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Contributions to Discrete Mathematics

Volume 20, Number 2, Pages 98–108
ISSN 1715-0868

DYCK PATHS WITH FIRST SOJOURN HIGHEST

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Abstract. A Dyck path is a lattice path in the first quadrant using steps $u = (1, 1)$ and $d = (1, -1)$, starting at the origin and ending on the x-axis. A Dyck sojourn is a Dyck path with only one return to the x-axis. Various subclasses of Dyck paths are shown to be counted by Catalan numbers: Dyck paths with weakly highest first sojourn and semilength $n+1$ are counted by twice the nth Catalan number and Dyck paths with semilength $n+1$ where the second sojourn is weakly highest are counted by the nth Catalan number. Also Dyck paths in which the first sojourn is strictly highest are equinumerous to Dyck paths with exactly one peak of maximum height. Some of these equivalences are proved using bijections. Generating functions for all these subclasses are also developed in the paper. The asymptotics as $n \to \infty$ for the case where the first sojourn is strictly highest are explored.

1. General introduction

A Dyck path is a lattice path that starts at the origin (0,0) where only up (i.e. $u = (1, 1)$) and down (i.e. $d = (1, -1)$) steps are allowed. A Dyck path is not allowed below the x-axis and must end on the x-axis. Therefore, a Dyck path with n up steps ends at the point $(2n, 0)$; see the definition in [5]. A Dyck sojourn is a Dyck path with only one return to the x-axis and a Dyck path is therefore a concatenation of a sequence of Dyck sojourns.

Interestingly, the underlying structure of Dyck paths entails the numerical equivalence of the various classes of Dyck path subtypes, a property that is explored in this paper. In section 2, we focus on Dyck paths in which the first sojourn is (weakly) highest. This leads to yet another appearance of the ubiquitous Catalan numbers, see the book by Stanley [12]. Sections 3 and 4 deal with the case where the first sojourn is strictly highest. The former deals with bijections and the latter with generating functions. Section 5 derives generating functions where the rth sojourn is weakly or strictly highest. Section 6 deals with asymptotic estimates for the number of Dyck paths with first sojourn strictly highest.

Received by the editors October 29, 2023, and in revised form November 1, 2023.
1991 *Mathematics Subject Classification.* Primary: 05A05, 05A15, 05A16; Secondary:
Key words and phrases. Dyck paths, generating functions, asymptotics.

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Sections 2, 3 and 5 of this paper primarily deal with bijections between different classes of Dyck paths, whereas the generating functions from which the number of these various classes is derived are dealt with in sections 4 and 5.

A related paper by Prodinger [11] studies the location of the first maximum in the first sojourn of a Dyck path. Recently, Dyck paths with a first return decomposition constrained recursively by height were studied by Baril, Kirgizov and Petrossian in [1].

2. First or second sojourn weakly highest

We first consider Dyck paths in which the highest point occurs in the first sojourn. And thereafter those Dyck paths in which the highest point is (weakly) in the second sojourn. Firstly, using two different bijections on this set, we show that

Proposition 2.1. *The number of Dyck paths of semilength $n+1$ with (weakly) highest point in the first sojourn is twice the nth Catalan number.*

Proof. Fix the semilength n of the Dyck paths. Consider their representation as a (possibly empty) ordered sequence of Dyck sojourns $D_1, D_2, \ldots, D_p$ with respective heights $h_1, h_2, \ldots, h_p$ and somewhere (nonvacuously) in this sequence another Dyck sojourn D of height h which is the leftmost largest of all the sojourn heights. So the Dyck paths under consideration have $p+1$ returns to the x-axis. We construct a bijection between the set of such Dyck paths of semilength n and Dyck paths of semilength $n+1$ having precisely one sojourn, by mapping each such path to itself preceded by a u step and followed by a d step.

We construct another bijection between this same set and Dyck paths of semilength $n+1$ having more than one sojourn as follows. Send the constituting sequence of Dyck sojourns where D was say the rth sojourn, i.e., $D_1, D_2, \ldots, D_{r-1}, D, D_{r+1}, \ldots, D_p$ to
$D, u, D_1, D_2, \ldots, D_{r-1}, d, D_{r+1}, \ldots, D_p$. If D was already the first sojourn of the Dyck path, this rule also applies because $D_1, D_2, \ldots, D_{r-1}$ is empty.

Note that all these images have more than one return to the x-axis and are therefore disjoint from the image of the first bijection. Moreover, by construction, the first sojourn of the image is the highest. And the mapping is onto the set of Dyck paths having semilength $n+1$, and more than one sojourn with the property that the first is the highest. It is also reversible simply by swapping the order of the first and second sojourns and then removing the ud pair at the beginning and end of the (new) first sojourn. It is well known that the number of Dyck paths of semilength n is given by the Catalan number $\frac{1}{n+1}\binom{2n}{n}$ and therefore the result follows. □

From this proof it follows that

Corollary 2.2. *Half of all Dyck paths having semilength $n > 1$ and weakly highest first sojourn have precisely one sojourn.*

Also we have,

Corollary 2.3. *The number of cases in which the second sojourn is (weakly) highest for Dyck paths of semilength $n+1$ is given by the nth Catalan number.*

Proof. Map the set of Dyck paths in which the second sojourn is weakly highest to the set of Dyck paths of semilength $n + 1$ in which the first sojourn is weakly highest by sending such a Dyck path $D_1, D, D_2, \ldots, D_{p-1}$ in which the second sojourn D is weakly highest to the Dyck path $D, D_1, D_2, \ldots, D_{p-1}$. This mapping is clearly a bijection and so the number of such Dyck paths of each type are equi-numerous. The previous corollary asserts that half of all Dyck paths with highest first sojourn have more than one sojourn and by Proposition 2.1 this number is given by the nth Catalan number.

□

Remark: The sequence obtained from Proposition 2.1 appears in [13] as $A068875$, described as the expansion of $(1 + xC)C$, where C is the Catalan generating function.

3. First sojourn strictly higher

Here, we consider various statistics connected to the case of Dyck paths in which the first sojourn is strictly the highest.

Proposition 3.1. *The number of Dyck paths of semilength n with the first sojourn being strictly the highest is equal to the number of Dyck paths of semilength n having exactly one peak of maximum height.*

Proof. Let us consider a Dyck path of semilength n comprised of the sojourns $D, D_1, D_2, \ldots, D_p$ in which D is strictly the highest. Obtain S from D by adding ud steps precisely at the rightmost maximum of D and then deleting the first and last steps of the resulting sojourn. $S = S_1 \ldots S_l$ is now a possibly multiple sojourn Dyck path of the same semilength as D in which its last sojourn S_l has a single strictly maximal point compared to the rest of S as well as $D_1, D_2, \ldots, D_p$. Map $D, D_1, D_2, \ldots, D_p$ to $S_1 \ldots S_l, D_1, D_2, \ldots, D_p$.

For the reverse map, let $S_1 \ldots S_l, D_1, D_2, \ldots, D_p$ be a Dyck path of semilength n made up of the sojourns shown in which S_l has a single peak of maximal height for the whole Dyck path. The peak in S_l is lowered by deletion (the converse operation in the forward map) and then $S_1 \ldots S_l$ has steps ud respectively pre- and postpended, becoming once again D, a single sojourn which is the strict highest in $D, D_1, D_2, \ldots, D_p$. The reverse mapping preserves the rest of the following order in $D, D_1, D_2, \ldots, D_p$. These two maps compose in either direction to the identity map and are therefore a bijection, establishing this proposition. □

Corollary 3.2. *The sum of the heights of Dyck paths of semilength n having exactly one peak of maximum height is equal to the sum of the heights of Dyck paths in which the first sojourn is strictly the highest.*

Proof. The bijection used in the previous proposition has the following properties:

(1) It maps a Dyck path with a highest first sojourn to a Dyck path having exactly one peak.

(2) The height of the first sojourn in its domain is the same as the height of the highest peak in its image.

This establishes our corollary. □

Corollary 3.3. *The number of Dyck paths of semilength n with the first sojourn being strictly the highest is equal to the number of peaks of maximum height in all Dyck paths of semilength $n-1$.*

Proof. First consider a Dyck path of semilength $n-1$. Mark each peak of maximum height in this path. Define a mapping from the set of such marked Dyck paths to the set of Dyck paths of semilength n having precisely one maximum point by sending the marked path to the same path except that the marked peak is made higher by the addition of an *ud* step precisely where the marked peak occurs. This mapping is clearly a bijection onto the set of Dyck paths of semilength n having exactly one peak of maximum height. Thus the total number of such paths is equal to the number of peaks of maximum height in Dyck paths of semilength $n-1$. By the previous proposition, the corollary follows. □

Remark: The number of Dyck paths of semilength n with first sojourn strictly higher is sequence A152880 in [13], described as the number of Dyck paths of semilength n having exactly one peak of maximum height.

4. Generating function where the first sojourn is strictly highest

We begin by referring to paper [11] by Prodinger on the first sojourn in Dyck paths. Using the notation from [11], we let $C(h)$ be the number of paths of height $\leq h$ and steps which follow all rules of Dyck paths except that they terminate at height h, and we let $A(h)$ be the number of Dyck paths of height $\leq h$ (which by definition end at height zero). It is shown in [11] that

$$C(h) := \frac{z^h\sqrt{1-4z^2}}{{\lambda_1}^{h+2} - {\lambda_2}^{h+2}} \tag{4.1}$$

and

$$A(h) := \frac{{\lambda_1}^{h+1} - {\lambda_2}^{h+1}}{{\lambda_1}^{h+2} - {\lambda_2}^{h+2}}, \tag{4.2}$$

where λ_1 and λ_2, are given by

$$\lambda_1 = \frac{1+\sqrt{1-4z^2}}{2}; \lambda_2 = \frac{1-\sqrt{1-4z^2}}{2}. \tag{4.3}$$

To obtain the generating function in which the first sojourn is strictly the highest, we decompose a Dyck path where the first sojourn is strictly the highest of height h into such a first sojourn followed by an arbitrary Dyck path of height less than or equal to $h-1$. First we need a generating function for a single sojourn of height h. Call this generating function $S_h(z)$.

So we have $S_h(z) = z^2(A(h-1)-A(h-2))$ because $A(h-1)-A(h-2)$ is the generating function for Dyck paths of height $h-1$ which is then preceded and followed by u and d steps tracked by z^2 in the generating function. This now needs to be followed by the generating function for Dyck paths of maximum height $h-1$, which is given by $A(h-1)$. Altogether we obtain the generating function $F_h(z)$ for Dyck paths in which the first sojourn is of height h and is strictly the highest:

$$F_h(z) := z^2(A(h-1) - A(h-2))A(h-1).$$

So the overall generating function in which the first sojourn is strictly the highest is

$$F(z) := \sum_{h=1}^{\infty} F_h(z) = \sum_{h=1}^{\infty} z^2(A(h-1) - A(h-2))A(h-1).$$

This gives for $F(z)$ the expression

$$\sum_{h=1}^{\infty} \frac{4^{1+h}\left(z^2\right)^h\left(1-4z^2\right)}{\left(-\left(1-\sqrt{X}\right)^h + \sqrt{X}\left(1-\sqrt{X}\right)^h + \left(1+\sqrt{X}\right)^h + \sqrt{X}\left(1+\sqrt{X}\right)^h\right)^2} \tag{4.4}$$

where $X := 1-4z^2$.

A simpler approach to obtain our generating function in which the first sojourn is the highest is to use the bijection to Dyck paths with a unique maximum. The generating function for this, $F_h(z)$, is obtained from that of the single peak generating function for Dyck paths of height h, namely $D_h(z)$. In its turn $D_h(z)$ is obtained from the generating function $C(h-1)$ from equation (4.1) where any of the peaks is raised by the insertion of ud steps at this point which contributes z^2 to this generating function. So

$$D_h(z) = z^2C^2(h-1)$$

and therefore the already defined

$$F(z) = \sum_{h=1}^{\infty} D_h(z) = \sum_{h=1}^{\infty} z^2C^2(h-1).$$

So we have proved:

Theorem 4.1. *The generating function for Dyck paths where z tracks the length and where the first sojourn is strictly highest is*

(4.5)
$$F(z) = \sum_{h=1}^{\infty} \frac{z^{2h}\left(1-4z^2\right)}{\left(-2^{-1-h}\left(1-\sqrt{1-4z^2}\right)^{1+h} + 2^{-1-h}\left(1+\sqrt{1-4z^2}\right)^{1+h}\right)^2}.$$

Remark: The series expansion for $F(z)$ starts

$$z^2 + z^4 + 3z^6 + 8z^8 + 23z^{10} + 71z^{12} + 229z^{14} + 759z^{16} + 2566z^{18} + 8817z^{20}.$$

The equality of (4.4) and (4.5) constitutes an identity.

To simplify the generating function, we substitute

$$z^2 = \frac{u}{(1+u)^2} \tag{4.6}$$

and obtain

$$F(z) = (1-u)^2 \sum_{h=1}^{\infty} \frac{u^h}{\left(1-u^{1+h}\right)^2}.$$

That is

$$F(z) = \frac{(1-u)^2}{u} \sum_{r=2}^{\infty} \frac{u^r}{\left(1-u^r\right)^2}. \tag{4.7}$$

4.1. **Total number of strictly highest first sojourn Dyck paths.** Here we will obtain an exact formula for the total number of strictly highest first sojourn Dyck paths. This will be in terms of a well-known arithmetic function, namely the sum of divisors function $\sigma(r)$. Note that

$$\sum_{r=2}^{\infty} \frac{u^r}{(1-u^r)^2} = \sum_{r=1}^{\infty} \sigma^*(r)\, u^r$$

where $\sigma^*(r) = \sigma(r) - r$ is the sum of the proper divisors of $r \geq 1$. We set $\sigma^*(r) = 0$ for $r \leq 0$.

To read off coefficients from equation (4.7), we observe that for any formal power series $f(z)$

$$[z^{2n}]f(z) = [u^n](1-u)(1+u)^{2n-1} f(z(u)).$$

This can be justified by using formal residue calculus, see for example [9]. Therefore

$$[z^{2n}]F(z) = [u^n](1-u)(1+u)^{2n-1}\frac{(1-u)^2}{u}\sum_{r=2}^{\infty}\frac{u^r}{(1-u^r)^2}$$

$$= [u^n](1-u)(1+u)^{2n-1}\sum_{r=1}^{\infty}(\sigma(r+1)-2\sigma(r)+\sigma(r-1))u^r$$

$$= \sum_{r=1}^{n}(\sigma(r+1)-2\sigma(r)+\sigma(r-1))\left(\binom{2n-1}{n-r}-\binom{2n-1}{n-r-1}\right). \tag{4.8}$$

Thus we have shown:

Theorem 4.3. *The total number of Dyck paths with first sojourn strictly highest, of semilength n is given by*

$$\sum_{r=1}^{n}(\sigma(r+1)-2\sigma(r)+\sigma(r-1))\left(\binom{2n-1}{n-r}-\binom{2n-1}{n-r-1}\right).$$

Remark: By redistributing the factors of $1-u$ in the computations of (4.8) one can obtain several alternative formulas such as

$$[z^{2n}]F(z) = \sum_{r=1}^{n}(-\sigma^*(r-2)+3\sigma^*(r-1)-3\sigma^*(r)+\sigma^*(1+r))\binom{2n-1}{n-r}.$$

5. Generating function where the the r-th sojourn is highest

First, we consider Dyck paths with exactly r sojourns in which the first is strictly the highest. Extend Equation (4.2) by defining $A(0) := 1$ and $A(r) := 0$ whenever $r < 0$. In the cases where $r \geq 2$, the first sojourn is of height $h \geq 2$ and the remaining $r-1$ sojourns are of height less than h. But when $r = 1$, there is the additional case of $h = 1$. From the explanation in the previous section, we obtain the generating function $V_r(z)$ in which the first sojourn is strictly highest is given by

$$V_1(z) = z^2 + \sum_{h=2}^{\infty} z^2(A(h-1)-A(h-2)), \tag{5.1}$$

and where $r \geq 2$,

$$V_r(z) = \sum_{h=2}^{\infty} z^2(A(h-1)-A(h-2))(z^2A(h-2))^{r-1}. \tag{5.2}$$

The term z^2 in (5.1) accounts for the additional case mentioned above. Similarly, we obtain the generating function $W_r(z)$ in which the first sojourn is weakly the highest (in this case its height may be 1), given by

$$W_r(z) = \sum_{h=1}^{\infty} z^2(A(h-1)-A(h-2))(z^2A(h-1))^{r-1}. \tag{5.3}$$

Here follow some examples of the series obtained for $W_r(z)$ for r values from 1 to 5:

$$z^2 + z^4 + 2z^6 + 5z^8 + 14z^{10} + 42z^{12} + 132z^{14} + 429z^{16} + O\left(z^{18}\right),$$
$$z^4 + z^6 + 3z^8 + 8z^{10} + 23z^{12} + 71z^{14} + 229z^{16} + 759z^{18} + O\left(z^{20}\right),$$
$$z^6 + z^8 + 4z^{10} + 12z^{12} + 36z^{14} + 114z^{16} + 377z^{18} + O\left(z^{20}\right),$$
$$z^8 + z^{10} + 5z^{12} + 17z^{14} + 54z^{16} + 176z^{18} + O\left(z^{20}\right),$$
$$z^{10} + z^{12} + 6z^{14} + 23z^{16} + 78z^{18} + O\left(z^{20}\right).$$

And now similar examples of the series obtained for $V_r(z)$:

$$z^2 + z^4 + 2z^6 + 5z^8 + 14z^{10} + 42z^{12} + 132z^{14} + 429z^{16} + O\left(z^{18}\right),$$
$$z^6 + 2z^8 + 6z^{10} + 19z^{12} + 61z^{14} + 200z^{16} + 671z^{18} + O\left(z^{20}\right),$$
$$z^8 + 2z^{10} + 7z^{12} + 25z^{14} + 86z^{16} + 294z^{18} + O\left(z^{20}\right),$$
$$z^{10} + 2z^{12} + 8z^{14} + 32z^{16} + 118z^{18} + O\left(z^{20}\right),$$
$$z^{12} + 2z^{14} + 9z^{16} + 40z^{18} + O\left(z^{20}\right).$$

And finally examples of the series obtained for $\sum_{i\geq r} W_i(z)$:

$$z^2 + 2z^4 + 4z^6 + 10z^8 + 28z^{10} + 84z^{12} + 264z^{14} + 858z^{16} + O\left(z^{18}\right),$$
$$z^4 + 2z^6 + 5z^8 + 14z^{10} + 42z^{12} + 132z^{14} + 429z^{16} + 1430z^{18} + O\left(z^{20}\right),$$
$$z^6 + 2z^8 + 6z^{10} + 19z^{12} + 61z^{14} + 200z^{16} + 671z^{18} + O\left(z^{20}\right),$$
$$z^8 + 2z^{10} + 7z^{12} + 25z^{14} + 86z^{16} + 294z^{18} + O\left(z^{20}\right),$$
$$z^{10} + 2z^{12} + 8z^{14} + 32z^{16} + 118z^{18} + O\left(z^{20}\right).$$

Remark: Moreover, we notice that $\sum_{r\geq 1} V_r(z)$ has the same sequence as $W_2(z)$. This gives rise to the following proposition.

Proposition 5.2. *The number of Dyck paths of semilength $n+1$ having exactly 2 sojourns with the first sojourn being weakly highest is equal to the number of Dyck paths of the semilength n having any number of sojourns in which the first is strictly the highest.*

Proof. Let the Dyck paths with exactly 2 sojourns and semilength $n+1$ be of the form DS in which D is the first weakly highest sojourn and S is the second sojourn. Let S_0 be a possibly empty sojourn which is obtained from S by deletion of the first and last steps. Define a bijection by sending DS to DS_0. The reverse map simply prepends the step u at the beginning of the possibly empty second sojourn and appends the step d at the end of the resulting lattice path. □

Remark: By comparing row r of the V_r table with the row $r+1$ of the $\sum_{i=r} W_i(z)$ table, we arrive at the following conjecture which we leave to readers for further consideration.

Conjecture. *The number of Dyck paths of semilength n having r sojourns with the first sojourn being strictly the highest is equal to the number of Dyck paths of the same semilength having at least $r+1$ sojourns in which the first is weakly highest.*

6. Asymptotics for proportion of Dyck paths in which the first sojourn is strictly highest

We will follow the approach used to study the height of planted plane trees by Prodinger in [9]. For related asymptotic calculations concerning the height of trees and lattice paths, see [7, 8, 10].

First, we extract the coefficients of z^n in $F(z)$. That is we find

$$[z^n]F(z) = [z^n]\frac{(1-u)^2}{u}\sum_{r=2}^{\infty}\frac{u^r}{(1-u^r)^2}. \tag{6.1}$$

When u is in terms of z^2, from (4.6) we obtain

$$u = \frac{1-2z^2-\sqrt{1-4z^2}}{2z^2}.$$

Thus the function $F(z)$ has its dominant singularity at $z = 1/2$ which is mapped to $u = 1$. To study this further, we set $u = e^{-t}$ and let $t \to 0$. Thus

$$\frac{(1-u)^2}{u} = e^t(1-e^{-t})^2 = t^2 + \frac{t^4}{12} + \frac{t^6}{360} + \cdots\ . \tag{6.2}$$

To estimate the harmonic sum $f_1(t) := \sum_{r=2}^{\infty}\frac{e^{-rt}}{(1-e^{-rt})^2}$ as $t \to 0$, we take the Mellin transform of $f_1(t)$, see [6], which is $f_1^*(s) := \int_0^{\infty} f_1(t)t^{s-1}\,dt$. Thus

$$f_1^*(s) = \Gamma(s)\zeta(s-1)(\zeta(s)-1), \text{ for } Re(s) > 2.$$

By using the Mellin inversion formula, we have $f_1(t) = \frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty} f_1^*(s)\,t^{-s}\,ds$ (again see [6]). By computing residues this yields

$$f_1(t) \sim \frac{\pi^2-6}{6t^2} - \frac{1}{2t} + \frac{1}{8} + \cdots\ . \tag{6.3}$$

Let

$$g_1(t) := e^t(1-e^{-t})^2\, f_1(t).$$

From (6.2) and (6.3)

$$g_1(t) \sim -1 + \frac{\pi^2}{6} - \frac{t}{2} + \left(\frac{1}{24} + \frac{\pi^2}{72}\right)t^2 + \cdots\ . \tag{6.4}$$

Let $y = \sqrt{1-4z^2}$ and writing $e^{-t} = u = \frac{1-y}{1+y}$, we find $t = -\log\frac{1-y}{1+y} = 2y + \frac{2y^3}{3} + \cdots$. In terms of the y variable, we therefore need to compute $g_1(2y + \frac{2y^3}{3} + \cdots)$. We obtain

$$g_1\left(2y + \frac{2y^3}{3} + \cdots\right) \sim \left(-1 + \frac{\pi^2}{6}\right) - y + \frac{1}{18}\left(3+\pi^2\right)y^2 - \frac{2y^3}{3} + \cdots .$$

Replacing y by $\sqrt{1-4z^2}$ gives

$$-1 + \frac{\pi^2}{6} - \sqrt{1-4z^2} + \frac{1}{18}\left(3+\pi^2\right)\left(1-4z^2\right) - \frac{2}{3}\left(1-4z^2\right)^{3/2} + \cdots .$$

To use singularity analysis, see [6], it is convenient to put $z^2 = x$, then we find the coefficient of x^n in the above expression as $n \to \infty$. It is asymptotically equal to

$$2^{2n}\left(\frac{1}{2n^{3/2}\sqrt{\pi}} - \frac{1}{2n^{5/2}\sqrt{\pi}} + \cdots\right). \tag{6.5}$$

To obtain the mean value we must divide by the total number of Dyck paths of semilength n, i.e., as $n \to \infty$

$$\frac{1}{n+1}\binom{2n}{n} = 2^{2n}\left(\frac{1}{n^{3/2}\sqrt{\pi}} - \frac{9}{8\ n^{5/2}\sqrt{\pi}}\right) + \cdots . \tag{6.6}$$

Hence, dividing (6.5) by (6.6) yields

Theorem 6.1. *The proportion of Dyck paths of semilength n with first sojourn strictly highest is*

$$\frac{1}{2} + O\left(\frac{1}{n}\right), \quad \text{as } n \to \infty.$$

We note that Theorem 6.1 without the big O estimate is already known. It appears as Corollary 3 in the paper [4] in an equivalent form, in terms of 132-avoiding permutations (which are in bijection with Dyck paths).

7. Further generalisations and future work

It is natural to extend this study from Dyck paths to Dyck prefixes. A Dyck prefix is defined to be a possibly empty Dyck path followed by further up and down steps in the first quadrant but ending at any positive height. This generalisation has been studied by the current authors in [2, 3] where the number of left-to-right maxima in Dyck paths and Dyck prefixes is studied, respectively. As in the current paper, focusing the study on Dyck prefixes in which the first sojourn has the highest point relative to the rest of the path is a good idea for future research.

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