

https://dspace.org/

Symmetry analysis of geometries in general relativity and mathematical models in quantitative finance

Obaidullah, Usaamah

2024

https://hdl.handle.net/10539/37407

Downloaded from a DSpace repository. https://dspace.org/

Symmetry Analysis of Geometries in General Relativity and Mathematical Models in Quantitative Finance

Mr Usaamah Obaidullah

Supervisor: Professor Sameerah Jamal



UNIVERSITY OF THE
WITWATERSRAND,
JOHANNESBURG

A thesis submitted to the Faculty of Science, University of the Witwatersrand, in requirement for the degree Doctor of Philosophy, March, 2023.

DECLARATION

I declare that the contents of this thesis are original, except where due references have been made. It is submitted for the degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It was not submitted before for any degree or examination to any other institution.

Usaamah Obaidullah

Signed at Johannesburg on the 27^{th} day of March 2023.

Abstract

This thesis may be divided into two themes. The first, consists of a mathematical analysis of selected models in cosmology, while the second, is in the field of quantitative finance. The pp-wave spacetime, Bianchi I spacetime, and the Bianchi II spacetime are the three universes that we examine. The latter two are studied in the framework of $f(R)$ theory of gravity, a plausible substitute for general relativity and a solution to the dark energy problem. We apply the Killing and homothetic vector fields to divide the spacetimes into classes or categories. Subsequently, potential functions are established using the geometry of the point symmetries of the space, while Noether's theorem provides the first integrals connected to each isometry. We take advantage of the geometric fact that the homothetic algebra of spacetimes yields the Noether point symmetries of geodesic Lagrangians. For the Bianchi spacetimes, the Wheeler-DeWitt equations are derived by the quantisation of the spacetime Lagrangians, and from the Lie point symmetries it admits, we sequentially find invariant solutions to solve for the universe's wave function. Finally, exact solutions to the field equations are also found. The research comprising of the second theme, investigates two nonlinear partial differential equations used in derivative pricing for financial markets. The point symmetries, invariant solutions, and conversation laws of these equations are found. In our analysis, we vary certain variables that change the nonlinearity of the models and thus give us unique symmetries and solutions. With various parameter settings, graphical solutions are investigated.

DEDICATION

For Abbu and Ammu

Per aspera ad astra

ACKNOWLEDGEMENTS

This PhD thesis is the product of a lifelong fascination with space, a yearning for an understanding of the cosmos, and a sense of child-like wonder that was sparked by many memories — a beloved one being the animated film Treasure Planet — and has evolved into both a labour of love and a scientific inquiry.

I should start by acknowledging the sacrifices my family have made so that I can engage in *all* of my intellectual pursuits. My father deserves a lot of credit for one of my first memories — introducing me to the elegance of algebra.

Second, I want to thank Professor Sameerah Jamal, who served as my academic advisor. Her advice, expertise and patience have been beneficial in teaching me how to conduct research and as a guide throughout this thesis.

Third, I acknowledge the Postgraduate Merit Award (PMA) from the University and the National Research Foundation (NRF) of South Africa (grant number 132750). Their funding have made this work possible.

My friends for making this a memorable period of my life and the many nights spent in the ins and outs of Johannesburg, a city I was foreign to but now will always consider a home.

Finally, if I didn't acknowledge Mr. Shashilan Singh's efforts to making every day of my postgraduate studies easier, including the brief but heartfelt conversations we always had in his office, I would feel true regret. I'm astonished to admit that Mr Shashi's cheerful demeanor is practically a given.

I owe many people a debt of gratitude, thank you all.

Contents

List of Publications **10**

List of Abbreviations **11**

List of Figures **13**

List of Tables **14**

1 Preface **15**

2 Notation and theory **23**

2.1 Lie symmetry analysis 23

2.2 Euler operators and conservation laws 25

2.3 Noether symmetries 26

2.4 Tensors and metric tensors 27

2.5 The Riemann curvature tensor, the Ricci tensor and the Ricci Scalar 28

2.6 Collineations 29

2.7 Potential functions and Hamiltonians 30

3 pp-wave potential functions: a complete study using Noether symmetries 33

3.1 Introduction 34

3.2 The spacetime 35

3.3 Category 1 37

3.4 Category 2 38

3.5 Category 3 40

3.6 Category 4 42

3.7 Category 5 42

3.8 Category 6 43

3.9 Category 7 45

3.10 Category 8 47

3.11 Category 9 49

3.12 Category 10 51

3.13 Category 11 53

3.14 Category 12 54

3.15 Category 13 54

3.16 Category 14 55

3.17 Discussion 55

4 Analytical field equation and wave function solutions of the Bianchi type I universe in vacuum f*(R)* gravity 57

4.1 Introduction 58

4.2 The Bianchi type I spacetime in vacuum f*(R)* gravity 60

4.2.1 Case (I) 64

4.2.2 Case (II) 67

4.2.3 Case (III) 68

4.2.4 Case (IV) 70

4.2.5 Case (V) 72

4.2.6 Case (VI) 73

4.2.7 Case (VII) 74

4.2.8 Case (VIII) 76

4.2.9 Case (IX) 77

4.2.10 Case (X) 78

4.2.11 Case (XI) 80

4.2.12 Case (XII) 81

4.2.13 Case (XIII) and (XIV) 82

4.3 Discussion 83

5 Classical solutions to Bianchi type II spacetimes in $f(R)$ theory of gravity 85

5.1 Introduction 86

5.2 The Hamiltonian, geodesics and Killing vectors 88

5.3 Analytical solutions: equations of motion and wave function 91

5.3.1 Case (I) $A = t^3$, $B = t^{-2}$, $C = t^{-3}$. 94

5.3.2 Case (II) $A = (k_1 t + k_2)^3$, $B = (k_1 t + k_2)$, $C = \frac{-2}{k_1^2}$, where $k_1, k_2 \in \Re$ $(k_1 \neq 0)$. 97

5.3.3 Case (III) $A = -\frac{1}{k_1^2}$, $B = (k_1 t + k_2)^{-1}$, $C = (k_1 t + k_2)$, where $k_1, k_2 \in \Re$ $(k_1 \neq 0)$. 100

5.3.4 Case (IV) $A = B = (2t^2 + k_1 t + k_2)^{\frac{1}{2}}$, $C = (2t^2 + k_1 t + k_2)$, where $k_1, k_2 \in \Re$ $(k_1 \neq 0)$. 103

5.3.5 Case (V) $A = C = (k_1 t + k_2)^{\frac{3}{2}}$, $B = (k_1 t + k_2)$, where $k_1 = \pm 2, k_2 \in \Re$. 106

5.3.6 Case (VI) $A = C = k_1$, $B = -2k_1^2 t^{-2}$, where $k_1 \in \Re \backslash \{0\}$. . . . 109

5.3.7 Case (VII) $A = B = \left(k_2 \cos \sqrt{\frac{2}{k_1}} t + k_3 \sin \sqrt{\frac{2}{k_1}} t\right)$, $C = k_1$, where $k_1, k_2, k_3 \in \Re$ $(k_1 \neq 0)$. 112

5.4 Discussion 116

6 Lie symmetry analysis and analytical solutions of the nonlinear Ivancevic options pricing model 118

6.1 Introduction 119

6.2 Symmetries and invariant solutions for the IOPM with a non-zero adaptive market potential 121

6.2.1 Conservation laws for $\beta \neq 0$ 122

6.2.2 Reductions and invariant solutions for $\beta \neq 0$ 125

6.3 Symmetries and invariant solutions for the IOPM with a zero adaptive market potential 127

6.3.1 Conservation laws for $\beta = 0$ 128

6.3.2 Reductions and invariant solutions for $\beta = 0$ 132

6.4 Discussion 139

7 Invariant solutions of a European option pricing equation under a continuous-time Capital Asset Pricing Model 143

7.1 Introduction 144

7.2 Analysis of the European option pricing model with a constant market beta 146

7.2.1 Reductions and invariant solutions for constant beta 146

7.3 Analysis of the European option pricing model with zero systematic risk 155

7.3.1 Reductions and invariant solutions for $\beta = 0$ 156

7.4 Discussion 162

Conclusion 167

Bibliography 168

List of Publications

Details of contributions to publications that form part of the research presented in this thesis are:

- *pp-wave potential functions: A complete study using Noether symmetries*, published in the International Journal of Geometric Methods in Modern Physics, 2021 [1].
- *Analytical field equation and wave function solutions of the Bianchi type I universe in vacuum f(R) gravity*, published in the International Journal of Geometric Methods in Modern Physics, 2022 [2].
- *Classical solutions to Bianchi type II spacetimes in f (R) theory of gravity*, published in the Indian Journal of Physics, 2022 [3].
- *Lie symmetry analysis and analytical solutions of the nonlinear Ivancevic options pricing model*, submitted, 2022.
- *Invariant Solutions of a European Option Pricing Equation under a Continuous-Time Capital Asset Pricing Model* , submitted, 2022.

List of Abbreviations

ODE ..... Ordinary Differential Equation

PDE ..... Partial Differential Equation

KVF ..... Killing Vector Field

HVF ..... Homothetic or Homothety Vector Field

IOPM ..... Ivancevic Options Pricing Model

CAPM ..... Capital Asset Pricing Model

List of Figures

1.1 First page of Einstein's paper on general relativity [10]. 18

5.1 Evolution of the analytical solutions of the WDW equation; we select the parameter values $C_1 = C_2 = 1$ in the solution (228). 96

5.2 Plot of (249) with we select the parameter values $C_1 = C_2 = 1$ and $k_1 = k_2 = 1$ in the solution. 102

5.3 Evolution of the solution (275), with we select the parameter values $C_1 = C_2 = 1$ and solution $k_1 = 1$. 110

6.1 Graphical plot of (306) with $C_1 = C_2 = 1$ and the range $0 \leq S \leq 10$. 126

6.2 Graphical plots of (317) for $C_1 = C_2 = C_3 = C_4 = 1$ and the ranges $0 \leq t \leq 10$ and $0 \leq S \leq 10$ 134

6.3 Graphical plots of (319) for $c_1 = c_2 = C_1 = C_2 = C_3 = C_4 = 1$ and the ranges $0 \leq t \leq 10$ and $0 \leq S \leq 10$ 135

6.4 Graphical plots for (321) for $c_1 = c_2 = C_1 = C_2 = C_3 = C_4 = 1$ and the ranges $0 \leq t \leq 10$ and $0 \leq S \leq 10$ 137

6.5 Graphical plots for (323) with $c_1 = c_2 = C_1 = C_2 = C_3 = C_4 = 1$ and the ranges $0 \leq t \leq 10$ and $0 \leq S \leq 10$ 138

6.6 Graphical plots for (325) with $c_1 = c_2 = C_1 = C_2 = 1$ the ranges $0 \leq t \leq 10$ and $0 \leq S \leq 10$ 140

6.7 Graphical plots for (327) with $c_1 = c_2 = C_1 = C_2 = C_3 = C_4 = 1$ the ranges $0 \leq t \leq 10$ and $0 \leq S \leq 10$ 141

7.1 Graphical plots for solutions (336), (337), (338), and (339) with varying parameters 149

7.2 Graphical plots for solutions (341), (342), and (343) with varying parameters 151

7.3 Graphical plot of solution (344) with varying parameter values 152

7.4 Graphical plot of solution (347) with varying parameter values 154

7.5 Graphical plots of solutions (357) and (358) with varying parameters 158

7.6 Graphical plots of solutions (360), (361), and (362) with varying parameters 159

7.7 Graphical plots of (364) and (365) with varying parameters 161

7.8 Graphical plots of solutions (366) and (367) with varied parameters . 162

List of Tables

3.1 Additional symmetries and potentials functions of Category 1. 38

3.2 Additional symmetries and potentials functions of Category 2. 41

3.3 Additional symmetries and potentials functions of Category 5. 43

3.4 Additional symmetries and potentials functions of Category 6. 46

3.5 Additional symmetries and potentials functions of Category 8. 50

3.6 Additional symmetries and potentials functions of Category 9. 52

6.1 Commutator table for the IOPM with $\beta \neq 0$. 123

6.2 Commutator table for the IOPM with $\beta = 0$ 128

7.1 Commutator table for the CAPM with $\beta = 0$. 156

Chapter 1

Preface

The works of two mathematical titans, Isaac Newton and Gottfried von Leibniz serve as the foundation for the study of differential equations, which since its inception in the middle of the 17th century, has had a considerable impact on our comprehension of natural phenomena. The majority of Leibnizian advancements in the 18th century extended the multi-variate form of the tradition, consolidating it and resulting in partial differential equations (PDEs). The contributions of several other notable mathematicians, such as Euler, Lagrange, Cauchy, and Laplace, to the development of the subject are also noteworthy. Arguably one of the most well-known applications of differential equations to physics would be Maxwell's equations, and now differential equations are fundamental to several areas of classical and quantum physics. In fact, the modelling of phenomena through these equations are seen in almost all disciplines of science and engineering. Beyond illustrating the rules of nature, differential equations explain several modern-day non-physical systems and behaviours such as economics, financial markets and social concepts.

With the foundations laid, the need to solve these equations and systems of differential equations became a focus in the field, which gave rise to a number of new branches of mathematics. Enter the concept of Lie symmetry analysis. The idea was first put forth by Norwegian mathematician Sophus Lie (1842–1899), who saw that the various approaches to solving these equations are only variations of a more general integration strategy based on the equation's invariance under a continuous group of symmetries [4]. A symmetry exists mathematically when there is invariance under transformation. Geometrically, a transformation is regarded as a symmetry if it preserves the object's structure, is a diffeomorphism, and maps the object to itself hence the original object and transformed image are identical [5]. A symmetry, as applied to a PDE, is a transformation that maps one solution to another solution of the same equation, where invariant solutions are those that map into themselves. This allows one to categorise the solutions' various symmetry classes. A differential equation's symmetries can be calculated and then used to methodically determine its analytical solutions. This technique also allows for the construction of new solutions from existing ones. As a result, one of the most popular methods for locating solutions to nonlinear differential equations and comprehending the physical implications of those solutions is symmetry analysis.

Related are Noether symmetries that bear the name of mathematician Emmy Noether. These are symmetries connected to differential equations with a Lagrangian. Noether established two theorems in 1918 that connected the symmetry groups of a variational integral to the properties of the corresponding Euler-Lagrange equations [6]. Noether was praised as a "creative mathematical genius" in her obituary by Albert Einstein, who also highlighted that she was not given credit for her substantial contributions. She revealed the connection between symmetries and conservation laws by proving that one implies the other. A conservation law in physics asserts that a quantifiable property of a physical system stays constant despite changes to the

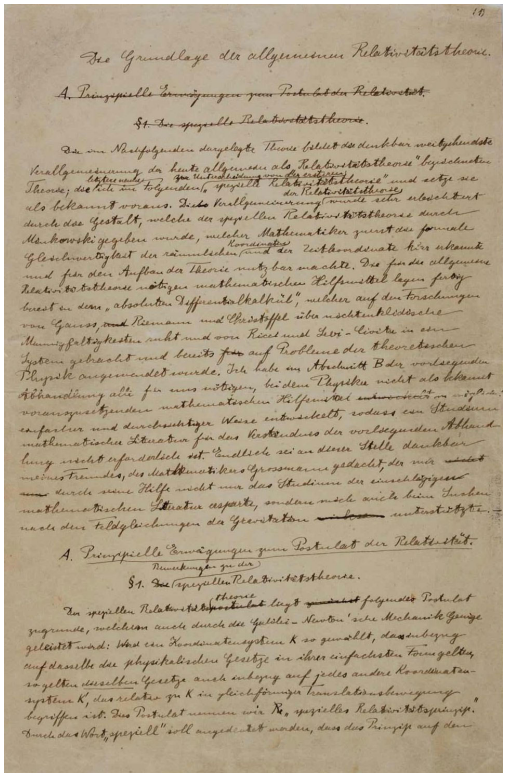
system. The relevance of these laws in the study of physics may be attributed to the fact that they explain physical qualities like the conservation of momentum and energy, amongst many others. Moreover, conservation laws play a key role in the analysis of differential equations, notably in relation to linearisation, integrability, constants of motion, solution analysis, and numerical solution techniques [7, 8, 9].

Albert Einstein's century-old discovery of general relativity has had a profound impact on science and has long since left a lasting impression on popular culture too. Black holes, wormholes, time dilation are all concepts thrown around in science fiction but are actually founded on Einstein's theory, which continues to advance state-of-the-art research, enabling mathematicians, cosmologists and physicists alike to query the nature of space and time, and in a grander way, study the past and future of the universe. Late in 1915, Einstein made multiple presentations of his novel theory of gravity to the Prussian Academy of Sciences. Figure (1.1) shows the first page of Einstein's general relativity manuscript. His field equations reads as

$$R_{\mu v} - \frac{1}{2} R g_{\mu v} + \Lambda g_{\mu v} = \frac{8\pi G}{c^4} T_{\mu v},$$

where $R_{\mu v}$ is the Ricci curvature tensor, R is the scalar curvature, $g_{\mu v}$ is a metric tensor, Λ is the cosmological constant, G is Newton's gravitational constant, c^4 is the speed of light in a vacuum, and finally $T_{\mu v}$ is the stress energy tensor. Interpreted more simply, the left hand side of the equation tells energy or matter how to curve spacetime, and the right hand side tells energy or matter how to move through spacetime. The field equations break down to ten nonlinear PDEs in four independent variables. When general relativity was initially published, a challenge emerged as it seemed that energy would not be conserved in substantially curved spacetime. As was already noted, Noether demonstrated that conserved quantities are connected to symmetries. Energy is always conserved, according to Noether's

Figure 1.1: First page of Einstein's paper on general relativity [10].



findings, who demonstrated that general relativity's symmetries between various reference frames ensure this. The conundrum was thus resolved, and Noether's significant contribution to this astonishing theory was solidified. These equations have among a plethora of other discoveries, predicted black holes which are a solution of these equations, postulated the existence of gravitational waves which have been recently observed, and even enabled us to use a Global Positioning System (GPS), without which the distance measurements to each satellite would be inaccurate by about 7 miles each day. It is hardly surprising that many people who are familiar with this theory consider it to be one of the most beautiful of physical theories.

Among many important nonlinear PDEs used in finance, the Black-Scholes equation

is arguably the one commonly adopted. In the 1970's, F. Black, M. Scholes and R.C. Merton introduced a mathematical method to price complex financial instruments [11], winning them a Nobel prize. We are able to ascertain the fair value of a stock option using the Black-Scholes model by taking into account the price of the underlying asset, the strike or exercise price of the option, the risk-free interest rate, time to expiration of the option, and volatility. Theoretically, option sellers may determine rational pricing for the options they are selling using these factors. In fact before Black and Scholes, no one had ever imagined that combining shares of stock and cash might be used to create an option. Furthermore, their work gave rise to the derivatives market - financial instruments that derive their value from other assets such as stocks. After the 2008 financial crisis, derivatives acquired a bad reputation, particularly collateralised debt obligations, which are obligations to pay investors a portion of the money that comes in from mortgage payments, for example. The PDEs used in finance, often of parabolic type, are highly nonlinear so solutions while difficult to come by, are sought after.

Outline of chapters

This thesis aims at conducting symmetry analysis of the several solutions of Einstein's field equations, namely the plane-fronted waves with parallel propagation i.e. pp-wave spacetime, the Bianchi I spacetime, and the Bianchi II spacetime. For each we provide potential functionals and use Noether's theorem to find the first integrals connected to each isometry. For the Bianchi spacetimes, we derive the Wheeler-DeWitt equations and subsequently derive invariant solutions to solve for the universe's wave function. Furthermore, we provide full symmetry analysis and novel exact solutions to two sets of nonlinear PDEs used in quantitative finance. This thesis is a collection of several publications.

The basic notation and concepts utilized throughout this thesis are described in chapter 2.

In chapter 3, we look at the functional forms of the potentials that result from the pp-wave spacetime's Lagrangian. This is done using Noether symmetries. Moreover, we derive additional potentials from the linear combinations of the vector fields and subalgebras.

The results of chapter 3 have been published in [1].

For chapter 4 we find the analytical solutions to the Bianchi type I universe in $f(R)$ gravity. We utilize the geometry of the point symmetries of the space to build potential functions and Noether's theorem to find the conservation laws connected to each isometry for each class. The Wheeler-DeWitt equation is derived by quantizing the spacetime Lagrangian, and from the Lie point symmetries it admits, we sequentially find invariant solutions for the universe's wave function. Finally, exact classical solutions to the field equations are determined.

Chapter 4 has been published in [2].

Chapter 5 considers the analytical solutions to the Bianchi type II universe in $f(R)$ theory of gravity. In the same manner as in chapter 4, the Lagrangian and Hamiltonian are calculated, and the conformal vector fields are given. We derive the potential functionals $V(t, x, y, z)$ and the first integrals for each homothetic and Killing vector field for each class. The Wheeler-DeWitt equation for this spacetime is then constructed, and we use it to find invariant solutions for the universe's wave function by applying the zero-order invariants of the admitted Lie point symmetries. In order to obtain the exact classical solutions to the field equations for all seven classes, we reduce the Hamiltonian system.

Chapter 5 has been published in [3].

Chapter 6 investigates the point symmetries, invariant solutions, and conversation laws of the nonlinear quantum-probability based, Schrödinger type Ivancevic options pricing model. A non-zero and zero adaptive market potential model is taken into account in our study. In both cases, we show that this model is invariant under a five-dimensional Lie algebra and a seven-dimensional Lie algebra, respectively. For both the non-zero and zero adaptive market potential models, we get reductions, solutions, and conservation laws. In specific situations, graphic solutions are offered.

Chapter 6 has been submitted for publication.

In chapter 7 we analyse a nonlinear European option pricing equation under a continuous-time Capital Asset Pricing Model. We specifically find its point symmetries using the Lie symmetry method in order to arrive at invariant analytical solutions. We change the beta coefficient, a gauge of an asset's systematic risk, in our research. We look at the model for both a non-zero and a zero market beta.

We demonstrate that this model is invariant under a smaller three-dimensional Lie algebra in the first case and a larger six-dimensional Lie algebra in the second. Graphical solutions with varied parameter values are investigated.

Chapter 7 has been submitted for publication.

Chapter 2

Notation and theory

2.1 Lie symmetry analysis

In general, a symmetry is a transformation, mapping all solutions of a given equation to another solution of the very same equation. An extraordinary contribution by Sophus Lie, this technique has amazing applications in physics and engineering. This procedure is described here. Consider q unknown functions u^a which depend on p independent variables x^i, i.e. $u = (u^1, \ldots, u^q)$, $x = (x^1, \ldots, x^p)$, with indices $\alpha = 1, \ldots, q$ and $i = 1, \ldots, p$. Let

$$G_\alpha(x, u^{(k)}) = 0, \tag{1}$$

be a system of nonlinear differential equations, where $u^{(k)}$ represents the k^{th} derivative of u with respect to x. A one-parameter Lie group of transformations with ϵ as the group parameter, invariant under (1), is given by

$$\bar{x} = \Xi(x, u; \epsilon) \quad \bar{u} = \Phi(x, u; \epsilon). \tag{2}$$

Invariance of (1) under the transformation (2) dictates that any solution $u = \Theta(x)$ of (1) maps into another solution $v = \Psi(x; \epsilon)$ of (1). Expanding (2) around the identity $\epsilon = 0$, generates the infinitesimal transformations:

$$\bar{x}^i = x^i + \epsilon \xi^i(x, u) + O(\epsilon^2), \tag{3}$$

$$\bar{u}^\alpha = u^\alpha + \epsilon \eta^\alpha(x, u) + O(\epsilon^2). \tag{4}$$

The action of the Lie group can be recovered from that of its infinitesimal generators acting on the space of independent and dependent variables. Hence, we consider the following vector field

$$X = \xi^i \partial_{x^i} + \eta^\alpha \partial_{u^\alpha}. \tag{5}$$

The infinitesimal criterion for invariance is given by

$$X[G_\alpha(x, u^{(k)})] = 0, \quad \text{when} \quad G_\alpha(x, u^{(k)}) = 0, \tag{6}$$

where X is extended to all derivatives appearing in the equation through an appropriate prolongation. The operator in Eq. (5) defines a Lagrange system or characteristic equation

$$\frac{dx^i}{\xi^i} = \frac{du^\alpha}{\eta^\alpha},$$

whose solution provides zero-order invariants

$$W^{[0]}(x^i, u^\alpha). \tag{7}$$

2.2 Euler operators and conservation laws

The generalized total differentiation operator D_i with respect to x^i is given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots .$$

The Euler-Lagrange equations or geodesic equations of motion, if they exist, is the system

$$\frac{\delta L}{\delta u^\alpha} = 0,$$

where $\delta/\delta u^\alpha$ is the Euler-Lagrange operator given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 \cdots i_s}} ,$$

and where L is a given Lagrangian.

Multipliers, $\Lambda^\alpha(x, u^{(k)})$, are integrating factors, satisfying the relation

$$D_i T^i = \Lambda^\alpha G_\alpha, \tag{8}$$

where Λ^α is derived by equations satisfying

$$\frac{\delta}{\delta u^\alpha}(\Lambda^\alpha G_\alpha) = 0. \tag{9}$$

Now having the multipliers, they give rise to a conserved vector $T = (T^1, ..., T^n)$ that satisfies the conservation relation

$$D_i T^i = 0, \tag{10}$$

along the solutions of a given equation.

2.3 Noether symmetries

In the work below, we illustrate the utility of point transformations, and for the reader's convenience we discuss some theory for the analysis which follows. Consider a system of second-order ordinary differential equations (ODEs)

$$\ddot{x}^i = \omega^i \left(s, x^j, \dot{x}^j\right), \tag{11}$$

where s is the independent variable and x^i denotes the dependent variables. We provide a discussion of this particular type of equation due to its relevance in later chapters - further details may be found in [12].

An one-parameter point transformation in the space $\{s, x^j\}$, such that $\{s, x^j\} \rightarrow \left\{\bar{s}\left(s, x^j, \epsilon\right), \bar{x}^{j'}\left(s, x^j, \epsilon\right)\right\}$ has the property of mapping solutions of (11) to themselves.

Let its generator X be defined as

$$X = \left.\frac{\partial \bar{s}}{\partial \varepsilon}\right|_{\varepsilon=0} \partial_s + \left.\frac{\partial \bar{x}^i}{\partial \varepsilon}\right|_{\varepsilon=0} \partial_i. \tag{12}$$

Suppose that the system (11) follows from the variation of the action integral

$$\mathcal{S} = \int L(s, x^j, \dot{x}^j) ds.$$

One of Noether's theorems states that when a (finite) group of transformations leaves the action form invariant, i.e. when

$$\mathcal{S}\left(s, x^j, ...\right) = \mathcal{S}\left(\bar{s}\left(s, x^j, \epsilon\right), \bar{x}^j\left(s, x^j, \epsilon\right), ...\right), \tag{13}$$

then a conserved quantity or first integral exists which can be constructed via the symmetry generator. In the case of a Lagrangian containing up to first order derivatives of the x^j's as is our case, the condition described by (13) is expressed in

infinitesimal form as

$$X^{[1]}L + L\frac{d}{ds}\left(\left.\frac{\partial \bar{s}}{\partial \epsilon}\right|_{\epsilon=0}\right) = \dot{b}, \tag{14}$$

where $b = b(s, x^j)$ is a gauge term, and the relative integral of motion is given by

$$I = \left(\dot{x}^j \frac{\partial L}{\partial \dot{x}^j} - L\right)\left.\frac{\partial \bar{s}}{\partial \epsilon}\right|_{\epsilon=0} - \left.\frac{\partial L}{\partial \dot{x}^j}\frac{\partial \bar{x}^j}{\partial \epsilon}\right|_{\epsilon=0} + b. \tag{15}$$

The generator X is called a Noether symmetry and it is straightforward to show that it is also a Lie symmetry of the equations of motion; however the inverse is not always true.

There are several methods to construct conservation laws, but Noether's theorems form beautiful and simple applications.

2.4 Tensors and metric tensors

A favored class of reference systems is utilized in Newtonian physics, these are the inertial systems where the Newtonian principles of dynamics are valid. Yet in actual use, identifying the inertial systems can be challenging. Hence, no reference system should be given preference in the formulation of the laws of physics. Tensors are objects designed without any special privileges for reference systems.

Let $V_p(M)$ be a finite dimensional vector space and let $V_p^*(M)$ denote its dual vector space. A tensor T of type (k, l) over $V_p(M)$ is a multi-linear map [13]

$$T : \underbrace{V_p(M) \times V_p(M) \times \cdots V_p(M)}_{k} \times \underbrace{V_p^*(M) \times V_p^*(M) \times \cdots V_p^*(M)}_{l} \to \Re. \tag{16}$$

Equation (16) demonstrates that with k dual vectors and l ordinary vectors, tensor T produces a number where if we fix all but one of the vectors or dual vectors, it is

a linear map on the remaining variable. With this definition, a tensor of type $(0,0)$ is a scalar, $(1,0)$ is an ordinary vector, and $(0,1)$ is a dual vector.

A metric tensor is a symmetric second-rank covariant tensor $g_{ij} = g_{(ij)}$. The metric or metric form is defined as [14]

$$ds^2 = g_{ij}dx^i dx^j. \tag{17}$$

This expression is a convenient shorthand notation, however, only under an integral can this expression, a scalar, make sense. The length of arc of a curve $x^i(\lambda)$ between the points $x^i(\lambda_0)$ and $x^i(\lambda_1)$ is

$$l_{\lambda_0\lambda_1} = \int_{\lambda_0}^{\lambda_1} ds = \int_{\lambda_0}^{\lambda_1} \left| g_{ij}(\lambda)\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} \right|^{1/2} d\lambda. \tag{18}$$

2.5 The Riemann curvature tensor, the Ricci tensor and the Ricci Scalar

The Riemann curvature tensor is defined as

$$R^{\alpha}_{\beta\gamma\delta} = \Gamma^{\alpha}_{\beta\delta,\gamma} - \Gamma^{\alpha}_{\beta\gamma,\delta} + \Gamma^{\alpha}_{\epsilon\gamma}\Gamma^{\epsilon}_{\beta\delta} - \Gamma^{\alpha}_{\epsilon\delta}\Gamma^{\epsilon}_{\beta\gamma}. \tag{19}$$

The Riemann curvature tensor in four dimensional space has 256 independent components, but its symmetries reduce it to 20. Its covariant form can be expressed as

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\epsilon}R^{\epsilon}_{\beta\gamma\delta}. \tag{20}$$

Let W_γ be a dual vector field, then the Ricci identity is defined by the relation

$$\nabla_\alpha\nabla_\beta W_\gamma - \nabla_\beta\nabla_\alpha W_\gamma = R^{\delta}_{\alpha\beta\gamma}W_\delta, \tag{21}$$

where $R^{\delta}_{\alpha\beta\gamma}$ is the Riemann curvature tensor.

The properties of the Riemann curvature tensor are

i. $R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}$, $R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma}$, and $R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$,

ii. $R_{\alpha[\beta\gamma\delta]} = 0$,

iii. the Bianchi identity $R_{\alpha\beta[\gamma\delta;\epsilon]} = \nabla_{\alpha} R_{\delta\epsilon\beta\gamma} + \nabla_{\gamma} R_{\delta\epsilon\alpha\beta} + \nabla_{\beta} R_{\delta\epsilon\gamma\alpha} = 0$,

iv. the Ricci tensor is defined as $R_{\gamma\delta} = R^{\epsilon}_{\gamma\epsilon\delta}$,

v. the Ricci scalar is defined as $R = g^{\alpha\beta} R_{\alpha\beta}$.

2.6 Collineations

Suppose we have the coordinates $\{x^i\}$ for an arbitrary space K^n of dimension n, and a one parameter point transformation

$$\tilde{x}^i = x^i + \epsilon X^i \left(x^k\right), \tag{22}$$

where the X^i are vector field components called the infinitesimal generator of Eq. (22). If Υ is a geometric object in K^n, under the action of (7), Υ transforms to $\Theta\left(\Upsilon^i, x^i, x^{i'}\right)$ and the transformation law is $\Upsilon^{j'} = \Theta^{j'}\left(\Upsilon^i, x^i, x^{i'}\right)$. The Lie derivative $\mathcal{L}_X$ of Υ is then [15]

$$\mathcal{L}_X \Upsilon = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \Big(\Theta \left(\Upsilon^i, x^i, x^{i'} \right) - \Upsilon \Big). \tag{23}$$

For functions $\Psi\left(x^i\right)$, the transformation law is $\Psi'\left(\tilde{x}^i\right) = \Psi\left(x^i\right)$, so that under the action of Eq. (7), one has

$$\tilde{\Psi}\left(\tilde{x}^i\right) = \Psi\left(x^i + \epsilon X^i\left(x^k\right)\right) = \Psi\left(x^i\right) + \epsilon \Psi_{,i} X^i + O\left(\epsilon^2\right).$$

Thus, by Eq. (23) we have

$$\mathcal{L}_X \Psi = \Psi_{,i} X^i. \tag{24}$$

If the right hand side of Eq. (24) vanishes, then $\Psi(x^i)$ is invariant under the point transformation (7), and X is known as a symmetry of $\Psi(x^i)$. If $\mathcal{L}_X \Upsilon = A$, where A is a tensor having the identical number and symmetries of the indices with Υ, the vector field X is called a collineation of Υ whose type depends on A.

In the context of general relativity, our geometric objects Υ can be defined in terms of a metric tensor. That is, let $\Upsilon = g_{ab}$ and $A = 2\psi g_{ab}$, then we have

$$\mathcal{L}_X g_{ab} = 2\psi g_{ab}, \tag{25}$$

and X is a conformal Killing vector field (CKVF) [16]. Here, $\psi = \psi(x^a)$ is a conformal factor. If $\psi_{;ab} \neq 0$, the CKVF is said to be proper.

$$\begin{aligned} \psi_{;ab} = 0 \quad &\Longleftrightarrow \quad X \text{ is a special CKVF (SCKVF)}, \\ \psi_a = 0 \quad &\Longleftrightarrow \quad X \text{ is a homothetic vector field (HVF)}, \\ \psi = 0 \quad &\Longleftrightarrow \quad X \text{ is a Killing vector field (KVF)}. \end{aligned} \tag{26}$$

These vector fields form a Lie algebra and the dimension of the conformal algebra is less than or equal to $\frac{1}{2}(n+1)(n+2)$. In general, collineations are the generators of one-parameter transformations which transform geometric objects under a particular rule [17].

2.7 Potential functions and Hamiltonians

Consider the line element of a space

$$ds^2 = g_{ij} dx^i dx^j,$$

where the metric tensor g_{ij} is symmetric, its determinant g is in general different from zero and it possesses an inverse g^{ij} [18].

Consider a particle moving in a Riemannian space with metric tensor g_{ij} under the influence of the potential $V(x^k)$. The Lagrangian describing the motion of the particle is

$$L = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j - V(x^k). \tag{27}$$

Suppose the Noether symmetry vector is of the form $X = \xi(s, x^k)\partial_s + \eta^i(s, x^k)\partial_{x^i}$, then by Noether's theorem, the symmetry conditions are

$$V_k \eta^k + V\xi_s = -b_s, \tag{28}$$

$$\eta^i_t g_{ij} - \xi_j V = b_j, \tag{29}$$

$$\mathcal{L}_{\eta^i} g_{ij} = 2\Big(\frac{1}{2}\xi_s\Big) g_{ij}, \tag{30}$$

$$\xi_k = 0. \tag{31}$$

Equation (31) implies $\xi = \xi(s)$, and from (30), it follows that η^i is a HVF. Therefore, the Noether symmetries of the Lagrangian (27) are generated from the homothetic algebra of the metric tensor g_{ij} of the space where the motion occurs [19]. If the HVFs are known, we will have the Noether symmetries, and one can then solve for the unknown autonomous potential $V(x^k)$.

The corresponding Hamiltonian to (27) is given by

$$E = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + V(x^k). \tag{32}$$

The Wheeler-DeWitt (WDW) equation is a Klein-Gordon equation which is defined by the conformal Laplacian operator, i.e.

$$\Delta\Psi + \frac{n-2}{4(n-1)} R(x^k)\Psi + V(x^k)\Psi = 0, \tag{33}$$

where Δ is the Laplacian operator, $\Delta = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}(\sqrt{|g|}g^{ij}\frac{\partial}{\partial x^i})$, Ψ is a wave function of the universe, R is the scalar curvature and n is the dimension of the space [20]. The WDW equation is a quantised version of the Hamiltonian of a given system written using conjugate momenta $p_i = \frac{\partial L}{\partial x^i}$.

The WKB approximation, named after Wentzel, Kramers, and Brillouin, is the search for solutions of the WDW equation of the form $u = A_n e^{if(x^k)}$, where $f(x^k)$ has to satisfy the Hamilton-Jacobi equations [21, 22],

$$g^{ij} f_i f_j + V(x^k) = 0, \tag{34}$$

which is a first-order, nonlinear PDE. Furthermore, similar to the momenta above, $p_i = \frac{\partial f}{\partial x^i}$, so that the Hamilton-Jacobi equations lead to a reduced Hamiltonian system which can be solved to find solutions to the field equations.

Chapter 3

pp-wave potential functions: a complete study using Noether symmetries

In this chapter we examine the functional forms of the potentials $W(u, v, x^A)$ that emerge from the Lagrangian of the pp-wave spacetime. To facilitate this investigation, Noether symmetries are employed as well as their linear combinations and subalgebras. We exploit the geometric fact that Noether point symmetries of geodesic Lagrangians are generated from the homothetic algebra of spacetimes. Thus, we provide a complete analysis of the potentials of this spacetime, which are split into fourteen isometry categories.

3.1 Introduction

The plane-fronted gravitational waves with parallel rays, abbreviated as pp-waves, are a prominent class of solutions to Einstein's equations in general relativity. The term pp-wave was coined by Ehlers and Kundt, although there exists alternative definitions within the literature [23, 24]. The spacetime admits a nowhere zero covariantly constant null bisector field [23]. In relation to this, they admit, in general, only one independent KVF. The Petrov classification [14] is N or O, and the rank of the Riemann tensor is two.

Numerous and detailed studies of this metric tensor have been performed - all isometry classes are known [23]. The research by Sippel and Goenner [26] lists all possible pp-wave isometries, but without any field equations being used. For many types of vacuum pp-waves, Aichelburg studied curvature collineations [27] and [23] examined vacuum pp-wave properties to conclude that the pp-wave maximal isometry group is a one-parameter group generated by the ray vector. In Minkowski spacetime, pp-waves generalize the characteristic property of having parallel rays of electromagnetic plane waves [26]. The spacetime also features as free radiation models in alternate theories to gravitation, for instance, in string theory [28] as well as in unified field theories such as supergravity [29].

In the present work, we use Noether point symmetries of the pp-wave Lagrangian to establish all possible potential functions of the space. Specifically, Noether point symmetries originate from the homothetic algebra of a space. We tie these two concepts together and allow geometry to dictate the functional forms of the potential function. That is, we utilise the Noether symmetry conditions of the metric tensor to analyse the potential function, on the premise that the Noether point symmetry is already known. Such a creative approach leads to an extensive classification of

the potentials. Furthermore, since a Noether symmetry is a Lie symmetry, one may use the Lie invariant method to reduce the order of the field equations and solve these equations, corresponding to any potential listed below.

This chapter is organised as follows. In the next section we introduce the pp-wave spacetime and its Lagrangian, from which we obtain the geodesic equations of motion for this spacetime. We proceed to focus on the primary motivation of this chapter. Fourteen symmetry classes of this spacetime, divided into categories, are studied and the numerous potentials in each class are presented. Tables 3.1, 3.2, 3.3, 3.4, 3.5 and 3.6 display the additional potentials derived from linear combinations of Noether symmetries and their subalgebras. Finally, we provide concluding remarks.

3.2 The spacetime

The line element of a pp-wave spacetime is [24]

$$ds^2 = -2\,dudv - 2\,H\left(u, x^A\right)du^2 + dy^2 + dz^2, \tag{35}$$

where $x^A = (y, z)$ are the Cartesian coordinates. The space is vacuum if $H_{,yy}+H_{,zz} = 0$ and conformally flat if $H_{,yy} = H_{,zz}$ and $H_{,yz} = 0$, where a subscript along with a comma implies partial derivative. We note that if each point has a neighborhood that can be mapped to flat space (its Riemann curvature tensor is everywhere zero) by a conformal transformation, a (pseudo-) Riemannian manifold is conformally flat.

The non-zero connection coefficients are

$$\Gamma^v_{uu} = H_{,u} \quad , \quad \Gamma^A_{uu} = H_{,A} \quad , \quad \Gamma^v_{Au} = H_{,A}. \tag{36}$$

Further information on connection coefficients can be found in [25].

The Lagrangian of the pp-wave spacetime is

$$L(u,v,x^A) = \frac{1}{2}(\dot{y}^2+\dot{z}^2) - H(u,y,z)\dot{u}^2 - \dot{u}\dot{v} - W(u,v,x^A), \tag{37}$$

so that the corresponding geodesic equations of motion for the pp-wave spacetime are

$$\ddot{u} - W_{,v} = 0, \tag{38}$$

$$\ddot{v} + H_{,u}\dot{u}^2 + 2H_{,y}\dot{y}\dot{u} + 2H_{,z}\dot{u}\dot{z} - W_{,u} + 2HW_{,v} = 0, \tag{39}$$

$$\ddot{y} + H_{,y}\dot{u}^2 + W_{,y} = 0, \tag{40}$$

$$\ddot{z} + H_{,z}\dot{u}^2 + W_{,z} = 0. \tag{41}$$

The energy functional or Hamiltonian of the dynamical system, E, associated with the Lagrangian (37) is

$$E \;=\; \frac{1}{2}(\dot{y}^2+\dot{z}^2) - H(u,y,z)\dot{u}^2 - \dot{u}\dot{v} + W(u,v,x^A). \tag{42}$$

The classes listed below arise from classifications by [26] (see also [30]). Also in the literature, [31] studied the Noether symmetries of pp-wave spaces with vanishing potential, [32] explored the Noether symmetries of vacuum classes of pp-waves and the wave equation, while [33] classified the symmetries of the Klein-Gordon equation of pp-wave spacetimes. Our study is unique for the following two reasons:

- The potential is non vanishing, i.e. our primary objective is to define the potential functions of pp-wave spacetimes.
- We do not consider the Noether symmetries that contain the arc length parameter s. Instead we focus on the Noether symmetries that are either from the homethetic or Killing algebra of pp-waves. Thus solidifying the geometric nature of this investigation.

3.3 Category 1

The first case is where the metric function is generic, namely $H\left(u, x^A\right)$. The Noether point symmetry admitted by the Lagrangian (37) is the KVF $X_1^1 = \partial_v$. In turn, from the symmetry conditions, the corresponding potential function $W(u, v, x^A)$ generated is

$$X_1^1 : \quad W(u, v, x^A) = pv + F\left(u, x^A\right), \tag{43}$$

where p is a constant. If the metric function is restricted to $H(u, z)$, two extra Noether symmetries are obtained,

$$X_2^1 = \partial_y, \text{and } X_3^1 = y\partial_v + u\partial_y.$$

For these symmetries the potentials are

$$X_2^1 : \quad W(u, v, x^A) = py + F(u, v, z), \tag{44}$$

$$X_3^1 : \quad W(u, v, x^A) = \frac{yp}{u} + F\left(u, -2\,uv + y^2, z\right). \tag{45}$$

Note that each Noether point symmetry above is a Lie point symmetry. We are able to find more cases for the potential $W(u, v, x^A)$ by considering two Lie group properties. Therefore, a construction of linear combinations of the vector fields will also yield a point symmetry. Furthermore, real subalgebras of the Lie algebra that has the Noether point symmetries as its basis elements, will provide additional potential functions. In Table 3.1, we list the additional forms of $W(u, v, x^A)$ for this case and where the c_i are arbitrary constants.

Table 3.1: Additional symmetries and potentials functions of Category 1.

Category	**Symmetry Combination/Subalgebra**	**Potential Function**
1	$c_1X_1^1 + c_2X_2^1 + c_3X_3^1$	$\frac{py}{uc_3+c_2} + \frac{pc_1}{c_3(uc_3+c_2)}$ $+F\left(u, \frac{1}{2}\frac{-2\,uvc_3+y^2c_3-2\,vc_2+2\,c_1y}{uc_3+c_2}, z\right)$
1	$\{X_1^1, X_2^1\}$	$F(u,z) + (v+y)\,p$
1	$\{X_1^1, X_3^1\}$	$pv - \frac{1}{2}\frac{py(y-2)}{u} + F(u,z)$

3.4 Category 2

Consider the function $H(u,r)$, to which we obtain the pair of Noether symmetries

$$X_1^1,\ X_2^2 = \partial_\theta.$$

The potential function for the latter is

$$X_2^2: \quad W(u,v,r,\theta) = p\theta + F(u,v,r). \tag{46}$$

In the special case of $H(u,r) = K(\alpha\,u+\beta)^q \ln(r)$, with $\beta \in \Re$ and $K, \alpha \in \Re^*$, we have the additional Noether symmetry

$$X_3^2 = \frac{2}{2\alpha+\alpha q}\left[(\beta+\alpha u)\,\partial_u + \left(2\alpha q v - \frac{K(q+2)(\beta+\alpha u)^{q+1}}{2(q+1)}\right)\partial_v + \alpha r\frac{q+2}{2}\partial_r\right],$$

and corresponding potential

$$\begin{aligned} X_3^2 : W(u,v,r,\theta) &= (\alpha\,u+\beta)^{-3/2\,q-3/2}\,F\left(A_1(u,v), r\,(\alpha\,u+\beta)^{-q/2-1}, \theta\right) \\ &\quad + \frac{1}{3}\frac{p\,(q+2)}{q+1}, \end{aligned} \tag{47}$$

where the function $A_1(u,v)$ is

$$A_1(u,v) = \frac{(2\,q^2 v - 2\,v)\,\alpha\,(\alpha\,u+\beta)^{-2\,q} - K\,(\alpha\,u+\beta)\,(q+2)\,(\alpha\,u+\beta)^{-q}}{(2\,q^2-2)\,\alpha}. \quad (48)$$

For $q=-1$ the symmetry becomes

$$X^2_{3(q=-1)} = \frac{2}{\alpha}\left[(\beta+\alpha u)\,\partial_u - \frac{K}{2}\ln(\beta+\alpha u)\partial_v + \alpha r\partial_r\right].$$

So that for the potential, we obtain

$$\begin{aligned} W\,(u,v,r,\theta) &= \frac{1}{3}\frac{1}{(\alpha\,u+\beta)^{3/2}}\left[(\alpha\,u+\beta)^{3/2}\,p\right. \\ &\left. +3\,F\left(\frac{1}{4}\frac{K\,(\ln(\alpha\,u+\beta))^2+4\,v\alpha}{\alpha}, \frac{r}{\alpha\,u+\beta}, \theta\right)\right]. \end{aligned} \quad (49)$$

A second special case, with $H\,(u,r) = K\mathrm{e}^{-\frac{\Theta\,u}{\beta}}\ln(r)$ where $\Theta \in \Re$ and $K,\beta \in \Re^*$, admits the extra Noether symmetry

$$X^2_4 = -\frac{2\beta}{\Theta}\partial_u + \left(\frac{\beta K e^{-\frac{\Theta u}{\beta}}}{\Theta} + 2v\right)\partial_v + r\partial_r,$$

and potential function

$$X^2_4: \quad W\,(u,v,r,\theta) = \frac{p}{2} + \mathrm{e}^{\frac{\Theta\,u}{\beta}}\,F\left(\frac{1}{2}\,Ku + v\mathrm{e}^{\frac{\Theta\,u}{\beta}}, r\mathrm{e}^{1/2\,\frac{\Theta\,u}{\beta}}, \theta\right). \quad (50)$$

Where $\Theta = 0$ the symmetry is $X^2_{4(\Theta=0)} = u\partial_u + (v-Ku)\partial_v + r\partial_r$ and

$$X^2_{4(\Theta=0)}: \quad W\,(u,v,r,\theta) = \frac{p}{2} + \frac{1}{u^2}F\left(\frac{K\ln(u)\,u+v}{u}, \frac{r}{u}, \theta\right). \quad (51)$$

Another case that is of interest is $H(u,r) = e^{g(u)}\ln(r)$ with $g(u) = -\ln(\rho\,u^2+\alpha\,u+\beta) - \Theta\int(\rho\,u^2+\alpha\,u+\beta)^{-1}du$ and $\rho,\alpha,\beta \in \Re^*$, $\Theta \in \Re$, but does not admit any extra Noether symmetries. In Table 3.2 we list some extra potentials functions and define

the function

$$\begin{aligned} A_2(u,v) &= \left(4\,\alpha\,q^3c_3 - 4\,\alpha\,qc_3\right)^{-1} \times \\ &\quad \Big(4\,(q-1)\,((vc_3 + 1/4\,c_1)\,q + 1/2\,c_1)\,(q+1)\,\alpha\,(\alpha\,u+\beta)^{-2\,q} \\ &\quad -2\,k\,(\alpha\,u+\beta)^{-q}\,qc_3\,(\alpha\,u+\beta)\,(q+2)\Big). \end{aligned} \tag{52}$$

3.5 Category 3

For this case, consider the form $H = u^{-2}W(s,t)$ where the coordinates $\{s,t\}$ are given by the transformation

$$\begin{aligned} s &= y\sin(c\ln(u)) - z\cos(c\ln(u)), \\ t &= y\cos(c\ln(u)) + z\sin(c\ln(u)). \end{aligned}$$

Two vector fields X_1^1 and

$$X_2^3 = u\partial_u - v\partial_v,$$

make up the Noether symmetries of the geodesic Lagrangian, where the potential for the latter symmetry is

$$W(u,v,s,t) = p\ln(u) + F(vu,s,t). \tag{53}$$

Also, the symmetry vector $c_1X_1^1 + c_2X_2^3$, admits the potential function

$$W(u,v,s,t) = \frac{1}{c_2}\left(p\ln(u) + F\left(\frac{(vc_2 - c_1)\,u}{c_2}, s, t\right)c_2\right). \tag{54}$$

Table 3.2: Additional symmetries and potentials functions of Category 2.

Category	**Symmetry Combination/Subalgebra**	**Potential Function**
2	$c_1X_1^1 + c_2X_2^2$	$\frac{vp}{c_1} + F\left(u, r, \frac{\theta\, c_1 - v c_2}{c_1}\right)$
2	$c_1X_1^1 + c_2X_2^2 + c_3X_3^2$	$(\alpha\, u + \beta)^{-3/2\, q - 3/2}\, F\Big(A_2(u,v),$ $r\,(\alpha\, u + \beta)^{-q/2-1},$ $\frac{1}{2}\,\frac{-c_2 \ln(\alpha\, u+\beta) q + 2\,\theta\, c_3 - 2\, c_2 \ln(\alpha\, u+\beta)}{c_3}\Big)$ $+\frac{1}{3}\,\frac{p(q+2)}{(q+1)c_3}$
2	$c_1X_1^1 + c_2X_2^2 + c_3X^2_{3(q=-1)}$	$\frac{1}{3}\,\frac{1}{c_3(\alpha\, u+\beta)^{3/2}}\Bigg[(\alpha\, u + \beta)^{3/2}\, p + 3\times$ $F\Bigg(\frac{1}{4}\,\frac{K(\ln(\alpha\, u+\beta))^2 c_3 + 4\, v\alpha\, c_3 - 2\, c_1 \ln(\alpha\, u+\beta)\alpha}{\alpha\, c_3},$ $\frac{r}{\alpha\, u+\beta}, \frac{1}{2}\,\frac{2\,\theta\, c_3 - c_2 \ln(\alpha\, u+\beta)}{c_3}\Bigg) c_3\Bigg]$
2	$\{X_1^1, X_2^2\}$	$F\,(u, r) + (v + \theta)\, p$
2	$c_1X_1^1 + c_2X_2^2 + c_3X_4^2$	$\frac{1}{2c_3}\Bigg[2\,\mathrm{e}^{\frac{\Theta\, u}{\beta}} F\Bigg(\frac{1}{2c_3}\left(uKc_3 + (2\, vc_3 + c_1)\mathrm{e}^{\frac{\Theta\, u}{\beta}}\right),$ $r\mathrm{e}^{\frac{1}{2}\frac{\Theta\, u}{\beta}}, \frac{1}{2}\,\frac{c_2\Theta\, u + 2\,\theta\,\beta\, c_3}{c_3\beta}\Bigg) c_3 + p\Bigg]$
2	$c_1X_1^1 + c_2X_2^2 + c_3X^2_{4(\Theta=0)}$	$\frac{p}{2c_3} + \frac{1}{u^2} F\Big(\frac{K\ln(u)c_3 u + vc_3 + c_1}{c_3 u}, \frac{r}{u},$ $-\frac{c_2\ln(u) - \theta\, c_3}{c_3}\Big)$

3.6 Category 4

Suppose we apply the transformations

$$\begin{aligned} \bar{s} &= y\sin\left(c\left(u\right)\right) - z\cos\left(c\ \ \left(u\right)\right), \\ \bar{t} &= y\cos\left(c\left(u\right)\right) + z\sin\left(c\left(u\right)\right), \end{aligned} \tag{55}$$

so that the metric function is defined as $H = W(\bar{s}, \bar{t})$. In this case, one finds the two Noether symmetries

$$X_1^1,\ X_2^4 = \partial_u.$$

The associated potential for X_2^4 is

$$W\left(u, v, \bar{s}, \bar{t}\right) = pu + F\left(v, \bar{s}, \bar{t}\right), \tag{56}$$

and a linear combination of vector fields $c_1 X_1^1 + c_2 X_2^4$, gives the interesting potential

$$W\left(u, v, \bar{s}, \bar{t}\right) = \frac{up}{c_2} + F\left(\frac{-uc_1 + vc_2}{c_2}, \bar{s}, \bar{t}\right). \tag{57}$$

The potential function generated from the subalgebra $\{X_1^1, X_2^4\}$ is

$$W\left(u, v, \bar{s}, \bar{t}\right) = F\left(\bar{s}, \bar{t}\right) + \left(u + v\right)p. \tag{58}$$

3.7 Category 5

Let $H(u, r) = u^{-2}W(r)$, to find the three Noether symmetres

$$X_1^1,\ X_2^5 = \partial_\theta,\ X_3^5 = u\partial_u - v\partial_v.$$

The corresponding potentials are

$$X_2^5 : W\left(u, v, r, \theta\right) = p\theta + F\left(u, v, r\right), \tag{59}$$

and

$$X_3^5 : W(u,v,r,\theta) = p\ln(u) + F(vu,r,\theta). \tag{60}$$

Several similar definitions for $H(u,r)$ exist, such as $H(u,r) = \zeta u^{-2}\ln r$ and $H(u,r) = u^{-2}(\delta r^{-\sigma} - \sigma(2-\sigma)^{-2}r^2)$, $|\sigma| \neq 0, 2$, but these do not result in any additional Killing or homothety symmetries - hence we do not explore these subcases. In Table 3.3 we list additional forms of $W(u,v,r,\theta)$.

Table 3.3: Additional symmetries and potentials functions of Category 5.

Category	**Symmetry Combination/Subalgebra**	**Potential Function**
5	$c_1X_1^1 + c_2X_2^5 + c_3X_3^5$	$\frac{p}{c_3}\ln(u)$ $+F\left(\frac{(vc_3-c_1)u}{c_3}, r, -\frac{c_2\ln(u)-\theta\, c_3}{c_3}\right)$
5	$\{X_1^1, X_2^5\}$	$F(u,r) + (v+\theta)\,p$
5	$\{X_2^5, X_3^5\}$	$p\theta + p\ln(u) + F(vu,r)$

3.8 Category 6

Where $H\left(u, x^A\right) = W(r)$, the spacetime (35) admits the three Noether symmetries

$$X_1^1,\ X_2^2,\ X_3^6 = \partial_u,$$

whose corresponding potentials are Eq. (46) and

$$X_3^6 : W(u,v,r,\theta) = pu + F(v,r,\theta). \tag{61}$$

When $W(r)$ takes the form $W(r) = \frac{1}{4}Nr^2 + \frac{\delta}{r^2}$, we do not get any additional Killing or homothetic symmetries, but when we let $N = 0$ i.e. $W(r) = \frac{\delta}{r^2}$, we get an additional symmetry:

$$X_4^6 = 2u\partial_u + r\partial_r,$$

whose potential is

$$X_4^6 : W(u, v, r, \theta) = \frac{p}{2} + \frac{1}{u}F\left(v, \frac{r}{\sqrt{u}}, \theta\right). \tag{62}$$

Whereas when $W(r) = \zeta \ln r$, the additional Noether point symmetry is

$$X_5^6 = u\partial_u + (v - \zeta u)\partial_v + r\partial_r,$$

and this symmetry results in the potential function

$$X_5^6 : \quad W(u, v, r, \theta) = \frac{p}{2} + \frac{1}{u^2}F\left(\frac{\ln(u)\, u\zeta + v}{u}, \frac{r}{u}, \theta\right). \tag{63}$$

The final form of $W(r)$ that will be considered is $W(r) = \frac{\delta}{r^s}$ with $s \neq 2, 0$. Here we find,

$$X_6^6 = \frac{\sigma + 2}{2}u\partial_u + \frac{2 - \sigma}{2}v\partial_v + r\partial_r,$$

as the extra Noether vector, whose potential function is

$$X_6^6 : W(u, v, r, \theta) = p/2 + u^{-4(\sigma+2)^{-1}}F\left(vu^{\frac{-2+\sigma}{\sigma+2}}, ru^{-2(\sigma+2)^{-1}}, \theta\right). \tag{64}$$

Table 3.4 lists the additional potentials for this case and within the table, we define the functions

$$B_1(u, v) = \frac{\ln(uc_4 + c_3)\, u\zeta\, c_4 + \ln(uc_4 + c_3)\, \zeta\, c_3 + vc_4 + c_3\zeta + c_1}{c_4(uc_4 + c_3)}, \tag{65}$$

$$
\begin{aligned}
C_1(u) &= \sigma\, uc_4 + 2\, uc_4 + 2\, c_3, && (66)\\
C_2(u,v) &= \frac{1}{c_4\,(-2+\sigma)} C_1(u)^{-4\,(2+\sigma)^{-1}} \left(\sigma^2 uv{c_4}^2 - 2\,\sigma\, uc_1c_4 + 2\,\sigma\, vc_3c_4 \right. \\
&\quad \left. -4\, uv{c_4}^2 - 4\, uc_1c_4 - 4\, vc_3c_4 - 4\, c_1c_3 \right). && (67)
\end{aligned}
$$

.

3.9 Category 7

Consider $H\left(u, x^A\right) = \mathrm{e}^{2\,\omega\,\theta} W\left(r\right)$, from which we attain the Noether symmetries

$$
X_1^1,\ X_2^7 = \omega u \partial_u - \omega v \partial_v - \partial_\theta,\ X_3^7 = \partial_u,
$$

and the potential functions that follow are

$$
X_2^7 : \quad W\left(u, v, r, \theta\right) = \frac{1}{\omega}\left(p \ln\left(u\right) + F\left(vu, r, \frac{\omega\,\theta + \ln\left(u\right)}{\omega} \right) \omega \right), \tag{68}
$$

and the potential for X_3^7 corresponds to (61).

A linear combination of vector fields $c_1 X_1^1 + c_2 X_2^7 + c_3 X_3^7$, yields the potential

$$
\begin{aligned}
W\left(u, v, r, \theta\right) &= \frac{1}{\omega\, c_2}\left[F\left(v\omega\, uc_2 - c_1 u + vc_3, r, \frac{\omega\,\theta + \ln\left(\omega\, uc_2 + c_3\right)}{\omega} \right) \omega\, c_2 \right. \\
&\quad \left. + p \ln\left(\omega\, uc_2 + c_3\right) \right]. && (69)
\end{aligned}
$$

Table 3.4: Additional symmetries and potentials functions of Category 6.

Category	**Symmetry Combination/Subalgebra**	**Potential Function**
6	$c_1X_1^1 + c_2X_2^2 + c_3X_3^6$	$\frac{pu}{c_3} + F\left(\frac{-c_1u+vc_3}{c_3}, r, \frac{\theta\, c_3 - c_2u}{c_3}\right)$
6	$\{X_1^1, X_2^2\}$	$F(u,r) + (v+\theta)\,p$
6	$\{X_1^1, X_3^6\}$	$F(r,\theta) + (u+v)\,p$
6	$\{X_2^2, X_3^6\}$	$F(v,r) + (u+\theta)\,p$
6	$\{X_1^1, X_2^2, X_3^6\}$	$F(r) + (u+v+\theta)\,p$
6	$c_1X_1^1 + c_2X_2^2 + c_3X_3^6 + c_4X_4^6$	$\frac{1}{2\,uc_4+c_3}\left[pu + F\left(-\frac{c_1\ln(2\,uc_4+c_3)-2\,vc_4}{2c_4}, \frac{r}{\sqrt{2\,uc_4+c_3}}, -\frac{c_2\ln(2\,uc_4+c_3)-2\,\theta\,c_4}{2c_4}\right)\right]$
6	$c_1X_1^1 + c_2X_2^2 + c_3X_3^6 + c_4X_5^6$	$\frac{1}{2}\frac{1}{(uc_4+c_3)^2}\times \left[pc_4u^2 + 2pc_3u + 2F\left(B_1(u,v), \frac{r}{uc_4+c_3} - \frac{c_2\ln(uc_4+c_3)-\theta c_4}{c_4}\right)\right]$
6	$c_1X_1^1 + c_2X_2^2 + c_3X_3^6 + c_4X_6^6$	$\frac{1}{2c_4}\left[\left(C_1(u)^{1-\frac{-2+\sigma}{2+\sigma}}p + 2\,F\left(C_2(u,v), rC_1(u)^{-2\,(2+\sigma)^{-1}}, -\frac{-\theta\,c_4\sigma+2\,c_2\ln(C_1(u))-2\,\theta\,c_4}{c_4(2+\sigma)}\right)c_4\right)\times C_1(u)^{-4\,(2+\sigma)^{-1}}\right]$

The potential from subalgebra $\{X_1^1, X_3^7\}$ can be recovered from line 3, Table 3.4.

3.10 Category 8

In this category, let $H = \mathrm{e}^{2\,\delta\,t'} W\left(s'\right)$ where $t' = \eta y + \sigma z$, $s' = \eta z - \sigma y$, $\delta = -\frac{c}{\eta^2+\sigma^2}$ and $c, \eta, \sigma \in \Re - \{\zeta^2 \equiv \eta^2 + \sigma^2 = 0\}$.

When $\delta \neq 0$, the spacetime (35) admits the three Noether vectors

$$X_1^1,\ X_2^8 = u\partial_u,\ X_3^8 = \delta\left(u\partial_u - v\partial_v\right) - \partial_{t'},$$

and for $\delta = 0$, it admits four symmetries, where the extra Noether symmetry is

$$X_4^8 = t'\partial_v + \left(\eta^2 + \sigma^2\right) u\partial_t'.$$

The potential functions relating to these symmetries are

$$X_2^8 : W\left(u, v, t', s'\right) = 2\,p + \frac{F\left(v, t', s'\right)}{\sqrt{u}}, \tag{70}$$

$$X_3^8 : W\left(u, v, t', s'\right) = \frac{1}{\delta}\left(F\left(vu, \frac{\delta\,t' + \ln\left(u\right)}{\delta}, s'\right)\delta + p\ln\left(u\right)\right), \tag{71}$$

$$X_4^8 : W\left(u, v, t', s'\right) = \frac{t'p}{\zeta^2 u} + F\left(u, -2\,\zeta^2 uv + t'^2, s'\right). \tag{72}$$

When $\delta = 0$ and

$$W\left(s\right) = Ks^{\gamma}, \tag{73}$$

where $K, \gamma \in \Re^*$ and $\gamma \neq \pm 2, 1$, the added Noether symmetry is

$$X_5^8 = (1 - \frac{\gamma}{2}) u\partial_u + \left(1 + \frac{\gamma}{2}\right) v\partial_v + s'\partial_{s'} + t'\partial_{t'}.$$

The potential function associated with X_5^8 is

$$W(u, v, s', t') = \frac{p}{2} + u^{4(-2+\gamma)^{-1}} F\left(vu^{\frac{2+\gamma}{-2+\gamma}}, s'u^{2(-2+\gamma)^{-1}}, t'u^{2(-2+\gamma)^{-1}}\right). \tag{74}$$

In (73), if we let $\gamma = 2$,this in turn gives us two additional Noether symmetries

$$X_6^8 = \frac{\sqrt{2K}}{\zeta} s' \cos\left(\zeta\sqrt{2K}u\right)\partial_v + \sin\left(\zeta\sqrt{2K}u\right)\partial_{s'}, \tag{75}$$

$$X_7^8 = \frac{\sqrt{2K}}{\zeta} s' \sin\left(\zeta\sqrt{2K}u\right)\partial_v - \cos\left(\zeta\sqrt{2K}u\right)\partial_{s'}. \tag{76}$$

The potential functions from these particular symmetries are

$$\begin{aligned} X_6^8 : W(u, v, s', t') \;=\; & \frac{1}{\sqrt{K}\sin\left(\sqrt{2}\sqrt{K}\zeta\, u\right)}\left[F\left(u, \frac{s'^2K - E_1(u,v)}{K}, t'\right) \times \right. \\ & \sqrt{K}\sin\left(\sqrt{2}\sqrt{K}\zeta\, u\right) \\ & \left. +p\sqrt{\frac{\sqrt{K}\sin\left(\sqrt{2}\sqrt{K}\zeta\, u\right)\zeta\sqrt{2}v}{\cos\left(\sqrt{2}\sqrt{K}\zeta\, u\right)} - E_1(u,v) + s'^2K}\;\right], \end{aligned} \tag{77}$$

where

$$E_1(u, v) = \sqrt{K}\tan\left(\sqrt{2}\sqrt{K}\zeta\, u\right)\zeta\sqrt{2}v, \tag{78}$$

and

$$X_7^8 : W\left(u, v, s', t'\right) = \frac{1}{E_2(u)}\left[F\left(u, \right.\right.$$
$$\left.\frac{\cos\left(\sqrt{2}\sqrt{K}\zeta\, u\right)\sqrt{2}\sqrt{K}\zeta\, v + s'^2 K \sin\left(\sqrt{2}\sqrt{K}\zeta\, u\right)}{K\sin\left(\sqrt{2}\sqrt{K}\zeta\, u\right)}, t'\right) E_2(u)$$
$$\left. -p\sqrt{K\left(\sin\left(\sqrt{2}\sqrt{K}\zeta\, u\right)\right)^2 s'^2}\right], \tag{79}$$

with

$$E_2(u) = \sqrt{K}\sin\left(\sqrt{2}\sqrt{K}\zeta\, u\right)\cos\left(\sqrt{2}\sqrt{K}\zeta\, u\right). \tag{80}$$

In Table 3.5, we list the additional potentials for this category, and where the functions $D_1(u, t')$ and $D_2(u, v)$ are given by

$$D_1(u, t') = \frac{-u\zeta^2 c_4 + t'\delta\, c_3 + t' c_2 + c_3 \ln(u)}{\delta\, c_3 + c_2}, \tag{81}$$

$$\begin{aligned} D_2(u, v) = {} & -\delta^2 u\zeta^2 c_3 {c_4}^2 + 2\, v\delta^4 {c_3}^3 + 3\, v\delta^3 c_2 {c_3}^2 + v\delta^2 {c_2}^2 c_3 - 2\, D_1(u, t')\delta^3 {c_3}^2 c_4 + \\ & 2\, \ln(u)\, {c_3}^2 c_4 \delta^2 - 3\, D_1(u, t')\delta^2 c_2 c_3 c_4 - 2\, \delta^3 c_1 {c_3}^2 + c_2 c_4 \ln(u)\, \delta\, c_3 \\ & -D_1(u, t')\delta\, {c_2}^2 c_4 - 3\, \delta^2 c_1 c_2 c_3 - 2\, {c_3}^2 c_4 \delta^2 - \delta\, c_1 {c_2}^2 - 3\, c_2 c_4 \delta\, c_3 - {c_2}^2 c_4. \end{aligned} \tag{82}$$

3.11 Category 9

Under this category $H = K\mathrm{e}^{\eta\, y - \sigma\, z}$ with $K, \eta, \sigma \in \Re - \{\zeta^2 \equiv \eta^2 + \sigma^2 = 0\}$. From this we acquire a five dimensional algebra of symmetries, viz. X_1^1 and the symmetries

Table 3.5: Additional symmetries and potentials functions of Category 8.

Category	**Symmetry Combination/Subalgebra**	**Potential Function**
8	$c_1X_1^1 + c_2X_2^8 + c_3X_3^8$	$\frac{1}{c_2}\Big[u^{-1/2\,\frac{c_2}{\delta\,c_3+c_2}}\,F\Big(\frac{\delta\,vc_3-c_1}{\delta\,c_3}u^{\frac{\delta\,c_3}{\delta\,c_3+c_2}}, s', \frac{t'\delta\,c_3+t'c_2+c_3\ln(u)}{\delta\,c_3+c_2}\Big)c_2 + 2\,p\Big]$
8	$c_1X_1^1 + c_2X_2^8 + c_3X_3^8$ $+c_4X_{4(\delta=0)}^8$	$\frac{1}{c_2}\Bigg(u^{-1/2\,\frac{c_2}{\delta\,c_3+c_2}}\,F\Big[s', D_1(u,t'), \frac{D_2(u,v)}{\delta^2c_3(2\,\delta^2{c_3}^2+3\,\delta\,c_2c_3+{c_2}^2)}u^{\frac{\delta\,c_3}{\delta\,c_3+c_2}}\Big]c_2 + 2\,p\Bigg)$
8	$\{X_1^1, \partial_{t'}\}$	$F\,(u, s') + (t'+v)\,p$
8	$\{X_1^1, X_3^8\}$	$pv - 1/2\,\frac{pt'(t'-2)}{\zeta^2u} + F\,(u, s')$

$$
\begin{aligned}
X_2^9 &= u\partial_u, \\
X_3^9 &= u\partial_u - v\partial_v + \frac{1}{\sigma}\partial_z, \\
X_4^9 &= (y + \frac{\eta z}{\sigma})\partial_v + u\partial_y + \frac{\eta}{\sigma}u\partial_z, \\
X_5^9 &= \partial_y + \frac{\eta}{\sigma}\partial_z. \qquad (83)
\end{aligned}
$$

Below we list the potential functions corresponding to each symmetry mentioned above:

$$X_2^9 : W\left(u, v, x^A\right) = 2\,p + \frac{F\left(v, x^A\right)}{\sqrt{u}}, \qquad (84)$$

$$X_3^9 : W\left(u, v, x^A\right) = p \ln\left(u\right) + F\left(vu, y, -\frac{-\sigma\, z + \ln\left(u\right)}{\sigma}\right), \tag{85}$$

$$\begin{aligned} X_4^9 : W\left(u, v, x^A\right) &= \frac{py}{u} + F\Big(u, \frac{-\eta\, y + \sigma\, z}{\sigma}, \\ &\qquad 1/2\, \frac{2\, v\sigma^2 u - 2\, \eta\, \left(-\eta\, y + \sigma\, z\right) y - y^2\eta^2 - y^2\sigma^2}{\sigma^2 u}\Big), \end{aligned} \tag{86}$$

$$X_5^9 : W\left(u, v, x^A\right) = py + F\left(u, v, \frac{-\eta\, y + \sigma\, z}{\sigma}\right). \tag{87}$$

Table 6 consists of the additional potentials under this category where we define

$$\begin{aligned} G_1(u, v) &= \frac{1}{\sigma^2 {c_3}^2\left(c_2 + 2\, c_3\right)} u^{\frac{c_3}{c_2 + c_3}} \Bigg[2\, v\sigma^2 {c_3}^3 \\ &\quad + \left(v\sigma^2 c_2 + \left(-2\, \eta\, \sigma\, z - 2\, \sigma^2 y + 2\, \eta\right) c_4 - 2\, \sigma^2 c_1\right) {c_3}^2 \\ &\quad + \Bigg(\left(\left(-\eta\, \sigma\, z - \sigma^2 y + \eta\right) c_4 - \sigma^2 c_1\right) c_2 + c_4\left(\eta^2 + \sigma^2\right)\left(u c_4 + 2\, c_5\right) \Bigg) c_3 \\ &\quad + c_4 c_5 c_2\left(\eta^2 + \sigma^2\right) \Bigg]. \end{aligned} \tag{88}$$

3.12 Category 10

When $H(u, x^A)$ has the form

$$H\left(u, x^A\right) = \frac{1}{2}(A\left(u\right) y^2 + C(u) z^2) + B\left(u\right) yz, \tag{89}$$

Table 3.6: Additional symmetries and potentials functions of Category 9.

Category	**Symmetry Combination/Subalgebra**	**Potential Function**
9	$c_1X_1^1 + c_2X_2^9 + c_3X_3^9$ $+c_4X_4^9 + c_5X_5^9$	$\frac{1}{c_2}\left[u^{-\frac{c_2}{2\,c_2+2\,c_3}}\,F\left(\frac{-c_5\ln(u)+(c_2+c_3)y-uc_4}{c_2+c_3}, \frac{(-\eta\,c_5-c_3)\ln(u)-\eta\,uc_4+z\sigma\,c_2+z\sigma\,c_3}{\sigma\,(c_2+c_3)}, G_1(u,v)\right)c_2+2\,p\right]$
9	$\{X_1^1, X_4^9\}$	$pv + F\left(u, \frac{-\eta\,y+\sigma\,z}{\sigma}\right)$ $+\frac{1}{2}\,\frac{p\left((-y+2)\sigma^2-2\,\eta\,\sigma\,z+\eta^2y\right)y}{\sigma^2u}$
9	$\{X_1^1, X_5^9\}$	$F\left(u, \frac{-\eta\,y+\sigma\,z}{\sigma}\right) + (v+y)\,p$
9	$\{X_3^9, X_5^9\}$	$py + p\ln(u) + F\left(vu, \frac{-\eta\,y+\sigma\,z-\ln(u)}{\sigma}\right)$

the spacetime (35) is the well known special case, called a plane wave spacetime.

The spacetime (35) with (89) is a vacuum when $A(u) = -C(u)$, and admits the symmetries

$$X_a^{10} = (y\dot{d}_a(u) + z\dot{e}_a(u))\partial_v + d_a(u)\partial_y + e_a(u)\partial_z, \tag{90}$$

where $\dot{d}_a(u) = \frac{d}{du}(d_a)$ and the functions $d_a(u), e_a(u)$ satisfy the relations

$$\ddot{d}_a + Cd_a + Be_a = 0, \tag{91}$$

$$\ddot{e}_a + Bd_a + Ae_a = 0. \tag{92}$$

The potential function generated by X_a^{10} is

$$W\left(u,v,x^A\right)=\frac{py}{d_a}+W\left(u,v-\frac{\dot{e_a}}{d_a}yz-\frac{y^2}{2d_a^2}(\dot{d_A}d-\dot{e_a}e_a),z-\frac{e_a}{d_{Aa}}y\right). \tag{93}$$

Another Noether symmetry admitted is $X_6^{10}=2v\partial_v+y\partial_y+z\partial_z$, and the potential function that follows is

$$W\left(u,v,y,z\right)=\frac{p}{2}+\frac{1}{v}F\left(u,\frac{y}{\sqrt{v}},\frac{z}{\sqrt{v}}\right). \tag{94}$$

Finally, the linear combination $c_1X_1^1+c_aX_a^{10}+c_6X_6^{(10)}$ yields the potential function

$$W=\left(c_ad_a+c_6y\right)^{-2}\left(pc_ayd_a+\frac{1}{2}c_6y^2p+W\left(u,g(v,u,y,z),\frac{c_5z+c_ae_a}{c_5(c_ad_a+c_5y)}\right)\right). \tag{95}$$

3.13 Category 11

In Eq. (89), if we let

$$A(u)=\alpha u^{-2},\ B(u)=\beta u^{-2},\ C(u)=\gamma u^{-2},$$

we get the added Noether symmetry

$$X_7^{11}=u\partial_u-v\partial_v,$$

which gives the potential

$$W(u, v, x^A) = p \ln(u) + F\left(vu, x^A\right). \tag{96}$$

3.14 Category 12

In this category, suppose that

$$A(u) = -cu^{-2}(\sin\phi + \lambda),\ B(u) = cu^{-2}(\cos\phi),\ C(u) = cu^{-2}(\sin\phi - \lambda),$$

in Eq. (89), where $\phi = 2\delta \ln u$.

From this we get another Noether symmetry $X_7^{12} = X_7^{11} + \delta\partial_\theta$, and potential function

$$W(u, v, r, \theta) = p \ln(u) + F(vu, r, -\delta \ln(u) + \theta). \tag{97}$$

3.15 Category 13

When the functions in Eq. (89) are arbitrary constants, such as

$$A(u) = \alpha,\ B(u) = \beta,\ C(u) = c,$$

the related Noether symmetry is $X_7^{13} = \partial_u$, which gives us the potential function

$$W(u, v, x^A) = pu + F\left(v, x^A\right). \tag{98}$$

3.16 Category 14

If we let

$$A(u) = -c\sin(2\delta u) + \lambda,\ B(u) = -c\cos(2\delta u),\ C(u) = c\sin(2\delta u) + \lambda,$$

in Eq. (89), $X_7^{14} = \partial_u$ is the extra Noether symmetry. This generates the potential function

$$W(u, v, r, \theta) = pu + F(v, r, \theta). \tag{99}$$

3.17 Discussion

In recent years, point-like Noether symmetries have garnered a lot of interest in physics, particularly in the field of cosmology. In [34] the Noether symmetry approach was utilized in extended teleparallel gravity to select the functional forms, first integrals and exact solutions in flat Friedman-Robertson-Walker cosmology. The paper [35] successfully applied Noether symmetries and used various potential functional forms in the study of scalar-tensor teleparallel wormholes. Minisuperspace quantum cosmology was investigated using the Noether symmetry approach in [36]. Indeed, Noether symmetries have been central to the study of gravitational fields and used to select physically consistent theories in cosmology. Further related applications can be found in [37, 38, 39, 40]. This study has successfully provided a complete analysis through Noether symmetries of the pp-wave spacetime, and presents the functional forms of the potential functions. In particular, we considered fourteen categories of this spacetime, and looked at any special case within these categories. We noted that linear combinations of the vector fields will also give us potential functions. In addition to this, numerous subalgebras in each category were examined, from which we obtained vital potential functions. These extra

potentials from the linear combinations and subalgebras appear in tables. Lastly, one may apply the Lie invariant method to reduce the order and solve the field equations, corresponding to any potential listed above.

Chapter 4

Analytical field equation and wave function solutions of the Bianchi type I universe in vacuum f*(R)* gravity

It has been proposed that $f(R)$ gravity is a viable proxy for general relativity and a means for an alternative to dark energy. Within this context, we present the analytical solutions to the Bianchi type I universe. For this spacetime, the KVFs and HVFs, categorise the space into fourteen classes. Noether's theorem is used to determine the first integrals associated with each isometry for every class and we use the geometry of the point symmetries of the space to establish potential functions. We also quantise the Lagrangian of the spacetime, to derive the WDW equation and successively determine invariant solutions for the wave function of the universe from the Lie point symmetries admitted by the equation. Finally for all cases, exact

classical solutions to the field equations are found.

4.1 Introduction

In 1929, Edwin Hubble's famous paper provided evidence that the red shift of galaxies, then called 'spiral nebulae', is directly proportional to the distance of the galaxy from the earth [41]. This information that the universe was expanding changed the playing field of cosmology altogether, where prior to this astronomers believed the universe was an unchanging one. In fact, Einstein initially included the cosmological constant, $'\Lambda'$, to his field equations to keep the universe static. To supplement this, towards the end of the 20^{th} century, observational data from ten high-redshift Type Ia supernovae (SNe Ia) provided proof for an accelerating universe [42]. These supernovae are uniform and extreme in their brightness and thus a great nominee for standard candles. Direct measurements revealed that distant supernovae appeared dimmer than predicted, ergo the conclusion that the universe is accelerating in it's expansion. Moreover, the Wilkinson Microwave Anisotropy Probe (WMAP) satellite measurements of the Cosmic Microwave Background (CMB) [43] and several other independent tests [44, 45] have yielded the same result. In an attempt to explain the image painted by these observations, 'dark energy' became the term used to describe a hypothetical energy that admits a negative pressure, explaining this accelerating expansion of the cosmos. Notwithstanding the clear observational data, our fundamental understanding of dark energy is still elementary, and further underdeveloped by the lack of experimental evidence.

Many have suggested that the problem of dark energy could be viewed as a shortfall in the theory of general relativity, which predicts a deceleration given standard non-exotic matter. Enter $f(R)$ theory of gravity, a modified theory of gravity first

introduced by Hans Adolph Buchdahl in 1970 [46], which explains the accelerating expansion without the addition of exotic matter, providing an entirely geometric alternative to dark energy. This theory deviates from general relativity by replacing the Ricci Scalar, R, in the Einstein-Hilbert action, with a scalar curvature function, $f(R)$ [47]. As opposed to general relativity, $f(R)$ gravity relaxes the rigid hypothesis that limits the gravitational action to a linear function of R. In fact, there is no theoretical rationale for imposing such restrictions, allowing the theory to be generalized to include any curvature invariant and/or any scalar fields [48]. In terms of scalar curvature, the resulting phase transition from one state of the vacuum to another can be viewed as a transition from the metastable state with high curvature to the stable state with low curvature [49]. The theory was also used to develop the Starobinsky model of cosmic inflation, demonstrating early-time inflation, by taking the square of R in the Einstein-Hilbert action [50]. Furthermore, early and late-time inflation have been unified within the context of $f(R)$ gravity [51]. A concise historical review of the can be found in [52]. Favourably, examination of the robustness of this modified theory of gravity through observational tests have been successful [53, 54].

Through solving the Einstein field equations the Bianchi spacetime was derived, modelling a spatially homogeneous anisotropic universe, and as a result, essential for explaining the large-scale behaviour of the cosmos. These spacetimes are critical in developing models that describe the universe's early stages. Additionally, Friedmann-Lemaître-Robertson-Walker (FLRW) universes, describing an expanding homogenous and isotropic universe, are a subclass of the Bianchi universe. The spacetime has been discussed in relation to anisotropies in a primordial universe and its evolution to the observed isotropy of the current era [55]. For this reason studying anisotropic Bianchi spacetimes are of value to us. Some recent explorations of Bianchi spaces appear in [56, 57, 58].

Such complex ideas and their analyses may be facilitated by imposing the theory of symmetry generators. In our analysis, we use known Noether symmetries of the metric tensor to find and classify the potential functions of the spacetime. As mentioned, a Noether point symmetry is a Lie point symmetry therefore the method of Lie invariance can be used to reduce the order and solve the field equations for any of the potentials mentioned below. The wide applicability of Noether symmetries in cosmology in particular can be seen in the previous chapter and the works of [59, 60, 61].

This chapter is organized as follows. Section 4.2 contains the primary results of the chapter, divided into fourteen subsections or cases, each containing the symmetry details, field equation solutions, conservation laws, potential functions and wave function solutions. In section 4.3, we conclude.

4.2 The Bianchi type I spacetime in vacuum f*(R)* gravity

The line element of a Bianchi type I spacetimes in the coordinates (t, x, y, z), given by (x^0, x^1, x^2, x^3) is

$$ds^2 = -dt^2 + H^2dx^2 + E^2dy^2 + W^2dz^2, \tag{100}$$

where $H = H(t),\ E = E(t),\ W = W(t)$ are non-zero functions of t.

The Ricci scalar, R, for the Bianchi I spacetime (100) is

$$R = 2\left[\frac{H''}{H} + \frac{E''}{E} + \frac{W''}{W} + \frac{H'E'}{HE} + \frac{E'W'}{EW} + \frac{W'H'}{WH}\right], \tag{101}$$

where prime denotes the derivative with respect to t. The Einstein field equations (without matter) is

$$F(R)R_{ab} - \frac{1}{2}f(R)g_{ab} - \nabla_a\nabla_b F(R) + g_{ab}\Box F(R) = 0, \tag{102}$$

where $f(R)$ is the function of the Ricci scalar R, $F(R) = \frac{d}{dR}f(R)$, and $\Box \equiv \nabla^a\nabla_a$ where ∇^a is the covariant derivative. We take the assumption from the literature [62] that

$$F(R) = f_0 R^m,$$

where $f_0,\ m \in \Re$, and thus for the Bianchi I spacetime in vacuum $f(R)$ gravity, we take Eq. (100) and substitute it into Eq. (102), to obtain a set of equations to solve for the metric functions $H(t),\ E(t),\ W(t)$. Such a procedure, takes into account an isometry classification that leads to fourteen cases, see [63] for further details.

The spacetime (100) yields the three KVFs:

$$X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial y},\ \text{and}\ X_3 = \frac{\partial}{\partial z}. \tag{103}$$

The Lagrangian of the spacetime (100) is

$$\mathcal{L}(t,x,y,z) = \frac{1}{2}H\left(t\right)^2\dot{x}^2 + \frac{1}{2}E\left(t\right)^2\dot{y}^2 + \frac{1}{2}W(t)^2\dot{z}^2 - \frac{1}{2}\,\dot{t}^2 - V(t,x,y,z), \tag{104}$$

where dot represents derivative with respect to the arclength parameter s and $V(t,x,y,z)$ is the potential function yet to be determined.

Using $\mathcal{L}(t,x,y,z)$ we derive the geodesic equations of motion:

$$\begin{aligned}
\ddot{t} &= -E(t)\,\dot{y}^2\frac{\mathrm{d}}{\mathrm{d}t}E(t) - W(t)\,\dot{z}^2\frac{\mathrm{d}}{\mathrm{d}t}W(t) - H(t)\,\dot{x}^2\frac{\mathrm{d}}{\mathrm{d}t}H(t) + \frac{\partial}{\partial t}V, \\
\ddot{x} &= \frac{-2\,H(t)\,\dot{x}\left(\frac{\mathrm{d}}{\mathrm{d}t}H(t)\right)\dot{t} - \frac{\partial}{\partial x}V}{(H(t))^2}, \\
\ddot{y} &= \frac{-2\,E(t)\,\dot{y}\left(\frac{\mathrm{d}}{\mathrm{d}t}E(t)\right)\dot{t} - \frac{\partial}{\partial y}V}{(E(t))^2}, \\
\ddot{z} &= \frac{-2\,W(t)\,\dot{z}\left(\frac{\mathrm{d}}{\mathrm{d}t}W(t)\right)\dot{t} - \frac{\partial}{\partial z}V}{(W(t))^2}.
\end{aligned} \tag{105}$$

The Hamiltonian of the dynamical system, associated with the Lagrangian (104) is

$$\mathcal{E}(t,x,y,z) = \frac{1}{2}H(t)^2\,\dot{x}^2 + \frac{1}{2}E(t)^2\,\dot{y}^2 + \frac{1}{2}W(t)^2\dot{z}^2 - \frac{1}{2}\,\dot{t}^2 + V(t,x,y,z). \tag{106}$$

We note that all the cases below include, at the minimum, the KVFs listed in (103). A geometric approach, involving symmetries, will yield the potential functions corresponding to each symmetry generator [59]. The potential functions, $V(t,x,y,z)$, that correspond to these KVFs in Eq.(103) are

$$X_1 : V(t,x,y,z) = px + F(t,y,z)\,, \tag{107}$$

$$X_2 : V(t,x,y,z) = py + F(t,x,z)\,, \tag{108}$$

and

$$X_3 : V(t,x,y,z) = pz + F(t,x,y)\,, \tag{109}$$

where p is a constant. Since these KVFs appear in all fourteen cases, the very same potential functions will be found for every case.

The WDW equation, commonly used in quantum cosmology, is a useful guide for comprehending the quantum properties of spacetime. In fact the equation is fundamental to most methods to quantize gravity such as loop quantum gravity, quantum geometrodynamics, etc [64]. The WDW equation is the wave functional of gravity, initially called the "Einstein-Schrödinger equation," by John Wheeler, and is used in quantum gravity as a tool for understanding the quantum properties of spacetime. The equation has been a subject of interest recently: in the analysis of a multiscalar interaction model [65], an exploration of scalar tensor cosmological model adopting a FLRW geometry with a perfect fluid [66], and in a similar approach to some axisymmetric quantum cosmologies with scalar fields [67]. The WDW equation can be achieved by a standard quantization, and under our conditions of the Bianchi type I spacetime in vacuum *f(R)* gravity, we derive the equation to be a second-order differential equation

$$\begin{aligned} 0 &= -U(t,x,y,z)\,V(t,x,y,z) - \frac{\partial^2}{\partial t^2}U(t,x,y,z) - \frac{\left(\frac{\partial}{\partial t}U(t,x,y,z)\right)\frac{\mathrm{d}}{\mathrm{d}t}W(t)}{W(t)} - \\ &\quad \frac{\left(\frac{\partial}{\partial t}U(t,x,y,z)\right)\frac{\mathrm{d}}{\mathrm{d}t}E(t)}{E(t)} - \frac{\left(\frac{\partial}{\partial t}U(t,x,y,z)\right)\frac{\mathrm{d}}{\mathrm{d}t}H(t)}{H(t)} + \frac{\frac{\partial^2}{\partial x^2}U(t,x,y,z)}{(H(t))^2} \\ &\quad + \frac{\frac{\partial^2}{\partial y^2}U(t,x,y,z)}{(E(t))^2} + \frac{\frac{\partial^2}{\partial z^2}U(t,x,y,z)}{(W(t))^2}, \end{aligned} \tag{110}$$

where $U(t,x,y,z)$ is the wave function of the universe.

The WDW equation admits the five Lie point symmetries:

$$\begin{aligned} W_1 &= X_1, \\ W_2 &= X_2, \\ W_3 &= X_3, \\ W_U &= U\partial_U, \\ W_Q &= Q(t,x,y,z)\partial_U, \end{aligned} \tag{111}$$

where the function $Q(t,x,y,z)$ is a solution to the WDW equation (110). Clearly, Eq. (110) is linear in derivatives, but with four independent variables, the solutions will be quite involved. That is, several Lie point symmetries are required for (110) to enact its reduction to an ODE, and thereafter solve the resulting equation to find an explicit solution. In the subsections to follow, we present the fourteen cases under study, with the focus of obtaining Lie invariant solutions to the WDW equation and the field equations of motion for vanishing potential. In pursuit of these solutions, we mention the Noether first integrals related to each KVF (which is a Noether symmetry of the field equations) and the functional form of some extra potential functions admitted by the spacetime.

4.2.1 Case (I)

Under the first case, we have $H(t)=(c_1t+c_2)^{-1}$, $E(t)=(c_1t+c_2)$, W is a constant and $c_1,c_2\in\Re,\ c_1\neq 0$.

In association with these three KVFs (103), the respective conservation laws are:

$$\begin{aligned} I_1^I &= -\frac{\dot{x}}{(c_1t+c_2)^2}, \\ I_2^I &= -\dot{y}\,(c_1t+c_2)^2\,, \\ I_3^I &= -\dot{z}. \end{aligned} \tag{112}$$

This spacetime admits a fourth symmetry which is a proper HVF, viz.

$$H_{\rm I} = \left(\frac{c_1t+c_2}{c_1}\right)\partial_t + 2x\partial_x + z\partial_z, \tag{113}$$

that admits the conservation law

$$\begin{aligned} I(H_I) &= \frac{1}{\left(c_1 t + c_2\right)^2 c_1} \times \Big({c_1}^3 t^3 \dot{t} - {c_1}^3 t^2 z \dot{z} + 3\, {c_1}^2 c_2 t^2 \dot{t} \\ &\quad -2\, {c_1}^2 c_2 t z \dot{z} + 3\, c_1 {c_2}^2 t \dot{t} - c_1 {c_2}^2 z \dot{z} + {c_2}^3 \dot{t} - 2\, \dot{x} x c_1 \Big). \end{aligned} \tag{114}$$

Corresponding to the above HVF, is the potential functional

$$\begin{aligned} V(t,x,y,z) &= \frac{1}{2\left(t c_1 + c_2\right)^2} \Big(p {c_1}^2 t^2 + 2\, p c_1 c_2 t + \\ &\quad 2\, F\left(\frac{x}{\left(t c_1 + c_2\right)^2}, y, \frac{z}{t c_1 + c_2} \right) \Big). \end{aligned} \tag{115}$$

Concerning the field equations for this spacetime, we solve the system of equations (105) to obtain

$$\begin{aligned} t(s) &= -\frac{c_2}{c_1} + \frac{t_0}{\sqrt{\frac{\mathrm{d}}{\mathrm{d}s} y(s)}}, \\ x(s) &= x_0, \\ y(s) &= -\frac{1}{2} \frac{\ln\left(y_1\, y_2 + y_1\, s + 4\, c_1\right)}{c_1} + \frac{1}{2} \frac{\ln\left(y_1\, y_2 + y_1\, s - 4\, c_1\right)}{c_1} + \\ &\quad y_0, \\ z(s) &= z_1\, s + z_0, \end{aligned} \tag{116}$$

where all parameters with numerical subscripts are arbitrary constants in this chapter, unless stated otherwise.

Turning our attention to the wave function, Case (I) admits an extra Lie point symmetry for the WDW equation (110), namely

$$W_I = 2 c_1\, W_U + c_1 H_I. \tag{117}$$

We utilize the method of Lie invariants and apply the symmetries $\{W_2, W_3, W_I\}$ to

reduce Eq. (110) to an ODE and thus obtain the solution for the wave function ,viz.

$$\begin{aligned} U_I^1(t,x,y,z) = \;=\; & C_1 \left(4\frac{{c_1}^2x^2}{(c_1\,t+c_2)^4}-1\right)^{1/12}\times \\ & LegendreP\left(-\frac{1}{4},\frac{1}{6},2\frac{c_1\,x}{(c_1\,t+c_2)^2}\right) \\ & +C2\,\left(4\frac{{c_1}^2x^2}{(c_1\,t+c_2)^4}-1\right)^{1/12}\times \\ & LegendreQ\left(-\frac{1}{4},\frac{1}{6},2\frac{c_1\,x}{(c_1\,t+c_2)^2}\right), \end{aligned} \tag{118}$$

where the C_1, C_2 are arbitrary constants, $LegendreP$ denotes the associated Legendre functions of the first kind, and $LegendreQ$ is the associated Legendre functions of the second kind. Note that this solution only contains the variables x, t, but for a solution dependent on all four variables, we apply the symmetry combinations $\{Y_1 = W_2 + W_U, Y_2 = W_3 + W_U, Y_3 = W_1 + W_U\}$ whose Lie brackets are zero. The solution we obtain, after successive reductions and solving the resultant ODE is

$$\begin{aligned} U_I^2(t,x,y,z) \;=\; & \mathrm{e}^{x+y+z}\Bigg(C_1\,\mathrm{e}^{-\frac{1}{2}\,c_1t^2-c_2t}\,(tc_1+c_2)^{\frac{1}{2}\sqrt{\frac{{c_1}^2+4}{{c_1}^2}}+\frac{1}{2}}\times \\ & HeunB\left(\sqrt{\frac{{c_1}^2+4}{{c_1}^2}},0,-{c_1}^{-1},0,-\frac{tc_1+c_2}{\sqrt{c_1}}\right)+ \\ & C_2\,\mathrm{e}^{-\frac{1}{2}\,c_1t^2-c_2t}\,(tc_1+c_2)^{-\frac{1}{2}\sqrt{\frac{{c_1}^2+4}{{c_1}^2}}+\frac{1}{2}}\times \\ & HeunB\left(-\sqrt{\frac{{c_1}^2+4}{{c_1}^2}},0,-{c_1}^{-1},0,-\frac{tc_1+c_2}{\sqrt{c_1}}\right)\Bigg), \end{aligned} \tag{119}$$

where the $HeunB$ function is the solution of the Heun Biconfluent equation.

4.2.2 Case (II)

The second case is similar to the first with $H(t) = (c_1 t + c_2)^{-1}$, $W(t) = (c_1 t + c_2)$, E is a constant and $c_1, c_2 \in \Re,\ c_1 \neq 0$. The HVF of this case is the same as those of Case (I).

The conservation laws associated Eq. (103) are

$$I_1^{II} = -\frac{\dot{x}}{(c_1 t + c_2)^2}, \tag{120}$$

$$I_2^{II} = -\dot{y}, \tag{121}$$

and

$$I_3^{II} = -{c_1}^2 t^2 \dot{z} - 2\, c_1 c_2 t \dot{z} - {c_2}^2 \dot{z}. \tag{122}$$

Finally, the first integral from Eq.(113) is

$$\begin{aligned} I^{II}(H_I) \;=\; & -\frac{1}{(c_1 t + c_2)^2\, c_1}\Big({c_1}^5 t^4 z\dot{z} + 4\,{c_1}^4 c_2 t^3 z\dot{z} + 6\,{c_1}^3 {c_2}^2 t^2 z\dot{z} + \\ & 4\,{c_1}^2 {c_2}^3 t z\dot{z} - {c_1}^3 t^3 \dot{t} + c_1 {c_2}^4 z\dot{z} - 3\,{c_1}^2 c_2 t^2 \dot{t} - 3\, c_1 {c_2}^2 t\dot{t} - \\ & {c_2}^3 \dot{t} + 2\, c_1 x\dot{x}\Big). \end{aligned} \tag{123}$$

Solving the system (105) we obtain, for the field equations

$$\begin{aligned} t(s) &= -\frac{c_2}{c_1} + \frac{t_0}{\sqrt{\frac{\mathrm{d}}{\mathrm{d}s} z\,(s)}}, \\ x(s) &= x_0, \\ y(s) &= y_1\, s + y_0, , \\ z(s) &= -\frac{1}{2}\,\frac{\ln\left(z_1\, z_2 + z_1\, s + 4\, c_1\right)}{c_1} + \frac{1}{2}\,\frac{\ln\left(z_1\, z_2 + z_1\, s - 4\, c_1\right)}{c_1} + \\ & \quad z_0. \end{aligned} \tag{124}$$

The WDW equation of this case, admits the same symmetries as Case (I), so as before, we first reduce the WDW equation with the point symmetries $\{W_2, W_3, W_I\}$ and obtain the wave function solution

$$\begin{aligned} U_{II}^{1}\,(t, x, y, z) &= C_1 + \frac{x C_2}{\left(t c_1 + c_2\right)^2} \times \\ & \quad \mathit{hypergeom}\Big(\left[\frac{1}{2}, \frac{3}{4}\right], \left[\frac{3}{2}\right], 4\,\frac{{c_1}^2 x^2}{\left(t c_1 + c_2\right)^4}\Big), \end{aligned} \tag{125}$$

where $hypergeom$ is the generalized hypergeometric function.

Lastly, a reduction by $\{Y_1, Y_2, Y_3\}$ gives us the solution identical to that given by (119).

4.2.3 Case (III)

With $E(t) = (c_1 t + c_2)^{-1}$, $W(t) = (c_1 t + c_2)$, and H a constant, this case has identical KVFs and HVFs as the previous two cases.
Below are the conservation laws linked to Eq. (103):

$$
\begin{aligned}
I_1^{III} &= -\dot{x}, \\
I_2^{III} &= -\frac{\dot{y}}{\left(c_1 t + c_2\right)^2}, \\
I_3^{III} &= -\dot{z}\left({c_1}^2 t^2 + 2\, c_1 c_2 t + {c_2}^2\right).
\end{aligned} \tag{126}
$$

and that associated with the proper HVF Eq.(113)

$$
I^{III}(H_I) - \frac{{c_1}^3 t^2 z\dot{z} + 2\,{c_1}^2 c_2 t z\dot{z} + c_1 {c_2}^2 z\dot{z} - c_1 t t_1 + 2\, c_1 x\dot{x} - c_2\dot{t}}{c_1}. \tag{127}
$$

The geodesic solutions from the system of equations (105) are

$$
\begin{aligned}
t(s) &= -\frac{c_2}{c_1} + \frac{t_0}{\sqrt{\frac{\mathrm{d}}{\mathrm{d}s} z\,(s)}}, \\
x(s) &= x_1 s + x_0, \\
y(s) &= y_0, \\
z(s) &= -\frac{1}{2}\frac{\ln\left(z_1\, z_2 + z_1\, s + 4\, c_1\right)}{c_1} + \frac{1}{2}\frac{\ln\left(z_1\, z_2 + z_1\, s - 4\, c_1\right)}{c_1} + \\
&\quad z_0.
\end{aligned} \tag{128}
$$

Using the Lie point symmetries $\{W_1, W_3, W_I\}$ to do a symmetry analysis for the WDW equation, the equation (110) admits the solution for the wave function

$$
\begin{aligned}
U_{III}^1\,(t, x, y, z) &= C_1 + \frac{y C_2}{(t c_1 + c_2)} \times \\
&\quad hypergeom\Big(\left[\frac{1}{2}, \frac{3}{4}\right], \left[\frac{3}{2}\right], 4\,\frac{{c_1}^2 y^2}{\left(t c_1 + c_2\right)^4}\Big),
\end{aligned} \tag{129}
$$

and with $\{Y_1, Y_2, Y_3\}$, once again we find a solution given by (119).

4.2.4 Case (IV)

For this case, let $H(t) = (c_1 t + c_2)^{-2}$, $E(t) = W(t) = (c_1 t + c_2)$.

The conservation laws in this spacetime associated with the KVFs in Eq. (103) are

$$I_1^{IV} = -\frac{\dot{x}}{(c_1 t + c_2)^4}, \tag{130}$$

$$I_2^{IV} = -\dot{y}\,(c_1 t + c_2)^2\,, \tag{131}$$

and

$$I_3^{IV} = -\dot{z}\,(c_1 t + c_2)^2\,. \tag{132}$$

Here we have a fourth extra KVF,

$$X_4^{\mathrm{IV}} \;=\; y\partial_z - z\partial_y, \tag{133}$$

which admits the Noether integral

$$I_4^{\mathrm{IV}} \;=\; -\,(c_1 t + c_2)^2\,(y\dot{z} - z\dot{y})\,, \tag{134}$$

and the potential function

$$V\,(t, x, y, z) = -p \arctan\left(\frac{y}{z}\right) + F\left(t, x, y^2 + z^2\right). \tag{135}$$

The fifth symmetry for this case is a proper HVF, given as

$$H_{\mathrm{IV}} = (c_1 t + c_2)\partial_t + 3c_1 x \partial_x, \tag{136}$$

and an integral

$$I(H_{\mathrm{IV}}) = \frac{(c_1 t + c_2)^5 \dot{t} - 3\, c_1 x x_1}{(c_1 t + c_2)^4}. \tag{137}$$

Using this HFV, we derive the potential function

$$V(t, x, y, z) = \frac{1}{2\,(t c_1 + c_2)^2}\left(p c_1 t^2 + 2\, p c_2 t + 2\, F\left(\frac{x}{(t c_1 + c_2)^3}, y, z\right)\right). \tag{138}$$

As before, we now give solutions for the equation of motion (105) for Case (IV):

$$\begin{aligned} t(s) &= -\frac{c_2}{c_1} + \frac{t_0}{\sqrt{\frac{\mathrm{d}}{\mathrm{d}s} z(s)}}, \\ x(s) &= x_0, \\ y(s) &= y_0, \\ z(s) &= -\frac{1}{2}\frac{\ln(z_1 z_2 + z_1 s + 4 c_1)}{c_1} + \frac{1}{2}\frac{\ln(z_1 z_2 + z_1 s - 4 c_1)}{c_1} + \\ & \quad z_0. \end{aligned} \tag{139}$$

Here we have an additional Lie point symmetry of the WDW equation (110) in relation to the fourth KVF (133), in the form of

$$W^1_{IV} = \ = \ -y\partial_z + z\partial_y, \tag{140}$$

As stated earlier, we will utilize the zeroth-order invariants of the Lie point symmetries to reduce the order of the WDW equation and find invariant solution. The

symmetries used are $\{W_1, W_2, W_3\}$ from which we obtain the solution (only dependent on the parameter t):

$$U_{IV}^{1}(t) = C_1\, t + C_2. \tag{141}$$

Furthermore, we use the set of Lie point symmetries $\{Y_1, Y_2, Y_3\}$ to get the wave function solution

$$\begin{aligned} U_{IV}^{2} &= \mathrm{e}^{x+y+z}\left(C_1 \sqrt{tc_1 + c_2}\, BesselJ\left(\frac{1}{6}\,\frac{\sqrt{{c_1}^2 + 8}}{c_1}, \left(\frac{i/3\,(tc_1 + c_2)^3}{c_1} \right) \right) + \right. \\ &\left. C_2 \sqrt{tc_1 + c_2}\, BesselY\left(\frac{1}{6}\,\frac{\sqrt{{c_1}^2 + 8}}{c_1}, \left(\frac{i/3\,(tc_1 + c_2)^3}{c_1} \right) \right) \right), \end{aligned} \tag{142}$$

where $BesselJ$ and $BesselY$ are the Bessel functions of the first and second kinds, respectively.

The next two cases, (V) and (VI), admit the same extra vector fields as this one.

4.2.5 Case (V)

In this scenario $E(t) = (c_1 t + c_2)^{-2}$, $H(t) = W(t) = (c_1 t + c_2)$.

The first integrals in this spacetime associated with the KVFs in Eq. (103) are

$$I_1^V = -\dot{x}\left({c_1}^2 t^2 + 2\, c_1 c_2 t + {c_2}^2 \right) \tag{143}$$

$$I_2^V = -\frac{\dot{y}}{(c_1 t + c_2)^4} \tag{144}$$

and

$$I_3^V = -\dot{z}\left({c_1}^2 t^2 + 2\,c_1 c_2 t + {c_2}^2\right) \tag{145}$$

Moreover, the KVF (133) admits the Noether integral

$$I_4^{\rm V} = \frac{-y\left(c_1 t + c_2\right)^6 \dot{z} + z\dot{y}}{\left(c_1 t + c_2\right)^4}, \tag{146}$$

and the HVF (136) admits

$$I(H_{\rm V}) = -3\left({c_1}^2 x t\dot{x} + c_1 c_2 x\dot{x} - \frac{1}{3}\,\dot{t}\right)\left(c_1 t + c_2\right). \tag{147}$$

For the field equation solutions for Case (V), we find that they are the same as that for Case (IV). In a similar fashion, the WDW solutions are also the same as that of Case (IV).

4.2.6 Case (VI)

In this scenario $W(t) = (c_1 t + c_2)^{-2}$, $H(t) = E(t) = (c_1 t + c_2)$.

The first integrals in this spacetime associated with the KVFs in Eq. (103) are

$$\begin{aligned} I_1^{VI} &= -\dot{x}\left({c_1}^2 t^2 + 2\,c_1 c_2 t + {c_2}^2\right), \\ I_2^{VI} &= -\dot{y}\left({c_1}^2 t^2 + 2\,c_1 c_2 t + {c_2}^2\right), \\ I_3^{VI} &= -\frac{\dot{z}}{\left(c_1 t + c_2\right)^4}. \end{aligned} \tag{148}$$

From KVF (133) we have the Noether integral

$$I_4^{\rm VI} = \frac{z\left(c_1 t + c_2\right)^6 \dot{y} - y\dot{z}}{\left(c_1 t + c_2\right)^4}, \tag{149}$$

and the HVF (136) admits the first integral (147).

We now provide solutions for the equation of motion, just as we did before:

$$\begin{aligned} t(s) &= -\frac{c_2}{c_1} + \frac{t_0}{\sqrt{\frac{\mathrm{d}}{\mathrm{d}s} y\,(s)}}, \\ x(s) &= x_0, \\ y(s) &= 1/2\,\frac{\ln\left(y_1\, y_2 + y_1\, s - 4\, c_1\right)}{c_1} - 1/2\,\frac{\ln\left(y_1\, y_2 + y_1\, s + 4\, c_1\right)}{c_1} + y_0, \\ z(s) &= z_0. \end{aligned} \tag{150}$$

The WDW solutions are identical to those in Case (IV).

4.2.7 Case (VII)

In this case suppose that $H(t)$ is a constant, whereas $E(t) = W(t) = t^{-\frac{1}{2}}$.
Under this case, the conservation laws associated with (103) are:

$$I_1^{VII} = -\dot{x}, \tag{151}$$

$$I_2^{VII} = -\frac{\dot{y}}{t}, \tag{152}$$

and

$$I_3^{VII} = -\frac{\dot{z}}{t}. \tag{153}$$

This spacetime has a fourth KVF given by (133) and corresponding first integral

$$I_4^{\mathrm{VII}} = -\frac{\dot{z}y - \dot{y}z}{t}, \tag{154}$$

and also, corresponding to this vector field is the potential function given by (135). Moreover, a fifth symmetry, a proper HVF given by

$$H_{\mathrm{VII}} = t\partial_t + x\partial_x + \frac{3}{2}y\partial_y + \frac{3}{2}z\partial_z, \tag{155}$$

with the conservation law

$$I(H_{\mathrm{VII}}) = \frac{1}{2}\frac{2\,t^2\dot{t} - 2\,\dot{x}xt - 3\,\dot{y}y - 3\,\dot{z}z}{t}. \tag{156}$$

The HVF yields a potential $V(t,x,y,z)$ of the form

$$V\left(t,x,y,z\right) = \frac{p}{2} + \frac{1}{t^2}F\left(\frac{x}{t}, \frac{y}{t^{3/2}}, \frac{z}{t^{3/2}}\right). \tag{157}$$

Below are the solutions for the geodesics of this case, that is, the solutions for the system (105):

$$\begin{aligned} t(s) &= \frac{1}{2}\frac{\sqrt[3]{4}\sqrt[3]{\left(\frac{\mathrm{d}^3}{\mathrm{d}s^3}z\left(s\right)\right)^2}\frac{\mathrm{d}}{\mathrm{d}s}z\left(s\right)}{\frac{\mathrm{d}^3}{\mathrm{d}s^3}z\left(s\right)}, \\ x(s) &= x_1\,s + x_0, \\ y(s) &= y_0, \\ z(s) &= \frac{1}{6}\,z_3\,s^3 + \frac{1}{2}\,z_2\,s^2 + z_1\,s + z_0. \end{aligned} \tag{158}$$

Next, we investigate the WDW equation (110) of this case. We discover the additional symmetries given by (140) and:

$$W_{VII} = H_{\mathrm{VII}}. \tag{159}$$

As we have seen, when invariant under one-parameter point transformations, the WDW equation can be solved by reduction, so similar to Case (I) but replacing W_I

with W_{VII}, the symmetries yield the invariant two parameter solution for the wave function

$$U^1_{VII}(t,x) = C_1 + \frac{xC_2}{t}\frac{1}{\sqrt{\frac{x^2}{t^2}-1}}. \tag{160}$$

A second solution, using the same technique, yields the solution of the WDW equation as

$$U^2_{VII}(t,x,y,z) = T(t), \tag{161}$$

where $T(t)$ satisfies the equation

$$2\,T(t)\,t^2 + T(t)\,t - \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}T(t)\right)t + \frac{\mathrm{d}}{\mathrm{d}t}T(t) = 0. \tag{162}$$

4.2.8 Case (VIII)

Assume that E is a constant, and $H(t) = W(t) = t^{-\frac{1}{2}}$. As this spacetime is quite similar to the previous case, we have the same KVFs and HVFs.
The Noether symmetries (103) admits the following conservation laws:

$$\begin{aligned} I_1^{VIII} &= -\frac{\dot{x}}{t}, \\ I_2^{VIII} &= -\dot{y}, \\ I_3^{VIII} &= -\frac{\dot{z}}{t}. \end{aligned} \tag{163}$$

From KVF (133) we have the Noether integral

$$I_4^{\mathrm{VIII}} = -\frac{-zt\dot{y} + y\dot{z}}{t}, \tag{164}$$

and from the HVF (155)

$$I(H_{\mathrm{VIII}}) = 1/2\,\frac{2\,t^2\dot{t} - 3\,ty\dot{y} - 2\,x\dot{x} - 3\,z\dot{z}}{t}. \tag{165}$$

The solutions for the equations of motion for this case, that is, the solutions for (105) are:

$$\begin{aligned} t(s) &= \frac{1}{2}\,\frac{\sqrt[3]{4}\sqrt[3]{\left(\frac{\mathrm{d}^3}{\mathrm{d}s^3}z\,(s)\right)^2}\,\frac{\mathrm{d}}{\mathrm{d}s}z\,(s)}{\frac{\mathrm{d}^3}{\mathrm{d}s^3}z\,(s)}, \\ x(s) &= x_0, \\ y(s) &= y_1 s + y_0, \\ z(s) &= \frac{1}{6}\,z_3\,s^3 + \frac{1}{2}\,z_2\,s^2 + z_1\,s + z_0. \end{aligned} \tag{166}$$

Similar to the reduction in (VII), we find the solution for (110) as

$$\begin{aligned} U^1_{VIII}\,(t,x) &= C_1 + \frac{x\,C_2}{t^{3/2}} \times \\ &\quad hypergeom\Big(\left[\frac{1}{2},\frac{7}{6}\right],\left[\frac{3}{2}\right],\frac{9}{4}\,\frac{x^2}{t^3}\Big), \end{aligned} \tag{167}$$

A second solution, $U^2_{VIII}\,(t,x,y,z)$ is identical to that of Case (VII) i.e. solution (161).

4.2.9 Case (IX)

With a spacetime where W is a constant, whereas $H(t) = E(t) = t^{-\frac{1}{2}}$, quite similar to the previous Case (VI) and (VII), we have the same vector fields.
From the KVFs (103) we obtain the Noether integrals:

$$\begin{aligned} I^{IX}_1 &= -\frac{\dot{x}}{t}, \\ I^{IX}_2 &= -\frac{\dot{y}}{t}, \\ I^{IX}_3 &= -\dot{z}. \end{aligned} \tag{168}$$

From KVF (133) we have the Noether integral

$$I_4^{\mathrm{IX}} = -\frac{yt\dot{z} - z\dot{y}}{t}, \tag{169}$$

and from the HVF (155)

$$I(H_{\mathrm{IX}}) = 1/2\,\frac{2\,t^2\dot{t} - 3\,tz\dot{z} - 2\,x\dot{x} - 3\,y\dot{y}}{t}. \tag{170}$$

The classical field equation solutions of (105) are:

$$\begin{aligned} t(s) &= \frac{1}{2}\,\frac{\sqrt[3]{4}\sqrt[3]{\left(\frac{\mathrm{d}^3}{\mathrm{d}s^3}y\left(s\right)\right)^2}\frac{\mathrm{d}}{\mathrm{d}s}y\left(s\right)}{\frac{\mathrm{d}^3}{\mathrm{d}s^3}y\left(s\right)}, \\ x(s) &= x_0, \\ y(s) &= \frac{1}{6}\,y_3\,s^3 + \frac{1}{2}\,y_2\,s^2 + y_1\,s + y_0, \\ z(s) &= z_1 s + z_0. \end{aligned} \tag{171}$$

For the WDW solutions of (110) for this case, solutions are given by the expressions in (167) and (161).

4.2.10 Case (X)

For the tenth case, let $H(t) = t^{-1}$ with $E(t)$ and $W(t)$ being constants. The first integrals associated with (103) are:

$$I_1^X = -\frac{\dot{x}}{t^2}, \tag{172}$$

$$I_2^X = -\dot{y}, \tag{173}$$

and

$$I_3^X = -\dot{z}. \tag{174}$$

Similar to Case VII this spacetime has a fourth KVF

$$X_4^{\mathrm{X}} = y\partial_z - z\partial_y, \tag{175}$$

and the same potential function (135). However, the conservation law corresponding to this symmetry is

$$I_4^{\mathrm{X}} = -\dot{z}y + \dot{y}z, \tag{176}$$

and a fifth symmetry, a proper HVF given by

$$H_{\mathrm{X}} = t\partial_t + 2x\partial_x + y\partial_y + z\partial_z. \tag{177}$$

The conservation law for this symmetry, with $V(t,x,y,z) = p(x+y+z)$, is

$$I(H_{\mathrm{X}}) = \frac{\dot{t}t^3 - \dot{y}yt^2 - \dot{z}zt^2 - 2\,\dot{x}x}{t^2}. \tag{178}$$

This has a corresponding potential of

$$V\left(t,x,y,z\right) = \frac{p}{2} + \frac{1}{t^2}F\left(\frac{x}{t^2}, \frac{y}{t}, \frac{z}{t}\right). \tag{179}$$

Below, we find the classical field equations solutions:

$$\begin{aligned}
t(s) &= \frac{\sqrt{2}\frac{\mathrm{d}}{\mathrm{d}s}x\left(s\right)}{\sqrt[4]{2\,\left(\frac{\mathrm{d}^3}{\mathrm{d}s^3}x\left(s\right)\right)\frac{\mathrm{d}}{\mathrm{d}s}x\left(s\right) - \left(\frac{\mathrm{d}^2}{\mathrm{d}s^2}x\left(s\right)\right)^2}}, \\
x(s) &= \frac{1}{8}\,x_1^4x_2^2\mathrm{e}^{\frac{x_3+s}{x_1}} - \frac{1}{2}\,x_1^2\mathrm{e}^{-\frac{x_3+s}{x_1}} - \frac{1}{2}\,x_1^2x_2\,s \\
&\quad + x_0, \\
y(s) &= y_1\,s + y_0, \\
z(s) &= z_1\,s + z_0.
\end{aligned} \tag{180}$$

For this case we have two additional WDW symmetries, namely the point symmetry (140) and

$$W_X = H_{\rm X}. \tag{181}$$

Performing a symmetry analysis for the WDW equation using the Lie point symmetries $\{W_1, W_3, W_2\}$, Eq. (110) admits the solution for the wave function

$$U_X^1(t,x) = C_2\, t^2 + C_1, \tag{182}$$

and a second, four variable invariant solution

$$\begin{aligned} U_X^2(t,x,y,z) &= \mathrm{e}^{x+y+z}\Big(C_1\, t^2\left(BesselI\left(0,\frac{1}{2}\,t^2\right) + BesselI\left(1,\frac{1}{2}\,t^2\right)\right) + \\ &\quad C_2\, t^2\left(BesselK\left(0,\frac{1}{2}\,t^2\right) - BesselK\left(1,\frac{1}{2}\,t^2\right)\right)\Big), \end{aligned} \tag{183}$$

where $BesselI$ and $BesselK$ are the modified Bessel functions of the first and second kinds, respectively.

4.2.11 Case (XI)

Similar to spacetime of the tenth case, let $E(t) = t^{-1}$ with $H(t)$ and $W(t)$ being constants. Subsequently, the conformal vector fields are the same.
Utilizing the KVFs (103) we calculate the first integrals as:

$$\begin{aligned} I_1^{XI} &= -\dot{x}, \\ I_2^{XI} &= -\frac{\dot{y}}{t^2}, \\ I_3^{XI} &= -\dot{z}. \end{aligned} \tag{184}$$

From KVF (133) we have the Noether integral

$$I_4^{\rm XI} = -\frac{yt^2\dot{z} - z\dot{y}}{t^2}, \tag{185}$$

and from the HVF (177)

$$I(H_{\mathrm{XI}}) \;=\; \frac{t^3\dot{t} - 2\,t^2 x\dot{x} - t^2 z\dot{z} - y\dot{y}}{t^2}. \tag{186}$$

Furthermore, we obtain the solutions of the geodesics:

$$\begin{aligned}
t(s) &= \frac{\sqrt{2}\frac{\mathrm{d}}{\mathrm{d}s}y\,(s)}{\sqrt[4]{2\,\left(\frac{\mathrm{d}^3}{\mathrm{d}s^3}y\,(s)\right)\frac{\mathrm{d}}{\mathrm{d}s}y\,(s) - \left(\frac{\mathrm{d}^2}{\mathrm{d}s^2}y\,(s)\right)^2}},\\
x(s) &= x_1\,s + x_0,\\
y(s) &= \frac{1}{8}\,y_1^4 y_2^2 \mathrm{e}^{\frac{y_3+s}{y_1}} - \frac{1}{2}\,y_1^2 \mathrm{e}^{-\frac{y_3+s}{y_1}} - \frac{1}{2}\,y_1^2 y_2\,s\\
&\quad + y_0,\\
z(s) &= z_1\,s + z_0.
\end{aligned} \tag{187}$$

Solving (110) for this case yields identical wave function solutions as those of Case (X) given by (182) and (183).

4.2.12 Case (XII)

With a spacetime where W is a constant, $H(t) = E(t) = t^{-\frac{1}{2}}$, again we have a similar situation as to the previous Case (VI) and (VII), that is, we have the same vector fields.

From the KVFs (103) we obtain the Noether integrals:

$$\begin{aligned}
I_1^{XII} &= -\dot{x},\\
I_2^{XII} &= -\dot{y},\\
I_3^{XII} &= -\frac{\dot{z}}{t^2}.
\end{aligned} \tag{188}$$

From KVF (133) we have the Noether integral

$$I_4^{\rm XII} = -\frac{-zt^2\dot{y}+y\dot{z}}{t^2}, \tag{189}$$

and from the HVF (177)

$$I(H_{\rm XII}) = \frac{t^3\dot{t}-2\,t^2x\dot{x}-t^2y\dot{y}-z\dot{z}}{t^2}. \tag{190}$$

The classical field equation solutions of (105) are:

$$\begin{aligned} t(s) &= \frac{\sqrt{2}\frac{\rm d}{{\rm d}s}z\left(s\right)}{\sqrt[4]{2\,\left(\frac{{\rm d}^3}{{\rm d}s^3}z\left(s\right)\right)\frac{\rm d}{{\rm d}s}z\left(s\right)-\left(\frac{{\rm d}^2}{{\rm d}s^2}z\left(s\right)\right)^2}}, \\ x(s) &= x_1\,s+x_0, \\ y(s) &= y_1\,s+y_0, \\ z(s) &= \frac{1}{8}\,z_1^4z_2^2{\rm e}^{\frac{z_3+s}{z_1}}-\frac{1}{2}\,z_1^2{\rm e}^{-\frac{z_3+s}{z_1}}-\frac{1}{2}\,z_1^2z_2\,s \\ &\quad +z_0. \end{aligned} \tag{191}$$

Just as above, solving (110) for the wave function of the universe yields solutions (182) and (183). Next we combine the last two cases.

4.2.13 Case (XIII) and (XIV)

These cases are the conformally flat and Minkowski space, with $H(t) = E(t) = W(t) = e^{k_1 t}$ and $H(t) = E(t) = W(t) =$constant, respectively. The conformal algebras of such cases are well-known, as is their first integrals and geodesic solutions. Potential functions in a lower dimension were compiled in [68], which may be extrapolated here, given the relation between the potential functions of the Klein-Gordon equation and the potentials of the geodesic Lagrangian. In one case, the WDW

equation is simply the standard (1+3) Klein-Gordon equation, whose solutions have been explored in multiple papers and contexts, and thus, a detailed analysis of these cases will not be revisited in this work.

4.3 Discussion

In this chapter we considered a class of homogeneous and anisotropic Bianchi models derived from the Einstein field equations, the Bianchi I spacetime. This spacetime was studied in *f(R)* gravity, a modification of general relativity, that has garnered a great deal of attention from the M-theory [69] and theoretical physics community as a whole. Favourably, many *f(R)* models pass experimental tests and fit cosmological data. We considered exact solutions to Bianchi I spacetimes in *f(R)* gravity and we presented explicit solutions to the wave function of the universe using Lie symmetry analysis of the WDW equation. Along with solutions to the field equations, the three-linearly independent KVFs (103) and any additional KVFs and HVFs, were used to calculate potential functionals and first integrals for all cases. The WDW equation for the Bianchi I spacetime in *f(R)* gravity and the five Lie point symmetries (111) that it admits, were derived. These were used to reduce and solve for the wave functions. From our calculations, it is evident that the infinitesimal generators of the WDW equation and the conformal algebra, particularly the HVF, are inextricably linked. Concerning the wave solutions, we find that they comprise interesting special functions in most cases. Moreover, associated with the WDW equation is the geometric implication that its point transformations provide Noetherian conservation laws for the classical field equations. This property can be used to examine the integrability of field equations and find closed-form solutions.

The WDW equation is capable, in principle, of recovering various late-time dynamics

as it lies at the epicentre of canonical approaches to quantum cosmology. It plays the role of the Schrödinger equation for the wave function of the universe, and its solutions are important in the problem of stability and the time evolution of the universe. Moreover, observations on energy decay related to the solution may prove the global nonlinear stability of the space-time [70].

In many modified theories, a configuration space called a minisuperspace is constructed. In this canonical quantisation approach, a point-like Lagrangian may be established from an action principle, from which Hamiltonian theory holds. This draws many ideas from classical mechanics, such as, in the WKB approximation, the Hamilton- Jacobi equation leads to a reduced Hamiltonian system. General relativity and $f(R)$ are both tensor theories, but since the latter includes higher powers of the Ricci scalar, this creates more complicated mathematical expressions, especially in the context of our analysis, as can be seen above. To pose new conditions for $f(R)$ gravity as an alternative standard cosmological model to the ΛCDM model, further refined investigations on homogeneous universes are needed. We leave these developments to further work.

Chapter 5

Classical solutions to Bianchi type II spacetimes in $f(R)$ theory of gravity

We consider the analytical solutions to the Bianchi type II universe in $f(R)$ theory of gravity, a modified gravity theory. The Lagrangian and Hamiltonian are derived and the conformal vector fields of the space, which are Noether point symmetries of the Lagrangian, are presented. This spacetime is categorised into seven isometry classes. For each class we establish the potential functionals $V(t, x, y, z)$, and the first integrals associated with each KVF and HVF via Noether's theorem. Subsequently, the Lagrangian of the Bianchi II spacetime is quantized yielding the WDW equation and we apply the zero-order invariants of the admitted Lie point symmetries to determine invariant solutions for the wave function of the universe. Lastly, we reduce the Hamiltonian system to acquire the exact classical solutions to the field equations for all seven classes.

5.1 Introduction

We consider the Bianchi type II spacetime in the coordinates (t, x, y, z), whose line element is

$$ds^2 = -dt^2 + A\,(t)\,dx^2 + B\,(t)\,dy^2 + [B\,(t)\,x^2 + C\,(t)]dz^2 - 2\,xB\,(t)\,dydz, \quad (192)$$

where $A = A(t),\ B = B(t),\ C = C(t)$ are nowhere zero functions of t.

The Ricci scalar, R, for the Bianchi II spacetime (192) is

$$R = -\frac{1}{2}\left[\frac{2A''}{A} + \frac{2B''}{B} + \frac{2C''}{C} - \frac{A'^2}{A^2} - \frac{B'^2}{B^2} - \frac{C'^2}{C^2} + \frac{A'B'}{AB} + \frac{B'C'}{BC} + \frac{C'A'}{CA} - \frac{B}{AC}\right], \quad (193)$$

where prime denotes the derivative with respect to t. However, our interest lies in analysing the model of Bianchi type II spacetimes in the $f(R)$ theory of gravity. The Einstein field equations in $f(R)$ theory of gravity is

$$F(R)R_{ab} - \frac{1}{2}f(R)g_{ab} - \nabla_a\nabla_b F(R) + g_{ab}\Box F(R) = kT_{ab}, \quad (194)$$

where $f(R)$ is the function of the Ricci scalar R, $F(R) \equiv \frac{d}{dR}f(R)$, k is the coupling constant, T_{ab} is the energy-momentum tensor and $\Box \equiv \nabla^a\nabla_a$ where ∇ is the covariant derivative. In a vacuum $T_{ab} = 0$, hence (194) can be rewritten in tensorial form as

$$G_{ab} = \frac{1}{F(R)}\left[\frac{f(R) - RF(R)}{2}g_{ab} + \nabla_a\nabla_b F(R) - g_{ab}\nabla^a\nabla_a F(R)\right], \quad (195)$$

where G_{ab} denotes the Einstein tensor. By substitution of Eq. (192) into (195) and taking into account the *ad hoc* relation from [71], the second author showed in a previous work [72] that there exist seven solutions to the resulting equations.

These seven solutions of the model in $f(R)$ theory compose our seven cases, Case (I)- (VII) in the work that follows.

In this chapter, in order to describe the possible potential functions of this spacetime in $f(R)$ gravity, we employ Noether point symmetries. In particular, we use known Noether symmetries of the metric tensor to find and classify the potential functions of the spacetime. As before, it is feasible to apply the Lie invariance approach to lower the order and solve the field equations corresponding to any potential listed below by taking use of the fact that a Noether point symmetry is a Lie point symmetry.

The chapter is organised as follows. Section 5.2 describes the heuristics involving the Lagrangian and Hamiltonian of the field, the corresponding equations of motion and the KVFs of the space, which are in fact Noether point symmetries of the Lagrangian. In section 5.3, we have the main results of our analysis, where we study the seven cases (I)- (VII) in detail. For each of the cases, we define the various potential functions of the field and construct the associated first integrals via Noether's theorem. Moreover, we derive the equation describing the wave function of the universe, i.e. the WDW equation and a reduced Hamiltonian system of equations. Specifically for the WDW equation, we select Lie point symmetries to reduce the equation in order and provide Lie invariant solutions. As for the reduced Hamiltonian system, these are utilized to provide solutions to the equations of motion in each of the seven cases. Finally, we conclude in section 5.4.

5.2 The Hamiltonian, geodesics and Killing vectors

From the space (192) of the model, we read that the dependent variables are $t(s), x(s), y(s), z(s)$.
Hence we assume the generator of the infinitesimal transformation to be

$$X = \xi(s,t,x,y,z)\,\partial_s + \eta_t(s,t,x,y,z)\,\partial_t + \eta_x(s,t,x,y,z)\,\partial_x \tag{196}$$

$$+\eta_y(s,t,x,y,z)\,\partial_y + \eta_z(s,t,x,y,z)\,\partial_z, \tag{197}$$

and the general form of the Noetherian conservation law for our problem is denoted as

$$I = f - \eta_t \frac{\partial \mathcal{L}}{\partial \dot{t}} - \eta_x \frac{\partial \mathcal{L}}{\partial \dot{x}} - \eta_y \frac{\partial \mathcal{L}}{\partial \dot{y}} - \eta_z \frac{\partial \mathcal{L}}{\partial \dot{z}}, \tag{198}$$

where $f(s,t,x,y,z)$ is a boundary function.

We continue with the determination of the model dynamics.

The Lagrangian of the spacetime (192) is

$$\begin{aligned} \mathcal{L}(t,x,y,z) &= \frac{1}{2}A\dot{x}^2 + \frac{1}{2}B\dot{y}^2 + \frac{1}{2}\left(Bx^2 + C\right)\dot{z}^2 - \\ &\quad \frac{1}{2}\dot{t}^2 - xB\dot{y}\dot{z} - V(t,x,y,z), \end{aligned} \tag{199}$$

where here, dot represents derivative with respect to the arclength parameter s and $V(t,x,y,z)$ is the potential function yet to be determined. We derive the corre-

sponding geodesic equations of motion:

$$\begin{aligned}
\ddot{t} &= \frac{\partial}{\partial t}V + 1/2\,\left(-x^2\dot{z}^2 + 2\,x\dot{y}\dot{z} - \dot{y}^2\right)\frac{\mathrm{d}}{\mathrm{d}t}B - 1/2\,\left(\frac{\mathrm{d}}{\mathrm{d}t}A\right)\dot{x}^2 - \\
&\quad 1/2\,\dot{z}^2\frac{\mathrm{d}}{\mathrm{d}t}C, \qquad (200)\\
\ddot{x} &= -\frac{-Bx\dot{z}^2 + B\dot{y}\dot{z} + \left(\frac{\mathrm{d}}{\mathrm{d}t}A\right)\dot{x}\dot{t} + \frac{\partial}{\partial x}V}{A}, \qquad (201)\\
\ddot{y} &= \frac{1}{CB}\Bigg(\left(-x^2B - C\right)\frac{\partial}{\partial y}V - \left(\frac{\partial}{\partial z}V\right)Bx + \\
&\quad C\dot{t}\left(x\dot{z} - \dot{y}\right)\frac{\mathrm{d}}{\mathrm{d}t}B + B\Big(-x\dot{t}\dot{z}\frac{\mathrm{d}}{\mathrm{d}t}C + \\
&\quad \dot{x}\left(\left(-x^2\dot{z} + x\dot{y}\right)B + C\dot{z}\right)\Big)\Bigg), \qquad (202)\\
\ddot{z} &= -\frac{1}{C}\Big(B\dot{x}\dot{z}x + \left(\frac{\mathrm{d}}{\mathrm{d}t}C\right)\dot{t}\dot{z} - B\dot{x}\dot{y} + \left(\frac{\partial}{\partial y}V\right)x \\
&\quad + \frac{\partial}{\partial z}V\Big). \qquad (203)
\end{aligned}$$

The Hamiltonian, also called the energy function, of the dynamical system, $\mathcal{E}$, associated with the Lagrangian (199) is

$$\begin{aligned}
\mathcal{E}(t,x,y,z) &= \frac{1}{2}\,A\dot{x}^2 + \frac{1}{2}\,B\dot{y}^2 + \frac{1}{2}\,\left(Bx^2 + C\right)\dot{z}^2 - \\
&\quad \frac{1}{2}\,\dot{t}^2 - xB\dot{y}\dot{z} + V(t,x,y,z). \qquad (204)
\end{aligned}$$

We define the momentum $p_i = \frac{\partial L}{\partial \dot{x}^i}$, to find that Eq. (204) has the form

$$\begin{aligned}
\mathcal{E} = \frac{1}{2}\,\frac{\left(\left(-p_t^2C + \left(p_yx + p_z\right)^2\right)B + p_y^2C\right)A + p_x^2BC}{BCA} \\
+V(t,x,y,z). \qquad (205)
\end{aligned}$$

Furthermore, the field equations are the Hamiltonian constraint (205) and the Hamilton equations

$$\dot{x} = \frac{p_x}{A}, \tag{206}$$

$$\dot{y} = \frac{p_y}{B} + \frac{xp_z}{C} + \frac{x^2 p_y}{C}, \tag{207}$$

$$\dot{z} = \frac{p_y x + p_z}{C}, \tag{208}$$

$$\dot{t} = -p_t, \tag{209}$$

$$\dot{p}_x = \frac{(p_y x + p_z)\, p_y}{C} + \frac{\partial}{\partial x} V, \tag{210}$$

$$\dot{p}_y = \frac{\partial}{\partial y} V,\ \dot{p}_z = \frac{\partial}{\partial z} V, \tag{211}$$

$$\dot{p}_t = \frac{1}{2B^2C^2A^2}\left(2\left(\frac{\partial}{\partial t}V\right)B^2C^2A^2 - A^2B^2\left(p_y x + p_z\right)^2 \frac{\mathrm{d}}{\mathrm{d}t}C \right.$$
$$\left. -C^2\left({p_x}^2\left(\frac{\mathrm{d}}{\mathrm{d}t}A\right)B^2 + \left(\frac{\mathrm{d}}{\mathrm{d}t}B\right){p_y}^2A^2\right)\right). \tag{212}$$

The spacetime (192) admits the three linearly-independent KVFs:

$$X_1 = \partial_y,\ X_2 = \partial_x + z\partial y, \text{ and } X_3 = \partial_z, \tag{213}$$

which represent the translational invariance in y, z and a scale invariance in x and y. The corresponding Lie bracket relations are $[X_1, X_2] = 0,\ [X_1, X_3] = 0,\ [X_2, X_3] = -X_1$. The conservation laws corresponding to these symmetries are linear momentum. In the sequel, we consider seven cases that exhibit the three KVFs given by Eq. (213). Within these cases, we identify the corresponding first integrals and the functional form of the potential function $V(t, x, y, z)$. We calculate the potentials that correspond to the three KVFs (X_i) in Eq.(213) to be:

$$X_1 : V(t, x, y, z) = py + F(t, x, z), \tag{214}$$

$$X_2 : V(t, x, y, z) = px + F(t, z, -xz + y), \tag{215}$$

and

$$X_3 : V(t,x,y,z) = pz + F(t,x,y). \tag{216}$$

Note that the above potentials occur in every case that follows.

5.3 Analytical solutions: equations of motion and wave function

The WDW equation for the Bianchi type II spacetime in $f(R)$ theory is

$$\begin{aligned} G: \quad & \frac{1}{2A\left(Bx^2+C\right)B}\Bigg[-2\left(Bx^2+C\right)AB\frac{\partial^2}{\partial t^2}U + \\ & \left(2\,x^2\left(B\right)^2+2\,CB\right)\frac{\partial^2}{\partial x^2}U + 2\,A\left(Bx^2+C\right)\frac{\partial^2}{\partial y^2}U + \\ & 2\left(\frac{\partial^2}{\partial z^2}U\right)AB + \Big(\left(-x^2\left(B\right)^2-CB\right)\frac{\mathrm{d}}{\mathrm{d}t}A \\ & -2\,A\left(\left(Bx^2+1/2\,C\right)\frac{\mathrm{d}}{\mathrm{d}t}B + 1/2\left(\frac{\mathrm{d}}{\mathrm{d}t}C\right)B\right)\Big)\frac{\partial}{\partial t}U + \\ & 2\,B\left(\left(\frac{\partial}{\partial x}U\right)xB\left(t\right) - 2\,AUV\left(Bx^2+C\right)\right)\Bigg] = 0, \end{aligned} \tag{217}$$

where $U = U(t,x,y,z)$ is the wave function of the universe.

The differential equation, $G = G\left(x^i, U, \frac{\partial}{\partial i}U, \frac{\partial^2}{\partial i \partial j}U\right)$, is invariant under the action of the one-parameter point transformation if there exists a function ρ such that [4]

$$W^{[2]}G = \rho G,$$

where $W^{[2]}$ is the second prolongation of W in the jet space $\left\{x^i, U, \frac{\partial}{\partial i}U, \frac{\partial^2}{\partial i \partial j}U\right\}$. When the above condition holds, we say that W is a Lie point symmetry of equation G. The symmetries W form a Lie algebra.

The WDW equation (217) is a second-order PDE defined in the space of the independent variables $x^i \to \{t, x, y, z\}$, where U is the dependent variable.

We include the symmetries and solutions for the WDW equation for spacetimes corresponding to the cases below. To solve for the WDW equation, point symmetries corresponding to it are required. In the same manner as before such symmetries can be used to systematically reduce the order of the WDW equation. Reduced equations will be reduced once more if they inherit any of the remaining symmetries of the Lie algebra. For our case, a four variable PDE makes up the WDW equation and, thus, several symmetries are required to reduce it to an ODE.
The form of the symmetry generator is

$$W = \eta\left(U,t,x,y,z\right)\partial_U + \xi_t\left(U,t,x,y,z\right)\partial_t + \xi_x\left(U,t,x,y,z\right)\partial_x \\ + \xi_y\left(U,t,x,y,z\right)\partial_y + \xi_z\left(U,t,x,y,z\right)\partial_z. \tag{218}$$

The WDW equation admits the four Lie point symmetries

$$\begin{aligned} W_1 &= X_1, \\ W_2 &= X_3, \\ W_U &= U\partial_U, \\ W_P &= P(t,x,y,z)\partial_U, \end{aligned} \tag{219}$$

where the function $P(t, x, y, z)$ is a solution to the WDW equation (217), when $V(t, x, y, z) = 0$ or $V(t, x, y, z) = \Psi(x)$. For arbitrary potential $V(t, x, y, z)$, only the point symmetries W_P and W_U are admitted. The vector field W_U is called a homogeneous symmetry, whereas W_P corresponds to the infinite number of solutions. Both these Lie symmetries are not useful to reduce the differential equation. However, they indicate that equation (217) is a linear second-order PDE.

Adopting the WKB approximation to the WDW equation or from (205), we have

$$\left(\left(-\left(\frac{\partial f}{\partial t}\right)^2 C+\left(\left(\frac{\partial f}{\partial y}\right)x+\frac{\partial f}{\partial z}\right)^2\right)B+\left(\frac{\partial f}{\partial y}\right)^2 C\right)A+ \\ +\left(\frac{\partial f}{\partial x}\right)^2 BC+2VABC=0. \tag{220}$$

where $f(t,x,y,z)$ describes the motion of a particle, and this equation is called the Hamilton-Jacobi equation. Since $p_t=\frac{\partial f}{\partial t}$, $p_x=\frac{\partial f}{\partial x}$, $p_y=\frac{\partial f}{\partial y}$, $p_z=\frac{\partial f}{\partial z}$, the solution of the Hamilton-Jacobi equation leads us to the following reduced Hamiltonian system (see Eq. (206)-(209)):

$$\begin{aligned} \dot{x}&=\frac{1}{A}\frac{\partial f}{\partial x},\\ \dot{t}&=-\frac{\partial f}{\partial t},\\ \dot{y}&=\frac{1}{B}\frac{\partial f}{\partial y}+\frac{x\frac{\partial f}{\partial z}}{C}+\frac{x^2\left(\frac{\partial f}{\partial y}\right)}{C},\\ \dot{z}&=\frac{\left(\frac{\partial f}{\partial y}\right)x+\frac{\partial f}{\partial z}}{C}. \end{aligned} \tag{221}$$

In the analysis below, we allow the potential V to vanish in obtaining the exact solutions of the field equations and the wave function, unless specified otherwise. We also apply the zero-order invariants of the Lie point symmetries for the WDW equation in order to reduce the equation and find the solution of the wave function $U(t,x,y,z)$. Moreover, by using the Hamilton-Jacobi equation, we provide analytical solutions to the field equations under study.

5.3.1 Case (I) $A = t^3$, $B = t^{-2}$, $C = t^{-3}$.

Associated with each KVF X_i, are the first integrals I_i listed below:

$$\begin{aligned} I_1^I &= \frac{x\dot{z} - \dot{y}}{t^2}, \\ I_2^I &= -\frac{t^5\dot{x} - xz\dot{z} + \dot{y}z}{t^2}, \\ I_3^I &= -\frac{tx^2\dot{z} - tx\dot{y} + \dot{z}}{t^3}. \end{aligned} \tag{222}$$

In addition to the KVFs in Eq. (213), a proper HVF is admitted:

$$H_{\rm I} \;=\; \frac{1}{2}t\partial_t - \frac{1}{4}x\partial_x + y\partial_y + \frac{5}{4}z\partial_z, \tag{223}$$

.

whose corresponding conservation law is

$$I(H_{\rm I}) = \frac{1}{4}\,\frac{t^6\dot{x}x + 2\,\dot{t}t^4 - 5\,tx^2z\dot{z} + 4\,txy\dot{z} + 5\,tx\dot{y}z - 4\,ty\dot{y} - 5\,z\dot{z}}{t^3}. \tag{224}$$

This HVF also gives the potential

$$H_{\rm I} : V\,(t,x,y,z) = p + \frac{1}{t^2}F\left(x\sqrt{t}, \frac{y}{t^2}, \frac{z}{t^{5/2}}\right). \tag{225}$$

The linear combination $d_iX_i + d_4H_{\rm I}$ gives the potential function

$$\begin{aligned} V\,(t,x,y,z) \;=\; & \frac{p}{d_4} + \frac{1}{t^2}F\Bigg(\frac{\sqrt{t}\,(d_4x - 4\,d_2)}{d_4}, 1/5\,\frac{5\,d_4z + 4\,d_3}{t^{5/2}d_4}, \\ & 1/5\,\frac{-4\,(5\,d_4z + 4\,d_3)\,d_2 + 5\,y{d_4}^2 + 5\,d_1d_4 - 4\,d_2d_3}{{d_4}^2t^2}\Bigg). \end{aligned} \tag{226}$$

This case also admits an extra Lie point symmetry for the WDW equation (217)

$$W_I = \frac{5}{4} W_U + 2H_I. \tag{227}$$

By the method of Lie invariants, we apply the symmetries $\{W_1, W_2, W_I\}$ to reduce Eq. (217) to an ODE and hence obtain the solution for the wave function, viz.

$$\begin{aligned} U_I(t,x,y,z) &= C_1 + \frac{C_2}{\sqrt{t^2x^4 - 3\,tx^2 - 4}}\Bigg(x^5t^{5/2} - 3\,x^3t^{3/2} - 4\,x\sqrt{t} + \\ &\quad 10\,i\sqrt{tx^2+1}\sqrt{-tx^2+4}\; E_F\left(ix\sqrt{t}, i/2\right) - \\ &\quad 36\,i\sqrt{tx^2+1}\sqrt{-tx^2+4}\; E_E\left(ix\sqrt{t}, i/2\right)\Bigg), \end{aligned} \tag{228}$$

where the C_1, C_2 are arbitrary constants, E_E denotes the incomplete and complete elliptic integrals of the second kind, and E_F is the incomplete elliptic integral of the first kind.

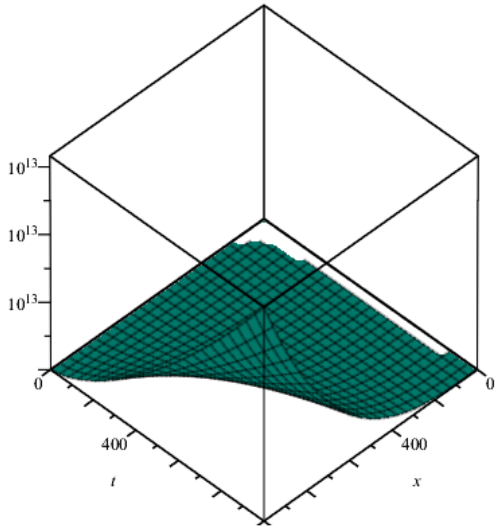
From above, we see the obvious geometric connection to the HVF. Thus, we note that the existence of the Lie point symmetry in the WDW equation, through its connection to the symmetries of the space, is also equivalent to the existence of a conserved quantity for the field equations. This may enable one to solve the field equations via this route. Turning our attention to the field equations, and considering another path to the solution, the solution of the Hamilton-Jacobi equation (220) is

$$f(t,x,y,z) = yc_3 + zc_4 + F_1(t,x) + f_0, \tag{229}$$

where c_j are constants, all variables with zero subscripts are arbitrary constants as well, and $F_1(t,x)$ satisfies the equation

$$\frac{-t^6x^2{c_3}^2 - 2\,t^6xc_3c_4 - t^6{c_4}^2 - t^5{c_3}^2 + \left(\frac{\partial}{\partial t}F_1\right)^2 t^3 - \left(\frac{\partial}{\partial x}F_1\right)^2}{t^3} = 0. \tag{230}$$

Figure 5.1: Evolution of the analytical solutions of the WDW equation; we select the parameter values $C_1 = C_2 = 1$ in the solution (228).



The expression (230) solves for $c_3 = 0$ to provide

$$F_1\left(t,x\right) = 2\,\frac{\epsilon\,c_2}{\sqrt{t}} + x\,c_2 + F_0,$$

and therefore we solve the reduced Hamiltonian system (221) to obtain the following solutions of the equations of motion for Case (I):

$$\begin{aligned}
t(s) &= \frac{1}{4}\left(80\,s\epsilon\,c_2 + 32\;t_0\right)^{2/5}, \\
x(s) &= \int \frac{c_2}{\left(t\left(s\right)\right)^3}\,\mathrm{d}s + x_0, \\
y(s) &= \int \left(t\left(s\right)\right)^2 \left(\left(x\left(s\right)\right)^2 t\left(s\right) c_3 + x\left(s\right) t\left(s\right) c_4 + c_3\right)\,\mathrm{d}s + y_0, \\
z(s) &= \int \left(c_3 x\left(s\right) + c_4\right)\left(t\left(s\right)\right)^3\,\mathrm{d}s + z_0,
\end{aligned} \tag{231}$$

where $\epsilon = \pm 1$ for our classical solution of the field equations.

5.3.2 Case (II) $A = (k_1 t + k_2)^3$, $B = (k_1 t + k_2)$, $C = \frac{-2}{k_1^2}$, where $k_1, k_2 \in \Re$ $(k_1 \neq 0)$.

Applying Noether's theorem, the conservation laws in this spacetime associated with the KVFs in Eq. (213) are

$$\begin{aligned} I_1^{II} &= k_1 t x\dot{z} - k_1 t\dot{y} + k_2 x\dot{z} - k_2\dot{y}, \\ I_2^{II} &= -\left(k_1 t + k_2\right)\left(\left(k_1 t + k_2\right)^2 \dot{x} - z\left(x\dot{z} - \dot{y}\right)\right), \\ I_3^{II} &= -\frac{{k_1}^3 t x^2\dot{z} - {k_1}^3 t x\dot{y} + {k_1}^2 k_2 x^2\dot{z} - {k_1}^2 k_2 x\dot{y} - 2\,\dot{z}}{{k_1}^2}. \end{aligned} \tag{232}$$

The fourth conformal vector field in this spacetime is the proper HVF

$$H_{\rm II} = 2\left(\frac{k_1 t + k_2}{k_1}\right)\partial_t - x\partial_x + y\partial_y + 2z\partial_z, \tag{233}$$

and the corresponding conservation law

$$\begin{aligned} I(H_{II}) = \frac{1}{k_1^2}\Big(& {k_1}^5 t^3 x\dot{x} + 3\,k_1^4 k_2 t^2 x\dot{x} + 3\,k_1^3 {k_2}^2 t x\dot{x} - 2\,k_1^3 t x^2 z\dot{z} + \\ & k_1^3 t x y\dot{z} + 2\,k_1^3 t x\dot{y} z + {k_1}^2 {k_2}^3 x\dot{x} - 2\,{k_1}^2 k_2 x^2 z\dot{z} - {k_1}^3 t y\dot{y} + \\ & {k_1}^2 k_2 x y\dot{z} + 2\,{k_1}^2 k_2 x\dot{y} z - {k_1}^2 k_2 y\dot{y} + 2\,{k_1}^2 t\dot{t} + \\ & 2\,k_1 k_2\dot{t} + 4\,z\dot{z}\Big). \end{aligned} \tag{234}$$

Associated with the HVF (233), is the potential

$$\begin{aligned} V(t,x,y,z) = \frac{1}{4}\,\frac{1}{\left(k_1 t + k_2\right)^2}\Bigg(& p{k_1}^2 t^2 + 2\,p k_1 k_2 t + \\ & 4\,F\left(x\sqrt{k_1 t + k_2}, \frac{y}{\sqrt{k_1 t + k_2}}, \frac{z}{k_1 t + k_2}\right)\Bigg). \end{aligned} \tag{235}$$

Lastly, the linear combination $d_i X_i + d_4 H_{\rm II}$ gives the potential function

$$V(t,x,y,z) = \frac{1}{4}\frac{1}{d_4\left(k_1 t + k_2\right)^2}\Bigg[{k_1}^2 p t^2 + 2\,k_1 k_2 p t + 4\,F\Bigg(\frac{\sqrt{k_1 t + k_2}\,(d_4 x - d_2)}{d_4}, 1/2\,\frac{2\,d_4 z + d_3}{d_4\left(k_1 t + k_2\right)}, \frac{y {d_4}^2 - z d_2 d_4 + d_1 d_4 - d_2 d_3}{\sqrt{k_1 t + k_2}\,{d_4}^2}\Bigg) d_4\Bigg]. \tag{236}$$

Next, we investigate the WDW equation (217) of this case. We discover the additional symmetry:

$$W_{II} = \frac{3}{4}\,k_1 W_U + \frac{k_1}{2} H_{II}. \tag{237}$$

The WDW equation, when invariant under one-parameter point transformations, can be solved by reduction as seen above. Thus the WDW solution, using the same technique as Case (I), but replacing W_I with W_{II}, is

$$U_{II}(t,x,y,z) = C_1 + \sqrt{k_1 t + k_2}\times C_2 \int \frac{1}{\sqrt{{k_1}^2 x^2\left(k_1 t + k_2\right) - 2}\left({k_1}^2 x^2\left(k_1 t + k_2\right) - 4\right)^{3/2}}\,\mathrm{d}x. \tag{238}$$

With a vanishing potential in (220) for this spacetime we obtain the functional that describes the motion of a particle

$$f(t,x,y,z) = y c_3 + z c_4 + F_2(t,x) + f_0, \tag{239}$$

where $F_2(t,x)$ satisfies the equation

$$\frac{1}{(k_1 t + k_2)^3}\Bigg(- {c_3}^2 {k_1}^5 t^3 x^2 - 2\, c_3 c_4 {k_1}^5 t^3 x - 3\, {c_3}^2 {k_1}^4 k_2 t^2 x^2 - {c_4}^2 {k_1}^5 t^3$$
$$-6\, c_3 c_4 {k_1}^4 k_2 t^2 x - 3\, {c_3}^2 {k_1}^3 {k_2}^2 t x^2 - 3\, {c_4}^2 {k_1}^4 k_2 t^2 - 6\, c_3 c_4 {k_1}^3 {k_2}^2 t x -$$
$${c_3}^2 {k_1}^2 {k_2}^3 x^2 - 2 \left(\frac{\partial}{\partial t} F_2\right)^2 {k_1}^3 t^3 - 3\, {c_4}^2 {k_1}^3 {k_2}^2 t - 2\, c_3 c_4 {k_1}^2 {k_2}^3 x -$$
$$6 \left(\frac{\partial}{\partial t} F_2\right)^2 {k_1}^2 k_2 t^2 - {c_4}^2 {k_1}^2 {k_2}^3 - 6 \left(\frac{\partial}{\partial t} F_2\right)^2 k_1 {k_2}^2 t +$$
$$2\, {c_3}^2 {k_1}^2 t^2 - 2 \left(\frac{\partial}{\partial t} F_2\right)^2 {k_2}^3 + 4\, {c_3}^2 k_1 k_2 t + 2\, {c_3}^2 {k_2}^2 +$$
$$2 \left(\frac{\partial}{\partial x} F_2\right)^2 \Bigg) = 0. \tag{240}$$

Suppose we let $k_2 = 0$ and $c_3 = 0$ in (240), so that we solve to find

$$F_2(t,x) = \epsilon \int \frac{1}{2} \frac{\sqrt{-2\, k_1 t \left({k_1}^5 t^3 {c_4}^2 - 2\, {c_2}^2\right)}}{{k_1}^2 t^2}\, \mathrm{d}t + x c_2 + F_0. \tag{241}$$

It is now possible to solve the reduced Hamiltonian system (221), to obtain the classcial field equation solutions

$$\begin{aligned} t(s) &= \frac{\sqrt[3]{2}\sqrt[3]{{c_2}^2 k_1 c_4}}{{k_1}^2 c_4}, \\ x(s) &= \frac{1}{2} \frac{{c_4}^2 {k_1}^2 s}{c_2} + x_0, \\ y(s) &= \int -\frac{1}{2}\, x(s)\, c_4 {k_1}^2\, \mathrm{d}s + y_0, \\ z(s) &= -\frac{1}{2}\, c_4 {k_1}^2 s + z_0. \end{aligned} \tag{242}$$

5.3.3 Case (III) $A = -\frac{1}{k_1^2}$, $B = (k_1 t + k_2)^{-1}$, $C = (k_1 t + k_2)$, where $k_1, k_2 \in \Re$ $(k_1 \neq 0)$.

The first integrals by Noether's theorem in this spacetime are:

$$\begin{aligned} I_1^{III} &= \frac{x\dot{z} - \dot{y}}{k_1 t + k_2}, \\ I_2^{III} &= \frac{{k_1}^2 x z\dot{z} - {k_1}^2 \dot{y} z + k_1 t\dot{x} + k_2\dot{x}}{{k_1}^2 (k_1 t + k_2)}, \\ I_3^{III} &= -\frac{{k_1}^2 t^2\dot{z} + 2\, k_1 k_2 t\dot{z} + {k_2}^2\dot{z} + x^2\dot{z} - x\dot{y}}{k_1 t + k_2}. \end{aligned} \tag{243}$$

This case admits the proper HVF

$$H_{\rm III} = \frac{2}{3}\frac{k_1 t + k_2}{k_1}\partial_t + \frac{2}{3}x\partial_x + y\partial_y + \frac{1}{3}z\partial_z, \tag{244}$$

for which the associated conservation law is

$$\begin{aligned} I(H_{\rm III}) = \frac{1}{3{k_1}^2 (k_1 t + k_2)}\Bigg(& -{k_1}^4 t^2 z\dot{z} + \left(-2\, k_2 t z\dot{z} + 2\, t^2\dot{t}\right){k_1}^3 + \\ & 2\, k_2 x\dot{x}\left(4\, k_2\dot{t}t - \dot{z}{k_2}^2 z - (xz - 3\, y)(x\dot{z} - \dot{y})\right){k_1}^2 + \\ & \left(2\,{k_2}^2\dot{t} + 2\, t x\dot{x}\right) k_1\Bigg). \end{aligned} \tag{245}$$

Additionally, (244) gives the potential

$$\begin{aligned} V(t, x, y, z) = \frac{1}{4}\frac{1}{(k_1 t + k_2)^2}\Bigg(& 3\, p{k_1}^2 t^2 + 6\, p k_1 k_2 t + \\ & 4\, F\left(\frac{x}{k_1 t + k_2}, \frac{y}{(k_1 t + k_2)^{3/2}}, \frac{z}{\sqrt{k_1 t + k_2}}\right)\Bigg). \end{aligned} \tag{246}$$

The linear combination $d_i X_i + d_4 H_{\rm III}$ produces the potential functional

$$\begin{aligned} V\left(t,x,y,z\right) &= \frac{1}{4}\,\frac{1}{d_4\left(k_1 t+k_2\right)^2}\Bigg(3\,{k_1}^2 p t^2+6\,k_1 k_2 p t+ \\ &\quad 4\,F\Bigg(1/2\,\frac{2\,d_4 x+3\,d_2}{d_4\left(k_1 t+k_2\right)},\frac{d_4 z+3\,d_3}{\sqrt{k_1 t+k_2}\,d_4}, \\ &\quad \frac{1}{2}\,\frac{2\,y{d_4}^2+3\,z d_2 d_4+2\,d_1 d_4+3\,d_2 d_3}{\left(k_1 t+k_2\right)^{3/2}\,{d_4}^2}\Bigg)d_4\Bigg). \end{aligned} \tag{247}$$

Here we have an additional Lie point symmetry in the case of the WDW equation (217)

$$W_{III} = \frac{3}{2}\,k_1 H_{III}. \tag{248}$$

As previously stated, we will apply the zeroth-order invariants of the Lie point symmetries to reduce the order of the WDW equation (217) to determine invariant solutions. The symmetries $W_1,\ W_2$ and W_{III} yield the invariant solution for the wave function

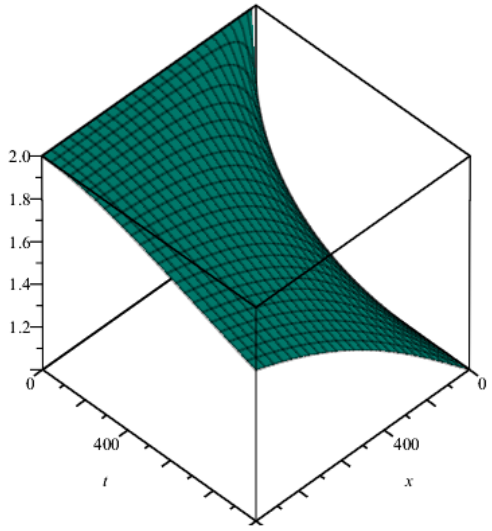
$$U_{III}\left(t,x,y,z\right) = C_1 + \frac{x}{k_1 t+k_2}\,\frac{C_2}{\sqrt{\frac{x^2}{\left(k_1 t+k_2\right)^2}+1}}. \tag{249}$$

The solution of the wave function appears in Figure 5.2.

From (220), we obtain the solution

$$f(t,x,y,z) = y c_3 + z c_4 + F_3(t,x) + f_0, \tag{250}$$

Figure 5.2: Plot of (249) with we select the parameter values $C_1 = C_2 = 1$ and $k_1 = k_2 = 1$ in the solution.



where $F_3(t, x)$ satisfies the constraint

$$\frac{1}{k_1 t + k_2}\left(\left(\frac{\partial}{\partial x} F_3\right)^2 {k_1}^3 t - {k_1}^2 t^2 {c_3}^2 + \left(\frac{\partial}{\partial x} F_3\right)^2 {k_1}^2 k_2 - 2\, k_1 k_2 t {c_3}^2 \right.$$
$$\left. + k_1 t \left(\frac{\partial}{\partial t} F_3\right)^2 - {c_3}^2 {k_2}^2 - x^2 {c_3}^2 + k_2 \left(\frac{\partial}{\partial t} F_3\right)^2 - 2\, x c_3 c_4 - {c_4}^2 \right) = 0. \tag{251}$$

We substitute $k_2 = 0$ and $c_3 = 0$ into constraint (251) to get

$$F_3\,(t, x) = 2\,\frac{\epsilon\, t c_4}{\sqrt{at}} + x c_2 + F_0. \tag{252}$$

Furthermore, with $c_2 = 0$ we solve the system (221) to obtain the solutions of the geodesics:

$$
\begin{aligned}
t(s) &= \frac{1}{4}\frac{\left(-12\,t_0\,k_1\epsilon\,c_4 - 12\,k_1\epsilon\,s c_4\right)^{2/3}}{k_1},\\
x(s) &= x_0,\\
y(s) &= \int \frac{x\,(s)\,c_4}{k_1 t\,(s)}\,\mathrm{d}s + y_0,\\
z(s) &= \int \frac{c_4}{k_1 t\,(s)}\,\mathrm{d}s + z_0.
\end{aligned}
\tag{253}
$$

We continue with the rest of the cases in a similar approach.

5.3.4 Case (IV) $A = B = (2t^2 + k_1 t + k_2)^{\frac{1}{2}}$, $C = (2t^2 + k_1 t + k_2)$, where $k_1, k_2 \in \Re$ $(k_1 \neq 0)$.

In this case, the only conformal vector fields are the ones given by Eq.(213). For these KVFs the Noether integrals in this case are:

$$
\begin{aligned}
I_1^{IV} &= (x\dot{z} - \dot{y})\,\sqrt{k_1 t + 2\,t^2 + k_2},\\
I_2^{IV} &= ((x\dot{z} - \dot{y})\,z - \dot{x})\,\sqrt{k_1 t + 2\,t^2 + k_2},\\
I_3^{IV} &= -\sqrt{k_1 t + 2\,t^2 + k_2}\left(x^2\dot{z} + \dot{z}\sqrt{k_1 t + 2\,t^2 + k_2} - x\dot{y}\right).
\end{aligned}
\tag{254}
$$

The linear combination of these vector fields $d_i X_i$ gives the corresponding potential

$$
\begin{aligned}
V\,(t,x,y,z) &= \frac{px}{d_2} + F\Big(t, \frac{-d_3 x + d_2 z}{d_2},\\
&\qquad 1/2\,\frac{-2\,\left(-d_3 x + d_2 z\right)x - x^2 d_3 - 2\,d_1 x + 2\,y d_2}{d_2}\Big).
\end{aligned}
\tag{255}
$$

Performing a symmetry analysis for the WDW equation using the Lie point symmetries in the case (217), a reduction with W_1 followed by W_2 gives the wave function

$$U_{IV}\,(t,x,y,z) = T(t,x),$$

where $T(t,x)$ satisfies the equation

$$
\begin{aligned}
-8\left(\frac{\partial^2}{\partial t^2}T\right)x^2t^2 - 4\left(\frac{\partial^2}{\partial t^2}T\right)x^2tk_1 - 8\sqrt{2\,t^2+tk_1+k_2}\left(\frac{\partial^2}{\partial t^2}T\right)t^2 \\
-4\sqrt{2\,t^2+tk_1+k_2}\left(\frac{\partial^2}{\partial t^2}T\,(t,x)\right)tk_1 + 4\sqrt{2\,t^2+tk_1+k_2}\left(\frac{\partial^2}{\partial x^2}T\right)x^2 \\
-4\left(\frac{\partial^2}{\partial t^2}T\right)x^2k_2 - 12\left(\frac{\partial}{\partial t}T\right)x^2t - 3\left(\frac{\partial}{\partial t}T\right)x^2k_1 \\
-4\sqrt{2\,t^2+tk_1+k_2}\left(\frac{\partial^2}{\partial t^2}T\right)k_2 - 16\sqrt{2\,t^2+tk_1+k_2}\left(\frac{\partial}{\partial t}T\right)t \\
-4\sqrt{2\,t^2+tk_1+k_2}\left(\frac{\partial}{\partial t}T\right)k_1 + 4\sqrt{2\,t^2+tk_1+k_2}\left(\frac{\partial}{\partial x}T\right)x \\
+8\left(\frac{\partial^2}{\partial x^2}T\right)t^2 + 4\left(\frac{\partial^2}{\partial x^2}T\right)tk_1 + 4\left(\frac{\partial^2}{\partial x^2}T\right)k_2 = 0. \qquad (256)
\end{aligned}
$$

An explicit solution is not possible here, as was in the previous cases. This is due to the fact that this case contains no extra symmetry, say in the form of a HVF. Thus, the best outcome is to reduce the four variable WDW equation (217) to a two variable PDE, for which the latter may be subjected to further analysis using some alternative method.

The solution of the Hamilton Jacobi equation, (220), for this particular case is

$$f(t,x,y,z) = yc_3 + zc_4 + F_4(t,x) + f_0, \qquad (257)$$

where $F_4(t,x)$ satisfies the relation

$$\frac{1}{(k_1 t + 2\,t^2 + k_2)^{3/2}}\left(k_1 t\left(\frac{\partial}{\partial t}F_4\right)^2\sqrt{k_1 t + 2\,t^2 + k_2} + 2\,t^2\left(\frac{\partial}{\partial t}F_4\right)^2\sqrt{k_1 t + 2\,t^2 + k_2} - x^2 {c_3}^2\sqrt{k_1 t + 2\,t^2 + k_2} - \left(\frac{\partial}{\partial x}F_4\right)^2 k_1 t - 2\left(\frac{\partial}{\partial x}F_4\right)^2 t^2 + k_2\left(\frac{\partial}{\partial t}F_4\right)^2\sqrt{k_1 t + 2\,t^2 + k_2} - k_1 t {c_3}^2 - 2\,t^2 {c_3}^2 - 2\,x c_3 c_4\sqrt{k_1 t + 2\,t^2 + k_2} - \left(\frac{\partial}{\partial x}F_4\right)^2 k_2 - k_2 {c_3}^2 - {c_4}^2\sqrt{k_1 t + 2\,t^2 + k_2}\right) = 0. \tag{258}$$

For $k_2 = 0$ and $c_3 = 0$ in (258), we have

$$\begin{aligned} F_4(t,x) &= \epsilon \int \frac{\sqrt{t\,(k_1 + 2\,t)\,\sqrt{t\,(k_1 + 2\,t)}\left({c_4}^2\sqrt{t\,(k_1 + 2\,t)} + {c_2}^2 k_1 t + 2\,t^2 {c_2}^2\right)}}{t\,(k_1 + 2\,t)\,\sqrt{t\,(k_1 + 2\,t)}}\,\mathrm{d}t \\ &\quad + x c_2 + F_0, \end{aligned}$$

and an exact solution to the reduced Hamiltonian system is found, namely

$$\begin{aligned} t(s) &= \frac{-\frac{1}{4}\,k_1 {c_2}^2 + \frac{1}{4}\,\sqrt{{k_1}^2 {c_2}^4 + 8\,{c_4}^4}}{{c_2}^2}, \\ x(s) &= -\frac{{c_2}^3 s}{{c_4}^2} + x_0, \\ y(s) &= \int \frac{{c_2}^4 x\,(s)}{{c_4}^3}\,\mathrm{d}s + y_0, \\ z(s) &= \frac{{c_2}^4 s}{{c_4}^3} + z_0. \end{aligned} \tag{259}$$

5.3.5 Case (V) $A = C = (k_1 t + k_2)^{\frac{3}{2}}$, $B = (k_1 t + k_2)$, where $k_1 = \pm 2, k_2 \in \Re$.

The first integrals of this case are:

$$\begin{aligned} I_1^V &= k_1 t x \dot{z} - k_1 t \dot{y} + k_2 x \dot{z} - k_2 \dot{y}, \\ I_2^V &= \sqrt{k_1 t + k_2} \left(-k_1 t \dot{x} - k_2 \dot{x} + z \left(x \dot{z} - \dot{y} \right) \sqrt{k_1 t + k_2} \right), \\ I_3^V &= -\left(x \left(x \dot{z} - \dot{y} \right) \sqrt{k_1 t + k_2} + \dot{z} \left(k_1 t + k_2 \right) \right) \sqrt{k_1 t + k_2}. \end{aligned} \tag{260}$$

In this spacetime all the conformal vector fields are KVFs, where the extra KVF is

$$X_4^V = z \partial_x + \left(\frac{x^2 - z^2}{2} \right) \partial_y - x \partial_z. \tag{261}$$

The associated conservation law is

$$\begin{aligned} I_4^V = \sqrt{k_1 t + k_2} \Big(& \frac{3}{2} \left(x^2 - \frac{1}{3} z^2 \right) \left(x \dot{z} - \dot{y} \right) \sqrt{k_1 t + k_2} + \\ & \left(x \dot{z} - \dot{x} z \right) \left(k_1 t + k_2 \right) \Big), \end{aligned} \tag{262}$$

and furthermore, this extra KVF generates the potential

$$V(t, x, y, z) = p \arctan\left(\frac{x}{z} \right) + F\left(t, x^2 + z^2, \frac{1}{2} x z + y \right). \tag{263}$$

The linear combination $d_i X_i + d_4 X_4^V$ yields the potential function

$$
\begin{aligned}
V\left(t,x,y,z\right) &= \frac{1}{\sqrt{{d_4}^2}}\Bigg[F\Bigg(t,-1/2\,d_4z^2-1/2\,d_4x^2-d_2z+xd_3,\\
&\quad 1/2\,\frac{1}{{d_4}^3}\Bigg(xz{d_4}^3+2f(x,z)d_1d_4-3f(x,z){d_2}^2+\\
&\quad f(x,z){d_3}^2-3\,xd_2{d_4}^2+2\,y{d_4}^3+z{d_3}{d_4}^2+d_2d_3d_4\Bigg)\Bigg)\sqrt{{d_4}^2}\\
&\quad +p\arctan\left(\frac{\sqrt{{d_4}^2}}{zd_4+d_2}\left(x-\frac{d_3}{d_4}\right)\right)\Bigg], \qquad (264)
\end{aligned}
$$

where $f(x,z)=\arctan\left(\frac{\sqrt{{d_4}^2}(-xd_4+d_3)}{d_4(zd_4+d_2)}\right)\sqrt{{d_4}^2}$.

Returning to the WDW equation (217), a reduction with W_1 followed by W_2 gives the wave function

$$U_V\left(t,x,y,z\right)=Q(t,x),$$

where $Q(t,x)$ satisfies the equation

$$
\begin{aligned}
&-4\left(\frac{\partial^2}{\partial t^2}Q\right)x^2k_1t\sqrt{tk_1+k_2}-4\left(\frac{\partial^2}{\partial t^2}Q\right)x^2k_2\sqrt{tk_1+k_2}\\
&-4\left(\frac{\partial^2}{\partial t^2}Q\right)t^2{k_1}^2-7\left(\frac{\partial}{\partial t}Q\right)k_1x^2\sqrt{tk_1+k_2}\\
&-8\left(\frac{\partial^2}{\partial t^2}Q\right)tk_1k_2-8\left(\frac{\partial}{\partial t}Q\right)t{k_1}^2\\
&-4\left(\frac{\partial^2}{\partial t^2}Q\right){k_2}^2-8\left(\frac{\partial}{\partial t}Q\right)k_1k_2\\
&+4\,x^2\frac{\partial^2}{\partial x^2}Q+4\left(\frac{\partial^2}{\partial x^2}Q\right)\sqrt{tk_1+k_2}+4\,x\frac{\partial}{\partial x}Q=0, \qquad (265)
\end{aligned}
$$

where we have a similar situation as in Case (IV). Even though this case has an extra KVF, it is not admitted by the WDW equation, nor is it related to any symmetry of the WDW equation.

Regarding the solution of (220) describing the motion of a particle, we obtain

$$\begin{aligned} f\left(t,x,y,z\right) &= \epsilon\left(\int\frac{\sqrt{\left(k_1t+k_2\right)^{5/2}\left({c_3}^2\left(k_1t+k_2\right)^{3/2}-2\,k_1tc_1-2\,k_2c_1\right)}}{\left(k_1t+k_2\right)^{5/2}}\,\mathrm{d}t\right. \\ &\quad -\frac{1}{4}\frac{\left(-2\,x{c_3}^2-2\,c_3c_4\right)\sqrt{-x^2{c_3}^2-2\,xc_3c_4-{c_4}^2-2\,c_1}}{{c_3}^2}- \\ &\quad \left.\frac{c_1}{\sqrt{{c_3}^2}}\arctan\left(\frac{\sqrt{{c_3}^2}}{\sqrt{-x^2{c_3}^2-2\,xc_3c_4-{c_4}^2-2\,c_1}}\left(x+\frac{c_4}{c_3}\right)\right)\right) \\ &\quad +yc_3+zc_4+f_0. \end{aligned} \tag{266}$$

Utilizing the system (221), we find the analytical solutions to the equations of motion:

$$\begin{aligned} t(s) &= \frac{-k_2{c_3}^4+4\,{c_1}^2}{k_1{c_3}^4}, \\ x(s) &= \frac{1}{c_3}\left[\left(c_1\left(\left(\cos\left(1/8\,\frac{{c_3}^7\epsilon\,\left(x_0+s\right)}{{c_1}^3}\right)\right)^2-1\right)\left(\cos\left(\frac{1}{8}\,\frac{{c_3}^7\epsilon\,\left(x_0+s\right)}{{c_1}^3}\right)\right)^{-4}\right)^{1/2}\right. \\ &\quad \left.\sqrt{2}\left(\cos\left(1/8\,\frac{{c_3}^7\epsilon\,\left(x_0+s\right)}{{c_1}^3}\right)\right)^2-c_4\right], \\ y(s) &= \int\frac{1}{8}\,\frac{x\left(s\right){c_3}^7\epsilon\,\left(x\left(s\right)c_3+2\,c_4\right)\gamma-8\,{c_1}^3c_4\left(\frac{\mathrm{d}}{\mathrm{d}s}x\left(s\right)\right)\left(x\left(s\right)c_3+c_4\right)}{\gamma\,{c_1}^3\epsilon\,c_3}\,\mathrm{d}s+ \\ &\quad \frac{1}{8}\,\frac{s{c_3}^5{c_4}^2}{{c_1}^3}+1/4\,\frac{s{c_3}^5}{{c_1}^2}+x_0, \\ z(s) &= -\frac{\gamma}{\epsilon\,c_3}+z_0, \end{aligned} \tag{267}$$

where $\gamma=\sqrt{-{c_3}^2\left(x\left(s\right)\right)^2-2\,c_3c_4x\left(s\right)-{c_4}^2-2\,c_1}$.

5.3.6 Case (VI) $A = C = k_1$, $B = -2k_1^2 t^{-2}$, where $k_1 \in \Re\backslash\{0\}$.

Here the conservation laws associated with (213) are:

$$\begin{aligned} I_1^{VI} &= -2\,\frac{{k_1}^2\,(x\dot{z}-\dot{y})}{t^2}, \\ I_2^{VI} &= -\frac{k_1\,(2\,k_1 x z\dot{z}-2\,k_1\dot{y}z+t^2\dot{x})}{t^2}, \\ I_3^{VI} &= \frac{k_1\,(2\,k_1 x^2\dot{z}-2\,k_1 x\dot{y}-t^2\dot{z})}{t^2}. \end{aligned} \tag{268}$$

This case possesses five conformal vector fields, where the fourth one is the KVF

$$X_4^{VI} = -z\partial_x - \left(\frac{x^2-z^2}{2}\right)\partial_y + x\partial_z, \tag{269}$$

whereas the fifth symmetry is the proper HVF given below

$$H_{VI} = \frac{1}{4}t\partial_t + \frac{1}{4}x\partial_x + \frac{1}{4}y\partial_y + \frac{1}{4}z\partial_z. \tag{270}$$

First integrals corresponding to the above symmetries are

$$\begin{aligned} I_4^{VI} &= 3\,\frac{k_1\,((x^2-1/3\,z^2)\,(x\dot{z}-\dot{y})\,k_1-1/3\,t^2\,(x\dot{z}-\dot{x}z))}{t^2}, \\ I(H_{VI}) &= \frac{1}{4}\,\frac{2\,(x\dot{z}-\dot{y})\,(xz-y)\,{k_1}^2-t^2\,(x\dot{x}+z\dot{z})\,k_1+\dot{t}t^3}{t^2}. \end{aligned} \tag{271}$$

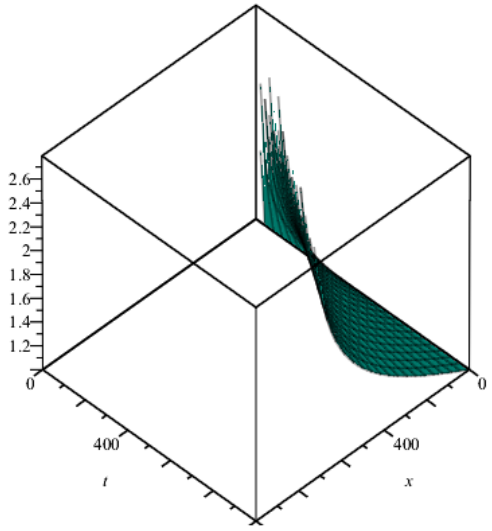
These additional vector fields (269) and (270) yield the following potential functions

$$V\,(t,x,y,z) = -p\arctan\left(\frac{x}{z}\right) + F\left(t, x^2+z^2, 1/2\,xz+y\right), \tag{272}$$

$$V\,(t,x,y,z) = \frac{8}{3}\,p + \frac{1}{t^{3/2}}F\left(\frac{x}{t},\frac{y}{t},\frac{z}{t}\right), \tag{273}$$

respectively.

Figure 5.3: Evolution of the solution (275), with we select the parameter values $C_1 = C_2 = 1$ and solution $k_1 = 1$.



For this case we have an additional WDW symmetry of Eq. (217)

$$W_{VI} = \frac{1}{2}W_U + 4H_{VI}. \tag{274}$$

Successive reductions of Eq. (217), with W_1, W_2 and W_{VI} yields a solution to WDW equation, as

$$\begin{aligned} U_{VI}(t,x,y,z) &= C_1 + \frac{C_2}{\sqrt{2\,\frac{{k_1}^2x^4}{t^4} - 3\,\frac{k_1x^2}{t^2} + 1}}\left(2\,\frac{x^3{k_1}^{3/2}}{t^3} - \frac{\sqrt{k_1}x}{t}\right. \\ &\left. E_E\left(\frac{x\sqrt{2}\sqrt{k_1}}{t}, 1/2\,\sqrt{2}\right)\sqrt{2}\sqrt{-2\,\frac{k_1x^2}{t^2}+1}\sqrt{-\frac{k_1x^2}{t^2}+1}\right), \end{aligned} \tag{275}$$

where E_E has been defined in case (I).

Suppose we consider potential function $V = px$, then the solution of (220) is given

by the expression

$$
\begin{aligned}
f(t,x,y,z) &= \epsilon\left(-\frac{1}{4}\,\frac{t\sqrt{-2\,t^2{c_3}^2-8\,k^2c_1}}{k}+\right.\\
&\quad \frac{kc_1\sqrt{2}}{\sqrt{{c_3}^2}}\arctan\left(\frac{\sqrt{2}\sqrt{{c_3}^2}t}{\sqrt{-2\,t^2{c_3}^2-8\,k^2c_1}}\right)\\
&\quad \left.+\int\sqrt{-x^2{c_3}^2-2\,kpx-2\,xc_3c_4-2\,kc_1-{c_4}^2}\,\mathrm{d}x\right)+\\
&\quad yc_3+zc_4+f_0, \qquad (276)
\end{aligned}
$$

Hence, we solve the system (221) to find the solution of the field equations

$$
\begin{aligned}
t(s) &= \pm 2\,\frac{k_1}{c_3}\sqrt{-\left(\tan\left(1/2\,\frac{\epsilon\sqrt{{c_3}^2}\,(t_0+s)\,\sqrt{2}}{k_1}\right)^2+1\right)c_1}\times\\
&\quad \tan\left(1/2\,\frac{\epsilon\sqrt{{c_3}^2}\,(t_0+s)\,\sqrt{2}}{k_1}\right)\left(\tan\left(1/2\,\frac{\epsilon\sqrt{{c_3}^2}\,(t_0+s)\,\sqrt{2}}{k_1}\right)^2+1\right)^{-1},\\
x(s) &= \frac{-k_1p-c_3c_4+\sqrt{{k_1}^2p^2+2\,k_1pc_3c_4-2\,k_1c_1{c_3}^2}}{{c_3}^2},\\
y(s) &= \int -\frac{1}{2}\,\frac{\left(t\left(s\right)\right)^2{c_3}^2+4\,x\left(s\right)p{k_1}^2+2\,x\left(s\right)c_4k_1c_3+4\,{k_1}^2c_1+2\,{c_4}^2k_1}{{k_1}^2c_3}\,\mathrm{d}s+\\
&\quad y_0,\\
z(s) &= \int\frac{x\left(s\right)c_3+c_4}{k_1}\,\mathrm{d}s+z_0. \qquad (277)
\end{aligned}
$$

5.3.7 Case (VII) $A = B = \left(k_2 \cos\sqrt{\frac{2}{k_1}}t + k_3 \sin\sqrt{\frac{2}{k_1}}t\right)$, $C = k_1$, where $k_1, k_2, k_3 \in \Re$ $(k_1 \neq 0)$.

The conservation laws associated with the three KVFs, (213), under these conditions are:

$$\begin{aligned}
I_1^{VII} &= (x\dot{z} - \dot{y})\left(k_2 \cos\left(\sqrt{2}\sqrt{{k_1}^{-1}}t\right) + k_3 \sin\left(\sqrt{2}\sqrt{{k_1}^{-1}}t\right)\right), \\
I_2^{VII} &= ((x\dot{z} - \dot{y})\, z - \dot{x})\left(k_2 \cos\left(\sqrt{2}\sqrt{{k_1}^{-1}}t\right) + k_3 \sin\left(\sqrt{2}\sqrt{{k_1}^{-1}}t\right)\right), \\
I_3^{VII} &= -k_1\dot{z} - k_2 x\,(x\dot{z} - \dot{y}) \cos\left(\sqrt{2}\sqrt{{k_1}^{-1}}t\right) - \\
&\quad k_3 x\,(x\dot{z} - \dot{y}) \sin\left(\sqrt{2}\sqrt{{k_1}^{-1}}t\right). \qquad (278)
\end{aligned}$$

The extra KVF generated in this case is

$$X_4^{VII} = z\partial_x + \left(\frac{k_1 x^2 - z^2}{2}\right)\partial_y - k_1 x \partial_z. \qquad (279)$$

Here we find the first integral is

$$\begin{aligned}
I^{\mathrm{VII}} &= 3/2\left(k_1 x^3 \dot{z} - k_1 x^2 \dot{y} - 1/3\, x z^2 \dot{z} + 1/3\, z\,(\dot{y} z - 2\,\dot{x})\right) \times \\
&\quad k_2 \cos\left(\sqrt{2}\sqrt{{k_1}^{-1}}t\right) + 3/2\left(k_1 x^3 \dot{z} - k_1 x^2 \dot{y} - 1/3\, x z^2 \dot{z} + \right. \\
&\quad \left. 1/3\, z\,(\dot{y} z - 2\,\dot{x})\right) k_3 \sin\left(\sqrt{2}\sqrt{{k_1}^{-1}}t\right) + {k_1}^2 x \dot{z}, \qquad (280)
\end{aligned}$$

and admitted potential function

$$\begin{aligned}
V(t, x, y, z) &= \frac{1}{\sqrt{k_1}}\left(p \arctan\left(\frac{x\sqrt{k_1}}{z}\right) + \right. \\
&\quad \left. F\left(t, k_1 x^2 + z^2, 1/2\, xz + y\right)\sqrt{k_1}\right). \qquad (281)
\end{aligned}$$

Moreover, the linear combination $d_i X_i + d_4 X_4^{VII}$ permits the potential

$$
\begin{aligned}
V\left(t,x,y,z\right) &= \frac{1}{\sqrt{k_1 {d_4}^2}}\Bigg[F\Bigg(t,-1/2\,k_1x^2d_4-1/2\,z^2d_4-d_2z+xd_3,\\
&\quad 1/2\,\frac{1}{\sqrt{k_1{d_4}^2}k_1{d_4}^2}\Big[x\left(zd_4+d_2\right)d_4k_1\sqrt{k_1{d_4}^2}+\\
&\quad 2\,y\sqrt{k_1{d_4}^2}k_1{d_4}^2-4\,d_2xd_4k_1\sqrt{k_1{d_4}^2}-2\,d_1g(x,z){d_4}^2k_1\\
&\quad +3g(x,z){d_2}^2d_4k_1-{d_3}^2g(x,z)d_4\\
&\quad +d_3\left(zd_4+d_2\right)\sqrt{k_1{d_4}^2}\Big]\Bigg)\sqrt{k_1{d_4}^2}\\
&\quad +p\arctan\left(\frac{\sqrt{k_1{d_4}^2}}{zd_4+d_2}\left(x-\frac{d_3}{k_1d_4}\right)\right)\Bigg], \qquad (282)
\end{aligned}
$$

where the function $g(x,z)$ has the form

$$
g(x,z)=\arctan\left(\frac{\sqrt{k_1{d_4}^2}\left(k_1xd_4-d_3\right)}{k_1d_4\left(zd_4+d_2\right)}\right). \qquad (283)
$$

In this case, the WDW equation (217), may be reduced with W_1 followed by W_2, to give the wave function

$$
U_{VII}\left(t,x,y,z\right)=M(t,x),
$$

where $M(t,x)$ satisfies the equation

$$
\begin{aligned}
&-2\left(\frac{\partial^2}{\partial t^2}M\right)\left(\cos\left(\frac{\sqrt{2}t}{\sqrt{k_1}}\right)\right)^2 x^2{k_2}^2\sqrt{k_1}\\
&+2\left(\frac{\partial^2}{\partial t^2}M\right)\left(\cos\left(\frac{\sqrt{2}t}{\sqrt{k_1}}\right)\right)^2 x^2{k_3}^2\sqrt{k_1}\\
&-4\left(\frac{\partial^2}{\partial t^2}M\right)\cos\left(\frac{\sqrt{2}t}{\sqrt{k_1}}\right)\sin\left(\frac{\sqrt{2}t}{\sqrt{k_1}}\right)x^2k_2k_3\sqrt{k_1}\\
&-6\left(\cos\left(\frac{\sqrt{2}t}{\sqrt{k_1}}\right)\right)^2\left(\frac{\partial}{\partial t}M\right)\sqrt{2}x^2k_2k_3\\
&+3\cos\left(\frac{\sqrt{2}t}{\sqrt{k_1}}\right)\sin\left(\frac{\sqrt{2}t}{\sqrt{k_1}}\right)\left(\frac{\partial}{\partial t}M\right)\sqrt{2}x^2{k_2}^2\\
&-3\cos\left(\frac{\sqrt{2}t}{\sqrt{k_1}}\right)\sin\left(\frac{\sqrt{2}t}{\sqrt{k_1}}\right)\left(\frac{\partial}{\partial t}M\right)\sqrt{2}x^2{k_3}^2\\
&-2\left(\frac{\partial^2}{\partial t^2}M\right)x^2{k_3}^2\sqrt{k_1}+2\cos\left(\frac{\sqrt{2}t}{\sqrt{k_1}}\right)\left(\frac{\partial^2}{\partial x^2}M\right)x^2k_2\sqrt{k_1}\\
&-2\cos\left(\frac{\sqrt{2}t}{\sqrt{k_1}}\right)\left(\frac{\partial}{\partial t}M\right)k_1\sqrt{2}k_3+2\sin\left(\frac{\sqrt{2}t}{\sqrt{k_1}}\right)\left(\frac{\partial^2}{\partial x^2}M\right)x^2k_3\sqrt{k_1}\\
&+2\sin\left(\frac{\sqrt{2}t}{\sqrt{k_1}}\right)\left(\frac{\partial}{\partial t}M\right)k_1\sqrt{2}k_2+3\left(\frac{\partial}{\partial t}M\right)\sqrt{2}x^2k_2k_3\\
&-2\left(\frac{\partial^2}{\partial t^2}M\right)\cos\left(\frac{\sqrt{2}t}{\sqrt{k_1}}\right){k_1}^{3/2}k_2-2\left(\frac{\partial^2}{\partial t^2}M\right)\sin\left(\frac{\sqrt{2}t}{\sqrt{k_1}}\right){k_1}^{3/2}k_3\\
&+2\cos\left(\frac{\sqrt{2}t}{\sqrt{k_1}}\right)\left(\frac{\partial}{\partial x}M\right)xk_2\sqrt{k_1}+2\sin\left(\frac{\sqrt{2}t}{\sqrt{k_1}}\right)\left(\frac{\partial}{\partial x}M\right)xk_3\sqrt{k_1}\\
&+2\left(\frac{\partial^2}{\partial x^2}M\right){k_1}^{3/2}=0. \qquad (284)
\end{aligned}
$$

Concerning the solution of (220), we find

$$f(t,x,y,z) = yc_3 + zc_4 + F_7(t,x) + f_0, \tag{285}$$

where $F_7(t,x)$ satisfies the constraint equation

$$\begin{aligned}
&\frac{1}{\cos\left(\sqrt{2}\sqrt{{k_1}^{-1}}t\right)k_2 + \sin\left(\sqrt{2}\sqrt{{k_1}^{-1}}t\right)k_3}\Bigg[-\cos\left(\sqrt{2}\sqrt{{k_1}^{-1}}t\right)k_2x^2{c_3}^2 \\
&-\sin\left(\sqrt{2}\sqrt{{k_1}^{-1}}t\right)k_3x^2{c_3}^2 + \left(\frac{\partial}{\partial t}F_7\right)^2\cos\left(\sqrt{2}\sqrt{{k_1}^{-1}}t\right)k_1k_2 + \\
&\left(\frac{\partial}{\partial t}F_7\right)^2\sin\left(\sqrt{2}\sqrt{{k_1}^{-1}}t\right)k_1k_3 - 2\cos\left(\sqrt{2}\sqrt{{k_1}^{-1}}t\right)k_2xc_3c_4 - \\
&2\sin\left(\sqrt{2}\sqrt{{k_1}^{-1}}t\right)k_3xc_3c_4 - {c_4}^2\cos\left(\sqrt{2}\sqrt{{k_1}^{-1}}t\right)k_2 - \\
&{c_4}^2\sin\left(\sqrt{2}\sqrt{{k_1}^{-1}}t\right)k_3 - \left(\frac{\partial}{\partial x}F_7\right)^2k_1 - {c_3}^2k_1\Bigg] = 0.
\end{aligned} \tag{286}$$

Unfortunately this case is challenging to solve and we may a very particular solution, for the parameter values $k_1 = k_3 = 1$, $k_2 = 0$ and $c_3 = 0$, viz.

$$F_7(t,x) = \epsilon\, c_4t + c_2x + F_0. \tag{287}$$

Moreover, for $c_2 = 0$, we may solve the system (221) to obtain the solutions of the equations of motion:

$$\begin{aligned}
t(s) &= -\epsilon\, sc_4 + t_0, \\
x(s) &= x_0, \\
y(s) &= x_0\, c_4s + y_0, \\
z(s) &= sc_4 + z_0.
\end{aligned} \tag{288}$$

5.4 Discussion

We see a good impetus to study Bianchi homogeneous and anisotropic cosmological models in current literature [73]. The modified $f(R)$ theory of gravity, a generalisation of general relativity from Einstein, has also been of prime significance in the field of theoretical cosmology. In recent years, it has piqued curiosity as a potential explanation for the observed acceleration of the universe, providing an alternative to dark energy. Aside from serving as a dark energy substitute, $f(R)$ gravity was subject to the investigation of traversable wormholes and associated energy conditions in its framework [74, 75, 76]. Theories on other modified gravity models can be found in, for example, [77, 78, 79].

Indeed there exist many works in the literature surrounding various theories of modified gravity. The crucial difference between our study and the existing work is that, to our knowledge, this is the first time that Lie and Noether symmetries have been used to shape an investigation for the Bianchi type II spacetimes in $f(R)$ theory of gravity. We have shown firstly, that novel solutions can be found cosmologically in this space. Secondly, we have calculated new potential functions of the space via a rigorous mathematical procedure. Specifically, in our study, KVFs and HVFs were the symmetries used to find the potential functions arising from the Lagrangian. More significantly, to our knowledge, this is a first time that several wave function solutions of the universe are obtained for the Bianchi type II spacetimes in $f(R)$ theory, whose dynamics are governed by the WDW equation, the quantised Hamiltonian constraint of the system. That is, when the WDW equation is invariant under a one-parameter point transformation, it may be reduced in order and an analysis of the invariant properties leads to some solutions for the wave functions for each of the seven cases. In one case, we showed that the wave function consists of elliptic integrals. The infinitesimal generators of the WDW

equation have strong connections to the conformal algebra, in particular, the HVF. We showed an important geometric implication, that for cases that do not admit a HVF, explicit solutions for the wave function cannot be found by reduction. These reasons provide compelling reasons to study the WDW equation in cosmology. In the relevant literature, neither the field solutions obtained above nor any connections to other reported findings have been found.

Chapter 6

Lie symmetry analysis and analytical solutions of the nonlinear Ivancevic options pricing model

This chapter studies the nonlinear quantum-probability based Schrödinger type, Ivancevic options pricing model (IOPM) using the method of Lie symmetries to determine its point symmetries, invariant analytical solutions and conversation laws. In our analysis we consider a non-zero and zero adaptive market potential model. We demonstrate that this model is invariant under a five-dimensional Lie algebra for the former, and invariant under a seven-dimensional Lie algebra in the latter case. We obtain reductions, exact solutions and conservation laws for both the non-zero and zero adaptive market potential models. We show that many exact solutions are expressible in terms of two transcendental functions, the Fresnel sine and cosine

integrals. Graphical solutions are provided in certain cases.

6.1 Introduction

The Black-Scholes model is defined as follows. $S = S(t)$ is the price function of the underlying asset at time t $(0 \leq t \leq T)$, that satisfies the stochastic differential equation describing geometric Brownian motion

$$dS = S(\mu dt + \sigma dW_t), \quad S \in [0, \infty), \tag{289}$$

where μ and σ are the drift parameter (in this case the rate of return of the asset price S) and volatility, respectively, and the standard Wiener process, W_t. The Black-Scholes PDE associated with (289) for an option value $V = V(S, t)$ and risk-free interest rate r can be expressed as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{290}$$

with $V(0, t) = 0$, $V(S, t) \to \infty$ as $S \to \infty$, $V(S, t) = max(S - E)$, E the strike price or exercise price is taken as a constant and the price function is

$$S(t) = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}, \; S_0 = S(0). \tag{291}$$

This model is primarily used to price European call and put options [80]. The main assumption of the Black-Scholes model are that μ and σ are constant. That is, over the life of the derivative, the underlying volatility remains constant and is unaffected by changes in the underlying asset's price level. This constrains the model since it is unable to account for long-observed characteristics of the implied volatility surface such as volatility smile and skew, which indicate that implied volatility varies with strike price and expiration [81]. In relation to physics, (290) resembles the

Fokker-Planck equation used to describe the Brownian motion of a particle in a fluid [82]. Additionally, using quantum mechanics [83] established that the Black-Scholes equation can be derived from the Schödinger equation. Analytical solutions to this model has been extensively studied in literature. In [84], the Black-Scholes equation with a non-smooth boundary condition was transformed into the heat equation using symmetries to find an exact solution. Lie symmetry methods have been utilised in the fractional Black-Scholes equation to find invariant solutions and conservation laws [85]. Other methods to find analytical solutions to the Black-Scholes model can be found in [86, 87, 88, 89]. Because of the Black-Scholes equation's limitations as a result of its assumptions, there has been a push to develop models that loosen these assumptions and provide a more realistic picture of the markets.

The IOPM is a nonlinear adaptive-wave model, describing controlled Brownian motion of financial markets, and a wave alternative to the standard Black-Scholes option-pricing model. The complex-valued equation, introduced by Vladimir G. Ivancevic [90], is defined by adaptive nonlinear Schrödinger (NLS) equations, in which the option-pricing wave function is expressed in terms of asset (stock) price and time. The wave function representing quantum probability amplitude, when taking its absolute square gives a probability density function. The IOPM is

$$i\frac{\partial \Psi}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 \Psi}{\partial S^2} + \beta|\Psi|^2\Psi = 0,\ i^2 = -1, \tag{292}$$

where $\Psi = \Psi(S,t)$ denotes the option pricing wave-function at time t, $|\Psi|^2 = |\Psi(S,t)|^2$ represents the probability density function (PDF) for the option price in terms of stock price and time, σ represents a constant or stochastic process as the dispersion frequency volatility coefficient, and β is the Landau coefficient representing adaptive market potential. This adaptive market potential in its simplest non-adaptive form is equal to the interest rate r. For the adaptive scenario, on the other hand, $\beta = \beta(r,w)$ depends on the set of adjustable parameters $\{w_j\}$. In this

adaptive case, the Landau coefficient can be linked to market temperature.

The use of Lie symmetry methods to study PDEs is highly effective and a plethora of work can be found in the literature [91, 92, 93]. In fact, the technique has been utilised in finding invariant solutions to other financial engineering models such as the Black-Scholes-Merton model for European options [94], the fractional Black-Scholes equation as mentioned above, and the Asian option [95] etc. Very few works on the exact solutions to the IOPM exist in literature. In [96], the He's Frequency Amplitude Formulation was utilised, whereas in [97], authors used the projected differential transform method. The paper [98] constructed dark wave, rogue wave and perturbation solutions via the trial function method. Solutions to the time-fractional IOPM was discussed in [81]. In our analysis we consider the deterministic form of the IOPM model with a constant volatility. We present novel invariant solutions for both the adaptive and non-adaptive market potentials. Where appropriate, we provide plots of these solutions. Finally, conservation laws for both cases are determined.

6.2 Symmetries and invariant solutions for the IOPM with a non-zero adaptive market potential

In this section we consider a non-zero adaptive market potential i.e. $\beta \neq 0$. We start by splitting the complex equation (292) into its real and imaginary counterparts by

taking $\Psi(S,t) = u(S,t) + iv(S,t)$, obtaining the system:

$$-\frac{\partial}{\partial t}v\left(S,t\right) + \frac{1}{2}\,\sigma^2\frac{\partial^2}{\partial S^2}u\left(S,t\right) + \beta\,\left(u\left(S,t\right)\right)^3 + \beta\,u\left(S,t\right)\left(v\left(S,t\right)\right)^2 = 0, \tag{293}$$

$$\frac{\partial}{\partial t}u\left(S,t\right) + \frac{1}{2}\,\sigma^2\frac{\partial^2}{\partial S^2}v\left(S,t\right) + \beta\,\left(u\left(S,t\right)\right)^2 v\left(S,t\right) + \beta\,\left(v\left(S,t\right)\right)^3 = 0. \tag{294}$$

The reduction of the number of independent variables in PDEs, is the subject of this chapter. The reductions can be used to obtain solutions known as invariant solutions. The value of symmetries is that they allow one to connect sophisticated equations or systems to simpler ones that may potentially be solved.

The IOPM system, (293)-(294), is invariant under a five-dimensional Lie algebra consisting of the symmetry vector fields, as determined by Lie theory:

$$X_1 = \frac{\partial}{\partial S}, \tag{295}$$

$$X_2 = \frac{\partial}{\partial t}, \tag{296}$$

$$X_3 = u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u}, \tag{297}$$

$$X_4 = \sigma^2 t\frac{\partial}{\partial S} + Su\frac{\partial}{\partial v} - Sv\frac{\partial}{\partial u}, \tag{298}$$

$$X_5 = -S\frac{\partial}{\partial S} - 2t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}. \tag{299}$$

The commutators of admitted point symmetries of the IOPM with a non-zero adaptive market potential are given in Table 1.

6.2.1 Conservation laws for $\beta \neq 0$

Suppose that the system (293)-(294) admits the multipliers of the form

$[,]$	X_1	X_2	X_3	X_4	X_5
X_1	0	0	0	X_3	$-X_1$.
X_2	0	0	0	$\sigma^2 X_1$	$-2X_2$
X_3	0	0	0	0	0
X_4	X_3	$\sigma^2 X_1$	0	0	X_4
X_5	$-X_1$	$-2X_2$	0	X_4	0

Table 6.1: Commutator table for $\beta \neq 0$.

$$\Lambda_1(S, t, u, v, v_S, u_S, v_{SS}, u_{SS}), \tag{300}$$

and

$$\Lambda_2(S, t, u, v, v_S, u_S, v_{SS}, u_{SS}), \tag{301}$$

that correspond to the conservation laws composed of T^t, the conserved density, and T^S, the conserved flux.

The system (293)-(294) with a non-zero adaptive market potential, $\beta \neq 0$, relating to the IOPM (292) admits the multipliers:

$$\Lambda_1 = \frac{\sigma^2 t u_S + Sv}{\sigma^2}, \tag{302}$$

$$\Lambda_2 = -\frac{-\sigma^2 t v_S + Su}{\sigma^2}. \tag{303}$$

These multipliers give rise to the conservation law vector components T_1^S and T^t:

$$\begin{aligned} T_1^S &= \frac{1}{4}\beta u^4 t + \frac{1}{2}\beta t u^2 v^2 + \frac{1}{4}\beta t v^4 - \frac{1}{2} S u v_S + \\ &\quad \frac{1}{4} u_S^2 \sigma^2 t + \frac{1}{2} S u_S v + \frac{1}{4}\sigma^2 t v_S^2 - \frac{1}{2} t u v_t + \\ &\quad \frac{1}{2} t u_t v, \\ T_1^t &= -\frac{1}{2}\frac{-\sigma^2 t u v_S + \sigma^2 t u_S v + Su^2 + Sv^2}{\sigma^2}. \end{aligned}$$

The above conservation laws can indeed be written in terms of the original variable Ψ:

$$\begin{aligned} T_1^S &= \frac{1}{4}\beta t(|\Psi|^2)^2 + \frac{1}{4}\sigma^2 t\Psi_S\overline{\Psi}_S - \frac{1}{4}iS(\Psi\overline{\Psi}_S - \overline{\Psi}\Psi_S) - \\ &\quad \frac{1}{4}it(\Psi\overline{\Psi}_t - \overline{\Psi}\Psi_t), \\ T_1^t &= -\frac{1}{4\sigma^2}(ti\sigma^2(\Psi\overline{\Psi}_S - \overline{\Psi}\Psi_S) + 2S|\Psi|^2), \end{aligned}$$

where $\overline{\Psi}$ is the conjugate of Ψ. The multipliers $\Lambda_1 = u_S$, and $\Lambda_2 = v_S$, lead to the conservation laws

$$\begin{aligned} T_2^S &= \frac{1}{4}\beta u^4 + \frac{1}{2}\beta u^2v^2 + \frac{1}{4}\beta v^4 + \frac{1}{4}u_S^2\sigma^2 + \\ &\quad \frac{1}{4}v_S^2\sigma^2 - \frac{1}{2}uv_t + \frac{1}{2}vu_t \\ &= \frac{1}{4}\beta(|\Psi|^2)^2 + \frac{1}{4}\sigma^2\Psi\overline{\Psi}_S - \frac{1}{4}i(\Psi\overline{\Psi}_t - \overline{\Psi}\Psi_t), \end{aligned}$$

and

$$\begin{aligned} T_2^t &= -\frac{1}{2}u_Sv + \frac{1}{2}uv_S \\ &= \frac{1}{4}i(\Psi\overline{\Psi}_S - \overline{\Psi}\Psi_S). \end{aligned}$$

Moreover, the admitted integrating factors that satisfy (8) and (9) are

$$\Lambda_1 = -\frac{2\beta u^2v + 2\beta v^3 + \sigma^2 v_{SS}}{\sigma^2}, \tag{304}$$

$$\Lambda_2 = \frac{2\beta u^3 + 2\beta uv^2 + \sigma^2 u_{SS}}{\sigma^2}, \tag{305}$$

with corresponding conservation laws

$$\begin{aligned} T_3^S &= -\frac{1}{2}vv_{St} + \frac{1}{2}v_Sv_t - \frac{1}{2}uu_{St} + \frac{1}{2}u_Su_t \\ &= -\frac{1}{4}(\Psi\overline{\Psi}_{St} + \overline{\Psi}\Psi_{St}) + \frac{1}{4}(\Psi_S\overline{\Psi}_t + \overline{\Psi}_S\Psi_t), \\ T_3^t &= \frac{1}{2}\frac{\beta u^4 + 2\beta u^2v^2 + \beta v^4 + \sigma^2 uu_{SS} + \sigma^2 vv_{SS}}{\sigma^2} \\ &= \frac{1}{4\sigma^2}(\sigma^2(\Psi\overline{\Psi}_{SS} + \overline{\Psi}\Psi_{SS}) + 2\beta(|\Psi|^2)^2). \end{aligned}$$

The last pair of multipliers for our system with $\beta \neq 0$ are $\Lambda_1 = -v$,$\Lambda_2 = u$. As a result, we obtain the conservation laws

$$\begin{aligned} T_4^S &= \frac{1}{2}uv_S\sigma^2 - \frac{1}{2}\sigma^2 u_S v \\ &= \frac{1}{4}\sigma^2 i(\Psi\overline{\Psi}_S - \overline{\Psi}\Psi_S), \\ T_4^t &= \frac{1}{2}v^2 + \frac{1}{2}u^2 \\ &= \frac{1}{2}(|\Psi|^2). \end{aligned}$$

We now construct some exact solutions to the investigated system using the acquired symmetries. The terms C_i denote arbitrary constants. Additionally, since $\Psi(S,t) = u(S,t) + iv(S,t)$, one may easily recover the solution to the IOPM equation (292), from the solutions of $u(S,t)$ and $v(S,t)$ obtained.

6.2.2 Reductions and invariant solutions for $\beta \neq 0$

A reduction with X_1 leads to the reduce system

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}K(t) + \beta\,(K(t))^2 W(t) + \beta\,(W(t))^3 = 0, \\ -\frac{\mathrm{d}}{\mathrm{d}t}W(t) + \beta\,(K(t))^3 + \beta\,K(t)\,(W(t))^2 = 0, \end{aligned}$$

where $u(S,t) = K(t)$ and $v(S,t) = W(t)$. The above system solves to give us a trivial (constant) solution.

A reduction using X_2 gives us a non-trivial solution of

$$\begin{aligned} u(S,t) = 0, \\ v(S,t) = C_2\, JacobiSN\left(\left(\sqrt{\beta}S + C_1\right)C_2, i\right), \end{aligned} \tag{306}$$

where the $JacobiSN$ is the inverse of elliptic integrals and doubly periodic elliptic functions. We plot the above solution in Figure 6.1 for different values of β.

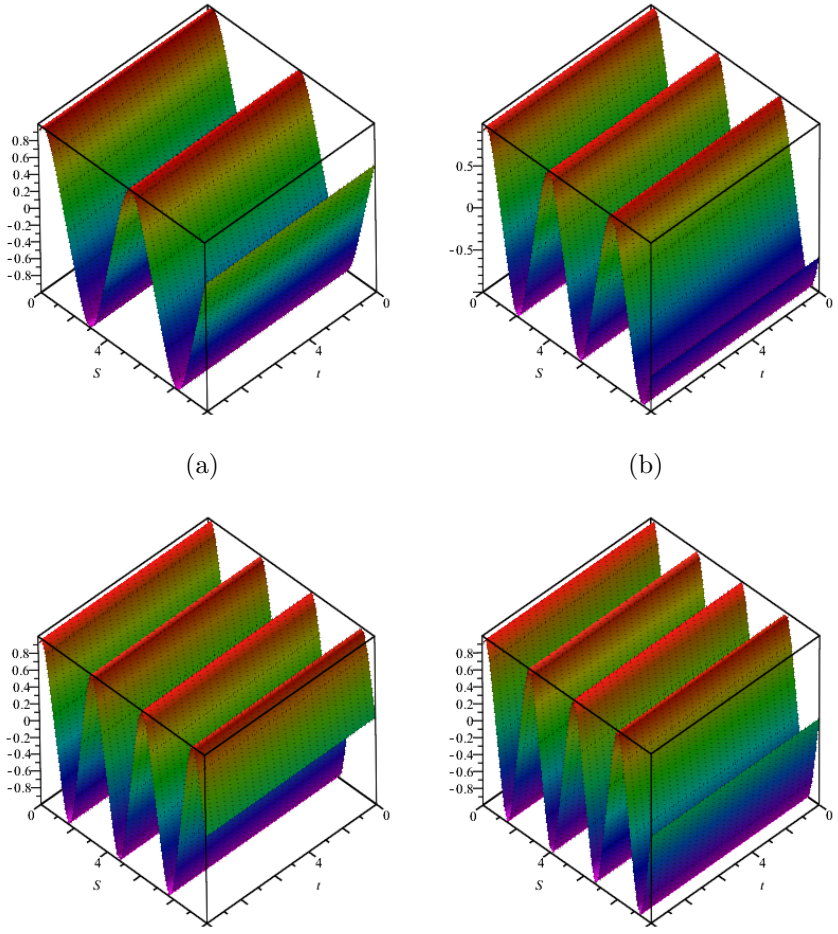


Figure 6.1: Graphical illustration of the analytical solutions are depicted, we select the parameter values $C_1 = C_2 = 1$ and the range $0 \leq S \leq 10$: (a) solution of (306) where $\beta = 1.0$; (b) solution of (306) where $\beta = 2.0$; (c) solution of (306) where $\beta = 3.0$; and, (d) solution of (306) where $\beta = 4.0$.

Lastly, reductions with X_5 yields

$$2\,\sigma^2\left(\frac{\mathrm{d}^2}{\mathrm{d}\frac{t}{S^2}^2}\kappa\left(\frac{t}{S^2}\right)\right)\frac{t}{S^2}^2+\beta\,\kappa\left(\frac{t}{S^2}\right)\left(\Omega\left(\frac{t}{S^2}\right)\right)^2+$$
$$\beta\left(\kappa\left(\frac{t}{S^2}\right)\right)^3+5\,\sigma^2\left(\frac{\mathrm{d}}{\mathrm{d}\frac{t}{S^2}}\kappa\left(\frac{t}{S^2}\right)\right)\frac{t}{S^2}+\sigma^2\kappa\left(\frac{t}{S^2}\right)-\frac{\mathrm{d}}{\mathrm{d}\frac{t}{S^2}}\Omega\left(\frac{t}{S^2}\right)=0,$$
$$2\,\sigma^2\left(\frac{\mathrm{d}^2}{\mathrm{d}\frac{t}{S^2}^2}\Omega\left(\frac{t}{S^2}\right)\right)\frac{t}{S^2}^2+\beta\left(\Omega\left(\frac{t}{S^2}\right)\right)^3+\beta\left(\kappa\left(\frac{t}{S^2}\right)\right)^2\Omega\left(\frac{t}{S^2}\right)+$$
$$5\,\sigma^2\left(\frac{\mathrm{d}}{\mathrm{d}\frac{t}{S^2}}\Omega\left(\frac{t}{S^2}\right)\right)\frac{t}{S^2}+\sigma^2\Omega\left(\frac{t}{S^2}\right)+\frac{\mathrm{d}}{\mathrm{d}\frac{t}{S^2}}\kappa\left(\frac{t}{S^2}\right)=0.$$

In the above system $u(S,t)=\kappa(\frac{t}{S^2})$ and $v(S,t)=\Omega(\frac{t}{S^2})$, where unfortunately an analytical solution is not found for this reduced system.

6.3 Symmetries and invariant solutions for the IOPM with a zero adaptive market potential

In the special case where we do not consider the adaptive market potential i.e. $\beta=0$, thus (292) becomes linear.

We use Lie's theory and find that under the action of the following seven-dimensional point transformation with generators given below, the IOPM system with $\beta=0$ is invariant:

$$X_1, X_2, X_3, X_4, \tag{307}$$

$$Y_1 = \frac{1}{2}S\frac{\partial}{\partial S} + t\frac{\partial}{\partial t}, \tag{308}$$

$$Y_2 = 2\sigma^2 St\frac{\partial}{\partial S} + 2\sigma^2 t^2\frac{\partial}{\partial t} + (-\sigma^2 tu - S^2 v)\frac{\partial}{\partial u} + (-\sigma^2 tv + S^2 u)\frac{\partial}{\partial v}, \tag{309}$$

$$Y_3 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}. \tag{310}$$

The commutators of the admitted Lie point symmetries of system (293)-(294), given a zero adaptive market potential are presented in Table 2.

$[,]$	X_1	X_2	X_3	X_4
Y_1	$\frac{1}{2}X_1$	X_2	0	$-\frac{1}{2}X_4$
Y_2	$2X_4$	$2\sigma^2 SX_1 + 4\sigma^2 tX_2 - \sigma^2 Y_3$	0	0
Y_3	0	0	0	0

Table 6.2: Commutator table for $\beta = 0$.

6.3.1 Conservation laws for $\beta = 0$

Employing the same mathematical method as detailed in the preceding section, we derive the conservation laws for our system (293)-(294), now using $\beta = 0$. This case is rich in conservation laws, possessing eight of them.

The first two multipliers derived are

$$\Lambda_1 = \frac{1}{2}v_{SS}\sigma^4 t^2 - \sigma^2 tSu_S - \frac{1}{2}\sigma^2 tu - \frac{1}{2}vS^2, \tag{311}$$

$$\Lambda_2 = -\frac{1}{2}u_{SS}\sigma^4 t^2 - \sigma^2 tSv_S - \frac{1}{2}\sigma^2 tv + \frac{1}{2}uS^2, \tag{312}$$

from which the conservation laws include

$$\begin{aligned} T_1^S &= \frac{1}{2}utS\sigma^2 v_t - \frac{1}{2}\sigma^2 vtSu_t - \frac{1}{4}v_S\sigma^4 t^2 v_t + \frac{1}{4}u\sigma^4 t^2 u_{St} - \\ &\quad \frac{1}{4}u_S\sigma^4 t^2 u_t - \frac{1}{4}S\sigma^4 t v_S^2 + \frac{1}{4}u\sigma^4 t u_S + \frac{1}{4}uS^2\sigma^2 v_S - \\ &\quad \frac{1}{4}u_S^2\sigma^4 tS - \frac{1}{4}u_S\sigma^2 vS^2 + \frac{1}{4}v\sigma^4 t v_S + \frac{1}{4}v\sigma^4 t^2 v_{St} \\ &= \frac{1}{4}S\sigma^2 it(\Psi\overline{\Psi}_t - \overline{\Psi}\Psi_t) - \frac{1}{4}\sigma^4 t^2\frac{1}{2}(\Psi_S\overline{\Psi}_t + \overline{\Psi}_S\Psi_t) + \\ &\quad \frac{1}{4}\sigma^4 t^2\frac{1}{2}(\Psi\overline{\Psi}_{St} + \overline{\Psi}\Psi_{St}) + \frac{1}{4}\sigma^4 t\frac{1}{2}(\Psi\overline{\Psi}_S + \overline{\Psi}\Psi_S) - \\ &\quad \frac{1}{4}\sigma^4 tS\Psi_S\overline{\Psi}_S + \frac{1}{4}\sigma^2 S^2\frac{1}{2}i(\Psi\overline{\Psi}_S - \overline{\Psi}\Psi_S), \end{aligned}$$

and

$$\begin{aligned} T_1^t &= -\frac{1}{4}vv_{SS}\sigma^4 t^2 + \frac{1}{2}v\sigma^2 tSu_S + \frac{1}{4}v^2S^2 - \frac{1}{4}\sigma^4 t^2 uu_{SS} - \\ &\quad \frac{1}{2}S\sigma^2 tuv_S + \frac{1}{4}u^2S^2 \\ &= -\frac{1}{4}\sigma^4 t^2(\frac{1}{2}(\Psi\overline{\Psi}_{SS} + \overline{\Psi}\Psi_{SS}) - \frac{1}{4}S\sigma^2 ti(\Psi\overline{\Psi}_S - \overline{\Psi}\Psi_S) + \\ &\quad \frac{1}{4}2S^2|\Psi|^2. \end{aligned}$$

Similarly, we obtained the multipliers

$$\Lambda_1 = -\sigma^2 tu_S - Sv, \tag{313}$$

$$\Lambda_2 = -\sigma^2 tv_S + Su, \tag{314}$$

with conservation laws

$$\begin{aligned} T_2^S &= \frac{1}{2}uS\sigma^2 v_S - \frac{1}{4}\sigma^4 t u_S^2 - \frac{1}{2}vS\sigma^2 u_S - \frac{1}{4}\sigma^4 t v_S^2 + \\ &\quad \frac{1}{2}\sigma^2 tuv_t - \frac{1}{2}\sigma^2 tvu_t \\ &= -\frac{1}{4}\sigma^4 t\Psi_S\overline{\Psi}_S + \frac{1}{4}iS\sigma^2(\Psi\overline{\Psi}_S - \overline{\Psi}\Psi_S) + \\ &\quad \frac{1}{4}\sigma^2 it(\Psi\overline{\Psi}_t - \overline{\Psi}\Psi_t), \\ T_2^t &= \frac{1}{2}\sigma^2 tvu_S + \frac{1}{2}Sv^2 - \frac{1}{2}\sigma^2 tuv_S + \frac{1}{2}Su^2 \\ &= -\frac{1}{4}\sigma^2 ti(\Psi\overline{\Psi}_S - \overline{\Psi}\Psi_S) + \frac{1}{2}S|\Psi|^2. \end{aligned}$$

Further, we apply the multipliers $\Lambda_1 = -v$, $\Lambda_2 = u$, to acquire the conservation laws

$$\begin{aligned} T_3^S &= \frac{1}{2}uv_S\sigma^2 - \frac{1}{2}\sigma^2 u_S v \\ &= \frac{1}{4}\sigma^2 i(\Psi\overline{\Psi}_S - \overline{\Psi}\Psi_S), \\ T_3^t &= \frac{1}{2}v^2 + \frac{1}{2}u^2 \\ &= \frac{1}{2}(|\Psi|^2). \end{aligned}$$

We deduce the multipliers

$$\Lambda_1 = -2\sigma^2 tv_{SS} + 2Su_S + u, \tag{315}$$

$$\Lambda_2 = 2\sigma^2 tu_{SS} + 2Sv_S + v, \tag{316}$$

from which we derive the conservation laws and express them in terms of Ψ:

$$
\begin{aligned}
T_4^S &= -\frac{1}{2}uu_S\sigma^2 + \frac{1}{2}S\sigma^2u_S^2 - \frac{1}{2}vv_S\sigma^2 + \frac{1}{2}S\sigma^2v_S^2 - \\
&\quad uSv_t - \sigma^2tvv_{St} + tv_S\sigma^2v_t - \sigma^2tuu_{St} + tu_S\sigma^2u_t + vSu_t \\
&= -\frac{1}{4}\sigma^2(\Psi\overline{\Psi}_S + \overline{\Psi}\Psi_S) + \frac{1}{2}S\sigma^2\Psi\overline{\Psi}_S - \frac{1}{2}\sigma^2t(\Psi\overline{\Psi}_{St} + \overline{\Psi}\Psi_{St}) - \\
&\quad \frac{1}{2}iS(\Psi\overline{\Psi}_t - \overline{\Psi}\Psi_t) + \frac{1}{2}\sigma^2t(\Psi_S\overline{\Psi}_t + \overline{\Psi}_S\Psi_t),
\end{aligned}
$$

and

$$
\begin{aligned}
T_4^t &= \sigma^2tuu_{SS} + \sigma^2tvv_{SS} + Suv_S - Su_Sv \\
&= \frac{1}{2}\sigma^2t(\Psi\overline{\Psi}_{SS} + \overline{\Psi}\Psi_{SS}) + \frac{1}{2}Si(\Psi\overline{\Psi}_S - \overline{\Psi}\Psi_S).
\end{aligned}
$$

The admitted multipliers also include $\Lambda_1 = u_S$, $\Lambda_2 = v_S$, leading to the corresponding conservation law

$$
\begin{aligned}
T_5^S &= \frac{1}{4}u_S^2\sigma^2 + \frac{1}{4}\sigma^2v_S^2 - \frac{1}{2}uv_t + \frac{1}{2}vu_t \\
&= \frac{1}{4}\sigma^4\Psi_S\overline{\Psi}_S - \frac{1}{4}i(\Psi\overline{\Psi}_t - \overline{\Psi}\Psi_t), \\
T_5^t &= -\frac{1}{2}vu_S + \frac{1}{2}uv_S \\
&= \frac{1}{4}i(\Psi\overline{\Psi}_S - \overline{\Psi}\Psi_S),
\end{aligned}
$$

followed by $\Lambda_1 = -v_{SS}$, and $\Lambda_2 = u_{SS}$, to get the conservation law

$$
\begin{aligned}
T_6^S &= -\frac{1}{2}vv_{St} + \frac{1}{2}v_Sv_t - \frac{1}{2}uu_{St} + \frac{1}{2}u_Su_t \\
&= -\frac{1}{4}(\Psi\overline{\Psi}_{St} + \overline{\Psi}\Psi_{St}) + \frac{1}{4}(\Psi_S\overline{\Psi}_t + \overline{\Psi}_S\Psi_t), \\
T_6^t &= \frac{1}{2}vv_{SS} + \frac{1}{2}uu_{SS} \\
&= \frac{1}{4}(\Psi\overline{\Psi}_{SS} + \overline{\Psi}\Psi_{SS}).
\end{aligned}
$$

The multipliers $\Lambda_1 = S$, $\Lambda_2 = 0$, yield conservation laws

$$\begin{aligned} T_7^S &= -\frac{1}{2}u\sigma^2 + \frac{1}{2}S\sigma^2 u_S \\ &= -\frac{1}{4}\sigma^2(\Psi + \overline{\Psi}) + \frac{1}{4}S\sigma^2(\Psi_S + \overline{\Psi}_S), \\ T_7^t &= -vS \\ &= \frac{1}{2}S(i\Psi - i\overline{\Psi}), \end{aligned}$$

and the multipliers $\Lambda_1 = 1$, $\Lambda_2 = 0$, lead to the conservation law

$$\begin{aligned} T_8^S &= \frac{1}{2}u_S\sigma^2 \\ &= \frac{1}{4}\sigma^2(\Psi_S + \overline{\Psi}_S), \\ T_8^t &= -v \\ &= \frac{1}{2}(i\Psi - i\overline{\Psi}). \end{aligned}$$

6.3.2 Reductions and invariant solutions for $\beta = 0$

Case 1: Reduction for Y_1

We obtain the following exact solution by applying Lie symmetry vector Y_1:

$$\begin{aligned} u(S,t) &= C_2 + C_3\, FresnelS\left(\frac{S}{\sigma\sqrt{\pi t}}\right) + \\ &\quad C_4\, FresnelC\left(\frac{S}{\sigma\sqrt{\pi t}}\right), \end{aligned} \tag{317}$$

$$v\left(S,t\right) = -FresnelC\left(\frac{S}{\sigma\sqrt{\pi t}}\right)C_3 + FresnelS\left(\frac{S}{\sigma\sqrt{\pi t}}\right)C_4 + C_1, \tag{318}$$

where $FresnelS$ and $FresnelC$ are the Fresnel sine integral and the Fresnel cosine integral respectively.

In Figure 6.2 we show the graphical solutions (317) and (318) for Case 1.

Case 2: Reduction via the linear combination $c_1X_1 + c_2X_2$

Reduction with X_1 and X_2, individually gives the solution for $\Psi(S,t)$ as a trivial (constant) solution. Instead, under this case we use the linear combinations of symmetries to get the analytical solutions

$$u\left(S,t\right) = C_2 + C_3\,\sin\left(2\,\frac{c_1\left(Sc_2 - tc_1\right)}{{c_2}^2\sigma^2}\right) + C_4\,\cos\left(2\,\frac{c_1\left(Sc_2 - tc_1\right)}{{c_2}^2\sigma^2}\right), \tag{319}$$

$$v\left(S,t\right) = \cos\left(2\,\frac{c_1\left(Sc_2 - tc_1\right)}{{c_2}^2\sigma^2}\right)C_3 - \sin\left(2\,\frac{c_1\left(Sc_2 - tc_1\right)}{{c_2}^2\sigma^2}\right)C_4 + C_1. \tag{320}$$

In Figure 6.3 we plot the solutions (319) and (320).

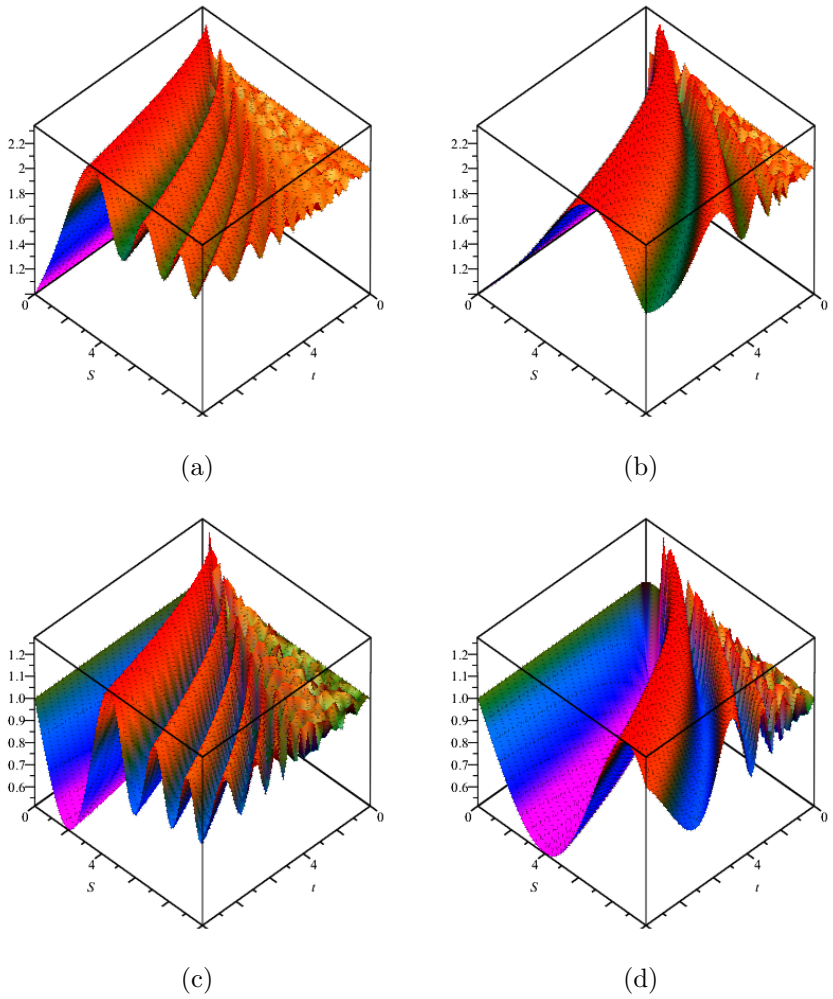


Figure 6.2: Graphical illustration of the analytical solutions are depicted, we select the parameter values $C_1 = C_2 = C_3 = C_4 = 1$ and the ranges $0 \leq t \leq 10$ and $0 \leq S \leq 10$: (a) solution of (317) where $\sigma = 0.5$; (b) solution of (317) where $\sigma = 1.0$; (c) solution of (318) where $\sigma = 0.5$; and, (d) solution of (318) where $\sigma = 1.0$.

Case 3: Reductions with $c_1X_1 + c_2Y_1$

Similar to Case 2 , we take a linear combination $c_1X_1 + c_2Y1$ which yields the invariant solutions below:

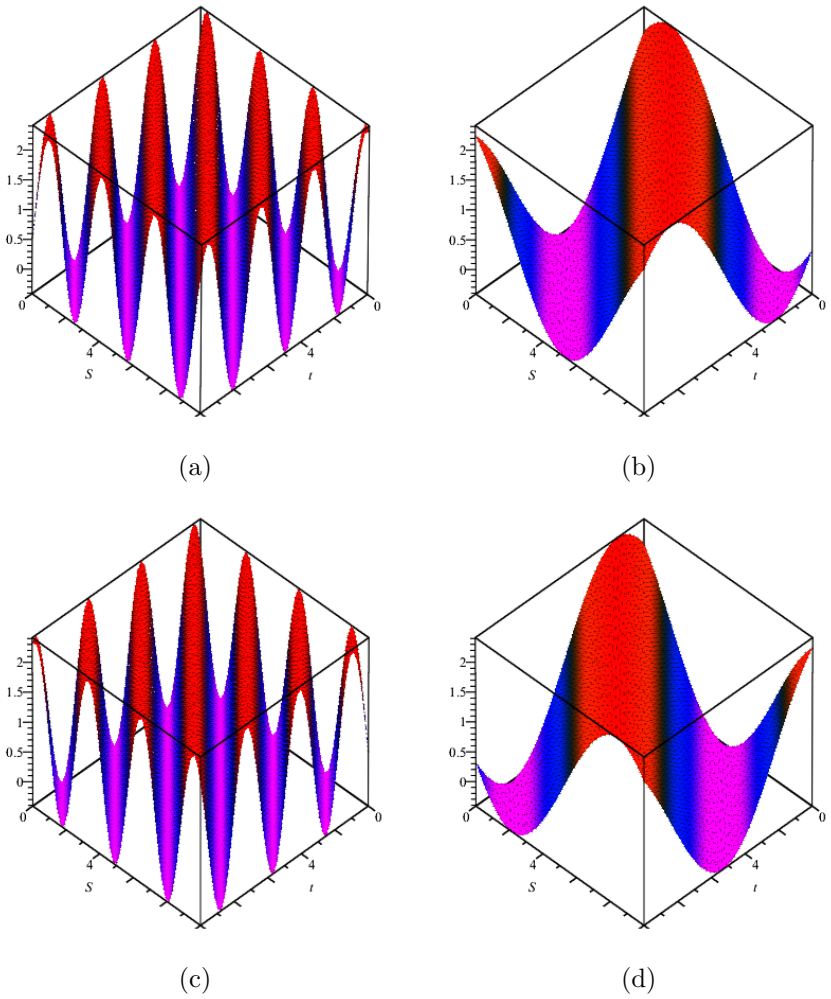


Figure 6.3: Graphical illustration of the analytical solutions are depicted, we select the parameter values $c_1 = c_2 = C_1 = C_2 = C_3 = C_4 = 1$ and the ranges $0 \leq t \leq 10$ and $0 \leq S \leq 10$: (a) solution of (319) where $\sigma = 1.0$; (b) solution of (319) where $\sigma = 2.0$; (c) solution of (320) where $\sigma = 1.0$; and, (d) solution of (320) where $\sigma = 2.0$.

$$u(S,t) = C_2 + C_3\, FresnelS\left(\frac{Sc_2 + 2\,c_1}{\sqrt{\pi t}c_2\sigma}\right) + C_4\, FresnelC\left(\frac{Sc_2 + 2\,c_1}{\sqrt{\pi t}c_2\sigma}\right), \tag{321}$$

$$v(S,t) = C_1 + FresnelS\left(\frac{Sc_2 + 2\,c_1}{\sqrt{\pi t}c_2\sigma}\right) C_4 - FresnelC\left(\frac{Sc_2 + 2\,c_1}{\sqrt{\pi t}c_2\sigma}\right) C_3. \quad (322)$$

In Figure 6.4 we plot the solutions (321) and (322).

Case 4: A reduction with $c_1X_2 + c_2Y_1$

We consider the linear combination $c_1X_2 + c_2Y1$ gives solutions:

$$u(S,t) = C_2 + C_3\, FresnelS\left(\frac{S\sqrt{c_2}}{\sigma\sqrt{\pi\ (tc_2 + c_1)}}\right) + C_4\, FresnelC\left(\frac{S\sqrt{c_2}}{\sigma\sqrt{\pi\ (tc_2 + c_1)}}\right), \quad (323)$$

$$v(S,t) = C_1 - FresnelC\left(\frac{S\sqrt{c_2}}{\sigma\sqrt{\pi\ (tc_2 + c_1)}}\right) C_3 + FresnelS\left(\frac{S\sqrt{c_2}}{\sigma\sqrt{\pi\ (tc_2 + c_1)}}\right) C4. \quad (324)$$

In Figure 6.5 we plot the solutions (323) and (324).

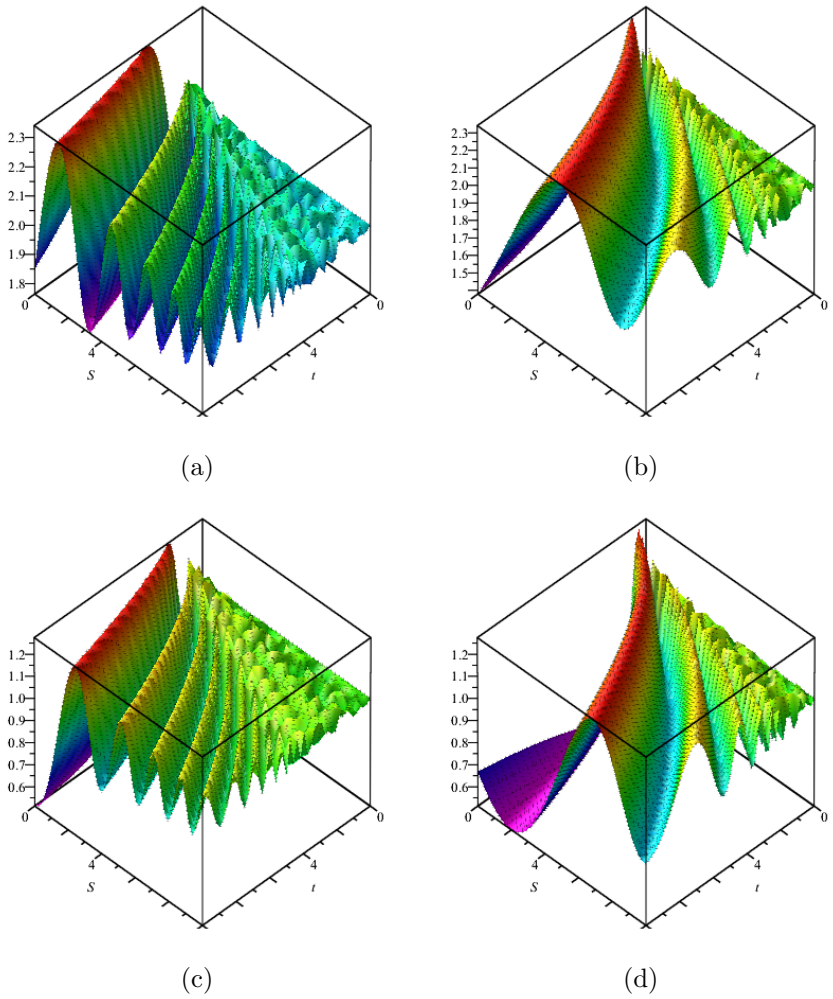


Figure 6.4: Graphical illustration of the analytical solutions are depicted, we select the parameter values $c_1 = c_2 = C_1 = C_2 = C_3 = C_4 = 1$ and the ranges $0 \leq t \leq 10$ and $0 \leq S \leq 10$: (a) solution of (321) where $\sigma = 0.5$; (b) solution of (321) where $\sigma = 1.0$; (c) solution of (322) where $\sigma = 0.5$; and, (d) solution of (322) where $\sigma = 1.0$.

Case 5: Reduction via $c_1 Y_3 + c_2 X_1$

Application of the linear combinations $c_1 Y_3 + c_2 X_1$ provides the following analytical solutions below:

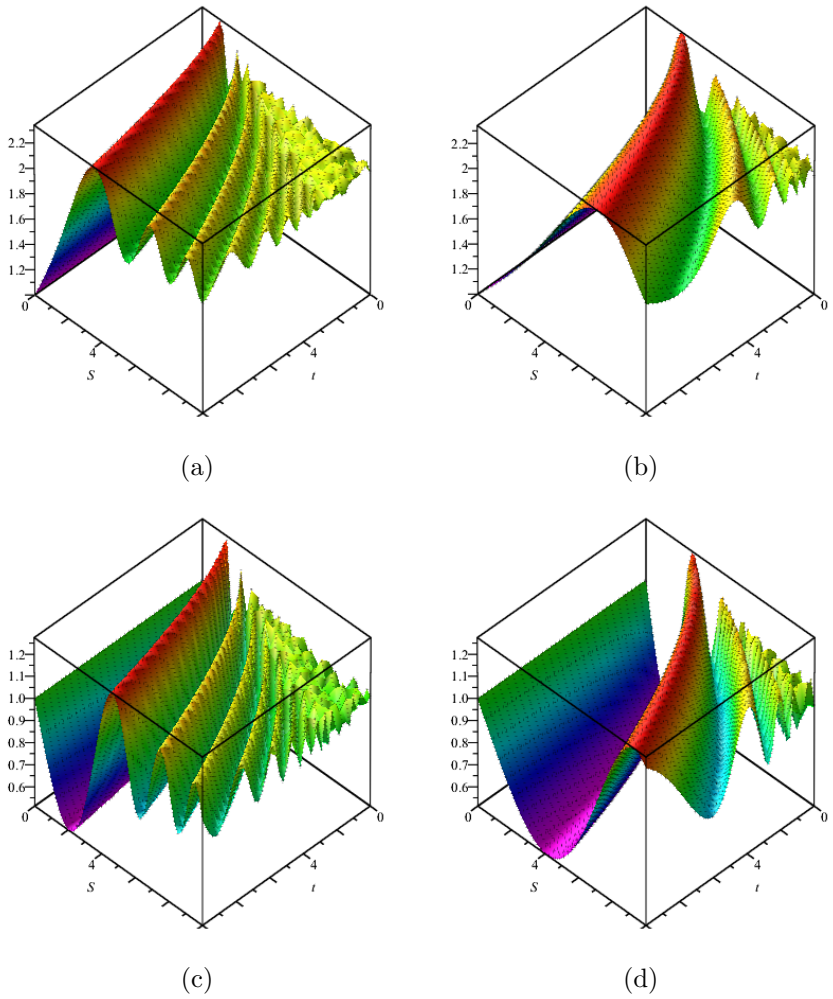


Figure 6.5: Graphical illustration of the analytical solutions are depicted, we select the parameter values $c_1 = c_2 = C_1 = C_2 = C_3 = C_4 = 1$ and the ranges $0 \leq t \leq 10$ and $0 \leq S \leq 10$: (a) solution of (323) where $\sigma = 0.5$; (b) solution of (323) where $\sigma = 1.0$; (c) solution of (324) where $\sigma = 0.5$; and, (d) solution of (324) where $\sigma = 1.0$.

$$u\left(S,t\right) = \mathrm{e}^{\frac{c_1 S}{c_2}} \left(C_1 \, \sin\left(\frac{1}{2} \frac{{c_1}^2 \sigma^2 t}{{c_2}^2} \right) + C_2 \, \cos\left(\frac{1}{2} \frac{{c_1}^2 \sigma^2 t}{{c_2}^2} \right) \right), \tag{325}$$

$$v\left(S,t\right)=\mathrm{e}^{\frac{c_1 S}{c_2}}\left(-\cos\left(\frac{1}{2}\,\frac{{c_1}^2\sigma^2 t}{{c_2}^2}\right)C_1+\sin\left(\frac{1}{2}\,\frac{{c_1}^2\sigma^2 t}{{c_2}^2}\right)C_2\right). \tag{326}$$

In Figure 6.6 we plot the solutions (325) and (326).

Case 6: $c_1 Y_3 + c_2 X_2$

Under this case we use the linear combinations of symmetries $c_1 Y_3 + c_2 X_2$ which yield the exact solutions below:

$$u\left(S,t\right)=\mathrm{e}^{\frac{c_1 t}{c_2}}\left(C_1\,\mathrm{e}^{-i\gamma}+C_2\,\mathrm{e}^{-\gamma}+C_3\,\mathrm{e}^{i\gamma}+C_4\,\mathrm{e}^{\gamma}\right), \tag{327}$$

$$v\left(S,t\right)=\mathrm{e}^{\frac{c_1 t}{c_2}}\left(\frac{\sqrt{-{c_1}^2{c_2}^2}}{c_2 c_1}\left(C_4\,\mathrm{e}^{\gamma}-C_1\,\mathrm{e}^{-i\gamma}+C_2\,\mathrm{e}^{-\gamma}-C_3\,\mathrm{e}^{i\gamma}\right)\right), \tag{328}$$

where $\gamma=\frac{\sqrt{2}\sqrt[4]{-{c_1}^2{c_2}^2}S}{c_2\sigma}$.

In Figure 6.7 we plot the solutions (327) and (328).

6.4 Discussion

Econophysics, as the name suggests is an interdisciplinary field, that deviates from mainstream economics taking concepts and models from the domain of physics to

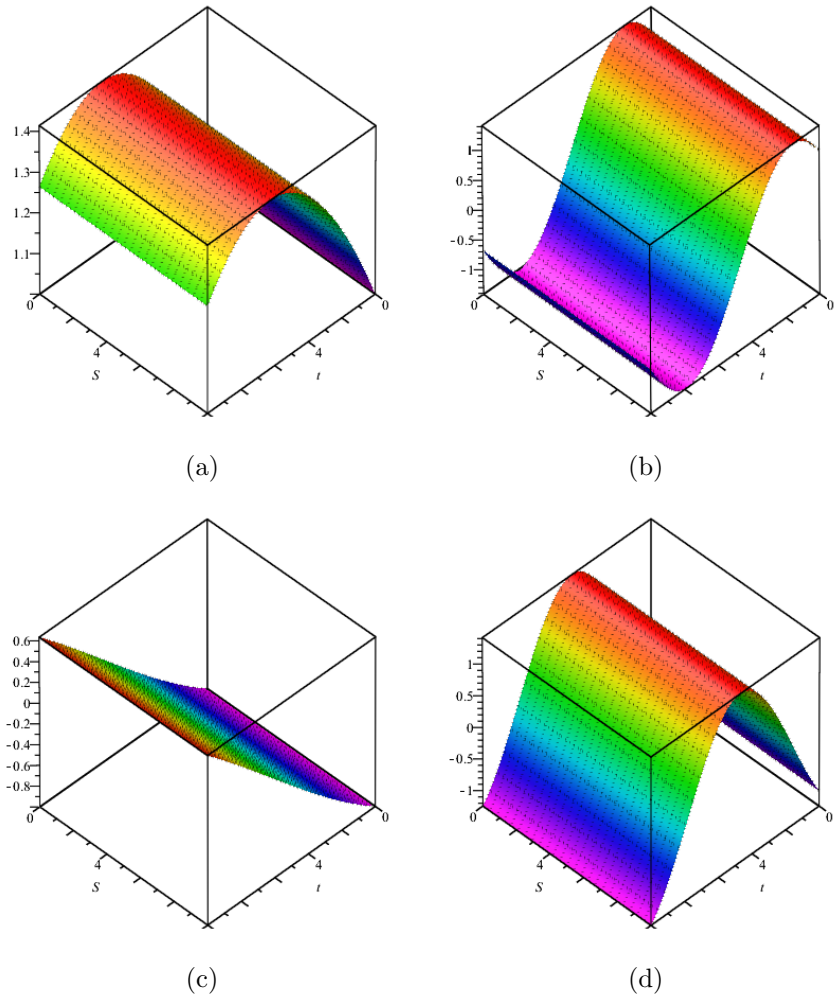


Figure 6.6: Graphical illustration of the analytical solutions are depicted, we select the parameter values $c_1 = c_2 = C_1 = C_2 = 1$ the ranges $0 \leq t \leq 10$ and $0 \leq S \leq 10$: (a) solution of (325) where $\sigma = 0.5$; (b) solution of (325) where $\sigma = 1.0$; (c) solution of (326) where $\sigma = 0.5$; and, (d) solution of (326) where $\sigma = 1.0$.

explain economic phenomena. Subsets of the field include quantum finance, thermoeconomics etc. In this chapter we have successfully performed a full symmetry analysis for the IOPM considering both $\beta \neq 0$ and $\beta = 0$, the Landau coefficient

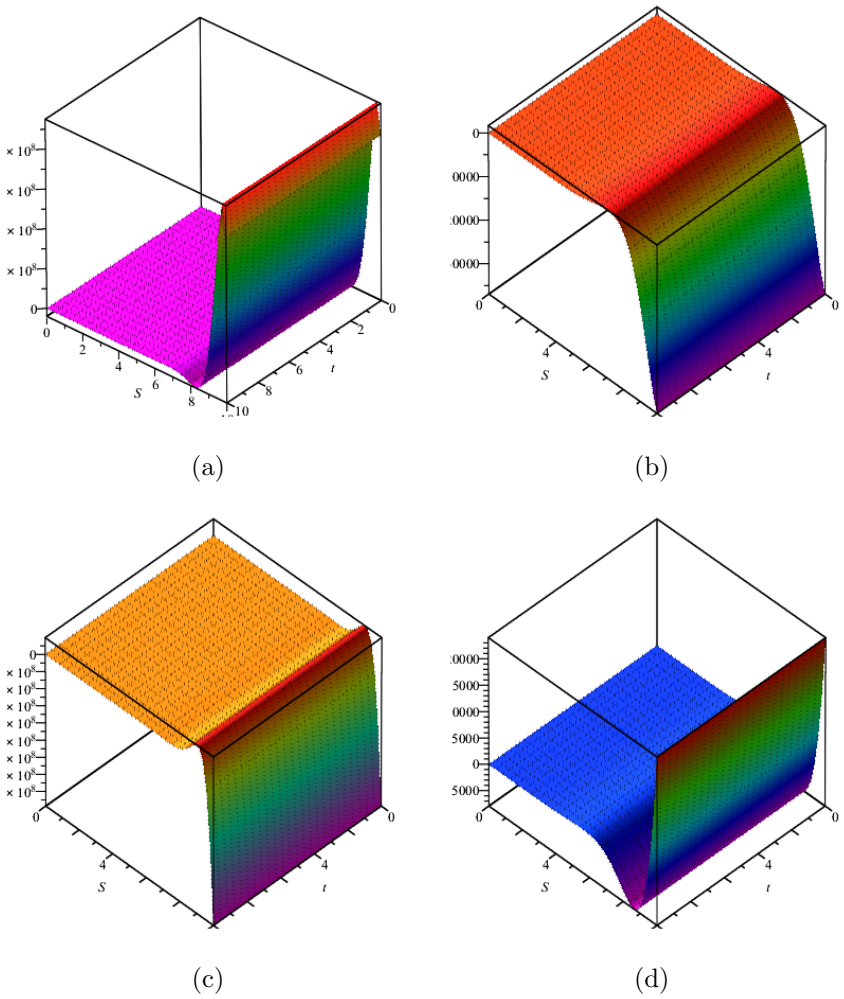


Figure 6.7: Graphical illustration of the analytical solutions are depicted, we select the parameter values $c_1 = c_2 = C_1 = C_2 = C_3 = C_4 = 1$ the ranges $0 \leq t \leq 10$ and $0 \leq S \leq 10$: (a) solution of (327) where $\sigma = 0.5$; (b) solution of (327) where $\sigma = 1.0$; (c) solution of (328) where $\sigma = 0.5$; and, (d) solution of (328) where $\sigma = 1.0$.

representing the adaptive market potential. We discovered that the IOPM admits several invariant solutions as well as conservation laws for the zero and nonzero adaptive market potential. We provided several plots for our solutions given differ-

ent levels of volatility. These solutions are novel and haven't appeared elsewhere in literature. In one case of a zero adaptive market potential, the IOPM wave function solution exhibits sinusoidal characteristics, with wave frequency positively correlated to the Landau coefficient.

Chapter 7

Invariant solutions of a European option pricing equation under a continuous-time Capital Asset Pricing Model

This work investigates a nonlinear European option pricing equation under a continuous-time Capital Asset Pricing Model (CAPM). In particular, we determine its point symmetries and thereby find invariant analytical solutions using the Lie symmetry approach. In our research, we vary the beta coefficient, a measure of an asset's systematic risk. We examine the model for both a non-zero and a zero market beta in our study. We show that in the first instance, this model is invariant under a three-dimensional Lie algebra, and in the second situation, it is invariant under a bigger six-dimensional Lie algebra. For both the non-zero and zero market beta models, we get reductions and exact solutions. With various parameter settings,

graphical solutions are investigated.

7.1 Introduction

H. Markowitz established the groundwork for asset pricing models being the first to devise a specific measure of portfolio risk and to calculate a portfolio's expected return and risk [99]. The CAPM is based on Markowitz's approach of portfolio selection and developed by W.F. Sharpe [100] and J. Lintner [101], becoming a significant instrument in pricing risky securities, evaluation of portfolio performance, portfolio diversification, valuation of investments, and portfolio strategy selection. Particularly, in determining whether or not to add assets to a well-diversified portfolio, the model can be used to determine the required rate of return of the asset. The Black-Scholes equation can be extended to give the price of an option under a continuous-time CAPM. In this work we analyse an option pricing PDE based on the continuous-time CAPM where the solution to this model gives the price of a European option. In the continuous-time CAPM the evolution of the market price, M_t, and asset price S_t are given as:

$$\frac{dM_t}{M_t} = \mu d_t + \sigma_m dW_t^{(1)}, \tag{329}$$

$$\frac{dS_t}{S_t} = \beta \frac{dM_t}{M} t + \sigma dW_t^{(2)}, \tag{330}$$

where μ is the expected rate of return, σ_m and σ are positive constant volatilities, β is the slope, and $W_t^{(1)}$ and $W_t^{(2)}$ are uncorrelated Brownian motions. From (329), we see that market price is governed by a geometric Brownian motion and in (330) the asset's expected return is proportional to the market's return. Under a continuous-time CAPM, the PDE derived by [102] depicts the evolution of the price

of a European option, $V := V(S, M, \tau)$, depending on underlying asset price , S, market price, M, and time-to-maturity τ:

$$\begin{aligned}\frac{\partial V}{\partial \tau} =& \frac{1}{2}(\sigma_m^2\beta^2 + \sigma^2)S^2\frac{\partial^2 V}{\partial S^2} + \beta\sigma_m^2 MS\frac{\partial^2 V}{\partial S \partial M} + \frac{1}{2}\sigma_m^2 M^2\frac{\partial^2 V}{\partial M^2} \\ &+ r(S\frac{\partial V}{\partial S} + M\frac{\partial V}{\partial M} - V),\ 0 < \tau \leq T,\ S, M \in \Re_+, \end{aligned} \tag{331}$$

where r is the risk-free interest.

The β of a stock is vital to the development and performance of stock portfolios showing the extent of the asset's risk in comparison to the market's systematic risk. β statistics for a single stock provides an investor an estimate of how much risk the stock will add to a supposedly well-diversified portfolio. A stock with β of 1.0 added to a portfolio does not raise risk, but it also does not enhance the chance of the portfolio providing an excess return. A β greater than 1.0 means that the security's price is more volatile than the market. Adding a security with said β will increase its risk but expected return may potentially be increased. New technology companies usually have a beta greater than one. With the same logic, a β less than 1.0 means the stock is less volatile than the market, therefore, adding the stock makes the portfolio less risky. Negative β's can occur, indicating that the investment moves in the opposite direction of the market measure.

The purpose of this work is to use method of Lie symmetries study the continuous-time CAPM options pricing model, equation (331), for β being a constant and β being equal to zero. We use this approach to identify invariant solutions to this PDE systematically, using first-order invariants that we construct and use to reduce the PDE. Section 7.2 incorporates a full symmetry analysis of (331) with a constant beta. The derivation and application of Lie point symmetries to (331), commutator relations, PDE reductions and novel invariant solutions of the model are presented here. Section 7.3 is dedicated to the analysis of this model with no systematic risk

i.e. $\beta = 0$, in the same manner as was done in section 7.2. Where appropriate, we provide plots of these solutions. Finally, in the last section we conclude and discuss our study.

7.2 Analysis of the European option pricing model with a constant market beta

The CAPM equation (331), is invariant under a three-dimensional Lie algebra consisting of the symmetry vector fields, as determined by Lie theory:

$$X_1 = M\frac{\partial}{\partial M}, \tag{332}$$

$$X_2 = S\frac{\partial}{\partial S}, \tag{333}$$

$$X_3 = V\frac{\partial}{\partial V}. \tag{334}$$

In addition to the above three symmetries, there is a fourth, the infinite number of solution symmetries (trivial symmetries). This will not be used for reduction, and so we omit it here and the sections that follow. For (332)-(334), calculations of the Lie brackets show that all commutators vanish.

7.2.1 Reductions and invariant solutions for constant beta

We include the solutions for the CAPM using the symmetries detailed above. To solve for the European option price function $V(S, M, \tau)$, point symmetries corre-

sponding to it are required. Such symmetries can be used to systematically reduce the order of the model. Reduced equations will be reduced once more if they inherit any of the remaining symmetries of the Lie algebra. For our case, a three-variable PDE makes up the European option price function and, thus, often two symmetries are required to reduce it to an ODE. Note that static solutions where V is not dependent on one of the independent variables are omitted.

Case 1: Reduction with X_1

We use the symmetry X_1 to reduce equation (331) to

$$S^2\left(\frac{\partial^2}{\partial S^2}K\left(\tau,S\right)\right)\beta^2a^2+S^2\left(\frac{\partial^2}{\partial S^2}K\left(\tau,S\right)\right)\sigma^2+2\,rS\frac{\partial}{\partial S}K\left(\tau,S\right)$$
$$-2\,rK\left(\tau,S\right)-2\,\frac{\partial}{\partial\tau}K\left(\tau,S\right), \tag{335}$$

where $V(S,M,\tau)=K\left(\tau,S\right)$. Reduced PDE (335), admits X_2 which we use to reduce again and get the solution

$$V(S,M,\tau)=C_1\mathrm{e}^{-r\tau}, \tag{336}$$

where C_1 is an arbitrary constant. This is a solution that we will see in the other cases that follow.

Secondly, equation (335), admits the vector field $\frac{\partial}{\partial\tau}$, which is then used to get the solution

$$V(S,M,\tau)=C_1S+C_2S^{-2\,\frac{r}{\beta^2\sigma_m{}^2+\sigma^2}}, \tag{337}$$

where the C_i are arbitrary constants. A more complicated symmetry vector field yielded by (335) is

$(-{\sigma_m}^2\beta^2\tau - \sigma^2\tau + 2\,r\tau + 2\,\ln(S))\,K\frac{\partial}{\partial K} + (-2\,\beta^2\tau{\sigma_m}^2 - 2\,\sigma^2\tau)\,X_2,$

giving the price of the European option as

$$V(S,M,\tau) = \frac{C_1}{\sqrt{\tau}}\mathrm{e}^{-1/8\,\frac{\tau\left(\beta^2{\sigma_m}^2+\sigma^2+2\,r\right)^2}{\beta^2{\sigma_m}^2+\sigma^2}}. \tag{338}$$

The application of a linear combination of symmetries yielded by (335), $X_2 + \frac{\partial}{\partial\tau}$, yields

$$\begin{aligned} V(S,M,\tau) =& C_1\,S^{\frac{{\sigma_m}^2\beta^2+\sigma^2-2\,r+\sqrt{\left({\sigma_m}^2\beta^2+\sigma^2+2\,r\right)^2}}{2\,{\sigma_m}^2\beta^2+2\,\sigma^2}} + \\ & C_2\,S^{\frac{{\sigma_m}^2\beta^2+\sigma^2-2\,r-\sqrt{\left({\sigma_m}^2\beta^2+\sigma^2+2\,r\right)^2}}{2\,{\sigma_m}^2\beta^2+2\,\sigma^2}}. \end{aligned} \tag{339}$$

In the same manner $X_2 + K\frac{\partial}{\partial K}$ gives (336). In Figure 7.1 we plot solutions (336)-(339) by varying either the risk-free interest rates or the β's.

Case 2: Reduction with X_2

Continuing our analysis in the same manner as in the previous case, X_2 reduces our model of study to get the reduced PDE

$$\begin{aligned} & {\sigma_m}^2M^2\frac{\partial^2}{\partial M^2}K(\tau,M) + 2\,rM\frac{\partial}{\partial M}K(\tau,M) - 2\,rK(\tau,M) - \\ & 2\,\frac{\partial}{\partial\tau}K(\tau,M), \end{aligned} \tag{340}$$

where $V(S,M,\tau) = K(\tau,M)$.

Application of $(-{\sigma_m}^2\tau + 2\,r\tau + 2\,\ln(M))\,K\frac{\partial}{\partial K} - 2\,X_1\tau{\sigma_m}^2$ admitted by (340) yields option price function

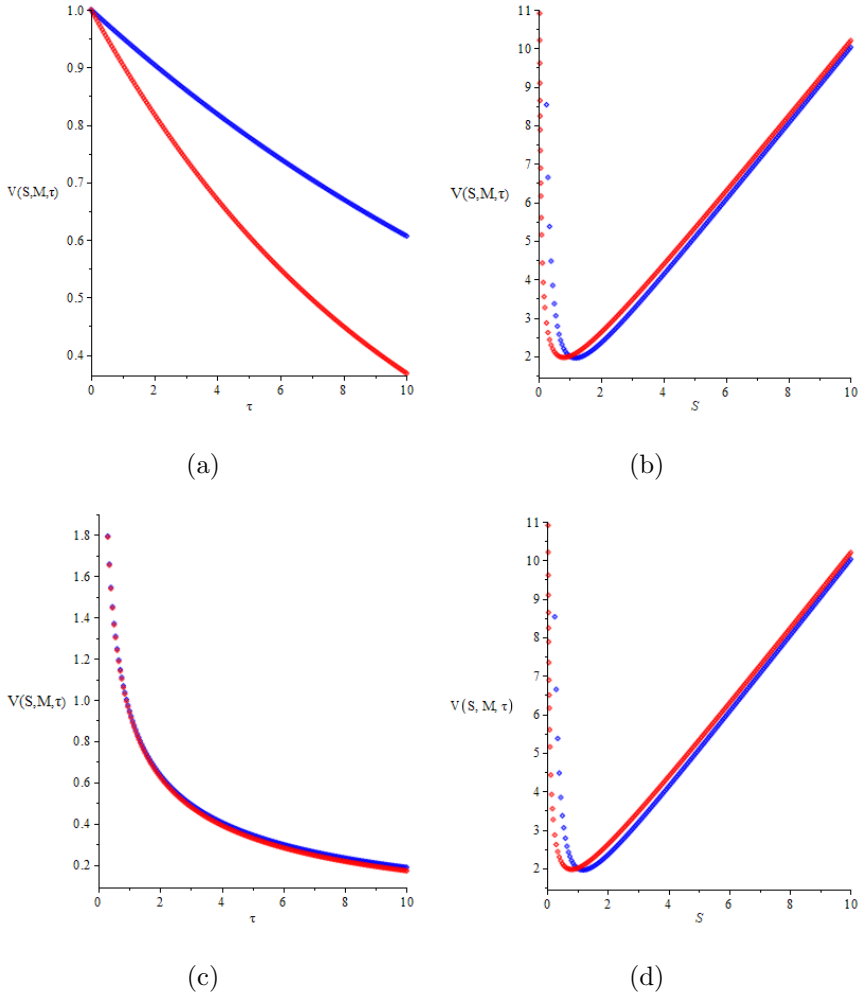


Figure 7.1: Graphical illustration of the analytical solutions are depicted, we select the parameter values $C_1 = C_2 = 1$, and the ranges $0 \leq \tau \leq 10$ and $0 \leq S \leq 10$. We choose the parameter values of $\sigma_m = 0.25$ and $\sigma = 0.05$ for the solutions graphed and a value of $r = 0.05$ except for the first plot: (a) solution of (336) where $r = 0.05$ (blue plot) and $r = 0.10$ (red plot); (b) solution of (337) where $\beta = 1.0$ (blue) and $\beta = 1.5$ (red); (c) solution of (338) where $\beta = 1.0$ (blue) and $\beta = 2.0$ (red); and, (d) solution of (339) where $\beta = 1.0$ (blue) and $\beta = 1.5$ (red).

$$V(S,M,\tau)=\frac{C_1}{\sqrt{\tau}}\mathrm{e}^{-1/8\,\frac{\left({\sigma_m}^2+2\,r\right)^2\tau}{{\sigma_m}^2}}. \tag{341}$$

Similarly using $\left({\sigma_m}^4+4\,r{\sigma_m}^2+4\,r^2\right)K\frac{\partial}{\partial K}-8\,\frac{\partial}{\partial\tau}{\sigma_m}^2$ we get

$$V(S,M,\tau)=C_1M^{1/2\,\frac{{\sigma_m}^2-2\,r}{{\sigma_m}^2}}+C_2M^{1/2\,\frac{{\sigma_m}^2-2\,r}{{\sigma_m}^2}}\ln\left(M\right). \tag{342}$$

Yet another symmetry of (340) we apply is $(\left({\sigma_m}^4+4\,r{\sigma_m}^2+4\,r^2\right)K\frac{\partial}{\partial K}-8\,\frac{\partial}{\partial\tau}{\sigma_m}^2)+K\frac{\partial}{\partial K}$ gives $V(S,M,\tau)$ of the form

$$V(S,M,\tau)=C_1\,M^{1/2\,\frac{{\sigma_m}^2-2\,r+i}{{\sigma_m}^2}}+C_2\,M^{-1/2\,\frac{-{\sigma_m}^2+2\,r+i}{{\sigma_m}^2}}. \tag{343}$$

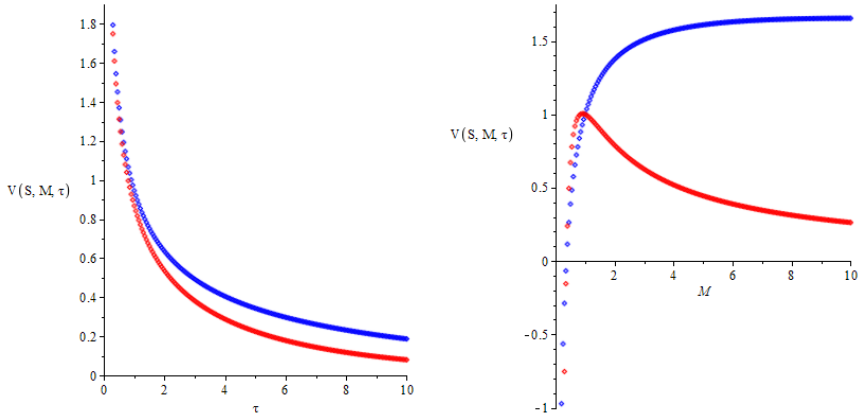
Solution (341) is repeated when symmetry vector field of (340),

$(\left(-{\sigma_m}^2\tau+2\,r\tau+2\,\ln\left(M\right)\right)K\frac{\partial}{\partial K}-2\tau X_1{\sigma_m}^2)+K\frac{\partial}{\partial K}$, is utilized. In Figure 7.2 we plot the unique solutions found in this case. For the next few cases in this section we employ linear combination of the symmetries (332)-(334).

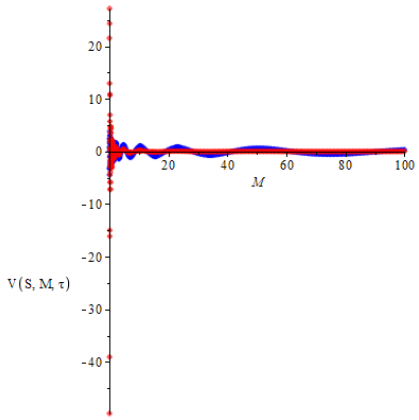
Case 3: Reduction for $c_1X_1+c_2X_2$

Using the reduced PDE (which is too lengthy to appear here) from the application of this linear combination of symmetry vector fields, then using $\frac{\partial}{\partial\tau}$ admitted by this reduced PDE and setting $c_1=0$ and $c_2=1$, we get the solution

$$\begin{aligned}V(S,M,\tau)=&C_1M^{1/2\,\frac{{\sigma_m}^2-2\,r+\sqrt{\left({\sigma_m}^2+2\,r\right)^2}}{{\sigma_m}^2}}+\\&C_2M^{-1/2\,\frac{-{\sigma_m}^2+2\,r+\sqrt{\left({\sigma_m}^2+2\,r\right)^2}}{{\sigma_m}^2}}.\end{aligned} \tag{344}$$



(a) (b)

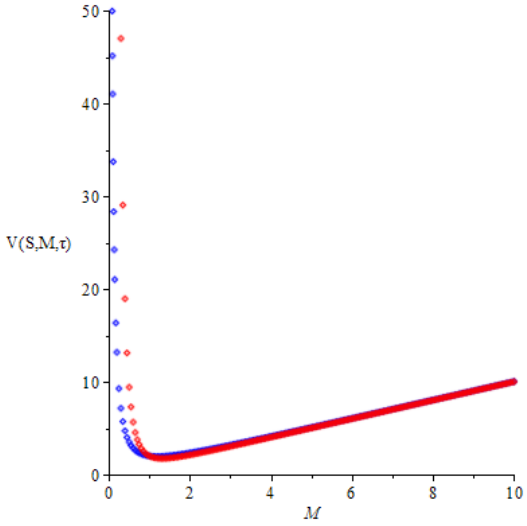


(c)

Figure 7.2: Plots of the solutions in Case 2 are depicted. We select the parameter values $C_1 = C_2 = 1$ and $\sigma_m = 0.25$, with the ranges $0 \leq t \leq 10$ and $0 \leq M \leq 100$, while changing r: (a) solution of (341) where $r = 0.05$ (blue) and $r = 0.10$ (red); (b) solution of (342) where $r = 0.05$ (blue) and $r = 0.10$ (red); (c) solution of (343) where $r = 0.05$ (blue) and $r = 0.10$ (red).

Option price function (344) can also be obtained by using $\frac{\partial}{\partial \tau} + K\frac{\partial}{\partial K}$, where $K\frac{\partial}{\partial K}$ is also a symmetry of this reduced equation. In Figure 7.3 we plot (344) to examine

Figure 7.3: Evolution of the analytical solution (344) in the range $0 \leq M \leq 10$; we select the parameter values $C_1 = C_2 = 1$ and $\sigma_m = 0.25$ in the solution while varying r, with $r = 0.05$ (blue) and $r = 0.10$ (red).



how it evolves.

Case 4: Reduction for $c_1X_1 + c_2X_3$

This reduces our model to

$$\left(\frac{\partial^2}{\partial S^2}K\left(\tau,S\right)\right){\sigma_m}^2\beta^2S^2{c_1}^2+\left(\frac{\partial^2}{\partial S^2}K\left(\tau,S\right)\right)\sigma^2S^2{c_1}^2+$$
$$2\left(\frac{\partial}{\partial S}K\left(\tau,S\right)\right){\sigma_m}^2\beta\,Sc_1c_2+2\left(\frac{\partial}{\partial S}K\left(\tau,S\right)\right)rS{c_1}^2-$$
$$K\left(\tau,S\right){\sigma_m}^2c_1c_2+K\left(\tau,S\right){\sigma_m}^2{c_2}^2-2\,K\left(\tau,S\right)r{c_1}^2+$$
$$2\,K\left(\tau,S\right)rc_1c_2-2\left(\frac{\partial}{\partial\tau}K\left(\tau,S\right)\right){c_1}^2, \tag{345}$$

where $V(S,M,\tau)=K\left(\tau,S\right)$. Then utilising X_2 that (345) exhibits gives

$$V(S,M,\tau)=C_1\mathrm{e}^{-1/2\,\frac{(c_1-c_2)\left(c_2{\sigma_m}^2+2\,rc_1\right)\tau}{{c_1}^2}}. \tag{346}$$

As this solution equates to (336) with a particular choice of the c_i we will not plot it, and the same will be true in the cases that follow.

Using a symmetry of (345), $\frac{\partial}{\partial\tau}$, $c_1=1$ and $c_2=0$ we get the invariant solution

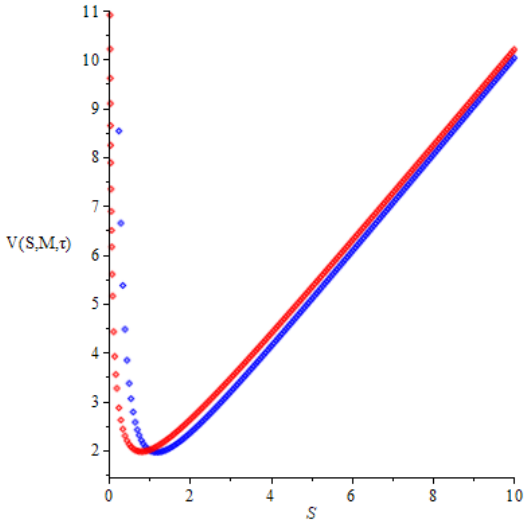
$$\begin{aligned}V(S,M,\tau)=&C_1S^{\frac{{\sigma_m}^2\beta^2+\sigma^2-2\,r+\sqrt{\left({\sigma_m}^2\beta^2+\sigma^2+2\,r\right)^2}}{2\,{\sigma_m}^2\beta^2+2\,\sigma^2}}+\\&C_2S^{\frac{{\sigma_m}^2\beta^2+\sigma^2-2\,r-\sqrt{\left({\sigma_m}^2\beta^2+\sigma^2+2\,r\right)^2}}{2\,{\sigma_m}^2\beta^2+2\,\sigma^2}}.\end{aligned} \tag{347}$$

Likewise, a linear combination of $\frac{\partial}{\partial\tau}+K\frac{\partial}{\partial K}$ gives (347). In Figure 7.4 we graph (347) by varying it's β from 1.0 to 1.5.

Case 5: Reduction for $c_1X_2+c_2X_3$

This linear combination reduces our European options pricing model (331) to

Figure 7.4: Evolution of the exact solution (347) in the range $0 \leq S \leq 10$; we select the parameter values $C_1 = C_2 = 1$ and $\sigma_m = 0.25; \sigma = 0.05; r = 0.05$ while changing β, with $\beta = 1.0$ (blue) and $\beta = 1.5$ (red).



$$
\begin{aligned}
&- K(\tau, M)\, a^2 \beta^2 c_1 c_2 + K(\tau, M)\, a^2 \beta^2 {c_2}^2 + 2 \left(\frac{\partial}{\partial M} K(\tau, M) \right) a^2 \beta\, M c_1 c_2 + \\
&\left(\frac{\partial^2}{\partial M^2} K(\tau, M) \right) a^2 M^2 {c_1}^2 - K(\tau, M)\, \sigma^2 c_1 c_2 + K(\tau, M)\, \sigma^2 {c_2}^2 + \\
&2 \left(\frac{\partial}{\partial M} K(\tau, M) \right) r M {c_1}^2 - 2\, K(\tau, M)\, r {c_1}^2 + 2\, K(\tau, M)\, r c_1 c_2 - \\
&2 \left(\frac{\partial}{\partial \tau} K(\tau, M) \right) {c_1}^2, \qquad (348)
\end{aligned}
$$

with $V(S, M, \tau) = K(\tau, M)$. This PDE has the symmetry vectors X_1, $\frac{\partial}{\partial \tau}$ and $K \frac{\partial}{\partial K}$. Firstly, we utilise X_1 to obtain the following invariant solution

$$
V(S, M, \tau) = C_1 \mathrm{e}^{-1/2 \frac{(c_1 - c_2)\left(\beta^2 c_2 {\sigma_m}^2 + \sigma^2 c_2 + 2 r c_1\right) \tau}{{c_1}^2}}. \qquad (349)
$$

Additionally, using $\frac{\partial}{\partial \tau}$ or the linear combination of symmetries of (348), $\frac{\partial}{\partial \tau} + K\frac{\partial}{\partial K}$, with $c_1 = 1$ and $c_2 = 0$ gives (344).

Case 6: Reduction for $c_1X_1 + c_2X_2 + c_3X_3$

Here the reduced PDE is invariant under the Lie symmetry $\alpha\frac{\partial}{\partial \alpha}$, where $\alpha = MS^{-\frac{c_1}{c_2}}$. This gives

$$V(S, M, \tau) = C_1 \mathrm{e}^{-1/2\,\frac{(c_2-c_3)\left(\beta^2 c_3 {\sigma_m}^2+\sigma^2 c_3+2\,r c_2\right)\tau}{{c_2}^2}}. \tag{350}$$

Then using $\frac{\partial}{\partial \tau}$ or $\frac{\partial}{\partial \tau} + K\frac{\partial}{\partial K}$ and letting $c_1 = 0$, $c_2 = 1$ and $c_3 = 0$ gives (344).

7.3 Analysis of the European option pricing model with zero systematic risk

In this section we let $\beta = 0$ in (331). This gives a zero-beta portfolio that has been designed to have no systematic risk. Risk-less assets have betas of zero. The projected return on a zero-beta portfolio would be the same as the risk-free rate. There is zero correlation with market movements. Due to the lack of market exposure, zero-beta portfolios are unlikely to attract investor attention in bull markets, as they would under-perform diversified market portfolios. But it may attract attention in bear markets. An example of such an asset is cash whose value remains constant given no inflation. Our model with this condition is now

$$-\frac{\partial V}{\partial t}+1/2\,\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+1/2\,{\sigma_m}^2 M^2\frac{\partial^2 V}{\partial M^2}+ \tag{351}$$
$$r\left(S\frac{\partial V}{\partial S}V+M\frac{\partial V}{\partial M}-V\right)=0$$

The model, (351), is invariant under a larger Lie algebra i.e. the six-dimensional Lie algebra consisting of the Lie point symmetries:

$$X_1, X_2, X_3, \tag{352}$$
$$X_4=\frac{\partial}{\partial \tau}, \tag{353}$$
$$Y_1=2\,X_1\tau{\sigma_m}^2+X_3\left({\sigma_m}^2\tau-2\,r\tau-2\,\ln\left(M\right)\right), \tag{354}$$
$$Y_2=-2\,X_2\tau\sigma^2+X_3\left(-\sigma^2\tau+2\,r\tau+2\,\ln\left(S\right)\right). \tag{355}$$

In Table 2, we present the Lie brackets of the Lie symmetries of (351) derived above.

$[,]$	X_1	X_2	X_3	X_4
X_4	0	0	0	0
Y_1	$2X_3$	0	0	$-2\sigma_m^2 X_1-(\sigma_m^2-2r)X_3$
Y_2	0	$-2X_3$	0	$2\sigma^2 X_2-(-\sigma^2+2r)X_3$

Table 7.1: Commutator table for $\beta=0$.

7.3.1 Reductions and invariant solutions for $\beta=0$

We derive the following reductions and invariant solutions using the same mathematical procedure as in the previous section.

Case 1: Reduction with X_1

We use the symmetry X_1 to reduce model (351) to

$$\sigma^2 S^2 \frac{\partial^2}{\partial S^2} K(\tau, S) + 2\, r S \frac{\partial}{\partial S} K(\tau, S) - 2\, r K(\tau, S) - 2 \frac{\partial}{\partial \tau} K(\tau, S), \tag{356}$$

where $V(S, M, \tau) = K(\tau, S)$ as a result of the reduction. From (356), it's symmetry X_2 is used to obtain yet another solution that is identical to (336) just as case was in the previous section. The third symmetry of the reduced equation (356) is $(\sigma^4 + 4\, r\sigma^2 + 4\, r^2) K \frac{\partial}{\partial K} - 8 \frac{\partial}{\partial \tau} \sigma^2$ from which the invariant solution found is

$$V(S, M, \tau) = C_1 S^{-1/2 \frac{-\sigma^2 + 2\, r}{\sigma^2}} + C_2 S^{-1/2 \frac{-\sigma^2 + 2\, r}{\sigma^2}} \ln(S). \tag{357}$$

Now (356) has a fourth symmetry vector field $(-\sigma^2 \tau + 2\, r\tau + 2 \ln(S)) K \frac{\partial}{\partial K} - 2 X_2 \tau \sigma^2$ with the following solution

$$V(S, M, \tau) = \frac{C_1}{\sqrt{\tau}} \mathrm{e}^{-1/8 \frac{\left(\sigma^2 + 2\, r\right)^2 \tau}{\sigma^2}}. \tag{358}$$

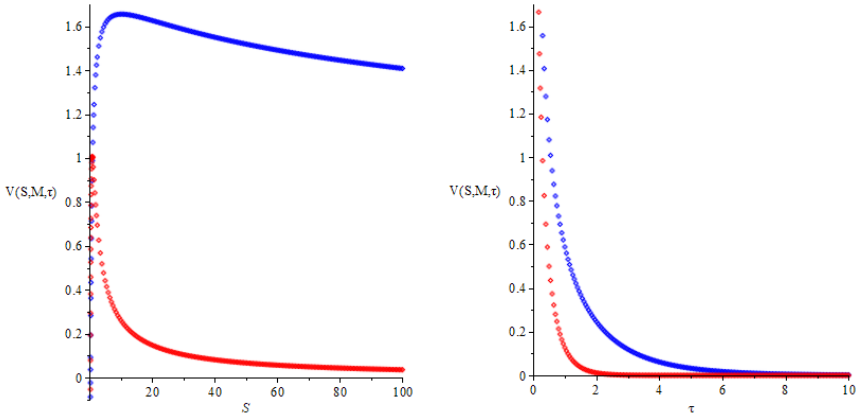
In Figure 7.5 we plot these two solutions.

Case 2: Reduction with X_4

The PDE is reduced for this case to

$$\begin{aligned} &\sigma^2 S^2 \frac{\partial^2}{\partial S^2} K(S, M) + {\sigma_m}^2 M^2 \frac{\partial^2}{\partial M^2} K(S, M) + \\ &2\, r S \frac{\partial}{\partial S} K(S, M) + 2\, r M \frac{\partial}{\partial M} K(S, M) - 2\, r K(S, M), \end{aligned} \tag{359}$$

with $V(S, M, \tau) = K(S, M)$. It admits X_1 and so the solution yielded is



(a) (b)

Figure 7.5: Graphical illustration of the analytical solutions are depicted, we select the parameter values $C_1 = C_2 = 1$ with the ranges $0 \leq \tau \leq 10$ and $0 \leq S \leq 100$: (a) solution of (357) with $\sigma = 0.25$ where the blue graph is for $r = 0.05$ and the red graph is for $r = 0.10$; (b) solution of (358) with $\sigma = 0.05$ where the blue graph is for $r = 0.05$ and the red graph is for $r = 0.10$.

$$V(S, M, \tau) = C_1 S + C_2 S^{-2\,\frac{r}{\sigma^2}}. \tag{360}$$

Secondly, it has symmetry X_2. The attached invariant form of $V(S, M, \tau)$ is

$$V(S, M, \tau) = C_1 M + C_2 M^{-2\,\frac{r}{\sigma_m{}^2}}. \tag{361}$$

Furthermore, we note that the reduced equation exhibits another Lie point symmetry of the form $\frac{\partial}{\partial K}$. We utilise this symmetry in linear combinations with the other symmetries to get a few more solutions. The first of this kind is $X_2 + K\frac{\partial}{\partial K}$ which gives (344). Whereas $X_1 + K\frac{\partial}{\partial K}$ gives

$$V(S, M, \tau) = C_1\, S^{1/2\,\frac{\sigma^2 - 2\,r + \sqrt{(\sigma^2+2\,r)^2}}{\sigma^2}} + C_2\, S^{-1/2\,\frac{-\sigma^2+2\,r+\sqrt{(\sigma^2+2\,r)^2}}{\sigma^2}}. \tag{362}$$

In Figure 7.6 shows the evolution of these solutions for selected parameter values.

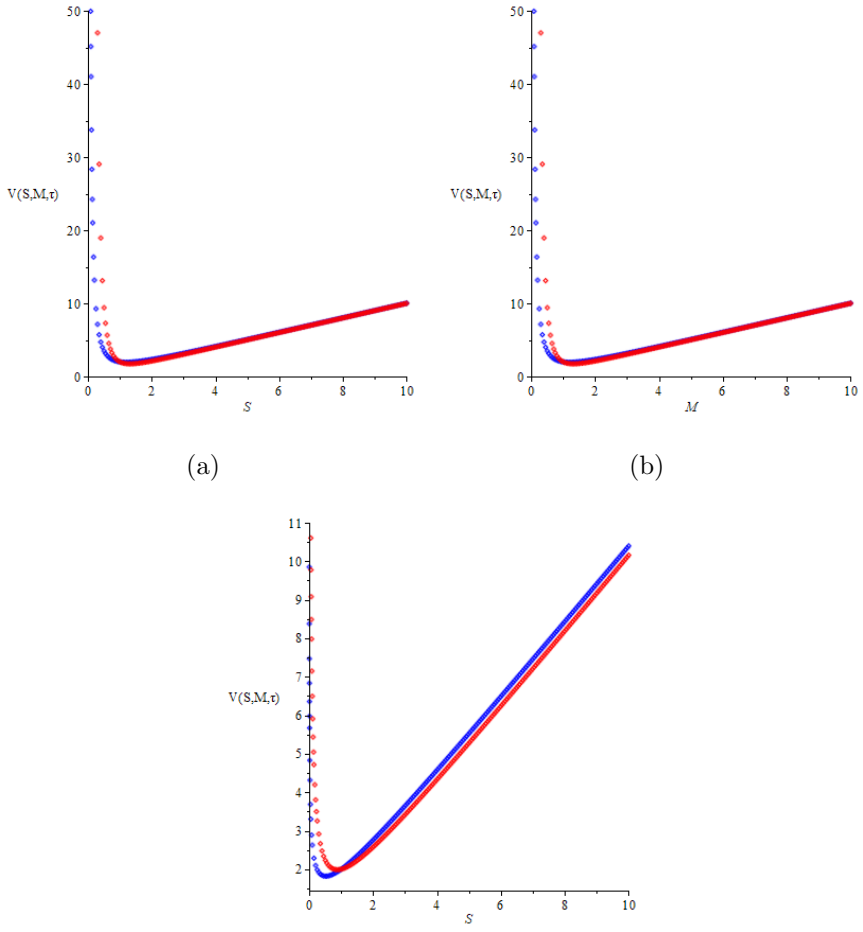


Figure 7.6: Plots of the invariant solutions for Case 3. We choose the parameter values $C_1 = C_2 = 1$ for the ranges $0 \leq M \leq 10$ and $0 \leq S \leq 10$: (a) solution of (360) with $\sigma = 0.25$ where the blue graph is for $r = 0.05$ and the red graph is for $r = 0.10$; (b) solution of (361) with $\sigma_m = 0.25$ where the blue graph is for $r = 0.05$ and the red graph is for $r = 0.10$; (c) solution of (362) with $\sigma = 0.5$ where the blue graph is for $r = 0.05$ and the red graph is for $r = 0.10$.

Case 3: Reduction with Y_2

$$
\begin{aligned}
&4\,{\sigma_m}^2 M^2 \left(\frac{\partial^2}{\partial M^2} K\left(\tau, M\right)\right) \sigma^2 \tau - \sigma^4 K\left(\tau, M\right) \tau + \\
&8\, r M \left(\frac{\partial}{\partial M} K\left(\tau, M\right)\right) \sigma^2 \tau - 4\, r K\left(\tau, M\right) \sigma^2 \tau - 4\, K\left(\tau, M\right) r^2 \tau \\
&- 8 \left(\frac{\partial}{\partial \tau} K\left(\tau, M\right)\right) \sigma^2 \tau - 4\, \sigma^2 K\left(\tau, M\right), \qquad (363)
\end{aligned}
$$

with $V(S, M, \tau) = K(\tau, M)$. Application of X_1 that this reduced PDE admits gives solution (358). Invariant solution employing another symmetry $\frac{K \frac{\partial}{\partial K}}{\tau} - 2\,\frac{\partial}{\partial \tau}$ of the reduced PDE:

$$
\begin{aligned}
V(S, M, \tau) = \quad & C_1 M^{1/2}\, \frac{\sqrt{(\sigma^2+{\sigma_m}^2)({\sigma^2}{\sigma_m}^2+4\,r^2)}+\left({\sigma_m}^2-2\,r\right)\sigma}{{\sigma_m}^2\sigma} \\
& + C_2 M^{1/2}\, \frac{-\sqrt{(\sigma^2+{\sigma_m}^2)({\sigma^2}{\sigma_m}^2+4\,r^2)}+\left({\sigma_m}^2-2\,r\right)\sigma}{{\sigma_m}^2\sigma}. \qquad (364)
\end{aligned}
$$

Lastly, from Lie point symmetry of the reduced PDE,

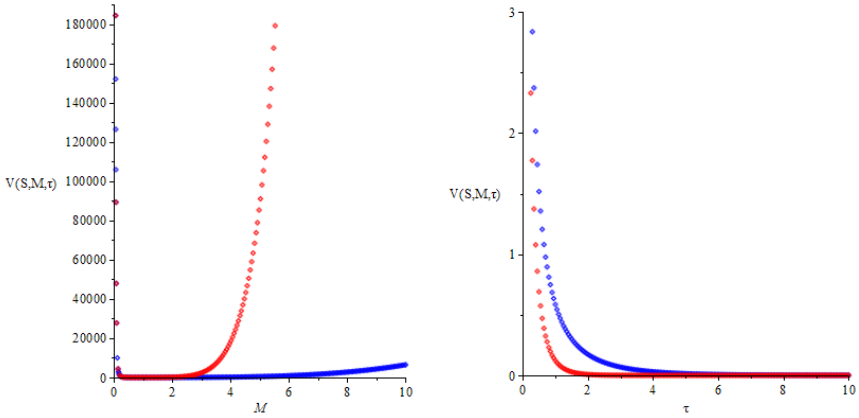
$\left(-K{\sigma_m}^2 t + 2\,K r \tau + 2\,K \ln\left(M\right)\right) \frac{\partial}{\partial K} - 2\tau X_1 {\sigma_m}^2$ we have

$$
V(S, M, \tau) = \frac{C_1}{\tau} \mathrm{e}^{-1/8\, \frac{\tau\left(\sigma^2+{\sigma_m}^2\right)\left({\sigma^2}{\sigma_m}^2+4\,r^2\right)}{{\sigma^2}{\sigma_m}^2}}. \qquad (365)
$$

In Figure 7.7 these invariant solutions are visually presented.

Case 4: Reduction with $c_1 X_2 + c_2 Y_2$

The reduced equation in this case exhibits symmetry X_1 from which we obtain



(a) (b)

Figure 7.7: Evolution of $V(S, M, \tau)$. We select the parameter values $C_1 = C_2 = 1$ for the ranges $0 \leq M \leq 10$ and $0 \leq \tau \leq 10$ with $\sigma_m = 0.25; \sigma = 0.05$: (a) solution of (364) where the blue graph is for $r = 0.05$ and the red graph is for $r = 0.10$; (b) solution of (365) where the blue graph is for $r = 0.05$ and the red graph is for $r = 0.10$.

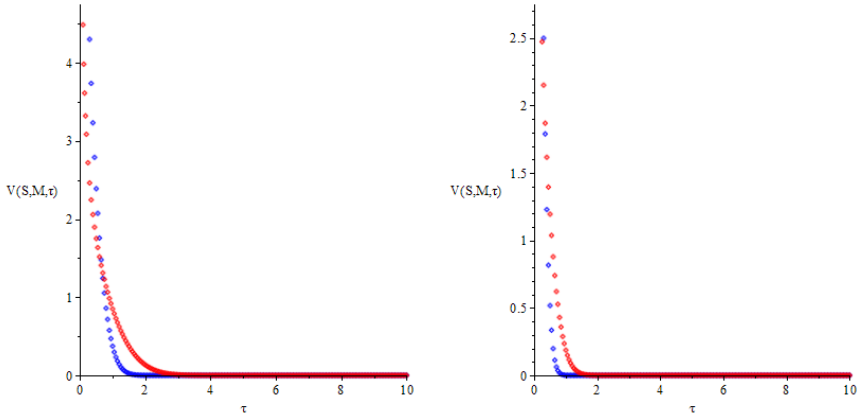
$$V(S, M, \tau) = \frac{1}{2} \frac{C_1 \sqrt{2}}{\sqrt{\sigma^2 \tau}} \mathrm{e}^{-1/8 \frac{\tau \left(\sigma^2 + 2r\right)^2}{\sigma^2}}. \tag{366}$$

Then using $X_1 + K\frac{\partial}{\partial K}$ we obtain

$$V(S, M, \tau) = \frac{1}{2} \frac{C_1 \sqrt{2}}{\sqrt{\sigma^2 \tau}} \mathrm{e}^{-1/2 \frac{\tau \left(1/2\,\sigma^2 + r\right)^2}{\sigma^2}}. \tag{367}$$

These solutions are plotted in Figure 7.8.

Case 5: Reduction with $c_1 X_1 + c_2 Y_1$



(a) (b)

Figure 7.8: Illustration of the solutions under Case 4. We let $C_1 = 1$ and $r = 0.05$ and the range $0 \leq \tau \leq 10$: (a) solution of (366) for $\sigma = 0.25$ (blue) and for $\sigma = 0.5$ (red); (b) solution of (367) for $\sigma = 0.25$ (blue) and for $\sigma = 0.5$ (red).

We can only use it's symmetry X_2 with $c_1 = 0; c_2 = 1$ in the final solution to obtain

$$V(S, M, \tau) = \frac{1}{2} \frac{C_1 \sqrt{2}}{\sqrt{{\sigma_m}^2 \tau}} \mathrm{e}^{-1/8 \frac{\tau \left({\sigma_m}^2 + 2r\right)^2}{{\sigma_m}^2}}. \tag{368}$$

It is apparent that this solution is similar to (366) and so we omit the plot.

7.4 Discussion

The CAPM is a critical component of portfolio management. Despite the presence of more advanced asset pricing and portfolio selection methodologies, the CAPM is widely utilized due to its ease of use. The European option price under a continuous-time CAPM can be derived. In this work we have successfully performed a full symmetry analysis for the European option pricing model under a continuous-time

CAPM, considering both constant and zero market betas. We discovered that this PDE admits several invariant solutions for zero and non-zero betas. Commutator relations where also derived in which, for beta equal to a constant, all commutators vanished. We provided several plots for our solutions given different levels of beta, risk-free interest rates, and volatilities.

When beta is constant in our model, we encounter the solution (336). This solution means the option price decreases exponentially over time for a given interest rate. This could be explained by time decay which states that as the time to the expiration date approaches, the value of an option decreases. In our solution, time decay accelerates as we increase τ because as the expiration date gets closer there is less time for an investor to profit from the option. In other graphs in our study we see that the option price generally increases with market price or with asset price. We conclude that symmetry analysis provides valuable insights into option pricing.

Conclusion

In our investigations, we clearly observe that symmetry analysis is a unique tool for comprehending and formulating solutions to differential equations. In this work, we have utilized the Lie point symmetries of the WDW equations of the corresponding universes to discover the invariant solutions to solve for their wave functions, as well as leverage the symmetries of the financial models to obtain unique invariant solutions to these nonlinear PDEs. When analyzing differential equations, particularly when linearising them and analyzing the solutions and their physical effects, conservation laws are crucial and frequently employed. As a result, they are the subject of several mathematical and physical publications. In our research, we looked at the conservation laws of several spacetimes, also referred to as first integrals. For the IOPM system, we derive multipliers that give rise to conservation law vector components. Not every symmetry presented here results in a physical conservation law. Nevertheless, these are equally helpful in practice, particularly in reducing the order of the underlying differential equation.

The middle chapters are dedicated to the study of various spacetimes derived from Einstein's field equations. The pp-wave spacetime is a spacetime with a single constant wave vector and solely radiation i.e. gravitational waves and electromagnetic waves. On the other hand, the Bianchi spacetimes which are spatially homogeneous

anisotropic cosmological models are useful in many ways, including describing the universe's large-scale behavior and simulating models explaining the early stages of the universe's evolution. We study the Bianchi universes in the framework of the modified $f(R)$ theory of gravity, a generalisation of general relativity from Einstein. In an effort to correct general relativity's flaws, this modified theory has gained widespread recognition. It has generated interest as a possible explanation for the apparent acceleration of the universe in recent years, offering an alternative to dark energy. The WDW equation makes an effort to mathematically integrate the concepts of general relativity with quantum mechanics, in an attempt to develop a theory of quantum gravity. Thus, the WDW equation can be used to comprehend the quantum characteristics of spacetime.

Each of these spacetimes were split into isometry classes i.e. KVFs and HVFs, to categorise the spacetime. Using Noether's theorem, we determine the potential functionals and the first integrals connected to each KVF and HVF for each class. We take advantage of the geometric fact that the homothetic algebra of spacetimes yields the Noether point symmetries of geodesic Lagrangians. We pointed out that linearly combining the vector fields is also useful and as a result, we offered more potential functions in the circumstances that were pertinent. Additionally, for the pp-wave spacetime we looked at a large number of subalgebras in each category and derived potential functions from them. For the Bianchi spacetimes our contribution was that of original analytical solutions to the spacetimes in the context of $f(R)$ theory of gravity, that have not appeared previously in literature. In order to properly deduce the geodesic equations of motion, we determined the Lagrangian of these two spacetimes. We also provide the system's Hamiltonian. Where feasible, we provide explicit solutions of the wave functions of the universes via Lie symmetry analysis of the WDW equation, and the solutions to the field equations through a reduced Hamiltonian system.

The last two chapters were devoted to the analysis and derivation of new solutions of mathematical models from quantitative finance, particularly variations of the Black-Scholes model used in options pricing.

The first of these financial models being the IOPM is an econophysics model, where in contrast to conventional economic theory, econophysics uses models from the realm of physics to explain economic phenomena. We have effectively completed a comprehensive symmetry analysis of the IOPM in this work, taking into account a varying Landau coefficient reflective of an adaptive market potential. This analysis has lead to us finding novel symmetries, solutions and conservation laws of this Schrödinger type equation. Our analysis has shown that when the adaptive market potential is non-zero, this model is invariant under a five-dimensional Lie algebra, and when it is zero, it is invariant under a seven-dimensional Lie algebra. Commutator tables of the admitted symmetries are also provided. Given various levels of volatility, we produced a number of graphs for our solutions.

In investigating a nonlinear European option pricing equation under a continuous-time CAPM, we varied the systematic risk variable, beta, to get invariant analytical solutions. We have successfully shown that under a three-dimensional Lie algebra for a constant beta and a six-dimensional Lie algebra with no systematic risk, the financial model is invariant. Consequently, we used these symmetries to reduce the order and solve for the model, thus accomplished a full symmetry analysis successfully. Given various beta, risk-free interest rates, and volatilities, we have plotted many graphs for our solutions. It is of interest to note that one of the solutions, (336), was was often obtained while beta remained constant, attributing it to the time decay of the option. Since there is less time left for an investor to benefit from the option as the expiry date approaches, time decay quickens as we raise τ. We can observe that the option price typically rises along with the market price or the

asset price in other graphs from our study. From our analysis, it is evident that symmetry analysis offers useful information to option pricing.

Indeed, symmetries are of paramount importance to the study of both differential equations and geometry. For ODEs, symmetries reduce the order, while for PDEs the number of independent variables is decreased by the invariance under a one-parameter Lie group of transformations, as was seen in this work. In terms of geometry they leave an object and its transformed image indistinguishable. This thesis is a culmination of a body of work that successfully utilised symmetries for the study of differential equations and the geometries of spacetimes.

Bibliography

[1] U. Obaidullah and S. Jamal, pp-wave potential functions: A complete study using Noether symmetries, *International Journal of Geometric Methods in Modern Physics* **18**(7) (2021) 2150108.

[2] U. Obaidullah, S. Jamal, and G. Shabbir, Analytical field equation and wave function solutions of the Bianchi type I universe in vacuum f(R) gravity, *International Journal of Geometric Methods in Modern Physics* **19**(09) (2022) 2250136.

[3] U. Obaidullah and S. Jamal, Classical solutions to Bianchi type II spacetimes in f(R) theory of gravity, *Indian Journal of Physics* **96**(12) (2022) 3675–3688.

[4] P. Olver, *Application of Lie Groups to Differential Equations,* Springer, New York (1993).

[5] P.E. Hydon, *Symmetry methods for differential equations: a beginner's guide*, Cambridge University Press, Cambridge (2000).

[6] E. Noether, Invariante Variationsprobleme, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* **2** (1918) 235–257.

[7] U. Obaidullah and S. Jamal, A Computational Procedure for Exact Solutions of Burgers' Hierarchy of Nonlinear Partial Differential Equations, *Journal of Applied Mathematics and Computing* **65** (2020) 541–551.

[8] U. Obaidullah and S. Jamal, On the Formulaic Solution of a (n+1)th Order Differential Equation, *International Journal of Applied and Computational Mathematics* **7**(58) (2021).

[9] S. Jamal, Approximate Conservation Laws of Nonvariational Differential Equations, *Mathematics* **7**(7) (2019) 574.

[10] A. Einstein, Die Grundlage der allgemeinen Relativitätstheorie, *Annalen der Physik* **354**(07) (1916) 769–822.

[11] F. Black and M. Scholes, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* **81**(03) (1973) 637–654.

[12] H. Stephani, *Differential Equations: Their Solutions Using Symmetry*, Cambridge University Press, New York (1989).

[13] R.M. Wald, *General Relativity*, Chicago University Press, Chicago (1984).

[14] J., Plebanski and A. Krasinski, *An Introduction to General Relativity and Cosmology*, Cambridge University Press, New York (2006).

[15] K. Yano, *The Theory of Lie Derivatives and Its Applications*, North Holland Publishing Co., Amsterdam (1956).

[16] G.H. Katzin, J. Levine, and W.R. Davis, Curvature Collineations: A Fundamental Symmetry Property of the Space-Times of General Relativity Defined by the vanishing Lie Derivative of the Riemann Curvature Tensor, *Journal of Mathematical Physics* **10** (1969) 617–629.

[17] G.W. Bluman and S. Kumei, *Symmetries and differential equations*, Springer-Verlag, New York (1989).

[18] H. Stephani, *General Relativity - An introduction to the theory of the gravitational field*, Cambridge University Press, Cambridge, (1990).

[19] M. Tsamparlis and A. Paliathanasis, Two-dimensional dynamical systems which admit Lie and Noether symmetries, *Journal of Physics A: Mathematical and Theoretical* **44**(17) (2011) 175202.

[20] B.S. DeWitt, *The many-worlds interpretation of quantum mechanics*, Princeton Series in Physics, Princeton University Press (1973).

[21] S. Capozziello, M. De Laurentis, and S.D. Odintsov, Hamiltonian dynamics and Noether symmetries in Extended Gravity Cosmology, *The European Physical Journal C* **72**(07) (2012) 2068.

[22] S.P. Kim, New asymptotic expansion method for the Wheeler-DeWitt equation, *Physical Review D* **52**(06) (1995) 3382–3391.

[23] J. Ehlers and W. Kundt, *In Gravitation: An Introduction to Current Research*, Wiley, New York (1962).

[24] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact solutions of Einstein's field equations* Cambridge University Press, Cambridge (2003).

[25] S. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*, Benjamin Cummings (2003).

[26] R. Sippel and H. Goenner, Symmetry classes of pp-waves, *General Relativity and Gravitation* **18** (1986) 1229–1243.

[27] P. C. Aichelburg, Curvature Collineations for gravitational pp waves, *Journal of Mathematical Physics* **11**(8) (1970) 2458–2462.

[28] D.G. Boulware and S. Deser, String-Generated Gravity Models, *Physical Review Letters* **55**(24) (1985) 2656–2660.

[29] C.M. Hull, Killing spinors and exact plane-wave solutions of extended supergravity, *Physical Review D* **30** (1984) 334–338.

[30] A.J. Keane and B.O.J. Tupper, Conformal symmetry classes for pp-wave spacetimes, *Classical and Quantum Gravity* **21**(8) (2004) 2037–2064.

[31] U. Camci and A. Yildirim, Noether gauge symmetry classes for pp-wave spacetimes, *International Journal of Geometric Methods in Modern Physics* **12**(10) (2015).

[32] S. Jamal and G. Shabbir, Noether symmetries of vacuum classes of pp-waves and the wave equation, *International Journal of Geometric Methods in Modern Physics* **13**(9) (2016) 1650109.

[33] A. Paliathanasis, M. Tsamparlis, and M.T. Mustafa, Classification of the Lie and Noether point symmetries for the Wave and the Klein-Gordon equations in pp-wave spacetimes, *Communications in Nonlinear Science and Numerical Simulation* **55** (2018) 68–83.

[34] S. Bahamonde, U. Camci, and S. Capozziello, Noether symmetries and boundary terms in extended Teleparallel gravity cosmology, *Classical and Quantum Gravity* **36**(6) (2019) 065013.

[35] S. Bahamonde, U.Camci, S. Capozziello, and M. Jamil, Scalar-tensor teleparallel wormholes by Noether symmetries, *Physical Review D* **98**(8) (2016) 084042.

[36] S. Capozziello and M. De Laurentis , Noether symmetries in extended gravity quantum cosmology, *International Journal of Geometric Methods in Modern Physics* **11**(2) (2014) 1460004.

[37] Z. Urban, F. Bajardi, and S. Capozziello, The Noether–Bessel-Hagen symmetry approach for dynamical systems, *International Journal of Geometric Methods in Modern Physics* **17**(14) (2020) 2050215.

[38] A.N. Nurbaki, S. Capozziello, and C. Deliduman, Spherical and cylindrical solutions in f(T) gravity by Noether symmetry approach, *The European Physical Journal C* **80**(108) (2020).

[39] K.F. Dialektopoulos and S. Capozziello, Noether Symmetries as a geometric criterion to select theories of gravity, *International Journal of Geometric Methods in Modern Physics* **15** (2018) 1840007.

[40] F. Bajardi and S. Capozziello, Equivalence of nonminimally coupled cosmologies by Noether symmetries, *International Journal of Modern Physics D* **29**(14) (2020) 2030015.

[41] E.P. Hubble, A Relation between Distance and Radial Velocity among Extra-Galactic Nebulae, *Proceedings of the National Academy of Sciences of the United States of America* **15**(3) (1929) 168–173.

[42] A.G. Riess, A.V. Filippenko, P. Challis, A. Clocchiatti, A. Diercks, P.M. Garnavich, R.L. Gilliland, C.J. Hogan, S. Jha, R.P. Kirshner, B. Leibundgut, M. M. Phillips, D. Reiss, B.P. Schmidt, R.A. Schommer, R.C. Smith, J. Spyromilio, C. Stubbs, N.B. Suntzeff, and J. Tonry, Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant, *The Astronomical Journal* **116**(3) (1998) 1009.

[43] C.L. Bennett, M. Halpern, G. Hinshaw, N. Jarosik, A. Kogut, M. Limon, S.S. Meyer, L. Page, D.N. Spergel, G.S. Tucker, E. Wollack, E.L. Wright, C. Barnes, M.R. Greason, R.S. Hill, E. Komatsu, M.R. Nolta, N. Odegard, H.V. Peiris, L. Verde, and J.L. Weiland, First-Year Wilkinson Microwave Anisotropy Probe (WMAP)* Observations: Preliminary Maps and Basic Results, *The Astrophysical Journal Supplement* **148** (2003) 1–27.

[44] C.B. Netterfield, P.A.R. Ade, J.J. Bock, J.R. Bond, J. Borrill, A. Boscaleri6, K. Coble, C.R. Contaldi, B.P. Crill, P. de Bernardis, P. Farese, K. Ganga, M. Giacometti, E. Hivon, V.V. Hristov, A. Iacoangeli, A.H. Jaffe11, W.C. Jones, A.E. Lange, L. Martinis, S. Masi, P. Mason, P.D. Mauskopf, A. Melchiorri, T. Montroy, E. Pascale, F. Piacentini, D. Pogosyan, F. Pongetti, S. Prunet, G. Romeo, J. E. Ruhl, and F. Scaramuzzi, A Measurement by BOOMERANG of Multiple Peaks in the Angular Power Spectrum of the Cosmic Microwave Background, *The Astrophysical Journal* **571**(2) (2002) 604–614.

[45] N.W. Halverson, E.M. Leitch, C. Pryke, J. Kovac, J.E. Carlstrom, W.L. Holzapfel, M. Dragovan, J. K. Cartwright, B.S. Mason, S. Padin, T.J. Pearson, A.C.S. Readhead, and M.C. Shepherd, Degree Angular Scale Interferometer First Results: A Measurement of the Cosmic Microwave Background Angular Power Spectrum, *The Astrophysical Journal* **568**(1) (2002) 38–45.

[46] H.A. Buchdahl, Non-Linear Lagrangians and Cosmological Theory, *Monthly Notices of the Royal Astronomical Society* **150** (1970) 1–8.

[47] S. Nojiri and S.D. Odintsov, Unified cosmic history in modified gravity: From F(R) theory to Lorentz non-invariant models, *Physics Reports* **505**(2-4) (2011) 59–144.

[48] S. Capozziello, R. Cianci, C. Stornaiolo, and S. Vignolo, f(R) gravity with torsion: a geometric approach within the $\mathcal{J}$–bundles framework, *International Journal of Geometric Methods in Modern Physics* **5**(5) (2008) 765–788.

[49] J.P. Duruisseau and R. Kerner, The effective gravitational Lagrangian and the energy-momentum tensor in the inflationary Universe, *Classical and Quantum Gravity* **3**(5) (1986) 817–824.

[50] A.A. Starobinsky, A new type of isotropic cosmological models without singularity, *Physics Letters B* **91**(1) (1980) 99–102.

[51] S.M. Carroll, V. Duvvuri, M. Trodden, and M.S. Turner, Is cosmic speed-up due to new gravitational physics?, *Physical Review D* **70**(4) (2004).

[52] H.J. Schmidt, Fourth order gravity: equations, history, and applications to cosmology, *International Journal of Geometric Methods in Modern Physics* **4**(02) (2007) 209–248.

[53] S. Capozziello, Dark energy models toward observational tests and data, *International Journal of Geometric Methods in Modern Physics* **4**(01) (2007) 53–78.

[54] S. Capozziello, V.F. Cardone, and A. Troisi, Reconciling dark energy models with f(R) theories, *Pysical Review D* **71**(4) (2005).

[55] K.C. Jacobs, Spatially Homogeneous and Euclidean Cosmological Models with Shear, *Astrophysical Journal* **153** (1968) 661.

[56] G. Shabbir, F. Hussain, A.H Kara, and M. Ramzan, A note on some perfect fluid Kantowski–Sachs and Bianchi type III spacetimes and their conformal vector fields in f(R) theory of gravity, *Modern Physics Letters A* **34**(11) (2019).

[57] F. Hussain, G. Shabbir, S. Jamal, and M. Ramzan, A note on some Bianchi type II spacetimes and their conformal vector fields in f(R) theory of gravity, *Modern Physics Letters A* **34**(38) (2019).

[58] S. Jamal and G. Shabbir, Geometric properties of the Kantowski-Sachs and Bianchi-type Killing algebra in relation to a Klein-Gordon equation, *The European Physical Journal Plus* **132**(70) (2017).

[59] S. Jamal, Quadratic integrals of a multi-scalar cosmological model, *Modern Physics Letters A* **35**(10) (2020) 2050068.

[60] S. Jamal, A group theoretical application of SO(4,1) in the de Sitter universe, *General Relativity and Gravitation*, **49**(88) (2017) 1–14.

[61] N. Dimakis, A. Giacomini, S. Jamal, G. Leon, and A. Paliathanasis, Noether symmetries and stability of ideal gas solutions in Galileon cosmology, *Physical Review D* **95** (2017) 064031.

[62] S. D. Odintsov and S. Nojiri, Modified gravity with negative and positive powers of curvature: Unification of inflation and cosmic acceleration, *Physical Review D* **68**(12) (2003) 123512.

[63] G. Shabbir, F. Hussain, S. Jamal, and M. Ramzan, Existence of conformal vector fields of Bianchi type I space-times in f(R) gravity, *International Journal of Geometric Methods in Modern Physics* **17**(08) (2020) 2050113.

[64] T.P. Shestakova, Is the Wheeler–DeWitt equation more fundamental than the Schrödinger equation?, *International Journal of Modern Physics D* **27**(6) (2018).

[65] S. Jamal, Multiscalar field models: Interaction potentials, wave functions and cosmological solutions, *International Journal of Modern Physics D* **29**(7) (2020).

[66] A. Paliathanasis, M. Tsamparlis, S. Basilakos, and J.D. Barrow, Dynamical analysis in scalar field cosmology, *Physical Reviews D* **91**(12) (2015) 123535.

[67] A. Zampeli, T. Pailas, P.A. Terzis, and T. Christodoulakis, Conditional symmetries in axisymmetric quantum cosmologies with scalar fields and the fate of the classical singularities, *Journal of Cosmology and Astroparticle Physics* **66**(05) (2016).

[68] S. Jamal, A. Paliathanasis, Group invariant transformations for the Klein–Gordon equation in three dimensional flat spaces, *Journal Geometry and Physics* **117** (2017) 50–59.

[69] S. Nojiri and S.D. Odintsov, Where new gravitational physics comes from: M-theory?, *Physics Letters B* **576**(1-2) (2003) 5–11.

[70] K. Yagdijian and A. Galstian, Fundamental solutions for the Klein–Gordon equation in de Sitter space- time, *Communications in Mathematical Physics* **285** (2009) 293–344.

[71] S. Ram and P. Singh, Bianchi type-II, VIII, and IX cosmological models with matter and electromagnetic fields, *Astrophysics and Space Science* **201** 29–33 (1993).

[72] F. Hussain, G. Shabbir, S. Jamal, and M. Ramzan, A note on some Bianchi type II spacetimes and their conformal vector fields in f(R) theory of gravity, *Modern Physics Letters A* **34**(38) (2019) 1950320.

[73] N. Godani, Locally rotationally symmetric Bianchi type-II cosmological model in f(R, T) gravity, *Indian Journal of Physics* **93**(7) (2019) 951—958.

[74] N.Godani and G.C Samanta, Traversable wormholes in $R + \alpha R^n$ gravity, *The European Physical Journal C* **80** (2020).

[75] G.C. Samanta and N. Godani, Validation of energy conditions in wormhole geometry within viable f(R) gravity, *The European Physical Journal C* **79**(623) (2019).

[76] N. Godani and G.C. Samanta, Static traversable wormholes in $f(R,T) = R + 2\alpha lnT$ gravity, *Chinese Journal of Physics* **62** (2019) 161–171.

[77] N. Godani and G.C. Samanta, Physical parameters for stable $f(R)$ models, *Indian Journal of Physics* **94**(8) (2020) 1303—1310.

[78] E. Elizalde, N. Godani, and G.C. Samanta, Cosmological dynamics in R^2 gravity with logarithmic trace term, *Physics of the Dark Universe* **30** (2020).

[79] N. Godani and G.C. Samanta, Estimation of cosmological parameters, stability analysis and energy conditions in viable modified gravity, *Chinese Journal of Physics* **66** (2020) 787–799.

[80] J.C. Hull, *Options, futures, and other derivatives*, Pearson, USA (2006).

[81] R.M. Jena, S. Chakraverty, and D. Baleanu, A novel analytical technique for the solution of time-fractional Ivancevic option pricing model, *Physica A: Statistical Mechanics and its Applications* **550** (2020) 124380.

[82] J. Voit, *The statistical mechanics of financial markets*, Springer, Berlin (2001).

[83] O. Vukovic, Interconnectedness of Schrodinger and Black-Scholes Equation, *Journal of Applied Mathematics and Physics* **3**(9) (2015) 1108–1113.

[84] A.H. Davison and S. Mamba, Symmetry methods for option pricing, *Communications in Nonlinear Science and Numerical Simulation* **47** (2017) 421–425.

[85] Y. Chatibi, E.H.E Kinani, and A. Ouhadan, Lie symmetry analysis and conservation laws for the time fractional Black–Scholes equation, *International Journal of Geometric Methods in Modern Physics* **17**(01) (2020) 2050010.

[86] M. Bohner and Y. Zheng, On analytical solutions of the Black–Scholes equation, *Applied Mathematics Letters* **22**(03) (2009) 309–313.

[87] P. Sawangtong, K. Trachoo, W. Sawangtong, and B. Wiwattanapataphee, The Analytical Solution for the Black-Scholes Equation with Two Assets in the Liouville-Caputo Fractional Derivative Sense, *Mathematics* **6**(08) (2018) 129.

[88] S.O. Edeki, O.O. Ugbebor, and E.A. Owoloko, Analytical Solutions of the Black–Scholes Pricing Model for European Option Valuation via a Projected Differential Transformation Method, *Entropy* **17**(11) (2015) 7510–7521.

[89] F. Mehrdoust and M. Mirzazadeh, On analytical solution of the black-scholes equation by the first integral method, *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics*, **76**(04) (2014) 85–90.

[90] V.G. Ivancevic, Adaptive-Wave Alternative for the Black-Scholes Option Pricing Model, *Cognitive Computation* **02** (2010) 17–30.

[91] S. Jamal and A.G. Johnpillai, Constitutive thermal laws and the exact solutions of Timoshenko systems, *Indian Journal of Physics* **94**(02) (2020) 233–242.

[92] S. Jamal, Imaging Noise Suppression: Fourth-Order Partial Differential Equations and Travelling Wave Solutions, *Mathematics* **08**(11) (2020).

[93] S. Jamal and A. Paliathanasis, Approximate symmetries and similarity solutions for wave equations on liquid films, *Applicable Analysis and Discrete Mathematics* **14**(02) (2020) 349–363.

[94] A. Paliathanasis, K. Krishnakumar, K.M. Tamizhmani, and P.G.L. Leach, Lie Symmetry Analysis of the Black-Scholes-Merton Model for European Options with Stochastic Volatility, *Mathematics* **4**(2) (2015).

[95] N.C. Caister, J.G. O'Hara, and K.S. Govinder, Solving the Asian option PDE using Lie symmetry methods, *International Journal of Theoretical and Applied Finance* **13**(08) (2010) 1256–1277.

[96] O. González-Gaxiola, S. O. Edeki, O. O. Ugbebor, and J.R. Ch'avez, Solving the Ivancevic Pricing Model Using the He's Frecuency Amplitude Formulation, *European Journal of Pure and Applied Mathematics* **10**(04) (2017) 631–637.

[97] S.O. Edeki, O.O. Ugbebor, and O. González-Gaxiola, Analytical solutions of the ivancevic option pricing model with a nonzero adaptive market potential, *International Journal of Pure and Applied Mathematics* **115**(01) (2017) 187–198.

[98] Y.Q. Chen, Y-H. Tang, J. Manafian, H. Rezazadeh, M.S. Osman, Dark wave, rogue wave and perturbation solutions of Ivancevic option pricing model, *Nonlinear Dynamics* **105** (2021) 2539–2548.

[99] H. Markowitz, Portfolio selection, *The Journal of Finance* **7**(1) (1952) 77–91.

[100] W.F. Sharpe, Capital asset prices: a theory of market equilibrium under conditions of risk, *The Journal of Finance* **19**(3) (1964) 425–442.

[101] J. Lintner, The valuation of risk assets and selection of risky investments in stock portfolios and capital budgets, *Review of Economics and Statistics* **47**(1) (1965) 13–37.

[102] A. Safdari-Vaighani, D. Ahmadian, and R. Javid-Jahromi, An Approximation Scheme for Option Pricing Under Two-State Continuous CAPM, *Computational Economics* **57** (2021) 1373–1385.