3}\left(\frac{1}{3} - c^2\right)\left(\frac{1}{15} - c^2\right)\ln\left(\frac{F+A}{A}\right) \to -\infty. \tag{9.20}$$

We note that $\ln\left(\frac{F+A}{A}\right)$ is complex for $F < -A$ where $-A$ is finite and negative and $-F^3$ only equals A^3 at $F = -A$. The logarithmic term dominates the behaviour as $F \to -\infty$. Thus, there will be a point $F = F_2 < 0$ for which $\left(\frac{dF}{d\xi}\right)^2 = 0$ where F_2 is the minimum fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ at $\xi = 0$.

Consider now the region $F > 0$. For sufficiently large F, $-F^3$ will dominate all terms in $\left(\frac{dF}{d\xi}\right)^2$ including the logarithmic term, and therefore, $\left(\frac{dF}{d\xi}\right)^2 \to -\infty$. Thus, there will be a point $F = F_1$ for which $\left(\frac{dF}{d\xi}\right)^2 = 0$ where F_1 is the maximum fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ at $\xi = 0$.

Case: $\frac{1}{15} < c^2 < \frac{1}{3}$

Through similar reasoning as was employed for the case above, we see that there is no point $F = F_2 < 0$ in the region $F < 0$ for which $\left(\frac{dF}{d\xi}\right)^2 = 0$ where F_2 is the minimum fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ at $\xi = 0$. There will be a point $F = F_1$ in the region $F > 0$ for which $\left(\frac{dF}{d\xi}\right)^2 = 0$ where F_1 is the maximum fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ at $\xi = 0$.

Case: $\frac{1}{3} < c^2 < 1$

In the region $F < 0$, consider the logarithmic term of $P(F)$ in (8.46) and let $B = -A = \frac{1}{\epsilon c}\left(c^2 - \frac{1}{3}\right)$. For sufficiently large F, $-F^3$ will dominate all terms in $\left(\frac{dF}{d\xi}\right)^2$ including the logarithmic term. As $F \to -\infty$, $-F^3 \to \infty$ and

$$\left(\frac{1}{3} - c^2\right)\left(\frac{1}{15} - c^2\right)\ln\left(\frac{B-F}{B}\right) \to -\infty. \tag{9.21}$$

Therefore, $\left(\frac{dF}{d\xi}\right)^2 \to \infty$ and there is, thus, a point $F = F_2$ for which $\left(\frac{dF}{d\xi}\right)^2 = 0$ where F_2 is the minimum fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ at $\xi = 0$.

In the region $F > 0$, again consider the logarithmic term above. For $F < B$ as $F \to$ B from the left,

$$\left(\frac{1}{3} - c^2\right)\left(\frac{1}{15} - c^2\right)\ln\left(\frac{B-F}{B}\right) \to -\infty. \tag{9.22}$$

We note that $\ln\left(\frac{B-F}{B}\right)$ is complex for $F > B$ where B is finite and positive and $-F^3$ only equals $-B^3$ at $F = B$. Therefore, $\left(\frac{dF}{d\xi}\right)^2 \to -\infty$. Thus, there is no point $F = F_1 > 0$ for which $\left(\frac{dF}{d\xi}\right)^2 = 0$ where F_1 is the maximum fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ at $\xi = 0$.

Case: $c^2 = 1$

In this case, a point of inflection occurs at $F = 0$. Thus, there is no bounded region.

Case: $1 < c^2 < \infty$

Using the same reasoning as was employed for the case $\frac{1}{3} < c^2 < 1$, we see that there is no point $F = F_2 < 0$ in the region $F < 0$ for which $\left(\frac{dF}{d\xi}\right)^2 = 0$ where F_2 is the minimum fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ at $\xi = 0$. There will be a point $F = F_1$ in the region $F > 0$ for which $\left(\frac{dF}{d\xi}\right)^2 = 0$ where F_1 is the maximum fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ at $\xi = 0$.

9.1.2.2 Cases (ii) and (iii): $c^2 = \frac{1}{15}$ and $c^2 = \frac{1}{3}$

Recall equations (9.13) and (9.17) where $P(F)$ takes the form

$$P(F) = -F^3 + \frac{3}{\epsilon c}\left(c^2 + \frac{1}{6}\right)F^2 - \frac{5}{\epsilon^2 c^2}\left(\frac{1}{15} - c^2\right)F.$$

The dominant behaviour as $F \to \infty$ is governed by term $-F^3$. Considering $F < 0$, there is no point $F = F_2$ for which $\left(\frac{dF}{d\xi}\right)^2 = 0$ where F_2 is the minimum fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ at $\xi = 0$. For $F > 0$ there will be a point $F = F_1 > 0$ for which $\left(\frac{dF}{d\xi}\right)^2 = 0$ where F_1 is the maximum fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ at $\xi = 0$.

9.2 Properties of the surface elevation of the wave $G(\xi)$

There are two existance conditions for surface elevation of the solitary wave $G(\xi)$ that need to be satisfied for a physically realistic result to occur. The first one relates to the interaction between $G(\xi)$ and $F(\xi)$. We recall that we cannot (with exception of the case $c^2 = \frac{1}{15}$) explicitly express $G(\xi)$ as a function of ξ, thus, we need to consider the relation $G(F)$ in order to analyse $G(\xi)$. If $F(\xi)$ does not produce physically realistic results in accordance with the existance framework outlined in Section 9.1 then the $G(\xi)$ cannot produce physically realistic results. Further to this, if $F(\xi)$ does produce physically realistic results, the interaction of $G(F)$ needs to be further analysed to determine whether $G(\xi)$ produces physically realistic results.

The second existance condition on $G(\xi)$ relates to the physical property inherent in wave theory, stating that all waves produced need to be above the seabed (i.e. no dry zones) [2]. We now determine the condition on $G(\xi)$ for a depression wave to be above the seabed. We make reference to depression waves as this condition is not relevant when studying excitation waves. For a depression wave to be above the seabed, the total wave depth needs to remain positive, i.e. $\eta' + h' > 0$, where $'$ denotes the dimensional form. As derived in Chapter 3, the free surface kinematic boundary condition occurs at $z' = \eta'$, whilst the seabed kinematic boundary condition occurs at $z' = -h'$, where $h' = h_0$ when h_0 is a constant.

A dry zone exists at $\eta' = -h'$, hence,

$$\eta' > -h'. \tag{9.23}$$

We now transform to dimensionless variables using the following scalings

$$\eta = \frac{\eta'}{a_0}, \qquad h = \frac{h'}{h_0}, \qquad \epsilon = \frac{a_0}{h_0}. \tag{9.24}$$

We note that, since we are considering a constant water depth $h' = h_0$, it follows that $h = 1$. Substituting (9.24) into (9.23) gives the following above seabed condition in dimensionless form

$$\epsilon G(\xi) > -1. \tag{9.25}$$

Using the existance condition framework detailed above, we analyse the properties of the surface elevation of the solitary wave $G(\xi)$ in this section. The results of this investigation will assist us in determining whether the solutions for $G(\xi)$ produce physically realistic results. This section is broken down into the following three subsections:

- Relation between the surface elevation of the solitary wave $G(\xi)$ and the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$, where we investigate the first existance condition on $G(\xi)$,

- Properties of $\frac{dG}{d\xi}$ to determine the turning points of the surface elevation of the solitary wave $G(\xi)$, where we investigate the first existance condition on $G(\xi)$,

- Analysis of the above the seabed condition $G(\xi) > -\frac{1}{\epsilon}$, where we investigate the second existance condition on $G(\xi)$.

9.2.1 Relation between the surface elevation of the solitary wave $G(\xi)$ and the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$

9.2.1.1 Case (i): $0 < c^2 < \infty$, $c^2 \neq \frac{1}{15}$ and $c^2 \neq \frac{1}{3}$

We consider the exact invariant solution for the surface elevation of the wave $G(F)$ expressed in terms of $F(\xi)$. Recall equation (8.48) given by

$$G(F) = -\frac{F\left(\frac{4c}{\epsilon} + F\right)}{6c\,(A+F)}, \text{ where } A = \frac{1}{\epsilon c}\left(\frac{1}{3} - c^2\right). \tag{9.26}$$

We use (9.26) in its current form for wave speeds in the range $\frac{1}{3} < c^2 < \infty$. Considering wave speeds in the range $\frac{1}{3} < c^2 < \infty$, we rewrite (9.26) as

$$G(F) = \frac{F\left(\frac{4c}{\epsilon} + F\right)}{6c\,(B-F)}, \text{ where } B = \frac{1}{\epsilon c}\left(c^2 - \frac{1}{3}\right). \tag{9.27}$$

We note that $G(\xi)$ is proportional to $F(\xi)$ for special cases of $F(\xi)$.

9.2.1.2 Case (ii): $c^2 = \frac{1}{15}$

We recall equation (8.58), representing the exact invariant solution for the surface elevation of the wave $G(\xi)$ given by

$$G(\xi) = -\frac{7}{4\epsilon}\operatorname{sech}^2\left(\frac{\sqrt{7}}{2\mu}\xi\right). \tag{9.28}$$

We rewrite (9.28) in terms of $F(\xi)$ and obtain the following relation between $G(\xi)$ and $F(\xi)$

$$G(F) = -\frac{F}{6c}. \tag{9.29}$$

This is the only case in which $G(\xi)$ is proportional to $F(\xi)$ for all values of $F(\xi)$, i.e. $A = \dfrac{4c}{\epsilon}$.

9.2.1.3 Case (iii): $c^2 = \frac{1}{3}$

We consider the exact invariant solution for the surface elevation of the wave $G(F)$ expressed in terms of $F(\xi)$. We recall equation (8.63) given by

$$G(F) = -\frac{\left(\frac{4c}{\epsilon} + F\right)}{6c}. \tag{9.30}$$

This is the only case in which a linear relation exists between $G(\xi)$ and $F(\xi)$. When $F = 0$, then (9.30) becomes

$$G(0) = -\frac{2}{3\epsilon}. \tag{9.31}$$

We note that $G(0) < 0$, thus, a depression wave is formed, but since $G(\xi)$ does not satisfy the condition $G(\xi) > -\frac{1}{\epsilon}$, no physically realistic wave occurs in this case. This is the only value of c^2 which does not satisfy the condition $G = 0$ where $F = 0$.

9.2.2 Properties of $\frac{dG}{d\xi}$ to determine the turning points of the surface elevation of the solitary wave $G(\xi)$

9.2.2.1 Case (i): $0 < c^2 < \infty$, $c^2 \neq \frac{1}{15}$ and $c^2 \neq \frac{1}{3}$

The turning points of the surface elevation of the solitary wave $G(\xi)$ are given by

$$\frac{dG}{d\xi} = \frac{dG}{dF}\frac{dF}{d\xi}. \tag{9.32}$$

Differentiating (9.26) with respect to $F(\xi)$ gives

$$\frac{dG}{dF} = -\frac{1}{6c\,(A+F)^2}\left(F^2 + 2AF + \frac{4c}{\epsilon}A\right), \text{ where } A = \frac{1}{\epsilon c}\left(\frac{1}{3} - c^2\right). \tag{9.33}$$

We now determine the roots of the term $F^2 + 2AF + \frac{4c}{\epsilon}A$ from (9.33), i.e.

$$F^2 + 2AF + \frac{4c}{\epsilon}A = 0. \tag{9.34}$$

The discriminant of (9.34) is given by

$$\triangle = \frac{20}{\epsilon^2 c^2}\left(\frac{1}{3} - c^2\right)\left(\frac{1}{15} - c^2\right). \tag{9.35}$$

Thus, we have three cases, namely $0 < c^2 < \frac{1}{15}$, $\frac{1}{15} < c^2 < \frac{1}{3}$ and $\frac{1}{3} < c^2 < \infty$.

We wish to determine the roots of (9.34) by using the quadratic formula. We do so by also making use of (9.35) so that we obtain the following roots

$$F_3 = -A + \frac{\sqrt{5\left(\frac{1}{3} - c^2\right)\left(\frac{1}{15} - c^2\right)}}{\epsilon c}, \tag{9.36}$$

and

$$F_4 = -A - \frac{\sqrt{5\left(\frac{1}{3} - c^2\right)\left(\frac{1}{15} - c^2\right)}}{\epsilon c}. \tag{9.37}$$

We note that F_4 is a non-real root, and $F_3 < 0$ provided

$$\begin{aligned} &\frac{\sqrt{5\left(\frac{1}{3} - c^2\right)\left(\frac{1}{15} - c^2\right)}}{\epsilon c} < A \\ &\Rightarrow c^2 > 0. \end{aligned} \tag{9.38}$$

Considering (9.36) and pulling out a common denominator gives

$$F_3 = -\frac{1}{\epsilon c}\left[\left(\frac{1}{3} - c^2\right) - \sqrt{5\left(\frac{1}{3} - c^2\right)\left(\frac{1}{15} - c^2\right)}\right]. \tag{9.39}$$

Thus, (9.32) can be written as

$$\frac{dG}{d\xi} = -\frac{1}{6c\,(A+F)^2}(F - F_3)(F - F_4)\frac{dF}{d\xi}. \tag{9.40}$$

For $F < 0$, there are two turning points, namely $\frac{dF}{d\xi} = 0$ and F_3. Thus, the wave takes the form of a bimodal solitary wave. In the region $F > 0$, there are no turning points in $G(\xi)$, aside from $\frac{dF}{d\xi} = 0$, thus, the wave takes the form of a unimodal solitary wave.

We now consider the discriminant (9.35), as well as F_3 and F_4 to analyse the three cases, namely $0 < c^2 < \frac{1}{15}$, $\frac{1}{15} < c^2 < \frac{1}{3}$ and $\frac{1}{3} < c^2 < \infty$.

Case: $0 < c^2 < \frac{1}{15}$

It can be seen easily, that $-A < F_3 < 0$ and that $F_4 < -A$. Thus, F_4 is a non-real root as it lies outside the physical range, defined in Section 9.1.2.1, where $F > -A$ is required for a physically realistic result to occur.

Using (9.36), (9.37) and (9.40) we can conclude that $\frac{dG}{d\xi}$ has the turning points

$$F = F_2 < 0 \text{ obtained from } \frac{dF}{d\xi} \text{ in Section 9.1.2.1,}$$
$$F = F_3 < 0 \text{ obtained from } \frac{dG}{dF},$$

provided $F_3 < F_2$ for $F_2 \le F \le 0$ and

$$F = F_1 > 0 \text{ obtained from } \frac{dF}{d\xi} \text{ in Section 9.1.2.1,}$$

for $0 \le F \le F_1$.

Consider the potential solitary wave in the region $F < 0$. We now return to the turning point in $G(\xi)$ derived from $\frac{dF}{d\xi}$ at $\xi = 0$ where $F = F_2 < 0$.

$$G(F_2) = -\frac{F_2}{6c}\frac{\left(\frac{4c}{\epsilon} + F_2\right)}{A + F_2}, \tag{9.41}$$

where $A = \frac{1}{\epsilon c}\left(\frac{1}{3} - c^2\right) > 0$. Now $G(F_2) > 0$ if $F_2 > -\frac{4c}{\epsilon}$ since $-\frac{F_2}{6c} > 0$ as $-F_2 < 0$ and $A + F_2 > 0$ for $0 < c^2 < \frac{1}{15}$. Similarly, $G(F_2) = 0$ if $F_2 = -\frac{4c}{\epsilon}$ and $G(F_2) < 0$ if $F_2 < -\frac{4c}{\epsilon}$.

Thus for $F < 0$, the surface elevation of the wave $G(\xi)$ can have one of three forms in the region $F_2 \le F \le 0$. To our knowledge, these are newly discovered forms of the solitary wave generated by the NB equations, namely

1. Bimodal excitation wave, formed when $G(\xi) > 0$ at $\xi = 0$ if $\frac{-4c}{\epsilon} < F_2 < 0$,
2. Bimodal transition wave, formed when $G(\xi) = 0$ at $\xi = 0$ if $F_2 = \frac{-4c}{\epsilon}$,
3. Bimodal depression wave, formed when $G(\xi) < 0$ at $\xi = 0$ if $F_2 < \frac{-4c}{\epsilon} < 0$.

Consider now $F > 0$, where $G(F)$ has two negative turning points F_3, F_4 but only $F_3 > -A$ lies inside the physical range, defined in Section 9.1.2.1, where $F > -A$ is required for a physically realistic result to occur. The turning point F_3 occurs in the region $-A < F_2 < F_3 < 0$, hence, there are no additional turning points in the region $0 \le F \le F_1$. Thus, for $0 \le F \le F_1$, the only turning point in $G(\xi)$ corresponds to $\frac{dF}{d\xi} = 0$, where $F = F_1$ at $\xi = 0$.

Case: $\frac{1}{15} < c^2 < \frac{1}{3}$

In this case the discriminant (9.35) is negative. The roots are complex conjugates, therefore, $\frac{dG}{dF}$ has no real roots. Thus, the only turning point in $G(\xi)$ corresponds to $\frac{dF}{d\xi} = 0$, where $F = F_1$ at $\xi = 0$.

Case: $\frac{1}{3} < c^2 < \infty$

In this case, the discriminant (9.35) is positive and produces two real roots. From (9.36) we get

$$F_3 = B + \frac{\sqrt{5\left(c^2 - \dfrac{1}{3}\right)\left(c^2 - \dfrac{1}{15}\right)}}{\epsilon c}, \tag{9.42}$$

where $F_3 > B$ is outside the physical range. From (9.37) we get

$$F_4 = B - \frac{\sqrt{5\left(c^2 - \dfrac{1}{3}\right)\left(c^2 - \dfrac{1}{15}\right)}}{\epsilon c}, \tag{9.43}$$

where $F_4 < 0$ provides

$$\begin{aligned} B &< \frac{\sqrt{5\left(c^2 - \dfrac{1}{3}\right)\left(c^2 - \dfrac{1}{15}\right)}}{\epsilon c} \\ &\Rightarrow c^2 > 0. \end{aligned} \tag{9.44}$$

Hence, $F_4 < 0$ and we will see later that F_4 is outside the physical range.

9.2.2.2 Cases (ii) and (iii): $c^2 = \frac{1}{15}$ and $c^2 = \frac{1}{3}$

Using (9.29) and (9.32) we get

$$\frac{dG}{d\xi} = -\frac{1}{6c}\frac{dF}{d\xi}, \tag{9.45}$$

and

$$\frac{d^2G}{d\xi^2} = -\frac{1}{6c}\frac{d^2F}{d\xi^2}. \tag{9.46}$$

From (9.46), we deduce that $G(\xi)$ is a minimum when $F(\xi)$ is a maximum. Considering (9.45), we know that $G(\xi)$ will have a turning point if and only if $F(\xi)$ has a turning point at $\xi = 0$ i.e. $F = F_1$.

9.2.3 Analysis of the above the seabed condition $G(\xi) > -\frac{1}{\epsilon}$

9.2.3.1 Cases (i) and (ii): $0 < c^2 < \infty$, $c^2 \neq \frac{1}{15}$ and $c^2 \neq \frac{1}{3}$ and $c^2 = \frac{1}{15}$

From (9.26) and (9.27), we consider two cases, namely $0 < c^2 < \frac{1}{3}$ and $\frac{1}{3} < c^2 < \infty$.

Case: $0 < c^2 < \frac{1}{3}$

Considering (9.26), we require the following condition, given by (9.25), to be satisfied in order for the solitary wave to be above the seabed

$$G(\xi) = G(F) = -\frac{F\left(\frac{4c}{\epsilon} + F\right)}{6c\,(A+F)} > -\frac{1}{\epsilon}, \text{ where } A = \frac{1}{\epsilon c}\left(\frac{1}{3} - c^2\right). \tag{9.47}$$

We can rewrite condition (9.47) as the following condition

$$F^2 - \frac{2c}{\epsilon}F - \frac{6cA}{\epsilon} < 0. \tag{9.48}$$

In order to determine the values of $F(\xi)$ for which the surface elevation of the wave $G(F)$ is above the seabed, we need to obtain the turning points of (9.48). We do this by calculating the discriminant of (9.48) given by

$$\triangle = \frac{20}{\epsilon^2}\left(\frac{2}{5} - c^2\right), \tag{9.49}$$

where $\triangle > 0$ since $0 < c^2 < \frac{1}{3}$. We note that the condition (9.48) is positive, thus, it will always have two real roots. We obtain the following expressions for the two real roots by applying the quadratic formula to (9.48)

$$F_5 = \frac{1}{\epsilon}\left(c + \sqrt{5}\sqrt{\frac{2}{5} - c^2}\right) > 0, \tag{9.50}$$

and

$$F_6 = \frac{1}{\epsilon}\left(c - \sqrt{5}\sqrt{\frac{2}{5} - c^2}\right) < 0. \tag{9.51}$$

where $F_6 < 0$ since $0 < c^2 < \frac{1}{3}$.

Let condition (9.48) be expressed as $Q(F)$ and written in terms of the newly found roots as follows

$$Q(F) = (F - F_5)(F - F_6) < 0. \tag{9.52}$$

In order for (9.52) to be satisfied for $0 < c^2 < \frac{1}{3}$, we require $F_1 < F_5$ and $F_2 > F_6$. To obtain the minimum value of the seabed $Q(F)$, we equate (9.52) to zero and differentiate the result with respect to $F(\xi)$ to get the corresponding value of $F(\xi)$ leading to a minimum value of the seabed $Q(F)$ given by

$$F(\xi) = \frac{c}{\epsilon}. \tag{9.53}$$

Evaluating (9.52) at $F = \frac{c}{\epsilon}$, we obtain the minimum value of the seabed $Q(F)$ given by

$$Q\left(\frac{c}{\epsilon}\right) = \frac{5}{\epsilon^2}\left(c^2 - \frac{2}{5}\right). \tag{9.54}$$

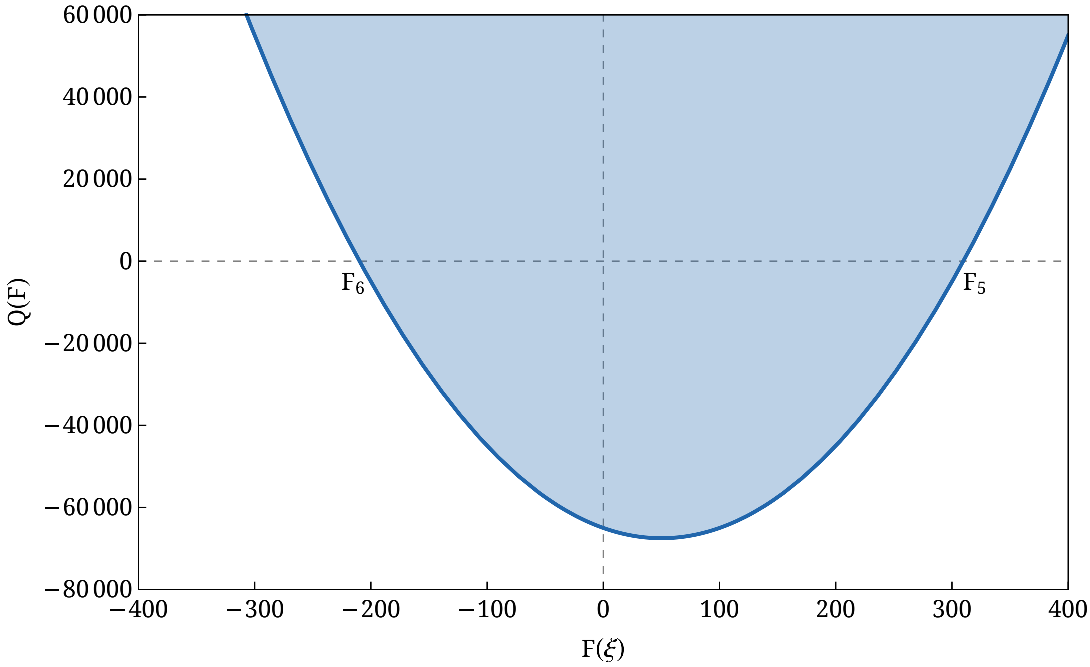

Figure 9.1: $Q(F)$, determining the $F(\xi)$ values for which the surface elevation of the solitary wave will be above the seabed for $c^2 = \frac{1}{17}$ where $\epsilon = 0.005$ and $\mu = 0.06$.

The shaded region in Figure 9.1 shows the range of $F(\xi)$ values for which the surface elevaton of the solitary wave is above the seabed i.e. $F_6 < F < F_5$ will be satisfied if

$$\begin{aligned} & F_2 > F_6 \quad \text{where} \quad F_2 \le F \le 0, \\ \text{or} \quad & F_1 < F_5 \quad \text{where} \quad 0 \le F \le F_1. \end{aligned} \tag{9.55}$$

Case: $\frac{1}{3} < c^2 < \infty$

In this case, we consider (9.27) and require the following condition, given by (9.25), to be satisfied in order for the solitary wave to be above the seabed

$$G(\xi) = G(F) = \frac{F\left(\frac{4c}{\epsilon} + F\right)}{6c\,(B - F)} > -\frac{1}{\epsilon}, \text{ where } B = \frac{1}{\epsilon c}\left(c^2 - \frac{1}{3}\right). \tag{9.56}$$

We can rewrite condition (9.56) as the following condition

$$F^2 - \frac{2c}{\epsilon}F + \frac{6cB}{\epsilon} > 0. \tag{9.57}$$

In order to determine the values of $F(\xi)$ for which the surface elevation of the wave $G(F)$ is above the seabed, we need to obtain the turning points of (9.57). We do this by calculating the discriminant of (9.57) given by

$$\triangle = \frac{20}{\epsilon^2}\left(\frac{2}{5} - c^2\right), \tag{9.58}$$

where $\triangle > 0$ provided $\frac{1}{3} < c^2 < \frac{2}{5}$. We note that the condition (9.57) is positive and it will always have two real roots provided $\frac{1}{3} < c^2 < \frac{2}{5}$. We obtain the following expressions for the two real roots by applying the quadratic formula to (9.57)

$$F_5 = \frac{1}{\epsilon}\left(c + \sqrt{5}\sqrt{\frac{2}{5} - c^2}\right) > 0, \tag{9.59}$$

and

$$F_6 = \frac{1}{\epsilon}\left(c - \sqrt{5}\sqrt{\frac{2}{5} - c^2}\right) > 0. \tag{9.60}$$

where $F_6 > 0$ since $c^2 > \frac{1}{3}$.

The shaded region in Figure 9.2 shows the range of $F(\xi)$ values for which the wave is above the seabed i.e. $F < F_6$ and $F > F_5$ will be satisfied if

$$\begin{aligned} & F_2 < F_6 \quad \text{where} \quad F_2 \le F \le 0, \\ \text{or} \quad & F_1 > F_5 \quad \text{where} \quad 0 \le F \le F_1. \end{aligned} \tag{9.61}$$

Now consider the case $\frac{2}{5} \le c^2 < \infty$. From (9.58), we note that the discriminate $\triangle < 0$, and thus, produces two complex roots. Let condition (9.57) be expressed as $R(F)$ as follows

$$R(F) = F^2 - \frac{2c}{\epsilon}F + \frac{6cB}{\epsilon} > 0. \tag{9.62}$$

To obtain the minimum value of the seabed $R(F)$, we equate (9.62) to zero and differentiate the result with respect to $F(\xi)$ to get the corresponding value of $F(\xi)$ leading to minimum value of the seabed $R(F)$ given by

$$F(\xi) = \frac{c}{\epsilon}. \tag{9.63}$$

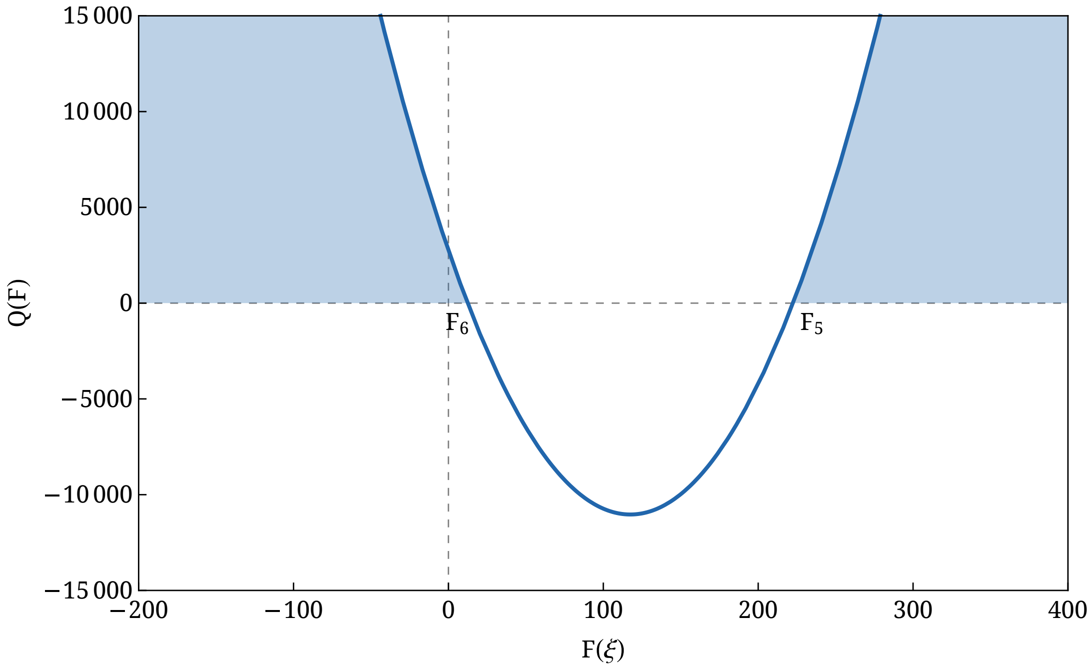

Figure 9.2: $Q(F)$, determining the $F(\xi)$ values for which the surface elevation of the solitary wave will be above the seabed for $c^2 = \frac{1}{2.9}$ where $\epsilon = 0.005$ and $\mu = 0.06$.

Evaluating (9.62) at $F = \frac{c}{\epsilon}$ gives

$$R\left(\frac{c}{\epsilon}\right) = \frac{5}{\epsilon^2}\left(c^2 - \frac{2}{5}\right). \tag{9.64}$$

Since $R\left(\frac{c}{\epsilon}\right) \geq 0$ for $c^2 \geq \frac{2}{5}$, $G(F) > -\frac{1}{\epsilon}$ for all $F(\xi)$.

In summary, for $\frac{1}{3} < c^2 < \frac{2}{5}$ the wave is above the seabed for all values where $F < F_6$ and $F > F_5$. Considering the case $\frac{2}{5} < c^2 < \infty$, both roots are complex i.e. $\triangle < 0$ and the surface elevation of the solitary wave is above the seabed for all values of $F(\xi)$.

Chapter 10

Case specific analysis for the NB equations

In this chapter, we plot and analyse the properties of the fluid velocity of the solitary wave in the horizontal direction and the surface elevation of the solitary wave in the two regions, namely $F_2 \leq F \leq 0$ and $0 \leq F \leq F_1$, considered in the previous two chapters for seven cases dependent on the wave speed c^2, namely $0 < c^2 < \frac{1}{15}$, $c^2 = \frac{1}{15}$, $\frac{1}{15} < c^2 < \frac{1}{3}$, $c^2 = \frac{1}{3}$, $\frac{1}{3} < c^2 < 1$, $c^2 = 1$ and $1 < c^2 < \infty$. In the study of these case specific properties, we obtain physically realistic results. To our knowledge, these are new results, used to enrich the analytical validation of numerical models for tsunami detection systems.

10.1 Case 1: $0 < c^2 < \frac{1}{15}$

10.1.1 Properties of the fluid velocity of the wave in the horizontal direction $F(\xi)$ relating to the exact invariant solution

We recall from Section 9.1 that there are two regions for case $0 < c^2 < \frac{1}{15}$, namely $F_2 \leq F \leq 0$ and $0 \leq F \leq F_1$, in which we potentially obtain a solitary wave. The shaded areas in Figure 10.1, the graph of $\left(\frac{dF}{d\xi}\right)^2$ versus $F(\xi)$, represent these regions of physical significance. It is important to note that the two solitary waves cannot be generated by the same event.

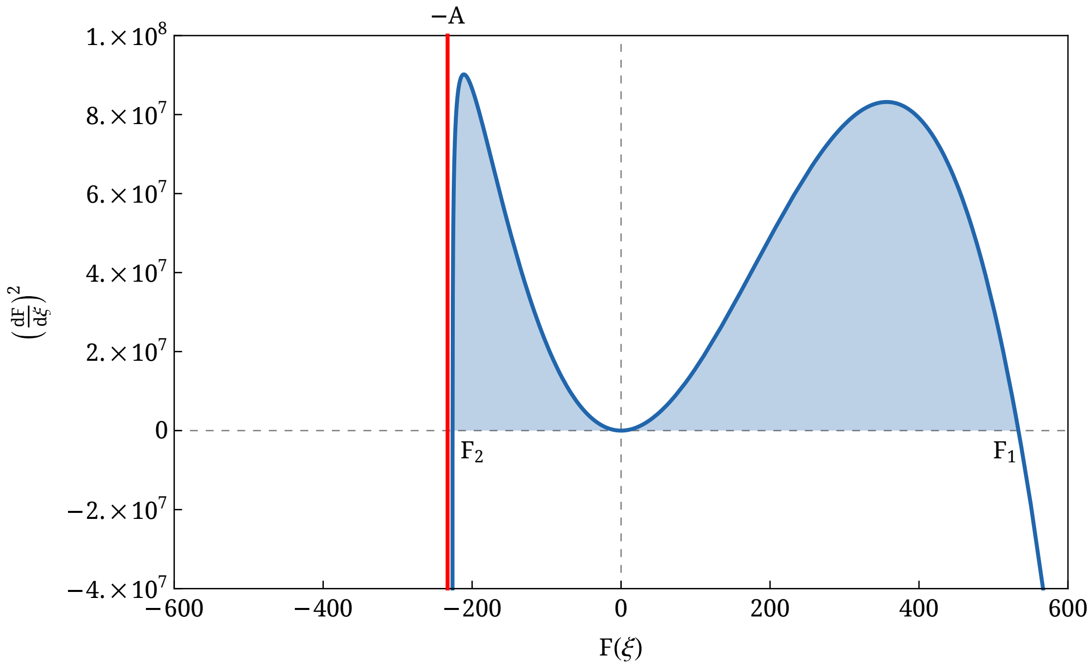

Figure 10.1: $\left(\frac{dF}{d\xi}\right)^2$ vs $F(\xi)$ at a constant wave speed $c^2 = \frac{1}{17}$ where $\epsilon = 0.005$ and $\mu = 0.06$.

We now present plots of the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ against ξ for varying c^2 values in the range $0 < c^2 < \frac{1}{15}$ where ϵ and μ are constants. An estimate of the width of the solitary wave is given by

$$\begin{aligned}\xi_B &= \pm\sqrt{\frac{3c\mu^2}{2\epsilon}} \int_{\frac{F_1}{2}}^{F} \frac{dF}{\sqrt{P(F)}} \\ &= \mathcal{O}\left(\left(\pm\sqrt{\frac{3c\mu^2}{2\epsilon}}\right)\right) \\ &= \mathcal{O}\left(\left(\sqrt{c}\right)\right),\end{aligned} \tag{10.1}$$

since $\mathcal{O}\left(\frac{\mu^2}{\epsilon}\right) = 1$. We note that the width of the wave ξ_B is directly proportional to the square root of the wave speed.

Figure 10.2 shows that the exact invariant solution for the fluid velocity of the solitary wave in the horizontal direction forms a unimodal depression fluid velocity profile in the region $F < 0$. The profile of the fluid velocity of the solitary wave in the horizontal direction steepens as the wave speed increases. We note that the minimum fluid velocity of the solitary wave in the horizontal direction F_2 is directly proportional to the wave speed. Refer to Table E.1 for results that can be used as a comparison for experimental or numerical investigations.

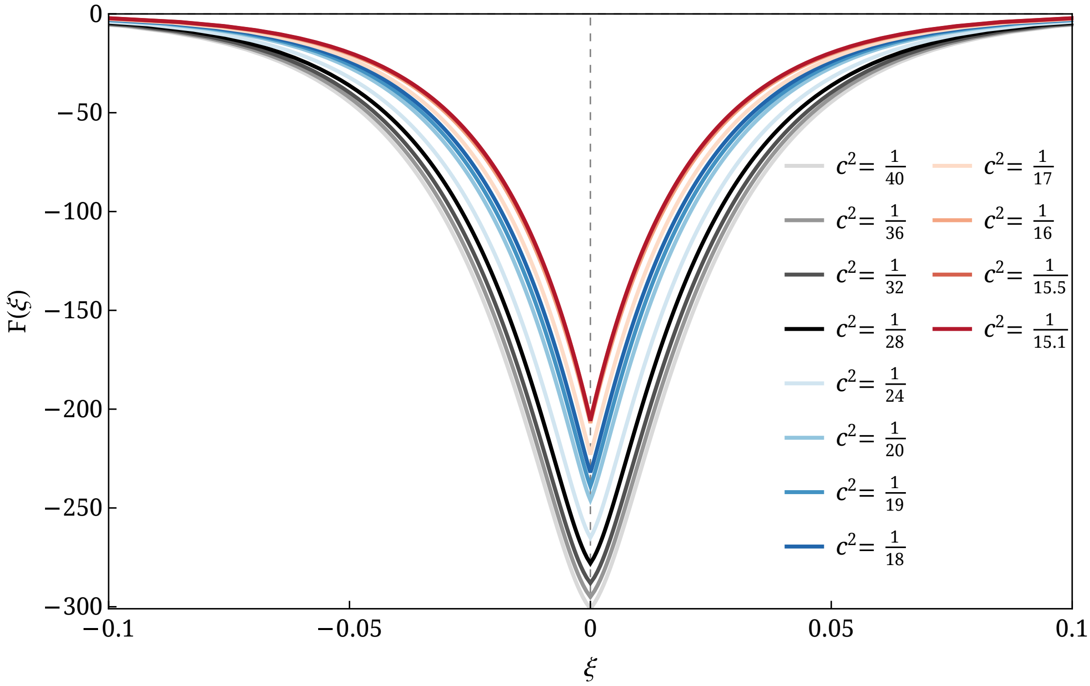

Figure 10.2: Fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ in the region $F < 0$ for varying wave speeds where $\epsilon = 0.005$ and $\mu = 0.06$.

Figure 10.3 shows that the exact invariant solution for the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ forms a unimodal excitation fluid velocity profile in the region $F > 0$. The profile of the fluid velocity of the solitary wave in the horizontal direction steepens as the wave speed increases. We note that the maximum fluid velocity of the solitary wave in the horizontal direction F_1 is directly proportional to the wave speed. Refer to Table E.2 for results that can be used as a comparison for experimental or numerical investigations.

10.1.2 Properties of the surface elevation of the solitary wave $G(\xi)$

Consider now the properties of the surface elevation of the solitary wave $G(\xi)$ analysed in Section 9.2.1.1. From (9.26), we have

$$G(F) = -\frac{F}{6c}\frac{\left(\frac{4c}{\epsilon}+F\right)}{A+F}, \text{ where } A = \frac{1}{\epsilon c}\left(\frac{1}{3}-c^2\right). \tag{10.2}$$

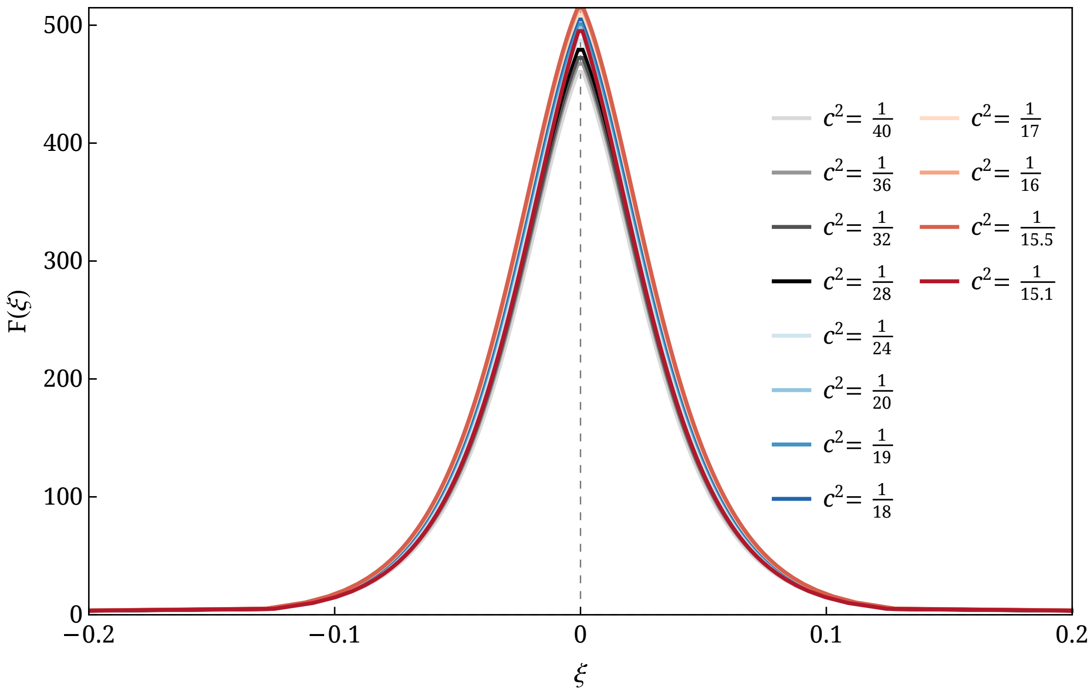

Figure 10.3: Fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ in the region $F > 0$ for varying wave speeds where $\epsilon = 0.005$ and $\mu = 0.06$.

Consider the potential solitary wave in the region $F < 0$. We now return to the turning point in $G(\xi)$ derived from $\frac{dF}{d\xi}$ at $\xi = 0$ where the minimum fluid velocity in the horizontal direction of the solitary wave F_2 is given by $F = F_2 < 0$,

$$G(F_2) = -\frac{F_2}{6c}\frac{\left(\frac{4c}{\epsilon} + F_2\right)}{A + F_2}, \text{ where } A = \frac{1}{\epsilon c}\left(\frac{1}{3} - c^2\right) > 0. \tag{10.3}$$

Now $G(F_2) > 0$ if $F_2 > -\frac{4c}{\epsilon}$ since $-\frac{F_2}{6c} > 0$ as $-F_2 < 0$ and $A + F_2 > 0$ for $0 < c^2 < \frac{1}{15}$. Similarly $G(F_2) = 0$ if $F_2 = -\frac{4c}{\epsilon}$ and $G(F_2) < 0$ if $F_2 < -\frac{4c}{\epsilon}$.

Thus, in the region $F < 0$ for the case $0 < c^2 < \frac{1}{15}$, the surface elevation of the solitary wave $G(\xi)$ can have one of three forms in the region $F_2 \leq F \leq 0$, two of which are shown in Figure 10.4. To our knowledge, these are newly discovered forms of the solitary wave generated by the NB equations, namely

1. Bimodal excitation wave, formed when $G(\xi) > 0$ at $\xi = 0$ if $\frac{-4c}{\epsilon} < F_2 < 0$,
2. Bimodal transition wave, formed when $G(\xi) = 0$ at $\xi = 0$ if $F_2 = \frac{-4c}{\epsilon}$,
3. Bimodal depression wave, formed when $G(\xi) < 0$ at $\xi = 0$ if $F_2 < \frac{-4c}{\epsilon} < 0$.

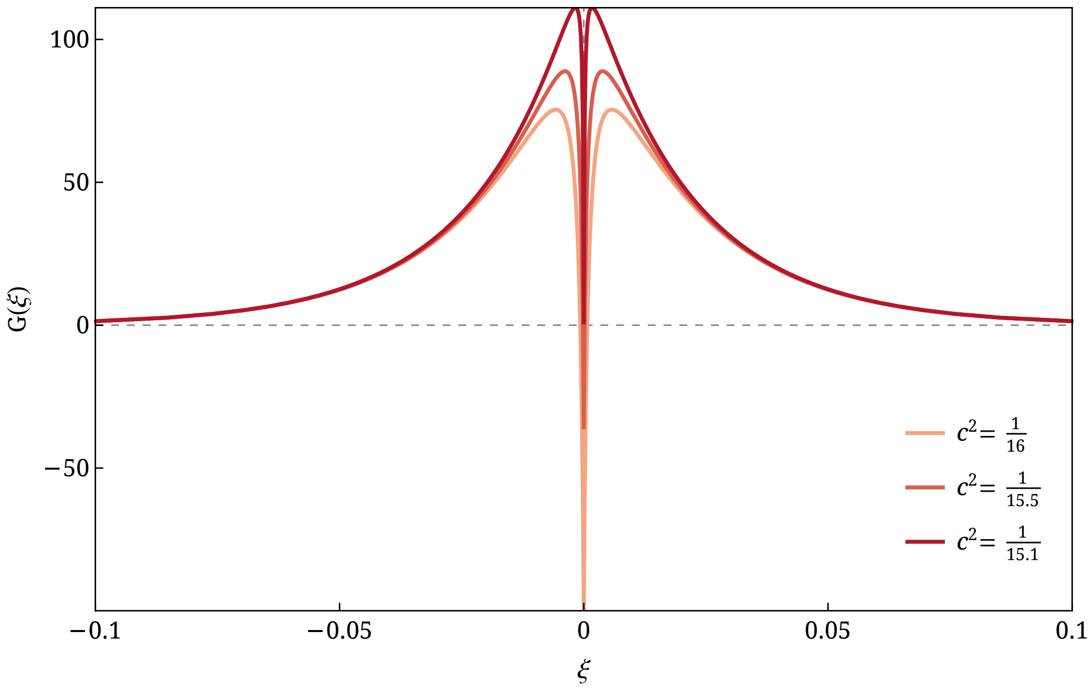

Figure 10.4: Surface elevation of the solitary wave $G(\xi)$ in the region $F < 0$ illustrating the effect the factor $\frac{4c}{\epsilon}$ has on the wave profile for $c^2 = \frac{1}{16}$, $c^2 = \frac{1}{15.5}$ and $c^2 = \frac{1}{15.1}$ where $\epsilon = 0.005$ and $\mu = 0.06$.

We now consider the solution for all previously stated c^2 values for the range $0 < c^2 < \frac{1}{15}$. Figure 10.5 shows that the exact invariant solution for the surface elevation of the solitary wave produces two different wave forms in the region $F < 0$, namely a bimodal transition wave and a bimodal depression wave. We note that the minimum surface elevation of the solitary wave G_2 is indirectly proportional to the wave speed for relatively small wave speeds i.e. $0 < c^2 < \frac{1}{17}$. From a wave speed of approximately $c^2 = \frac{1}{17}$, the minimum surface elevation of the solitary wave is directly proportional to the wave speed i.e. $\frac{1}{17} \leq c^2 < \frac{1}{15}$. Refer to Table E.1 for results that can be used as a comparison for experimental or numerical investigations.

Figure 10.6 shows that the exact invariant solution for the surface elevation of the solitary wave forms a unimodal depression wave in the region $F > 0$. For $0 \leq F \leq F_1$, the only turning point in $G(\xi)$ corresponds to the maximum fluid velocity in the horizontal direction of the solitary wave turning point $F = F_1$ at $\xi = 0$ where $\frac{dF}{d\xi} = 0$. Therefore, a unimodal depression wave is formed since $G(\xi) < 0$ at $\xi = 0$. The profile of the solitary wave flattens as the wave speed increases. We note that the minimum surface elevation of the solitary

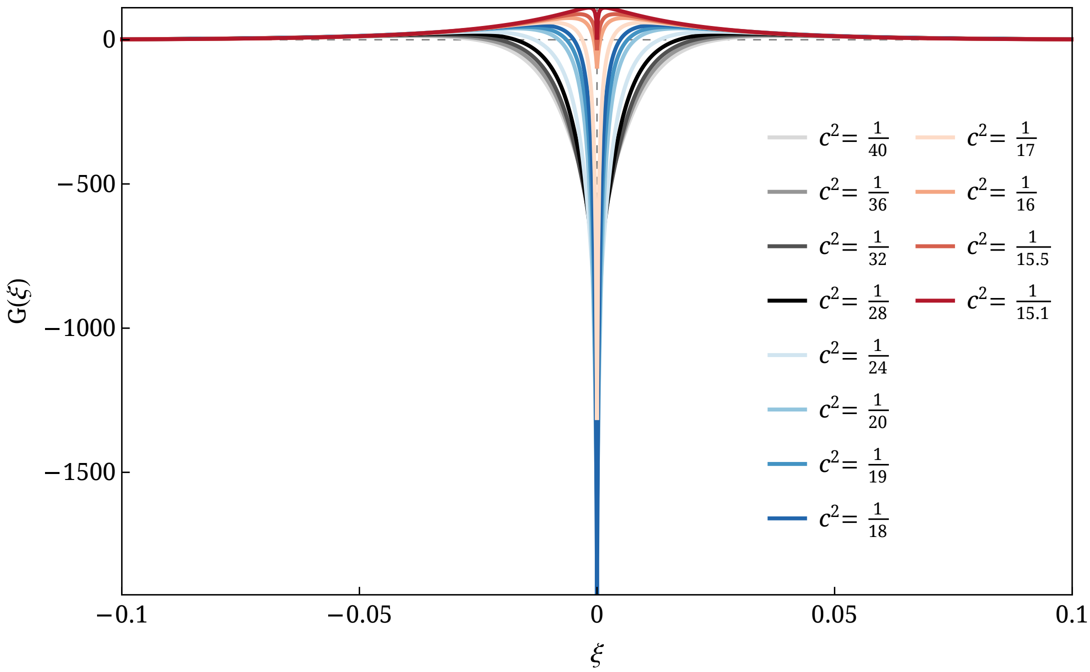

Figure 10.5: Surface elevation of the solitary wave $G(\xi)$ in the region $F < 0$ for varying wave speeds where $\epsilon = 0.005$ and $\mu = 0.06$.

wave G_1 is directly proportional to the wave speed. Refer to Table E.2 for results that can be used as a comparison for experimental or numerical investigations.

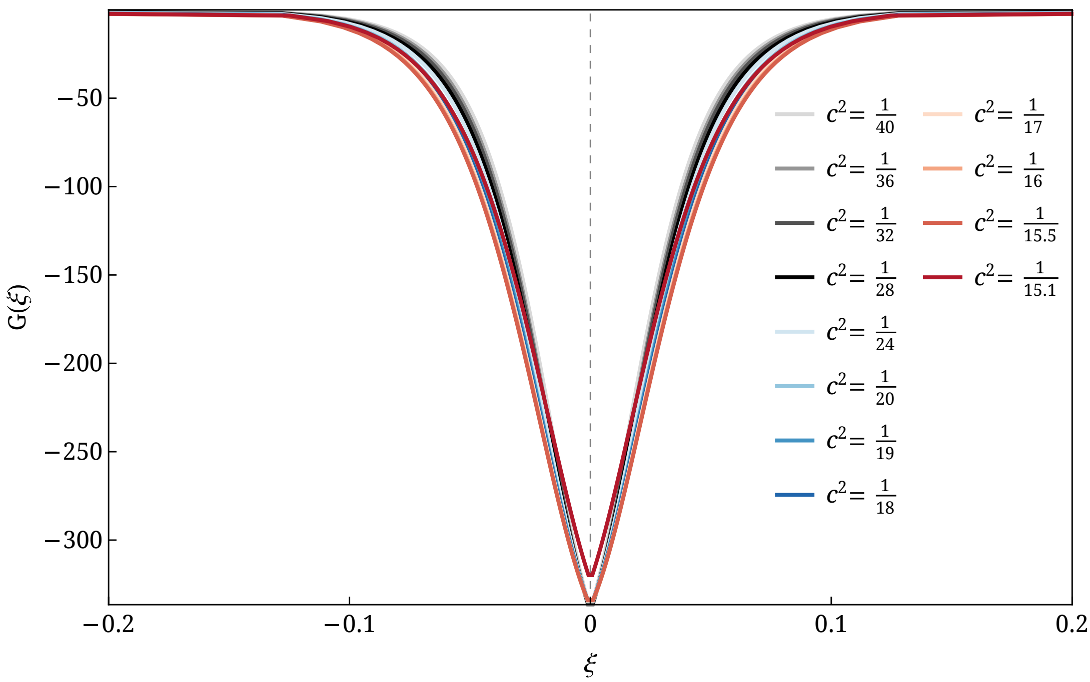

Figure 10.6: Surface elevation of the solitary wave $G(\xi)$ in the region $F > 0$ for varying wave speeds where $\epsilon = 0.005$ and $\mu = 0.06$.

Further investigations of the bimodal depression wave, in region $F < 0$, and the unimodal depression wave, in the region $F > 0$, are needed to determine the condition for which these waves are above the seabed i.e. $G(\xi) > -\frac{1}{\epsilon}$. From condition (9.52) in Section 9.2.3.1, the following condition needs to be satisfied in order for the wave to be above the seabed

$$Q(F) = (F - F_5)(F - F_6) < 0, \tag{10.4}$$

where

$$F_5 = \frac{1}{\epsilon}\left(c + \sqrt{5}\sqrt{\frac{2}{5} - c^2}\right) > 0, \tag{10.5}$$

and

$$F_6 = \frac{1}{\epsilon}\left(c - \sqrt{5}\sqrt{\frac{2}{5} - c^2}\right) < 0. \tag{10.6}$$

The range of $F(\xi)$ values for which the wave is above the seabed i.e. $F_6 < F < F_5$ will be satisfied if

$$\begin{aligned} & F_2 > F_6 \quad \text{where} \quad F_2 \le F \le 0, \\ \text{or} \quad & F_1 < F_5 \quad \text{where} \quad 0 \le F \le F_1. \end{aligned} \tag{10.7}$$

We now determine the values for c^2 that satisfy the above conditions for which $G(\xi)$ is above the seabed. Considering $F < 0$, referring to Table E.1, it is evident that the only physically realistic waves are for cases $c^2 = \frac{1}{16}$, $c^2 = \frac{1}{15.5}$ and $c^2 = \frac{1}{15.1}$. These waves are the only waves that satisfy the condition for the wave to be above the seabed i.e. $F_2 - F_6 > 0$ where $G(\xi) = G(F) > -\frac{1}{\epsilon} = -200$ at $\xi = 0$. The cases where $c^2 = \frac{1}{16}$ and $c^2 = \frac{1}{15.5}$ both form bimodal depression waves i.e. the minimum fluid velocity in the horizontal direction of the solitary wave $F_2 < -\frac{4c}{\epsilon}$ and $G(\xi) < 0$ at $\xi = 0$. The case where $c^2 = \frac{1}{15.1}$ is the only case in which a bimodal transition wave is formed i.e. the minimum fluid velocity in the horizontal direction of the solitary wave $F_2 = -\frac{4c}{\epsilon}$ and $G(\xi) = 0$ at $\xi = 0$. It is important to note that for the case $0 < c^2 < \frac{1}{15}$, in the region $F < 0$, no physically realistic bimodal elevation waves exist.

Now considering $F > 0$, referring to Table E.2, the waves formed are unimodal depressive waves, but none are physically realistic as the condition for the wave to be above the seabed is not satisfied i.e. $F_5 - F_1 < 0$ where $G(\xi) = G(F) < -\frac{1}{\epsilon} = -200$ at $\xi = 0$.

In conclusion, the only physically realistic waves shown in Figures 10.7 and 10.8 formed are for cases $c^2 = \frac{1}{16}$, $c^2 = \frac{1}{15.5}$ and $c^2 = \frac{1}{15.1}$ in the region $F < 0$. To our knowledge, these are new results in the form of a bimodal depression wave and a bimodal transition wave.

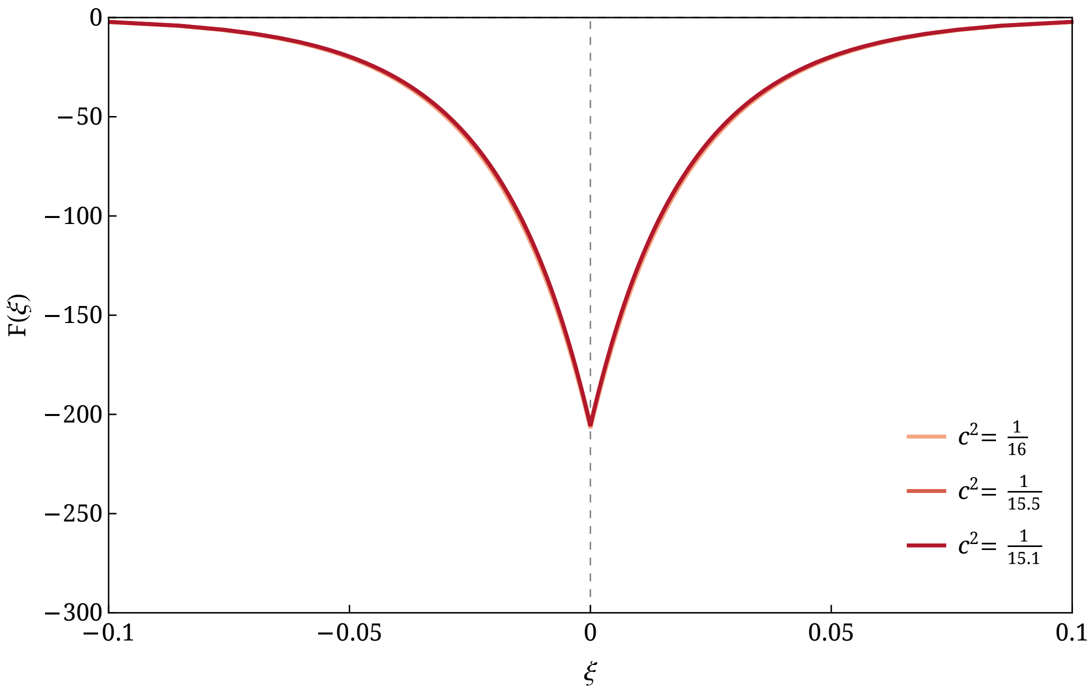

Figure 10.7: Physically realistic fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ in the region $F < 0$ for $c^2 = \frac{1}{16}$, $c^2 = \frac{1}{15.5}$ and $c^2 = \frac{1}{15.1}$ where $\epsilon = 0.005$ and $\mu = 0.06$.

10.1.3 Comparision of the results obtained using the approximate solutions $F^{A4}(\xi)$ and $G^{A4}(\xi)$ to the exact solutions $F(\xi)$ and $G(\xi)$

In this section, we derive the approximate solutions for the NB equations by making use of a pertubation method. We then compare results of the approximate solutions of the fluid velocity of the solitary wave in the horizontal direction $F^{A4}(\xi)$ and the surface elevation of the solitary wave $G^{A4}(\xi)$ to that of the exact solutions of the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ and the surface elevation of the solitary wave $G(\xi)$, all governed by the NB equations.

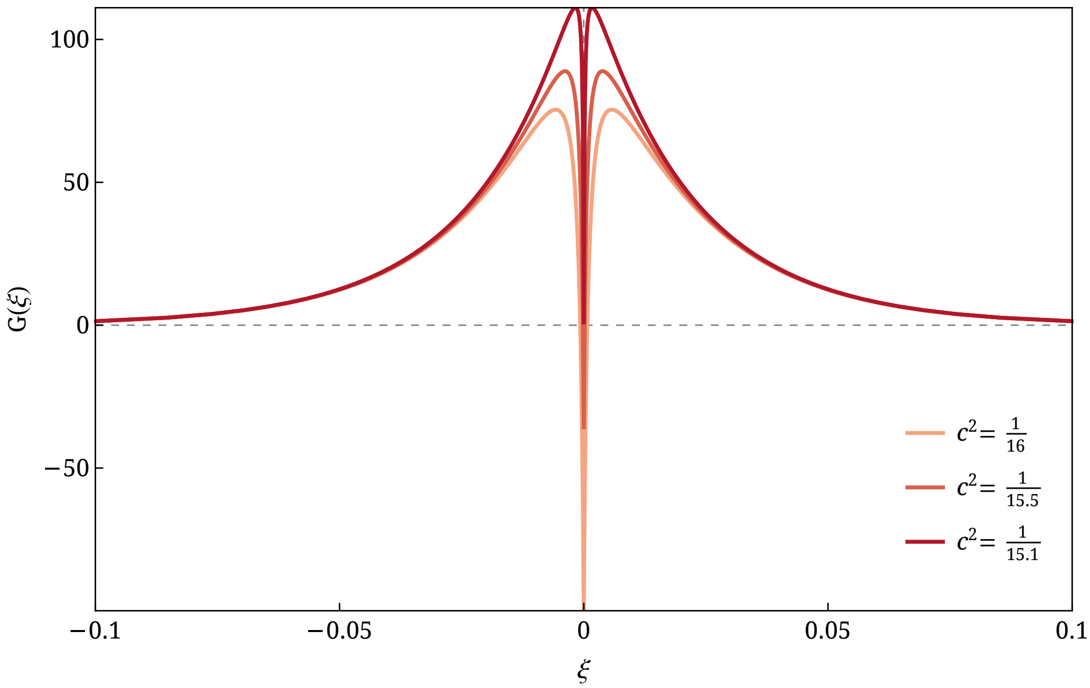

Figure 10.8: Physically realistic surface elevation of the solitary wave $G(\xi)$ in the region $F<0$ for $c^2=\frac{1}{16}$, $c^2=\frac{1}{15.5}$ and $c^2=\frac{1}{15.1}$ where $\epsilon=0.005$ and $\mu=0.06$.

10.1.3.1 Derivation of approximate solutions $F^{A4}(\xi)$ and $G^{A4}(\xi)$

We consider a straightforward Taylor series expansion to approximate the term $\ln(1+\epsilon)$ up to order $\mathcal{O}(4)$

$$\ln(1+\epsilon)=\epsilon-\frac{\epsilon^2}{2}+\frac{\epsilon^3}{3}-\frac{\epsilon^4}{4}+\mathcal{O}(\epsilon^5). \tag{10.8}$$

A fourth-order approximation is used, as it is the only approximation that gives us two bounded regions when considering $\left(\frac{dF}{d\xi}\right)^2$ vs $F(\xi)$. The lower order approximations only produce a bounded region for $F>0$. We now consider the values of c^2 for which the approximate solution best compares to the exact solution.

Consider equation (8.46) given by

$$P(F)=-F^3+\frac{3}{\epsilon c}\left(c^2+\frac{1}{6}\right)F^2-\frac{5}{\epsilon^2c^2}\left(\frac{1}{15}-c^2\right)F+\frac{5}{\epsilon^3c^3}\left(\frac{1}{3}-c^2\right)\left(\frac{1}{15}-c^2\right)\ln\left(\frac{F+A}{A}\right), \tag{10.9}$$

where

$$A=\frac{1}{\epsilon c}\left(\frac{1}{3}-c^2\right). \tag{10.10}$$

Substituting (10.8) into (10.9), we obtain

$$\begin{aligned} P_4(F) = & -F^3 + \frac{3}{\epsilon c}\left(c^2+\frac{1}{6}\right)F^2 - \frac{5}{\epsilon^2 c^2}\left(\frac{1}{15}-c^2\right)F \\ & + \frac{5}{\epsilon^3 c^3}\left(\frac{1}{3}-c^2\right)\left(\frac{1}{15}-c^2\right)\left[\frac{F}{A}-\frac{1}{2}\left(\frac{F}{A}\right)^2+\frac{1}{3}\left(\frac{F}{A}\right)^3-\frac{1}{4}\left(\frac{F}{A}\right)^4\right]. \end{aligned} \tag{10.11}$$

Equating terms of the same order of F in (10.11) results in the following systems of equations

$$F: -\frac{5}{\epsilon^2 c^2}\left(\frac{1}{15}-c^2\right)+\frac{5}{\epsilon^3 c^3}\left(\frac{1}{3}-c^2\right)\left(\frac{1}{15}-c^2\right)\left[\frac{\epsilon c}{\frac{1}{3}-c^2}\right]=0, \tag{10.12}$$

$$F^2: \frac{3}{\epsilon c}\left(c^2+\frac{1}{6}\right)+\frac{5}{\epsilon^3 c^3}\left(\frac{1}{3}-c^2\right)\left(\frac{1}{15}-c^2\right)\left[-\frac{1}{2}\left(\frac{\epsilon c}{\frac{1}{3}-c^2}\right)^2\right]=\frac{3c^2}{\epsilon c\left(\frac{1}{3}-c^2\right)}\left(1-c^2\right), \tag{10.13}$$

$$F^3: -1+\frac{5}{\epsilon^3 c^3}\left(\frac{1}{3}-c^2\right)\left(\frac{1}{15}-c^2\right)\left[\frac{1}{3}\left(\frac{1}{A}\right)^3\right]=\frac{-c^2\left(1+c^2\right)}{\left(\frac{1}{3}-c^2\right)^2}, \tag{10.14}$$

$$F^4: \frac{5}{\epsilon^3 c^3}\left(\frac{1}{3}-c^2\right)\left(\frac{1}{15}-c^2\right)\left[-\frac{1}{4}\left(\frac{1}{A}\right)^4\right]=-\frac{5\epsilon c}{4}\frac{\left(\frac{1}{15}-c^2\right)}{\left(\frac{1}{3}-c^2\right)^3}. \tag{10.15}$$

Substituting (10.12)-(10.15) into (10.11) results in the following simplified expression

$$\begin{aligned} P_4(F) &= -\frac{5\epsilon c}{4}\frac{\left(\frac{1}{15}-c^2\right)}{\left(\frac{1}{3}-c^2\right)^3}F^4-\frac{c^2\left(1+c^2\right)}{\left(\frac{1}{3}-c^2\right)^2}F^3+\frac{3c^2}{\epsilon c\left(\frac{1}{3}-c^2\right)}\left(1-c^2\right)F^2 \\ &= \frac{5\epsilon c}{4}\frac{\left(\frac{1}{15}-c^2\right)}{\left(\frac{1}{3}-c^2\right)^3}F^2\left[-F^2-\frac{4c^2}{5\epsilon c}\frac{\left(1+c^2\right)\left(\frac{1}{3}-c^2\right)}{\left(\frac{1}{15}-c^2\right)}F+\frac{12c^2}{5\epsilon^2 c^2}\left(1-c^2\right)\frac{\left(\frac{1}{3}-c^2\right)^2}{\left(\frac{1}{15}-c^2\right)}\right]. \end{aligned} \tag{10.16}$$

We can rewrite (10.16) as

$$P_4(F)=\left(-F^2-MF+N\right)\frac{5\epsilon c}{4}\frac{\left(\frac{1}{15}-c^2\right)}{\left(\frac{1}{3}-c^2\right)^3}F^2, \tag{10.17}$$

where

$$M=\frac{4c^2}{5\epsilon c}\frac{\left(1+c^2\right)\left(\frac{1}{3}-c^2\right)}{\left(\frac{1}{15}-c^2\right)}, \text{ and } N=\frac{12c^2}{5\epsilon^2 c^2}\left(1-c^2\right)\frac{\left(\frac{1}{3}-c^2\right)^2}{\left(\frac{1}{15}-c^2\right)}. \tag{10.18}$$

Consider the polynomial in (10.17)

$$-F^2-MF+N=0, \tag{10.19}$$

with discriminate

$$\triangle=M^2+4N. \tag{10.20}$$

Substituting (10.18) into (10.20) we get

$$M^2+4N=\frac{16\left(c^6+17c^4-15c^2+1\right)}{\left(\frac{1}{15}-c^2\right)^2}. \tag{10.21}$$

We now check that $M^2+4N>0$ for $0<c^2<\frac{1}{15}$

$$\begin{aligned}&c^6+17c^4-15c^2+1>0\\ \Rightarrow &c^4\left(c^2+17\right)+15\left(\frac{1}{15}-c^2\right)>0.\end{aligned} \tag{10.22}$$

Therefore, the roots of $F^2+MF-N=0$ are

$$F=-\frac{-M\pm\sqrt{M^2+4N}}{2}. \tag{10.23}$$

Substituting (10.17) and (10.18) into (10.23) we get

$$F=\frac{2}{5\epsilon}\frac{\left(\frac{1}{3}-c^2\right)}{\left(\frac{1}{15}-c^2\right)}\left[-c\left(1+c^2\right)\pm\sqrt{c^6+17c^4-15c^2+1}\right]. \tag{10.24}$$

We check that

$$\begin{aligned}&\sqrt{c^6+17c^4-15c^2+1}>-c\left(1+c^2\right)\\ \Rightarrow &\left(1-c^2\right)\left(\frac{1}{15}-c^2\right)>0,\end{aligned} \tag{10.25}$$

for $0<c^2<\frac{1}{15}$. Thus, the positive root is valid and

$$P_4(F)=\frac{5\epsilon c}{4}\frac{\left(\frac{1}{15}-c^2\right)}{\left(\frac{1}{3}-c^2\right)^3}(F)^2\left(F_{4,1}-F\right)\left(F-F_{4,2}\right), \tag{10.26}$$

where

$$F_{4,1}=\frac{2}{5\epsilon}\frac{\left(\frac{1}{3}-c^2\right)}{\left(\frac{1}{15}-c^2\right)}\left[\sqrt{c^2(1+c^2)^2+15(1-c^2)\left(\frac{1}{15}-c^2\right)}-c\left(1+c^2\right)\right], \tag{10.27}$$

and

$$F_{4,2}=-\frac{2}{5\epsilon}\frac{\left(\frac{1}{3}-c^2\right)}{\left(\frac{1}{15}-c^2\right)}\left[\sqrt{c^2(1+c^2)^2+15(1-c^2)\left(\frac{1}{15}-c^2\right)}+c\left(1+c^2\right)\right]. \tag{10.28}$$

We have two solution spaces in $F(\xi)$ for the case $0<c^2<\frac{1}{15}$. Thus, ξ takes the following form

$$\xi=\pm\sqrt{\frac{3c\mu^2}{2\epsilon}}\int_{F_{4,n}}^{F}\frac{dF}{\sqrt{P_4(F)}}$$

$$\Rightarrow \xi = \pm\sqrt{\frac{6}{5}\frac{\mu}{\epsilon}}\frac{\left(\frac{1}{3}-c^2\right)^{\frac{3}{2}}}{\left(\frac{1}{15}-c^2\right)^{\frac{1}{2}}}\int_{F_{4,n}}^{F}\frac{dF}{F\left[\left(F_{4,1}-F\right)\left(F-F_{4,2}\right)\right]^{\frac{1}{2}}}, \tag{10.29}$$

where $F_{4,n}=F_{4,1}>0$ when $n=1$ and $F_{4,n}=F_{4,2}<0$ when $n=2$, where

$$F_{4,1}=\frac{2}{5\epsilon}\frac{\left(\frac{1}{3}-c^2\right)}{\left(\frac{1}{15}-c^2\right)}\left[\sqrt{c^2(1+c^2)^2+15(1-c^2)\left(\frac{1}{15}-c^2\right)}-c\left(1+c^2\right)\right], \tag{10.30}$$

and

$$F_{4,2}=-\frac{2}{5\epsilon}\frac{\left(\frac{1}{3}-c^2\right)}{\left(\frac{1}{15}-c^2\right)}\left[\sqrt{c^2(1+c^2)^2+15(1-c^2)\left(\frac{1}{15}-c^2\right)}+c\left(1+c^2\right)\right]. \tag{10.31}$$

Consider $F<0$, we simplify (10.29) to give

$$\xi = \pm\sqrt{\frac{6}{5}\frac{\mu}{\epsilon}}\frac{\left(\frac{1}{3}-c^2\right)^{\frac{3}{2}}}{\left(\frac{1}{15}-c^2\right)^{\frac{1}{2}}}\int_{F_{4,2}}^{0}\frac{dF^{A4}}{F^{A4}\left[\left(F_{4,1}-F^{A4}\right)\left(F^{A4}-F_{4,2}\right)\right]^{\frac{1}{2}}}. \tag{10.32}$$

Let

$$\begin{aligned} I_2 &= \int_{F_{4,2}}^{0}\frac{dF^{A4}}{F^{A4}\left[\left(F_{4,1}-F^{A4}\right)\left(F^{A4}-F_{4,2}\right)\right]^{\frac{1}{2}}} \\ \Rightarrow I_2 &= \int_{F_{4,2}}^{0}\frac{dF^{A4}}{F^{A4}\left(F^{A4}-F_{4,2}\right)}\left(\frac{F^{A4}-F_{4,2}}{F_{4,1}-F^{A4}}\right)^{\frac{1}{2}}. \end{aligned} \tag{10.33}$$

Now we introduce a variable s and express F^{A4} in terms of s. Let

$$s^2=\frac{F^{A4}-F_{4,2}}{F_{4,1}-F^{A4}}, \tag{10.34}$$

solving for F^{A4} we get

$$F^{A4}=\frac{F_{4,1}s^2+F_{4,2}}{1+s^2}. \tag{10.35}$$

Differentiating F^{A4} with respect to s gives

$$\frac{dF^{A4}}{ds}=\frac{2s}{(1+s^2)^2}\left(F_{4,1}-F_{,2}\right). \tag{10.36}$$

Rewriting I_2 in terms of s, we obtain the following

$$I_2=2\int_{s_2}^{0}\frac{ds}{F_{4,1}s^2-B}, \tag{10.37}$$

where $B=-F_{4,2}$. Let

$$s=\sqrt{\frac{B}{F_{4,1}}}\tanh\theta \tag{10.38}$$

$$\Rightarrow ds = \sqrt{\frac{B}{F_{4,1}}}\operatorname{sech}^2\theta\, d\theta. \tag{10.39}$$

Rewrite (10.37) in terms of θ to get

$$I_2 = 2\int_{\theta_2}^{0} \frac{\sqrt{\frac{B}{F_{4,1}}}\operatorname{sech}^2\theta\, d\theta}{B\left(\tanh^2\theta - 1\right)}. \tag{10.40}$$

Using the trigonometric identity $\operatorname{sech}^2\theta + \tanh^2\theta = 1$, (10.40) becomes

$$\begin{aligned} I_2 &= -\frac{2}{\sqrt{F_{4,1}B}}\int_{\theta_2}^{0} d\theta \\ \Rightarrow I_2 &= -\frac{2}{\sqrt{F_{4,1}B}}\theta + k, \end{aligned} \tag{10.41}$$

where k is a constant. Thus, (10.32) becomes

$$\xi = \pm\sqrt{\frac{6}{5}\frac{\mu}{\epsilon}}\frac{\left(\frac{1}{3}-c^2\right)^{\frac{3}{2}}}{\left(\frac{1}{15}-c^2\right)^{\frac{1}{2}}}\left(-\frac{2}{\sqrt{F_{4,1}B}}\theta + k\right), \tag{10.42}$$

where $k = 0$ as $F^{A4} \to F_{4,2}$ when $\xi = 0$. Solving for θ gives

$$\theta = \pm\sqrt{\frac{5}{24}\frac{\epsilon}{\mu}}\frac{\left(\frac{1}{15}-c^2\right)^{\frac{1}{2}}}{\left(\frac{1}{3}-c^2\right)^{\frac{3}{2}}}\sqrt{F_{4,1}B}\,\xi. \tag{10.43}$$

Substituting (10.38) into (10.35), we get the following approximate solution for the fluid velocity of the wave

$$F^{A4} = -\frac{B\operatorname{sech}^2\theta}{1+\frac{B}{F_{4,1}}\tanh^2\theta}, \tag{10.44}$$

where

$$\theta = \pm\sqrt{\frac{5}{24}\frac{\epsilon}{\mu}}\frac{\left(\frac{1}{15}-c^2\right)^{\frac{1}{2}}}{\left(\frac{1}{3}-c^2\right)^{\frac{3}{2}}}\sqrt{F_{4,1}B}\,\xi, \tag{10.45}$$

and

$$F_{4,1} = \frac{2}{5\epsilon}\frac{\left(\frac{1}{3}-c^2\right)}{\left(\frac{1}{15}-c^2\right)}\left[\sqrt{c^2(1+c^2)^2 + 15(1-c^2)\left(\frac{1}{15}-c^2\right)} - c\left(1+c^2\right)\right], \tag{10.46}$$

and

$$F_{4,2} = -\frac{2}{5\epsilon}\frac{\left(\frac{1}{3}-c^2\right)}{\left(\frac{1}{15}-c^2\right)}\left[\sqrt{c^2(1+c^2)^2 + 15(1-c^2)\left(\frac{1}{15}-c^2\right)} + c\left(1+c^2\right)\right]. \tag{10.47}$$

By substituting (10.44) into the exact solution for $G(\xi)$, we obtain the approximate solution for the surface elevation of the solitary wave given by

$$G^{A4}(\xi) = -\frac{F^{A4}\left(4c + \epsilon F^{A4}\right)}{2c\,(1 - 3c^2 + 3c\epsilon F^{A4})}. \tag{10.48}$$

Now consider $F > 0$, we simplify (10.29) to give

$$\xi = \pm\sqrt{\frac{6}{5}\frac{\mu}{\epsilon}\frac{\left(\frac{1}{3} - c^2\right)^{\frac{3}{2}}}{\left(\frac{1}{15} - c^2\right)^{\frac{1}{2}}}} \int_0^{F_{4,1}} \frac{dF}{F\left[\left(F_{4,1} - F\right)\left(F - F_{4,2}\right)\right]^{\frac{1}{2}}}. \tag{10.49}$$

Let

$$I_1 = \int_0^{F_{4,1}} \frac{dF}{F\left[\left(F_{4,1} - F\right)\left(F - F_{4,2}\right)\right]^{\frac{1}{2}}}$$

$$\Rightarrow I_1 = \int_0^{F_{4,1}} \frac{dF}{F\left(F_{4,1} - F\right)} \left(\frac{F_{4,1} - F}{F - F_{4,2}}\right)^{\frac{1}{2}}. \tag{10.50}$$

Now we introduce a variable s, where $s = F^{A4}$ and express F in terms of s. Let

$$s^2 = \frac{F_{4,1} - F}{F - F_{4,2}}, \tag{10.51}$$

solving for F^{A4} we get

$$F^{A4} = \frac{F_{4,1} + F_{4,2}s^2}{1 + s^2}. \tag{10.52}$$

Differentiating F^{A4} with respect to s, we get

$$\frac{dF^{A4}}{ds} = \frac{2s}{(1+s^2)^2}\left(F_{4,2} - F_{4,1}\right). \tag{10.53}$$

Rewriting I_1 in terms of s, we obtain the following

$$I_1 = -2\int_0^{s_1} \frac{ds}{F_{4,1} - Bs^2}, \tag{10.54}$$

where $B = -F_{4,2}$. Let

$$s = \sqrt{\frac{F_{4,1}}{B}}\tanh\theta \tag{10.55}$$

$$\Rightarrow ds = \sqrt{\frac{F_{4,1}}{B}}\operatorname{sech}^2\theta\, d\theta. \tag{10.56}$$

Rewrite (10.54) in terms of θ

$$I_1=-2\int_0^{\theta_1}\frac{\sqrt{\dfrac{F_{4,1}}{B}}\operatorname{sech}^2\theta\,d\theta}{F_{4,1}\left(1-\tanh^2\theta\right)}. \tag{10.57}$$

Making use of the trigonometric identity $\operatorname{sech}^2\theta+\tanh^2\theta=1$, (10.57) becomes

$$\begin{aligned} I_1&=-\frac{2}{\sqrt{F_{4,1}B}}\int_0^{\theta_1}d\theta\\ \Rightarrow I_1&=-\frac{2}{\sqrt{F_{4,1}B}}\theta+k_1, \end{aligned} \tag{10.58}$$

where k_1 is a constant. Thus, (10.49) becomes

$$\xi=\pm\sqrt{\frac{6}{5}\frac{\mu}{\epsilon}\frac{\left(\frac{1}{3}-c^2\right)^{\frac{3}{2}}}{\left(\frac{1}{15}-c^2\right)^{\frac{1}{2}}}}\left(-\frac{2}{\sqrt{F_{4,1}B}}\theta+k_1\right), \tag{10.59}$$

where $k_1=0$ as $F^{A4}\longrightarrow F_{4,1}$ when $\xi=0$. Solving for θ gives

$$\theta=\pm\sqrt{\frac{5}{24}\frac{\epsilon}{\mu}\frac{\left(\frac{1}{15}-c^2\right)^{\frac{1}{2}}}{\left(\frac{1}{3}-c^2\right)^{\frac{3}{2}}}}\sqrt{F_{4,1}B}\,\xi. \tag{10.60}$$

Now, using (10.51) and (10.55), we obtain the following approximate solution for the fluid velocity of the wave

$$F^{A4}(\xi)=\frac{F_{4,1}\operatorname{sech}^2\theta}{1+\frac{F_{4,1}}{B}\tanh^2\theta}, \tag{10.61}$$

where

$$\theta=\pm\sqrt{\frac{5}{24}\frac{\epsilon}{\mu}\frac{\left(\frac{1}{15}-c^2\right)^{\frac{1}{2}}}{\left(\frac{1}{3}-c^2\right)^{\frac{3}{2}}}}\sqrt{F_{4,1}B}\,\xi, \tag{10.62}$$

and

$$F_{4,1}=\frac{2}{5\epsilon}\frac{\left(\frac{1}{3}-c^2\right)}{\left(\frac{1}{15}-c^2\right)}\left[\sqrt{c^2(1+c^2)^2+15(1-c^2)\left(\frac{1}{15}-c^2\right)}-c\left(1+c^2\right)\right], \tag{10.63}$$

and

$$F_{4,2}=-\frac{2}{5\epsilon}\frac{\left(\frac{1}{3}-c^2\right)}{\left(\frac{1}{15}-c^2\right)}\left[\sqrt{c^2(1+c^2)^2+15(1-c^2)\left(\frac{1}{15}-c^2\right)}+c\left(1+c^2\right)\right]. \tag{10.64}$$

The approximate solution for the surface elevation of the solitary wave is given by

$$G^{A4}(\xi)=-\frac{F^{A4}\left(4c+\epsilon F^{A4}\right)}{2c\left(1-3c^2+3c\epsilon F^{A4}\right)}. \tag{10.65}$$

10.1.3.2 Comparision of the results obtained using the approximate solutions $F^{A4}(\xi)$ and $G^{A4}(\xi)$ to the exact solutions $F(\xi)$ and $G(\xi)$

We now analyse the results obtained from the approximate solutions $F^{A4}(\xi)$ and $G^{A4}(\xi)$. From Figures 10.9 and 10.10, where Figure 10.10 provides a closer view of the phenomenon represented in Figure 10.9, the exact and approximate profiles of $\left(\frac{dF}{d\xi}\right)^2$ for relatively small c^2 values is of a similar form. The shaded areas represent the regions of physical significance.

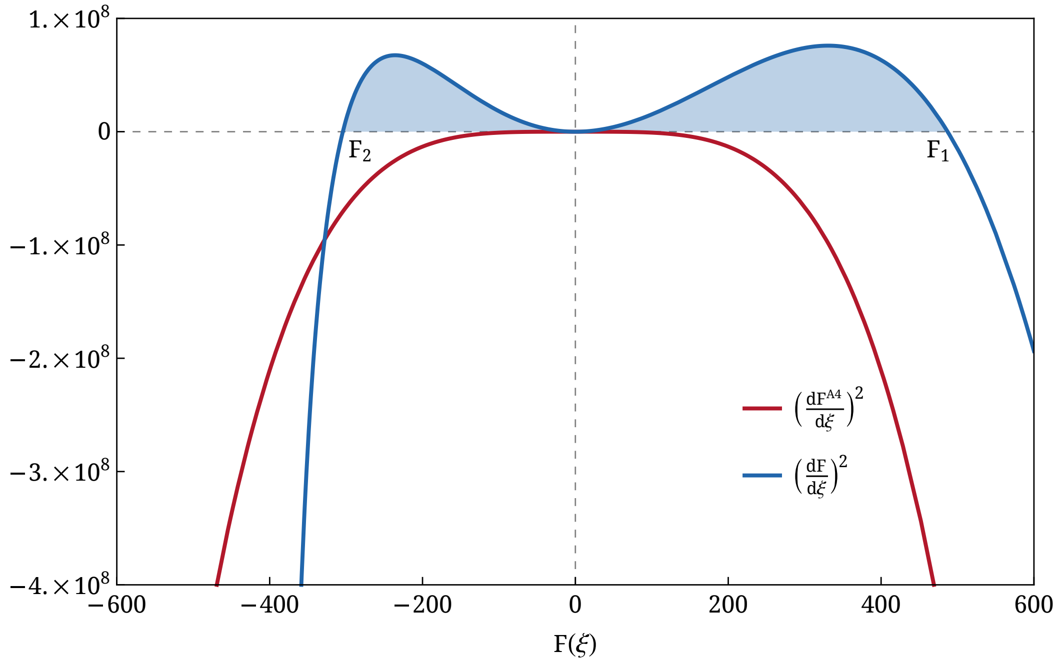

Figure 10.9: $\left(\frac{dF}{d\xi}\right)^2, \left(\frac{dF^{A4}}{d\xi}\right)^2$ vs $F(\xi)$ at a constant wave speed $c^2 = \frac{1}{40}$ where $\epsilon = 0.005$ and $\mu = 0.06$.

Now consider Figures 10.11 and 10.12, where Figure 10.12 provides a closer view of the phenomenon represented in Figure 10.11 where the shaded area represents the regions of physical significance. It is evident that as c^2 increases the exact profile of $\left(\frac{dF}{d\xi}\right)^2$ steepens while the approximate profile remains gradual. Thus, the approximate solution is a more accurate representation of the exact solution for smaller values of c^2.

Consider the behaviour of the fourth-order approximate solutions for $F < 0$. The uni-modal depression fluid velocity profile of the approximate solution $F^{A4}(\xi)$, given by Figure 10.13, is identical to that of the exact solution $F(\xi)$ given by Figure 10.2. We note that the

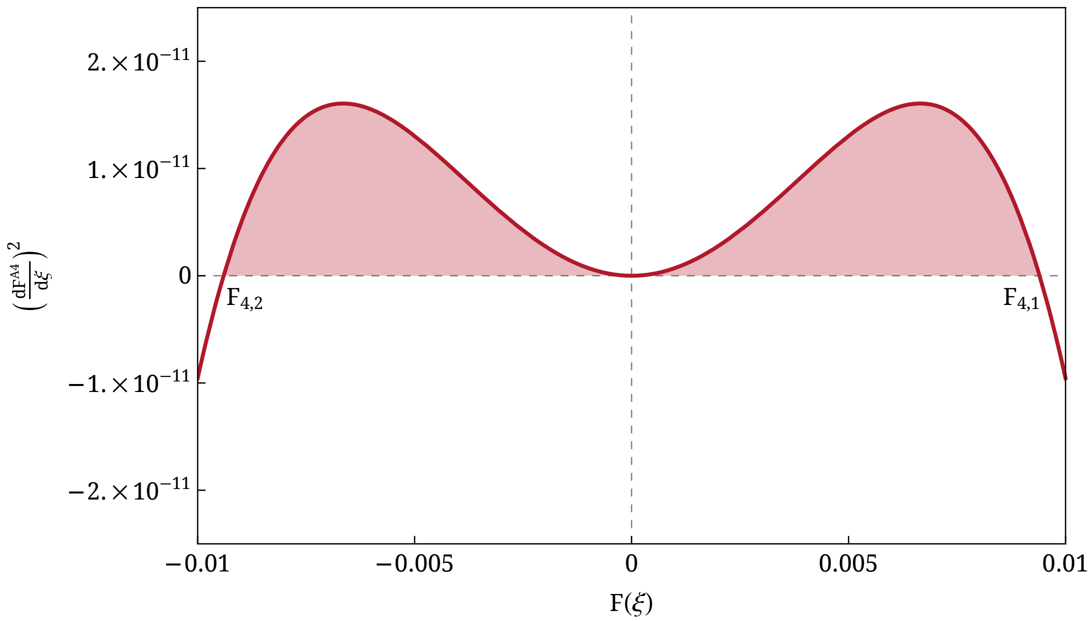

Figure 10.10: A zoomed in view of $\left(\frac{dF}{d\xi}\right)^2$, $\left(\frac{dF^{A4}}{d\xi}\right)^2$ vs $F(\xi)$ at a constant wave speed $c^2 = \frac{1}{40}$ where $\epsilon = 0.005$ and $\mu = 0.06$.

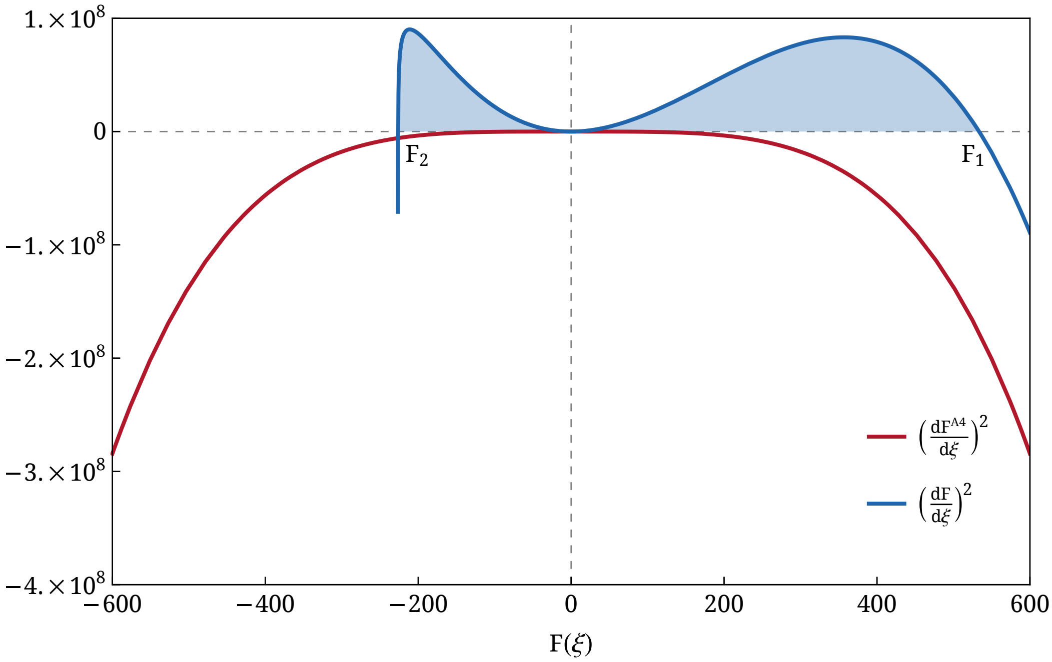

Figure 10.11: $\left(\frac{dF}{d\xi}\right)^2$, $\left(\frac{dF^{A4}}{d\xi}\right)^2$ vs $F(\xi)$ at a constant wave speed $c^2 = \frac{1}{17}$ where $\epsilon = 0.005$ and $\mu = 0.06$.

relationship between the wave speed and the minimum fluid velocity of the solitary wave $F_{4,2}$ is inversely proportional, but when considering the exact solution, the same relationship is directly proportional. Thus, qualitatively the approximate solution is an inaccurate representation of the exact solution. The approximate solution grossly underestimates the

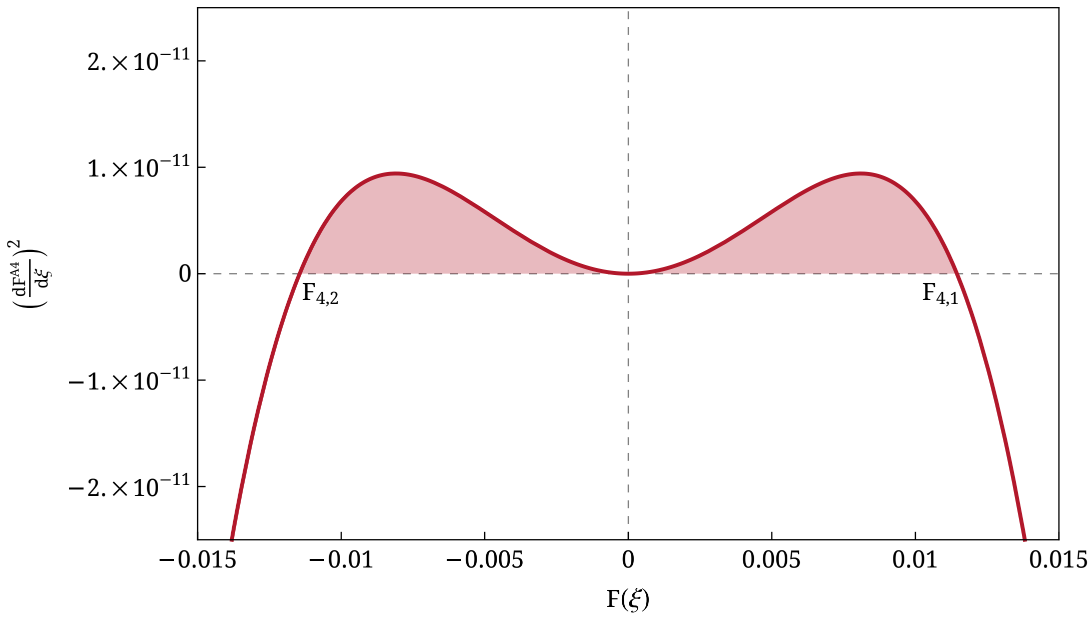

Figure 10.12: A zoomed in view of $\left(\frac{dF}{d\xi}\right)^2$, $\left(\frac{dF^{A4}}{d\xi}\right)^2$ vs $F(\xi)$ at a constant wave speed $c^2 = \frac{1}{17}$ where $\epsilon = 0.005$ and $\mu = 0.06$.

amplitude of the fluid velocity of the solitary wave in the horizontal direction, thus, quantitatively, the approximate solution is an inaccurate representation of the exact solution.

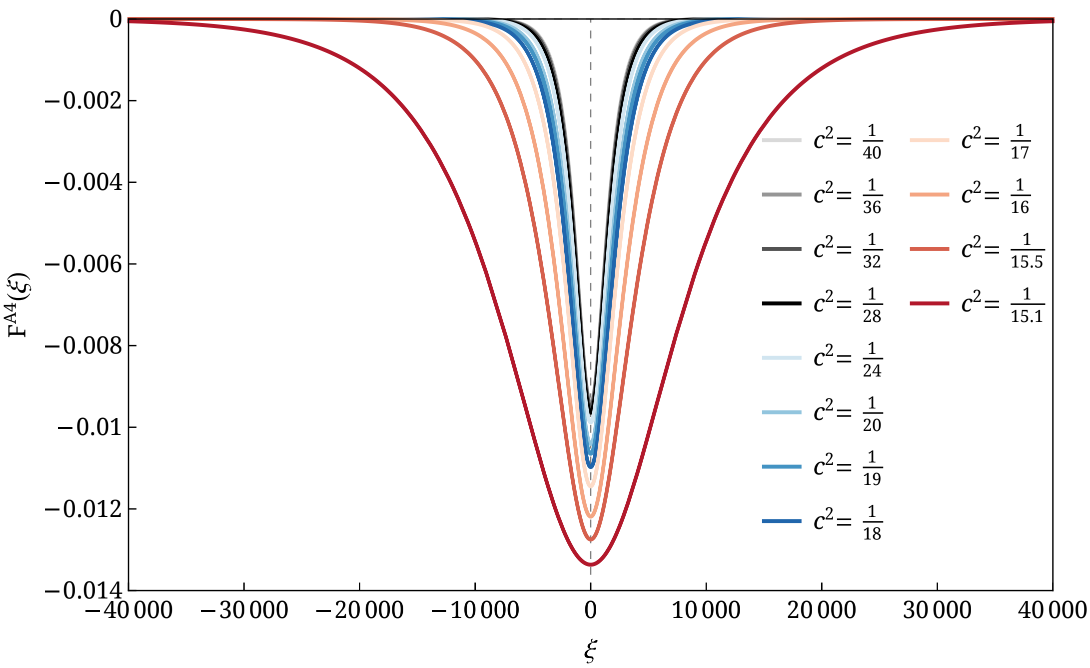

Figure 10.13: Fluid velocity of the solitary wave in the horizontal direction $F^{A4}(\xi)$ in the region $F < 0$, using a fourth-order approximation, for varying wave speeds where $\epsilon = 0.005$ and $\mu = 0.06$.

Similarly, when considering the approximate solution for the surface elevation of the solitary wave, $G^{A4}(\xi)$, given by Figure 10.14, in comparison to the exact solution given by Figure 10.5, the approximate solution is qualitatively and quantitatively an inaccurate approximation. The profile of the approximate solution is of a unimodal excited form whilst the exact solution is of a bimodal transitional or depressive form. The approximate solution also underestimates the amplitude of the surface elevation of the solitary wave. We note that the relationship between the wave speed and the maximum surface elevation of the solitary wave $G_{4,2}$ is directly proportional. Considering the exact solution, initially the maximum surface elevation of the solitary wave decreases as the wave speed increases, but at a wave speed approximately $c^2 = \frac{1}{17}$ the maximum surface elevation of the solitary wave increases as the wave speed increases.

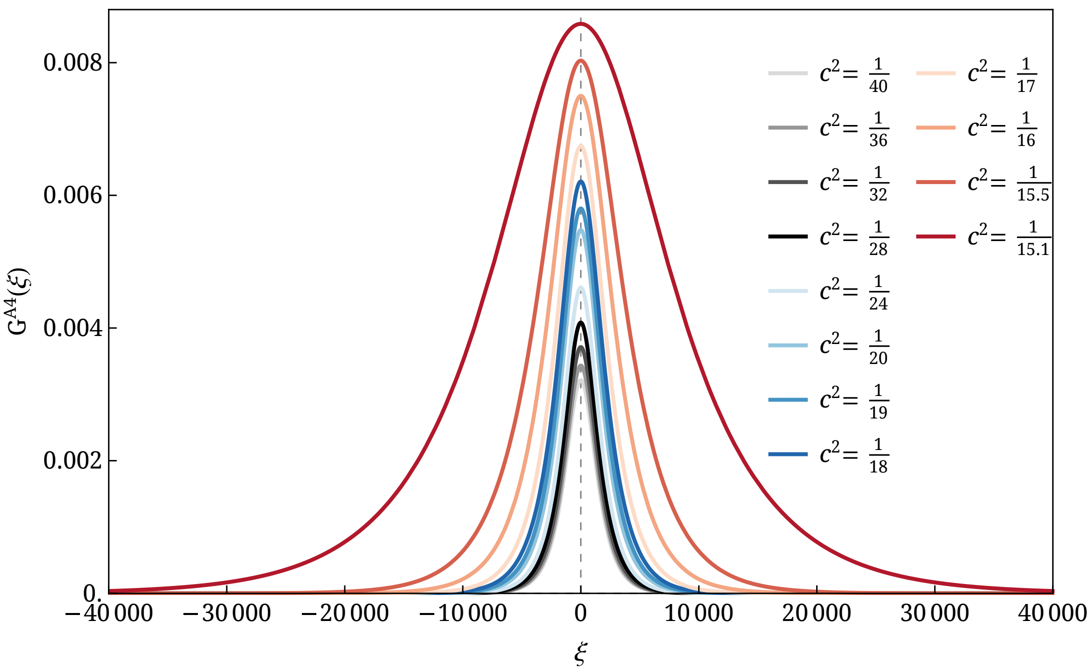

Figure 10.14: Surface elevation of the solitary wave $G^{A4}(\xi)$ in the region $F < 0$, using a fourth-order approximation, for varying wave speeds where $\epsilon = 0.005$ and $\mu = 0.06$.

Now consider the behaviour of the fourth-order approximate solutions for $F > 0$. The unimodal depression fluid velocity profile of the approximate solution $F^{A4}(\xi)$, given by Figure 10.15, is identical to that of the exact solution $F(\xi)$ given by Figure 10.3. We note that the relationship between the wave speed and the maximum fluid velocity of the solitary wave $F_{4,1}$ is directly proportional. This is identical to that of the exact solution. Thus, qualitatively, the approximate solution is an accurate representation of the exact solution. The approximate solution grossly underestimates the amplitude of the fluid velocity of the solitary wave in the horizontal direction. Thus, quantitatively, the approximate solution is an inaccurate representation of the exact solution.

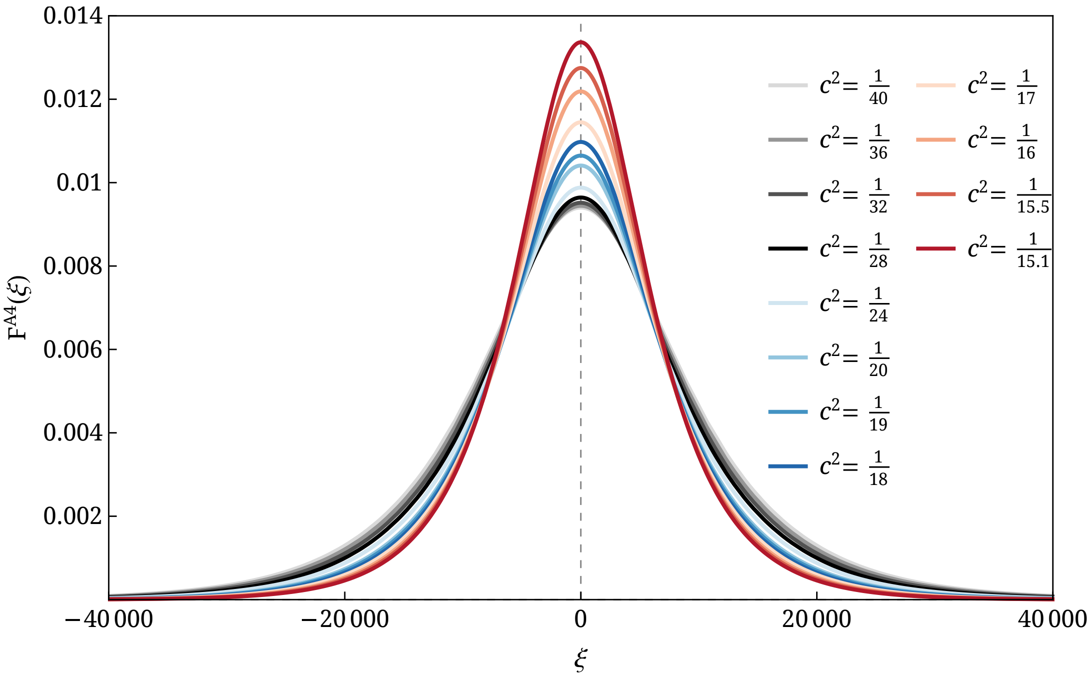

Figure 10.15: Fluid velocity of the solitary wave in the horizontal direction $F^{A4}(\xi)$ in the region $F > 0$, using a fourth order approximation, for varying wave speeds where $\epsilon = 0.005$ and $\mu = 0.06$.

We observe a similar occurrence when considering the approximate solution for the surface elevation of the solitary wave, $G^{A4}(\xi)$, given by Figure 10.16, in comparison to the exact solution given by Figure 10.6. The profile of the approximate solution is of a unimodal depressed form, identical to that of the exact solution. We note that the relationship between the wave speed and the minimum surface elevation of the solitary wave $G_{4,1}$ is inversely proportional, but when considering the exact solution, the same relationship is directly proportional. The approximate solution also underestimates the amplitude of the surface elevation of the solitary wave, thus, quantitatively, the approximate solution is an inaccurate representation of the exact solution.

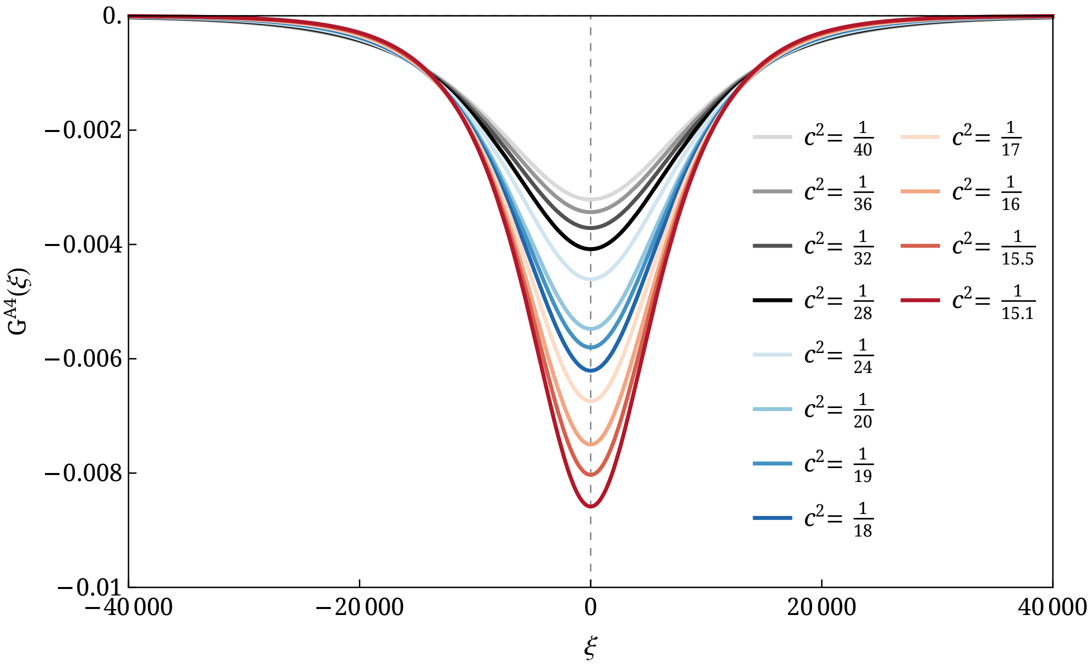

Figure 10.16: Surface elevation of the solitary wave $G^{A4}(\xi)$ in the region $F > 0$, using a fourth-order approximation, for varying wave speeds where $\epsilon = 0.005$ and $\mu = 0.06$.

10.1.4 Concluding remarks

The only physically realistic waves shown in Figures 10.7 and 10.8 are for cases $c^2 = \frac{1}{16}$, $c^2 = \frac{1}{15.5}$ and $c^2 = \frac{1}{15.1}$. To our knowledge, these are new results, used to enrich the analytical validation of numerical models for tsunami detection systems. We observe that two new wave forms are observed, namely a bimodal depression wave and a bimodal transition wave. However, the fourth-order approximate solutions are qualitatively and quantitatively inaccurate.

10.2 Case 2: $c^2 = \frac{1}{15}$

10.2.1 Properties of the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ relating to the exact invariant solution

We recall from Section 9.1 that there is one region for the special case (ii) $c^2 = \frac{1}{15}$, namely $0 \leq F \leq F_1$, in which we potentially obtain a solitary wave. The shaded area in Figure 10.17, the graph of $\left(\frac{dF}{d\xi}\right)^2$ versus $F(\xi)$, represents the region of physical significance.

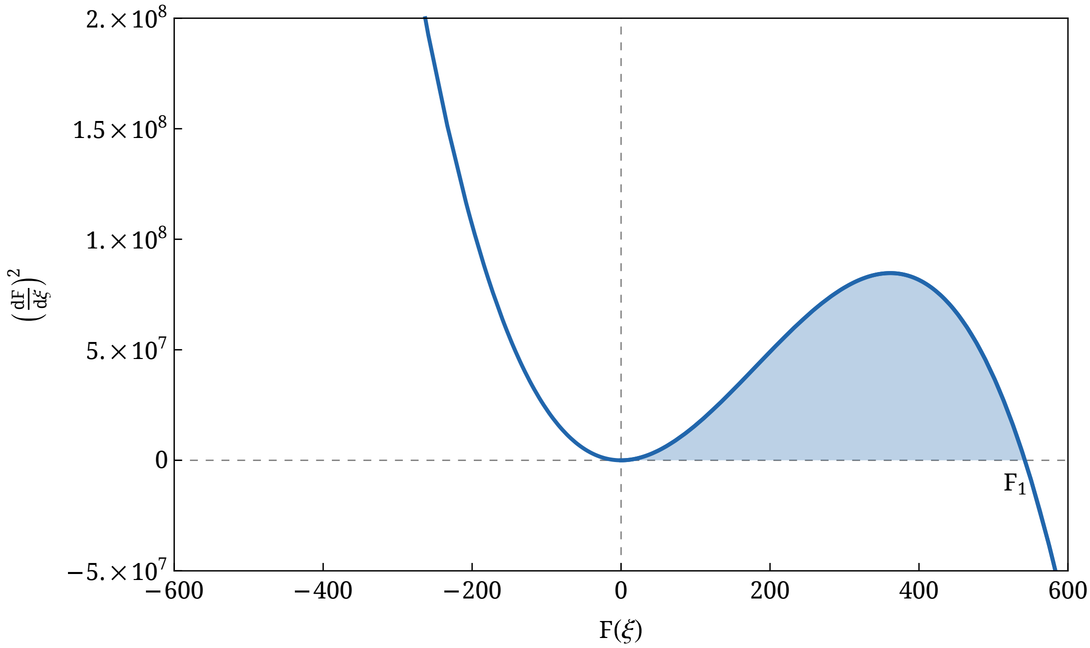

Figure 10.17: $\left(\frac{dF}{d\xi}\right)^2$ vs $F(\xi)$ at a constant wave speed $c^2 = \frac{1}{15}$ where $\epsilon = 0.005$ and $\mu = 0.06$.

We now present the plot of the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ against ξ for $c^2 = \frac{1}{15}$ where ϵ and μ are constants. An estimate of the width of the solitary wave is given by

$$\begin{aligned} \xi_B &= \sqrt{\frac{2\mu}{7}} \\ &= \mathcal{O}(1). \end{aligned} \tag{10.66}$$

We note that the width of the wave ξ_B is directly proportional to the square root of the dispersion of the wave μ. The maximum fluid velocity in the horizontal direction of the solitary wave F_1 is given by $F_1 = \frac{7}{10\epsilon c}$. Figure 10.18 shows that the fluid velocity of the solitary wave in the horizontal direction has a unimodal excitation profile in the region $F > 0$.

10.2.2 Properties of the surface elevation of the solitary wave $G(\xi)$

Consider now the properties of the surface elevation of the solitary wave $G(\xi)$ analysed in Section 9.2.1.2. From (9.29), we consider the relation between $G(\xi)$ and $F(\xi)$ given by

$$G(F) = -\frac{F}{6c}. \tag{10.67}$$

This is the only case in which $G(\xi)$ is proportional to $F(\xi)$ for all values of F i.e. $\frac{4c}{\epsilon} = A$.

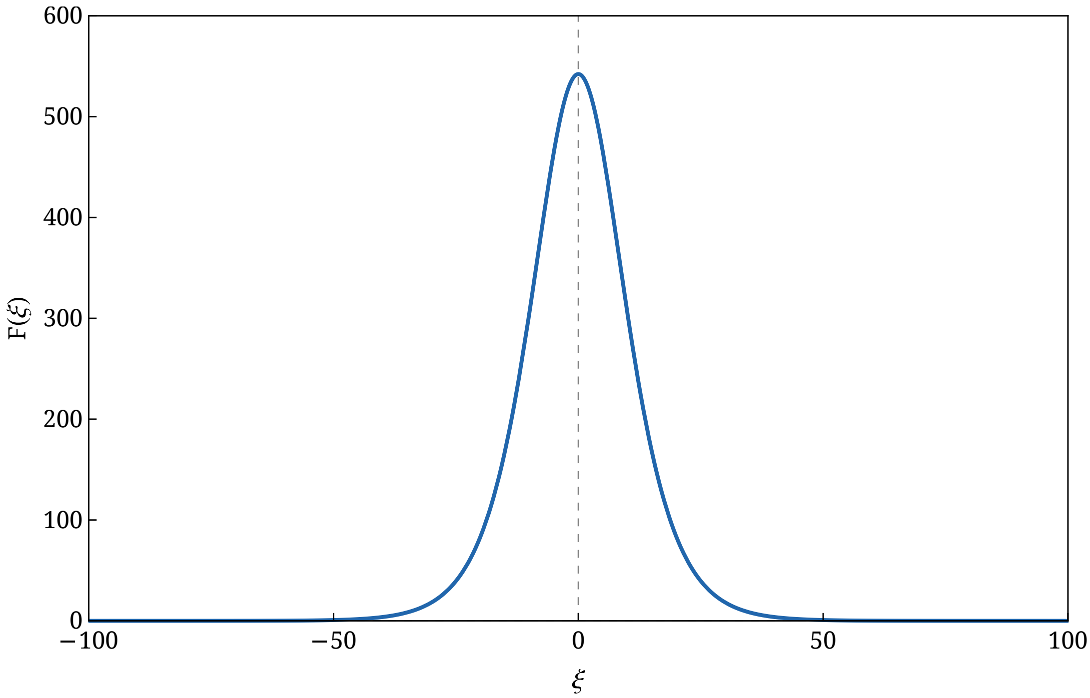

Figure 10.18: Fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ in the region $F > 0$ for $c^2 = \frac{1}{15}$ where $\epsilon = 0.005$ and $\mu = 0.06$.

From Section 9.2.2.2, considering $\frac{d^2G}{d\xi^2}$ we can say that G_1 is a minimum when F_1 is a maximum. Considering $\frac{dG}{d\xi}$ we also know that $G(\xi)$ will have a turning point if and only if $F(\xi)$ has a turning point at $\xi = 0$ i.e. at the maximum fluid velocity in the horizontal direction of the solitary wave $F_1 = \frac{7}{10\epsilon c}$. For $0 \leq F \leq F_1$, the only turning point in $G(\xi)$ corresponds to the maximum fluid velocity in the horizontal direction of the solitary wave turning point $F_1 = \frac{7}{10\epsilon c}$ at $\xi = 0$ where $\frac{dF}{d\xi} = 0$. Thus, the turning point is given by

$$\begin{aligned} \frac{dG}{d\xi} &= -\frac{1}{6c}\frac{dF}{d\xi}, \\ &= -\frac{1}{6c}\left(\frac{7}{10\epsilon c}\right), \\ &= -350, \end{aligned} \tag{10.68}$$

where $F_1 = \frac{7}{10\epsilon c}$ at $\xi = 0$ with $\epsilon = 0.005$.

We now consider the potential solitary wave that forms in the region $F > 0$. Figure 10.19 shows that the surface elevation of the solitary wave has a unimodal depression profile.

Further investigation of the unimodal depression wave in region $F > 0$ is needed to determine the condition for which these waves are above the seabed i.e. $G(\xi) > -\frac{1}{\epsilon}$.

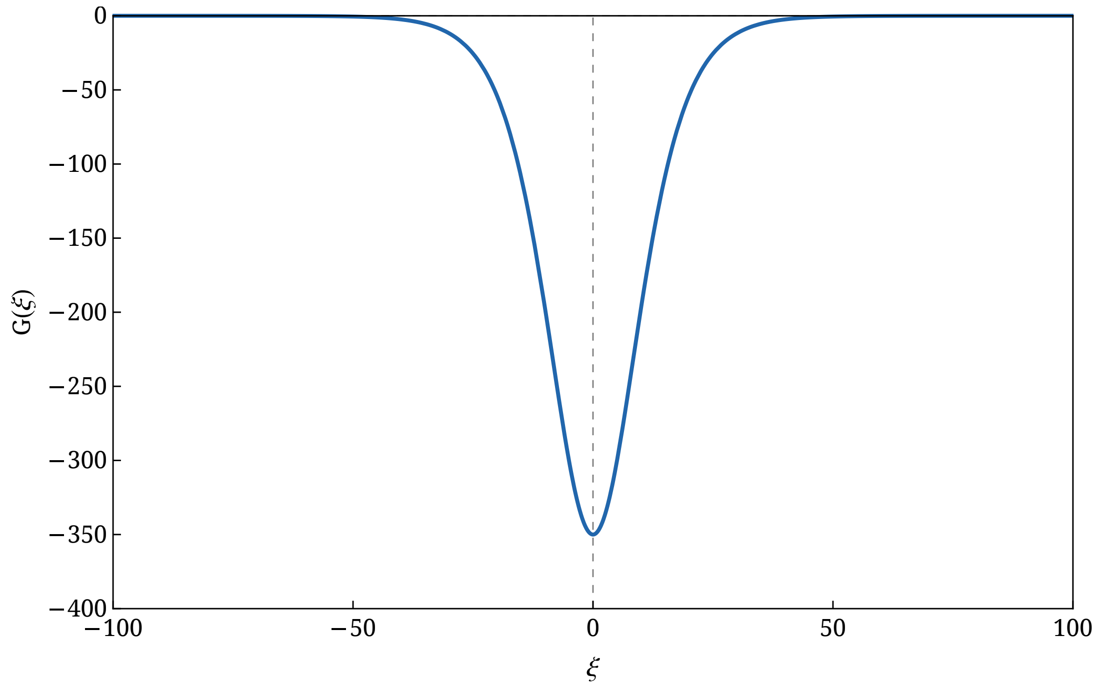

Figure 10.19: Surface elevation of the solitary wave $G(\xi)$ in the region $F > 0$ for $c^2 = \frac{1}{15}$ where $\epsilon = 0.005$ and $\mu = 0.06$.

From (10.68), we see that the minimum surface elevation of the solitary wave G_1 is given by

$$G_1 = -\frac{1}{6c}\left(\frac{7}{10\epsilon c}\right) = -350 \ngtr -\frac{1}{\epsilon} = -200, \tag{10.69}$$

where with $\epsilon = 0.005$. Thus, the case $c^2 = \frac{1}{15}$ is not a physically realistic solution for the surface elevation of the solitary wave, as the surface elevation is below the seabed.

10.2.3 Concluding remarks

This case does not produce physically realistic waves, thus, the case $c^2 = \frac{1}{15}$ does not form part of the solution set. To our knowledge, these are new results. Readers can refer to the work done by Dutykh and Dias [2], who perform an alternative analysis on the same set of solutions for this specific case.

10.3 Case 3: $\frac{1}{15} < c^2 < \frac{1}{3}$

10.3.1 Properties of the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ relating to the exact invariant solution

We recall from Section 9.1 that there is one region for case $\frac{1}{15} < c^2 < \frac{1}{3}$, namely $0 \le F \le F_1$, in which we potentially obtain a solitary wave. The shaded area in Figure 10.20, the graph of $\left(\frac{dF}{d\xi}\right)^2$ versus $F(\xi)$, represents this region of physical significance.

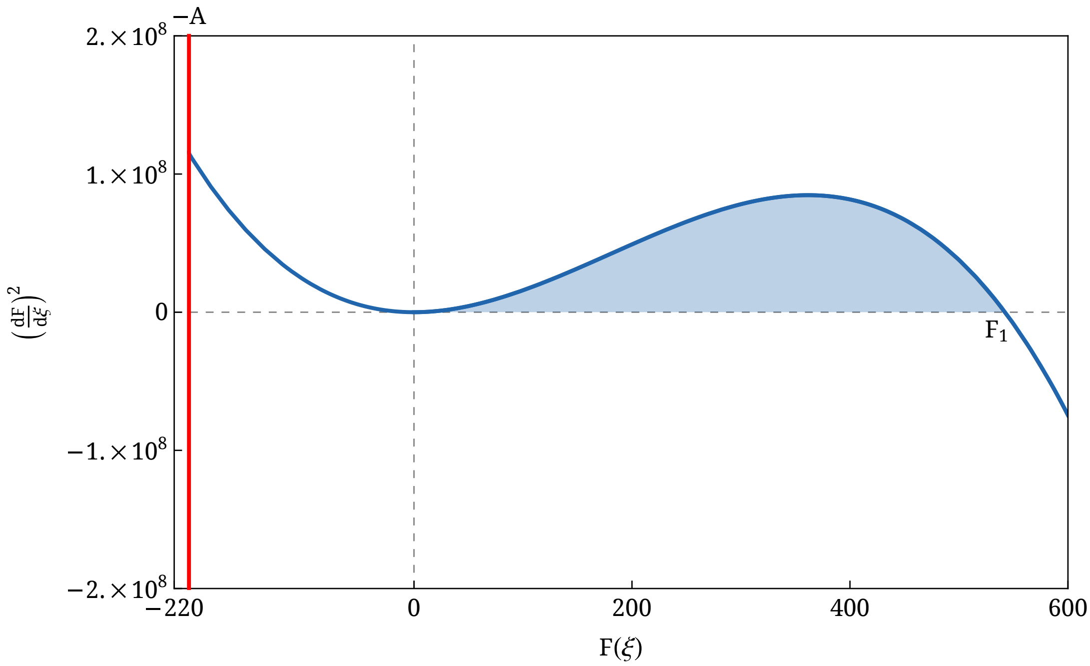

Figure 10.20: $\left(\frac{dF}{d\xi}\right)^2$ vs $F(\xi)$ at a constant wave speed $c^2 = \frac{1}{15} + \delta$ where $\delta = 10^{-4}$, $\epsilon = 0.005$ and $\mu = 0.06$.

We now present plots of the potential solution for the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ against ξ for varying c^2 values in the range $\frac{1}{15} < c^2 < \frac{1}{3}$, where ϵ and μ are constants. An estimate of the width of the solitary wave is given by

$$\begin{aligned}
\xi_B &= \pm\sqrt{\frac{3c\mu^2}{2\epsilon}} \int_{\frac{F_1}{2}}^{F} \frac{dF}{\sqrt{P(F)}} \\
&= \mathcal{O}\left(\left(\pm\sqrt{\frac{3c\mu^2}{2\epsilon}}\right)\right) \\
&= \mathcal{O}\left(\left(\sqrt{c}\right)\right), \qquad (10.70)
\end{aligned}$$

since $\mathcal{O}\left(\frac{\mu^2}{\epsilon}\right)=1$. We note that the width of the wave ξ_B is directly proportional to the square root of the wave speed.

Figure 10.21 shows that the fluid velocity of the solitary wave in the horizontal direction has a unimodal excitation profile. We note that the maximum fluid velocity of the solitary wave in the horizontal direction F_1 is directly proportional to the wave speed. Refer to Table E.3 for results which can be used as a comparison for experimental or numerical investigations.

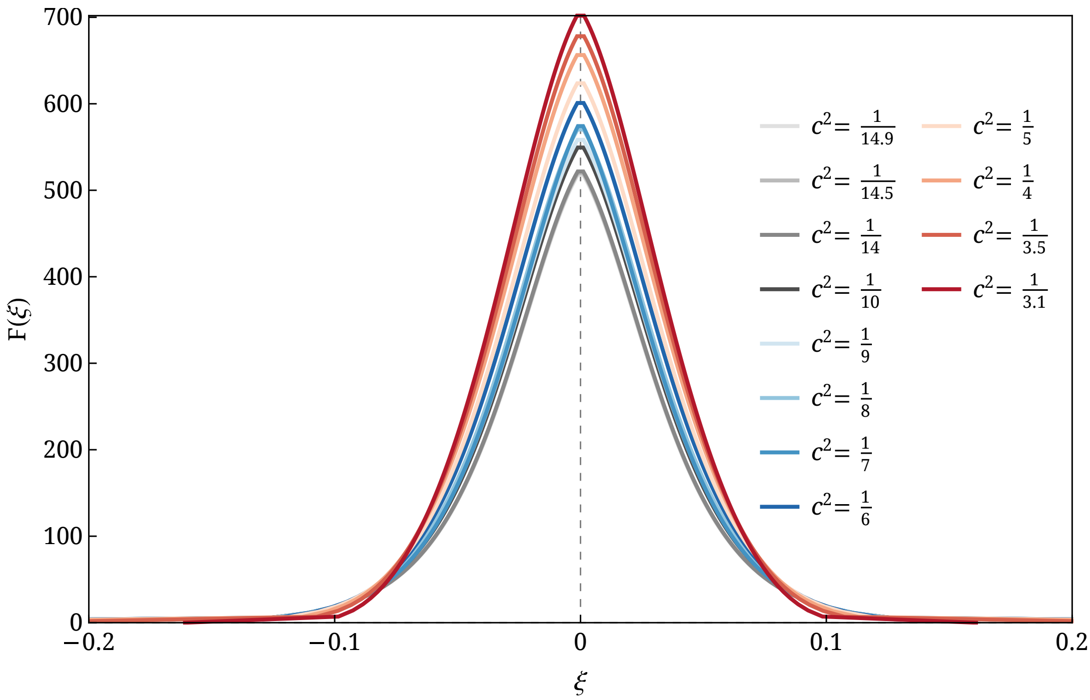

Figure 10.21: Fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ in the region $F>0$ for varying wave speeds where $\epsilon=0.005$ and $\mu=0.06$.

10.3.2 Properties of the surface elevation of the solitary wave $G(\xi)$

Consider now the properties of the surface elevation of the solitary wave $G(\xi)$ analysed in Section 9.2.1.1. From (9.26), we have

$$G(F)=-\frac{F}{6c}\frac{\left(\frac{4c}{\epsilon}+F\right)}{A+F}, \text{ where } A=\frac{1}{\epsilon c}\left(\frac{1}{3}-c^2\right). \tag{10.71}$$

The turning points determined by $\frac{dG}{d\xi}$ were fully analysed in Section 9.2.2.1. It was found that $\frac{dG}{dF}$ has no real roots. Thus, the only turning point in $G(\xi)$ corresponds to the maximum fluid velocity in the horizontal direction of the solitary wave turning point $F = F_1$ at $\xi = 0$ where $\frac{dF}{d\xi} = 0$.

Figure 10.22 shows that the surface elevation of the solitary wave has a unimodal depression profile. For $0 \leq F \leq F_1$, the only turning point occurs at the origin at the maximum fluid velocity in the horizontal direction of the solitary wave $F = F_1$ at $\xi = 0$. We note that the minimum surface elevation of the solitary wave G_1 is inversely proportional to the wave speed. Refer to Table E.3 for results which can be used as a comparison for experimental or numerical investigations.

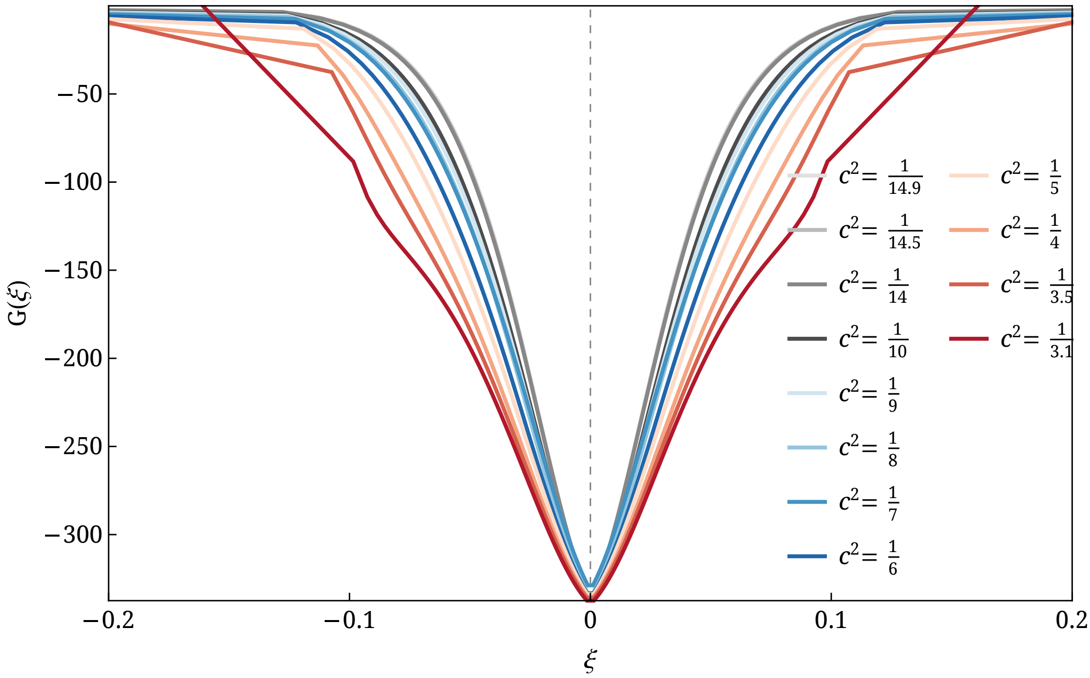

Figure 10.22: Surface elevation of the solitary wave $G(\xi)$ in the region $F > 0$ for varying wave speeds where $\epsilon = 0.005$ and $\mu = 0.06$.

Further investigation of the unimodal depression wave, in the region $F > 0$, is needed to determine the condition for which these waves are above the seabed i.e. $G(\xi) > -\frac{1}{\epsilon}$. From condition (9.52) in Section 9.2.3.1, the following condition needs to be satisfied in order for the wave to be above the seabed

$$Q(F) = (F - F_5)(F - F_6) < 0, \tag{10.72}$$

where

$$F_5 = \frac{1}{\epsilon}\left(c + \sqrt{5}\sqrt{\frac{2}{5} - c^2}\right) > 0, \text{ and } F_6 = \frac{1}{\epsilon}\left(c - \sqrt{5}\sqrt{\frac{2}{5} - c^2}\right) < 0. \tag{10.73}$$

The range of F values for which the wave is above the seabed i.e. $F_6 < F < F_5$ will be satisfied if

$$\begin{aligned} & F_1 \leq F_5 \quad \text{where} \quad 0 < F < F_1, \\ \text{or} \quad & F_2 \geq F_6 \quad \text{where} \quad F_2 < F < 0. \end{aligned} \tag{10.74}$$

We now determine the values for c^2 satisfying the above conditions for which $G(\xi)$ is above the seabed. Considering $F > 0$, from Table E.3 the waves formed are unimodal depression waves, but none are physically realistic, as the condition for the wave to be above the seabed is not satisfied i.e. $F_5 - F_1 < 0$ where $G(\xi) < -\frac{1}{\epsilon} = -200$ at $\xi = 0$.

In conclusion, the case $\frac{1}{15} < c^2 < \frac{1}{3}$ is not a physically realistic solution for the surface elevation of the solitary wave, as the surface elevation is below the seabed. Thus, the case $\frac{1}{15} < c^2 < \frac{1}{3}$ does not form part of the solution set.

10.3.3 Derivation of approximate solutions $F^{An}(\xi)$ and $G^{An}(\xi)$ for $n = 1, 2, 3$

We also employed the pertubation method in order to derive the approximate solutions of the fluid velocity of the solitary wave in the horizontal direction $F^{An}(\xi)$ and surface elevation of the solitary wave $G^{An}(\xi)$ for the NB equations, but did not present the results because the solutions are not physically meaningful.

10.3.4 Concluding remarks

The case $\frac{1}{15} < c^2 < \frac{1}{3}$ is not a physically realistic solution for the surface elevation of the solitary wave, as the surface elevation is below the seabed. Thus, the case $\frac{1}{15} < c^2 < \frac{1}{3}$ does not form part of the solution set.

10.4 Case 4: $c^2 = \frac{1}{3}$

10.4.1 Properties of the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ relating to the exact invariant solution

We recall from Section 9.1 that there is one region for the special case (iii) $c^2 = \frac{1}{3}$, namely $0 \le F \le F_1$, in which we potentially obtain a solitary wave. The shaded area in Figure 10.23, the graph of $\left(\frac{dF}{d\xi}\right)^2$ versus $F(\xi)$, represents the region of physical significance.

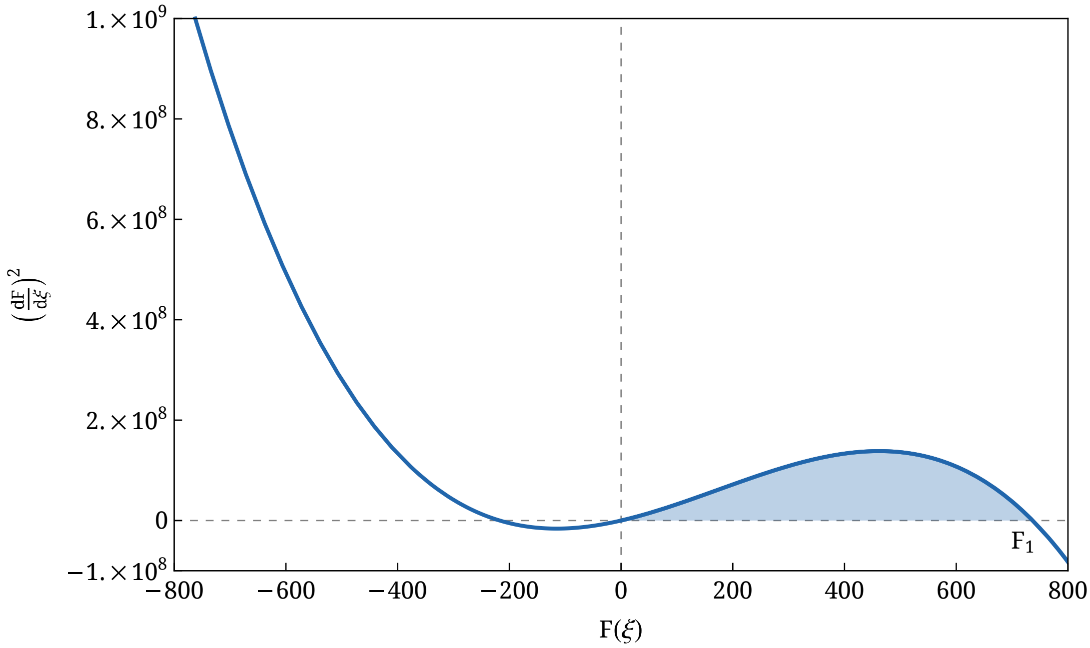

Figure 10.23: $\left(\frac{dF}{d\xi}\right)^2$ vs $F(\xi)$ at a constant wave speed $c^2 = \frac{1}{3}$ where $\epsilon = 0.005$ and $\mu = 0.06$.

We now present the plot of the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ against ξ for $c^2 = \frac{1}{3}$, where ϵ and μ are constants. An estimate of the width of the solitary wave is given by

$$\begin{aligned} \xi_B &= \sqrt{\frac{2}{\sqrt{3}}} \\ &= \mathcal{O}(1), \end{aligned} \tag{10.75}$$

since $\mathcal{O}\left(\frac{\mu^2}{\epsilon}\right) = 1$.

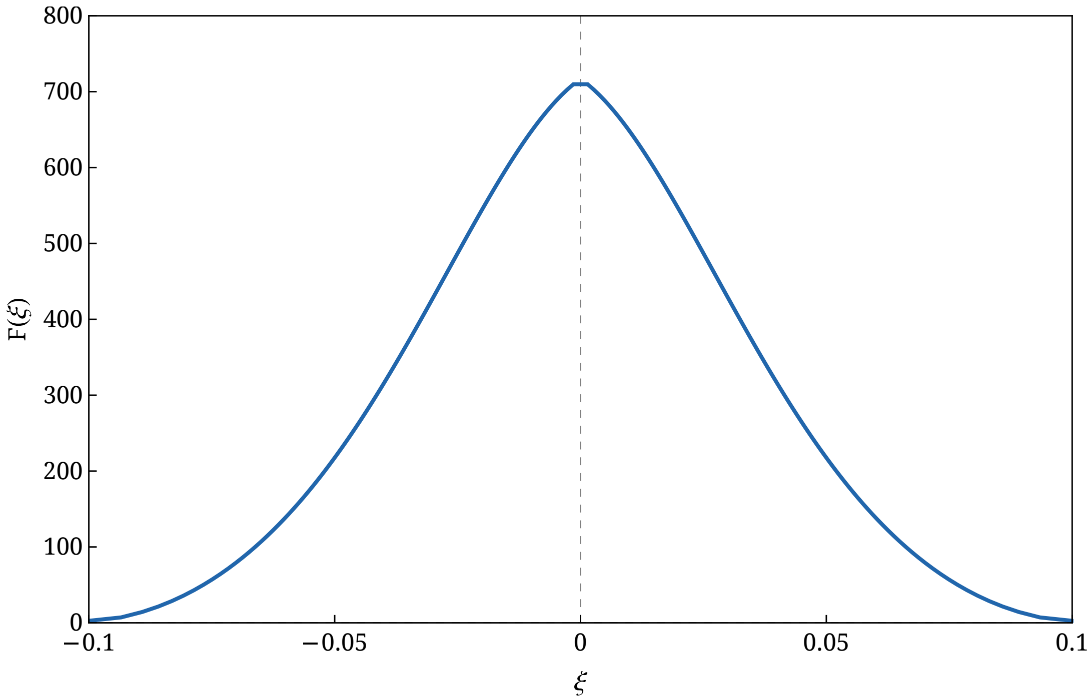

Figure 10.24: Fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ in the region $F > 0$ for $c^2 = \frac{1}{3}$ where $\epsilon = 0.005$ and $\mu = 0.06$.

The maximum fluid velocity in the horizontal direction of the solitary wave F_1 is given by $F = F_1 = \frac{3.684}{\epsilon}$. Figure 10.24 shows that the fluid velocity of the solitary wave in the horizontal direction has a unimodal excitation profile in the region $F > 0$. In this case, ξ is finite, therefore, the boundary conditions at negative and positive infinity are not satisfied. Thus, the case $c^2 = \frac{1}{3}$ does not form part of the solution set.

10.4.2 Properties of the surface elevation of the solitary wave $G(\xi)$

For completeness we consider the properties of the surface elevation of the solitary wave $G(\xi)$ analysed in Section 9.2.1.3. From (9.30), we have

$$G = -\frac{\left(\frac{4c}{\epsilon} + F\right)}{6c}. \tag{10.76}$$

This is the only case in which there is a linear relation between $G(\xi)$ and $F(\xi)$. When $F = 0$, then (10.76) becomes

$$G = -\frac{2}{3\epsilon}. \tag{10.77}$$

We note that $G < 0$ at $F = 0$. Thus, a depression solitary wave is formed. This is the only value of c^2 for which the condition $G = 0$ where $F = 0$ is not satisfied.

From Section 9.2.2.2, considering $\frac{d^2G}{d\xi^2}$ we can say that $G(\xi)$ is a minimum when $F(\xi)$ is a maximum. Considering $\frac{dG}{d\xi}$, we also know that $G(\xi)$ will have a turning point if and only if $F(\xi)$ has a turning point for $\xi = 0$ i.e. at the maximum fluid velocity in the horizontal direction of the solitary wave $F = F_1 = \frac{3.684}{\epsilon}$. For $0 \le F \le F_1$, the only turning point in $G(\xi)$ corresponds to the maximum fluid velocity in the horizontal direction of the solitary wave turning point $F = F_1 = \frac{3.684}{\epsilon}$ at $\xi = 0$ where $\frac{dF}{d\xi} = 0$. Thus, the turning point is given by

$$\begin{aligned}\frac{dG}{d\xi} &= -\frac{1}{6c}\frac{dF}{d\xi} \\ &= -\frac{1}{6c}\left(\frac{3.684}{\epsilon}\right) \\ &= -344, \end{aligned} \tag{10.78}$$

where $F_1 = \frac{3.684}{\epsilon}$ at $\xi = 0$ with $\epsilon = 0.005$.

We now consider the potential solitary wave that forms in the region $F > 0$. Figure 10.25 shows that the surface elevation of the solitary wave has a unimodal depression profile. In this case, ξ is finite, therefore, the boundary conditions at negative and positive infinity are not satisfied. Thus, the case $c^2 = \frac{1}{3}$ does not form part of the solution set.

Further investigation of the unimodal depression wave, in region $F > 0$, is needed to determine the condition for which these waves are above the seabed i.e. $G(\xi) > -\frac{1}{\epsilon}$.

From (10.68), we see that the minimum surface elevation of the solitary wave G_1 given by

$$G_1 = -\frac{1}{6c}\left(\frac{7}{10\epsilon c}\right) = -344 \not> -\frac{1}{\epsilon} = -200, \tag{10.79}$$

where with $\epsilon = 0.005$. We once again reach the conclusion that $c^2 = \frac{1}{3}$ is not a physically realistic solution for the surface elevation of the solitary wave, as the surface elevation is below the seabed.

10.4.2.1 Concluding remarks

This case does not produce physically realistic waves, as ξ is finite, and thus, does not satisfy the boundary conditions at negative and positive infinity. In addition to the above

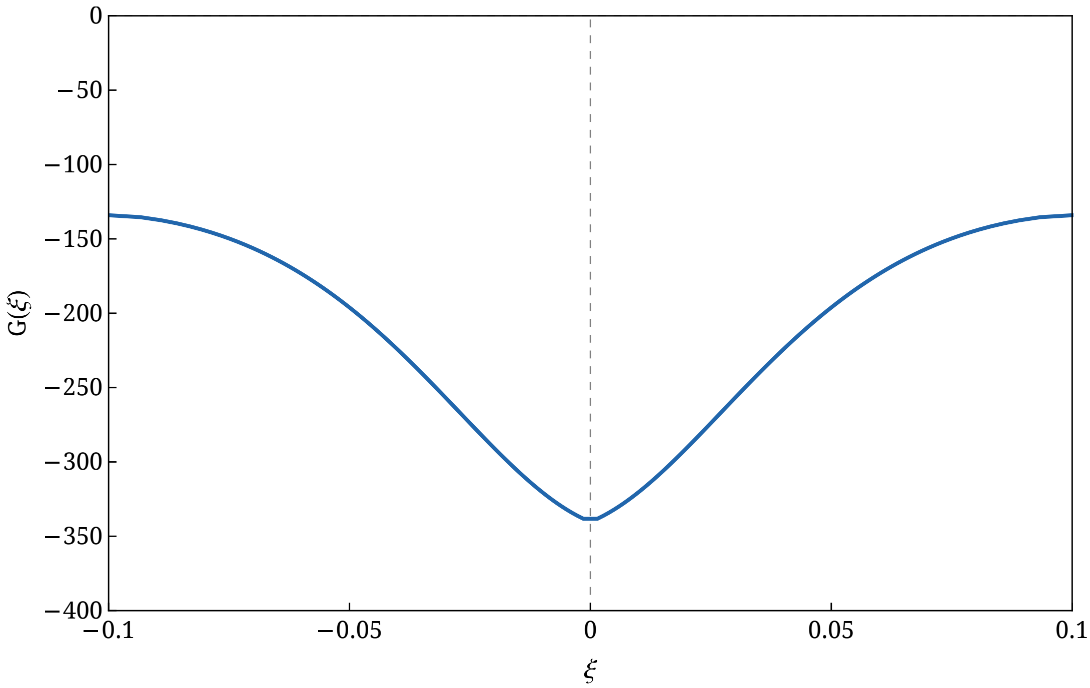

Figure 10.25: Surface elevation of the solitary wave $G(\xi)$ in the region $F > 0$ for $c^2 = \frac{1}{3}$ where $\epsilon = 0.005$ and $\mu = 0.06$.

reasoning, the surface elevation of the solitary wave was also discovered to be below the seabed for this case. Therefore, the case $c^2 = \frac{1}{3}$ does not form part of the solution set.

10.5 Case 5: $\frac{1}{3} < c^2 < 1$

10.5.1 Properties of the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ relating to the exact invariant solution

We recall from Section 9.1 that there is no region of physical significance for case $\frac{1}{3} < c^2 < 1$. Figure 10.26, the graph of $\left(\frac{dF}{d\xi}\right)^2$ versus $F(\xi)$, illustrates this finding. Thus, this case does not produce physically realistic waves and does not form part of the solution set.

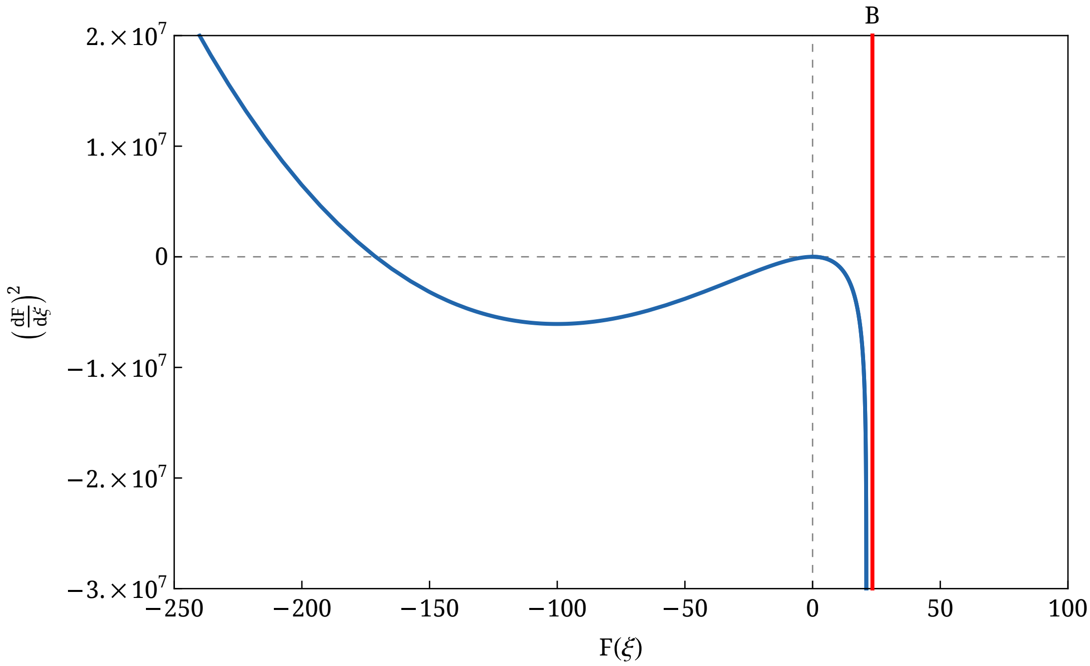

Figure 10.26: $\left(\frac{dF}{d\xi}\right)^2$ vs $F(\xi)$ at a constant wave speed $c^2 = \frac{1}{2.5}$ where $\epsilon = 0.005$ and $\mu = 0.06$.

10.6 Case 6: $c^2 = 1$

10.6.1 Properties of the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ relating to the exact invariant solution

We recall from Section 9.1 that there is no region of physical significance for case $c^2 = 1$, as a point of inflection occurs at $F = 0$, and therefore, there is no bounded region for $F < 0$ or $F > 0$. Figure 10.27, the graph of $\left(\frac{dF}{d\xi}\right)^2$ versus $F(\xi)$, illustrates this finding. Thus, this case does not produce physically realistic waves and does not form part of the solution set.

10.7 Case 7: $1 < c^2 < \infty$

10.7.1 Properties of the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ relating to the exact invariant solution

We recall from Section 9.1 that there is one region for case $\frac{1}{15} < c^2 < \frac{1}{3}$, namely $0 \le F \le F_1$, in which we potentially obtain a solitary wave. The shaded area in Figure 10.28, the graph

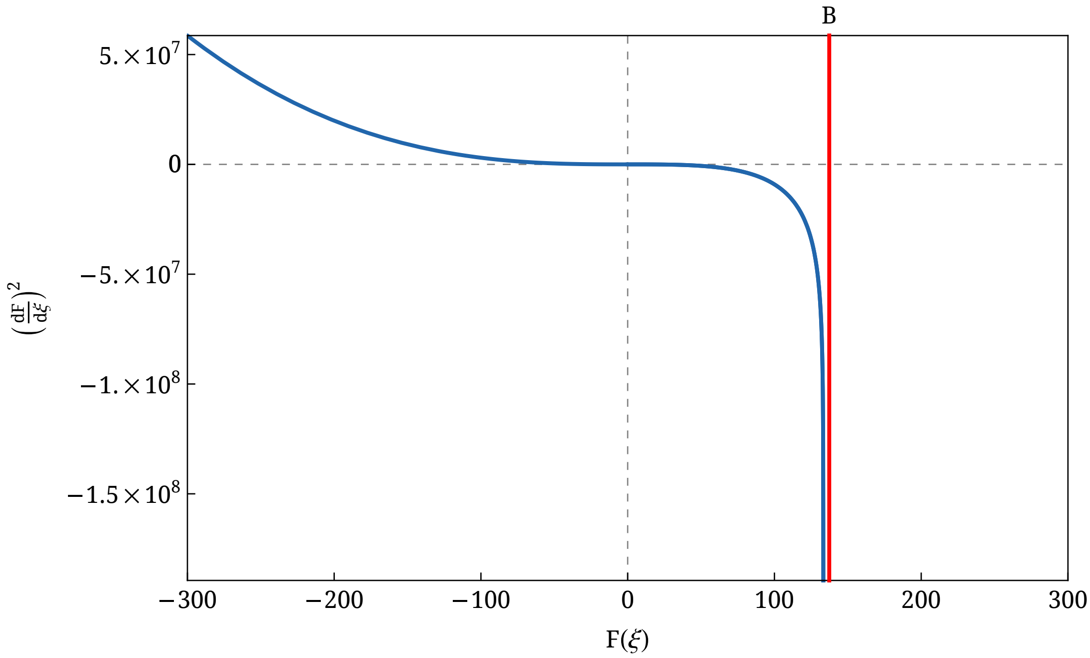

Figure 10.27: $\left(\frac{dF}{d\xi}\right)^2$ vs $F(\xi)$ at a constant wave speed $c^2 = 1$ where $\epsilon = 0.005$ and $\mu = 0.06$.

of $\left(\frac{dF}{d\xi}\right)^2$ versus $F(\xi)$, represents the region of physical significance.

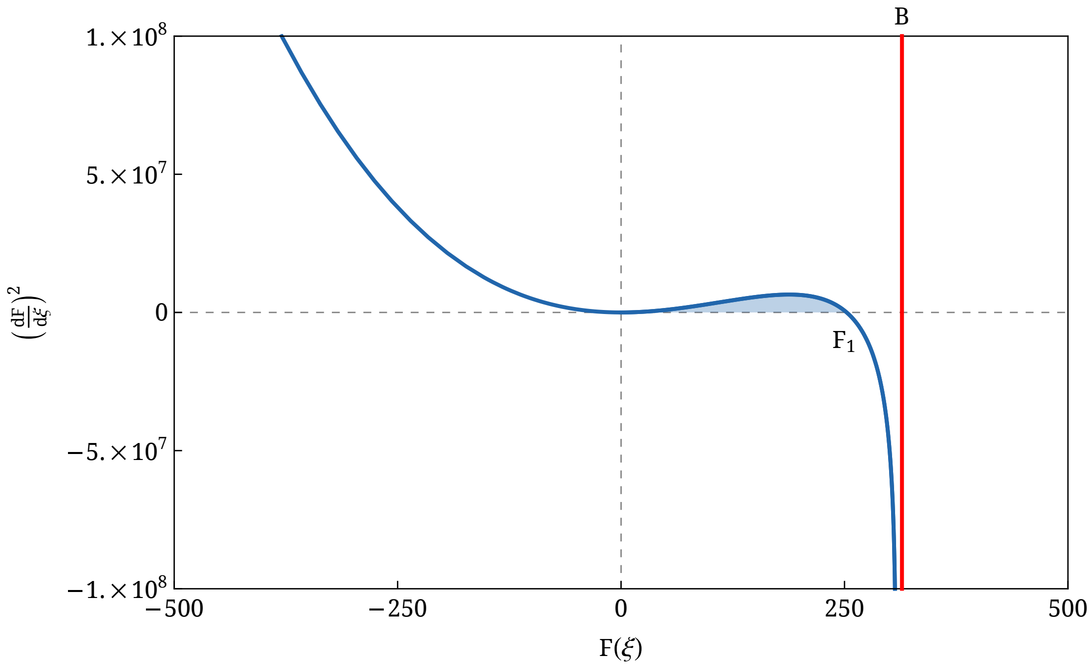

Figure 10.28: $\left(\frac{dF}{d\xi}\right)^2$ vs $F(\xi)$ at a constant wave speed $c^2 = 3$ where $\epsilon = 0.005$ and $\mu = 0.06$.

We now present plots of the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ against ξ for varying c^2 values in the range $1 < c^2 < \infty$, where ϵ and μ are constants.

An estimate of the width of the solitary wave is given by

$$\begin{aligned}
\xi_B &= \pm\sqrt{\frac{3c\mu^2}{2\epsilon}}\int_{\frac{F_1}{2}}^{F}\frac{dF}{\sqrt{P(F)}} \\
&= \mathcal{O}\left(\left(\pm\sqrt{\frac{3c\mu^2}{2\epsilon}}\right)\right) \\
&= \mathcal{O}\left(\left(\sqrt{c}\right)\right), \qquad (10.80)
\end{aligned}$$

since $\mathcal{O}\left(\frac{\mu^2}{\epsilon}\right) = 1$. The width of the wave ξ_B is directly proportional to the square root of the wave speed, $\sqrt{c}$.

Figure 10.29 shows that the fluid velocity of the solitary wave in the horizontal direction has a unimodal excitation profile. We note that the maximum fluid velocity of the solitary wave in the horizontal direction F_1 is directly proportional to the wave speed. Refer to Table E.4 for results that can be used as a comparison for experimental or numerical investigations.

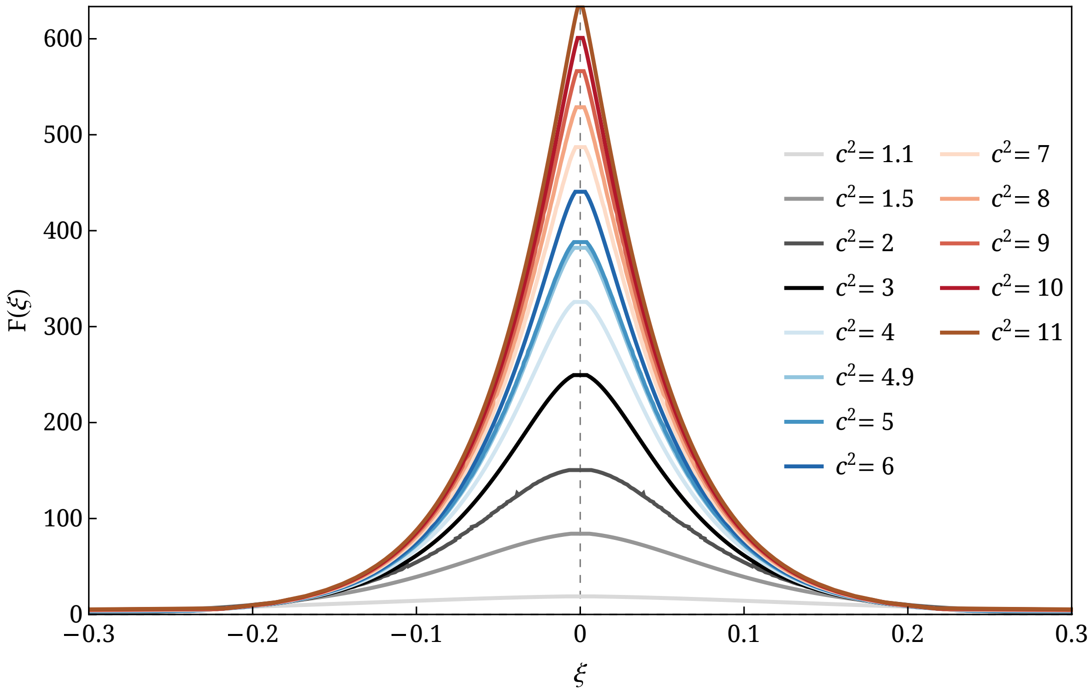

Figure 10.29: Fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ in the region $F > 0$ for varying wave speeds where $\epsilon = 0.005$ and $\mu = 0.06$.

10.7.2 Properties of the surface elevation of the solitary wave $G(\xi)$

Consider now the properties of the surface elevation of the solitary wave $G(\xi)$ analysed in Section 9.2.1.1. From (9.27),

$$G(F) = \frac{F\left(\frac{4c}{\epsilon} + F\right)}{6c\,(B - F)}, \text{ where } B = \frac{1}{\epsilon c}\left(c^2 - \frac{1}{3}\right). \tag{10.81}$$

The turning points determined by $\frac{dG}{d\xi}$ were fully analysed in Section 9.2.2.1. It was found that $\frac{dG}{dF}$ has two real roots, namely

$$F_3 = B + \frac{\sqrt{5\left(c^2 - \frac{1}{3}\right)\left(c^2 - \frac{1}{15}\right)}}{\epsilon c} \text{ and } F_4 = B - \frac{\sqrt{5\left(c^2 - \frac{1}{3}\right)\left(c^2 - \frac{1}{15}\right)}}{\epsilon c}, \tag{10.82}$$

where $F_3 > B$ and $F_4 < 0$ are outside the physical range. Thus for $0 \le F \le F_1$, the only turning point in $G(\xi)$ corresponds to the maximum fluid velocity in the horizontal direction of the solitary wave turning point $F = F_1$ at $\xi = 0$ where $\frac{dF}{d\xi} = 0$.

Figure 10.30 shows that the surface elevation of the solitary wave has a unimodal excitation profile. We note that the maximum surface elevation of the solitary wave G_1 is directly proportional to the wave speed. Refer to Table E.4 for results that can be used as a comparison for experimental or numerical investigations.

In conclusion, the case $1 < c^2 < \infty$ forms part of the solution set. All waves in this case will produce unimodal excitation waves. To our knowledge, these are new results.

10.7.3 Comparision of the results obtained using the approximate solutions $F^{An}(\xi)$ and $G^{An}(\xi)$ to the exact solutions $F(\xi)$ and $G(\xi)$ for $n = 1, 2, 3$

In this section, we derive the approximate solutions for the NB equations by making use of a pertubation method. We then compare results of the approximate solutions of the fluid velocity of the solitary wave in the horizontal direction $F^{An}(\xi)$ and surface elevation of the solitary wave $G^{An}(\xi)$ to that of the exact solutions of the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ and surface elevation of the solitary wave $G(\xi)$, all governed by the NB equations.

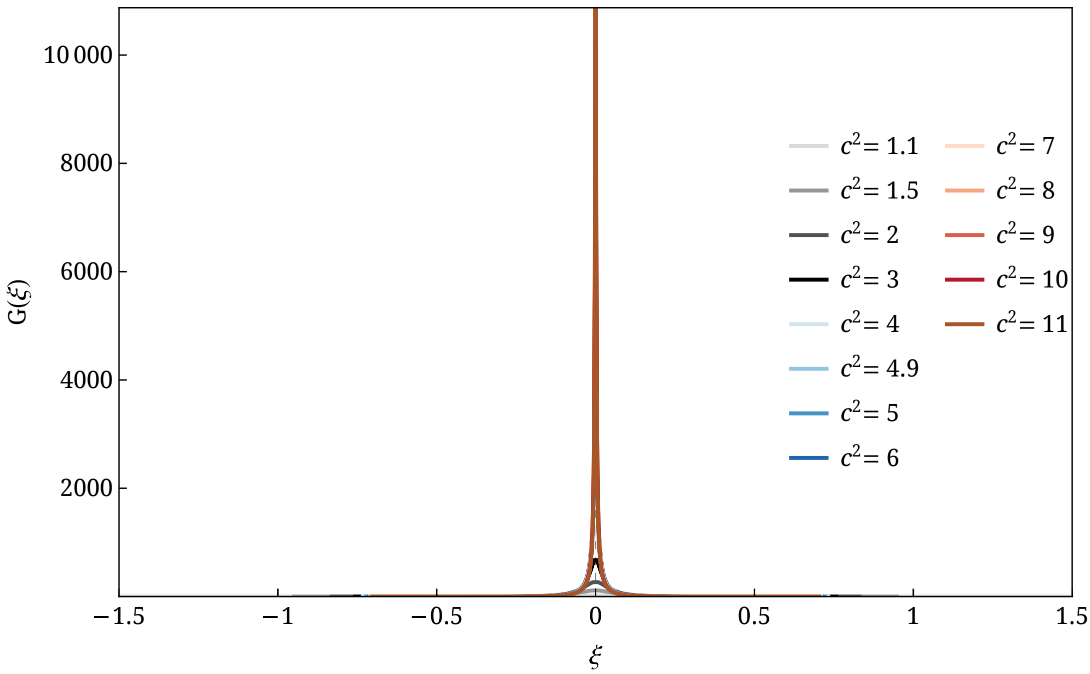

Figure 10.30: Surface elevation of the solitary wave $G(\xi)$ in the region $F > 0$ for varying wave speeds where $\epsilon = 0.005$ and $\mu = 0.06$.

10.7.3.1 Derivation of approximate solutions $F^{An}(\xi)$ and $G^{An}(\xi)$ for $n = 1,2,3$

We consider a straightforward Taylor series expansion to approximate the term $\ln(1+\epsilon)$ up to order $\mathcal{O}(3)$

$$\ln(1+\epsilon) = \epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{3} + \mathcal{O}(\epsilon^4). \tag{10.83}$$

A first, second and third-order approximation is used in this case, as this is the highest approximation that produces a positive bounded region when considering $\left(\frac{dF}{d\xi}\right)^2$ vs F. We investigate the approximation that best mimics the exact invariant solutions. We now consider the values of c^2 for which the approximate solution best compares to the exact solution.

Consider (8.46) given by

$$P(F) = -F^3 + \frac{3}{\epsilon c}\left(c^2 + \frac{1}{6}\right)F^2 - \frac{5}{\epsilon^2 c^2}\left(\frac{1}{15} - c^2\right)F + \frac{5}{\epsilon^3 c^3}\left(\frac{1}{3} - c^2\right)\left(\frac{1}{15} - c^2\right)\ln\left(\frac{F+A}{A}\right), \tag{10.84}$$

where

$$A = \frac{1}{\epsilon c}\left(\frac{1}{3} - c^2\right). \tag{10.85}$$

Let $B=-A=\dfrac{1}{\epsilon c}\left(c^2-\dfrac{1}{3}\right)$, resulting in the following expression

$$P(F)=-F^3+\frac{3}{\epsilon c}\left(c^2+\frac{1}{6}\right)F^2-\frac{5}{\epsilon^2c^2}\left(\frac{1}{15}-c^2\right)F+\frac{5}{\epsilon^3c^3}\left(\frac{1}{3}-c^2\right)\left(\frac{1}{15}-c^2\right)\ln\left(1-\frac{F}{B}\right). \quad (10.86)$$

Substituting (10.83) into (10.86), we obtain

$$P_3(F)=-F^3+\frac{3}{\epsilon c}\left(c^2+\frac{1}{6}\right)F^2-\frac{5}{\epsilon^2c^2}\left(\frac{1}{15}-c^2\right)F+\frac{5}{\epsilon^3c^3}\left(\frac{1}{3}-c^2\right)\left(\frac{1}{15}-c^2\right)\left[-\frac{F}{B}+\frac{1}{2}\left(\frac{F}{B}\right)^2-\frac{1}{3}\left(\frac{F}{B}\right)^3\right]. \quad (10.87)$$

Equating terms of the same order of F in (10.87) results in the following systems of equations

$$F:-\frac{5}{\epsilon^2c^2}\left(\frac{1}{15}-c^2\right)+\frac{5}{\epsilon^3c^3}\left(\frac{1}{3}-c^2\right)\left(\frac{1}{15}-c^2\right)\left[-\frac{\epsilon c}{c^2-\frac{1}{3}}\right]=0, \quad (10.88)$$

$$\begin{aligned}F^2:&\frac{3}{\epsilon c}\left(c^2+\frac{1}{6}\right)+\frac{5}{\epsilon^3c^3}\left(\frac{1}{3}-c^2\right)\left(\frac{1}{15}-c^2\right)\left[\frac{1}{2}\left(\frac{\epsilon c}{c^2-\frac{1}{3}}\right)^2\right]\\ &=\frac{3}{\epsilon c}\left(c^2+\frac{1}{6}\right)+\frac{5}{2\epsilon c}\frac{\left(c^2-\frac{1}{15}\right)}{\left(c^2-\frac{1}{3}\right)},\end{aligned} \quad (10.89)$$

$$F^3:-1+\frac{5}{\epsilon^3c^3}\left(\frac{1}{3}-c^2\right)\left(\frac{1}{15}-c^2\right)\left[-\frac{1}{3}\left(\frac{1}{B}\right)^3\right]=\frac{-c^2\left(1+c^2\right)}{\left(c^2-\frac{1}{3}\right)^2}. \quad (10.90)$$

Substituting (10.88) into (10.87) results in the following first-order approximation

$$\begin{aligned}P_1(F)&=-F^3+\frac{3}{\epsilon c}\left(c^2+\frac{1}{6}\right)F^2\\ \Rightarrow P_1(F)&=\lambda_1F^2\left[F_{1,1}-F\right],\end{aligned} \quad (10.91)$$

where $F_{1,1}=\dfrac{3}{\epsilon c}\left(c^2+\frac{1}{6}\right)$ and $\lambda_1=1$.

Substituting (10.88) and (10.89) into (10.87) results in the following second-order approximation

$$\begin{aligned}P_2(F)&=-F^3+\left[\frac{3}{\epsilon c}\left(c^2+\frac{1}{6}\right)+\frac{5}{2\epsilon c}\frac{\left(c^2-\frac{1}{15}\right)}{\left(c^2-\frac{1}{3}\right)}\right]F^2\\ \Rightarrow P_2(F)&=\lambda_2F^2\left[F_{2,1}-F\right],\end{aligned} \quad (10.92)$$

where $F_{2,1}=\dfrac{3c^4+2c^2-\frac{1}{3}}{c\epsilon\left(c^2-\frac{1}{3}\right)}$ and $\lambda_2=1$.

Substituting (10.88)-(10.90) into (10.87) results in the following third-order approximation

$$P_3(F)=\frac{-c^2\left(1+c^2\right)}{\left(c^2-\frac{1}{3}\right)^2}F^3+\left[\frac{3}{\epsilon c}\left(c^2+\frac{1}{6}\right)+\frac{5}{2\epsilon c}\frac{\left(c^2-\frac{1}{15}\right)}{\left(c^2-\frac{1}{3}\right)}\right]F^2$$

$$\Rightarrow P_3(F) = F^2 \frac{c^2\left(1+c^2\right)}{\left(c^2-\frac{1}{3}\right)^2}\left[F_{3,1}-F\right], \tag{10.93}$$

where $F_{3,1} = \dfrac{\left(3c^4+2c^2-\frac{1}{3}\right)\left(c^2-\frac{1}{3}\right)}{\epsilon c^3(1+c^2)}$ and $\lambda_3 = \dfrac{c^2\left(1+c^2\right)}{\left(c^2-\frac{1}{3}\right)^2}$.

Equations (10.91)-(10.93) are all of the form

$$P_n(F) = \lambda_n F^2\left[F_{n,1}-F\right], \tag{10.94}$$

for $n = 1, 2, 3$.

Again, ξ takes the following form

$$\xi = \pm\sqrt{\frac{3c\mu^2}{2\epsilon}}\int_0^{F_{n,1}} \frac{dF}{\sqrt{P_n(F)}}, \tag{10.95}$$

for $n = 1, 2, 3$.

Substituting (10.94) into (10.95) we get

$$\xi = \pm\sqrt{\frac{3c\mu^2}{2\epsilon\lambda_n}}\int_0^{F_{n,1}} \frac{dF}{F^{An}\sqrt{F_{n,1}-F^{An}}}. \tag{10.96}$$

Let

$$F^{An} = F_{n,1}\mathrm{sech}^2\theta = \frac{F_{n,1}}{\cosh^2\theta}. \tag{10.97}$$

Differentiating (10.97) with respect to θ gives

$$dF = -2F_{n,1}\frac{\sinh\theta}{\cosh^3\theta}d\theta. \tag{10.98}$$

Substituting (10.97) and (10.98) into (10.96) and writing the result in terms of θ we get

$$\xi = \pm\sqrt{\frac{3c\mu^2}{2\epsilon\lambda_n}}\int_0^{\theta_{n,1}}\left(-2F_{n,1}\frac{\sinh\theta}{\cosh^3\theta}\right)\left(\frac{1}{F_{n,1}\mathrm{sech}^2\theta}\right)\left(\frac{1}{\sqrt{F_{n,1}-F_{n,1}\mathrm{sech}^2\theta}}\right)d\theta$$

$$\Rightarrow \xi = \pm\sqrt{\frac{3c\mu^2}{2\epsilon\lambda_n}}\int_0^{\theta_{n,1}}\left(-2F_{n,1}\frac{\sinh\theta}{\cosh^3\theta}\right)\left(\frac{1}{F_{n,1}\mathrm{sech}^2\theta}\right)\left(\frac{1}{\sqrt{F_{n,1}}\tanh\theta}\right)d\theta$$

$$\Rightarrow \xi = \pm\sqrt{\frac{6c\mu^2}{\epsilon\lambda_n F_{n,1}}}\int_0^{\theta_{n,1}} d\theta$$

$$\Rightarrow \xi = \pm\sqrt{\frac{6c\mu^2}{\epsilon\lambda_n F_{n,1}}}\theta + k. \tag{10.99}$$

Solving for θ gives

$$\theta = \pm\sqrt{\frac{\epsilon\lambda_n F_{n,1}}{6c\mu^2}}\xi + k. \tag{10.100}$$

Since cosh is an even function and $F^{An} = F_{n,1}$ at $\xi = 0$, we deduce that $k = 0$, and therefore, (10.97) simplifies to give the following approximate solution for the fluid velocity of the wave

$$F^{An} = F_{n,1}\mathrm{sech}^2\theta = F_{n,1}\mathrm{sech}^2\left[\left(\sqrt{\frac{\epsilon\lambda_n F_{n,1}}{6c\mu^2}}\right)\xi\right], \tag{10.101}$$

giving

$$\begin{aligned}
F^{A1}(\xi) &= \frac{3}{\epsilon c}\left(c^2+\frac{1}{6}\right)\mathrm{sech}^2\left[\left(\sqrt{\frac{c^2+\frac{1}{6}}{2c^2\mu^2}}\right)\xi\right],\\
F^{A2}(\xi) &= \frac{3c^4+2c^2-\frac{1}{3}}{c\epsilon\left(c^2-\frac{1}{3}\right)}\mathrm{sech}^2\left[\left(\sqrt{\frac{3c^4+2c^2-\frac{1}{3}}{6c^2\mu^2\left(c^2-\frac{1}{3}\right)}}\right)\xi\right],\\
F^{A3}(\xi) &= \frac{\left(3c^4+2c^2-\frac{1}{3}\right)\left(c^2-\frac{1}{3}\right)}{\epsilon c^3(1+c^2)}\mathrm{sech}^2\left[\left(\sqrt{\frac{3c^4+2c^2-\frac{1}{3}}{6c^2\mu^2\left(c^2-\frac{1}{3}\right)}}\right)\xi\right].
\end{aligned} \tag{10.102}$$

By substituting (10.101) into the exact solution for $G(\xi)$ we obtain the approximate solution for the surface elevation of the solitary wave given by

$$G^{An} = \frac{F^{An}}{6c}\left(\frac{\frac{4c}{\epsilon}+F^{An}}{B-F^{An}}\right), \tag{10.103}$$

where $B = \dfrac{1}{\epsilon c}\left(c^2-\frac{1}{3}\right)$ and $n = 1,2,3$, for $1 < c^2 < \infty$.

10.7.3.2 Comparision of the results obtained using the approximate solutions $F^{An}(\xi)$ and $G^{An}(\xi)$ to the exact solutions $F(\xi)$ and $G(\xi)$ for $n = 1,2,3$

We now analyse the results obtained from the approximate solutions $F^{An}(\xi)$ and $G^{An}(\xi)$. From Figure 10.31, it is clear that the second order approximate profile of $\left(\frac{dF}{d\xi}\right)^2$ for relatively small c^2 values is of a similar form as the exact profile. The shaded areas represent the regions of physical significance.

Now consider Figure 10.32 where the shaded areas represent the regions of physical significance. The second-order approximate profile of $\left(\frac{dF}{d\xi}\right)^2$ for relatively large c^2 values is very similar to the exact profile. In fact, the second-order approximation for F_1 is the most

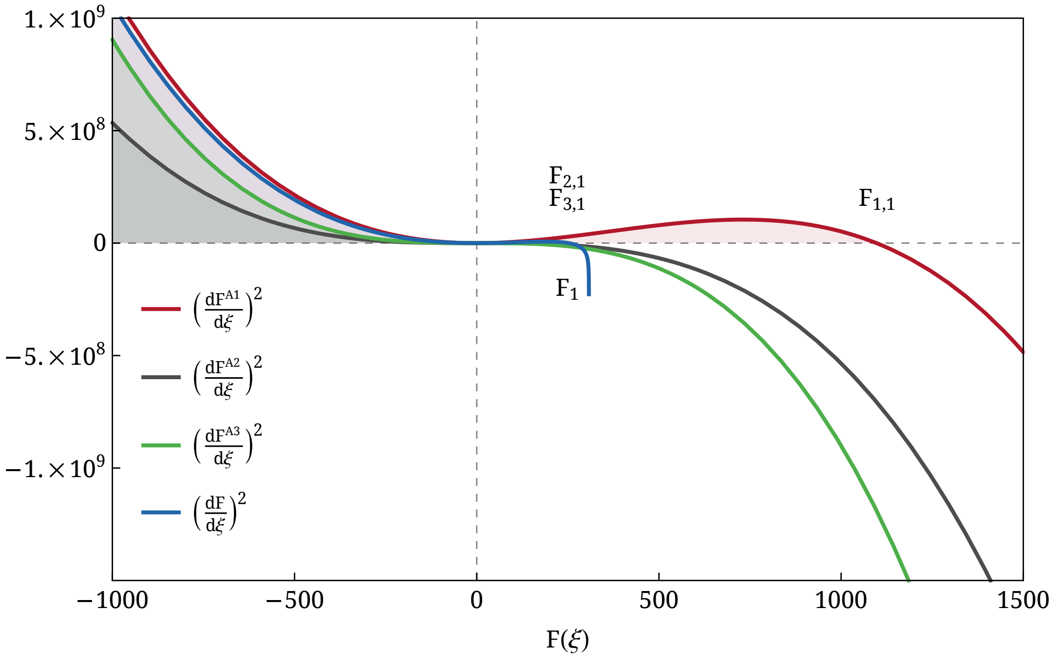

Figure 10.31: $\left(\frac{dF}{d\xi}\right)^2$, $\left(\frac{dF^{A3}}{d\xi}\right)^2$, $\left(\frac{dF^{A2}}{d\xi}\right)^2$, $\left(\frac{dF^{A1}}{d\xi}\right)^2$ vs $F(\xi)$ at a constant wave speed $c^2 = 3$ where $\epsilon = 0.005$ and $\mu = 0.06$.

accurate approximation. Thus, we use a second-order approximation for the rest of this analysis. Note the second-order approximate solution is a more accurate representation of the exact solution for smaller values of c^2.

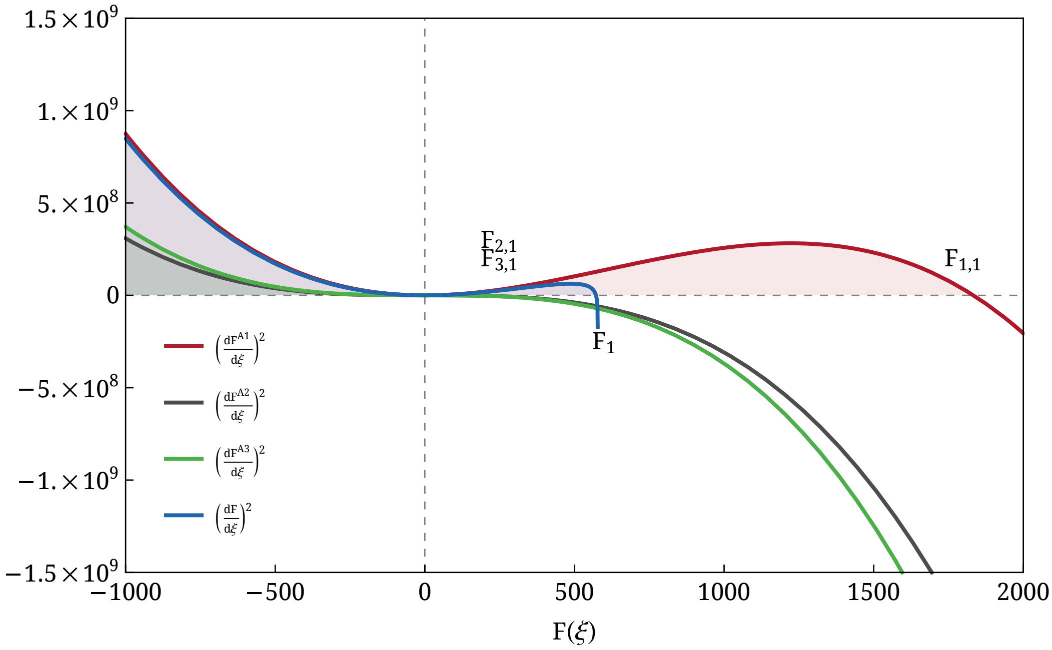

Figure 10.32: $\left(\frac{dF}{d\xi}\right)^2$, $\left(\frac{dF^{A3}}{d\xi}\right)^2$, $\left(\frac{dF^{A2}}{d\xi}\right)^2$, $\left(\frac{dF^{A1}}{d\xi}\right)^2$ vs $F(\xi)$ at a constant wave speed $c^2 = 9$ where $\epsilon = 0.005$ and $\mu = 0.06$.

Consider the behaviour of the second-order approximate solution when $F > 0$. The unimodal excitation fluid velocity profile of the approximate solution $F^{A2}(\xi)$, given by Figure 10.33, is identical to that of the exact solution $F(\xi)$ given by Figure 10.5. We note that the relationship between the wave speed and the maximum fluid velocity of the solitary wave $F_{2,1}$ is directly proportional. This is identical to that of the exact solution. Thus, qualitatively, the approximate solution is an accurate representation of the exact solution. However, the approximate solution grossly underestimates the amplitude of the fluid velocity of the solitary wave in the horizontal direction. Thus, quantitatively, the approximate solution is an inaccurate representation of the exact solution.

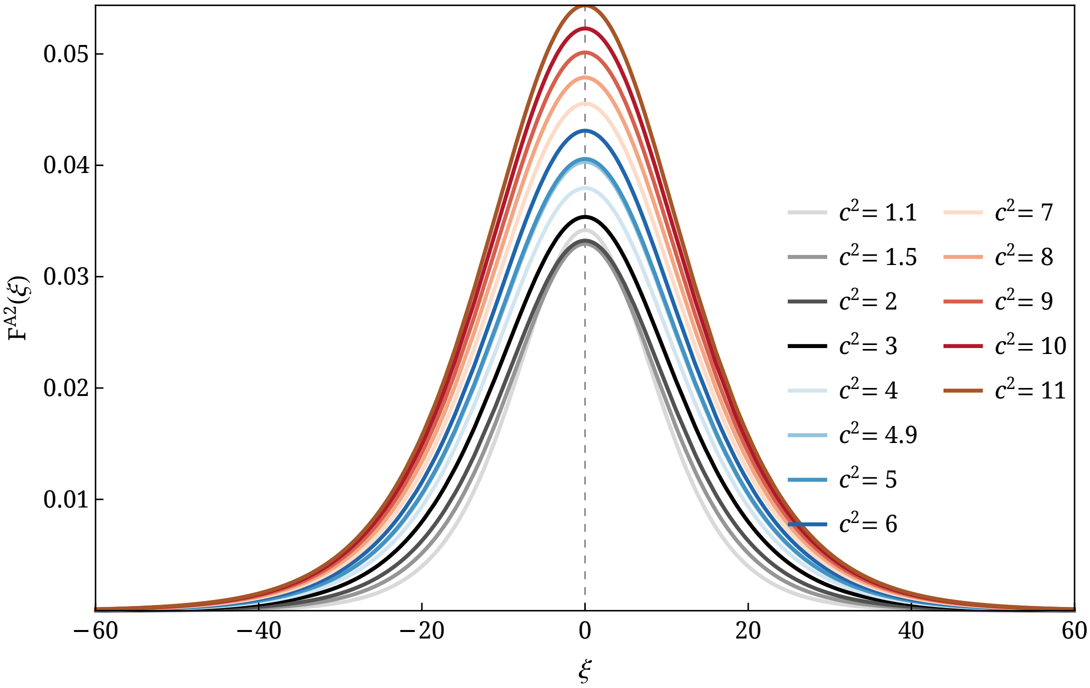

Figure 10.33: Fluid velocity of the solitary wave in the horizontal direction $F^{A2}(\xi)$ in the region $F > 0$, using a second-order approximation, for varying wave speeds where $\epsilon = 0.005$ and $\mu = 0.06$.

We now consider the approximate solution for the surface elevation of the solitary wave, $G^{A2}(\xi)$, given by Figure 10.34, in comparison to the exact solution given by Figure 10.30. Although the profile of the approximate and exact solutions are of a unimodal excited form, the exact solution has a steepened profile whilst the approximate solution has a more gradual profile. The approximate solution also underestimates the amplitude of the surface elevation of the solitary wave. We note that the relationship between the wave speed and the maximum surface elevation of the solitary wave G_{12} is inversely proportional, but when considering the exact solution, the same relationship is directly proportional.

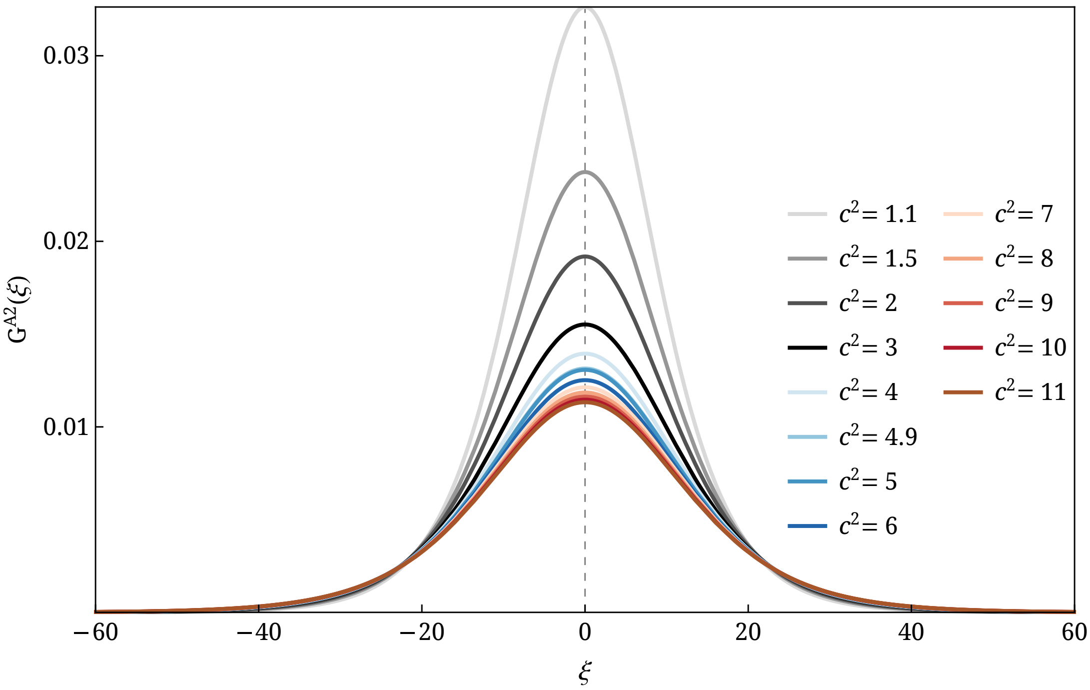

Figure 10.34: Surface elevation of the solitary wave $G^{A2}(\xi)$ in the region $F > 0$, using a second-order approximation, for varying wave speeds where $\epsilon = 0.005$ and $\mu = 0.06$.

10.7.4 Concluding remarks

The case $1 < c^2 < \infty$ forms part of the solution set. All waves produced in this case are unimodal excitation waves. To our knowledge, these are newly discovered solutions and results that can be used to enrich the analytical validation of numerical models for tsunami detection systems. The second-order approximate solution for the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ is accurate, qualitatively, but inaccurate quantitatively. The second-order approximate solution for the surface elevation of the solitary wave $G(\xi)$ is qualitatively and quantitatively inaccurate.

Chapter 11

Conclusion

The aim of this thesis is to contribute to the known solutions of the NB equations, which effectively describe solitary wave motion in shallow water (typically used to describe tsunami wave motion), using a systematic symmetry approach. We have been successful in achieving our aim. The results we have obtained enrich the analytical validation of numerical models for tsunami detection systems.

The (1+1)D Boussinesq equations investigated in this thesis are considered for the equations' ability to model tsunamis via solitary wave motion. Three models are investigated, namely the (1+1)D Boussinesq equations for a moving bathymetry, the LB equations at a constant water depth and the NB equations at a constant water depth.

11.1 LB equations at a constant water depth

Conservation laws and physical conserved quantities for the LB equations at a constant water depth are presented. The equations produced six conserved quantities. Conserved mass, energy and momentum quantities are found in a more systematic way than that employed in [6].

In addition to these conserved quantities, we are able to obtain two, as yet, unobserved conserved quantities, namely a conserved mass flow rate and a conserved energy flux density. Through the work we have undertaken, we have also been able to observe a conserved quantity which we, as yet, cannot relate to any known physical quantities.

The complete Lie point symmetries for the LB equations at a constant water depth are obtained and used in an attempt to determine the invariant solutions in terms of the fluid velocity of the solitary wave in the horizontal direction and surface elevation of a solitary wave. A linear combination of the time and spatial symmetries leads to the invariant travelling wave solutions. The travelling wave solutions do not satisfy the lateral boundary conditions and, therefore, are not physically realistic solutions. We discover that linearizing the NB equations does not lead to physically realistic solutions. Lastly, the associated Lie point symmetry is derived and we deduce that all the conserved quantities are associated to the travelling wave solutions, which do not satisfy the lateral boundary conditions. The conserved quantities are not needed to solve for the constants in an attempt to obtain the linear solutions.

The only other attempt at solving these equations analytically can be seen in Dutykh and Dias [2], where they use the analytical solutions found to validate their numerical computations. The solutions they obtain are not physically realistic, as the fluid velocity is negative where it is expected to be positive for a solitary wave propagating from right to left.

11.2 NB equations at a constant water depth

Conservation laws and physical conserved quantities for the NB equations at a constant water depth are presented. The equations produce two physical conserved quantities, namely the conservation of mass flow rate and the conservation of mass.

The complete Lie point symmetries for the NB equations at a constant water depth are obtained and used to determine the invariant solutions in terms of the fluid velocity of the solitary wave in the horizontal direction and surface elevation of a solitary wave. We derive the associated Lie point symmetry and deduce that all conserved vectors are associated with the linear combination of the time and spatial symmetries.

A linear combination of the time and spatial symmetries leads to the invariant exact travelling wave solutions. We obtain solutions for three cases of the NB equations dependent on the wave speed c^2, namely (i) $0 < c^2 < \infty$, $c^2 \neq \frac{1}{15}$ and $c^2 \neq \frac{1}{3}$; (ii) $c^2 = \frac{1}{15}$ and (iii) $c^2 = \frac{1}{3}$.

To our knowledge, the invariant exact travelling wave solutions for the NB equations for cases (i) and (iii) are new, whilst case (ii) is identical to that displayed in the work of Dutykh and Dias [2]. Lastly, we provide physically realistic explanations of the results relating to the horizontal component of the fluid velocity and surface elevation of the solitary wave for the invariant exact solutions. We present and discuss the approximate solutions for cases $0 < c^2 < \frac{1}{15}$, $\frac{1}{15} < c^2 < \frac{1}{3}$ and $1 < c^2 < \infty$ for the NB equations by making use of a perturbation method. To our knowledge, these are newly discovered solutions and results.

The case $0 < c^2 < \frac{1}{15}$ forms physically realistic waves where $c^2 = \frac{1}{16}$, $c^2 = \frac{1}{15.5}$ and $c^2 = \frac{1}{15.1}$. To our knowledge, these are new results and two new wave forms are observed, namely a bimodal depression wave and a bimodal transition wave. The fourth-order approximate solutions are qualitatively and quantitatively inaccurate.

The special case, $c^2 = \frac{1}{15}$, does not produce physically realistic waves, and therefore, does not form part of the solution set. To our knowledge, these are new results. Readers can refer to the work done by Dutykh and Dias [2], who perform alternative analysis on the same set of solutions and for this specific case.

The case $1 < c^2 < \infty$ forms part of the solution set. All waves produced in this case are unimodal excitation waves. To our knowledge, these are new results. The second-order approximate solution for the fluid velocity of the solitary wave in the horizontal direction $F(\xi)$ is accurate, qualitatively, but inaccurate quantitatively. The second-order approximate solution for the surface elevation of the solitary wave $G(\xi)$ is qualitatively and quantitatively inaccurate.

Appendices

Appendix A

Multipliers and conserved vectors for the LB equations

This appendix contains detailed calculations used to determine the multipliers and conserved vectors for the LB equations by making use of the Multiplier approach.

We recall the system of dimensional LB equations given by

$$\frac{\partial \eta}{\partial t} + h\frac{\partial u}{\partial x} = \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}, \tag{A.1}$$

$$\frac{\partial u}{\partial t} + g\frac{\partial \eta}{\partial x} = \frac{h^2}{2}\frac{\partial^3 u}{\partial t \partial x^2}, \tag{A.2}$$

where h is constant water depth, g is gravitational acceleration, u is the fluid velocity of the solitary wave in the horizontal direction in the x-direction and η is the surface elevation of the solitary wave.

A.1 Conservation laws considering multipliers of the form $\Lambda_{1,2}(t,x,u,\eta)$

In order to derive the conservation laws for the system given by (4.1) and (4.2), we consider multipliers $\Lambda_1(t,x,u,\eta)$ and $\Lambda_2(t,x,u,\eta)$. By properties of multipliers we have

$$\Lambda_1(\eta_t + hu_x - \frac{h^3}{6}u_{xxx}) + \Lambda_2(u_t + g\eta_x - \frac{h^2}{2}u_{xxt}) = D_t T^1 + D_x T^2, \tag{A.3}$$

where $T=(T^1,T^2)$ are the components of the conserved vector and the total derivative operators D_t and D_x are defined by

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + \eta_t\frac{\partial}{\partial \eta} + u_{tt}\frac{\partial}{\partial u_t} + \eta_{tt}\frac{\partial}{\partial \eta_t} + u_{tx}\frac{\partial}{\partial u_x} + \eta_{tx}\frac{\partial}{\partial \eta_x} + \ldots, \tag{A.4}$$

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + \eta_x\frac{\partial}{\partial \eta} + u_{xx}\frac{\partial}{\partial u_x} + \eta_{xx}\frac{\partial}{\partial \eta_x} + u_{xt}\frac{\partial}{\partial u_t} + \eta_{xt}\frac{\partial}{\partial \eta_t} + \ldots. \tag{A.5}$$

In order to derive the determining equations we make use of the Euler operator to annihilate the divergence of the conservation law.

The determining equations can be found by expanding

$$E_u[\Lambda_1(\eta_t + hu_x - \frac{h^3}{6}u_{xxx}) + \Lambda_2(u_t + g\eta_x = \frac{h^2}{2}u_{xxt})], \tag{A.6}$$

$$E_\eta[\Lambda_1(\eta_t + hu_x - \frac{h^3}{6}u_{xxx}) + \Lambda_2(u_t + g\eta_x = \frac{h^2}{2}u_{xxt})], \tag{A.7}$$

where the Euler operator is defined as

$$E_u = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_t^2\frac{\partial}{\partial u_{tt}} + D_x^2\frac{\partial}{\partial u_{xx}} + D_tD_x\frac{\partial}{\partial u_{tx}} + \ldots, \tag{A.8}$$

$$E_\eta = \frac{\partial}{\partial \eta} - D_t\frac{\partial}{\partial \eta_t} - D_x\frac{\partial}{\partial \eta_x} + D_t^2\frac{\partial}{\partial \eta_{tt}} + D_x^2\frac{\partial}{\partial \eta_{xx}} + D_tD_x\frac{\partial}{\partial \eta_{tx}} + \ldots. \tag{A.9}$$

By expanding equation (A.7) we obtain the following determining equation

$$\begin{aligned}&\frac{\partial\Lambda_1}{\partial\eta}\left(\eta_t + hu_x - \frac{h^3}{6}u_{xxx}\right) + \frac{\partial\Lambda_2}{\partial\eta}\left(u_t + g\eta_x - \frac{h^2}{2}u_{xxt}\right)\\ &-\frac{\partial\Lambda_1}{\partial t} - u_t\frac{\partial\Lambda_1}{\partial u} - \eta_t\frac{\partial\Lambda_1}{\partial\eta} - g\frac{\partial\Lambda_2}{\partial x} - gu_x\frac{\partial\Lambda_2}{\partial u} - g\eta_x\frac{\partial\Lambda_2}{\partial\eta} = 0.\end{aligned} \tag{A.10}$$

We split equation (A.10) by the following third order derivatives of u

$$u_{xxx}: \frac{h^3}{6}\frac{\partial\Lambda_1}{\partial\eta} = 0, \tag{A.11}$$

$$u_{xxt}: \frac{h^2}{2}\frac{\partial\Lambda_2}{\partial\eta} = 0. \tag{A.12}$$

From (A.11) and (A.12) we note that Λ_1 and Λ_2 are independent of η.

Next, we separate by the following first order derivatives of u

$$u_x: g\frac{\partial\Lambda_2}{\partial u} = 0, \tag{A.13}$$

$$u_t: \frac{\partial\Lambda_1}{\partial u} = 0. \tag{A.14}$$

From (A.13) and (A.14), we see that Λ_1 and Λ_2 are independent of u. We are left with the following remainder

$$1:\frac{\partial\Lambda_1}{\partial t}+g\frac{\partial\Lambda_2}{\partial x}=0. \tag{A.15}$$

Now by expanding (A.6) and substituting $\Lambda_1=\Lambda_1(t,x)$ and $\Lambda_2=\Lambda_2(t,x)$ we obtain the following determining equation

$$-\frac{\partial\Lambda_2}{\partial t}-h\frac{\partial\Lambda_1}{\partial x}+\frac{h^2}{2}\frac{\partial^3\Lambda_2}{\partial x^2\partial t}+\frac{h^3}{6}\frac{\partial^3\Lambda_1}{\partial x^3}=0. \tag{A.16}$$

Thus, we are left with (A.15) and (A.16). Consider $\Lambda_1=\Lambda_1(x)$ and $\Lambda_2=\Lambda_2(t)$ then (A.15) is identically satisfied and (A.16) becomes

$$-\frac{\partial\Lambda_2}{\partial t}-h\frac{\partial\Lambda_1}{\partial x}+\frac{h^3}{6}\frac{\partial^3\Lambda_1}{\partial x^3}=0. \tag{A.17}$$

By separation of variables, each side is a constant. Thus

$$\frac{h^3}{6}\frac{\partial^3\Lambda_1}{\partial x^3}-h\frac{\partial\Lambda_1}{\partial x}=\frac{\partial\Lambda_2}{\partial t}=a_1. \tag{A.18}$$

Hence

$$\Lambda_2(t)=a_1t+a_2, \tag{A.19}$$

and we are left with an ordinary differential equation

$$\frac{\partial^3\Lambda_1}{\partial x^3}-\frac{6}{h^2}\frac{\partial\Lambda_1}{\partial x}=\frac{6}{h^3}a_1. \tag{A.20}$$

We solve (A.20) by considering a complementary and particular solution as follows. Let

$$\frac{\partial\Lambda_1}{\partial x}=w, \tag{A.21}$$

then (A.20) becomes

$$\frac{\partial^2 w}{\partial x^2}-\frac{6}{h^2}w=\frac{6}{h^3}a_1. \tag{A.22}$$

Consider a complementary solution of the form $w_c=e^{\lambda x}$. Substituting this solution, as well as its derivatives, into (A.22) and equating the right hand side of the equation to zero gives

$$e^{\lambda x}(\lambda^2-\frac{6}{h^2})=0. \tag{A.23}$$

Thus, $\lambda = \pm\frac{\sqrt{6}}{h}$ and hence the complementary solution is given by

$$\Lambda_1(x) = c_3 e^{\frac{\sqrt{6}}{h}x} + c_4 e^{-\frac{\sqrt{6}}{h}x} + c_5. \tag{A.24}$$

Now, consider a particular solution of the form $w_p = P$ where P denotes a constant. Substituting this solution, as well as its derivatives, into (A.22) gives

$$-\frac{6P}{h^2} = \frac{6}{h^3}a_1. \tag{A.25}$$

Thus, the particular solution is given by

$$\Lambda_1(x) = -\frac{a_1}{h}x + a_6. \tag{A.26}$$

Combining (A.24) and (A.26) gives the general solution

$$\Lambda_1(x) = a_3 e^{\frac{\sqrt{6}}{h}x} + a_4 e^{-\frac{\sqrt{6}}{h}x} - \frac{a_1}{h}x + a_7. \tag{A.27}$$

The set of multipliers of the form $\Lambda_{1,2}(t,x,u,\eta)$ are thus given by

$$\Lambda_1(x) = a_3 e^{\frac{\sqrt{6}}{h}x} + a_4 e^{-\frac{\sqrt{6}}{h}x} - \frac{a_1}{h}x + a_7, \tag{A.28}$$

$$\Lambda_2(t) = a_1 t + a_2. \tag{A.29}$$

The conserved vectors can now be obtained by using the Multiplier method.

Case 1: $a_2 = 1, a_1 = 0, a_7 = 0, a_3 = 0, a_4 = 0$

$$\Lambda_1(x) = 0, \tag{A.30}$$

$$\Lambda_2(t) = 1. \tag{A.31}$$

By properties of multipliers the following must be satisfied

$$\begin{aligned}&\Lambda_1\left(\eta_t + h u_x - \frac{h^3}{6}u_{xxx}\right) + \Lambda_2\left(u_t + g\eta_x - \frac{h^2}{2}u_{xxt}\right) = D_t T^1 + D_x T^2\\ &\Rightarrow u_t + g\eta_x - \frac{h^2}{2}u_{xxt} = D_t T^1 + D_x T^2.\end{aligned} \tag{A.32}$$

When $u(t,x)$ and $\eta(t,x)$ are solutions of the system of equations (A.1) and (A.2), we have

$$D_t[u] + D_x\left[g\eta - \frac{h^2}{2}u_{xt}\right] = 0. \tag{A.33}$$

Hence the elementary conserved vector $T=(T^1,T^2)$ is given by

$$T^1 = u, \tag{A.34}$$

$$T^2 = g\eta - \frac{h^2}{2}u_{xt}. \tag{A.35}$$

Case 2: $a_7=1, a_1=0, a_2=0, a_3=0, a_4=0$

$$\Lambda_1(x) = 1, \tag{A.36}$$

$$\Lambda_2(t) = 0. \tag{A.37}$$

By properties of multipliers the following must be satisfied

$$\begin{aligned} &\Lambda_1(\eta_t + hu_x - \frac{h^3}{6}u_{xxx}) + \Lambda_2(u_t + g\eta_x - \frac{h^2}{2}u_{xxt}) = D_t T^1 + D_x T^2 \\ \Rightarrow &\eta_t + hu_x - \frac{h^3}{6}u_{xxx} = D_t T^1 + D_x T^2. \end{aligned} \tag{A.38}$$

When $u(t,x)$ and $\eta(t,x)$ are solutions of the system of equations (A.1) and (A.2), we have

$$D_t[\eta] + D_x\left[hu - \frac{h^3}{6}u_{xx}\right] = 0. \tag{A.39}$$

Hence the elementary conserved vector $T=(T^1,T^2)$ is given by

$$T^1 = \eta, \tag{A.40}$$

$$T^2 = hu - \frac{h^3}{6}u_{xx}. \tag{A.41}$$

Case 3: $a_1=1, a_2=0, a_3=0, a_4=0, a_7=0$

$$\Lambda_1(x) = -\frac{x}{h}, \tag{A.42}$$

$$\Lambda_2(t) = t. \tag{A.43}$$

By properties of multipliers the following must be satisfied

$$\begin{aligned} &\Lambda_1(\eta_t + hu_x - \frac{h^3}{6}u_{xxx}) + \Lambda_2(u_t + g\eta_x - \frac{h^2}{2}u_{xxt}) = D_t T^1 + D_x T^2 \\ \Rightarrow &-\frac{x}{h}(\eta_t + hu_x - \frac{h^3}{6}u_{xxx}) + t(u_t + g\eta_x - \frac{h^2}{2}u_{xxt}) = D_t T^1 + D_x T^2. \end{aligned} \tag{A.44}$$

When $u(t,x)$ and $\eta(t,x)$ are solutions of the system of equations (A.1) and (A.2), we have

$$D_t\left[tu-\frac{1}{h}x\eta\right]+D_x\left[-xu+gt\eta-\frac{h^2}{2}tu_{xt}+\frac{h^2}{6}[xu_{xx}-u_x]\right]=0. \tag{A.45}$$

Hence the conserved vector $T=(T^1,T^2)$ is given by

$$T^1=tu-\frac{1}{h}x\eta, \tag{A.46}$$

$$T^2=-xu+gt\eta-\frac{h^2}{2}tu_{xt}+\frac{h^2}{6}[xu_{xx}-u_x]. \tag{A.47}$$

Case 4: $a_3=1, a_4=1, a_1=0, a_2=0, a_7=0$

$$\Lambda_1(x)=e^{\alpha\frac{\sqrt{6}}{h}x}, \tag{A.48}$$

$$\Lambda_2(t)=0, \tag{A.49}$$

where $\alpha=\pm1$.

By properties of multipliers the following must be satisfied

$$\begin{aligned}&\Lambda_1(\eta_t+hu_x-\frac{h^3}{6}u_{xxx})+\Lambda_2(u_t+g\eta_x-\frac{h^2}{2}u_{xxt})=D_tT^1+D_xT^2\\ \Rightarrow\,&e^{\alpha\frac{\sqrt{6}}{h}x}(\eta_t+hu_x-\frac{h^3}{6}u_{xxx})=D_tT^1+D_xT^2.\end{aligned} \tag{A.50}$$

When $u(t,x)$ and $\eta(t,x)$ are solutions of the system of equations (A.1) and (A.2), we have

$$D_t\left[e^{\epsilon\frac{\sqrt{6}}{h}x}\eta\right]+D_x\left[\frac{h^3}{6}e^{\epsilon\frac{\sqrt{6}}{h}x}\left[\epsilon\frac{\sqrt{6}}{h}u_x-u_{xx}\right]\right]=0. \tag{A.51}$$

Hence the conserved vector $T=(T^1,T^2)$ is given by

$$T^1=e^{\epsilon\frac{\sqrt{6}}{h}x}\eta, \tag{A.52}$$

$$T^2=\frac{h^3}{6}e^{\epsilon\frac{\sqrt{6}}{h}x}\left[\epsilon\frac{\sqrt{6}}{h}u_x-u_{xx}\right]. \tag{A.53}$$

We have obtained additional multipliers of the form $\Lambda_{1,2}(t,x,u,\eta)$ using Maple software [50]. An identical Multiplier approach was implemented and we obtained the following multipliers as a function of the dependent and independent variables:

$$\Lambda_1(t,x,u,\eta)=\frac{1}{2}\frac{1}{\sqrt{c_2}h}\Big[\sqrt{2}\sqrt{g(3c_2h+6g+\sqrt{9h^2c_2^2+12ghc_2+36g^2})}e^{-\frac{1}{2}\frac{-2\sqrt{c_2}tgh+\sqrt{2}\sqrt{g(3c_2h+6g+\sqrt{9h^2c_2^2+12ghc_2+36g^2})}x}{gh}}C3C5$$

$$
\begin{aligned}
&+\sqrt{2}\sqrt{g(3c_2h+6g+\sqrt{9h^2c_2^2+12ghc_2+36g^2})}\,e^{\frac{1}{2}\frac{-2\sqrt{c_2}tgh+\sqrt{2}\sqrt{g(3c_2h+6g+\sqrt{9h^2c_2^2+12ghc_2+36g^2})}}{gh}x}C4C6\\
&-\sqrt{2}\sqrt{g(3c_2h+6g+\sqrt{9h^2c_2^2+12ghc_2+36g^2})}\,e^{-\frac{1}{2}\frac{2\sqrt{c_2}tgh+\sqrt{2}\sqrt{g(3c_2h+6g+\sqrt{9h^2c_2^2+12ghc_2+36g^2})}}{gh}x}C3C6\\
&-\sqrt{2}\sqrt{g(3c_2h+6g+\sqrt{9h^2c_2^2+12ghc_2+36g^2})}\,e^{\frac{1}{2}\frac{2\sqrt{c_2}tgh+\sqrt{2}\sqrt{g(3c_2h+6g+\sqrt{9h^2c_2^2+12ghc_2+36g^2})}}{gh}x}C4C5\\
&+\sqrt{2}\Big(\Big(e^{-\frac{1}{2}\frac{-2\sqrt{c_2}tgh+\sqrt{2}\sqrt{g(3c_2h+6g+\sqrt{9h^2c_2^2+12ghc_2+36g^2})}}{gh}x}C1\\
&-e^{\frac{1}{2}\frac{2\sqrt{c_2}tgh+\sqrt{2}\sqrt{g(3c_2h+6g+\sqrt{9h^2c_2^2+12ghc_2+36g^2})}}{gh}x}C2\Big)C5\\
&+\Big(e^{\frac{1}{2}\frac{-2\sqrt{c_2}tgh+\sqrt{2}\sqrt{g(3c_2h+6g+\sqrt{9h^2c_2^2+12ghc_2+36g^2})}}{gh}x}C2\\
&-e^{-\frac{1}{2}\frac{2\sqrt{c_2}tgh+\sqrt{2}\sqrt{g(3c_2h+6g+\sqrt{9h^2c_2^2+12ghc_2+36g^2})}}{gh}x}C1\Big)C6\Big)\sqrt{g(3c_2h+6g+\sqrt{9h^2c_2^2+12ghc_2+36g^2})}\\
&+2h\sqrt{c_2}\Big(C7+C8e^{\frac{\sqrt{6}}{h}x}+C9e^{\frac{-\sqrt{6}}{h}x}\Big)\Big]\Big],
\end{aligned}
\tag{A.54}
$$

$$
\begin{aligned}
\Lambda_2(t,x,u,\eta)=\frac{1}{e^{\sqrt{c_2}t}}\Big[\Big(&C3e^{-\frac{1}{2}\frac{\sqrt{2}\sqrt{g(3c_2h+6g+\sqrt{9h^2c_2^2+12ghc_2+36g^2})}}{gh}x}\\
&+C4e^{\frac{1}{2}\frac{\sqrt{2}\sqrt{g(3c_2h+6g+\sqrt{9h^2c_2^2+12ghc_2+36g^2})}}{gh}x}\\
&+C1e^{-\frac{1}{2}\frac{\sqrt{2}\sqrt{g(3c_2h+6g-\sqrt{9h^2c_2^2+12ghc_2+36g^2})}}{gh}x}\\
&+C2e^{\frac{1}{2}\frac{\sqrt{2}\sqrt{g(3c_2h+6g-\sqrt{9h^2c_2^2+12ghc_2+36g^2})}}{gh}x}\Big)[(e^{\sqrt{c_2}t})^2C5+C6]\Big].
\end{aligned}
\tag{A.55}
$$

Using Table A.1 and assuming $c_2=1$ we obtain 8 sets of multipliers given in Table A.2 where

$$
c_1=\frac{\sqrt{g(3h+6g-\sqrt{36g^2+12gh+9h^2})}}{\sqrt{2}hg},
\tag{A.56}
$$

$$
c_3=\frac{\sqrt{g(3h+6g+\sqrt{36g^2+12gh+9h^2})}}{\sqrt{2}hg}.
\tag{A.57}
$$

Considering Table A.2, since the multipliers in Case (5(a)-8(a)) are identical in form to those in Case (5(b)-8(b)) respectively, we introduce a constant to reduce the number of cases we need to consider. Thus the resulting additional multipliers of the form $\Lambda_{1,2}(t,x,u,\eta)$ are given in Table A.3 , where

Case	5(a)	6(a)	7(a)	8(a)	5(b)	6(b)	7(b)	8(b)
C1	**1**	**1**	0	0	0	0	0	0
C2	0	0	**1**	**1**	0	0	0	0
C3	0	0	0	0	**1**	**1**	0	0
C4	0	0	0	0	0	0	**1**	**1**
C5	**1**	0	**1**	0	**1**	0	**1**	0
C6	0	**1**	0	**1**	0	**1**	0	**1**
C7	0	0	0	0	0	0	0	0
C8	0	0	0	0	0	0	0	0
C9	0	0	0	0	0	0	0	0
C10	0	0	0	0	0	0	0	0

Table A.1: Coefficients of multipliers of the form $\Lambda_{1,2}(t,x,u,\eta)$ using Maple software [50].

Case	Multiplier
5(a)	$\Lambda_1 = c_1 g\ e^{-c_1 x + ght}$ $\Lambda_2 = e^{-c_1 x + t}$
6(a)	$\Lambda_1 = -c_1 g\ e^{-c_1 x - ght}$ $\Lambda_2 = e^{-c_1 x - t}$
7(a)	$\Lambda_1 = -c_1 g\ e^{c_1 x + ght}$ $\Lambda_2 = e^{c_1 x + t}$
8(a)	$\Lambda_1 = c_1 g\ e^{c_1 x - ght}$ $\Lambda_2 = e^{c_1 x - t}$
5(b)	$\Lambda_1 = c_3 g\ e^{-c_3 x + ght}$ $\Lambda_2 = e^{-c_3 x + t}$
6(b)	$\Lambda_1 = -c_3 g\ e^{-c_3 x - ght}$ $\Lambda_2 = e^{-c_3 x - t}$
7(b)	$\Lambda_1 = -c_3 g\ e^{c_3 x + ght}$ $\Lambda_2 = e^{c_3 x + t}$
8(b)	$\Lambda_1 = c_3 g\ e^{c_3 x - ght}$ $\Lambda_2 = e^{c_3 x - t}$

Table A.2: Multipliers of the form $\Lambda_{1,2}(t,x,u,\eta)$.

$$c_4 = \frac{\sqrt{g(3h + 6g + \alpha\sqrt{36g^2 + 12gh + 9h^2})}}{\sqrt{2}hg}, \tag{A.58}$$

Case	Multiplier
5	$\Lambda_1 = c_4 g\ e^{-c_4 x + ght}$ $\Lambda_2 = e^{-c_4 x + t}$
6	$\Lambda_1 = -c_4 g\ e^{-c_4 x - ght}$ $\Lambda_2 = e^{-c_4 x - t}$
7	$\Lambda_1 = -c_4 g\ e^{c_4 x + ght}$ $\Lambda_2 = e^{c_4 x + t}$
8	$\Lambda_1 = c_4 g\ e^{c_4 x - ght}$ $\Lambda_2 = e^{c_4 x - t}$

Table A.3: Reduced multipliers of the form $\Lambda_{1,2}(t,x,u,\eta)$.

where $\alpha = \pm 1$.

Using the same Maple code [50], we were able to obtain multipliers of the form $\Lambda_{1,2}(t,x,u,\eta,u_t,\eta_t,u_x,\eta_x)$, producing identical sets to the multipliers of the form $\Lambda_{1,2}(t,x,u,\eta)$ - Table A.3.

The conserved vectors can now be obtained for Cases (5-8) by using the Multiplier method.

Case 5

$$\Lambda_1 = c_4 g e^{-c_4 x + ght}, \tag{A.59}$$

$$\Lambda_2 = e^{-c_4 x + t}. \tag{A.60}$$

By properties of multipliers the following must be satisfied

$$\begin{aligned} &\Lambda_1(\eta_t + hu_x - \frac{h^3}{6}u_{xxx}) + \Lambda_2(u_t + g\eta_x - \frac{h^2}{2}u_{xxt}) = D_t T^1 + D_x T^2 \\ \Rightarrow &c_4 g\ e^{-c_4 x + ght}(\eta_t + hu_x - \frac{h^3}{6}u_{xxx}) + e^{-c_4 x + t}(u_t + g\eta_x - \frac{h^2}{2}u_{xxt}) = D_t T^1 + D_x T^2. \end{aligned} \tag{A.61}$$

When $u(t,x)$ and $\eta(t,x)$ are solutions of the system of equations (A.1) and (A.2), we have

$$D_t\left[e^{-c_4 x + t}\left[g\eta_x - 2u\right]\right] + D_x\left[e^{-c_4 x + t}\eta_t\left[-g - \frac{1}{c_4}u_{xx} - u_x - \frac{3}{{c_4}^2}u_{xt} - \frac{3}{c_4}u_t + \int 2u\, dx\right]\right] = 0.$$

Hence the conserved vector $T=(T^1,T^2)$ is given by

$$T^1 = e^{-c_4x+t}\left[g\eta_x - 2u\right], \tag{A.62}$$

$$T^2 = e^{-c_4x+t}\eta_t\left[-g - \frac{1}{c_4}u_{xx} - u_x - \frac{3}{{c_4}^2}u_{xt} - \frac{3}{c_4}u_t + \int 2u\,dx\right]. \tag{A.63}$$

where $gh=1$ and $c_4=\alpha\frac{\sqrt{6}}{h}$, with $\alpha=\pm1$.

Case 6

$$\Lambda_1 = -c_4 g\ e^{-c_4x-ght}, \tag{A.64}$$

$$\Lambda_2 = e^{-c_4x-t}. \tag{A.65}$$

By properties of multipliers the following must be satisfied

$$\begin{aligned}
&\Lambda_1(\eta_t + hu_x - \frac{h^3}{6}u_{xxx}) + \Lambda_2(u_t + g\eta_x - \frac{h^2}{2}u_{xxt}) = D_tT^1 + D_xT^2\\
\Rightarrow &- c_4 g\ e^{-c_4x-ght}(\eta_t + hu_x - \frac{h^3}{6}u_{xxx}) + e^{-c_4x-t}(u_t + g\eta_x - \frac{h^2}{2}u_{xxt}) = D_tT^1 + D_xT^2.
\end{aligned} \tag{A.66}$$

When $u(t,x)$ and $\eta(t,x)$ are solutions of the system of equations (A.1) and (A.2), we have

$$D_t\left[e^{-c_4x-t}\left[-g\eta_x - 2u\right]\right] + D_x\left[e^{-c_4x-t}\left[g\eta_t + \frac{1}{c_4}u_{xx} + u_x - \frac{3}{{c_4}^2}u_{xt} - \frac{3}{c_4}u_t - \int 2u\,dx\right]\right] = 0.$$

Hence the conserved vector $T=(T^1,T^2)$ is given by

$$T^1 = e^{-c_4x-t}\left[-g\eta_x - 2u\right], \tag{A.67}$$

$$T^2 = e^{-c_4x-t}\left[g\eta_t + \frac{1}{c_4}u_{xx} + u_x - \frac{3}{{c_4}^2}u_{xt} - \frac{3}{c_4}u_t - \int 2u\,dx\right]. \tag{A.68}$$

where $gh=1$ and $c_4=\alpha\frac{\sqrt{6}}{h}$, with $\alpha=\pm1$.

Case 7

$$\Lambda_1 = -c_4 g\ e^{c_4x+ght}, \tag{A.69}$$

$$\Lambda_2 = e^{c_4x+t}. \tag{A.70}$$

By properties of multipliers the following must be satisfied

$$\Lambda_1(\eta_t + hu_x - \frac{h^3}{6}u_{xxx}) + \Lambda_2(u_t + g\eta_x - \frac{h^2}{2}u_{xxt}) = D_tT^1 + D_xT^2$$

$$\Rightarrow -c_4 g\ e^{c_4 x+ght}(\eta_t + hu_x - \frac{h^3}{6}u_{xxx}) + e^{c_4 x+t}(u_t + g\eta_x - \frac{h^2}{2}u_{xxt}) = D_t T^1 + D_x T^2. \tag{A.71}$$

When $u(t,x)$ and $\eta(t,x)$ are solutions of the system of equations (A.1) and (A.2), we have

$$D_t\left[e^{c_4 x+t}\left[g\eta_x - 2u\right]\right] + D_x\left[e^{c_4 x+t}\left[-g\eta_t + \frac{1}{c_4}u_{xx} - u_x - \frac{3}{{c_4}^2}u_{xt} + \frac{3}{c_4}u_t + \int 2u\,dx\right]\right] = 0.$$

Hence the conserved vector $T = (T^1, T^2)$ is given by

$$T^1 = e^{c_4 x+t}\left[g\eta_x - 2u\right], \tag{A.72}$$

$$T^2 = e^{c_4 x+t}\left[-g\eta_t + \frac{1}{c_4}u_{xx} - u_x - \frac{3}{{c_4}^2}u_{xt} + \frac{3}{c_4}u_t + \int 2u\,dx\right]. \tag{A.73}$$

where $gh = 1$ and $c_4 = \alpha\frac{\sqrt{6}}{h}$, with $\alpha = \pm 1$.

Case 8

$$\Lambda_1 = c_4 g\ e^{c_4 x-ght}, \tag{A.74}$$

$$\Lambda_2 = e^{c_4 x-t}. \tag{A.75}$$

By properties of multipliers the following must be satisfied

$$\Lambda_1(\eta_t + hu_x - \frac{h^3}{6}u_{xxx}) + \Lambda_2(u_t + g\eta_x - \frac{h^2}{2}u_{xxt}) = D_t T^1 + D_x T^2$$

$$\Rightarrow c_4 g\ e^{c_4 x-ght}(\eta_t + hu_x - \frac{h^3}{6}u_{xxx}) + e^{c_4 x-t}(u_t + g\eta_x - \frac{h^2}{2}u_{xxt}) = D_t T^1 + D_x T^2. \tag{A.76}$$

When $u(t,x)$ and $\eta(t,x)$ are solutions of the system of equations (A.1) and (A.2), we have

$$D_t\left[e^{c_4 x-t}\left[g\eta_x - 2u\right]\right] + D_x\left[e^{c_4 x-t}\left[g\eta_t - \frac{1}{c_4}u_{xx} - u_x - \frac{3}{{c_4}^2}u_{xt} + \frac{3}{c_4}u_t + \int 2u\,dx\right]\right] = 0.$$

Hence the conserved vector $T = (T^1, T^2)$ is given by

$$T^1 = e^{c_4 x-t}\left[g\eta_x - 2u\right], \tag{A.77}$$

$$T^2 = e^{c_4 x-t}\left[g\eta_t - \frac{1}{c_4}u_{xx} - u_x - \frac{3}{{c_4}^2}u_{xt} + \frac{3}{c_4}u_t + \int 2u\,dx\right]. \tag{A.78}$$

where $gh = 1$ and $c_4 = \alpha\frac{\sqrt{6}}{h}$, with $\alpha = \pm 1$.

A.2 Conservation laws considering multipliers of the form $\Lambda_{1,2}(t,x,u,\eta,u_t,\eta_t,u_x,\eta_x,u_{tt},u_{tx},u_{xx},\eta_{tt},\eta_{tx},\eta_{xx})$

Consider multipliers of the form $\Lambda_1(t,x,u,\eta,u_t,\eta_t,u_x,\eta_x,u_{tt},u_{tx},u_{xx},\eta_{tt},\eta_{tx},\eta_{xx})$ and $\Lambda_2(t,x,u,\eta,u_t,\eta_t,u_x,\eta_x,u_{tt},u_{tx},u_{xx},\eta_{tt},\eta_{tx},\eta_{xx})$. The following multipliers were obtained using Maple software [50]:

$$\begin{aligned}
&\Lambda_1(t,x,u,\eta,u_t,\eta_t,u_x,\eta_x,u_{xx},u_{xt},u_{tt},\eta_{xx},\eta_{xt},\eta_{tt}) = -\frac{3}{h^3 e^{\sqrt{c_2}}}\Bigg[-\frac{1}{6}e^{-\frac{1}{2}\frac{-2\sqrt{c_2}thg+\sqrt{2}\sqrt{g(3c_2h+6g-\sqrt{9h^2c_2^2+12ghc_2+36g^2})}}{gh}x}\sqrt{2}\\
&\sqrt{g(3c_2h+6g-\sqrt{9h^2c_2^2+12ghc_2+36g^2})}C9C13h^2\\
&-\frac{1}{6}e^{\frac{1}{2}\frac{-2\sqrt{c_2}thg+\sqrt{2}\sqrt{g(3c_2h+6g-\sqrt{9h^2c_2^2+12ghc_2+36g^2})}}{gh}x}\sqrt{2}\sqrt{g(3c_2h+6g-\sqrt{9h^2c_2^2+12ghc_2+36g^2})}C10C14h^2\\
&+\frac{1}{6}e^{-\frac{1}{2}\frac{2\sqrt{c_2}thg+\sqrt{2}\sqrt{g(3c_2h+6g-\sqrt{9h^2c_2^2+12ghc_2+36g^2})}}{gh}x}\sqrt{2}\sqrt{g(3c_2h+6g-\sqrt{9h^2c_2^2+12ghc_2+36g^2})}C9C14h^2\\
&+\frac{1}{6}e^{\frac{1}{2}\frac{2\sqrt{c_2}thg+\sqrt{2}\sqrt{g(3c_2h+6g-\sqrt{9h^2c_2^2+12ghc_2+36g^2})}}{gh}x}\sqrt{2}\sqrt{g(3c_2h+6g-\sqrt{9h^2c_2^2+12ghc_2+36g^2})}C10C13h^2\\
&-\frac{1}{6}e^{-\frac{1}{2}\frac{-2\sqrt{c_2}thg+\sqrt{2}\sqrt{g(3c_2h+6g+\sqrt{9h^2c_2^2+12ghc_2+36g^2})}}{gh}x}\sqrt{2}\sqrt{g(3c_2h+6g+\sqrt{9h^2c_2^2+12ghc_2+36g^2})}C11C13h^2\\
&-\frac{1}{6}e^{\frac{1}{2}\frac{2\sqrt{c_2}thg+\sqrt{2}\sqrt{g(3c_2h+6g+\sqrt{9h^2c_2^2+12ghc_2+36g^2})}}{gh}x}\sqrt{2}\sqrt{g(3c_2h+6g+\sqrt{9h^2c_2^2+12ghc_2+36g^2})}C12C14h^2\\
&+\frac{1}{6}e^{-\frac{1}{2}\frac{2\sqrt{c_2}thg+\sqrt{2}\sqrt{g(3c_2h+6g+\sqrt{9h^2c_2^2+12ghc_2+36g^2})}}{gh}x}\sqrt{2}\sqrt{g(3c_2h+6g+\sqrt{9h^2c_2^2+12ghc_2+36g^2})}C11C14h^2\\
&+\frac{1}{6}e^{\frac{1}{2}\frac{2\sqrt{c_2}thg+\sqrt{2}\sqrt{g(3c_2h+6g+\sqrt{9h^2c_2^2+12ghc_2+36g^2})}}{gh}x}\sqrt{2}\sqrt{g(3c_2h+6g+\sqrt{9h^2c_2^2+12ghc_2+36g^2})}C12C13h^2\\
&+\Big(-\frac{1}{3}C17e^{-\frac{\sqrt{6}}{h}x}h^3-\frac{1}{3}C16e^{\frac{\sqrt{6}}{h}x}h^3+\frac{1}{6}C8h^5u_{xx}+(((-\frac{1}{3}u_{xx}x+\frac{4}{3}tu_{xt}+\frac{1}{3}u_x)C1+\frac{1}{6}u_{xx}tC2+\frac{1}{3}u_{xt}C4\\
&+\frac{1}{3}C3u_{xx})g+\frac{2}{3}C6u_{xt}-\frac{1}{3}C8u+\frac{2}{3}u_{xx}C5-\frac{1}{3}C15)h^3+(((t\eta tt+\eta xtx+\eta_t)C1+(\frac{2}{3}t\eta xt+\frac{1}{2}\eta_x)C2-\frac{1}{3}\eta C7)g\\
&+C2u_{xt}x+\eta_{xt}C5+\eta_{tt}C6)h^2+(-2C1g^2t\eta+u(2C1x+C2t)g+\frac{2}{3}\eta_{tt}xC2)h-C2gx\eta\Big)\sqrt{c_2}\Bigg],
\end{aligned} \tag{A.79}$$

$$\begin{aligned}
&\Lambda_2(t,x,u,\eta,u_t,\eta_t,u_x,\eta_x,u_{xx},u_{xt},u_{tt},\eta_{xx},\eta_{xt},\eta_{tt}) = \frac{1}{6h^2e^{\sqrt{c_2}t}}\Bigg[6C9h^2\Big(C13(e^{\sqrt{c_2}t})^2+C14\Big)\\
&C1e^{-\frac{1}{2}\frac{\sqrt{2}\sqrt{g(3c_2h+6g-\sqrt{9h^2c_2^2+12ghc_2+36g^2})}}{gh}x}\\
&+6C10h^2\Big(C13(e^{\sqrt{c_2}t})^2+C14\Big)C1e^{\frac{1}{2}\frac{\sqrt{2}\sqrt{g(3c_2h+6g-\sqrt{9h^2c_2^2+12ghc_2+36g^2})}}{gh}x}\\
&+6C11h^2\Big(C13(e^{\sqrt{c_2}t})^2+C14\Big)C1e^{-\frac{1}{2}\frac{\sqrt{2}\sqrt{g(3c_2h+6g+\sqrt{9h^2c_2^2+12ghc_2+36g^2})}}{gh}x}
\end{aligned}$$

$$+6C12h^2\Big(C13(e^{\sqrt{c_2}\,t})^2+C14\Big)C1e^{\frac{1}{2}\frac{\sqrt{2}\sqrt{g(3c_2h+6g+\sqrt{9h^2c_2^2+12ghc_2+36g^2})}}{gh}x}$$
$$+6\Big(-\frac{1}{6}h^4u_{xx}C7-h^3u_{xx}tgC1+(((t\eta_{xt}+\eta_{xx}x+\eta_x)g+2tu_{tt}+\eta_t)C1$$
$$+(tu_{xt}+u_x-(1/2)u_{xx}x+(1/2)\eta_{xx}tg)C2+C3u_{xt}+C4u_{tt}$$
$$+C5\eta_{xx}+C6\eta_{xt}+C7u+C8\eta)h^2+(6C1gtu+(\frac{3}{2}(\eta_{xt}x+\eta_t))C2)h-6\eta xgC1$$
$$+3C2(-gt\eta+xu)\Big)e^{\sqrt{c_2}\,t}\Big]. \tag{A.80}$$

Coefficient	Case 9	Case 10	Case 11	Case 12	Case 13	Case 14
C1	0	0	0	0	0	0
C2	0	0	0	0	0	0
C3	**1**	0	0	0	0	0
C4	0	**1**	0	0	0	0
C5	0	0	**1**	0	0	0
C6	0	0	0	**1**	0	0
C7	0	0	0	0	**1**	0
C8	0	0	0	0	0	**1**
C9	0	0	0	0	0	0
C10	0	0	0	0	0	0
C11	0	0	0	0	0	0
C12	0	0	0	0	0	0
C13	0	0	0	0	0	0
C14	0	0	0	0	0	0
C15	0	0	0	0	0	0
C16	0	0	0	0	0	0
C17	0	0	0	0	0	0

Table A.4: Coefficients of multipliers of the form $\Lambda_{1,2}(t,x,u,\eta,u_t,\eta_t,u_x,\eta_x,u_{tt},u_{tx},u_{xx},\eta_{tt},\eta_{tx},\eta_{xx})$ using Maple software [50].

Using Table A.4 and assuming $c_2 = 1$, we obtain an additional six sets of multipliers- refer to Table A.5. The conserved vectors can now be obtained for Cases (9-14) by using the Multiplier method.

Case	Multiplier
9	$\Lambda_1 = -g u_{xx}$ $\Lambda_2 = u_{xt}$
10	$\Lambda_1 = -g u_{xt}$ $\Lambda_2 = u_{tt}$
11	$\Lambda_1 = -2u_{xx} - \frac{3}{h}\eta_{xt}$ $\Lambda_2 = u_{xx}$
12	$\Lambda_1 = -2u_{xt} - \frac{3}{h^2}\eta_{tt}$ $\Lambda_2 = \eta_{xt}$
13	$\Lambda_1 = \frac{g}{h}\eta$ $\Lambda_2 = u - \frac{1}{6}h^2 u_{xx}$
14	$\Lambda_1 = u - \frac{1}{2}h^2 u_{xx}$ $\Lambda_2 = \eta$

Table A.5: Multipliers of the form $\Lambda_{1,2}(t,x,u,\eta,u_t,\eta_t,u_x,\eta_x,u_{tt},u_{tx},u_{xx},\eta_{tt},\eta_{tx},\eta_{xx})$

Case 9

$$\Lambda_1 = -g u_{xx}, \tag{A.81}$$

$$\Lambda_2 = u_{xt}. \tag{A.82}$$

By properties of multipliers the following must be satisfied

$$\begin{aligned}
&\Lambda_1(\eta_t + h u_x - \frac{h^3}{6}u_{xxx}) + \Lambda_2(u_t + g\eta_x - \frac{h^2}{2}u_{xxt}) = D_t T^1 + D_x T^2 \\
\Rightarrow &-g u_{xx}(\eta_t + h u_x - \frac{h^3}{6}u_{xxx}) + u_{xt}(u_t + g\eta_x - \frac{h^2}{2}u_{xxt}) = D_t T^1 + D_x T^2.
\end{aligned} \tag{A.83}$$

When $u(t,x)$ and $\eta(t,x)$ are solutions of the system of equations (A.1) and (A.2), we have

$$D_t\left[-g u_{xx}\eta\right] + D_x\left[\frac{1}{2}\left[\frac{h^3}{6}g(u_{xx})^2 - g h(u_x)^2 - \frac{h^2}{2}(u_{xt})^2 + (u_t)^2\right] + g u_{xt}\eta\right] = 0.$$

Hence the conserved vector $T = (T^1, T^2)$ is given by

$$T^1 = -g u_{xx}\eta, \tag{A.84}$$
$$T^2 = \frac{1}{2}\left[\frac{h^3}{6}g(u_{xx})^2 - gh(u_x)^2 - \frac{h^2}{2}(u_{xt})^2 + (u_t)^2\right] + g u_{xt}\eta. \tag{A.85}$$

Case 10

$$\Lambda_1 = -g u_{xt}, \tag{A.86}$$
$$\Lambda_2 = u_{tt}. \tag{A.87}$$

By properties of multipliers the following must be satisfied

$$\begin{aligned}
&\Lambda_1(\eta_t + h u_x - \frac{h^3}{6}u_{xxx}) + \Lambda_2(u_t + g\eta_x - \frac{h^2}{2}u_{xxt}) = D_t T^1 + D_x T^2 \\
\Rightarrow &-g u_{xt}(\eta_t + h u_x - \frac{h^3}{6}u_{xxx}) + u_{tt}(u_t + g\eta_x - \frac{h^2}{2}u_{xxt}) = D_t T^1 + D_x T^2.
\end{aligned} \tag{A.88}$$

When $u(t,x)$ and $\eta(t,x)$ are solutions of the system of equations (A.1) and (A.2), we have

$$\begin{aligned}
&D_t\left[\frac{1}{2}\left[-\frac{gh^3}{6}(u_{xx})^2 + (u_t)^2 - gh(u_x^2) + \frac{h^2}{2}(u_{xt})^2\right] + g u_t \eta_x\right] \\
&+ D_x\left[\frac{1}{2}\left[\frac{gh^3}{3}u_{xt}u_{xx} - h^2 u_{tt}(u_{xt})\right] - g u_t \eta_t\right] = 0.
\end{aligned} \tag{A.89}$$

Hence the conserved vector $T = (T^1, T^2)$ is given by

$$T^1 = \frac{1}{2}\left[-\frac{gh^3}{6}(u_{xx})^2 + (u_t)^2 - gh(u_x)^2 + \frac{h^2}{2}(u_{xt})^2\right] + g u_t \eta_x, \tag{A.90}$$
$$T^2 = \frac{1}{2}\left[\frac{gh^3}{3}u_{xt}u_{xx} - h^2 u_{tt}(u_{xt})\right] - g u_t \eta_t. \tag{A.91}$$

Case 11

$$\Lambda_1 = -2u_{xx} - \frac{3}{h}\eta_{xt}, \tag{A.92}$$

$$\Lambda_2 = u_{xx}. \tag{A.93}$$

By properties of multipliers the following must be satisfied

$$\begin{aligned}
&\Lambda_1(\eta_t + hu_x - \frac{h^3}{6}u_{xxx}) + \Lambda_2(u_t + g\eta_x - \frac{h^2}{2}u_{xxt}) = D_t T^1 + D_x T^2 \\
\Rightarrow &(-2u_{xx} - \frac{3}{h}\eta_{xt})(\eta_t + hu_x - \frac{h^3}{6}u_{xxx}) + u_{xx}(u_t + g\eta_x - \frac{h^2}{2}u_{xxt}) = D_t T^1 + D_x T^2.
\end{aligned} \tag{A.94}$$

When $u(t,x)$ and $\eta(t,x)$ are solutions of the system of equations (A.1) and (A.2), we have

$$\begin{aligned}
&D_t\left[\frac{1}{2}\left[-\frac{h^2}{2}(u_{xx})^2 - (u_x)^2\right]\right] \\
&+ D_x\left[\frac{h^3}{6}(u_{xx})^2 - \frac{3}{2h}(\eta_t)^2 - h(u_x)^2 + \frac{h^2}{2}\left[\eta_{xt}u_{xx} - \eta_{xxt}u_x + \eta_{xxxt}u\right]\right. \\
&\left. + u_x u_t + g u_{xx}\eta - u\eta_{xt} - 2u_x\eta_t + \int\left(-\frac{h^2}{2}\eta_{xxxt}u - g u_{xxx}\eta + u\eta_{xxt}\right)dx\right] = 0.
\end{aligned} \tag{A.95}$$

Hence the conserved vector $T = (T^1, T^2)$ is given by

$$T^1 = \frac{1}{2}\left[-\frac{h^2}{2}(u_{xx})^2 - (u_x)^2\right], \tag{A.96}$$

$$\begin{aligned}
T^2 = &\frac{h^3}{6}(u_{xx})^2 - \frac{3}{2h}(\eta_t)^2 - h(u_x)^2 + \frac{h^2}{2}\left[\eta_{xt}u_{xx} - \eta_{xxt}u_x + \eta_{xxxt}u\right] \\
&+ u_x u_t + g u_{xx}\eta - u\eta_{xt} - 2u_x\eta_t + \int\left(-\frac{h^2}{2}\eta_{xxxt}u - g u_{xxx}\eta + u\eta_{xxt}\right)dx.
\end{aligned} \tag{A.97}$$

Case 12

$$\Lambda_1 = -2u_{xt} - \frac{3}{h^2}\eta_{tt}, \tag{A.98}$$

$$\Lambda_2 = \eta_{xt}. \tag{A.99}$$

By properties of multipliers the following must be satisfied

$$\Lambda_1(\eta_t + hu_x - \frac{h^3}{6}u_{xxx}) + \Lambda_2(u_t + g\eta_x - \frac{h^2}{2}u_{xxt}) = D_t T^1 + D_x T^2$$

$$\Rightarrow(-2u_{xt}-\frac{3}{h^2}\eta_{tt})(\eta_t+hu_x-\frac{h^3}{6}u_{xxx})+\eta_{xt}(u_t+g\eta_x-\frac{h^2}{2}u_{xxt})=D_tT^1+D_xT^2. \tag{A.100}$$

When $u(t,x)$ and $\eta(t,x)$ are solutions of the system of equations (A.1) and (A.2), we have

$$\begin{aligned}&D_t\left[\frac{g}{2}(\eta_x)^2-h(u_x)^2-\frac{3}{2h^2}(\eta_t)^2-3u_{xt}\eta-\frac{h^2}{2}\eta_{xxxt}u-\frac{h^3}{3}uu_{xxxx}+\frac{h}{2}\eta_tu_{xxx}-\frac{h}{2}\eta u_{xxxt}\right]\\&+D_x\left[\eta_tu_t-\frac{h^2}{2}\eta_{xt}u_{xt}+\frac{h^2}{2}\eta_{xxt}u_t+\frac{h^3}{3}u_tu_{xxx}-\frac{3}{h}u\eta_{tt}\right.\\&\left.+\int\left(\frac{3}{h}u\eta_{ttx}+3u_{xtt}\eta+\frac{h^2}{2}\eta_{xxxtt}u+\frac{h^3}{3}uu_{xxxxt}+\frac{h}{2}\eta u_{xxxtt}\right)dx\right]=0.\end{aligned} \tag{A.101}$$

Hence the conserved vector $T=(T^1,T^2)$ is given by

$$T^1=\frac{g}{2}(\eta_x)^2-h(u_x)^2-\frac{3}{2h^2}(\eta_t)^2-3u_{xt}\eta-\frac{h^2}{2}\eta_{xxxt}u-\frac{h^3}{3}uu_{xxxx}+\frac{h}{2}\eta_tu_{xxx}-\frac{h}{2}\eta u_{xxxt}, \tag{A.102}$$

$$\begin{aligned}T^2=&\eta_tu_t-\frac{h^2}{2}\eta_{xt}u_{xt}+\frac{h^2}{2}\eta_{xxt}u_t+\frac{h^3}{3}u_tu_{xxx}-\frac{3}{h}u\eta_{tt}\\&+\int\left(\frac{3}{h}u\eta_{ttx}+3u_{xtt}\eta+\frac{h^2}{2}\eta_{xxxtt}u+\frac{h^3}{3}uu_{xxxxt}+\frac{h}{2}\eta u_{xxxtt}\right)dx.\end{aligned} \tag{A.103}$$

Case 13

$$\Lambda_1=\frac{g}{h}\eta, \tag{A.104}$$

$$\Lambda_2=u-\frac{1}{6}h^2u_{xx}. \tag{A.105}$$

By properties of multipliers the following must be satisfied

$$\begin{aligned}&\Lambda_1(\eta_t+hu_x-\frac{h^3}{6}u_{xxx})+\Lambda_2(u_t+g\eta_x-\frac{h^2}{2}u_{xxt})=D_tT^1+D_xT^2\\&\Rightarrow\frac{g}{h}\eta(\eta_t+hu_x-\frac{h^3}{6}u_{xxx})+(u-\frac{1}{6}h^2u_{xx})(u_t+g\eta_x-\frac{h^2}{2}u_{xxt})=D_tT^1+D_xT^2.\end{aligned} \tag{A.106}$$

When $u(t,x)$ and $\eta(t,x)$ are solutions of the system of equations (A.1) and (A.2), we have

$$D_t\left[\frac{g}{2h}\eta^2+\frac{1}{2}u^2+\frac{h^4}{24}{u_{xx}}^2+\frac{h}{4}(u_x)^2+\frac{h^2}{12}(u_x)^2\right]$$

$$+D_x\left[g\eta u-\frac{gh^2}{6}u_{xx}\eta+\frac{h}{2}u_x u_t+\frac{h^2}{6}u_x u_t\right]=0. \tag{A.107}$$

Hence the conserved vector $T=(T^1,T^2)$ is given by

$$T^1=\frac{g}{2h}\eta^2+\frac{1}{2}u^2+\frac{h^4}{24}(u_{xx})^2+\frac{h}{4}(u_x)^2+\frac{h^2}{12}(u_x)^2, \tag{A.108}$$

$$T^2=g\eta u-\frac{gh^2}{6}u_{xx}\eta+\frac{h}{2}u_x u_t+\frac{h^2}{6}u_x u_t. \tag{A.109}$$

Case 14

$$\Lambda_1=u-\frac{1}{2}h^2u_{xx}, \tag{A.110}$$

$$\Lambda_2=\eta. \tag{A.111}$$

By properties of multipliers the following must be satisfied

$$\begin{aligned}&\Lambda_1(\eta_t+hu_x-\frac{h^3}{6}u_{xxx})+\Lambda_2(u_t+g\eta_x-\frac{h^2}{2}u_{xxt})=D_tT^1+D_xT^2\\ \Rightarrow&(u-\frac{1}{2}h^2u_{xx})(\eta_t+hu_x-\frac{h^3}{6}u_{xxx})+\eta(u_t+g\eta_x-\frac{h^2}{2}u_{xxt})=D_tT^1+D_xT^2.\end{aligned} \tag{A.112}$$

When $u(t,x)$ and $\eta(t,x)$ are solutions of the system of equations (A.1) and (A.2), we have

$$D_t\left[\eta u-\frac{h^2}{2}u_{xx}\eta-\frac{h^3}{4}{u_x}^2\right]+D_x\left[\frac{h}{2}u^2-\frac{h^3}{6}uu_{xx}+\frac{h^3}{12}{u_{xx}}^2+\frac{h^5}{24}{u_{xx}}^2+\frac{g}{2}\eta^2\right]=0.$$

Hence the conserved vector $T=(T^1,T^2)$ is given by

$$T^1=\eta u-\frac{h^2}{2}u_{xx}\eta-\frac{h^3}{4}(u_x)^2, \tag{A.113}$$

$$T^2=\frac{h}{2}u^2-\frac{h^3}{6}uu_{xx}+\frac{h^3}{12}(u_{xx})^2+\frac{h^5}{24}(u_{xx})^2+\frac{g}{2}\eta^2. \tag{A.114}$$

Appendix B

Lie point symmetries for the LB equations

The extended infinitesimals for (5.8) are given by

$$\begin{aligned}
\varphi_t^2 &= D_t\left(\psi^2\right) - \eta_t D_t\left(\xi^1\right) - \eta_x D_t\left(\xi^2\right) \\
&= \frac{\partial \psi^2}{\partial t} + \frac{\partial \psi^2}{\partial u} u_t + \frac{\partial \psi^2}{\partial \eta}\eta_t - \frac{\partial \xi^1}{\partial t}\eta_t - \frac{\partial \xi^1}{\partial u}\eta_t u_t - \frac{\partial \xi^1}{\partial \eta}\left(\eta_t\right)^2 - \frac{\partial \xi^2}{\partial t}\eta_x \\
&- \frac{\partial \xi^2}{\partial u}\eta_x u_t - \frac{\partial \xi^2}{\partial \eta}\eta_t \eta_x,
\end{aligned} \tag{B.1}$$

and similarly

$$\begin{aligned}
\varphi_x^1 &= D_x\left(\psi^1\right) - u_t D_x\left(\xi^1\right) - u_x D_x\left(\xi^2\right) \\
&= \frac{\partial \psi^1}{\partial x} + \frac{\partial \psi^1}{\partial u} u_x + \frac{\partial \psi^1}{\partial \eta}\eta_x - \frac{\partial \xi^1}{\partial x} u_t - \frac{\partial \xi^1}{\partial u} u_t u_x - \frac{\partial \xi^1}{\partial \eta} u_t \eta_x - \frac{\partial \xi^2}{\partial x} u_x \\
&- \frac{\partial \xi^2}{\partial u}(u_x)^2 - \frac{\partial \xi^2}{\partial \eta} u_x \eta_x,
\end{aligned} \tag{B.2}$$

and finally

$$\begin{aligned}
\varphi_{xxx}^1 &= D_x\left(\varphi_{xx}^1\right) - u_{xxt} D_x\left(\xi^1\right) - u_{xxx} D_x\left(\xi^2\right) \\
&= \left(\frac{\partial^3 \psi^1}{\partial x^3} + \frac{\partial^3 \psi^1}{\partial x^2 \partial u} u_x + \frac{\partial^3 \psi^1}{\partial x^2 \partial \eta}\eta_x\right) + \left(2\frac{\partial^3 \psi^1}{\partial x^2 \partial u} u_x + 2\frac{\partial^3 \psi^1}{\partial u^2 \partial x}(u_x)^2 + 2\frac{\partial^3 \psi^1}{\partial u \partial x \partial \eta} u_x \eta_x + 2\frac{\partial^2 \psi^1}{\partial u \partial x} u_{xx}\right) \\
&+ \left(2\frac{\partial^3 \psi^1}{\partial x^2 \partial \eta}\eta_x + 2\frac{\partial^3 \psi^1}{\partial u \partial \eta \partial x} u_x \eta_x + 2\frac{\partial^3 \psi^1}{\partial \eta^2 \partial x}(\eta_x)^2 + 2\frac{\partial^2 \psi^1}{\partial \eta \partial x}\eta_{xx}\right) \\
&+ \left(\frac{\partial^3 \psi^1}{\partial u^2 \partial x}(u_x)^2 + \frac{\partial^3 \psi^1}{\partial u^3}(u_x)^3 + \frac{\partial^3 \psi^1}{\partial u^2 \partial \eta}(u_x)^2 \eta_x + 2\frac{\partial^2 \psi^1}{\partial u^2} u_x u_{xx}\right) \\
&+ \left(\frac{\partial^3 \psi^1}{\partial \eta^2 \partial x}(\eta_x)^2 + \frac{\partial^3 \psi^1}{\partial \eta^2 \partial u} u_x (\eta_x)^2 + \frac{\partial^3 \psi^1}{\partial \eta^3}(\eta_x)^3 + 2\frac{\partial^2 \psi^1}{\partial \eta^2}\eta_x \eta_{xx}\right) \\
&+ \left(2\frac{\partial^3 \psi^1}{\partial u \partial \eta \partial x} u_x \eta_x + 2\frac{\partial^3 \psi^1}{\partial u^2 \partial \eta}(u_x)^2 \eta_x + 2\frac{\partial^3 \psi^1}{\partial \eta^2 \partial u} u_x (\eta_x)^2 + 2\frac{\partial^2 \psi^1}{\partial u \partial \eta}\eta_x u_{xx} + 2\frac{\partial^2 \psi^1}{\partial u \partial \eta} u_x \eta_{xx}\right)
\end{aligned}$$

$$+\left(\frac{\partial^2\psi^1}{\partial u\partial x}u_{xx}+\frac{\partial^2\psi^1}{\partial u^2}u_xu_{xx}+\frac{\partial^2\psi^1}{\partial u\partial\eta}u_{xx}\eta_x+\frac{\partial\psi^1}{\partial u}u_{xxx}\right)$$
$$+\left(\frac{\partial^2\psi^1}{\partial\eta\partial x}\eta_{xx}+\frac{\partial^2\psi^1}{\partial u\partial\eta}u_x\eta_{xx}+\frac{\partial^2\psi^1}{\partial\eta^2}\eta_{xx}\eta_x+\frac{\partial\psi^1}{\partial\eta}\eta_{xxx}\right)$$
$$-\left(\frac{\partial^3\xi^1}{\partial x^3}u_t+\frac{\partial^3\xi^1}{\partial x^2\partial u}u_tu_x+\frac{\partial^3\xi^1}{\partial x^2\partial\eta}u_t\eta_x+\frac{\partial^2\xi^1}{\partial x^2}u_{xt}\right)$$
$$-\left(\frac{\partial^3\xi^2}{\partial x^3}u_x+\frac{\partial^3\xi^2}{\partial x^2\partial u}(u_x)^2+\frac{\partial^3\xi^2}{\partial x^2\partial\eta}u_x\eta_x+\frac{\partial^2\xi^2}{\partial x^2}u_{xx}\right)$$
$$-\left(2\frac{\partial^3\xi^1}{\partial x^2\partial u}u_tu_x+2\frac{\partial^3\xi^1}{\partial u^2\partial x}u_t(u_x)^2+2\frac{\partial^3\xi^1}{\partial x\partial u\partial\eta}u_tu_x\eta_x+2\frac{\partial^2\xi^1}{\partial x\partial u}u_{xx}u_t+2\frac{\partial^2\xi^1}{\partial x\partial u}u_{xt}u_x\right)$$
$$-\left(2\frac{\partial^3\xi^1}{\partial x^2\partial\eta}u_t\eta_x+2\frac{\partial^3\xi^1}{\partial x\partial u\partial\eta}u_tu_x\eta_x+2\frac{\partial^3\xi^1}{\partial\eta^2\partial x}u_t(\eta_x)^2+2\frac{\partial^2\xi^1}{\partial x\partial\eta}u_{xt}\eta_x+2\frac{\partial^2\xi^1}{\partial x\partial\eta}u_t\eta_{xx}\right)$$
$$-\left(2\frac{\partial^2\xi^1}{\partial x^2}u_{xt}+2\frac{\partial^2\xi^1}{\partial x\partial u}u_{xt}u_x+2\frac{\partial^2\xi^1}{\partial x\partial\eta}u_{xt}\eta_x+3\frac{\partial\xi^1}{\partial x}u_{xxt}\right)$$
$$-\left(\frac{\partial^3\xi^1}{\partial u^2\partial x}u_t(u_x)^2+\frac{\partial^3\xi^1}{\partial u^3}u_t(u_x)^3+\frac{\partial^3\xi^1}{\partial u^2\partial\eta}u_t(u_x)^2\eta_x+\frac{\partial^2\xi^1}{\partial u^2}u_{xt}(u_x)^2+2\frac{\partial^2\xi^1}{\partial u^2}u_xu_{xx}u_t\right)$$
$$-\left(\frac{\partial^3\xi^1}{\partial\eta^2\partial x}u_t(\eta_x)^2+\frac{\partial^3\xi^1}{\partial\eta^2\partial u}u_tu_x(\eta_x)^2+\frac{\partial^3\xi^1}{\partial\eta^3}u_t(\eta_x)^3+\frac{\partial^2\xi^1}{\partial\eta^2}u_{xt}(\eta_x)^2+2\frac{\partial^2\xi^1}{\partial\eta^2}\eta_x\eta_{xx}u_t\right)$$
$$-\left(\frac{\partial^3\xi^2}{\partial u^2\partial x}(u_x)^3+\frac{\partial^3\xi^2}{\partial u^3}(u_x)^4+\frac{\partial^3\xi^2}{\partial u^2\partial\eta}(u_x)^3\eta_x+3\frac{\partial^2\xi^2}{\partial u^2}(u_x)^2u_{xx}\right)$$
$$-\left(2\frac{\partial^3\xi^2}{\partial u\partial\eta\partial x}(u_x)^2\eta_x+2\frac{\partial^3\xi^2}{\partial u^2\partial\eta}(u_x)^3\eta_x+2\frac{\partial^3\xi^2}{\partial\eta^2\partial u}(u_x)^2(\eta_x)^2+2\frac{\partial^2\xi^2}{\partial u\partial\eta}\eta_{xx}(u_x)^2+4\frac{\partial^2\xi^2}{\partial u\partial\eta}u_xu_{xx}\eta_x\right)$$
$$-\left(\frac{\partial^3\xi^2}{\partial\eta^2\partial x}u_x(\eta_x)^2+\frac{\partial^3\xi^2}{\partial\eta^2\partial u}(u_x)^2(\eta_x)^2+\frac{\partial^3\xi^2}{\partial\eta^3}u_x(\eta_x)^3+\frac{\partial^2\xi^2}{\partial\eta^2}u_{xx}(\eta_x)^2+2\frac{\partial^2\xi^2}{\partial\eta^2}\eta_x\eta_{xx}u_x\right)$$

$$-\left(2\frac{\partial^3\xi^1}{\partial u\partial x\partial\eta}u_tu_x\eta_x+2\frac{\partial^3\xi^1}{\partial u^2\partial\eta}u_t(u_x)^2\eta_x+2\frac{\partial^3\xi^1}{\partial\eta^2\partial u}u_tu_x(\eta_x)^2+2\frac{\partial^2\xi^1}{\partial u\partial\eta}u_{xt}u_x\eta_x\right.$$
$$\left.+2\frac{\partial^2\xi^1}{\partial u\partial\eta}u_tu_{xx}\eta_x+2\frac{\partial^2\xi^1}{\partial u\partial\eta}u_tu_x\eta_{xx}\right)$$
$$-\left(\frac{\partial^2\xi^1}{\partial x\partial u}u_tu_{xx}+\frac{\partial^2\xi^1}{\partial u^2}u_tu_{xx}u_x+\frac{\partial^2\xi^1}{\partial u\partial\eta}u_tu_{xx}\eta_x+\frac{\partial\xi^1}{\partial u}u_{xt}u_{xx}+\frac{\partial\xi^1}{\partial u}u_tu_{xxx}\right)$$
$$-\left(\frac{\partial^2\xi^1}{\partial\eta\partial x}u_t\eta_{xx}+\frac{\partial^2\xi^1}{\partial\eta\partial u}u_tu_x\eta_{xx}+\frac{\partial^2\xi^1}{\partial\eta^2}u_t\eta_{xx}\eta_x+\frac{\partial\xi^1}{\partial\eta}u_{xt}\eta_{xx}+\frac{\partial\xi^1}{\partial\eta}u_t\eta_{xxx}\right)$$
$$-\left(2\frac{\partial^2\xi^1}{\partial x\partial u}u_xu_{xt}+2\frac{\partial^2\xi^1}{\partial u^2}(u_x)^2u_{xt}+2\frac{\partial^2\xi^1}{\partial u\partial\eta}u_xu_{xt}\eta_x+2\frac{\partial\xi^1}{\partial u}u_{xx}u_{xt}+3\frac{\partial\xi^1}{\partial u}u_xu_{xxt}\right)$$
$$-\left(2\frac{\partial^2\xi^1}{\partial\eta\partial x}u_{xt}\eta_x+2\frac{\partial^2\xi^1}{\partial\eta\partial u}u_{xt}u_x\eta_x+2\frac{\partial^2\xi^1}{\partial\eta^2}u_{xt}(\eta_x)^2+3\frac{\partial\xi^1}{\partial\eta}u_{xxt}\eta_x+2\frac{\partial\xi^1}{\partial\eta}u_{xt}\eta_{xx}\right)$$
$$-\left(2\frac{\partial^3\xi^2}{\partial x^2\partial u}(u_x)^2+2\frac{\partial^3\xi^2}{\partial u^2\partial x}(u_x)^3+2\frac{\partial^3\xi^2}{\partial x\partial u\partial\eta}(u_x)^2\eta_x+4\frac{\partial^2\xi^2}{\partial x\partial u}u_xu_{xx}\right)$$
$$-\left(2\frac{\partial^3\xi^2}{\partial x^2\partial\eta}u_x\eta_x+2\frac{\partial^3\xi^2}{\partial x\partial u\partial\eta}(u_x)^2\eta_x+2\frac{\partial^3\xi^2}{\partial\eta^2\partial x}u_x(\eta_x)^2+2\frac{\partial^2\xi^2}{\partial x\partial\eta}u_{xx}\eta_x+2\frac{\partial^2\xi^2}{\partial x\partial\eta}u_x\eta_{xx}\right)$$
$$-\left(2\frac{\partial^2\xi^2}{\partial x^2}u_{xx}+2\frac{\partial^2\xi^2}{\partial x\partial u}u_{xx}u_x+2\frac{\partial^2\xi^2}{\partial x\partial\eta}u_{xx}\eta_x+3\frac{\partial\xi^2}{\partial x}u_{xxx}\right)$$

$$
\begin{aligned}
&-\left(3\frac{\partial^2\xi^2}{\partial x\partial u}u_x u_{xx}+3\frac{\partial^2\xi^2}{\partial u^2}(u_x)^2u_{xx}+3\frac{\partial^2\xi^2}{\partial u\partial\eta}u_x u_{xx}\eta_x+3\frac{\partial\xi^2}{\partial u}(u_{xx})^2+4\frac{\partial\xi^2}{\partial u}u_x u_{xxx}\right)\\
&-\left(2\frac{\partial^2\xi^2}{\partial x\partial\eta}u_{xx}\eta_x+2\frac{\partial^2\xi^2}{\partial u\partial\eta}u_x u_{xx}\eta_x+2\frac{\partial^2\xi^2}{\partial\eta^2}u_{xx}(\eta_x)^2+3\frac{\partial\xi^2}{\partial\eta}u_{xxx}\eta_x+2\frac{\partial\xi^2}{\partial\eta}u_{xx}\eta_{xx}\right)\\
&-\left(\frac{\partial^2\xi^2}{\partial\eta\partial x}u_x\eta_{xx}+\frac{\partial^2\xi^2}{\partial u\partial\eta}(u_x)^2\eta_{xx}+\frac{\partial^2\xi^2}{\partial\eta^2}u_x\eta_{xx}\eta_x+\frac{\partial\xi^2}{\partial\eta}u_{xx}\eta_{xx}+\frac{\partial\xi^2}{\partial\eta}u_x\eta_{xxx}\right).
\end{aligned}
\tag{B.3}
$$

The first determining equation is given by

$$
\begin{aligned}
0=&\left(\frac{\partial\psi^2}{\partial t}+\frac{\partial\psi^2}{\partial u}u_t+\frac{\partial\psi^2}{\partial\eta}\eta_t-\frac{\partial\xi^1}{\partial t}\eta_t-\frac{\partial\xi^1}{\partial u}\eta_t u_t-\frac{\partial\xi^1}{\partial\eta}(\eta_t)^2-\frac{\partial\xi^2}{\partial t}\eta_x-\frac{\partial\xi^2}{\partial u}\eta_x u_t-\frac{\partial\xi^2}{\partial\eta}\eta_t\eta_x\right)\\
&+h\left(\frac{\partial\psi^1}{\partial x}+\frac{\partial\psi^1}{\partial u}u_x+\frac{\partial\psi^1}{\partial\eta}\eta_x-\frac{\partial\xi^1}{\partial x}u_t-\frac{\partial\xi^1}{\partial u}u_t u_x-\frac{\partial\xi^1}{\partial\eta}u_t\eta_x-\frac{\partial\xi^2}{\partial x}u_x-\frac{\partial\xi^2}{\partial u}(u_x)^2-\frac{\partial\xi^2}{\partial\eta}u_x\eta_x\right)\\
&-\frac{h^3}{6}\Bigg[\left(\frac{\partial^3\psi^1}{\partial x^3}+\frac{\partial^3\psi^1}{\partial x^2\partial u}u_x+\frac{\partial^3\psi^1}{\partial x^2\partial\eta}\eta_x\right)+\left(2\frac{\partial^3\psi^1}{\partial x^2\partial u}u_x+2\frac{\partial^3\psi^1}{\partial u^2\partial x}(u_x)^2+2\frac{\partial^3\psi^1}{\partial u\partial x\partial\eta}u_x\eta_x+2\frac{\partial^2\psi^1}{\partial u\partial x}u_{xx}\right)\\
&+\left(2\frac{\partial^3\psi^1}{\partial x^2\partial\eta}\eta_x+2\frac{\partial^3\psi^1}{\partial u\partial\eta\partial x}u_x\eta_x+2\frac{\partial^3\psi^1}{\partial\eta^2\partial x}(\eta_x)^2+2\frac{\partial^2\psi^1}{\partial\eta\partial x}\eta_{xx}\right)\\
&+\left(\frac{\partial^3\psi^1}{\partial u^2\partial x}(u_x)^2+\frac{\partial^3\psi^1}{\partial u^3}(u_x)^3+\frac{\partial^3\psi^1}{\partial u^2\partial\eta}(u_x)^2\eta_x+2\frac{\partial^2\psi^1}{\partial u^2}u_x u_{xx}\right)\\
&+\left(\frac{\partial^3\psi^1}{\partial\eta^2\partial x}(\eta_x)^2+\frac{\partial^3\psi^1}{\partial\eta^2\partial u}u_x(\eta_x)^2+\frac{\partial^3\psi^1}{\partial\eta^3}(\eta_x)^3+2\frac{\partial^2\psi^1}{\partial\eta^2}\eta_x\eta_{xx}\right)\\
&+\left(2\frac{\partial^3\psi^1}{\partial u\partial\eta\partial x}u_x\eta_x+2\frac{\partial^3\psi^1}{\partial u^2\partial\eta}(u_x)^2\eta_x+2\frac{\partial^3\psi^1}{\partial\eta^2\partial u}u_x(\eta_x)^2+2\frac{\partial^2\psi^1}{\partial u\partial\eta}\eta_x u_{xx}+2\frac{\partial^2\psi^1}{\partial u\partial\eta}u_x\eta_{xx}\right)\\
&+\left(\frac{\partial^2\psi^1}{\partial u\partial x}u_{xx}+\frac{\partial^2\psi^1}{\partial u^2}u_x u_{xx}+\frac{\partial^2\psi^1}{\partial u\partial\eta}u_{xx}\eta_x+\frac{\partial\psi^1}{\partial u}u_{xxx}\right)\\
&+\left(\frac{\partial^2\psi^1}{\partial\eta\partial x}\eta_{xx}+\frac{\partial^2\psi^1}{\partial u\partial\eta}u_x\eta_{xx}+\frac{\partial^2\psi^1}{\partial\eta^2}\eta_{xx}\eta_x+\frac{\partial\psi^1}{\partial\eta}\eta_{xxx}\right)\\
&-\left(\frac{\partial^3\xi^1}{\partial x^3}u_t+\frac{\partial^3\xi^1}{\partial x^2\partial u}u_t u_x+\frac{\partial^3\xi^1}{\partial x^2\partial\eta}u_t\eta_x+\frac{\partial^2\xi^1}{\partial x^2}u_{xt}\right)\\
&-\left(\frac{\partial^3\xi^2}{\partial x^3}u_x+\frac{\partial^3\xi^2}{\partial x^2\partial u}(u_x)^2+\frac{\partial^3\xi^2}{\partial x^2\partial\eta}u_x\eta_x+\frac{\partial^2\xi^2}{\partial x^2}u_{xx}\right)\\
&-\left(2\frac{\partial^3\xi^1}{\partial x^2\partial u}u_t u_x+2\frac{\partial^3\xi^1}{\partial u^2\partial x}u_t(u_x)^2+2\frac{\partial^3\xi^1}{\partial x\partial u\partial\eta}u_t u_x\eta_x+2\frac{\partial^2\xi^1}{\partial x\partial u}u_{xx}u_t+2\frac{\partial^2\xi^1}{\partial x\partial u}u_{xt}u_x\right)\\
&-\left(2\frac{\partial^3\xi^1}{\partial x^2\partial\eta}u_t\eta_x+2\frac{\partial^3\xi^1}{\partial x\partial u\partial\eta}u_t u_x\eta_x+2\frac{\partial^3\xi^1}{\partial\eta^2\partial x}u_t(\eta_x)^2+2\frac{\partial^2\xi^1}{\partial x\partial\eta}u_{xt}\eta_x+2\frac{\partial^2\xi^1}{\partial x\partial\eta}u_t\eta_{xx}\right)\\
&-\left(2\frac{\partial^2\xi^1}{\partial x^2}u_{xt}+2\frac{\partial^2\xi^1}{\partial x\partial u}u_{xt}u_x+2\frac{\partial^2\xi^1}{\partial x\partial\eta}u_{xt}\eta_x+3\frac{\partial\xi^1}{\partial x}u_{xxt}\right)\\
&-\left(\frac{\partial^3\xi^1}{\partial u^2\partial x}u_t(u_x)^2+\frac{\partial^3\xi^1}{\partial u^3}u_t(u_x)^3+\frac{\partial^3\xi^1}{\partial u^2\partial\eta}u_t(u_x)^2\eta_x+\frac{\partial^2\xi^1}{\partial u^2}u_{xt}(u_x)^2+2\frac{\partial^2\xi^1}{\partial u^2}u_x u_{xx}u_t\right)\\
&-\left(\frac{\partial^3\xi^1}{\partial\eta^2\partial x}u_t(\eta_x)^2+\frac{\partial^3\xi^1}{\partial\eta^2\partial u}u_t u_x(\eta_x)^2+\frac{\partial^3\xi^1}{\partial\eta^3}u_t(\eta_x)^3+\frac{\partial^2\xi^1}{\partial\eta^2}u_{xt}(\eta_x)^2+2\frac{\partial^2\xi^1}{\partial\eta^2}\eta_x\eta_{xx}u_t\right)\\
&-\left(\frac{\partial^3\xi^2}{\partial u^2\partial x}(u_x)^3+\frac{\partial^3\xi^2}{\partial u^3}(u_x)^4+\frac{\partial^3\xi^2}{\partial u^2\partial\eta}(u_x)^3\eta_x+3\frac{\partial^2\xi^2}{\partial u^2}(u_x)^2u_{xx}\right)
\end{aligned}
$$

$$-\left(2\frac{\partial^3\xi^2}{\partial u\partial\eta\partial x}(u_x)^2\eta_x+2\frac{\partial^3\xi^2}{\partial u^2\partial\eta}(u_x)^3\eta_x+2\frac{\partial^3\xi^2}{\partial\eta^2\partial u}(u_x)^2(\eta_x)^2+2\frac{\partial^2\xi^2}{\partial u\partial\eta}\eta_{xx}(u_x)^2+4\frac{\partial^2\xi^2}{\partial u\partial\eta}u_x u_{xx}\eta_x\right)$$

$$-\left(\frac{\partial^3\xi^2}{\partial\eta^2\partial x}u_x(\eta_x)^2+\frac{\partial^3\xi^2}{\partial\eta^2\partial u}(u_x)^2(\eta_x)^2+\frac{\partial^3\xi^2}{\partial\eta^3}u_x(\eta_x)^3+\frac{\partial^2\xi^2}{\partial\eta^2}u_{xx}(\eta_x)^2+2\frac{\partial^2\xi^2}{\partial\eta^2}\eta_x\eta_{xx}u_x\right)$$
$$-\left(2\frac{\partial^3\xi^1}{\partial u\partial x\partial\eta}u_t u_x\eta_x+2\frac{\partial^3\xi^1}{\partial u^2\partial\eta}u_t(u_x)^2\eta_x+2\frac{\partial^3\xi^1}{\partial\eta^2\partial u}u_t u_x(\eta_x)^2+2\frac{\partial^2\xi^1}{\partial u\partial\eta}u_{xt}u_x\eta_x\right.$$
$$\left.+2\frac{\partial^2\xi^1}{\partial u\partial\eta}u_t u_{xx}\eta_x+2\frac{\partial^2\xi^1}{\partial u\partial\eta}u_t u_x\eta_{xx}\right)$$
$$-\left(\frac{\partial^2\xi^1}{\partial x\partial u}u_t u_{xx}+\frac{\partial^2\xi^1}{\partial u^2}u_t u_{xx}u_x+\frac{\partial^2\xi^1}{\partial u\partial\eta}u_t u_{xx}\eta_x+\frac{\partial\xi^1}{\partial u}u_{xt}u_{xx}+\frac{\partial\xi^1}{\partial u}u_t u_{xxx}\right)$$
$$-\left(\frac{\partial^2\xi^1}{\partial\eta\partial x}u_t\eta_{xx}+\frac{\partial^2\xi^1}{\partial\eta\partial u}u_t u_x\eta_{xx}+\frac{\partial^2\xi^1}{\partial\eta^2}u_t\eta_{xx}\eta_x+\frac{\partial\xi^1}{\partial\eta}u_{xt}\eta_{xx}+\frac{\partial\xi^1}{\partial\eta}u_t\eta_{xxx}\right)$$
$$-\left(2\frac{\partial^2\xi^1}{\partial x\partial u}u_x u_{xt}+2\frac{\partial^2\xi^1}{\partial u^2}(u_x)^2u_{xt}+2\frac{\partial^2\xi^1}{\partial u\partial\eta}u_x u_{xt}\eta_x+2\frac{\partial\xi^1}{\partial u}u_{xx}u_{xt}+3\frac{\partial\xi^1}{\partial u}u_x u_{xxt}\right)$$
$$-\left(2\frac{\partial^2\xi^1}{\partial\eta\partial x}u_{xt}\eta_x+2\frac{\partial^2\xi^1}{\partial\eta\partial u}u_{xt}u_x\eta_x+2\frac{\partial^2\xi^1}{\partial\eta^2}u_{xt}(\eta_x)^2+3\frac{\partial\xi^1}{\partial\eta}u_{xxt}\eta_x+2\frac{\partial\xi^1}{\partial\eta}u_{xt}\eta_{xx}\right)$$
$$-\left(2\frac{\partial^3\xi^2}{\partial x^2\partial u}(u_x)^2+2\frac{\partial^3\xi^2}{\partial u^2\partial x}(u_x)^3+2\frac{\partial^3\xi^2}{\partial x\partial u\partial\eta}(u_x)^2\eta_x+4\frac{\partial^2\xi^2}{\partial x\partial u}u_x u_{xx}\right)$$
$$-\left(2\frac{\partial^3\xi^2}{\partial x^2\partial\eta}u_x\eta_x+2\frac{\partial^3\xi^2}{\partial x\partial u\partial\eta}(u_x)^2\eta_x+2\frac{\partial^3\xi^2}{\partial\eta^2\partial x}u_x(\eta_x)^2+2\frac{\partial^2\xi^2}{\partial x\partial\eta}u_{xx}\eta_x+2\frac{\partial^2\xi^2}{\partial x\partial\eta}u_x\eta_{xx}\right)$$
$$-\left(2\frac{\partial^2\xi^2}{\partial x^2}u_{xx}+2\frac{\partial^2\xi^2}{\partial x\partial u}u_{xx}u_x+2\frac{\partial^2\xi^2}{\partial x\partial\eta}u_{xx}\eta_x+3\frac{\partial\xi^2}{\partial x}u_{xxx}\right)$$
$$-\left(3\frac{\partial^2\xi^2}{\partial x\partial u}u_x u_{xx}+3\frac{\partial^2\xi^2}{\partial u^2}(u_x)^2u_{xx}+3\frac{\partial^2\xi^2}{\partial u\partial\eta}u_x u_{xx}\eta_x+3\frac{\partial\xi^2}{\partial u}(u_{xx})^2+4\frac{\partial\xi^2}{\partial u}u_x u_{xxx}\right)$$
$$-\left(2\frac{\partial^2\xi^2}{\partial x\partial\eta}u_{xx}\eta_x+2\frac{\partial^2\xi^2}{\partial u\partial\eta}u_x u_{xx}\eta_x+2\frac{\partial^2\xi^2}{\partial\eta^2}u_{xx}(\eta_x)^2+3\frac{\partial\xi^2}{\partial\eta}u_{xxx}\eta_x+2\frac{\partial\xi^2}{\partial\eta}u_{xx}\eta_{xx}\right)$$
$$-\left(\frac{\partial^2\xi^2}{\partial\eta\partial x}u_x\eta_{xx}+\frac{\partial^2\xi^2}{\partial u\partial\eta}(u_x)^2\eta_{xx}+\frac{\partial^2\xi^2}{\partial\eta^2}u_x\eta_{xx}\eta_x+\frac{\partial\xi^2}{\partial\eta}u_{xx}\eta_{xx}+\frac{\partial\xi^2}{\partial\eta}u_x\eta_{xxx}\right)\Bigg]. \qquad \text{(B.4)}$$

Appendix C

Multipliers for the NB equations

We recall the system of dimensionless NB equations given by

$$\frac{\partial \eta}{\partial t}+\frac{\partial u}{\partial x}+\epsilon\frac{\partial}{\partial x}\left(u\eta\right)=\frac{\mu^2}{6}\frac{\partial^3 u}{\partial x^3}, \tag{C.1}$$

$$\frac{\partial u}{\partial t}+\epsilon u\frac{\partial u}{\partial x}+\frac{\partial \eta}{\partial x}=\frac{\mu^2}{2}\frac{\partial^3 u}{\partial x^2\partial t}, \tag{C.2}$$

where u is the fluid velocity of the solitary wave in the x-direction and η is the surface elevation of the solitary wave.

C.1 Multipliers of the form $\Lambda_{1,2}(t,x,u,\eta,u_t,\eta_t,u_x,\eta_x)$

In order to derive the conservation laws for the system given by (C.1)-(C.2), we consider multipliers $\Lambda_1(t,x,u,\eta,u_t,u_x,\eta_t,\eta_x)$ and $\Lambda_2(t,x,u,\eta,u_t,u_x,\eta_t,\eta_x)$. By properties of multipliers we have

$$\Lambda_1\left(\eta_t+u_x+\epsilon\eta_x u+\epsilon\eta u_x-\frac{\mu^2}{6}u_{xxx}\right)+\Lambda_2\left(u_t+\epsilon u_x u+\eta_x-\frac{\mu^2}{2}u_{xxt}\right)=D_t T^1+D_x T^2, \tag{C.3}$$

where $T=(T^1,T^2)$ are the components of the conserved vector and D_t and D_x represent the total derivative operators.

In order to derive the determining equations, we make use of the Euler operator to annihilate the divergence of the conservation law.

The determining equations can be found by expanding

$$E_u\left[\Lambda_1\left(\eta_t+u_x+\epsilon\eta_x u+\epsilon\eta u_x-\frac{\mu^2}{6}u_{xxx}\right)+\Lambda_2\left(u_t+\epsilon u_x u+\eta_x-\frac{\mu^2}{2}u_{xxt}\right)\right]=0, \tag{C.4}$$

$$E_\eta\left[\Lambda_1\left(\eta_t+u_x+\epsilon\eta_x u+\epsilon\eta u_x-\frac{\mu^2}{6}u_{xxx}\right)+\Lambda_2\left(u_t+\epsilon u_x u+\eta_x-\frac{\mu^2}{2}u_{xxt}\right)\right]=0, \tag{C.5}$$

where E_u and E_η represent the Euler operators.

By expanding (C.5) we obtain the following determining equation

$$\begin{aligned}
&\frac{\partial\Lambda_1}{\partial\eta}\left(\eta_t+u_x+\epsilon\eta_x u+\epsilon\eta u_x-\frac{\mu^2}{6}u_{xxx}\right)+\Lambda_1\epsilon u_x\\
&+\frac{\partial\Lambda_2}{\partial\eta}\left(u_t+\epsilon u_x u+\eta_x-\frac{\mu^2}{2}u_{xxt}\right)\\
&-\left[\frac{\partial^2\Lambda_1}{\partial\eta_t\partial t}\left(\eta_t+u_x+\epsilon\eta_x u+\epsilon\eta u_x-\frac{\mu^2}{6}u_{xxx}\right)+\frac{\partial\Lambda_1}{\partial t}\right.\\
&\left.+\frac{\partial^2\Lambda_2}{\partial\eta_t\partial t}\left(u_t+\epsilon u_x u+\eta_x-\frac{\mu^2}{2}u_{xxt}\right)\right]\\
&-u_t\left[\frac{\partial^2\Lambda_1}{\partial\eta_t\partial u}\left(\eta_t+u_x+\epsilon\eta_x u+\epsilon\eta u_x-\frac{\mu^2}{6}u_{xxx}\right)+\frac{\partial\Lambda_1}{\partial\eta_t}\epsilon\eta_x+\frac{\partial\Lambda_1}{\partial u}\right.\\
&\left.+\frac{\partial^2\Lambda_2}{\partial\eta_t\partial u}\left(u_t+\epsilon u_x u+\eta_x-\frac{\mu^2}{2}u_{xxt}\right)+\frac{\partial\Lambda_2}{\partial\eta_t}\epsilon u_x\right]\\
&-\eta_t\left[\frac{\partial^2\Lambda_1}{\partial\eta_t\partial\eta}\left(\eta_t+u_x+\epsilon\eta_x u+\epsilon\eta u_x-\frac{\mu^2}{6}u_{xxx}\right)+\frac{\partial\Lambda_1}{\partial\eta_t}\epsilon u_x+\frac{\partial\Lambda_1}{\partial\eta}\right.\\
&\left.+\frac{\partial^2\Lambda_2}{\partial\eta_t^2}\left(u_t+\epsilon u_x u+\eta_x-\frac{\mu^2}{2}u_{xxt}\right)+\frac{\partial\Lambda_2}{\partial\eta_t}(0)\right]\\
&-u_{tt}\left[\frac{\partial^2\Lambda_1}{\partial\eta_t\partial u_t}\left(\eta_t+u_x+\epsilon\eta_x u+\epsilon\eta u_x-\frac{\mu^2}{6}u_{xxx}\right)+\frac{\partial\Lambda_1}{\partial\eta_t}(0)+\frac{\partial\Lambda_1}{\partial u_t}\right.\\
&\left.+\frac{\partial^2\Lambda_2}{\partial\eta_t\partial u_t}\left(u_t+\epsilon u_x u+\eta_x-\frac{\mu^2}{2}u_{xxt}\right)+\frac{\partial\Lambda_2}{\partial\eta_t}(1)\right]\\
&-u_{xt}\left[\frac{\partial^2\Lambda_1}{\partial\eta_t\partial u_x}\left(\eta_t+u_x+\epsilon\eta_x u+\epsilon\eta u_x-\frac{\mu^2}{6}u_{xxx}\right)+\frac{\partial\Lambda_1}{\partial\eta_t}(1+\epsilon\eta)+\frac{\partial\Lambda_1}{\partial u_x}\right.\\
&\left.+\frac{\partial^2\Lambda_2}{\partial\eta_t\partial u_x}\left(u_t+\epsilon u_x u+\eta_x-\frac{\mu^2}{2}u_{xxt}\right)+\frac{\partial\Lambda_2}{\partial\eta_t}\epsilon u\right]\\
&-\eta_{tt}\left[\frac{\partial^2\Lambda_1}{\partial\eta_t^2}\left(\eta_t+u_x+\epsilon\eta_x u+\epsilon\eta u_x-\frac{\mu^2}{6}u_{xxx}\right)+\frac{\partial\Lambda_1}{\partial\eta_t}(1)+\frac{\partial\Lambda_1}{\partial\eta_t}\right.\\
&\left.+\frac{\partial^2\Lambda_2}{\partial\eta_t^2}\left(u_t+\epsilon u_x u+\eta_x-\frac{\mu^2}{2}u_{xxt}\right)+\frac{\partial\Lambda_2}{\partial\eta_t}(0)\right]\\
&-\eta_{xt}\left[\frac{\partial^2\Lambda_1}{\partial\eta_t\partial\eta_x}\left(\eta_t+u_x+\epsilon\eta_x u+\epsilon\eta u_x-\frac{\mu^2}{6}u_{xxx}\right)+\frac{\partial\Lambda_1}{\partial\eta_t}\epsilon u+\frac{\partial\Lambda_1}{\partial\eta_x}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\partial^2\Lambda_2}{\partial\eta_t\partial\eta_x}\left(u_t+\epsilon u_x u+\eta_x-\frac{\mu^2}{2}u_{xxt}\right)+\frac{\partial\Lambda_2}{\partial\eta_t}(1)\Bigg]\\
&-u_{xxxt}\left[-\frac{\mu^2}{6}\frac{\partial\Lambda_1}{\partial\eta_t}\right]-u_{xxtt}\left[-\frac{\mu^2}{2}\frac{\partial\Lambda_2}{\partial\eta_t}\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\Bigg[\frac{\partial^2\Lambda_1}{\partial\eta_x\partial x}\left(\eta_t+u_x+\epsilon\eta_x u+\epsilon\eta u_x-\frac{\mu^2}{6}u_{xxx}\right)+\frac{\partial\Lambda_1}{\partial x}\epsilon u\\
&+\frac{\partial^2\Lambda_2}{\partial\eta_x\partial x}\left(u_t+\epsilon u_x u+\eta_x-\frac{\mu^2}{2}u_{xxt}\right)+\frac{\partial\Lambda_2}{\partial x}\Bigg]\\
&-u_x\Bigg[\frac{\partial^2\Lambda_1}{\partial\eta_x\partial u}\left(\eta_t+u_x+\epsilon\eta_x u+\epsilon\eta u_x-\frac{\mu^2}{6}u_{xxx}\right)+\frac{\partial\Lambda_1}{\partial\eta_x}\epsilon\eta_x+\frac{\partial\Lambda_1}{\partial u}\epsilon u+\Lambda_1\epsilon\\
&+\frac{\partial^2\Lambda_2}{\partial\eta_x\partial u}\left(u_t+\epsilon u_x u+\eta_x-\frac{\mu^2}{2}u_{xxt}\right)+\frac{\partial\Lambda_2}{\partial\eta_x}\epsilon u_x+\frac{\partial\Lambda_2}{\partial u}\Bigg]\\
&-\eta_x\Bigg[\frac{\partial^2\Lambda_1}{\partial\eta_x\partial\eta}\left(\eta_t+u_x+\epsilon\eta_x u+\epsilon\eta u_x-\frac{\mu^2}{6}u_{xxx}\right)+\frac{\partial\Lambda_1}{\partial\eta_x}\epsilon u_x+\frac{\partial\Lambda_1}{\partial\eta}\epsilon u\\
&+\frac{\partial^2\Lambda_2}{\partial\eta_x\partial\eta}\left(u_t+\epsilon u_x u+\eta_x-\frac{\mu^2}{2}u_{xxt}\right)+\frac{\partial\Lambda_2}{\partial\eta}\Bigg]\\
&-u_{xt}\Bigg[\frac{\partial^2\Lambda_1}{\partial\eta_x\partial u_t}\left(\eta_t+u_x+\epsilon\eta_x u+\epsilon\eta u_x-\frac{\mu^2}{6}u_{xxx}\right)+\frac{\partial\Lambda_1}{\partial u_t}\epsilon u\\
&+\frac{\partial^2\Lambda_2}{\partial\eta_x\partial u_t}\left(u_t+\epsilon u_x u+\eta_x-\frac{\mu^2}{2}u_{xxt}\right)+\frac{\partial\Lambda_2}{\partial\eta_x}(1)+\frac{\partial\Lambda_2}{\partial u_t}\Bigg]\\
&-u_{xx}\Bigg[\frac{\partial^2\Lambda_1}{\partial\eta_x\partial u_x}\left(\eta_t+u_x+\epsilon\eta_x u+\epsilon\eta u_x-\frac{\mu^2}{6}u_{xxx}\right)+\frac{\partial\Lambda_1}{\partial\eta_x}(1+\epsilon\eta)+\frac{\partial\Lambda_1}{\partial u_x}\\
&+\frac{\partial^2\Lambda_2}{\partial\eta_x\partial u_x}\left(u_t+\epsilon u_x u+\eta_x-\frac{\mu^2}{2}u_{xxt}\right)+\frac{\partial\Lambda_2}{\partial\eta_x}\epsilon u+\frac{\partial\Lambda_2}{\partial u_x}\Bigg]\\
&-\eta_{tx}\Bigg[\frac{\partial^2\Lambda_1}{\partial\eta_t\partial\eta_x}\left(\eta_t+u_x+\epsilon\eta_x u+\epsilon\eta u_x-\frac{\mu^2}{6}u_{xxx}\right)+\frac{\partial\Lambda_1}{\partial\eta_x}(1)+\frac{\partial\Lambda_1}{\partial\eta_t}\epsilon u\\
&+\frac{\partial^2\Lambda_2}{\partial\eta_t\partial\eta_x}\left(u_t+\epsilon u_x u+\eta_x-\frac{\mu^2}{2}u_{xxt}\right)+\frac{\partial\Lambda_2}{\partial\eta_x}(0)+\frac{\partial\Lambda_2}{\partial\eta_t}\Bigg]\\
&-\eta_{xx}\Bigg[\frac{\partial^2\Lambda_1}{\partial\eta_x^2}\left(\eta_t+u_x+\epsilon\eta_x u+\epsilon\eta u_x-\frac{\mu^2}{6}u_{xxx}\right)+\frac{\partial\Lambda_1}{\partial\eta_x}\epsilon u+\frac{\partial\Lambda_1}{\partial\eta_x}\epsilon u\\
&+\frac{\partial^2\Lambda_2}{\partial\eta_x^2}\left(u_t+\epsilon u_x u+\eta_x-\frac{\mu^2}{2}u_{xxt}\right)+\frac{\partial\Lambda_2}{\partial\eta_x}(1)+\frac{\partial\Lambda_2}{\partial\eta_x}\Bigg]\\
&-u_{xxxt}\left[-\frac{\mu^2}{2}\frac{\partial\Lambda_2}{\partial\eta_x}\right]-u_{xxxx}\left[-\frac{\mu^2}{6}\frac{\partial\Lambda_1}{\partial\eta_x}\right]=0.
\end{aligned}
\tag{C.6}
$$

Equation (C.6) can be split into derivative products and powers of u and η as follows

$$
u_{xxxx}:\frac{\mu^2}{6}\frac{\partial\Lambda_1}{\partial\eta_x}=0. \tag{C.7}
$$

Thus

$$\Lambda_1 = \Lambda_1(t, x, u, \eta, u_t, u_x, \eta_t), \tag{C.8}$$

$$\Lambda_2 = \Lambda_2(t, x, u, \eta, u_t, u_x, \eta_t, \eta_x). \tag{C.9}$$

Next, we separate by the following derivative of u

$$u_{xxtt} : \frac{\mu^2}{2} \frac{\partial \Lambda_2}{\partial \eta_t} = 0. \tag{C.10}$$

Thus

$$\Lambda_1 = \Lambda_1(t, x, u, \eta, u_t, u_x, \eta_t), \tag{C.11}$$

$$\Lambda_2 = \Lambda_2(t, x, u, \eta, u_t, u_x, \eta_x). \tag{C.12}$$

Next, we separate by the following derivative of u

$$u_{xxxt} : \frac{\partial \Lambda_1}{\partial \eta_t} + 3 \frac{\partial \Lambda_2}{\partial \eta_x} = 0. \tag{C.13}$$

Next, we separate by the following derivative of u

$$\begin{aligned} u_{xxx} : -\frac{\mu^2}{6} \Bigg(& \frac{\partial \Lambda_1}{\partial \eta} - \frac{\partial^2 \Lambda_1}{\partial \eta_t \partial t} - u_t \frac{\partial^2 \Lambda_1}{\partial \eta_t \partial u} - \eta_t \frac{\partial^2 \Lambda_1}{\partial \eta \partial \eta_t} - u_{tt} \frac{\partial^2 \Lambda_1}{\partial u_t \partial \eta_t} \\ & - u_{xt} \frac{\partial^2 \Lambda_1}{\partial u_x \partial \eta_t} - \eta_{tt} \frac{\partial^2 \Lambda_1}{\partial \eta_t^2} \Bigg) = 0. \end{aligned} \tag{C.14}$$

Next, we separate (C.14) by the following derivative of u

$$u_{tt} : \frac{\partial}{\partial u_t} \left(\frac{\partial \Lambda_1}{\partial \eta_t} \right) = 0, \tag{C.15}$$

and by the following derivative of u

$$u_{xt} : \frac{\partial}{\partial u_x} \left(\frac{\partial \Lambda_1}{\partial \eta_t} \right) = 0, \tag{C.16}$$

and by the following derivative of η

$$\eta_{tt} : \frac{\partial}{\partial \eta_t} \left(\frac{\partial \Lambda_1}{\partial \eta_t} \right) = 0. \tag{C.17}$$

From equation (C.15) we know

$$\frac{\partial}{\partial u_t} \left(\frac{\partial \Lambda_1}{\partial \eta_t} (t, x, u, \eta, u_t, u_x, \eta_x) \right) = 0, \tag{C.18}$$

and from equations (C.15), (C.16) and (C.17) we know

$$\frac{\partial \Lambda_1}{\partial \eta_t} = \frac{\partial \Lambda_1}{\partial \eta_t}\big(t, x, u, \eta\big), \tag{C.19}$$

therefore, we have

$$\Lambda_1 = \eta_t f\big(t, x, u, \eta\big) + g\big(t, x, u, \eta, u_t, u_x\big). \tag{C.20}$$

Also, we have the following remaining from (C.14)

$$\frac{\partial \Lambda_1}{\partial \eta} - \frac{\partial^2 \Lambda_1}{\partial \eta_t \partial t} - u_t \frac{\partial^2 \Lambda_1}{\partial \eta_t \partial u} - \eta_t \frac{\partial^2 \Lambda_1}{\partial \eta \partial \eta_t} = 0. \tag{C.21}$$

Substituting (C.20) into (C.21) we get

$$\frac{\partial g}{\partial \eta}(t, x, u, \eta, u_t, u_x) - \frac{\partial f}{\partial t}(t, x, u, \eta) - u_t \frac{\partial f}{\partial u}(t, x, u, \eta) = 0. \tag{C.22}$$

Substituting (C.20) into (C.13) and integrating with respect to η_x we get

$$\Lambda_2(t, x, u, \eta, u_t, u_x, \eta_x) = -\frac{1}{3}\eta_x f(t, x, u, \eta) + m(t, x, u, \eta, u_t, u_x). \tag{C.23}$$

Thus

$$\Lambda_1(t, x, u, \eta, u_t, u_x, \eta_t) = \eta_t f(t, x, u, \eta) + g(t, x, u, \eta, u_t, u_x), \tag{C.24}$$

$$\Lambda_2(t, x, u, \eta, u_t, u_x, \eta_x) = -\frac{1}{3}\eta_x f(t, x, u, \eta) + m(t, x, u, \eta, u_t, u_x). \tag{C.25}$$

Next, we separate by the following derivative of u

$$\begin{aligned} u_{xxt} : \frac{\mu^2}{2}\bigg(\frac{\partial \Lambda_2}{\partial \eta} + \frac{\partial^2 \Lambda_2}{\partial x \partial \eta_x} + u_x \frac{\partial^2 \Lambda_2}{\partial u \partial \eta_x} + \eta_x \frac{\partial^2 \Lambda_2}{\partial \eta \partial \eta_x} \\ + u_{xt} \frac{\partial^2 \Lambda_2}{\partial u_t \partial \eta_x} + u_{xx} \frac{\partial^2 \Lambda_2}{\partial u_x \partial \eta_x} + \eta_{xx} \frac{\partial^2 \Lambda_2}{\partial \eta_x^2}\bigg) = 0. \end{aligned} \tag{C.26}$$

Next, we separate (C.26) by the following derivative of u

$$u_{xx} : \frac{\partial}{\partial u_x}\left(\frac{\partial \Lambda_2}{\partial \eta_x}\right) = 0, \tag{C.27}$$

and by the following derivative of u

$$u_{xt} : \frac{\partial}{\partial u_t}\left(\frac{\partial \Lambda_2}{\partial \eta_x}\right) = 0, \tag{C.28}$$

and by the following derivative of η

$$\eta_{xx}: \frac{\partial}{\partial \eta_x}\left(\frac{\partial \Lambda_2}{\partial \eta_x}\right) = 0, \tag{C.29}$$

where equations (C.27), (C.28) and (C.29) are all identically satisfied.

Also, we have the following remaining from (C.26)

$$\frac{\partial \Lambda_2}{\partial \eta} + \frac{\partial^2 \Lambda_2}{\partial x \partial \eta_x} + u_x \frac{\partial^2 \Lambda_2}{\partial u \partial \eta_x} + \eta_x \frac{\partial^2 \Lambda_2}{\partial \eta \partial \eta_x} = 0. \tag{C.30}$$

Substituting (C.25) into (C.30) we get

$$\begin{aligned} &\eta_x \frac{\partial f}{\partial \eta}(t,x,u,\eta) + \frac{\partial f}{\partial x}(t,x,u,\eta) + u_x \frac{\partial f}{\partial u}(t,x,u,\eta) + \eta_x \frac{\partial f}{\partial \eta}(t,x,u,\eta) \\ &-3\frac{\partial m}{\partial \eta}(t,x,u,\eta,u_t,u_x) = 0. \end{aligned} \tag{C.31}$$

Next, we separate (C.31) by the following derivative of u

$$u_x: \frac{\partial f}{\partial u}(t,x,u,\eta) = 0, \tag{C.32}$$

and by the following derivative of η

$$\eta_x: \frac{\partial f}{\partial \eta}(t,x,u,\eta) = 0, \tag{C.33}$$

(which simplifies $f = f(t,x)$) and by the following remainder

$$\begin{aligned} &1: \frac{\partial f}{\partial x}(t,x) - 3\frac{\partial m}{\partial \eta}(t,x,u,\eta,u_t,u_x) = 0 \\ &\Rightarrow m(t,x,u,\eta,u_t,u_x) = \frac{\eta}{3}\frac{\partial f}{\partial x}(t,x) + N(t,x,u,u_t,u_x). \end{aligned} \tag{C.34}$$

Thus

$$\Lambda_1(t,x,u,\eta,u_t,u_x,\eta_t) = \eta_t f(t,x) + g(t,x,u,\eta,u_t,u_x), \tag{C.35}$$

$$\Lambda_2(t,x,u,\eta,u_t,u_x,\eta_x) = -\frac{1}{3}\eta_x f(t,x) + \frac{\eta}{3}\frac{\partial f}{\partial x}(t,x) + N(t,x,u,u_t,u_x). \tag{C.36}$$

Considering (C.6) and substituting the results obtained in (C.35) and (C.36), we obtain the following

$$\begin{aligned} &\frac{\partial g}{\partial \eta}(\eta_t + u_x + \epsilon \eta_x u + \epsilon \eta u_x) + \epsilon u_x(\eta_t f + g) \\ &+ \frac{1}{3}\frac{\partial f}{\partial x}(u_t + \epsilon u u_x + \eta_x) - \frac{\partial f}{\partial t}(\eta_t + u_x + \epsilon \eta_x u + \epsilon \eta u_x) \end{aligned}$$

$$
\begin{aligned}
&-3\eta_t\frac{\partial f}{\partial t}-\frac{\partial g}{\partial t}-u_t\left(\epsilon f\eta_x+\frac{\partial g}{\partial u}\right)-\eta_t\left(f\epsilon u_x+\frac{\partial g}{\partial \eta}\right)\\
&-u_{tt}\frac{\partial g}{\partial u_t}-u_{xt}\left[f(1+\epsilon\eta)+\frac{\partial g}{\partial u_x}\right]-\eta_{tt}2f-\eta_{xt}\epsilon f u\\
&-\left[\left(\eta_t\frac{\partial f}{\partial x}+\frac{\partial g}{\partial x}\right)\epsilon u-\frac{1}{3}\frac{\partial f}{\partial x}(u_t+\epsilon u_x u+\eta_x)-\frac{1}{3}\eta_x\frac{\partial f}{\partial x}+\frac{\eta}{3}\frac{\partial^2 f}{\partial x^2}+\frac{\partial N}{\partial x}\right]\\
&-u_x\left[\frac{\partial g}{\partial u}\epsilon u+(\eta_t f+g)\epsilon-\frac{1}{3}f\epsilon u_x+\frac{\partial N}{\partial u}\right]-\eta_x\left(\frac{\partial g}{\partial \eta}\epsilon u+\frac{1}{3}\frac{\partial f}{\partial x}\right)\\
&-u_{xt}\left(\frac{\partial g}{\partial u_t}\epsilon u-\frac{1}{3}f+\frac{\partial N}{\partial u_t}\right)-u_{xx}\left(\frac{\partial g}{\partial u_x}\epsilon u-\frac{1}{3}f\epsilon u+\frac{\partial N}{\partial u_x}\right)\\
&-\eta_{tx}f\epsilon u+\frac{2}{3}\eta_{xx}f=0.
\end{aligned}
\tag{C.37}
$$

Equation (C.37) can be split into derivative products and powers of u and η as follows

$$u_{tt}:\frac{\partial g}{\partial u_t}=0, \tag{C.38}$$

and by the following derivative of u

$$u_{xt}:f\left(\frac{2}{3}-\epsilon\eta\right)+\frac{\partial g}{\partial u_x}+\epsilon u\frac{\partial g}{\partial u_t}+\frac{\partial N}{\partial u_t}=0, \tag{C.39}$$

and by the following derivative of u

$$u_{xx}:-\frac{\partial g}{\partial u_x}\epsilon u+\frac{1}{3}f\epsilon u-\frac{\partial N}{\partial u_x}=0, \tag{C.40}$$

and by the following derivative of η

$$
\begin{aligned}
\eta_{tt}:&-2f=0\\
&\Rightarrow f=0,
\end{aligned}
\tag{C.41}
$$

and by the following derivative of η

$$
\begin{aligned}
\eta_{xt}:&-2f\epsilon u=0\\
&\Rightarrow f=0,
\end{aligned}
\tag{C.42}
$$

and by the following derivative of η

$$
\begin{aligned}
\eta_{xx}:&\frac{2}{3}f=0\\
&\Rightarrow f=0.
\end{aligned}
\tag{C.43}
$$

Thus

$$\Lambda_1 = g(t, x, u, \eta, u_x), \tag{C.44}$$

$$\Lambda_2 = N(t, x, u, u_t, u_x), \tag{C.45}$$

where

$$\frac{\partial g}{\partial u_x} + \frac{\partial N}{\partial u_t} = 0, \tag{C.46}$$

$$\epsilon u \frac{\partial g}{\partial u_x} + \frac{\partial N}{\partial u_x} = 0, \tag{C.47}$$

$$\frac{\partial g}{\partial \eta}\left[\left(1+\epsilon\eta\right)u_x\right] - \frac{\partial g}{\partial t} - \frac{\partial g}{\partial u}u_t - \frac{\partial g}{\partial x}\epsilon u - \frac{\partial N}{\partial x} - \frac{\partial g}{\partial u}\epsilon u u_x - \frac{\partial N}{\partial u}u_x = 0. \tag{C.48}$$

Now by expanding (C.4) and substituting (C.44) and (C.45) we obtain the following determining equation

$$\begin{aligned}
&\frac{\partial g}{\partial u}\left(\eta_t + u_x + \epsilon\eta_x u + \epsilon\eta u_x - \frac{\mu^2}{6}u_{xxx}\right) + g\epsilon\eta_x \\
&+ \frac{\partial N}{\partial u}\left(u_t + \epsilon u_x u + \eta_x - \frac{\mu^2}{2}u_{xxt}\right) + N\epsilon u_x \\
&- \left[\frac{\partial^2 N}{\partial u_t \partial t}\left(u_t + \epsilon u_x u + \eta_x - \frac{\mu^2}{2}u_{xxt}\right) + \frac{\partial N}{\partial t}\right] \\
&- u_t\left[\frac{\partial^2 N}{\partial u_t \partial u}\left(u_t + \epsilon u_x u + \eta_x - \frac{\mu^2}{2}u_{xxt}\right) + \frac{\partial N}{\partial u_t}\epsilon u_x + \frac{\partial N}{\partial u}\right] \\
&- \eta_t[0] - u_{tt}\left[\frac{\partial^2 N}{\partial u_t^2}\left(u_t + \epsilon u_x u + \eta_x - \frac{\mu^2}{2}u_{xxt}\right) + \frac{\partial N}{\partial u_t}(1)\right] \\
&- u_{xt}\left[\frac{\partial^2 N}{\partial u_t \partial u_x}\left(u_t + \epsilon u_x u + \eta_x - \frac{\mu^2}{2}u_{xxt}\right) + \frac{\partial N}{\partial u_t}(\epsilon u) + \frac{\partial N}{\partial u_x}\right] \\
&- \eta_{tt}[0] - \eta_{xt}\left[\frac{\partial N}{\partial u_t}(1)\right] - u_{xxxt}[0] - u_{xxtt}\left[-\frac{\mu^2}{2}\frac{\partial N}{\partial u_t}\right]
\end{aligned}$$

$$\begin{aligned}
&- \left[\frac{\partial^2 g}{\partial u_x \partial x}\left(\eta_t + u_x + \epsilon\eta_x u + \epsilon\eta u_x - \frac{\mu^2}{6}u_{xxx}\right) + \frac{\partial g}{\partial x}\left(1+\epsilon\eta\right)\right. \\
&\left. + \frac{\partial^2 N}{\partial u_x \partial x}\left(u_t + \epsilon u_x u + \eta_x - \frac{\mu^2}{2}u_{xxt}\right) + \frac{\partial N}{\partial x}\epsilon u\right] \\
&- u_x\left[\frac{\partial^2 g}{\partial u_x \partial u}\left(\eta_t + u_x + \epsilon\eta_x u + \epsilon\eta u_x - \frac{\mu^2}{6}u_{xxx}\right) + \frac{\partial g}{\partial u_x}\epsilon\eta_x + \frac{\partial g}{\partial u}\left(1+\epsilon\eta\right)\right. \\
&\left. + \frac{\partial^2 N}{\partial u_x \partial u}\left(u_t + \epsilon u_x u + \eta_x - \frac{\mu^2}{2}u_{xxt}\right) + \frac{\partial N}{\partial u_x}\epsilon u_x + \frac{\partial N}{\partial u}\epsilon u + \epsilon N\right] \\
&- \eta_x\left[\frac{\partial^2 g}{\partial u_x \partial \eta}\left(\eta_t + u_x + \epsilon\eta_x u + \epsilon\eta u_x - \frac{\mu^2}{6}u_{xxx}\right) + \frac{\partial g}{\partial u_x}\epsilon u_x + \frac{\partial g}{\partial \eta}\left(1+\epsilon\eta\right) + g\epsilon\right]
\end{aligned}$$

$$-u_{tx}\left[\frac{\partial^2 N}{\partial u_x \partial u_t}\left(u_t+\epsilon u_x u+\eta_x-\frac{\mu^2}{2}u_{xxt}\right)+\frac{\partial N}{\partial u_x}(1)+\frac{\partial N}{\partial u_t}\epsilon u\right]$$

$$-u_{xx}\left[\frac{\partial^2 g}{\partial u_x^2}\left(\eta_t+u_x+\epsilon\eta_x u+\epsilon\eta u_x-\frac{\mu^2}{6}u_{xxx}\right)+\frac{\partial g}{\partial u_x}\left(1+\epsilon\eta\right)+\frac{\partial g}{\partial u_x}\left(1+\epsilon\eta\right)\right.$$

$$\left.+\frac{\partial^2 N}{\partial u_x^2}\left(u_t+\epsilon u_x u+\eta_x-\frac{\mu^2}{2}u_{xxt}\right)+\frac{\partial N}{\partial u_x}\epsilon u+\frac{\partial N}{\partial u_x}\epsilon u\right]$$

$$-\eta_{tx}\left[\frac{\partial g}{\partial u_x}(1)\right]-u_{xxxx}\left[-\frac{\mu^2}{6}\frac{\partial g}{\partial u_x}\right]-u_{xxxt}\left[-\frac{\mu^2}{2}\frac{\partial N}{\partial u_x}\right]$$

$$+\frac{\mu^2}{6}\left[\frac{\partial^3 g}{\partial x^3}+u_x\frac{\partial^3 g}{\partial u_x\partial x^2}+\eta_x\frac{\partial^3 g}{\partial x^2\partial\eta}+u_{xx}\frac{\partial^3 g}{\partial x^2\partial u_x}\right.$$

$$+u_x\left(\frac{\partial^3 g}{\partial x^2\partial u}+u_x\frac{\partial^3 g}{\partial u^2\partial x}+\eta_x\frac{\partial^3 g}{\partial u\partial\eta\partial x}+u_{xx}\frac{\partial^3 g}{\partial u\partial u_x\partial x}\right)$$

$$+\eta_x\left(\frac{\partial^3 g}{\partial x^2\partial\eta}+u_x\frac{\partial^3 g}{\partial u\partial\eta\partial x}+\eta_x\frac{\partial^3 g}{\partial x\partial\eta^2}+u_{xx}\frac{\partial^3 g}{\partial u_x\partial\eta\partial x}\right)$$

$$+u_{xx}\left(\frac{\partial^3 g}{\partial x^2\partial u_x}+\frac{\partial^2 g}{\partial x\partial u}+u_x\frac{\partial^3 g}{\partial u\partial u_x\partial x}+\eta_x\frac{\partial^3 g}{\partial\eta\partial u_x\partial x}+u_{xx}\frac{\partial^3 g}{\partial u_x^2\partial x}\right)+\eta_{xx}\left(\frac{\partial^2 g}{\partial x\partial\eta}\right)$$

$$+u_x\left[\frac{\partial^3 g}{\partial x^2\partial u}+u_x\frac{\partial^3 g}{\partial x\partial u^2}+\eta_x\frac{\partial^3 g}{\partial x\partial\eta\partial u}+u_{xx}\frac{\partial^3 g}{\partial x\partial u_x\partial u}\right.$$

$$+u_x\left(\frac{\partial^3 g}{\partial x\partial u^2}+u_x\frac{\partial^3 g}{\partial u^3}+\eta_x\frac{\partial^3 g}{\partial u^2\partial\eta}+u_{xx}\frac{\partial^3 g}{\partial u^2\partial u_x}\right)$$

$$+\eta_x\left(\frac{\partial^3 g}{\partial x\partial\eta\partial u}+u_x\frac{\partial^3 g}{\partial u^2\partial\eta}+\eta_x\frac{\partial^3 g}{\partial u\partial\eta^2}+u_{xx}\frac{\partial^3 g}{\partial u\partial u_x\partial\eta}\right)$$

$$\left.+u_{xx}\left(\frac{\partial^3 g}{\partial x\partial u_x\partial u}+u_x\frac{\partial^3 g}{\partial u^2\partial u_x}+\eta_x\frac{\partial^3 g}{\partial\eta\partial u_x\partial u}+u_{xx}\frac{\partial^3 g}{\partial u_x^2\partial u}\right)+\eta_{xx}\left(\frac{\partial^2 g}{\partial\eta\partial u}\right)\right]$$

$$+\eta_x\left[\frac{\partial^3 g}{\partial x^2\partial\eta}+u_x\frac{\partial^3 g}{\partial u_x\partial x\partial\eta}+\eta_x\frac{\partial^3 g}{\partial x\partial\eta^2}+u_{xx}\frac{\partial^3 g}{\partial x\partial u_x\partial\eta}\right.$$

$$+u_x\left(\frac{\partial^3 g}{\partial x\partial u\partial\eta}+u_x\frac{\partial^3 g}{\partial u^2\partial\eta}+\eta_x\frac{\partial^3 g}{\partial u\partial\eta^2}+u_{xx}\frac{\partial^3 g}{\partial u\partial u_x\partial\eta}\right)$$

$$+\eta_x\left(\frac{\partial^3 g}{\partial x\partial\eta^2}+u_x\frac{\partial^3 g}{\partial u\partial\eta^2}+\eta_x\frac{\partial^3 g}{\partial\eta^3}+u_{xx}\frac{\partial^3 g}{\partial u_x\partial\eta^2}\right)$$

$$\left.+u_{xx}\left(\frac{\partial^3 g}{\partial x\partial u_x\partial\eta}+\frac{\partial^2 g}{\partial u\partial\eta}+u_x\frac{\partial^3 g}{\partial u\partial u_x\partial\eta}+\eta_x\frac{\partial^3 g}{\partial\eta^2\partial u_x}+u_{xx}\frac{\partial^3 g}{\partial u_x^2\partial\eta}\right)+\eta_{xx}\left(\frac{\partial^2 g}{\partial\eta^2}\right)\right]$$

$$+u_{tx}(0)+u_{xx}\left[\frac{\partial^3 g}{\partial x^2\partial u_x}+\frac{\partial^2 g}{\partial u_x\partial x}+u_x\frac{\partial^3 g}{\partial u\partial u_x\partial x}+\eta_x\frac{\partial^3 g}{\partial x\partial\eta\partial u_x}+u_{xx}\frac{\partial^3 g}{\partial x\partial u_x^2}\right.$$

$$+\left(\frac{\partial^2 g}{\partial x\partial u}+u_x\frac{\partial^2 g}{\partial u^2}+\eta_x\frac{\partial^2 g}{\partial u\partial\eta}+u_{xx}\frac{\partial^2 g}{\partial u\partial u_x}\right)$$

$$+u_x\left(\frac{\partial^3 g}{\partial x\partial u\partial u_x}+\frac{\partial^2 g}{\partial u^2}+u_x\frac{\partial^3 g}{\partial u^2\partial u_x}+\eta_x\frac{\partial^3 g}{\partial u\partial\eta\partial u_x}+u_{xx}\frac{\partial^3 g}{\partial u\partial u_x^2}\right)$$

$$+\eta_x\left(\frac{\partial^3 g}{\partial x\partial\eta\partial u_x}+\frac{\partial^2 g}{\partial u\partial\eta}+u_x\frac{\partial^3 g}{\partial u\partial\eta\partial u_x}+\eta_x\frac{\partial^3 g}{\partial\eta^2\partial u_x}+u_{xx}\frac{\partial^3 g}{\partial u_x^2\partial\eta}\right)$$

$$
\begin{aligned}
&+u_{xx}\left(\frac{\partial^3 g}{\partial x\partial u_x^2}+2\frac{\partial^2 g}{\partial u\partial u_x}+u_x\frac{\partial^3 g}{\partial u\partial u_x^2}+\eta_x\frac{\partial^3 g}{\partial\eta\partial u_x^2}+u_{xx}\frac{\partial^3 g}{\partial u_x^3}\right)+\eta_{xx}\left(\frac{\partial^2 g}{\partial\eta\partial u_x}\right)\Bigg]\\
&+\eta_{tx}(0)+\eta_{xx}\left[\frac{\partial^2 g}{\partial x\partial\eta}+u_x\left(\frac{\partial^2 g}{\partial u\partial\eta}\right)+\left(\frac{\partial^2 g}{\partial x\partial\eta}+u_x\frac{\partial^2 g}{\partial u\partial\eta}+\eta_x\frac{\partial^2 g}{\partial\eta^2}+u_{xx}\frac{\partial^2 g}{\partial u_x\partial\eta}\right)\right.\\
&\left.+\eta_x\left(\frac{\partial^2 g}{\partial\eta^2}\right)+u_{xx}\left(\frac{\partial^2 g}{\partial\eta\partial u_x}\right)\right]\Bigg]\\
&+\frac{\mu^2}{2}\left[\frac{\partial^3 N}{\partial x^2\partial t}+u_t\frac{\partial^3 N}{\partial x^2\partial u}+u_{tt}\frac{\partial^3 N}{\partial x^2\partial u_t}+u_{xt}\frac{\partial^3 N}{\partial x^2\partial u_x}\right.\\
&+u_x\left(\frac{\partial^3 N}{\partial t\partial u\partial x}+u_t\frac{\partial^3 N}{\partial u^2\partial x}+u_{tt}\frac{\partial^3 N}{\partial u_t\partial u\partial x}+u_{xt}\frac{\partial^3 N}{\partial u\partial u_x\partial x}\right)\\
&+u_{tx}\left(\frac{\partial^3 N}{\partial t\partial u_t\partial x}+\frac{\partial^2 N}{\partial x\partial u}+u_t\frac{\partial^3 N}{\partial u\partial u_t\partial x}+u_{tt}\frac{\partial^3 N}{\partial u_t^2\partial t}+u_{xt}\frac{\partial^3 N}{\partial u_x\partial u_t\partial x}\right)\\
&+u_{xx}\left(\frac{\partial^3 N}{\partial t\partial u_x\partial x}+u_t\frac{\partial^3 N}{\partial u\partial u_x\partial x}+u_{tt}\frac{\partial^3 N}{\partial u_x\partial u_t\partial x}+u_{xt}\frac{\partial^3 N}{\partial u_x^2\partial x}\right)\\
&+u_x\left[\frac{\partial^3 N}{\partial x\partial t\partial u}+u_t\frac{\partial^3 N}{\partial x\partial u^2}+u_{tt}\frac{\partial^3 N}{\partial x\partial u_t\partial u}+u_{xt}\frac{\partial^3 N}{\partial x\partial u_x\partial u}\right.\\
&+u_x\left(\frac{\partial^3 N}{\partial t\partial u^2}+u_t\frac{\partial^3 N}{\partial u^3}+u_{tt}\frac{\partial^3 N}{\partial u_t\partial u^2}+u_{xt}\frac{\partial^3 N}{\partial u^2\partial u_x}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+u_{tx}\left(\frac{\partial^3 N}{\partial t\partial u_t\partial u}+\frac{\partial^2 N}{\partial u^2}+u_t\frac{\partial^3 N}{\partial u^2\partial u_t}+u_{tt}\frac{\partial^3 N}{\partial u_t^2\partial u}+u_{xt}\frac{\partial^3 N}{\partial u\partial u_x\partial u_t}\right)\\
&\left.+u_{xx}\left(\frac{\partial^3 N}{\partial t\partial u_x\partial u}+u_t\frac{\partial^3 N}{\partial u^2\partial u_x}+u_{tt}\frac{\partial^3 N}{\partial u\partial u_x\partial u_t}+u_{xt}\frac{\partial^3 N}{\partial u\partial u_x^2}\right)\right]+\eta_x(0)\\
&+u_{tx}\left[\left(\frac{\partial^3 N}{\partial x\partial t\partial u_t}+\frac{\partial^2 N}{\partial x\partial u}+u_t\frac{\partial^3 N}{\partial u_t\partial x\partial u}+u_{tt}\frac{\partial^3 N}{\partial x\partial u_t^2}+u_{xt}\frac{\partial^3 N}{\partial x\partial u_x\partial u_t}\right)\right.\\
&+u_x\left(\frac{\partial^3 N}{\partial u_t\partial t\partial u}+\frac{\partial^2 N}{\partial u^2}+u_t\frac{\partial^3 N}{\partial u^2\partial u_t}+u_{tt}\frac{\partial^3 N}{\partial u_t^2\partial u}+u_{xt}\frac{\partial^3 N}{\partial u\partial u_x\partial u_t}\right)\\
&+u_{tx}\left(\frac{\partial^3 N}{\partial t\partial u_t^2}+2\frac{\partial^2 N}{\partial u\partial u_t}+u_t\frac{\partial^3 N}{\partial u\partial u_t^2}+u_{tt}\frac{\partial^3 N}{\partial u_t^3}+u_{xt}\frac{\partial^3 N}{\partial u_x\partial u_t^2}\right)\\
&\left.+u_{xx}\left(\frac{\partial^3 N}{\partial t\partial u_x\partial u_t}+\frac{\partial^2 N}{\partial u\partial u_x}+u_t\frac{\partial^3 N}{\partial u\partial u_x\partial u_t}+u_{tt}\frac{\partial^3 N}{\partial u_x\partial u_t^2}+u_{xt}\frac{\partial^3 N}{\partial u_x^2\partial u_t}\right)\right]\\
&+u_{xx}\left[\frac{\partial^3 N}{\partial x\partial t\partial u_x}+u_t\frac{\partial^3 N}{\partial x\partial u\partial u_x}+u_{tt}\frac{\partial^3 N}{\partial x\partial u_t\partial u_x}+u_{xt}\frac{\partial^3 N}{\partial x\partial u_x^2}\right.\\
&+\left(\frac{\partial^2 N}{\partial t\partial u}+u_t\frac{\partial^2 N}{\partial u^2}+u_{tt}\frac{\partial^2 N}{\partial u_t\partial u}+u_{xt}\frac{\partial^2 N}{\partial u\partial u_x}\right)\\
&+u_x\left(\frac{\partial^3 N}{\partial t\partial u\partial u_x}+u_t\frac{\partial^3 N}{\partial u^2\partial u_x}+u_{tt}\frac{\partial^3 N}{\partial u_t\partial u\partial u_x}+u_{xt}\frac{\partial^3 N}{\partial u\partial u_x^2}\right)\\
&+u_{tx}\left(\frac{\partial^3 N}{\partial t\partial u_t\partial u_x}+\frac{\partial^2 N}{\partial u\partial u_x}+u_t\frac{\partial^3 N}{\partial u\partial u_t\partial u_x}+u_{tt}\frac{\partial^3 N}{\partial u_t^2\partial u_x}+u_{xt}\frac{\partial^3 N}{\partial u_x^2\partial u_t}\right)\\
&\left.+u_{xx}\left(\frac{\partial^3 N}{\partial t\partial u_x^2}+u_t\frac{\partial^3 N}{\partial u\partial u_x^2}+u_{tt}\frac{\partial^3 N}{\partial u_x^2\partial u_t}+u_{xt}\frac{\partial^3 N}{\partial u_x^3}\right)\right]\\
&+\eta_{tx}(0)+\eta_{xx}(0)+u_{xxxx}(0)+u_{xxxt}(0)\Big]=0. \qquad \text{(C.49)}
\end{aligned}
$$

Next, we separate (C.49) by the following derivative of u

$$u_{xxtt}:\frac{\mu^2}{2}\frac{\partial N}{\partial u_t}=0, \tag{C.50}$$

and by the following derivative of u

$$u_{xxxx}:\frac{\mu^2}{6}\frac{\partial g}{\partial u_x}=0, \tag{C.51}$$

and by the following derivative of u

$$u_{xxxt}:\frac{\mu^2}{2}\frac{\partial N}{\partial u_x}=0. \tag{C.52}$$

Thus

$$\Lambda_1 = g(t,x,u,\eta), \tag{C.53}$$

$$\Lambda_2 = N(t,x,u), \tag{C.54}$$

where (C.46) and (C.47) are identically satisfied, and (C.48) remains unchanged. Next, we separate (C.48) by the following derivative of u

$$u_t:\frac{\partial g}{\partial u}=0, \tag{C.55}$$

and by the following derivative of u

$$u_x:\frac{\partial g}{\partial \eta}\left(1+\epsilon\eta\right)-\frac{\partial N}{\partial u}=0, \tag{C.56}$$

and by the following remiander

$$1:\frac{\partial g}{\partial t}+\frac{\partial g}{\partial x}\epsilon u+\frac{\partial N}{\partial x}=0. \tag{C.57}$$

Thus

$$\Lambda_1 = g(t,x,\eta), \tag{C.58}$$

$$\Lambda_2 = N(t,x,u), \tag{C.59}$$

where

$$\frac{\partial g}{\partial \eta}\left(1+\epsilon\eta\right)-\frac{\partial N}{\partial u}=0, \tag{C.60}$$

$$\frac{\partial g}{\partial t}+\frac{\partial g}{\partial x}\epsilon u+\frac{\partial N}{\partial x}=0. \tag{C.61}$$

Returning to (C.49) we separate by the following derivative of u

$$u_{xxt}:\frac{\partial N}{\partial u}=0. \tag{C.62}$$

Thus

$$\Lambda_1=g(t,x,\eta), \tag{C.63}$$

$$\Lambda_2=N(t,x), \tag{C.64}$$

where (C.60) simplifies to

$$\frac{\partial g}{\partial \eta}=0, \tag{C.65}$$

and (C.61) remains as

$$\frac{\partial g}{\partial t}+\frac{\partial g}{\partial x}\epsilon u+\frac{\partial N}{\partial x}=0. \tag{C.66}$$

Thus

$$\Lambda_1=g(t,x), \tag{C.67}$$

$$\Lambda_2=N(t,x), \tag{C.68}$$

where

$$\frac{\partial g}{\partial t}+\frac{\partial g}{\partial x}\epsilon u+\frac{\partial N}{\partial x}=0. \tag{C.69}$$

Next, we separate (C.69) by u

$$u:\frac{\partial g}{\partial x}=0. \tag{C.70}$$

Thus

$$\Lambda_1=g(t), \tag{C.71}$$

$$\Lambda_2=N(t,x), \tag{C.72}$$

where

$$\frac{\partial g}{\partial t}+\frac{\partial N}{\partial x}=0. \tag{C.73}$$

By substituting (C.71) and (C.72) into (C.4) we have

$$-\frac{\partial N}{\partial t}-\epsilon u\frac{\partial N}{\partial x}+\frac{\mu^2}{2}\frac{\partial^3 N}{\partial x^2\partial t}=0. \tag{C.74}$$

We separate (C.74) by u

$$u:\frac{\partial N}{\partial x}=0. \tag{C.75}$$

Thus

$$\Lambda_1 = g(t), \tag{C.76}$$

$$\Lambda_2 = N(t), \tag{C.77}$$

where (C.69) simplifies to

$$\frac{\partial g}{\partial t}=0, \tag{C.78}$$

and equation (C.74) simplifies to

$$\frac{\partial N}{\partial t}=0. \tag{C.79}$$

Thus

$$\Lambda_1 = k_1, \tag{C.80}$$

$$\Lambda_2 = k_2, \tag{C.81}$$

where k_1 and k_2 are constants.

Appendix D

Lie point symmetries for the NB equations

The extended infinitesimals for (7.9) are given by

$$\begin{aligned}
\varphi^1_t &= D_t\left(\psi^1\right) - u_t D_t\left(\xi^1\right) - u_x D_t\left(\xi^2\right)\\
&= \frac{\partial \psi^1}{\partial t} + \frac{\partial \psi^1}{\partial u} u_t + \frac{\partial \psi^1}{\partial \eta}\eta_t - \frac{\partial \xi^1}{\partial t} u_t - \frac{\partial \xi^1}{\partial u}(u_t)^2 - \frac{\partial \xi^1}{\partial \eta} u_t \eta_t - \frac{\partial \xi^2}{\partial t} u_x\\
&- \frac{\partial \xi^2}{\partial u} u_t u_x - \frac{\partial \xi^2}{\partial \eta}\eta_t u_x,
\end{aligned} \tag{D.1}$$

and

$$\begin{aligned}
\varphi^1_x &= D_x\left(\psi^1\right) - u_t D_x\left(\xi^1\right) - u_x D_x\left(\xi^2\right)\\
&= \frac{\partial \psi^1}{\partial x} + \frac{\partial \psi^1}{\partial u} u_x + \frac{\partial \psi^1}{\partial \eta}\eta_x - \frac{\partial \xi^1}{\partial x} u_t - \frac{\partial \xi^1}{\partial u} u_t u_x - \frac{\partial \xi^1}{\partial \eta} u_t \eta_x - \frac{\partial \xi^2}{\partial x} u_x\\
&- \frac{\partial \xi^2}{\partial u}(u_x)^2 - \frac{\partial \xi^2}{\partial \eta} u_x \eta_x,
\end{aligned} \tag{D.2}$$

and similarly

$$\begin{aligned}
\varphi^2_x &= D_x\left(\psi^2\right) - \eta_t D_x\left(\xi^1\right) - \eta_x D_x\left(\xi^2\right)\\
&= \frac{\partial \psi^2}{\partial x} + \frac{\partial \psi^2}{\partial u} u_x + \frac{\partial \psi^2}{\partial \eta}\eta_x - \frac{\partial \xi^1}{\partial x}\eta_t - \frac{\partial \xi^1}{\partial u} u_x \eta_t - \frac{\partial \xi^1}{\partial \eta}\eta_t \eta_x - \frac{\partial \xi^2}{\partial x}\eta_x\\
&- \frac{\partial \xi^2}{\partial u}\eta_x u_x - \frac{\partial \xi^2}{\partial \eta}\left(\eta_x\right)^2,
\end{aligned} \tag{D.3}$$

and finally

$$\varphi^1_{xxt} = D_t\left(\varphi^1_{xx}\right) - u_{xxt} D_t\left(\xi^1\right) - u_{xxx} D_t\left(\xi^2\right)$$

$$
\begin{aligned}
&= \frac{\partial^3\psi^1}{\partial t\partial x^2} + \frac{\partial^3\psi^1}{\partial u\partial x^2}u_t + \frac{\partial^3\psi^1}{\partial\eta\partial x^2}\eta_t + \frac{\partial^3\psi^1}{\partial t\partial x\partial u}u_x + \frac{\partial^3\psi^1}{\partial x\partial u^2}u_t u_x + \frac{\partial^3\psi^1}{\partial x\partial u\partial\eta}\eta_t u_x \\
&+ \frac{\partial^2\psi^1}{\partial x\partial u}u_{xt} + \frac{\partial^3\psi^1}{\partial t\partial x\partial\eta}\eta_x + \frac{\partial^3\psi^1}{\partial x\partial u\partial\eta}u_t\eta_x + \frac{\partial^3\psi^1}{\partial x\partial\eta^2}\eta_t\eta_x + \frac{\partial^2\psi^1}{\partial x\partial\eta}\eta_{xt} + \frac{\partial^3\psi^1}{\partial t\partial x\partial u}u_x \\
&+ \frac{\partial^3\psi^1}{\partial x\partial u^2}u_t u_x + \frac{\partial^3\psi^1}{\partial x\partial u\partial\eta}\eta_t u_x + \frac{\partial^2\psi^1}{\partial x\partial u}u_{xt} + \frac{\partial^3\psi^1}{\partial t\partial u^2}(u_x)^2 + \frac{\partial^3\psi^1}{\partial u^3}u_t(u_x)^2 + \frac{\partial^3\psi^1}{\partial u^2\partial\eta}\eta_t(u_x)^2 \\
&+2\frac{\partial^2\psi^1}{\partial u^2}u_x u_{xt} + \frac{\partial^3\psi^1}{\partial t\partial u\partial\eta}\eta_x u_x + \frac{\partial^3\psi^1}{\partial\eta\partial u^2}u_t\eta_x u_x + \frac{\partial^3\psi^1}{\partial\eta^2\partial u}\eta_t\eta_x u_x + \frac{\partial^2\psi^1}{\partial\eta\partial u}\eta_{xt}u_x + \frac{\partial^2\psi^1}{\partial\eta\partial u}\eta_x u_{xt} \\
&+ \frac{\partial^2\psi^1}{\partial t\partial u}u_{xx} + \frac{\partial^2\psi^1}{\partial u^2}u_t u_{xx} + \frac{\partial^2\psi^1}{\partial\eta\partial u}\eta_t u_{xx} + \frac{\partial\psi^1}{\partial u}u_{xxt} + \frac{\partial^3\psi^1}{\partial t\partial x\partial\eta}\eta_x + \frac{\partial^3\psi^1}{\partial x\partial u\partial\eta}u_t\eta_x \\
&+ \frac{\partial^3\psi^1}{\partial x\partial\eta^2}\eta_t\eta_x + \frac{\partial^2\psi^1}{\partial x\partial\eta}\eta_{xt} + \frac{\partial^3\psi^1}{\partial t\partial u\partial\eta}u_x\eta_x + \frac{\partial^3\psi^1}{\partial u^2\partial\eta}u_t u_x\eta_x + \frac{\partial^3\psi^1}{\partial u\partial\eta^2}u_x\eta_t\eta_x + \frac{\partial^2\psi^1}{\partial u\partial\eta}u_{xt}\eta_x \\
&+ \frac{\partial^2\psi^1}{\partial u\partial\eta}u_x\eta_{xt} + \frac{\partial^3\psi^1}{\partial t\partial\eta^2}\left(\eta_x\right)^2 + \frac{\partial^3\psi^1}{\partial u\partial\eta^2}u_t\left(\eta_x\right)^2 + \frac{\partial^3\psi^1}{\partial\eta^3}\eta_t\left(\eta_x\right)^2 + 2\frac{\partial^2\psi^1}{\partial\eta^2}\eta_x\eta_{xt} + \frac{\partial^2\psi^1}{\partial t\partial\eta}\eta_{xx} \\
&+ \frac{\partial^2\psi^1}{\partial u\partial\eta}u_t\eta_{xx} + \frac{\partial^2\psi^1}{\partial\eta^2}\eta_t\eta_{xx} + \frac{\partial\psi^1}{\partial\eta}\eta_{xxt} - \frac{\partial^3\xi^1}{\partial t\partial x^2}u_t - \frac{\partial^3\xi^1}{\partial u\partial x^2}(u_t)^2 - \frac{\partial^3\xi^1}{\partial x^2\partial\eta}\eta_t u_t \\
&- \frac{\partial^2\xi^1}{\partial x^2}u_{tt} - \frac{\partial^3\xi^1}{\partial t\partial u\partial x}u_x u_t - \frac{\partial^3\xi^1}{\partial u^2\partial x}(u_t)^2 u_x - \frac{\partial^3\xi^1}{\partial x\partial u\partial\eta}u_x u_t\eta_t - \frac{\partial^2\xi^1}{\partial u\partial x}u_{xt}u_t - \frac{\partial^2\xi^1}{\partial u\partial x}u_x u_{tt} \\
&- \frac{\partial^3\xi^1}{\partial t\partial x\partial\eta}\eta_x u_t - \frac{\partial^3\xi^1}{\partial x\partial u\partial\eta}(u_t)^2\eta_x - \frac{\partial^3\xi^1}{\partial x\partial\eta^2}\eta_t\eta_x u_t - \frac{\partial^2\xi^1}{\partial x\partial\eta}\eta_{xt}u_t - \frac{\partial^2\xi^1}{\partial x\partial\eta}\eta_x u_{tt} - \frac{\partial^2\xi^1}{\partial t\partial x}u_{tx} \\
&- \frac{\partial^2\xi^1}{\partial u\partial x}u_t u_{tx} - \frac{\partial^2\xi^1}{\partial x\partial\eta}\eta_t u_{tx} - \frac{\partial\xi^1}{\partial x}u_{ttx} - \frac{\partial^3\xi^1}{\partial t\partial x\partial u}u_t u_x - \frac{\partial^3\xi^1}{\partial x\partial u^2}(u_t)^2 u_x - \frac{\partial^3\xi^1}{\partial x\partial u\partial\eta}u_t u_x\eta_t \\
&- \frac{\partial^2\xi^1}{\partial x\partial u}u_{tt}u_x - \frac{\partial^2\xi^1}{\partial x\partial u}u_t u_{xt} - \frac{\partial^3\xi^1}{\partial t\partial u^2}u_t(u_x)^2 - \frac{\partial^3\xi^1}{\partial u^3}(u_t)^2(u_x)^2 - \frac{\partial^3\xi^1}{\partial\eta\partial u^2}u_t(u_x)^2\eta_t \\
&- \frac{\partial^2\xi^1}{\partial u^2}u_{tt}(u_x)^2 - 2\frac{\partial^2\xi^1}{\partial u^2}u_t u_x u_{xt} - \frac{\partial^3\xi^1}{\partial t\partial u\partial\eta}\eta_x u_t u_x - \frac{\partial^3\xi^1}{\partial u^2\partial\eta}\eta_x(u_t)^2 u_x - \frac{\partial^3\xi^1}{\partial u\partial\eta^2}\eta_t\eta_x u_t u_x \\
&- \frac{\partial^2\xi^1}{\partial u\partial\eta}\eta_{xt}u_t u_x - \frac{\partial^2\xi^1}{\partial u\partial\eta}\eta_x u_{tt}u_x - \frac{\partial^2\xi^1}{\partial u\partial\eta}\eta_x u_t u_{xt} - \frac{\partial^2\xi^1}{\partial t\partial u}u_{tx}u_x - \frac{\partial^2\xi^1}{\partial u^2}u_t u_{tx}u_x - \frac{\partial^2\xi^1}{\partial u\partial\eta}u_{tx}u_x\eta_t \\
&- \frac{\partial\xi^1}{\partial u}u_{ttx}u_x - \frac{\partial\xi^1}{\partial u}(u_{tx})^2 - \frac{\partial^2\xi^1}{\partial t\partial u}u_t u_{xx} - \frac{\partial^2\xi^1}{\partial u^2}(u_t)^2 u_{xx} - \frac{\partial^2\xi^1}{\partial u\partial\eta}\eta_t u_t u_{xx} - \frac{\partial\xi^1}{\partial u}u_{tt}u_{xx} \\
&- \frac{\partial\xi^1}{\partial u}u_t u_{xxt} - \frac{\partial^3\xi^1}{\partial t\partial x\partial\eta}u_t\eta_x - \frac{\partial^3\xi^1}{\partial x\partial u\partial\eta}(u_t)^2\eta_x - \frac{\partial^3\xi^1}{\partial x\partial\eta^2}u_t\eta_x\eta_t - \frac{\partial^2\xi^1}{\partial x\partial\eta}u_{tt}\eta_x - \frac{\partial^2\xi^1}{\partial x\partial\eta}u_t\eta_{xt} \\
&- \frac{\partial^3\xi^1}{\partial t\partial u\partial\eta}u_x u_t\eta_x - \frac{\partial^3\xi^1}{\partial u^2\partial\eta}u_x(u_t)^2\eta_x - \frac{\partial^3\xi^1}{\partial u\partial\eta^2}u_x u_t\eta_x\eta_t - \frac{\partial^2\xi^1}{\partial u\partial\eta}u_{xt}u_t\eta_x - \frac{\partial^2\xi^1}{\partial u\partial\eta}u_x u_{tt}\eta_x
\end{aligned}
$$

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\begin{aligned}
&- \frac{\partial^2\xi^1}{\partial u\partial\eta}u_x u_t\eta_{xt} - \frac{\partial^3\xi^1}{\partial t\partial\eta^2}u_t\left(\eta_x\right)^2 - \frac{\partial^3\xi^1}{\partial u\partial\eta^2}(u_t)^2\left(\eta_x\right)^2 - \frac{\partial^3\xi^1}{\partial\eta^3}u_t\left(\eta_x\right)^2\eta_t - \frac{\partial^2\xi^1}{\partial\eta^2}u_{tt}\left(\eta_x\right)^2 \\
&-2\frac{\partial^2\xi^1}{\partial\eta^2}u_t\eta_x\eta_{xt} - \frac{\partial^2\xi^1}{\partial t\partial\eta}u_{tx}\eta_x - \frac{\partial^2\xi^1}{\partial u\partial\eta}u_t u_{tx}\eta_x - \frac{\partial^2\xi^1}{\partial\eta^2}u_{tx}\eta_x\eta_t - \frac{\partial\xi^1}{\partial\eta}u_{ttx}\eta_x - \frac{\partial\xi^1}{\partial\eta}u_{tx}\eta_{xt} \\
&- \frac{\partial^2\xi^1}{\partial t\partial\eta}u_t\eta_{xx} - \frac{\partial^2\xi^1}{\partial u\partial\eta}(u_t)^2\eta_{xx} - \frac{\partial^2\xi^1}{\partial\eta^2}u_t\eta_{xx}\eta_t - \frac{\partial\xi^1}{\partial\eta}u_{tt}\eta_{xx} - \frac{\partial\xi^1}{\partial\eta}u_t\eta_{xxt} - \frac{\partial^3\xi^2}{\partial t\partial x^2}u_x \\
&- \frac{\partial^3\xi^2}{\partial u\partial x^2}u_t u_x - \frac{\partial^3\xi^2}{\partial x^2\partial\eta}\eta_t u_x - \frac{\partial^2\xi^2}{\partial x^2}u_{xt} - \frac{\partial^2\xi^2}{\partial t\partial x\partial u}(u_x)^2 - \frac{\partial^3\xi^2}{\partial x\partial u^2}u_t(u_x)^2 - \frac{\partial^3\xi^2}{\partial x\partial u\partial\eta}\eta_t(u_x)^2
\end{aligned}
$$

$$
\begin{aligned}
&-2\frac{\partial^2\xi^2}{\partial x\partial u}u_x u_{xt}-\frac{\partial^3\xi^2}{\partial t\partial x\partial \eta}\eta_x u_x-\frac{\partial^3\xi^2}{\partial x\partial u\partial \eta}u_t\eta_x u_x-\frac{\partial^3\xi^2}{\partial x\partial \eta^2}\eta_x u_x\eta_t-\frac{\partial^2\xi^2}{\partial x\partial \eta}\eta_{xt}u_x-\frac{\partial^2\xi^2}{\partial x\partial \eta}\eta_x u_{xt}\\
&-\frac{\partial^2\xi^2}{\partial x\partial t}u_{xx}-\frac{\partial^2\xi^2}{\partial u\partial x}u_t u_{xx}-\frac{\partial^2\xi^2}{\partial x\partial \eta}\eta_t u_{xx}-\frac{\partial\xi^2}{\partial x}u_{xxt}-\frac{\partial^3\xi^2}{\partial t\partial x\partial u}(u_x)^2-\frac{\partial^3\xi^2}{\partial x\partial u^2}u_t(u_x)^2\\
&-\frac{\partial^3\xi^2}{\partial x\partial u\partial \eta}\eta_t(u_x)^2-2\frac{\partial^2\xi^2}{\partial x\partial u}u_x u_{xt}-\frac{\partial^3\xi^2}{\partial t\partial u^2}(u_x)^3-\frac{\partial^3\xi^2}{\partial u^3}u_t(u_x)^3-\frac{\partial^3\xi^2}{\partial u^2\partial \eta}\eta_t(u_x)^3\\
&-3\frac{\partial^2\xi^2}{\partial u^2}(u_x)^2u_{xt}-\frac{\partial^3\xi^2}{\partial t\partial u\partial \eta}\eta_x(u_x)^2-\frac{\partial^3\xi^2}{\partial u^2\partial \eta}u_t\eta_x(u_x)^2-\frac{\partial^3\xi^2}{\partial u\partial \eta^2}\eta_t\eta_x(u_x)^2-\frac{\partial^2\xi^2}{\partial u\partial \eta}\eta_{xt}(u_x)^2\\
&-2\frac{\partial^2\xi^2}{\partial u\partial \eta}\eta_x u_x u_{xt}-2\frac{\partial^2\xi^2}{\partial u\partial t}u_x u_{xx}-2\frac{\partial^2\xi^2}{\partial u^2}u_t u_x u_{xx}-2\frac{\partial^2\xi^2}{\partial u\partial \eta}\eta_t u_x u_{xx}-2\frac{\partial\xi^2}{\partial u}u_{xt}u_{xx}-2\frac{\partial\xi^2}{\partial u}u_x u_{xxt}\\
&-\frac{\partial^3\xi^2}{\partial t\partial x\partial \eta}u_x\eta_x-\frac{\partial^2\xi^2}{\partial x\partial \eta\partial u}u_t u_x\eta_x-\frac{\partial^3\xi^2}{\partial x\partial \eta^2}\eta_t u_x\eta_x-\frac{\partial^2\xi^2}{\partial x\partial \eta}u_{xt}\eta_x-\frac{\partial^2\xi^2}{\partial x\partial \eta}u_x\eta_{xt}\\
&-\frac{\partial^3\xi^2}{\partial t\partial u\partial \eta}(u_x)^2\eta_x-\frac{\partial^3\xi^2}{\partial u^2\partial \eta}u_t(u_x)^2\eta_x-\frac{\partial^3\xi^2}{\partial u\partial \eta^2}(u_x)^2\eta_x\eta_t-2\frac{\partial^2\xi^2}{\partial u\partial \eta}u_x u_{xt}\eta_x-\frac{\partial^2\xi^2}{\partial u\partial \eta}(u_x)^2\eta_{xt}\\
&-\frac{\partial^3\xi^2}{\partial \eta^2\partial t}u_x\left(\eta_x\right)^2-\frac{\partial^3\xi^2}{\partial \eta^2\partial u}u_t u_x\left(\eta_x\right)^2-\frac{\partial^3\xi^2}{\partial \eta^3}\eta_t u_x\left(\eta_x\right)^2-\frac{\partial^2\xi^2}{\partial \eta^2}u_{xt}\left(\eta_x\right)^2-2\frac{\partial^2\xi^2}{\partial \eta^2}u_x\eta_x\eta_{xt}\\
&-\frac{\partial^2\xi^2}{\partial t\partial \eta}u_{xx}\eta_x-\frac{\partial^2\xi^2}{\partial u\partial \eta}u_t u_{xx}\eta_x-\frac{\partial^2\xi^2}{\partial \eta^2}\eta_t u_{xx}\eta_x-\frac{\partial\xi^2}{\partial \eta}u_{xxt}\eta_x-\frac{\partial\xi^2}{\partial \eta}u_{xx}\eta_{xt}-\frac{\partial^2\xi^2}{\partial t\partial \eta}u_x\eta_{xx}\\
&-\frac{\partial^2\xi^2}{\partial u\partial \eta}u_t u_x\eta_{xx}-\frac{\partial^2\xi^2}{\partial \eta^2}\eta_t u_x\eta_{xx}-\frac{\partial\xi^2}{\partial \eta}u_{xt}\eta_{xx}-\frac{\partial\xi^2}{\partial \eta}u_x\eta_{xxt}-\frac{\partial^2\xi^1}{\partial t\partial x}u_{xt}-\frac{\partial^2\xi^1}{\partial u\partial x}u_t u_{xt}\\
&-\frac{\partial^2\xi^1}{\partial \eta\partial x}\eta_t u_{xt}-\frac{\partial\xi^1}{\partial x}u_{xtt}-\frac{\partial^2\xi^1}{\partial t\partial u}u_x u_{xt}-\frac{\partial^2\xi^1}{\partial u^2}u_t u_x u_{xt}-\frac{\partial^2\xi^1}{\partial u\partial \eta}\eta_t u_x u_{xt}-\frac{\partial\xi^1}{\partial u}(u_{xt})^2\\
&-\frac{\partial\xi^1}{\partial u}u_x u_{xtt}-\frac{\partial^2\xi^1}{\partial t\partial \eta}\eta_x u_{xt}-\frac{\partial^2\xi^1}{\partial \eta\partial u}u_t\eta_x u_{xt}-\frac{\partial^2\xi^1}{\partial \eta^2}\eta_t\eta_x u_{xt}-\frac{\partial\xi^1}{\partial x}\eta_{xt}u_{xt}-\frac{\partial\xi^1}{\partial \eta}\eta_x u_{xtt}\\
&-\frac{\partial^2\xi^2}{\partial t\partial x}u_{xx}-\frac{\partial^2\xi^2}{\partial u\partial x}u_t u_{xx}-\frac{\partial^2\xi^2}{\partial x\partial \eta}\eta_t u_{xx}-\frac{\partial\xi^2}{\partial x}u_{xxt}-\frac{\partial^2\xi^2}{\partial u\partial t}u_x u_{xx}-\frac{\partial^2\xi^2}{\partial u^2}u_t u_x u_{xx}\\
&-\frac{\partial^2\xi^2}{\partial u\partial \eta}\eta_t u_x u_{xx}-\frac{\partial\xi^2}{\partial u}u_{xt}u_{xx}-\frac{\partial\xi^2}{\partial u}u_x u_{xxt}-\frac{\partial^2\xi^2}{\partial \eta\partial t}\eta_x u_{xx}-\frac{\partial^2\xi^2}{\partial \eta\partial u}u_t\eta_x u_{xx}-\frac{\partial^2\xi^2}{\partial \eta^2}\eta_t\eta_x u_{xx}\\
&-\frac{\partial\xi^2}{\partial \eta}\eta_{xt}u_{xx}-\frac{\partial\xi^2}{\partial \eta}\eta_x u_{xxt}-\frac{\partial\xi^1}{\partial t}u_{xxt}-\frac{\partial\xi^1}{\partial u}u_t u_{xxt}-\frac{\partial\xi^1}{\partial \eta}\eta_t u_{xxt}-\frac{\partial\xi^2}{\partial t}u_{xxx}-\frac{\partial\xi^2}{\partial u}u_t u_{xxx}\\
&-\frac{\partial\xi^2}{\partial \eta}\eta_t u_{xxx}.
\end{aligned}
\tag{D.4}
$$

The first determining equation is given by

$$
\begin{aligned}
0=\;&\epsilon\psi^1u_x+\frac{\partial\psi^1}{\partial t}+\frac{\partial\psi^1}{\partial u}u_t+\frac{\partial\psi^1}{\partial \eta}\eta_t-\frac{\partial\xi^1}{\partial t}u_t-\frac{\partial\xi^1}{\partial u}(u_t)^2-\frac{\partial\xi^1}{\partial \eta}u_t\eta_t-\frac{\partial\xi^2}{\partial t}u_x-\frac{\partial\xi^2}{\partial u}u_t u_x-\frac{\partial\xi^2}{\partial \eta}\eta_t u_x\\
&+\epsilon u\left(\frac{\partial\psi^1}{\partial x}+\frac{\partial\psi^1}{\partial u}u_x+\frac{\partial\psi^1}{\partial \eta}\eta_x-\frac{\partial\xi^1}{\partial x}u_t-\frac{\partial\xi^1}{\partial u}u_t u_x-\frac{\partial\xi^1}{\partial \eta}u_t\eta_x-\frac{\partial\xi^2}{\partial x}u_x-\frac{\partial\xi^2}{\partial u}(u_x)^2-\frac{\partial\xi^2}{\partial \eta}u_x\eta_x\right)\\
&+\frac{\partial\psi^2}{\partial x}+\frac{\partial\psi^2}{\partial u}u_x+\frac{\partial\psi^2}{\partial \eta}\eta_x-\frac{\partial\xi^1}{\partial x}\eta_t-\frac{\partial\xi^1}{\partial u}u_x\eta_t-\frac{\partial\xi^1}{\partial \eta}\eta_t\eta_x-\frac{\partial\xi^2}{\partial x}\eta_x-\frac{\partial\xi^2}{\partial u}\eta_x u_x-\frac{\partial\xi^2}{\partial \eta}\left(\eta_x\right)^2\\
&-\frac{1}{2}\mu^2\left(\frac{\partial^3\psi^1}{\partial t\partial x^2}+\frac{\partial^3\psi^1}{\partial u\partial x^2}u_t+\frac{\partial^3\psi^1}{\partial \eta\partial x^2}\eta_t+\frac{\partial^3\psi^1}{\partial t\partial x\partial u}u_x+\frac{\partial^3\psi^1}{\partial x\partial u^2}u_t u_x+\frac{\partial^3\psi^1}{\partial x\partial u\partial \eta}\eta_t u_x\right.
\end{aligned}
$$

$$+\frac{\partial^2\psi^1}{\partial x\partial u}u_{xt}+\frac{\partial^3\psi^1}{\partial t\partial x\partial\eta}\eta_x+\frac{\partial^3\psi^1}{\partial x\partial u\partial\eta}u_t\eta_x+\frac{\partial^3\psi^1}{\partial x\partial\eta^2}\eta_t\eta_x+\frac{\partial^2\psi^1}{\partial x\partial\eta}\eta_{xt}+\frac{\partial^3\psi^1}{\partial t\partial x\partial u}u_x$$

$$+\frac{\partial^3\psi^1}{\partial x\partial u^2}u_tu_x+\frac{\partial^3\psi^1}{\partial x\partial u\partial\eta}\eta_tu_x+\frac{\partial^2\psi^1}{\partial x\partial u}u_{xt}+\frac{\partial^3\psi^1}{\partial t\partial u^2}(u_x)^2+\frac{\partial^3\psi^1}{\partial u^3}u_t(u_x)^2+\frac{\partial^3\psi^1}{\partial u^2\partial\eta}\eta_t(u_x)^2$$

$$+2\frac{\partial^2\psi^1}{\partial u^2}u_xu_{xt}+\frac{\partial^3\psi^1}{\partial t\partial u\partial\eta}\eta_xu_x+\frac{\partial^3\psi^1}{\partial\eta\partial u^2}u_t\eta_xu_x+\frac{\partial^3\psi^1}{\partial\eta^2\partial u}\eta_t\eta_xu_x+\frac{\partial^2\psi^1}{\partial\eta\partial u}\eta_{xt}u_x+\frac{\partial^2\psi^1}{\partial\eta\partial u}\eta_xu_{xt}$$

$$+\frac{\partial^2\psi^1}{\partial t\partial u}u_{xx}+\frac{\partial^2\psi^1}{\partial u^2}u_tu_{xx}+\frac{\partial^2\psi^1}{\partial\eta\partial u}\eta_tu_{xx}+\frac{\partial\psi^1}{\partial u}u_{xxt}+\frac{\partial^3\psi^1}{\partial t\partial x\partial\eta}\eta_x+\frac{\partial^3\psi^1}{\partial x\partial u\partial\eta}u_t\eta_x$$

$$+\frac{\partial^3\psi^1}{\partial x\partial\eta^2}\eta_t\eta_x+\frac{\partial^2\psi^1}{\partial x\partial\eta}\eta_{xt}+\frac{\partial^3\psi^1}{\partial t\partial u\partial\eta}u_x\eta_x+\frac{\partial^3\psi^1}{\partial u^2\partial\eta}u_tu_x\eta_x+\frac{\partial^3\psi^1}{\partial u\partial\eta^2}u_x\eta_t\eta_x+\frac{\partial^2\psi^1}{\partial u\partial\eta}u_{xt}\eta_x$$

$$+\frac{\partial^2\psi^1}{\partial u\partial\eta}u_x\eta_{xt}+\frac{\partial^3\psi^1}{\partial t\partial\eta^2}\left(\eta_x\right)^2+\frac{\partial^3\psi^1}{\partial u\partial\eta^2}u_t\left(\eta_x\right)^2+\frac{\partial^3\psi^1}{\partial\eta^3}\eta_t\left(\eta_x\right)^2+2\frac{\partial^2\psi^1}{\partial\eta^2}\eta_x\eta_{xt}+\frac{\partial^2\psi^1}{\partial t\partial\eta}\eta_{xx}$$

$$+\frac{\partial^2\psi^1}{\partial u\partial\eta}u_t\eta_{xx}+\frac{\partial^2\psi^1}{\partial\eta^2}\eta_t\eta_{xx}+\frac{\partial\psi^1}{\partial\eta}\eta_{xxt}-\frac{\partial^3\xi^1}{\partial t\partial x^2}u_t-\frac{\partial^3\xi^1}{\partial u\partial x^2}(u_t)^2-\frac{\partial^3\xi^1}{\partial x^2\partial\eta}\eta_tu_t$$

$$-\frac{\partial^2\xi^1}{\partial x^2}u_{tt}-\frac{\partial^3\xi^1}{\partial t\partial u\partial x}u_xu_t-\frac{\partial^3\xi^1}{\partial u^2\partial x}(u_t)^2u_x-\frac{\partial^3\xi^1}{\partial x\partial u\partial\eta}u_xu_t\eta_t-\frac{\partial^2\xi^1}{\partial u\partial x}u_{xt}u_t-\frac{\partial^2\xi^1}{\partial u\partial x}u_xu_{tt}$$

$$-\frac{\partial^3\xi^1}{\partial t\partial x\partial\eta}\eta_xu_t-\frac{\partial^3\xi^1}{\partial x\partial u\partial\eta}(u_t)^2\eta_x-\frac{\partial^3\xi^1}{\partial x\partial\eta^2}\eta_t\eta_xu_t-\frac{\partial^2\xi^1}{\partial x\partial\eta}\eta_{xt}u_t-\frac{\partial^2\xi^1}{\partial x\partial\eta}\eta_xu_{tt}-\frac{\partial^2\xi^1}{\partial t\partial x}u_{tx}$$

$$-\frac{\partial^2\xi^1}{\partial u\partial x}u_tu_{tx}-\frac{\partial^2\xi^1}{\partial x\partial\eta}\eta_tu_{tx}-\frac{\partial\xi^1}{\partial x}u_{ttx}-\frac{\partial^3\xi^1}{\partial t\partial x\partial u}u_tu_x-\frac{\partial^3\xi^1}{\partial x\partial u^2}(u_t)^2u_x-\frac{\partial^3\xi^1}{\partial x\partial u\partial\eta}u_tu_x\eta_t$$

$$-\frac{\partial^2\xi^1}{\partial x\partial u}u_{tt}u_x-\frac{\partial^2\xi^1}{\partial x\partial u}u_tu_{xt}-\frac{\partial^3\xi^1}{\partial t\partial u^2}u_t(u_x)^2-\frac{\partial^3\xi^1}{\partial u^3}(u_t)^2(u_x)^2-\frac{\partial^3\xi^1}{\partial\eta\partial u^2}u_t(u_x)^2\eta_t$$

$$-\frac{\partial^2\xi^1}{\partial u^2}u_{tt}(u_x)^2-2\frac{\partial^2\xi^1}{\partial u^2}u_tu_xu_{xt}-\frac{\partial^3\xi^1}{\partial t\partial u\partial\eta}\eta_xu_tu_x-\frac{\partial^3\xi^1}{\partial u^2\partial\eta}\eta_x(u_t)^2u_x-\frac{\partial^3\xi^1}{\partial u\partial\eta^2}\eta_t\eta_xu_tu_x$$

$$-\frac{\partial^2\xi^1}{\partial u\partial\eta}\eta_{xt}u_tu_x-\frac{\partial^2\xi^1}{\partial u\partial\eta}\eta_xu_{tt}u_x-\frac{\partial^2\xi^1}{\partial u\partial\eta}\eta_xu_tu_{xt}-\frac{\partial^2\xi^1}{\partial t\partial u}u_{tx}u_x-\frac{\partial^2\xi^1}{\partial u^2}u_tu_{tx}u_x-\frac{\partial^2\xi^1}{\partial u\partial\eta}u_{tx}u_x\eta_t$$

$$-\frac{\partial\xi^1}{\partial u}u_{ttx}u_x-\frac{\partial\xi^1}{\partial u}(u_{tx})^2-\frac{\partial^2\xi^1}{\partial t\partial u}u_tu_{xx}-\frac{\partial^2\xi^1}{\partial u^2}(u_t)^2u_{xx}-\frac{\partial^2\xi^1}{\partial u\partial\eta}\eta_tu_tu_{xx}-\frac{\partial\xi^1}{\partial u}u_{tt}u_{xx}$$

$$-\frac{\partial\xi^1}{\partial u}u_tu_{xxt}-\frac{\partial^3\xi^1}{\partial t\partial x\partial\eta}u_t\eta_x-\frac{\partial^3\xi^1}{\partial x\partial u\partial\eta}(u_t)^2\eta_x-\frac{\partial^3\xi^1}{\partial x\partial\eta^2}u_t\eta_x\eta_t-\frac{\partial^2\xi^1}{\partial x\partial\eta}u_{tt}\eta_x-\frac{\partial^2\xi^1}{\partial x\partial\eta}u_t\eta_{xt}$$

$$-\frac{\partial^3\xi^1}{\partial t\partial u\partial\eta}u_xu_t\eta_x-\frac{\partial^3\xi^1}{\partial u^2\partial\eta}u_x(u_t)^2\eta_x-\frac{\partial^3\xi^1}{\partial u\partial\eta^2}u_xu_t\eta_x\eta_t-\frac{\partial^2\xi^1}{\partial u\partial\eta}u_{xt}u_t\eta_x-\frac{\partial^2\xi^1}{\partial u\partial\eta}u_xu_{tt}\eta_x$$

$$-\frac{\partial^2\xi^1}{\partial u\partial\eta}u_xu_t\eta_{xt}-\frac{\partial^3\xi^1}{\partial t\partial\eta^2}u_t\left(\eta_x\right)^2-\frac{\partial^3\xi^1}{\partial u\partial\eta^2}(u_t)^2\left(\eta_x\right)^2-\frac{\partial^3\xi^1}{\partial\eta^3}u_t\left(\eta_x\right)^2\eta_t-\frac{\partial^2\xi^1}{\partial\eta^2}u_{tt}\left(\eta_x\right)^2$$

$$-2\frac{\partial^2\xi^1}{\partial\eta^2}u_t\eta_x\eta_{xt}-\frac{\partial^2\xi^1}{\partial t\partial\eta}u_{tx}\eta_x-\frac{\partial^2\xi^1}{\partial u\partial\eta}u_tu_{tx}\eta_x-\frac{\partial^2\xi^1}{\partial\eta^2}u_{tx}\eta_x\eta_t-\frac{\partial\xi^1}{\partial\eta}u_{ttx}\eta_x-\frac{\partial\xi^1}{\partial\eta}u_{tx}\eta_{xt}$$

$$-\frac{\partial^2\xi^1}{\partial t\partial\eta}u_t\eta_{xx}-\frac{\partial^2\xi^1}{\partial u\partial\eta}(u_t)^2\eta_{xx}-\frac{\partial^2\xi^1}{\partial\eta^2}u_t\eta_{xx}\eta_t-\frac{\partial\xi^1}{\partial\eta}u_{tt}\eta_{xx}-\frac{\partial\xi^1}{\partial\eta}u_t\eta_{xxt}-\frac{\partial^3\xi^2}{\partial t\partial x^2}u_x$$

$$-\frac{\partial^3\xi^2}{\partial u\partial x^2}u_tu_x-\frac{\partial^3\xi^2}{\partial x^2\partial\eta}\eta_tu_x-\frac{\partial^2\xi^2}{\partial x^2}u_{xt}-\frac{\partial^2\xi^2}{\partial t\partial x\partial u}(u_x)^2-\frac{\partial^3\xi^2}{\partial x\partial u^2}u_t(u_x)^2-\frac{\partial^3\xi^2}{\partial x\partial u\partial\eta}\eta_t(u_x)^2$$

$$-2\frac{\partial^2\xi^2}{\partial x\partial u}u_xu_{xt}-\frac{\partial^3\xi^2}{\partial t\partial x\partial\eta}\eta_xu_x-\frac{\partial^3\xi^2}{\partial x\partial u\partial\eta}u_t\eta_xu_x-\frac{\partial^3\xi^2}{\partial x\partial\eta^2}\eta_xu_x\eta_t-\frac{\partial^2\xi^2}{\partial x\partial\eta}\eta_{xt}u_x-\frac{\partial^2\xi^2}{\partial x\partial\eta}\eta_xu_{xt}$$

$$
\begin{aligned}
&-\frac{\partial^2\xi^2}{\partial x\partial t}u_{xx}-\frac{\partial^2\xi^2}{\partial u\partial x}u_t u_{xx}-\frac{\partial^2\xi^2}{\partial x\partial\eta}\eta_t u_{xx}-\frac{\partial\xi^2}{\partial x}u_{xxt}-\frac{\partial^3\xi^2}{\partial t\partial x\partial u}(u_x)^2-\frac{\partial^3\xi^2}{\partial x\partial u^2}u_t(u_x)^2\\
&-\frac{\partial^3\xi^2}{\partial x\partial u\partial\eta}\eta_t(u_x)^2-2\frac{\partial^2\xi^2}{\partial x\partial u}u_x u_{xt}-\frac{\partial^3\xi^2}{\partial t\partial u^2}(u_x)^3-\frac{\partial^3\xi^2}{\partial u^3}u_t(u_x)^3-\frac{\partial^3\xi^2}{\partial u^2\partial\eta}\eta_t(u_x)^3\\
&-3\frac{\partial^2\xi^2}{\partial u^2}(u_x)^2u_{xt}-\frac{\partial^3\xi^2}{\partial t\partial u\partial\eta}\eta_x(u_x)^2-\frac{\partial^3\xi^2}{\partial u^2\partial\eta}u_t\eta_x(u_x)^2-\frac{\partial^3\xi^2}{\partial u\partial\eta^2}\eta_t\eta_x(u_x)^2-\frac{\partial^2\xi^2}{\partial u\partial\eta}\eta_{xt}(u_x)^2\\
&-2\frac{\partial^2\xi^2}{\partial u\partial\eta}\eta_x u_x u_{xt}-2\frac{\partial^2\xi^2}{\partial u\partial t}u_x u_{xx}-2\frac{\partial^2\xi^2}{\partial u^2}u_t u_x u_{xx}-2\frac{\partial^2\xi^2}{\partial u\partial\eta}\eta_t u_x u_{xx}-2\frac{\partial\xi^2}{\partial u}u_{xt}u_{xx}-2\frac{\partial\xi^2}{\partial u}u_x u_{xxt}\\
&-\frac{\partial^3\xi^2}{\partial t\partial x\partial\eta}u_x\eta_x-\frac{\partial^2\xi^2}{\partial x\partial\eta\partial u}u_t u_x\eta_x-\frac{\partial^3\xi^2}{\partial x\partial\eta^2}\eta_t u_x\eta_x-\frac{\partial^2\xi^2}{\partial x\partial\eta}u_{xt}\eta_x-\frac{\partial^2\xi^2}{\partial x\partial\eta}u_x\eta_{xt}\\
&-\frac{\partial^3\xi^2}{\partial t\partial u\partial\eta}(u_x)^2\eta_x-\frac{\partial^3\xi^2}{\partial u^2\partial\eta}u_t(u_x)^2\eta_x-\frac{\partial^3\xi^2}{\partial u\partial\eta^2}(u_x)^2\eta_x\eta_t-2\frac{\partial^2\xi^2}{\partial u\partial\eta}u_x u_{xt}\eta_x-\frac{\partial^2\xi^2}{\partial u\partial\eta}(u_x)^2\eta_{xt}\\
&-\frac{\partial^3\xi^2}{\partial\eta^2\partial t}u_x\left(\eta_x\right)^2-\frac{\partial^3\xi^2}{\partial\eta^2\partial u}u_t u_x\left(\eta_x\right)^2-\frac{\partial^3\xi^2}{\partial\eta^3}\eta_t u_x\left(\eta_x\right)^2-\frac{\partial^2\xi^2}{\partial\eta^2}u_{xt}\left(\eta_x\right)^2-2\frac{\partial^2\xi^2}{\partial\eta^2}u_x\eta_x\eta_{xt}-\frac{\partial\xi^2}{\partial\eta}\eta_t u_{xxx}\\
&-\frac{\partial^2\xi^2}{\partial t\partial\eta}u_{xx}\eta_x-\frac{\partial^2\xi^2}{\partial u\partial\eta}u_t u_{xx}\eta_x-\frac{\partial^2\xi^2}{\partial\eta^2}\eta_t u_{xx}\eta_x-\frac{\partial\xi^2}{\partial\eta}u_{xxt}\eta_x-\frac{\partial\xi^2}{\partial\eta}u_{xx}\eta_{xt}-\frac{\partial^2\xi^2}{\partial t\partial\eta}u_x\eta_{xx}\\
&-\frac{\partial^2\xi^2}{\partial u\partial\eta}u_t u_x\eta_{xx}-\frac{\partial^2\xi^2}{\partial\eta^2}\eta_t u_x\eta_{xx}-\frac{\partial\xi^2}{\partial\eta}u_{xt}\eta_{xx}-\frac{\partial\xi^2}{\partial\eta}u_x\eta_{xxt}-\frac{\partial^2\xi^1}{\partial t\partial x}u_{xt}-\frac{\partial^2\xi^1}{\partial u\partial x}u_t u_{xt}\\
&-\frac{\partial^2\xi^1}{\partial\eta\partial x}\eta_t u_{xt}-\frac{\partial\xi^1}{\partial x}u_{xtt}-\frac{\partial^2\xi^1}{\partial t\partial u}u_x u_{xt}-\frac{\partial^2\xi^1}{\partial u^2}u_t u_x u_{xt}-\frac{\partial^2\xi^1}{\partial u\partial\eta}\eta_t u_x u_{xt}-\frac{\partial\xi^1}{\partial u}(u_{xt})^2\\
&-\frac{\partial\xi^1}{\partial u}u_x u_{xtt}-\frac{\partial^2\xi^1}{\partial t\partial\eta}\eta_x u_{xt}-\frac{\partial^2\xi^1}{\partial\eta\partial u}u_t\eta_x u_{xt}-\frac{\partial^2\xi^1}{\partial\eta^2}\eta_t\eta_x u_{xt}-\frac{\partial\xi^1}{\partial x}\eta_{xt}u_{xt}-\frac{\partial\xi^1}{\partial\eta}\eta_x u_{xtt}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\partial^2\xi^2}{\partial t\partial x}u_{xx}-\frac{\partial^2\xi^2}{\partial u\partial x}u_t u_{xx}-\frac{\partial^2\xi^2}{\partial x\partial\eta}\eta_t u_{xx}-\frac{\partial\xi^2}{\partial x}u_{xxt}-\frac{\partial^2\xi^2}{\partial u\partial t}u_x u_{xx}-\frac{\partial^2\xi^2}{\partial u^2}u_t u_x u_{xx}-\frac{\partial\xi^2}{\partial u}u_t u_{xxx}\\
&-\frac{\partial^2\xi^2}{\partial u\partial\eta}\eta_t u_x u_{xx}-\frac{\partial\xi^2}{\partial u}u_{xt}u_{xx}-\frac{\partial\xi^2}{\partial u}u_x u_{xxt}-\frac{\partial^2\xi^2}{\partial\eta\partial t}\eta_x u_{xx}-\frac{\partial^2\xi^2}{\partial\eta\partial u}u_t\eta_x u_{xx}-\frac{\partial^2\xi^2}{\partial\eta^2}\eta_t\eta_x u_{xx}\\
&-\frac{\partial\xi^2}{\partial\eta}\eta_{xt}u_{xx}-\frac{\partial\xi^2}{\partial\eta}\eta_x u_{xxt}-\frac{\partial\xi^1}{\partial t}u_{xxt}-\frac{\partial\xi^1}{\partial u}u_t u_{xxt}-\frac{\partial\xi^1}{\partial\eta}\eta_t u_{xxt}-\frac{\partial\xi^2}{\partial t}u_{xxx}\Bigg).
\end{aligned}
\tag{D.5}
$$

Appendix E

Table of results for the NB equations

In this appendix, newly discovered results relating to the horizontal component of the fluid velocity and the surface elevation for the dimensionless NB equations representing a solitary wave propagating from left to right are presented. These results can be used as a comparison to numerical and experimental results. In investigating the solutions, we consider the effect of varying the wave speed c^2 whilst keeping the non-linear parameter ϵ and the dispersive parameter μ constant.

E.1 Case 1: $0 < c^2 < \frac{1}{15}$

c^2	F_2	F_6	$F_2 - F_6$	G_2	$F_{4,2}$	$G_{4,2}$
$\frac{1}{40}$	-304	-242	-59	-622	-0.0094	0.0032
$\frac{1}{36}$	-298	-240	-58	-665	-0.0094	0.0034
$\frac{1}{32}$	-291	-236	-55	-740	-0.0095	0.0037
$\frac{1}{28}$	-281	-232	-49	-841	-0.0096	0.0041
$\frac{1}{24}$	-268	-227	-41	-1 059	-0.0099	0.0046
$\frac{1}{20}$	-249	-220	-29	-1 658	-0.0104	0.0055
$\frac{1}{19}$	-242	-218	-24	-1 687	-0.0106	0.0058
$\frac{1}{18}$	-235	-215	-20	-1 925	-0.0110	0.0062
$\frac{1}{17}$	-226	-213	-13	-1 319	-0.0114	0.0067
$\frac{1}{16}$	-210	-216	6	-100	-0.0122	0.0075
$\frac{1}{15.5}$	-208	-211	3	-36	-0.0127	0.0080
$\frac{1}{15.1}$	-206	-207	1	0	-0.0134	0.0086

Table E.1: Results for the fluid velocity in the horizontal direction and surface elevation of the solitary wave in the region $F < 0$ for varying wave speeds where $\epsilon = 0.005$ and $\mu = 0.06$.

c^2	F_1	F_5	$F_5 - F_1$	G_1	$F_{4,1}$	$G_{4,1}$
$\frac{1}{40}$	487	305	-182	-355	0.0094	-0.0032
$\frac{1}{36}$	492	306	-186	-354	0.0094	-0.0034
$\frac{1}{32}$	497	307	-190	-352	0.0095	-0.0037
$\frac{1}{28}$	504	308	-196	-351	0.0096	-0.0041
$\frac{1}{24}$	512	309	-203	-350	0.0099	-0.0046
$\frac{1}{20}$	523	310	-214	-349	0.0104	-0.0055
$\frac{1}{19}$	526	310	-217	-348	0.0106	-0.0058
$\frac{1}{18}$	530	310	-220	-348	0.0110	-0.0062
$\frac{1}{17}$	534	310	-224	-348	0.0114	-0.0067
$\frac{1}{16}$	538	310	-228	-347	0.0122	-0.0075
$\frac{1}{15.5}$	540	310	-230	-347	0.0127	-0.0080
$\frac{1}{15.1}$	542	310	-232	-347	0.0134	-0.0086

Table E.2: Results for the fluid velocity in the horizontal direction and surface elevation of the solitary wave in the region $F > 0$ for varying wave speeds where $\epsilon = 0.005$ and $\mu = 0.06$.

E.2 Case 3: $\frac{1}{15} < c^2 < \frac{1}{3}$

c^2	F_1	F_5	$F_5 - F_1$	G_1
$\frac{1}{14.9}$	543	310	-233	-347
$\frac{1}{14.5}$	545	310	-235	-348
$\frac{1}{14}$	547	310	-237	-349
$\frac{1}{10}$	575	308	-267	-367
$\frac{1}{9}$	584	307	-277	-373
$\frac{1}{8}$	596	305	-291	-380
$\frac{1}{7}$	610	302	-308	-389
$\frac{1}{6}$	627	298	-329	-400
$\frac{1}{5}$	650	289	-361	-415
$\frac{1}{4}$	683	273	-410	-436
$\frac{1}{3.5}$	705	258	-447	-450
$\frac{1}{3.1}$	729	238	-491	-465

Table E.3: Results for the fluid velocity in the horizontal direction and surface elevation of the solitary wave in the region $F > 0$ for varying wave speeds where $\epsilon = 0.005$ and $\mu = 0.06$.

E.3 Case 7: $1 < c^2 < \infty$

c^2	F_1	G_1	$F_{2,1}$	$G_{2,1}$
1.1	19	20	0.034	0.033
1.5	85	115	0.033	0.024
2	152	267	0.033	0.019
3	252	672	0.035	0.016
4	329	1 276	0.038	0.014
4.9	386	2 034	0.040	0.013
5	392	2 148	0.041	0.013
6	445	3 251	0.043	0.013
7	492	4 735	0.046	0.012
8	534	6 463	0.048	0.012
9	572	8 116	0.050	0.012
10	607	9 495	0.052	0.011
11	640	10 874	0.054	0.011

Table E.4: Results for the fluid velocity in the horizontal direction and surface elevation of the solitary wave in the region $F > 0$ for varying wave speeds where $\epsilon = 0.005$ and $\mu = 0.06$.

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