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An interaction field model for immiscible fluids

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To Dad

Abstract

An alternative to the classical surface tension model is proposed for the engineering analysis of macroscopic fluid systems comprised of a number of domains containing fluids which have dissimilar physical properties and which are subject to internal interactions. The Young-Laplace equation, which represents the effects of the internal interactions in the classical theory, is known to fail when the curvature of the interfaces between the domains becomes infinite, and admits a number of unphysical equilibrium configurations for the fluid. The interaction field model derived in this work reduces under suitable conditions to the classical surface tension theory, but where the classical theory fails, it continues to produce physically plausible predictions for the static and dynamic behaviour of the fluid.

The interactions within the fluid system are represented as an integral over a volume form, called the interaction kernel, defined on the product structure $\mathcal{M}^n$ derived from n copies of the space $\mathcal{M}$ occupied by the fluid. The lowest-order interaction field model capable of reproducing the predictions of the classical surface tension model requires $n = 2$. The interaction kernel need not be related to the actual interactions present in the particular physical system under study, but must possess certain symmetries which ensure compatibility with the macroscopic structure of the fluid. Construction of an algebra based on an interpretation of $\mathcal{M}^n$ as a fibre bundle allows the domain structure of the fluid on $\mathcal{M}$ to be transferred to $\mathcal{M}^n$. A group theoretical analysis of the fluid structure on $\mathcal{M}^n$ then shows that equilibrium configurations of the fluid system are related to the symmetry of the interaction kernel.

Direct equivalence between the classical and interaction field models is obtained where the interaction kernel has limitingly small range. If the range of the internal interactions is allowed to remain finite, then the interaction field approach does not generate the singularities associated with the Young-Laplace equation, and the false equilibria predicted by the classical approach are also avoided. Re-analysis, using the interaction field approach, of Plateau's problem for the intersection of three fluid membranes and the linear dynamics of planar, cylindrical and spherical interfaces about their equilibrium configurations shows additional structure when compared to the predictions of the classical theory. This structure may have physical relevance but also may be strongly dependent on the particular form used for the interaction kernel.

A system of domains with topology other than that of an equilibrium configuration of the fluid evolves towards equilibrium by ejection of domains of dissimilar fluid which occur within, or transect another, domain of fluid. This ejection occurs by the process of film-draining, and extrudes any configuration with arbitrary topology into the minimum-energy configuration permitted by the interactions within the system.

Declaration by the Candidate

I, Richard Ewart Brown, declare that this thesis is my own work, that its substance, neither in whole nor in part, has been submitted to, or is to be submitted to, the examining bodies appointed by any institution other than the University of the Witwatersrand, Johannesburg.

Richard E Brown

Signature of candidate

10th of December 1996.

AN INTERACTION FIELD MODEL FOR IMMISCIBLE FLUIDS

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could be reconciled with an interfacial structure in which the fluid properties varied continuously between the properties of the fluids on either side of the interface. Fowler [29] and Kirkwood and Buff [42] later showed, under various assumptions, how the surface tension could be related directly to the structure of the interface and to the specific form of the interactions between the particles comprising the fluid. The analyses beginning with the work of Yvon [90], following the dual paths adopted by Zwanzig *et al.* [91] and Born and Green [10], showed that the internal structure of the fluid (including of course the structure at the interfaces within the fluid) should itself depend on the specific form of the interactions within the fluid[5]. Specification of the interactions between the particles comprising the fluid, as well as the kinetic energy of the particles, should thus be sufficient to determine the evolution of the fluid system. The resultant Hamiltonian model for the dynamics of the N particles comprising the fluid is described for instance by Kirkwood [40], and Irving and Kirkwood [36]. This approach leads to the derivation of the N-fold hierarchy of initial-boundary value equations, known as the Bogoliubov [7] -Born-Green [10] -Kirkwood [41] -Yvon [90] (or BBGKY) series, for the evolution of the structure of the fluid. The value of N is usually too large for practical calculation, and an algebraic closure condition is often imposed on the fluid structure at some level of the hierarchy, thus truncating the system of equations to a manageable size.

The unifying feature of the approaches described in this section is that they rely on an inherently microscopic view of the fluid system as a collection of particles. For the remainder of this work, the fluid dynamical model based on this microscopic approach will be termed the *molecular model*.

[5]This approach has been confirmed by molecular dynamic simulations such as those of Barker and Watts [5] and Rahman and Stillinger [70].

tinuous, and the magnitude of the discontinuity, Δp, is given by the Young-Laplace equation [50]

$$\Delta p = \sigma \kappa \tag{1.1}$$

where κ is the mean curvature of the interfacial membrane. This equation, coupled with a model for the dynamics of the fluid within the domains on either side of the interface, has been used with various degrees of success to analyse quantitatively the dynamics of a fluid system containing interfaces. Some of the applications of this approach are reviewed by Levich and Krylov [51]. For the purposes of this work, the macroscopic dynamics of the fluid away from the interfaces are assumed to be governed by the Navier-Stokes equations, augmented by the assumption of incompressibility of the fluid.

Alternatively, the surface tension can be interpreted as the work associated with the creation or destruction of area of the interfacial membrane. For an interface with positive surface tension, energy is liberated by the destruction of surface area of the interface. This leads to the concept of minimisation of surface area as the guiding principle for a qualitative analysis, using surface tension theory, of the dynamics of a fluid system containing interfaces.

For the remainder of this work, this inherently macroscopic fluid dynamical model based on the membrane analogy for the behaviour of interfaces will be termed the *classical model* or the *surface tension-based model*.

1.3 Internal Interactions

The origin of the interfacial properties in the interactions between the particles comprising the fluid was realised by Plateau [68]. The thermodynamic theory of interfaces was eventually developed (principally[4] by Gibbs [31]) to show that the mechanical model of the interface as a membrane under tension

[4]Although a more modern thermodynamic treatment is given by Melrose [55].

[63], while some of the industrial applications of this class of fluid systems are reviewed by Dickinson [22], Waldie [86], Drake [23] and Bastress [6].

In particular, this work is concerned with the qualitative and quantitative dynamical behaviour of fluid systems with interfaces as they move towards equilibrium, or as they oscillate about some equilibrium configuration. To this end, a new dynamic model of such systems is shown to lead to some particular advantages over previous analyses when applied in the macroscopic regime of most current engineering interest. Previous analyses of the dynamics of fluid systems with interfaces can be divided into two practically disjoint approaches[2], one *macroscopic* and the other *microscopic*[3] in its treatment of the fluid's properties. A brief review of the history of the two approaches is the subject of the following two sections.

1.2 Surface Tension Theory

In certain fluid systems with internal interfaces, for example where the fluids on either side of the interfaces are of different species or phase, a pressure difference is measured across the interfaces. The behaviour of these interfaces from a macroscopic, mechanical, standpoint as though they consisted of two homogeneous fluid domains separated by a uniformly-stretched membrane of infinitesimal thickness was pointed out by Young [89]. A property of the fluids on either side of the interface, called the surface tension, σ, giving the tensile stress across an arbitrary cut in the interfacial membrane, could be defined in the case where the fluid on either side of the interface was of uniform composition.

According to the membrane model, the pressure across an interface is discon-

[2]The review article by Koplik and Banavar [44] is important, however, as it documents the progress made in reconciling the two approaches.

[3]Perhaps the most practical cutoff between 'macroscopic' and 'microscopic' scales of a fluid system is that given by Schowalter [73] as the scale at which Brownian motion within the system ceases to be important.

Chapter 1

Introduction

1.1 Overview

This work is concerned with the macroscopic dynamics of classical fluid systems where the space occupied by the fluid is divided into a number of distinct domains. These domains are separated by interfaces across which certain of the fluid's chemical or physical properties may be discontinuous. Systems such as this are widely encountered in nature, and include foams, emulsions, collections of fluid droplets moving through gas or vapour, and liquid films on a variety of substrates. This vast field has been reviewed by Bogy [8], Tjahjadi *et al.* [80], Prosperetti [69], Schowalter [73], Sountag and Strenge [75], Kraynik [45] and Anwander *et al.* [1], to cite only those reviews most relevant to the work to follow. Some flavour of the engineering applications of such fluid systems is obtained by considering that an understanding of foams and emulsions is crucial to the paint and food industries, the production and control of droplet systems in jet and rocket propulsion, and new technology[1] for the small printer industry. Film-flow systems are extensively employed in the cooling and fabrication of electronic devices. The relevance of the physical properties of these systems to industrial applications is discussed by Ottino

[1] See for instance the January 1977 issue of the IBM Journal for Research and Development which is exclusively devoted to ink-jet printing technology.

Integral Operators

$\int_M$	:	Integral over manifold M.
$\sharp_M$	:	fibrewise integration over manifold M.
$\int_{\epsilon_0}^{\epsilon_1}$	:	Integral between limits ϵ_0 and ϵ_1.

Transformations

ψ_ϵ	:	One-parameter transformation, Parameterised differential form.
$\psi \circ \psi'$	:	Composition of transformations ψ and ψ'.
${}^*\psi_\epsilon$	:	Transformation induced from ψ_ϵ.
ψ_ϵ^{-1}	:	Inverse of transformation ψ_ϵ.
$\psi : A \rightarrow B$	:	ψ maps set A to set B.
$\psi : x \mapsto y$	:	ψ maps element x to element y.
$\omega_{\triangleright M}$	:	Inclusion of form ω into manifold M.
$\omega\|_M$	:	Restriction of form ω to manifold M.

Matrices

$[\,\omega\,]$	:	Matrix function ω.
$[\,\omega\,]^T$	:	Diagonal transpose of matrix function ω.
$[\,\overline{\omega}\,]$	:	Column transpose of matrix function ω.

Miscellaneous Operators

$O(\,\cdot\,)$	:	Estimate of truncation error.

Algebraic Operators

$\sum$	:	Summation operator.
$\prod$	:	Product operator.
$+$	:	Direct addition.
$\times$	:	Direct product.
$-$	:	Direct subtraction.
$/$	:	Direct division.
$\wedge$	:	Exterior product.
$\lrcorner$	:	Interior product.
$\mathbf{d}$	:	Exterior derivative.
$\mathbf{d}^2$	:	Laplace-DeRham operator.
$\#$	:	Dual operator.
$*$	:	Hodge star operator.
$\otimes$	:	Composition product.
$\oplus$	:	Composition sum, coalescence sum.
$\lvert \cdot \rvert$	:	Metric operator.

Differential Operators

$\frac{\mathbf{d}}{\mathbf{d}x}$	:	Functional derivative with respect to form x.
$\frac{d}{d\epsilon}$	:	Total derivative with respect to parameter ϵ.
$\frac{\partial}{\partial\epsilon}$	:	Partial derivative with respect to parameter ϵ.
$\mathcal{L}_\xi$	:	Lie derivative with respect to $\xi \triangleright \mathcal{X}M$.
$\xi(\cdot)$	:	Differential operator $\xi \triangleright \mathcal{X}M$.
δ	:	Variational operator.

$A \,\&\, B$	:	Sum of set A and set B.
$B \supset A$	:	Superset B of set A.
$B \subset A$	:	Subset B of set A.
$x \in A$	:	Element x of set A.
$A \cup B$	:	Union of set A and set B.
$A \cap B$	:	Intersection of set A and set B.
$\mathrm{diff}(A, B)$	:	Difference of sets: $(A \cup B) \sim (A \cap B)$.
$\overline{A}$	:	Closure of set A.
∂M	:	Boundary of set M.

Set Operations

$\mathrm{meas}_\Omega(M)$	:	Measure of set M with respect to volume form Ω.
$\mathrm{supp}(\omega)$	:	Support of differential form ω.
$\dim_{\mathbb{R}}(M)$	:	Dimension of manifold M.
$\mathrm{card}(A)$	:	Cardinal number of set A.
$\sup(A)$	:	Least upper bound of set A.
$\inf(A)$	:	Greatest lower bound of set A.

Logical and Assignment Relations

$a \equiv b$	:	a is identical or isomorphic to b.
$a = b$	:	a is equal to b.
$a \neq b$	:	a is not equal to b.
$a > b$	:	a is greater than b.
$a < b$	:	a is less than b.
$a \approx b$	:	a is approximately equal to b.
$a \Rightarrow b$	:	a implies b.
$a : b$	:	a on condition that b.
$\omega \triangleright \mathcal{B}$	:	Section ω of bundle $\mathcal{B}$.

List of Symbols

Sets

$\mathbb{R}^n$	:	Real space of dimension n.
$\mathbb{R}^+$	:	Positive real line.
$\mathbb{R}^-$	:	Negative real line.
$\mathbf{Z}^n$	:	Integer space of dimension n.
$[a, b]$	:	Closed interval from a to b.
$(a, \ldots, b)$	:	Ordered list of elements a through b.
$\{a, \ldots, b\}$	:	Set of elements a through b.
$\{\emptyset\}$	:	Null set.

Manifolds

$\bigwedge^n M$	:	Bundle of n-forms over manifold M.
ΩM	:	Bundle of volume forms over manifold M.
$\mathcal{X}M$	:	Bundle of one-forms over manifold M.
$\mathsf{T}_x M$	:	Tangent space at $x \in M$.
$\mathsf{T}M$	:	Tangent bundle over manifold M.
$\mathcal{V}M$	:	Vector bundle over manifold M.
$\mathsf{T}_x^* M$	:	Co-tangent space at $x \in M$.
$\mathsf{T}^* M$	:	Co-tangent bundle over manifold M.

Set Relations

$A \sim B$	:	Complement of set B in set A.

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1.9.3 Integral Relations

Fragmentation of Manifolds

Any fragmentation of a manifold M into the open sets $\{M_i\}_{i=1,\ldots,n}$ such that

$$\bigcup_{i=1}^{n} \overline{M}_i = \overline{M} \tag{1.17}$$

(where $\overline{N}$ denotes the closure of set N) and

$$M_i \cap M_j = \begin{cases} 0 & \text{if } i \neq j \\ M_i & \text{if } i = j \end{cases} \tag{1.18}$$

for all $i, j = 1, \ldots, n$ will be termed an *M-complete* fragmentation[11] of M. The concept of complete fragmentation of manifolds is closely related to that of partition of manifolds, the only difference being in the treatment of the boundaries of the elements of the fragmentation or partition.

Continuity of Forms

The Hodge operator on M allows the continuity of differential forms on M to be defined via the metric operator

$$\begin{aligned} |\cdot| \; &: \; \textstyle\bigwedge^{m'} M \rightarrow \mathbb{R}^+ \\ &: \; \omega \mapsto |\omega| = *(\omega \wedge *\omega) \end{aligned} \tag{1.19}$$

Let $\omega \in \bigwedge^{m'} M$ be continuous on M if, for all $\xi \in \mathcal{X}M$ and each $\epsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that $|\xi| < \delta$ implies $|\mathcal{L}_\xi \omega| < \epsilon$.

[11] Of course, M is an M-complete fragmentation of itself.

Total Derivative

Let $\omega_\epsilon \triangleright \Lambda^n M$ be a parameterised field. If ψ_ϵ is an automorphism of M then the derivative of ω_ϵ following the flow of ψ_ϵ, or *total derivative*, is defined by

$$\begin{aligned}
\tfrac{d}{d\epsilon}\omega_0 &= \lim_{\epsilon\to 0}\tfrac{1}{\epsilon}({}^*\psi_\epsilon\omega_\epsilon - \omega_0) \\
&= \lim_{\epsilon\to 0}\tfrac{1}{\epsilon}({}^*\psi_\epsilon(\,\omega_0 + \tfrac{d}{d\epsilon}\omega_0\,\epsilon + O(\epsilon^2)\,) - \omega_0) \\
&= \tfrac{\partial}{\partial\epsilon}\omega_0 + \mathcal{L}_\xi\omega_0 \qquad (1.13)
\end{aligned}$$

using definition (1.7). Rearrangement of definition (1.13) leads to the linearisation

$${}^*\psi_\epsilon\omega_\epsilon = \omega_0 + \tfrac{d}{d\epsilon}\omega_0\,\epsilon + O(\epsilon^2) \qquad (1.14)$$

Comparing definitions (1.12) and (1.13), if ω_ϵ is transported by the flow of ψ_ϵ, then $\frac{d}{d\epsilon}\omega_\epsilon = 0$.

Finally, if $\omega_\epsilon \triangleright \Omega M$, define the derivative

$$\begin{aligned}
\tfrac{d}{d\epsilon}\int_M \omega &= \lim_{\epsilon\to 0}\tfrac{1}{\epsilon}(\int_{\psi_\epsilon M}\omega_\epsilon - \int_M \omega_0) \\
&= \lim_{\epsilon\to 0}\tfrac{1}{\epsilon}\int_M({}^*\psi_\epsilon\omega_\epsilon - \omega_0) \\
&= \int_M \tfrac{d}{d\epsilon}\omega_0 \qquad (1.15)
\end{aligned}$$

using (1.6) and (1.13). An important special case of this result occurs when ω is invariant with respect to ϵ. Then

$$\begin{aligned}
\tfrac{d}{d\epsilon}\int_M \omega &= \int_M \mathcal{L}_\xi\omega \\
&= \int_M \mathrm{d}(*\xi\,*\omega) \qquad (1.16)
\end{aligned}$$

using (1.10).

Using (1.3) the Cartan identity expressing the Lie derivative in terms of the exterior derivative and the inner product can easily be re-expressed in terms of the exterior derivative, the exterior product and the Hodge star:

$$\begin{aligned} \mathcal{L}_\xi \omega &= \#\xi \lrcorner \, \mathsf{d}\omega + \mathsf{d}(\#\xi \lrcorner \, \omega) \\ &= *(\xi \wedge *\mathsf{d}\omega) + \mathsf{d}*(\xi \wedge *\omega) \end{aligned} \tag{1.8}$$

Two specific cases of the Cartan identity are used repeatedly in the analysis to follow. If $\omega \triangleright \bigwedge^0 M$, then the Cartan identity reduces to

$$\begin{aligned} \mathcal{L}_\xi \omega &= *(\xi \wedge *\mathsf{d}\omega) \\ &= \xi(\omega) \end{aligned} \tag{1.9}$$

This simplified notation is introduced since $*(\xi \wedge *\mathsf{d})$ can be interpreted as a linear differential operator on zero-forms[10]. If $\omega \triangleright \Omega M$, then the Cartan identity reduces to

$$\mathcal{L}_\xi \omega = \mathsf{d}(*\xi \; *\omega) \tag{1.10}$$

An important special case of (1.10) arises if $\mathsf{d}*\xi = 0$ and $\omega = *\phi$ where $\phi \triangleright \bigwedge^n M$. After some manipulation (1.10) can be written

$$\begin{aligned} \mathcal{L}_\xi *\phi &= **(\xi \wedge *\mathsf{d}\phi) \\ &= *\xi(\phi) \end{aligned} \tag{1.11}$$

on substitution of the definition of ξ as a linear differential operator.

Transport by the Flow of a Map

A parameterised form $\omega_t \triangleright \bigwedge^n M$ is said to be *transported by the flow* of ψ_t if

$$^*\psi_t \omega_t = \omega_0 \tag{1.12}$$

for all t.

[10] This follows since $*(\xi \wedge *\mathsf{d}) = \#\xi$, and the vector $\#\xi$ can be defined as a linear differential operator on zero-forms.

1.9.2 Transformations on a Manifold

Let the one-parameter map $\psi_t : M \to M$ be an automorphism of manifold M. The linearisation

$$\psi_\epsilon : x \mapsto x + \epsilon \, \#\xi + O(\epsilon^2) \tag{1.4}$$

for all $x \in M$ defines the *flow field*[8] $\xi \triangleright \mathcal{X}M$ of the map ψ_t.

The map ${}^*\psi_t : \bigwedge^n M \to \bigwedge^n M$ can then be induced from ψ_t. The induced map has the properties

$${}^*\psi_t f = f \circ \psi_t \tag{1.5}$$

for $f \triangleright \bigwedge^0 M$, and

$$\int_{\psi_t M} \omega = \int_M {}^*\psi_t \omega \tag{1.6}$$

for $\omega \triangleright \Omega M$.

Lie and Exterior Derivative

Define the Lie derivative[9] of $\omega \triangleright \bigwedge^n M$ with respect to the transformation ψ_t by

$$\mathcal{L}_\xi \omega = \lim_{t \to 0} \frac{1}{t} ({}^*\psi_t \omega - \omega) \tag{1.7}$$

where $\xi \triangleright \mathcal{X}M$ is the flow field of ψ_t.

[8] This definition is consistent with the idea of replacing vectors with their dual one-forms. Conventionally $\#\xi \triangleright \mathcal{V}M$ would be defined as the flow field of ψ_t.

[9] Again, in conventional notation the Lie derivative would be written $\mathcal{L}_{\#\xi}$.

Duality Between Vectors and One-Forms

The assumption of an Euclidean structure on a manifold M allows the distinction between one-form and vector fields to be relaxed. Given manifold M with Euclidean metric g, define the dual operator $\# : \mathcal{X}M \to \mathcal{V}M$ such that $\mathrm{g}(\#\omega) = \omega$ for any $\omega \triangleright \mathcal{X}M$. In the following analysis, minimal use is made of vector fields since the action of vectors on forms requires the use of the inner product $\lrcorner : \mathcal{V}M \times \bigwedge^n M \to \bigwedge^{n-1} M$, and the algebra associated with this operator is relatively undeveloped. As far as possible, a section of $\mathcal{V}M$ is replaced with its dual section of $\mathcal{X}M$. This replacement allows extensive use of Cartan's exterior algebra[7].

Hodge Star Operator

The metric structure on a manifold M allows the definition of the Hodge star operator $* : \bigwedge^{m'} M \to \bigwedge^{m-m'} M$, where $m = \dim(M)$ and $m' \leq m$. Throughout the following analysis it is assumed that $*1 = \Omega_M$, the Euclidean volume form, or *Hodge form* on M. The definition of the star operator on other forms follows from recursion on the definition

$$*(\beta \wedge \alpha) = \#\alpha \lrcorner *\beta \tag{1.2}$$

after appropriate definition of the forms α and β.

The Hodge star allows the inner product between vectors and forms to be replaced with the exterior product between the one-form dual to the vector and the original form:

$$\#\xi \lrcorner \omega = *(\xi \wedge *\omega) \tag{1.3}$$

for $\xi \triangleright \mathcal{X}M$ and $\omega \triangleright \bigwedge^{m'} M$, $m' \leq \dim(M)$.

[7]Standard references to the exterior algebra are the books by Burke [13] and Crampin and Pirani [18].

and analysed as a replacement for the classical theory's concept of minimisation of surface area as a guiding principle for the qualitative analysis of the dynamics of a fluid system with domains or interfaces.

Finally, in Chapter 12 some conclusions are drawn regarding the success and applicability of the techniques and models developed in this work, and some proposals for the extension and clarification of this work are put forward.

The remainder of this chapter is used to introduce some of the mathematical formalism that is used throughout the work which follows. An approach through the exterior algebra and calculus is used almost exclusively. This approach has seldom been used in the engineering fluid dynamics community, and one of the subsidiary aims of this work is to demonstrate to this community the simplicity and clarity of this approach when applied to fluid dynamic problems, especially perhaps to those classical problems where full understanding has been hampered at least partially by archaic and opaque mathematical formalism.

1.9 Notation and Conventions

1.9.1 Differential Forms

Denote the bundle of n-forms over a manifold M by $\bigwedge^n M$. The notation $\omega \triangleright \bigwedge^n M$ is used to indicate that the form-field ω is a section of the bundle $\bigwedge^n M$. Two bundles over M are of particular significance in the analysis which follows, and are given their own notation: denote the bundle of one-forms $\bigwedge^1 M$ by $\mathcal{X}M$ and the bundle of forms of maximal rank[8], $\bigwedge^m M$, where $m = \dim(M)$, by ΩM. Denote the tangent space at $x \in M$ by $\mathsf{T}_x M$ and the tangent bundle over M by $\mathsf{T}M$. Similarly denote the co-tangent space at $x \in M$ by $\mathsf{T}^*_x M$ and the co-tangent bundle over M by T^*M. On occasion the vector bundle over M is distinguished from the tangent bundle and denoted by $\mathcal{V}M$.

[8] Or 'volume forms'.

be related directly to the symmetry groups associated with the interactions present within the fluid. The Lie group approach is incomplete, however, in some cases of physical relevance.

It remains to be shown that the interaction field based approach is indeed consistent with the classical surface tension based approach. In Chapter 9 a quantity analogous to the classical surface tension is shown to be associated with most forms of interaction within the fluid. This analogous quantity, however, only results in a Young-Laplace equation for the pressure change across interfaces when the interactions within the fluid are of sufficiently short range. The classical example of the pressure at the centre of a spherical droplet is investigated, and the interaction field approach is shown not to exhibit the singularity associated with the classical approach as the droplet's radius goes to zero. This result is particularly relevant to an understanding of the analyses of Chapter 10, where a number of classical cases for the linearised dynamics of systems with interfaces about their equilibrium configurations are re-analysed using the interaction field based approach. The interaction field approach and the classical approach are shown to yield consistent predictions when the characteristic dimensions of the fluid system are large compared to the range of the interactions present in the system. Alternatively, in some cases the classical theory is shown to be a low-order approximation to the interaction field model. Where the characteristic dimensions of the fluid system are comparable to or smaller than the range of the interactions present in the system, the classical cases are shown to possess some interesting additional structure if the interaction field model rather than the surface tension based approach is employed in their analysis.

The analysis of Chapter 8 shows that the number of topologies available to the equilibrium configurations of any fluid system composed of distinct domains is small compared to the number of topologies available to non-equilibrium configurations of the system. An integral geometric approach is employed in Chapter 11 to show how configurations with arbitrary topology attain equilibrium configurations which are compatible with the results of Chapter 8. Most interestingly, a concept based on the ejection of topological defects is proposed

After the introduction of some statistical mechanical formalism, it is shown in Chapter 6 how the effects of the interactions between the particles comprising the fluid can be incorporated into a macroscopic model of the internal interactions within the fluid. The natural space for the analysis of the internal interactions is shown in general not to be the space occupied by the fluid, but to be a product structure constructed from this space. A qualitative form for the internal interactions, which is consistent with previous assumptions made with respect to the macroscopic structure of the fluid, is then proposed. It remains to be shown in later chapters that, under suitable conditions, such a form does indeed lead to correct predictions when compared to the predictions of the surface tension theory.

In Chapter 7, a second set of evolution equations is derived using the Lagrangian approach employed in Chapter 3 and Chapter 4. By enforcing compatibility between the analyses of Chapter 4 and Chapter 7, the two sets of equations are made to be equivalent representations of the dynamics of the fluid system. Whereas the equations of Chapter 4 apply on the space occupied by the fluid, the equations of Chapter 7 are based on a decomposition of the distortions to which the fluid is subjected into a set of distortion modes, and hence apply on a suitably defined space of deformations. It is shown that the distortion modes can be transferred to the product structure on which the internal interactions are defined, and hence that the resulting system of equations is compatible with the product structure that is shown in Chapter 6 to be required for description of the internal interactions within the fluid.

Several methods for the enumeration of possible configurations of the fluid system when it is in a state of static equilibrium are discussed in Chapter 8. As a first example of the application of the interaction field model derived in the previous chapters, Plateau's classical problem concerning the equilibrium of intersecting fluid membranes is shown to possess some interesting additional structure if the interaction field model, rather than a surface tension based analysis, is used for its solution. The most interesting approach to the enumeration of equilibria is that which employs the symmetry techniques of Lie. Some of the equilibrium configurations of a fluid with interfaces are shown to

itative and quantitative dynamical behaviour of fluid systems with interfaces as they move towards equilibrium or as they oscillate about some equilibrium configuration. This restriction has primarily been made to limit the scope of this work to manageable proportions. There is no reason to believe that the analyses presented here should not be valid in a far broader class of dynamical cases.

1.8 Synopsis

In Chapter 2 a physical model for the fluid system is introduced. It is assumed that the fluid density in each of the domains comprising the fluid system is invariant as the configuration of the fluid evolves. The resulting kinematic constraints on the fluid velocity and other physical properties of the fluid are then derived. As a prerequisite to a Lagrangian approach to the dynamics of the fluid, in Chapter 3 the kinetic energy, dissipation, and potential energy in the case where there are internal interactions within the fluid are written as integrals over the space occupied by the fluid. In Chapter 4 the results of the previous two chapters are used to derive a form of the Navier-Stokes equations which is applicable to the evolution of a fluid system where there are interfaces and internally generated interactions. Compatibility and continuity conditions on the velocity and pressure at the interfaces within the fluid are derived, and these results are used extensively in the chapters which follow. In the absence of dissipation, the Navier-Stokes equations can be written in terms of a velocity potential on the space occupied by the fluid. This is done in Chapter 5. Compatibility conditions on the fluid properties at interfaces are again derived for use in the analysis of Chapter 10. An alternate formulation of the dynamics in terms of vorticity is also derived. Although this formalism will not be taken further than this chapter, it is shown how this approach might be used to analyse some fluid configurations, in particular those with fractal interfaces, which are not accessible to an approach via the classical surface tension theory.

formulation, results in a system which is as readily, if not more, accessible to analysis than the approach based on the classical surface tension theory.

- A prime component of the integral formulation is a spatially-distributed potential field associated with the interactions between the various domains constituting the space occupied by the fluid system. It is not, however, necessary to make the direct connection between this potential field and the interactions between the individual particles comprising the fluid, as is suggested by the molecular approaches.

- The potential field associated with the interactions within the fluid can be incorporated into a macroscopic interaction field type model, and in a suitable limit such an approach does indeed reproduce the predictions of the classical surface tension theory.

- The interaction field based approach will yield insights into the physical behaviour of some fluid systems which surface tension based analysis cannot. Indeed, some systems which are not amenable at all to analysis via surface tension theory can be analysed using the interaction field based approach.

1.7 Scope

In common to all theoretical treatments, it is by definition impossible, without reference to experiment, to distinguish between effects which are spurious artifacts of the model and those effects which are true properties of the physical system that is being modelled. No attempt has been made to conduct the requisite experimentation which would support the predictions of the model derived in this work. The necessity to validate any divergences between the predictions of the model derived here and the predictions of the classical surface tension model is acknowledged, however.

As mentioned previously, the results presented in this work relate to the qual-

a model of the macroscopic effects of the interactions within the fluid on the dynamics of the system can be constructed. Of necessity such a macroscopic model would require the suppression of most of the microscopic detail within the system, and a particularly simple form would thus have to be assumed for the internal structure of the fluid. The remainder of the dynamic effects in the system should however be accessible from a direct macroscopic analysis of the fluid's behaviour. Compatibility between the classical analysis and such a hybrid approach would require that, in the absence of internal interactions, the hybrid fluid model and the classical model should revert to a common system of equations applicable to the dynamics of the fluid system. If properly constructed, this hybrid model should not suffer from the failings of the classical approach, but, in a certain limit, or under certain non-stringent conditions, should yield equivalent results to the surface tension theory.

For the remainder of this work, any macroscopic fluid model derived from a combination of a microscopic interaction model and a macroscopic dynamic model and using a simplified model for the internal structure of the fluid will be termed a *macroscopic interaction field model*, or simply an interaction field model. The remainder of this work deals with the construction and verification of such a model for the engineering analysis of fluid systems containing interfaces.

1.6 Hypothesis

The following statements form the basic hypothesis for the work to follow:

- An integral formulation of the dynamics of the system removes the problems associated with the classical theory, since these problems spring from local issues of differentiability and connectedness of the various interfaces within the fluid system.
- The shift in emphasis to the domains within the fluid, rather than their boundaries, associated with the change from a differential to an integral

1.4 Limitations of the Classical Model

The classical model is widely accepted within the engineering community, but is known to possess a number of limitations. The principal failing of the classical model arises from the singularity in the pressure which the Young-Laplace equation (1.1) admits as the curvature of the interfacial membrane becomes infinite. As is discussed in more detail in Chapter 9, there are a number of physical systems of interest which, when viewed on a macroscopic scale, do indeed have interfaces with near-singular curvature. There must be some doubt as to the accuracy of any analysis of such systems based on the classical model. This concern has been expressed by various authors, including Bogy [8] and Tjahjadi *et al.* [80].

In addition, there are a number of cases, described more fully in Chapters 9 and 11, where the surface tension-based model predicts that no evolution of the fluid system should take place. If additional terms, however, are introduced into the equations of motion for the fluid, and these terms are related in some way to the presence of long-range interactions within the fluid, then the system does indeed evolve and improved predictions of experimental data over those of the classical theory are obtained. In this regard, the original analyses of Derjaguin and Kusakov [21] and Verwey and Overbeek [85] and the more recent applications of this work by Erneux and Davis [26], Joosten [49] and Ruckenstein and Jain [72] should be consulted.

1.5 Hybrid Model

The ideal fluid model, for engineering applications, would be macroscopic in its applicability, as is the classical surface tension-based model, but would not suffer from the failings described in the previous section.

Comparison of the approaches described in the previous sections suggests that the molecular model of internal interactions can be used as a basis from which

a Lagrangian map and in terms of a differential formulation of the equations of motion, and again in Chapter 7 to derive a set of equations for the evolution of the fluid system which is based on the set of allowable deformations of the space occupied by the fluid.

3.2 Interaction Field Model

Define the interaction field model via the Lagrangian map

$$L = T - V + \int_{t_0}^{t_0+t} D_{pdt'} \tag{3.3}$$

where T and V are respectively the kinetic energy and potential energy of the fluid system, and D is the rate of energy dissipation in the system.

3.2.1 System Integrals

In this section, the various terms in the Lagrangian map (3.3) are defined in terms of integrals over the fluid domain $\mathcal{M}$. As such, the Lagrangian map can be interpreted here as a map $L : \mathcal{M} \to \mathbb{R}$ rather than as the map from the tangent space to the real line defined in Section 3.1. This redefinition has advantages, especially in the ease with which the resulting integrals can be calculated when the interaction field model is applied to cases of physical relevance. Unfortunately this redefinition leads to an element of algebraic clumsiness in the definitions of the Lagrangian operators in Chapter 4 and Chapter 7 where the dynamic equations for the interaction field model are derived via the Lagrangian approach.

Chapter 3

Interaction Field Model

3.1 Overview

In this chapter the interaction field model is defined in terms of a Lagrangian map $L : TM \rightarrow R$ from the tangent space of a suitably defined manifold M to the real line. The equations of motion for the fluid system can be obtained from the Lagrangian map of the system by invoking Hamilton's principle that the action

$$I = \int_{t_0}^{t_1} L dt \tag{3.1}$$

is stationary for the evolution of the system over the arbitrary time interval $t_0 \leq t \leq t_1$ or, in the conventional notation of the variational calculus,

$$\begin{aligned} \delta I &= \int_{t_0}^{t_1} \delta L dt \\ &= 0 \end{aligned} \tag{3.2}$$

where δ is a suitably defined[1] variational operator. This approach is used in Chapter 4 to show equivalence between the definition of the system in terms of

[1] For an account of the construction of these operators, see the books by Lanczos [48] and especially Burke [13].

interactions within the fluid which is consistent, in a fairly weak sense, with the very simple model for the interfacial structure assumed here.

It should be pointed out that more general macroscopic models for the fluid structure have been derived, for example the director-field models of Oseen [62]. These more sophisticated models allow properties such as the relative orientation of the particles comprising the fluid to be accounted for, and can be used to describe a more general class of fluid systems, including liquid crystal systems, than that described by the model presented here. The selection of a macroscopic model for the structure of the fluid amounts to an arbitrary restriction of the applicability of the particular fluid model to a particular class of physical fluid systems. This philosophy is dealt with in some detail by various authors, particularly by Truesdell [83]. It should be remembered that the aim of the present work is not to derive a model which is applicable to as broad a class of fluid systems as possible, but is rather, in as minimal a sense as possible, to obtain a fluid model which reduces, in appropriate circumstances, to the classical surface tension model described in Section 1.2. In the following chapters, the model defined here for the macroscopic structure of the fluid is shown to be a necessary component of an interaction field model which does indeed admit this reduction.

Since the volume of any domain interface is zero for all time, (2.15) is satisfied by the stronger condition

$$(*\nu|_{\partial\mathcal{M}_\alpha})|_{\partial_{\alpha\beta}} - (*\nu|_{\partial\mathcal{M}_\beta})|_{\partial_{\alpha\beta}} = 0 \tag{2.16}$$

for all $\alpha, \beta \in \mathcal{Z}$, which expresses the condition that the normal components of the velocities on either side of an internal interface must be equal.

By generalisation of (2.13), any $\xi \triangleright \mathcal{KM}$ obeying the restriction

$$(*\xi|_{\partial\mathcal{M}_\alpha})|_{\partial_{\alpha\beta}} - (*\xi|_{\partial\mathcal{M}_\beta})|_{\partial_{\alpha\beta}} = 0 \tag{2.17}$$

or, on rearranging,

$$\begin{aligned} (*\xi|_{\partial\mathcal{M}_\alpha})|_{\partial_{\alpha\beta}} &= \xi^{\mathrm{Land}} \\ (*\xi|_{\partial\mathcal{M}_\beta})|_{\partial_{\alpha\beta}} &= \xi^{\mathrm{Land}} \end{aligned} \tag{2.18}$$

where $\xi^{\mathrm{Land}} \triangleright \Omega\partial_{\alpha\beta}$, for all $\alpha, \beta \in \mathcal{Z}$, will be said to obey the *no-rupture* condition on the internal domain interfaces.

2.4 Applicability of the Fluid Model

In this chapter, a particularly simple model of the macroscopic fluid structure has been proposed. It might initially appear that the assumption *a priori* of immiscibility and incompressibility contradicts the underlying philosophy of the molecular models described in Section 1.3 that the structure of the fluid is governed on the microscopic level by the interactions between the particles of the various species present in the fluid. According to the molecular approaches, the species compositions and density variations of the fluid near the internal interfaces should in principle be determined from a knowledge of the interactions between the fluid particles near the interfaces. The apparent contradiction is partially removed in Chapter 6 by selecting a form for the

where

$$\partial_{\alpha\beta} = \partial\mathcal{M}_\alpha \cap \partial\mathcal{M}_\beta \tag{2.11}$$

for $\alpha, \beta \in \mathcal{Z}$ are interfaces located within $\mathcal{M}$, while the external interfaces ∂_α, $\alpha \in \mathcal{Z}$, are segments of the boundary $\partial\mathcal{M}$ as shown in Figure 2.1.

The domains of integration in (2.9) can be rearranged using these definitions to give

$$\sum_{\alpha\in\mathcal{Z}} \int_{\partial_\alpha} *\nu + \sum_{\alpha,\beta\in\mathcal{Z}} \int_{\partial_{\alpha\beta}} (*\nu|_{\partial\mathcal{M}_\alpha} - *\nu|_{\partial\mathcal{M}_\beta}) = 0 \tag{2.12}$$

on accounting for the opposite orientations on $\partial\mathcal{M}_\alpha, \partial\mathcal{M}_\beta \subset \partial_{\alpha\beta}$, $\alpha \neq \beta$ induced by the Hodge form $\Omega_\mathcal{M}$. Since the flow φ_t was defined as an automorphism of $\mathcal{M}$,

$$(*\nu)|_{\partial_\alpha} = 0 \tag{2.13}$$

for all $\alpha \in \mathcal{Z}$. By generalisation of (2.13), any $\xi \triangleright \mathcal{M}$ obeying the restriction

$$(*\xi)|_{\partial_\alpha} = 0 \tag{2.14}$$

for all $\alpha \in \mathcal{Z}$ will be said to obey the *automorphism condition* on the boundary of $\mathcal{M}$.

Equation 2.13 can be used to reduce (2.12) to the condition

$$\sum_{\alpha,\beta\in\mathcal{Z}} \int_{\partial_{\alpha\beta}} (*\nu|_{\partial\mathcal{M}_\alpha} - *\nu|_{\partial\mathcal{M}_\beta}) = 0 \tag{2.15}$$

Equation 2.15 implies that there is no net motion of fluid into any of the internal domain interfaces, and hence that the creation of volume in the fluid system by rupture or interpenetration of adjoining boundaries does not occur.

Since $*\rho$ is transported by the flow of φ_t, from Section 1.9.2,

$$\begin{aligned} \tfrac{d}{dt}*\rho &= *\tfrac{d}{dt}\rho \\ &= 0 \end{aligned} \tag{2.5}$$

since the Hodge form on $\mathcal{M}$ is time-invariant. Equivalently $(\frac{d}{dt}*\rho)|_{\mathcal{M}_\alpha} = 0$ for all $\alpha \in \mathcal{Z}$. Using definition (1.13) and the Cartan identity (1.8) then yields

$$\begin{aligned} [\tfrac{\partial}{\partial t}*\rho + \mathsf{d}(\rho\, *\boldsymbol{\nu})]|_{\mathcal{M}_\alpha} &= [\rho\, \mathsf{d}*\boldsymbol{\nu}]|_{\mathcal{M}_\alpha} \\ &= 0 \end{aligned} \tag{2.6}$$

after substitution of the restrictions (2.3) and (2.4) on the form of the fluid density. Hence, if $\rho|_{\mathcal{M}_\alpha} \neq 0$,

$$\mathsf{d}*\boldsymbol{\nu}|_{\mathcal{M}_\alpha} = 0 \tag{2.7}$$

for all $\alpha \in \mathcal{Z}$. The incompressibility condition (2.7) can be integrated over the manifold $\mathcal{M}_\alpha$ to give

$$\int_{\partial\mathcal{M}_\alpha} *\boldsymbol{\nu} = 0 \tag{2.8}$$

for all $\alpha \in \mathcal{Z}$, on applying Stokes' theorem.

2.3.1 Interfacial Conditions

From (2.8),

$$\sum_{\alpha\in\mathcal{Z}} \int_{\partial\mathcal{M}_\alpha} *\boldsymbol{\nu} = 0 \tag{2.9}$$

Let

$$\partial\mathcal{M}_\alpha = \partial_r \bigcup_{\beta\in\mathcal{Z}} \partial_{\alpha\beta} \tag{2.10}$$

Transport of Fragmentations

It follows that if ρ and η are transported by the flow, then for discontinuities in these fields to remain on the interfaces $\partial\mathcal{M}_\alpha$, $\alpha \in \mathcal{Z}$ for all time, each element $\mathcal{M}_\alpha$, $\alpha \in \mathcal{Z}$, of the $\mathcal{M}$-complete fragmentation $\mathcal{M}_\mathcal{Z}$ must also be transported by the flow of φ_t.

Hodge Forms on the Fragmentation

Each $\mathcal{M}_\alpha$, $\alpha \in \mathcal{Z}$, of the $\mathcal{M}$-complete fragmentation $\mathcal{M}_\mathcal{Z}$ is assumed to inherit the Hodge form $\Omega_\mathcal{M} \triangleright \Omega\mathcal{M}$ from $\mathcal{M}$ by restriction.

2.2.2 Fluid Species

Where appropriate, the index $\alpha \in \mathcal{Z}$ defines the *species* of the fluid in $\mathcal{M}_\alpha$. Note that the condition (1.18) for $\mathcal{M}$-completeness of $\{\mathcal{M}_\alpha\}_{\alpha\in\mathcal{Z}}$ amounts to an assumption of the immiscibility of the fluid species comprising the fluid[2].

2.3 Incompressibility Relations

For the moment, it is assumed that discontinuities may exist in the velocity $\nu \triangleright \mathcal{X}\mathcal{M}$, but that any such discontinuities are confined, as in the case of the density, to the interfaces $\partial\mathcal{M}_\alpha$, $\alpha \in \mathcal{Z}$. Note that this assumption places no further restrictions on the structure of the fluid system other than perhaps to require that the fragmentation of $\mathcal{M}$ be changed to incorporate discontinuities in the velocity as well as the density. The question of continuity of the velocity is addressed in greater detail in Section 4.5.

[2]Hence physical processes resulting from diffusion of one species into another are beyond the scope of the following analysis. This restriction has interesting consequences for the topology of the fluid on $\mathcal{M}$, as discussed in Chapter 11.

and also

$$\frac{\partial}{\partial t}{}^*\rho|_{\mathcal{M}_\alpha} = 0 \quad \text{and} \quad \frac{\partial}{\partial t}\eta|_{\mathcal{M}_\alpha} = 0 \tag{2.4}$$

Spatial discontinuities in ${}^*\rho$ and η are thus restricted to the interfaces $\partial\mathcal{M}_\alpha$, $\alpha \in \mathcal{Z}$. Note though that not all elements of the interfaces need be points at which the density or viscosity is discontinuous. It is advantageous in some cases (see Section 4.5) to introduce a complete fragmentation of $\mathcal{M}$ in which the locations of some of the interfaces are arbitrary and do not necessarily partition the fluid domain into regions of different density or viscosity.

A schematic of the resulting structure of the fluid domain is shown in Figure 2.1.

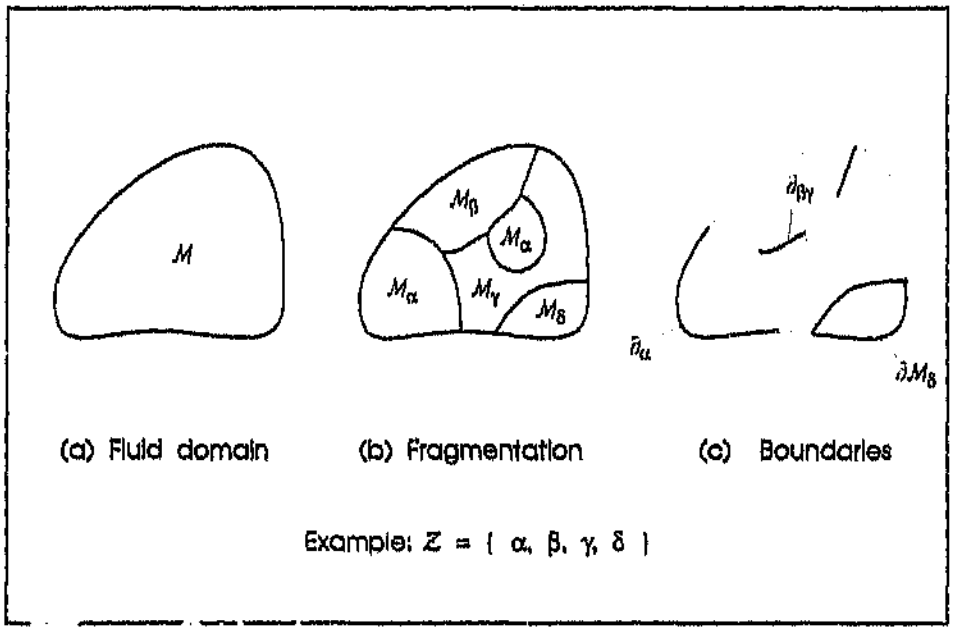


Figure 2.1: Schematic of the structure of the fluid domain.

2.2 Fluid Density and Viscosity

Let $*\rho \triangleright \Omega\mathcal{M}$ represent the density of the fluid, such that the total mass of fluid in $\mathcal{M}$ is

$$m = \int_{\mathcal{M}} *\rho \tag{2.1}$$

If it is assumed that m is invariant with time, then

$$\begin{aligned} \int_{\mathcal{M}} *\rho_0 &= \int_{\varphi_t\mathcal{M}} *\rho_t \\ &= \int_{\mathcal{M}} {}^*\varphi_t *\rho_t \end{aligned} \tag{2.2}$$

using (1.6). Hence $*\rho_0 = {}^*\varphi_t *\rho_t$, and thus from definition (1.12) the fluid density is transported by the flow of φ_t.

Similarly, define $\eta \triangleright \Lambda^0\mathcal{M}$ as the viscosity of the fluid. It is assumed that η is also transported by the flow of φ_t, although there is no justification for such an assumption based on a physical invariance as there is in the case of the density. In the work to follow, the condition that $\rho, \eta : \mathcal{M} \to \mathbb{R}^+$ is assumed to hold.

2.2.1 Continuity of the Density and Viscosity

Assume *a priori* that $*\rho$ and η are piecewise constant in $\mathcal{M}$. The assumption of piecewise continuity, and hence the possible presence of discontinuities in the density and viscosity fields on $\mathcal{M}$, leads to difficulties in interpreting elements of the exterior calculus on $\mathcal{M}$. These problems can be circumvented by defining an $\mathcal{M}$-complete fragmentation[1] $\{\mathcal{M}_\alpha\}_{\alpha\in\mathcal{Z}} \equiv \mathcal{M}_\mathcal{Z}$ (see Section 1.9.3) where $\mathcal{Z}$ is isomorphic to a subset of the integers, such that within each domain $\mathcal{M}_\alpha$,

$$\mathrm{d}\rho|_{\mathcal{M}_\alpha} = 0 \quad \text{and} \quad \mathrm{d}\eta|_{\mathcal{M}_\alpha} = 0 \tag{2.3}$$

[1] Implicit is that $\mathrm{meas}_{\mathcal{M}}\mathcal{M}_\alpha \neq 0$ for all $\alpha \in \mathcal{Z}$, and hence that the fragmentation is not degenerate.

Chapter 2

Fluid Model

In this chapter a macroscopic model for the structure of the fluid system is proposed. The fluid is modelled on an Euclidean manifold divided into a number of domains on which the density of the fluid is assumed to be constant throughout the evolution of the fluid system. The interfaces of the classical theory are interpreted as the boundaries between these domains. The assumption of constant density induces a number of kinematic constraints on the velocity of the fluid on the domains and their boundaries, and these constraints are derived in this chapter. The chapter closes with a discussion of the applicability of the proposed model to the description of the physical fluid systems of interest in this work.

2.1 The Domain of the Fluid

Assume that the space occupied by the fluid is adequately modelled by a two- or three-dimensional real Hausdorff manifold $\mathcal{M}$, endowed with a time-independent Euclidean metric tensor $\mathbf{g}$. Denote the Hodge form on $\mathcal{M}$ induced by the metric by $*1 = \Omega_{\mathcal{M}}$. Define the automorphism $\varphi_t : \mathcal{M} \to \mathcal{M}$, where t is time, such that the flow field $\nu \triangleright \mathfrak{X}\mathcal{M}$ of φ_t is the velocity of the fluid in $\mathcal{M}$.

Measure of a Set

If $S \subset M$, define the *measure* of S with respect to a volume form Ω on M by the integral

$$\text{meas}_\Omega(S) = \int_S \Omega \tag{1.23}$$

This definition can be used to define the measure of any $\omega \triangleright \bigwedge^{m'} M$, $m' \leq \dim(M)$ with respect to Ω by

$$\text{meas}_\Omega(\omega) = \int_{\text{supp}(\omega)} \Omega \tag{1.24}$$

where $\text{supp}(\omega)$ is the support of ω. Using definitions (1.23) and (1.24) it follows that if $N \subset M$ and $N \subset \text{supp}(\omega)$ then $\text{meas}_\Omega(\omega) > \text{meas}_\Omega(N)$ and if $\text{supp}(\omega) = N$ then $\text{meas}_\Omega(\omega) = \text{meas}_\Omega(N)$.

Stokes' Theorem

Let $m = \dim(M)$ and $\omega \triangleright \bigwedge^{m-1} M$. If an M-complete fragmentation $\{M_i\}_{i=1,\ldots,n}$ can be found such that the restrictions $\mathrm{d}(\omega|_{M_i})$ are continuous[12] for all $i = 1, \ldots, n$, then

$$\begin{aligned} \int_M \mathrm{d}\omega &= \sum_{i=1}^{n} \int_{M_i} \mathrm{d}\omega \\ &= \sum_{i=1}^{n} \int_{\partial M_i} \omega \end{aligned} \tag{1.20}$$

Note that in (1.20) the restriction operator on the integrand has been omitted for clarity. Restriction of the integrand to the domain of integration is implicit in all similar cases to follow.

Midpoint Theorem

If $f \triangleright \bigwedge^0 M$ then

$$\int_M *f = f(x) \int_M *1 \tag{1.21}$$

for some $x \in M$.

Integration by Parts

Let $\omega_\epsilon \triangleright \bigwedge^n \mathcal{M}$ and $\chi_\epsilon \triangleright \bigwedge^{n'} \mathcal{M}$, $n + n' \leq \dim(\mathcal{M})$, be two parameterised forms, and let $\dot{\omega}_\epsilon = \frac{d}{d\epsilon}\omega_\epsilon$ and $\dot{\chi}_\epsilon = \frac{d}{d\epsilon}\chi_\epsilon$ as defined in (1.13). Then

$$\int_{\epsilon_0}^{\epsilon_1} (\dot{\omega}_\epsilon \wedge \chi_\epsilon) d\epsilon = [\omega_\epsilon \wedge \chi_\epsilon]_{\epsilon_0}^{\epsilon_1} - \int_{\epsilon_0}^{\epsilon_1} (\omega_\epsilon \wedge \dot{\chi}_\epsilon) d\epsilon \tag{1.22}$$

on integrating by parts.

[12] Implicit is that a metric structure on M does exist so that continuity of forms on M is defined.

of the variations allows (4.21) to be decoupled into the independent equations

$$(p|_{\partial\mathcal{M}_\alpha})|_{\partial_{\alpha\beta}} - (p|_{\partial\mathcal{M}_\beta})|_{\partial_{\alpha\beta}} = 0$$

$$(p * \delta\mathbf{x})|_{\partial_\alpha} = 0$$

$$(\eta * \mathbf{d}\nu)|_{\partial_\alpha} = 0$$

$$((\eta\ \delta\mathbf{x} \wedge * \mathbf{d}\nu)|_{\partial\mathcal{M}_\alpha})|_{\partial_{\alpha\beta}} - ((\eta\ \delta\mathbf{x} \wedge * \mathbf{d}\nu)|_{\partial\mathcal{M}_\beta})|_{\partial_{\alpha\beta}} = 0 \tag{4.22}$$

for all $\alpha, \beta \in \mathcal{Z}$. The first equation in this set shows that the pressure p is continuous across internal domain interfaces while the second equation is automatically satisfied by the automorphism condition on $\delta\mathbf{x}$.

The third and fourth equations in this set impose important restrictions on the dynamics of the system only in the case where the viscosity η is not identically zero. Define the vorticity $\varsigma = *\mathbf{d}\nu$. By taking the exterior derivative of (4.18), the vorticity transport equation

$$[\ \rho\ \dot{\varsigma} - \eta\ \mathbf{d}^2\varsigma\ ]|_{\mathcal{M}_\alpha} = 0 \tag{4.23}$$

is obtained. The third equation in 4.22 implies that the restriction of the vorticity to the exterior boundary $\partial\mathcal{M}$ (since the $\partial_\alpha \subset \partial\mathcal{M}$) must never become non-zero. Via the following physical argument, it is easily seen that this restriction yields an important constraint on the time over which (4.18) is a valid description for the evolution of the fluid system.

Let $\text{meas}_{*1}(\varsigma)$ be the measure of ς with respect to the Euclidean volume form $*1$ on $\mathcal{M}$ (see Section 2.1). Assume that initially $\text{supp}(\varsigma) \subset \mathcal{M}$, so that the third equation in (4.22) is satisfied and $0 < \text{meas}_{*1}(\varsigma) < \text{meas}_{*1}(\mathcal{M})$. On $\partial\,\text{supp}(\varsigma)$, $\varsigma = 0$ but $\mathbf{d}^2\varsigma$ cannot be zero. It follows from (4.23) that $\dot{\varsigma}|_{\partial\text{supp}(\varsigma)}$ is non-zero, and hence $\frac{d}{dt}\text{meas}_{*1}(\varsigma) > 0$ for all time, implying from Section 1.9.3 the growth of $\text{supp}(\varsigma)$ as time progresses. At some time t^*, $\text{supp}(\varsigma) \cap \partial\mathcal{M}$ will thus become non-empty. (Note that by the assumption of φ_t an automorphism on $\mathcal{M}$, the vorticity in $\mathcal{M}$ will not be transported to the boundary of $\mathcal{M}$ in finite time by the flow of φ_t. The encroachment of $\text{supp}(\varsigma)$ on $\partial\mathcal{M}$ is solely a

4.5 Compatibility Conditions on Interfaces

The boundary integrals in (4.16) provide an additional set of equations constraining the evolution of the fluid system:

$$\begin{aligned}\delta\vartheta &\equiv \sum_{\alpha\in\mathcal{Z}}\int_{\partial\mathcal{M}_\alpha}[\,\eta\,\delta\mathbf{x}\wedge *\mathrm{d}\nu + p\,*\delta\mathbf{x}\,]\\ &= 0\end{aligned}\tag{4.19}$$

As in Section 2.3.1, the integral can be rewritten in terms of the factorisation of $\partial\mathcal{M}_\alpha$, $\alpha\in\mathcal{Z}$ into internal domain interfaces and exterior boundaries as

$$\begin{aligned}\delta\vartheta &= \sum_{\alpha\in\mathcal{Z}}\int_{\partial_\alpha}[\eta\,\delta\mathbf{x}\wedge *\mathrm{d}\nu + p\,*\delta\mathbf{x}\,]\\ &\quad+\sum_{\alpha,\beta\in\mathcal{Z}}\int_{\partial_{\alpha\beta}}[(\eta\,\delta\mathbf{x}\wedge *\mathrm{d}\nu)|_{\partial\mathcal{M}_\alpha}-(\eta\,\delta\mathbf{x}\wedge *\mathrm{d}\nu)|_{\partial\mathcal{M}_\beta}]\\ &\quad+\sum_{\alpha,\beta\in\mathcal{Z}}\int_{\partial_{\alpha\beta}}[(p\,*\delta\mathbf{x})|_{\partial\mathcal{M}_\alpha}-(p\,*\delta\mathbf{x})|_{\partial\mathcal{M}_\beta}\,]\\ &= 0\end{aligned}\tag{4.20}$$

Exploiting the no-rupture condition on the variation $\delta\mathbf{x}$,

$$\begin{aligned}\delta\vartheta &= \sum_{\alpha\in\mathcal{Z}}\int_{\partial_\alpha}[\eta\,\delta\mathbf{x}\wedge *\mathrm{d}\nu + p\,*\delta\mathbf{x}\,]\\ &\quad+\sum_{\alpha,\beta\in\mathcal{Z}}\int_{\partial_{\alpha\beta}}[(\eta\,\delta\mathbf{x}\wedge *\mathrm{d}\nu)|_{\partial\mathcal{M}_\alpha}-(\eta\,\delta\mathbf{x}\wedge *\mathrm{d}\nu)|_{\partial\mathcal{M}_\beta}]\\ &\quad+\sum_{\alpha,\beta\in\mathcal{Z}}\int_{\partial_{\alpha\beta}}(p|_{\partial\mathcal{M}_\alpha}-p|_{\partial\mathcal{M}_\beta})\delta\mathbf{x}^{\perp\alpha\beta}\\ &= 0\end{aligned}\tag{4.21}$$

where $\delta\mathbf{x}^{\perp\alpha\beta}=(*\delta\mathbf{x}|_{\mathcal{M}_\alpha})|_{\partial_{\alpha\beta}}=(*\delta\mathbf{x}|_{\mathcal{M}_\alpha})|_{\partial_{\alpha\beta}}\triangleright\Omega\vartheta_{\alpha\beta}$.

The variations $\delta\mathbf{x}|_{\partial_\alpha}$, $*\delta\mathbf{x}|_{\partial_\alpha}$ and $\delta\mathbf{x}^{\perp\alpha\beta}$, for all $\alpha,\beta\in\mathcal{Z}$, can be imposed on the fluid system independently of each other and of the variations $(\delta\mathbf{x}|_{\partial\mathcal{M}_\alpha})|_{\partial_{\alpha\beta}}$, $\alpha,\beta\in\mathcal{Z}$. The relationship between the variations $(\delta\mathbf{x}|_{\partial\mathcal{M}_\alpha})|_{\partial_{\alpha\beta}}$, $\alpha,\beta\in\mathcal{Z}$, is determined later in this section. Exploiting the independence and arbitrariness

where (4.3) has been used to replace $\delta\boldsymbol{\nu}$ with $\delta\dot{\mathbf{x}}$. On exploiting (1.22) and the condition that the variations vanish at t_0 and t_1,

$$\begin{aligned}\delta I &= \sum_{\alpha\in\mathcal{Z}}\int_{t_0}^{t_1}\int_{\mathcal{M}_\alpha}\delta\mathbf{x}\wedge *[\,\rho\dot{\boldsymbol{\nu}}+\mathsf{d}\Phi+\mathsf{d}p-\eta *\mathsf{d}*\mathsf{d}\boldsymbol{\nu}\,]dt\\ &\quad+\sum_{\alpha\in\mathcal{Z}}\int_{t_0}^{t_1}\int_{\partial\mathcal{M}_\alpha}[\,\eta\,\delta\mathbf{x}\wedge *\mathsf{d}\boldsymbol{\nu}+p\,*\delta\mathbf{x}\,]dt\\ &= 0\end{aligned}\tag{4.16}$$

since, from (2.5), $\dot{\rho}|_{\mathcal{M}_\alpha}=0$ for all $\alpha\in\mathcal{Z}$. Since the variations $\delta\mathbf{x}|_{\mathcal{M}_\alpha}$, $\alpha\in\mathcal{Z}$ are arbitrary and independent of each other (up to the imposition of the no-rupture condition on the boundaries $\partial\mathcal{M}_\alpha$),

$$[\,\rho\dot{\boldsymbol{\nu}}+\mathsf{d}\Phi+\mathsf{d}p-\eta *\mathsf{d}*\mathsf{d}\boldsymbol{\nu}\,]|_{\mathcal{M}_\alpha}=0\tag{4.17}$$

for all $\alpha\in\mathcal{Z}$. Since from (2.7) $\mathsf{d}*\boldsymbol{\nu}|_{\mathcal{M}_\alpha}=0$, $\mathsf{d}*\mathsf{d}*\boldsymbol{\nu}|_{\mathcal{M}_\alpha}=0$ and hence

$$[\,\rho\dot{\boldsymbol{\nu}}+\mathsf{d}\Phi+\mathsf{d}p-\eta\mathsf{d}^2\boldsymbol{\nu}\,]|_{\mathcal{M}_\alpha}=0\tag{4.18}$$

for all $\alpha\in\mathcal{Z}$, where $\mathsf{d}^2=*\mathsf{d}*\mathsf{d}+(-1)^m\mathsf{d}*\mathsf{d}*$, $m=\dim(\mathcal{M})$, is the conventional Laplace-DeRham operator.

If the Lagrange multiplier p is interpreted as the pressure in the fluid, (4.18) should be recognised as the Navier-Stokes momentum equation[1] except that the body-force term[2] $\mathsf{d}\Phi$ contains a contribution from the internal interactions within the fluid.

[1] See, for instance, the book by Landau and Lifshitz [40].

[2] Note that the quantity $\mathsf{d}(\Phi+p)$ in (4.18) can be interpreted as the velocity-independent component of the force acting on the fluid. This interpretation forms the basis of the analysis of Chapter 9.

and the variation of the dissipation function is then

$$\delta D = \sum_{\alpha\in\mathcal{Z}} \int_{\mathcal{M}_\alpha} \eta \, \mathrm{d}(\delta\nu_t) \wedge *\mathrm{d}\nu_t \tag{4.12}$$

through an analysis similar to that in the case of the kinetic energy, by assumption of the translational invariance of $\eta|_{\mathcal{M}_\alpha}$ for all $\alpha \in \mathcal{Z}$. Exploiting this assumption and the $\wedge$-antiderivation property of the exterior derivative,

$$\begin{aligned} \delta D &= \sum_{\alpha\in\mathcal{Z}} \int_{\mathcal{M}_\alpha} \{ \, \mathrm{d}(\eta \, \delta\nu_t \wedge *\mathrm{d}\nu_t) \; + \; \eta \, \delta\nu_t \wedge \mathrm{d}*\mathrm{d}\nu_t \, \} \\ &= \sum_{\alpha\in\mathcal{Z}} \int_{\mathcal{M}_\alpha} \eta \, \delta\nu_t \wedge \mathrm{d}*\mathrm{d}\nu_t + \sum_{\alpha\in\mathcal{Z}} \int_{\partial\mathcal{M}_\alpha} \eta \, \delta\nu_t \wedge *\mathrm{d}\nu_t \end{aligned} \tag{4.13}$$

using Stokes' theorem.

Finally, the variation of the potential energy is

$$\begin{aligned} \delta V &= \sum_{\alpha\in\mathcal{Z}} \int_{\mathcal{M}_\alpha} \mathcal{L}_{\delta\mathbf{x}} * \Phi \\ &= \sum_{\alpha\in\mathcal{Z}} \int_{\mathcal{M}_\alpha} \delta\mathbf{x} \wedge *\mathrm{d}\Phi \end{aligned} \tag{4.14}$$

on fragmenting the integral and using (4.4).

4.4 Navier Stokes Analogue

Substituting (4.10), (4.13) and (4.14) into (3.2), the variation of the action

$$\begin{aligned} \delta I &= \sum_{\alpha\in\mathcal{Z}} \int_{t_0}^{t_1} \int_{\mathcal{M}_\alpha} [\, \delta\dot{\mathbf{x}} \wedge (\, p*\nu - \eta \int_{t_0}^{t_0+t} \mathrm{d}*\mathrm{d}\nu_t dt' \,) \,] dt \\ &\quad - \int_{t_0}^{t_1} \int_{\mathcal{M}_\alpha} [\, \delta\mathbf{x} \wedge (\, *\mathrm{d}\Phi + *\mathrm{d}p \,) \,] dt \\ &\quad + \sum_{\alpha\in\mathcal{Z}} \int_{t_0}^{t_1} \int_{\partial\mathcal{M}_\alpha} [\, p \, *\delta\mathbf{x} - \eta \, \delta\dot{\mathbf{x}} \wedge \int_{t_0}^{t_0+t} *\mathrm{d}\nu_t dt' \,] dt \\ &= 0 \end{aligned} \tag{4.15}$$

on applying Stokes' theorem.

4.3.1 Variations of the System Integrals

The variations of the various terms in the Lagrangian map as a result of the variations $\delta\mathbf{x}$ and $\delta\boldsymbol{\nu}$ are now calculated using the variational operator defined in (4.1).

Expression 3.6 for the kinetic energy of the system can be rewritten in terms of the fragmentation of $\mathcal{M}$ introduced in Section 2.2.1 as

$$T = \sum_{\alpha\in\mathcal{Z}} \tfrac{1}{2}\int_{\mathcal{M}_\alpha} \rho\, \boldsymbol{\nu}\wedge *\boldsymbol{\nu} \tag{4.9}$$

The variation of the kinetic energy of the system is then

$$\begin{aligned} \delta T &= \sum_{\alpha\in\mathcal{Z}} \int_{\mathcal{M}_\alpha} [\, \rho\, \delta\boldsymbol{\nu}\wedge *\boldsymbol{\nu} \;+\; \mathcal{L}_{\delta\mathbf{x}}(\tfrac{1}{2}\rho)\, \boldsymbol{\nu}\wedge *\boldsymbol{\nu}\,] \\ &= \sum_{\alpha\in\mathcal{Z}} \int_{\mathcal{M}_\alpha} [\, \rho\, \delta\boldsymbol{\nu}\wedge *\boldsymbol{\nu}\,] \end{aligned} \tag{4.10}$$

since, using the Cartan identity (1.9), $\mathcal{L}_{\delta\mathbf{x}}\rho|_{\mathcal{M}_\alpha} = *(\delta\mathbf{x}\wedge *\mathrm{d}\rho)|_{\mathcal{M}_\alpha} = 0$ by the assumption of translational invariance of $\rho|_{\mathcal{M}_\alpha}$. The advantage of using the approach via the fragmentation of $\mathcal{M}$ into domains upon which the density and viscosity is translationally invariant should now be clear: the problem of possibly having to interpret $\mathrm{d}\rho$ (and, in the case of the dissipation function, $\mathrm{d}\eta$) at discontinuities on the boundaries $\partial\mathcal{M}_\alpha$, $\alpha\in\mathcal{Z}$ has been avoided. Use of the fragmentation of $\mathcal{M}$ has in fact deferred the classification of the discontinuities in the various fields on $\mathcal{M}$ to the interpretation of a number of compatibility equations on the interfaces $\partial\mathcal{M}_\alpha$, $\alpha\in\mathcal{Z}$ (see Section 4.5).

Expression (3.8) for the dissipation function can be rewritten in terms of the fragmentation of $\mathcal{M}$ as

$$D = \sum_{\alpha\in\mathcal{Z}} \tfrac{1}{2}\int_{\mathcal{M}_\alpha} \eta\, \mathrm{d}\boldsymbol{\nu}\wedge *\mathrm{d}\boldsymbol{\nu} \tag{4.11}$$

itself obey the no-rupture condition. Similarly, $\delta\mathbf{x}$ and $\delta\nu$ must both obey the automorphism condition (2.14) on the exterior interfaces.

Within $\mathcal{M}_\alpha$, $\alpha \in \mathcal{Z}$, the assumption of incompressibility (2.7) induces the additional conditions on the variations that

$$\begin{aligned} \mathsf{d}*\delta\mathbf{x}|_{\mathcal{M}_\alpha} &= 0 \\ \mathsf{d}*(\nu+\delta\nu)|_{\mathcal{M}_\alpha} &= 0 \end{aligned} \tag{4.4}$$

or, since ν is assumed already to obey (2.7),

$$\mathsf{d}*\delta\nu|_{\mathcal{M}_\alpha} = 0 \tag{4.5}$$

for all $\alpha \in \mathcal{Z}$.

4.3 Modification of Hamilton's Principle

Hamilton's principle can be amended to include condition (4.4) by modifying the Lagrangian ma; (.3) for the system to the form

$$L = T - V + \int_{t_0}^{t_0+t} D_t dt' + \mathrm{II} \tag{4.6}$$

where

$$\mathrm{II} = \sum_{\alpha\in\mathcal{Z}} \int_{\mathcal{M}_\alpha} p\, \mathsf{d}*\delta\mathbf{x} \tag{4.7}$$

and $p \triangleright \Lambda^0\mathcal{M}$ is a Lagrange multiplier. Note that since $\delta\mathbf{x}$ might contain discontinuities on the interfaces $\partial\mathcal{M}_\alpha$, $\alpha \in \mathcal{Z}$, so might p. Consideration of the continuity of p follows in Section 4.5.

Alternatively, using the $\wedge$-antiderivation property of the exterior derivative

$$\begin{aligned} \mathrm{II} &= \sum_{\alpha\in\mathcal{Z}} \int_{\mathcal{M}_\alpha} \{\, \mathsf{d}(p\,*\delta\mathbf{x}) - \delta\mathbf{x}\wedge *\mathsf{d}p \,\} \\ &= \sum_{\alpha\in\mathcal{Z}} \int_{\partial\mathcal{M}_\alpha} p\,*\delta\mathbf{x} - \sum_{\alpha\in\mathcal{Z}} \int_{\mathcal{M}_\alpha} \delta\mathbf{x}\wedge *\mathsf{d}p \end{aligned} \tag{4.8}$$

to the velocity field $\boldsymbol{\nu}$, such that $\delta\mathbf{x}_{t_0} = \delta\mathbf{x}_{t_1} = 0$ and $\delta\boldsymbol{\nu}_{t_0} = \delta\boldsymbol{\nu}_{t_1} = 0$, that is, such that the variations vanish at the beginning and end-points of the motion.

These distortions to the fluid system yield

$$\delta = \delta\boldsymbol{\nu}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\nu}} + \delta\mathbf{x}\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}} \tag{4.1}$$

as the variational operator for the system (see Section 3.1). The operators $\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\nu}}$ and $\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}}$ are functional derivatives of the variational calculus obeying the rules

$$\begin{aligned}
\delta\mathbf{x}\,\tfrac{\mathrm{d}}{\mathrm{d}\mathbf{x}}\boldsymbol{\nu} &= 0 \\
\delta\mathbf{x}\,\tfrac{\mathrm{d}}{\mathrm{d}\mathbf{x}}\omega &= \mathcal{L}_{\delta\mathbf{x}}\omega \\
\delta\boldsymbol{\nu}\,\tfrac{\mathrm{d}}{\mathrm{d}\boldsymbol{\nu}}\boldsymbol{\nu} &= \delta\boldsymbol{\nu} \\
\delta\boldsymbol{\nu}\,\tfrac{\mathrm{d}}{\mathrm{d}\boldsymbol{\nu}}\omega &= 0
\end{aligned} \tag{4.2}$$

where $\boldsymbol{\nu}$ is the fluid velocity as previously interpreted, and $\omega \triangleright \bigwedge^n\mathcal{M}$, $n \leq \dim(\mathcal{M})$, distinct from $\boldsymbol{\nu}$. The messiness of the algebraic behaviour of the variational operator is a consequence of the variations $\delta\boldsymbol{\nu}$ and $\delta\mathbf{x}$ actually being linked via the kinematic compatibility condition

$$\begin{aligned}
\delta\boldsymbol{\nu}_t &= \tfrac{\mathrm{d}}{\mathrm{d}t}\delta\mathbf{x}_t \\
&= \delta\dot{\mathbf{x}}_t
\end{aligned} \tag{4.3}$$

The notation where the superscript dot represents the total derivative with respect to the time parameter t is used throughout the following sections.

The analysis which follows is based on the complete fragmentation of $\mathcal{M}$ into the sets $\mathcal{M}_\alpha$, $\alpha \in \mathfrak{Z}$, as described in Section 2.2.1. In this case, the restrictions $\delta\mathbf{x}|_{\mathcal{M}_\alpha}$ and $\delta\boldsymbol{\nu}|_{\mathcal{M}_\alpha}$ of the variations can be considered to be independent for all $\alpha \in \mathfrak{Z}$, but for the variations to be consistent with the restrictions on the kinematics of the system imposed by the macroscopic structure of the fluid on $\mathcal{M}$, $\boldsymbol{\nu} + \delta\boldsymbol{\nu}$ and $\delta\mathbf{x}$ must satisfy the no-rupture condition (2.18) on all internal interfaces in the fluid. Since the no-rupture condition is a linear operator on one-forms, it follows, since $\boldsymbol{\nu}$ already obeys the condition, that $\delta\boldsymbol{\nu}$ must

Chapter 4

Fluid Dynamic Equations

4.1 Overview

In this chapter, the form adopted for the Lagrangian in the definition of the interaction field model of the previous chapter is shown to yield a differential equation on the fluid domain $\mathcal{M}$. This equation closely resembles the Navier Stokes equation which is assumed to govern the fluid mechanics in the domains of the classical surface tension model. In addition, the model adopted for the macroscopic structure of the fluid, derived in Chapter 2, is shown to result in a set of dynamic compatibility conditions which must be enforced at the boundaries between the domains in the interaction field model. These conditions allow the continuity of the velocity $\nu \triangleright \mathcal{M}$ to be investigated, and permit the issues of compatibility between the interaction field approach and the classical model raised in Section 1.5 and Section 2.4 to be partially resolved.

4.2 Variational Analysis

Assume the fluid system to be subject to time-parameterised variations $\delta\chi \triangleright \mathcal{M}$, representing a distortion to $\mathcal{M}$, and $\delta\nu \triangleright \mathcal{M}$, representing a distortion

tions which better accommodate the form of the interaction potential derived in Chapter 6.

but is independent of the fluid's velocity. Included in this term are assumed to be contributions to the potential energy of the fluid from effects generated outside the fluid and also from internal interactions generated within the fluid itself.

Dissipation Function

The dissipation function is in general a time-parameterised map $D_t : \mathcal{TM} \rightarrow \mathbb{R}$, the specific form of which is dependent on the local structure of the fluid. Consistent with the neglect of the internal and rotational modes in the kinetic energy term, and with the isotropic form of the fluid density assumed in Section 2.2, the simplest, isotropic, form for the dissipation function is assumed:

$$D = \tfrac{1}{2} \int_{\mathcal{M}} \eta \, \mathrm{d}\nu \wedge *\mathrm{d}\nu \tag{3.8}$$

where η is the viscosity of the fluid defined in Section 2.2.

3.3 Summary

In this chapter a Lagrangian approach to the dynamics of the fluid system has been followed: The Lagrangian (3.3), defined in terms of the energy and rate of energy dissipation within the system, together with the specific forms (3.6), (3.7) and (3.8) and the structural assumptions of Chapter 2 define the interaction field model. In the following chapter it is shown that one particular manifestation of this model is a dynamic equation on $\mathcal{M}$ which is analogous to the Navier-Stokes equations for an incompressible fluid. The form of the interaction potential Φ in (3.7) has yet to be related to the configuration of the fluid system. The construction of this relationship is the subject of Chapter 6, where the Navier-Stokes analogue on $\mathcal{M}$ is shown not to be the most natural dynamical manifestation of the interaction field model. The results of this chapter will thus be used again in Chapter 7 to derive a set of dynamic equa-

Kinetic Energy

Define[2]

$$T = \tfrac{1}{2} \int_{\mathcal{M}} \rho \, \nu \wedge *\nu \tag{3.6}$$

This form accounts for the kinetic energy contained in the fluid by virtue of its velocity ν but, to be consistent with the structural assumptions made for the fluid in Chapter 2, neglects possible contributions to the kinetic energy from other sources such as the rotational motion of individual fluid particles or motion with respect to internal degrees of freedom of the fluid particles.

Potential Energy

The potential energy of the fluid is assumed to have the form

$$V = \int_{\mathcal{M}} *\Phi_t \tag{3.7}$$

where the time-parameterised form $\Phi_t \triangleright \bigwedge^0 \mathcal{M}$ is the potential associated with the interactions within the fluid. As is shown in Chapter 6 and in the chapters which follow, a suitable definition of this potential leads to compatibility between the predictions of the interaction field model and those of the classical surface tension theory.

The general form of the interaction potential given here assumes that the potential energy of the system is dependent on the configuration of the fluid,

[2]The kinetic energy could equivalently have been defined as

$$T = \tfrac{1}{2} \int_{\mathcal{M}} \rho \, \mathrm{g}(\#\nu, \#\nu) \, \Omega_{\mathcal{M}} \tag{3.4}$$

using the metric tensor on $\mathcal{M}$, or

$$T = \tfrac{1}{2} \int_{\mathcal{M}} \rho \, \#\nu \rfloor \nu \, \Omega_{\mathcal{M}} \tag{3.5}$$

using the inner product. The form (3.6) is to be preferred for reasons discussed in Section 1.9.1.

Kinetic Energy

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using the inner product. The form (3.6) is to be preferred for reasons discussed in Section 1.9.1.

on combining (5.12), (5.18) and (5.19) and hence

$$*(\ \mathrm{d}(\rho^{-1}) \wedge \mathrm{d}(p+\Phi)\ ) \ = \ -\tfrac{1}{\epsilon}\Delta^{(\epsilon)}\rho^{-1} \times \tfrac{\partial}{\partial x^3}(p+\Phi_{\alpha\beta})\ dx^2$$
$$+\tfrac{1}{\epsilon}\Delta^{(\epsilon)}\rho^{-1} \times \tfrac{\partial}{\partial x^2}(p+\Phi_{\alpha\beta})\ dx^3 \qquad (5.21)$$

where in (5.20) and (5.21) the right-hand side is to be evaluated at $(0, x^2, x^3)$.

Notice that the derivative normal to the interface of both the pressure and the potential has disappeared, obviating the problems with the continuity of the pressure and potential that were discussed earlier in this section.

The second term in the vorticity source

$$\mathrm{d}(\eta/\rho) \wedge \mathrm{d}^2\nu \ = \ \tfrac{1}{\epsilon}\Delta^{(\epsilon)}(\eta/\rho)\ dx^1 \wedge \tfrac{\partial}{\partial x^2}\tfrac{\partial}{\partial x^3}[\overline{v}_2 dx^2 + \overline{v}_3 dx^3]$$
$$+\tfrac{1}{\epsilon}\Delta^{(\epsilon)}(\eta/\rho)\ dx^1 \wedge \tfrac{\partial}{\partial x^3}\tfrac{\partial}{\partial x^2}[\overline{v}_2 dx^2 + \overline{v}_3 dx^3] \qquad (5.22)$$

on combining (5.12) and (5.15), and hence

$$*(\ \mathrm{d}(\eta/\rho) \wedge \mathrm{d}^2\nu\ ) \ = \ -\tfrac{1}{\epsilon}\Delta^{(\epsilon)}(\eta/\rho) \times (\tfrac{\partial}{\partial x^2}\tfrac{\partial}{\partial x^2}\overline{v}_3 + \tfrac{\partial}{\partial x^3}\tfrac{\partial}{\partial x^3}\overline{v}_3)\ dx^2$$
$$+\tfrac{1}{\epsilon}\Delta^{(\epsilon)}(\eta/\rho) \times (\tfrac{\partial}{\partial x^3}\tfrac{\partial}{\partial x^3}\overline{v}_2 + \tfrac{\partial}{\partial x^3}\tfrac{\partial}{\partial x^3}\overline{v}_2)\ dx^3 \qquad (5.23)$$

where again in (5.22) and (5.23) the right-hand side is to be evaluated at $(0, x^2, x^3)$.

It is a simple matter to reconstruct the geometric form of the vorticity source from (5.21) and (5.23). By inspection

$$\sigma|_{\partial_{\alpha\beta}} = \tfrac{1}{\epsilon}\Delta^{(\epsilon)}\rho^{-1}\ *\mathrm{d}\ (p|_{\partial_{\alpha\beta}} + \Phi_{\alpha\beta}) + \tfrac{1}{\epsilon}\Delta^{(\epsilon)}(\eta/\rho)\ *\mathrm{d}^2\nu_{\alpha\beta} \qquad (5.24)$$

where $\Phi_{\alpha\beta} \triangleright \bigwedge^0 \partial_{\alpha\beta}$ and, from (5.14),

$$\nu_{\alpha\beta} = \frac{1}{2}[\ (\nu|_{\mathcal{M}_\alpha})|_{\partial_{\alpha\beta}} + (\nu|_{\mathcal{M}_\beta})|_{\partial_{\alpha\beta}}\ ] \qquad (5.25)$$

showing that the vorticity source becomes singular as the thickness of the interfacial volume is reduced to zero, or, in other words, in the case where

on substituting (5.14) for the velocity at the interface into definition (5.9), and hence on the interface

$$\begin{aligned} \mathrm{d}^2\nu &= \tfrac{\partial}{\partial x^2}\tfrac{\partial}{\partial x^1}[\overline{v}_1 dx^1 + \overline{v}_2 dx^2 + \overline{v}_3 dx^3]|_{(0,x^2,x^3)} \\ &\quad + \tfrac{\partial}{\partial x^3}\tfrac{\partial}{\partial x^2}[\overline{v}_1 dx^1 + \overline{v}_2 dx^2 + \overline{v}_3 dx^3]|_{(0,x^2,x^3)} \end{aligned} \tag{5.17}$$

In Section 4.5 it was shown that the pressure is continuous through an internal fluid interface. Hence on the interface

$$\mathrm{d}p = (\ \tfrac{1}{\epsilon}\Delta^{(\epsilon)}p\ dx^1 + \tfrac{\partial}{\partial x^2}p\ dx^2 + \tfrac{\partial}{\partial x^3}p\ dx^3\ )|_{(0,x^2,x^3)} \tag{5.18}$$

in the limit as $\epsilon \to 0$. As in the case of the velocity, the generalisation of the normal derivative has been used since continuity of the pressure is insufficient for the normal derivative to exist. It is shown shortly, however, that the first term on the right hand side of this expression cancels in the vorticity equation and hence the problem of the existence of the pressure derivative normal to the interface can be avoided.

The exterior derivative of the potential Φ on the interface should similarly be generalised as

$$\begin{aligned} \mathrm{d}\Phi &= (\ \tfrac{1}{\epsilon}\Delta^{(\epsilon)}\Phi\ dx^1 + \tfrac{\partial}{\partial x^2}\Phi\ dx^2 + \tfrac{\partial}{\partial x^3}\Phi\ dx^3\ )|_{(0,x^2,x^3)} \\ &= (\ \tfrac{1}{\epsilon}\Delta^{(\epsilon)}\Phi\ dx^1 + \tfrac{\partial}{\partial x^2}\Phi_{\alpha\beta}\ dx^2 + \tfrac{\partial}{\partial x^3}\Phi_{\alpha\beta}\ dx^3\ )|_{(0,x^2,x^3)} \end{aligned} \tag{5.19}$$

as $\epsilon \to 0$, where $\Phi_{\alpha\beta}$ is the potential on the interface $\partial_{\alpha\beta}$. This form follows since, as is implied by the analysis of Chapter 6, the potential Φ is in general discontinuous across the interface.

The results obtained above can now be combined to obtain a coordinate expression for the vorticity source in the interfacial region.

The first term in the vorticity source

$$\begin{aligned} \mathrm{d}(\rho^{-1}) \wedge \mathrm{d}(p+\Phi) &= \tfrac{1}{\epsilon}\Delta^{(\epsilon)}\rho^{-1}\ dx^1 \wedge \tfrac{\partial}{\partial x^2}(p+\Phi_{\alpha\beta})\ dx^2 \\ &\quad + \tfrac{1}{\epsilon}\Delta^{(\epsilon)}\rho^{-1}\ dx^1 \wedge \tfrac{\partial}{\partial x^3}(p+\Phi_{\alpha\beta})\ dx^3 \end{aligned} \tag{5.20}$$

Using the dual basis introduced in the neighbourhood of P, the velocity

$$\nu = v_1 dx^1 + v_2 dx^2 + v_3 dx^3 \tag{5.13}$$

where $v_1, v_2, v_3 \triangleright \bigwedge^0 \mathcal{M}$. Let the velocity in the interfacial volume be given by

$$\nu = (\overline{v}_1 dx^1 + \overline{v}_2 dx^2 + \overline{v}_3 dx^3)|_{(0,x^2,x^3)} \tag{5.14}$$

in the limit as $\epsilon \to 0$. This form is suggested by Friedrichs [30] in the case of an interface between two immiscible domains of fluid and is consistent with the no-rupture condition on the internal fluid interfaces developed in Section 2.3.1.

On the interface the Laplace-deRham derivative of the velocity is thus

$$\begin{aligned}
\mathrm{d}^2\nu &= \tfrac{1}{\epsilon^2}(\Delta^{(\epsilon)})^2[v_1 dx^1 + v_2 dx^2 + v_3 dx^3]|_{(0,x^2,x^3)} \\
&\quad + \tfrac{\partial}{\partial x^2}\tfrac{\partial}{\partial x^2}[v_1 dx^1 + v_2 dx^2 + v_3 dx^3]|_{(0,x^2,x^3)} \\
&\quad + \tfrac{\partial}{\partial x^3}\tfrac{\partial}{\partial x^3}[v_1 dx^1 + v_2 dx^2 + v_3 dx^3]|_{(0,x^2,x^3)} \\
&= \tfrac{1}{\epsilon^2}(\Delta^{(\epsilon)})^2[v_1 dx^1 + v_2 dx^2 + v_3 dx^3]|_{(0,x^2,x^3)} \\
&\quad + \tfrac{\partial}{\partial x^2}\tfrac{\partial}{\partial x^2}[\overline{v}_1 dx^1 + \overline{v}_2 dx^2 + \overline{v}_3 dx^3]|_{(0,x^2,x^3)} \\
&\quad + \tfrac{\partial}{\partial x^3}\tfrac{\partial}{\partial x^3}[\overline{v}_1 dx^1 + \overline{v}_2 dx^2 + \overline{v}_3 dx^3]|_{(0,x^2,x^3)}
\end{aligned} \tag{5.15}$$

using (5.13) and (5.14) and abusing the notation slightly, in the limit as $\epsilon \to 0$. This form follows since, as discussed in Section 2.3, the components v_2 and v_3 of the velocity parallel to the interface may be discontinuous across the interface. The no-rupture condition of Section 2.3.1 of course requires the component v_1 of the velocity normal to the interface to be continuous across the interface, but continuity is insufficient for the derivative $\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^1}v_1$ to exist. Hence in (5.15), the derivatives with respect to x^1 of all the velocity components must be generalised using the limiting procedure described earlier. Note however that on the interface

$$(\Delta^{(\epsilon)})^2[v_1 dx^1 + v_2 dx^2 + v_3 dx^3]|_{(0,x^2,x^3)} = 0 \tag{5.16}$$

infinitesimally thick, discontinuous interface follows as $\epsilon \to 0$. To make further progress it is necessary to model the variation of the fluid properties through the interfacial volume. The best model is obviously obtained if the assumptions made during the modelling of the interfacial properties have no effect on the form of the final equations derived. As will be shown, this independence of the final model on assumptions regarding the structure of the interface can be achieved with some success.

The relationship between the density and viscosity within the interfacial volume and that on either side of the interfacial volume will not be specified other than to assume that the interfacial properties bear a *constant relationship*[3] to the properties of the fluid on either side of the interface. Since the density and viscosity are assumed to be constant on either side of the interface (see Section 2.2.1), it follows that in the limit as $\epsilon \to 0$, $\rho|_{\partial_{\alpha\beta}}$ and $\eta|_{\partial_{\alpha\beta}}$ should then also be constant, or, in terms of the coordinates introduced onto the interface,

$$\begin{aligned} \tfrac{\partial}{\partial x^2}\rho|_{(0,x^2,x^3)} = \tfrac{\partial}{\partial x^3}\rho|_{(0,x^2,x^3)} &= 0 \\ \tfrac{\partial}{\partial x^2}\eta|_{(0,x^2,x^3)} = \tfrac{\partial}{\partial x^3}\eta|_{(0,x^2,x^3)} &= 0 \end{aligned} \tag{5.11}$$

Using (5.11), since the density and viscosity of the fluid may be discontinuous across the interface $\partial_{\alpha\beta}$,

$$\begin{aligned} \mathrm{d}(\rho^{-1}) &= \tfrac{1}{\epsilon}\Delta^{(\epsilon)}\rho^{-1}|_{(0,x^2,x^3)}\, dx^1 \\ \mathrm{d}(\eta/\rho) &= \tfrac{1}{\epsilon}\Delta^{(\epsilon)}(\eta/\rho)|_{(0,x^2,x^3)}\, dx^1 \end{aligned} \tag{5.12}$$

at any point on the interface $\partial_{\alpha\beta}$ in the limit as $\epsilon \to 0$.

[3] Note that in some physical systems the viscosity *on* the interface differs significantly from the viscosity in the fluid on either side of the interface: see for instance the paper by Sørensen and Hennenberg [76]. It would appear from the simple form of the vorticity source that results from the analysis presented here that the fluid properties in these systems are not 'sufficiently smooth' for (5.5) and (5.7) to capture adequately their variations across the interfacial volume. This situation could be remedied, and compatibility with the structural assumptions of Chapter 2 could be enforced, by including the interfacial volume, with its constant interfacial viscosity and density, into the $\mathcal{M}$-complete fragmentation $\mathcal{M}_3$ of the fluid domain. Note though that it is implicit in this case that the interfacial volume should not reduce to a set of zero measure with respect to the volume form on $\mathcal{M}$ but should retain a finite thickness.

If ω is 'sufficiently smooth' then[2]

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \Delta^{(\epsilon)} \omega|_{(0,x^2,x^3)} = \frac{\partial}{\partial x^1} \omega|_{(0,x^2,x^3)} \tag{5.6}$$

Likewise define

$$(\Delta^{(\epsilon)})^2 \omega|_{(0,x^2,x^3)} = \omega|_{(\frac{\epsilon}{2},x^2,x^3)} - 2\,\omega|_{(0,x^2,x^3)} + \omega|_{(-\frac{\epsilon}{2},x^2,x^3)} \tag{5.7}$$

If ω is sufficiently smooth then of course

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} (\Delta^{(\epsilon)})^2 \omega|_{(0,x^2,x^3)} = \frac{\partial}{\partial x^1} \frac{\partial}{\partial x^1} \omega|_{(0,x^2,x^3)} \tag{5.8}$$

Finally define the interfacial average of ω,

$$\overline{\omega}^{(\epsilon)}|_{(0,x^2,x^3)} = \frac{1}{2}\left(\omega|_{(\frac{\epsilon}{2},x^2,x^3)} + \omega|_{(-\frac{\epsilon}{2},x^2,x^3)} \right) \tag{5.9}$$

Substituting this definition into (5.7) gives the alternate definition

$$(\Delta^{(\epsilon)})^2 \omega|_{(0,x^2,x^3)} = 2\left(\overline{\omega}|_{(0,x^2,x^3)} - \omega|_{(0,x^2,x^3)} \right) \tag{5.10}$$

These results enable the derivatives of the fluid properties at the internal interfaces to be generalised in the case where the fluid properties may be discontinuous at the interfaces.

Interfacial Model

For the purposes of the following analysis, assume the existence of an interfacial volume, comprised of the thin volume of fluid, characterised by thickness ϵ, surrounding the surface ∂_{int}. In this interfacial volume the fluid propoerties may differ from those on either side of the interface. The analysis for the

[2]See Footnote 3.

5.2.1 Vorticity Source

At the fluid interfaces, however, there may indeed be sources of vorticity. The analysis of the vorticity equation in the neighbourhood of the internal interfaces proceeds here via the introduction of coordinates in the neighbourhood of an arbitrary point located in one of the internal interfaces. The geometric form of the vorticity equation is then reconstituted by examination of the resulting coordinate form of the expression for the vorticity source at the interface.

Coordinate Analysis

The following analysis assumes $\dim(\mathcal{M}) = 3$, but the two-dimensional case can be handled in a similar fashion. Let (x^1, x^2, x^3) be orthogonal coordinates valid in the neighbourhood of a point $P \in \mathcal{M}$ located in one of the internal fluid interfaces $\partial_{\alpha\beta}$, $\alpha, \beta \in \mathcal{Z}$ (see Section 2.3.1 for the definitions of external and internal fluid interfaces). Let P have coordinates $(0,0,0)$, and let the coordinate x^1 be normal to $\partial_{\alpha\beta}$ in the sense that $(dx^1)|_{\partial_{\alpha\beta}} = 0$, where (dx^1, dx^2, dx^3) is the dual basis induced by the coordinates in the neighbourhood of P. This condition on x^1 and the orthogonality of coordinates implies that the coordinates x^2 and x^3 are tangential to the interface at P.

Derivatives at the Interface

Let $\omega \triangleright \bigwedge^0 \mathcal{M}$, or in other words $\omega : (x^1, \ldots, x^m) \to \mathbb{R}$ for all m-tuples $(x^1, \ldots, x^m)$ in the chart induced by the coordinates defined in the neighbourhood of P. Define

$$\Delta^{(\epsilon)}\omega|_{(0,x^2,x^3)} = \omega|_{(\frac{\epsilon}{2},x^2,x^3)} - \omega|_{(-\frac{\epsilon}{2},x^2,x^3)} \tag{5.5}$$

5.2 Vorticity Transport Equation

Define the vorticity ς as[1]

$$\begin{aligned} \varsigma &= *\mathrm{d}\nu \\ &\triangleright \Lambda^0\mathcal{M} \end{aligned} \tag{5.1}$$

As shown already in Section 4.5, the vorticity transport equation on the fluid domains $\mathcal{M}_\alpha$, $\alpha \in \mathcal{Z}$ can be derived very simply by taking the exterior derivative of the momentum equation (4.18). Here a similar procedure is followed, but (4.18) is first assumed to be valid throughout the fluid domain $\mathcal{M}$. Then

$$\dot{\varsigma} + (\eta/\rho)\mathrm{d}^2\varsigma = \sigma \tag{5.2}$$

where the vorticity source $\sigma \triangleright \Lambda^0\mathcal{M}$ is given by

$$\sigma = *[\ \mathrm{d}(\rho^{-1}) \wedge \mathrm{d}(p+\Phi) + \mathrm{d}(\eta/\rho) \wedge \mathrm{d}^2\nu\ ] \tag{5.3}$$

since exterior differentiation and the Hodge star commute with the total derivative with respect to time, the Lie derivative and the Laplace De Rham operator, and exploiting definition (5.1).

Away from the fluid interfaces $\partial\mathcal{M}_\alpha$, $\alpha \in \mathcal{Z}$, $\sigma = 0$, by the assumption that ρ and η are piecewise constant in $\mathcal{M}$, and by definition constant in each $\mathcal{M}_\alpha$. Hence away from fluid interfaces the vorticity transport equation reduces to the linear transport-diffusion equation

$$[\ \dot{\varsigma} + (\eta/\rho)\mathrm{d}^2\varsigma\ ]|_{\mathcal{M}_\alpha} = 0 \tag{5.4}$$

for all $\alpha \in \mathcal{Z}$, derived in Section 4.5.

[1] The conventional definition of the vorticity is of course the vector field $\#(*\mathrm{d}\nu)$, but, as described in Section 1.9.1, use of the dual section will admit the more powerful algebra.

Chapter 5

Alternative Formulations

5.1 Overview

In this chapter, two alternative formulations, based on the Navier-Stokes equations, for the dynamics of the fluid system are derived. The vorticity transport equation derived in Section 5.2 yields additional insight into the structure of the deformations of internal interfaces under the influence of the interaction potential and the pressure within the fluid. The vorticity transport equation is shown to contain a source term which is non-zero only on the internal interfaces within the fluid , and in Section 5.2.1 an explicit form for this source is obtained. This form for the source term is used in Section 5.3 to complete the analysis of Section 4.5 of the continuity of the velocity of the fluid by supplying a result for the case where the fluid has identically zero viscosity. The analysis of the vorticity transport equations is concluded by an examination of the practicality of the technique in the analysis of fluid systems with non-differentiable boundaries - a class of fluid systems which is not amenable to surface tension based analysis. Finally in Section 5.5 a second set of evolution equations, based on the existence of a potential for the velocity in the case where the fluid has zero vorticity, is derived. These equations are used extensively in Chapter 10 where the correspondence between the dynamical predictions of the classical surface tension model and the interaction field based approach is examined.

the interaction field approach in the absence of internal interactions is thus demonstrated.

It is also shown that the macroscopic structure of the fluid adopted in Chapter 2 leads to a set of conditions on the continuity of the velocity which are imposed on the boundaries of the domains comprising the fluid by the interaction field approach. This analysis is applied in Chapter 11 where the effect of the topology of the fluid configuration on the evolution of the fluid system towards equilibrium is discussed. The discussion of the continuity of the velocity has not yet been completed, however, since the compatibility equations (4.22) yield no information on the velocity continuity in the important case of identically zero viscosity. The analysis of the continuity of the velocity in this case must await the results of the following chapter, where two alternative dynamical formulations for the fluid system, based on the Navier-Stokes equation (4.18), are derived.

conditions (4.22) hold, except that the condition of pressure continuity across the internal domain boundaries is replaced with the condition that

$$(\overline{p}|_{\partial\mathcal{M}_\alpha})|_{\partial_{\alpha\beta}} - (\overline{p}|_{\partial\mathcal{M}_\beta})|_{\partial_{\alpha\beta}} = \sigma_{\alpha\beta}\kappa_{\alpha\beta} \tag{4.30}$$

for all $\alpha, \beta \in \mathcal{Z}$, using the Young-Laplace equation presented in Section 1.2. In this expression $\sigma_{\alpha\beta}$ is the surface tension, and $\kappa_{\alpha\beta}$ is the mean curvature, of the interface $\partial_{\alpha\beta}$.

4.7 Conclusion

In this chapter, the structural assumptions of Chapter 2 and the forms for the kinetic energy, potential energy and dissipation assumed in Chapter 3 are shown to lead to the dynamics of the fluid on $\mathcal{M}$, under the interaction field model, being governed by an analogue to the Navier-Stokes equation. Importantly, the issue of compatibility between the classical surface tension theory and the interaction field model, discussed in Section 1.5 and again in Section 2.4, can be resolved trivially: If there are no internal interactions present within the fluid, then the term $\mathbf{d}\Phi$ in (4.18) can be interpreted as the usual body force term resulting from interactions between the fluid and a field generated externally to the fluid, and hence (4.18) and (4.29) are identical models for the dynamics of the fluid within the domains $\mathcal{M}_\alpha$, $\alpha \in \mathcal{Z}$, if it can be shown that $p \equiv \overline{p}$ in the absence of internal interactions. In the classical model, from Section 1.3, absence of internal interactions requires the surface tension $\sigma_{\alpha\beta} = 0$ on all interfaces within $\mathcal{M}$. In this case (4.30) reduces to

$$(\overline{p}|_{\partial\mathcal{M}_\alpha})|_{\partial_{\alpha\beta}} - (\overline{p}|_{\partial\mathcal{M}_\beta})|_{\partial_{\alpha\beta}} = 0 \tag{4.31}$$

for all $\alpha, \beta \in \mathcal{Z}$, which is identical to the pressure condition in (4.22) which is imposed on the boundaries by the interaction field approach. Hence p is trivially equivalent to $\overline{p}$ and compatibility between the classical theory and

and that

$$(\delta \mathbf{x}|_{\partial \mathcal{M}_\alpha})|_{\partial_{\alpha\beta}} = (\delta \mathbf{x}|_{\partial \mathcal{M}_\beta})|_{\partial_{\alpha\beta}} \tag{4.28}$$

on all internal domain boundaries $\partial_{\alpha\beta}$. Equation 4.27 is a condition on the continuity of the vorticity $*\mathrm{d}\boldsymbol{\nu}$ on internal domain boundaries, while (4.28) shows that the variation $\delta\mathbf{x}$ must be continuous on internal domain boundaries. Via the kinematic compatibility condition (4.3), $\delta\boldsymbol{\nu}$ must thus also be continuous on internal domain boundaries. Since the variations represent physically realisable deformations to the system, it must be concluded that the flow field $\boldsymbol{\nu}$ must also be continuous on all internal boundaries induced by the $\mathcal{M}$-complete fragmentation $\mathcal{M}_{\mathcal{Z}}$. Note though that the result just obtained holds also in the case where $\mathcal{M}_{\mathcal{Z}}$ is an arbitrary complete fragmentation of $\mathcal{M}$ in which the viscosity is not necessarily discontinuous everywhere on the internal domain boundaries (see Section 2.2.1). Hence the flow field $\boldsymbol{\nu}$ must in fact be continuous *throughout* $\mathcal{M}$.

4.6 Surface Tension Model

As discussed in Section 1.2, the classical surface tension model adopted in this work assumes that the fluid dynamics within the domains comprising the fluid is governed by Navier-Stokes equations analogous to (4.18):

$$[\ \rho\dot{\boldsymbol{\nu}} + \mathrm{d}\Phi + \mathrm{d}\overline{p} - \eta\mathrm{d}^2\boldsymbol{\nu}\ ]|_{\mathcal{M}_\alpha} = 0 \tag{4.29}$$

for all $\alpha \in \mathcal{Z}$, where Φ, instead of representing the potential arising from internal interactions and interactions with external fields, must be interpreted as the potential arising solely from interactions with *external* fields. The pressure $\overline{p}$ in (4.29) has a somewhat different interpretation to fluid variable p occurring in (4.18). The reason for this difference in interpretation will become apparent in Chapter 9 when the static predictions of the classical theory and the interaction field model are reconciled. On the domain boundaries the interfacial

result of viscous diffusion.) Hence, for any time $t \geq t^*$ the condition imposed by the third equation in (4.22) will be violated. This equation thus imposes a limit to the time for which the system of equations derived here remains a valid description of the dynamics of the system. An alternate viewpoint on this restriction is obtained by recalling that the velocity ν can be obtained by integration of the vorticity field over the domain of the fluid, as shown, for example, by Wu and Thompson [88]. This restriction can then be understood by realising that the velocity on $\mathcal{M}$ resulting from a vorticity field ς with $\mathrm{supp}(\varsigma) \supset \mathcal{M}$ cannot be accounted for by a description of the fluid system based on integration over $\mathcal{M}$ alone.

It remains to interpret the constraint on the evolution of the system implied by the fourth equation of (4.22). Define the sections

$$\begin{aligned}
\zeta_\alpha &\equiv ((\eta * \mathrm{d}\nu)|_{\partial\mathcal{M}_\alpha})|_{\partial_{\alpha\beta}} \\
\zeta_\beta &\equiv ((\eta * \mathrm{d}\nu)|_{\partial\mathcal{M}_\beta})|_{\partial_{\alpha\beta}} \\
\delta\mathbf{x}_\alpha &\equiv (\delta\mathbf{x}|_{\partial\mathcal{M}_\alpha})|_{\partial_{\alpha\beta}} \\
\delta\mathbf{x}_\beta &\equiv (\delta\mathbf{x}|_{\partial\mathcal{M}_\beta})|_{\partial_{\alpha\beta}}
\end{aligned} \tag{4.24}$$

of $\mathcal{K}\partial_{\alpha\beta}$ such that the fourth equation of (4.22) can be rewritten as

$$\zeta_\alpha \wedge \delta\mathbf{x}_\alpha = \zeta_\beta \wedge \delta\mathbf{x}_\beta \tag{4.25}$$

If $\zeta_\alpha \neq \zeta_\beta$ then this equation can be solved to give

$$\begin{aligned}
\delta\mathbf{x}_\alpha &= k\,\zeta_\beta \\
\delta\mathbf{x}_\beta &= -k\,\zeta_\alpha
\end{aligned} \tag{4.26}$$

where $k \in \mathbb{R}$, and hence the variations $\delta\mathbf{x}$ on the internal domain boundaries are not arbitrary, as assumed, but are forced to have a particular orientation related to the fluid velocity at the boundaries. If $\zeta_\alpha = \zeta_\beta$, however, then $\delta\mathbf{x}_\alpha = \delta\mathbf{x}_\beta = \xi$ where ξ is an *arbitrary* section of $\mathcal{K}\mathcal{M}$. Arbitrariness of the variations $\delta\mathbf{x}$ must also be obtained by requiring that

$$((\eta * \mathrm{d}\nu)|_{\partial\mathcal{M}_\alpha})|_{\partial_{\alpha\beta}} = ((\eta * \mathrm{d}\nu)|_{\partial\mathcal{M}_\beta})|_{\partial_{\alpha\beta}} \tag{4.27}$$

Differentiation

A consequence of definition (6.4) is that

$$\mathsf{d}(\omega_{\mathcal{F}\gg\mathcal{T}}) = (\mathsf{d}\omega_{\mathcal{F}})_{\gg\mathcal{T}} \tag{6.14}$$

since $\omega_{\mathcal{F}\gg\mathcal{T}}$ is invariant transverse to the fibres $i_f(\mathcal{F}, b)$, $b \in B$. A similar result holds for $\omega_{\mathcal{B}\gg\mathcal{T}}$, of course.

Using this result

$$\mathsf{d}(\omega_{\mathcal{B}} \oplus \omega_{\mathcal{F}}) = (\mathsf{d}\omega_{\mathcal{B}}) \oplus (\mathsf{d}\omega_{\mathcal{F}}) \tag{6.15}$$

and

$$\mathsf{d}(\omega_{\mathcal{B}} \otimes \omega_{\mathcal{F}}) = (\mathsf{d}\omega_{\mathcal{B}}) \otimes \omega_{\mathcal{F}} + (-1)^p \omega_{\mathcal{B}} \otimes (\mathsf{d}\omega_{\mathcal{F}}) \tag{6.16}$$

where $p = \dim(\mathcal{F})$.

It is easily shown using the Cartan identity (1.10) and the results (6.14), (6.9) and (6.10) that

$$\mathcal{L}_{(\xi_{\mathcal{B}} \oplus \xi_{\mathcal{F}})}(\omega_{\mathcal{B}} \otimes \omega_{\mathcal{F}}) = (\mathcal{L}_{\xi_{\mathcal{B}}}\omega_{\mathcal{B}}) \otimes \omega_{\mathcal{F}} + \omega_{\mathcal{B}} \otimes (\mathcal{L}_{\xi_{\mathcal{F}}}\omega_{\mathcal{F}}) \tag{6.17}$$

Equivalently, if $\xi_{\mathcal{B}}$ and $\xi_{\mathcal{F}}$ are interpreted as differential operators, then

$$\xi_{\mathcal{B}} \oplus \xi_{\mathcal{F}} : (\omega_{\mathcal{B}} \otimes \omega_{\mathcal{F}}) \mapsto \xi_{\mathcal{B}}(\omega_{\mathcal{B}}) \otimes \omega_{\mathcal{F}} + \omega_{\mathcal{B}} \otimes \xi_{\mathcal{F}}(\omega_{\mathcal{F}}) \tag{6.18}$$

Finally, since differentiation with respect to a parameter commutes with the composition product,

$$\tfrac{d}{dt}(\omega_{\mathcal{B}} \otimes \omega_{\mathcal{F}}) = (\tfrac{d}{dt}\omega_{\mathcal{B}}) \otimes \omega_{\mathcal{F}} + \quad (\tfrac{d}{dt}\omega_{\mathcal{F}}) \tag{6.19}$$

It is important to realise that, from their definition in terms of the inclusion operator, neither the composition product nor the composition sum are commutative.

Hodge Star Operator

Let Ω_B and Ω_F be the Hodge forms on $\mathcal{B}$ and $\mathcal{F}$ respectively. These forms induce a Hodge form

$$\Omega_P = \Omega_B \otimes \Omega_F \tag{6.8}$$

on $\mathcal{P}$ such that

$$*(\omega_{B\geq} \ ) = (*\omega_B) \otimes \Omega_F \tag{6.9}$$

while

$$*(\omega_{F>P}) = \Omega_B \otimes (*\omega_F) \tag{6.10}$$

Note too that, since the fibres derived from $\mathcal{F}$ and $\mathcal{B}$ are orthogonal at every point in $\mathcal{X}$,

$$\omega_{B>P} \wedge *(\omega_{F>P}) = 0 \tag{6.11}$$

and similarly

$$\omega_{F>P} \wedge *(\omega_{B>P}) = 0 \tag{6.12}$$

It follows from (6.9) and (6.10) that

$$*(\omega_B \flat \omega_F) = (*\omega_B) \otimes \Omega_F + \Omega_B \otimes (*\omega_F) \tag{6.13}$$

while no simple result holds for the composition product since the Hodge star operator does not distribute over the exterior product.

6.2.2 Inclusion Operator

The induced map

$$^{*}i_f : \bigwedge^n \mathcal{F} \times \mathcal{B} \to \bigwedge^n \mathcal{P} \tag{6.3}$$

for any $n \leq \dim(\mathcal{F})$, can be used to define the inclusion operators

$$\begin{aligned} > \; &: \; \bigwedge^n \mathcal{F} \to \bigwedge^n \mathcal{P} \\ &: \; \omega_{\mathcal{F}} \mapsto \omega_{\mathcal{F}>\mathcal{P}} \end{aligned} \tag{6.4}$$

for all $n \leq \dim(\mathcal{F})$, by construction of $\omega_{\mathcal{F}>\mathcal{P}}$ such that

$$\omega_{\mathcal{F}>\mathcal{P}} - {}^{*}i_f(\omega_{\mathcal{F}}, b) = 0 \;\; \forall\, b \in \mathcal{B} \tag{6.5}$$

Note that by transposing the roles of $\mathcal{B}$ and $\mathcal{F}$ in the interpretation of $\mathcal{X}$ as a trivial fibre bundle, the inclusion operators $>: \bigwedge^{n'} \mathcal{B} \to \bigwedge^{n'} \mathcal{P}$, for all $n' \leq \dim(\mathcal{B})$, can also be defined.

6.2.3 Composite Forms

Let $\omega_{\mathcal{B}} \triangleright \bigwedge^{n'} \mathcal{B}$ and $\omega_{\mathcal{F}} \triangleright \bigwedge^{n} \mathcal{F}$. Also define $\xi_{\mathcal{B}} \triangleright \mathcal{K}\mathcal{B}$ and $\xi_{\mathcal{F}} \triangleright \mathcal{K}\mathcal{F}$.

Composition Operators

Define the *composition product* by

$$\begin{aligned} \otimes \; &: \; \bigwedge^{n'} \mathcal{B} \times \bigwedge^{n} \mathcal{F} \to \bigwedge^{n'+n} \mathcal{P} \\ &: \; (\omega_{\mathcal{B}}, \omega_{\mathcal{F}}) \mapsto \omega_{\mathcal{B}} \otimes \omega_{\mathcal{F}} \equiv \omega_{\mathcal{B}>\mathcal{P}} \wedge \omega_{\mathcal{F}>\mathcal{P}} \end{aligned} \tag{6.6}$$

and, in the case where $n' = n$, the *composition sum* by

$$\begin{aligned} \oplus \; &: \; \bigwedge^{n} \mathcal{B} \times \bigwedge^{n} \mathcal{F} \to \bigwedge^{n} \mathcal{P} \\ &: \; (\omega_{\mathcal{B}}, \omega_{\mathcal{F}}) \mapsto \omega_{\mathcal{B}} \oplus \omega_{\mathcal{F}} \equiv \omega_{\mathcal{B}>\mathcal{P}} + \omega_{\mathcal{F}>\mathcal{P}} \end{aligned} \tag{6.7}$$

over a potential $\mathcal{F}$ defined on $\mathcal{M}^n$, where n is the number of particles in the largest cluster for which the energy of interaction is considered significant. It is demonstrated in Section 6.4.6 that the potential $\Phi \triangleright \bigwedge^0 \mathcal{M}$ can be obtained from the potential $\mathcal{F} \triangleright \bigwedge^0 \mathcal{M}^n$ via a suitable projection onto $\mathcal{M}$.

The transfer from the microscopic view of the fluid system adopted in Section 6.4 to the macroscopic perspective required by the interaction field approach forms the subject of Section 6.5. In this section the potential energy associated with interactions on a microscopic level between the particles comprising the fluid is transformed into an energy associated solely with the macroscopic configuration of the fluid. This transformation follows by generalisation of the form derived in Section 6.4 for the potential $\mathcal{F}$. Finally, a form for $\mathcal{F}$ which is consistent with the macroscopic structure of the fluid detailed in Chapter 2 is derived in Section 6.5.1.

6.2 Structure of Product Spaces

6.2.1 Trivial Fibre Bundles

Any product space $\mathcal{P} = \mathcal{B} \times \mathcal{F}$ can be interpreted as a trivial fibre bundle $\mathcal{P}$ over $\mathcal{B}$ by defining the projection

$$\pi : \mathcal{P} \to \mathcal{B} \tag{6.1}$$

such that there exists the isomorphism between fibres

$$\begin{aligned} i_f \; &: \; \mathcal{F} \times \mathcal{B} \to \mathcal{P} \\ &: \; (\mathcal{F}, b) \mapsto \pi^{-1}(b) \;\; \forall \, b \in \mathcal{B} \end{aligned} \tag{6.2}$$

Chapter 6

Potential Energy

6.1 Overview

In Chapter 3 the potential energy of the system was written as an integral over the potential Φ defined on the domain of the fluid $\mathcal{M}$. In this chapter, the dependence of the potential $\Phi \in \bigwedge^0 \mathcal{M}$ on the configuration of the fluid system is determined for a fluid where there are both internal interactions within the fluid and also interactions between the fluid and some externally-generated field.

In Section 6.2 some preliminary results regarding the algebra and calculus of differential forms on product spaces are derived. These results are used in Section 6.3 to transform some of the results of Chapter 2 for the structure of the fluid system from expressions on the fluid domain $\mathcal{M}$ into analogous results on the product structure $\mathcal{M}^n = \mathcal{M} \times \cdots \times \mathcal{M}$ (n factors), where $n \in \mathbb{Z}^+$. These results are used in Section 6.4 and again in Chapter 7. In Section 6.4 a microscopic view of the fluid system as a collection of particles is taken and a statistical mechanical approach is then used to derive a form for the potential energy which is explicitly dependent on the energies associated with the interaction between clusters of the particles comprising the fluid. It is shown that the most natural description of the potential energy is as an integral

5.6 Summary

In this chapter, two additional dynamic formulations for the fluid system are derived from the Navier-Stokes equation. The vorticity-based approach has enabled the analysis of the continuity of the velocity to be completed, while the utility of the approach based on the velocity potential is demonstrated in Chapter 10.

In all the dynamical formulations derived so far, the interaction potential Φ has appeared, but no explicit form for this potential in terms of the configuration of the fluid system has yet been derived. This derivation is t ubject of the following chapter.

and hence

$$\begin{aligned} \varsigma &= *\mathrm{d}\mathrm{d}\lambda \\ &\equiv 0 \end{aligned} \tag{5.30}$$

An important property of the velocity potential arises from the incompressibility condition (2.7). Since $\mathrm{d}*\mathrm{d}*\omega = 0$ for any $\omega \triangleright \Lambda^0\mathcal{M}$,

$$\mathrm{d}^2\lambda|_{\mathcal{M}_\alpha} = 0 \tag{5.31}$$

for all $\alpha \in \mathcal{Z}$, and hence the velocity potential is harmonic in each of the fluid domains.

The no-rupture condition (2.16) on the fluid velocity can be written in terms of the velocity potential as

$$(\ (*\mathrm{d}\lambda)|_{\mathcal{M}_\alpha}\ )|_{\partial_{\alpha\beta}} = (\ (*\mathrm{d}\lambda)|_{\mathcal{M}_\beta}\ )|_{\partial_{\alpha\beta}} \tag{5.32}$$

on all internal interfaces $\partial_{\alpha\beta}$, while the automorphism condition (2.13) on the fluid velocity can be written as

$$(*\mathrm{d}\lambda)|_{\partial_\alpha} = 0 \tag{5.33}$$

on all segments ∂_α of the boundary $\partial\mathcal{M}$.

Finally, an evolution equation for the velocity potential can be derived by substituting (5.29) into the equation of motion (4.18):

$$[\ \mathrm{d}(\rho\dot{\lambda} + p + \Phi)\ ]|_{\mathcal{M}_\alpha} = 0 \tag{5.34}$$

for all $\alpha \in \mathcal{Z}$, since exterior differentiation commutes with the total derivative and $\mathrm{d}\rho|_{\mathcal{M}_\alpha} = 0$. On integrating

$$[\ \rho\dot{\lambda} + p + \Phi\ ]|_{\mathcal{M}_\alpha} = c_\alpha \tag{5.35}$$

for all $\alpha \in \mathcal{Z}$, where $\mathrm{d}c_\alpha|_{\mathcal{M}_\alpha} = 0$.

term for the vorticity. It appears though that the pressure could be found at any time-step of the calculation algorithm by the conventional approach of substituting the requisite derivatives of the velocity field into the equation of motion (4.18). Such an approach would neatly avoid the issues of smoothness of the velocity and pressure at the fractal boundaries. The pressure calculated using this technique would however be based on a velocity which has been calculated from the averaged vorticity, and may not be sufficiently accurate to resolve the fine structure which, as shown earlier in this section, is associated with the vorticity source on a fractal boundary. In addition, it remains to be demonstrated (possibly via the techniques of Panagiotopoulos [64] or Panagiotopoulos *et al.* [65]) that the approximate dynamics described here does indeed converge to that of the fractal system.

The work presented in this section has been somewhat speculative in nature, but it is hoped that the analysis provided here might in future be put to good use. In this work, however, the analysis of fluid systems in terms of the vorticity contained within the fluid domain will not be continued any further. The remainder of this chapter is used to summarise an extremely useful formulation of the equations of motion of the system which follows from (4.18) if the fluid is assumed to have zero viscosity.

5.5 Velocity Potential Formulation

If the viscosity of the fluid is identically zero, then the vorticity transport equation (5.4) reduces to

$$\dot{\varsigma}|_{\mathcal{M}_\alpha} = 0 \tag{5.28}$$

for all $\alpha \in \mathcal{Z}$, implying that if $\varsigma|_{\mathcal{M}_\alpha} = 0$ at some time, then it is zero for all time. This suggests the introduction of a velocity potential $\lambda \triangleright \bigwedge^0 \mathcal{M}$ in this case such that

$$\nu = \mathrm{d}\lambda \tag{5.29}$$

approximation to the Sierpinski Gasket. Above the diagrams of geometry is shown the strength of the vorticity source on the boundary as a fraction of the maximum vorticity source σ_{max} on the configuration. Points at which there is zero vorticity source have been suppressed to retain clarity of the diagram. For the simple geometries and pressure-potential field assumed here, the fractal boundary between the two fluid domains can be divided into three sets: those points at which the vorticity source is zero, and those points at which the vorticity source is either equal to σ_{max} or $-\sigma_{max}$. This analysis demonstrates that the vorticity source on a fractal boundary can be constructed for any level of approximation of the fractal geometry of the boundary, and hence that the vorticity formulation shows potential as a technique for the analysis of fluid systems with non-differentiable boundaries.

Computational Analysis

The vorticity-based analysis may be particularly suitable for implementation as a computational technique. Box averaging or renormalisation of the vorticity near the fractal boundaries followed by an integral approach to the calculation of the associated velocity on the space occupied by the fluid (as described by Wu and Thompson [88] and Wang and Wu [87]) will lead to an approach to the dynamics of the fluid system which is similar to the vortex-cloud type approach advocated by Chorin [16]. Such an approach may even allow the constraints on the evolution of the fluid system to be determined via an analysis of topological features of the flow such as the helicity or the degree of tangling of the vortex lines within the domain of the fluid. Such approaches have been employed successfully in the analysis of somewhat simpler flows by Chorin [17], Moffatt [59] and especially by Arnold and Khesin [4]. Alternatively, it would appear that the transport and dissipation of the box-averaged vorticity is particularly amenable to calculation using a conservative transport algorithm such as that of Toro and Brown [82] or Brown [12].

The principal obstacle to an effective implementation of the vorticity-based analysis as a calculation method is the presence of the pressure in the source

tems since the associated non-differentability of interfaces renders the Young-Laplace equation, which as shown in Section 1.2 relies on the existence of the curvature of the interfaces, useless.

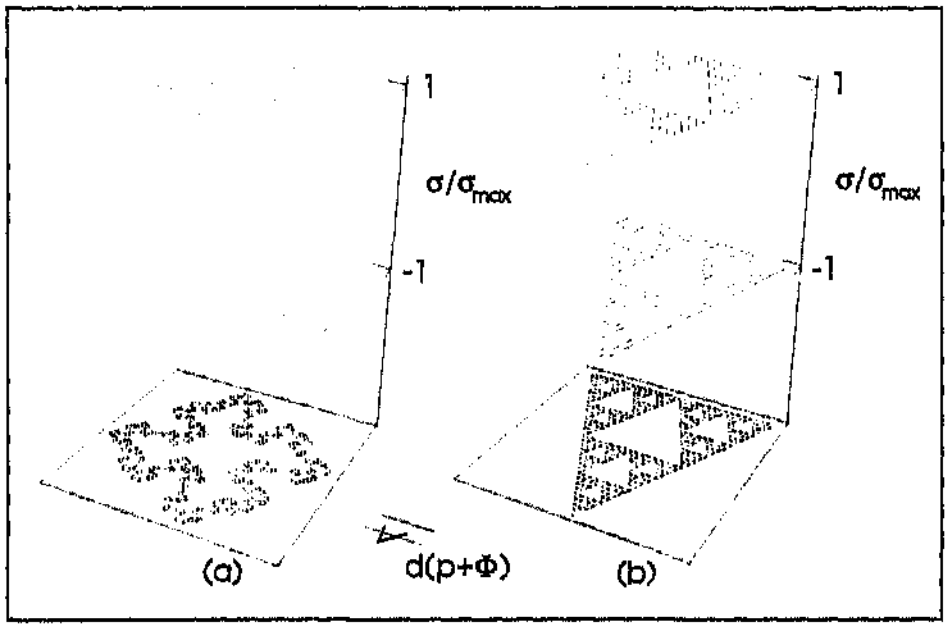


Figure 5.1: Generation of vorticity on fractal boundaries. (a) Quadric Koch Island (b) Sierpinski Gasket.

Equation 5.26 implies that the vorticity generated in the fluid system is closely coupled to the geometry of the system, and in the case of a system containing fractal boundaries, it appears that the vorticity generated on these boundaries will itself be fractal. Figure 5.1 shows an approximation to the vorticity source on the boundary between two domains of fluid of differing density, calculated for two well-known fractal geometries using (5.24) and assuming $d(p+\Phi)$ constant in the direction indicated. The lower part of the diagram shows the geometry[4] of the fractal boundary: on the left is shown a four-level approximation to the Quadric Koch Island and on the right is shown a six-level

[4]Standard references to the construction and analysis of fractal shapes are the books by Mandelbrot [54], Peitgen and Saupe [66], and Falconer [27].

case of identically zero viscosity the vorticity produced on the boundaries of the domains will remain there for all time. This implies that if the velocity is initially continuous within the domains $\mathcal{M}_\alpha$, $\alpha \in \mathcal{Z}$, it will remain continuous within these domains throughout the evolution of the system.

This result completes the analysis of the continuity on $\mathcal{M}$ of the velocity of the fluid. These results form the basis of the work of Chapter 11.

5.4 Applications

Apart from providing information on the continuity of the velocity of the fluid, it appears that the vorticity equations may be useful in two special circumstances. The broad description in this section of the possible applications of the vorticity based analysis forms somewhat of an aside to the main progression of this work.

5.4.1 Qualitative Dynamics

The rule of thumb that surfaces containing vorticity tend to move transversally to the vorticity gradient on the surface could possibly be used to gain qualitative insight into the dynamics, or at least of the incipient motion, of a system of interfaces. Unfortunately this rule cannot be relied upon in all cases, for example where the vortex surface bounds a compact domain and hence the dynamics of the surface is constrained by conservation of the volume bounded by the surface.

5.4.2 Fractal Domains

The most useful application of the formulation in terms of the vorticity of the fluid would appear to be in the analysis of systems with fractal domains or boundaries. Surface tension theory cannot be used to analyse fractal sys-

the density and viscosity of the fluid may be discontinuous at the internal interfaces within the fluid. Additionally

$$(*\boldsymbol{\sigma})|_{\partial_{\alpha\beta}} = 0 \tag{5.26}$$

implying that the vorticity source lies in the co-tangent bundle of the interface.

An expression for the interfacial potential $\Phi_{\alpha\beta}$ can be obtained by enforcing compatibility between the vorticity equation (5.2) and the Navier-Stokes equation (4.18). Using definition (5.1) and subtracting the Navier-Stokes equations on the domains on either side of the interface $\partial_{\alpha\beta}$,

$$\Phi_{\alpha\beta} = \frac{((\rho\Phi)|_{\mathcal{M}_\alpha})|_{\partial_{\alpha\beta}} - ((\rho\Phi)|_{\mathcal{M}_\beta})|_{\partial_{\alpha\beta}}}{(\rho|_{\mathcal{M}_\alpha})|_{\partial_{\alpha\beta}} - (\rho|_{\mathcal{M}_\beta})|_{\partial_{\alpha\beta}}} \tag{5.27}$$

5.3 Velocity Continuity

Equations 5.24 and 5.26 can be used to complete the analysis of the continuity of the velocity of the fluid, begun in Section 4.5, by providing results for the case of identically zero viscosity.

Continuity of the velocity normal to any interface within the fluid domain is ensured by the no-rupture condition (2.16), and any discontinuity in the velocity tangential to an interface is associated, via definition (5.1), with non-zero (but singular) vorticity. In the case of zero viscosity, from (5.24), any source of vorticity must be located at a point in one of the boundaries $\partial\mathcal{M}_\alpha$, $\alpha \in \mathcal{Z}$, at which the density is discontinuous. Hence if there are initially no discontinuities in the velocity, only tangential discontinuities in the velocity might develop at any later time, and these discontinuities will develop on the domain boundaries where the density is discontinuous.

Since, from Section 2.2.1, the domains $\mathcal{M}_\alpha$ and their boundaries $\partial\mathcal{M}_\alpha$, $\alpha \in \mathcal{Z}$ are all transported by the flow of φ_t and, from (5.4), the vorticity away from the boundaries of the domains is also transported by the flow of φ_t, in the

is known as the *n-tuple probability density function* for the species contained in $\sigma \in \mathcal{Z}^n$. As mentioned in Chapter 1, once the form of the dynamics of the particles under their mutual interaction is specified, it is possible to derive a hierarchy of initial-boundary value equations[1], known as the Bogoliubov-Born-Green-Kirkwood-Yvon (or BBGKY) series, for the evolution of each of the forms f_σ, $\sigma \in \mathcal{Z}^n$. The difficulty with this sequence of equations is that the equation for the evolution of any n-tuple probability density function always contains terms involving n'-tuple probability density functions, where $n < n' \leq N$. It follows that unless the sequence is truncated at some $n < N$, usually by substitution of algebraic forms for the n'-tuple probability density functions, the BBGKY approach yields no advantage (at least in terms of computational or descriptive effort) over the use of the evolution equation for the probability density function f governing the behaviour of the complete set of particles comprising the fluid system.

6.4.5 Constitutive Relation

In this work, an algebraic approach similar to that conventionally used to truncate the BBGKY hierarchy is employed to model the species probability density functions. The simplest approach to the constitution of the species probability density functions f_σ, $\sigma \in \mathcal{Z}^n$ is to assume that

$$f_\sigma = \rho_\sigma \gamma_\sigma \tag{6.63}$$

where $\gamma_\sigma \triangleright \bigwedge^n \mathcal{M}^n$ is known conventionally as the *n-tuple correlation function* for the combination of species denoted by $\sigma \in \mathcal{Z}^n$. This form has the important property that if particles with the species associated with $\sigma \in \mathcal{Z}^n$ are not present, then the probability of encountering these particles is zero, as it should be. The form of the correlation functions is of course determined by the symmetries of the microscopic structure of the fluid system.

[1] For example, the BBGKY hierarchy which results from the common assumption that the motion of the particles is Hamiltonian is derived by Cercignani [11].

where $V_\sigma \triangleright \Lambda^0\mathcal{M}^n$ is the contribution to the potential energy as a result of n-particle interactions for the given combination of species.

Rewrite

$$(\mathcal{N}:n) = \bigcup_{\sigma\in\mathcal{Z}^n} (\mathcal{N}:n)_\sigma \tag{6.56}$$

on defining

$$(\mathcal{N}:n)_\sigma = \{e \in (\mathcal{N}:n) : \sigma(e) = \sigma \in \mathcal{Z}^n\} \tag{6.57}$$

Using (6.55) and (6.56), (6.52) can then be rewritten as

$$*\ddot{V} = \sum_{\sigma\in\mathcal{Z}^n} \sum_{e\in(\mathcal{N}:n)_\sigma} \Omega_{\mathcal{N}\sim e} \otimes {}^*\mathrm{i}_e(*V_\sigma) \tag{6.58}$$

Substituting this expression into (6.50) gives

$$V = \sum_{\sigma\in\mathcal{Z}^n} \sum_{e\in(\mathcal{N}:n)_\sigma} \int_{\mathcal{M}_\mathcal{N}} f\, \Omega_{\mathcal{N}\sim e} \otimes {}^*\mathrm{i}_e(*V_\sigma) \tag{6.59}$$

which can be rewritten using the fibre integral as

$$V = \sum_{\sigma\in\mathcal{Z}^n} \sum_{e\in(\mathcal{N}:n)_\sigma} \int_{\mathcal{M}_e} {}^*\mathrm{i}_e(*V_\sigma) \oint_{\mathcal{M}_{\mathcal{N}\sim e}} *f \tag{6.60}$$

Alternatively, using the isomorphism i_e to change the integrating variables,

$$\begin{aligned} V &= \sum_{\sigma\in\mathcal{Z}^n} \sum_{e\in(\mathcal{N}:n)_\sigma} \int_{\mathcal{M}^n} {}^*\mathrm{i}_e^{-1}{}^*\mathrm{i}_e(*V_\sigma)\, {}^*\mathrm{i}_e^{-1} \oint_{\mathcal{M}_{\mathcal{N}\sim e}} *f \\ &= \sum_{\sigma\in\mathcal{Z}^n} \int_{\mathcal{M}^n} *V_\sigma\, f_\sigma \end{aligned} \tag{6.61}$$

where

$$\begin{aligned} f_\sigma &= \sum_{e\in(\mathcal{N}:n)_\sigma} {}^*\mathrm{i}_e^{-1} \oint_{\mathcal{M}_{\mathcal{N}\sim e}} *f \\ &\triangleright \Lambda^0\mathcal{M}^n \end{aligned} \tag{6.62}$$

It is common practice to assume that $\tilde{V}$ can be decomposed such that

$$*\tilde{V} = \sum_{i=1}^{N} \sum_{e \in (\mathcal{N}:i)} \Omega_{\mathcal{N} \sim e} \otimes (*\tilde{V}_e) \tag{6.51}$$

where $\tilde{V}_e \in \bigwedge^0 \mathcal{M}_e$, $e \in (\mathcal{N} : n)$ represents the contribution to the potential energy of the system as a result of interactions between the n particles with indices in e.

For present purposes, it is assumed that the decomposition can be terminated for some $n < N$, in which case (6.51) can be recast as

$$*\tilde{V} = \sum_{e \in (\mathcal{N}:n)} \Omega_{\mathcal{N} \sim e} \otimes (*\overline{V}_e) \tag{6.52}$$

where

$$\begin{aligned} *\overline{V}_e &= \sum_{i=1}^{n} \sum_{e' \in (e:i)} \Omega_{e \sim e'} \otimes (*\tilde{V}_{e'}) \\ &\triangleright \ \Omega \mathcal{M}_e \end{aligned} \tag{6.53}$$

This form assumes that the contribution to the potential energy from n'-particle interactions, where $n' > n$, can be neglected compared to the contribution from interactions between smaller numbers of particles.

6.4.4 Interchangeability of Particles

Assume the existence of a (most likely many-to-one) map

$$\sigma : (\mathcal{N} : n) \to \mathcal{Z}^n \tag{6.54}$$

for all $n \in [1, N]$, where $\mathcal{Z}$ is the set of chemical species introduced in Section 2.2.2, such that for each $e \in (\mathcal{N} : n)$

$$*\overline{V}_e = {}^{\sigma}\mathbf{i}_e(*V_{\sigma(e)}) \tag{6.55}$$

mechanical context, this assumption implies that the potential energy of the system must be solely dependent on the location of the particles comprising the fluid, and must be independent of the linear and angular momenta, and of the internal state, of the particles comprising the fluid. In this case the potential energy of the system of particles, $\tilde{V} \triangleright \Lambda^0 \mathcal{M}_N$.

Let the *probability density function* $f \triangleright \Lambda^0 \mathcal{M}_N$ be defined such that the probability of finding the configuration of the system of particles within $\mathcal{R} \subset \mathcal{M}_N$ is

$$\int_{\mathcal{R}} *f \tag{6.48}$$

and hence

$$\int_{\mathcal{M}_N} *f = 1 \tag{6.49}$$

If the system of particles is assumed to evolve through the configurations available to it on a timescale which is much shorter than the timescale over which a typical measurement of the state of the system is made, then the measured (or 'macroscopically observeable') value of the potential energy of the system, V, is the weighted mean

$$V = \int_{\mathcal{M}_N} *f \, \tilde{V} \tag{6.50}$$

Various symmetry properties can now be introduced to reduce the dimension of the manifold on which the potential energy is defined.

6.4.3 Decomposition of Potentials

Define the set $(\mathcal{E} : n)$, where $\mathcal{E} \subset \mathcal{N}$ and $n \in [1, N]$, as the set of all n-tuples $e = \{i_1, \ldots, i_n\}$ composed of distinct elements $i_1, \ldots, i_n \in \mathcal{N}$ such that if $e_1, e_2 \in (\mathcal{E} : n)$, $e_1 \cap e_2 = 0$.

(with index $i \in \mathcal{N}$) can be described by a point on a manifold $\mathcal{M}_{(i)}$ which is isomorphic to the system manifold $\mathcal{M}$, the simultaneous location of all the particles in the fluid can be described by a point on the manifold

$$\begin{aligned} \mathcal{M}_{\mathcal{N}} &= \prod_{i \in \mathcal{N}} \mathcal{M}_{(i)} \\ &= \prod_{i \in \mathcal{N}} \mathrm{i}_{(i)} \mathcal{M} \end{aligned} \tag{6.45}$$

where $\mathrm{i}_{(i)} : \mathcal{M} \to \mathcal{M}_{(i)}$ is the isomorphism referred to above. This approach to the description of the configuration of the fluid is a reduction of the approach proposed by Gibbs [32].

Define the manifold

$$\begin{aligned} \mathcal{M}_e &= \prod_{i \in e} \mathcal{M}_{(i)} \\ &= \prod_{i \in e} \mathrm{i}_{(i)} \mathcal{M} \\ &\equiv \mathrm{i}_e \mathcal{M}^n \end{aligned} \tag{6.46}$$

for any $e \subset \mathcal{N}$. It follows that $\mathrm{i}_e : \mathcal{M}^n \to \mathcal{M}_e$ is an isomorphism and $n = \mathrm{card}(e)$. Obviously if $e' \subset e \subset \mathcal{N}$, then from (6.46),

$$\begin{aligned} \mathcal{M}_e &= \mathcal{M}_{e'} \times \mathcal{M}_{e \sim e'} \\ &= \mathrm{i}_{e'} \mathcal{M}^{n'} \times \mathrm{i}_{e \sim e'} \mathcal{M}^{n-n'} \end{aligned} \tag{6.47}$$

where $n = \mathrm{card}(e)$ and $n' = \mathrm{card}(e')$.

Note that the Hodge form $\Omega_{\mathcal{M}} \equiv \Omega^1$ on $\mathcal{M}$ induces the Hodge form $\Omega^n = \Omega^1 \otimes \Omega^{n-1}$ on the product manifolds $\mathcal{M}^n$, $n > 1$ and hence via the isomorphism the Hodge form $\Omega_e = {}^*\mathrm{i}_e \Omega^n$ on $\mathcal{M}_e$.

6.4.2 Potential Energy

In Section 3.2.1 it was assumed *a priori* that the potential energy of the system is solely dependent on the configuration of the fluid system. In a statistical

holds for all $\mathcal{M}^n_\sigma \in \mathcal{M}^n_\Sigma$. For consistency, condition (6.42) should of course hold directly from the compressibility condition (2.7) on $\nu \triangleright \mathcal{KM}$ via definition (6.38).

This can easily be demonstrated. From (6.38) and (6.13),

$$\begin{aligned} *\nu_n &= *(\nu \oplus \nu_{n-1}) \\ &= (*\nu) \otimes \Omega^{n-1} + \Omega^1 \otimes (*\nu_{n-1}) \end{aligned} \tag{6.43}$$

using (6.13), where Ω^1 and Ω^{n-1} are the Hodge forms on $\mathcal{M}$ and $\mathcal{M}^{n-1}$. Using (6.16)

$$\begin{aligned} \mathrm{d}*\nu_n &= (\mathrm{d}*\nu) \otimes \Omega^{n-1} + \mathrm{d}(\Omega^1 \otimes (*\nu_{n-1})) \\ &= (-1)^p \Omega^1 \otimes (\mathrm{d}*\nu_{n-1}) \end{aligned} \tag{6.44}$$

where $p = \dim(\mathcal{M}^n)$, using (2.7) and since the exterior derivative of a volume form is zero. Hence, by induction, condition (6.42) does hold as a direct consequence of condition (2.7).

Finally, since in Section 2.3 it was assumed that discontinuities in ν are confined to the boundaries $\partial\mathcal{M}_\alpha$, $\mathcal{M}_\alpha \in \mathcal{M}_\Sigma$, it follows (since the set product maps boundaries to boundaries) from (6.25) that discontinuities in ν_n will be confined to the boundaries $\partial\mathcal{M}^n_\sigma$, $\mathcal{M}^n_\sigma \in \mathcal{M}^n_\Sigma$.

6.4 Statistical Analysis

The analysis in this section follows very closely that of Kirkwood [40], although somewhat more modern notation will be employed here.

6.4.1 Manifold Structure

Assume the fluid to be comprised of N particles, each indexed by an unique element of the set $\mathcal{N} = [1, N] \subset \mathbf{Z}$. Since the location of a single particle

in Chapter 8, where the conditions required for the existence of equilibrium configurations of the fluid system are investigated.

6.3.5 Transport of Occupation Functions

In Section 2.1 the flow field $\nu \triangleright \mathcal{KM}$ of the map φ_t was defined. Let $\nu_1 \equiv \nu$. It then follows from the analysis of Section 6.2.3 that φ_t induces a flow field

$$\begin{aligned} \nu_n &= \nu \oplus \nu_{n-1} \\ &= \nu \oplus (\nu \oplus (\cdots \oplus \nu \cdots)) \\ &\triangleright \mathcal{K}\Lambda_t{}^n \end{aligned} \tag{6.38}$$

on the manifolds $\mathcal{M}^n$, $n > 1$, and hence, by restriction, a flow field $\nu_n|_{\mathcal{M}^n_\sigma}$ on each $\mathcal{M}^n_\sigma \in \mathcal{M}^n_\Sigma$. Furthermore, since from Section 2.3 $(\frac{d}{dt}*\rho)|_{\mathcal{M}_n} = 0$ for all $\alpha \in \mathcal{Z}$, it follows from (6.31), (6.32) and (6.19) that

$$(\tfrac{d}{dt}*\rho_\sigma)|_{\mathcal{M}^n_\sigma} = 0 \tag{6.39}$$

for all $\mathcal{M}^n_\sigma \in \mathcal{M}^n_\Sigma$ and $\rho_\sigma \in \mathcal{O}^n_\Sigma$, and hence that each $\rho_\sigma \in \mathcal{O}^n_\Sigma$ is transported by the flow of φ_t. The restrictions (2.3) and (2.4) on $*\rho$ translate via (6.31) and (6.32) to the restrictions

$$(\mathrm{d}\rho_\sigma)|_{\mathcal{M}^n_\sigma} = 0 \tag{6.40}$$

and

$$(\tfrac{\partial}{\partial t}*\rho_\sigma)|_{\mathcal{M}^n_\sigma} = 0 \tag{6.41}$$

on all $\rho_\sigma \in \mathcal{O}^n_\Sigma$. Hence, through a similar analysis to that used to obtain (2.7), the incompressibility condition

$$\mathrm{d}*\nu_n|_{\mathcal{M}^n_\sigma} = 0 \tag{6.42}$$

$$\mathcal{M}^n_{\mathcal{Z}} \mapsto \mathcal{M}^n_{\mathcal{Z}}$$

$$\mathcal{O}^n_{\mathcal{Z}} \mapsto \mathcal{O}^n_{\mathcal{Z}} \tag{6.36}$$

if $\alpha = \beta$. Physically the action of the coalescence operator is to remove any distinction between the elements associated with the indices α and β. This is done in the case where $\alpha \neq \beta$ by rearranging $\mathcal{Z}^n$ into a smaller set and by combining the separate domains and occupation functions associated with α and β into single objects. It is important to realise though that not all sets created using the coalescence sum are constructible via definitions (6.35) and (6.36) from some set $\mathcal{Z} \subset \mathbf{Z}$. Those sets which *are* will have special physical significance, and are discussed further in Section 8.5.3.

Isomorphism

By definition of the notation for the elements of the sets $\{\mathcal{Z}^n\}_{n\in\mathbf{Z}^+}$, $\{\mathcal{M}^n_{\mathcal{Z}}\}_{n\in\mathbf{Z}^+}$ and $\{\mathcal{O}^n_{\mathcal{Z}}\}_{n\in\mathbf{Z}^+}$, there is trivially an isomorphism between $\{\mathcal{Z}^n\}_{n\in\mathbf{Z}^+}$ equipped with the set composition operator, $\{\mathcal{M}^n_{\mathcal{Z}}\}_{n\in\mathbf{Z}^+}$ equipped with the set product, and $\{\mathcal{O}^n_{\mathcal{Z}}\}_{n\in\mathbf{Z}^+}$ equipped with the composition product.

Note also that, by construction, there exists a bijection between the sets $\mathcal{Z}^n$, $\mathcal{M}^n_{\mathcal{Z}}$ and $\mathcal{O}^n_{\mathcal{Z}}$, for any given $n \geq 1$. By definition of the notation for the elements of each of these sets, it follows quite simply that the definition of the coalescence sum given above also induces an isomorphism, this time between the spaces $\mathcal{Z}^n$, $\mathcal{M}^n_{\mathcal{Z}}$ and $\mathcal{O}^n_{\mathcal{Z}}$ themselves, and as a result the coalescence sum must be associative over the composition product:

$$(\alpha\otimes\beta)\oplus(\alpha\otimes\beta') = \alpha\otimes(\beta\oplus\beta') \tag{6.37}$$

Note though that the composition product is not associative over the coalescence sum.

These isomorphisms allow operations involving the occupation functions and their associated fragmentations of manifolds to be expressed in terms of manipulations of indices. The resulting simplification of calculations is exploited

The set $\mathcal{O}_{\mathcal{Z}}$ can be used to define occupation functions on the product structures $\mathcal{M}^n$, $n > 1$, via the composition product defined in Section 6.2.3. Given $\mathcal{O}^1_{\mathcal{Z}} \equiv \mathcal{O}_{\mathcal{Z}}$, define the sets

$$\mathcal{O}^n_{\mathcal{Z}} = \{\rho_{\alpha\oplus\beta} \equiv \rho_\alpha \otimes \rho_\beta : \rho_\alpha \in \mathcal{O}^1_{\mathcal{Z}}, \rho_\beta \in \mathcal{O}^{n-1}_{\mathcal{Z}}\} \tag{6.32}$$

by recursion for all $n > 1$. From the definition of the composition product, the elements of $\mathcal{O}^n_{\mathcal{Z}}$ are all sections of $\Lambda^0\mathcal{M}^n$, and, if $\sigma \in \mathcal{Z}^n$,

$$\text{supp}(\rho_\sigma) = \mathcal{M}^n_\sigma \in \mathcal{M}^n_{\mathcal{Z}} \tag{6.33}$$

and

$$\rho_\sigma : \mathcal{M}^n_\sigma \to 1 \tag{6.34}$$

Hence the elements of $\mathcal{O}^n_{\mathcal{Z}}$, $n > 1$ are binary maps on $\mathcal{M}^n$.

6.3.4 Coalescence Sum

Note that the sum of two distinct occupation functions defined on the same space is a binary map from that space to the real line and hence may also be interpreted as an occupation function. This leads to the definition of the *coalescence sum* : If $\alpha, \beta \in \mathcal{Z}^n$ define

$$\begin{aligned}
&{}^{\oplus(\alpha,\beta)}: \\
\mathcal{Z}^n &\mapsto (\ \mathcal{Z}^n \sim \{\alpha,\beta\}\ ) \,\&\, \{\alpha\oplus\beta\} \\
\mathcal{M}^n_{\mathcal{Z}} &\mapsto (\ \mathcal{M}^n_{\mathcal{Z}} \sim \{\mathcal{M}^n_\alpha, \mathcal{M}^n_\beta\}\ ) \,\&\, \{\mathcal{M}^n_\alpha \cup \mathcal{M}^n_\beta \cup (\partial\mathcal{M}^n_\alpha \cap \partial\mathcal{M}^n_\beta)\} \\
\mathcal{O}^n_{\mathcal{Z}} &\mapsto (\ \mathcal{O}^n_{\mathcal{Z}} \sim \{\rho_\alpha, \rho_\beta\}\ ) \,\&\, \{\rho_\alpha + \rho_\beta\}
\end{aligned} \tag{6.35}$$

if $\alpha \neq \beta$ and

$$\begin{aligned}
&{}^{\oplus(\alpha,\beta)}: \\
\mathcal{Z}^n &\mapsto \mathcal{Z}^n
\end{aligned}$$

Then, by induction, since $\bigcup_{\alpha\in\mathcal{Z}} \overline{\mathcal{M}}_\alpha = \overline{\mathcal{M}}$,

$$\begin{aligned} \bigcup_{\sigma\in\mathcal{Z}^n} \overline{\mathcal{M}}^n_\sigma &= \overline{\mathcal{M}} \times \overline{\mathcal{M}}^{n-1} \\ &= \overline{\mathcal{M}}^n \end{aligned} \tag{6.27}$$

Hence the remaining condition (1.17) for $\mathcal{M}^n_{\mathcal{Z}}$ to be $\mathcal{M}^n$-complete is also satisfied.

6.3.3 Occupation Functions

Finally, define the set of *occupation functions* on $\mathcal{M}$ associated with the fragmentation $\mathcal{M}_{\mathcal{Z}}$,

$$\mathcal{O}_{\mathcal{Z}} = \{\rho_\alpha \triangleright \Lambda^0\mathcal{M}\}_{\alpha\in\mathcal{Z}} \tag{6.28}$$

as the set of binary maps defined such that

$$\mathrm{supp}(\rho_\alpha) = \mathcal{M}_\alpha \tag{6.29}$$

and

$$\rho_\alpha : \mathcal{M}_\alpha \to 1 \tag{6.30}$$

Physically the occupation functions can be thought of as behaving as 'tracers' for the presence of the various domains of fluid distinguished by the fragmentation $\mathcal{M}_{\mathcal{Z}}$.

The fluid density $*\rho \triangleright \Omega\mathcal{M}$ introduced in Section 2.2 can then be written as

$$*\rho = *\rho_\alpha \overline{\rho}^\alpha, \qquad \alpha \in \mathcal{Z} \tag{6.31}$$

(summation over repeated indices assumed) since under the assumption of piecewise constant fluid density $\rho|_{\mathcal{M}_\alpha} : \mathcal{M}_\alpha \to \overline{\rho}^\alpha \in \mathbb{R}$ for all $\alpha \in \mathcal{Z}$.

6.3 Product Structures in the Fluid

6.3.1 Enumerating Sets

In Section 2.2.1, the set $\mathcal{Z}$ used to enumerate the elements of the $\mathcal{M}$-complete fragmentation $\mathcal{M}_{\mathcal{Z}} = \{\mathcal{M}_\alpha\}_{\alpha\in\mathcal{Z}}$ was defined simply as some set isomorphic to a subset of the integers. In this section, this idea is extended to the definition of complete fragmentations of the product spaces $\mathcal{M}^n = \mathcal{M} \times \cdots \times \mathcal{M}$ (n factors) , where $n \in \mathbf{Z}^+$. Given $\mathcal{Z} \equiv \mathcal{Z}^1$, define the sets

$$\mathcal{Z}^n = \{\alpha\otimes\beta : \alpha \in \mathcal{Z}, \beta \in \mathcal{Z}^{n-1}\} \tag{6.24}$$

by recursion for all $n > 1$, where the set composition operator $\otimes$ is non-commutative. The set $\mathcal{Z}^n$ created in this manner is of course isomorphic to $\mathcal{Z} \times \cdots \times \mathcal{Z}$ (n factors).

6.3.2 Complete Fragmentations

Given the $\mathcal{M}$-complete fragmentation $\mathcal{M}_{\mathcal{Z}} \equiv \mathcal{M}^1_{\mathcal{Z}}$, define

$$\mathcal{M}^n_{\mathcal{Z}} = \{\mathcal{M}^n_{\alpha\otimes\beta} \equiv \mathcal{M}_\alpha \times \mathcal{M}^{n-1}_\beta : \mathcal{M}_\alpha \in \mathcal{M}_{\mathcal{Z}}, \mathcal{M}^{n-1}_\beta \in \mathcal{M}^{n-1}_{\mathcal{Z}}\} \tag{6.25}$$

by recursion for all $n > 1$.

It follows from the following argument that $\mathcal{M}^n_{\mathcal{Z}}$ is $\mathcal{M}^n$-complete. Condition (1.18) for completeness of $\mathcal{M}^n_{\mathcal{Z}}$ is satisfied by the product structure of definition (6.25) since the elements of $\mathcal{Z}^n$ are distinct. Also

$$\begin{aligned}\bigcup_{\sigma\in\mathcal{Z}^n} \overline{\mathcal{M}}^n_\sigma &= \bigcup_{\substack{\alpha\in\mathcal{Z}\\ \beta\in\mathcal{Z}^{n-1}}} \overline{\mathcal{M}}_\alpha \times \overline{\mathcal{M}}^{n-1}_\beta \\ &= \bigcup_{\alpha\in\mathcal{Z}} \overline{\mathcal{M}}_\alpha \times \bigcup_{\beta\in\mathcal{Z}^{n-1}} \overline{\mathcal{M}}^{n-1}_\beta \end{aligned} \tag{6.26}$$

A simple consequence of this result is that if $\psi^{\mathcal{B}}_t : \mathcal{B} \to \mathcal{B}$ and $\psi^{\mathcal{F}}_t : \mathcal{F} \to \mathcal{F}$ are one-parameter automorphisms with flow fields $\xi_{\mathcal{B}} \triangleright \mathcal{X}\mathcal{B}$ and $\xi_{\mathcal{F}} \triangleright \mathcal{X}\mathcal{F}$ respectively, then these transformations induce an automorphism $\psi^{\mathcal{P}}_t$ on $\mathcal{P}$ with flow field $\xi_{\mathcal{B}} \oplus \xi_{\mathcal{F}} \triangleright \mathcal{X}\mathcal{P}$.

6.2.4 Fibrewise Integration

If $\omega_{\mathcal{P}} \triangleright \Omega\mathcal{P}$ then it is always possible to write $\omega_{\mathcal{P}} = \omega_{\mathcal{B}} \otimes \Omega_{\mathcal{F}}\, g_{\mathcal{P}}$ for some $g_{\mathcal{P}} \triangleright \bigwedge^0 \mathcal{P}$, $\omega_{\mathcal{B}} \triangleright \Omega\mathcal{B}$ and $\Omega_{\mathcal{F}}$ the Hodge form on $\mathcal{F}$. Now ${}^*i_f(\Omega_{\mathcal{F}}, b)\; g_{\mathcal{P}}|_{\pi^{-1}(b)}$ is a volume form on the fibre $\pi^{-1}(b)$, and hence can be integrated over the fibre. Using the isomorphism from $\mathcal{F}$ to the fibres to change the integrating variables

$$\begin{aligned}\int_{\pi^{-1}(b)} {}^*i_f(\Omega_{\mathcal{F}}, b)\; g_{\mathcal{P}}|_{\pi^{-1}(b)} &= \int_{\mathcal{F}} \Omega_{\mathcal{F}}\, {}^*i_f^{-1}(g_{\mathcal{P}}|_{\pi^{-1}(b)}, b) \\ &= \int_{\mathcal{F}} *{}^*i_f^{-1}(g_{\mathcal{P}}|_{\pi^{-1}(b)}, b) \\ &\in \mathbb{R} \end{aligned} \tag{6.20}$$

Hence define the *fibre integral*

$$\begin{aligned}\fint_{\mathcal{F}} *g_{\mathcal{P}} \;&:\; \mathcal{B} \to \mathbb{R} \\ &:\; b \mapsto \int_{\mathcal{F}} * \,{}^*i_f^{-1}(g_{\mathcal{P}}|_{\pi^{-1}(b)}, b)\end{aligned} \tag{6.21}$$

The fibre integral is thus a section of $\bigwedge^0\mathcal{B}$. This definition can be used to rewrite the integral $\int_{\mathcal{P}} \omega_{\mathcal{P}}$ using the trivial fibre structure of $\mathcal{X}$ as

$$\begin{aligned}\int_{\mathcal{P}} \omega_{\mathcal{P}} &= \int_{\mathcal{P}} \omega_{\mathcal{B}} \otimes \Omega_{\mathcal{F}}\, g_{\mathcal{P}} \\ &= \int_{\mathcal{B}} \omega_{\mathcal{B}} \fint_{\mathcal{F}} *g_{\mathcal{P}}\end{aligned} \tag{6.22}$$

since, from the definition of the inclusion operator, $\omega_{\mathcal{B}}$ is invariant on the fibres isomorphic to $\mathcal{F}$. Of course, if $\omega_{\mathcal{B}}$ is the Hodge form on $\mathcal{B}$ then

$$\int_{\mathcal{P}} \omega_{\mathcal{P}} = \int_{\mathcal{B}} * \fint_{\mathcal{F}} *g_{\mathcal{P}} \tag{6.23}$$

Note that the fibre integral can be interpreted as a projection operator onto the base space of a trivial fibration of a product structure.

Interaction Kernels

Since the interaction kernels $\mathcal{F}^\sigma \triangleright \bigwedge^0 \mathcal{M}^n$, $\sigma \in \mathcal{Z}^n$ are independent of the configuration of the fluid system (see Section 6.4.5),

$$\psi_\vartheta : \mathcal{F}^\sigma \mapsto \mathcal{F}^\sigma \tag{7.6}$$

for all $\sigma \in \mathcal{Z}^n$.

7.3 Dynamical System on $\mathcal{D}$

Evolution from a Reference State

Assume that the evolution of the fluid system can be described by a connected path $\vartheta : \mathbb{R} \to \mathcal{D}$, and hence that the configuration of the system at any time can be found on the curve $\gamma \subset \mathcal{D}$ defined by

$$\gamma = \{\vartheta(t) : t \in \mathbb{R}\} \tag{7.7}$$

where t can be interpreted as the elapsed time. Let $\dot{\vartheta}(t) \in \mathsf{T}_{\vartheta(t)}\mathcal{D}$ denote the tangent vector to γ at the point $\vartheta(t) \in \gamma$.

Distortion Modes

The quickest route to obtaining the equations of motion of the system on $\mathcal{D}$ is via the adoption of local coordinates $(a_1, \ldots, a_p)$ in a neighbourhood $\mathsf{C}_\vartheta\mathcal{D}$ of each $\vartheta \in \mathcal{D}$. The curve γ representing the evolution of the system is then given locally by paths $a_i : \mathbb{R} \to \mathsf{C}_\vartheta\mathcal{D}$, $i = 1, \ldots, p$. The coordinate form of the tangent vector to γ at $\vartheta \in \mathcal{D}$ is thus

$$\dot{\vartheta} = \dot{a}_1(t)\tfrac{\partial}{\partial a_1} + \cdots + \dot{a}_p(t)\tfrac{\partial}{\partial a_p} \tag{7.8}$$

$$: \quad \hat{\mathcal{M}}_\alpha \mapsto \mathcal{M}_\alpha = \psi_\vartheta \hat{\mathcal{M}}_\alpha \tag{7.3}$$

For $\mathcal{M}_\mathcal{Z}$ to be $\mathcal{M}$-complete for all $\vartheta \in \mathcal{D}$, then, as described in Section 7.4, the flow field of ψ_ϑ must obey the no-rupture condition of Section 2.3.1. The $\mathcal{M}^n$-completeness of the fragmentations $\mathcal{M}^n_\mathcal{Z}$ for any $n \in \mathbf{Z}^+$ follows from the $\mathcal{M}$-completeness of $\mathcal{M}_\mathcal{Z}$ via the analysis of Section 6.3.2.

If, for consistency with the analysis of Section 2.2.1, discontinuities in the density and viscosity fields are to remain on the boundaries of the elements of the set $\mathcal{M}_\mathcal{Z}$ for all $\vartheta \in \mathcal{D}$, and the density and viscosity are also to be piecewise constant and ϑ-invariant on the elements of $\mathcal{M}_\mathcal{Z}$ for all $\vartheta \in \mathcal{D}$, then the fluid density and viscosity must be transported by the flow field of ψ_ϑ. Hence from the definition of Section 1.9.2,

$$\begin{aligned} \psi_\vartheta \quad &: \quad *\hat{\rho} \mapsto *\rho = {}^*\psi_\vartheta^{-1} *\hat{\rho} \\ &: \quad \hat{\eta} \mapsto \eta = {}^*\psi_\vartheta^{-1} \hat{\eta} \end{aligned} \tag{7.4}$$

Occupation Functions

Let $\hat{\mathcal{O}}^n_\mathcal{Z} = \{\hat{\rho}_\alpha\}_{\alpha \in \mathcal{Z}^n}$ be the set of occupation functions on $\mathcal{M}^n$ induced by the fragmentation $\hat{\mathcal{M}}_\mathcal{Z}$ of the reference state (see Section 6.3.3). Then

$$\begin{aligned} \psi_\vartheta \quad &: \quad \hat{\mathcal{O}}^n_\mathcal{Z} \to \mathcal{O}^n_\mathcal{Z} \\ &: \quad \hat{\rho}_\alpha \mapsto \rho_\alpha = {}^*\psi_\vartheta^{-1} \hat{\rho}_\alpha \end{aligned} \tag{7.5}$$

where $\mathcal{O}^n_\mathcal{Z}$ is the set of occupation functions on $\mathcal{M}^n$ induced by the fragmentation $\psi_\vartheta \hat{\mathcal{M}}_\mathcal{Z}$. This follows since from their definition in Section 6.3.3 the occupation functions must be piecewise constant and ϑ-invariant on the elements of $\mathcal{M}^n_\mathcal{Z}$ for all $\vartheta \in \mathcal{D}$, and hence must be transported by the flow of ψ_ϑ.

the density and viscosity of the fluid. As in Section 2.2.1, $\mathcal{Z}$ is isomorphic to a subset of the integers and $\mathcal{M}_{\mathcal{Z}} = \{\mathcal{M}_\alpha\}_{\alpha\in\mathcal{Z}}$ is an $\mathcal{M}$-complete fragmentation such that discontinuities in ρ and η are restricted to $\partial\mathcal{M}_\alpha$, $\alpha \in \mathcal{Z}$. Denote the set of all such states by Σ.

Select some privileged state

$$\begin{aligned}\hat{\mathcal{R}} &= (\mathcal{M}, \hat{\rho}, \hat{\eta}, \mathcal{Z}, \hat{\mathcal{M}}_{\mathcal{Z}}) \\ &\in \Sigma \end{aligned} \tag{7.1}$$

to be the *reference state* for the fluid system.

Automorphisms of the Reference State

Assume the existence of a map

$$\begin{aligned}\psi &: \hat{\mathcal{D}} \to \Sigma \\ &: \vartheta \mapsto \mathcal{R} = \psi_\vartheta \hat{\mathcal{R}}\end{aligned} \tag{7.2}$$

for some suitable space $\hat{\mathcal{D}}$. Let $\mathcal{D} \subset \hat{\mathcal{D}}$ be defined such that $\{\psi_\vartheta\hat{\mathcal{R}} : \vartheta \in \mathcal{D}\}$ is some subset of all fluid states obtainable from the reference state via a transformation which is compatible with the kinematic and dynamic constraints on the behaviour of the fluid derived in Section 2.3.1 and Section 4.5. This implies from the continuity of the velocity in the fluid system (as discussed in Section 7.1) and for consistency with the assumptions made in Section 2.1 that $\psi : \vartheta \mapsto \psi_\vartheta\mathcal{M} = \mathcal{M}$ for all $\vartheta \in \mathcal{D}$ and hence that $\mathcal{D}$ is isomorphic to a subset[1] of the group of piecewise-continuous automorphisms of $\mathcal{M}$, $\mathsf{aut}_0(\mathcal{M})$. Term the space $\mathcal{D}$ the *configuration manifold* for the fluid system.

The action of ψ on each of the elements of the reference state is now examined. The action of ψ on the elements of the fragmentation $\hat{\mathcal{M}}_{\mathcal{Z}}$ can be written as

$$\psi_\vartheta \; : \; \hat{\mathcal{M}}_{\mathcal{Z}} \to \mathcal{M}_{\mathcal{Z}}$$

[1] It is implicitly assumed that $\mathcal{D}$ is in fact isomorphic to a *submanifold* of $\mathsf{aut}_0(\mathcal{M})$. This assumption of underlying continuity is required before the dynamical system (7.7) can be defined on the group.

equations of motion developed in Chapters 4 and 5 require, however, that the potential energy of the internal interactions be expressed as an integral over a volume form on $\mathcal{M}$. Reduction of the representation on $\mathcal{M}^n$ to an equivalent representation on $\mathcal{M}$ can be avoided by developing the equations of motion on a configuration manifold which is isomorphic to an appropriate subset of $\mathsf{aut}_0(\mathcal{M})$.

In Section 7.2 it is shown how the assumptions made in Chapter 2 regarding the structure of the fluid allow the motion of the fluid to be described as a dynamical system on a configuration manifold which is isomorphic to a subset of $\mathsf{aut}_0(\mathcal{M})$. In Section 7.3 the evolution of the system is defined in terms of the growth of a set of suitably defined distortion modes, and in Section 7.4 the kinematic and dynamic constraints on the motion of the fluid derived in Chapters 2, 4 and 5 are then rewritten in terms of these modes. Following re-definition in Section 7.5 of the kinetic energy, potential energy and energy dissipation within the fluid system in terms of the distortion modes, the Lagrangian approach of Chapter 3 is used in Sections 7.6 and 7.7 to derive the equations of motion of the system on the configuration manifold. In Section 7.8 a linear version of this equation, valid in the neighbourhood of any point in the configuration manifold, is derived for use in Chapters 8 and 10, while in Section 7.9 some ideas for a powerful formulation of the dynamic equations for the interaction field approach, based on the theory of geodesics on a manifold, are put forward. The analysis of this chapter draws much of its inspiration from the work of Arnold [3], Arnold and Khesin [4], and Ebin and Marsden [25].

7.2 Evolution from a Reference State

Reference State

Define a *state* of the fluid system as a list $\mathcal{R} = (\mathcal{M}, \rho, \eta, \mathcal{Z}, \mathcal{M}_Z)$. $\mathcal{M}$ is the fluid domain, and $*\rho \triangleright \Omega\mathcal{M}$ and $\eta \triangleright \Lambda^0\mathcal{M}$ are piecewise-constant fields representing

Chapter 7

Dynamics on a Configuration Manifold

7.1 Overview

The analyses of the continuity of the velocity $\nu \triangleright \mathcal{X}\mathcal{M}$ conducted in Section 4.5 and Section 5.3 can be combined to show that, if there are no initial discontinuities in the velocity, then the velocity will remain continuous within each $\mathcal{M}_\alpha \subset \mathcal{M}$, $\alpha \in \mathcal{Z}$ for all time. If the fluid has non-zero viscosity, then the velocity will be continuous throughout $\mathcal{M}$ for all time. This result implies that the transformation $\varphi_t : \mathcal{M} \to \mathcal{M}$ associated with ν is a piecewise-continuous automorphism on $\mathcal{M}$ and a continuous transformation on each of the elements $\mathcal{M}_\alpha$, $\alpha \in \mathcal{Z}$, of the $\mathcal{M}$-complete fragmentation $\mathcal{M}_\mathcal{Z}$ defined in Section 2.2.1.

An elegant approach to the equations of motion of the system is obtained by constructing them on the infinite-dimensional Lie group $\mathrm{aut}_0(\mathcal{M})$ of piecewise continuous automorphisms of $\mathcal{M}$. This approach has particular economy in the case where there are mutual interactions between the fluid particles. It was shown in Chapter 6 that the potential energy of internal interactions within the fluid is most naturally expressed as an integral over a section of $\Omega\mathcal{M}^n$, where $n > 1$, rather than as an object associated with $\mathcal{M}$ itself. The

Explicit Form for $\mathcal{F}$

In preparation for the analysis of the following chapters, the explicit prototype

$$\mathcal{F}^{\sigma}_{\mathrm{iso}(\mathbb{R}^3)} : (x, y, z, x_*, y_*, z_*)$$
$$\mapsto q_\sigma \exp[\, -k_\sigma^2((x - x_*)^2 + (y - y_*)^2 + (z - z_*)^2)\,] \tag{6.74}$$

for an interaction kernel on $\mathcal{M}^2$ in the absence of external interactions is defined here. The parameter $k_\sigma \in \mathbb{R}$ is a measure of the range of the interaction kernel, in the sense that, as $k \to \infty$, the range of the interaction kernel goes to zero, while q_σ is of course the amplitude of the interaction kernel. This form, illustrated schematically in Figure 6.1, is used extensively in the calculations of the chapters which follow, principally because of the ease with which it can be manipulated, but, as is shown in the examples of Chapters 9 and 10, also because it has particularly simple asymptotic properties.

6.6 Conclusion

The natural representation of the potential energy as an object on $\mathcal{M}^n$, rather than on $\mathcal{M}$, derived in this chapter suggests that analysis on $\mathcal{M}^n$ may in fact yield a dynamic equation for the fluid system which is better adapted to the analysis of fluid systems which are subject to internal interactions than the equations on $\mathcal{M}$ derived in Chapters 4 and 5. The derivation of such an equation is the subject of the following chapter.

respect to the symmetry group $\mathrm{iso}(\mathbb{R}^6)$ must be of the form[6]

$$\mathcal{F}_{\mathrm{iso}(\mathbb{R}^6)} : (x, y, z, x_*, y_*, z_*)$$
$$\mapsto F\big((x - x_*)^2 + (y - y_*)^2 + (z - z_*)^2 \big) \tag{6.73}$$

where $F : \mathbb{R} \to \mathbb{R}$, on interpreting the zero form $\mathcal{F}^0$ as a map from the chart induced by the coordinates on $\mathcal{M}^2$ to the real line. The physical interpretation of such a form is that the interaction between two infinitesimally small domains within the fluid is dependent on the distance between the domains, but is not dependent on the relative orientation of the domains or on their location within the fluid.

Note that a kernel of the form of (6.73) naturally possesses the projection symmetry required for uniqueness of the potential $\Phi \triangleright \bigwedge^0 \mathcal{M}$.

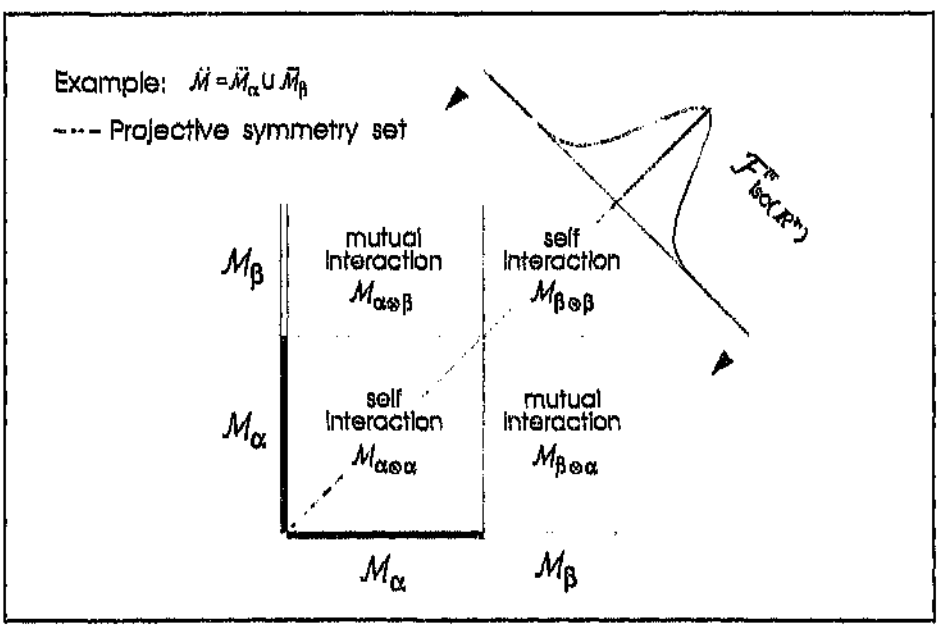


Figure 6.1: Schematic of the prototype interaction kernel.

[6]This is easily proved using the standard techniques described by Ibragimov [35].

structure of the fluid, any interaction kernel on $\mathcal{M}^n$, $n > 1$ describing solely an *internal* interaction within the fluid should itself be isotropic in the sense that it should be invariant with respect to translation and rotation of the fluid domain $\mathcal{M}$. In other words the interaction kernels describing internal interactions should possess the full set of rotational and translational symmetries permitted by the value of n, since any reduction of the symmetry of the interaction kernels would require some account to be kept of the evolving internal structure of the system. It is important to realise that the symmetry of the interaction kernels can of course be broken in the case where a component of the interactions is generated externally to the fluid. This situation is investigated further in Chapter 8.

Thus, in the absence of external interactions, the kernels $\mathcal{F}^\sigma \triangleright \bigwedge^0 \mathcal{M}^2$, $\sigma \in \mathcal{Z}^2$ should be invariant with respect to the infinitesimal generators $\xi_i \triangleright \mathcal{XM}^2$, $i = 1, \ldots, 9$ given by the translation operators

$$\begin{aligned} \xi_1 &= dx + dx_* \\ \xi_2 &= dy + dy_* \\ \xi_3 &= dz + dz_* \end{aligned} \tag{6.71}$$

and the rotation operators

$$\begin{aligned} \xi_4 &= (y - y_*)dx - (x - x_*)dy \\ \xi_5 &= (z - z_*)dy - (y - y_*)dz \\ \xi_6 &= (x - x_*)dz - (z - z_*)dx \\ \xi_7 &= (y - y_*)dx_* - (x - x_*)dy_* \\ \xi_8 &= (z - z_*)dy_* - (y - y_*)dz_* \\ \xi_9 &= (x - x_*)dz_* - (z - z_*)dx_* \end{aligned} \tag{6.72}$$

Denote the symmetry group generated from the basis $\xi_i \triangleright \mathcal{XM}^2$, $i = 1, \ldots, 9$ by $\mathrm{iso}(\mathbb{R}^6)$. Any interaction kernel $\mathcal{F}_{\mathrm{iso}(\mathbb{R}^6)} \triangleright \bigwedge^0 \mathcal{M}^2$ which is invariant with

$\mathcal{M}$ to be isomorphic[9] to $\mathbb{R}^3$, and let (x, y, z) and (x_*, y_*, z_*) be equivalent, uniformly valid, rectangular coordinates for $\mathcal{M}$. Then (x, y, z, x_*, y_*, z_*) are coordinates for $\mathcal{M}^2$ induced by the coordinates on $\mathcal{M}$. Denote the dual basis induced by the coordinates on $\mathcal{M}^2$ by $(dx, dy, dz, dx_*, dy_*, dz_*)$.

Projective Symmetries of $\mathcal{F}$

If the potential Φ is to be determined uniquely from the interaction kernel $\mathcal{F}$ via the projection (6.67), then the species interaction kernels must possess the projective symmetry

$$\mathcal{F}^\sigma(x, y, z, x_*, y_*, z_*) = \mathcal{F}^\sigma(x_*, y_*, z_*, x, y, z) \tag{6.68}$$

for all $\sigma \in \mathcal{Z}^2$, on interpreting $\mathcal{F}^\sigma$ as a map from the coordinates on $\mathcal{M}^2$ to the real line.

For later use let $\mathrm{pr}_i : \mathcal{M}^n \to \mathcal{M}$ be the canonical projection onto the i'th factor in the product $\mathcal{M}^n = \mathcal{M} \times \cdots \times \mathcal{M}$. Define $\mathcal{S} \subset \mathcal{M}^n$ as the *projective symmetry set*

$$\mathcal{S} = \{\mathbf{x} \in \mathcal{M}^n : \mathrm{pr}_i(\mathbf{x}) = \mathrm{pr}_j(\mathbf{x}) \ \forall\, i, j \in \{1, \ldots, n\}\} \tag{6.69}$$

In terms of the coordinates on $\mathcal{M}^2$ defined above, the projective symmetry set $\mathcal{S} \subset \mathcal{M}^2$ is the set

$$\mathcal{S} = \{(x, y, z, x_*, y_*, z_*) : x = x_*, y = y_*, z = z_*\} \tag{6.70}$$

Point Symmetries of $\mathcal{F}$

To be consistent with the isotropic form (2.3) for the density within the domains comprising the fluid and with the absence of any model for the internal

[9]The case where $\mathcal{M}$ is isomorphic to $\mathbb{R}^2$ follows in similar fashion.

As discussed in Chapter 1, a macroscopic model of any particular fluid system cannot by definition contain a detailed description of the microscopic physical properties of the system. The microscopic analysis of Section 6.4 must thus be carried through into the macroscopic interaction field model for the fluid system via the following two assertions:

- For the purposes of generating a macroscopic model for the dynamics of the fluid system, the order-n interaction kernels need not bear any direct relationship to the interactions between the particles within the system, but can be generalised to be any one-form field on $\mathcal{M}^n$ consistent with the symmetries of the *macroscopic* model adopted for the structure of the fluid system.
- For the purposes of generating a macroscopic interaction field model which is consistent with the classical surface tension theory, it is unnecessary to model higher than pairwise-order interactions in the expression for the potential energy of the fluid system.

Of course the validity of these assertions has still to be demonstrated. This demonstration forms the subject of Chapters 9 and 10, where it is shown that some additional conditions must, in fact, be imposed on the form of the interaction kernels if consistency with the classical surface tension theory is to be obtained.

6.5.1 Appropriate Form for $\mathcal{F}$

It remains to determine a general form for the interaction kernels which is consistent with the model of the macroscopic structure of the fluid system adopted in Chapter 2.

Using the second assertion made in Section 6.5 to reduce the scope of the present analysis, only the form of the interaction kernel on $\mathcal{M}^2$ is considered here. The forms of interaction kernels on the higher dimensional product structures follow by similar arguments to those presented in this section. Assume

Note that there are n distinct ways in which $\mathcal{F}$ can be projected via the fibre integral from $\mathcal{M}^n$ onto $\mathcal{M}$ to obtain the potential Φ. If the potential Φ is to be uniquely determined from $\mathcal{F}$ then the interaction kernels $\mathcal{F}^\sigma$, $\sigma \in \mathcal{Z}$ must possess certain projective symmetries, as described in Section 6.5.1.

The result (6.67) shows that the approaches to the dynamics of the fluid system employed in Chapters 4 and 5 will become extremely awkward if there are interactions between the fluid particles of pairwise or higher order, or in other words if there are any internal interactions within the fluid whatsoever. If $n > 1$, the dependence of Φ on the occupation functions on $\mathcal{M}^n$, and hence through (6.32) and (6.31) on the density $*\rho \triangleright \Omega\mathcal{M}$, is obscured by formulating the potential term on $\mathcal{M}$ rather than on $\mathcal{M}^n$. The structure of (6.65) shows that in general it is not possible[2] to write an evolution equation for Φ solely in terms of fields and operations on $\mathcal{M}$, and hence that any attempt to follow this route must lead to an overcomplicated and unnatural time-parametrisation of Φ, or must require substantial simplification of the dynamics involved[3]. Some previous attempts to follow this route are reviewed in Chapter 9.

6.5 Generalisation

A large body of literature exists[4] in which specific forms for the interactions within specific fluid systems and detailed information on the correlation functions and their dependence on such factors as the rate of deformation of the fluid and proximity to interfaces within the fluid are described. These descriptions are usually derived from the results of calculations and experiments on the molecular level of description of the fluid system which are often intricate and extremely difficult to perform.

[2]Except of course when $n = 1$ in the case where the only interaction in the system originates in an external field.

[3]See Chapter 11 though, where an integral geometric analysis of $\Phi \triangleright \bigwedge^0 \mathcal{M}$ is used to obtain some of the *qualitative* features of a simplified dynamic for the fluid system.

[4]As reviewed by various authors including Boon and Yip [9], Temperley and Trevena [79], Croxton [19] and in the standard reference by Hirschfelder, Curtiss and Bird [34].

Substituting (6.63) into (6.61),

$$\begin{aligned} V &= \sum_{\sigma \in \mathcal{Z}^n} \int_{\mathcal{M}^n} {}^*\rho_\sigma \gamma_\sigma V_\sigma \\ &\equiv \int_{\mathcal{M}^n} {}^*\mathcal{F} \end{aligned} \tag{6.64}$$

where the *order-n interaction kernel*

$$\begin{aligned} \mathcal{F} &= \rho_\sigma \mathcal{F}^\sigma , \quad \sigma \in \mathcal{Z}^n \\ &\triangleright \Lambda^0 \mathcal{M}^n \end{aligned} \tag{6.65}$$

(summation over repeated indices assumed). The forms $\mathcal{F}^\sigma \triangleright \Lambda^0 \mathcal{M}^n$, $\sigma \in \mathcal{Z}^n$, will be termed the order-n interaction kernels for the particular species contained in $\sigma \in \mathcal{Z}^n$. It is implicit in the constitutive assumption that the entire dependence of the potential energy on the configuration of the fluid system is contained within the occupation functions, and hence that the interaction kernels for the species are themselves independent of the configuration of the fluid system.

6.4.6 Reconciliation

Expression 6.64 for the potential energy can be reconciled with the form assumed in Section 3.2.1 by using the fibre integral to expoit the fibre structure of $\mathcal{M}^n$:

$$\begin{aligned} V &= \int_{\mathcal{M}^n} {}^*\mathcal{F} \\ &= \int_{\mathcal{M}^n} \Omega^1 \otimes \Omega^{n-1} \mathcal{F} \\ &= \int_{\mathcal{M}} {}^* \oint_{\mathcal{M}^{n-1}} {}^*\mathcal{F} \end{aligned} \tag{6.66}$$

using (6.23).

Hence, comparing (6.66) and (3.7),

$$\begin{aligned} \Phi &= \oint_{\mathcal{M}^{n-1}} {}^*\mathcal{F} \\ &\triangleright \Lambda^0 \mathcal{M} \end{aligned} \tag{6.67}$$

7.10 Conclusion

In this chapter it is shown that an evolution equation can be derived on a suitably-defined configuration manifold for the fluid system. This approach avoids the need to project the potential energy of the fluid system from an object defined on $\mathcal{M}^n$, $n > 1$, onto an object on $\mathcal{M}$, as is required by the form of the equations of motion derived in Chapters 4 and 5. Note though that since the same Lagrangian is used in the derivation of the results of both this chapter and of Chapter 4, and since each variation defined on the configuration manifold $\mathcal{D}$ in Section 7.6 can be mapped using (7.10) and the compatibility conditions of Section 7.1 to a variation on the domain of the fluid $\mathcal{M}$ of the type defined in Section 4.2, the analyses of Section 4.2 and Section 7.6 are equivalent and hence (4.18) and (7.38) are merely two alternate manifestations of the interaction field model defined in Chapter 3.

The analysis of this chapter concludes the definition of the interaction field model. In the remaining chapters of this work, the properties of this model are explored in various static and dynamic contexts, and, in addition, the interaction field approach is validated against the predictions of the classical surface tension model.

Maupertuis' variational principle (7.14) for the fluid system can be written as

$$\begin{aligned}\delta\int_{t_0}^{t_1} T dt &= \delta\int_{t_0}^{t_1}\sqrt{T}\sqrt{T}dt \\ &= \delta\int_{t_0}^{t_1}[\,(E_0-V)\mathsf{T}(\dot\vartheta,\dot\vartheta)\,]^{1/2}dt \\ &= 0 \end{aligned} \tag{7.17}$$

on substitution of (7.16) and the constraint on the physical path that $T = E_0 - V$ under the assumption of zero dissipation. Note that

$$\hat{\mathsf{T}} = (E_0 - V)\mathsf{T} \tag{7.18}$$

is also a metric tensor on $\mathcal{D}$, and is conformally related to the metric tensor T. Using this definition (7.17) becomes

$$\delta\int_{t_0}^{t_1}[\,\hat{\mathsf{T}}(\dot\vartheta,\dot\vartheta)\,]^{1/2}dt = 0 \tag{7.19}$$

which shows that the trajectories γ (see Section 7.3) of the non-dissipative system are geodesics on $\mathcal{D}$ under the metric $\hat{\mathsf{T}}$. This metric, known as the Jacobi metric, on the configuration space of the system has been used in the analysis of the dynamics of simple mechanical systems[6] and would appear to be admirably suited to the analysis of non-dissipative fluid systems of the type discussed in this work. In particular, there exist a number of theorems on the global properties of geodesics on a manifold[7] which may be used to yield more general results for the dynamics of fluid systems with interfaces than can be obtained via the analyses presented in this work. Unfortunately, very little appears to be known about the global structure of the group manifold $\mathsf{aut}_0(\mathcal{M})$ on which $\mathcal{D}$ is based, and this is a major impediment to progress via an analysis of the geodesics on $\mathcal{D}$. Further progress may thus be forced to await further developments in the theory of groups or in the theory of manifolds.

[6] See the paper by Pin [67].

[7] See for instance the books by Cheeger and Ebin [15], and Klingenberg [43].

7.9.2 Maupertuis' Principle

An alternative, but equivalent, formulation of Hamilton's variational principle (see Section 3.1) which is valid if the dissipation in the fluid system is identically zero is that due to Maupertuis[4]. The action

$$A = \int_{t_0}^{t_1} T dt \tag{7.44}$$

is assumed to be stationary for the evolution of the system over the arbitrary time interval $t_0 \leq t \leq t_1$ and the variational paths are restricted to those satisfying the total energy condition

$$T + V = E_0 \tag{7.45}$$

where E_0 is the sum of the kinetic and potential energies of the of the system at $t = t_0$.

7.9.3 Dynamic Metric on $\mathcal{D}$

Note that (7.18) for the kinetic energy of the system is identical to the equation

$$T = \mathsf{T}(\dot{\vartheta}, \dot{\vartheta}) \tag{7.46}$$

written in terms of the coordinates on $\mathcal{D}$, if T is the second-order symmetric definite[5] tensor field on $\mathcal{D}$ with components $\mathsf{T}_{ij} = M^{ij}$ in the coordinate basis for $\mathcal{D}$ adopted in Section 7.3. The tensor T thus can assume the role of a metric on $\mathcal{D}$.

[4]See, for instance, the book by Abraham and Marsden [2].

[5]Symmetry and definiteness follow from the properties of the exterior product and the Hodge Star, and the tensorial nature of T follows from the transformation law for a change of coordinates on $\mathcal{D}$.

using Stokes' Theorem.

Substituting (7.39) and (7.40) into (7.38) and retaining only those terms of linear order in the a_i gives

$$M^{ij}\,\ddot{a}_i + C^{ij}\,\dot{a}_i + K^{ij}\,a_i - F^j = 0 \tag{7.42}$$

7.9 Geodesic Flow on $\mathcal{D}$

A particularly compact formulation of the dynamic equation in the case of identically zero viscosity of the fluid can be obtained by realising that, if $C^{ij} \equiv 0$ and $F^j \equiv 0$ for all $i, j = 1, \ldots, p$, then (7.38) is the coordinate form of an equation governing a geodesic flow on $\mathcal{D}$. Although the analysis leading to the formulation of the dynamic equation as a geodesic flow on the configuration space for the system is presented in this section, for reasons to be discussed here this approach will be taken no further in this work.

7.9.1 Geodesics on a Manifold

Given a manifold M with metric $\mathbf{g}$, a geodesic connecting points x_1 and x_2 in M is a curve in M (generated by the path $\zeta : \mathbb{R} \to M$) of minimal length passing through x_1 and x_2. That is, along a geodesic the path length

$$\int_{\epsilon_0}^{\epsilon_1} [\, \mathbf{g}(\dot{\zeta}(\epsilon), \dot{\zeta}(\epsilon))\,]^{1/2} d\epsilon \tag{7.43}$$

is stationary, where ϵ is the parameter along the path such that $\zeta(\epsilon_0) = x_0$, $\zeta(\epsilon_1) = x_1$, and $\dot{\zeta}(\epsilon)$ is the tangent vector to the geodesic at $\zeta(\epsilon) \in M$.

for $j = 1, \ldots, p$.

Note though that

$$\frac{d}{dt} M^{ij} = M^{ij,k}\, \dot{a}_k \tag{7.37}$$

Hence the evolution of the system is given by

$$M^{ij}\, \ddot{a}_i + M^{ij,k}\, \dot{a}_i\, \dot{a}_k + C^{ij}\, \dot{a}_i - F^j = 0 \tag{7.38}$$

for $j = 1, \ldots, p$, on partial differentiation of the first term of (7.36).

7.8 Linearised Equation of Motion

The equation of motion (7.38) can be linearised about any point in $\mathcal{D}$. Define coordinates $(a_1, \ldots, a_p)$ in the neighbourhood of $\vartheta \in \mathcal{D}$ such that ϑ has coordinates $(0, \ldots, 0)$. In the neighbourhood of ϑ,

$$\begin{aligned} M^{ij} &= M^{ij} + M^{ij,k}\, a_k + O((a_k)^2) \\ M^{ij,k} &= M^{ij,k} + M^{ij,kl}\, a_l + O((a_l)^2) \\ C^{ij} &= C^{ij} + C^{ij,k}\, a_k + O((a_k)^2) \end{aligned} \tag{7.39}$$

and

$$F^j = F^j - K^{ij}\, a_i + O((a_i)^2) \tag{7.40}$$

where the *stiffness coefficients*

$$\begin{aligned} K^{ij} &\equiv F^{ji} \\ &= \mathcal{L}_{v_n^i} \int_{\mathcal{M}^n} \mathcal{L}_{v_n^j} (*\mathcal{F}) \\ &= \int_{\mathcal{M}^n} \mathsf{d}(*v_n^i * \mathsf{d}(\mathcal{F} * v_n^j)) \\ &= \sum_{\sigma \in \mathcal{B}^n} \int_{\sigma \mathcal{M}_\sigma^n} *v_n^i * \mathsf{d}(\mathcal{F} * v_n^j) \end{aligned} \tag{7.41}$$

from (7.17) and (7.22). Hence from definition (1.13)

$$\begin{aligned} P^j &= \int_{\mathcal{M}^n} [\, *\hat{p}_\sigma \, \tfrac{\partial}{\partial a_j}(\, {}^*\psi_\theta \mathcal{F}^\sigma \,) + \mathcal{L}_{\mathcal{V}_n^j}(\, *\mathcal{F} \,) \,] \\ &= \int_{\mathcal{M}^n} \mathcal{L}_{\mathcal{V}_n^j}(\, *\mathcal{F} \,) \end{aligned} \tag{7.32}$$

since, from (7.6), ${}^*\psi_\theta \mathcal{F}^\sigma = \mathcal{F}^\sigma$ and hence $\frac{\partial}{\partial a_j}(\, {}^*\psi_\theta \mathcal{F}^\sigma \,) = 0$. Alternatively, using the Cartan identity (1.11),

$$\begin{aligned} P^j &= \int_{\mathcal{M}^n} \mathcal{V}_n^j \wedge *\mathsf{d}(\mathcal{F}) \\ &= \int_{\mathcal{M}^n} *\mathcal{V}_n^j(\mathcal{F}) \end{aligned} \tag{7.33}$$

7.7 Equation of Motion

Since the sections $\mathcal{V}^i$, $i = 1, \ldots, p$ already obey the incompressibility restriction (7.12), Equation 4.6 for the Lagrangian density can be used without the inclusion of the multipliers[3] required in the derivations of Section 4.2. The variation of the Lagrangian map for the system is thus

$$\begin{aligned} \delta L &= M^{ij} \, \dot{a}_i \, \delta \dot{a}_j \\ &\quad + \delta \dot{a}_j \int_{t_0}^{t_0+t} C^{ij}_{\mathcal{V}(t')} \, \dot{a}_i(t') \, dt' \\ &\quad - P^j \, \delta a_j \end{aligned} \tag{7.34}$$

Applying Hamilton's principle (3.2),

$$\int_{t_0}^{t_1} \delta a_j \, [\, \tfrac{d}{dt}(M^{ij} \, \dot{a}_i) + C^{ij} \, \dot{a}_i - P^j \,] dt = 0 \tag{7.35}$$

for $j = 1, \ldots, p$, on integrating by parts using (1.22). Since the variations δa_j, $j = 1, \ldots, p$ are arbitrary and independent of each other,

$$\tfrac{d}{dt}(M^{ij} \, \dot{a}_i) + C^{ij} \, \dot{a}_i - P^j = 0 \tag{7.36}$$

[3]The absence of the pressure in the resulting equation of motion is another major advantage of the present approach over the analysis of Chapter 4.

Using this operator, the variation of the kinetic energy is

$$\delta T = M^{ij}\,\dot{a}_i\,\delta a_j \tag{7.25}$$

The variation of the dissipation function is similarly

$$\delta D = C^{ij}_{v(t')}\,\dot{a}_i(t')\,\delta a_j(t) \tag{7.26}$$

The variation of the potential energy is now determined. Let $v^j_1 \equiv v^j$, $j = 1,\ldots,p$. Then, following the analysis of Section 6.2.3, the map ψ_0, together with the coordinates $(a_1,\ldots,a_p)$ for $\mathcal{D}$ induce distortion modes

$$\begin{aligned} v^j_n &= v^j \circ v^j_{n-1} \\ &\triangleright \mathcal{X}\mathcal{M}^n \end{aligned} \tag{7.27}$$

on the manifolds $\mathcal{M}^n$, $n > 1$ such that the flow field $\nu_n \triangleright \mathcal{X}^{\mathcal{M}^n}$ of ψ_0 on $\mathcal{M}^n$ is given by

$$\nu_n = \dot{a}_i\, v^i_n \tag{7.28}$$

From (6.42) and a similar analysis to that used to obtain (7.12), it follows that

$$(\mathsf{d}*v^i_n)|_{\mathcal{M}^n_0} = 0 \tag{7.29}$$

for all $i = 1,\ldots,p$ and $\mathcal{M}^n_0 \subset \mathcal{M}^n_\xi$. Hence

$$\delta V = F^j\,\delta a_j \tag{7.30}$$

where the *force coefficients*

$$\begin{aligned} F^j &= \frac{d}{da_j} V \\ &= \int_{\mathcal{M}^n} \frac{d}{da_j}\left(*\hat{\rho}_n \, {}^*v_n \mathcal{F}^n \right) \end{aligned} \tag{7.31}$$

Potential Energy

Finally, if the potential energy of the system is now assumed to be of the form of (6.64) rather than of the form of (3.7):

$$\begin{aligned} V &= \int_{\mathcal{M}^n} *\mathcal{F} \\ &= \int_{\mathcal{M}^n} *\rho_\sigma \mathcal{F}^\sigma \\ &= \int_{\mathcal{M}^n} *\hat{\rho}_\sigma \, {}^*\psi_0 \mathcal{F}^\sigma \\ &= \int_{\mathcal{M}^n} *\hat{\rho}_\sigma \mathcal{F}^\sigma \end{aligned} \tag{7.22}$$

(summation over $\sigma \in \mathcal{Z}^n$ assumed) using (7.4) and (7.6).

7.6 Variational Analysis

At any point $\vartheta \in \mathcal{D}$, the curve γ along which the system evolves is specified by the point in $T\mathcal{D}$ with coordinates $(a_1, \ldots, a_p, \dot{a}_1, \ldots, \dot{a}_p)$. Assume variations $\delta a_i(t)$ and $\delta \dot{a}_i(t)$, $i = 1, \ldots, p$ to these coordinates. The variations are subject to the kinematical constraints

$$\delta \dot{a}_i = \frac{\partial}{\partial t} \delta a_i \tag{7.23}$$

and the end-point conditions $\delta \dot{a}_i(t_0) = \delta \dot{a}_i(t_1) = 0$ and $\delta a_i(t_0) = \delta a_i(t_1) = 0$ for all $i = 1, \ldots, p$.

The variation operator for this system[2] is then

$$\delta = \delta \dot{a}_i \frac{\partial}{\partial \dot{a}_i} + \delta a_i \frac{\partial}{\partial a_i} \tag{7.24}$$

[2] As in Section 3.1, the reader is referred to the texts by Lanczos [48] and Burke [13] for the derivation of the variation operator.

$$= \int_{\mathcal{M}^m} \frac{d}{da_i}\omega_j \tag{7.17}$$

using the definition of the total derivative given in Section 1.9.2. In the following analysis, the abbreviated notation $I_{,i}$ or $I^{,i}$ is used to denote the derivative of the integral I with respect to the parameter a_i.

Kinetic Energy

On combining (7.10) and (3.6), the kinetic energy of the fluid system is

$$T = \tfrac{1}{2} M^{ij}\, \dot{a}_i\, \dot{a}_j \tag{7.18}$$

where the *inertia coefficient*

$$\begin{aligned} M^{ij} &= \int_{\mathcal{M}} \rho\, \upsilon^i \wedge *\upsilon^j \\ &= \int_{\mathcal{M}} \hat{\rho}\, {}^*\psi_\vartheta(\, \upsilon^i \wedge *\upsilon^j\,) \end{aligned} \tag{7.19}$$

using (7.4).

Dissipation Function

Similarly, on combining (7.10) and (3.8), the dissipation function for the system is

$$D = \tfrac{1}{2} C^{ij}\, \dot{a}_i\, \dot{a}_j \tag{7.20}$$

where the *damping coefficient*

$$\begin{aligned} C^{ij} &= \int_{\mathcal{M}} \eta\, \mathsf{d}\upsilon^i \wedge *\mathsf{d}\upsilon^j \\ &= \int_{\mathcal{M}} \hat{\eta}\, {}^*\psi_\vartheta(\, \mathsf{d}\upsilon^i \wedge *\mathsf{d}\upsilon^j\,) \end{aligned} \tag{7.21}$$

again using (7.4).

non-zero viscosity the distortion modes must be continuous everywhere, while if the viscosity is identically zero, then the definition of the distortion modes must take into account the possibility that the velocity tangential to the internal interfaces $\partial_{\alpha\beta}$, $\alpha, \beta \in \mathcal{Z}$ may be discontinuous where the density at the interface is discontinuous.

7.5 System Integrals

Given a reference state

$$\hat{\mathcal{R}} = (\mathcal{M}, \hat{\rho}, \hat{\eta}, \mathcal{Z}, \hat{\mathcal{M}}_{\mathcal{Z}}, \hat{\mathcal{O}}_{\mathcal{Z}}) \tag{7.15}$$

let

$$\psi_\vartheta \hat{\mathcal{R}} = (\mathcal{M}, \rho, \eta, \mathcal{Z}, \mathcal{M}_{\mathcal{Z}}) \tag{7.16}$$

The system integrals defined in Section 3.2.1 can then be re-expressed in terms of the distortion modes $v^i \triangleright \mathcal{X}\mathcal{M}$, $i = 1, \ldots, p$. The following result is needed for the analysis to follow:

Differentiation with respect to a Parameter

Let $\tilde{\vartheta} : \mathbb{R} \to \mathcal{D}$ be a one-parameter path such that $\tilde{\vartheta}(0)$ is the point $\vartheta \in \mathcal{D}$ with coordinates $(a_1, \ldots, a_p)$ and $\tilde{\vartheta}(\epsilon)$ is the point in $\mathcal{D}$ with coordinates $(a_1, \ldots, a_i + \epsilon, \ldots, a_p)$. The derivative of the integral $I = \int_{\psi_\vartheta \mathcal{M}^m} \omega_\vartheta$, where $\omega_\vartheta \triangleright \Omega\mathcal{M}^m$, for any $m \geq 1$, with respect to the parameter a_i can then be defined as

$$\begin{aligned} \frac{d}{da_i} I &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\int_{\psi_{\tilde{\vartheta}(\epsilon)} \mathcal{M}^m} \omega_{\tilde{\vartheta}(\epsilon)} - \int_{\psi_{\tilde{\vartheta}(0)} \mathcal{M}^m} \omega_{\tilde{\vartheta}(0)} \right) \\ &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\mathcal{M}_0^m} \left({}^*\psi_{\tilde{\vartheta}(\epsilon)} \omega_{\tilde{\vartheta}(\epsilon)} - {}^*\psi_{\tilde{\vartheta}(0)} \omega_{\tilde{\vartheta}(0)} \right) \end{aligned}$$

Kinematic Constraints

A kinematic constraint on the form of the distortion modes is implied by the incompressibility condition (2.7). Since

$$\begin{aligned} \mathrm{d}{*}(\dot{a}_i\, \boldsymbol{v}^i)|_{\psi_\vartheta \hat{\mathcal{M}}_\alpha} &= \dot{a}_i\, \mathrm{d}{*}\boldsymbol{v}^i|_{\psi_\vartheta \hat{\mathcal{M}}_\alpha} \\ &= 0 \end{aligned} \tag{7.11}$$

for all $\alpha \in \mathcal{Z}$ and $\vartheta \in \mathcal{D}$, or, since the $\dot{a}_i$, $i = 1, \ldots, p$ are in principle arbitrary and independent,

$$\mathrm{d}{*}\boldsymbol{v}^i|_{\psi_\vartheta \hat{\mathcal{M}}_\alpha} = 0 \tag{7.12}$$

for all $i = 1, \ldots, p$, $\alpha \in \mathcal{Z}$ and $\vartheta \in \mathcal{D}$.

As in Section 2.3, this result can be integrated over each of the elements of $\mathcal{M}_\mathcal{Z}$, giving

$$\int_{\psi_\vartheta \partial \hat{\mathcal{M}}_\alpha} {*}\boldsymbol{v}^i = 0 \tag{7.13}$$

for all $i = 1, \ldots, p$, $\alpha \in \mathcal{Z}$ and $\vartheta \in \mathcal{D}$, using Stokes' theorem.

The no-rupture condition (2.16) leads, by a similar analysis, to the second kinematic constraint on the distortion modes that

$$({*}\boldsymbol{v}^i|_{\psi_\vartheta \partial \hat{\mathcal{M}}_\alpha})|_{\partial_{\alpha\beta}} - ({*}\boldsymbol{v}^i|_{\psi_\vartheta \partial \hat{\mathcal{M}}_\beta})|_{\partial_{\alpha\beta}} = 0 \tag{7.14}$$

where $\partial_{\alpha\beta} = \psi_\vartheta \partial \hat{\mathcal{M}}_\alpha \cap \psi_\vartheta \partial \hat{\mathcal{M}}_\beta$. If this constraint is satisfied by the distortion modes then $\psi_\vartheta \hat{\mathcal{M}}_\mathcal{Z}$ is $\psi_\vartheta \hat{\mathcal{M}}$-complete if $\hat{\mathcal{M}}_\mathcal{Z}$ is $\hat{\mathcal{M}}$-complete.

Dynamic Constraints

Dynamic constraints on the forms of the distortion modes are implied by the interfacial conditions derived in Section 4.5 and Section 5.3. In the case of

where $(\frac{\partial}{\partial a_1}, \ldots, \frac{\partial}{\partial a_p})$ is the coordinate basis for $\mathsf{T}_\vartheta\mathcal{D}$ and the superscript dots on the right hand side of the equation signify differentiation with respect to $t \in \mathbb{R}$. These definitions can of course be extended in the standard way to the whole of $\gamma \subset \mathcal{D}$ via an atlas on $\mathcal{D}$.

Using local coordinates, $\psi_{\vartheta(t)}$ has the linearisation

$$\varphi_{\vartheta(t)} : \mathbf{x} \mapsto \mathbf{x} + t\, \dot{a}_i(t)\, \#\boldsymbol{v}^i + O(t^2) \tag{7.9}$$

(where summation over repeated indices is implied) for all $\mathbf{x} \in \mathcal{M}$. Assume that the maps φ_t and $\psi_{\vartheta(t)}$ are equivalent representations of the motion of the fluid. The equivalence condition

$$\begin{aligned} \boldsymbol{\nu} &= \dot{a}_i\, \boldsymbol{v}^i \\ &\triangleright \quad \mathcal{K}\mathcal{M} \end{aligned} \tag{7.10}$$

on the flow fields generated by the two automorphisms (see Section 2.1) must then be satisfied.

The forms $\boldsymbol{v}^i \triangleright \mathcal{K}\mathcal{M}$, $i = 1, \ldots, p$, will be termed the *distortion modes* associated with a particular selection of coordinates for $\mathcal{D}$ in the neighbourhood of a particular point $\vartheta \in \mathcal{D}$. The transformation laws for the distortion modes follow immediately from those for a change of coordinates on $\mathcal{D}$.

7.4 Compatibility Conditions

If compatibility with the macroscopic model for the fluid structure adopted in Chapter 2 is to be ensured, then a number of constraints on the form of the distortion modes must be enforced.

Define the column-transpose transformation by

$$\begin{aligned} \leftarrow \; &: \; [M, I\!R]_n^m \to [M, I\!R]_n^m \\ &: \; [\,\mathbf{g}\,] \mapsto [\,\overleftarrow{\mathbf{g}}\,] = [\,\mathbf{g}\,]\,[\,\%\,] \end{aligned} \tag{8.19}$$

where the transpose matrix

$$\begin{aligned} [\,\%\,] \; &= \; \begin{bmatrix} 0 & & 1 \\ & \cdot & \\ 1 & & 0 \end{bmatrix} \\ &\in \; [\,I\!R\,]_n^n \end{aligned} \tag{8.20}$$

The diagonal transpose is defined in the usual way by

$$\begin{aligned} T \; &: \; [M, I\!R]_n^m \to [M, I\!R]_m^n \\ &: \; [\,g_{ij}\,]_{(m,n)} \mapsto [\,g_{ij}\,]_{(m,n)}^T = [\,g_{ji}\,]_{(n,m)} \end{aligned} \tag{8.21}$$

Linear Independence

Define the matrix function $[\,\mathbf{g}^*\,] \in [\,M, I\!R\,]_n$ to be *linearly independent* if

$$\begin{aligned} [\,\boldsymbol{\lambda}\,]\,[\,\mathbf{g}^*\,]^T \; &= \; [\,\vec{\mathbf{0}}\,],\; [\,\boldsymbol{\lambda}\,] \in [\,I\!R\,]_n \\ &\Rightarrow \; [\,\boldsymbol{\lambda}\,] = [\,\mathbf{0}\,] \end{aligned} \tag{8.22}$$

where $[\,\mathbf{0}\,]$ is the zero of $[\,I\!R\,]_n$ and $[\,\vec{\mathbf{0}}\,] : M \to [\,\mathbf{0}\,]$ is the zero of $[\,M, I\!R\,]_n$. (Note that this definition immediately implies that none of the elements of $[\,\mathbf{g}^*\,]$ are the null form $\vec{\mathbf{0}} : M \to 0$.) Let $[\,M, I\!R\,]_n^* \subset [\,M, I\!R\,]_n$ be the set of linearly independent matrix functions in $[\,M, I\!R\,]_n$.

From this definition, if $[\,\mathbf{f}\,] \in [\,M, I\!R\,]_n$, and $[\,\mathbf{g}^*\,] \in [\,M, I\!R\,]_n$ is linearly independent, then, since $[\,\mathbf{f}\,] : M \to [\,I\!R\,]_n$,

$$\begin{aligned} [\,\mathbf{f}\,]\,[\,\mathbf{g}^*\,]^T \; &= \; [\,\vec{\mathbf{0}}\,] \\ &\Rightarrow \; [\,\mathbf{f}\,] = [\,\vec{\mathbf{0}}\,] \end{aligned} \tag{8.23}$$

where the distortion mode v_n^l has been generalised to some $\tau_n \triangleright \mathcal{K}\mathcal{M}^n$. Restrictions on τ_n for solutions to (8.16) to correspond to physically-valid solutions of the equilibrium equation (8.15) are developed in the analysis which follows. In the following sections, a matrix-based technique for the solution of (8.16) in terms of the occupation forms $\rho_n \in \mathcal{O}_2^n$ is developed. In Section 8.6 it is shown how this approach allows the equilibria of a particular fluid system consisting of a number of distinct domains to be determined by largely algebraic means.

8.4.2 Notation and Definitions

Matrix Functions on a Manifold

Define $[\ I\!R\ ]_n^m$ as the set of all $m \times n$ real-valued matrices. Use the notation $[\ \lambda_{ij}\ ]_{(m,n)}$ to denote the composition of $[\ \boldsymbol{\lambda}\ ] \in [\ I\!R\ ]_n^m$ in terms of its elements. For later convenience define $[\ I\!R\ ]_n \equiv [\ I\!R\ ]_n^1$ and $[\ \lambda_j\ ]_{(n)} \equiv [\ \lambda_{1j}\ ]_{(1,n)}$. Define the set $[\ M, I\!R\ ]_n^m$ of real valued $m \times n$ matrix functions on a manifold M such that if $[\ \mathbf{g}\ ] = [\ g_{ij}\ ]_{(m,n)} \in [\ M, I\!R\ ]_n^m$, each $g_{ij} \triangleright \bigwedge^0 M$ and hence $[\ \mathbf{g}\ ] : M \rightarrow [\ I\!R\ ]_n^m$. As before, define abbreviated notation for the case $m = 1$.

Define the matrix product of two matrix functions $[\ \mathbf{f}\ ] = [\ f_{ij}\ ]_{(m,l)} \in [\ M, I\!R\ ]_l^m$ and $[\ \mathbf{g}\ ] = [\ g_{ij}\ ]_{(l,n)} \in [\ M, I\!R\ ]_n^l$ such that

$$\begin{aligned} [\ \mathbf{f}\ ]\,[\ \mathbf{g}\ ] &= [\ f_{ik}\ g_{kj}\ ]_{(m,n)} \\ &\in [\ M, I\!R\ ]_n^m \end{aligned} \tag{8.17}$$

where summation over repeated indices is assumed.

Also, if $\xi \triangleright \mathcal{K}M$ can be interpreted as a linear differential operator on M and $[\ \mathbf{f}\ ] = [\ f_{ij}\ ]_{(m,n)} \in [M, I\!R]_n^m$, define

$$\begin{aligned} \xi([\ \mathbf{f}\ ]) &= [\ \xi(f_{ij})\ ]_{(m,n)} \\ &\in [M, I\!R]_n^m \end{aligned} \tag{8.18}$$

figures 8.3, 8.4 and 8.5, where the asymptotic agreement between the analysis based on the interaction field approach and Plateau's classical analysis can be more clearly seen.

8.4 Equilibrium via Symmetry Groups

Using (7.33), Equation 8.6 for static equilibrium of the reference state of the fluid system can be rewritten as

$$\begin{aligned} P^j &= \int_{\mathcal{M}^n} *v_n^j(\mathcal{F}) \\ &= \int_{\mathcal{M}^n} *v_n^j(\rho_\sigma \mathcal{F}^\sigma) \,, \quad \sigma \in \mathcal{Z}^n \\ &= 0 \end{aligned} \tag{8.14}$$

for all $j = 1, \ldots, p$.

8.4.1 Strong Condition for Equilibrium

Equation 8.14 is satisfied in the strong sense if

$$v_n^j(\rho_\sigma \mathcal{F}^\sigma) = 0 \,, \quad \sigma \in \mathcal{Z}^n \tag{8.15}$$

for all $j = 1, \ldots, p$. This equilibrium equation should be recognised as the generating equation for the point symmetries of $\rho_\sigma \mathcal{F}^\sigma$, $\sigma \in \mathcal{Z}^n$. It is assumed that the interaction kernels $\mathcal{F}^\sigma$, $\sigma \in \mathcal{Z}^n$, representing the interactions within the fluid, are known forms on $\mathcal{M}^n$ and that the equilibrium configuration of the system is given once the forms $\rho_\sigma \in \mathcal{O}_2^n$ satisfying (8.15) are determined.

Since the differential operator is a derivation with respect to multiplication, (8.15) can be rewritten as

$$\mathcal{F}^\sigma \, r_n(\rho_\sigma) + \rho_\sigma \, r_n(\mathcal{F}^\sigma) = 0 \,, \quad \sigma \in \mathcal{Z}^n \tag{8.16}$$

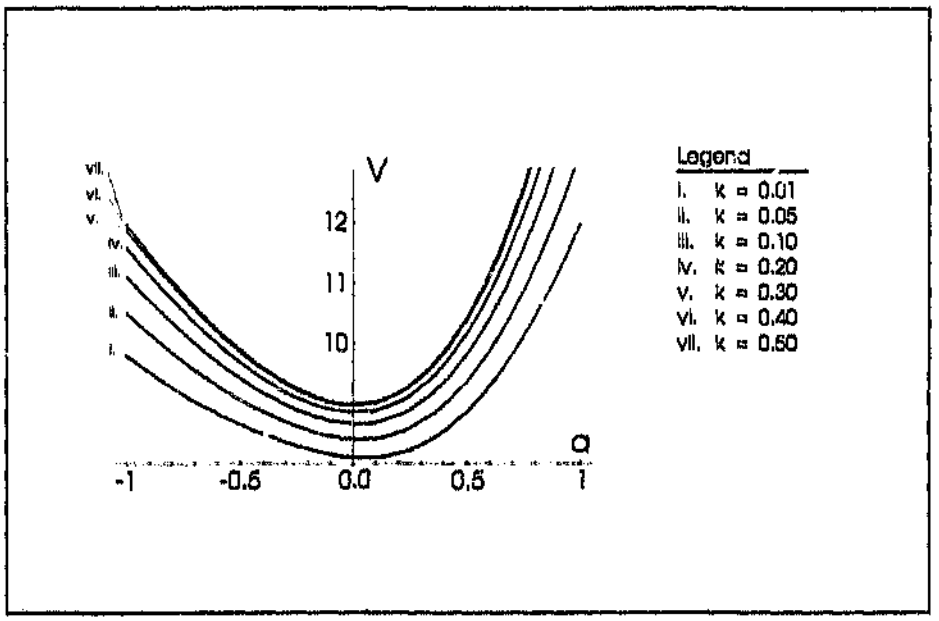


Figure 8.5: Energy variation for Plateau's problem ($k \ll 1$).

interactions[1] between the segments comprising $\mathcal{M}_\alpha$, particularly between the segments oy and oz when the angle γ is particularly acute. Indeed, over a small interval of k a second stable equilibrium is encountered where the included angle between the segments oy and oz is approximately $\pi/10$. Comparing the two contour diagrams for $\theta_c = \pi/3$ and $\theta_c = \pi/6$, this second equilibrium would appear to be a structurally stable feature of the system. The comments of Section 8.1.1 as to the physical relevance of results such as those discovered here are relevant and in this particular case it is surmised that the conditions required for differentiation of the two nearby equilibria in an experimental situation would be somewhat specialised.

For completeness, the variations of the potential energy with the geometric parameter a in each of the regimes $k \gg 1$, $k \approx 1$ and $k \ll 1$ are shown in

[1] As is discussed in Chapter 9, long-range interactions are excluded from analyses based on the classical surface tension based theory.

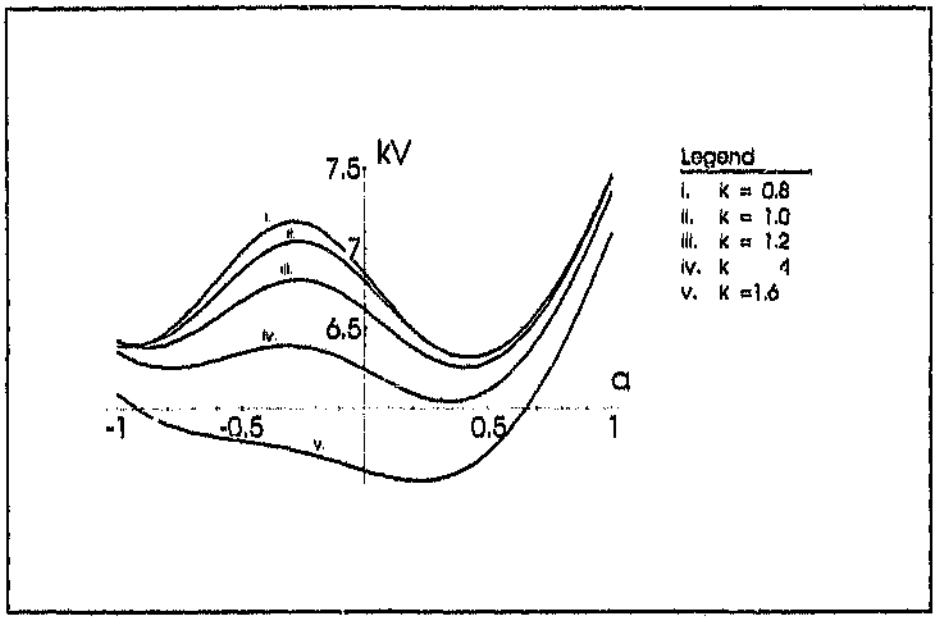


Figure 8.4: Energy variation for Plateau's problem ($k \approx 1$).

The potential energy of the system then follows from (6.64) as

$$V = \int_{\mathcal{M}^2} *\mathcal{F} \tag{8.13}$$

Figure 8.2 shows the relation of the potential energy of the system to the geometric parameter a and to the parameter k representing the range of the interaction kernel $\mathcal{F}$. Energy contours for $\theta_c = \pi/3$ and $\theta_c = \pi/6$ are shown, and the stable equilibria of the system (assuming k to be fixed) have been identified.

It appears that in the limits as $k \to 0$ and as $k \to \infty$, that is, as the range of the interaction becomes respectively large or small compared to the radius of the circle xyz, Plateau's result is regained, while where the range of the interaction is comparable to the radius of xyz, significant deviation from the classical result is found. This deviation would appear to result from the long-range

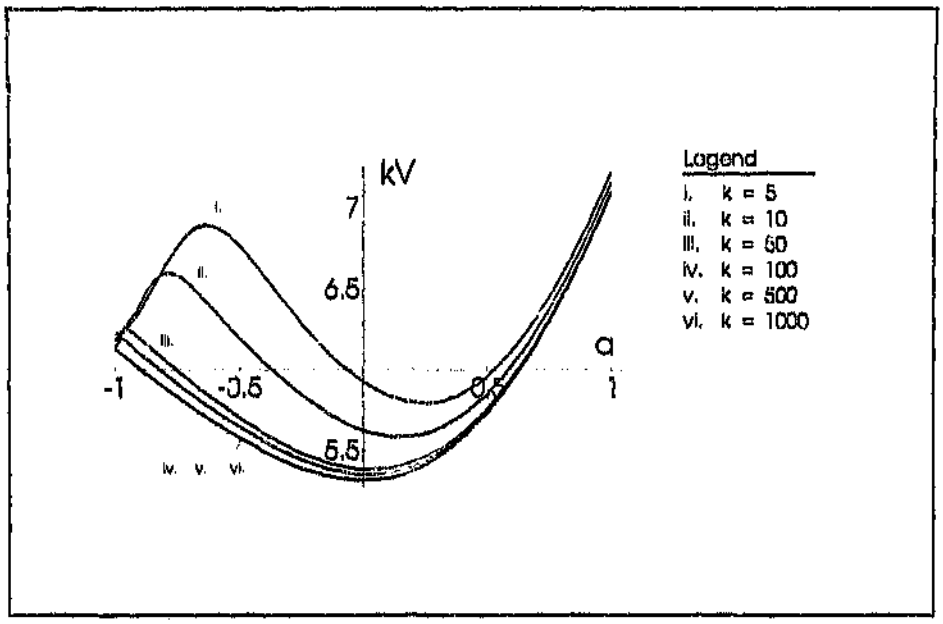


Figure 8.3: Energy variation for Plateau's problem ($k \gg 1$).

the interaction kernel $\mathcal{F} \triangleright \bigwedge^0 \mathcal{M}^2$ such that

$$\mathcal{F}|_{\mathcal{M}\otimes\mathcal{M}} = \mathcal{F}^{pr}_{iso(\mathbb{R}^4)} \tag{8.10}$$

but

$$\mathcal{F}|_{\mathcal{M}_\infty\otimes\mathcal{M}} = \mathcal{F}|_{\mathcal{M}\otimes\mathcal{M}_\infty} = \mathcal{F}|_{\mathcal{M}_\infty\otimes\mathcal{M}_\infty} = 0 \tag{8.11}$$

where $\mathcal{F}^{pr}_{iso(\mathbb{R}^4)}$ is the four-dimensional analogue to the kernel defined in Section 6.5.1. This kernel can be written as

$$\mathcal{F}^{pr}_{iso(\mathbb{R}^4)} : (r, \theta, r_*, \theta_*) \mapsto -\exp[\ -k^2(\ r^2 + r_*^2 - 2rr_*cos(\theta - \theta_*)\ )\ ] \tag{8.12}$$

using equivalent, uniformly valid, polar coordinates (r, θ) and (r_*, θ_*) for $\mathcal{M}$.

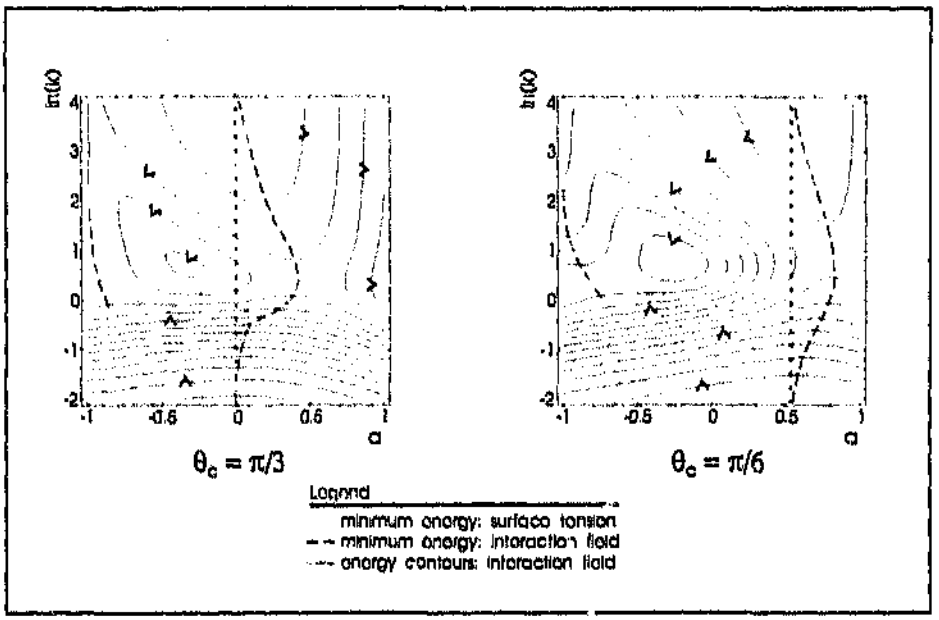


Figure 8.2: Energy contours for Plateau's problem.

ox, oy and oz representing the fluid membranes in Plateau's analysis, and $\mathcal{M}_\beta$ consisting of the complement of $\mathcal{M}_\alpha$ in $\mathcal{M}$. The thickness of the fluid membranes is assumed to be irrelevant to the analysis. The extremities x, y and z of the segments comprising $\mathcal{M}_\alpha$ are constrained to lie on the circle xyz of unit radius, while the point o is free to move along the diameter passing through x. The displacement of o from the centre of the circle xyz is given by the parameter a as shown, while the locations of the extremities are governed by the angle θ_c.

By simple trigonometry, the included angle γ between the segments oy and oz is related to the geometric parameter a by

$$a = \cos\theta_c - \frac{\sin\theta_c}{\tan\frac{\gamma}{2}} \tag{8.9}$$

The fluid in $\mathcal{M}_\alpha$ is assumed to be subject to a pairwise interaction with itself, while the fluid in $\mathcal{M}_\beta$ is assumed not to be subject to any interactions. Define

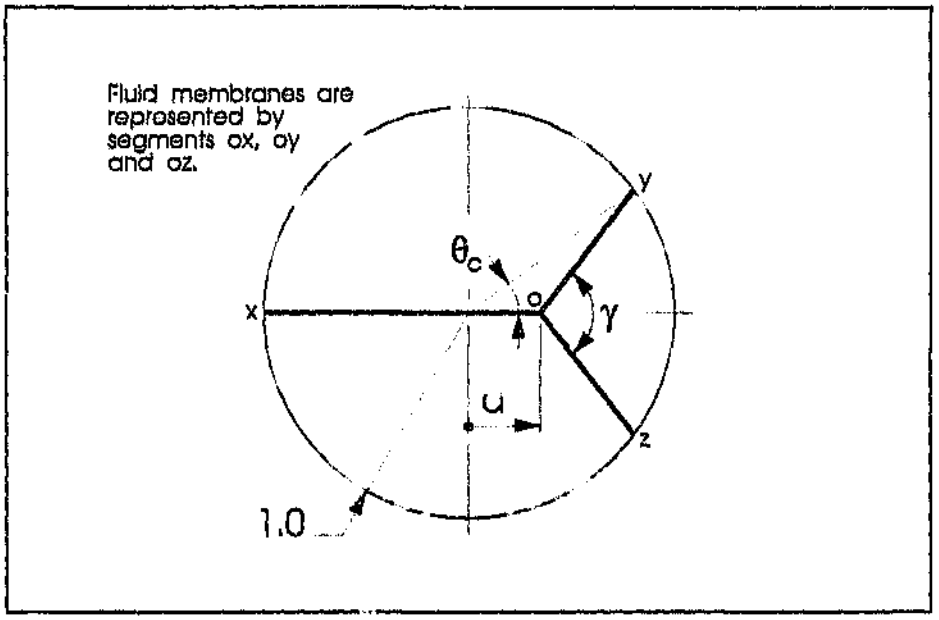


Figure 8.1: Geometry for Plateau's problem.

The equilibrium condition derived in this section will not be used again in this work, except indirectly in Chapter 11 and, by way of example, to determine the equilibrium configurations of the following well-known classical system.

8.3.1 Plateau's Problem

Plateau [68] showed, using surface tension theory, that three thin fluid membranes in stable equilibrium always intersect such that the angle between adjacent membranes is $2\pi/3$ radians. As a simple illustration of the use of the system energy to locate equilibrium configurations of a fluid system, Plateau's problem is re-analysed using the interaction field approach. Consider the geometry shown in Figure 8.1. Let the fluid domain $\mathcal{M}$ be isomorphic to $I\!R^2$ and let $\mathcal{M}$ consists of the two domains $\mathcal{M}_\alpha$, comprised of the three segments

[3]See for instance the text by Arnold [3].

bourhood surrounding the point about which the equations have been linearised. Hence the linearisation (7.42) of the equation of motion on the configuration manifold $\mathcal{D}$ of the fluid system can be used to locate points on the configuration manifold which correspond to equilibrium configurations of the fluid system.

If the point $\vartheta \in \mathcal{D}$ is to correspond to a static equilibrium configuration of the fluid system, then, from (8.1) and (7.10), $\ddot{a}_i(0) = 0$ and $\dot{a}_i(0) = 0$ for all $i = 1, \ldots, p$. (It has already been assumed during the derivation of the linearised equations that $a_i(0) = 0$ fi. all $i = 1, \ldots, p$).

Hence, from (7.42) and definition (7.31), the fluid system is in static equilibrium if the force coefficients

$$\begin{aligned} P^j &= \tfrac{d}{da_j} V \\ &= 0 \end{aligned} \tag{8.6}$$

for all $j = 1, \ldots, p$. Equivalently, if $(da_1, \ldots, da_p)$ is the dual basis on the configuration manifold induced by the coordinates in the neighbourhood of ϑ, then

$$\mathrm{d}V = \tfrac{d}{da_1} V da_1 + \cdots + \tfrac{d}{da_p} V da_p \tag{8.7}$$

on interpreting the potential energy V as a section of $\bigwedge^0 \mathcal{D}$, and hence (8.6) is satisfied if $\mathrm{d}V = 0$.

Define the set of critical points of some $f \triangleright \bigwedge^0 M$ as

$$C_M(f) = \{x \in M : (\mathrm{d}f)|_x = 0\} \tag{8.8}$$

There is thus a 1:1 correspondence between the elements of the set $C_{\mathcal{D}}(V)$ of critical points of the potential energy $V \triangleright \bigwedge^0 \mathcal{D}$ and the set of equilibrium configurations of the fluid system. It can be shown[3] that those critical points where the energy is locally a minimum correspond to *stable* equilibrium configurations of the system.

experimental confirmation of the predictions of any system which claims to be a model of the physical world, and the generality of analysis based on a few explicit examples, should however be borne in mind throughout the analysis of the following chapters.

8.2 Equilibrium via Equation on $\mathcal{M}$

A simple substitution of (8.1) into the equation of motion (4.18) and the interfacial conditions (4.22) yields the conditions

$$\mathrm{d}(p - \Phi)|_{\mathcal{M}_\alpha} = 0 \tag{8.2}$$

subject to interfacial conditions

$$(p|_{\partial\mathcal{M}_\alpha})|_{\partial_{\alpha\beta}} - (p|_{\partial\mathcal{M}_\beta})|_{\partial_{\alpha\beta}} = 0 \tag{8.3}$$

for all $\alpha, \beta \in \mathcal{Z}$, for static equilibrium of the fluid system. Integrating (8.2) over $\mathcal{M}_\alpha$ gives

$$(p - \Phi)|_{\mathcal{M}_\alpha} = c_\alpha \tag{8.4}$$

where $c_\alpha \in \mathbb{R}$, and on substituting this result into (8.3) the interfacial conditions can be rewritten as

$$(\Phi|_{\partial\mathcal{M}_\alpha})|_{\partial_{\alpha\beta}} - (\Phi|_{\partial\mathcal{M}_\beta})|_{\partial_{\alpha\beta}} = c_{\alpha\beta} \tag{8.5}$$

for all $\alpha, \beta \in \mathcal{Z}$, where $c_{\alpha\beta} = c_\beta - c_\alpha \in \mathbb{R}$.

8.3 Equilibrium via System Energy

Any linearisation of the dynamic equations for the fluid system is a valid description of the dynamics of the system confined to a limitingly small neigh-

equilibrium configurations of the fluid system which is based on the theory of transformation groups. Although this method fails in a number of circumstances, it is perhaps the most elegant approach and yields some of the deepest insights into the characteristics of static equilibria in fluid systems which are subjected to internally-generated interactions.

8.1.1 Physical Interpretation

In this chapter the first comparisons of the predictions of the interaction field model derived in the previous chapters with the predictions of the classical surface tension theory are made. The interaction field model is treated as a fluid dynamic model which has potentially greater validity than merely as a surrogate for the classical surface tension based approach, and all predictions of the interaction field model, even where they contradict the results of the classical approach, or where no direct equivalence between the interaction field model and the surface tension based approach can be shown, are analysed for their physical relevance and plausibility.

It is essential that agreement between the predictions of the classical approach and those of the interaction field based approach should exist in some suitable limit if the validity of the interaction field based approach as a physical fluid model is to be supported. This follows since the applicability of the surface tension model to a wide variety of physical systems is experimentally well established[1] . The more interesting cases occur of course where the predictions of the two approaches do not agree. It is known that the classical approach is not without its problems[2], and in these cases it is argued that the interaction field approach yields the more valid prediction, or at least that the interaction field model shows additional physical insight into the fluid system under analysis, and hence is a description of the physics of macroscopic fluid systems with interfaces which is superior in certain respects to the classical surface tension model. The comments made in Chapter 1, regarding the necessity for

[1] See for instance the review article by Bogy [8] mentioned in Chapter 1.

[2] See the discussion of Chapter 1 and also of Section 9.2.

Chapter 8

Equilibrium

8.1 Overview

In this chapter, the conditions under which a fluid system governed by an interaction field model exists in static equilibrium, that is, in a state where

$$\begin{aligned} \nu &\equiv 0 \\ \dot{\nu} &\equiv 0 \end{aligned} \tag{8.1}$$

for all time, are investigated. Three approaches are adopted. In Section 8.2, the equations of motion derived in Chapter 4 are used to derive the conditions for static equilibrium in terms of the balance between the pressure and interaction potential in the fluid induced by the compatibility conditions derived in Section 4.5 on the fluid properties on the internal domain boundaries. This result is used in Chapter 9 when the static predictions of the interaction field theory are reconciled with those of the classical surface tension theory. A second approach, described in Section 8.3, follows directly from the modal analysis developed in Chapter 7, and is perhaps the most powerful and generally-applicable method for determining the equilibrium states of the fluid system. The third approach, described in Section 8.4, uses the linearised equations developed in Section 7.8 to derive a method for enumerating the static

8.6.1 Symmetries of the Fluid Interfaces

Let $\mathcal{M}$ be isomorphic to $\mathbb{R}^3$ and let (x, y, z) and (x_*, y_*, z_*) be identical, globally-valid, rectangular coordinates for $\mathcal{M}$. These coordinates can then be used to generate rectangular coordinates (x, y, z, x_*, y_*, z_*) for the product structure $\mathcal{M}^2$.

Any symmetry field $\tau \triangleright \mathcal{X}\mathcal{M}$ of the occupation functions on $\mathcal{M}$ can be written using the dual basis (dx, dy, dz) induced by the coordinates (x, y, z) for $\mathcal{M}$ as

$$\tau = u(x, y, z)dx + v(x, y, z)dy + w(x, y, z)dz \tag{8.61}$$

or equivalently, using the dual basis (dx_*, dy_*, dz_*) induced by the coordinates (x_*, y_*, z_*) as

$$\tau = u(x_*, y_*, z_*)dx_* + v(x_*, y_*, z_*)dy_* + w(x_*, y_*, z_*)dz_* \tag{8.62}$$

where $u, v, w \triangleright \bigwedge^0 \mathcal{M}$. The physically realisable fields $\tau \oplus \tau \triangleright \mathcal{X}\mathcal{M}^2$ must then be of the form

$$\begin{aligned} \tau \oplus \tau \;=\; & u(x, y, z)dx + v(x, y, z)dy + w(x, y, z)dz \\ & + u(x_*, y_*, z_*)dx_* + v(x_*, y_*, z_*)dy_* + w(x_*, y_*, z_*)dz_* \end{aligned} \tag{8.63}$$

when written in terms of the dual basis $(dx, dy, dz, dx_*, dy_*, dz_*)$ induced by the coordinates (x, y, z, x_*, y_*, z_*) on $\mathcal{M}^2$.

Since

$$\begin{aligned} \mathrm{d}\mathcal{F}_{\mathrm{Iso}(\mathbb{R}^6)} \;=\; & 2\,[\,(x - x_*)(dx - dx_*) + (y - y_*)(dy - dy_*) \\ & + (z - z_*)(dz - dz_*)\,]\,\mathcal{F}'_{\mathrm{Iso}(\mathbb{R}^6)}(r) \end{aligned} \tag{8.64}$$

where

$$r = (x - x_*)^2 + (y - y_*)^2 + (z - z_*)^2 \tag{8.65}$$

If two fields $r_1, r_2 \triangleright \mathcal{KM}$ can be found such that r_1 and r_2 both satisfy the equilibrium equation (8.59) and generate a foliation of $\mathcal{M}$ as described in Section 8.5.1, then (8.59) implies that

$$(\mathrm{d}\Phi)|_{\mathfrak{I}_r \dot{\mathcal{M}}_\alpha} = 0 \tag{8.60}$$

for all $\alpha \in \dot{\mathcal{Z}}$, and hence condition (8.57) required for (8.56) to hold will be satisfied if $\mathcal{M}_\alpha \in \mathcal{M}_{\mathcal{Z}}$. Thus any configuration of the fluid system in which the locations of the boundaries of the minimal fragmentation of $\mathcal{M}$ are related to the symmetries of the interaction kernels via (3.53) will satisfy the equilibrium condition (8.14) expressed in terms of the distortion modes available to the fluid system.

8.6 Equilibrium Configurations

Although the techniques developed in the previous sections can account for arbitrary-order interactions between the particles of the fluid, and can account for an arbitrary number of species present in the fluid, there is little point in introducing unnecessary complexity into the present work for its own sake, especially if added complexity results in no additional understanding of the behaviour of the fluid system.

The principal goal of the following analysis is to derive some initial results demonstrating the validity of the assertions made in Section 6.5 regarding the properties required of the interaction kernels in an interaction field model which, under suitable conditions, yields equivalent predictions to those of the classical surface tension model. To this end, it is assumed in the examples which follow that the interaction between the various domains comprising the fluid can be characterised by a kernel defined on $\mathcal{M}^2$, and that each of the kernels for the various species present in the system has the same symmetry group as the kernel $\mathcal{F}_{\mathrm{iso}\,\mathbb{R}^6}$ defined in Section 6.3.1.

Since $\upsilon_n^j \triangleright \mathcal{KM}^n$ is constructible from $\upsilon^j \triangleright \mathcal{KM}$, projection via the fibre integral yields

$$\int_{\mathcal{M}^n} *\upsilon_n^j(\mathcal{F}) = n \int_{\mathcal{M}} *\upsilon^j \wedge \mathrm{d}\Phi \tag{8.54}$$

for all $j = 1, \ldots, p$, where

$$\Phi = \oint_{\mathcal{M}^{n-1}} *\mathcal{F} \tag{8.55}$$

is the potential on $\mathcal{M}$ defined in Section 6.4.6. Hence, using Stokes' theorem and the incompressibility condition (7.12) on the distortion modes, the equilibrium condition on $\mathcal{M}^n$ can be projected onto $\mathcal{M}$ as

$$\sum_{\alpha \in \mathcal{Z}} n \int_{\partial \mathcal{M}_\alpha} \Phi \, *\upsilon^j = 0 \tag{8.56}$$

From the incompressibility condition (7.13), Equation 8.56 is satisfied[8] by any of the distortion modes $\upsilon^j \triangleright \mathcal{KM}$, $j = 1, \ldots, p$, if

$$(\mathrm{d}\Phi)|_{\partial \mathcal{M}_\alpha} = 0 \tag{8.57}$$

for all $\alpha \in \mathcal{Z}$, or in other words if Φ is constant on each of the domain boundaries.

Again using the fibre integral, if $\tau_n \triangleright \mathcal{KM}^n$ is constructible from $\tau \triangleright \mathcal{KM}$, then the generalisation of the equilibrium condition (8.14) in terms of the arbitrary operator τ_n can be transferred from $\mathcal{M}^n$ to $\mathcal{M}$ as

$$\begin{aligned} \int_{\mathcal{M}^n} *\tau_n(\mathcal{F}) &= \tau \int_{\mathcal{M}} *\tau \wedge \mathrm{d}\Phi \\ &= 0 \end{aligned} \tag{8.58}$$

implying in the strong case that

$$*\tau \wedge \mathrm{d}\Phi = 0 \tag{8.59}$$

[8] Note that this is a stronger condition than the interfacial condition derived in Section 8.2 from the equation of motion on $\mathcal{M}$.

the minimal fragmentation of $\mathcal{M}$, *each* interface $\partial\mathcal{M}_a$, $a \in \hat{\mathcal{Z}}$ in the $\mathcal{M}$-complete fragmentation $\mathcal{M}_{\hat{\mathcal{Z}}}$ of the equilibrium configuration of the fluid must be isomorphic to one of the surfaces $\mathcal{S} \in \mathsf{F}_{r_1,r_2}\mathcal{M}$.

The tangency condition on the two operators can be expressed equivalently as

$$(*(r_1 \wedge r_2))|_{\mathcal{S}} = 0 \tag{8.50}$$

for every surface $\mathcal{S} \in \mathsf{F}_{r_1,r_2}\mathcal{M}$. Since a foliation of $\mathcal{M}$ can be defined pointwise as

$$\mathsf{F}_{r_1,r_2}\mathcal{M} = \{ \mathcal{S}_a = \{x \in \mathcal{M} : S_{r_1,r_2}(x) = a\} : a \in I\!R \} \tag{8.51}$$

for some $S_{r_1,r_2} : \mathcal{M} \rightarrow I\!R$, it follows on interpreting S_{r_1,r_2} as a section of $\bigwedge^0\mathcal{M}$ that

$$(\mathsf{d}S_{r_1,r_2})|_{\mathcal{S}_a} = 0 \tag{8.52}$$

for all $a \in I\!R$. Combining (8.52) and (8.50),

$$*(r_1 \wedge r_2) = k\, \mathsf{d}S_{r_1,r_2} \tag{8.53}$$

where $k \in \bigwedge^0\mathcal{M}$. Hence for the operators r_1 and r_2 to define a fibration of $\mathcal{M}$, the Hodge star of their exterior product should be an *exact* section of $\bigwedge^1\mathcal{M}$, at least up to multiplication by a scalar field on $\mathcal{M}$.

8.5.5 Distortion Modes

In this section the analysis of Section 8.4.3 is completed by determining the conditions under which the equilibrium equation (8.14) is satisfied by the distortion modes w_a^j, $j = 1, \ldots, p$, in addition to being satisfied by the symmetry operators of the interaction kernels.

This form of the equilibrium condition accounts both for the requirement that the equilibrium configuration of the system be physically realisable and for the restriction on the symmetries of the equilibrium configuration of the system imposed by the condition that r_n be an operator in the group $\mathcal{Q}$.

The set of fields r satisfying (8.48) can be determined algebraically and can be arranged to form a basis of generators for the symmetry group of the elements of $\mathcal{O}_{\underline{\mathcal{Z}}}$. The elements of $\mathcal{O}_{\underline{\mathcal{Z}}}$ can then be found by inspection or by solving the characteristic equations generated by this basis, again using the standard procedure described by Ibragimov. Finally, the $\mathcal{M}$-complete fragmentation $\mathcal{M}_{\underline{\mathcal{Z}}}$ isomorphic to $\mathcal{O}_{\underline{\mathcal{Z}}}$ via the coalescence sum and the composition product defined in Section 6.3.4 and Section 6.2.3 can be obtained, and this set defines the equilibrium configuration of the fluid system in terms of the locations of the various domains comprising the system.

8.5.4 Solutions Defined by Foliations of $\mathcal{M}$

If $\mathcal{M}$ is three-dimensional, then the interfaces in the fluid are all two-surfaces[7]. In this section, a technique for deriving these two-surfaces from the physically-realisable symmetry operators obtained via the analysis of the previous section is developed.

Let $r_1, r_2 \triangleright \mathcal{K}\mathcal{M}$ be two physically-realisable symmetry operators satisfying (8.48). Under suitable restrictions on the forms of these operators, a unique foliation $\mathsf{F}_{r_1,r_2}\mathcal{M}$ of $\mathcal{M}$ can be defined by the operators r_1 and r_2 by requiring that both operators be everywhere tangential to the surfaces $\mathcal{S}$ of the foliation. Then, since (8.45) projects onto $\mathcal{M}$ as

$$(*r)|_{\partial\mathcal{M}_\alpha} = 0 \tag{8.49}$$

for all $\alpha \in \tilde{\mathcal{Z}}$, or in other words since any operator $r \triangleright \mathcal{K}\mathcal{M}$ satisfying the equilibrium conditions must be tangent to the boundaries of the elements of

[7] A similar analysis to that presented here follows in the case where $\mathcal{M}$ is two-dimensional, except that only a single symmetry operator is required to define the foliation on $\mathcal{M}$.

since the elements of $[\,\mathcal{F}_*\,]$ are a subset of the elements of $[\,\mathcal{F}\,]$. Rewriting (8.16) using the exterior derivative (see Section 1.9.2) allows (8.16) to be interpreted as requiring

$$*r_n \wedge \mathrm{d}\mathcal{F}_*^i = 0 \tag{8.17}$$

for all $i = 1, \ldots, r'$.

8.5.3 Physically Realisable Solutions

It should be noted that the minimal fragmentations of $\mathcal{M}^n$ generated by procedure described in Section 8.5.1 do not all correspond to some physically realisable configuration of the fluid system. In particular, those fragmentations which do not yield a set of occupation functions which are constructible via the composition product from some set of occupation functions on $\mathcal{M}$ (see Section 6.3.4) must be rejected since they do not correspond to any physical configuration of the fluid on $\mathcal{M}$. Assume that the set $\{\tilde{\rho}_k\}_{k=1,\ldots,k'}$ of occupation functions on the minimal fragmentation of $\mathcal{M}^n$ introduced in Section 8.5.1 is indeed physically realisable, or in other words is constructible via the composition product from some set $\mathcal{O}_{\mathcal{Z}}$ of occupation functions on $\mathcal{M}$.

It is assumed that the group of symmetries $\mathcal{Q}$ common to all the interaction kernels $\mathcal{F}_\sigma$, $\sigma \in \mathcal{Z}^n$ is known *a priori*, or that the group $\mathcal{Q}$ can be determined by performing a standard analysis of the invariants of the set of interaction kernels. Let $\mathcal{F}_\mathcal{Q} \triangleright \bigwedge^0 \mathcal{M}^n$ be any form which is invariant under the action of $\mathcal{Q}$. $\mathcal{F}_\mathcal{Q}$ can be found by solving the characteristic equations generated by a basis of generators of $\mathcal{Q}$ as described in the standard reference by Ibragimov [35].

Let $r \triangleright \mathcal{X}\mathcal{M}$ be everywhere tangential to the discontinuities in the occupation functions in $\mathcal{O}_{\mathcal{Z}}$. Then, following the analysis of Section 6.3.5, r induces the field $r \oplus r_{n-1} \triangleright \mathcal{X}\mathcal{M}^n$ on defining $r_1 \equiv r$. On recursion, $r \oplus r_{n-1} = r \oplus \cdots \oplus r$ (n factors), allowing the equilibrium condition (8.17) to be recast as

$$*(r \oplus \cdots \oplus r) \wedge \mathrm{d}\mathcal{F}_\mathcal{Q} = 0 \tag{8.18}$$

8.5 Structure of Solutions

8.5.1 Minimal Fragmentation

Let $\{\tilde{\mathcal{M}}^n_k\}_{k=1,\ldots,k'}$ be a $\mathcal{M}^n$-complete fragmentation such that $(\mathrm{d}\rho'_i)|_{\tilde{\mathcal{M}}^n_k} = 0$ for all $i = 1,\ldots,r'$ and $k = 1,\ldots,k'$. Let the fragmentation be minimal in the sense that each $x \in \partial\tilde{\mathcal{M}}^n_k$ is a point at which one of the forms ρ'_i, $i = 1,\ldots,r'$, is discontinuous. Such a fragmentation is constructed from any arbitrary complete fragmentation of $\mathcal{M}^n$ by using the coalescence sum defined in Section 6.3.4 to combine elements of the arbitrary fragmentation on either sides of any internal boundary at which there is no discontinuity in any of the fields ρ'_i, $i = 1,\ldots,r'$. In Section 6.3.4 it was shown that the coalescence sum induces an isomorphism between occupation functions and fragmentations. Let $\{\tilde{\rho}_k\}_{k=1,\ldots,k'}$ be the set of occupation functions on the minimal fragmentation generated from the set $\mathcal{O}^n_2$ of occupation functions on the arbitrary fragmentation via this isomorphism. Definition (8.38) can then be rewritten in terms of the matrix of occupation functions on the minimal fragmentation $[\,\tilde{\rho}\,] = [\,\tilde{\rho}_k\,]_{(k')}$ as

$$[\,\rho'\,] = [\,\tilde{\rho}\,][\,\tilde{\mathbf{L}}'\,] \tag{8.44}$$

for some matrix $[\,\tilde{\mathbf{L}}'\,] \in [\,I\!R\,]^{k'}_{r'}$.

8.5.2 Geometric Interpretation

Equation 8.37 can be interpreted as requiring $\mathbf{r}_n$ to be parallel to all discontinuities in ρ'_i, $i = 1,\ldots,r'$, that is

$$(*\boldsymbol{\tau}_n)|_{\partial\tilde{\mathcal{M}}^n_k} = 0 \tag{8.45}$$

for all $k = 1,\ldots,k'$. Equation 8.40 implies that

$$\boldsymbol{\tau}_n(\,[\,\mathcal{F}_n\,]\,) = [\,\bar{0}\,] \tag{8.46}$$

The set of equations (8.37) and (8.40) is now in soluble form. If $\mathcal{F}^\sigma$, $\sigma \in \mathcal{Z}^n$ admits the group of point symmetries $\mathcal{Q}_\sigma$, then any τ_n satisfying (8.40) is an infinitesimal operator of the group

$$\mathcal{Q} = \bigcap_{\sigma \in \mathcal{Z}^n} \mathcal{Q}_\sigma \tag{8.41}$$

where the intersection is to be interpreted as the largest subgroup common to the groups $\mathcal{Q}_\sigma$, $\sigma \in \mathcal{Z}^n$.

The static equilibrium configurations of the fluid system are then obtained by defining the elements of $[\,\rho'\,]$ in terms of the occupation functions on $\mathcal{M}^n$ such that (8.37) is satisfied, in other words such that all the elements of $[\,\rho'\,]$ are invariant under the symmetry group $\mathcal{Q}$.

Note that

$$\begin{aligned} \rho_\sigma \mathcal{F}^\sigma &= [\,\rho\,]\,[\,\mathcal{F}\,]^T \\ &= [\,\rho\,]\,[\,\mathbf{L}'\,][\,\mathcal{F}_*\,]^T \\ &= [\,\rho'\,]\,[\,\mathcal{F}_*\,]^T \end{aligned} \tag{8.42}$$

using definitions (8.33) and (8.38). Hence (8.14) can be recast as

$$\int_{\mathcal{M}^n} *\upsilon_n^j(\,\rho_i'\,\mathcal{F}_*^i\,) = 0\,, \quad i = 1, \ldots, r' \tag{8.43}$$

if $[\,\mathcal{F}_*\,] = [\,\mathcal{F}_*^i\,]_{(r')}$ and $[\,\rho'\,] = [\,\rho_i'\,]_{(r')}$. It has been shown that (8.43) and hence (8.14) is satisfied if υ_n^j is an operator corresponding to a point symmetry of the interaction kernels, but it remains to be shown that this equation is satisfied by each of the distortion modes υ_n^j, $j = 1, \ldots, p$ defined in Section 7.6 if the set of occupation functions ρ_i', $i = 1, \ldots r'$ is suitably defined. This demonstration requires the additional analysis of the following section, and is deferred to Section 8.5.5.

If in addition it is assumed that no element of $[\mathcal{F}]$ can be written as a linear combination[6] of the elements of $[\rho]$, the matrix $[\mathbf{L}]$ in (8.28) can be written in the form

$$[\mathbf{L}] = \begin{bmatrix} \mathbf{L}' & | & 0 \\ \cdots & & \cdots \\ 0 & | & \mathbf{I} \end{bmatrix} \tag{8.35}$$

using matrix structure notation, where $[\mathbf{I}] \in [\mathbb{R}]^r_r$ is the identity matrix, and the elements in the upper right and lower left partitions of $[\mathbf{L}]$ are all zero.

Hence

$$[\tilde{\mathcal{G}}]\,[\mathbf{L}] = \left[\rho \;\vdots\; \tilde{\mathcal{F}}\right] \begin{bmatrix} \mathbf{L}' & | & 0 \\ \cdots & & \cdots \\ 0 & | & \mathbf{I} \end{bmatrix} \tag{8.36}$$

The structure of $[\mathbf{L}]$ allows (8.36) to be decomposed by quadrants as

$$r_n([\rho']) = [\bar{0}] \tag{8.37}$$

where

$$\begin{aligned} [\rho'] &= [\rho][\mathbf{L}'] \\ &\in [\mathcal{M}^n, \mathbb{R}]_{r'} \end{aligned} \tag{8.38}$$

and

$$r_n([\tilde{\mathcal{F}}][\mathbf{I}]) = [\bar{0}] \tag{8.39}$$

or equivalently, since the column transpose commutes with r_n and $r_n([\mathbf{I}]) = [\bar{0}]$,

$$r_n([\mathcal{F}]) = [\bar{0}] \tag{8.40}$$

[6]The case where there is linear dependence between the elements of these matrices may even be considered to be pathological.

and

$$[\mathcal{F}_{\cdot}\quad [\mathcal{F}^{\sigma_j}]_{(r)} \tag{8.30}$$

It can be shown very simply that $[\rho]$ is linearly independent: Let $[\lambda] = [\lambda_{\sigma_j}]_{(r)} \in [\mathbb{R}]_r$. Since each $\rho_{\sigma_j} \triangleright \Lambda^0\mathcal{M}^n$ in $\mathcal{M}^n_{\mathcal{Z}}$ has non-zero support, and since $\mathcal{M}^n_{\mathcal{Z}}$ is $\mathcal{M}^n$-complete (see Section 6.3.2), $\text{supp}\rho_{\sigma_i} \cap \text{supp}\rho_{\sigma_j} = \mathcal{M}^n_{\sigma_i} \cap \mathcal{M}^n_{\sigma_j} = \{\emptyset\}$ if $i \neq j$ (see Section 1.9.3). Thus

$$\begin{aligned} [\lambda][\rho] &= [\bar{0}] \\ &\Rightarrow \lambda_\sigma = 0 \; \forall \, \sigma \in \mathcal{Z}^n \\ &\Rightarrow [\lambda] = [0] \end{aligned} \tag{8.31}$$

and hence $[\rho]$ is linearly independent. Using matrix structure notation[5]

$$[G] = \left[\begin{array}{c:c} \mathcal{F} & \tilde{\rho} \end{array}\right] \tag{8.32}$$

and from the previous argument, if $[G]$ is not linearly independent, then it is because there is a linear dependence between the elements of the matrix $[\mathcal{F}]$. If there is linear dependence, then r' elements of $[\mathcal{F}]$ can be written as a linear combination of the remaining $r - r'$ elements. Hence

$$[\mathcal{F}]^T = [L'][\mathcal{F}_*]^T \tag{8.33}$$

where the *interaction matrix* for the fluid system $[L'] \in [\mathbb{R}]^r_{r-r'}$ and $[\mathcal{F}_*] \in [\mathcal{M}^n, \mathbb{R}]^*_{r-r'}$, giving

$$\begin{aligned} [G^*] &= \left[\begin{array}{c:c} \mathcal{F}_* & \tilde{\rho} \end{array}\right] \\ &\in [\mathcal{M}^n, \mathbb{R}]^*_{2r-r'} \end{aligned} \tag{8.34}$$

in matrix structure notation.

[5]The convention is used that dotted lines denote partitions of the matrix into equal numbers of rows or columns, while solid lines denote less symmetric partitions of the matrix structure.

and

$$[\mathcal{F}] = [\mathcal{F}^{\sigma_j}]_{(r)} \tag{8.30}$$

It can be shown very simply that $[\rho]$ is linearly independent: Let $[\lambda] = [\lambda_{\sigma_j}]_{(r)} \in [\mathbb{R}]_r$. Since each $\rho_{\sigma_i} \triangleright \Lambda^0\mathcal{M}^n$ in $\mathcal{M}^n_{\mathcal{Z}}$ has non-zero support, and since $\mathcal{M}^n_{\mathcal{Z}}$ is $\mathcal{M}^n$-complete (see Section 6.3.2), $\text{supp}\rho_{\sigma_i} \cap \text{supp}\rho_{\sigma_j} = \mathcal{M}^n_{\sigma_i} \cap \mathcal{M}^n_{\sigma_j} = \{\emptyset\}$ if $i \neq j$ (see Section 1.9.3). Thus

$$\begin{aligned} [\lambda][\rho] &= [\bar{0}] \\ &\Rightarrow \lambda_\sigma = 0 \; \forall \, \sigma \in \mathcal{Z}^n \\ &\Rightarrow [\lambda] = [0] \end{aligned} \tag{8.31}$$

and hence $[\rho]$ is linearly independent. Using matrix structure notation[5]

$$[G] = \left[\mathcal{F} \;\vdots\; \tilde{\rho} \right] \tag{8.32}$$

and from the previous argument, if $[G]$ is not linearly independent, then it is because there is a linear dependence between the elements of the matrix $[\mathcal{F}]$. If there is linear dependence, then r' elements of $[\mathcal{F}]$ can be written as a linear combination of the remaining $r - r'$ elements. Hence

$$[\mathcal{F}]^T = [L'][\mathcal{F}_*]^T \tag{8.33}$$

where the *interaction matrix* for the fluid system $[L'] \in [\mathbb{R}]^r_{r-r'}$ and $[\mathcal{F}_*] \in [\mathcal{M}^n, \mathbb{R}]^*_{r-r'}$, giving

$$\begin{aligned} [G^*] &= \left[\mathcal{F}_* \;|\; \tilde{\rho} \right] \\ &\in [\mathcal{M}^n, \mathbb{R}]^*_{2r-r'} \end{aligned} \tag{8.34}$$

in matrix structure notation.

[5]The convention is used that dotted lines denote partitions of the matrix into equal numbers of rows or columns, while solid lines denote less symmetric partitions of the matrix structure.

8.4.3 Solving the Strong Equilibrium Equation

Equation 8.16 can be rewritten in the matrix notation described above as

$$r_n([\underleftarrow{\mathbf{G}}]) [\mathbf{G}]^T = [\bar{0}] \tag{8.24}$$

if $[\mathbf{G}] = [G_j]_{(2r)} \in [\mathcal{M}^n, I\!R]_{2r}$ is defined in terms of its elements by

$$G_j = \begin{cases} \mathcal{F}^{\sigma_j} & \text{if } 1 \leq j \leq r \\ \rho_{\sigma_{2r-j+1}} & \text{if } r < j \leq 2r \end{cases} \tag{8.25}$$

where $\mathcal{Z}^n = \{\sigma_1, \ldots, \sigma_r\}$.

In general, $[\mathbf{G}]$ is not linearly independent. It is possible however to write

$$[\mathbf{G}]^T = [\mathbf{L}] [\mathbf{G}^*]^T \tag{8.26}$$

for some $[\mathbf{L}] \in [I\!R]^{2r}_m$ and some linearly independent $[\mathbf{G}^*] \in [\mathcal{M}^n, I\!R]^*_m$, where $m \leq 2r$.

Hence

$$\begin{aligned} r_n([\underleftarrow{\mathbf{G}}]) [\mathbf{G}]^T &= r_n([\underleftarrow{\mathbf{G}}]) [\mathbf{L}][\mathbf{G}^*]^T \\ &= r_n([\underleftarrow{\mathbf{G}}] [\mathbf{L}]) [\mathbf{G}^*]^T \end{aligned} \tag{8.27}$$

since $r_n([\mathbf{L}]) = [\bar{0}]$.

Then, since $[\mathbf{G}^*]$ is linearly independent, the solution to (8.24) is given by the matrix equation

$$r_n([\underleftarrow{\mathbf{G}}] [\mathbf{L}]) = [\bar{0}] \tag{8.28}$$

Further progress can be made by examining the structure of the matrix $[\mathbf{L}]$ which follows from the definitions of the occupation functions and the interaction kernels on $\mathcal{M}^n$. Define

$$[\rho] = [\rho_{\sigma_j}]_{(r)} \tag{8.29}$$

functions on $\mathcal{M}^2$ obtained in the stronger case *is* constructible as a subset of $\mathcal{O}_{\mathcal{Z}}^3$, where $\mathcal{O}_{\mathcal{Z}} = \{\rho_\alpha, \rho_\beta, \rho_\gamma\}$. The equilibrium configurations of the fluid in this case are thus obtained when $\partial\mathcal{M}_\alpha$, $\partial\mathcal{M}_\beta$ and $\partial\mathcal{M}_\gamma$ are all isomorphic to elements of one of the fibrations of $\mathcal{M}$ enumerated in Section 8.6.1.

8.6.3 Properties of the Strong Solutions

The process of obtaining the static equilibrium configurations of the fluid system via the strong form (8.15) of the equilibrium condition (8.11) has some interesting properties:

Structural Instability

Consider a fluid system in equilibrium within the domain $\mathcal{M}$. The introduction of a new domain anywhere within $\mathcal{M}$ containing a fluid with a different symmetry group to that of the fluids in the original system will, according to the analysis of Section 8.4.3, collapse the set of equilibrium configurations available to the fluid system into the smaller set generated by the intersection of the symmetry group of the original fluids and that of the introduced fluid. The fact that the domain of introduced fluid may be arbitrarily small, or be located at an arbitrarily large distance from the domain boundaries in the original system, shows that the solutions generated by the strong procedure are in fact structurally unstable in the sense that the solutions may be destroyed by arbitrarily small perturbations to the structure of the fluid system.

Incompleteness of Solutions

It is easily shown by example that the strong form (8.15) does not in general yield all the static equilibria of any given fluid system.

Consider the following example. Let the fluid domain $\mathcal{M}$ be isomorphic to $\mathbb{R}^2$, and let $\mathcal{M}$ be comprised of two distinct domains such that $\mathcal{Z} = \{\alpha, \beta\}$. Let

In this example let there be three distinct (that is, linearly independent) interactions: the self-interactions between the fluids in the domains $\mathcal{M}_\beta$ and $\mathcal{M}_\gamma$ characterised by interaction kernels $\mathcal{F}^{\beta\beta}$ and $\mathcal{F}^{\gamma\gamma}$, and the mutual interaction between the fluids in the domains $\mathcal{M}_\beta$ and $\mathcal{M}_\gamma$, characterised by the interaction kernel $\mathcal{F}^{\beta\gamma}$. Assume each interaction kernel to have the same symmetries as $\mathcal{F}_{\mathrm{iso}(\mathbb{R}^6)}$. Thus

$$[\,\mathcal{F}_*\,] = \left[\begin{array}{ccc} \mathcal{F}^{\beta\beta} & \mathcal{F}^{\beta\gamma} & \mathcal{F}^{\gamma\gamma} \end{array}\right] \tag{8.101}$$

and the interaction matrix is

$$[\,\mathbf{L}'\,] = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \tag{8.102}$$

Then

$$[\,\rho'\,] = [\,\rho\,][\,\mathbf{L}'\,] = \left[\begin{array}{ccc} \rho_{\beta\otimes\beta} & \rho_{\beta\otimes\gamma} + \rho_{\gamma\otimes\beta} & \rho_{\gamma\otimes\gamma} \end{array}\right] \tag{8.103}$$

Now the set of occupation functions $\{\rho_{\beta\otimes\beta}, \rho_{\beta\otimes\gamma} + \rho_{\gamma\otimes\beta}, \rho_{\gamma\otimes\gamma}\}$ on $\mathcal{M}^2$ derived from the elements of $[\,\rho'\,]$ is not constructible from any set of occupation functions on $\mathcal{M}$. Note though that if (8.37) holds in the stronger case where

$$[\,\rho'\,] = \left[\begin{array}{cccc} \rho_{\beta\otimes\beta} & \rho_{\beta\otimes\gamma} & \rho_{\gamma\otimes\beta} & \rho_{\gamma\otimes\gamma} \end{array}\right] \tag{8.104}$$

then, since the differential operator commutes with summation, (8.37) holds for $[\,\rho'\,]$ given by (8.103). The set $\{\rho_{\beta\otimes\beta}, \rho_{\beta\otimes\gamma}, \rho_{\gamma\otimes\beta}, \rho_{\gamma\otimes\gamma}\}$ of occupation

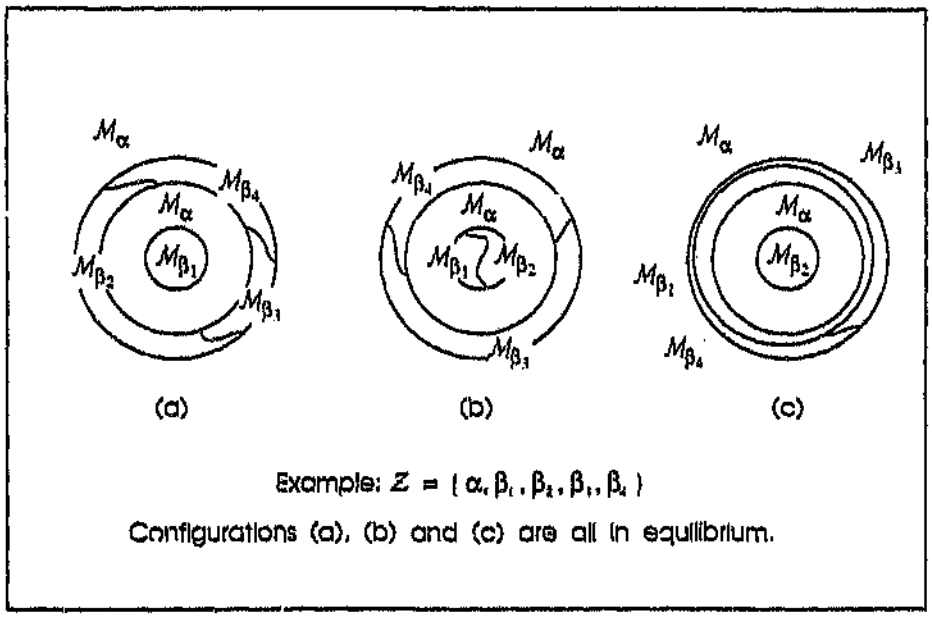


Figure 8.6: Schematic of multi-domain equilibrium.

Example 8

In this example it is demonstrated how certain combinations of interaction kernel can lead to an equilibrium configuration on $\mathcal{M}^2$ which is not physically realisable since the occupation functions on $\mathcal{M}^2$ satisfying the equilibrium equations cannot be constructed from a set of occupation functions on $\mathcal{M}$.

Assume the fluid system on $\mathcal{M}$ to be comprised of three distinct domains, such that $\mathcal{Z} = \{\alpha, \beta, \gamma\}$. Again, assuming pairwise interactions to be the highest order interactions present in the system, the analysis takes place on $\mathcal{M}^2$ with

$$\mathcal{Z}^2 = \{\alpha\otimes\alpha, \alpha\otimes\beta, \alpha\otimes\gamma, \beta\otimes\alpha, \beta\otimes\beta, \beta\otimes\gamma, \gamma\otimes\alpha, \gamma\otimes\beta, \gamma\otimes\gamma\} \tag{8.99}$$

such that

$$[\,\rho\,] = \left[\rho_{\alpha\otimes\alpha}\ \rho_{\alpha\otimes\beta}\ \rho_{\alpha\otimes\gamma}\ \rho_{\beta\otimes\alpha}\ \rho_{\beta\otimes\beta}\ \rho_{\beta\otimes\gamma}\ \rho_{\gamma\otimes\alpha}\ \rho_{\gamma\otimes\beta}\ \rho_{\gamma\otimes\gamma}\right] \tag{8.100}$$

From (8.38)

$$[\,\rho'\,] = [\,\rho\,][\,\mathbf{L}'\,] = \left[\rho_{\beta_1 \otimes \beta_1} + \cdots + \rho_{\beta_r \otimes \beta_r}\right] \tag{8.97}$$

In this case the set of occupation functions $\{\rho_{\beta_1\beta_1} + \cdots + \rho_{\beta_r\beta_r}\}$ on $\mathcal{M}^4$ obtained as the set of elements of the matrix $[\,\rho'\,]$ is physically constructible from the set $\mathcal{O}_{\underline{\beta}} = \{\rho_\alpha, \rho_{\beta_1 \oplus \cdots \oplus \beta_r}\}$ of occupation functions on $\mathcal{M}$ since

$$\begin{aligned}
\{\rho_{\beta_1 \otimes \beta_1} + \cdots + \rho_{\beta_r \otimes \beta_r}\} &= \{\rho_{\beta_1} \otimes \rho_{\beta_1} + \cdots + \rho_{\beta_r} \otimes \rho_{\beta_r}\} \\
&= \{(\rho_{\beta_1} + \cdots + \rho_{\beta_r}) \otimes (\rho_{\beta_1} + \cdots + \rho_{\beta_r})\} \\
&= \{(\rho_{\beta_1} \oplus \cdots \oplus \rho_{\beta_r}) \otimes (\rho_{\beta_1} \oplus \cdots \oplus \rho_{\beta_r})\} \\
&= \{(\rho_{\beta_1 \oplus \cdots \oplus \beta_r}) \otimes (\rho_{\beta_1 \oplus \cdots \oplus \beta_r})\} \\
&\subset \{\rho_\alpha, \rho_{\beta_1 \oplus \cdots \oplus \beta_r}\}^2
\end{aligned} \tag{8.98}$$

using the coalescence sum defined in Section 6.3.4. Performing the isomorphic operations on the set $\mathcal{M}_{\underline{\beta}}$ yields $\mathcal{M}_{\underline{\beta}} = \{\mathcal{M}_\alpha, \mathcal{M}_{\beta_1 \oplus \cdots \oplus \beta_r}\}$. Hence from Section 8.5.3, $\partial\mathcal{M}_\alpha$ and $\partial\mathcal{M}_{\beta_1 \oplus \cdots \oplus \beta_r}$ are both isomorphic to one of the surfaces enumerated in Section 8.6.1. This equilibrium case is illustrated schematically in Figure 8.6, where three of the various equilibrium configurations available to an example system (on assuming a particular concentric-sphere equilibrium configuration of the interfaces $\partial\mathcal{M}_\alpha$ and $\partial\mathcal{M}_{\beta_1 \oplus \cdots \oplus \beta_r}$) are shown.

Note that one way of interpreting the interactions present in this example is that the fluids in the domains $\mathcal{M}_{\beta_1}, \ldots \mathcal{M}_{\beta_r}$ are all of the same species, since the self-interactions and the mutual interactions between the domains are all governed by the same interaction kernel. It is of course to be expected that an analysis for multiple domains of fluid all containing the same species of fluid should reduce, as has been demonstrated, to an analysis for a single domain of fluid consisting of the union of the multiple domains.

Example 2

The following example has relevance in Chapter 11, where the equilibria of systems having arbitrary topology at some stage during their evolution, rather than the simple topologies of the configurations enumerated in Section 8.6.1, are considered. The fluid system on $\mathcal{M}$ is assumed to be comprised of $r+1$ distinct domains. Let $\mathcal{Z} = \{\alpha, \beta_1, \ldots, \beta_r\}$, hence $\mathcal{O}_{\mathcal{Z}} = \{\rho_\alpha, \rho_{\beta_1}, \ldots, \rho_{\beta_r}\}$ and $\mathcal{M}_{\mathcal{Z}} = \{\mathcal{M}_\alpha, \mathcal{M}_{\beta_1}, \ldots, \mathcal{M}_{\beta_r}\}$. As in the previous example, interactions of greater than pairwise order are assumed not to be present. Again analysis takes place on $\mathcal{M}^2$, but this time with

$$\mathcal{Z}^2 = \{\alpha\otimes\alpha, \alpha\otimes\beta_1, \ldots, \alpha\otimes\beta_r, \beta_1\otimes\beta_1, \ldots, \beta_r\otimes\beta_r\} \tag{8.93}$$

and hence

$$[\,\rho\,] = \left[\begin{array}{ccccccc} \rho_{\alpha\otimes\alpha} & \rho_{\alpha\otimes\beta_1} & \cdots & \rho_{\alpha\otimes\beta_r} & \rho_{\beta_1\otimes\beta_1} & \cdots & \rho_{\beta_r\otimes\beta_r} \end{array}\right] \tag{8.94}$$

Again assume the interactions to be characterised by the single interaction kernel $\mathcal{F}_{\mathrm{iso}(\mathbb{R}^n)}$ such that

$$[\,\mathcal{F}_*\,] = \left[\,\mathcal{F}_{\mathrm{iso}(\mathbb{R}^n)}\,\right] \tag{8.95}$$

but in this case let the interactions in the fluid consist of both self-interactions within, and mutual interactions between, all the domains $\mathcal{M}_{\beta_1}, \ldots, \mathcal{M}_{\beta_r}$ such that the interaction matrix for the system must be written as

$$[\,\Gamma'\,] = \left[\begin{array}{c} 0 \\ 0_1 \\ \vdots \\ 0_r \\ 1_1 \\ \vdots \\ 1_r \end{array}\right] \tag{8.96}$$

Assume that the interactions within the fluid are characterised by the single interaction kernel $\mathcal{F}_{\mathrm{iso}(\mathbb{R}^0)}$. Then

$$[\,\mathcal{F}_{\bullet}\,] = \left[\,\mathcal{F}_{\mathrm{iso}(\mathbb{R}^0)}\,\right] \tag{8.89}$$

Furthermore let the interaction consist solely of the interaction of the fluid in $\mathcal{M}_\alpha$ with itself. This assumption allows the interaction matrix for the fluid system to be written as

$$[\,\mathbf{L}'\,] = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{8.90}$$

Hence from (8.38)

$$[\,\rho'\,] = [\,\rho\,][\,\mathbf{L}'\,] = \left[\rho_{\alpha\circledast\alpha}\right] \tag{8.91}$$

Note though that the set $\{\rho_{\alpha\circledast\alpha}\}$ of occupation functions on $\mathcal{M}^2$ obtained as the set of elements of the matrix $[\,\rho'\,]$ is physically constructible from the set $\mathcal{O}_{\mathfrak{F}} = \{\rho_\alpha, \rho_\beta\}$ of occupation functions on $\mathcal{M}$, since

$$\begin{aligned} \{\rho_{\alpha\circledast\alpha}\} &= \{\rho_\alpha \circledast \rho_\alpha\} \\ &\subset \{\rho_\alpha, \rho_\beta\}^2 \end{aligned} \tag{8.92}$$

from the definition of the occupation functions on $\mathcal{M}^2$ in terms of the occupation functions on $\mathcal{M}$. The set $\mathcal{M}_{\mathfrak{F}} = \{\mathcal{M}_\alpha, \mathcal{M}_\beta\}$ is of course the $\mathcal{M}$-complete fragmentation which is isomorphic to $\mathcal{O}_{\mathfrak{F}}$. The analysis of Section 8.5.4 thus implies that $\partial\mathcal{M}_\alpha$ and $\partial_*\mathcal{M}_\beta$ are both isomorphic to one of the surfaces enumerated in Section 8.6.1. This prototype example shows that the interfaces in a two-domain system in equilibrium, where one of the domains is subject to a pairwise self-interaction, can adopt only those geometries which are permitted by the foliations of $\mathcal{M}$ induced by the symmetries of the interaction kernels.

8.6.2 Examples

Some examples of the application of the symmetry techniques developed in the previous sections to the enumeration of the static equilibrium configurations of a fluid with internal interactions of the form assumed in Section 8.6 are now given. The first example illustrates the simplest case of two domains of fluid in contact along a common interface. The second illustrates the case where there are multiple domains of fluid, but all share the same interaction kernel. Example 3 illustrates the case where the form of the interaction kernels limits the number of equilibrium configurations available to the fluid since a matrix of occupation functions is generated which cannot be constructed via the composition sum from the matrix of occupation functions on the fluid domain $\mathcal{M}$.

Example 1

In this example the simplest case where the fluid domain $\mathcal{M}$ is composed of two distinct domains is analysed. Let $\mathcal{Z} = \{\alpha, \beta\}$. The set of occupation functions on $\mathcal{M}$ is thus $\mathcal{O}_{\mathcal{Z}} = \{\rho_\alpha, \rho_\beta\}$, corresponding to a $\mathcal{M}$-complete fragmentation $\mathcal{M}_{\mathcal{Z}} = \{\mathcal{M}_\alpha, \mathcal{M}_\beta\}$. Since the highest order interactions within the fluid are assumed to be pairwise interactions, the equilibrium analysis takes place on $\mathcal{M}^2$. Then

$$\mathcal{Z}^2 = \{\alpha\otimes\alpha, \alpha\otimes\beta, \beta\otimes\alpha, \beta\otimes\beta\} \tag{8.86}$$

and hence

$$\begin{aligned}\mathcal{O}_{\mathcal{Z}}^2 &= \{\rho_\alpha\otimes\rho_\alpha, \rho_\alpha\otimes\rho_\beta, \rho_\beta\otimes\rho_\beta, \rho_\beta\otimes\rho_\alpha\} \\ &= \{\rho_{\alpha\otimes\alpha}, \rho_{\alpha\otimes\beta}, \rho_{\beta\otimes\alpha}, \rho_{\beta\otimes\beta}\}\end{aligned} \tag{8.87}$$

Hence from (8.29)

$$[\,\rho\,] = \begin{bmatrix} \rho_{\alpha\otimes\alpha} & \rho_{\alpha\otimes\beta} & \rho_{\beta\otimes\alpha} & \rho_{\beta\otimes\beta} \end{bmatrix} \tag{8.88}$$

8.6.2 Examples

Some examples of the application of the symmetry techniques developed in the previous sections to the enumeration of the static equilibrium configurations of a fluid with internal interactions of the form assumed in Section 8.6 are now given. The first example illustrates the simplest case of two domains of fluid in contact along a common interface. The second illustrates the case where there are multiple domains of fluid, but all share the same interaction kernel. Example 3 illustrates the case where the form of the interaction kernels limits the number of equilibrium configurations available to the fluid since a matrix of occupation functions is generated which cannot be constructed via the composition sum from the matrix of occupation functions on the fluid domain $\mathcal{M}$.

Example 1

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$$\mathcal{Z}^2 = \{\alpha\otimes\alpha, \alpha\otimes\beta, \beta\otimes\alpha, \beta\otimes\beta\} \tag{8.86}$$

and hence

$$\begin{aligned} \mathcal{O}_{\mathcal{Z}}^2 &= \{\rho_\alpha\otimes\rho_\alpha, \rho_\alpha\otimes\rho_\beta, \rho_\beta\otimes\rho_\beta, \rho_\beta\otimes\rho_\alpha\} \\ &= \{\rho_{\alpha\otimes\alpha}, \rho_{\alpha\otimes\beta}, \rho_{\beta\otimes\alpha}, \rho_{\beta\otimes\beta}\} \end{aligned} \tag{8.87}$$

Hence from (8.29)

$$[\,\rho\,] = \begin{bmatrix} \rho_{\alpha\otimes\alpha} & \rho_{\alpha\otimes\beta} & \rho_{\beta\otimes\alpha} & \rho_{\beta\otimes\beta} \end{bmatrix} \tag{8.88}$$

case 3:

$$rw - ta \neq 0,\ u = 0,\ v = 0,\ b = 0,\ c = 0 \tag{8.80}$$

in which case (8.73) reduces to

$$XdX + YdY = 0 \tag{8.81}$$

on the elements of the foliation. This expression solves to give

$$\mathsf{F}_{r_1,r_2}\mathcal{M} = \{\ \mathcal{S}_q = \{(X,Y,Z) : X^2 + Y^2 = q\} : q \in I\!R\ \} \tag{8.82}$$

in other words a set of arbitrarily located, coaxial, cylinders all with the same, arbitrary, orientation within $\mathcal{M}$.

case 4:

$$r \neq 0,\ t = 0,\ u = kb,\ v = kc,\ w = 0,\ b \neq 0 \text{ or } c \neq 0 \tag{8.83}$$

in which case (8.73) reduces to

$$XdX + YdY + (Z + k)dZ = 0 \tag{8.84}$$

on the elements of the foliation. This expression solves to give

$$\mathsf{F}_{r_1,r_2}\mathcal{M} =$$

$$\{\ \mathcal{S}_q = \{(X,Y,Z) : X^2 + Y^2 + (Z + k)^2 = q\} : q \in I\!R\ \} \tag{8.85}$$

which is a set of concentric spheres, arbitrarily located within $\mathcal{M}$.

Degenerate Solution

Another solution to the equilibrium equations of course occurs when there are no domain boundaries within the fluid at all and hence (8.45) or (8.49) is identically satisfied by any symmetry operator satisfying (8.48).

The exact two-forms defining the foliations of $\mathcal{M}$ can then be obtained by arranging the parameters r, t, u, v, w, a, b, c such that groups of terms in the above expression cancel. The various combinations of parameters which yield exact two-forms are

case 1:

$$r \neq 0,\ t = 0,\ u \neq 0 \text{ or } v \neq 0,\ w = 0,\ b = 0,\ c = 0 \tag{8.74}$$

in which case (8.73) reduces to

$$dZ = 0 \tag{8.75}$$

on the elements of the foliation. This expression solves to give

$$\mathsf{F}_{r_1,r_2}\mathcal{M} = \{\ S_a = \{(X,Y,Z) : Z = q\} : q \in I\!R\ \} \tag{8.76}$$

that is, a set of parallel planes with arbitrary orientation within $\mathcal{M}$.

case 2:

$$r = 0,\ t \neq 0,\ u \neq 0 \text{ or } v \neq 0,\ a = 0,\ b = 0,\ c = 0 \tag{8.77}$$

In which case (8.73) reduces to

$$v dX - u dY = 0 \tag{8.78}$$

on the elements of the foliation. This expression solves to give

$$\mathsf{F}_{r_1,r_2}\mathcal{M} =$$

$$\{\ \mathcal{S}_a = \{(X,Y,Z) : vX - uY = q, u, v \in I\!R\} : q \in I\!R\ \} \tag{8.79}$$

which is again a set of parallel planes with arbitrary orientation within $\mathcal{M}$.

and uniform scaling of the coordinates (x, y, z) for $\mathcal{M}$, the actual transformation being dependent of course on the parameters $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{u}$, $\vec{v}$ and $\vec{w}$. Of particular importance is that the map from the parameter space to the cotangent bundle $\mathsf{T}^*\mathcal{M}$ induced by the coordinate transform is a surjection, and hence the symmetry operators of the system are linear combinations of a rotation operator centred upon an *arbitrary* position $x \in \mathcal{M}$ and oriented along an *arbitrary* axis defined by the one-form η, and a translation operator directed along the same axis.

Foliations Yielding Equilibrium Configurations

It remains to establish the foliations of $\mathcal{M}$ which are generated by pairs of symmetry operators of the form given in (8.70), and hence to enumerate the equilibrium configurations of the interfaces in the minimal fragmentation of the fluid system.

To simplify the resulting algebra, let one of the pair of symmetry operators be defined in canonical coordinates:

$$r_1 = r(XdY - YdX) + tdZ \tag{8.71}$$

The other operator will then have the general form

$$\begin{aligned} r_2 \;=\; & udX + vdY + wdZ \\ & + a(XdY - YdX) + b(ZdX - XdZ \qquad\quad ZdY - YdZ) \end{aligned} \tag{8.72}$$

when expressed in the same canonical coordinates. The Hodge star of the exterior product of the two operators is then

$$\begin{aligned} *(r_1 \wedge r_2) = & \\ & ((rw - ta)X - rX(bX + cY) - tv - tcZ \qquad\quad)dX \\ & +((rw - ta)Y - rY(bX + cY) + tu + tbZ \qquad\quad)dY \\ & +(\qquad\quad -rZ(bX + cY) \qquad\quad -r(uX + vY))dZ \end{aligned} \tag{8.73}$$

and

$$\mathcal{F}'_{\mathrm{iso}(\mathbb{R}^6)}(r) = \frac{\partial}{\partial r}\mathcal{F}_{\mathrm{iso}(\mathbb{R}^6)}(r) \tag{8.66}$$

Equation 8.48 becomes, after some simplification,

$$\begin{aligned}
&(x - x_*)(u(x,y,z) - u(x_*,y_*,z_*)) \\
&+(y - y_*)(v(x,y,z) - v(x_*,y_*,z_*)) \\
&+(z - z_*)(w(x,y,z) - w(x_*,y_*,z_*)) \;=\; 0
\end{aligned} \tag{8.67}$$

This equation has the non-trivial solution

$$\begin{aligned}
u(x,y,z) &= \bar{u} + \bar{a}y + \bar{b}z \\
v(x,y,z) &= \bar{v} - \bar{a}x + \bar{c}z \\
w(x,y,z) &= \bar{w} - \bar{b}x - \bar{c}y
\end{aligned} \tag{8.68}$$

by inspection or by substitution of a polynomial trial solution, where $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{u}$, $\bar{v}$ and $\bar{w}$ are arbitrary real constants. Thus any symmetry field $r \triangleright \mathcal{XM}$ of the configuration of the fluid system at equilibrium is of the form

$$\begin{aligned}
r \;=\; &\bar{u}dx + \bar{v}dy + \bar{w}dz \\
&+\bar{a}(xdy - ydx) + \bar{b}(zdx - xdz) + \bar{c}(zdy - ydz)
\end{aligned} \tag{8.69}$$

This is not however the most useful form in which to analyse the symmetries of the equilibrium configurations of the fluid system. It is easy (but somewhat tedious) to show that any field of the form of (8.69) can be brought into the canonical form

$$r(\eta, x, \bar{r}, \bar{l}) = \bar{r}(XdY - YdX) + \bar{l}dZ \tag{8.70}$$

for some one-form $\eta = dZ \in \mathsf{T}^*_x\mathcal{M}$ and constant $\bar{r}$ and $\bar{l}$. The coordinates (X, Y, Z), and hence η and x, are obtained by a suitable translation, rotation

(x, y, z, x_*, y_*, z_*) is a system of globally valid coordinates for $\mathcal{M}^2$.

Assume the fluid domain $\mathcal{M}$ to be divided into two domains $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$, each of infinite extent, by the interface $\partial_{\alpha\beta} = \partial\mathcal{M}_\alpha \cap \partial\mathcal{M}_\beta$. If

$$\partial_{\alpha\beta} = \{(x, y, z) : z = Z(x, y)\} \tag{9.1}$$

(with an equivalent definition in terms of the coordinates (x_*, y_*, z_*)) then

$$\begin{aligned} \mathcal{M}_\alpha &= \{(x, y, z) : z > Z(x, y)\} \\ \mathcal{M}_\beta &= \{(x, y, z) : z < Z(x, y)\} \end{aligned} \tag{9.2}$$

and hence, using the analysis of Section 6.3.3, the occupation functions[3]

$$\begin{aligned} \rho_\alpha(x, y, z) &= \begin{cases} 1 & \text{if } z > Z(x, y) \\ 0 & \text{if } z < Z(x, y) \end{cases} \\ \rho_\beta(x, y, z) &= \begin{cases} 1 & \text{if } z < Z(x, y) \\ 0 & \text{if } z > Z(x, y) \end{cases} \\ &\triangleright \ \Lambda^0\mathcal{M} \end{aligned} \tag{9.3}$$

can be defined. These in turn induce the occupation functions

$$\begin{aligned} \rho_{\alpha\otimes\alpha} &= \rho_\alpha \otimes \rho_\alpha &&= \rho_\alpha(x, y, z)\, \rho_\alpha(x_*, y_*, z_*) \\ \rho_{\alpha\otimes\beta} &= \rho_\alpha \otimes \rho_\beta &&= \rho_\alpha(x, y, z)\, \rho_\beta(x_*, y_*, z_*) \\ \rho_{\beta\otimes\alpha} &= \rho_\beta \otimes \rho_\alpha &&= \rho_\beta(x, y, z)\, \rho_\alpha(x_*, y_*, z_*) \\ \rho_{\beta\otimes\beta} &= \rho_\beta \otimes \rho_\beta &&= \rho_\beta(x, y, z)\, \rho_\beta(x_*, y_*, z_*) \\ &\triangleright \ \Lambda^0\mathcal{M}^2 \end{aligned} \tag{9.4}$$

and so on for each $\mathcal{M}^n$, $n > 2$.

This topological structure for $\mathcal{M}$ is of course by no means general, and in Section 9.4.3 the effects of one of the fluid domains having finite, rather than infinite, extent is investigated.

[3]The abuse of notation committed by confusing a map and its image is continued throughout this chapter as it simplifies the coordinate-based presentation of the following sections.

a consequence of bounded range of the interaction kernel.

Physical systems exhibiting the effects of long-range interactions are relatively common. It is known, for example, that a force can be generated between two nearby water droplets by virtue of their long-range mutual interaction, and a similar situation occurs in thin film flows. Traditionally these effects have been accounted for by the addition of extra terms into a classical surface tension model for the behaviour of the system. Introduction of the so-called disjoining pressure, beginning with the analysis of Derjaguin and Kusakov [21] and including the analyses of Ernoux and Davis [26] and Joosten [39], to account for the interaction between the opposing surfaces of a thin liquid film would appear to be an unnatural attempt to introduce some elements of the global structure of an interaction field type approach into the inherently local classical theory. The analysis of Ruckenstein and Jain [72] where the interactions between opposing surfaces *are* accounted for by a body-force type term rather than by a disjoining pressure goes some way to introducing some of the structure of the interaction field type approach. Unfortunately the integral dependence of the internal interactions on the overall configuration of the fluid as suggested by the analysis of Chapter 6 is not accounted for, and hence even this analysis fails to account properly for the global structure of the interactions within the fluid. The various enhancements to the classical surface tension theory proposed in the various analyses cited here all appear as somewhat unnatural attempts to shore up the failings of the classical approach[2] especially when, as shown in Chapter 6, a concise description of the effects described here can be obtained using an interaction field model with a suitably defined interaction kernel.

9.3 Preliminary Results

Let $\mathcal{M}$ be isomorphic to $I\!R^3$ and let (x, y, z) and (x_*, y_*, z_*) be equivalent, globally valid, rectangular coordinates for $\mathcal{M}$. Then, as in Section 8.6.1,

[2] Unfortunately the allusion to the work by Kuhn [46] can hardly be avoided.

termining fluid properties such as the pressure. The analysis of Section 5.4.2 even suggests that the interaction field model can be used to analyse systems where the curvature is not defined at any point on any of the interfaces in the fluid. If reconciliation between the predictions of the classical approach and the predictions of the interaction field model can be achieved in all situations where the classical model does not break down, then this continuation of the good behaviour of the interaction field model into situations which are inaccessible to the classical model yields strong evidence to suggest that the interaction field model is superior to the classical approach as a model for the dynamics of fluid systems which have the structure derived in Chapter 2.

The second failing of the classical theory as a consequence of its inherently local description of the interactions within the fluid system is its failure to predict effects resulting from long-range interaction between nearby, but possibly non-connected, domains of fluid. Effects of this type have already been encountered in the analysis of Plateau's problem in the previous chapter, where the interaction field approach shows effects which are not predicted by the classical surface tension theory, and in particular where it is claimed that some of the structure of the potential energy surface of the system is indeed an effect of the long-range character of the interactions within the system. The origin of the structural instability of the solutions to the strong form of the equilibrium equation derived in Section 8.4 can also be traced to the effects of finite range of the internal interactions on the behaviour of the fluid system. It is implicit in the derivation of the strong solutions to the equilibrium equation that any symmetry operator of the interaction kernel is uniquely defined at each point in $\mathcal{M}^n$ by (8.48). If the interaction kernel $\mathcal{F}_\mathcal{Q}$ has bounded range, however, in the sense that $\text{supp}(\mathcal{F}_\mathcal{Q}) \cap \mathcal{M}^n \subset \mathcal{M}^n$, then outside $\text{supp}(\mathcal{F}_\mathcal{Q})$ *any* operator satisfies (8.48). This implies that if the interaction kernel has bounded range then the introduction of a new domain into the fluid, following the analysis of Section 8.6.3, need not lead to changes to the equilibrium structure of the fluid except within some bounded neighbourhood of the new domain. This issue is addressed again in Chapter 11 where the possibility of coexistence of multiple, independent, equilibrium configurations within the fluid system is shown to be

of the generalised occupation functions has zero support, physically the most plausible reduction of Φ to a point property occurs if the species interaction kernels $\mathcal{F}^\sigma$, $\sigma \in \mathcal{Z}^n$ have vanishing support, and in addition $\text{supp}(\mathcal{F}^\sigma) \subset S$ for all $\sigma \in \mathcal{Z}^n$, where S is the projective symmetry set of $\mathcal{M}^n$ defined in Section 6.5.1. The physical implication of these requirements is that the interactions between the fluid particles must have infinitesimal range. This argument generalises the work by Kirkwood and Buff [42], where the surface tension is related to the internal interactions within the fluid under the *assumption* of vanishing range of the internal interactions. Here it has been argued that vanishing of the support of the interaction kernels is in fact *necessary* for the classical surface tension theory to exist.

The most obvious failing of the pointwise treatment of the interactions within the fluid by the classical surface tension model is found in the behaviour of the Young-Laplace equation. A primary problem with the Young-Laplace equation is that it becomes singular as the mean curvature of the interface becomes infinite, and hence the Young-Laplace equation can allow the pressure in the fluid to become infinite. Systems with large interfacial curvature are encountered frequently in nature, for example in aerosol droplets, at the intersections between the liquid films comprising foams, and during the breakup and coalescence of liquid droplets. The physical existence of pressure singularities in such systems is more than doubtful, and analysis using the Young-Laplace equation, given its inherently pathological behaviour, might quite easily lead to incorrect physical insights in these systems. Included here are, for example, the analyses of the instabilities of thin liquid films and threads of Rayleigh [71] and Taylor [77], [78] and many of the analyses of physical systems reviewed by Bogy [8] and more recently by Tjahjadi *et al.* [80].

The integral approach of the interaction field model contradicts the classical approach by implying that infinite curvature of the interfaces within the system should not lead to singularities, and, since the interaction potential is determined by integration over a volume form defined on the fluid domain, that in fact the curvature of the interfaces has only a secondary effect in de-

on the fluid properties within an infinitesimal neighbourhood of $\mathbf{x}$.

surface tension in the case where the characteristic dimension of the fluid system approaches the range of the internal interactions are obtained in the case of spherical interfacial geometry.

9.2 Limitations of the Classical Model

Various authors have noted that the measured surface tension at the interface between two fluids in a system characterised by a very small length scale differs, in some cases quite considerably, from the surface tension measured at the interface between the same two fluids in a system with larger length scale. This dependence has been examined by several authors, notably by Tolman [81] in his study of the variation of the surface tension of a spherical droplet as the radius of the droplet is varied. Stated differently, the existence of these systems shows that the surface tension is not only a function of the properties of the fluids at the interface, as is assumed by the classical model (see Section 1.2), but is a function also of the geometry of the system. The classical approach thus appears to incorporate a fundamental oversimplification of the interactions that occur within the fluid system. On the other hand, the interaction field approach accounts in a very natural way for those interfacial effects which are induced by the overall geometry of the fluid system. This follows since, through the integral relationships of Section 6.4.6, the interaction potential Φ, or equivalently the potential energy, for the system is dependent on the generalised occupation functions for the system, and hence is dependent on the configuration of the fluid system as a whole.

Direct reconciliation between the classical surface tension model and the interaction field approach is only possible if the potential Φ describing the interactions within the fluid system in the interaction field model can be made to be independent of the overall configuration of the fluid. This independence is achieved only in the case where the form of the integrand in definition (6.67) reduces Φ to a point property of the fluid[1]. In the general case where none

[1] That is, to a section of $\bigwedge^0 \mathcal{M}$ the value of which at any point $x \in \mathcal{M}$ is dependent only

Chapter 9

Reconciliation with Surface Tension Theory

9.1 Overview

The analysis of this chapter shows how the predictions of the classical surface tension model and the interaction field model can be reconciled when the fluid system is in static equilibrium. Reconciliation of the dynamic predictions of the two approaches is considered in Chapters 4 and 11. In Section 9.2 the introductory discussion of Section 1.4 on the limitations of the classical surface tension theory is continued, and various previous attempts to correct the shortcomings of the classical theory are reviewed. In Section 9.4.1 it is shown that an interaction field analogue to the classical surface tension can be defined if the interaction kernel exhibits sufficiently rapid decay, but that equivalence with the classical notion of the surface tension is only obtained in the limit as the range of the interaction kernel approaches zero. In Section 9.4.2 its is shown that even more rigorous constraints on the range of the internal interactions must be enforced for the interaction field approach to reproduce the Young-Laplace equation of the classical theory. Finally, in Section 9.4.3, results similar to those of Tolman [81] and Melrose [55] for the reduction of

each equilibrium configuration for the fluid system generated by the approach of Section 8.5.1 using the interaction kernels defined in Section 8.6, there is an isomorphic equilibrium configuration generated by the classical surface tension theory. This amounts to a partial validation of the assertions made in Section 6.5 that an interaction kernel on $\mathcal{M}^2$ with the symmetries derived in Section 6.5.1 can be used to create an interaction field model which reproduces the predictions of the classical theory.

Unfortunately the enumeration of all two-surfaces which have constant mean curvature and are embedded in $I\!R^3$ has not not yet been completed[10] and hence it cannot yet be determined whether the strong approach yields all equilibrium configurations of the fluid system, even in the absence of external fields.

8.7 Conclusion

It is extremely interesting that predictions of the equilibrium configurations of the fluid system which are equivalent to those of the classical theory are obtained from an interaction field model where only the symmetry properties, and not the exact forms, of the interaction kernel have been specified. This generality does not carry through to the dynamics of the fluid system: As is shown in Chapter 10, the dynamic predictions of the interaction field model are indeed affected by the exact form of the interaction kernel, and agreement with the classical theory is only obtained under restrictions on the form of the interaction kernel. In the following chapter, however, the analysis of this chapter is continued, and several general results concerning restrictions on the form of the interaction kernel which must be obeyed for reconciliation of the static predictions of the interaction field approach and the classical surface tension theory are derived.

[10] This situation exists despite considerable effort over nearly two centuries. See the historical review by Fomenko [28].

or translational invariance of potential equilibrium configurations would seem to be innocuous enough, however.

Additionally, in certain cases there may exist special solutions $\gamma_1, \gamma_2 \triangleright \mathcal{XM}$ of (8.48) which do not foliate $\mathcal{M}$, but nevertheless generate a *single* surface embedded within $\mathcal{M}$ which satisfies the equilibrium condition (8.49). The presence of these solutions is suggested by the various analyses of self-gravitating fluids interacting with external fields conducted by Lichtenstein [52], Greenspan [33] and Lyttleton [53] amongst others, and by the appearance of the volume of the fluid domains in expressions such as (8.112) for the interaction kernels on $\mathcal{M}^2$ when interactions with external fields are present.

Solutions in the absence of External Fields

From the analysis of Section 4.6, if there are no interactions with external fields, then the classical surface tension theory predicts the fluid system to be in static equilibrium if

$$(\mathrm{d}\overline{p})|_{\mathcal{M}_\alpha} = 0 \tag{8.123}$$

or equivalently if

$$\overline{p}|_{\mathcal{M}_\alpha} = c_\alpha \tag{8.124}$$

for all $\alpha \in \mathcal{Z}$ where $c_\alpha \in \mathbb{R}$ is a constant. Hence from the interfacial condition (4.30), if the surface tension is non-zero on the internal interfaces $\partial_{\alpha\beta}$, $\alpha, \beta \in \mathcal{Z}$ within the fluid, then the mean curvature of the internal interfaces

$$\kappa_{\alpha\beta} = \text{constant} \tag{8.125}$$

for all $\alpha, \beta \in \mathcal{Z}$.

Note that all the surfaces generated in Section 8.6.1 as equilibrium configurations for the fluid system do indeed have constant mean curvature. Hence, for

for all x and y. Hence the strong form of the equilibrium equation requires the external field G to be constant on $\mathcal{M}$, implying that the external field will have no effect on the dynamics of the system, or the symmetry operators of the equilibrium configurations to be of the form

$$\begin{aligned} r &= v(x,y)dy + v(x_*,y_*)dy_* \\ &\triangleright \mathcal{X}\mathcal{M}^2 \end{aligned} \tag{8.121}$$

or equivalently

$$\begin{aligned} r &= v(x,y)dy \\ &\triangleright \mathcal{X}\mathcal{M} \end{aligned} \tag{8.122}$$

implying that the fluid interfaces at equilibrium all lie parallel to the y coordinate. Hence the strong form of the equilibrium equation will omit solutions in a fairly general range of circumstances[9].

8.6.4 Generality of Strong-Form Solutions

Solutions in the Presence of External Fields

The analysis of Section 8.6.3 shows that the breaking of the symmetry of the equilibrium configurations of the fluid system which is precipitated by the introduction of the external field into the strong analysis is excessively severe in certain cases. This flaw in the strong approach might be corrected by defining a reduced strong form of the equilibrium equation, perhaps by eliminating some of the terms in the coordinate form of (8.48) which through certain symmetry arguments can be shown to disappear on integration over $\mathcal{M}^n$. An approach such as this would suffer from the obvious drawback that some *a priori* knowledge of the equilibrium configuration of the system, or at least of some of the symmetries of the equilibrium configuration, would be required for its implementation. Use of assumptions such as the axisymmetry

[9] A similar analysis can of course be performed when $\mathcal{M}$ has dimension greater than two.

$$= \int_{x_*=-a}^{a}\int_{y_*=-l(x_*)}^{l(x_*)}\int_{x=-a}^{a}\int_{y=-l(x)}^{l(x)} 2(\tfrac{1}{a^2}-\tfrac{1}{b^2})(x_*y - xy_*)dydxdy_*dx_*$$

$$= 0 \tag{8.115}$$

by direct integration where $l(x) = b\sqrt{1-x^2/a^2}$, and hence the symmetry operator of the ellipse *does* satisfy the weak form of the equilibrium equation.

Strong Form Solutions in an External Field

The analysis can be taken further by generalising the form of the pairwise and external interaction kernels: Let

$$\mathcal{F} : (x, y, x_*, y_*)$$

$$\mapsto \quad F((x-x_*)^2 + (y-y_*)^2) + G(x) + G(x_*) \tag{8.116}$$

A physically-constructible symmetry operator on $\mathcal{M}$ is of the form

$$r^2 = u(x,y)dx + v(x,y)dy + u(x_*,y_*)dx_* + v(x_*,y_*)dy_* \tag{8.117}$$

and hence the strong form of the equilibrium equation becomes

$$2F' \left[(x-x_*)(\, u(x,y) - u(x_*,y_*)\,) \right]$$

$$+2F' \left[(y-y_*)(\, v(x,y) - v(x_*,y_*)\,) \right]$$

$$+G'(x)\, u(x,y) + G'(x_*)\, u(x_*,y_*) \;=\; 0 \tag{8.118}$$

where $F' = \frac{\partial}{\partial r}F(r)$ and $G' = \frac{\partial}{\partial r}G(r)$. Consider the behaviour of this equation on the projective symmetry set (see Section 6.5.1) of $\mathcal{M}^2$ where $x = x_*$ and $y = y_*$. Then

$$2\, G'(x)\, u(x,y) = 0 \tag{8.119}$$

implying that

$$G'(x) = 0 \;\text{ or }\; u(x,y) = 0 \tag{8.120}$$

By inspection, the ellipse has the single symmetry operator

$$r = \tfrac{y}{b^2}dx - \tfrac{x}{a^2}dy \tag{8.111}$$

It is now shown that the strong-system analysis of Section 8.4.3 does not generate the equilibrium shown to exist by the direct integral approach. Using the analysis of Section 6.4.5, the pairwise interaction and the external field acting on the fluid in $\mathcal{M}_\alpha$ generate the single interaction kernel $\mathcal{F} \triangleright \bigwedge^0 \mathcal{M}^2$ such that

$$\begin{aligned} &\mathcal{F} : (x, y, x_*, y_*) \\ &\quad \mapsto \ -[\,(x - x_*)^2 + (y - y_*)^2\,] + \frac{k}{\pi ab}(x^2 + x_*^2) \end{aligned} \tag{8.112}$$

since $\int_{\mathcal{M}_\alpha} *1 = \pi ab$. The symmetry operator of the ellipse, transferred to $\mathcal{M}^2$ using the composition sum, becomes

$$\begin{aligned} r \ &= \ \tfrac{y}{b^2}dx - \tfrac{x}{a^2}dy + \tfrac{y_*}{b^2}dx_* - \tfrac{x_*}{a^2}dy_* \\ &\triangleright \ \mathcal{K}\mathcal{M}^2 \end{aligned} \tag{8.113}$$

giving

$$r(\mathcal{F}) = 2(\tfrac{1}{a^2} - \tfrac{1}{b^2})(x_* y - x y_*) \tag{8.114}$$

on substitution of (8.110). This strong form of the equilibrium equation has the solution $a = b$, implying $k = 0$ or in other words that the external field should be of zero strength.

The symmetry operator of the ellipse where $a \neq b$ is not a solution of the strong form of the equilibrium equation. It is easily shown that the omission of this case as an equilibrium configuration is a direct result of adopting the strong rather than the weak form of the equilibrium equation. Returning to the weak form of the equilibrium equation

$$\int_{\mathcal{M}^2} r(\rho_\alpha \otimes_\alpha \mathcal{F})$$

the fluid in $\mathcal{M}_\alpha$ be subject to a pairwise self-interaction such that the kernel for this interaction has the explicit form

$$\begin{aligned} \mathcal{F}^{\alpha\alpha} &= -[\,(x-x_*)^2+(y-y_*)^2\,] \\ &\triangleright \ \Lambda^0\mathcal{M}^2 \end{aligned} \tag{8.105}$$

where (x,y) and (x_*,y_*) are equivalent, globally valid rectangular coordinates for $\mathcal{M}$. In addition, let the fluid in $\mathcal{M}$, be subject to an interaction with an external field, such that the kernel for this interaction is

$$\begin{aligned} \mathcal{F}^{\alpha} &= kx^2 \\ &\triangleright \ \Lambda^0\mathcal{M} \end{aligned} \tag{8.106}$$

where $k \in \mathbb{R}$. The fluid in $\mathcal{M}_\beta$ will not interact with itself or with the fluid in $\mathcal{M}_\alpha$, nor will it interact with the external field.

If $\mathcal{M}_\alpha$ is the elliptical domain

$$\mathcal{M}_\alpha = \{(x,y) : \tfrac{x^2}{a^2}+\tfrac{y^2}{b^2} < 1,\ a,b \in \mathbb{R}\} \tag{8.107}$$

then the potential $\Phi \triangleright \Lambda^0\mathcal{M}$ can be obtained by direct integration:

$$\begin{aligned} \Phi &= \mathcal{F}^\alpha + \oint_{\mathcal{M}_\alpha} *\mathcal{F}^{\alpha\alpha} \\ &= kx^2 - \pi ab\,(\,\tfrac{a^2+b^2}{4}+x^2+y^2\,) \end{aligned} \tag{8.108}$$

on substituting (8.105), (8.106) and the limits on the integration implied by (8.107). From (8.5), for equilibrium $\Phi|_{\partial\mathcal{M}_\alpha}$ is constant since $\Phi|_{\partial\mathcal{M}_\beta} = 0$. Now

$$\Phi|_{\partial\mathcal{M}_\alpha} = kx^2 - \pi ab\,(\,\tfrac{a^2+b^2}{4}+b^2+(1-\tfrac{b^2}{a^2})\,x^2\,) \tag{8.109}$$

from (8.107), where $-a \leq x \leq a$, implying that

$$k = \pi ab\,(1-\tfrac{b^2}{a^2}) \tag{8.110}$$

for equilibrium of the system.

of $(0,0,0)$, but in (9.40) this expansion is integrated over the entire fluid domain. This contradiction is resolved if the support of the interaction kernel $\mathcal{F}$ is confined to a sufficiently small neighbourhood of $(0,0,0)$. Equivalently, since the Taylor expansion contains infinite powers of both x_* and y_*, $\mathcal{F}$ must converge to zero away from $(0,0,0)$ at a faster than polynomial rate[8] to ensure that the error in (9.40) and in the expressions to follow remains finite.

Exploiting the symmetries of the interaction kernel, x_* and y_* can be interchanged and hence both integrals on the right hand side of (9.40) are equal. Writing

$$I_{1/2}(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{F}(x,y,z,x_*,y_*,0)x_*^2 dy_* dx_* \tag{9.41}$$

allows (9.40) to be rewritten as

$$\Phi(x,y,z) = \phi_{1/2}(z) + (c_1+c_2)I_{1/2}(x,y,z) + \mathcal{E}(x,y,z) \tag{9.42}$$

Pressure in the Fluid

Substitution of (9.42) into the equilibrium equation (8.4) gives the pressure along the axis $x = y = 0$ as

$$p(0,0,z) - p_\infty = 0 \tag{9.43}$$

if $z > 0$, where $p_\infty \in \mathbb{R}$, while if $z < 0$,

$$\begin{aligned} p(0,0,z) - p_\infty &= \phi_{1/2}(0) + (c_1+c_2)I_{1/2}(0,0,0) \\ &\quad -(\ \phi_{1/2}(z) + (c_1+c_2)I_{1/2}(0,0,z)\ ) \end{aligned}$$

[8]This implies practically that $\mathrm{supp}(\mathcal{F}) \cap \mathcal{M}^2 \subset \mathcal{M}^3$, which is of course a more rigorous condition on the asymptotic behaviour of the interaction kernel than that implied by (9.7). In addition, if the error in (9.40) is to disappear, then the measure of $\mathrm{supp}(\mathcal{F})$ with respect to the volume form on $\mathcal{M}^2$ must in fact be vanishingly small. This is a similar condition to that derived in Section 9.2 for equivalence between the interaction field approach and the classical surface tension theory.

$$= \rho_\beta(x,y,z)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{c_1x_*^2+c_2y_*^2+\ldots}\mathcal{F}(x,y,z,x_*,y_*,z_*)dz_*dy_*dx_* \tag{9.37}$$

on substituting (9.10) and (9.4) with $Z(x,y)$ given by (9.35). Hence if $z > c_1x^2 + c_2y^2 + \ldots$ then $\Phi(x,y,z) = 0$, while if $z < c_1x^2 + c_2y^2 + \ldots$ then

$$\begin{aligned}\Phi(x,y,z) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{c_1x_*^2+c_2y_*^2+\ldots}\mathcal{F}(x,y,z,x_*,y_*,z_*)dz_*dy_*dx_* \\ &\quad +\phi_{1/2}(z)\end{aligned} \tag{9.38}$$

by decomposing the integration domain into the half-space $z \leq 0$ plus a second region between the plane $z = 0$ and the surface $z = c_1x^2 + c_2y^2 + \ldots$ Note that $\phi_{1/2}$ is the interaction potential generated by the system when the interface is planar (see Section 9.4), and hence the first term on the right hand side of (9.38) is the contribution to the interaction field from the distortion of the interface.

By Taylor expansion

$$\begin{aligned}\int_0^m f(x)dx &= F(m) - F(0) \\ &= (F(0) + f(0)m + O(m^2)) - F(0) \\ &= f(0)m + O(m^2)\end{aligned} \tag{9.39}$$

where $F(x) = \int f(x)dx$. Using this result to evaluate the integral with respect to z_*, the potential when $z < c_1x^2 + c_2y^2 + \ldots$ can be written as

$$\begin{aligned}\Phi(x,y,z) &= \phi_{1/2}(x,y,z) \\ &\quad +c_1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{F}(x,y,z,x_*,y_*,0)x_*^2dy_*dx_* \\ &\quad +c_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{F}(x,y,z,x_*,y_*,0)y_*^2dy_*dx_* \\ &\quad +\mathcal{E}(x,y,z)\end{aligned} \tag{9.40}$$

where $\mathcal{E}$ symbolically denotes the error introduced by the truncation of the integral. It would appear that a contradiction has been introduced: The Taylor expansion of the interfacial geometry is only valid within some neighbourhood

of the point with coordinates $(0,0,0)$ as

$$Z(x,y) = c_1x^2 + c_2y^2 + O(x^3,y^3) \tag{9.35}$$

where $c_1 = \frac{1}{2}\frac{\partial^2}{\partial x^2}Z|_{(0,0)}$ and $c_2 = \frac{1}{2}\frac{\partial^2}{\partial y^2}Z|_{(0,0)}$. This form relies on suitable orientation of the coordinates: The zero-order coefficient in the expansion is made zero by allowing the interface to pass through the point with coordinates $(0,0,0)$ and the coefficients of x and y are made zero by allowing the x- and y-axes to be tangent to the interface. Finally, the coefficient of xy is made zero by suitable orientation of the x- and y-axes. Note that $2c_1$ and $2c_2$ are the principal curvatures of the interface at $(0,0,0)$, and hence the mean curvature

$$\kappa_{\alpha\beta}(0,0) = 2(c_1 + c_2) \tag{9.36}$$

Reconciliation of Pressures

Reconciliation between the Young-Laplace equation of the classical surface tension theory and the analogue derived using the interaction field model requires the assumption that the pressure $\overline{p}$ acting on a surface $S \subset \mathcal{M}$ in the surface tension based approach is the sum of of two components which are considered separately in interaction field approach. The first component p of the pressure in the classical model is simply the pressure in the fluid required to maintain equilibrium against the action of the forces arising from the internal interaction, while the second contribution p_i to the pressure in the classical case results directly from the internal interaction across the surface S.

Interaction Potential in the Fluid

The determination of the interaction potential in the fluid follows closely the analysis of Section 9.4.1. Using (9.16), the potential in the fluid is

$$\Phi(x,y,z) \triangleright \Lambda^0\mathcal{M}$$

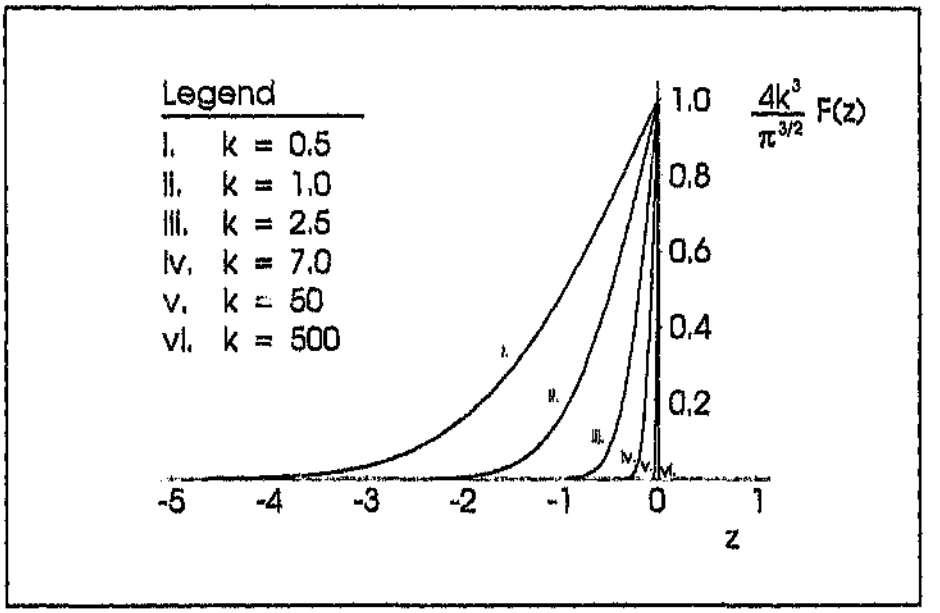


Figure 9.1: Force near a planar interface.

the classical analysis may differ from the interpretation given to the pressure in the interaction field based approach.

In this section it is shown that under suitable interpretation of the pressure within the fluid system, and under suitable restrictions on the range of the interaction kernel, the pressure change across the internal fluid interfaces predicted by the interaction field model indeed obeys the Young-Laplace equation (9.34).

The Curved Interface

Aga'n assume the configuration of the fluid system to be as described in Section 9.3. If the interface $\partial_{\alpha\beta}$ is sufficiently smooth then a Taylor series expansion can be used to write the equation of the interface in the neighbourhood

where $Q : \mathbb{R} \to \mathbb{R}$. Hence, from (9.27)

$$\sigma_p = \int_{-\infty}^{\infty} F(z)dz \tag{9.32}$$

where, as shown in the previous section, the integrand

$$F(z) = \begin{cases} 0 & \text{if } z > 0 \\ \frac{\pi^{3/2}}{4k^3}(\ \mathrm{erf}(kz) + 1\ ) & \text{if } z < 0 \end{cases} \tag{9.33}$$

can be interpreted as the force exerted across a dividing plane placed normal to the interface. This function is plotted in Figure 9.1 for various values of k. It is easily seen how, as the range of the interaction kernel is decreased, the force transmitted across the dividing plane effectively concentrates at the interface and hence how the mechanical view of the surface tension as an interfacial stress (see Section 1.2) is obtained in the limit as the range of the internal interactions within the fluid system goes to zero.

9.4.2 The Young-Laplace Equation

The classical surface tension model for the fluid configuration defined in Section 9.3 is based on the Young-Laplace equation

$$\begin{aligned} \Delta\overline{p} &= (\overline{p}|_{\partial\mathcal{M}_\alpha})|_{\partial_{\alpha\beta}} - (\overline{p}|_{\partial\mathcal{M}_\beta})|_{\partial_{\alpha\beta}} \\ &= \sigma_p \kappa_{\alpha\beta} \end{aligned} \tag{9.34}$$

for the change in pressure across the fluid interface $\partial_{\alpha\beta}$, where $\kappa_{\alpha\beta} \triangleright \bigwedge^0 \partial_{\alpha\beta}$ is the mean curvature of the interface. Since the surface tension in the classical approach is assumed to be a property of the fluids at the interface, and thus to be independent of the configuration of the fluid, assume σ_p to be equal to the equivalent surface tension for the system derived in the case of the planar interface. If the fluid system is in static equilibrium then, from (8.124), $\overline{p}|_{\partial\mathcal{M}_\alpha}$ and $\overline{p}|_{\partial\mathcal{M}_\beta}$ are in fact the pressures measured throughout the domains $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$. Overbars are used to signify that the interpretation of the pressure in

or, from definitions (9.9) and (9.10),

$$\sigma_p = \frac{1}{2}\int_{-\infty}^{0}\int_{-\infty}^{0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{F}(0,0,z,x_*,y_*,z_*)dx_*dy_*dz_*dz$$
$$-\frac{1}{2}\int_{-\infty}^{0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{F}(0,0,z,x_*,y_*,z_*)dx_*dy_*dz_*dz \qquad (9.29)$$

or, by manipulation of the limits of integration,

$$\sigma_p = -\frac{1}{2}\int_{-\infty}^{0}\int_{0}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{F}(0,0,z,x_*,y_*,z_*)dx_*dy_*dz_*dz \qquad (9.30)$$

where, since ϕ_1 and $\phi_{1/2}$ are independent of x and y, x and y have been set equal to zero arbitrarily. The reasons for this will become clear when this result and the result obtained for the surface tension in the case of a curved interface are reconciled.

Equation 9.30 defines a stress associated with the interaction field approach, which, through the manner of its derivation, is analogous to the surface tension defined in the classical approach. For this reason this stress will be termed the *equivalent surface tension* associated with the kernel of the interaction field model. It has yet to be shown however that the properties of the equivalent surface tension endow the interaction field approach with any of the dynamic or static characteristics of the classical surface tension model. In the following example, the reduction of the equivalent surface tension, as the range of the internal interaction approaches zero, to an object which in a sense exists within the interface, and hence conforms to the classical notion of the surface tension, is demonstrated.

Model Problem

Assume the explicit form given by (9.8) for the interaction kernel. By direct integration using definition (9.10)

$$\phi_{1/2}(z) = \frac{\pi^{3/2}}{2k^3}(\,\mathrm{erf}(kz) - 1\,) \qquad (9.31)$$

on substituting (9.18), where

$$P = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_\infty dydz \tag{9.24}$$

Reconciliation

In the interaction field approach, the total normal force f_{p+i} transmitted across S is the sum of the normal forces resulting from the internal interactions and the pressure. Thus

$$\begin{aligned} f_{p+i} &= f_i + f_p \\ &= -\int_{-\infty}^{0}\int_{-\infty}^{\infty}(\tfrac{1}{2}\phi_{1/2}(z) - \phi_{1/2}(0))dydz + P \end{aligned} \tag{9.25}$$

on substituting (9.22) and (9.23).

Using the classical surface tension approach, the normal force transmitted across S is

$$f_t = -\int_{-\infty}^{\infty}\sigma_p dy + P \tag{9.26}$$

where σ_p is the surface tension for the planar interface. Reconciliation of the surface tension and interaction field approaches finally follows by setting $f_t = f_{p+i}$. Hence, by comparing (9.25) and (9.26),

$$\sigma_p = \int_{-\infty}^{0}(\tfrac{1}{2}\phi_{1/2}(z) - \phi_{1/2}(0))dz \tag{9.27}$$

Note that the symmetries of the interaction kernel $\mathcal{F}$ have led to a value of the surface tension which in this simple case is constant everywhere on the planar interface.

Equation 9.27 for the surface tension can be rewritten in terms of the interaction kernel itself. Note that, from (9.13),

$$\sigma_p = \tfrac{1}{2}\int_{-\infty}^{0}(\phi_{1/2}(z) - \phi_1)dz \tag{9.28}$$

The force[4] on the fluid arising from this potential is thus zero if $z > 0$, but if $z < 0$, the force is

$$-\mathrm{d}\Phi_L = -\frac{\partial}{\partial x}\phi_{1/4}(x,z)dx - \frac{\partial}{\partial y}\phi_{1/4}(x,z)dy - \frac{\partial}{\partial z}\phi_{1/4}(x,z)dz \tag{9.20}$$

The component of the force on the fluid acting in a direction normal to the dividing surface is thus

$$-\frac{\partial}{\partial x}\phi_{1/4}(x,z)dx \tag{9.21}$$

if $z < 0$ and zero if $z > 0$.

The total normal force on the fluid to the right of the dividing surface arising from the internal interaction with the fluid to the left of the dividing surface is then

$$\begin{aligned} f_i &= -\int_{-\infty}^{0}\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{\partial}{\partial x}\phi_{1/4}(x,z)dxdydz \\ &= -\int_{-\infty}^{0}\int_{-\infty}^{\infty}(\phi_{1/4}(\infty,z) - \phi_{1/4}(0,z))dydz \\ &= \frac{1}{2}\int_{-\infty}^{0}\int_{-\infty}^{\infty}\phi_{1/2}(z)dydz \end{aligned} \tag{9.22}$$

on evaluating the integral over x and exploiting the symmetries of $\phi_{1/4}$.

Pressure Force across the Dividing Surface

The normal force transmitted across S as a consequence of the pressure in the fluid is simply

$$\begin{aligned} f_p &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}p(z)dydz \\ &= \int_{-\infty}^{0}\int_{-\infty}^{\infty}(\phi_{1/2}(0) - \phi_{1/2}(z))dydz + P \end{aligned} \tag{9.23}$$

[4]See Footnote 2 in Chapter 4.

9.4.1 Surface Tension of a Planar Interface

Let the interface $\partial_{\alpha\beta}$ between the two domains of fluid $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ defined in Section 9.3 be given by (9.1) in the case where $Z(x,y) = 0$.

Equilibrium Potential and Pressure

In this case, using (9.16), the potential in the fluid is

$$\begin{aligned}\Phi(x,y,z) &= \rho_\beta(x,y,z)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{0}\mathcal{F}(x,y,z,x_*,y_*,z_*)dz_*dy_*dx_* \\ &= \begin{cases} 0 & \text{if } z>0 \\ \phi_{1/2}(z) & \text{if } z<0 \end{cases} \end{aligned} \tag{9.17}$$

on substituting (9.10) and (9.4).

The equilibrium equation (8.4) then gives the pressure in the fluid as

$$p(z) - p_\infty = \begin{cases} 0 & \text{if } z>0 \\ \phi_{1/2}(0) - \phi_{1/2}(z) & \text{if } z<0 \end{cases} \tag{9.18}$$

where $p_\infty \in \mathbb{R}$, on applying the pressure matching condition (8.3).

Internal Force Across a Dividing Surface

If the dividing surface is chosen arbitrarily to be the plane $S = \{(x,y,z) : x = 0\}$ then the potential resulting from the fluid to the left of the dividing surface is

$$\begin{aligned}\Phi_L &= \rho_\beta(x,y,z)\int_{-\infty}^{0}\int_{-\infty}^{\infty}\int_{-\infty}^{0}\mathcal{F}(x,y,z,x_*,y_*,z_*)dz_*dx_*dy_* \\ &= \begin{cases} 0 & \text{if } z>0 \\ \phi_{1/4}(x,z) & \text{if } z<0 \end{cases} \end{aligned} \tag{9.19}$$

on substituting (9.11).

Interaction Potential

From (6.67) and (6.65), the potential

$$\begin{aligned}\Phi &= \oint_{\mathcal{M}} {}^*\rho_{\beta\otimes\beta}\mathcal{F} \\ &= \rho_\beta(x,y,z)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho_\beta(x_*,y_*,z_*)\mathcal{F}(x,y,z,x_*,y_*,z_*)dz_*dx_*dy_* \\ &\triangleright\ \Lambda^0\mathcal{M}\end{aligned} \tag{9.15}$$

on substituting (9.4). This result can be rewritten using (9.3) as

$$\Phi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{Z(x_*,y_*)}\mathcal{F}(x,y,z,x_*,y_*,z_*)dz_*dx_*dy_* \tag{9.16}$$

if $z < Z(x,y)$ and $\Phi = 0$ if $z > Z(x,y)$.

9.4 Surface Tension in terms of Interaction Fields

In this section, the mechanical analogy of Fowler [29] and Kirkwood and Buff [42] is used to reconcile the surface tension and interaction field approaches for planar and curved interfaces. In a static fluid this reconciliation relies on determining the force transmitted across an arbitrary dividing surface placed normal to the interface. In the classical approach this force is attributed to the action of the pressure across the dividing surface and an additional contribution from an effective tension existing in the interfacial surface. In the interaction field approach, the force transmitted across the arbitrary dividing surface has a contribution f_i arising from the internal interactions within the fluid as well as another contribution f_p from the pressure field required to maintain equilibrium against the internal forces.

where the fibre integral is defined in Section 6.2.4, and the related integrals

$$\phi_{1/2} = \int_{-\infty}^{0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{F}(x,y,z,x_*,y_*,z_*)dx_*dy_*dz_* \tag{9.10}$$

and

$$\phi_{1/4} = \int_{-\infty}^{0}\int_{-\infty}^{\infty}\int_{-\infty}^{0} \mathcal{F}(x,y,z,x_*,y_*,z_*)dx_*dy_*dz_* \tag{9.11}$$

If these integrals exist then ϕ_1, $\phi_{1/2}$ and $\phi_{1/4}$ inherit the symmetries of the interaction kernel $\mathcal{F}$ such that

$$\begin{aligned} \phi_1 &= \text{constant} \\ \phi_{1/2} &= \phi_{1/2}(z) \\ \phi_{1/4} &= \phi_{1/4}(x,z) \end{aligned} \tag{9.12}$$

In addition, the symmetries of the interaction kernel and the asymptotic condition (9.7) result in a number of simple relationships between ϕ_1, $\phi_{1/2}$ and $\phi_{1/4}$ at various points in $\mathcal{M}$:

$$\begin{aligned} \phi_{1/2}(0) &= \tfrac{1}{2}\phi_1 \\ \phi_{1/2}(\infty) &= 0 \\ \phi_{1/2}(-\infty) &= \phi_1 \end{aligned} \tag{9.13}$$

and

$$\begin{aligned} \phi_{1/4}(x,\infty) &= 0 \\ \phi_{1/4}(\infty,z) &= 0 \\ \phi_{1/4}(0,z) &= \tfrac{1}{2}\phi_{1/2}(z) \end{aligned} \tag{9.14}$$

In all cases the proofs follow by realising that the value of the interaction kernel is not affected by reflection of any of the coordinate axes.

Assume that the fluid occupying $\mathcal{M}_\alpha$ is not subject to any interactions, either with itself, with an external field, or with the fluid occupying $\mathcal{M}_\beta$. Assume, however, that the fluid in region $\mathcal{M}_\beta$ interacts with itself, and that this interaction has a kernel $\mathcal{F} \triangleright \bigwedge^0 \mathcal{M}^2$ which has the same symmetry group as the kernel $\mathcal{F}_{\mathrm{iso}(\mathbb{R}^3)}$ defined in Section 6.5.1. Hence, as in Section 6.5.1,

$$\mathcal{F}(x, y, z, x_*, y_*, z_*) = \mathcal{F}(r^2) \tag{9.5}$$

where

$$r^2 = (x - x_*)^2 + (y - y_*)^2 + (z - z_*)^2 \tag{9.6}$$

In addition, assume that $\mathcal{F}$ obeys the asymptotic condition

$$\lim_{r \to \infty} \mathcal{F}(r^2) = 0 \tag{9.7}$$

In the examples which follow, where an explicit form for the interaction kernel is required, assume that

$$\begin{aligned} \mathcal{F} &= \mathcal{F}^{\mathrm{pr}}_{\mathrm{iso}(\mathbb{R}^3)} \\ &= -\exp(\, k^2 r^2 \,) \end{aligned} \tag{9.8}$$

where $\mathcal{F}^{\mathrm{pr}}_{\mathrm{iso}(\mathbb{R}^3)}$ is the explicit form defined in Section 6.5.1 and $k \in \mathbb{R}$. Note that if $k \neq 0$ then this form does indeed satisfy the asymptotic range condition implied by (9.7).

Integral Symmetries of the Interaction Kernel

For later convenience, define

$$\begin{aligned} \phi_1 &= \oint_{\mathcal{M}} {}^*\mathcal{F} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{F}(x, y, z, x_*, y_*, z_*) dx_* dy_* dz_* \end{aligned} \tag{9.9}$$

Chapter 10

Linear Dynamics

10.1 Overview

In this chapter the reconciliation between the predictions of the classical surface tension theory and the interaction field approach is continued by rederiving the classical results for the small-amplitude oscillatory behaviour and stability of several near-equilibrium systems using the interaction field approach. The linearised equations derived in Section 7.8 are used throughout this chapter to model the dynamics of the fluid system using the interaction field approach, and an equivalent linearised system is derived for the classical surface tension model. It may be argued that the full, but non-linear, equation of motion (7.38) should be used in the verification of the dynamic predictions of the interaction field model. From Section 7.8, however, the major effect which is eliminated by the linearisation process is the modal coupling related to possibly non-zero M^{ijk} terms in the full dynamic equation. Since this is an inertial effect rather than one related to the interactions present in the system, the verification of the interaction field approach through use of the linearised equations of motion for the fluid system would appear to be sufficient.

equation, where bubble formation in bulk fluid is thought to be suppressed by the large vapour pressures required to initiate the process.

9.5 Conclusion

Having derived the properties required of the interaction kernel for reconciliation to be possible between the static predictions of the interaction field approach and the classical surface tension theory, attention is turned in the following chapter to the reconciliation of the dynamic predictions of the two approaches.

$(R, 0, 0)$. Finally, a result in the form of (9.74) can be constructed by combining the two interaction field contributions to the pressure in the classical approach:

$$p|_C + p_i = \frac{2\sigma_c}{R} + p|_A \tag{9.79}$$

This result effectively defines the surface tension σ_c for the spherical interface. The absence of internal interactions within $\mathcal{M}_\alpha$ yields $p|_A = \bar{p}_\infty$ for any point $A \in \mathcal{M}_\alpha$.

Combining (9.74) and (9.79)

$$\frac{\sigma_c}{\sigma_p} = \frac{p|_C + p_i - p|_A}{\sigma_p} \tag{9.80}$$

This equation, evaluated using the explicit form of the interaction kernel given in (9.8), is plotted in Figure 9.2 and shows how, as the radius of the sphere becomes large compared to the range of the interaction kernel, the ratio σ_c/σ_p approaches unity and the classical Young-Laplace equation becomes increasingly valid. Where the radius of the sphere becomes comparable to the range of the interaction potential, however, the equivalent surface tension of the curved interface reduces below that of the surface tension of the planar interface, and tends to zero as the radius of the sphere tends to zero. This result is in accordance with the results presented by Tolman [81] and by Melrose [55]. The structure assumed for the fluid in Chapter 2 allows the results presented here to be given directly in terms of the range of the interaction kernel, however, rather than in terms of the less-easily accessible position of the Gibbs surface as is given in the analysis by both Tolman and Melrose.

Note too that the interaction field approach shows the pressure at the centre of the droplet going to zero as the radius goes to zero, whereas the Young-Laplace equation allows the pressure to become singular as the radius goes to zero. The results of the analysis given here have crucial implication for the simpler theories of nucleation[7] based on application of the Young-Laplace

[7] See for instance the paper by Crum [20].

can be evaluated using (9.73). By direct integration

$$\begin{aligned}\sigma_p &= \frac{\pi}{2}\int_0^\infty r_*^3 \exp[-k^4 r_*^2]dr_* \\ &= \frac{\pi}{4k^4}\end{aligned} \tag{9.75}$$

where, as before, k is a measure of the range of the interaction kernel. The interaction field techniques developed in the previous section can also be used to find the pressure at the centre of the spherical domain $\mathcal{M}_\beta$, simply by modifying the limits on the integrations to account for the finite extent of $\mathcal{M}_\beta$. To avoid unnecessary and tedious repetition, these results will merely be summarised. The potential of the internal interaction follows from (9.16) as

$$\begin{aligned}\Phi|_{\mathcal{M}_\alpha} &= 0 \\ \Phi|_{\mathcal{M}_\beta} &= \oint\!\!\!\!\!\!-_{\mathcal{M}_\beta} *\mathcal{F}\end{aligned} \tag{9.76}$$

from which the pressure p throughout $\mathcal{M}$ can be determined (up to the addition of the arbitrary constant p_∞) using the equilibrium conditions (8.4) and (8.3). The force exerted at the centre of the sphere as a result of the internal interactions must then be calculated after choosing the dividing surface across which the net force is to be evaluated as the plane $\{(x,y,z) : x = 0\}$. Let $L = \{(x,y,z) : x < 0\}$. The potential Φ_L resulting from the fluid to the left of the dividing surface follows from (9.16) as

$$\begin{aligned}(\Phi_L)|_{\mathcal{M}_\alpha} &= 0 \\ (\Phi_L)|_{\mathcal{M}_\beta} &= \oint\!\!\!\!\!\!-_{\mathcal{M}_\beta \cap L} *\mathcal{F}\end{aligned} \tag{9.77}$$

The additional component of the classical pressure at the centre of the sphere which can be attributed to the internal interactions is then

$$p_i = (\Phi_L)|_{\mathcal{C}} - (\Phi_L)|_{\mathcal{R}} \tag{9.78}$$

on integrating over the infinitesimal element surrounding the line $\{(x,y,z) : 0 < x < R,\ y = 0,\ z = 0\}$ and defining $\mathcal{R}$ as the point with coordinates

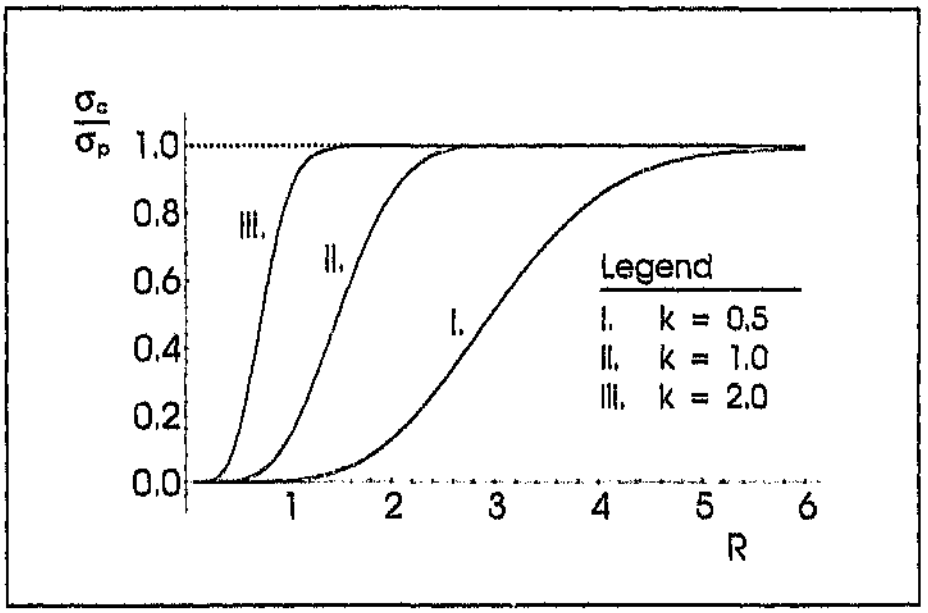


Figure 9.2: Equivalent surface tension for a spherical droplet.

As in Section 9.3, let the fluid domain $\mathcal{M}$ be isomorphic to $I\!R^3$, and let (x, y, z) and (x_*, y_*, z_*) be eqivalent, globally valid, rectangular coordinates for $\mathcal{M}$. Let $\mathcal{M}$ consist of two domains $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ defined respectively as the regions exterior and interior to the sphere, radius R, located with its centre at the point $C \in \mathcal{M}$ with coordinates $(0, 0, 0)$. Let the interactions for the two domains be as defined in Section 9.3 and let the kernel $\mathcal{F}$ have the explicit form given by (9.8). The pressure $\bar{p}_c$ at the centre of the spherical domain $\mathcal{M}_\beta$ predicted by the classical surface tension theory follows from (9.34). Since $\kappa_{\alpha\beta} = -2/R$,

$$\bar{p}_c = \frac{2\sigma_p}{R} + \bar{p}_\infty \tag{9.74}$$

where $\bar{p}_\infty \in I\!R$. The surface tension σ_p of the interface is assumed, as in the previous section, to be independent of the configuration of the fluid system and hence is set equal to the equivalent surface tension associated with the interaction field model for the planar interface. The equivalent surface tension

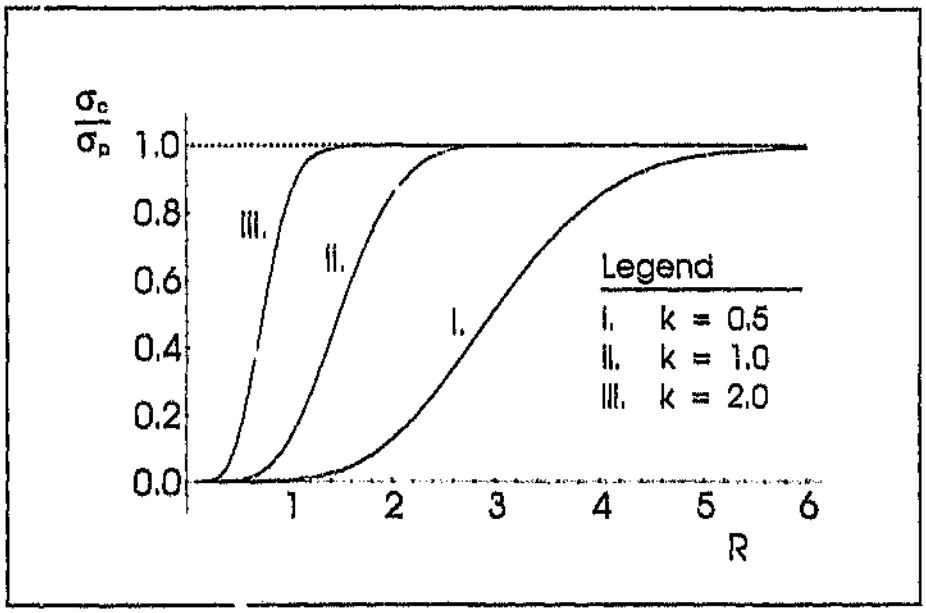


Figure 9.2: Equivalent surface tension for a spherical droplet.

As in Section 9.3, let the fluid domain $\mathcal{M}$ be isomorphic to $\mathbb{R}^3$, and let (x, y, z) and (x_*, y_*, z_*) be eqivalent, globally valid, rectangular coordinates for $\mathcal{M}$. Let $\mathcal{M}$ consist of two domains $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ defined respectively as the regions exterior and interior to the sphere, radius R, located with its centre at the point $\mathcal{C} \in \mathcal{M}$ with coordinates $(0, 0, 0)$. Let the interactions for the two domains be as defined in Section 9.3 and let the kernel $\mathcal{F}$ have the explicit form given by (9.8). The pressure $\overline{p}_c$ at the centre of the spherical domain $\mathcal{M}_\beta$ predicted by the classical surface tension theory follows from (9.34). Since $\kappa_{\alpha\beta} = -2/R$.

$$\overline{p}_c = \frac{2\sigma_p}{R} + \overline{p}_\infty \tag{9.74}$$

where $\overline{p}_\infty \in \mathbb{R}$. The surface tension σ_p of the interface is assumed, as in the previous section, to be independent of the configuration of the fluid system and hence is set equal to the equivalent surface tension associated with the interaction field model for the planar interface. The equivalent surface tension

Let $r_*^2 + u^2 = a$. Then by the chain rule

$$\begin{aligned}\frac{d}{d(r_*^2)}\mathcal{F}(a) &= \frac{d}{da}\mathcal{F}(a)\frac{\partial}{\partial(r_*^2)}a \\ &= \frac{d}{da}\mathcal{F}(a) \qquad (9.71)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial u}\mathcal{F}(a) &= \frac{d}{da}\mathcal{F}(a)\frac{\partial}{\partial u}a \\ &= 2u\frac{d}{da}\mathcal{F}(a) \qquad (9.72)\end{aligned}$$

Hence (9.70) can be written

$$\begin{aligned}\sigma_p &= -\frac{\pi}{2}\int_0^\infty r_*^3\int_{-\infty}^0 \frac{\partial}{\partial u}\mathcal{F}(a)dudr_* \\ &= -\frac{\pi}{2}\int_0^\infty r_*^3(\mathcal{F}(r_*^2) - \mathcal{F}(\infty))dr_* \\ &= -\frac{\pi}{2}\int_0^\infty r_*^3\mathcal{F}(r_*^2)dr_* \qquad (9.73)\end{aligned}$$

on evaluating the integral over u and exploiting the asymptotic property (9.7).

Comparing (9.64) and (9.73) shows that σ_c and σ_p are in fact the same object and hence that the two forms of the Young-Laplace equation, (9.34) and (9.58), one obtained from the classical surface tension theory and the other from the interaction field approach, are indeed equivalent under suitable restrictions on the interaction kernel $\mathcal{F}$.

9.4.3 Pressure at the Centre of a Spherical Droplet

As discussed in Section 9.2, there are certain fluid configurations where the pressures in the fluid system are strongly influenced by the geometry of the system. In this section, the approach of the previous sections is used to derive a result for the variation of the pressure at the centre of a spherical domain of fluid as the radius of the domain is varied.

on converting to polar coordinates and integrating over θ_*, where

$$\begin{aligned}\mathcal{G}(x_*^2+y_*^2) &= \int_{-\infty}^{0}\int_{0}^{\infty}\mathcal{F}(0,0,y,z,x_*,y_*,z_*)dz_*dz \\ &= \int_{-\infty}^{0}\int_{0}^{\infty}\mathcal{F}(x_*^2+y_*^2+(z-z_*)^2)dz_*dz \qquad (9.66)\end{aligned}$$

and again the assumed form of $\mathcal{F}$ and rectangular coordinates have been used to simplify the expression.

Note that, for given x_* and y_*, the integrand in (9.66) is invariant along the lines on which $z-z_*$ is constant. This suggests the change of variables

$$\begin{aligned}u &= z-z_* \\ v &= z+z_* \qquad (9.67)\end{aligned}$$

Noting that the Jacobian of the transformation from (z,z_*) to (u,v) coordinates is $1/2$, (9.66) can be rewritten as

$$\begin{aligned}\mathcal{G}(x_*^2+y_*^2) &= \frac{1}{2}\int_{-\infty}^{0}\int_{-u}^{u}\mathcal{F}(x_*^2+y_*^2+u^2)dvdu \\ &= \frac{1}{2}\int_{-\infty}^{0}2u\mathcal{F}(x_*^2+y_*^2+u^2)du \qquad (9.68)\end{aligned}$$

on exploiting the invariance of the integrand with respect to v. Hence, by combining (9.65) and (9.68) and integrating by parts

$$\begin{aligned}\sigma_p &= \frac{\pi}{2}\mathcal{G}(r_*^2)r_*^2\,|_0^\infty - \pi\int_0^\infty r_*^3\frac{d}{d(r_*^2)}\mathcal{G}(r_*^2)dr_* \\ &= -\pi\int_0^\infty r_*^3\frac{d}{d(r_*^2)}\mathcal{G}(r_*^2)dr_* \qquad (9.69)\end{aligned}$$

since $\mathcal{G}(r_*^2)r_*^2=0$ at $r_*=0$ and, if $\mathcal{G}$ exists, by the asymptotic property (9.7). Hence, in terms of the kernel $\mathcal{F}$, from (9.68),

$$\begin{aligned}\sigma_p &= -\frac{\pi}{2}\int_0^\infty r_*^3\frac{d}{d(r_*^2)}\int_{-\infty}^{0}2u\mathcal{F}(r_*^2+u^2)dudr_* \\ &= -\frac{\pi}{2}\int_0^\infty r_*^3-\int_{-\infty}^{0}2u\frac{d}{d(r_*^2)}\mathcal{F}(r_*^2+u^2)dudr_* \qquad (9.70)\end{aligned}$$

hence that the equivalent surface tensions associated with the interaction field models for the curved interface and for the planar interface are identical.

By definition (9.41),

$$\begin{aligned} I_{1/2}(0,0,0) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_*^2 \mathcal{F}(0,0,0,x_*,y_*,0)dy_*dx_* \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y_*^2 \mathcal{F}(0,0,0,x_*,y_*,0)dy_*dx_* \end{aligned} \tag{9.60}$$

by previous argument. Hence, from (9.59)

$$\sigma_c = -\frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_*^2+y_*^2)\mathcal{F}(0,0,0,x_*,y_*,0)dy_*dx_* \tag{9.61}$$

Now, from (9.5) and (9.6), in rectangular coordinates

$$\mathcal{F}(0,0,0,x_*,y_*,0) = \mathcal{F}(x_*^2+y_*^2) \tag{9.62}$$

hence, on changing to polar coordinates

$$\begin{aligned} r_*^2 &= x_*^2+y_*^2 \\ x_* &= r_*\cos\theta_* \\ y_* &= r_*\sin\theta_* \end{aligned} \tag{9.63}$$

(9.61) transforms to

$$\begin{aligned} \sigma_c &= -\frac{1}{4}\int_0^{\infty}\int_0^{2\pi} r_*^3\mathcal{F}(r_*^2)d\theta_*dr_* \\ &= -\frac{\pi}{2}\int_0^{\infty} r_*^3\mathcal{F}(r_*^2)dr_* \end{aligned} \tag{9.64}$$

on evaluating the integral over θ_*.

From (9.30), the surface tension in the case of a planar interface can be written as

$$\begin{aligned} \sigma_p &= -\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{G}(x_*^2+y_*^2)dx_*dy_* \\ &= -\frac{1}{2}\int_0^{\infty}\int_0^{2\pi}\mathcal{G}(r_*^2)r_*d\theta_*dr_* \\ &= -\pi\int_0^{\infty}\mathcal{G}(r_*^2)r_*dr_* \end{aligned} \tag{9.65}$$

The change in pressure $\Delta\overline{p}$ across the interface $\delta_{\alpha\beta}$ in the classical surface tension model for the fluid configuration can be interpreted as the quantity $\overline{p}(\infty) - \overline{p}(-\infty)$. From (9.55)

$$\begin{aligned}\Delta\overline{p} &= \tfrac{1}{2}\phi_{1/2}(-\infty) - \phi_{1/2}(0) \\ &\quad +(c_1+c_2)(\ \tfrac{1}{2}I_{1/2}(0,0,-\infty) - I(0,0,0)\ ) \\ &\quad +\mathcal{E} \\ &= -(c_1+c_2)I_{1/2}(0,0,0) + \mathcal{E} \qquad (9.56)\end{aligned}$$

by exploiting the symmetries of $\phi_{1/2}$ and since, as a result of the symmetries of the kernel $\mathcal{F}$,

$$I_{1/2}(0,0,-\infty) = 0 \qquad (9.57)$$

Hence

$$\Delta\overline{p} = \kappa_{\alpha\beta}\sigma_c \qquad (9.58)$$

where the mean curvature of the interface is defined in (9.36). The quantity σ_c can be interpreted as the equivalent surface tension associated with the interaction field model for the *curved* interface, and is given by

$$\sigma_c = -\frac{1}{2}I_{1/2}(0,0,0) \qquad (9.59)$$

Comparing (9.34) and (9.58), the surface tension based approach and the approach through the interaction field can be completely reconciled if it can be shown that $\sigma_c \equiv \sigma_p$.

Surface Tension for Curved and Planar Interfaces

It is now shown by change of variables that the integrals for σ_c and for σ_p in terms of the interaction kernel can both be brought to a common form, and

and hence, on evaluating the integral,

$$p_i(0,0,z) = \Phi_L(0,0,z) - \Phi_L(\infty,0,z) \tag{9.50}$$

or, on substituting (9.46),

$$\begin{aligned} p_i(0,0,z) \;\; = \;\; & \phi_{1/4}(0,z) - \phi_{1/4}(\infty,z) \\ & +(c_1+c_2)(\; I_{1/4}(0,0,z) - I_{1/4}(\infty,0,z)\;) \\ & +\mathcal{E}(0,0,z) \end{aligned} \tag{9.51}$$

Note that

$$I_{1/4}(0,0,z) = \tfrac{1}{2} I_{1/2}(0,0,z) \tag{9.52}$$

an *I*

$$I_{1/4}(\infty,0,z) = 0 \tag{9.53}$$

as a result of the symmetries of $\mathcal{F}$, and hence, on exploiting the symmetries of $\phi_{1/4}$, (9.51) can be rewritten as

$$p_i(0,0,z) = \tfrac{1}{2}\phi_{1/2}(z) + \tfrac{1}{2}(c_1+c_2) I_{1/2}(0,0,z) + \mathcal{E}(0,0,z) \tag{9.54}$$

Reconciliation

As described earlier in this section, the quantity $p + p_i$ should be interpreted as the pressure $\overline{p}$ in the classical surface tension model. Thus along the axis $x = y = 0$, $\overline{p}(z) = p_\infty$ if $z > 0$, while if $z < 0$, from (9.44) and (9.54),

$$\begin{aligned} \overline{p}(z) - p_\infty \;\; = \;\; & \phi_{1/2}(0) - \tfrac{1}{2}\phi_{1/2}(z) \\ & +(c_1+c_2)(\; I(0,0,0) - \tfrac{1}{2} I_{1/2}(0,0,z)\;) \\ & +\mathcal{E}(z) \end{aligned} \tag{9.55}$$

$$+\mathcal{E}(0,0,z) \tag{9.44}$$

on applying the pressure matching condition (8.3).

Arbitrarily define the dividing surface S across which the contibution to the classical pressure from the internal interactions is calculated as the plane $\{(x,y,z) : x = 0\}$. The interaction potential of the fluid to the left of the dividing surface is then

$$\Phi_L(x,y,z) = \int_{-\infty}^{0}\int_{-\infty}^{\infty}\int_{-\infty}^{c_1x_*^2+c_2y_*^2+\cdots} \mathcal{F}(x,y,z,x_*,y_*,z_*)dz_*dy_*dx_* \tag{9.45}$$

Following the same procedure as before,

$$\Phi_L(x,y,z) = \phi_{1/4}(x,z) + (c_1+c_2)I_{1/4}(x,y,z) + \mathcal{E}(x,y,z) \tag{9.46}$$

where $\phi_{1/4}$, defined in (9.11), is the interaction potential generated by the fluid to the left of the dividing surface in the case of a planar interface, and

$$I_{1/4}(x,y,z) = \int_{-\infty}^{0}\int_{-\infty}^{\infty} \mathcal{F}(x,y,z,x_*,y_*,0)x_*^2dy_*dx_* \tag{9.47}$$

is the contribution to the interaction field from the distortion of the interface.

The internal force[6] transmitted across the infinitesimal area element S^* surrounding the point $(0,0,z)$ on the dividing surface is then the component normal to S of the internal force exerted by the fluid to the left of the dividing surface on the fluid in the infinitesimal volume element R^* surrounding the ray $\{(x,y,z) : x > 0,\ y = 0,\ z = \text{constant}\}$.

Since the component normal to the dividing surface of the internal force is

$$-\tfrac{\partial}{\partial x}\Phi_L(x,y,z)dx \tag{9.48}$$

from (9.45) the total internal force transmitted across S^* is

$$p_i(0,0,z)dydz = -dydz\int_0^{\infty}\tfrac{\partial}{\partial x}\Phi_L(x,0,z)dx \tag{9.49}$$

[6]Strictly, the force arising from the internal interactions within the fluid.

System Dynamics

The results of Section 10.2 can be used to examine the stability of the system. Note that the inertia and stiffness coefficients of the system are already in diagonal form, and hence (10.8) can be used to analyse the stability of the system.

In the case where the system stiffness results from the gravitational potential, the eigenvalues of the motion of the interface are given by

$$(\lambda_{(i,j)})^2 = \frac{(\overline{\rho}^{\alpha} - \overline{\rho}^{\beta})}{(\overline{\rho}^{\alpha} + \overline{\rho}^{\beta})}\, g(i^2 + j^2)^{1/2} \tag{10.36}$$

which is identical to the classical result given by Lamb [17], and shows the expected instability when $\overline{\rho}^{\alpha} > \overline{\rho}^{\beta}$, that is, when a domain with greater density lies above another with smaller density.

The case where the system stiffness arises from the internal interactions is more interesting. Here the eigenvalues of the motion of the system are given by

$$(\lambda_{(i,j)})^2 = -\frac{\pi}{k^2(\overline{\rho}^{\alpha} + \overline{\rho}^{\beta})}(1 - e^{-(i^2+j^2)/(4k^2)})(i^2 + j^2)^{1/2} \tag{10.37}$$

showing the system to be unconditionally stable[1]. Note that the first term in a Taylor expansion in $(i^2 + j^2)^{1/2}$ about $(i^2 + j^2)^{1/2} = 0$ of (10.37) is

$$\begin{aligned}(\lambda_{(i,j)})^2 &= -\frac{\pi}{4k^4(\overline{\rho}^{\alpha} + \overline{\rho}^{\beta})}(i^2 + j^2)^{3/2} \\ &= -\frac{\sigma_p}{(\overline{\rho}^{\alpha} + \overline{\rho}^{\beta})}(i^2 + j^2)^{3/2}\end{aligned} \tag{10.38}$$

where, as shown in Section 9.4.1, σ_p is the equivalent surface tension associated with the interaction kernel assumed here. Equation 10.38 is identical to the classical result given by Lamb [17].

[1]Note that if the opposite sign for $\mathcal{F}$ is assumed, the analysis leads to unconditional instability.

Hence, from (10.10), the stiffness coefficients can be written as

$$K^{(i,j)(i',j')} = G^{(i,j)(i',j')} + I^{(i,j)(i',j')} \tag{10.29}$$

where

$$\begin{aligned} G^{(i,j)(i',j')} &= \int_{\partial_{m\beta}} *\upsilon^{(i,j)}|_{\mathcal{M}_\beta} * \mathsf{d}(\phi_{1/2} * \upsilon^{(i',j')}|_{\mathcal{M}_\beta}) \\ &= (i^2+j^2)\ \tfrac{\partial}{\partial z}\phi_{1/2}|_{z=0} \int_{\partial_{m\beta}} *\mathsf{H}_{+1}^{(i,j)} \mathsf{H}_{+1}^{(i',j')} \end{aligned} \tag{10.30}$$

and

$$\begin{aligned} \phi_{1/2} &= \oint_{\mathcal{M}_\beta} \mathcal{F} \\ &= \frac{\pi^{3/2}}{2k^3}(\ \mathrm{erf}(kz) - 1\ ) \end{aligned} \tag{10.31}$$

from Section 9.4.1. Then, since $\frac{\partial}{\partial z}\phi_{1/2} = \pi/k^2$,

$$G^{(i,j)(i',j')} = \begin{cases} 0 & \text{if } (i,j) \neq (i',j') \\ \frac{\pi I_0}{k^2}(i^2+j^2) & \text{if } (i,j) = (i',j') \end{cases} \tag{10.32}$$

The second component of the stiffness integral is

$$\begin{aligned} I^{(i,j)(i',j')} &= \int_{\partial_{m\beta}} \oint_{\partial_{m\beta}} \mathcal{F}\ (*\mathsf{d}\mathsf{H}_{+1}^{(i,j)})|_{\partial_{m\beta}} \otimes (*\mathsf{d}\mathsf{H}_{+1}^{(i',j')})|_{\partial_{m\beta}} \\ &= \begin{cases} 0 & \text{if } (i,j) \neq (i',j') \\ -\frac{\pi I_0}{k^2}(i^2+j^2)e^{-(i^2+j^2)/(4k^2)} & \text{if } (i,j) = (i',j') \end{cases} \end{aligned} \tag{10.33}$$

by direct integration on substitution of (10.25) and (10.17), noting that

$$\begin{aligned} &(*\mathsf{d}\mathsf{H}_{+1}^{(i,j)})|_{\partial_{m\beta}} \otimes (*\mathsf{d}\mathsf{H}_{+1}^{(i',j')})|_{\partial_{m\beta}} \\ &\quad = \mathsf{H}_{+1}^{(i,j)}(x,y,0)\ \mathsf{H}_{+1}^{(i',j')}(x_*,y_*,0)\ dx\,dy\,dx_*\,dy_* \end{aligned} \tag{10.34}$$

Hence

$$K^{(i,j)(i',j')} = \begin{cases} 0 & \text{if } (i,j) \neq (i',j') \\ \frac{\pi I_0}{k^2}(i^2+j^2)(1-e^{-(i^2+j^2)/(4k^2)}) & \text{if } (i,j) = (i',j') \end{cases} \tag{10.35}$$

$$\triangleright \quad \Lambda^0\mathcal{M} \tag{10.25}$$

where $g \in I\!R$, ρ is the fluid density, and ρ_σ, $\sigma \in \{\alpha, \beta\}$ are the occupation functions on $\mathcal{M}$ defined in Section 6.3.3.

Since $\mathcal{F}_0 \triangleright \Lambda^0\mathcal{M}$, the stiffness coefficients for the system can be calculated via an analysis on $\mathcal{M}$. From (7.41),

$$\begin{aligned}
K^{(i,j)(i',j')} &= \overline{\rho}^\beta g \int_{\partial_{\alpha\beta}} *v^{(i,j)}|_{\mathcal{M}_\beta} * \mathsf{d}(z * v^{(i',j')}|_{\mathcal{M}_\beta}) \\
&\quad -\overline{\rho}^\alpha g \int_{\partial_{\alpha\beta}} *v^{(i,j)}|_{\mathcal{M}_\alpha} * \mathsf{d}(z * v^{(i',j')}|_{\mathcal{M}_\alpha}) \\
&= \overline{\rho}^\beta (i^2 + j^2) g \int_{\partial_{\alpha\beta}} *(\mathsf{H}_{+1}^{(i,j)} \mathsf{H}_{+1}^{(i',j')}) \\
&\quad -\overline{\rho}^\alpha (i^2 + j^2) g \int_{\partial_{\alpha\beta}} *(\mathsf{H}_{-1}^{(i,j)} \mathsf{H}_{-1}^{(i',j')}) \\
&= \begin{cases} 0 & \text{if } (i,j) \neq (i',j') \\ L(\overline{\rho}^\beta - \overline{\rho}^\alpha) g (i^2 + j^2) & \text{if } (i,j) = (i',j') \end{cases}
\end{aligned} \tag{10.26}$$

where the negative sign arises since the Hodge form induced on $\partial_{\alpha\beta}$ by the Hodge form on $\mathcal{M}_\alpha$ has opposite orientation compared to the Hodge form induced on $\partial_{\alpha\beta}$ by the Hodge form on $\mathcal{M}_\beta$.

Internal Interaction

As in Section 9.3, assume that the fluid in $\mathcal{M}_\alpha$ is not subject to any interaction, either with itself, with an external field, or with the fluid in $\mathcal{M}_\beta$. Assume however that the fluid in region $\mathcal{M}_\beta$ is subject to a pairwise interaction with itself. Let the kernel $\mathcal{F} \triangleright \Lambda^0\mathcal{M}^2$ of this interaction be of the same form as that used in the examples of Chapter 9:

$$\begin{aligned}
&\mathcal{F}|_{\mathcal{M}_\beta \otimes \beta}(x, y, z, x_*, y_*, z_*) \\
&\quad = \mathcal{F}^{pr}_{\mathsf{iso}(I\!R^6)} \\
&\quad = -\exp[\, k^2((x - x_*)^2 + (y - y_*)^2 + (z - z_*)^2)\,]
\end{aligned} \tag{10.27}$$

where, as $k \to \infty$, the range of the interaction kernel approaches zero, while

$$\mathcal{F}|_{\mathcal{M}_\alpha \otimes \alpha} = \mathcal{F}|_{\mathcal{M}_\alpha \otimes \beta} = \mathcal{F}|_{\mathcal{M}_\beta \otimes \alpha} = 0 \tag{10.28}$$

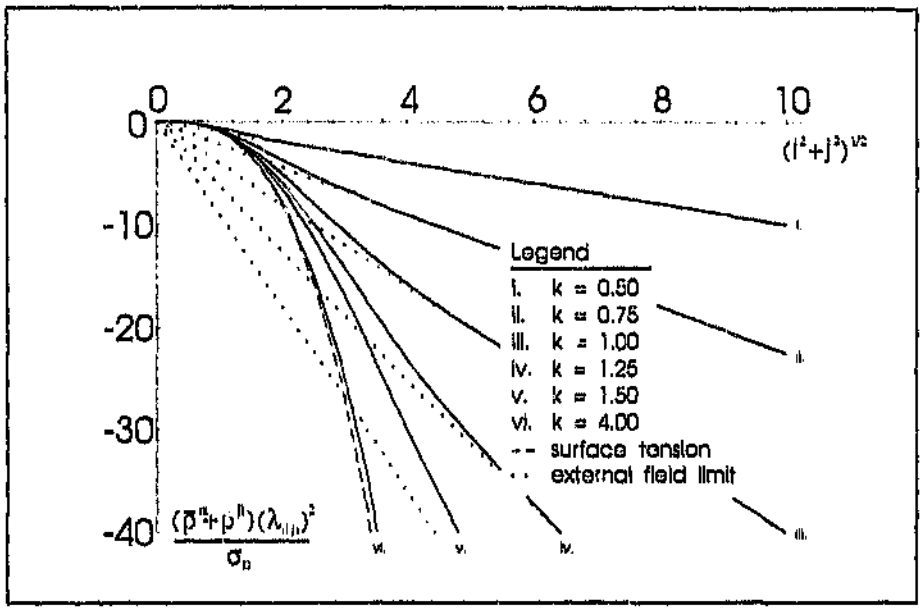


Figure 10.1: Eigenvalue-wavenumber relation for the planar interface with pairwise internal interactions.

$$= \begin{cases} 0 & \text{if } (i,j) \neq (i',j') \\ L(\bar{\rho}^{\alpha}+\bar{\rho}^{\beta})(i^2+j^2)^{1/2} & \text{if } (i,j) = (i',j') \end{cases} \tag{10.24}$$

by direct integration after substitution of (10.17).

The stiffness coefficients for the system can be calculated once the interactions within the fluid have been specified. As mentioned earlier, two cases are analysed here:

Gravitational Potential

Assume that the fluid in both domains $\mathcal{M}_{\alpha}$ and $\mathcal{M}_{\beta}$ is subject to an external interaction $\mathcal{F} \triangleright \Lambda^0\mathcal{M}$ modelling a constant gravitational field, but that no higher-order interactions are present. Let the interaction kernel

$$\mathcal{F}_g = \rho g z$$

$$= \bar{\rho}^{\gamma}\rho_{\sigma} g z$$

The dynamics of the system can be transferred to the configuration manifold $\mathcal{D}$ defined in Section 7.2 by comparing (10.21) with (7.10). If coordinates for $\mathcal{D}$ in the neighbourhood of the reference condition are $(a_0, \ldots, a_\infty)$, then the tangent vector at the reference condition to the curve representing the evolution of the system is

$$\begin{aligned}\dot{\vartheta} &= \dot{a}_0 \tfrac{\partial}{\partial a_0} + \cdots + \dot{a}_\infty \tfrac{\partial}{\partial a_\infty} \\ &= c_{(0,0)} \tfrac{\partial}{\partial a_0} + \cdots + c_{(\infty,\infty)} \tfrac{\partial}{\partial a_\infty}\end{aligned} \tag{10.22}$$

after identifying distortion modes $\upsilon^{(i,j)} \triangleright \mathcal{XM}$, where $i, j \in \{0, \ldots, \infty\}$, such that

$$\begin{aligned}\upsilon^{(i,j)}|_{\mathcal{M}_\alpha} &= \mathsf{d}\mathsf{H}_{-1}^{(i,j)} \\ \upsilon^{(i,j)}|_{\mathcal{M}_\beta} &= -\mathsf{d}\mathsf{H}_{+1}^{(i,j)}\end{aligned} \tag{10.23}$$

and identifying each of the coefficients $c_{(i,j)}$ uniquely with one of the coefficients $\dot{a}_l$ of the tangent vector.

System Integrals

Take the condition of the undistorted planar interface as the reference condition corresponding to the point ϑ in the configuration manifold. The system integrals required for the linear analysis of the system's dynamics about ϑ can then be calculated using the results of Section 7.5.

From (7.19), the inertia coefficients for the fluid system are

$$\begin{aligned}M^{(i,j)(i',j')} &= \int_{\mathcal{M}} \rho \upsilon^{(i,j)} \wedge *\upsilon^{(i',j')} \\ &= \int_{\mathcal{M}_\alpha} \overline{\rho}^\alpha \upsilon^{(i,j)}|_{\mathcal{M}_\alpha} \wedge *\upsilon^{(i',j')}|_{\mathcal{M}_\alpha} \\ &\quad + \int_{\mathcal{M}_\beta} \overline{\rho}^\beta \upsilon^{(i,j)}|_{\mathcal{M}_\beta} \wedge *\upsilon^{(i',j')}|_{\mathcal{M}_\beta} \\ &= \int_{\mathcal{M}_\alpha} \overline{\rho}^\alpha \mathsf{d}\mathsf{H}_{-1}^{(i,j)} \wedge *\mathsf{d}\mathsf{H}_{-1}^{(i',j')} + \int_{\mathcal{M}_\beta} \overline{\rho}^\beta \mathsf{d}\mathsf{H}_{+1}^{(i,j)} \wedge *\mathsf{d}\mathsf{H}_{+1}^{(i',j')}\end{aligned}$$

where summation over all repeated indices is implied. Then, from (5.29),

$$\begin{aligned} \nu|_{\mathcal{M}_+} &= \overline{c}^k_{(i,j)} \mathsf{dH}^{(i,j)}_k \\ \nu|_{\mathcal{M}_-} &= c^k_{(i,j)} \mathsf{dH}^{(i,j)}_k \end{aligned} \tag{10.16}$$

where the exterior derivatives can be evaluated in terms of the elements of $\mathrm{harm}^{\mathrm{plan}}(I\!R^3)$:

$$\begin{aligned} \mathsf{dH}^{(2i,2j)}_k &= -\mathsf{H}^{(2i+1,2j+1)}_k\,(i\,dx + j\,dy) + \mathsf{H}^{(2i,2j)}_k\,k(i^2+j^2)^{1/2}\,dz \\ \mathsf{dH}^{(2i+1,2j+1)}_k &= \mathsf{H}^{(2i,2j)}_k\,(i\,dx + j\,dy) + \mathsf{H}^{(2i+1,2j+1)}_k\,k(i^2+j^2)^{1/2}\,dz \end{aligned} \tag{10.17}$$

where (dx, dy, dz) is the dual basis induced on $\mathcal{M}$ by the coordinates. The no-rupture and automorphism conditions of Section 7.4 can be used to reduce the number of independent coefficients in the expressions for the velocity. To satisfy the automorphism condition $*d\lambda = 0$ on the boundaries $\{(x,y,z) : z = \infty\}$ and $\{(x,y,z) : z = -\infty\}$ of $\mathcal{M}$,

$$\begin{aligned} \overline{c}^{+1}_{(i,j)} &\equiv 0 \\ c^{-1}_{(i,j)} &\equiv 0 \end{aligned} \tag{10.18}$$

In addition, the no-rupture condition on $\partial_{\alpha\beta}$ requires

$$c^{+1}_{(i,j)} = -\overline{c}^{-1}_{(i,j)} \equiv c_{(i,j)} \tag{10.19}$$

on substituting (10.17) into (10.16), and using (10.14) since

$$\mathsf{H}^{(i,j)}_{-1}|_{\partial_{\alpha\beta}} = \mathsf{H}^{(i,j)}_{+1}|_{\partial_{\alpha\beta}} \tag{10.20}$$

Hence

$$\begin{aligned} \nu|_{\mathcal{M}_+} &= c_{(i,j)} \mathsf{dH}^{(i,j)}_{-1} \\ \nu|_{\mathcal{M}_-} &= -c_{(i,j)} \mathsf{dH}^{(i,j)}_{+1} \end{aligned} \tag{10.21}$$

Harmonic Forms

The complete set of harmonic zero-forms

$$\mathrm{harm}^{\mathrm{plan}}(I\!R^3) = \{\mathsf{H}_k^{(i,j)}\}_{i,j\in\{0,\ldots,\infty\},\ k\in\{-1,1\}} \tag{10.12}$$

where

$$\begin{aligned}
\mathsf{H}_k^{(2i,2j)}(x,y,z) &= \tfrac{1}{\sqrt{2\pi}} e^{k(i^2+j^2)^{1/2}z}\cos(ix+jy)\\
\mathsf{H}_k^{(2i+1,2j+1)}(x,y,z) &= \tfrac{1}{\sqrt{2\pi}} e^{k(i^2+j^2)^{1/2}z}\sin(ix+jy)
\end{aligned} \tag{10.13}$$

(where the wavenumbers $i,j \in \{0,\ldots,\infty\}$) can be used to analyse disturbances on the planar interface, since this set of forms obeys the integral condition

$$\int_{\partial_{\alpha\beta}} *\mathsf{H}_k^{(i,j)}\mathsf{H}_{k'}^{(i',j')} = \begin{cases} 0 & \text{if } (i,j)\neq(i',j') \text{ or } k\neq k' \\ L & \text{if } (i,j)=(i',j') \text{ and } k=k' \end{cases} \tag{10.14}$$

and hence is orthogonal on $\partial_{\alpha\beta}\times I\!R = I\!R^3$. In (10.14), $L = \int_{\partial_{\alpha\beta}}\Omega_{\alpha\beta}/\int_{\mathcal{S}'}\Omega_{\alpha\beta}$ and $\mathcal{S}' = \{(x,y,z) : z=0,\ c<x<2\pi+c,\ d<y<2\pi+d,\ c,d\in I\!R\}\subset\partial_{\alpha\beta}$. Depending on context, $\Omega_{\alpha\beta}$ is the Hodge form on $\partial_{\alpha\beta}$ induced by either the Hodge form on $\mathcal{M}_\alpha$ or the Hodge form on $\mathcal{M}_\beta$. The factor L appears since the elements of $\mathrm{harm}^{\mathrm{plan}}(I\!R^3)$ are strictly orthogonal on $\mathcal{S}'\times I\!R\subset I\!R^3$ rather than on $I\!R^3$ itself. Note that the periodicity implied by orthogonality over this subset of $I\!R^3$ can be varied quite simply by rescaling the coordinates adopted for $\mathcal{M}$.

Distortion Modes

Since, from Section 5.5, the velocity potential is harmonic in both $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$,

$$\begin{aligned}
\Lambda|_{\mathcal{M}_\alpha} &= \bar{r}^k_{(i,j)}\mathsf{H}_k^{(i,j)}\\
\Lambda|_{\mathcal{M}_\beta} &= \bar{c}^k_{(i,j)}\mathsf{H}_k^{(i,j)}
\end{aligned} \tag{10.15}$$

interaction kernel. On the other hand it would appear that, although certain elements of the dynamics of the system in the case of non-infinitesimal range of the interactions are indeed form-specific, there appear to be certain elements of the dynamics that are shared between the examples presented here, and there is hence strong support for the proposition that these elements are indeed universal characteristics of the dynamics of fluid systems modelled using the interaction field approach.

10.4.1 The Planar Interface

Let (x, y, z) and (x_*, y_*, z_*) be equivalent, uniformly valid, rectangular coordinates for $\mathcal{M}$. Define

$$\begin{aligned} \mathcal{M}_\alpha &= \{(x, y, z) : z > 0\} \\ \mathcal{M}_\beta &= \{(x, y, z) : z < 0\} \end{aligned} \tag{10.11}$$

then $\partial_{\alpha\beta}$ is the plane $\{(x, y, z) : z = 0\}$.

Two particularly well-known classical analyses for the linear motion of this planar interface between the two dissimilar domains of fluid exist. The first is the so-called gravity-wave case, where the fluid is subject to a gravitational field acting normal to the undisturbed interface. The second case is the so-called capillary-wave analysis, where disturbances on the interface are driven by the surface tension in the interface. Both cases are re-analysed here using the interaction field approach. In the gravity wave case, the analysis follows directly by defining an interaction kernel which is equivalent to the gravitational potential on $\mathcal{M}$. For consistency with the results of Chapters 9 and 8 and the assertions of Section 6.5, the capillary wave case is modelled in the situation where a pairwise internal interaction exists within $\mathcal{M}_\beta$ and hence an interaction kernel on $\mathcal{M}^2$ can be defined.

and any harmonic zero-form on M can be written as a linear combination of the elements of any one of the sets $\mathrm{harm}^n(M)$.

Stiffness Integrals on $\mathcal{M}^2$

If the interaction kernel $\mathcal{F} \triangleright \Lambda^0\mathcal{M}^2$, then expression (7.41) for the stiffness coefficients of the fluid system can be specialised as

$$\begin{aligned} K^{ij} &= 2\sum_{\sigma,\sigma'\in\mathcal{Z}} \int_{\partial\mathcal{M}_\sigma} \oint_{\mathcal{M}_{\sigma'}} *(\upsilon^i \circledast \upsilon^i) * \mathrm{d}(\mathcal{F} * (\upsilon^j \circledast \upsilon^j)) \\ &= 2\sum_{\sigma,\sigma'\in\mathcal{Z}} \int_{\partial\mathcal{M}_\sigma} *\upsilon^i * \mathrm{d}\Big(*\upsilon^j \oint_{\mathcal{M}_{\sigma'}} *\mathcal{F} \Big) \\ &\quad + 2\sum_{\sigma,\sigma'\in\mathcal{Z}} \int_{\partial\mathcal{M}_\sigma} \oint_{\partial\mathcal{M}_{\sigma'}} \mathcal{F}(*\upsilon^i) \otimes (*\upsilon^j) \end{aligned} \tag{10.10}$$

since $\upsilon^i_2 = \upsilon^i \circledast \upsilon^i$ and $\partial\mathcal{M}^2 = (\partial\mathcal{M} \times \mathcal{M}) \cup (\mathcal{M} \times \partial\mathcal{M})$.

Topology of the Fluid System

In all cases to follow the fluid domain $\mathcal{M}$ is assumed to be isomorphic to $\mathbb{R}^3$ and to consist of two domains $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ separated by an interface $\partial_{\alpha\beta} = \partial\mathcal{M}_\alpha \cap \partial\mathcal{M}_\beta$.

10.4 Comparative Results

In this section the results for some of the well-known classical interfacial configurations are compared to the predictions of the interaction field approach. The analysis which follows is not general, but is presented in the form of a series of examples based on the assumption of specific forms for the interactions within the fluid. This reduction of generality is required in most cases since explicit results for the dynamics of the system are reliant, in the case of non-infinitesimal range of the interaction kernel, on the explicit form of the

in a number of references, for example in the books by Lamb [47] and by Landau and Lifschitz [...] to which the reader is referred for the derivation of the classical results for the dynamics of the various example systems to follow.

Principal use is made of the analysis presented by Lamb [47], based on the theory developed by Rayleigh [71], throughout which it was assumed that the fluid was inviscid. To match these results, throughout this chapter it is assumed that $\eta \equiv 0$ throughout $\mathcal{M}$. Through (7.21) this implies that the damping coefficients C^{ij} for the system are zero and hence any diagonalisation of the eigenvalue equations is of the form

$$(\lambda_i)^2 = -\frac{K}{M} \tag{10.8}$$

implying instability of the mode $\boldsymbol{v}^i$ if $K/M < 0$ and stability of the system if $K/M \geq 0$ for all modes.

In addition, the assumption of an inviscid system allows the analysis in terms of the velocity potential, derived in Section 5.5, to be used to establish suitable forms for the distortion modes of the fluid system.

10.3.1 Preliminary Results

The following definitions and results are used repeatedly in the analysis to follow.

Harmonic Zero-Forms

Let $\text{harm}^n(M) = \{\mathsf{H}^i\}_{i\in\mathcal{I}}$, where $\mathcal{I} \subset \mathbf{Z}$ and each $\mathsf{H}^i \triangleright \bigwedge^0 M$, denote the n-th set of harmonic zero-forms which is orthogonal and complete on M. Then (if such a set does exist)

$$\int_M *(\mathsf{H}^i\mathsf{H}^j) = \begin{cases} 0 & \text{if } i \neq j \\ r \in \mathbb{R} \sim \{0\} & \text{if } i = j \end{cases} \tag{10.9}$$

The analysis simplifies if (10.2) can be brought into diagonal form such that

$$M(\lambda_i)^2 + C\lambda_i + K = 0 \tag{10.4}$$

(where M, C and K are possibly different for each eigenvalue), in which case

$$\lambda_i = \frac{-C \pm \sqrt{(C)^2 - 4MK}}{2M} \tag{10.5}$$

10.3 Classical Results

The classical surface tension-based theory of the dynamics of fluids with interfaces is described in Section 4.6. This classical model can also be linearised about some equilibrium reference condition for the fluid system (see Section 7.2). Let $\tilde{\mathcal{M}}_{\mathcal{Z}}$ be the $\mathcal{M}$-complete fragmentation at the reference condition. The condition imposed by the Young-Laplace equation (4.30) on the pressure at the domain boundaries is transferred to the locations of the undistorted boundaries as

$$(\overline{p}|_{\partial\tilde{\mathcal{M}}_\alpha})|_{\tilde{\partial}_{\alpha\beta}} - (\overline{p}|_{\partial\tilde{\mathcal{M}}_\beta})|_{\tilde{\partial}_{\alpha\beta}} = \sigma_{\alpha\beta}\tilde{\kappa}_{\alpha\beta} \tag{10.6}$$

for all $\alpha, \beta \in \mathcal{Z}$, where $\tilde{\partial}_{\alpha\beta} = \partial\tilde{\mathcal{M}}_\alpha \cap \partial\tilde{\mathcal{M}}_\beta$ and $\tilde{\kappa}_{\alpha\beta} \triangleright \Lambda^0\tilde{\partial}_{\alpha\beta}$. Away from the interfaces, the linearised form

$$[\ \rho\tfrac{\partial}{\partial t}\nu + \mathrm{d}\overline{\mathcal{P}} + \mathrm{d}\overline{p} - \eta\mathrm{d}^2\nu\ ]|_{\tilde{\mathcal{M}}_\alpha} = 0 \tag{10.7}$$

of (4.29) is assumed to govern the dynamics of the system. Philosophical reconciliation between the classical approach and the the interaction field approach is obtained by realising that the pressure entering the classical equations is in fact the 'effective pressure' defined in Section 9.4.2 which contains within it a contribution from the internal interactions within the fluid. The linear dynamic theory based on the surface tension approach is presented in detail

10.2 Stability of Equilibria

The small-amplitude dynamics and stability of a particular near-equilibrium configuration can be determined from the linearised dynamic equation (7.42). Let $\vartheta \in \mathcal{D}$, where $\mathcal{D}$ is the configuration space of the fluid system defined in Section 7.2, correspond to an equilibrium configuration of the fluid system. Let $(a_1, \ldots, a_p)$ be coordinates for $\mathcal{D}$ in the neighbourhood of ϑ, and let the distortion modes associated with the choice of the equilibrium configuration as the reference condition for the system be $\boldsymbol{v}^i$, where $i = 1, \ldots, p$.

Assume the coordinates along the path $\vartheta(t)$ of the fluid system to vary with time according to

$$a_i = A_i e^{\lambda_i t} \tag{10.1}$$

(no summation intended). The constants A_i and the eigenvalues λ^i are possibly complex, but combine in (10.1) to yield real coordinates a_i, $i = 1, \ldots, p$. Substituting (10.1) into the linear equation (7.42)

$$M^{ij}(\lambda_i)^2 + C^{ij}\lambda_i + K^{ij} = 0 \tag{10.2}$$

where the inertia, damping and stiffness coefficients are given by (7.19), (7.21) and (7.41), and since from (8.6), at equilibrium

$$F^i = 0 \tag{10.3}$$

Equation 10.2 can be solved for the eigenvalues λ_i. The equilibrium configuration is unstable if any mode can be found such that the real part of its associated eigenvalue is positive, indicating a mode with displacement which increases exponentially with time. On the other hand, if no mode with positive or zero real part of its associated eigenvalue can be found, then the equilibrium configuration is (linearly) stable since all sufficiently small displacements to the equilibrium configuration of the system will disappear after sufficient time.

of the spherical interface, calculated numerically using the same interaction kernel as adopted in Section 10.4.1 and Section 10.4.2, for various values of the interaction range parameter k. Numerical experiments confirm the diagonality of the system and the independence of the eigenvalues of the system on the modal index j. (Interestingly, neither the inertia coefficients nor the stiffness coefficients are themselves independent of j.) Once again it is seen that as the range of the interaction kernel becomes small compared to the radius of the interface, the range of modal indices over which agreement between the eigenvalues obtained using the interaction field analysis and those obtained using the surface tension approach becomes larger, until in the limit it would appear that the two approaches yield identical results.

10.5 Conclusion

In Chapter 9 it was shown that the surface tension theory could be reconciled with the predictions of the interac ion field approach, in the sense that an expression for the surface tension of a static interface could be obtained even where the interaction kernel had non-infinitesimal range. In the limit as the range of the kernel became small compared to the curvature of the interface, the Young-Laplace equation governing the pressure change across the interface could be extracted. The examples of this chapter suggest that in dynamic fluid systems, at least where the dynamics is governed by the linearised equations derived in Section 7.8, a similar situation holds: Reconciliation between the predictions of classical surface tension theory and the interaction field approach is obtained in the limit as the range of the interactions within the fluid system becomes small compared to the characteristic dimensions of the system. Additionally, several cases have been shown where the results of the interaction field approach suggest that the classical analysis may be inappropriate in physical situations of interest. Of course, the statements made in Section 8.1.1 should once again be borne in mind when considering the interaction field approach as a replacement for the classical theory. As in Chapters 8 and 9, the results presented here are highly suggestive, but the analysis of the present

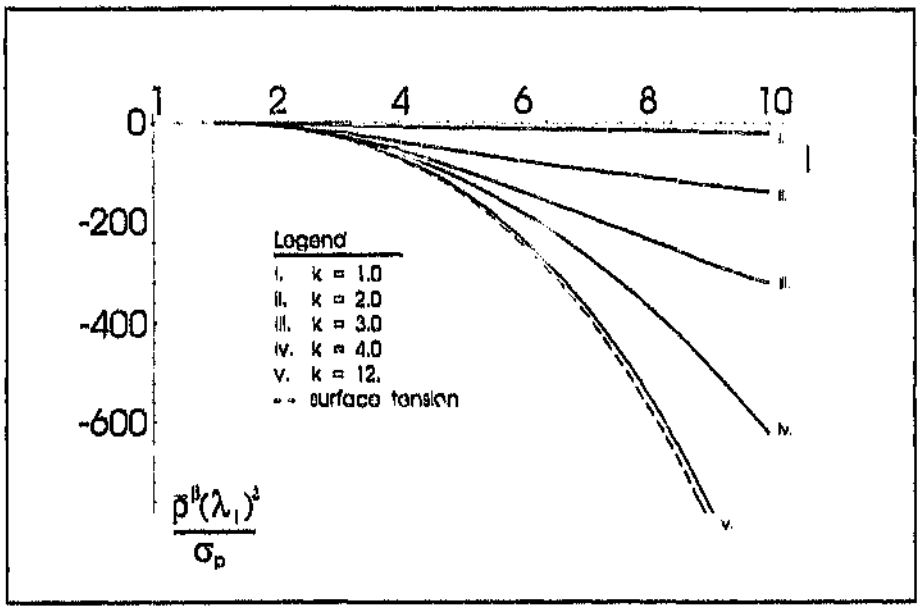


Figure 10.7: Eigenvalue-modal relationship for the spherical interface.

System Integrals

The momentum and stiffness coefficients for the system can be calculated by integration over the distortion modes, following the same procedure as used in Section 10.4.1 and Section 10.4.2. Since numerical evaluation of the resulting forms is required in most cases, explicit formulations for the coefficients will not be given here.

The classical analysis by Landau and Lifschitz [49], using surface tension theory, gives

$$(\lambda_{(i,j)})^4 = \frac{\sigma_p}{\bar{\rho}^{ij}} i\,(i-1)\,(i+2) \tag{10.60}$$

which is valid for $i \geq 1$, and where σ_p and $\bar{\rho}^{ij}$ are defined as before.

Figure 10.7 shows the eigenvalue-wavenumber relationship for the dynamics

Harmonic Forms

The set of harmonic zero-forms

$$\mathrm{harm}^{0\mathrm{ph}}(\mathbb{R}^3) = \{\mathsf{H}^i\}_{i\in\{0,\ldots,\infty\}} \tag{10.56}$$

on defining

$$\begin{aligned}\mathsf{H}^{2i}(r,\theta,\phi) &= r^i P_i^j(\cos\theta)\cos(j\phi)\\ \mathsf{H}^{2i+1}(r,\theta,\phi) &= r^i P_i^j(\cos\theta)\sin(j\phi)\end{aligned} \tag{10.57}$$

(no summation intended) where the modal indices $i,j \in \{0,\ldots,\infty\}$ and the $P_i^j \flat \Lambda^0\mathcal{M}$ are the associated Legendre polynomials, can then be used as a complete basis for the velocity potential $\lambda|_{\mathcal{M}_\beta}$ inside the sphere, under the restriction that the potential be finite at the origin of the coordinates. It is easily confirmed that the elements of $\mathrm{harm}^{0\mathrm{ph}}(\mathbb{R}^3)$ are orthogonal on $\partial_{\infty\beta}\times\mathbb{R}^+ = \mathbb{R}^3$.

Distortion Modes

The dynamics of the system follows once again by defining distortion modes

$$\upsilon^i = \mathsf{d}\mathsf{H}^i \tag{10.58}$$

such that the fluid velocity

$$\begin{aligned}\nu|_{\mathcal{M}_\beta} &= (\mathsf{d}\lambda)|_{\mathcal{M}_\beta}\\ &= \dot{a}_i\upsilon^i|_{\mathcal{M}_\beta}\end{aligned} \tag{10.59}$$

where again the $\dot{a}_i$ are the coefficients of the tangent vector to the curve representing the evolution of the system on the configuration manifold associated with the distortion modes.

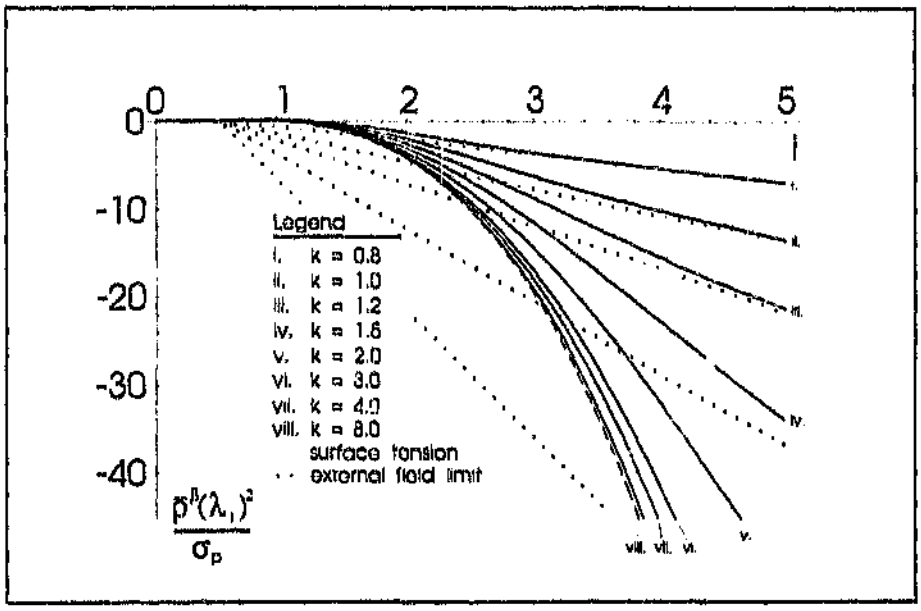


Figure 10.6: Eigenvalue-wavenumber relation for the cylindrical interface for large wavenumbers.

Let (r, θ, ϕ) and (r_*, θ_*, ϕ_*) be equivalent, globally-valid, spherical coordinates for $\mathcal{M}$, defined in the conventional sense. (Hence the Hodge form on $\mathcal{M}$ is $*1 = r^2 \sin\theta dr \wedge d\theta \wedge d\phi$ or equivalently $*1 = r_*^2 \sin\theta_* dr_* \wedge d\theta_* \wedge d\phi_*$.) Define

$$\begin{aligned} \mathcal{M}_\beta &= \{(r, \theta, \phi) : r < 1\} \\ \mathcal{M}_\alpha &= \{(r, \theta, \phi) : r > 1\} \end{aligned} \tag{10.55}$$

then $\partial_{\alpha\beta}$ is the sphere $\{(r, \theta, \phi) : r = 1\}$. The analysis for spheres of different radius follows as before by rescaling the coordinates.

As in the case of the cylindrical interface, the momentum of the fluid exterior to the cylinder is neglected to obtain closest agreement with Rayleigh's classical analysis. This leads to the same simplifications as in the case of the cylindrical interface.

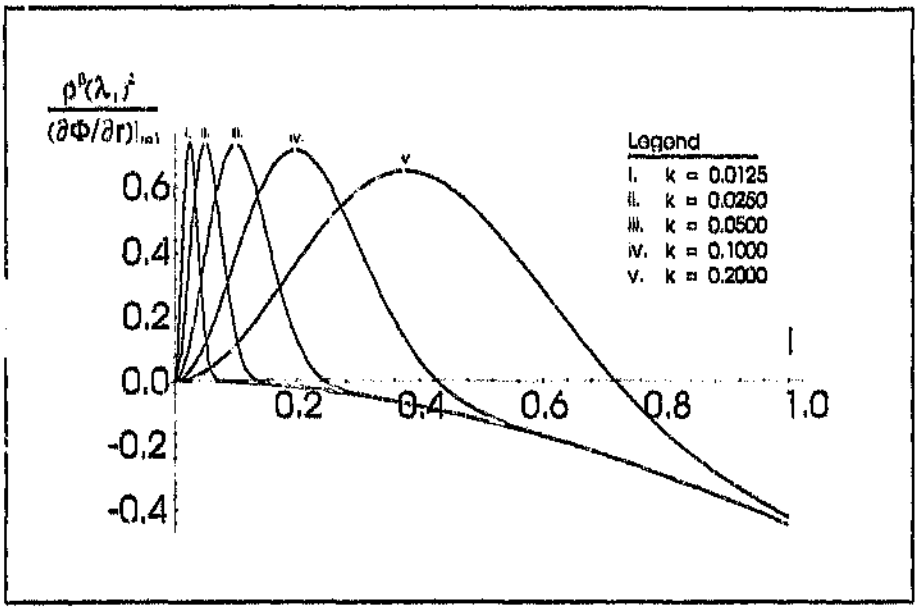


Figure 10.5: Eigenvalue-wavenumber relation: cylindrical interface regime iii.

(10.51) asymptotes to

$$(\lambda_i)^2 \approx -\left(i - \tfrac{1}{2}\right) \frac{\partial}{\partial r}\Phi|_{r=1} \tag{10.54}$$

for large i a. 1 finite k, and hence, as in the case of the planar interface, the dynamics of the cylindrical interface tends to an external-field type limit, governed by the field generated by the equilibrium configuration of the system, as the wavenumber of the distortion becomes large.

10.4.3 The Spherical Interface

The final case to be considered here (more for completeness than for any additional insight that this case might yield) is that for the dynamics of a liquid droplet suspended within an infinitely large domain of fluid.

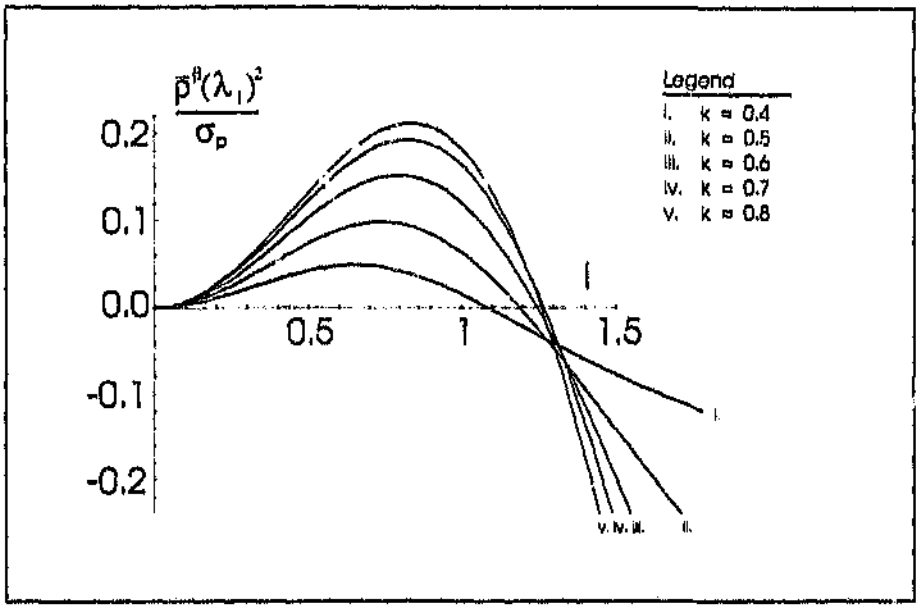


Figure 10.4: Eigenvalue-wavenumber relation: cylindrical interface regime ii.

this third regime may of course have important implications for the behaviour of the very thin threads of fluid encountered, for example, in the context of the rupture of cylindrical tubes of fluid as described by Tjahjadi [80] and others.

Figures 10.3, 10.4 and 10.5 show the eigenvalue-wavenumber relation for the system within each of the regimes i, ii, and iii of Figure 10.2. Of particular interest is Figure 10.3 which shows the asymptotic approach, as k becomes large, of the eigenvalue-wavenumber relation calculated using the interaction field approach to the relation obtained using surface tension theory. Finally, Figure 10.6 shows the large-wavenumber behaviour of the eigenvalue-wavenumber relation for the cylindrical interface. Since for large i

$$I_0(i) \approx \frac{1}{\sqrt{2\pi i}} e^i \tag{10.53}$$

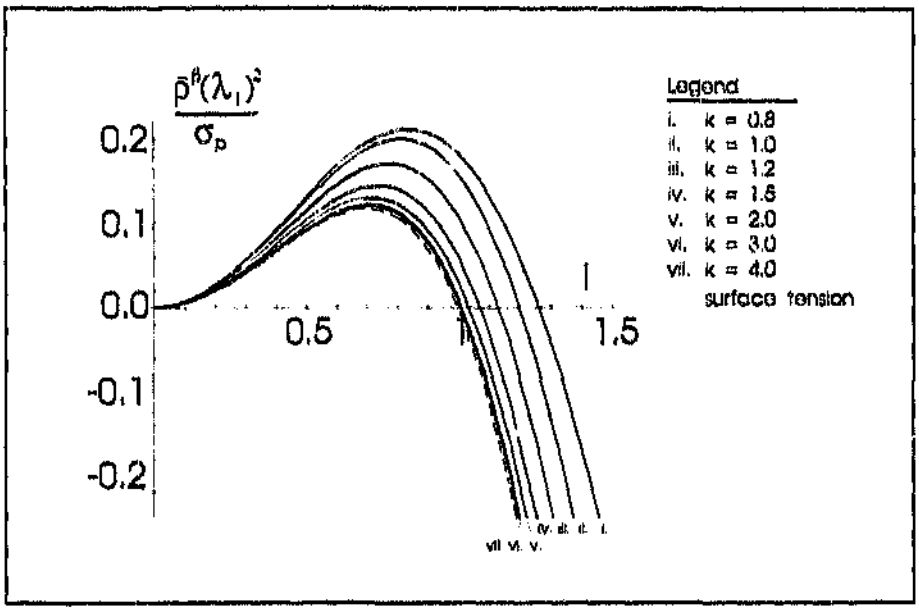


Figure 10.3: Eigenvalue-wavenumber relation: cylindrical interface regime i.

The classical result is well known to predict instability of all distortions to the interface which have a wavelength greater than the circumference of the cylinder (that is, a wavenumber less than unity). Figure 10.2 shows that, if the effects of non-infinitesimal range of the interaction kernel are taken into account, then the stability boundaries for the system have a more complicated behaviour than the classical theory would suggest. As the range of the interaction kernel becomes limitingly small compared to the radius of the cylinder (that is, $k \to \infty$, or regime i. of Figure 10.2) the classical result is regained, but for ranges comparable to the radius of the cylinder ($k \approx 1$ or regime ii. of Figure 10.2) the range of wavelengths for which the system is unstable is somewhat greater than that predicted by the classical theory. Finally, as the range of the interaction kernel becomes large compared to the radius of the cylinder (that is, $k \to 0$ or regime iii. of Figure 10.2), the range of wavelengths over which the system is unstable shrinks until at the limit the system is stable to distortions of all wavelengths. The behaviour of the stability boundaries in

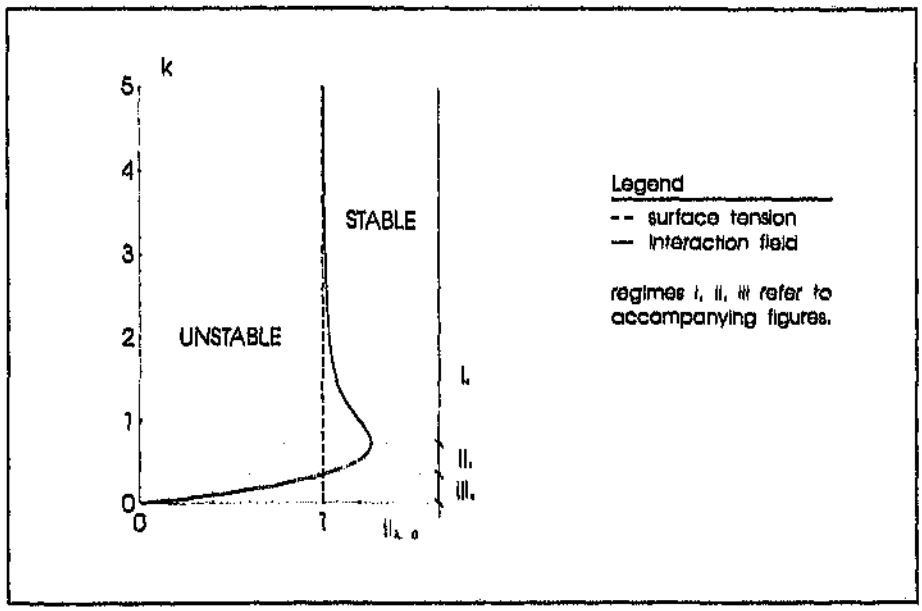


Figure 10.2: Stability-wavenumber relation for the cylindrical interface.

after manipulation of the Bessel functions. Explicit calculation of the eigenvalues using this relationship would appear to require numerical evaluation of the integrals for the potential and for the function $R(k)$.

The analysis[2] by Rayleigh [71] using surface tension theory yields the classical result

$$(\lambda_i)^2 = \frac{\sigma_p}{\bar{\rho}^d} \frac{i I_0'(i)}{I_0(i)} (1 - i^2) \tag{10.52}$$

where, as before, σ_p an be interpreted as the surface tension for the planar interface. As in the case of the dynamics of the planar interface, comparison of the classical analysis and the result derived above using the interaction field approach follows equation of σ_p to the value calculated in Section 9.4.3 for the interaction kernel $\mathcal{F}$ defined in (10.27).

[2]See also Drazin [24] for a more modern approach to the problem.

where the scaling parameter associated with the interval of orthogonality has been suppressed and the density of the fluid in $\mathcal{M}_\beta$ is $\overline{p}^\beta$.

If the same interaction kernel $\mathcal{F} \triangleright \bigwedge^0 \mathcal{M}^2$ is assumed as in the case of the planar interface subject to internal interactions, then the stiffness coefficients of the system can be calculated by substitution of (10.42) into (10.10) and direct integration as

$$K^{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 2\pi^2 i^2 I_0'(i)^2 (\frac{\partial}{\partial r}\Phi|_{r=1} - R(k)e^{-i^2/(4k^2)}) & \text{if } i = j \end{cases} \tag{10.47}$$

where

$$R(k) = \frac{\sqrt{\pi}}{k} \int_0^{2\pi} e^{2k^2(\cos u - 1)} du \tag{10.48}$$

and once again the scaling parameter associated with the interval of orthogonality has been suppressed. The potential

$$\Phi = \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^1 \mathcal{F}(r, \theta, z, r_*, \theta_*, z_*) r_* dr_* d\theta_* dz_* \tag{10.49}$$

follows from the expression

$$\begin{aligned} &\mathcal{F}|_{\mathcal{M}_{\beta \otimes \beta}}(r, \theta, z, r_*, \theta_*, z_*) \\ &= -\exp[\, k^2(r^2 + r_*^2 - 2rr_* \cos(\theta - \theta_*) + (z - z_*)^2)\,] \end{aligned} \tag{10.50}$$

for the interaction kernel in cylindrical coordinates.

System Dynamics

As once again the system integrals are in diagonal form, (10.8) can be used directly to calculate the eigenvalues of the fluid motion as

$$\begin{aligned} (\lambda_i)^2 &= -\frac{I_0'(i)^2}{\overline{p}^\beta \int_0^1 (r I_0'(ir)^2 + I_0(ir)^2) dr} (\tfrac{\partial}{\partial r}\Phi|_{r=1} - R(k)e^{-i^2/(4k^2)}) \\ &= -\frac{i I_0'(i)}{\overline{p}^\beta I_0(i)} (\tfrac{\partial}{\partial r}\Phi|_{r=1} - R(k)e^{-i^2/(4k^2)}) \end{aligned} \tag{10.51}$$

axisymmetric velocity potential $\Lambda|_{\mathcal{M}_\beta}$ inside the cylinder. This follows since it is easily shown that the elements of this set are orthogonal on $S^1 \times \mathbb{R}^+ \subset \mathbb{R}^3$ where

$$\begin{aligned} S^1 &= \{(r,\theta,z) : r = 1,\ s < z < 2\pi + s,\ s \in \mathbb{R}\} \\ &\subset \partial_{\alpha\beta} \end{aligned} \tag{10.43}$$

Note that the hyperbolic Bessel functions of the second kind can be eliminated from the basis since they do not result in finite velocity at the centre of the cylinder.

Distortion Modes

The dynamics of the system on the configuration manifold follows as before by defining distortion modes

$$\upsilon^i = \mathrm{d}\mathsf{H}^i \tag{10.44}$$

such that the fluid velocity

$$\begin{aligned} \nu|_{\mathcal{M}_\beta} &= (\mathrm{d}\Lambda)|_{\mathcal{M}_\beta} \\ &= \dot{a}_i \upsilon^i|_{\mathcal{M}_\beta} \end{aligned} \tag{10.45}$$

where the $\dot{a}_i$ are the coefficients of the tangent vector to the curve representing the evolution of the system on the configuration manifold associated with the distortion modes.

System Integrals

The inertia coefficients of the system can be evaluated by substitution of (10.42) into (7.19) and direct integration as

$$M^{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 2\pi^2 \bar{\rho}^\beta i^2 \int_0^1 (r I_0'(ir)^2 + I_0(ir)^2) dr & \text{if } i = j \end{cases} \tag{10.46}$$

10.4.2 The Cylindrical Interface

This example is most interesting since the classical theory predicts a transition from stability to instability of disturbances which are rotationally symmetric with respect to the axis of the cylinder as the wavelength of the disturbance is increased.

Let (r, θ, z) and (r_*, θ_*, z_*) be equivalent, globally-valid, cylindrical coordinates for $\mathcal{M}$, defined in the conventional sense. (Hence the Hodge form on $\mathcal{M}$ is $*1 = r dr \wedge d\theta \wedge dz$ or equivalently $*1 = r_* dr_* \wedge d\theta_* \wedge dz_*$.) Define

$$\begin{aligned} \mathcal{M}_\beta &= \{(r, \theta, z) : r < 1\} \\ \mathcal{M}_\alpha &= \{(r, \theta, z) : r > 1\} \end{aligned} \tag{10.40}$$

then $\partial_{\alpha\beta}$ is the cylinder $\{(r, \theta, z) : r = 1\}$. The analysis for cylinders of different radius follows of course by rescaling the coordinates.

Harmonic Forms

To obtain closest agreement with Rayleigh's classical analysis, the momentum of the fluid exterior to the cylinder is neglected. Only the flow within the cylinder then needs to be represented in terms of a suitable velocity potential, and in addition there is no need to apply the no-rupture condition on $\partial_{\alpha\beta}$. Assume that the flow within the cylinder is symmetric with respect to arbitrary rotation about the z-axis. The set of harmonic zero-forms

$$\mathrm{harm}^{cyl}(\mathbb{R}^3) = \{\mathsf{H}^i\}_{i \in \{0, \ldots, \infty\}} \tag{10.41}$$

where

$$\begin{aligned} \mathsf{H}^{2i}(r, \theta, z) &= I_0(ir)\cos(iz) \\ \mathsf{H}^{2i+1}(r, \theta, z) &= I_0(ir)\sin(iz) \end{aligned} \tag{10.42}$$

where the wavenumber $i \in \{0, \ldots, \infty\}$ and $I_0 \triangleright \Lambda^0\mathcal{M}$ is the order-zero hyperbolic Bessel function of the first kind, can then be used as a basis for the

The eigenvalue-wavenumber relations for the planar interface given by (10.37) and (10.38) are shown in Figure 10.1. The manner in which reconciliation between the result based on the interaction field approach and the result from classical analysis is achieved is somewhat surprising. In Section 9.4.1, reconciliation between the two approaches is achieved in the limit as the range of the interaction potential goes to zero. Here no such limit is necessary, and the classical theory is shown to bear a first order relationship to the interaction field approach even in the case where the range of the internal interactions is non-infinitesimal. Hence the limiting process is encountered in a different way: the classical theory becomes increasingly valid as the wavenumber of the distortion to the interface becomes small, and hence as the wavelength of the distortion becomes large, compared to the range of the interaction kernel. Note too that, on comparing (10.36) and (10.38), the limit for large wavenumbers i and j is identical to the result which would have been obtained if the interaction kernel was

$$\begin{aligned} \mathcal{F} &= \tfrac{\partial}{\partial z}\phi_{1/2}|_{z=0}\, z \\ &\triangleright \Lambda^0 \mathcal{M} \end{aligned} \tag{10.39}$$

in other words, if the interaction originated from a gravity-like external field rather than an internal interaction. This large-wavenumber behaviour can be explained by analysing the contribution of the components $G^{(i,j)(i',j')}$ and $I^{(i,j)(i',j')}$ to the stiffness of the system. $G^{(i,j)(i',j')}$ can be thought of as the contribution to the stiffness from distortions to the interface under the influence of the potential field generated by the unperturbed fluid at equilibrium. $I^{(i,j)(i',j')}$ is the direct contribution to the stiffness from the potential changes accompanying the distortion to the interface. As the wavenumber of the distortion becomes large, the external field-like behaviour becomes more apparent as the effects on the potential energy induced by the distortion modes becomes increasingly immersed in the field generated by the bulk fluid.

11.8 Critical Point Topology of $\Phi \triangleright \Lambda^0\mathcal{M}$

In this section the critical point structure of the potential on $\mathcal{M}$ generated by the interactions within a fluid system with the simple structure defined in Section 11.5 is analysed. The results obtained here are used to support the derivations of the qualitative dynamic results of Section 11.9.1.

11.8.1 Containment of Critical Points of Φ

Let the fluid domain $\mathcal{M}$ have the simple structure assumed in Section 11.5, and let the interaction kernel $\mathcal{F} \triangleright \Lambda^0\mathcal{M}^2$ have the form specified in Section 11.5.1. The following result shows that the critical points of the potential on $\mathcal{M}$ generated as a result of the interactions within this simple fluid system are all confined to within the convex hull of the union $\mathcal{U}$ of the interacting fluid domains.

Convex Hull

Given the rectangular coordinates (x^1, x^2, x^3) on $\mathcal{M}$ define the sets

$$\begin{aligned} P_i^+(a) &= \{(x^1, x^2, x^3) \in \mathcal{M} : a \leq x^i\} \\ P_i^+(a) &= \{(x^1, x^2, x^3) \in \mathcal{M} : x^i \leq a\} \end{aligned} \tag{11.9}$$

where $a \in \mathbb{R}$. Then define

$$\begin{aligned} a_i^+ &= \inf\{a \in \mathbb{R} : P_i^+(a) \cap \mathcal{U} = \{\emptyset\}\} \\ a_i^- &= \sup\{a \in \mathbb{R} : P_i^-(a) \cap \mathcal{U} = \{\emptyset\}\} \end{aligned} \tag{11.10}$$

and finally the sets

$$H_i = P_i^+(a_i^+) \cup P_i^-(a_i^-) \tag{11.11}$$

with respect to the volume form on $\mathcal{M}$. Then, for any finite $\omega \triangleright \Omega\mathcal{M}$,

$$\int_S \omega = \int_{S \sim \mu} \omega = \int_{S'} \omega \tag{11.5}$$

Let $S_\epsilon \subset \mathcal{M}$ be an *ϵ-perturbation* of S if

$$\text{meas}_n(S) = \text{meas}_n(S_\epsilon) \tag{11.6}$$

that is, the two sets have the same volume, but

$$\text{meas}_n(\ \text{diff}(S, S_\epsilon)\ ) < \epsilon \tag{11.7}$$

It then follows very simply from the midpoint theorem that

$$\int_{(S_\epsilon)^n} \omega = \int_{S^n} \omega + O(n\ \epsilon) \tag{11.8}$$

for any finite $\omega \triangleright \Omega\mathcal{M}^n$. Define S_ϵ as an ϵ-perturbation of S *based on* μ if $S \sim S_\epsilon$ has measure-zero set μ as a deformation retract.

The notion of an ϵ-perturbation based on a set of measure zero can be used to define the 'nearness' to equilibrium of the configuration of a system of fluid domains which might have different homology to that of the equilibrium configurations associated with the interactions present within the fluid. This follows since, by combination of (11.5) and the definitions of Section 7.5, in the limit as $\epsilon \to 0$, any ϵ-perturbation of an equilibrium configuration reverts to a system with the same variation of energy and hence dynamical characteristics as the equilibrium configuration, but possibly with a different topology, while (11.8) shows that the difference in the variation of the energy of any ϵ-perturbation of the equilibrium configuration and that of the equilibrium configuration itself is bounded. By defining the structure of the fluid system in terms of an ϵ-perturbation to some reference configuration of the fluid, the energy characteristics of a wide variety of fluid systems of physical interest, for example foams and systems with fractal boundaries, can be calculated by referring these systems back to other systems with simpler topology. This approach is adopted in the analysis presented in Section 11.9.1.

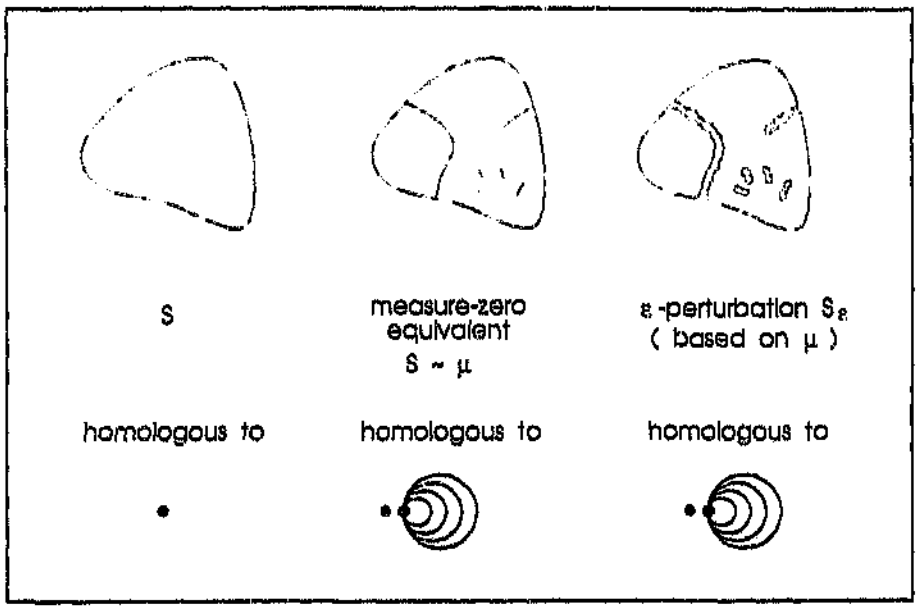


Figure 11.1: Schematic showing perturbed topologies of domain S.

two configurations of the fluid system which applies irrespective of whether the two configurations of the fluid system have the same topology or not. Considering the approach adopted in this work, where the dynamic equations for the system are defined in terms of a number of integrals over the fluid domain, the most relevant approach to the generation of a fluid configuration with perturbed topology compared to another configuration would appear to involve the removal of a point set with zero measure with respect to the volume form on the fluid domain. This follows since the removal of a set of zero measure from the fluid domain does not change the value of any integral over any finite volume form on the fluid domain, but can be used to modify the homology groups associated with the fluid configuration.

Formally, let $S \subset \mathcal{M}$ be some open set. Define the open set $S' \subset S$ be *measure-zero equivalent* to S if $S' = S \sim \mu$ where $\mu \subset \mathcal{M}$ is of measure zero

11.6 Critical Point Topology of $V \triangleright \Lambda^0\mathcal{D}$

If, in addition, $\mathcal{F}$ is invariant with respect to the symmetry group $\mathsf{iso}(\mathbb{R}^n)$ defined in Section 6.5.1, then, from the analysis of Section 8.6 and in particular the second example of Section 8.6.2, it would appear that the equilibrium configuration with lowest energy, resulting in a domain with finite volume, is obtained when $\mathcal{U}$ is the interior of a sphere with finite radius. For given volume of $\mathcal{U}$, it is easily shown using the techniques of Chapters 8 and 10 that the concentric spherical shell-type equilibrium configurations admitted by the symmetries of $\mathcal{F}$ have greater potential energy than the spherical equilibrium configuration, and are unstable to perturbations to the concentricity of the shells. The dynamical analysis of Section 11.4 thus suggests that this spherical equilibrium is an attractor for *all other* configurations of the system where $\mathcal{U}$ has finite volume.

Similarly, in the case of systems where $\mathcal{U}$ has unbounded volume, it is simple to show that the minimum potential energy is obtained when $\tilde{\mathcal{U}} = \{\emptyset\}$, or in other words when the whole of the fluid domain is filled with the interacting fluid. It would appear that there are no other stable equilibria, and hence that this state is an attractor for all other configurations of the system where $\mathcal{U}$ has unbounded volume.

Having postulated this rather bleak picture for the final state of any fluid system with the structure described in Section 11.5, the remainder of this chapter is concerned with establishing those principal characteristics of the evolution of a system with arbitrary topology which eventually allow it to reach one of the two stable equilibrium configurations described here.

11.7 Perturbed Topology

To make any progress in the analysis of the approach to equilibrium of systems with arbitrary topology, it is necessary to define some notion of 'nearness' of

Physically this structure models any fluid system in which a number of domains of a single, interacting, fluid are immersed in a second, inert, fluid domain. This class of fluid systems should be recognised as the same class of systems as that used in Chapters 9 and 10 to reconcile the static and dynamic predictions of the classical surface tension model and the interaction field approach. The generalisation of the results presented here to fluid systems with more complicated interactions is not at all straightforward and will not be attempted in this work.

11.5.1 Form of $\mathcal{F}$

Let (x^1, x^2, x^3) and (x_*^1, x_*^2, x_*^3) be equivalent, globally valid, rectangular coordinates for $\mathcal{M}$. Then $(x^1, x^2, x^3, x_*^1, x_*^2, x_*^3)$ are suitable coordinates for $\mathcal{M}^2$.

Assume $\mathcal{F} : \mathcal{M}^2 \to I\!R^-$. In addition, assume this map to have a single (degenerate) critical point on the projective symmetry set where $x^i = x_*^i$ for all $i = 1, 2, 3$, and to asymptote to zero as $|x^i - x_*^i| \to \infty$ for each $i = 1, 2, 3$. Assume $\mathcal{F}$ to be sufficiently smooth for these conditions to be expressed equivalently as

$$\frac{\partial}{\partial x^i}\mathcal{F}(x^1, x^2, x^3, x_*^1, x_*^2, x_*^3) \begin{cases} < 0 & \text{if} \quad x^i - x_*^i < 0 \\ = 0 & \text{if} \quad x^i - x_*^i = 0 \\ \to 0 & \text{if} \quad |x^i - x_*^i| \to 0 \\ > 0 & \text{if} \quad x^i - x_*^i > 0 \end{cases} \tag{11.4}$$

for all $i = 1, 2, 3$. These restrictions on the form of the interaction kernel imply additional constraints on the asymptotic properties of the interaction kernel used in the derivation of the results of Chapter 9. Note that the prototype interaction kernel $\mathcal{F}^r_{\mathrm{Iso}(I\!R^3)}$, defined in Section 6.5.1 and used extensively in the examples of the previous chapter, is of the form described here if the amplitude of the kernel is restricted to be an element of $I\!R^-$.

Note that (11.1) is identical to the equation

$$\mathsf{C}(\dot{\vartheta}) - \mathrm{d}V = 0 \tag{11.2}$$

written in terms of the coordinates on the configuration manifold $\mathcal{D}$, if C is the second-order symmetric definite[5] tensor field on $\mathcal{D}$ with components $\mathsf{C}_{ij} = C^{ij}$ in the coordinate basis for $\mathcal{D}$ adopted in Section 7.3. This form follows since, as described in Section 8.3, the force coefficients F^i, $i = 1, \ldots, p$, are the components of $\mathrm{d}V$, if the potential energy V is considered as a zero-form on $\mathcal{D}$. If the distortion modes can be arranged to remain constant throughout the evolution of the system, then C can assume the role of a time-invariant metric on $\mathcal{D}$. Equation 11.2 can then be written equivalently as

$$\dot{\vartheta} = \#\mathrm{d}V \tag{11.3}$$

using the dual operator defined in Section 1.9.1. This equation shows that the curves in $\mathcal{D}$ along which the fluid system evolves are the streamlines on $\mathcal{D}$ generated under the metric C by the potential energy $V \in \bigwedge^0\mathcal{D}$. Since these streamlines must necessarily terminate at the critical points of $V \in \bigwedge^0\mathcal{D}$, the long-term evolution of the system involves motion away from the maxima and towards the minima in the potential energy of the system, or in other words away from the unstable equilibria of the system towards the stable equilibria.

11.5 Fluid Structure

For the remainder of this chapter, a particularly simple structure to the fluid domain $\mathcal{M}$ is assumed. Let $\mathcal{M}$ be homeomorphic to $\mathbb{R}^3$ and let $\mathcal{M}$ contain a number of connected domains $\mathcal{M}_\alpha$, $\alpha \in \mathcal{Z}^+$ all subject to mutual and self interactions with *the same* interaction kernel $\mathcal{F} \in \bigwedge^0\mathcal{M}^2$. Define $\mathcal{U} = \bigcup_{\alpha \in \mathcal{Z}^+}\mathcal{M}_\alpha$ and $\dot{\mathcal{U}} = \mathcal{M} \sim \mathcal{U}$. Assume the fluid in $\dot{\mathcal{U}}$ not to be subject to any interactions.

[5] As in Section 7.9.3, symmetry and definiteness follow from definition (7.21) via the properties of the exterior product and the Hodge star operator, and the tensorial nature of C follows from the transformation law for a change of coordinates on $\mathcal{D}$.

The presence of the circle components to the deformation retract in three dimensions allows for complex tangling of domains (since circles do not admit any notion of 'exterior' or 'interior') whereas no such tangling is possible in two dimensions. In three dimensions it is partially this tangling which leads to the problems, associated with the relaxation of the system to an equilibrium configuration, described in the introduction to this chapter. In two or three dimensions an additional impediment to the relaxation process arises from the entrapment of all voids located within the domains comprising the fluid system.

11.4 Simplified Evolution Equations

For the purposes of studying the motion of any fluid system towards its final equilibrium configuration, the equations derived in Chapter 7 contain too much detail of the initial dynamics of the fluid system. A simplified set of equations applicable to the final dynamics of the system can be derived if it is assumed that the initial dynamics of the system (perhaps following some perturbation from equilibrium) is governed by the interaction between the kinetic and potential energy terms in the equations of motion, but as time progresses, the kinetic energy of the system is dissipated through the action of the fluid's viscosity. In this case the initial dynamic eventually decays into a motion where the kinetic energy of the system is small enough to be neglected, and where the dynamics of the fluid is governed in the long term by the interaction between the potential energy and the dissipation.

The equations of motion for this final, dissipation-dominated, phase of the dynamics are easily derived from the results of Chapter 7. If the kinetic energy of the system can be neglected, then (7.38) reduces to

$$C^{ij}\dot{a}_j - f^i = 0 \tag{11.1}$$

where $i, j = 1, \ldots, p$.

11.3 Specification of Topology

When calculating of the potential associated with a particular fluid configuration, specification of the topology of the system via the homology groups[3] of the domains $\mathcal{M}_\alpha$, $\alpha \in \mathcal{Z}$ is usually sufficient. This follows since the structure of the fluid domains, in terms of the number of enclosed voids, is captured by these topological invariants. It is shown in Section 11.8.2 that the critical point topology of the potential $\Phi \triangleright \bigwedge^0 \mathcal{M}$ is strongly governed by the number of voids in each domain within the fluid and hence that the critical point topology of Φ strongly influences the route which the fluid configuration follows in its motion towards its final equilibrium.

11.3.1 Homology

By suitable definition, each of the domains $\mathcal{M}_\alpha$, $\alpha \in \mathcal{Z}$ can be arranged to be a connected subset of the fluid domain $\mathcal{M}$. If $\mathcal{M}$ is two-dimensional, then each connected domain $\mathcal{M}_\alpha$, $\alpha \in \mathcal{Z}$ has a deformation retract[4] which is homeomorphic to an object $\mathcal{H}^2(c)$ consisting of $c \in \mathbf{Z}$ circles intersecting at a single point. If $\mathcal{M}$ is three-dimensional then matters are somewhat more complicated. Each connected fluid domain has a deformation retract which is homeomorphic to an object $\mathcal{H}^3(s, c_0, \ldots, c_s)$ consisting of $s \in \mathbf{Z}$ spheres S_i, $i = 1, \ldots, s$, nested such that S_i is interior to S_{i+1} and such that all the spheres intersect at a single point P, together with $s+1$ sets of circles $\mathcal{C}_i = \{C_j\}_{j=1,\ldots,c_i}$, $i = 0, \ldots, s$, nested such that the elements of $\mathcal{C}_i$ are exterior to the sphere S_i but interior to the sphere S_{i+1} (on defining an imaginary, non-physical, sphere S_0 to lie in the interior of S_1) and intersect at P. This approach, although not nearly as elegant as the usual approach to homology through group theory, is adopted here as it gives a more concrete physical picture of the configuration of the fluid system than that obtained through the usual approach.

[3] A full account of homotopy and homology theory in the context of the mechanics of continuous media is given by Mermin [56].

[4] A discussion of the relation between the homology groups and the deformation retract of a point set is contained in the book by Nash and Sen [61].

of the transformation associated with the fluid's velocity[2].

From the analysis of Chapter 8, it would appear that the set of domain topologies associated with the equilibrium configurations of the fluid system, given the symmetry group of the interaction kernel, is in general only a small part of the set of all possible topologies that the domains within the fluid might be able to adopt. Since, via the argument presented above, the topology of the fluid domains cannot change within finite time, this suggests that if the topology of all the domains $\mathcal{M}_\alpha$, $\alpha \in \mathcal{Z}$, comprising $\mathcal{M}$ does not conform to one of the topologies associated with the equilibrium configurations, given interaction kernels for all the species present in the fluid, then the topology of the domains may act in some fashion to restrain the fluid from attaining an equilibrium configuration within finite time. This idea has been explored by Moffatt [57], [58] in the context of the relaxation of magnetic systems and incompressible fluids to their equilibrium configurations. Some of Moffatt's ideas are continued in the work presented in this chapter.

It is possible that the evolution of the fluid system will drive the configuration of the fluid to within an appropriately-defined neighbourhood of one of the equilibrium configurations generated by the particular form of the interactions within the system, but it is not immediately obvious which of the equilibrium configurations associated with the interactions serves as the attractor for an arbitrary system with arbitrary topology. In this chapter it is shown how the dynamics of a fluid configuration, the topology of which can be considered as a perturbation to the topology of a given equilibrium configuration of the fluid, can be reconstructed from a knowledge of the potential associated with the unperturbed system, and hence how a fluid system comprised of domains with essentially arbitrary topology attains an equilibrium configuration.

[2]The most general approach to the various limitations to topological change in continuum systems is possibly that adopted by Truesdell and Toupin [84].

Section 11.5 to the class of fluid systems for which the classical results of Chapter 10 were derived. In Section 11.6 it is shown that the set of final equilibrium configurations adopted by this class of fluid systems may be even more limited than the analysis of Chapter 8 would suggest. In Section 11.7, a method for relating the integral properties of two fluid configurations which are nearby in a certain sense is derived, and, after derivation of some supporting results in Section 11.8, this analysis is used in Section 11.9 to derive a mechanism by which a system comprised of domains having arbitrary topology might, given sufficient time, attain its final equilibrium configuration. Finally, the same integral geometric approach is used in Section 11.10 to resolve the reasons for the persistence of those classically-modelled systems, mentioned in Chapter 9, which would appear to have a non-equilibrium configuration when analysed using the interaction field approach.

11.2 Topological Change

From the analyses of Section 4.5 and Section 5.3, the velocity $\nu \triangleright \mathcal{XM}$ in the fluid is continuous within each $\mathcal{M}_\alpha \subset \mathcal{M}$, $\alpha \in \mathcal{Z}$ for all time provided that there are no initial discontinuities in the velocity. In the case of an identically inviscid fluid, the vorticity, and hence any discontinuity in the velocity, in the fluid system is confined to the interfaces within the fluid, while if the system has non-zero viscosity then the velocity will be continuous throughout $\mathcal{M}$ for all time.

These results imply that the transformations $\varphi_t : \mathcal{M} \to \mathcal{M}$ and $\psi_t : \mathcal{M} \to \mathcal{M}$ associated with the velocity ν (see Section 2.1 and Section 7.2) are continuous on each $\mathcal{M}_\alpha$, and hence, if the velocity is assumed to be finite everywhere, are homeomorphisms *over finite times*. This implies that the domains $\mathcal{M}_\alpha$, $\alpha \in \mathcal{Z}$, cannot undergo any topological change in finite time[1] under the action

[1]Observed topological change in finite time in some fluid systems is most likely associated with bulk transfer processes such as diffusion and evaporation. These physical effects were excluded from the present analysis at an early stage: see Chapter 2.

Chapter 11

Domains with Arbitrary Topology

11.1 Overview

In the previous chapter, linearised dynamic results were obtained for a number of fluid systems subject to pairwise interactions with a kernel having the symmetry group $\mathrm{iso}(\mathbb{R}^6)$ defined in Chapter 8. In each case, the topology of the fluid system coincided with the topology of one of the equilibrium configurations enumerated in Chapter 8 for systems subjected to $\mathrm{iso}(\mathbb{R}^6)$-symmetric pairwise interactions. In this chapter, some heuristic arguments enabling a qualitative understanding of the dynamics associated with the motion towards equilibrium of fluid systems composed of domains which have more arbitrary topology are derived.

The results derived in Chapters 4 and 5 for the continuity of the velocity of the fluid are shown in Section 11.2 to impose strong constraints on the evolution of the topology of the domains comprising the fluid system. After the introduction of some preliminary results in Section 11.3, a simplified evolution equation, valid during the final decay of a system towards its equilibrium configuration, is derived in Section 11.4. The analysis is then specialised ir

chapter lacks complete generality through the assumption of a single specific form for the interaction kernel. It should thus be realised that an interaction field model based on an interaction kernel other than $\mathcal{F}^{\mathrm{pr}}_{\mathrm{iso}(\mathbb{R}^n)}$ will, if the kernel has non-infinitesimal range, in all likelihood yield detailed predictions of the dynamics of the fluid system which are somewhat different to those presented here.

In this chapter, the dynamics of fluid systems with fairly simple topology have been considered. It remains to extend the results developed here to the dynamics of systems with more general topology. Some progress towards this generalisation, at least in a qualitative sense, is presented in the following chapter.

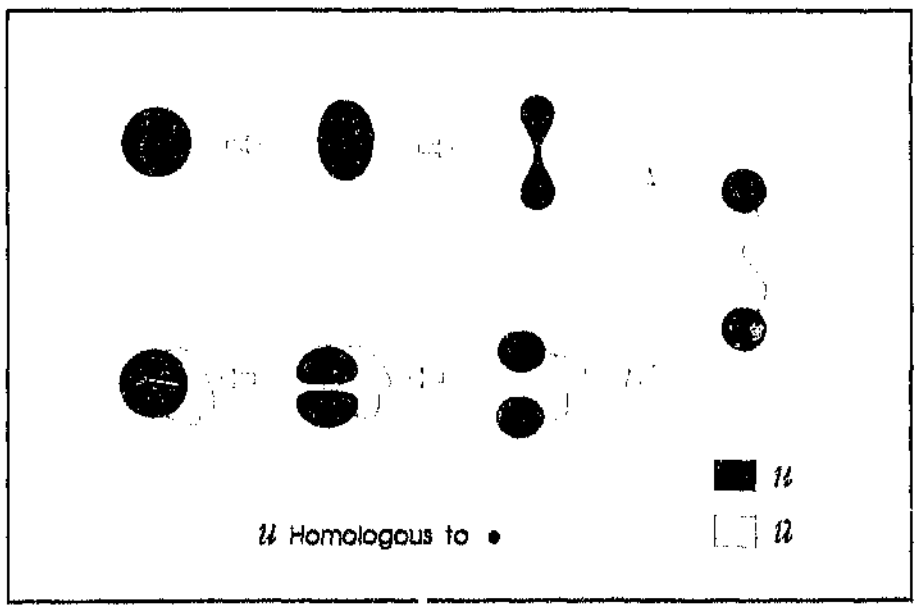


Figure 11.6: Relaxation to equilibrium: Rupturing droplet.

with the motion of a set of domains in $\mathcal{M}$ where the sets in $\mathcal{M}^n$ representing the mutual interactions all lie outside S is small. From the analysis of Section 11.4, the configuration of the system is drawn towards one of minimum energy at a rate which is proportional to dV, and hence in the case of a slowly-varying interaction kernel there are likely to exist some modes, principally those governing the motion of the separate domains relative to each other, in which the system evolves slowly[11] compared to the evolution of the separate domains towards independent equilibrium configurations. The evolution towards independent equilibrium configurations is of course governed by the changes in energy associated with deformations of the sets in $\mathcal{M}^n$ representing the self-interactions within the system, and, since these sets all intersect with

[11]Note that since the modes associated with the expulsion of defects fall into this category, the ejection mechanism described in Section 11.9.1, especially in the case of spheroidal defects, will also be inhibited in a system where the interactions have the characteristics described here.

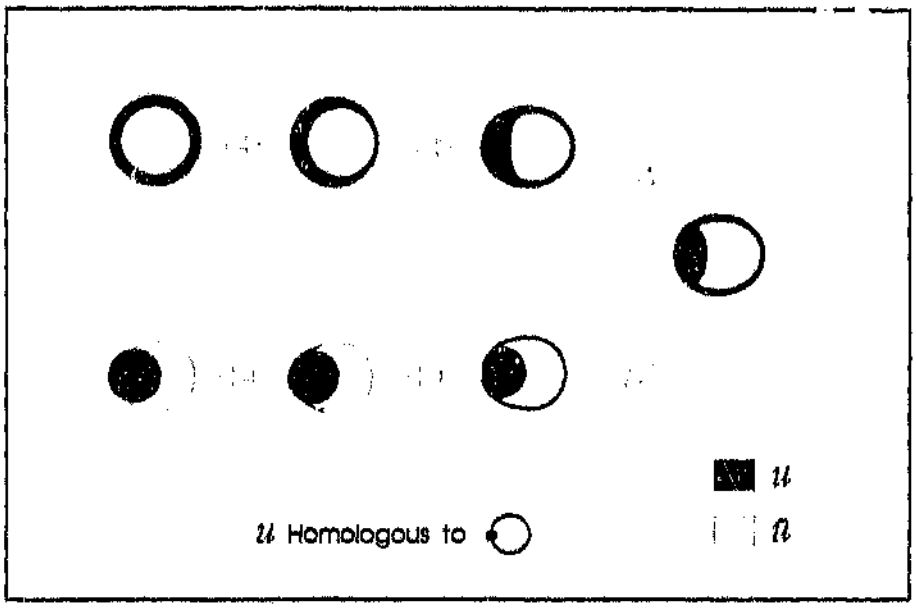


Figure 11.5: Relaxation to equilibrium: The bursting bubble.

of $\mathcal{M}^n$ (see Section 6.5.1 and Section 9.2). Again assume the simple structure for $\mathcal{M}$ given in Section 11.5. In the case of restricted support for $\mathcal{F}$, as shown schematically in Figure 11.8, if the separation between the domains comprising $\mathcal{U}$ becomes greater than a certain value, then the contributions to the potential energy from the mutual interactions between the domains becomes zero (that is, when the sets in $\mathcal{M}^n$ representing the mutual interactions all lie outside $\text{supp}(\mathcal{F})$). In this case the only contributions to the potential energy are due to the self-interactions within the domains, and the various domains behave essentially as though they were independent of each other.

Physically the more likely case (and the case consistent with the assumptions made in Section 11.5.1 regarding the form of the interaction kernel) is that $\mathcal{F}$ is not zero anywhere within $\mathcal{M}^n$, but outside some $S \subset \mathcal{M}^n$ enclosing the projective symmetry set of $\mathcal{M}^n$ the interaction kernel is *slowly-varying* in the sense that $d\mathcal{F}$ is small. In this case, the potential energy changes associated

likely be drawn into that domain.

The results presented here assume a rather constrained form for the interactions in the fluid. The generalisation of these results to more complicated interactions will not be conducted here, but any such generalisation to the case where both the defect and the domain containing the inclusion are both subject to interact' :s should of course show that the ejection mechanism disappears if the mutual and self-interactions in the inclusion and in its containing domain all have the same form.

Some examples showing how a fluid system with arbitrary topology might relax to an equilibrium configuration via the ejection mechanism are presented[10] in Figures 11.5, 11.6 and 11.7.

11.10 Coexistence of Equilibria

As described in Chapter 9, there appears to be an essential contradiction between the predictions of the surface tension and interaction field based approaches. The interaction field approach predicts the existence of very few equilibria, towards which, as shown in Section 11.4 the system will evolve in the long term. In the classical analysis there are however certain systems, the configurations of which do not conform to any of these equilibrium configurations, but for which the analysis based on surface tension theory and the Young-Laplace equation predicts no further evolution.

Reconciliation follows from the analysis of Chapter 9 and Chapter 10, where it is shown that the interaction field and surface tension based approaches are fully compatible effectively only if the support of the interaction kernel $\mathcal{F} \triangleright \Lambda^0 \mathcal{M}^n$ is restricted to a finite neighbourhood of the projective symmetry set

[10]Note that these figures are highly schematic. The actual sequence of distortions which the system undergoes is not only dependent on the range of the internal interations within the system, but also, of course, on the viscosities and densities of the various fluids comprising the system.

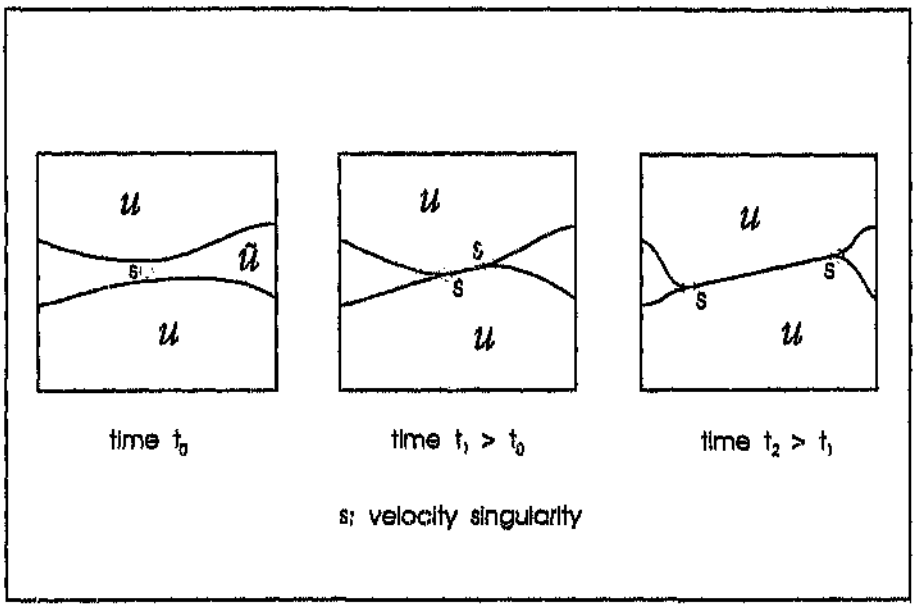


Figure 11.4: The draining mechanism for ejection of defects.

Generalisation

A similar argument follows where $\mathcal{U} = \mathcal{M}_\alpha \cup \mathcal{I}$ but $\mathcal{M}_\alpha \cap \mathcal{I} = \{\emptyset\}$. The analogous equation to (11.24) in this case is

$$V = \int_{(\mathcal{M}_\alpha)^2} *\mathcal{F} + \int_{\mathcal{I}^2} *\mathcal{F} + 2A\Phi|_x \tag{11.25}$$

where $A = \int_{\mathcal{I}} *1$ is the volume of the defect and x is again some point in $\mathcal{I}$. The same tendency towards an equilibrium configuration is indicated by the first two terms on the right hand side of the equation, but the change in sign of the third term indicates, through analogous arguments to those of the previous section, that the defect will be drawn towards a minimum, rather than a maximum, in the potential generated by the domain $\mathcal{M}_\alpha$. Hence, from the containment result and the analysis of Section 11.8.2, defects lying near the boundary of a nearby domain with the same form of interaction will most

that is, $\int_{\mathcal{I}} *\Phi / \int_{\mathcal{I}} *1$ is maximised. Since Φ is a maximum at the boundary of $\mathcal{M}_a$, this average is of course maximised if $\mathcal{I}$ is itself located at the boundary of $\mathcal{M}_a$. Hence, the distortion of an extended defect in such a way as to shift its volume towards the boundary of the domain containing the defect will be energetically favoured. Of course, the instability associated with extended defects suggested by the analysis of Section 10.4.2 may lead to the formation of a series of spheroidal defects within the fluid before this process is completed, but the resultant defects will in turn will be subject to the ejection mechanism as already discussed.

The somewhat heuristic analysis presented above is of course strongly dependent on there being a dynamic mechanism, or, in the terminology of Chapters 7 and 10, a distortion mode, present which allows the fluid to deform in such a way as to expel the defect. The dynamics would most likely involve the film-draining mechanism[8] described in detail by various authors[9] including Mysels *et al.* [60], Isenberg [37] and Johannes and Whitaker [38]. It appears that this mechanism is capable of transporting a defect with arbitrary topology to the boundary of its containing domain simply, in the case of a spheroidal defect, by a process of extrusion of the fluid separating the defect from the boundary, or, in the case of an extended defect, by extrusion of the fluid in the defect itself as illustrated in Figure 11.4. Note that the action of the draining mechanism on any domain in $\mathcal{M}$ is to convert the domain into its deformation retract. This implies that under the action of the draining mechanism, every fluid domain in the final configuration of the system will have the same homology groups as it had in the initial configuration of the fluid system. It is important to realise that the draining mechanism is singular, and hence, as discussed in Section 11.2, in general the expulsion of defects from a domain will require an infinite amount of time to run to completion. Equivalently, the draining mechanism satisfies the implicit condition on any evolution process acting on the fluid, derived in Section 11.2, that no topological change within the system should occur within finite time.

[8] Also known as 'marginal regeneration' in the foam community.

[9] See also the fractal version of this mechanism described by Brenner *et al.* [11] and Shi *et al.* [74].

where $A = \int_{\mathcal{I}} *1$ is the volume of the defect and x is some point *in* $\mathcal{I}$. Equation 11.21 shows two competing effects: The first two terms on the right-hand side arise from the interaction of the fluid in $\mathcal{M}_a$ with itself, and, from the first example of Section 8.6.2, drive the configuration of both $\mathcal{M}_a$ and the defect to an equilibrium configuration related to the form of $\mathcal{F}$.

The second term in (11.24) shows the potential energy to be minimised when x is such that $\Phi|_x$ is maximised, and hence motion of x in the direction of increasing potential Φ is favoured energetically. Remembering that $x \in \mathcal{I}$, motion of x can only be achieved by a combination of distortion and convection of $\mathcal{I}$.

For limitingly small defects where $\mathcal{I}$ reduces to the infinitesimal neighbourhood of x, only the transport mechanism can be operational. For larger spheroidal defects the distortion mechanism will be inhibited by the previously-mentioned tendency of the configuration towards an equilibrium configuration, which, if $\mathcal{I}$ is compact, is one with spherical boundary if, as in Section 11.6, the interaction kernel is assumed to be invariant with respect to the symmetry group iso$'$).

Motion o wards a maximum in Φ in the case of spheroidal defects will thus in general be accompanied by a motion of the defect itself. The spheroidal defect will be driven towards the increase in potential associated with any other defects (see Section 11.8.2) and this process would lead to a coalescence of defects. Alternatively, since as shown in Section 11.8.2 the potential generated by a kernel of the form specified in Section 11.5.1 generally increases towards the boundary of $\mathcal{M}_a$, eventually a single defect or a coalescence of defects will be ejected onto the boundary of the domain $\mathcal{M}_a$. Of course, once having reached the boundary, the driving potential gradient is no longer present and the final stages of the ejection process will be dominated by viscous and inertial effects.

In the case of extended defects the convecti e mechanism is inhibited by the attachments of the defect to the boundary of $\mathcal{M}_a$. In this case, motion of x must occur primarily by distortion of the defect. The second term of (11.24) can be reinterpreted to imply that V is minimised if the *average* of Φ over $\mathcal{I}$,

11.9.1 Dynamics of Defects

Let $\mathcal{S} \subset \mathcal{M}$. A set $\mathcal{I} \subset \mathcal{S}$ will be termed a *spheroidal defect* in $\mathcal{S}$ if $\mathcal{I} \cap \partial\mathcal{S} = \{\emptyset\}$, while if $\mathcal{I} \cap \partial\mathcal{S} \neq \{\emptyset\}$, then $\mathcal{I}$ will be termed an *extended defect* in $\mathcal{S}$. Note that it is not necessarily implicit that $\mathcal{I}$ have a volume which is small compared to that of $\mathcal{S}$.

In this section, the mechanism by which any system which is a perturbation of an equilibrium configuration attains measure-zero equivalence to the equilibrium configuration is established by considering the dynamics of a defect immersed somewhere within one of the domains comprising the fluid system.

Ejection Mechanism

To model the dynamics associated with defects, assume $\mathcal{M}$ to have the simple structure assumed in Section 11.5. Let $\mathcal{U}$ consist of a single connected domain $\mathcal{M}_\alpha \sim \mathcal{I}$, where the defect domain $\mathcal{I} \subset \mathcal{M}_\alpha$. As before, let $\tilde{\mathcal{U}} = \mathcal{M} \sim \mathcal{U}$. Let the interaction kernel for the fluid in $\mathcal{U}$ obey the restrictions on form required for the containment result of Section 11.8.1 to apply. Then, from (6.66), the potential energy of the system is

$$\begin{aligned} V &= \int_{(\mathcal{M}_\alpha \sim \mathcal{I})^2} *\mathcal{F} \\ &= \int_{(\mathcal{M}_\alpha)^2} *\mathcal{F} + \int_{\mathcal{I}^2} *\mathcal{F} - \int_{\mathcal{M}_\alpha \times \mathcal{I}} *\mathcal{F} - \int_{\mathcal{I} \times \mathcal{M}_\alpha} *\mathcal{F} \\ &= \int_{(\mathcal{M}_\alpha)^2} *\mathcal{F} + \int_{\mathcal{I}^2} *\mathcal{F} - 2\int_{\mathcal{I}} *\Phi \end{aligned} \tag{11.22}$$

where

$$\Phi = \oint_{\mathcal{M}_\alpha} *\mathcal{F} \tag{11.23}$$

is the potential on $\mathcal{M}$ created by the fluid in $\mathcal{M}_\alpha$ when the defect is not present. Finally, using the midpoint theorem,

$$V \cong \int_{(\mathcal{M}_\alpha)^2} *\mathcal{F} + \int_{\mathcal{I}^2} *\mathcal{F} = 2A\Phi|_x \tag{11.24}$$

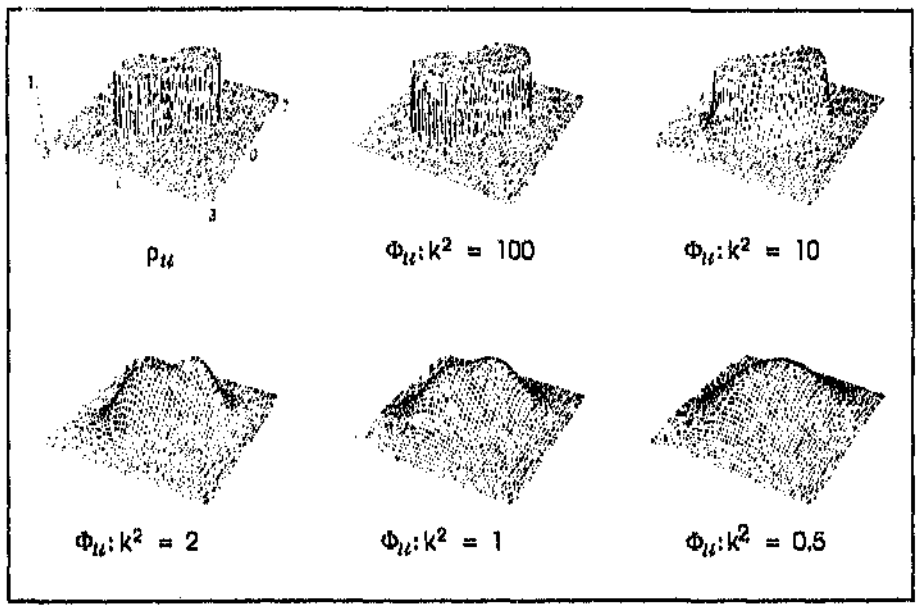


Figure 11.3: The interaction kernel as a smoothing function.

11.9 Evolution to Equilibrium

The critical point structure of Φ associated with some fluid configuration can be used to obtain a qualitative description of the dynamics of any perturbation to that configuration. In Section 11.4 it was argued that the long-term evolution of the fluid system terminates at the stable equilibrium associated with the minimum potential energy which the fluid system can attain. In this section it is postulated that the principle mode of evolution to equilibrium of a system with arbitrary topology is via the ejection of domains of material, leading to a final configuration which is *measure-zero equivalent* to the minimum-energy equilibrium configuration of the system admitted by the interactions present in the system.

$\Phi \triangleright \bigwedge^0 \mathcal{M}$ induced on $\mathcal{M}$ by this kernel[7] is, from (11.13),

$$\Phi = q\, \rho_{\mathcal{U}} \tag{11.21}$$

after substituting (11.20) into (11.13).

By inspection, the number of maxima in this potential is equal to the minimum number of connected domains from which $\tilde{\mathcal{U}}$ can be constructed, while the number of minima in the potential is equal to the minimum number of connected domains from which $\mathcal{U}$ can be constructed. In the case of non-zero range of the interaction kernel, the monotone nature of the derivatives of $\mathcal{F}$ imply that (11.15) can be interpreted as a convolution of the occupation function $\rho_{\mathcal{U}}$ with a *smoothing function* given by the interaction kernel itself, or, by substitution of (11.21), as a smoothing operation on the potential obtained in the limit of zero range of the interactions present in the system. This smoothing operation of course modifies the critical point topology of the zero-range potential through both the creation and destruction of maxima and minima, but the smoothing is concentrated in those parts of $\mathcal{M}$ where the characteristic dimensions of the domains $\mathcal{U}$ and $\tilde{\mathcal{U}}$ are of the same order as, or are smaller than, the range of the interaction kernel. This smoothing of the zero-range potential, and the associated generation of additional critical points, is illustrated in the example of Figure 11.3 which was calculated using the kernel $\mathcal{F}^{pr}_{\mathrm{iso}(\mathbb{R}^4)}$ defined in Section 8.3.1 as a four-dimensional analogue to the kernel $\mathcal{F}^{pr}_{\mathrm{iso}(\mathbb{R}^3)}$ defined in Section 6.5.1. Most importantly for the analysis which follows, if the interaction kernel is of the form specified in Section 11.5.1, then the potential on $\mathcal{U}$ is usually at or near its maximum value close to the boundary of $\mathcal{U}$.

[7] An interesting aside is that if the interaction kernel was indeed given by (11.20), then from (11.21) and (3.7) the potential energy of the system $V = q\ \mathrm{meas}_{\bullet 1}\rho_{\mathcal{U}} = q\ \mathrm{meas}_{\bullet 1}\mathcal{U}$ which remains constant as the fluid system undergoes a volume-preserving deformation. Hence, in the context of the dynamics of the system, a kernel of the form given by (11.20) would lead to a lower order approximation than even the surface tension based theory, since all configurations of the system would be in neutral equilibrium!

where $i \in [1,2,3]$. If $\infty < x^i \leq a_i^-$, then $x^i - x_*^i < 0$ and from (11.4) the integrand in (11.19) is always negative. Hence the integral itself cannot be zero. Similarly if $a_i^+ \leq x^i < \infty$, then $x^i - x_*^i > 0$, the integrand is always positive and the integral again cannot be zero. Hence any zeros of this integral, if there are any, must lie in $\mathcal{M} \sim H_i$ or at $|x^i| = \infty$. Combining similar analyses for each $i \in [1,2,3]$, the zeros of $\mathrm{d}\Phi_{\mathcal{U}}$ must lie in H or on the infinite boundary of $\mathcal{M}$. Hence, combining the analyses for all possible rectangular coordinates on $\mathcal{M}$, the critical points of $\Phi_{\mathcal{U}}$, and hence from (11.16), except for the degenerate critical point occupying the whole of $\hat{\mathcal{U}}$, the critical points of the potential Φ generated by the interactions in $\mathcal{U}$ must lie in the convex hull of $\mathcal{U}$ or on the boundary of $\mathcal{M}$.

11.8.2 Estimating the Critical Point Topology of Φ

The number and type of critical points associated with the interactions within the fluid can be estimated if the interaction kernel has the form given in Section 11.5.1. If the range of the interaction kernel is identically zero, then the only kernel which is compatible with the restrictions of Section 11.5.1 is (up to scaling by some value $q \in \mathbb{R}^-$) the delta kernel

$$\begin{aligned} \mathcal{F}_\delta &: \mathcal{M}^2 \to \mathbb{R} \\ &: (x^1, x^2, x^3, x_*^1, x_*^2, x_*^3) \mapsto \prod_{i=1,2,3} \delta(x^i - x_*^i) \end{aligned} \tag{11.20}$$

using the rectangular coordinates on $\mathcal{M}^2$ defined in Section 11.5.1, where $\delta : \mathbb{R} \to \mathbb{R}$ is the usual delta function on the real line.

If $\mathcal{M}$ has the simple structure assumed in Section 11.5 then the potential

$$= \rho_{\mathcal{U}} \oint_{\mathcal{U}} *\mathcal{F} \tag{11.13}$$

where the occupation function

$$\rho_{\mathcal{U}} = \sum_{\alpha \in \mathcal{Z}'} \rho_\alpha \tag{11.14}$$

and $\rho_\alpha \triangleright \bigwedge^0 \mathcal{M}$, $\alpha \in \mathcal{Z}'$ are the occupation functions associated with the domains $\mathcal{M}_\alpha$, $\alpha \in \mathcal{Z}'$, defined in Section 6.3.3. Define

$$\begin{aligned} \Phi_{\mathcal{U}} &= \oint_{\mathcal{M}} \rho_{\mathcal{U}} * \mathcal{F} \\ &= \oint_{\mathcal{U}} * \mathcal{F} \end{aligned} \tag{11.15}$$

Then

$$\begin{aligned} \Phi|_{\mathcal{U}} &= \Phi_{\mathcal{U}} \\ \Phi|_{\bar{\mathcal{U}}} &= 0 \end{aligned} \tag{11.16}$$

Equivalently, in coordinate form,

$$\Phi_{\mathcal{U}}(x^1, x^2, x^3) = \int_{(x_*^1, x_*^2, x_*^3) \in \mathcal{U}} \mathcal{F}(x^1, x^2, x^3, x_*^1, x_*^2, x_*^3)\, dx_*^1 dx_*^2 dx_*^3 \tag{11.17}$$

and the potential gradient

$$\begin{aligned} & \mathrm{d}\Phi_{\mathcal{U}}(x^1, x^2, x^3) \\ &= dx^i \frac{\partial}{\partial x^i} \int_{(x_*^1, x_*^2, x_*^3) \in \mathcal{U}} \mathcal{F}(x^1, x^2, x^3, x_*^1, x_*^2, x_*^3)\, dx_*^1 dx_*^2 dx_*^3 \\ &= dx^i \int_{(x_*^1, x_*^2, x_*^3) \in \mathcal{U}} \frac{\partial}{\partial x^i} \mathcal{F}(x^1, x^2, x^3, x_*^1, x_*^2, x_*^3)\, dx_*^1 dx_*^2 dx_*^3 \end{aligned} \tag{11.18}$$

where summation over $i = 1, 2, 3$ is implied and $(dx^1, dx^2, dx^3, dx_*^1, dx_*^2, dx_*^3)$ is the dual basis induced by the coordinates on $\mathcal{M}^2$.

The bounds a_i^-, a_i^+ and the sets H_i and H associated with the set $\mathcal{U}$ can then be obtained for the particular rectangular coordinates used to obtain (11.18). Consider the integral

$$\int_{(x_*^1, x_*^2, x_*^3) \in \mathcal{U}} \frac{\partial}{\partial x^i} \mathcal{F}(x^1, x^2, x^3, x_*^1, x_*^2, x_*^3)\, dx_*^1 dx_*^2 dx_*^3 \tag{11.19}$$

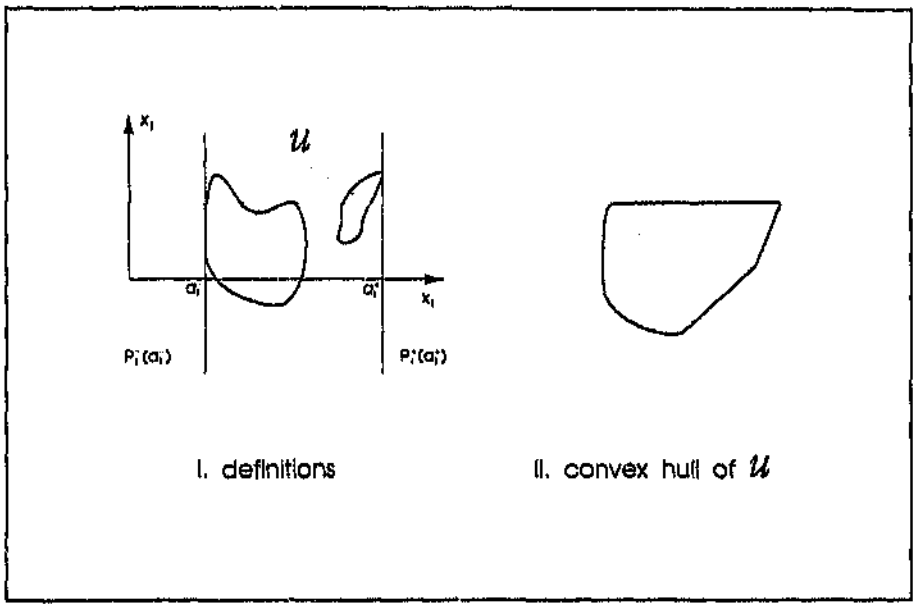


Figure 11.2: Schematic illustrating the convex hull of a domain $\mathcal{U}$.

and

$$H = \bigcup_{i=1,2,3} H_i \tag{11.12}$$

Let H^* be the union of all sets H obtained from all possible rectangular coordinates on $\mathcal{M}$. Then $\mathcal{M} \sim H^*$ is defined[6] as the *convex hull* of $\mathcal{U}$.

Potential on $\mathcal{M}$

The potential on $\mathcal{M}$ resulting from the interactions in $\mathcal{U}$ is, from (6.67),

$$\Phi = \oint_{\mathcal{M}} \rho_{ll} \circledast \rho_{ll} * \mathcal{F}$$

[6]The definition given here is not the most economical one, but does allow some reduction of complexity in the analysis to follow.

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as described in Section 7.9, should yield an extremely elegant approach to the dynamics of inviscid systems if certain elements of the pure mathematics can be resolved. Most importantly, the few examples presented here using an explicit form for the interaction kernel have satisfied the objective of proving the validity of the interaction field approach, but the range of physical behaviour encompassed by the interaction field approach, only hinted at in the examples presented here, has not yet been fully explored. Such exploration is probably best conducted together with an experimental investigation into the physical systems of interest. This would allow a range of physically relevant forms for the interaction kernel to be established. It is in its ability to incorporate a range of physical effects that cannot be handled by the classical surface tension theory that the major contribution of the interaction field approach may indeed be made.

It is hoped that the analysis and ideas presented in this work will be of use to researchers studying the behaviour of fluid systems with interfaces, and especially to engineers engaged in the exploitation of the industrial and technological potential of this particular class of fluid systems.

over which the system is unstable when the radius of the interface is comparable to the range of the interaction, and the subsequent shrinkage of this range as the radius becomes small compared to the range of the interaction which the interaction field approach predicts, may have important consequences for the physical interpretation of many previous studies of the coalescence and breakup of fluid systems with cylindrical interfaces.

12.6 Qualitative Dynamics

Although the number of equilibrium configurations, and hence the equilibrium topologies, accessible to a fluid system is limited by the symmetry group of the interaction kernels, the number of topologies available to the domains comprising the fluid system is comparatively very large. Analysis of the long-term dynamics shows that a system comprised of domains with arbitrary topology will attain the lowest-energy equilibrium configuration permitted by the interactions within the system. The ejection of domains of dissimilar fluid which occur within, or transect another, domain of fluid is energetically favoured. It is postulated that this ejection occurs by the process of film-draining, or marginal regeneration, whereby any configuration with arbitrary topology is extruded into the minimum-energy configuration for the system. This process is the interaction field analogue of the drive towards minimum surface area which governs the qualitative behaviour of a fluid system under the classical surface tension model.

12.7 Extensions

A number of ideas put forward in this work have not yet been exploited to their full potential. Development of the analysis of Section 5.4 on fractal boundaries into a computer algorithm seems immediately feasible, and might have application to the study of industrially-relevant systems such as foams and colloids. The analysis in terms of geodesics on the configuration manifold of the system,

the classical Young-Laplace equation for the behaviour of the pressure near the interface between two domains, then the kernel must in fact have limitingly small range. On the other hand, it is in fact this imposition of a condition of limitingly short range on the interactions within the fluid in the classical approach that precipitates the problems encountered by the classical surface tension model. If the range of the interaction kernel in the interaction field model is allowed to remain finite, then the interaction field approach does not generate the singularities associated with the Young-Laplace equation, and the false equilibria predicted by the classical approa ' are also avoided.

The interaction field model with finite-range interaction kernel yields additional structure in solutions to some classical problems, and this additional structure may have physical relevance. A re-analysis, using the interaction field approach, of the behaviour of the pressure at the centre of a spherical droplet as the radius of the droplet is varied confirms existing analyses that the pressure, rather than becoming singular as the droplet radius goes to zero, should in fact become zero. This result is relevant to studies of droplet nucleation. In the re-analysis of Plateau's problem for the intersection of three thin fluid membranes using an interaction-field approach, an additional equilibrium configuration for the system is identified. The additional structure observed in these systems may be strongly dependent on the particular form of the interaction kernel used in the interaction field approach.

These observations of the behaviour of the classical and interaction field models carry through into the analysis of non-static fluid systems. In Chapter 10 the well-known classical results for the linear dynamics of planar, cylindrical and spherical interfaces about their equilibrium configurations are re-analysed using the interaction field model. It is shown that the classical results are first-order approximations to the dynamics predicted by the interaction field approach, or, alternately, as in the static analyses, that the classical results are regained in the limit of infinitesimal range of the interaction kernel. The interaction field analysis of the stability of the cylindrical interface suggests that the classical analysis yields a particularly extreme oversimplification of the physical structure of the system. The growth of the range of wavenumbers

12.4 Equilibrium

In this work, particular emphasis is placed on the behaviour of fluid systems as the structure of their domains approaches an equilibrium configuration. In Chapter 8, an approach through Lie's transformation group theory shows that equilibrium configurations of the fluid system are obtained if the domains comprising $\mathcal{M}$ are all invariant with respect to a set of symmetries which is related to the largest symmetry group common to the interactions present within the system. If the interactions within the system are expressed as a section of the bundle of one-forms over $\mathcal{M}^n$, where $n > 1$, then an additional constraint on the equilibrium configuration of the fluid system on $\mathcal{M}$ is imposed by the requirement that the symmetry group on $\mathcal{M}$ of the equilibrium configuration transfer from $\mathcal{M}$ to $\mathcal{M}^n$ using the composition sum. Equivalently, the domain structure on $\mathcal{M}$ implied by the symmetry group must transfer from $\mathcal{M}$ to $\mathcal{M}^n$ using the composition operators. The Lie group approach, when coupled to the composition algebras, yields a powerful algebraic approach to the enumeration of the equilibrium configurations of the fluid system. This approach is most concisely expressed in terms of operations on matrices whose elements are sections of the bundle one-forms over $\mathcal{M}^n$. Unfortunately, it is easily shown that the Lie group approach is incomplete in the sense that it can be shown that there are equilibrium configurations for the fluid system which are not predicted by the transformation group analysis.

12.5 Reconciliation

The Lie group analysis of Chapter 8 suggests that possession of a suitable symmetry group is all that is required of the interaction kernel in an interaction field model which is to reproduce the equilibrium configurations predicted by the classical surface tension model. In Chapter 9 it is shown that an analogue to the classical surface tension can be defined in an interaction field model if the strength of the interaction kernel decays sufficiently quickly with distance, but if the kernel is to endow the interaction field model with an analogue to

between the domains.

The space occupied by the fluid, $\mathcal{M}$, does not provide the the most convenient manifold for the analysis of the dynamics of the fluid system if the potential energy of the system is most naturally defined on $\mathcal{M}^n$ where $n > 1$. Again the interpretation of $\mathcal{M}^n$ as a fibre bundle over $\mathcal{M}$ can be used to define an algebra in terms of two operators, here called the composition product and the composition sum, which can be used to transfer the properties of the fluid from $\mathcal{M}$ to $\mathcal{M}^n$. Under the composition product, the domain structure on $\mathcal{M}$ has an analogue on $\mathcal{M}^n$ which is preserved throughout the evolution of the fluid system. This follows since the velocity on $\mathcal{M}^n$, which is created via the composition sum from the velocity on $\mathcal{M}$, preserves the property of incompressibility which was shown to hold for the velocity on $\mathcal{M}$, given the structure of the fluid system assumed here.

Using the same expressions for the kinetic energy, potential energy and dissipation which were used to derive the Navier-Stokes equation on $\mathcal{M}$ for the interaction field model, an equivalent dynamic equation can be derived on a suitably-defined configuration space $\mathcal{D}$ for the fluid system. This is done in Chapter 7. Constraints on the continuity of the velocity field which are imposed by the structure of the fluid and by the forms for the energies of the fluid assumed here imply that, in the absence of viscosity anywhere in the fluid, this configuration space is the space of automorphisms of $\mathcal{M}$ which are piecewise continuous on the domains comprising the fluid. If the viscosity of the fluid is non-zero, then $\mathcal{D}$ is the space of continuous automorphisms of $\mathcal{M}$. On assuming coordinates, the equation of motion for the system on $\mathcal{D}$ takes on a particularly simple form in terms of a number of sections of the bundle one-forms over $\mathcal{D}$ defined as integrals over the energies contained in the distortion modes available to the fluid system. This equation can be linearised about some reference condition for the fluid system through a particularly simple analysis.

classical approach. In addition, the potential term contains a contribution from the internal interactions occurring within the fluid. In the absence of these interactions, the equations of motion of the classical and interaction field models reduce to a common form.

Reformulation of the dynamic equations for the interaction field model in terms of the vorticity of the flow is used in Chapter 5 to show that the interaction field approach may allow analysis of fluid systems where the interfaces between the domains comprising the fluid are nowhere differentiable, and hence are not amenable to a classical analysis using the Young-Laplace equation.

In Chapter 6 an explicit form for the potential energy of the fluid system is derived. The fluid is assumed to consist of a number of particles and a classical statistical mechanics approach is used to write the potential energy of the system as an integral over a volume form defined on the product structure obtained by set multiplication of as many copies of the space occupied by the fluid as there are particles in the fluid. An integral projection technique based on the interpretation of this product structure as a fibre bundle is used to determine the resulting potential which appears in the Navier-Stokes equation for the interaction field based approach. If interactions between groups of greater than n particles can be neglected, then the most natural space for the representation of the potential energy is in fact the product space formed by set multiplication of n copies of $\mathcal{M}$. This work shows that $n = 2$ for the lowest-order interaction field model capable of reproducing the predictions of the classical surface tension model. In addition the form on $\mathcal{M}^n$ representing the internal interaction within the fluid, here called the interaction kernel, need not bear any relationship to the actual interactions present in the particular physical system under study, but in fact can be generalised to be any form possessing certain symmetry requirements which ensure compatibility with the assumed macroscopic structure for the fluid system. For the particularly simple fluid structure assumed here, the interaction kernel should be invariant under all rotations and translations permitted by the dimension of $\mathcal{M}$, and hence the potential energy associated with the interaction between two infinitesimal domains of fluid should be dependent only on the distance

of the dynamic and static properties of such systems, but is known to fail in a number of important practical cases. The classical theory is based on the Young-Laplace equation relating the pressure difference measured across an interface between domains to the curvature of the interface. This pressure difference is unbounded as the curvature of the interface becomes infinite. In addition, the classical approach allows the co-existence in equilibrium of certain fluid configurations which are known from physical experiment to evolve further.

A hybrid model, microscopic in its application but based largely on the analysis of the microscopic interactions between the particles comprising the fluid, is proposed in this work. The principal feature of this model is that it can be shown to reduce under suitable conditions to the classical surface tension theory, but where the classical theory fails, it can also be shown to continue to produce physically plausible predictions for the static behaviour and dynamic evolution of the fluid systems described above.

12.3 The Interaction Field Model

This hybrid model, proposed in this work as an alternative to the classical surface tension model, is called here the interaction field model. In the absence of internal interactions within the fluid, the surface tension model and the interaction field model should reduce to identical models for the dynamics of the fluid. That this reduction does occur is shown in Chapter 4. Here it is assumed that, in the classical approach, the dynamics of the fluid in the various domains comprising $\mathcal{M}$, the space occupied by the fluid, is governed by the Navier-Stokes equation which, in this work, is interpreted as a restriction on a section of the bundle of one-forms over $\mathcal{M}$. By defining suitable forms for the kinetic energy, potential energy and dissipation in the fluid, the interaction field approach yields a dynamic equation which is identical to the Navier-Stokes equation, except that the term interpreted as the pressure in this equation has a somewhat different interpretation to that attached to the pressure in the

Chapter 12

Conclusion

12.1 Introduction

It remains to summarise the major contributions of this work in the light of the hypotheses stated in the introductory chapter, and to put forward a number of ideas for the extension of the analysis presented here.

12.2 The Fluid System

This work has considered the macroscopic dynamics of classical fluid systems where the space occupied by the fluid can be decomposed into a number of domains. By assumption, within each of these domains the density and the viscosity of the fluid is invariant with time, and the action of the velocity of the fluid is such that the volume of the interfaces between the various domains comprising the fluid remains zero for all time.

The physical properties of the fluids in each of the various domains may not all be the same. A classical theory, based on a mechanical model of the interfaces between the domains as a membrane subjected to a tensile stress, called here the classical surface tension theory, has been successful in predicting many

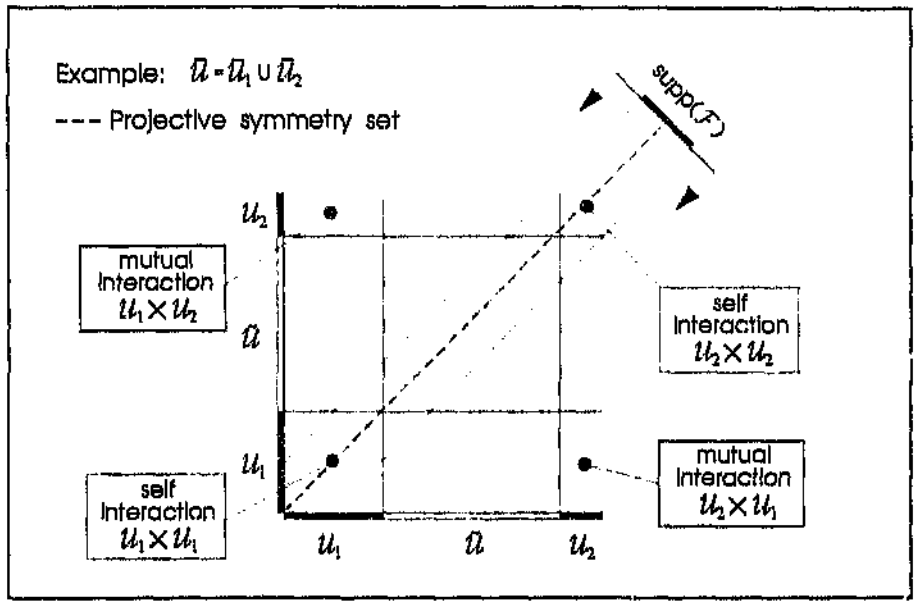


Figure 11.8: Interaction kernel with bounded support.

domains comprising the fluid system have arbitrarily complex topology, the fluid system evolves, in the limit of infinite time, to a configuration which, except for the possible absence from the configuration of a point-set with zero volume, is identical to the lowest-energy equilibrium configuration allowed by the symmetries of the interaction kernel for the system. The set with zero volume is the only artifact of the original topology of the fluid system which is retained by the equilibrium configuration of the fluid system. The mechanism of ejection of domains of dissimilar fluid via film-draining (or marginal regeneration) is proposed as the mode through which the fluid system evolves to its final equilibrium configuration. The principle of the expulsion of defects via this mechanism, as developed in this chapter, can be interpreted as the interaction field analogue for the principle of minimisation of surface area which is available for the qualitative analysis of the dynamics of fluid systems which are governed by the classical surface tension model.

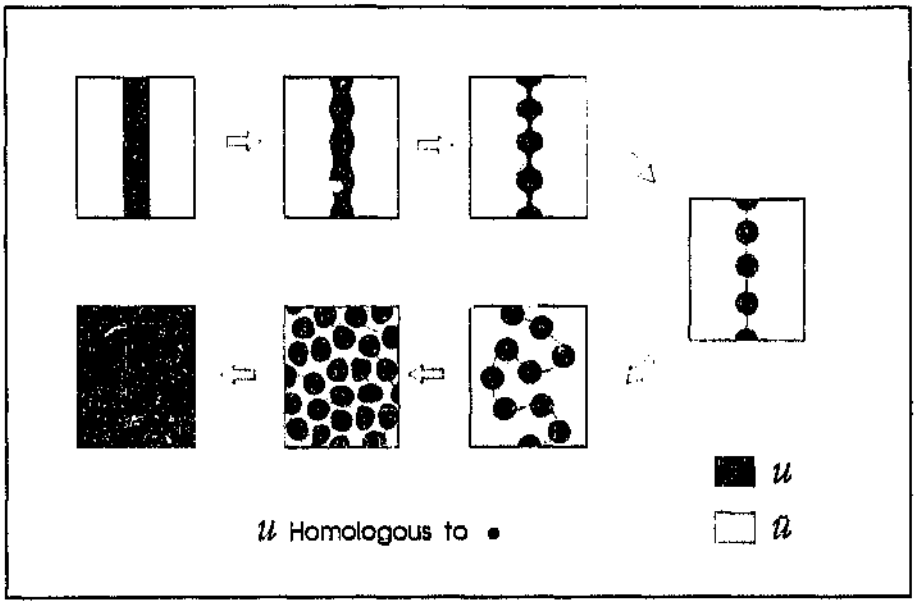


Figure 11.7: Relaxation to equilibrium: The rupturing cylinder.

the projective symmetry set of $\mathcal{M}^n$, some part of these sets must lie within S where $\mathcal{F}$ is not slowly varying.

Incidentally, the surface tension model is a good approximation to observation: it appears that few experimentalists have had the patience required to keep these slowly-evolving systems under observation through to the eventual formation of the equilibrium configuration predicted by the interaction field based approach!

11.11 Summary

In this chapter an integral-geometric approach has been used to show how the classical fluid systems containing a single type of interacting fluid immersed in another, inert fluid relax towards an equilibrium configuration. Even if the

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