

https://dspace.org/

Kripke completeness of predicate modal logics

Bowditch, Zachary

2024
https://hdl.handle.net/10539/37425

Downloaded from a DSpace repository. https://dspace.org/

DISSERTATION

Kripke Completeness of Predicate Modal Logics

Author:
Zachary BOWDITCH (1414769)

Supervisor:
Prof. Dmitry SHKATOV



UNIVERSITY OF THE
WITWATERSRAND,
JOHANNESBURG

submitted to
the Faculty of Science, in fulfilment of the requirements for the degree of
Masters of Science

in the

School of Computer Science and Applied Mathematics

June 9, 2023

Declaration of Authorship

I, Zachary BOWDITCH (1414769), declare that this dissertation titled, "Kripke Completeness of Predicate Modal Logics" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this dissertation has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this dissertation is entirely my own work.
- I have acknowledged all main sources of help.
- Where the dissertation is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed: 

Date: 9th June 2023

UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG

Abstract

Faculty of Science

School of Computer Science and Applied Mathematics

Masters of Science

Kripke Completeness of Predicate Modal Logics

by Zachary BOWDITCH (1414769)

Kripke semantics is well-known for predicate and propositional modal logic. While the study of Kripke semantics has produced numerous completeness results for propositional modal logics, there are substantially fewer examples of Kripke complete predicate modal logics. This is dually attributable to inadequacies of predicate Kripke semantics as well as to the inherent technical difficulties that accompany domains in semantics of predicate logic. There is presently little literature in the area of Kripke completeness for predicate modal logics, which motivates a comprehensive survey of existing results, recent innovations as well as an investigation into what insights are required to produce more general results. The study of canonicity - the property of a modal logic where it is validated by its canonical frame - has led to wide coverage of Kripke completeness results in propositional modal logic. Canonicity, however, does not always persist for predicate extensions of propositional modal logics. We survey a number of propositional and predicate modal logics, both canonical and non-canonical, demonstrating techniques which show Kripke completeness for these logics, and discuss the extent to which the canonical model technique is useful to this end.

Acknowledgements

I'd like to acknowledge and thank Prof. Dmitry Shkatov, for his patience and willingness to supervise this research and to provide ample guidance with the choice of literature and areas of focus; my parents, for having given me a rich education and for supporting my choice of studies; my wife, for her love, continued encouragement and companionship; our Lord, from whom all blessings, salvation and satisfaction flow and in whose image we were created, for having created an intelligible universe and for endowing us with the ability to rationally consider and ponder its metaphysical substrate. *"For from Him and through him and to Him are all things. To Him be glory forever. Amen."* - Romans 11:36

Contents

Declaration of Authorship i

Abstract ii

Acknowledgements iii

1 Introduction 1

1.1 A Brief Overview of Modal Logic . . . 1

1.2 Motivation and Outline . . . 3

2 Syntax and Semantics for Propositional Modal Logic 6

2.1 Language . . . 6

2.2 Syntax . . . 6

2.3 Kripke Semantics . . . 10

2.3.1 General Kripke Frames . . . 13

2.3.2 First-Order Translations and Definability . . . 15

3 Kripke Completeness for Propositional Modal Logics 17

3.1 Soundness . . . 17

3.2 Canonical Models . . . 21

3.3 Canonicity and Completeness . . . 26

3.3.1 Propositional Modal Logics Alt_n . . . 30

3.3.2 General Canonical Frames . . . 31

4 Syntax and Semantics for Predicate Modal Logic 34

4.1 Language . . . 34

4.2 Syntax . . . 35

4.3 Predicate Kripke Semantics 38

5 Kripke Completeness for Predicate Modal Logics **41**

5.1 Some Preliminary Results 42

5.2 Canonical Models for Predicate Modal Logics 43

5.2.1 Henkin Theories 44

5.2.2 Canonical Models 50

5.3 Canonicity and Completeness 54

5.3.1 Canonical Logics 56

5.3.2 Non-Canonical Logics 57

5.3.3 Quasi-Canonical Logics 58

5.4 Constant Domain Frames 62

5.4.1 General Predicate Kripke Frames 66

6 Closing Remarks and Future Work **68**

Bibliography **70**

List of Abbreviations

iff	if and only if
PTC	pseudo-transitive or closed
w.r.t	with respect to
wff	well-formed formula
resp.	respectively

List of Symbols

$\supset$	Logical implication (in a wff)
$\sim$	Logical negation
$\vee$	Logical disjunction
$\wedge$	Logical conjunction
$\Box$	Modal operator of necessity
$\Diamond$	Modal operator of possibility
$\forall$	Universal quantifier
$\exists$	Existential quantifier
$\rightarrow$	Logical implication (in metalanguage)
$\subset$	Set-theoretic strict inclusion
$\subseteq$	Set-theoretic non-strict inclusion
$-$	Set-theoretic complement
$\cup$	Set-theoretic union
$\cap$	Set-theoretic intersection
$\varnothing$	Empty set
$\vdash$	Provable
$\vdash_L$	L-derivable
$\Vdash$	Semantic consequence
$\restriction$	Restriction to
ω	First countable infinite ordinal number
$\aleph_0$	Cardinality of the natural numbers

Chapter 1

Introduction

1.1 A Brief Overview of Modal Logic

Over the many centuries of mathematical and philosophical inquiry, various studies in logic have emerged, each motivated by its own set of philosophical persuasions, concerned with the underlying nature of truth and reason. What unites many of these disciplines, however, is the notion of "formal logic": the study of using rules of inference to make deductions from a set of purely abstract or symbolic statements. Among these categories of formal logic are several areas distinguished by their application, including Mathematical and Philosophical Logic, where scholars of the former are concerned primarily with extending formal logical systems to areas of mathematical interest (e.g. Set Theory) or applying mathematical techniques to other areas of logic, while scholars of the latter are primarily concerned with innovating formal descriptions of intuitively true statements in natural language. Different schools of thought have provided different bases of assumptions for these studies, characterised by the set of propositional tautologies and rules of inference they permit (or omit). In Classical Logic, the Law of Excluded Middle is one such tautology. This law formally states that a proposition is either true or false (in the traditional syntax: $\sim p \vee p$, or equivalently, $\sim\sim p \supset p$). While this is axiomatic in classical logic, this luxury is not available in other foundations such as Intuitionistic Logic.

Firmly mounted on these logical foundations, different syntaxes and semantics have been proposed and defended for different applications, depending on the descriptional function

assigned to the chosen language and the desired degree of expressivity. One particular area of logic, Modal Logic, developed as an extension to classical propositional logic to accommodate modalities in natural language. One example of a modality in language might be applied to the statement "This is a sentence.". With a modality applied, one might obtain: "This is *necessarily* a sentence." or "This is *possibly* a sentence.". These two sentences are examples of linguistic modalities known as *alethic* modalities [9], which accounts for the particular metaphysical modalities of truth: necessity and possibility. A common syntax used for formally denoting these modalities is "$\Box$" for "it is necessary that" and "$\Diamond$" for "it is possible that". These symbols operate unarily on other formulas (e.g. $\Box(p \vee \Diamond q)$ which, alethically, would be read as "it is necessarily true that either p is true or q is possibly true") and constitute the linguistic extension to classical logic, making up Modal Logic. The alethic interpretation is one of several interpretations given to the "$\Box$" and "$\Diamond$" operators, defined dually, i.e. $\Box p$ if and only if $\sim\Diamond\sim p$ (yielding, again under an alethic interpretation, "if p is necessarily true, then it is not the case that p is possibly false"). Other interpretations of the modalities include *deontic* modalities, where $\Box$ means "obligatorily" and $\Diamond$ means "permissibly", and *temporal* modalities, interpreting the same operators respectively as "is the case at every future time" and "is the case at some future time" respectively.

An extension of their underlying classical or intuitionistic logics, propositional modal logics are often defined as sets of well-formed formulas in the combined language of propositional logic together with the modalities, closed under certain rules, including rules that involve modalities. Hilbert-style proof systems in modal logic were pioneered by C. I. Lewis, often referred to as the founder of modal logic, in 1912 [11] and formed part of an effort to develop a new kind of logic, known as Conditional Logic, free of classical theorems that are counter-intuitively false: the so-called "Paradoxes of Material Implication". Since Lewis, many more modal proof systems have been proposed and studied, each carrying with it different peculiarities and behaviours with respect to different semantics. Cresswell and Hughes [8] present numerous proof systems of modal logic, of which many are *normal* modal logics (closed under the rule of necessitation, as defined later). These proof systems are typically presented in propositional modal languages whose atomic symbols are propositional letters (whose values are assigned true or false in the usual semantics). With appropriate linguistic additions,

these propositional modal logics can be extended to predicate (first-order) modal logics, whose atomic symbols are n-ary predicate letters with variables as arguments. In classical predicate semantics, originally developed by Alfred Tarski, the truth value of an atomic symbol is determined by the assignment of "individuals" to the constituent variables of the atomic symbol.

An important consideration in the broader field of logic surrounds the definition of an appropriate semantics for a formal language and the criteria for semantically valid well-formed formulas of the language, often accompanied by the desire to find a proof system which (1) proves only semantically valid well-formed formulas and (2) in which every semantically valid well-formed formula is provable within the system (or, if we think of logics as sets, if every semantically valid well-formed formula belongs to the logic). (1) and (2) are the definitions of *soundness* and *completeness*, respectively. To accomplish such a task, one must first define appropriate notions of truth and validity in the chosen semantics, respectful of the motivating intuitions behind the language. In this thesis, we consider the "possible world" semantics developed by Saul Kripke in the 1950s [9], hereafter referred to as "Kripke semantics". In this semantics, the modal operators are non-truth-functional - that is, the truth value of the resulting well-formed formula, when the modality is applied, is not deduced solely from the truth values of the operand formula. Kripke semantics is formulated around the notion of a "frame", an object comprising a set of "possible worlds" and a binary relation on the set. Truth of a well-formed formula in the semantics is then defined for every possible world, for a given model. A modal logic is "Kripke complete" if its provable well-formed formulas are exactly those that are valid on every frame in some class of frames. It is demonstrating this phenomenon that we discuss in the context of propositional modal logic, and then in predicate modal logic, whose related work we discuss more in the introduction to Chapter 5.

1.2 Motivation and Outline

Unlike the case in propositional modal logic, there has up to this point in time been relatively few results in the area of Kripke completeness for predicate modal logics. To date, the most complete and comprehensive survey of Kripke completeness for predicate modal logic

is found in [10], where much technical detail is delineated and aimed at readers who are well-acquainted with at least propositional modal logic. Even a brief look at this book reveals, however, that few results are in fact known and, due to the technical difficulty involved, few papers have surfaced demonstrating Kripke completeness for predicate modal logics, let alone more general results, which are numerous in propositional modal logic, including notable efforts such as [18]. We, therefore, produce a survey of several seminal Kripke completeness results for predicate modal logics and pinpoint the gaps within the field. Additionally, we intend to describe several results known to researchers in the field of modal logic but which aren't as of yet widely known.

The outline of this thesis is as follows:

- Chapter 1 (the current chapter) gives a brief overview of modalities and modal logic, as well as a discussion on the motivation for this thesis.
- Chapter 2 lays the foundation of syntax in propositional modal logic as an extension to classical propositional logic, defining the propositional modal well-formed formulas and proof systems for these formulas. We list some theorems in propositional modal logic and then discuss Kripke frames, models and relevant operations, giving definitions of validity, soundness and completeness. We then define and discuss general Kripke frames and then state results surrounding first-order definability of modal logic.
- Chapter 3 discusses soundness and completeness for several propositional modal logics, demonstrating the use of canonical models in proving Kripke completeness, and the combined utility of canonicity with soundness in demonstrating completeness w.r.t particular frame classes. We then revisit general frames, stating more theorems, anticipating their application later in the thesis.
- Chapter 4 introduces predicate modal logic, extending the language of propositional modal logic, as well as several needed theorems. We then introduce predicate Kripke frames as extensions to propositional Kripke frames and define truth in predicate Kripke models, giving definitions of validity, soundness and completeness. We introduce the Barcan formula, a predicate modal axiom, anticipating its

use in a later section on constant domain frames.

- Chapter 5 starts with a brief summary of a few important results in Kripke complete for predicate modal logic. We then define some technical machinery involving modal theories, formulas with constants and canonicity, derived from the combined use of the canonical model technique from Chapter 3 with the Henkin method of proving completeness for classical predicate logic. We go on to discuss quasi-canonicity for non-canonical logics, feeding into the final section on constant domain frames, wherein the work done on general frames culminates in a general completeness theorem.
- Chapter 6 concludes the thesis with a summary of what was hoped it would highlight to the reader.

Chapter 2

Syntax and Semantics for Propositional Modal Logic

We start by introducing syntax and semantics for propositional modal logic.

2.1 Language

First, we define the propositional modal language to contain the following *primitives*:

A countable set $PL = \{p_1, p_2, \dots\}$ of *propositional letters*,

The *classical propositional symbols* "$\bot$", "$\supset$",

the *modal symbol* "$\Box$",

and the punctuation marks, "(", ")".

2.2 Syntax

We now have a language with which to construct expressions (strings of symbols of the language). We now impose constraints on the kinds of expressions we permit by defining a set, closed under certain rules.

Definition 2.2.1. The propositional modal *well-formed formulas* (wffs) are expressions of the language of propositional modal logic defined by recursion as follows:

- Every propositional letter is a wff,
- $\bot$ is a wff,

- If α and β are wffs, then so is $(\alpha \supset \beta)$,
- If α is a wff, then so is $\Box\alpha$.

We use the following standard syntactic abbreviations:

[abbr. $\top$] $\top$ abbreviates $\bot \supset \bot$,

[abbr. $\sim$] $\sim\alpha$ abbreviates $\alpha \supset \bot$,

[abbr. $\vee$] $\alpha \vee \beta$ abbreviates $\sim\alpha \supset \beta$,

[abbr. $\wedge$] $\alpha \wedge \beta$ abbreviates $\sim(\alpha \supset \sim\beta)$,

[abbr. $\Diamond$] $\Diamond\alpha$ abbreviates $\sim\Box\sim\alpha$.

If α is a wff, then we let $PL(\alpha)$ denote the set of all propositional letters occurring in α. We use $\Box^n\alpha$ to abbreviate $\underbrace{\Box \ldots \Box}_{n \text{ times}}\alpha$ and $\Diamond^n\alpha$ to denote $\underbrace{\Diamond \ldots \Diamond}_{n \text{ times}}\alpha$.

Definition 2.2.2. A wff is called *closed* if it does not contain any propositional letters.

Definition 2.2.3. A *uniform substitution instance* of a propositional modal wff α is any wff β obtained by uniformly substituting letters $p_1, \ldots, p_n$ with wffs $\gamma_1, \ldots, \gamma_n$ in α. We denote this operation by $\beta = [p_1, \ldots, p_n/\gamma_1, \ldots, \gamma_n]\alpha$.

With our syntax defined, we now provide definitions that further restrict the set of wffs to those belonging to a "logic" by specifying an initial set of wffs, and closing the set under certain rules. While this definition is a rather abstract formulation of logic, it will prove convenient in later definitions and applications. Now, let K denote the following wff (named as such in honour of Saul Kripke):

$$(K) \qquad \Box(p \supset q) \supset (\Box p \supset \Box q)$$

K is also called the "distribution axiom", due to how $\Box$ distributes over $\supset$. We let **CL** denote the set of all classical propositional (boolean) tautologies.

Definition 2.2.4. A propositional modal logic is any set of wffs S including **CL**, containing K, and closed under the following rules of inference:

[MP] *Modus Ponens*. If $\alpha \in S$ and $\alpha \supset \beta \in S$, then $\beta \in S$.

[US] *Uniform Substitution*. If $\alpha \in S$ and the wff β is a uniform substitution instance of α, then $\beta \in S$.

Additionally, S is *normal* if S is closed under the following rule of inference:

$[N]$ *Rule of Necessitation*. If $\alpha \in S$, then $\Box\alpha \in S$.

The smallest logic satisfying Definition 2.2.4 is called **K**. Hence, throughout this thesis, by "a propositional modal logic", we mean an extension of **K**, i.e. logics which extend **K** with additional wffs called *axioms*. When interpreted alethically, the modal logic **K** is considered weak since it does not provide a sufficiently meaningful account for necessity [11] which many logicians and philosophers might require in using modal logics, thus limiting its philosophical applicability as an axiomatic basis while unaccompanied by other modal axioms. Some common axioms are listed in Table 2.1. In this thesis, we will consider only propositional modal logics that are normal and that are representable as Hilbert-style proof systems with finitely many axioms.

Definition 2.2.5. Let H be a Hilbert-style proof system for propositional modal logic. A *proof* of a wff α in H is a sequence of wffs $\alpha_1, \ldots, \alpha_n$, with $\alpha_n = \alpha$, such that each α_i (for $1 \leq i \leq n$) satisfies at least one of the following conditions:

- α_i is an axiom of H,
- α_i is obtained from some α_j, α_k $(1 \leq j < k < i)$ using $[MP]$,
- α_i is obtained from some α_j $(1 \leq j < i)$ using $[N]$ or $[US]$.

If S is a propositional modal logic and α is a wff, then $S \vdash \alpha$ will denote that α is provable in any Hilbert-style proof system that proves exactly the members of S.

Lemma 2.2.6. Let S be a propositional modal logic and let α be a wff. Then, $S \vdash \alpha$ iff $\alpha \in S$.

Proof. Trivial, since α is proved in S using the same rules of inference under which S is closed. ■

Axiom	Scheme
T	$\Box p \supset p$
D	$\Diamond \top$
4	$\Box p \supset \Box\Box p$
B	$p \supset \Box\Diamond p$
5	$\Diamond p \supset \Box\Diamond p$

TABLE 2.1: Some Standard Axioms

If S is a propositional modal logic and Γ is a set of wffs, then $S + \Gamma$ will denote the set $S \cup \Gamma$, closed under the rules of inference in Definition 2.2.4. Hence, $S + \Gamma$ is also a propositional modal logic. If α is a wff, then $S + \alpha$ will abbreviate $S + \{\alpha\}$. If both S and S' are propositional modal logics, it suffices to know that S contains the axioms of S' to establish $S' \subseteq S$. For example, since $\boldsymbol{K}$T contains the axiom K, it is clear that $\boldsymbol{K}$T proves the theorems of logic $\boldsymbol{K}$ and therefore $\boldsymbol{K} \subseteq \boldsymbol{K}$T. We call any member of a propositional modal logic S a *theorem* of S and adopt the convention of denoting specific logics of interest by the sets of axioms they contain in bold (e.g. the logic $\boldsymbol{K}$T contains axioms K and T), except for traditional names for certain logics such as **S5**. We now prove several useful theorems for the following sections.

Theorem 2.2.7. Let S be a propositional modal logic and let α and β be wffs. If $S \vdash \alpha \supset \beta$ then, $S \vdash \Diamond\alpha \supset \Diamond\beta$.

Proof. By **CL**, $S \vdash \sim\beta \supset \sim\alpha$. Then, by $[N]$, K and $[MP]$, $S \vdash \Box\sim\beta \supset \Box\sim\alpha$. Again, by **CL**, $S \vdash \sim\Box\sim\alpha \supset \sim\Box\sim\beta$, i.e. $S \vdash \Diamond\alpha \supset \Diamond\beta$. ■

Theorem 2.2.8. Let S be a propositional modal logic and let $\alpha_1, \ldots, \alpha_n$ be wffs. Then,

$$\text{if } S \vdash \Box \bigwedge_{i\geq 1}^{n} \alpha_i, \text{ then } S \vdash \bigwedge_{i\geq 1}^{n} \Box\alpha_i.$$

Proof. Suppose $S \vdash \Box \bigwedge_{i\geq 1}^{n} \alpha_i$. By **CL**, for each $1 \leq j \leq n$, $S \vdash \bigwedge_{i\geq 1}^{n} \alpha_i \supset \alpha_j$. By $[N]$, $S \vdash \Box(\bigwedge_{i\geq 1}^{n} \alpha_i \supset \alpha_j)$. Thus, by K and $[MP]$, $S \vdash \Box\alpha_j$, for each $1 \leq j \leq n$. Hence, by **CL**, $S \vdash \Box\alpha_1 \wedge \cdots \wedge \Box\alpha_n$, i.e. $S \vdash \bigwedge_{1\geq i}^{n} \Box\alpha_i$. ■

Theorem 2.2.9. If α is a wff, then $\boldsymbol{K}$B $\vdash \Diamond\Box\alpha \supset \alpha$.

Proof. Let α be a wff. By $[US]$, $\boldsymbol{K}$B $\vdash \sim\alpha \supset \Box\Diamond\sim\alpha$. Then, by **CL**, it follows that $\boldsymbol{K}$B $\vdash \sim\Box\Diamond\sim\alpha \supset \alpha$, i.e. $\boldsymbol{K}$B $\vdash \Diamond\Box\alpha \supset \alpha$. ■

Theorem 2.2.10. Let S be a propositional modal logic such that $\boldsymbol{K}$B $\subseteq S$ and let α and β be wffs. If $S \vdash \Diamond\alpha \supset \beta$, then $S \vdash \alpha \supset \Box\beta$.

Proof. Let S be a propositional logic including $\boldsymbol{K}$B and suppose that $S \vdash \Diamond\alpha \supset \beta$. By $[N]$, $S \vdash \Box(\Diamond\alpha \supset \beta)$. And, by K and $[MP]$, (1) $S \vdash \Box\Diamond\alpha \supset \Box\beta$. Now, since S includes $\boldsymbol{K}$B, (2) $S \vdash \alpha \supset \Box\Diamond\alpha$. From (1) and (2), by **CL**, it follows that $S \vdash \alpha \supset \Box\beta$. ■

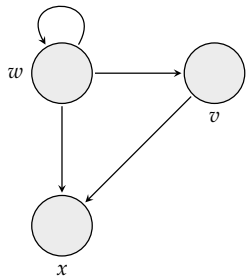


FIGURE 2.1: A Kripke frame defined by $W = \{w, v, x\}$, $R = \{(w, w), (w, v), (w, x), (v, x)\}$.

2.3 Kripke Semantics

The dual modalities "$\Box$" and "$\Diamond$" are amenable to various interpretations (as stated earlier, this thesis will use the alethic interpretation). Given an interpretation of the modalities, the syntactic modal language is amenable to various *semantics*, each respecting the modal interpretation in one way or another. For example, provability logic - wherein "$\Box$" assigns the property "is provable in S" to the operand, where S is usually a mathematical theory containing Peano Arithmetic - has been given possible world semantics (our chosen semantics for this thesis) as well as topological semantics (wherein propositions are considered as subsets of a topological space) [20]. Other semantics based on different interpretations of the modalities include those for temporal logic (propositional truth values at different points in time), deontic logic (moral obligation and permission), doxastic logic (reasoning about belief) and epistemic (reasoning about knowledge) [11]. Our focus will be on *Kripke Semantics*, i.e. "possible world" semantics (sometimes referred to as "relational semantics" or "frame semantics"), which we now formally introduce.

Definition 2.3.1. A *Kripke frame* is a pair $\langle W, R \rangle$ where W is a non-empty set and $R \subseteq W \times W$.

For a frame $\langle W, R \rangle$, we will call elements of W "possible worlds" (or just "worlds") and R an "accessibility relation". We say that "a world v is accessible from a world w" if wRv and define $R(X) := \{v \in W \mid wRv, \text{for some } w \in X\}$. If X is a singleton, with say $X = \{w\}$, then we may write $R(w)$ instead.

If $F = \langle W, R \rangle$ is a frame and $W' \subseteq W$, then we define the restriction of R to W' by

$$R \restriction W' := R \cap (W' \times W').$$

Definition 2.3.2. Let $F = \langle W, R\rangle$ be a frame and W' be a non-empty subset of W. Then, the frame,

$$F' := \langle W',\ R' \restriction W'\rangle,$$

is called a *subframe* of F. If, in addition, $R(W') \subseteq W'$, then F' is a *generated subframe* of F.

Definition 2.3.3. A *Kripke model* is a triple $M = \langle W, R, V\rangle$, with $\langle W, R\rangle$ a frame and $V : PL \to 2^W$, a valuation function. The truth of a wff α at a world w in a model M, denoted by $M, w \Vdash \alpha$, is defined recursively as follows:

$[VL]$ $M, w \Vdash p_i$ if $w \in V(p_i)$,

$[V\bot]$ $M, w \nVdash \bot$,

$[V\supset]$ $M, w \Vdash \alpha \supset \beta$ if $M, w \nVdash \alpha$ or $M, w \Vdash \beta$,

$[V\Box]$ $M, w \Vdash \Box\alpha$ if $M, v \Vdash \alpha$ for every $v \in R(w)$

For notational convenience, lest the sole usage of primitive logical symbols becomes burdensome, we state additional model valuation rules for each of the syntactic abbreviations:

$[V\top]$ $M, w \Vdash \top$,

$[V\sim]$ $M, w \Vdash \sim\alpha$ if $M, w \nVdash \alpha$,

$[V\vee]$ $M, w \Vdash \alpha \vee \beta$ if $M, w \Vdash \alpha$ or $M, w \Vdash \beta$

$[V\wedge]$ $M, w \Vdash \alpha \wedge \beta$ if both $M, w \Vdash \alpha$ and $M, w \Vdash \beta$

$[V\Diamond]$ $M, w \Vdash \Diamond\alpha$ if there exists $v \in R(w)$ such that $M, v \Vdash \alpha$

For a model $M = \langle W, R, V\rangle$, while the domain of V is PL, we may extend it to the set of all propositional modal wffs in the following way:

- $V(\bot) = \emptyset$,
- $V(\alpha \supset \beta) = (W - V(\alpha)) \cup V(\beta)$,
- $V(\Box\alpha) = \{w \in W \mid R(w) \subseteq V(\alpha)\}$,
- $V(\top) = W$,
- $V(\sim\alpha) = W - V(\alpha)$,
- $V(\alpha \vee \beta) = V(\alpha) \cup V(\beta)$,
- $V(\alpha \wedge \beta) = V(\alpha) \cap V(\beta)$,
- $V(\Diamond\alpha) = \{w \in W \mid R(w) \cap V(\alpha) \neq \emptyset\}$.

Thus, $V(\alpha) = \{w \in W \mid M, w \Vdash \alpha\}$.

From now on, we may refer to Kripke frames as "frames" and Kripke models as "models".

Definition 2.3.4. We say a model $M = \langle W, R, V \rangle$ is *based* on the frame $\langle W, R \rangle$.

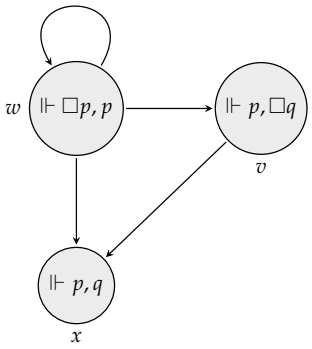


FIGURE 2.2: A Kripke model, with highlighted wffs, whose underlying Kripke frame is taken from Figure 2.1.

Definition 2.3.5. Let α be a wff and let S be a propositional modal logic. Then,

- If M is a model, then α is *valid* in M if $M, w \Vdash \alpha$ for every $w \in W$. We denote this by $M \Vdash \alpha$.
- If F is a frame, then α is *valid* on F if $M \Vdash \alpha$ for every model M based on F. We denote this by $F \Vdash \alpha$.
- If C is a class of frames, then α is *valid* on C if $F \Vdash \alpha$ for every $F \in C$. We denote this by $C \Vdash \alpha$.

If C is a class of frames, let $\mathrm{Log}(C) := \{\alpha \mid C \Vdash \alpha,\ \alpha \text{ is a wff}\}$.

- S is *validated* by C if $S \subseteq \mathrm{Log}(C)$ (soundness).
- S is *complete* w.r.t C if $S \supseteq \mathrm{Log}(C)$ (completeness).
- S is *determined* by C if $S = \mathrm{Log}(C)$. Thus, S is determined by C if S is validated by and complete w.r.t C.

Definition 2.3.6. A propositional modal logic is *Kripke complete* if it is determined by some class of Kripke frames.

These particular logics correspond to (are validated by) frames bearing distinct first-order definable graph-theoretic conditions w.r.t the accessibility relation. In Table 2.2, these conditions are listed with their corresponding first-order conditions on the accessibility relation. The other logics listed in Table 2.3 correspond to unique combinations of the conditions listed in

Table 2.2, commonly referred to throughout the literature (for example, the introductory text [8]).

Reflexive	$\forall w\ (wRw)$
Serial	$\forall w\ \exists v\ (wRv)$
Symmetric	$\forall w\ \forall v\ (wRv \rightarrow vRw)$
Transitive	$\forall w\ \forall v\ \forall u\ (wRv\ \&\ vRu \rightarrow wRu)$
Euclidean	$\forall w\ \forall v\ \forall u\ (wRv\ \&\ wRu \rightarrow vRu)$

TABLE 2.2: Some First-Order Frame Conditions

Logic	Axioms	Frame condition
KT	T	reflexive
KD	D	serial
KB	B	symmetric
K4	4	transitive
K5	5	Euclidean
KTB	T, B	reflexive, symmetric
S4	T, 4	reflexive, transitive
S5	T, 5 or T, 4, B	reflexive, symmetric, transitive

TABLE 2.3: Some Standard Modal Logics

2.3.1 General Kripke Frames

We now present definitions and several results relating to general Kripke frames, anticipating their use in later chapters. For a more in-depth discussion on general Kripke frames, see Chagrov and Zakharyaschev [4] and Gabbay, Shehtman, and Skvortsov [10].

Definition 2.3.7. A *General Kripke frame* is a triple $\langle W, R, \mathcal{A}\rangle$ where $W \neq \emptyset$, $R \subseteq W \times W$ and $\mathcal{A} \subseteq 2^W$ is a set of "admissible sets" which contains $\emptyset$ and is closed under $\cup$, $-$ and the operation $\Box$, defined for every $X \subseteq W$ by

$$\Box X := \{w \in W \mid R(w) \subseteq X\}.$$

From now on, we will call general Kripke frames "general frames". Note that $\mathcal{A}$ in Definition 2.3.7 is also a *modal algebra*, as defined in [10, Definition 1.2.5 on p.14] (the authors define the more general N-modal algebra; in this thesis, N will always be 1).

Definition 2.3.8. A model $M = \langle W, R, V \rangle$ based on a general frame $\mathcal{F} = \langle W, R, \mathcal{A} \rangle$ is *admissible* in $\mathcal{F}$ if for every $p_i \in PL$, $V(p_i) \in \mathcal{A}$.

Proposition 2.3.9. Let $\mathcal{F} = \langle W, R, \mathcal{A} \rangle$ be a general frame, $M = \langle W, R, V \rangle$ be a model based on $\mathcal{F}$. If M is admissible in $\mathcal{F}$, then for every wff α, $V(\alpha) \in \mathcal{A}$.

Proof. We prove the statement by induction on the complexity of α. By definition, the statement holds for $\bot$ and every $p_i \in PL$. We show that the statement holds for $[V \supset]$ and $[V\Box]$.

($\supset$) Let $\alpha = \beta \supset \gamma$. $V(\beta \supset \gamma) = (W - V(\beta)) \cup V(\gamma)$. Since $\mathcal{A}$ is closed under $\cup$ and $-$, $V(\beta \supset \gamma) \in \mathcal{A}$.

($\Box$) Let $\alpha = \Box\beta$. $V(\Box\beta) = \Box V(\beta)$. Since $\mathcal{A}$ is closed under $\Box$, $V(\beta \supset \gamma) \in \mathcal{A}$.

■

Definition 2.3.10. If $M = \langle W, R, V \rangle$ is a model, then the *set of all definable sets for* M is defined by $\mathcal{A}_M := \{V(\alpha) \mid \alpha \text{ is a wff}\}$.

If $M = \langle W, R, V \rangle$ is a model, then we define $GF(M) := \langle W, R, \mathcal{A}_M \rangle$.

Lemma 2.3.11. Let $M = \langle W, R, V \rangle$. Then, $GF(M)$ is a general frame.

Proof. Let $GF(M) = \langle W, R, \mathcal{A} \rangle$. Since M is a model, $W \neq \emptyset$ and $R \subseteq W \times W$. By definition, $\mathcal{A}_M \subseteq 2^W$. Moreover, since $V(\bot) = \emptyset$, it follows that $\emptyset \in \mathcal{A}$. We show that $\mathcal{A}_M$ satisfies the closure rules of general frames. Let α and β be wffs. Then, $V(\alpha)$, $V(\beta)$, $V(\alpha \vee \beta)$, $V(\sim\alpha)$, $V(\Box\alpha) \in \mathcal{A}_M$. Now, $V(\alpha \vee \beta) = V(\alpha) \cup V(\beta)$. Hence, $\mathcal{A}_M$ is closed under $\cup$. Furthermore, $V(\sim\alpha) = W - V(\alpha)$. Hence, $\mathcal{A}_M$ is closed under $-$. Finally, $V(\Box\alpha) = \{w \in W \mid R(w) \in V(\alpha)\}$. Hence, $\mathcal{A}_M$ is closed under $\Box$. ■

Definition 2.3.12. A wff α is *valid* in a general frame $\mathcal{F} = \langle W, R, \mathcal{A} \rangle$ if $M, w \Vdash \alpha$ for every model $M = \langle W, R, V \rangle$ admissible in $\mathcal{F}$, at every $w \in W$.

Note that a Kripke frame is just a general frame with $\mathcal{A} = 2^W$.

Definition 2.3.13. A general frame $\mathcal{F} = \langle W, R, \mathcal{A} \rangle$ is called

- *differentiated*, if for every $w, v \in W$, there exists $X \in \mathcal{A}$ such that $w \in X$ and $v \notin X$,
- *tight*, if for every $X \in \mathcal{A}$ and every $w, v \in W$,

$$\text{if } (w \in \Box X \to v \in X), \text{ then } wRv.$$

- *refined*, if $\mathcal{F}$ is both differentiated and tight.
- *compact*, if for every $\mathcal{B} \subseteq \mathcal{A}$ with the finite intersection property, i.e.

  $$\forall \mathcal{K} \subseteq \mathcal{B} \; (\; \mathcal{K} \text{ is finite } \to \bigcap \mathcal{K} \neq \emptyset),$$

  then $\bigcap \mathcal{B} \neq \emptyset$.
- *descriptive*, if $\mathcal{F}$ is both refined and compact.

Definition 2.3.14. A set of wff Γ is *r-persistent* if for every refined general frame $\langle W, R, \mathcal{A} \rangle$,

$$\text{If } \langle W, R, \mathcal{A} \rangle \Vdash \Gamma, \text{ then } \langle W, R \rangle \Vdash \Gamma.$$

Similarly, a set of wff Γ is *d-persistent* if the above condition holds for every descriptive general frame $\langle W, R, \mathcal{A} \rangle$.

2.3.2 First-Order Translations and Definability

Well-established in the literature is the syntactic and semantic correspondence between propositional modal wffs and (non-modal) classical predicate sentences, e.g. propositional modal wffs can be translated to classical predicate wffs via the *standard translation* [10, p. 48-49]. In this section, we highlight a few main results in this area, due to van Benthem et al, anticipating their later use in this thesis, in light of other results we demonstrate in following sections.

Let $\mathcal{L}_1$ be a classical predicate language with equality and a binary predicate letter R (corresponding to the accessibility relation in a frame). Thus, classical $\mathcal{L}_1$-structures are Kripke frames. Predicate wffs built from $\mathcal{L}_1$ will be called $\mathcal{L}_1$-wffs. We denote the truth of an $\mathcal{L}_1$-wff A in a frame F by $F \Vdash A$.

Definition 2.3.15. A propositional modal wff α *corresponds* to an $\mathcal{L}_1$-wff A if for every frame F,

$$F \Vdash \alpha \text{ iff } F \Vdash A.$$

Definition 2.3.16. An $\mathcal{L}_1$-wff is called *universal* if every variable it contains is universally quantified.

Definition 2.3.17. A class of classical $\mathcal{L}_1$ structures is called *universal* if it is a class of classical models of a set of universal classical $\mathcal{L}_1$-wffs.

Definition 2.3.18. A propositional modal logic S is *universal* if the class of all frames validating S is universal.

If Σ is a set of classical predicate sentences, then we define $Mod(\Sigma) := \{F \mid F \Vdash \Sigma\}$, the set of all classical models of Σ.

Definition 2.3.19. A class C of $\mathcal{L}_1$-structures is called Δ-*elementary* if there exists a set of classical predicate sentences Σ such that $C = Mod(\Sigma)$.

Definition 2.3.20. A propositional modal logic is called Δ-*elementary* if it is determined by a Δ-elementary class of frames.

Theorem 2.3.21 (Fine-Benthem)**.** Every Δ-elementary propositional modal logic is d-persistent.

Proof. See Benthem [2] or Chagrov and Zakharyaschev [4]. ■

Definition 2.3.22. A propositional modal logic S is called *subframe* if for every frame F validating S, every subframe of F also validates S.

The following theorem, forming part of a larger theorem (see [10, Theorem 1.12.8, p. 62]), while only be used at the end of Chapter 5, is included here so that it can be understood in the context of the definitions on which it relies. It is shown in [10] that a Δ-elementary propositional modal logic is universal iff it is subframe, and so we only present the part of the result that demonstrates that a logic is r-persistent if it is universal and Kripke complete.

Theorem 2.3.23. If S is a universal Kripke complete propositional modal logic, then S is r-persistent.

Proof. Since S is universal, it is clearly Δ-elementary. Thus, by Theorem 2.3.21, S is d-persistent. Now, let $\mathcal{F} = \langle W, R, \mathcal{A}\rangle$ be a refined general frame validating S. Via ultrafilter extension (see [2]), $\mathcal{F}$ has a descriptive extension $\mathcal{F}' = \langle W', R', \mathcal{A}'\rangle$ validating S. Since S is d-persistent, it follows that $\langle W', R'\rangle \Vdash S$. Since S is a subframe logic and $\langle W, R\rangle$ is a subframe of $\langle W', R'\rangle$, it follows that $\langle W, R\rangle \Vdash S$. Hence, S is r-persistent. ■

Chapter 3

Kripke Completeness for Propositional Modal Logics

Even with the backdrop of Blok's Dichotomy, which implies the existence of a continuum many incomplete propositional modal logics (see [22] for more discussion), Kripke complete logics are nonetheless commonly found, given their relative naturality when compared to the contrived logics, manufactured for the purpose of demonstrating incompleteness. With our goal set on demonstrating Kripke completeness results for predicate modal logics, we now proceed to lay the needed foundations in the context of *propositional* modal logics and Kripke frames. The literature in the area of Kripke completeness results for propositional modal logic is substantial and plentiful. One seminal contribution to this area is the Sahlqvist Theorem [18] (see Section 3.3.2), which actually implies all of the completeness and soundness results preceding it in this chapter. The theorem, equivalently, states that propositional modal logics axiomatised by wffs of a certain form are canonical and correspond to first-order definable frames. Further to this, algorithms have been developed to obtain these first-order correspondents (see Conradie and Palmigiano [5]).

3.1 Soundness

We now present soundness results for the propositional modal logics ***K***, ***K***T, ***K***D, ***K***B, ***K***TB, ***K*4**, ***K*5**, ***S*4** and ***S*5**, which are all representable as Hilbert-style deductive systems with finitely many modal axioms. For any propositional modal logic S, soundness w.r.t a class C of frames

can be shown by demonstrating that axioms of S are validated by C and that the rules of inference preserve validity of inferred theorems on C, thus showing that every theorem of S is validated by C.

Lemma 3.1.1. **CL** is valid on all frames.

Proof. Let α be a propositional tautology and $F = \langle W, R\rangle$ a frame. Suppose $F \nVdash \alpha$. Then, $M, w \nVdash \alpha$ for some model $M = \langle F, V\rangle$ and some $w \in W$. Let V' be a classical propositional assignment such that p_i is true under V' iff $M, w \Vdash p_i$, for each $p_i \in PL(\alpha)$. Since $M, w \nVdash \alpha$, it follows that α is false under V', contradicting the assumption that α is a tautology. ■

Theorem 3.1.2. K is validated by every frame.

Proof. Recall axiom $K : \Box(p \supset q) \supset (\Box p \supset \Box q)$. Let $F = \langle W, R\rangle$ be a frame, $M = \langle F, V\rangle$ a model and $w \in W$. Suppose $M, w \Vdash \Box(p \supset q)$ and $M, w \Vdash \Box p$. Then, by $[V\Box]$, $M, w' \Vdash p \supset q$, for every $w' \in R(w)$. By $[V\Box]$, $M, w' \Vdash p$, for every $w' \in R(w)$. Thus, by $[V\supset]$, $w' \Vdash q$. Then, by $[V\Box]$, $M, w \Vdash \Box q$ and thus, by $[V\supset]$, $M, w \Vdash \Box p \supset \Box q$. Finally, by $[V\supset]$, it follows that $M, w \Vdash \Box(p \supset q) \supset (\Box p \supset \Box q)$. ■

Lemma 3.1.3. The rules of inference $[MP]$, $[US]$ and $[N]$ preserve validity on frames.

Proof. Let $F = \langle W, R\rangle$ be a frame. We show for $[MP]$, $[US]$ and $[N]$ that the inferred theorems are also valid in F:

> $[MP]$: Suppose $F \Vdash \alpha \supset \beta$ and $F \Vdash \alpha$. Let $M = \langle F, V\rangle$ be a model based on F and let $w \in W$. Then, $M, w \Vdash \alpha$ and $M, w \Vdash \alpha \supset \beta$. By $[V\supset]$, it follows that $M, w \Vdash \beta$. Since M and w are arbitrary, it follows that $F \Vdash \beta$.
>
> $[US]$: Let $F \Vdash \alpha$ and let the wff γ be the result of uniformly substituting wffs $\beta_1, \ldots, \beta_n$ for $p_1, \ldots, p_n \in PL(\alpha)$. Suppose $F \nVdash \gamma$. Then, there exists a model $M = \langle F, V\rangle$ such that $M, w \nVdash \gamma$, for some $w \in W$. Let $M' = \langle F, V'\rangle$ be a model such that $M', w' \Vdash p_i$ iff $M, w \Vdash \beta_i$ for each $1 \leq i \leq n$, for some $w' \in W$. Simple induction shows that $M', w' \nVdash \alpha$. Hence, $F \nVdash \alpha$.
>
> $[N]$: Let $F \Vdash \alpha$. Let $M = \langle F, V\rangle$ be a model and let $w \in W$. Then, $M, w \Vdash \alpha$ and $M, v \Vdash \alpha$, for every $v \in R(w)$. Thus, by $[V\Box]$, it follows that $M, w \Vdash \Box\alpha$. Since M and w are arbitrary, it follows that $F \Vdash \Box\alpha$.

■

Corollary 3.1.4. The propositional modal logic $\boldsymbol{K}$ is validated by all frames.

Proof. Immediate from Lemmas 3.1.1, 3.1.2 and 3.1.3. ■

Lemma 3.1.5. Let Γ be a set of wffs, $S = \boldsymbol{K} + \Gamma$ a propositional modal logic and C a class of frames. Then, $C \Vdash S$ iff $C \Vdash \Gamma$.

Proof. Immediate from Lemma 3.1.3 and Corollary 3.1.4. ■

If $\langle W, R\rangle$ is a frame and $w, v \in W$, then we write $wR^n w'$ if there exist $w_1, \ldots, w_{n-1} \in W$ such that wRw_1, w_1Rw_2, $\ldots$, $w_{n-2}Rw_{n-1}$, $w_{n-1}Rv$. We define R^0 to be the identity relation on W, i.e. wR^0v iff $w = v$.

The propositional modal axioms listed in Table 2.1 have the following form (for $k, l, m, n \geq 0$), a generalisation of the Church-Rosser formula:

(*Gen*) $$\Diamond^k\Box^l p \supset \Box^m\Diamond^n p$$

The following theorem, due to van Bentham [1], yields several soundness results all at once, needed for demonstrating Kripke completeness later on.

Theorem 3.1.6. For $k, l, m, n \geq 0$, the axioms *Gen* are valid on a frame iff the frame satisfies

(1) $$\forall w\, \forall w'\, \forall w''\, (wR^k w' \wedge wR^m w'' \rightarrow \exists w'''\, (w'R^l w''' \wedge w''R^n w''')).$$

Proof. For "if", we argue by contraposition. Let $F = \langle W, R\rangle$. Suppose there exists a model $M = \langle F, V\rangle$ and $w \in W$ such that $M, w \nVdash \Diamond^k\Box^l p \supset \Box^m\Diamond^n p$. By $[V \supset]$, it follows that (2) $M, w \Vdash \Diamond^k\Box^l p$ but $M, w \nVdash \Box^m\Diamond^n p$. By $[V\sim]$ we can express the latter, equivalently, as (3) $M, w \Vdash \Diamond^m\Box^n\sim p$. So, by (2) and $[V\Diamond]$, there exists $w' \in R^k(w)$ such that $M, u \Vdash p$, for every $u \in R^l(w')$. And, by (3), there exists $M, w'' \in R^m(w)$ such that $M, v \Vdash \sim p$, for every $v \in R^n(w'')$. Hence, there cannot exist $w''' \in R^l(w') \cap R^n(w'')$. Or, expressed as a first-order condition:

$$\exists w\, \exists w'\, \exists w''\, (wR_S^k w' \;\&\; wR_S^m w'' \rightarrow \forall w'''\, \sim(w'R_S^l w''' \;\&\; w''R_S^n w''')),$$

which is the negation of (1).

For "only if", we again argue by contraposition. Let $F = \langle W, R\rangle$ be a frame such that (*) $F \Vdash \Diamond^k\Box^l p \supset \Box^m\Diamond^n p$. Suppose for some $w, w', w'' \in W$ that wR^kw' and wR^mw'' but that for every $w''' \in W$, $w''' \notin R^l(w')$ or $w''' \notin R^n(w'')$. Let $M = \langle F, V\rangle$ be a model based on F such that $V(p) = R^l(w') - R^n(w'')$. Then, (1) $M, w' \Vdash \Box^l p$ and (2) $M, w'' \nVdash \Diamond^n p$. By (1) and $[V\Diamond]$, it follows that (3) $M, w \Vdash \Diamond^k\Box^l p$ and by (2) and $[V\Box]$, it follows that (4) $M, w \nVdash \Box^m\Diamond^n p$. By (*), (3) and $[V \supset]$, it follows that $M, w \Vdash \Box^m\Diamond^n p$, contradicting (4). ■

Corollary 3.1.7. The axioms T, B and 4 are validated by reflexive, symmetric and transitive frames respectively.

Proof. T, B and 4 are of the form *Gen*. By Theorem 3.1.6, therefore, they are validated by frames satisfying the theorem's first-order condition. It is easy to check that the values of k, l, m and n corresponding to T, B and 4 yield the first-order frame conditions of reflexivity, symmetry and transitivity respectively. ■

Corollary 3.1.8. Frames validating T, B and 4 satisfy reflexivity, symmetry and transitivity respectively.

Proof. The axioms T, B and 4 are of the form *Gen*. By Theorem 3.1.6, therefore, the frames satisfying the first-order condition with the appropriate values for k, l, m and n for reflexivity, symmetry and transitivity, each validate T, B and 4, obtained with the same values of k, l, m and n. ■

Corollary 3.1.9. The propositional modal logics *K*T, *K*B and *K*4 are validated by reflexive, symmetric and transitive frames respectively.

Proof. Immediate from Corollary 3.1.7 and Lemma 3.1.3. ■

Theorem 3.1.10. Axiom D is validated by serial frames.

Proof. Recall axiom D := $\Diamond\top$. Let $F = \langle W, R\rangle$ be a serial frame, let $M = \langle F, V\rangle$ be a model and let $w \in W$. By $[V\top]$, $M, w \Vdash \top$. Since F is serial, there exists $v \in R(w)$. Again, by $[V\top]$, $M, v \Vdash \top$. Thus, by $[V\Diamond]$, it follows that $M, w \Vdash \Diamond\top$. Since M and w were arbitrary, it follows that $F \Vdash \Diamond\top$. ■

Theorem 3.1.11. Let S and S' be propositional modal logics validated by classes C and C', respectively. Then, the propositional modal logic $S + S'$ is validated by $C \cap C'$.

Proof. Trivial. ■

3.2 Canonical Models

A canonical model M_S of a propositional modal logic S bears its significance in that it validates all and only the theorems of S, i.e. $M_S \Vdash \alpha$ iff $S \vdash \alpha$, for every wff α. In this section, we provide the machinery needed to construct such a model; define canonicity and show that the logics for which we proved soundness results possess this property, and are therefore Kripke complete; prove that these same logics are complete w.r.t their validating classes.

Definition 3.2.1. Let S be a propositional modal logic, Γ a set of wffs, and α a wff. An *S-derivation* of α from Γ is a sequence $\alpha_1, \ldots, \alpha_n$, with $\alpha_n = \alpha$, such that each α_i (for $1 \leq i \leq n$) satisfies at least one of the following conditions:

- $\alpha_i \in \Gamma$,
- $S \vdash \alpha_i$,
- α_i is obtained from some α_j, α_k $(1 \leq j < k < i)$ using $[MP]$,

We denote that α has an S-derivation from Γ by $\Gamma \vdash_S \alpha$. We abbreviate $\emptyset \vdash_S \alpha$ with $\vdash_S \alpha$. It is easy to check that $\vdash_S \alpha$ is equivalent to $S \vdash \alpha$.

Lemma 3.2.2. Let S be a propositional modal logic, Γ a set of wffs, and $\alpha, \beta \in \Gamma$. If $\Gamma \cup \{\alpha\} \vdash_S \beta$, then $\Gamma \vdash_S \alpha \supset \beta$.

Proof. Standard, by induction on the length of an S-derivation of β from $\Gamma \cup \{\alpha\}$. Observe that $[N]$ is not a derivation rule in Definition 3.2.1. ■

Definition 3.2.3. If S is a propositional modal logic, we say a set of wffs Γ is *S-inconsistent* if $\Gamma \vdash_S \bot$. Otherwise, Γ is *S-consistent*.

Lemma 3.2.4. Let S be a propositional modal logic, Γ a set of wff and $\alpha \in \Gamma$. Then, $\Gamma \cup \{\alpha\}$ is S-consistent iff $\Gamma \nvdash_S \sim\alpha$.

Proof. For "if", suppose otherwise, i.e. that $\Gamma \cup \{\alpha\}$ is S-inconsistent. By definition, this means $\Gamma \cup \{\alpha\} \vdash_S \bot$. By Lemma 3.2.2, $\Gamma \vdash_S \alpha \supset \bot$, i.e. $\Gamma \vdash_S \sim\alpha$. For "only if", suppose otherwise, i.e.

that $\Gamma \vdash_S \sim\alpha$, i.e. $\Gamma \vdash_S \alpha \supset \alpha$. Then, it follows that $\Gamma \cup \{\alpha\} \vdash_S \alpha \supset \bot$. Since $\Gamma \cup \{\alpha\} \vdash_S \alpha$, it follows by $[MP]$ that $\Gamma \cup \{\alpha\} \vdash_S \bot$, i.e. $\Gamma \cup \{\alpha\}$ is S-inconsistent. ■

Lemma 3.2.5. Let S be a propositional modal logic, Γ a set of wffs and α a wff. Then, $\Gamma \vdash_S \alpha$ iff there exist $\beta_1, \ldots, \beta_n \in \Gamma$ such that $\vdash_S \bigwedge\limits_{i\geq 1}^{n} \beta_i \supset \alpha$.

Proof. Suppose $\Gamma \vdash_S \alpha$. Then, there exists an S-derivation of α from Γ. Let $\beta_1, \ldots, \beta_n$ be all the wff from Γ used in this S-derivation. Then, $\{\beta_1, \ldots, \beta_n\} \vdash_S \alpha$. By **CL**, it follows that $\bigwedge\limits_{i\geq 1}^{n} \beta_i \vdash_S \alpha$. Thus, by Lemma 3.2.2, it follows that $\vdash_S \bigwedge\limits_{i\geq 1}^{n} \beta_i \supset \alpha$. Conversely, suppose there exist $\beta_1, \ldots, \beta_n \in \Gamma$ such that $\vdash_S \bigwedge\limits_{i\geq 1}^{n} \beta_i \supset \alpha$. Clearly, $\Gamma \vdash_S \beta_i$, for each $1 \leq i \leq n$. By **CL**, $\Gamma \vdash_S \bigwedge\limits_{i\geq 1}^{n} \beta_i$. Since $\vdash_S \bigwedge\limits_{i\geq 1}^{n} \beta_i \supset \alpha$, it follows by $[MP]$ that $\Gamma \vdash_S \alpha$. ■

If Γ be a set of wffs, then we define $\Box\Gamma := \{\Box\alpha \mid \alpha \in \Gamma\}$.

Lemma 3.2.6. Let S be a propositional modal logic, Γ a set of wffs, and α a wff. If $\Gamma \vdash_S \alpha$, then $\Box\Gamma \vdash_S \Box\alpha$.

Proof. Suppose $\Gamma \vdash_S \alpha$. Then by Lemma 3.2.5, there exist $\beta_1, \ldots, \beta_n \in \Gamma$ such that $S \vdash \bigwedge\limits_{i\geq 1}^{n} \beta_i \supset \alpha$. By $[N]$, $S \vdash \Box(\bigwedge\limits_{i\geq 1}^{n} \beta_i \supset \alpha)$. Then, by K and $[MP]$, we obtain $S \vdash \Box(\bigwedge\limits_{i\geq 1}^{n} \beta_i) \supset \Box\alpha$. Now, by Theorem 2.2.8 and **CL**, we obtain $S \vdash (\bigwedge\limits_{i\geq 1}^{n} \Box\beta_i) \supset \Box\alpha$. Finally, since $\Box\beta_1, \ldots, \Box\beta_n \in \Box\Gamma$, by Lemma 3.2.5, it follows that $\Box\Gamma \vdash_S \Box\alpha$. ■

Definition 3.2.7. Let S be a propositional modal logic and let Γ be an S-consistent set. We say that Γ is *S-complete* if there exists no S-consistent set Ω such that $\Gamma \subset \Omega$.

Lemma 3.2.8. Let S be a propositional modal logic and Γ a set of wffs. Then, Γ is S-complete iff for every wff α, either $\alpha \in \Gamma$ or $\sim\alpha \in \Gamma$, but not both.

Proof. We first show "if". Let Γ' be a set of wffs such that $\Gamma \subset \Gamma'$. Suppose for every α, either $\alpha \in \Gamma$ or $\sim\alpha \in \Gamma$, but not both. Since $\Gamma \subset \Gamma'$, there must exist a wff α such that $\alpha \in \Gamma'$ but $\alpha \notin \Gamma$. But then $\sim\alpha \in \Gamma$ and, by set inclusion, $\sim\alpha \in \Gamma'$. Thus, by **CL**, $\Gamma' \vdash_S \bot$ and, hence, Γ' is S-inconsistent. Since Γ' was an arbitrary proper superset of Γ, Γ is S-complete.

We now show "only if". For "either or", suppose that neither $\alpha \in \Gamma$ nor $\sim\alpha \in \Gamma$, for some $\alpha \in \Gamma$. Then, since Γ is S-complete, both $\Gamma \cup \{\alpha\}$ and $\Gamma \cup \{\sim\alpha\}$ are L-inconsistent. Since $\Gamma \cup \{\alpha\}$ is S-inconsistent, by Lemma 3.2.4, $\Gamma \vdash_S \sim\alpha$. And, since $\Gamma \cup \{\sim\alpha\}$ is S-inconsistent, $\Gamma \vdash_S \sim\sim\alpha$,

i.e. $\Gamma \vdash_S \alpha$. So, by **CL**, $\Gamma \vdash_S \bot$ and, hence, Γ is S-inconsistent. For "not both", suppose $\alpha \in \Gamma$ and $\sim\alpha \in \Gamma$. Then, by **CL**, $\Gamma \vdash_S \bot$ and, hence, Γ is again S-inconsistent. ■

Lemma 3.2.9. Let S be a propositional modal logic and Γ an S-complete set. Then,

(a) $\Gamma \vdash_S \alpha$ iff $\alpha \in \Gamma$.

(b) Γ is closed under $[MP]$,

Proof. (a) "If" holds by definition of $\vdash_S$. For "only if", suppose that $\Gamma \vdash_S \alpha$ but $\alpha \notin \Gamma$. Since Γ is S-complete, $\Gamma \cup \{\alpha\}$ is S-inconsistent. By Lemma 3.2.4, it follows that $\Gamma \vdash_S \sim\alpha$, i.e. $\Gamma \vdash_S \alpha \supset \bot$. Then, by $[MP]$, $\Gamma \vdash_S \bot$, contrary to the S-consistency of Γ.

(b) Suppose $\alpha, \alpha \supset \beta \in \Gamma$. Then, by $[MP]$, $\Gamma \vdash_S \beta$. By (a), therefore, $\beta \in \Gamma$.

■

Lemma 3.2.10. Let S be a propositional modal logic and Ω an S-consistent set. Then, Ω can be extended to an S-complete set Γ.

Proof. Let $\alpha_1, \alpha_2, \ldots$ be the enumeration of all propositional modal wffs. We define a sequence $\Gamma_0, \Gamma_1, \ldots$ of sets of wffs as follows:

$\Gamma_0 = \Omega$,

$\Gamma_{i+1} = \Gamma_i \cup \{\alpha_{i+1}\}$ if $\Gamma_i \cup \{\alpha_{i+1}\}$ is S-consistent; otherwise, $\Gamma_{i+1} = \Gamma_i \cup \{\sim\alpha_{i+1}\}$

Next, we show by induction that every Γ_i is consistent, with Γ_0 the inductive basis. Suppose that Γ_i is S-consistent but that neither $\Gamma_i \cup \{\alpha_{i+1}\}$ nor $\Gamma_i \cup \{\sim\alpha_{i+1}\}$ is S-consistent. Then, $\Gamma_i \cup \{\alpha_{i+1}\} \vdash_S \bot$ and $\Gamma_i \cup \{\sim\alpha_{i+1}\} \vdash_S \bot$. Since $\Gamma_i \cup \{\alpha_{i+1}\}$, by Lemma 3.2.4, it follows that $\Gamma_i \vdash_S \sim\alpha_{i+1}$. And, since $\Gamma_i \cup \{\sim\alpha_{i+1}\}$, by Lemma 3.2.4, it follows that $\Gamma_i \vdash_S \sim\sim\alpha_{i+1}$, i.e. $\Gamma_i \vdash_S \alpha_{i+1}$. Hence, by **CL**, $\Gamma_i \vdash_S \bot$, contrary to the assumption that Γ_i is S-consistent. Now, set

$$\Gamma := \bigcup_{i<\omega} \Gamma_i.$$

We now show that Γ is S-consistent. Suppose otherwise, i.e. $\Gamma \vdash_S \bot$. By Lemma 3.2.5, there exist $\beta_1, \ldots, \beta_n \in \Gamma$ such that $\vdash_S \bigwedge_{i\geq 1}^{n} \beta_i \supset \bot$. Since $\beta_1, \ldots, \beta_n$ is finite, it follows that for some $k < \omega$, $\beta_1, \ldots, \beta_n \in \Gamma_k$. And so, $\Gamma_k \vdash_S \bot$, contrary to the S-consistency of Γ_k. Finally, we show

that Γ is S-complete. By construction, for every $1 \leq i \leq \omega$, if $\alpha_i \notin \Gamma$, then $\sim\alpha_i \in \Gamma$. Hence, by Lemma 3.2.8, Γ is S-complete.

■

If Γ is a set of wffs, then we define $\Box^{-}\Gamma := \{\alpha : \Box\alpha \in \Gamma\}$.

Lemma 3.2.11. Let S be a propositional modal logic and Γ an S-consistent set such that for some wff α, $\sim\Box\alpha \in \Gamma$. Then, $\Box^{-}\Gamma \cup \{\sim\alpha\}$ is S-consistent.

Proof. Suppose not, i.e. that $\Box^{-}\Gamma \cup \{\sim\alpha\}$ is S-inconsistent. Then, by Lemma 3.2.4, $\Box^{-}\Gamma \vdash_S \sim\sim\alpha$, and so $\Box^{-}\Gamma \vdash_S \alpha$. Then, by Lemma 3.2.6, $\Box\Box^{-}\Gamma \vdash_S \Box\alpha$. But, since $\Box\Box^{-}\Gamma \subseteq \Gamma$, it follows that $\Gamma \vdash_S \Box\alpha$ which, by Lemma 3.2.9, implies that $\Box\alpha \in \Gamma$. Then, by Lemma 3.2.8, Γ is S-consistent, contradicting the assumption. ■

We now have enough to proceed with defining the canonical model for a propositional modal logic.

Definition 3.2.12. Let S be a propositional modal logic. The *canonical frame* of S is a Kripke frame $F_S = \langle W_S, R_S \rangle$ where

- $W_S := \{\Gamma \mid \Gamma \text{ is an } S\text{-complete set}\}$,
- $R_S := \{\langle \Gamma, \Omega \rangle \in W_S \times W_S \mid \Box^{-}\Gamma \subseteq \Omega\}$.

Definition 3.2.13. The *canonical model* of S is a model $M_S = \langle F_S, V_S \rangle$, based on the canonical frame F_S of S, where for every $p_i \in PL$,

$$V_S(p_i) := \{\Gamma \in W_S \mid p_i \in \Gamma\}.$$

We now prove that a wff's membership of a world corresponds to the wff's truth at that world.

Lemma 3.2.14 (Existence Lemma)**.** Let S be a propositional modal logic, F_S the canonical frame of S, $\Gamma \in W_S$ and $\Diamond\alpha \in \Gamma$, for some wff α. Then, there exists $\Omega \in R_S(\Gamma)$ such that $\alpha \in \Omega$.

Proof. Equivalently, suppose $\sim\Box\sim\alpha \in \Gamma$. By Lemma 3.2.11, $\Delta := \Box^{-}\Gamma \cup \{\alpha\}$ is S-consistent. Hence, by Lemma 3.2.10, there exists an S-complete set $\Omega \in W_S$ such that $\Delta \subseteq \Omega$. Since $\Box^{-}\Gamma \subseteq \Omega$, by definition of R_S, $\Omega \in R_S(\Gamma)$. Hence, $\alpha \in \Omega$. ■

Theorem 3.2.15 (Propositional Canonical Model Theorem)**.** Let S be a propositional modal logic and $M_S = \langle W_S, R_S, V_S \rangle$ the canonical model of S. Then, for every wff α and every $\Gamma \in W_S$:

$$\alpha \in \Gamma \text{ iff } M_S, \Gamma \Vdash \alpha.$$

Proof. We prove the theorem by induction on the complexity of α. By definition of W_S, the theorem holds for $\bot$, since every $\Gamma \in W_S$ is S-consistent. Furthermore, by definition of V_S, the theorem holds for every $p_i \in PL$.

($\supset$) Let $\alpha = \beta \supset \gamma$. Then, $\beta \supset \gamma \in \Gamma$ iff, by Lemma 3.2.9 (b), either $\beta \notin \Gamma$ or $\gamma \in \Gamma$ iff, by induction hypothesis, either $M_S, \Gamma \nVdash \beta$ or $M_S, \Gamma \Vdash \gamma$ iff, by $[V \supset]$, $M_S, \Gamma \Vdash \beta \supset \gamma$.

($\Box$) Let $\alpha = \Box\beta$. For "if", suppose $\Box\beta \notin \Gamma$. Then, by Lemma 3.2.8, $\sim\Box\beta \in \Gamma$, i.e. $\Diamond\sim\beta \in \Gamma$. By Lemma 3.2.14, there exists $\Omega \in R_S(\Gamma)$ such that $\sim\beta \in \Omega$. Then, by Lemma 3.2.8, $\beta \notin \Omega$. By induction hypothesis, therefore, $M_S, \Omega \nVdash \beta$. By $[V\Box]$, $M_S, \Gamma \nVdash \Box\beta$. For "only if", suppose $\Box\beta \in \Gamma$. By definition of R_S, $\Box^-\Gamma \subseteq \Omega$, for every $\Omega \in R_S(\Gamma)$. Hence, for every $\Omega \in R_S(\Gamma)$, $\beta \in \Omega$ and, therefore, by induction hypothesis, $M_S, \Omega \Vdash \beta$. Hence, By $[V\Box]$, $M_S, \Gamma \Vdash \Box\beta$.

■

We need one more lemma if a logic's canonical model is to be of any utility.

Theorem 3.2.16. Let S be a propositional modal logic and let $M_S = \langle W_S, R_S, V_S \rangle$ be the canonical model for S. Then,

$$S \vdash \alpha \text{ iff } M_S \Vdash \alpha.$$

Proof. For "if", suppose $S \nvdash \alpha$. Then, $S \nVdash \sim\sim\alpha$. Thus, $\{\sim\alpha\}$ is S-consistent. By Lemma 3.2.10, $\{\sim\alpha\}$ can be extended to an S-complete set Γ. Therefore, by Lemma 3.2.15, $M_S, \Gamma \Vdash \sim\alpha$ and so, by $[V\sim]$, $M_S, \Gamma \nVdash \alpha$. Hence, $M_S \nVdash \alpha$. For "only if", by Lemma 3.2.9 (a), if $S \vdash \alpha$, then $\alpha \in \Gamma$, for every $\Gamma \in W_S$. ■

3.3 Canonicity and Completeness

We now have at our disposal enough machinery in establishing completeness for the logics for which we presented soundness results. As we will now see, these logics share the property of being valid on their canonical frame, called *canonicity*.

Definition 3.3.1. Let S be a propositional modal logic and $F_S = \langle W_S, R_S \rangle$ the canonical frame of S. We say that S is *canonical* if

$$F_S \Vdash S.$$

Canonicity for a logic S immediately yields Kripke completeness (the class $\{F_S\}$ determines S). Moreover, if the canonical frame of S belongs to a class C of frames, then S is complete w.r.t C. We now introduce the notion of strong Kripke completeness.

Definition 3.3.2. A propositional modal logic S is *strongly Kripke complete* if every S-consistent set is satisfied in a model over a frame validating S.

Theorem 3.3.3. Let S be a propositional modal logic, $F_S = \langle W_S, R_S \rangle$ be the canonical frame of S and C some class of frames. If $F_S \in C$, then S is strongly Kripke complete.

Proof. Immediate from Definition 3.3.1. ■

Proposition 3.3.4. ***K*** is canonical.

Proof. By Theorem 3.1.2, ***K*** is validated by every frame and therefore, a fortiori, ***K*** is validated by its canonical frame. ■

Corollary 3.3.5. ***K*** is strongly Kripke complete.

Lemma 3.3.6. Let S and S' be propositional modal logics such that $S \subseteq S'$ and Γ an S'-consistent set of wffs. Then, Γ is S-consistent.

Proof. Suppose that Γ is S-inconsistent, i.e. $\Gamma \vdash_S \bot$. Since $S \subseteq S'$, every S-derivation from Γ is also an S'-derivation from Γ. Hence, $\Gamma \vdash_{S'} \bot$, contradicting the S'-consistency of Γ. ■

Lemma 3.3.7. Let S and S' be propositional modal logics such that $S \subseteq S'$ and Γ a set of wffs. Then,

$$\Gamma \text{ is } S'\text{-complete iff } S' \subseteq \Gamma \text{ and } \Gamma \text{ is } S\text{-complete.}$$

Proof. For "if", suppose $S' \subseteq \Gamma$ and Γ is S-complete. By Lemma 3.2.8, for every wff α, either $\alpha \in \Gamma$ or $\sim\alpha \in \Gamma$, but not both. Hence, to show that Γ is S'-complete, it suffices to show that Γ is S'-consistent. Suppose that Γ is S'-inconsistent. Then, by Definition 3.2.3, $\Gamma \vdash_{S'} \bot$. By Lemma 3.2.5, there exist $\beta_1, \ldots, \beta_n \in \Gamma$ such that $\vdash_{S'} \bigwedge_{i=0}^{n} \beta_i \supset \bot$. Since $S' \subseteq \Gamma$, by Lemma 2.2.6, it follows that $\bigwedge_{i=1}^{n} \beta_i \supset \bot \in \Gamma$ and, therefore, by definition of $\vdash_S$, it follows that $\Gamma \vdash_S \bigwedge_{i=1}^{n} \beta_i \supset \bot$. But, since $\beta_1, \ldots, \beta_n \in \Gamma$, it follows that $\Gamma \vdash_S \bigwedge_{i=1}^{n} \beta_i$. Thus, by $[MP]$, $\Gamma \vdash_S \bot$. So Γ is S-inconsistent, contrary to the assumption. For "only if", suppose Γ is S'-complete. Let $\alpha \in S$. Thus, $\vdash_S \alpha$ and so, $\Gamma \vdash_S \alpha$. Since Γ is S'-complete, by Lemma 3.2.9, it follows that $\alpha \in \Gamma$. Hence, $S' \subseteq \Gamma$. Now, by Lemma 3.2.8, for every wff α, either $\alpha \in \Gamma$ or $\sim\alpha \in \Gamma$, but not both. Thus, any proper superset of Γ is S-inconsistent. Furthermore, by Lemma 3.3.6, Γ is S-consistent. Thus, by definition, Γ is S-complete. ■

Lemma 3.3.8. Let S be a propositional modal logic, α a wff, F a frame and F' a generated subframe of F. If $F \Vdash \alpha$, then $F' \Vdash \alpha$.

Proof. See [10, Proposition 1.3.35 on p. 29]. ■

Lemma 3.3.9. Let S and S' be propositional modal logics such that $S \subseteq S'$. Then, $F_{S'}$ is a generated subframe of F_S.

Proof. By Lemma 3.3.7, $W_{S'} \subseteq W_S$ and so $F_{S'}$ is a subframe of F_S. We now show that $F_{S'}$ is a generated subframe of F_S. Let $\Gamma \in W_{S'}$ and suppose $\Omega \in R_S(\Gamma)$. By $[N]$, $\Box S' \subseteq S'$. Hence, $\Box S' \subseteq \Gamma$. By definition of R_S, $\Box^-(\Gamma) \subseteq \Omega$. Thus, $S' \subseteq \Omega$. Hence, by Lemma 3.3.7, Ω is S'-complete and, therefore, $\Omega \in W'_S$. ■

Definition 3.3.10. A wff α is *canonical* if the propositional modal logic $\boldsymbol{K} + \alpha$ is canonical.

Lemma 3.3.11. Let S be a canonical propositional modal logic and X be a set of canonical formulas. Then, the propositional modal logic $S + X$ is canonical.

Proof. Let $S' = S + X$. Since $S \subseteq S'$, by Lemma 3.3.9, $F_{S'}$ is a generated subframe of F_S. By Lemma 3.3.8, $F_{S'} \Vdash S$. Now, for each $\alpha \in X$, the propositional modal logic $\boldsymbol{K} + \alpha$ is canonical. Therefore, for each $\alpha \in X$, since $\boldsymbol{K} + \alpha \subseteq S'$, by Lemma 3.3.9, it follows that $F_{S'}$ is a generated subframe of $\boldsymbol{K} + \alpha$. By the definition of canonicity and Lemma 3.3.8, $F_{S'} \Vdash X$. By Lemma 3.1.3, $F_{S'} \Vdash S + X$, i.e. $F_{S'} \Vdash S'$. Hence, S' is canonical. ■

We next show that the logics ***K***, ***K***T, ***K***D, ***K***B, ***K***TB, ***S4*** and ***S5*** are canonical.

Definition 3.3.12. A *one-way Pseudo Transitive or Closed* (or, one-way PTC) formula is any wff that is closed or of the form

$$\Box\alpha \supset \Box^n\alpha,$$

for $n \geq 0$ and α a wff.

Definition 3.3.13. A propositional modal logic S is *one-way PTC* if $S = K + X$, for some set of one-way PTC formulas.

If $F = \langle W, R\rangle$, then we define $R^n := \{\langle w, v\rangle \in W \times W \mid \exists w_1, \ldots w_{n-1} \in W\,(wRw_1, \ldots, w_{n-1}Rv)\}$. Furthermore, for any $w \in W$, $R^n(w) := \{v \in W \mid wR^nv\}$.

Theorem 3.3.14. For $n \geq 0$, the axioms $\Box\alpha \supset \Box^n\alpha$ are validated by all frames $\langle W, R\rangle$ such that $R^n \subseteq R$.

Proof. Let $F = \langle W, R\rangle$, $M = \langle F, V\rangle$ a model based on F and α be a wff. Suppose $R^n \subseteq R$ and that $M, w \nVdash \Box\alpha \supset \Box^n\alpha$, for some $w \in W$. Then, $M, w \Vdash \Box\alpha$ but $M, w \Vdash \Box^n\alpha$. By $[V\Box]$ n times, there exists $v \in R^n(w)$ such that $M, v \nVdash \alpha$. Since $R^n \subseteq R$, $v \in R$. Hence, by $[V\Box]$, it follows that $M, w \nVdash \Box\alpha$, contradicting our assumption. ■

Theorem 3.3.15. Every closed wff is canonical.

Proof. Let α be a closed wff, $S = \boldsymbol{K} + \alpha$ be a propositional modal logic, $F_S = \langle W_S, R_S\rangle$ be the canonical frame of S and $M_S = \langle F_S, V_S\rangle$ be the canonical model of S. Since α does not contain any propositional letters, the truth of α at any world in W_S is model-invariant. By Lemma 3.2.16, $M_S \Vdash \alpha$ and so, $F_S \Vdash \alpha$, i.e. α is canonical. ■

Theorem 3.3.16. Every propositional one-way PTC logic is canonical.

Proof. Let S be a propositional modal logic containing $\Box p \supset \Box^n p$ and let $M_S = \langle W_S, R_S, V_S\rangle$ be the canonical model of S. In light of Theorem 3.3.14, it is enough to show

$$R_S^n \subseteq R_S.$$

Let $\Gamma \in W_S$ and let α be a wff. Suppose $\Omega \in R_S^n(\Gamma)$, for some $\Omega \in W_S$, and that $\Box\alpha \in \Gamma$. Then, since $\Box\alpha \supset \Box^n\alpha \in \Gamma$, it follows that $\Box^n\alpha \in \Gamma$. Since $\Gamma R_S^n\Omega$, by definition of R_S (n times), it follows that $\alpha \in \Omega$. By definition of R_S again, $\Gamma R_S\Omega$. ■

Corollary 3.3.17. The logics ***K***T, ***K4***, and ***S4*** are canonical.

Proof. Each of these logics are axiomatised by one-way PTC formulas. Hence, by Lemma 3.3.4, 3.3.11 and Theorem 3.3.16, they are are canonical. ■

Of course, not every logic we've looked at is PTC. Thus, we demonstrate canonicity case by case for the remaining logics.

Proposition 3.3.18. ***K***D is canonical.

Proof. Let $S = \mathbf{K}\mathrm{D}$, let $F_S = \langle W_S, R_S \rangle$ be the canonical frame of S and let $M_S = \langle F_S, V_S \rangle$ be the canonical model of S. By Theorem 3.1.10, ***K***D is validated by serial frames. Thus, it is enough to show that F_S is serial. Let $\Gamma \in W_S$. By D and Lemma 3.2.9 (a), $\Diamond\top \in \Gamma$. By Theorem 3.2.16, it follows that $M_S, \Gamma \Vdash \Diamond\top$. By Theorem 3.2.15 and $[V\Diamond]$, it follows that there exists $\Omega \in R_S(\Gamma)$ such that $M_S, \Omega \Vdash \top$. Hence, F_S is serial. ■

Proposition 3.3.19. ***K***B is canonical.

Proof. Let $S = \mathbf{K}\mathrm{B}$, let $F_S = \langle W_S, R_S \rangle$ be the canonical frame of S and let $M_S = \langle F_S, V_S \rangle$ be the canonical model of S. By Theorem 3.2.16, $M_S \Vdash S$. Thus, by Corollary 3.1.8, F is symmetric. By Corollary 3.1.7, $F \Vdash S$. Hence, S is canonical. ■

Proposition 3.3.20. ***K***TB is canonical.

Proof. ***K***TB can be axiomatised by T and B. By Corollary 3.3.17 and Proposition 3.3.19, both T and B are canonical formulas. Thus, by Lemma 3.3.11, ***K***TB is canonical. ***K***TB is also complete w.r.t the class of all reflexive and symmetric frames. ■

Proposition 3.3.21. ***S5*** is canonical.

Proof. ***S5*** can be axiomatised by T, B and 4. By Corollary 3.3.17 and Proposition 3.3.19, T, B and 4 are canonical formulas. Thus, by Lemma 3.3.11, ***S5*** is canonical. Since $F_{\mathbf{S5}} \Vdash B$, by Corollary 3.1.8, $F_{\mathbf{S5}}$ is symmetric. Furthermore, since $\mathbf{S4} \subseteq \mathbf{S5}$, by Lemma 3.3.9, $F_{\mathbf{S5}}$ is a generated subframe of $F_{\mathbf{S4}}$. Thus, $F_{\mathbf{S5}}$ is also reflexive and transitive. Hence, ***S5*** is complete w.r.t the class of all reflexive, transitive and symmetric frames. ■

Corollary 3.3.22. The propositional modal logics ***K***, ***K***T, ***K***D, ***K***B, ***K***TB, ***K5***, ***S4*** and ***S5*** are Kripke complete.

3.3.1 Propositional Modal Logics $\mathbf{Alt}_n$

Our form of the axiom Alt_n is as follows:

$$\mathrm{Alt}_n := \sim \bigwedge_{i\geq 0}^{n} \Diamond(p_i \wedge \bigwedge_{j\neq i}^{n} \sim p_j) \quad (n \geq 1)$$

We will show that Alt_n is determined by the class of frames whose worlds have at most n successors:

$$C_{alt_n} := \{\langle W, R\rangle \mid \forall w \in W(\ |R(w)| \leq n)\}.$$

Proposition 3.3.23. ***K*.Alt$_n$** is validated by C_{alt_n}.

Proof. Let $F = \langle W, R\rangle$ be a frame. Suppose $F \in C_{alt_n}$ but $F \nVdash \mathrm{Alt}_n$. Then, for some model $M = \langle F, V\rangle$ and $w \in W$,

$$w \nVdash \sim \bigwedge_{i\geq 0}^{n} \Diamond(p_i \wedge \bigwedge_{j\neq i}^{n} \sim p_j)$$

By $[V\sim]$, we obtain,

$$w \Vdash \bigwedge_{i\geq 0}^{n} \Diamond(p_i \wedge \bigwedge_{j\neq i}^{n} \sim p_j)$$

Then, for each $0 \leq i \leq n$, by $[V\Diamond]$, there exists $w' \in R(w)$ such that,

$$w' \Vdash p_i \wedge \bigwedge_{j\neq i}^{n} \sim p_j,$$

and, therefore, by $[V\wedge]$ and $[V\sim]$, it follows that

$$w' \Vdash p_i \text{ and } w' \nVdash p_j, \text{ for each } 0 \leq j \neq i \leq n.$$

Clearly, by the above, it follows that since $0 \leq i \leq n$, w must have at least $n+1$ successors, i.e. $|R(w)| > n$, contradicting $F \in C_{alt_n}$. ■

Remark 3.3.24. It is useful to note that for each $w, v \in W_S$ such that $w \neq w$, and for a propositional modal logic S, there exists a wff α such that $\alpha \in w$ but $\alpha \notin v$.

Proposition 3.3.25. The logics ***K*.Alt$_n$** are canonical.

Proof. Let $S = \mathbf{K.Alt}_n$, and $M_S = \langle F_S, V_S \rangle$ be the canonical model of S with $F_S = \langle W_S, R_S \rangle$ the canonical frame of S. Let $w \in W_S$.

Suppose there are distinct $w_0, \ldots, w_n \in R_S(w)$. Hence, there exist wffs α_{ij} such that $\alpha_{ij} \notin w_i$ but $\alpha_{ij} \in w_j$ for each $i \neq j$ For $0 \leq i \leq n$, define

$$\beta_i := \bigwedge_{j \neq i}^{n} (\alpha_{ji} \wedge {\sim}\alpha_{ij}).$$

Then, it is clear that $\beta_i \in w_j$ iff $i = j$. Hence,

$$\bigwedge_{i \geq 0}^{n} \Diamond(\beta_i \wedge \bigwedge_{j \neq i}^{n} {\sim}\beta_j) \in w.$$

By $[US]$ and Alt_n, ${\sim} \bigwedge_{i \geq 0}^{n} \Diamond(\beta_i \wedge \bigwedge_{j \neq i}^{n} {\sim}\beta_j) \in w$. By Lemma 3.2.8, we obtain a contradiction. ■

Corollary 3.3.26. $\mathbf{K.Alt}_n$ is strongly Kripke complete.

3.3.2 General Canonical Frames

Lemma 3.3.27. If M is a model and α a wff, then

$$GF(M) \Vdash \alpha \text{ iff for every uniform substitution instance } \beta \text{ of } \alpha,\ M \Vdash \beta.$$

Proof. Let $F = \langle W, R \rangle$ be a frame and $M = \langle F, V \rangle$ be a model based on F. Suppose $PL(\alpha) = \{p_1, \ldots, p_n\}$. Now, let $\gamma_1, \ldots, \gamma_n$ be wffs and let $M' = \langle F, V' \rangle$ be a model such that $V'(p_i) = V(\gamma_i)$, for each $1 \leq i \leq n$. By simple induction, it follows that for every wff α,

$$V'(\alpha) = V([\gamma_1, \ldots, \gamma_n / p_1, \ldots, p_n]\, \alpha).$$

Now, suppose $GF(M) \Vdash \alpha$. By the equality above, for every uniform substitution instance β of α, there exists an admissible model $M' = \langle F, V' \rangle$ in $GF(M)$ such that $V(\beta) = V'(\alpha)$. ■

Definition 3.3.28. If S be a propositional modal logic, then the *general canonical frame* for S is $GF(M_S)$.

Theorem 3.3.29. If S is a propositional modal logic, then for every wff α,

$$GF(M_S) \Vdash \alpha \text{ iff } S \vdash \alpha.$$

Proof. By Lemma 3.3.27, for every wff α, if $GF(M_S) \Vdash \alpha$, then for every uniform substitution instance β of α, $M_S \Vdash \beta$. By definition, S is closed under uniform substitution instances. Hence, by Theorem 3.2.16, $S \vdash \alpha$. ■

Lemma 3.3.30. Every general canonical frame is descriptive.

Proof. See [10, Lemma 1.7.21, p. 47] for proof and surrounding discussion on modal algebras. ■

Theorem 3.3.31. Every d-persistent propositional modal logic is canonical and therefore Kripke complete.

Proof. Let S be a d-persistent propositional modal logic. By Theorem 3.3.29, $GF(M_S) \Vdash S$. By Lemma 3.3.30, therefore, $GF(M_S)$ is descriptive. Hence, $F_S \Vdash S$, i.e. S is canonical and therefore Kripke complete. ■

Definition 3.3.32. A *Sahlqvist antecedent* is a wff satisfying the following:

- $\Box^n p_i$ is a Sahlqvist antecedent, for $p_i \in PL$ and $n \geq 0$,
- $\top$ and $\bot$ are Sahlqvist antecdents,
- If α is a Sahlqvist antecedent, then so is $\Diamond\alpha$,
- If α and β are Sahlqvist antecedents, then so is $\alpha \wedge \beta$.

Definition 3.3.33. A wff is called *positive* if is not of the form $\sim\alpha$, for some wff α.

Definition 3.3.34. A *simple Sahlqvist formula* is a wff of the form $\alpha \supset \beta$ where α is a Sahlqvist antecedent and β is a positive wff.

Definition 3.3.35. A *Sahlqvist formula* is a wff satisfying the following:

- Every simple Sahlqvist formula is a Sahlqvist formula,
- If α is a Sahlqvist formula, then so is $\Box\alpha$,
- If α and β are Sahlqvist formulas, then so is $\alpha \vee \beta$.

Theorem 3.3.36 (Sahlqvist Theorem)**.** Every propositional modal logic axiomatised by Sahlqvist formulas is d-persistent and Δ-elementary.

Proof. See Blackburn, Rijke, and Venema [3, Theorem 10.19, p. 344]. ■

Corollary 3.3.37. Every propositional modal logic axiomatised by Sahlqvist formulas is canonical and therefore strongly Kripke complete.

To date, the Sahlqvist Theorem for propositional modal logics is unmatched in its generality by any results in the area of Kripke completeness for predicate modal logic. The next few chapters aim to illuminate part of why this generality is not importable from the ideas presented in the thesis so far.

Chapter 4

Syntax and Semantics for Predicate Modal Logic

4.1 Language

To define the language for predicate modal logic, we extend the language of classical predicate logic [1], adding the modalities.

Definition 4.1.1. The *language* $\mathcal{L}$ for predicate modal logic contains the following symbols:

- A countable set of *bound variables* denoted by "$x_1, x_2, \ldots$" (or "y")
- A countable set of *free variables* denoted by "$a_1, a_2, \ldots$" (or "b")
- A countable set of n-ary *predicate letters* ($n \geq 0$) denoted by "P_1^n", "P_2^n", ... (or "Q"), 0-ary predicate letters are propositional letters.
- The *classical connectives* "$\bot$", "$\supset$"
- The *modal* symbol "$\Box$",
- The *quantifier* symbol "$\forall$",
- And the punctuation marks "(", ")".

[1] In this thesis, we omit the inclusion of function symbols and constant symbols, since we would otherwise have incurred an unnecessary technical overhead, unrelated to the central issues addressed by the thesis.

4.2 Syntax

Just as in the propositional language, we impose restrictions on the kinds of expressions we can form with $\mathcal{L}$. Before we do so, we define an operation over the variables in an expression.

Definition 4.2.1. A *substitution* of a bound (resp. free) variable y (resp. b) for a bound (resp. free) variable x (resp. a) in an expression A is the uniform replacement of x (resp. a) with y (resp. b) in A. We denote this operation by $[y/x]A$ (resp. $[b/a]A$).

Definition 4.2.2. A predicate modal *well-formed formula* in $\mathcal{L}$ ($\mathcal{L}$-wff) is an expression defined recursively:

- If P is an n-ary predicate letter and $a_1, \ldots, a_n$ are free variables, then $P(a_1, \ldots, a_n)$ is an $\mathcal{L}$-wff.
- $\bot$ is an $\mathcal{L}$-wff,
- If A and B are $\mathcal{L}$-wffs, then so is $(A \supset B)$.
- If A is an $\mathcal{L}$-wff, then so is $\Box A$.
- If $A(a)$ is an $\mathcal{L}$-wff and x is a bound variable, then $\forall x[x/a]A(x)$ is an $\mathcal{L}$-wff.

We use the following syntactic abbreviations:

- [abbr. $\top$] $\top$ abbreviates $A \supset A$,
- [abbr. $\sim$] $\sim A$ abbreviates $A \supset \bot$,
- [abbr. $\vee$] $A \vee B$ abbreviates $\sim A \supset B$,
- [abbr. $\wedge$] $A \vee B$ abbreviates $\sim(A \supset \sim B)$,
- [abbr. $\Diamond$] $\Diamond A$ abbreviates $\sim\Box\sim A$,
- [abbr. $\exists$] $\exists x A(x)$ abbreviates $\sim\forall x\sim A(x)$,
- [abbr. $\equiv$] $A \equiv B$ abbreviates $(A \supset B) \wedge (B \supset A)$.

If we state $A(a)$ and then state $A(x)$ in the same context (e.g. a proof or definition) then the reader may assume that by $A(x)$ we mean $[x/a]A(a)$. We will use $\overline{x}$ to denote a *list* of variables, i.e. $x_1, \ldots, x_n$. We use $[\overline{x}/\overline{y}]$ to denote the substitutions $[x_1/y_1, \ldots, x_n/y_n]$ (the reader may assume that the lists $\overline{x}$ and $\overline{y}$ are of equal length).

FVar(A) and BVar(A) will denote the set of all the free variables and the set of all bound variables, respectively in an $\mathcal{L}$-wff A. We let $Sen(\mathcal{L}) := \{A \mid A \text{ is an } \mathcal{L}\text{-wff } \& \ FVar(A) = \emptyset\}$.

Members of $Sen(\mathcal{L})$ will be called *sentences* in $\mathcal{L}$.

We also permit the performance of a *predicate substitution*, an instance of which can be one of two general cases. In the first case, we substitute an $\mathcal{L}$-wff into a classical propositional wff by uniformly replacing a designated propositional variable by the same $\mathcal{L}$-wff. In the second case, we may substitute an $\mathcal{L}$-wff into another, conditioned on rules, rigorously defined in [10, Section 2.5], which we omit here due to their unwieldy technical involvement. An example of this kind of substitution is moving from the $\mathcal{L}$-wff, "$\forall x P(x)$", to the $\mathcal{L}$-wff, "$\forall x \exists y (Q(x,y) \supset R(x))$".

We denote the set of all the theorems of the classical predicate logic by **QCL**. Recall, **QCL** contains the following:

$[Q1]$ $\forall x A(x) \supset A(a)$,

$[Q2]$ $\forall x (A \supset B(x)) \supset (A \supset \forall x B(x))$,

$[Q3]$ $A(a) \supset \exists x A(x)$,

$[Q4]$ $\forall x (A(x) \supset B) \supset (\exists x A(x) \supset B)$,

$[Q5]$ predicate substitution instances of every propositional tautology.

We use the following definition for a predicate modal logic which includes the rule of necessitation $[N]$, meaning that the logics we deal with in this section will all be *normal*.

Definition 4.2.3. A *predicate modal logic* is a set L of $\mathcal{L}$-wffs including **QCL** and containing K, closed under the following conditions (rules of inference):

$[MP]$ If $A \supset B \in L$ and $A \in L$, then $B \in L$,

$[N]$ If $A \in L$, then $\Box A \in L$,

$[G]$ If $A(a) \in L$, then $\forall x [x/a] A \in L$,

[Sub] If $A \in L$ and if B is a predicate substitution instance of A, then $B \in L$.

Definition 4.2.4. If S be a propositional modal logic, then the *minimal predicate extension* of S, denoted by $\boldsymbol{Q.S}$, is the minimal predicate modal logic including S.

Note that the above definition relies on the fact that propositional modal wffs can be rewritten as predicate modal $\mathcal{L}$-wffs simply by replacing each unique propositional letter by a unique 0-ary predicate letter.

As in the propositional case, we will work with for predicate modal logics representable by Hilbert-style proof systems. These proof systems will include $[Q1] - [Q5]$ (enough axioms for **QCL**), modal axioms, and the inference rules: $[MP]$, $[N]$, $[G]$ and [Sub]. It should be noted that, under Definition 4.2.3, not all predicate modal logics are representable as proof systems. Hence, we restrict our study to those logics which are.

Definition 4.2.5. If H is a Hilbert-style proof system for predicate modal logic, then a *proof* of an $\mathcal{L}$-wff A in H is a sequence $A_1, \ldots, A_n$ of $\mathcal{L}$-wffs, with $A_n = A$, such that each A_i (for $1 \leq i \leq n$) satisfies at least one of the following conditions:

- A_i is an axiom of H,
- $A_i = \Box A_j$ for some j $(1 \leq j < i)$,
- A_i is obtained from some A_j, A_k $(1 \leq j < k < i)$ using $[MP]$,
- A_i is obtained from some A_j $(1 \leq j < i)$ using $[G]$, $[N]$ or [Sub].

If L is a predicate modal logic and A is an $\mathcal{L}$-wff, then $L \vdash A$ will denote that A is provable from any Hilbert-style proof system that proves exactly the members of L.

We now recall the definitions of the *Barcan formula* and its converse:

(BF) $$\forall x \Box A(x) \supset \Box \forall x A(x)$$

(CBF) $$\Box \forall x A(x) \supset \forall x \Box A(x)$$

Theorem 4.2.6. *CBF* is contained in every predicate modal logic.

Proof. Let L be a predicate modal logic. By $[Q1]$ and $[N]$, $L \vdash \Box(\forall x A(x) \supset A(a))$. Then, by K and $[MP]$, $L \vdash \Box \forall x A(x) \supset \Box A(a)$. And finally, by $[G]$, $[Q2]$ and $[MP]$, $L \vdash \Box \forall x A(x) \supset \forall x \Box A(x)$. ■

Theorem 4.2.7. $BF \in \mathbf{Q.K}\text{B}$.

Proof. Let $L = \mathbf{Q.K}\text{B}$. Recall axiom B: $p \supset \Box \Diamond p$.

By Theorem 2.2.9, $\mathbf{K}\text{B} \vdash \Diamond \Box p \supset p$. By $[Q1]$, $L \vdash \forall x \Box A(x) \supset \Box A(a)$. Then, by Theorem 2.2.7, (1) $L \vdash \Diamond \forall x \Box A(x) \supset \Diamond \Box A(a)$. By Theorem 2.2.9, (2) $L \vdash \Diamond \Box A(a) \supset A(a)$. Thus, by (1) and

(2), $L \vdash \Diamond\forall x\Box A(x) \supset A(a)$. Then, by [Q2], $L \vdash \Diamond\forall x\Box A(x) \supset \forall x A(x)$. Finally, by Theorem 2.2.10, $L \vdash \forall x\Box A(x) \supset \Box\forall x A(x)$. ■

4.3 Predicate Kripke Semantics

Definition 4.3.1. A *predicate Kripke frame* $\boldsymbol{F}$ is a pair $\langle F, D\rangle$ where F is a Kripke frame $\langle W, R\rangle$, and $D : W \to \mathcal{D}$ (for some non-empty set of individuals $\mathcal{D}$) is a domain function, such that for every $w \in W$, $D(w) \neq \emptyset$ and for every $w, v \in W$, if wRv then $D(w) \subseteq D(v)$. We abbreviate $D(w)$ with D_w.

From now on, we will call predicate Kripke frames "predicate frames". The condition that $D(w) \subseteq D(v)$, for every $w \in W$ and every $v \in R(w)$, is known as the *expanding domain* condition [2]. Furthermore, for a frame $F = \langle W, R, D\rangle$, if $D(w) = D(v)$, for every $w \in W$ and every $v \in R(w)$, we say that the domains of F are *locally constant*.

Definition 4.3.2. A predicate frame $\boldsymbol{F}$ is *over* a frame F if $\boldsymbol{F} = \langle F, D\rangle$, for some domain function D.

Definition 4.3.3. A *predicate Kripke model* M is a pair $\langle \boldsymbol{F}, V\rangle$, with $\boldsymbol{F}$ a predicate frame and V a valuation function such that for every $w \in W$,

$$V(P^0, w) \in \{0, 1\},$$
$$V(P^n, w) \subseteq (D(w))^n, \text{ if } n > 0.$$

At each $w \in W$ in M, we evaluate D_w-*sentences*, obtained by replacing all the free variables in $\mathcal{L}$-wffs with members of $\underline{D}_w$, a set of constants defined by $\underline{D}_w = \{\underline{d} \mid d \in D_w\}$.

$[VP^0]$ $M, w \Vdash P^0$ if $V(P^0, w) = 1$,

$[VP^n]$ $M, w \Vdash P^n(\underline{d}_1, \ldots, \underline{d}_n)$ if $<d_1, \ldots, d_n> \in V(P^n, w)$,

$[V\bot]$ $M, w \nVdash \bot$,

$[V\supset]$ $M, w \Vdash A \supset B$ if $M, w \nVdash A$ or $M, w \Vdash B$,

$[V\Box]$ $M, w \Vdash \Box A$ if $M, v \Vdash A$ for every $v \in R(w)$,

$[V\forall]$ $M, w \Vdash \forall x A(x)$ if $M, w \Vdash A(\underline{d})$ for every $d \in D_w$.

[2] In this thesis, we have chosen to restrict our study of predicate Kripke frames to those with expanding domains, allowing us to retain conventional semantics for classical predicate logic, particularly when it comes to the scope of quantifiers.

We infer a few additional model valuation rules for the logical connective and quantifier abbreviations:

$[V\top]$ $M, w \Vdash \top$,

$[V\sim]$ $M, w \Vdash \sim A$ if $M, w \nVdash A$,

$[V\vee]$ $M, w \Vdash A \vee B$ if $M, w \Vdash A$ or $M, w \Vdash B$,

$[V\wedge]$ $M, w \Vdash A \wedge B$ if both $M, w \Vdash A$ and $M, w \Vdash B$,

$[V\Diamond]$ $M, w \Vdash \Diamond A$ if there exists $v \in R(w)$ such that $M, v \Vdash A$,

$[V\exists]$ $M, w \Vdash \exists x A(x)$ if there exists $d \in D_w$ such that $M, w \Vdash A(\underline{d})$.

From now on, we will call predicate Kripke models "models".

Definition 4.3.4. If $M = \langle \boldsymbol{F}, V \rangle$ is a model based on a predicate frame $\boldsymbol{F} = \langle W, R, D \rangle$, then we define for all sentences A,

$$V(A) := \{w \in W \mid M, w \Vdash A\}.$$

Definition 4.3.5. If A is an $\mathcal{L}$-wff and $FVar(A) = \{a_1, \ldots, a_n\}$, then *standard universal closure* of A is defined by $\overline{\forall} A := \forall x_1 \ldots \forall x_n [x_1/a_1] \ldots [x_n/a_n] A(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are the first n bound variables not occurring in A.

Definition 4.3.6. Let A be an $\mathcal{L}$-wff and L be a predicate modal logic. Then:

- If M is a model, then A is *true* at a world $w \in W$ if $M, w \Vdash \overline{\forall} A$.
- If M is a model, then A is *valid* in M if $M, w \Vdash \overline{\forall} A$, for every $w \in W$. We denote this by $M \Vdash A$.
- If $\boldsymbol{F}$ is a predicate frame, then A is *valid* on $\boldsymbol{F}$ if $M \Vdash A$, for every model based on $\boldsymbol{F}$. We denote this by $\boldsymbol{F} \Vdash A$.
- If $\boldsymbol{C}$ is a class of predicate frames, then A is *valid* on $\boldsymbol{C}$ if $F \Vdash A$, for every $F \in \boldsymbol{C}$. We denote this by $\boldsymbol{C} \Vdash A$.
- If F is a (propositional) frame, then A is *valid* on F if A is valid on every predicate frame over F.

If $\boldsymbol{C}$ is a class of predicate frames, let $\mathrm{Log}(\boldsymbol{C}) := \{A \mid \boldsymbol{C} \Vdash A,\ A \text{ is an } \mathcal{L}\text{-wff}\}$.

- L is *validated* by $\boldsymbol{C}$ if $L \subseteq \mathrm{Log}(\boldsymbol{C})$ (soundness).
- L is *complete* w.r.t $\boldsymbol{C}$ if $L \supseteq \mathrm{Log}(\boldsymbol{C})$ (completeness).

- L is *determined* by $\boldsymbol{C}$ if $L = \mathrm{Log}(\boldsymbol{C})$. Thus, L is determined by $\boldsymbol{C}$ if L is validated by and complete w.r.t $\boldsymbol{C}$.

Definition 4.3.7. A predicate modal logic is *Kripke complete* if it is determined by some class of predicate frames.

Kripke completeness is also called *weak* Kripke completeness. In Section 5.3, we introduce the definition of *strong* Kripke completeness for predicate modal logics, once other needed definitions have been introduced.

If $\boldsymbol{F} = \langle W, R, D\rangle$ and $W' \subseteq W$, we define the restriction of D to W' by

$$D \restriction W' := \{<w, D_w> \mid w \in W'\}.$$

Definition 4.3.8. If $\boldsymbol{F} = \langle F, D\rangle$ is a predicate frame based on a frame $F = \langle W, R\rangle$ and $F' = \langle W', R'\rangle$ is a subframe of F, then the frame

$$\boldsymbol{F}' := \langle F',\ D \restriction W'\rangle$$

is called a *subframe* of $\boldsymbol{F}$. If, in addition, F' is a generated subframe of F, then $\boldsymbol{F}'$ is a *generated subframe* of $\boldsymbol{F}$.

Theorem 4.3.9. CBF is valid on every predicate frame.

Proof. Let $M = \langle \boldsymbol{F}, V\rangle$ be a model on some predicate frame $\boldsymbol{F}$ and let $w \in W$. Suppose $w \Vdash \Box\forall x A(x)$. Then, by $[V\Box]$ $w' \Vdash \forall x A(x)$ for every w' such that wRw'. And by $[V\forall]$ $w' \Vdash A(\underline{d})$ for every $d \in D_{w'}$. Since wRw', we have $D_w \subseteq D_{w'}$ and thus $w \Vdash \Box A(\underline{d})$ for every $d \in D_w$. But again by $[V\forall]$, we have $w \Vdash \forall x \Box A(x)$. ■

Theorem 4.3.10. Every frame with constant domains validates BF.

Proof. Let $M = \langle \boldsymbol{F}, V\rangle$ be a model based on a predicate frame $\boldsymbol{F} = \langle F, D\rangle$ with constant domains. Let $w \in W$ and $v \in R(w)$. Suppose $w \Vdash \forall x \Box A(x)$. Then, by $[V\forall]$ and $[V\Box]$, $w' \Vdash A(\underline{d})$, for every $d \in D_w$ (this is possible since $D_w \subseteq D_{w'}$). Thus, $w' \Vdash A(\underline{d})$, for every $d \in D_{w'}$. Hence, by $[V\forall]$, $w' \Vdash \forall x A(x)$ and finally by $[V\Box]$, $w \Vdash \Box\forall x A(x)$. ■

Chapter 5

Kripke Completeness for Predicate Modal Logics

When moving from the context of propositional modal logics to that of predicate modal logics, the landscape of completeness changes strikingly. Kripke incomplete predicate modal logics - predicate modal logics which are not Kripke complete - are numerous and few completeness theorems are known [10, p. 214]. To salvage the techniques used in demonstrating soundness and completeness in propositional modal logic, they require some modification to accommodate the peculiarities inherent in predicate modal logic. Namely, the handling of individuals of domains at each world in predicate Kripke semantics. One fruitful technique, adapting the canonical model technique in Section 3.2, adopts a method devised by Henkin [12] for proving the completeness of the classical predicate logic. This technique, as we will see, is enough to prove completeness for the predicate extensions of several standard logics discussed thus far.

Kripke completeness for other, stronger logics has undergone study such as in Corsi and Ghilardi [7] who established completeness results for certain extensions of ***Q.S4.2*** which equals $\boldsymbol{S4} + \Diamond\Box p \supset \Box\Diamond p$ w.r.t certain subclasses of directed frames with expanding or constant domains. A frame $F = \langle W, R\rangle$ is called *directed* if R is reflexive, transitive and such that for any $w, v \in W$ there exists $u \in R(w) \cap R(v)$.

Corsi [6] showed that predicate modal logics ***Q.K4.3.***D.X (with axioms $3 := \Box(\Box p \wedge p \supset q) \vee \Box(\Box q \wedge q \supset p)$ and $X := \Box\Box p \supset \Box p$) and ***Q.K4.3*** are complete w.r.t to predicate frames

$\langle Q^+, <, D\rangle$ and $\langle Q^+, \leq, D\rangle$, whose worlds are the positive rational numbers. Furthermore, she demonstrated completeness of $\boldsymbol{C}_n.\boldsymbol{Q.K4.3}$.D.X w.r.t reflexive towers of at most n with expanding domains. We omit the more elaborate definition of the predicate axiom C_n as well as the descriptions of the frame conditions. Axiom 3 corresponds to weak connectedness: $\forall w\ \forall v\ \forall u\ (wRv \wedge wRu) \rightarrow (vRu \vee uRv \vee (v = u))$. Axiom X ($\Box\Box p \supset \Box p$) corresponds to dense frames: $\forall w\ \forall v\ \forall u\ (wRv \rightarrow \exists u\ (wRu \wedge uRv))$. More recently, Shehtman [19] has demonstrated Kripke completeness results for predicate modal logics based on $\boldsymbol{Q.K5}$, namely that $\boldsymbol{Q.K5}$ is Kripke incomplete but $\boldsymbol{Q.K5}$ + $\Box$BF is complete w.r.t Euclidean frames.

In addition to the work done on completeness, various studies have been conducted on the algorithmic properties of predicate modal logics in the context of Kripke semantics. Among these studies are investigations of decidability and undecidability of particular predicate modal and intuitionistic logics, subject to restrictions on the language as well as the classes of frames. Wolter and Zakharyaschev [21] identified decidable fragments of predicate modal logics, while Kontchakov, Kurucz, and Zakharyaschev [13] and Rybakov and Shkatov [15, 17, 16] identified undecidable fragments.

5.1 Some Preliminary Results

We now present a few fundamental results of predicate Kripke frames including the preservation of validity of propositional wff on frames when these frames are given domains, converting them to predicate frames.

Lemma 5.1.1. **QCL** is valid on every predicate frame.

Proof. It is well known that **QCL** is sound. That is, in the standard classical predicate symantics, **QCL** is valid on any model $\langle D, V\rangle$ where D is a domain and V is a valuation. Let $M = \langle W, R, D, V\rangle$ be a predicate model. It is clear that each $\langle D_w, V \restriction w\rangle$, for $w \in W$, is a classical predicate model. Hence, **QCL** is valid on $\boldsymbol{F}$. ■

Lemma 5.1.2. If F is a propositional frame and α a propositional modal wff, then for every predicate frame $\boldsymbol{F}$ over F,

$$F \Vdash \alpha \text{ iff } \boldsymbol{F} \Vdash \alpha.$$

Proof. By Definition 4.3.3, $M, w \Vdash P^0$ iff $V(P^0, w) = 1$, for every model based on $\boldsymbol{F}$ for every $w \in W$. It follows by induction that $F \Vdash \alpha$ iff $\boldsymbol{F} \Vdash \alpha$. ■

Corollary 5.1.3. Let S be a propositional modal logic and F be a frame such that $F \Vdash S$. Then, $\boldsymbol{F} \Vdash S$, for every predicate frame $\boldsymbol{F}$ over F.

Lemma 5.1.4. The rules of inference $[MP]$, $[N]$, $[G]$ and [Sub] preserve validity on frames and predicate frames.

Proof. Due to the technical density of [Sub] and the consequent length of this proof, we omit it here and refer the reader to [10]. ■

Lemma 5.1.5 (Soundness)**.** Let F be a frame and S a propositional modal logic. Then, for every predicate frame $\boldsymbol{F}$ over F,

$$F \Vdash S \text{ iff } \boldsymbol{F} \Vdash \boldsymbol{Q.S}.$$

Proof. Let $\boldsymbol{F}$ be a predicate frame over F. Suppose $F \Vdash S$. By Lemma 5.1.1, $\boldsymbol{F} \Vdash \mathbf{QCL}$. By Lemma 5.1.2, $\boldsymbol{F} \Vdash S$. Hence, by Lemma 5.1.4, it follows that $\boldsymbol{F} \Vdash \boldsymbol{Q.S}$. ■

Lemma 5.1.6. $\boldsymbol{Q.K}$ is valid on every predicate frame.

Proof. By Theorem 3.1.2, the propositional logic $\boldsymbol{K}$ is valid on every frame. Thus, by Lemma 5.1.5, $\boldsymbol{Q.K}$ is valid on every predicate frame over F. ■

5.2 Canonical Models for Predicate Modal Logics

In order to demonstrate completeness for predicate extensions to propositional logics, we will modify our method for constructing a (propositional) logic's canonical model using S-complete sets to accommodate domains and quantification over variables. We present a well-known approach that combines the method by Henkin [12], for proving completeness of classical predicate logic, with some additional technique for the construction of canonical models, in order to demonstrate completeness of predicate modal logics.

5.2.1 Henkin Theories

An essential tool in this section is the use of *constants*, denoted by "$c_1, c_2, \ldots$" (or "d"). These behave syntactically similarly to free variables in $\mathcal{L}$-wffs.

Definition 5.2.1. If A is an $\mathcal{L}$-wff, $\mathcal{C}$ a set a constants, $\overline{a}$ a list of all the free variables in FVar(A) and $\overline{c}$ a list of constants of the same length as $\overline{a}$, then we call $[\overline{c}/\overline{a}]A$ a *$\mathcal{C}$-sentence*.

If $\mathcal{C}$ is a set of constants, then the set of constants in a $\mathcal{C}$-sentence will be denoted by $\mathcal{C}(A)$ and the set of all possible $\mathcal{C}$-sentences will be denoted by $Sen(\mathcal{C})$. Note that $Sen(L) \subseteq Sen(\mathcal{C})$, for every set of constants $\mathcal{C}$. The term "sentence" will denote any $\mathcal{L}$-wff not containing any free variables or any $\mathcal{C}$-sentence, for some set of constants $\mathcal{C}$.

Definition 5.2.2. If L is a predicate modal logic, $\mathcal{C}$ a set of constants and $A \in Sen(\mathcal{C})$, then we say that A is *L-provable* if there exists $A' \in L$ such that $A = [\overline{c}/\overline{a}]A'$, for a list of constants $\overline{c} = c_1, \ldots, c_n$ and a list of free variables $\overline{a} = a_1, \ldots, a_n$. We denote that A is L-provable by $\vdash_L A$.

Definition 5.2.3. If $\mathcal{C}$ is a set of constants, then we call a subset of $Sen(\mathcal{C})$ a modal *theory*.

We denote the set of constants occurring in a theory Γ by $\mathcal{C}_\Gamma$.

Definition 5.2.4. If L is a predicate modal logic, Γ a modal theory and $A \in Sen(\mathcal{C}_\Gamma)$, then an *$L$-derivation* of A from Γ is a (finite) sequence $A_1, \ldots, A_n$, with $A_n = A$, such that each A_i (for $1 \leq i \leq n$) satisifes at least one of the following conditions:

(a) $A_i \in \Gamma$

(b) $\vdash_L A_i$

(c) A_i is obtained from some $A_j, A_j \supset A_i$ $(1 \leq j < i)$ using $[MP]$

(d) A_i for some A_j $(1 \leq j < i)$ using $[G]$

We denote that A has an L-derivation from Γ by $\Gamma \vdash_L A$. Note that (b) is effective since L is representable as a Hilbert-style proof system and the substitution of constants for free variables for each inference in a given proof is also effective.

Lemma 5.2.5. Let L is a predicate modal logic and A a sentence. Then,

$$\emptyset \vdash_L A \text{ iff } \vdash_L A.$$

Proof. Suppose $\vdash_L A$. Then, by Definition 5.2.4, it follows immediately that $\emptyset \vdash_L A$. Conversely, suppose $\emptyset \vdash_L A$. Then, there is an L-derivation, $A_1, \ldots, A_n$, of A from $\emptyset$. Clearly, $\vdash_L A_1$. Thus, it suffices to show that $[MP]$ and $[G]$ preserve L-provability:

$[MP]$ Suppose that A_i is obtained from A_j, $A_j \supset A_i$ for some j $(1 \leq j < i \leq n)$ using $[MP]$ and that $\vdash_L A_j$ and $\vdash_L A_j \supset A_i$. So, there exists $A'_j \in L$ such that $A_j = [\overline{c}/\overline{a}]A'_j$, for a list of constants $\overline{c} = c_1, \ldots c_n$ and a list of free variables $\overline{a} = a_1, \ldots a_n$. Furthermore, there exists $A'_j \supset A'_i \in L$ such that $A_j \supset A_i = [\overline{c} + \overline{d}/\overline{a} + \overline{b}](A'_j \supset A'_i)$, for additional lists of constants and free variables, $\overline{d} = d_1, \ldots d_m$ and $\overline{b} = b_1, \ldots, b_m$ respectively. Hence, $A_i = [\overline{c} + \overline{d}/\overline{a} + \overline{b}]A'_i$. Hence, $\vdash_L A_i$.

$[G]$ Suppose $A_i = \forall x[x/a]A_j(a)$ for some j $(1 \leq j < i \leq n)$ and that $\vdash_L A_j(a)$. Then, there exists $A'_j(a) \in L$ such that $A_j(a) = [\overline{d}/\overline{b}]A'_j(a)$. Now, by $[G]$, $\forall x[x/a]A'_j(a) \in L$. Thus, $A_i = [\overline{d}/\overline{b}](\forall x[x/a]A'_j(a))$. Hence, $\vdash_L A_i$.

■

Lemma 5.2.6 (Deduction Theorem)**.** Let L be a predicate modal logic, Γ a modal theory and $A, B \in Sen(\mathcal{C}_\Gamma)$. If $\Gamma \cup \{A\} \vdash_L B$, then $\Gamma \vdash_L A \supset B$.

Proof. This is a standard proof of the deduction theorem for predicate classical logic, by induction on the length of an L-derivation of B in $\Gamma \cup \{A\}$, since $[N]$ is not a derivation rule in Definition 5.2.4. ■

Definition 5.2.7. Let L be a predicate modal logic. We say a theory Γ is *L-inconsistent* if $\Gamma \vdash_L \bot$. Otherwise, we say Γ is *L-consistent*.

Lemma 5.2.8. Let L be a predicate modal logic, Γ a modal theory and $A \in Sen(\mathcal{C}_\Gamma)$. Then, $\Gamma \cup \{A\}$ is L-consistent iff $\Gamma \nvdash_L \sim A$.

Proof. Similar to the proof of Lemma 3.2.4, using Definition 5.2.7 and Lemma 5.2.6. ■

Lemma 5.2.9. Let L be a predicate modal logic, Γ a modal theory and $A \in Sen(\mathcal{C}_\Gamma)$. Then, $\Gamma \vdash_L A$ iff there exist $B_1, \ldots B_n \in \Gamma$ such that $\vdash_L \bigwedge_{i=0}^{n} B_i \supset A$.

Proof. Suppose $\Gamma \vdash_L A$. Then, there exists an L-derivation of A from Γ. Let $B_1, \ldots, B_n \in \Gamma$ denote all the wff from Γ used in this L-derivation. Then, $\{B_1, \ldots, B_n\} \vdash_L A$. By **QCL**, it

follows that $\{\bigwedge_{i\geq 1}^{n} B_i\} \vdash_L A$. Trivially, $\emptyset \cup \{\bigwedge_{i\geq 1}^{n} B_i\} \vdash_L A$. Thus, by Lemma 5.2.6, it follows that $\emptyset \vdash_L \bigwedge_{i\geq 1}^{n} B_i \supset A$. Now, by Lemma 5.2.5, $\vdash_L \bigwedge_{i\geq 1}^{n} B_i \supset A$. Conversely, suppose there exist $B_1, \ldots, B_n \in \Gamma$ such that $\vdash_L \bigwedge_{i\geq 1}^{n} B_i \supset A$. By Definition 5.2.2, $\Gamma \vdash_L \bigwedge_{i\geq 1}^{n} B_i \supset A$. Furthermore, by Definition 5.2.2, $\Gamma \vdash_L B_i$, for $1 \geq i \geq n$. By **QCL**, $\Gamma \vdash_L \bigwedge_{1\geq i}^{n} B_i$. Hence, by $[MP]$, $\Gamma \vdash_L A$. ■

If Γ is a modal theory, then we define $\Box\Gamma := \{\Box A \mid A \in \Gamma\}$.

The following lemma is proved in a similar way to Lemma 3.2.6, but requires the careful handling of constants to avoid technical error, and so we prove it here fully.

Lemma 5.2.10. Let L be a predicate modal logic, Γ a modal theory and $A \in Sen(\mathcal{C}_\Gamma)$. If $\Gamma \vdash_L A$, then $\Box\Gamma \vdash_L \Box A$.

Proof. Suppose $\Gamma \vdash_L A$. Then by Lemma 5.2.9, there exist $B_1, \ldots, B_n \in \Gamma$ such that $\vdash_L \bigwedge_{i\geq 1}^{n} B_i \supset A$. By definition of $\vdash_L$, there exists $\bigwedge_{i\geq 1}^{n} B_i' \supset A'$ such that $\bigwedge_{i\geq 1}^{n} B_i' \supset A' \in L$ and

$$\bigwedge_{i\geq 1}^{n} B_i \supset A = [\overline{c}/\overline{a}](\bigwedge_{i\geq 1}^{n} B_i' \supset A'),$$

By $[N]$, $\Box(\bigwedge_{i\geq 1}^{n} B_i' \supset A') \in L$. Then, by K and $[MP]$, we obtain $\Box(\bigwedge_{i\geq 1}^{n} B_i') \supset \Box A' \in L$. Now, by Theorem 2.2.8 and **QCL**, $(\bigwedge_{i\geq 1}^{n} \Box B_i') \supset \Box A' \in L$. Then, $\vdash_L [\overline{c}/\overline{a}](\bigwedge_{i\geq 1}^{n} \Box B_i' \supset \Box A')$. Finally, by Lemma 5.2.9, $\{\bigwedge_{i\geq 1}^{n} \Box B_i\} \vdash_L \Box A$. Since, $\Box B_1, \ldots, \Box B_n \in \Box\Gamma$, it follows that $\Box\Gamma \vdash_L \Box A$. ■

Lemma 5.2.11 (Fresh Constant Lemma)**.** Let L be a predicate modal logic, Γ a modal theory, and $A(c)$ be a sentence containing a single constant $c \notin \mathcal{C}_\Gamma$. If $\Gamma \vdash_L A(c)$, then $\Gamma \vdash_L \forall x A(x)$.

Proof. Let $A_1, \ldots, A_n$ be an L-derivation of $A(c)$ from Γ. Since $c \notin \mathcal{C}_\Gamma$, it follows that $A(c) \notin \Gamma$. Let $B_1, \ldots, B_n$ be a sequence such that, for b a free variable, $B_i = [b/c]A_i$. Then, observe that $B_1, \ldots, B_n$ is an L-derivation of B_n, i.e. $A(b)$, from Γ: Indeed, for $1 \leq i \leq n$:

(a) If $A_i \in \Gamma$ then, since $c \notin \Gamma$, $[b/c]A_i = B_i = A_i$. Thus, $B_i \in \Gamma$.

(b) If $\vdash_L A_i$ then $A_i = [\overline{c}/\overline{a}]A_i'$ for some $A_i' \in L$. Thus, $B_i = [\overline{b}/\overline{c}][\overline{c}/\overline{a}]A_i' = [\overline{b}/\overline{a}]A_i'$. Since **QCL** permits the renaming of free variables (provided no free variable becomes quantified as a result of the renaming), $B_i \in L$. Thus, $\vdash_L B_i$.

(c) If A_i is obtained from some $A_j, A_j \supset A_i$ $(1 \leq j < i)$ using $[MP]$, then B_i is obtained from $B_j, B_j \supset B_i$ using $[MP]$.

(d) If $A_i = \forall x[x/a]A_j(a)$ for some j $(1 \leq j < i)$, then $B_i = \forall x[x/a]B_j(a)$.

Hence, by $[G]$ in Definition 5.2.4, $\Gamma \vdash_L \forall x A(x)$. ■

Definition 5.2.12. Let L be a predicate modal logic and Γ an L-consistent theory. We say that Γ is *L-complete* if there exists no L-consistent theory Ω such that $\Gamma \subset \Omega$.

Lemma 5.2.13. Let L be a predicate modal logic and let Γ be an L-complete theory. Then, for every $A \in Sen(\mathcal{C}_\Gamma)$, either $A \in \Gamma$ or $\sim A \in \Gamma$.

Proof. Let $A \in Sen(\mathcal{C}_\Gamma)$ and suppose that $A \notin \Gamma$ and $\sim A \notin \Gamma$. Then since Γ is L-complete, both $\Gamma \cup \{A\}$ and $\Gamma \cup \{\sim A\}$ are L-inconsistent. Since $\Gamma \cup \{A\}$ is L-inconsistent, $\Gamma \vdash_L \sim A$, i.e. $\Gamma \vdash_L A \supset \bot$. Since $\Gamma \cup \{\sim A\}$ is L-inconsistent, by Lemma 5.2.8, $\Gamma \vdash_L \sim\sim A$, and so $\Gamma \vdash_L A$. Thus, by $[MP]$, $\Gamma \vdash_L \bot$, contradicting the L-consistency of Γ. ■

Lemma 5.2.14. Let L be a predicate modal logic and let Γ be a L-complete theory. Then,

(a) If $A \in Sen(\mathcal{C}_\Gamma)$, then $A \in \Gamma$ iff $\Gamma \vdash_L A$,

(b) Γ is closed under $[MP]$.

Proof. (a) Let $A \in Sen(\mathcal{C}_\Gamma)$. For "if", suppose $A \notin \Gamma$. Since Γ is L-complete, $\Gamma \cup \{A\}$ is L-inconsistent. Thus, by Lemma 3.2.8, $\sim A \in \Gamma$. Hence, $\Gamma \cup \{\sim A\}$ is L-consistent and therefore, by Lemma 5.2.8, $\Gamma \nvdash_L \sim\sim A$, i.e. $\Gamma \nvdash_L A$. "Only if" follows immediately from the definition of $\vdash_L$ in Definition 5.2.4.

(b) Suppose $A, A \supset B \in \Gamma$. By (a), $\Gamma \vdash_L A$ and $\Gamma \vdash_L A \supset B$. By $[MP]$, $\Gamma \vdash_L B$. Hence, by (a), $B \in \Gamma$.

■

Lemma 5.2.15. Let L be a predicate modal logic and let Γ be an L-complete modal theory. Then, $Sen(L) \subseteq \Gamma$,

Proof. Let $A \in Sen(L)$. Then, $A \in Sen(\mathcal{C}_\Gamma)$. Thus, $\vdash_L A$ and so $\Gamma \vdash_L A$. Hence, by Lemma 5.2.14 (a), $A \in \Gamma$. ■

Lemma 5.2.16. Let L be a predicate modal logic and let $\Gamma_1 \subseteq \Gamma_2 \subseteq \ldots$ be a chain of L-consistent theories. Then, $\Gamma_\omega = \bigcup_{i<\omega} \Gamma_i$ is L-consistent.

Proof. Assume otherwise. Then, $\Gamma_\omega \vdash_L \bot$. Since this L-derivation for $\bot$ from Γ is finite, it must use a finite number of sentences in Γ. Hence, there exists $k \leq \omega$ such that $\Gamma_k \vdash_L \bot$, contradicting the L-consistency of Γ_k. ■

Lemma 5.2.17 (Lindenbaum)**.** Let L be a predicate modal logic and let Γ be an L-consistent theory. Then, there exists an L-complete theory Ω such that $\Gamma \subseteq \Omega \subseteq Sen(\mathcal{C}_\Gamma)$.

Proof. Let

$$\mathcal{P} := \{\Omega' \mid \Omega' \text{ is } L\text{-consistent } \& \ \Gamma \subseteq \Omega' \subseteq Sen(\mathcal{C}_\Gamma)\}$$

and let $\mathcal{H}$ be a chain in $\mathcal{P}$. By Lemma 5.2.16, $\bigcup \mathcal{H}$ is L-consistent. So $\mathcal{H} \in \mathcal{P}$. Thus, by Zorn's Lemma, $\mathcal{P}$ has a maximal element Ω. It should be clear that $\Gamma \subseteq \Omega \subseteq Sen(\mathcal{C}_\Gamma)$. ■

Lemma 5.2.18. Let L be a predicate modal logic, Γ an L-consistent theory and $A(c)$ a sentence containing a single constant $c \notin \mathcal{C}_\Gamma$. Then, $\Gamma \cup \{\exists x A(x) \supset A(c)\}$ is L-consistent.

Proof. Suppose otherwise, i.e. that $\Gamma \cup \{\exists x A(x) \supset A(c)\}$ is L-inconsistent. Then, by Lemma 5.2.8, $\Gamma \vdash_L \sim\{\exists x A(x) \supset A(c)\}$. By **QCL**, $\Gamma \vdash_L \exists x A(x)$ and (*) $\Gamma \vdash_L \sim A(c)$. Since $c \notin \mathcal{C}_\Gamma$, it follows by Lemma 5.2.11 and (*) that $\Gamma \vdash_L \forall x \sim A(x)$. Then, by **QCL**, $\Gamma \vdash_L \sim\exists x A(x)$, contradicting the L-consistency of Γ. ■

Definition 5.2.19. Let L be a predicate modal logic and let Γ be an L-consistent theory. Γ is called *Henkin* if for every $\exists x B(x) \in Sen(\mathcal{C}_\Gamma)$, there exists $c \in \mathcal{C}_\Gamma$ such that $\exists x B(x) \supset B(c) \in \Gamma$.

Definition 5.2.20. If $\mathcal{S}$ is a fixed infinite set of constants and $\mathcal{C}$ is a set of constants, then we say that $\mathcal{C}$ is $\mathcal{S}$-*small* if $|\mathcal{S} - \mathcal{C}| = |\mathcal{S}|$.

If Γ is a modal theory, then we define

$$\mathcal{E}\Gamma := \{A \in \Gamma \mid A \text{ is of the form } \exists x[x/a]B(x), FVar(B) = \{a\}\}$$

Proposition 5.2.21. Let Γ be a modal theory. Then, $|Sen(\mathcal{C}_\Gamma)| = |\mathcal{E}Sen(\mathcal{C}_\Gamma)| = \max\{|\mathcal{C}_\Gamma|, \aleph_0\}$.

From now on, unless otherwise defined in the context, we will take $\mathcal{S}$ to be the "universal" infinite set of constants. We denote the (infinite) cardinality of $\mathcal{S}$ by κ.

For a given predicate modal logic L, the following lemma will ensure that an L-consistent

theory Γ is included in some Henkin theory. This will be used later in the construction of a canonical model for L, ensuring that each world's domain is adequately populated for the sentences belonging to the world.

Lemma 5.2.22 (Henkin Lemma)**.** Let L be a predicate modal logic and let Γ_0 be an L-consistent modal theory such that $\mathcal{C}_{\Gamma_0}$ is $\mathcal{S}$-small. Then, Γ_0 can be extended to an L-consistent Henkin theory with an $\mathcal{S}$-small set of constants.

Proof. Let $\kappa_0 = \max\{|\mathcal{C}_{\Gamma_0}|, \aleph_0\}$. By Proposition 5.2.21, $|Sen(\mathcal{C}_{\Gamma_0})| = \kappa_0$. Since $\mathcal{C}_{\Gamma_0}$ is $\mathcal{S}$-small and $|S| = \kappa$, it follows that $\kappa_0 \leq \kappa$. Using the Axiom of Choice, we extend Γ_0 to an L-complete Henkin theory Γ_ω with $|Sen(\mathcal{C}_{\Gamma_\omega})| = \kappa_0$. Let S' and S'' be infinite sets partitioning $\mathcal{S} - \mathcal{C}_{\Gamma_0}$ such that $|S'| = |S''| = \kappa$. This is possible since $|S - \mathcal{C}_{\Gamma_0}| = \kappa$ is infinite. To guarantee that $\mathcal{C}_{\Gamma_\omega}$ is $\mathcal{S}$-small, it is enough to ensure that $\mathcal{C}_{\Gamma_\omega} \cap S'' = \emptyset$, since $|S''| = |\mathcal{S}| = \kappa$. We obtain a countable increasing sequence of modal theories $\Gamma_0 \subseteq \Gamma_1 \subseteq \ldots$ such that $\mathcal{C}_{\Gamma_i} = \kappa_0$ for $i < \omega$. Let S_0, S_1, $\ldots$ be disjoint subsets of S' with $|S_i| = \kappa_0$ for each $i < \omega$. For convenience, we use each S_i as an index set. Now, we define

$$\Gamma_{i+1} := \Gamma_i \cup \{\exists x_c A_c(x_c) \supset A_c(c) \mid c \in S_i\}.$$

Note: if k_0 is uncountable, then for some $c \neq c'$, $x_c = x'_c$. Now, $|\mathcal{C}_{\Gamma_{i+1}}| = |\mathcal{C}_{\Gamma_i} \cup S_i| = \kappa_0 + \kappa_0 = \kappa_0$. Finally, we set

$$\Gamma_\omega := \bigcup_{i<\omega} \Gamma_i.$$

We now show that Γ_ω is L-consistent, $\mathcal{S}$-small and Henkin. For L-consistency, we first show by induction on i that $\mathcal{C}_{\Gamma_i} \cap S_i = \emptyset$ for each $i < \omega$. Our base case, $\mathcal{C}_{\Gamma_0} \cap S_0 = \emptyset$, follows from the fact that $S_0 \subset S' \subset S - C_{\Gamma_0}$. Our induction step, $\mathcal{C}_{\Gamma_i} \cap S_i = \emptyset$, works by the definition that S_0, S_1, $\ldots$ are disjoint and $\mathcal{C}_{\Gamma_0} \cap S_i$, for each $i < \omega$. By assumption, Γ_0 is L-consistent (base case). Then, assuming Γ_i is L-consistent, for some $i < \omega$, it follows by induction on c (used as an index in S_i) and repeated application of Lemma 5.2.18 that Γ_{i+1} is L-consistent. Thus, by Lemma 5.2.16, Γ_ω is L-consistent. For $\mathcal{S}$-smallness, observe that

$$\mathcal{C}_{\Gamma_\omega} = \mathcal{C}_{\Gamma_0} \cup (\bigcup_{i<\omega} S_i).$$

Then, $|\mathcal{C}_{\Gamma_\omega}| = \kappa_0$ and so $|Sen(\mathcal{C}_{\Gamma_\omega})| = |Sen(\mathcal{C}_{\Gamma_0})|$. Now, $S_i \subset S'$ for each $i < \omega$. Since, by definition, $S' \cap S'' = \emptyset$, it follows that $\mathcal{C}_{\Gamma_\omega} \cap S'' = \emptyset$. Thus, $|S''| = \kappa$, it follows that $\mathcal{C}_{\Gamma_\omega}$ is

$\mathcal{S}$-small.

Lastly, to show that Γ_ω is Henkin, let $\exists x A(x) \in \mathcal{E}Sen(\mathcal{C}_{\Gamma_\omega})$. Since $\exists x A(x)$ contains finitely many constants, there exists $i < \omega$ such that $\exists x A(x) \in Sen(\mathcal{C}_{\Gamma_i})$. By construction of Γ_ω, there exists $c \in S_i$ such that $\exists x A(x) \supset A(c) \in \Gamma_{i+1}$. Since $\Gamma_{i+1} \subseteq \Gamma_\omega$, surely, $\exists x A(x) \supset A(c) \in \Gamma_\omega$. Hence, Γ_ω is Henkin. ■

Lemma 5.2.23. Let L be a predicate modal logic, Ω an L-consistent Henkin theory and Γ an L-complete theory such that $\Omega \subseteq \Gamma \subseteq Sen(\mathcal{C}_\Omega)$. Then, Γ is also Henkin.

Proof. Let $\exists x A(x) \in \mathcal{E}Sen(\mathcal{C}_\Gamma)$. Clearly, $Sen(\mathcal{C}_\Omega) = Sen(\mathcal{C}_\Gamma)$ and thus $\mathcal{E}Sen(\mathcal{C}_\Omega) = \mathcal{E}Sen(\mathcal{C}_\Gamma)$. Hence, $\exists x A(x) \in Sen(\mathcal{C}_\Omega)$. Since Ω is Henkin, there exists $c \in \mathcal{C}_\Omega$ such that $\exists x A(x) \supset A(c) \in \Omega$. Moreover, since $\Omega \subseteq \Gamma$, $\exists x A(x) \supset A(c) \in \Gamma$. Finally, since $\exists x A(x)$ was arbitrary, it follows that Γ is Henkin. ■

Definition 5.2.24. A modal theory Γ is *satisfiable* in a model $M = \langle W, R, D, V \rangle$ if there exists $\Omega \in W$ and a map $\phi : \mathcal{C}_\Gamma \to \mathcal{C}_\Omega$ such that for every $A \in \Gamma$,

$$\text{If } \mathcal{C}(A) = \{c_1, \ldots, c_n\}, \text{ then } M, \Omega \Vdash [\phi(c_1), \ldots, \phi(c_n)/c_1, \ldots, c_n]A.$$

5.2.2 Canonical Models

Lemma 5.2.25. Let Γ be an L-complete Henkin theory and let $\exists x A(x) \in Sen(\mathcal{C}_\Gamma)$. Then, $\exists x A(x) \in \Gamma$ iff there exists $c \in \mathcal{C}_\Gamma$ such that $A(c) \in \Gamma$.

Proof. For "if", suppose $A(c) \in \Gamma$. By **QCL**, $\vdash_L A(c) \supset \exists x A(x)$ and, therefore, $\Gamma \vdash_L A(c) \supset \exists x A(x)$. Hence, by Lemma 5.2.14 (a), $A(c) \supset \exists x A(x) \in \Gamma$ and therefore by Lemma 5.2.14 (b), $\exists x A(x) \in \Gamma$. For "only if", suppose $\exists x A(x) \in \Gamma$. Then, since Γ is Henkin, there exists $c \in \mathcal{C}_\Gamma$ such that $\exists x A(x) \supset A(c) \in \Gamma$. Hence, by Lemma 5.2.14 (b), $A(c) \in \Gamma$. ■

In defining the canonical model $\langle F_S, V_S \rangle$ for a propositional modal logic S, we took W_S to be the set of all S-complete sets. For a predicate modal logic L, we must ensure that each $\Gamma \in W_L$ possesses some additional properties.

If Γ is a modal theory, then $\Box^- \Gamma := \{A \mid \Box A \in \Gamma\}$.

Lemma 5.2.26. Let L be a predicate modal logic, Γ be an L-complete Henkin theory and $A \in Sen(\mathcal{C}_\Gamma)$. Let $\sim\Box A \in \Gamma$. Then $\Box^{-}\Gamma \cup \{\sim A\}$ is L-consistent.

Proof. Similar to the proof of Lemma 3.2.11, using Lemmas 5.2.8, 5.2.10, 5.2.14 (a) and 5.2.13. ■

Definition 5.2.27. If L be a predicate modal logic, then an $(L, \mathcal{S})$-*place* is any L-complete Henkin theory whose set of constants is $\mathcal{S}$-small.

Lemma 5.2.28. Let Ω be an L-consistent theory with an $\mathcal{S}$-small set of constants. Then, there exists an $(L, \mathcal{S})$-place that includes Ω.

Proof. By Lemma 5.2.22, we can extend Ω to an L-consistent Henkin theory Ω'. And, by Lemma 5.2.17 and Lemma 5.2.23, we can then extend Ω' to an L-complete Henkin theory Γ that includes Ω and whose set of constants is $\mathcal{S}$-small. ■

Definition 5.2.29. Let L be a predicate modal logic. A *canonical predicate frame* for L is a tuple $\langle W_{L,\mathcal{S}}, R_{L,\mathcal{S}}, D_{L,\mathcal{S}} \rangle$, where:

- $W_{L,\mathcal{S}} := \{\Gamma \mid \Gamma \text{ is an } (L, \mathcal{S})\text{-place}\}$,
- for every $\Gamma, \Omega \in W_{L,\mathcal{S}}$,
$$\text{if } \Box^{-}\Gamma \subseteq \Omega, \text{ then } \Gamma R_{L,\mathcal{S}} \Omega,$$
- $D_{L,\mathcal{S}}(\Gamma) := \mathcal{C}_\Gamma$, for every $\Gamma \in W_{L,\mathcal{S}}$.

Note: we must still demonstrate that a canonical predicate frame for L, as defined above, is indeed a predicate Kripke frame. Specifically, we must show that if $\Gamma R_{L,\mathcal{S}} \Omega$ then $\mathcal{C}_\Gamma \subseteq \mathcal{C}_\Omega$, for every $\Gamma, \Omega \in W_{L,\mathcal{S}}$. To accomplish this, we need the following lemmas.

If Γ is a modal theory, then we define $\Diamond\Gamma := \{\Diamond A \mid A \in \Omega\}$.

Lemma 5.2.30. Let L be a predicate modal logic. Then for every $\Gamma, \Omega \in W_{L,\mathcal{S}}$:

$$\Gamma R_{L,\mathcal{S}} \Omega \text{ iff both } \mathcal{C}_\Gamma \subseteq \mathcal{C}_\Omega \text{ and } \Diamond\Omega \cap Sen(\mathcal{C}_\Gamma) \subseteq \Gamma$$

Proof. Suppose $\Gamma R_{L,\mathcal{S}} \Omega$. We first show that $\mathcal{C}_\Gamma \subseteq \mathcal{C}_\Omega$. Let $c \in \mathcal{C}_\Gamma$ and let P be a 1-ary predicate letter. Clearly, $\vdash_L \Box(P(c) \supset P(c))$. Thus, by Lemma 5.2.14 (a), $\Box(P(c) \supset P(c)) \in \Gamma$. By definition of $R_{L,\mathcal{S}}$, it follows that $P(c) \supset P(c) \in \Omega$. Hence, $c \in \mathcal{C}_\Omega$ and therefore $\mathcal{C}_\Gamma \subseteq \mathcal{C}_\Omega$.

Secondly, we show $\Diamond\Omega \cap Sen(\mathcal{C}_\Gamma) \subseteq \Gamma$. Let $A \in Sen(\mathcal{C}_\Gamma)$ and suppose that $\Diamond A \in \Diamond\Omega \cap Sen(\mathcal{C}_\Gamma)$ but $\Diamond A \notin \Gamma$. Thus, $A \in \Omega$. And since $\Diamond A \notin \Gamma$, by Lemma 5.2.13, $\sim\Diamond A \in \Gamma$. And so, $\Box\sim A \in \Gamma$. Now, since $A \in \Omega$, by Lemma 5.2.13, $\sim A \notin \Gamma$. Hence, $\Box^-\Gamma \not\subseteq \Omega$. By definition of $R_{L,\mathcal{S}}$, therefore, it follows that $\Omega \notin R_{L,\mathcal{S}}(\Gamma)$, contradicting our assumption.

Conversely, assume $\mathcal{C}_\Gamma \subseteq \mathcal{C}_\Omega$ and $\Diamond\Omega \cap Sen(\mathcal{C}_\Gamma) \subseteq \Gamma$. Suppose for some $A \in Sen(\mathcal{C}_\Gamma)$ that $\Box A \in \Gamma$ but $A \notin \Omega$. Then, by Lemma 5.2.13, $\sim A \in \Omega$. Then, $\Diamond\sim A \in \Diamond\Omega \cap Sen(\mathcal{C}_\Gamma)$. Hence, $\Diamond\sim A \in \Gamma$. And so, $\sim\Box A \in \Gamma$, contrary to the L-consistency of Γ. ■

Corollary 5.2.31. Let L be a predicate modal logic. Then a canonical predicate frame $\langle W_{L,\mathcal{S}}, R_{L,\mathcal{S}}, D_{L,\mathcal{S}}\rangle$ of L is a Kripke frame.

Proof. It is clear from Definition 5.2.29 and Lemma 5.2.30 that $\langle W_{L,\mathcal{S}}, R_{L,\mathcal{S}}, D_{L,\mathcal{S}}\rangle$ satisfies the requirements for a Kripke frame. ■

Lemma 5.2.32. Let L be a predicate modal logic, Γ be an L-consistent theory and $\Diamond A \in \Gamma$. Then, there exists an L-complete theory $\Omega \subseteq Sen(\mathcal{C}_\Gamma)$ containing A.

Proof. Since $\Diamond A \in \Gamma$, i.e. $\sim\Box\sim A \in \Gamma$, it follows by Lemma 5.2.26 that $\Box^-\Gamma \cup \{\sim\sim A\} = \Box^-\Gamma \cup \{A\}$ is L-consistent. Thus, by Lemma 5.2.17, there exists an L-complete theory Ω such that $\Box^-\Gamma \cup \{A\} \subseteq \Omega$. ■

We prove one last lemma before we define canonical models for predicate modal logics, which guarantees an existence property similar to Lemma 3.2.14.

Lemma 5.2.33 (Existence Lemma)**.** Let L be a predicate modal logic, $\langle W_{L,\mathcal{S}}, R_{L,\mathcal{S}}, D_{L,\mathcal{S}}\rangle$ be a canonical predicate frame of L, $\Gamma \in W_{L,\mathcal{S}}$ and $\Diamond A \in \Gamma$, for some $A \in Sen(\mathcal{C}_\Gamma)$. Then, there exists $\Omega \in R_{L,\mathcal{S}}(\Gamma)$ such that $A \in \Omega$.

Proof. Let $\Diamond A \in \Gamma$, i.e. $\sim\Box\sim A \in \Gamma$. By Lemma 5.2.26, $\Delta := \Box^-\Gamma \cup \{A\}$ is L-consistent. Since $\mathcal{C}_\Delta \subseteq \mathcal{C}_\Gamma$, it follows that $\mathcal{C}_\Delta$ is $\mathcal{S}$-small. Hence, by Lemma 5.2.28, there exists an $(L,\mathcal{S})$-place $\Omega \in W_{L,\mathcal{S}}$ with $\Delta \subseteq \Omega$ and so $A \in \Omega$. Thus, $\Box^-\Gamma \subseteq \Omega$ and thus, by definition of $R_{L,\mathcal{S}}$, it follows that $\Gamma R_{L,\mathcal{S}}\Omega$. ■

Definition 5.2.34. Let L be a predicate modal logic. Then, a *canonical model* of L, denoted by $M_{L,\mathcal{S}} = \langle W_{L,\mathcal{S}}, R_{L,\mathcal{S}}, D_{L,\mathcal{S}}, V_{L,\mathcal{S}}\rangle$, is a canonical predicate frame together with a valuation function $V_{L,S}$ defined, for every $\Gamma \in W_{L,\mathcal{S}}$, by

$$V(P^n, \Gamma) := \{<c_1, \ldots, c_n> \mid P^n(c_1, \ldots, c_n) \in \Gamma\}.$$

We now demonstrate the correspondence between membership of a world $\Gamma \in W_{L,\mathcal{S}}$ and the behaviour of $\Vdash$ at Γ by showing a result similar to Theorem 3.2.15.

Theorem 5.2.35 (Canonical Model Theorem)**.** Let L be a predicate modal logic and $M_{L,\mathcal{S}} = \langle W_{L,\mathcal{S}}, R_{L,\mathcal{S}}, D_{L,\mathcal{S}}, V_{L,\mathcal{S}} \rangle$ a canonical model for L. Then, for every $A \in Sen(\mathcal{C}_\Gamma)$ and for every $\Gamma \in W_{L,\mathcal{S}}$:

$$A \in \Gamma \text{ iff } \Gamma \Vdash A$$

Proof. We prove the theorem by induction on the complexity of a sentence A. By definition of $W_{L,\mathcal{S}}$, the theorem holds for $\bot$, since every $\Gamma \in W_{L,\mathcal{S}}$ is L-consistent. Furthermore, by definition of $V_{L,\mathcal{S}}$, it immediately follows that for any n-ary predicate letter P and constants $c_1, \ldots, c_n$,

$$P^n(c_1, \ldots, c_n) \in \Gamma \text{ iff } \Gamma \Vdash P^n(c_1, \ldots, c_n).$$

Thus, it is enough to show that the theorem holds for $[V \supset]$, $[V\Box]$ and $[V\forall]$ (since the other model valuation rules are for abbreviated expressions):

($\supset$) Similar to the proof of Theorem 3.2.15, using Lemma 5.2.14 (b).

($\forall$) Let $A = \forall x B(x)$. Then, $\forall x B(x) \notin \Gamma$ iff, by Lemma 5.2.13, $\sim\forall x B(x) \in \Gamma$ iff $\exists x \sim B(x) \in \Gamma$ iff, by Lemma 5.2.25, there exists $c \in \mathcal{C}_\Gamma$ such that $\sim B(c) \in \Gamma$ iff, by Lemma 5.2.13, $B(c) \notin \Gamma$ iff, by induction hypothesis, $\Gamma \nVdash B(c)$ iff, by $[V\forall]$, $\Gamma \nVdash \forall x B(x)$.

($\Box$) Let $A = \Box B$. For "if", suppose $\Box B \notin \Gamma$. Then, by Lemma 5.2.13, $\sim\Box B \in \Gamma$, i.e. $\Diamond\sim B \in \Gamma$. By Lemma 5.2.33, there exists $\Omega \in R_{L,\mathcal{S}}(\Gamma)$ such that $\sim B \in \Omega$. Then, by Lemma 5.2.13, $B \notin \Omega$. Thus, by induction hypothesis, $\Omega \nVdash B$ and so, by $[V\Box]$, $\Gamma \nVdash \Box A$. For "only if", suppose $\Gamma \nVdash \Box B$. Then, by $[V\Box]$, there exists $\Omega \in R_{L,\mathcal{S}}(\Gamma)$ such that $\Omega \nVdash B$. By induction hypothesis, $B \notin \Omega$ and so, by Lemma 5.2.13, $\sim B \in \Omega$. By Lemma 5.2.25, $\Diamond\Omega \cap Sen(\mathcal{C}_\Gamma) \subseteq \Gamma$ and thus $\Diamond\sim B \in \Gamma$, i.e. $\sim\Box B \in \Gamma$.

■

5.3 Canonicity and Completeness

In this section, we introduce the concepts of canonicity and strong Kripke completeness. Canonicity in predicate modal logic is less common than in propositional modal logic, where it can be used to demonstrate Kripke completeness for many logics. As such, we also consider the weaker property of *quasi-canonicity* for several non-canonical logics, as introduced in Shehtman [19] (see also, Rybakov, Shehtman, and Shkatov [14], once published and available).

Theorem 5.3.1. Let L be a predicate modal logic and let $\boldsymbol{F}_{L,\mathcal{S}}$ be a canonical predicate frame of L. If $\boldsymbol{F}_{L,\mathcal{S}} \in \boldsymbol{C}$, for some class $\boldsymbol{C}$, then L is complete w.r.t $\boldsymbol{C}$.

Proof. Let $\boldsymbol{M}_{L,\mathcal{S}}$ be a canonical model for L whose underlying frame is $\boldsymbol{F}_{L,\mathcal{S}}$ and let A be a sentence such that $A \notin L$. Then, the set $\{\sim A\}$ is L-consistent. By Lemma 5.2.28, $\{\sim A\}$ can be extended to an $(L,\mathcal{S})$-place Γ. By definition of $W_{L,\mathcal{S}}$, it follows that $\Gamma \in W_{L,\mathcal{S}}$. By Theorem 5.2.35, $\boldsymbol{M}_{L,\mathcal{S}}, \Gamma \Vdash \sim A$. By $[V\sim]$, $\boldsymbol{M}_{L,\mathcal{S}}, \Gamma \nVdash A$. Hence, $\boldsymbol{M}_{L,\mathcal{S}} \nVdash A$ and so $\boldsymbol{F}_{L,\mathcal{S}} \nVdash A$. Hence, a fortiori, $\boldsymbol{C} \nVdash A$. ■

Definition 5.3.2. A predicate modal logic L is *canonical* if $\boldsymbol{F}_{L,S} \Vdash L$ for some infinite set of constants $\mathcal{S}$.

Definition 5.3.3. A predicate modal logic L is *strongly Kripke complete* if every L-consistent theory Γ is satisfied in a model over a frame validating L.

Lemma 5.3.4. Let L be a canonical predicate modal logic. Then, L is strongly complete.

Proof. Let Ω be an L-consistent modal theory. Let $\mathcal{S}$ be large enough for $\mathcal{C}_\Omega$ to be $\mathcal{S}$-small. By Lemma 5.2.28, we can extend Ω to an $(L,\mathcal{S})$-place Γ. Then, by definition of $W_{L,\mathcal{S}}$, it follows that $\Gamma \in W_{L,\mathcal{S}}$. Now, let Ψ be an identity map on $\mathcal{C}_\Omega$. Then, by Theorem 5.2.35, it follows that $\boldsymbol{M}_{L,\mathcal{S}}, \Gamma \Vdash \Psi(A)$, for every $A \in \Omega$. Since L is canonical, $\boldsymbol{F}_{L,\mathcal{S}} \Vdash L$, from which it follows that Ω is satisfied in a model over a frame validating L. Hence, L is strongly complete. ■

Theorem 5.3.5. $\boldsymbol{Q.K}$ is canonical and therefore strongly Kripke complete.

Proof. By Theorem 5.1.6, $\boldsymbol{Q.K}$ is validated by every predicate frame. In particular, $\boldsymbol{Q.K}$ is validated by the canonical predicate frame of $\boldsymbol{Q.K}$. Hence, $\boldsymbol{Q.K}$ is canonical. By Lemma 5.3.4, therefore, $\boldsymbol{Q.K}$ is strongly Kripke complete. ■

Definition 5.3.6. A wff A is *canonical* if the predicate modal logic $\boldsymbol{Q.K} + A$ is canonical.

Lemma 5.3.7. Let L and L' be predicate modal logics such that $L \subseteq L'$ and let Γ be an L'-consistent theory. Then, Γ is also L-consistent.

Proof. Suppose that Γ is L-inconsistent. Then, $\Gamma \vdash_L \bot$. Since $L \subseteq L'$, every L-derivation from Γ is also an L'-derivation from Γ. Hence, $\Gamma \vdash_{L'} \bot$, contrary to the L-consistency of Γ. ■

Lemma 5.3.8. Let L and L' be predicate modal logics such that $L \subseteq L'$ and let Γ be a modal theory. Then,

$$\Gamma \text{ is } L'\text{-complete iff } Sen(L') \subseteq \Gamma \text{ and } \Gamma \text{ is } L\text{-complete.}$$

Proof. Suppose Γ is L'-complete. By Lemma 5.2.15, $Sen(L') \subseteq \Gamma$. Since Γ is L'-complete, by Lemma 5.2.13, for every $A \in Sen(\mathcal{C}_\Gamma)$, either $A \in \Gamma$ or $\sim A \in \Gamma$. Thus, any proper superset of Γ is L-inconsistent. Furthermore, by Lemma 5.3.7, Γ is L-consistent. Thus, by definition, Γ is L-complete.

Now, suppose $Sen(L') \subseteq \Gamma$ and Γ is L-complete. By Lemma 5.2.13, for every $A \in Sen(\mathcal{C}_\Gamma)$, either $A \in \Gamma$ or $\sim A \in \Gamma$. Hence, to show that Γ is L'-complete, it suffices to show that Γ is L'-consistent. Suppose otherwise, i.e. that Γ is L'-inconsistent. Then, by Definition 5.2.7, $\Gamma \vdash_{L'} \bot$. By Lemma 5.2.9, there exist $B_1, \ldots, B_n \in \Gamma$ such that $\vdash_{L'} \bigwedge_{i=0}^{n} B_i \supset \bot$, i.e. $\vdash_{L'} \sim \bigwedge_{i=1}^{n} B_i$. Let $A := \sim \bigwedge_{i=0}^{n} B_i$. Then, there exists $A' \in L'$ such that $A = [\overline{c}/\overline{a}]A'$. Thus, by $[G]$, $\overline{\forall} A' \in L'$. Now, since $\overline{\forall} A' \in Sen(L')$, it follows that $\overline{\forall} A' \in \Gamma$. But then $\Gamma \vdash_L \overline{\forall} A'$. Now, by **QCL**, $\Gamma \vdash_L \overline{\forall} A' \supset A$. Hence, by $[MP]$, it follows that $\Gamma \vdash_L A$. But $\Gamma \vdash_L \sim A$. Thus, by $[MP]$, $\Gamma \vdash_L \bot$. So, Γ is L-inconsistent, contrary to our assumption that Γ is L-complete. ■

Lemma 5.3.9. Let L be a predicate modal logic, A a sentence, $\boldsymbol{F}$ a predicate frame and $\boldsymbol{F}'$ a generated subframe of $\boldsymbol{F}$. If $\boldsymbol{F} \Vdash A$, then $\boldsymbol{F}' \Vdash A$.

Proof. Immediate from Lemma 3.3.8, Lemma 5.1.2 and Lemma 5.1.4. ■

Lemma 5.3.10. Let L and L' be predicate modal logics with canonical predicate frames $\boldsymbol{F}_{L,\mathcal{S}}$ and $\boldsymbol{F}_{L',\mathcal{S}}$ respectively such that $L \subseteq L'$. Then, $\boldsymbol{F}_{L',\mathcal{S}}$ is a generated subframe of $\boldsymbol{F}_{L,\mathcal{S}}$.

Proof. Let $\boldsymbol{F}_{L,\mathcal{S}} = \langle W_{L,\mathcal{S}}, R_{L,\mathcal{S}}, D_{L,\mathcal{S}} \rangle$, $\boldsymbol{F}_{L',\mathcal{S}} = \langle W_{L',\mathcal{S}}, R_{L',\mathcal{S}}, D_{L',\mathcal{S}} \rangle$ and let $\Gamma \in W_{L',\mathcal{S}}$. We show that $\Gamma \in W_{L,\mathcal{S}}$ (and thus $\boldsymbol{F}_{L,\mathcal{S}}$ is a subframe of $\boldsymbol{F}_{L',\mathcal{S}}$) and that $R_{L,\mathcal{S}}(\Gamma) \subseteq W_{L',\mathcal{S}}$ ($\boldsymbol{F}_{L,\mathcal{S}}$ is generated). By definition of $W_{L',\mathcal{S}}$, Γ is L'-complete, $\mathcal{S}$-small and Henkin. Thus, by Lemma 5.3.8,

Γ is L-complete and $Sen(L') \subseteq \Gamma$. So, by definition of $W_{L,\mathcal{S}}$, it follows that $\Gamma \in W_{L,\mathcal{S}}$. Hence, $\boldsymbol{F}_{L,\mathcal{S}}$ is a subframe of $\boldsymbol{F}_{L',\mathcal{S}}$. To show $R_{L,\mathcal{S}}(\Gamma) \subseteq W_{L',\mathcal{S}}$, suppose further that $\Omega \in R_{L,\mathcal{S}}(\Gamma)$. By $[N]$, $\Box Sen(L') \subseteq Sen(L')$ and therefore, $\Box Sen(L') \subseteq \Gamma$. By definition of $R_{L,\mathcal{S}}$, $\Box^{-}\Gamma \subseteq \Omega$ and so $Sen(L') \subseteq \Omega$. Moreover, since $\Omega \in W_{L,\mathcal{S}}$, Ω is L-complete. Hence, by Lemma 5.3.8, Ω is L'-complete. Ω is Henkin and $\mathcal{S}$-small and so, $\Omega \in W_{L',\mathcal{S}}$. Hence, $R_{L,\mathcal{S}}(\Gamma) \subseteq W_{L',\mathcal{S}}$ and so $\boldsymbol{F}_{L',\mathcal{S}}$ is generated. ■

Lemma 5.3.11. Let L be a canonical predicate modal logic and X a set of canonical formulas. Then, the predicate modal logic $L + X$ is also canonical.

Proof. Let $L' = L + X$. Since $L \subseteq L'$, Since L is canonical, it follows by Lemma 5.3.10 that $\boldsymbol{F}_{L',\mathcal{S}}$ is a generated subframe of $\boldsymbol{F}_{L,\mathcal{S}}$. By Lemma 5.3.9, $\boldsymbol{F}_{L',\mathcal{S}} \Vdash L$. Now, for each $A \in X$, the predicate modal logic $\boldsymbol{Q.K} + A$ is canonical. Since $\boldsymbol{Q.K} + A \subseteq L'$, it follows by Lemma 5.3.10, that $\boldsymbol{F}_{L',\mathcal{S}}$ is a generated subframe of $\boldsymbol{Q.K} + A$, for each $A \in X$. Hence, by Lemma 5.3.9, $\boldsymbol{F}_{L',\mathcal{S}} \Vdash A$, for every $A \in X$. Finally, from Lemma 3.1.3, it follows that $\boldsymbol{F}_{L',\mathcal{S}} \Vdash L'$, i.e. L' is canonical. ■

5.3.1 Canonical Logics

Lemma 5.3.12. All one-way PTC formulas of the form $\Box A \supset \Box^n A$ are canonical.

Proof. Let $A \supset \Box^n A$ be a one-way PTC formula, $L = \boldsymbol{Q.K} + A$ and $\boldsymbol{F}_{L,\mathcal{S}} = \langle W_{L,\mathcal{S}}, R_{L,\mathcal{S}}, D_{L,\mathcal{S}} \rangle$ be a canonical predicate frame for L. We show that $\boldsymbol{F}_{L,\mathcal{S}} \Vdash \Box A \supset \Box^n A$. By Theorem 3.3.14, $\Box A \supset \Box^n A$ is validated by every frame $F = \langle W, R \rangle$ satisfying $R^n \subseteq R$. Then, by Lemma 5.1.2, $F \Vdash \alpha$ iff $\boldsymbol{F} \Vdash \alpha$, for every predicate frame $\boldsymbol{F}$ over F. Thus, it suffices to show that $R^n_{L,\mathcal{S}} \subseteq R_{L,\mathcal{S}}$. Let $\Gamma, \Omega \in W_{L,\mathcal{S}}$. Suppose $\Gamma R^n_{L,\mathcal{S}} \Omega$ and that $\Box A \in \Gamma$. Then, by Lemma 5.2.14 (b), $\Box^n A \in \Gamma$. Since $\Gamma R^n_{L,\mathcal{S}} \Omega$, it follows by definition of $R_{L,\mathcal{S}}$ (n times) that $A \in \Omega$. Thus, by definition of $R_{L,\mathcal{S}}$ again, $\Gamma R_{L,\mathcal{S}} \Omega$. Hence, $R^n_{L,\mathcal{S}} \subseteq R_{L,\mathcal{S}}$. ■

Lemma 5.3.13. Every closed formula is canonical.

Proof. Similar to the proof of Theorem 3.3.15. ■

Theorem 5.3.14. Every predicate one-way PTC logic is canonical and therefore strongly Kripke complete.

Proof. Let X be a set of one-way PTC formulas and let $L = \mathbf{Q.K} + X$ be a predicate modal logic. By Lemmas 5.3.12 and 5.3.13, each member of X is canonical. Thus, by Lemma 5.3.11, $\boldsymbol{F}_{L,\mathcal{S}} \Vdash X$. Hence, L is canonical. ■

Corollary 5.3.15. The logics **Q.K**T, **Q.K**D, **Q.K4**, **Q.K4D** and **Q.S4** are canonical and therefore strongly Kripke complete.

Proof. Each of these logics are axiomatised by one-way PTC formulas. Thus, by Theorem 5.3.14, they are all canonical. Hence, by Lemma 5.3.4, the logics are strongly Kripke complete. ■

5.3.2 Non-Canonical Logics

We now present a few examples of *non-canonical* predicate modal logics. We demonstrate non-canonicity by finding a model M over $\boldsymbol{F}_{L,\mathcal{S}}$ that refutes an axiom of L at some $\Gamma \in W_{L,\mathcal{S}}$.

Proposition 5.3.16. **Q.K**B is not canonical.

Proof. Let $L = \mathbf{Q.K}$B and $\boldsymbol{M}_{L,\mathcal{S}} = \langle \boldsymbol{F}_{L,\mathcal{S}}, V_{L,\mathcal{S}} \rangle$ be a canonical model for L, with $\boldsymbol{F}_{L,\mathcal{S}} = \langle W_{L,\mathcal{S}}, R_{L,\mathcal{S}}, D_{L,\mathcal{S}} \rangle$. We show that $R_{L,\mathcal{S}}$ is not symmetric which, by Theorem 3.1.8, implies that $\boldsymbol{F}_{L,\mathcal{S}} \nVdash L$. Observe that the sentence $\Diamond\top$ is L-consistent since it is satisfiable in any frame with two worlds w, v such that wRv. Therefore, by Lemmas 5.2.17 and 5.2.22, we can extend $\{\Diamond\top\}$ to an $(L,\mathcal{S})$-place Γ. Thus, $\Gamma \in W_{L,\mathcal{S}}$. By Lemma 5.2.33, there exists $\Omega \in R_{L,\mathcal{S}}(\Gamma)$. Now, let $c \in S - \mathcal{C}_\Omega$ and P be a 1-ary predicate letter. Then, by **QCL**, $\vdash_L P(c) \supset P(c)$. Thus, $\Omega \cup \{P(c) \supset P(c)\}$ is L-consistent, and so by Lemma 5.2.17 and Lemma 5.2.22, we can extend $\Omega \cup \{P(c) \supset P(c)\}$ to an $(L,\mathcal{S})$-place $\Delta \in W_{L,\mathcal{S}}$. Since, by definition of $R_{L,\mathcal{S}}$, $\Box^-\Gamma \subseteq \Omega$, it follows that $\Box^-\Gamma \subseteq \Delta$ and so $\Gamma R_{L,\mathcal{S}}\Delta$. But, since $c \notin \mathcal{C}_\Omega$, it follows by Lemma 5.2.30 that $c \notin \mathcal{C}_\Gamma$. Thus, $\mathcal{C}_\Gamma \nsubseteq \mathcal{C}_\Delta$. Hence, by Lemma 5.2.30, it follows that $\Gamma \notin R_{L,\mathcal{S}}(\Delta)$, i.e. $\boldsymbol{F}_{L,\mathcal{S}}$ is not symmetric. ■

Proposition 5.3.17. **Q.S5** is not canonical.

Proof. **Q.S5** can be axiomatised by **Q.K** and the axioms T, B and 4. Since B $\in$ **Q.S5**, by the same argument in the proof of Proposition 5.3.16, it follows that $\boldsymbol{F}_{L,\mathcal{S}} \nVdash$ **Q.S5**. ■

Recall the definition of Alt_n:

$$\mathrm{Alt}_n := \sim \bigwedge_{i\geq 0}^{n} \Diamond(p_i \wedge \bigwedge_{j\neq i}^{n} \sim p_j) \quad (n \geq 1)$$

Proposition 5.3.18. Logics $\boldsymbol{Q.K.Alt}_n$ are not canonical.

Proof. Let $L = \boldsymbol{Q.K.Alt}_n$ and let $\boldsymbol{F}_{L,\mathcal{S}} = \langle W_{L,\mathcal{S}}, R_{L,\mathcal{S}}, D_{L,\mathcal{S}}\rangle$. be a canonical predicate frame for L. The sentence $\Diamond\top$ is L-consistent. Therefore, by Lemmas 5.2.17 and 5.2.22, we can extend $\{\Diamond\top\}$ to an $(L,\mathcal{S})$-place Γ. Thus, $\Gamma \in W_{L,\mathcal{S}}$. By Lemma 5.2.33, there exists $\Omega_0 \in R_{L,\mathcal{S}}(\Gamma)$. Since $\mathcal{C}_{\Omega_0}$ is $\mathcal{S}$-small, there exist distinct constants $c_1, \ldots, c_n \in \mathcal{S} - \mathcal{C}_{\Omega_0}$. Let P be a 1-ary predicate letter. Then, the following is a tautology, $\vdash_L P(c_i) \supset P(c_i)$, for each $1 \leq i \leq n$. Let

$$\Delta_i := \Omega_0 \cup \{P(c_i) \supset P(c_i)\}.$$

Then, since Ω_0 is L-consistent, it follows that each Δ_i is L-consistent, for $1 \leq i \leq n$. Let

$$\mathcal{S}_i := \mathcal{S} - \{c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n\}.$$

So, $c_i \in S_j$ iff $i = j$, for $i \leq i \leq n$. By Lemmas 5.2.17 and 5.2.22, each Δ_i, for $1 \leq i \leq n$, can be extended to an $(L,\mathcal{S}_i)$-place, say Ω_i. Since $\mathcal{S}_i \subseteq \mathcal{S}$ and $\{c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n\}$ is finite, Ω_i is also an $(L,\mathcal{S})$-place. It follows that $c_i \in \Omega_j$ iff $i = j$ and, therefore, $P(c_i) \supset P(c_i) \in \Omega_j$ iff $i = j$, meaning each Ω_i is distinct. Thus, $\Omega_i = \Omega_j$ iff $i = j$. By definition of $R_{L,\mathcal{S}}$, $\Box^-\Gamma \subseteq \Omega_0$ and so, since $\Omega_0 \subseteq \Omega_i$, it follows that $\Box^-\Gamma \subseteq \Omega_i$, for $1 \leq i \leq n$. Hence, both $\Omega_0 \in R_{L,\mathcal{S}}(\Gamma)$ and $\Omega_i \in R_{L,\mathcal{S}}(\Gamma)$, for $1 \leq i \leq n$. Thus, $|R_{L,\mathcal{S}}(\Gamma) \geq n+1|$. By Remark 5.3.26, this means $\boldsymbol{F}_{L,\mathcal{S}} \nVdash \mathrm{Alt}_n$. Hence, by Lemma 5.1.5, $\boldsymbol{F}_{L,\mathcal{S}} \nVdash L$, i.e. L is not canonical. ■

5.3.3 Quasi-Canonical Logics

As shown in the previous section, certain predicate extensions of canonical propositional modal logics are rendered non-canonical. This is due to the introduction of world domains, as per predicate semantics, together with the condition of expanding domains.

In this section, we remedy this deficiency by extracting, according to some criteria (e.g. constant domains), a subframe of the canonical predicate frame of the (non-canonical) logic that validates the logic. This is achieved by (recursively) selecting worlds from the canonical predicate frame. The method, termed *selection*, is discussed more in [19] and [14], and is similar to selective filtration in Kripke semantics for propositional modal logic. In the absence of

canonicity, certain logics still possess the weaker condition of quasi-canonicity (when a logic is validated by a subframe of a canonical predicate frame of the logic), whereby we may prove strong Kripke completeness.

Definition 5.3.19. A model $M' = \langle W', R', D', V' \rangle$ is a *weak* submodel of a model $M = \langle W, R, D, V \rangle$ if $W' \subseteq W$, $R' \subseteq R$ and for every $w \in W'$, $D'_w = D_w$ and $V'(P, w) = V(P, w)$, for every predicate letter P. Furthermore, if for every $w \in W'$ and every D_w-sentence A,

$$\text{If } M, w \Vdash \Diamond A \text{ then there exists } v \in R'(w) \text{ such that } M, v \Vdash A,$$

then M' is a *selective weak* submodel of M.

Theorem 5.3.20. Let $M = \langle W, R, D, V \rangle$ be a predicate model and let $M' = \langle W', R', D', V' \rangle$ be a selective weak submodel of M and $w \in W'$. Then, for every $A \in Sen(\mathcal{C}_w)$,

$$M, w \Vdash A \text{ iff } M', w \Vdash A.$$

Proof. We prove the theorem by induction on the complexity of a sentence A. By $[V\bot]$, the theorem clearly holds for $\bot$. Furthermore, since $V'(P, w) = V(P, w)$, for every predicate letter P, it immediately follows that for any n-ary predicate letter P and constants $c_1, \ldots, c_n$,

$$M, w \Vdash P^n(c_1, \ldots, c_n) \text{ iff } M', w \Vdash P^n(c_1, \ldots, c_n).$$

($\supset$) Let $A = B \supset C$ be a D_w-sentence. Then, $M, w \Vdash B \supset C$ iff, by $[V\supset]$, $M, w \nVdash B$ or $M, w \Vdash C$ iff, by induction hypothesis, $M', w \nVdash B$ or $M', w \Vdash C$.

($\forall$) Let $A = \forall x B(x)$. Since M' is a weak submodel of M, $D'_w = D_w$. Then, $M, w \Vdash \forall x B(x)$ iff, by $[V\forall]$, $M, w \Vdash B(\underline{d})$ for every $d \in D_w$ iff, by induction hypothesis, $M', w \Vdash B(\underline{d})$ for every $d \in D_w$ iff, by $[V\forall]$, $M', w \Vdash \forall x B(x)$.

($\Box$) Let $A = \Box B$. Suppose $M, w \nVdash \Box B$. Since M' is a weak selective submodel of M, there exists $v \in R'(w)$ such that $M, v \nVdash B$. By induction hypothesis, $M', v \nVdash B$. By $[V\Box]$, it follows that $M', w \nVdash \Box B$. Conversely, suppose $M', w \nVdash \Box B$. By $[V\Box]$, there exists $v \in R'(w)$ such that $M', v \nVdash B$. So, by induction hypothesis, $M, v \nVdash B$. Since M' is a weak submodel of M, $R' \subseteq R$ and so, $v \in R(w)$. Hence, by $[V\Box]$, it follows that $M, w \nVdash \Box B$.

■

Definition 5.3.21. A *quasi-canonical model* for a predicate modal logic L is a selective weak submodel of a canonical model for L.

Theorem 5.3.22. Let L be a predicate modal logic and let $M' = \langle W', R', D', V' \rangle$ be a quasi-canonical model for L. Then, for every $\Gamma \in W'$,

$$M', \Gamma \Vdash A \text{ iff } A \in \Gamma.$$

Proof. By Theorem 3.2.15, it follows that $M_{L,\mathcal{S}}, \Gamma \Vdash A$ iff $A \in \Gamma$. Then by Lemma 5.3.20, $M_{L,\mathcal{S}}, \Gamma \Vdash A$ iff $M', \Gamma \Vdash A$. Thus, $M', \Gamma \Vdash A$ iff $A \in \Gamma$. ■

Definition 5.3.23. A predicate modal logic L is *quasi-canonical* if, for every $(L, \mathcal{S})$-place Γ, there exists a quasi-canonical model over a frame $\boldsymbol{F} = \langle W, R, D \rangle$ such that $\boldsymbol{F} \Vdash L$ and $\Gamma \in W$.

Theorem 5.3.24. Let L be a quasi-canonical predicate modal logic. Then, L is strongly complete.

Proof. Let Ω be an L-consistent modal theory. Let $\mathcal{S}$ be large enough for $\mathcal{C}_\Omega$ to be $\mathcal{S}$-small. By Lemma 5.2.28, we can extend Ω to an $(L, \mathcal{S})$-place Γ. Since L is quasi-canonical, there exists a quasi-canonical model $M = \langle \boldsymbol{F}, V \rangle$ over a frame $\boldsymbol{F} = \langle W, R, D \rangle$ such that $\boldsymbol{F} \Vdash L$ and $\Gamma \in W$. Let $A \in \Gamma$. By Theorem 5.3.22, $M, \Gamma \Vdash A$. Now, let Ψ be an identity map on $\mathcal{C}_\Omega$. Then, $M, \Gamma \Vdash \Psi(A)$. Thus, Ω is satisfied in a model over a frame validating L. Hence, L is strongly Kripke complete. ■

Recall the following class of Kripke frames that determines $\mathbf{K.Alt}_n$:

$$C_{alt_n} := \{\langle W, R \rangle \mid \forall w(\, |R(w)| \leq n)\}.$$

We show that the logics $\boldsymbol{Q.K.Alt}_n$ are quasi-canonical, and complete w.r.t $\boldsymbol{C}_{alt_n}$, by extracting a quasi-canonical predicate frame validating $\boldsymbol{Q.K.Alt}_n$ from a canonical predicate frame for $\boldsymbol{Q.K.Alt}_n$. To achieve this, we will need the following lemma.

Lemma 5.3.25. Let $L = \boldsymbol{Q.K.Alt}_n$ and let $M_{L,\mathcal{S}} = \langle W_{L,\mathcal{S}}, R_{L,\mathcal{S}}, D_{L,\mathcal{S}}, V_{L,\mathcal{S}} \rangle$ be a canonical model for L. Then, for any $\Gamma \in W_{L,\mathcal{S}}$, let

$$\Lambda := \{\Omega \mid \Omega \text{ is an } L\text{-complete theory } \& \, \Box^- \Gamma \subseteq \Omega \, \& \, \mathcal{C}_\Omega = \mathcal{C}_\Gamma\}.$$

Then, $|\Lambda| \leq n$.

Proof. Suppose there are distinct $\Omega_0, \ldots, \Omega_n \in \Lambda$. Since each Ω_i is L-complete and $\mathcal{C}_{\Omega_i} = \mathcal{C}_\Gamma$, it follows that for $0 \leq i, j \leq n$ $(i \neq j)$, there exists $A_{ij} \in Sen(\mathcal{C}_\Gamma)$ such that $A_{ij} \notin \Omega_i$, but $A_{ij} \in \Omega_j$. For $0 \leq i \leq n$, define

$$B_i := \bigwedge_{j \neq i}^{n} (A_{ji} \wedge \sim A_{ij}).$$

Then, $B_i \in \Omega_j$ iff $i = j$. Hence,

$$\bigwedge_{i \geq 0}^{n} \Diamond (B_i \wedge \bigwedge_{j \neq i}^{n} \sim B_j) \in \Gamma$$

Since L is closed under predicate substitutions, the above contradicts a predicate substitution instance of Alt_n (the propositional wff), contradicting the L-consistency of Γ. ■

Remark 5.3.26. Note that the argument in Proposition 3.3.25 demonstrates that if $M = \langle F, V \rangle$ is a model validating $\boldsymbol{K}.\mathbf{Alt}_n$ based on a frame F, then $F \in C_{altn}$.

Theorem 5.3.27. Every logic $\boldsymbol{Q}.\boldsymbol{K}.\mathbf{Alt}_n$ is quasi-canonical and therefore strongly Kripke complete.

Proof. Let $L = \boldsymbol{Q}.\boldsymbol{K}.\mathbf{Alt}_n$ and let $M_{L,\mathcal{S}} = \langle W_{L,\mathcal{S}}, R_{L,\mathcal{S}}, D_{L,\mathcal{S}}, V_{L,\mathcal{S}} \rangle$ be a canonical model for L. Let $\Gamma_0 \in W_{L,\mathcal{S}}$. We extract a selective weak submodel $M = \langle \boldsymbol{F}, V \rangle$ of $M_{L,\mathcal{S}}$ over a frame $\boldsymbol{F} = \langle W, R, D \rangle$ such that $\Gamma_0 \in W$ and $\boldsymbol{F} \Vdash L$, defining W and R recursively.
Let $W_0 = \emptyset$, $W_1 = \{\Gamma_0\}$, and $R_0 = R_1 = \emptyset$. Before we define W_{k+1} and R_{k+1}, we consider for each $\Gamma \in W_k \backslash W_{k-1}$ the following set:

$$\Lambda_\Gamma := \{\Omega \mid \Omega \text{ is an } L\text{-complete theory } \& \Box^- \Gamma \subseteq \Omega \,\&\, \mathcal{C}_\Omega = \mathcal{C}_\Gamma\}$$

By Lemma 5.3.25, $|\Lambda_\Gamma| \leq n$. Furthermore, by Lemma 5.2.17, there exists for each $\Omega \in \Lambda_\Gamma$ a set $\Omega' \in W_{L,\mathcal{S}}$ such that $\Omega \subseteq \Omega'$. Let Π_Γ denote the set containing exactly one $\Omega' \in W_{L,\mathcal{S}}$ for each $\Omega \in \Lambda_\Gamma$. Clearly, $|\Pi_\Gamma| \leq n$.

(1) Now, for any $A \in Sen(\mathcal{C}_\Gamma)$, if $\Diamond A \in \Gamma$, then by Lemma 5.2.32 and the definition of Λ_Γ, there exists $\Omega \in \Lambda_\Gamma$ such that $A \in \Omega$. And so, by the definition of Π_Γ, there exists $\Omega' \in \Pi_\Gamma$ such that $A \in \Omega'$. Now, let

$$W_{k+1} = W_k \cup \bigcup_{\Gamma \in W_k \backslash W_{k-1}} \Pi_\Gamma$$
$$R_{k+1} = R_k \cup \bigcup_{\Gamma \in W_k \backslash W_{k-1}} (\{\Gamma\} \times \Pi_\Gamma)$$

Since $|\Pi_\Gamma| \leq n$, for every $\Gamma \in W_k \backslash W_{k-1}$, it follows that $|R_{k+1}(\Gamma)| \leq n$, for every such Γ. Finally, we let

$$W = \bigcup_{k \geq 0} W_k \text{ and } R = \bigcup_{k \geq 0} R_k.$$

By (1), for every $\Gamma \in W$ and for every $A \in Sen(\mathcal{C}_\Gamma)$, (2) if $\Diamond A \in \Gamma$, then there exists $\Omega \in R(\Gamma)$ such that $A \in \Omega$. Observe that $|R(\Gamma)| \leq n$, for every $\Gamma \in W$, and that $R \subseteq R_{L,\mathcal{S}}$. Thus, by Theorem 3.3.23, $\langle W, R \rangle \Vdash \mathbf{Alt}_n$. Now, let $M = \langle W, R, D, V \rangle$ to be $M_{L,\mathcal{S}}$ restricted to W. Thus, by (2) and Lemma 5.2.35, M is a selective weak submodel of $\boldsymbol{M}_{L,\mathcal{S}}$. Let $\boldsymbol{F} = \langle W, R, D \rangle$. By Lemma 5.1.2, $\boldsymbol{F} \Vdash \mathbf{Alt}_n$ and, furthermore, by Lemma 5.1.5, $\boldsymbol{F} \Vdash L$. Since $\boldsymbol{F} \in \boldsymbol{C}_{alt_n}$, L is complete w.r.t $\boldsymbol{C}_{alt_n}$. And, from Theorem 5.3.24, L is strongly Kripke complete. ■

5.4 Constant Domain Frames

In showing the non-canonicity of logics $\boldsymbol{Q}.\boldsymbol{K}\mathrm{B}$ and $\boldsymbol{Q}.\boldsymbol{S5}$, we refuted an axiom in the canonical model by exploiting the fact that for any $\Gamma \in W_{L,\mathcal{S}}$, we can find $\Omega \in R(\Gamma)$ such that $\mathcal{C}_\Gamma \subset \mathcal{C}_\Omega$. While the canonical predicate frames for these logics do not have constant domains, the logics are indeed valid on constant domain frames [8]. Thus, it seems sensible to somehow restrict the canonical predicate frame to worlds which satisfy $\mathcal{C}_\Gamma = \mathcal{C}_\Omega$ whenever $\Gamma R \Omega$. Or, more generally, to restrict the canonical predicate frame to worlds Γ such that $\mathcal{C}_\Gamma = \mathcal{C}_*$, for some fixed countable set of constants $\mathcal{C}_*$ (see [10, Chapter 7] for more discussion).

Definition 5.4.1. If $\mathcal{C}_*$ is a set of constants, then a *constant domain predicate frame* is a predicate frame $\boldsymbol{F}_{\mathcal{C}_*} = \langle W_{\mathcal{C}_*}, R_{\mathcal{C}_*}, D_{\mathcal{C}_*} \rangle$ such that $D_{\mathcal{C}_*}(w) = \mathcal{C}_*$, for every $w \in W$.

We can extract a subframe with constant domains from any predicate frame $\boldsymbol{F} = \langle W, R, D \rangle$ as follows:

Let $w \in W$ and let $\mathcal{C}_* := D_w$. Then, we define

$$W_{\mathcal{C}_*} := \{v \in W \mid \mathcal{C}_v = \mathcal{C}_*\},$$
$$R_{\mathcal{C}_*} := R \cap (W_{\mathcal{C}_*} \times W_{\mathcal{C}_*}),$$
$$D_{\mathcal{C}_*} := \{\langle v, \mathcal{C}_v \rangle \mid v \in W_{\mathcal{C}_*}\}.$$

Definition 5.4.2. If L be a predicate modal logic, $\boldsymbol{F}_{L,\mathcal{S}}$ is the canonical predicate frame for L and $\mathcal{C}_*$ is a countable set of constants, then we define a *constant domain canonical predicate frame* for L to be

$$\boldsymbol{F}_{L,\mathcal{C}_*} := \langle W_{L,\mathcal{C}_*}, R_{L,\mathcal{C}_*}, D_{L,\mathcal{C}_*} \rangle,$$

obtained from $\boldsymbol{F}_{L,\mathcal{S}}$ by the selection process described above.

Before we can be sure that applying a valuation to $\langle W_{L,\mathcal{C}_*}, R_{L,\mathcal{C}_*}, D_{L,\mathcal{C}_*} \rangle$ results in a predicate Kripke model, we need to ensure that an analogue of Lemma 5.2.33 (Existence Lemma) is preserved by the selection process in obtaining $W_{L,\mathcal{C}_*}$ and $R_{L,\mathcal{C}_*}$.

Lemma 5.4.3. Let L be a predicate modal logic containing BF and Γ an L-complete Henkin theory. If $\Diamond A \in \Gamma$, then there exists an L-complete Henkin theory Ω such that $\Box^{-}\Gamma \subseteq \Omega$, $\mathcal{C}_\Omega = \mathcal{C}_\Gamma$ and $A \in \Omega$.

Proof. Let $\exists x_1 A_1(x_1)$, $\exists x_2 A_2(x_2)$, $\ldots$ be an enumeration of $\mathcal{C}_\Gamma$-sentences of the form $\exists x A(x)$. Then, we construct a sequence of finite modal theories Γ_1, Γ_2, $\ldots$ such that for each $i > 0$,

$$\Diamond(\bigwedge \Gamma_i) \in \Gamma.$$

Let $\Gamma_0 := \{A\}$. Then, we show that we may obtain each Γ_i, $i \leq \omega$ by

$$\Gamma_i := \Gamma_{i-1} \cup \{\exists x A_i(x) \supset A_i(c)\} \text{ for some } c \in \mathcal{C}_\Gamma \text{ such that } \Diamond(\bigwedge \Gamma_i) \in \Gamma.$$

We first show that such a c exists for Γ_i. By inductive hypothesis, (1) $\Diamond(\bigwedge \Gamma_{i-1}) \in \Gamma$. Now, by **QCL**, $\vdash_L \exists y(\exists x A_i(x) \supset A_i(y))$, for a variable y not occurring in Γ_{i-1}. Moreover, by $[N]$, (2) $\vdash_L \Box\exists y(\exists x A_i(x) \supset A_i(y))$. From (1) and (2), it follows that

$$\Diamond\exists y(\bigwedge \Gamma_{i-1} \wedge (\exists x A_i(x) \supset A_i(y))) \in \Gamma.$$

Now, recall BF: $\forall x \Box A(x) \supset \Box \forall x A(x)$. This is logically equivalent to

$$\Diamond\exists x A(x) \supset \exists x \Diamond A(x).$$

Thus, by BF,

$$\exists y \Diamond(\bigwedge \Gamma_{i-1} \wedge (\exists x A_i(x) \supset A_i(y))) \in \Gamma$$

Now, since Γ is Henkin, there exists $c \in \mathcal{C}_\Gamma$ such that

$$\Diamond(\bigwedge \Gamma_{i-1} \wedge (\exists x A_i(x) \supset A_i(c))) \in \Gamma.$$

Let $\Gamma_\omega := \bigcup_{i<\omega} \Gamma_i$. Now, we show that $\Box^- \Gamma \cup \Gamma_\omega$ is L-consistent. Suppose otherwise, i.e. $\Box^- \Gamma \cup \Gamma_\omega \vdash_L \bot$. By Lemma 5.2.8 and **QCL**, it follows that $\Box^- \Gamma \vdash_L \sim(\bigwedge \Gamma_i)$ for some $i < \omega$. But, by Lemma 5.2.10, $\Box\Box^- \Gamma \vdash_L \Box\sim(\bigwedge \Gamma_i)$ and so $\Box\Box^- \Gamma \vdash_L \sim\Diamond(\bigwedge \Gamma_i)$. Then, since $\Box\Box^- \Gamma \subseteq \Gamma$, it follows that $\Gamma \vdash_L \sim\Diamond(\bigwedge \Gamma_i)$, contradicting the L-consistency of Γ since $\Diamond(\bigwedge \Gamma_i) \in \Gamma$. By construction, $\Box^- \Gamma \cup \Gamma_\omega$ is Henkin and therefore, by Lemma 5.2.17, there exists an L-complete Henkin theory Ω such that $\Box^- \Gamma \cup \Gamma_\omega \subseteq \Omega$. Lastly, since $A \in \Gamma_\omega$, this means $\Box^- \Gamma \cup \{A\} \subseteq \Omega$. ■

Definition 5.4.4. If L is a predicate modal logic and $\boldsymbol{F}_{L,\mathcal{C}_*} = \langle W_{L,\mathcal{C}_*}, R_{L,\mathcal{C}_*}, D_{L,\mathcal{C}_*} \rangle$ is a constant domain canonical predicate frame for L, then we define the *constant domain canonical model* over $\boldsymbol{F}_{L,\mathcal{C}_*}$ to be

$$\boldsymbol{M}_{L,\mathcal{C}_*} := \langle \boldsymbol{F}_{L,\mathcal{C}_*}, V_{L,\mathcal{C}_*} \rangle,$$

where $V_{L,\mathcal{C}_*} := V_{L,\mathcal{S}} \restriction W_{L,\mathcal{C}_*}$.

Lemma 5.4.5. Let L be a predicate modal logic and $\boldsymbol{M}_{L,\mathcal{C}_*} = \langle W_{L,\mathcal{C}_*}, R_{L,\mathcal{C}_*}, D_{L,\mathcal{C}_*}, V_{L,\mathcal{C}_*} \rangle$ a constant domain canonical model for L. Then, $\boldsymbol{M}_{L,\mathcal{C}_*}$ is a selective weak submodel of $\boldsymbol{M}_{L,\mathcal{S}} = \langle W_{L,\mathcal{S}}, R_{L,\mathcal{S}}, D_{L,\mathcal{S}}, V_{L,\mathcal{S}} \rangle$.

Proof. The definition of $\boldsymbol{F}_{L,\mathcal{C}_*}$ requires $W_{L,\mathcal{C}_*} \subseteq W_{L,\mathcal{S}}, R_{L,\mathcal{C}_*} \subseteq R_{L,\mathcal{S}}, D_{L,\mathcal{C}_*}(\Gamma) = D_{L,\mathcal{C}_*}(\Gamma)$ for every $\Gamma \in W_{L,\mathcal{C}_*}$ and $V_{L,\mathcal{C}_*}(P,\Gamma) = V_{L,\mathcal{S}}(P,\Gamma)$ for every predicate letter P. Thus, $\boldsymbol{M}_{L,\mathcal{C}_*}$ is a weak submodel of $\boldsymbol{M}_{L,\mathcal{S}}$. We now show that $\boldsymbol{M}_{L,\mathcal{C}_*}$ is a selective weak submodel of $\boldsymbol{M}_{L,\mathcal{S}}$. Let $\Gamma \in W_{L,\mathcal{C}_*}$ and suppose $\Diamond A \in \Gamma$. By Theorem 5.2.35, $\boldsymbol{M}_{L,\mathcal{S}}, \Gamma \Vdash A$. Since Γ is L-complete and Henkin, it follows by Lemma 5.4.3 that there exists an L-complete Henkin theory Ω such that $\Box^- \Gamma \subseteq \Omega$, $\mathcal{C}_\Omega = \mathcal{C}_\Gamma$, i.e. $\mathcal{C}_\Omega = \mathcal{C}_*$, and $A \in \Omega$. Since $\mathcal{C}_\Omega = \mathcal{C}_*$, Ω is S-small and, therefore, an $(L,\mathcal{S})$-place. Thus, $\Omega \in W_{L,\mathcal{S}}$. By Theorem 5.2.35, $\boldsymbol{M}_{L,\mathcal{S}}, \Omega \Vdash A$. Moreover, since $\Box^- \Gamma \subseteq \Omega$, it follows that $\Omega \in R_{L,\mathcal{S}}(\Gamma)$. Finally, by the definitions of $W_{L,\mathcal{C}_*}$ and $R_{L,\mathcal{C}_*}$, it follows that $\Omega \in W_{L,\mathcal{C}_*}$ and $\Omega \in R_{L,\mathcal{C}_*}(\Gamma)$. ■

Theorem 5.4.6. ***Q.K*B** is quasi-canonical.

Proof. Let $L =$ ***Q.K*B**, $\boldsymbol{F}_{L,\mathcal{C}_*}$ a constant domain canonical predicate frame of L. We show that $\boldsymbol{F}_{L,\mathcal{C}_*}$ is symmetric.

Let $\Gamma \in W_{L,\mathcal{C}_*}$, $A \in \Gamma$ and suppose there exists $\Omega \in R_{L,\mathcal{C}_*}(\Gamma)$. Since $\vdash_L A \supset \Box\Diamond A$, it follows that $\Gamma \vdash_L A \supset \Box\Diamond A$. Thus, by Theorem 5.2.14 (a), $A \supset \Box\Diamond A \in \Gamma$. Then, by Lemma 5.2.14 (b), $\Box\Diamond A \in \Gamma$. Thus, $\Diamond A \in \Omega$. Since A is arbitrary, it follows that $\Diamond\Gamma \cap Sen(\mathcal{C}_\Omega) \subseteq \Omega$. Thus, by Lemma 5.2.30 and $\mathcal{C}_\Gamma = \mathcal{C}_\Omega$, it follows that $\Omega R_{L,\mathcal{C}_*}\Gamma$. Since Γ and Ω are arbitrary, $\boldsymbol{F}_{L,\mathcal{C}_*}$ is symmetric. So, $\boldsymbol{Q.K}$B is complete w.r.t the class of all symmetric frames. By Corollary 3.1.7, $\boldsymbol{F}_{L,\mathcal{C}_*} \Vdash$KB. Thus, by Lemma 5.1.5, $\boldsymbol{F}_{L,\mathcal{S}} \Vdash L$. Hence, L is quasi-canonical. ■

Lemma 5.4.7. Let $\Box^n A \supset A$ be a one-way PTC formula, $L = \boldsymbol{Q.K} + A$, $\boldsymbol{F}$ a predicate frame such that $\boldsymbol{F} \Vdash L$ and $\boldsymbol{F}_{\mathcal{C}_*} = \langle W_{\mathcal{C}_*}, R_{\mathcal{C}_*}, D_{\mathcal{C}_*}\rangle$ a constant domain predicate frame. Then, $\boldsymbol{F}_{\mathcal{C}_*} \Vdash L$.

Proof. In light of Lemmas 3.3.14 and 5.1.2, it suffices to show that $R^n_{\mathcal{C}_*} \subseteq R_{\mathcal{C}_*}$. Let $\boldsymbol{F} = \langle W, R, D\rangle$ be a canonical predicate frame of L. By Lemma 5.3.14, $R^n \subseteq R$. We show that the selection by $R_{\mathcal{C}_*}$ preserves this inclusion. Let $w \in W_{\mathcal{C}_*}$ and suppose there exists $v \in R^n_{\mathcal{C}_*}(\Gamma)$. Then, surely, $v \in R^n(w)$, and there exist $u_1, \ldots, u_{n-1} \in W_{\mathcal{C}_*}$ such that $wR_{\mathcal{C}_*}u_1, \ldots, u_{n-1}R_{\mathcal{C}_*}v$. By the selection of worlds by $R_{\mathcal{C}_*}$,

$$\mathcal{C}_w = \mathcal{C}_{u_1} = \cdots = \mathcal{C}_{u_{n-1}} = \mathcal{C}_v = \mathcal{C}_*.$$

Moreover, since $R^n \subseteq R$, it follows that $v \in R(w)$. Now, since $\mathcal{C}_w = \mathcal{C}_v = \mathcal{C}_*$, it follows that $v \in R_{\mathcal{C}_*}(w)$. Hence, $R^n_{\mathcal{C}_*} \subseteq R_{\mathcal{C}_*}$. ■

Theorem 5.4.8. $\boldsymbol{Q.K}$TB is quasi-canonical.

Proof. Let $L = \boldsymbol{Q.K}$TB and let $\boldsymbol{F}_{L,\mathcal{C}_*}$ be a constant domain canonical predicate frame of L. By the same argument in Theorem 5.4.6, $\boldsymbol{F}_{L,\mathcal{C}_*} \Vdash \boldsymbol{Q.K}$B. Since $\boldsymbol{Q.K}$T $\subseteq L$, it follows by Lemma 5.3.10 that $\boldsymbol{F}_{L,\mathcal{S}}$ is a generated subframe of $\boldsymbol{F}_{\boldsymbol{Q.K}\text{T},\mathcal{S}}$. By Lemma 5.3.14, $\boldsymbol{Q.K}$T is canonical and so, by Lemma 5.3.9, $\boldsymbol{F}_{L,\mathcal{C}_*} \Vdash \boldsymbol{Q.K}$T. Furthermore, axiom T is one-way PTC and so, by Lemma 5.4.7, $\boldsymbol{F}_{L,\mathcal{C}_*} \Vdash \boldsymbol{Q.K}$T. Hence, by Lemma 5.1.4, $\boldsymbol{F}_{L,\mathcal{C}_*} \Vdash L$. ■

Theorem 5.4.9. $\boldsymbol{Q.S5}$ is quasi-canonical.

Proof. Let $L = \boldsymbol{Q.S5}$ and let $\boldsymbol{F}_{L,\mathcal{C}_*}$ be a constant domain canonical predicate frame of L. Recall that $\boldsymbol{S5}$ can be axiomatised by the axioms 4, T and B. By similar arguments in the proof of Theorem 5.4.8, $\boldsymbol{F}_{L,\mathcal{C}_*} \Vdash \boldsymbol{Q.K}$TB. Since $\boldsymbol{Q.K4} \subseteq L$, it follows by Lemma 5.3.10 that $\boldsymbol{F}_{L,\mathcal{S}}$ is a generated subframe of $\boldsymbol{F}_{\boldsymbol{Q.K4},\mathcal{S}}$. By Lemma 5.3.14, $\boldsymbol{Q.K4}$ is canonical and so, by Lemma 5.3.9,

$\boldsymbol{F}_{L,\mathcal{S}} \Vdash \boldsymbol{Q.K4}$. Furthermore, both axioms T and 4 are one-way PTC and so, by Lemma 5.4.7, $\boldsymbol{F}_{L,\mathcal{C}_*} \Vdash \boldsymbol{Q.K4}$. Hence, by Lemma 5.1.4, $\boldsymbol{F}_{L,\mathcal{C}_*} \Vdash L$. ■

Corollary 5.4.10. $\boldsymbol{Q.K}$B, $\boldsymbol{Q.K}$TB and $\boldsymbol{Q.S5}$ are strongly complete.

Proof. By Theorems 5.4.6, 5.4.8 and 5.4.9, these logics are quasi-canonical. Hence, by Theorem 5.3.24, they are strongly complete. ■

5.4.1 General Predicate Kripke Frames

In this section, we demonstrate how the concepts and results in Sections 2.3.1 and 2.3.2 yield a general result (Theorem 5.4.16) in Kripke completeness for predicate modal logics w.r.t constant domain frames due to Y. Tanaka and H. Ono, as proved in [10, Theorem 7.4.7 on p. 590]. Before we can apply these results, we need to fill in some gaps.

Lemma 5.4.11. If L is a predicate modal logic containing BF and $\langle W_{L,\mathcal{C}_*}, R_{L,\mathcal{C}_*}, D_{L,\mathcal{C}_*}, V_{L,\mathcal{C}_*}\rangle$ is a constant domain canonical model for L, then the following is a general Kripke frame:

$$\mathcal{F}_{L,\mathcal{C}_*} := \langle W_{L,\mathcal{C}_*}, R_{L,\mathcal{C}_*}, \mathcal{A}_{L,\mathcal{C}_*}\rangle,$$

where $\mathcal{A}_{L,\mathcal{C}_*} := \{V_{L,\mathcal{C}_*}(A) \mid A \in Sen(\mathcal{C}_*)\}$. We call $\mathcal{F}_{L,\mathcal{C}_*}$ a *predicate canonical general frame* of L.

Proof. Since each world has the same set of constants, this proof is similar to that of Proposition 2.3.9. ■

Remark 5.4.12. Lemma 5.4.11 requires the frame $\langle W_{L,\mathcal{C}_*}, R_{L,\mathcal{C}_*}, D_{L,\mathcal{C}_*}\rangle$ to have constant domains since otherwise we would violate the closure conditions of $\mathcal{A}_{L,\mathcal{C}_*}$, the set of admissible sets ($\mathcal{A}_{L,\mathcal{C}_*}$ would not be a modal algebra). To see this, let $\langle W_{L,\mathcal{S}}, R_{L,\mathcal{S}}, D_{L,\mathcal{S}}\rangle$ be a predicate frame, $\Gamma, \Omega \in W_{L,\mathcal{S}}$ and P be a 1-ary predicate letter. Suppose there exists $c \in \mathcal{S}$ such that $c \in \mathcal{C}_\Gamma$ but $c \notin \mathcal{C}_\Omega$. Then, $V_{L,\mathcal{S}}(\sim P(c)) \neq W_{L,\mathcal{S}} - V_{L,\mathcal{S}}(P(c))$.

Lemma 5.4.13. Every predicate canonical general frame is refined.

Proof. Let L be a predicate modal logic containing BF, $\mathcal{F}_{L,\mathcal{C}_*}$ a predicate canonical general frame of L and $\Gamma, \Omega \in W_{L,\mathcal{C}_*}$. Since Γ and Ω are both L-complete, it follows that $\Gamma \neq \Omega$ iff $\Gamma \not\subseteq \Omega$. Hence, by Theorem 5.4.5, $\Gamma \in V_{L,\mathcal{C}_*}(A)$ and $\Omega \notin V_{L,\mathcal{C}_*}(A)$, for some $A \in Sen(\mathcal{C}_*)$. Hence, $\mathcal{F}_{L,\mathcal{C}_*}$ is differentiated. By definition of $R_{L,\mathcal{C}_*}$, if $\Box^-(\Gamma) \subseteq \Omega$, then $\Gamma R_{L,\mathcal{C}_*} \Delta$. Hence, by Theorems 5.4.5 and 5.3.20, it follows that

$$\text{if } \Gamma \Vdash \Box A \text{ implies } \Delta \Vdash A, \text{ for every } A \in Sen(\mathcal{C}_*), \text{ then } \Gamma R_{L,\mathcal{C}_*} \Delta.$$

Hence, $\mathcal{F}_{L,\mathcal{C}_*}$ is tight. Thus, by definition, $\mathcal{F}_{L,\mathcal{C}_*}$ is refined. ■

Lemma 5.4.14. If L is a predicate modal logic, then every predicate canonical general frame for L validates every propositional wff in L.

Proof. Let $\mathcal{F}_{L,\mathcal{C}_*} = \langle W_{L,\mathcal{C}_*}, R_{L,\mathcal{C}_*}, \mathcal{A}_{L,\mathcal{C}_*} \rangle$ be a predicate canonical general frame for L. The argument is similar to that in the proof of Theorem 3.3.27. Let $\alpha \in L$ be a propositional modal wff and let M be an admissible model in $\mathcal{F}_{L,\mathcal{C}_*}$ with valuation V. Suppose $PL(\alpha) = \{p_1, \ldots, p_n\}$. Let $\gamma_1, \ldots, \gamma_n$ be propositional wffs such that $V(p_i) = V_{L,\mathcal{C}_*}(\gamma_i)$, for each $1 \leq i \leq n$. Let $\beta = [\gamma_1, \ldots, \gamma_n / p_1, \ldots, p_n]\, \alpha$. Then, $\beta \in L$. Hence. by Theorem 3.3.29, $V(\alpha) = V_{L,\mathcal{C}_*}(\beta) = W_{L,\mathcal{C}_*}$. ■

Lemma 5.4.15. If S is an r-persistent propositional modal logic, then $\boldsymbol{Q.S}$ + BF is quasi-canonical.

Proof. Let $L = \boldsymbol{Q.S}$ + BF and let $\mathcal{F}_{L,\mathcal{C}_*} = \langle W_{L,\mathcal{C}_*}, R_{L,\mathcal{C}_*}, \mathcal{A}_{L,\mathcal{C}_*} \rangle$ be a predicate canonical general frame for L. By Lemma 5.4.14, $\mathcal{F}_{L,\mathcal{C}_*} \Vdash S$. Then, by Lemma 5.4.13, $\mathcal{F}_{L,\mathcal{C}_*}$ is refined. Since S is r-persistent, $\langle W_{L,\mathcal{C}_*}, R_{L,\mathcal{C}_*} \rangle \Vdash S$ and, therefore, by Lemma 5.1.5, $\boldsymbol{F}_{L,\mathcal{C}_*} \Vdash \boldsymbol{Q.S}$. Furthermore, by Theorem 4.3.10, $\boldsymbol{F}_{L,\mathcal{C}_*} \Vdash$ BF. Hence, by Lemma 5.1.4, $\boldsymbol{F}_{L,\mathcal{C}_*} \Vdash L$. ■

At this point, we can now present the culmination of these definitions and lemmas:

Theorem 5.4.16. If S is a universal propositional modal logic, then $\boldsymbol{Q.S}$ + BF is quasi-canonical and therefore strongly Kripke complete.

Proof. By Theorem 2.3.23, S is r-persistent. Hence, by Lemma 5.4.15, $\boldsymbol{Q.S}$ + BF is quasi-canonical. ■

Chapter 6

Closing Remarks and Future Work

What we've shown and surveyed in this paper can be summarised in the following list: syntax and Kripke semantics for propositional modal logic; the property of canonicity and its utility in proving Kripke completeness of propositional modal logics; a natural semantic extension of propositional Kripke frames and models to predicate Kripke frames and models; how these semantics break completeness results for predicate extensions to propositional modal logics; how we can remedy this deficiency by extracting subframes/submodels possessing useful properties to prove quasi-canonicity for a logic.

The number of results in the area of Kripke completeness for predicate modal logic are considerably outweighed by the ocean of results in Kripke completeness for propositional modal logics. One reason for this is the added technical difficulty that is introduced with varying world domains, even if we further require expanding domains. So far, there are a few general results, and none whose coverage compares to that of the Sahlqvist theorem (Theorem 3.3.36) in propositional modal logic. The result that comes closest to this degree of generality is the Tanaka-Ono theorem (Theorem 5.4.16), which requires constant domains to utilise general Kripke frames. Even with constant domains, general Kripke frames based on predicate Kripke models are not descriptive (they are not compact [10]) and thus, there is not yet a known analogue of the Sahlqvist theorem for predicate modal logics.

For the more idiosyncratic cases in predicate modal logics, such as $\boldsymbol{Q.K.Alt}_n$, whose propositional counterparts may even be Kripke complete in the propositional Kripke semantics,

demonstrating Kripke completeness has so far required that we manipulate the logic's canonical frame to yield a useful subframe. It is not yet known if there exists a more general technique which accommodates the extra demand with varying world domains. It is also not known whether there are canonical predicate modal logics whose domains are $\mathcal{S}$-small where $\mathcal{S}$ is countable.

Bibliography

[1] J. F. A. K. van Benthem. "Modal Reduction Principles". In: *Journal of Symbolic Logic* 41.2 (1976), pp. 301–312.

[2] Johan van Benthem. *Modal Logic and Classical Logic*. Distributed in the U.S.A. By Humanities Press, 1983.

[3] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2001.

[4] Alexander V. Chagrov and Michael Zakharyaschev. "Modal Logic". In: *Oxford logic guides*. Clarendon Press, 1997.

[5] Willem Conradie and Alessandra Palmigiano. "Algorithmic correspondence and canonicity for distributive modal logic". In: *Annals of Pure and Applied Logic* 163.3 (2012), pp. 338–376.

[6] Giovanna Corsi. "Quantified Modal Logics of Positive Rational Numbers and Some Related Systems". In: *Notre Dame Journal of Formal Logic* 34.2 (1993), pp. 263–283.

[7] Giovanna Corsi and Silvio Ghilardi. "Directed frames". In: *Archive for Mathematical Logic* 29 (1989), pp. 53–67.

[8] M.J. Cresswell and G.E. Hughes. *A New Introduction to Modal Logic*. 1996.

[9] Melvin Fitting and Richard L. Mendelsohn. *First-Order Modal Logic*. USA: Kluwer Academic Publishers, 1999.

[10] Dov Gabbay, Valentin Shehtman, and Dmitrij Skvortsov. *Quantification in Nonclassical Logic, Volume 1*. Vol. 153. Studies in Logic and the Foundations of Mathematics. Elsevier, 2009.

[11] James Garson. "Modal Logic". In: *The Stanford Encyclopedia of Philosophy*. Ed. by Edward N. Zalta. Summer 2021. Metaphysics Research Lab, Stanford University, 2021.

[12] Leon Henkin. "The Completeness of the First-Order Functional Calculus". In: *The Journal of Symbolic Logic* 14.3 (1949), pp. 159–166.

[13] Roman Kontchakov, Agi Kurucz, and Michael Zakharyaschev. "Undecidability of First-Order Intuitionistic and Modal Logics with Two Variables". In: *The Bulletin of Symbolic Logic* 11.3 (2005), pp. 428–438.

[14] Mikhail Rybakov, Valentin Shehtman, and Dmitry Shkatov. *Mathematical Introduction to First-Order Modal and Intuitionistic Logic, Pre-Draft*.

[15] Mikhail Rybakov and Dmitry Shkatov. "Algorithmic properties of first-order modal logics of finite Kripke frames in restricted languages". In: *Journal of Logic and Computation* 30.7 (2020), pp. 1305–1329.

[16] Mikhail Rybakov and Dmitry Shkatov. "Algorithmic properties of first-order modal logics of linear Kripke frames in restricted languages". In: *Journal of Logic and Computation* 31.5 (2021), pp. 1266–1288.

[17] Mikhail Rybakov and Dmitry Shkatov. *Undecidability of First-Order Modal and Intuitionistic Logics with Two Variables and One Monadic Predicate Letter*. 2018.

[18] Henrik Sahlqvist. "Completeness and Correspondence in the First and Second Order Semantics for Modal Logic (revised version of parts of the author's thesis submitted to the University of Oslo, Dept. of Mathematics, Spring 1973)". In: *Proceedings of the Third Scandinavian Logic Symposium*. Ed. by Stig Kanger. Vol. 82. Studies in Logic and the Foundations of Mathematics. Elsevier, 1975, pp. 110–143.

[19] Valentin Shehtman. "On Kripke completeness of modal predicate logics around quantified K5". In: *Annals of Pure and Applied Logic* (2023), p. 103202.

[20] Rineke (L.C.) Verbrugge. "Provability Logic". In: *The Stanford Encyclopedia of Philosophy*. Ed. by Edward N. Zalta. Fall 2017. Metaphysics Research Lab, Stanford University, 2017.

[21] Frank Wolter and Michael Zakharyaschev. "Decidable Fragments of First-Order Modal Logics". In: *The Journal of Symbolic Logic* 66.3 (2001), pp. 1415–1438.

[22] Frank Wolter and Michael Zakharyaschev. "Modal decision problems". In: *Handbook of Modal Logic*. Ed. by Patrick Blackburn, Johan Van Benthem, and Frank Wolter. Vol. 3. Studies in Logic and Practical Reasoning. Elsevier, 2007, pp. 427–489.