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Self interaction of gluons within the M.I.T. bag model

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SELF INTERACTION OF GLUONS WITHIN THE M.I.T. BAG MODEL

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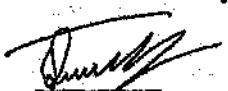
Abstract

One consequence of the non-abelian nature of Quantum Chromodynamic (QCD) gauge theory is that its gauge particles, or gluons as they are called, must be able to interact with each other. It has long been suggested that this must give rise to a complicated QCD vacuum structure, and many attempts to describe and model such a problem can be found in the literature. Another consequence of the self-interaction of the gluons is the effect of colour confinement, which specifies that only colour-charge neutral objects may exist in free space. One of the most successful models in QCD so far takes into account this confinement effect directly, by means of a phenomenological confining potential or bag. The aim of this thesis is to consider the structure of the QCD vacuum within the confines of such a bag.

Initially, the interaction amongst the QCD gauge particles is considered self-consistently in the finite M.I.T. bag model, using a Boglioubov-type approximation. We assume the gluons have condensed into the lowest single particle state of the bag, forming a boson condensate. A new set of single particle states is then found by diagonalising the self-energies of gluons propagating in the presence of such a condensate. From the true single particle spectrum that is found, the energies of possible gluonium states are then estimated. The vacuum condensate parameter $(\alpha_s \sum_a \hat{F}_a^{\mu\nu}(x^\kappa)\hat{F}_{a\mu\nu}(x^\kappa))$ is also explained as an effect of the presence of the gluon condensate.

Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the degree of Doctor of Philosophy in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.



J.F.H.Quick

31st day of October, 1990

To Shannon

Notation

We work in the so called 'natural' units

$$\hbar = c = 1,$$

throughout. In these units inverse lengths are equivalent to energies and most quantities such as the energy, momentum and time etc. have units of length (or mass) to some power. We note that

$$\hbar c = 197.3\,\text{MeV fm}.$$

Since the radius R is the natural unit of length within a bag, we will work mainly in units of $1/R$. We use Minkowski space-time where the metric $g^{\mu\nu}$ has signature

$$g^{\mu\nu} = g_{\mu\nu} = \text{diag}\{1, -1, -1, -1\},$$

for four vectors $x_\mu = (t, -\vec{x})$ and $x^\mu = (t, \vec{x})$. We also use the repeated index summation convention for all four vector indices μ etc. unless otherwise indicated. All sums over other indices, such as the colour indices 'a' etc., are shown explicitly. The $3j$ and $6j$ symbols are those defined according to the conventions of Edmonds [11]. Finally we note that in the literature the word Lagrangian is spelt Lagrangean and Lagrangian with roughly equal frequency. We will adopt the latter convention.

Acknowledgements

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Chapter 1

Introduction

The concept of a gauge field in Physics, namely a field whose action is preserved under some local gauge transformation, has long been established in the literature. For example in Electrodynamics, the electromagnetic energy is invariant under global, and indeed local variations in the phase of the corresponding electromagnetic fields or, equivalently, of the four-vector potential that defines them. Such variations in phase, correspond to the simplest gauge transformation, defined by the U(1) gauge group. Quantisation of such gauge fields then yields the familiar quantum gauge field theories and defines the concept of gauge particles, such as the photon of Quantum Electrodynamics (QED). The gauge field concept has been extended to more complicated gauge groups with some success, notably the electroweak unification model (U(1)×SU(2)) of Glashow, Weinberg and Salam [14] and others.

Nucleonic component models have focussed attention on yet another gauge group, SU(3) colour, formalised as Quantum Chromodynamics (QCD). This new theory differs markedly from QED in that the gauge group is non-abelian, so that its generators are by definition non-commutative. The gauge particles must therefore be able to interact with themselves, resulting in a highly non-linear problem. The long range nature of the interaction between two colour charges, produced by these self-interacting particles, is such that it diverges as the distance separating the two charge centres increases. One is thus lead to the conclusion that all observable particles must be neutral with respect to

this colour charge, i.e. confinement of all coloured constituents [8].

The QCD world-picture would thus seem to consist of small areas of perturbative vacuum, within which coloured constituent particles move freely, embedded in a non-perturbative vacuum where the colour charge is zero. These microscopic areas (often modelled by so-called bag models), corresponding to observable particles, are taken to be bound states of coloured constituents, including both quarks and the gauge particles or gluons, which exert an outward pressure on the walls to match the collective collapsing effect of the surrounding colour-free non-perturbative vacuum. Clearly we would expect there to be observable particles consisting of gauge particles alone, the so-called glueballs of QCD. In fact it has been suggested by Hansson, Johnson and Peterson [16], that the QCD vacuum itself might be a condensate of such glueball states. The identification and analysis of such glueballs is a topic that has enjoyed much speculation in the literature.

As the full QCD Lagrangian is highly non-linear, the exact modelling of such gluonic states is difficult to perform. Although QCD as a gauge theory is in principle renormalisable, no clear method to perform such a renormalisation has been found, mainly as there is no clear length scale in QCD. Attempts to perform such a renormalisation in terms of effective gluon masses by Cornwall [9], Thoma, Lüst and Mang [23] and others (defining glueballs as coupled pairs of such gluons), suffer from the drawback that extreme truncations of the diagrammatic series are necessary in order to make the problem tractable. The problem has also been approached via Lattice QCD (see for example the work of Fernandez and Marinari [12]), where rough bounds on constituent masses are found by considering the time dependence of plaquette correlation functions. Such lattice calculations, from their very nature, are dependent on the size of the lattice used and no truly effective method has yet been found to determine the true continuum limit of such results. Due to the success of effective Lagrangian methods in nuclear physics, there have also been several attempts to model QCD similarly (see for example the work by Gomm, Jain, Johnson and Schechter [15], where an effective chiral Lagrangian is found in

terms of a single heavy gluonium field). Unfortunately it is difficult to extract any of the mediating glueball's properties from such effective models.

The only models to have systematically produced any reasonable results for gluonium are the bag models, based on the original concept of Chodos, Jaffe, Johnson, Thorn and Weisskopf [6, 7], which reproduced the hadronic sector so well [10]. Simple calculations by Jaffe and Johnson [19], reveal that there should be a whole plethora of such gluonic states and the problem of finding or explaining such states away has yet to be solved. However the lowest of these gluonic states is in very good agreement with the predictions of the other models mentioned above. These calculations are, however, made without reference to the vacuum properties of QCD (see the work by Pascual and Tarrach [21] for examples of these) and usually only use effective gluon-gluon interactions. The advantages of such bag models over other methods are clear. The finite nature of the bag immediately resolves the problem of infra-red divergences, though there is some measure of renormalisation required in order to relate the model predictions to free perturbative QCD. Attempts have been made to model the vacuum as some condensate of these glueball states by Hansson, Johnson and Peterson [16] and to reproduce the full QCD vacuum properties directly instead.

In this thesis we will attempt to calculate gluon self-interactions within an MIT bag model, in the presence of some background condensate of gluons, which will generate some of the macroscopic properties of the vacuum even within the confines of our perturbative bag. From these self-interactions we hope to find a new set of self-consistent bag states and hence recalculate the glueball spectrum. One advantage of such a procedure is to enable us to determine self-consistent wavefunctions for these gluons and thus in principle to predict the expectation values of any operator we might require. In practice, the symmetry breaking nature of the bag leads to renormalisation difficulties as discussed above, though perhaps the method of Hansson, Johnson and Peterson [16] might be used to resolve these.

In our model, we have attempted to model the gluons self-consistently,

making use of the non-perturbative condensate vacuum picture of Beliaev [2]. This thesis attempts to present a clear logical derivation of such a model from basic principles. In Chapter 2, the problem of QCD within a finite system in the presence of a background condensate is discussed. Chapter 3 contains the details of the self-interactions and the formal structure of the Schwinger-Dyson equations that have to be solved. The connection between QCD vacuum properties and our condensate is considered in Chapter 4, and some numerical calculations of self-consistent gluon states are presented. Finally we compare these results to values found using the other models in the literature and discuss a few conclusions.

We have chosen to details all the more difficult calculational details in the appendices, so as not to interrupt the flow of physical ideas in the course of the thesis. The appendices thus, in no way represent a summary of standard results, but rather contain the calculational heart of the thesis. Appendix A contains details of the MIT bag model together with derivations of the second-quantisation procedure and the corresponding wavefunctions and propagators used in the text. Appendix B deals with the parameterisation of the background condensate, and explains how the quantities which describe the condensate are calculated. Appendix C contains explanations of the colour factors implicit in the gluonic self-energies, including an interesting result detailing the coupling of two octet fields to a colour singlet. Appendix D details the somewhat complex calculation of the integrals found in the model. These appendices tend to be mathematically exhaustive, but this is intentional in order to clarify some of the notational and approximative details of the calculation.

In summary then, we find that this model gives rise to a series of multi-gluon (or gluonium) states, in direct analogy with those of the standard MIT bag model. The lowest of these is a 0^+, 2^+ $(1M1)^2$ which lies at an energy that varies from 1313 MeV (the standard model value) down to 1073 MeV. We also find a 1^-, 2^-, 3^- state with energy between 1309 MeV and 1511 MeV, a 0^-, 1^-, 2^- state with energy between 1498 MeV and 1617 MeV, a 0^+, 2^+, 4^+ state with energy between 1531 MeV and 1700 MeV, and

an infinite series of higher states. Although there is no definite signature for gluonium states, elimination of those states known to be quark-antiquark pairs from the experimental spectrum [20], suggests that the $f_0(1240)0^{++}$, $\eta(1440)0^{-+}$, $f_0(1590)0^{++}$ and $f_2(1720)2^{++}$ resonances may well be multi-gluon states. These values compare very favourably with the predictions of our model.

Chapter 2

Formalism

2.1 The QCD Lagrangian

The basic form of the Lagrange density $\hat{\mathcal{L}}(x^\kappa)$ for free electromagnetic fields is [24]

$$\hat{\mathcal{L}}(x^\kappa) = -\frac{1}{4}\hat{F}^{\mu\nu}(x^\kappa)\hat{F}_{\mu\nu}(x^\kappa), \tag{2.1}$$

where $\hat{F}^{\mu\nu}(x^\kappa)$ is the field tensor defined by

$$\hat{F}^{\mu\nu}(x^\kappa) = \partial^\mu \hat{A}^\nu(x^\kappa) - \partial^\nu \hat{A}^\mu(x^\kappa), \tag{2.2}$$

in term of the four-potential $\hat{A}^\mu(x^\kappa)$ of the fields. The zeroth component of the canonical momenta defined by such a Lagrange density is identically equal to zero, which means that we cannot quantize a Lagrange density of this form. Instead we are led to introduce an extra, so called, *gauge-fixing* term [24], as follows :

$$\hat{\mathcal{L}}(x^\kappa) = -\frac{1}{4}\hat{F}^{\mu\nu}(x^\kappa)\hat{F}_{\mu\nu}(x^\kappa) - \frac{\lambda}{2}\left(\partial_\mu \hat{A}^\mu(x^\kappa)\right)^2, \tag{2.3}$$

where the factor λ can be seen as a Lagrange multiplier, enforcing the covariant Lorentz gauge condition $\partial_\mu \hat{A}^\mu(x^\kappa) = 0$. We note that in adding this term, we have already chosen a gauge and, to some extent, transferred the idea of gauge independence to the arbitrariness of the value of λ.

The corresponding Hamiltonian $\hat{\mathcal{H}}_0$ is

$$\begin{aligned}\hat{\mathcal{H}}_0 &= \int d^3x \hat{F}^{0\rho}(x^\kappa)\left(\partial_\rho \hat{A}^0(x^\kappa) - \partial^0 \hat{A}_\rho(x^\kappa)\right) \\ &+ \frac{1}{4}\int d^3x \hat{F}^{\mu\nu}(x^\kappa)\hat{F}_{\mu\nu}(x^\kappa) \\ &+ \text{appropriate gauge fixing terms .} \end{aligned} \tag{2.4}$$

If we now consider the case of the SU(3) chromomagnetic fields of QCD, then we will need eight independent fields $\hat{A}^\mu_a(x^\kappa)$ ($a = 1 \ldots 8$), corresponding to the eight generators of SU(3). The full field operator is then

$$\hat{\mathbf{A}}^\mu(x^\kappa) = \sum_{a=1}^{8} \hat{A}^\mu_a(x^\kappa)\lambda_a, \tag{2.5}$$

in terms of the Gellmann matrices λ_a. Due to the non-abelian nature of this field operator, the field tensor in QCD is more complicated :

$$\hat{F}^{\mu\nu}_a(x^\kappa) = \partial^\mu \hat{A}^\nu_a(x^\kappa) - \partial^\nu \hat{A}^\mu_a(x^\kappa) + g\sum_{bc} f_{abc}\hat{A}^\mu_b(x^\kappa)\hat{A}^\nu_c(x^\kappa), \tag{2.6}$$

where g is the dimensionless QCD coupling constant, and where the f_{abc} denote the structure constants of the SU(3) group, defined by

$$[\lambda_a, \lambda_b] = 2i\sum_c f_{abc}\lambda_c. \tag{2.7}$$

This results in further terms in the Hamiltonian

$$\begin{aligned}\hat{\mathcal{H}} &= \sum_{a=1}^{8}\hat{\mathcal{H}}_0^{(a)} + \sum_{abcde}\int d^3x \frac{1}{4}g^2 f_{abc}f_{ade}\hat{A}^\mu_b(x^\kappa)\hat{A}^\nu_c(x^\kappa)\hat{A}_{d\mu}(x^\kappa)\hat{A}_{e\nu}(x^\kappa) \\ &+ \cdots, \end{aligned} \tag{2.8}$$

where the remaining terms are trilinear in the fields $\hat{A}^\mu_a(x^\kappa)$. Since we wish to calculate the effects of self-interaction within the M.I.T. bag, where, to some extent, the trilinear terms are responsible for the confinement, we will limit our study to the four-point term only, which takes the form of a simple two-body contact interaction.

2.2 The Non-Interacting Field Equations

We wish to work within a finite system, specifically the M.I.T. bag model, so instead of using ordinary Fourier transforms, we will expand in terms of the discrete spectrum $A^{\mu}_{n}(x^{\kappa})$ of the finite system. From the Lagrange density given in equation (2.3) for the case of free fields, we may derive, using the Euler-Lagrange relations, the equation of motion

$$\Box \hat{A}^{\nu}_{a}(x^{\kappa}) - (1-\lambda)\partial^{\nu}\partial_{\mu}\hat{A}^{\mu}_{a}(x^{\kappa}) = 0, \tag{2.9}$$

subject to the Lorentz condition $\partial_{\mu}\hat{A}^{\mu}_{a}(x^{\kappa}) = 0$. This set of equations, together with the bag boundary condition, gives rise to a set of discrete states $A^{\mu}_{n}(x^{\kappa})$ (see Appendix A) which, since the problem is colour symmetric, will be independent of the colour 'a'. We will work in the Fermi-Feynman gauge where λ is taken to be 1, simplifying the equation of motion (2.9) above. We now expand the field operator in terms of these discrete states

$$\hat{A}^{\mu}_{a}(x^{\kappa}) = \sum_{n} A^{*\mu}_{n}(x^{\kappa})\, c^{\dagger}_{na} + \sum_{n} A^{\mu}_{n}(x^{\kappa})\, c_{na}, \tag{2.10}$$

where $c^{\dagger}_{na}$ and c_{na} are the usual boson creation and annihilation operators respectively, with commutator

$$[\, c_{ma}, c^{\dagger}_{nb}\,] = \delta_{ab}\delta_{mn}, \tag{2.11}$$

and where m, n are indices denoting the full quantum numbers of the states, which usually consists of a radial quantum number, the usual angular momentum quantum numbers and one further quantum number representing the polarisation of the gluonic wavefunction.

Taking the corresponding eigenenergies to be ω_n, we obtain for the Hamiltonian $\hat{\mathcal{H}}$

$$\hat{\mathcal{H}} = \sum_{a} \sum_{n(\text{TRANSVERSE})} \omega_n\, c^{\dagger}_{na} c_{na}$$

$$+\frac{g^2}{4}\sum_{\pm m\pm n p q}\left(\int d^4x A^{*\mu}_m(x^\kappa)A^{*\nu}_n(x^\kappa)A_{p\mu}(x^\kappa)A_{q\nu}(x^\kappa)\right)$$
$$\times\sum_{abcde} f_{abc}f_{ade}\,c^\dagger_{mb}c^\dagger_{nc}c_{pd}c_{qe}$$
$$+\cdots \tag{2.12}$$
$$= \hat{\mathcal{H}}^{\Sigma}_0+\hat{\mathcal{H}}_I+\hat{\mathcal{H}}_{res}.$$

The sums over $\pm n$ etc. denote that a sum over both positive and negative energies is intended, bearing in mind that a gauge quanta is its own anti-particle, so that this is merely a shorthand notation to include both the wavefunction and its conjugate in one sum (see Appendix A).

2.3 Perturbative QCD in a Condensate

We now consider the one particle Green's function of the full Hamiltonian $\hat{\mathcal{H}}$ in the Heisenberg representation

$$i\tilde{D}^{\mu\nu}_{ab}(x,x') = \langle\,\phi^{\mathbf{H}}_0\,|\,\mathrm{T}\{\hat{A}^{\mu}_{\mathbf{H}a}(x)\hat{A}^{\nu}_{\mathbf{H}b}(x')\}\,|\,\phi^{\mathbf{H}}_0\rangle, \tag{2.13}$$

where the expectation value is taken with respect to the full interacting ground state. If we move to the usual interaction representation, then formally we have [13]

$$i\tilde{D}^{\mu\nu}_{ab}(x,x') = \langle\,\mathrm{T}\{\hat{A}^{\mu}_{\mathbf{I}a}(x)\hat{A}^{\nu}_{\mathbf{I}b}(x')\hat{S}\}\,\rangle\,/\,\langle\hat{S}\rangle, \tag{2.14}$$

in terms of the $\hat{S}$ matrix (taking $\hat{\mathcal{H}}_{res}$ to be insignificant)

$$\hat{S} = \mathrm{T}\left\{\exp\left(-i\frac{g^2}{4}\sum_{abcde}f_{abc}f_{ade}\int d^4x\hat{A}^{\mu}_{\mathbf{I}b}(x)\hat{A}^{\nu}_{\mathbf{I}c}(x)\hat{A}_{\mathbf{I}d\mu}(x)\hat{A}_{\mathbf{I}e\nu}(x)\right)\right\}, \tag{2.15}$$

where now the expectation value is taken with respect to the non-interacting ground state of the Hamiltonian $\hat{\mathcal{H}}_0$, where all the particles are condensed into the lowest single particle state.

We separate off the operators acting on this lowest single particle state explicitly as follows :

$$\hat{A}^{\mu}_{\mathrm{a}}(x^{\kappa}) = \hat{A}^{0\mu}_{\mathrm{a}}(x^{\kappa}) + \hat{A}'^{\mu}_{\mathrm{a}}(x^{\kappa}), \tag{2.16}$$

where

$$\hat{A}^{0\mu}_{\mathrm{a}}(x^{\kappa}) = A^{*\mu}_{0}(x^{\kappa})\,\mathbf{c}^{\dagger}_{0\mathrm{a}} + A^{\mu}_{0}(x^{\kappa})\,\mathbf{c}_{0\mathrm{a}}. \tag{2.17}$$

We then consider the following two propagators separately :

$$i\tilde{D}^{0\mu\nu}_{\mathrm{ab}}(x,x') = \langle\, \mathrm{T}\{\hat{A}^{0\mu}_{\mathrm{Ia}}(x)\hat{A}^{0\nu}_{\mathrm{Ib}}(x')\hat{\mathcal{S}}\}\,\rangle \,/\, \langle\hat{\mathcal{S}}\rangle, \tag{2.18}$$

and

$$i\tilde{D}'^{\mu\nu}_{\mathrm{ab}}(x,x') = \langle\, \mathrm{T}\{\hat{A}'^{\mu}_{\mathrm{Ia}}(x)\hat{A}'^{\nu}_{\mathrm{Ib}}(x')\hat{\mathcal{S}}\}\,\rangle \,/\, \langle\hat{\mathcal{S}}\rangle. \tag{2.19}$$

Following the method of Beliaev [2], we also divide the two operations $\mathrm{T}\{\cdots\}$ and $\langle\cdots\rangle$ into two successive operations, the first acting only upon $\hat{A}'^{\mu}_{\mathrm{Ia}}(x^{\kappa})$, and the second only upon $\hat{A}^{0\mu}_{\mathrm{Ia}}(x^{\kappa})$. Thus we define

$$\mathrm{T}\{\cdots\} = \mathrm{T}^{0}\{\mathrm{T}'\{\cdots\}\} \quad , \quad \langle\cdots\rangle = \langle\langle\cdots\rangle'\rangle^{0}, \tag{2.20}$$

where T^{0} and $\langle\cdots\rangle^{0}$ act on $\hat{A}^{0\mu}_{\mathrm{Ia}}(x^{\kappa})$. Specifically we find that the propagator equation (2.19) now becomes

$$i\tilde{D}'^{\mu\nu}_{\mathrm{ab}}(x,x') = \langle\, \mathrm{T}^{0}\{i\tilde{\Theta}^{\mu\nu}_{\mathrm{ab}}(x,x')\}\,\rangle^{0} \,/\, \langle\hat{\mathcal{S}}\rangle, \tag{2.21}$$

where

$$i\tilde{\Theta}^{\mu\nu}_{\mathrm{ab}}(x,x') = \langle\, \mathrm{T}'\{\hat{A}'^{\mu}_{\mathrm{Ia}}(x)\hat{A}'^{\nu}_{\mathrm{Ib}}(x')\hat{\mathcal{S}}\}\,\rangle'. \tag{2.22}$$

Now equation (2.22) has the same structure as the numerator of equation (2.19), but with the operators $\hat{A}^{0\mu}_{\mathrm{Ia}}(x^{\kappa})$ appearing in $\hat{\mathcal{S}}$ now to be treated as parameters. The expectation value in equation (2.22) is taken with respect to the vacuum state of the operators $\mathbf{c}^{\dagger}_{n\mathrm{a}}$ and $\mathbf{c}_{n\mathrm{a}}$ $(n \neq 0)$, and so the formalism of Feynman graphs can be used to calculate $i\tilde{\Theta}^{\mu\nu}_{\mathrm{ab}}(x,x')$.

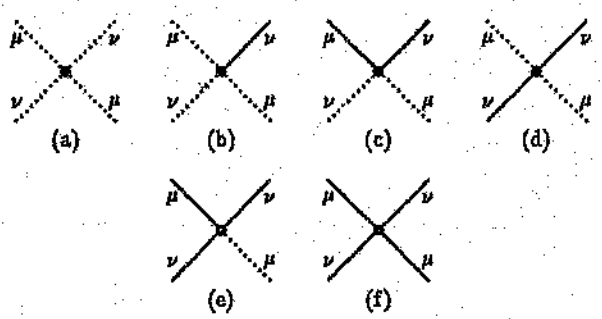


Figure 2.1: The Elementary Vertices of the Four-Point Interaction

We represent each contact interaction by a single dark point with four legs. The free propagator

$$i\tilde{D}'^{(0)\mu\nu}_{ab}(x,x') = \langle\, \mathrm{T}'\{\hat{A}'^{\mu}_{\mathrm{I}a}(x)\hat{A}'^{\nu}_{\mathrm{I}b}(x')\}\,\rangle', \tag{2.23}$$

is depicted by a solid line connecting the points x and x', and each factor $\hat{A}^{0\mu}_{\mathrm{I}a}(x^\kappa)$ by a dotted leg at that vertex. The six basic vertices are shown in Figure 2.1. The calculation of diagrams then proceeds analogously to the ordinary case, apart from the presence of the $\hat{A}^{0\mu}_{\mathrm{I}a}(x^\kappa)$ operators as parameters. For example, one diagram that contributes to $i\tilde{\Theta}^{\mu\nu}_{ab}(x,x')$ is shown in Figure 2.2, and the corresponding integral is

$$\begin{aligned}\hat{\mathcal{M}}_2(x_1,x_4) \;=\; & \int i\tilde{D}'^{(0)\mu\mu'}_{aa'}(x_1,x_2)\,\hat{A}^{0\sigma}_{c}(x_2)\left(-i\bar{F}^{a'c\sigma\rho}_{\mu'\sigma\rho\kappa}(x_2)\right) i\tilde{D}'^{(0)\kappa\rho'}_{ce'}(x_2,x_3) \\ & \times\; i\tilde{D}'^{(0)\kappa\kappa'}_{ff'}(x_2,x_3)\left(-i\bar{F}^{f'e'b'd}_{\kappa'\rho'\nu'\lambda}(x_3)\right) i\tilde{D}'^{(0)\nu'\nu}_{b'b}(x_3,x_4)\,\hat{A}^{0\lambda}_{d}(x_3) \\ & \times\, d^4x_2 d^4x_3,\end{aligned} \tag{2.24}$$

where $-i\bar{F}^{abcd}_{\mu\nu\lambda\sigma}(x^\kappa)$ denotes the four-point interaction vertex function that we will handle more explicitly in Chapter 3.

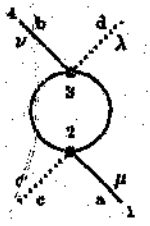


Figure 2.2: A Typical Second-order Feynman Diagram : see expression (2.24)

Now let $\hat{\mathcal{M}}$ be any graph, contributing to equation (2.22), that contains no disconnected parts or vacuum loops. Together with $\hat{\mathcal{M}}$, we consider all graphs that differ from $\hat{\mathcal{M}}$ by the addition of vacuum loops. The totality of such graphs gives $\hat{\mathcal{M}}$ multiplied by a factor that is just the vacuum expectation value of the $\hat{S}$ matrix, namely $\langle\hat{S}\rangle'$ in our case, since we take the expectation value with respect to the vacuum of the $\hat{A}^{\mu}_{\mathrm{In}}(x^{\kappa})$ operators. Thus the inclusion of vacuum loops changes $\hat{\mathcal{M}}$ into

$$\hat{\mathcal{M}}(x;x')\,\langle\hat{S}\rangle'. \tag{2.25}$$

Substituting this into equation (2.21) gives us a term that goes as

$$\langle \mathrm{T}^{0}\{\hat{\mathcal{M}}(x;x')\,\langle\hat{S}\rangle'\}\rangle^{0} \,/\, \langle\hat{S}\rangle, \tag{2.26}$$

where the operation T^{0} acts on the factors $\hat{A}^{0\mu}_{\mathrm{Io}}(x^{\kappa})$ etc. appearing in $\hat{\mathcal{M}}$ and in $\langle\hat{S}\rangle'$. Now suppose that $\hat{\mathcal{M}}$ contains m pairs of $\hat{A}^{0\mu}_{\mathrm{I_o}}(x^{\kappa})$ operators. Then

$$\begin{aligned}\hat{\mathcal{M}}(x;x') \;=\; \int M^{\mu_1\ldots\mu_m\nu_1\ldots\nu_m}_{\mathrm{a_1}\ldots\mathrm{a_m},\mathrm{b_1}\ldots\mathrm{b_m}}(x;x';x_1\ldots x_m;x'_1\ldots x'_m)\hat{A}^{0+}_{\mathrm{Ia_1}\mu_1}(x_1)\ldots \\ \ldots\hat{A}^{0+}_{\mathrm{Ia_m}\mu_m}(x_m)\hat{A}^{0}_{\mathrm{Ib_1}\nu_1}(x'_1)\ldots\hat{A}^{0}_{\mathrm{Ib_m}\nu_m}(x'_m)(d^4x_i)(d^4x'_i),\end{aligned} \tag{2.27}$$

and so (2.26) becomes

$$\int M^{\mu_1\ldots\mu_m,\nu_1\ldots\nu_m}_{a_1\ldots a_m,b_1\ldots b_m} i\tilde{D}^{0\,a_1\ldots a_m,b_1\ldots b_m}_{\mu_1\ldots\mu_m,\nu_1\ldots\nu_m}(x_1\ldots x_m;x'_1\ldots x'_m)(d^4x_i)(d^4x'_i), \qquad (2.28)$$

where

$$i\tilde{D}^{0\,a_1\ldots a_m,b_1\ldots b_m}_{\mu_1\ldots\mu_m,\nu_1\ldots\nu_m}(x_1\ldots x_m;x'_1\ldots x'_m)$$
$$= \frac{\langle \mathrm{T}\{\hat{A}^{0*}_{\mathrm{I}a_1\mu_1}(x_1)\ldots\hat{A}^{0*}_{\mathrm{I}a_m\mu_m}(x_m)\hat{A}^{0}_{\mathrm{I}b_1\nu_1}(x'_1)\ldots\hat{A}^{0}_{\mathrm{I}b_m\nu_m}(x'_m)\hat{S}\}\rangle}{\langle\hat{S}\rangle}, \qquad (2.29)$$

is the m-particle propagator of the condensed phase, equation (2.18) being the special case $m = 1$.

The graphs for the propagator in (2.21) thus coincide with the graphs for $i\tilde{\Theta}^{\mu\nu}_{ab}(x,x')$, only the $2m$ factors $\hat{A}^{0\mu}_{\mathrm{I}a}(x^\kappa)$ are replaced by the corresponding m-particle propagator for the condensed phase. For example, in the integral of equation (2.24), the factor $\hat{A}^{0\sigma}_{c}(x_2)\hat{A}^{0\lambda}_{d}(x_3)$ is replaced by $i\tilde{D}^{0\,\sigma\lambda}_{cd}(x_2,x_3)$.

We need not consider graphs with disconnected parts, since these are already included in the appropriate $i\tilde{D}^0(m)$.

2.4 The Propagators of the Condensed Phase

We now consider the condensed phase propagator in its single particle form

$$\begin{aligned} i\tilde{D}^{0\,\mu\nu}_{ab}(x,x') &= \langle \mathrm{T}^0\{\hat{A}^{0\mu}_{\mathrm{I}a}(x)\hat{A}^{0\nu}_{\mathrm{I}b}(x')\langle\hat{S}\rangle'\}\rangle^0 / \langle\hat{S}\rangle \qquad (2.30)\\ &= \langle \mathrm{T}^0\{A^{\mu}_0(x^\kappa)A^{\nu}_0(x'^{\kappa'})\,\mathrm{c}_{\mathrm{I}0a}(t)\mathrm{c}_{\mathrm{I}0b}(t')\langle\hat{S}\rangle'\}\rangle^0 / \langle\hat{S}\rangle\\ &+ \langle \mathrm{T}^0\{A^{\mu}_0(x^\kappa)A^{*\nu}_0(x'^{\kappa'})\,\mathrm{c}_{\mathrm{I}0a}(t)\mathrm{c}^\dagger_{\mathrm{I}0b}(t')\langle\hat{S}\rangle'\}\rangle^0 / \langle\hat{S}\rangle\\ &+ \langle \mathrm{T}^0\{A^{*\mu}_0(x^\kappa)A^{\nu}_0(x'^{\kappa'})\,\mathrm{c}^\dagger_{\mathrm{I}0a}(t)\mathrm{c}_{\mathrm{I}0b}(t')\langle\hat{S}\rangle'\}\rangle^0 / \langle\hat{S}\rangle\\ &+ \langle \mathrm{T}^0\{A^{*\mu}_0(x^\kappa)A^{*\nu}_0(x'^{\kappa'})\,\mathrm{c}^\dagger_{\mathrm{I}0a}(t)\mathrm{c}^\dagger_{\mathrm{I}0b}(t')\langle\hat{S}\rangle'\}\rangle^0 / \langle\hat{S}\rangle. \end{aligned}$$

The total number of particles is a good quantum number since the four-point term conserves particle number. Thus only the middle two terms will contribute to the above expression. Colour will also be conserved which will give rise to a δ_{ab} dependence. Now, given a large number of particles in the lowest single particle state, the addition of one particle to this state, and the removal of another particle from this state, occur essentially independently of each other. As a result, the propagator must be separable in the times of two such events as well as being explicitly spatially separable i.e.

$$i\tilde{D}^{0\,\mu\nu}_{ab}(x,x') = K^{\mu}(x)K^{*\nu}(x')\,\delta_{ab} + K^{*\mu}(x)K^{\nu}(x')\,\delta_{ab}.$$

Since the propagator may only be dependent on the time difference, s led to the following form for K^{μ} and $K^{*\mu}$:

$$\begin{aligned} K^{\mu}(x) &= \sqrt{n_0}\,A_0^{\mu}(\vec{x})\,e^{-i\mu_0 t}, \\ K^{*\mu}(x) &= \sqrt{n_0}\,A_0^{*\mu}(\vec{x})\,e^{+i\mu_0 t}. \end{aligned} \tag{2.32}$$

The propagator is thus

$$\begin{aligned} i\tilde{D}^{0\,\mu\nu}_{ab}(x,x') &= n_0\,\delta_{ab}A_0^{\mu}(\vec{x})A_0^{*\nu}(\vec{x}')\,e^{-i\mu_0(t-t')} \\ &+ n_0\,\delta_{ab}A_0^{*\mu}(\vec{x})A_0^{\nu}(\vec{x}')\,e^{+i\mu_0(t-t')}. \end{aligned} \tag{2.33}$$

If we compare this form with the original definition in equation (2.18), we find that

$$n_0 = \lim_{t'\to t^+} \langle\, \phi_0^{\mathrm{H}} \mid \mathrm{T}\{c_{\mathrm{H}0a}(t)\,c^{\dagger}_{\mathrm{H}0a}(t')\} \mid \phi_0^{\mathrm{H}} \,\rangle. \tag{2.34}$$

Thus n_0 is the mean occupation number of the lowest single particle state in the full interacting condensate ground state. To determine the nature of the other parameter μ_0, we consider the first term of equation (2.33), which is related to

$$\begin{aligned} &\langle\, \phi_0^{\mathrm{H}}(N) \mid c_{\mathrm{H}0a}(t)c^{\dagger}_{\mathrm{H}0a}(t') \mid \phi_0^{\mathrm{H}}(N) \,\rangle = \\ &\quad \langle\, \phi_0^{\mathrm{H}}(N) \mid c_{\mathrm{H}0a}(t) \mid \phi_0^{\mathrm{H}}(N+1) \,\rangle \langle\, \phi_0^{\mathrm{H}}(N+1) \mid c^{\dagger}_{\mathrm{H}0a}(t') \mid \phi_0^{\mathrm{H}}(N) \,\rangle + \end{aligned}$$

$$\sum_{s\neq 0}\langle\phi_0^{\mathrm{H}}(N)\mid c_{\mathrm{H}0a}(t)\mid\phi_s^{\mathrm{H}}(N+1)\rangle\langle\phi_s^{\mathrm{H}}(N+1)\mid c_{\mathrm{H}0a}^{\dagger}(t')\mid\phi_0^{\mathrm{H}}(N)\rangle, \tag{2.35}$$

in terms of the eigenstates of the interacting $N+1$-particle system. Since in general

$$c_{\mathrm{H}a}(t)=e^{i\hat{\mathcal{H}}t}c_{\mathrm{S}a}e^{-i\hat{\mathcal{H}}t}, \tag{2.36}$$

and

$$c_{\mathrm{H}a}^{\dagger}(t)=e^{i\hat{\mathcal{H}}t}c_{\mathrm{S}a}^{\dagger}e^{-i\hat{\mathcal{H}}t}, \tag{2.37}$$

thus the first term of equation (2.35) will have time dependence

$$e^{-i(E_0^{N+1}-E_0^N)(t-t')}$$

and the other terms will go, respectively, as

$$e^{-i(E_s^{N+1}-E_0^N)(t-t')}.$$

Comparison of this exact result (2.35) with our approximate result, given by the first term of equation (2.33), reveals that, in the limit of large N, only one term survives, and we conclude that the parameter μ_0 must go as

$$\mu_0=E_0^{N+1}-E_0^N. \tag{2.38}$$

Thus μ_0 is the chemical potential of the condensate. The parameters n_0 and μ_0 are not totally independent but are related by the expression

$$\mu_0=\sum_{(W)}n_0^{m-1}W_m(\mu_0\ldots\mu_0;\mu_0\ldots\mu_0), \tag{2.39}$$

(see Appendix B) where $W_m()$ denotes that component of the connected vacuum loop contribution to $\langle\hat{S}\rangle'$, which contains m pairs of $\hat{A}^0_{\mathrm{I}a}(x^\kappa)$ operators, and where the sum runs not only over geometric structure, but also over the position of a special vertex as explained fully in Appendix B.

Equation (2.39) may be considered as an equation for $\mu_0(n_0)$. The original problem is specified in terms of one free parameter, the total number of particles n. Thus both μ_0 and n_0 ought to be expressible in terms of n. However n does not appear explicitly in our equations. It is convenient rather to consider n_0 as the free parameter instead of n, and to express all other quantities as a functions of n_0. The connection between n_0 and n can be established after the problem is solved. From this standpoint equation (2.32) completely determines K^μ and $K^{*\mu}$ and, consequently, all the propagators of the condensed phase.

Chapter 3

Propagating Gluons in the Presence of a Condensate

3.1 The Dyson Equation for the Model

We look now at the propagator

$$i\tilde{D}'^{\mu\nu}_{ab}(x,x') = \langle \phi_0^H \mid T\{\hat{A}'^{\mu}_{Ha}(x)\hat{A}'^{\nu}_{Hb}(x')\} \mid \phi_0^H \rangle, \tag{3.1}$$

and its Fourier transform, defined by

$$i\tilde{D}'^{\mu\nu}_{ab}(x,x') = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')}\, i\tilde{D}'^{\mu\nu}_{ab}(\vec{x},\vec{x}',\omega). \tag{3.2}$$

Instead of transforming into momentum space as well, we expand rather, in terms of eigenstates of the M.I.T. Bag [6, 7, 10]

$$i\tilde{D}'^{\mu\nu}_{ab}(\vec{x},\vec{x}',\omega) = \sum_{mn} A^{\mu}_{m}(\vec{x})\, i\tilde{D}'_{mnab}(\omega)\, A^{*\nu}_{n}(\vec{x}'). \tag{3.3}$$

We now consider the general structure of any graph which contributes to the propagator (3.1). Every graph contributing to $i\tilde{D}'_{mnab}(\omega)$ has the form of a chain of separate irreducible parts, connected to each other by only one continuous line. There are only three different types of irreducible parts, which differ in the number of outgoing and incoming continuous lines as shown in Figure 3.1. Σ^{11} describes processes in which the number of particles in the con-

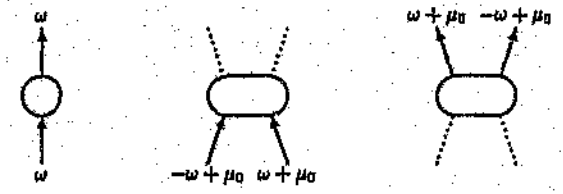


Figure 3.1: The Three Irreducible Contributions to $i\tilde{D}'^{\omega}_{ab}(x,x')$

densed phase is conserved. Σ^{02} and Σ^{20} describe the absorption and emission, respectively, of two particles out of the condensed phase. In these processes, two particles must be simultaneously absorbed, or emitted, and the energy required to do this (twice the chemical potential of the condensate) must be taken into account. We are thus lead to define

$$\begin{aligned}
-i\Sigma^{11}_{mn;ab}(\omega_1,\omega_2) &= -i\Sigma^{11}_{mn;ab}(\omega_1)\delta(\omega_1-\omega_2),\\
-i\Sigma^{02}_{mn;ab}(\omega_1,\omega_2) &= -i\Sigma^{02}_{mn;ab}(\omega_2)\delta(\omega_1+\omega_2-2\mu_0),\\
-i\Sigma^{20}_{mn;ab}(\omega_1,\omega_2) &= -i\Sigma^{20}_{mn;ab}(\omega_1)\delta(\omega_1+\omega_2-2\mu_0),
\end{aligned} \tag{3.4}$$

again transforming to the energy/eigenstate representation.

The Dyson equation for the ordinary propagator then turns out to be

$$\begin{aligned}
&i\tilde{D}'_{mnab}(\omega+\mu_0)\\
&\quad= i\tilde{D}'^{(0)}_{mnab}(\omega+\mu_0)\\
&\quad+ i\tilde{D}'^{(0)}_{mpac}(\omega+\mu_0)(-i)\Sigma^{11}_{pqcd}(\omega+\mu_0)i\tilde{D}'_{qndb}(\omega+\mu_0)\\
&\quad+ i\tilde{D}'^{(0)}_{mpac}(\omega+\mu_0)(-i)\Sigma^{20}_{pqcd}(\omega+\mu_0)i\overline{\tilde{D}'}_{qndb}(\omega+\mu_0),
\end{aligned} \tag{3.5}$$

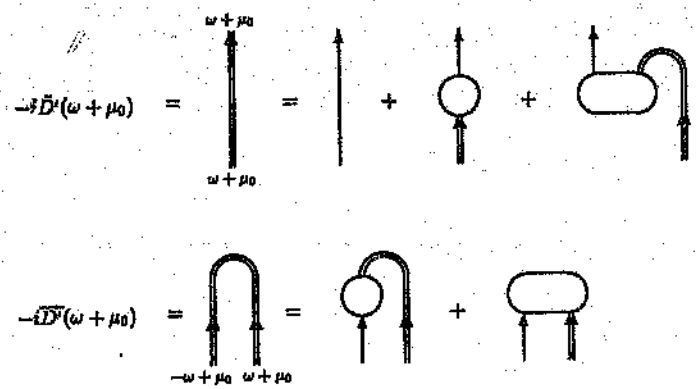


Figure 3.2: The Dyson Equations defining the Full Propagator

where $i\widetilde{D'}_{mnab}(\omega)$ is the anomalous propagator with equation

$$\begin{aligned} & i\widetilde{D'}_{mnab}(\omega+\mu_0) \\ & \quad = \; i\tilde{D}'^{(0)}_{mpac}(-\omega+\mu_0)(-i)\Sigma^{11}_{pqcd}(-\omega+\mu_0)i\widetilde{D'}_{qndb}(\omega+\mu_0) \\ & \quad + \; i\tilde{D}'^{(0)}_{mpac}(-\omega+\mu_0)(-i)\Sigma^{02}_{pqcd}(\omega+\mu_0)i\tilde{D}'_{qndb}(\omega+\mu_0), \end{aligned} \tag{3.6}$$

as described graphically in Figure 3.2, where the double lines, joining two points, denote the two full propagators, $i\tilde{D}'_{mnab}(\omega)$ and $i\widetilde{D'}_{mnab}(\omega)$.

3.2 Solution of the Dyson Equations

The Dyson equations may be solved explicitly, though the solutions turn out to be quite complicated :

$$i\tilde{D}'(+) \;=\; i\frac{\left[\Sigma^{-1}_{02}(+)\tilde{D}'^{-1}_{(0)}(-)\Sigma^{-1}_{11}(-)\Sigma_{02}(+)-1\right]\Sigma_{11}(-)}{\text{Denominator}},$$

$$i\bar{D}'(+) = i\frac{\Sigma_{11}^{-1}(-)\Sigma_{02}(+)\Sigma_{11}(-)}{\text{Denominator}}, \tag{3.7}$$

where the common denominator is

$$\left[\tilde{D}'^{-1}_{(0)}(+) - \Sigma_{11}(+)\right]\left[\Sigma_{02}^{-1}(+)\tilde{D}'^{-1}_{(0)}(-)\Sigma_{11}^{-1}(-)\Sigma_{02}(+) - 1\right]\Sigma_{11}(-)$$
$$- \Sigma_{20}(+)\left[\Sigma_{11}^{-1}(-)\Sigma_{02}(+)\Sigma_{11}(-)\right], \tag{3.8}$$

and where $(+)$ denotes an argument of $\omega+\mu_0$, and $(-)$ an argument of $-\omega+\mu_0$. Here the matrix indices in eigen- and colour-space have been suppressed for clarity. The inverses are taken with respect to both these indices simultaneously.

3.3 Evaluation of the Self-Energies and the Non-Interacting Propagator

The non-interacting propagator $i\tilde{D}'^{(0)}_{mnab}(\omega)$ is given by (see Appendix A)

$$i\tilde{D}'^{(0)}_{mnab}(\omega) = -i\delta_{mn}\,\delta_{ab}\,\frac{2\omega_m^0}{\omega^2 - {\omega_m^0}^2 + i0+}, \tag{3.9}$$

in terms of the unperturbed single particle eigenenergies ω_n^0, associated with those M.I.T. bag states $A_m^\mu(\vec{x})$ with respect to which we expand.

To evaluate the self energies, we start from the simplest general interaction diagram, shown in Figure 3.3, which evaluates to [24]

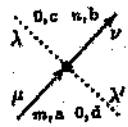
$$-ig^2\sum_c\{f_{abe}f_{cde}(g_{\lambda\nu}g_{\mu\sigma} - g_{\lambda\sigma}g_{\mu\nu}) + f_{ace}f_{bde}(g_{\lambda\mu}g_{\nu\sigma} - g_{\lambda\sigma}g_{\mu\nu})$$
$$+ f_{ade}f_{cbe}(g_{\lambda\nu}g_{\mu\sigma} - g_{\lambda\mu}g_{\sigma\nu})\}$$
$$\times\int d^3x A_m^{*\lambda}(\vec{x})A_q^{*\sigma}(\vec{x})A_p^{\nu}(\vec{x})A_n^{\mu}(\vec{x}). \tag{3.10}$$

Now the lowest order contributions to Σ^{11} are shown in Figure 3.4. Due to the condensation of the gluons into the lowest state, the first of the two diagrams



Figure 3.3: The Basic Interaction Diagram

$$-i\Sigma^{11}_{mnab}(\omega+\mu_0) \quad =$$



$$+$$



$$(p \neq 0)$$

Figure 3.4: Lowest Order Contributions to the Self-Energy Σ^{11}

should dominate. In this diagram we then tie the λ, λ' and 'c','d' indices together with the condensate propagator. Since this propagator is diagonal in colour, this eliminates two of the six terms and we are left with

$$\begin{aligned}\Sigma^{11}_{mnab}(\omega+\mu_0) = g^2 \sum_{ce} \{ & f_{cae} f_{cbe} \int d^3x\, i\tilde{D}^{0\,\lambda\lambda'}_{cc}(\vec{x},\vec{x}) g_{\lambda\lambda'} A^{\mu}_{m}(\vec{x}) A^{*}_{n\mu}(\vec{x}) \\ & - f_{cae} f_{cbe} \int d^3x\, A_{m\mu}(\vec{x}) i\tilde{D}^{0\,\mu\nu}_{cc}(\vec{x},\vec{x}) A^{*}_{n\nu}(\vec{x}) \\ & + f_{cbe} f_{cae} \int d^3x\, i\tilde{D}^{0\,\lambda\lambda'}_{cc}(\vec{x},\vec{x}) g_{\lambda\lambda'} A^{\mu}_{m}(\vec{x}) A^{*}_{n\mu}(\vec{x}) \\ & - f_{cbe} f_{cae} \int d^3x\, A^{*}_{n\nu}(\vec{x}) i\tilde{D}^{0\,\nu\mu}_{cc}(\vec{x},\vec{x}) A_{m\mu}(\vec{x}) \}\,. \end{aligned} \tag{3.11}$$

The colour nature of Σ^{11} is easily determined, as the condensate propagator is colour independent. Thus each term has colour dependence

$$\sum_{ce} f_{ace} f_{bce} = C_A\, \delta_{ab}, \tag{3.12}$$

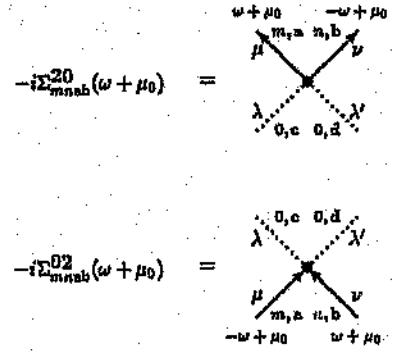


Figure 3.5: Lowest Order Contributions to the Anomalous Self-Energies

where C_A is a constant, which, for the SU(3) group, has value 3. Thus we obtain, substituting for $i\tilde{D}^{0\mu\nu}_{ab}(\vec{x},\vec{x})$ (see equation (2.33)),

$$\Sigma^{11}_{mnab}(\omega+\mu_0) = 2n_0\, g^2 C_A \delta_{ab}\Big\{2\int d^3x A^{\mu}_0(\vec{x}) A^{*}_{0\mu}(\vec{x}) A^{\nu}_m(\vec{x}) A^{*}_{n\nu}(\vec{x}) - \int d^3x A^{\mu}_m(\vec{x}) A_{0\mu}(\vec{x}) A^{*\nu}_0(\vec{x}) A^{*}_{n\nu}(\vec{x}) - \int d^3x A^{\mu}_n(\vec{x}) A^{*}_{0\mu}(\vec{x}) A^{\nu}_0(\vec{x}) A^{*}_{m\nu}(\vec{x})\Big\}. \tag{3.13}$$

The anomalous self energies are shown to lowest order in Figure 3.5. Again we tie the λ, λ' and 'c','d' indices together with a condensate propagator. However this time the propagator is anomalous, and so the colour indices must now be coupled to a colour singlet by some coupling coefficient c_{cd} say. We have then

$$\Sigma^{02}_{mnab}(\omega+\mu_0) = g^2 \sum_{cde}\{\, f_{cae} f_{bde} \int d^3x A_{n\nu}(\vec{x}) i\overline{D^{0\nu\mu}_{cd}}(\vec{x},\vec{x}) A_{m\mu}(\vec{x}) - f_{cae} f_{bde} \int d^3x i\overline{D^{0\lambda\lambda'}_{cd}}(\vec{x},\vec{x}) g_{\lambda\lambda'} A^{\mu}_m(\vec{x}) A_{n\mu}(\vec{x}) + f_{cbe} f_{ade} \int d^3x A_{m\mu}(\vec{x}) i\overline{D^{0\mu\nu}_{cd}}(\vec{x},\vec{x}) A_{n\nu}(\vec{x})$$

$$
\begin{aligned}
&- f_{\mathrm{dbe}} f_{\mathrm{ade}} \int d^3x \overline{iD^0_{\mathrm{cd}}}^{\lambda\lambda'}(\vec{x},\vec{x}) g_{\lambda\lambda'} A^{\mu}_{\mathrm{m}}(\vec{x}) A_{\mathrm{n}\mu}(\vec{x}) \\
&+ f_{\mathrm{cde}} f_{\mathrm{bae}} \int d^3x A_{\mathrm{n}\nu}(\vec{x}) \overline{iD^0_{\mathrm{cd}}}^{\nu\mu}(\vec{x},\vec{x}) A_{\mathrm{m}\mu}(\vec{x}) \\
&- f_{\mathrm{cde}} f_{\mathrm{bae}} \int d^3x A_{\mathrm{m}\mu}(\vec{x}) \overline{iD^0_{\mathrm{cd}}}^{\mu\nu}(\vec{x},\vec{x}) A_{\mathrm{n}\nu}(\vec{x}) \Big\} .
\end{aligned}
\tag{3.14}
$$

Analogous to $i\tilde{D}^{0\,\mu\nu}_{\mathrm{ab}}(x,x')$ in Chapter 2, we may obtain the anomalous condensate propagator

$$
\overline{iD^0_{\mathrm{ab}}}^{\mu\nu}(x,x') = n_0\, c_{\mathrm{ab}}\, A^{*\mu}_0(\vec{x})\, A^{*\nu}_0(\vec{x}')\, e^{i\mu_0(t+t')}, \tag{3.15}
$$

which gives us finally

$$
\begin{aligned}
\Sigma^{02}_{\mathrm{mnab}}(\omega+\mu_0) = 2n_0 g^2 m_{\mathrm{ab}} \Big\{ &\int d^3x A_{\mathrm{m}\mu}(\vec{x}) A^{*\mu}_0(\vec{x}) A^{*\nu}_0(\vec{x}) A_{\mathrm{n}\nu}(\vec{x}) \\
&- \int d^3x A^{*\mu}_0(\vec{x}) A^{*}_{0\mu}(\vec{x}) A^{\nu}_{\mathrm{m}}(\vec{x}) A_{\mathrm{n}\nu}(\vec{x}) \Big\},
\end{aligned}
\tag{3.16}
$$

in terms of some colour coupling constant

$$
m_{\mathrm{ab}} = \sum_{\mathrm{cde}} f_{\mathrm{ace}}\, c_{\mathrm{cd}}\, f_{\mathrm{dbe}} = -\frac{1}{2} C_{\mathrm{A}}\, \delta_{\mathrm{ab}}, \tag{3.17}
$$

as shown explicitly in Appendix C. Examination of the various indices in equation (3.14) in the light of Figure 3.5 reveals immediately that

$$
\Sigma^{20}_{\mathrm{mnab}}(\omega+\mu_0) = \left[\Sigma^{02}_{\mathrm{mnab}}(\omega+\mu_0)\right]^* . \tag{3.18}
$$

The lowest order contribution to μ_0 may be found in an analogous manner using the formalism of Appendix B, and as shown there we obtain

$$
\mu_0 = 2n_0 g^2 C_{\mathrm{A}} N_{\mathrm{C}} \Big\{ 2 \int d^3x A^{\mu}_0(\vec{x}) A^{*}_{0\mu}(\vec{x}) A^{\nu}_{0'}(\vec{x}) A^{*}_{0'\nu}(\vec{x})
$$

$$- \int d^3x A_0^{\mu}(\vec{x}) A_{0'\mu}^{\nu}(\vec{x}) A_0^{\lambda\nu}(\vec{x}) A_{0'\nu}(\vec{x})$$

$$- \int d^3x A_0^{\nu\mu}(\vec{x}) A_{0'\mu}^{\nu}(\vec{x}) A_0^{\nu}(\vec{x}) A_{0'\nu}(\vec{x}) \Big\}, \tag{3.19}$$

where the notation 0 and 0' is used to denote the two residual m-sums which remain implicitly in this integral forms Here $N_C = 8$ is the number of possible bicolours which can be carried by the gluons, and C_A is the SU(3) constant defined above.

The evaluation of the integrals in the expressions above for the eigenstates of the M.I.T. bag is a complicated procedure, shown explicitly in Appendix D. The integrals reduce, in the end, to combinations of radial integrals over four spherical Bessel functions of varying arguments, which we are only able to evaluate numerically.

Chapter 4

Results of Numerical Calculations and Conclusions

4.1 Analysis of the Full Propagator

In order to analyse the full propagator, we make use of the fact that it may be expanded in terms of its eigenvalues and eigenstates as follows (cf. the unperturbed propagator in Appendix A)

$$i\tilde{D}'^{\mu\nu}_{ab}(\vec{x},\vec{x}',\omega) = -i\,\delta_{ab}\sum_k \frac{2\omega_k \mathcal{A}^\mu_k(\vec{x})\mathcal{A}^{*\nu}_k(\vec{x}')}{\omega^2-\omega_k^2+i0+}, \tag{4.1}$$

where $\mathcal{A}^\mu_k(\vec{x})$ denotes the eigenstate with eigenvalue ω_k. (Note the distinction between the full eigenstates $\mathcal{A}^\mu_k(\vec{x})$, and the unperturbed eigenstates $A^\mu_n(\vec{x})$.) Now the unperturbed eigenstates form a complete set (cf. equation (A.29))

$$\sum_m 2\omega^0_m A^\mu_m(\vec{x})A^{*\nu}_m(\vec{x}') = g^{\mu\nu}\,\delta^3(\vec{x}-\vec{x}'), \tag{4.2}$$

so that, using the definition (3.3) in Chapter 3, we obtain for the propagator

$$i\tilde{D}'_{mnab}(\omega) = -i\,\delta_{ab}\,2\omega^0_m 2\omega^0_n \sum_k \frac{2\omega_k \langle m|k'\rangle\langle k'|n\rangle}{\omega^2-\omega_k^2+i0+}. \tag{4.3}$$

In the previous chapter, we determined the propagator to be

$$i\tilde{D}'(+) = i\,\frac{\left[\Sigma^{-1}_{02}(+)\tilde{D}'^{-1}_{(0)}(-)\Sigma^{-1}_{11}(-)\Sigma_{02}(+)-1\right]\Sigma_{11}(-)}{\text{Denominator}}, \tag{4.4}$$

where the denominator is

$$\left[\tilde{D}'^{-1}_{(0)}(+) - \Sigma_{11}(+)\right]\left[\Sigma^{-1}_{02}(+)\tilde{D}'^{-1}_{(0)}(-)\Sigma^{-1}_{11}(-)\Sigma_{02}(+) - 1\right]\Sigma_{11}(-) - \Sigma_{20}(+)\left[\Sigma^{-1}_{11}(-)\Sigma_{02}(+)\Sigma_{11}(-)\right], \tag{4.5}$$

and where $(+)$ denotes an argument of $\omega+\mu_0$, and $(-)$ an argument of $-\omega+\mu_0$, with the matrix indices in eigen- and colour-space suppressed for clarity. Thus the poles of the propagator will be given by the values of ω for which this denominator is singular. The true eigenvalues are thus determined by requiring that

$$\det(\text{Denominator})\big|_{\omega+\mu_0=\omega_k+\mu_0} = 0. \tag{4.6}$$

We may also obtain the overlap integrals of the full eigenstates with the unperturbed eigenstates, from the appropriate residue of the propagator :

$$\begin{aligned} |\langle m\,|\,k'\rangle|^2 &= -\frac{1}{(2\omega^0_m)^2}\lim_{\omega+\mu_0\to\omega_k+\mu_0}(\omega-\omega_k)\tilde{D}'_{mmaa}(\omega+\mu_0) \\ &= -\frac{1}{(2\omega^0_m)^2}\left[\frac{\text{Numerator}\times\text{adj(Denominator)}}{\frac{\partial}{\partial\omega}\det(\text{Denominator})}\bigg|_{\omega=\omega_k}\right]_{ma;ma} \end{aligned} \tag{4.7}$$

where we take the mth diagonal element in eigenspace, the colour space being trivial. Unfortunately all information on the relative phases of these overlaps is lost. Here we use that, given any matrix $A(\omega)$ such that

$$\det A(\omega_k) = 0, \tag{4.8}$$

one may show that

$$\begin{aligned} \lim_{\omega\to\omega_k}(\omega-\omega_k)A^{-1}(\omega) &= \lim_{\omega\to\omega_k}(\omega-\omega_k)\frac{\text{adj}A(\omega)}{\det A(\omega)} \\ &= \lim_{\omega\to\omega_k}\text{adj}A(\omega)\left(\frac{\det A(\omega)-\det A(\omega_k)}{\omega-\omega_k}\right)^{-1} \\ &= \frac{\text{adj}A(\omega)}{\frac{\partial}{\partial\omega}\det A(\omega)}\bigg|_{\omega=\omega_k}. \end{aligned} \tag{4.9}$$

4.2 Parameterisation of n_0

If we look at the self-energies that contribute to the propagator, it is easily seen that they all go as $n_0 g^2$. Now, although, the variable n_0 is well defined in our model, it is not something we can easily measure. We would thus like to relate it to the QCD coupling constant g. In order to do this, we consider the usual parameterisation of gluon condensation in the vacuum, given by the expectation value

$$\bar{\mathcal{F}} = \frac{g^2}{4\pi} \sum_a \langle \hat{F}_{a\mu\nu}(0) \hat{F}^{\mu\nu}_a(0) \rangle = 0.0377\,\mathrm{GeV}^4, \tag{4.10}$$

as obtained from QCD sum-rule analysis [22]. Now we know that

$$\hat{F}^{\mu\nu}_a(x^\kappa) = \partial^\mu \hat{A}^\nu_a(x^\kappa) - \partial^\nu \hat{A}^\mu_a(x^\kappa) + g \sum_{bc} f_{abc} \hat{A}^\mu_b(x^\kappa) \hat{A}^\nu_c(x^\kappa), \tag{4.11}$$

and thus one identifies the condensation parameter as

$$\bar{\mathcal{F}} = \frac{1}{4\pi} \sum_{abcde} f_{abe} f_{ade} \langle g^4 \hat{A}^\mu_b(0) \hat{A}^\nu_c(0) \hat{A}_{d\mu}(0) \hat{A}_{e\nu}(0) \rangle. \tag{4.12}$$

Assuming that the coherent part of this last expression dominates, we find, after equating 'b' with 'd' and 'c' with 'e', that [5]

$$\bar{\mathcal{F}} = \frac{C_A}{4\pi} \frac{1}{N_C} \left[\sum_b \langle g^2 \hat{A}^\mu_b(0) \hat{A}_{b\mu}(0) \rangle \right]^2. \tag{4.13}$$

Thus

$$\sum_b \langle g^2 \hat{A}^\mu_b(0) \hat{A}_{b\mu}(0) \rangle = \left(\frac{4\pi \times 0.0377\,\mathrm{GeV}^4 N_C}{C_A} \right)^{\frac{1}{2}} = 1.124\,\mathrm{GeV}^2, \tag{4.14}$$

since $N_C = 8$ and $C_A = 3$ for SU(3). We now expand our field operators as follows

$$\hat{A}^\mu_a(x^\kappa) = \hat{A}^{0\mu}_a(x^\kappa) + \hat{A}'^\mu_a(x^\kappa), \tag{4.15}$$

where

$$\hat{A}^{\mu}_{a}(x^{\kappa}) = A^{\pi\mu}_{0}(x^{\kappa})\, c^{\dagger}_{0a} + A^{\mu}_{0}(x^{\kappa})\, c_{0a}, \tag{4.16}$$

and

$$\hat{A}'^{\mu}_{a}(x^{\kappa}) = \sum_{k\neq 0} \mathcal{A}^{\pi\mu}_{k}(x^{\kappa})\, c'^{\dagger}_{ka} + \sum_{k\neq 0} \mathcal{A}^{\mu}_{k}(x^{\kappa})\, c'_{ka}, \tag{4.17}$$

in terms of the full eigenstates $\mathcal{A}^{\mu}_{k}(x^{\kappa})$ of the problem. Note that the lowest state is equal to its non-interacting form, as the self-energies of this state are zero. Thus we have

$$\begin{aligned}\sum_{b}\langle g^2 \hat{A}^{\mu}_{b}(0)\hat{A}_{b\mu}(0)\rangle &= g^2 A^{\mu}_{0}(0)A^{*}_{0\mu}(0)\,\langle \sum_{b}(c_{0b}+c^{\dagger}_{0b})(c_{0b}+c^{\dagger}_{0b})\rangle \\ &\quad + \sum_{k\neq 0} g^2 \mathcal{A}^{\mu}_{k}(0)\mathcal{A}^{*}_{k\mu}(0)\,\langle \sum_{b} c'_{kb}c'^{\dagger}_{kb}\rangle. \end{aligned}\tag{4.18}$$

Consideration of the unperturbed eigenstates (MIT bag model states) reveals that only the L0 states are non-zero at the origin. Since our interaction conserves angular momentum, and these are the only $l = 0$ states, thus those full eigenstates that will contribute to the sum above must be super-positions of L0 states. Furthermore, since the states are energy normalised, the lowest of these states will dominate the sum. Thus we will consider only the lowest energy state, denoted by k_0, which must evolve from the lowest L0 state as the interaction is turned on. Since the mixing with higher radial excitations is also small, we may take the lowest L0 state to be the main component of the new state, so that

$$(k'_0|\mathrm{L0}) = (2\omega_{k_0} 2\omega^{0}_{\mathrm{L0}})^{-\frac{1}{2}}. \tag{4.19}$$

Now the lowest L0 eigenstate evaluated at the origin is (from the general form in equation (A.44))

$$\begin{aligned} A^{\mu}_{\mathrm{L0}}(0)A^{*}_{\mathrm{L0}\mu}(0) &= \frac{1}{R^3} j^2_0(0) Y^2_{00}(0)\left[\omega^{0}_{\mathrm{L0}} j^2_0(\omega^{0}_{\mathrm{L0}}R)\right]^{-1} \\ &= \frac{1}{4\pi R^2}\left[\omega^{0}_{\mathrm{L0}}R\, j^2_0(\omega^{0}_{\mathrm{L0}}R)\right]^{-1}. \end{aligned}\tag{4.20}$$

Thus we have for the full eigenstate, at the origin,

$$\begin{aligned} \mathcal{A}^{\mu}_{k_0}(0)\mathcal{A}^{*}_{k_0\mu}(0) &= (2\omega^0_{L0})^2(k'_0|L0)^2 A^{\mu}_{L0}(0)A^{*}_{L0\mu}(0) \\ &= \frac{1}{4\pi R^2}\left[\omega_{k_0}R j_0^2(\omega^0_{L0}R)\right]^{-1}. \end{aligned} \tag{4.21}$$

The energy of the lowest L0 state is given by (see Appendix A)

$$\omega^0_{L0}R = 4.4934, \tag{4.22}$$

and thus substituting, we find that

$$\mathcal{A}^{\mu}_{k_0}(0)\mathcal{A}^{*}_{k_0\mu}(0) = \frac{1.6863}{\omega_{k_0}R^3}. \tag{4.23}$$

Thus we have, from equation (4.18), that

$$\sum_b \langle g^2 \hat{A}^{\mu}_b(0)\hat{A}_{b\mu}(0)\rangle = g^2 N_C \frac{1.6863}{\omega_{k_0}R^3} = 1.124\,\mathrm{GeV}^2, \tag{4.24}$$

and we identify the dimensionless quantity

$$\frac{\omega_{k_0}R}{g^2} = 0.6721\,(R[\mathrm{fm}])^{-2}, \tag{4.25}$$

with the bag radius R expressed in fm. Here we have used that

$$\langle \sum_b c'_{kb}c'^{\dagger}_{kb}\rangle = N_C = 8. \tag{4.26}$$

In order to calculate the full-propagator exactly, we would need to solve the matrix problem in the full infinite set of unperturbed eigenstates. Although the freedom in the angular-momentum projection of the states allows us to represent each block of projections as a single state (that with zero projection) with no loss of generality, we still have an infinite matrix problem. Thus it is clear that we must truncate our basis and consider a finite set of states, starting with the lowest energy states first, as the energy normalisation ensures that

$n_0 g^2$	ω_{k_0}	g	n_0
0.0000	4.493	3.1011	0.00000
2.5094	5.155	3.3217	0.22743
4.6670	5.633	3.4723	0.38709
6.8285	6.129	3.6219	0.52053
8.9881	6.562	3.7477	0.63995
11.1476	6.968	3.8619	0.74746
	fm^{-1}		

Table 4.1: Sample Calculations of determining n_0 ($R = 1\mathrm{fm}$)
For each value of $n_0 g^2$, the full Hamiltonian is diagonalised to find the energy, ω_{k_0}, of the state which evolves out of the lowest L0 state. From this energy, using the prescription (4.25), we may determine the value of g to which each calculation corresponds, and hence infer the value of n_0 in each case.

these states should dominate the low energy spectrum of the full propagator. We will investigate more fully what the effect of this truncation is, later in this chapter. In order to numerically evaluate n_0 as a function of g, we now proceed by diagonalising the full Hamiltonian for various values of $n_0 g^2$. After identifying the energy, ω_{k_0}, of the state which develops out of the lowest L0 state, from the prescription (4.25) we may determine the appropriate value of g. (Note that after a level repulsion, one follows the L0 nature across into the other state.) Since we now know both $n_0 g^2$ and g, we may easily determine the corresponding n_0 value in each case. Some example calculations are shown in Table 4.1. The value of n_0 turns out to have an approximately linear dependence on g, and a linear regression analysis over the range of g for which the model holds reveals that

$$n_0(g) = 1.0130g\,(R[\mathrm{fm}])^{-3} - 3.1752\,(R[\mathrm{fm}])^{-2}, \qquad (4.27)$$

with a confidence level of over 99.9%. We then use this result to determine the true eigenenergies as functions now only of g and R. Note that, since by definition $n_0 \geq 0$, we thus have a lower bound on the QCD coupling constant

g, corresponding to the non-interacting lowest L0 state

$$g \geq 3.1345\,(R[\text{fm}])\,. \tag{4.28}$$

Note also that we are lead to a modified QCD coupling constant that *scales as the bag radius* R, and that, correspondingly the value of n_0 scales as $1/R^2$, so that the interaction strength $n_0 g^2$ is independent of R. Thus the true eigenenergies ω_k will scale like $1/R$, just as the non-interacting eigenenergies ω_n^0 do. The scaling of the coupling constant is a result of the implicit infra-red renormalisation that occurs as a result of the finite size of the system we are considering. All our results are plotted in terms of the invariant $n_0 g^2$, in order to resolve any ambiguity such a scaling might otherwise introduce.

4.3 Basis Truncation Effects

As discussed above, we calculate the true eigenstates with respect to a finite basis of unperturbed states, instead of the full infinite set. We must, therefore, investigate the effect of this finiteness of the basis on our results. Fortunately our interaction breaks up into sub-blocks, allowing us to consider the effect of truncation on any one of these blocks separately, and hence to infer the total effect. As shown in section D.4 of appendix D, each of the transverse magnetic states, together with all its radial excitations, forms a closed sub-block with respect to our interaction.

Consider thus the sub-block of M1 states shown in Table 4.2, calculated for some arbitrary value of $n_0 g^2$ using varying numbers of these states as a basis. The lowest state moves in total as follows

$$\omega_{k_{\text{M1}}} : (2.5694 \rightarrow 2.5653)\ \frac{1}{R}, \tag{4.29}$$

as the rest of the basis is included. We see clearly that the truncation effects are small, being of the order of 0.2%. Furthermore, we need only include one radial excitation over and above those we wish to determine for the effect to become totally negligible. Note also how only the lowest radial excitation

Basis Size	(2.7437)	(6.1166)	(9.3166)	(12.4859)	(15.6439)
1	2.5694				
2	2.5653	6.1380			
3	2.5653	6.1364	9.3391		
4	2.5653	6.1364	9.3383	12.5047	
5	2.5653	6.1364	9.3383	12.5042	15.6595
All Energies in units $1/R$					

Table 4.2: Dependence of M1 Eigenenergies on Calculational Basis Size
The energies of the first five M1 radial excitations for some arbitrary value of n_0g^2 are shown in the horizontal direction, labelled at the top by their corresponding unperturbed energies. In the vertical direction the number of M1 states included in the basis, upon which the Hamiltonian was diagonalised, is increasing. In each case, the number of eigenvalues produced is equal to the actual basis size.

moves to lower energies. This is true for all the M states and indeed for the E states too, though the lowest radial excitation rises along with the rest in the case of the L states as the interaction is turned on (see Figure 4.1). The effects of truncation are, however, consistently negligible for the second to last radial excitation included in the basis in each of these three cases.

4.4 Gluonium in the MIT Bag Model

If we consider the full single particle eigenenergies as a function of the QCD coupling constant, we find that the second lowest state moves steadily downwards till eventually it is lower than our original lowest state, which has a constant energy in our formalism. Clearly the whole model would break down at such a point, and hence we obtain an effective upper bound on the value of g, below which our picture remains valid. This limiting value is depicted by the dashed vertical line in Figure 4.1 where the full eigenenergies are plotted against n_0g^2 over the appropriate range. Disentangling the value of g from the limit shown in the figure, we find that g should lie somewhere in the range (

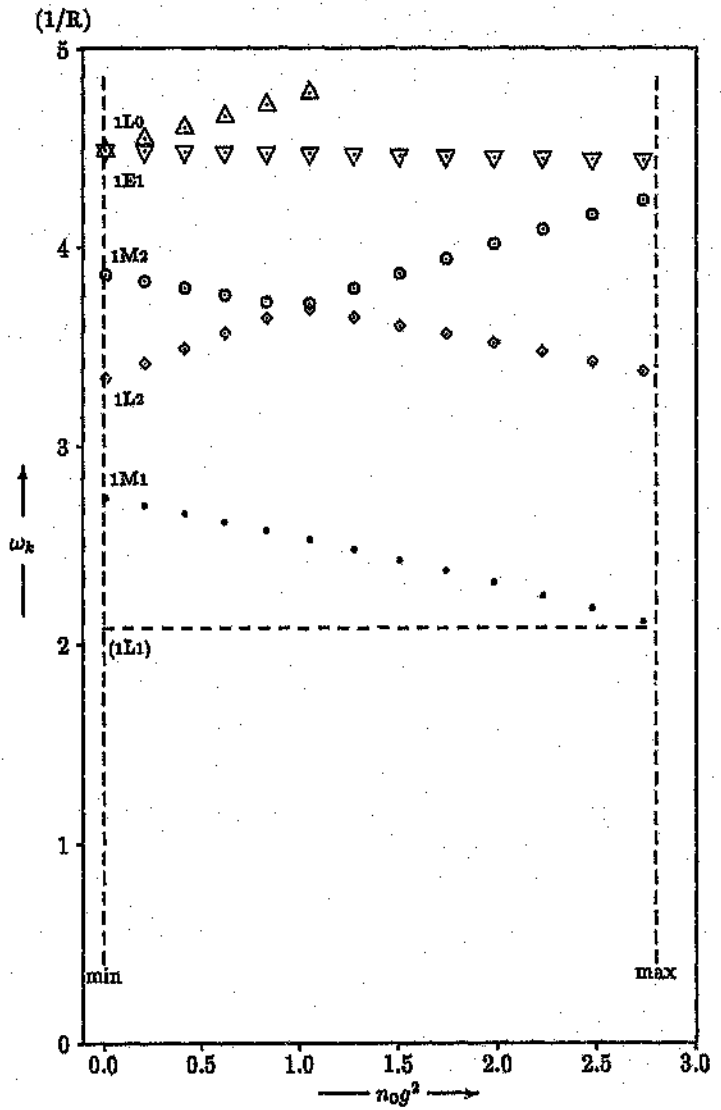


Figure 4.1: Full Single Particle Eigenenergies as functions of n_0g^2
The single particle states are identified by the quantum numbers $i\mathbf{P}j$, where i is the radial, $\mathbf{P}$ the polarisation ($\mathbf{L}$, $\mathbf{M}$ or $\mathbf{E}$), and j the spin quantum numbers of the state, respectively. The lowest state condensate is included to show the critical crossover where the model breaks down, at $n_0g^2 = 2.803$.

using also the relation (4.28))

$$g : (3.134 \rightarrow 3.344)\ (R[\text{fm}]). \tag{4.30}$$

Now the mass of any MIT bag state is found by minimising with respect to the bag radius R, the following energy [10]

$$m(R) = \frac{4}{3}\pi B R^3 + \sum_{k\neq 0} n_k \frac{\omega_k}{R}, \tag{4.31}$$

consisting of a bag pressure, which acts so as to collapse the bag, and of the energy of the particles making up the bag state, defined in terms of the occupation numbers n_k, and eigenenergies ω_k (expressed in units $\frac{1}{R}$) of the true eigenstates k. The particles here could be quarks or gluons, depending on the bag state one wishes to calculate. Since the condensate in the lowest L1 state has, by construction, the quantum numbers of our new single particle vacuum, states of gluonium in this picture will consist of two or more higher state gluons in a colour singlet, coupled to the appropriate spin and parity. Note that although the lowest state has an occupation number n_0, and thus would seem to contribute to the energy, it forms part of the interacting vacuum, and thus should be left out.

Now, in general, to minimise $m(R)$ with respect to R, we require, using the usual derivative prescription for a minimum, that

$$4\pi B R_{\min}^2 - \sum_{k\neq 0} n_k \frac{\omega_k}{R_{\min}^2} = 0, \tag{4.32}$$

i.e.

$$R_{\min} = \left(\sum_{k\neq 0} n_k \frac{\omega_k}{4\pi B} \right)^{\frac{1}{4}}. \tag{4.33}$$

Thus the energy of the bag state is given by

$$m(R_{\min}) = \frac{4}{3}(4\pi B)^{\frac{1}{4}} \left(\sum_{k\neq 0} n_k \omega_k \right)^{\frac{3}{4}}. \tag{4.34}$$

n_0g^2	ω_{1M_1}	$\sum_{k\neq 0} n_k\omega_k$	R_{min}	$m(R_{min})$
0.000	2.7415	5.4830	1.0985	1313.3
2.803	2.0944	4.1888	1.0270	1073.2
	$1/R$	$1/R$	fm	MeV

Table 4.3: Energy Range of the $(1M1)^2$ $0^+, 2^+$ Gluonium State

n_0g^2	ω_{1M_1}	ω_{1M_2}	$\sum_{k\neq 0} n_k\omega_k$	R_{min}	$m(R_{min})$
0.000	2.7415	3.8634	6.6099	1.1570	1510.9
2.803	2.0944	3.3647	5.4591	1.0973	1309.0
	$1/R$	$1/R$	$1/R$	fm	MeV

Table 4.4: Energy Range of the $[(1M1)(1M2)]$ $1^-, 2^-, 3^-$ Gluonium State

These two tables show the steps taken in determining the radius and energy of the two lowest gluonium states at both the minimum and maximum allowed values of the interaction strength n_0g^2. The energies are calculated using the method detailed in equation (4.34).

Using the single particle gluon spectrum as a function of n_0g^2, we may now calculate the energies of the low lying gluonium state lso as functions of n_0g^2. The results are summarised in Figure 4.2. The lowest state is a $(1M1)^2$ state, which has spin and parity 0^+ or 2^+, the third possibility 1^+ being ruled since the gluons obey Bose statistics. The energy range of this state, together with the range of equilibrated bag radii, is summarised in Table 4.3. Similar results for the $[(1M1)(1M2)]$, 1^-, 2^-, 3^- state appear in Table 4.4. The value of B we have used in our calculations is defined by [10]

$$B^{\frac{1}{4}} = 146\,\text{MeV}, \tag{4.35}$$

4.5 Conclusions

We have considered the interaction of QCD gauge particles within a finite bag, assuming that they will form a condensate in the lowest single particle state

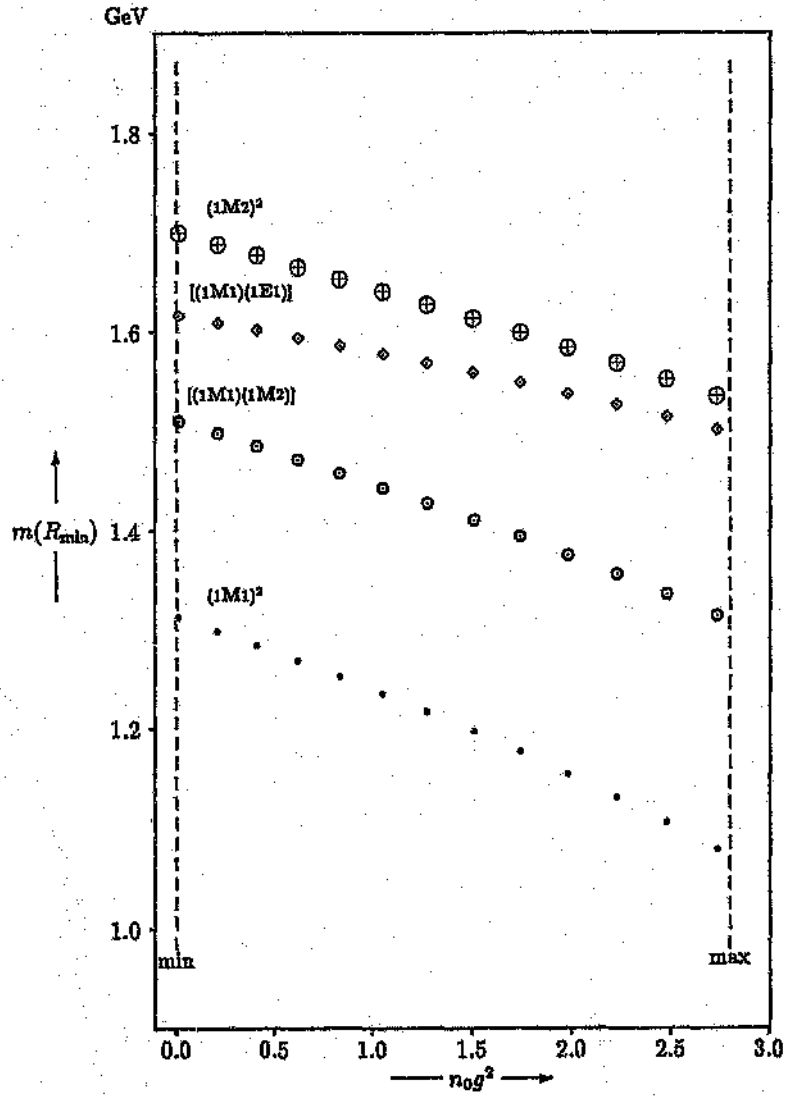


Figure 4.2: Low-lying States of Gluonium as a function of n_0g^2

The lowest-lying gluonium states consist of two gluons coupled to a colour singlet. Spin quantum numbers are suppressed as all the members of a multiplet will be degenerate. Each gluonium state is labelled by the two occupied single particle states.

of the system. This many-body ground state or condensate is then taken to be the true vacuum of QCD, and all physical gluons are considered as single particle excitations thereof. Now the choice of interaction for such a self-consistent problem will have a direct effect on the final end result. Bearing this in mind, we have chosen to consider the exact interaction found in the QCD Lagrangian rather than one of the many effective gluon-gluon interactions that abound in the literature. Now it is clear that, in working within a bag model, we have already built in some of the interaction in terms of explicit confinement. Thus to work with the full interaction given in the Lagrangian would involve a certain degree of overkill. For practical reasons it is clear that a number conserving term would be much easier to handle, as this enables us to use a fixed n-body approach. In this spirit we have considered the four-gluon point interaction term of the Lagrangian only. Besides it is the three-gluon term, which does not conserve gluon-number, that is believed to be responsible for the confinement effect.

Since we wish to consider the many-body condensate as the QCD vacuum, we have attempted to fix the parameters of our model by considering the value of vacuum parameterisations as evaluated on our ground state. Our description in terms of a finite bag introduces an intrinsic unit of length into the problem, and, as a result, the connection between the value of the QCD coupling constant g within the bag and the equivalent coupling strength in free space becomes somewhat blurred. We chose to work within a bag model as it immediately resolves the problem of infra-red divergences, which would otherwise wreak havoc with a lowest-state condensate model. In so doing, we have, in principle, renormalised the problem, and the scaling of g appears as a result. We present all results in terms of the effective coupling $n_0 g^2$, which is free of any such scaling.

Over the range for which our model is valid, we find that the lowest gluonium state (formed by a two particle excitation of our new vacuum state) lies at energies that range from 1313 MeV down to 1073 MeV as the effective interaction increases. The range of energies of the other low-lying states is

	$(1M1)^2$	$[(1M1)(1M2)]$	$[(1M1)(1E1)]$	$(1M2)^2$	
Spin/Parity	$0^+, 2^+$	$1^-, 2^-, 3^-$	$0^-, 1^-, 2^-$	$0^+, 2^+, 4^+$	J^P
Energy Range	1073-1313	1309-1511	1498-1617	1531-1700	MeV
Experimental	$f_0(1240)$?		$\eta(1440)$?	$f_0(1599)$?	(MeV)

Table 4.5: Energies and Radii of the Low-lying Gluonium States
We present the spin and parity quantum numbers for each of the gluonium states depicted in Figure 4.2, along with the energy range of each state. Possible experimental candidates are identified after eliminating all quark-antiquark states from the spectrum. See also Figure 4.3 for a visual summary of this table.

shown in Table 4.5. Although there is no definite gluonium signature, some possible experimental candidates lying in the appropriate range may be inferred by eliminating those states which represent quark-antiquark pairs from the spectrum [20], and identifying the spin and parity quantum numbers of the remaining states. See Figure 4.3 for a summary of the results.

The model we have developed is an extension of the familiar M.I.T. bag model of DeGrand, Jaffe, Johnson and Kiskis [10], as extended by Jaffe and Johnson [19], but using more complicated self-consistent single particle states as a basis. The results turn out to be very similar in nature to the results of a straightforward bag model calculation, evolving out of such results as the effective interaction is turned on. We have neglected the centre of mass term of [10], and after the addition of this term, which would shift our values down by 330 to 370 MeV, we find results in close agreement with Jaffe and Johnson. Our results are also in close agreement with those of Thoma, Lüst and Mang [23] who obtain an energy of 1050 ± 120 MeV for the degenerate 0^+, 2^+ lowest state of gluonium. We also find good quantitative agreement with the results of Carlson, Hansson and Peterson [4] and Hess and Viollier [17].

One of the features of this model is that it introduces a measure of the full self-interaction of the gluons in a simple way. It is reasonably easy to calculate, with few approximations or assumptions necessary to produce results that are not only self-consistent, but also in good agreement with other predictions.

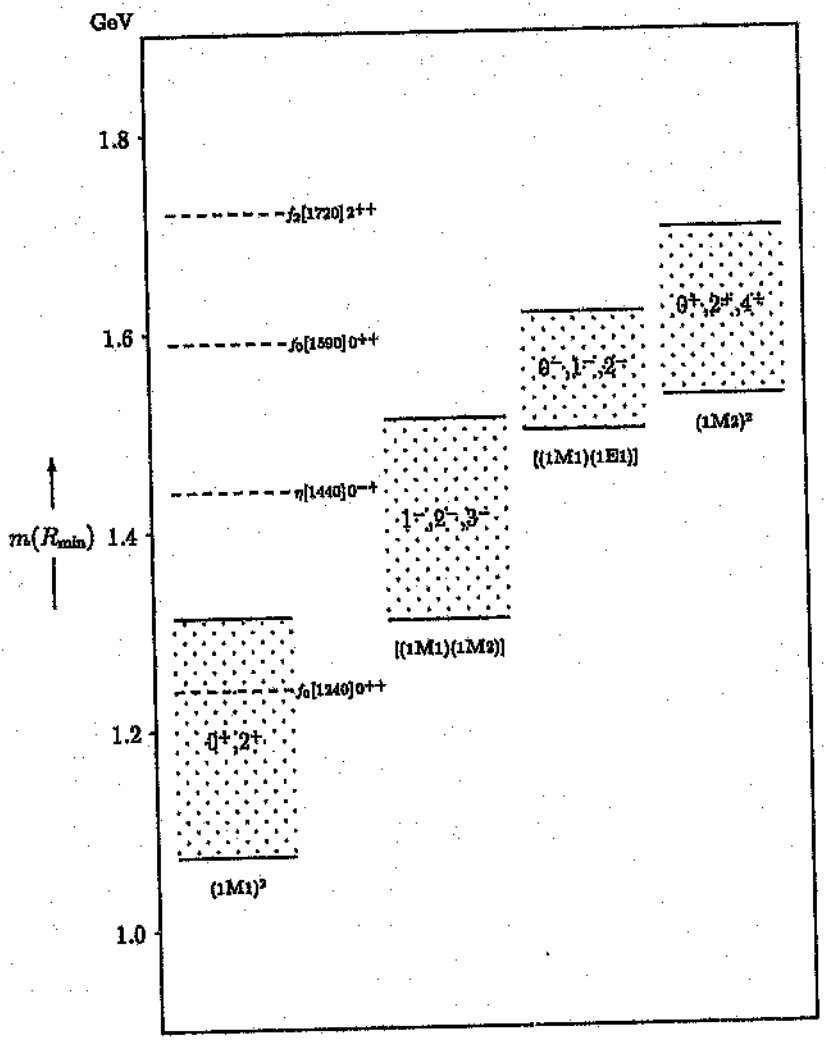


Figure 4.3: Comparison of Theoretical and Experimental Gluonium Energies
Theoretical gluonium energies are plotted, showing the possible range that the model predicts for each state. The experimental candidates for gluonium are also shown and are in good agreement with the theoretical results. The states are labelled as before, only we now include spin/parity J^{PC}.

No renormalisation of the self-energies is required either, as the diagrammatic sums found in this model converge absolutely. The biggest benefit of this model is that it provides us with a way to introduce some measure of vacuum correlation effects back into the simple bag model, without seriously increasing the technical difficulties.

Appendix A

Non-Interacting Gauge Fields within the M.I.T. bag

A.1 The Equations of Motion for Free Non-Interacting Gauge Fields

The Lagrange density for non-interacting QCD gauge fields is given in general by [24]

$$\hat{\mathcal{L}}(x^\kappa) = -\frac{1}{4}\sum_a \hat{F}^{\mu\nu}_a(x^\kappa)\hat{F}_{a\mu\nu}(x^\kappa) - \frac{\lambda}{2}\sum_a \left(\partial_\mu \hat{A}^\mu_a(x^\kappa)\right)^2, \qquad \text{(A.1)}$$

where

$$\hat{F}^{\mu\nu}_a(x^\kappa) = \partial^\mu \hat{A}^\nu_a(x^\kappa) - \partial^\nu \hat{A}^\mu_a(x^\kappa). \qquad \text{(A.2)}$$

The Euler-Lagrange relations state that, given a Lagrange density $\hat{\mathcal{L}}(x^\kappa)$ dependent on co-ordinates $q_i(x^\kappa)$ and their derivatives $q_{i,\mu}$, the equation of motion for the co-ordinate q_i is, formally,

$$\frac{\partial}{\partial x^\mu}\frac{\delta\hat{\mathcal{L}}(x^\kappa)}{\delta q_{i,\mu}} = \frac{\delta\hat{\mathcal{L}}(x^\kappa)}{\delta q_i}. \qquad \text{(A.3)}$$

In our case, the generalised co-ordinates are the fields $\hat{A}_{a\mu}(x^\kappa)$. Since our Lagrange density is dependent only upon derivatives of these fields, and not explicitly on the fields themselves, the right hand side of the above expression

evaluates to zero. The classical equation of motion for each field is, as a result, of the form

$$\frac{\partial}{\partial x^\mu}\frac{\delta\hat{\mathcal{L}}(x^\kappa)}{\delta(\partial_\mu\hat{A}_{a\nu}(x^\kappa))}=0. \tag{A.4}$$

Introducing the explicit form of our Lagrange density this becomes

$$\partial_\mu\left[-\left(\partial^\mu\hat{A}^\nu_a(x^\kappa)-\partial^\nu\hat{A}^\mu_a(x^\kappa)\right)-\lambda\partial_\rho\hat{A}^\rho_a(x^\kappa)\,g^{\mu\nu}\right]=0\,, \tag{A.5}$$

i.e.

$$\partial_\mu\partial^\mu\hat{A}^\nu_a(x^\kappa)-(1-\lambda)\,\partial^\nu\partial_\mu\hat{A}^\mu_a(x^\kappa)=0. \tag{A.6}$$

Thus in the Fermi-Feynman gauge ($\lambda=1$), in which we work throughout this thesis, the equation of motion reduces to a simple Helmholtz equation in each of the fields $\hat{A}^\mu_a(x^\kappa)$,

$$\Box\hat{A}^\mu_a(x^\kappa)=0. \tag{A.7}$$

A.2 Second Quantization of Non-Interacting Free Gauge Fields

We consider the four-potential of a free travelling chromomagnetic field. Now since there are no interactions, and we are in free space where there are no colour charges, the scalar potential is zero, and we need only consider the vector potential $\vec{A}_a(\vec{r},t)$, which will then obey

$$\partial_t^2\vec{A}_a-\vec{\nabla}^2\vec{A}_a=0\quad,\quad\vec{\nabla}.\vec{A}_a=0. \tag{A.8}$$

Expanding now in terms of the discrete states of a finite box (volume $V=1$ for simplicity), we have

$$\vec{A}_a(\vec{r},t)=\sum_k\vec{a}^a_k e^{i\vec{k}.\vec{r}}+\sum_k\vec{a}^{*a}_k e^{-i\vec{k}.\vec{r}}, \tag{A.9}$$

where the coefficients $\vec{a}^a_k$ are functions of time

$$\vec{a}^a_k\sim e^{-i\omega_k t}\quad,\quad\omega_k=|\,\vec{k}\,|\,. \tag{A.10}$$

From equation (A.8) we see that $\vec{a}^{\,a}_k$ must satisfy

$$\vec{a}^{\,a}_k.\vec{k} = 0. \tag{A.11}$$

Before we quantize, it is necessary first to transform to suitable canonical variables [3]

$$\begin{aligned} \vec{Q}^a_k &= \vec{a}^{\,a}_k + \vec{a}^{*a}_k \\ \vec{P}^a_k &= -i\omega_k\left(\vec{a}^{\,a}_k - \vec{a}^{*a}_k\right) = \partial_t \vec{Q}^a_k. \end{aligned} \tag{A.12}$$

Expandir in terms of these new variables, we have

$$\vec{A}_a(\vec{r},t) = \sum_k \left\{\vec{Q}^a_k \cos(\vec{k}.\vec{r}) - \frac{1}{\omega_k}\vec{P}^a_k \sin(\vec{k}.\vec{r})\right\}. \tag{A.13}$$

The fields generated by the potential are thus

$$\begin{aligned} \vec{E}^a &= -\partial_t \vec{A}^a = -\sum_k \left\{\vec{P}^a_k \cos(\vec{k}.\vec{r}) + \omega_k \vec{Q}^a_k \sin(\vec{k}.\vec{r})\right\}, \\ \vec{B}^a &= -\vec{\nabla}\times\vec{A}^a = -\sum_k \vec{k}\times\left\{\vec{Q}^a_k \sin(\vec{k}.\vec{r}) + \frac{1}{\omega_k}\vec{P}^a_k \cos(\vec{k}.\vec{r})\right\}, \end{aligned} \tag{A.14}$$

and so the Hamiltonian $\mathcal{H}$ is, classically,

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}\sum_a \int d^3r \left(\vec{E}^{a\,2} + \vec{B}^{a\,2}\right) \\ &= \sum_a \int d^3r \left[\sum_k \left\{\vec{P}^a_k \cos(\vec{k}.\vec{r}) + \omega_k \vec{Q}^a_k \sin(\vec{k}.\vec{r})\right\}\right]^2 \\ &= \frac{1}{2}\sum_{a,k}\left(\vec{P}^{a2}_k + \omega_k^2 \vec{Q}^{a2}_k\right), \end{aligned} \tag{A.15}$$

since

$$\int d^3r \cos(\vec{k}.\vec{r})\cos(\vec{k}'.\vec{r}) = \frac{1}{2}\delta^3(\vec{k}-\vec{k}'),$$

$$\int d^3r \sin(\vec{k}.\vec{r}) \sin(\vec{k}'.\vec{r}) = \frac{1}{2}\delta^3(\vec{k}-\vec{k}'),$$
$$\int d^3x \cos(\vec{k}.\vec{r}) \sin(\vec{k}'.\vec{r}) = 0.$$

We now expand both $\vec{Q}^{\rm a}_k$ and $\vec{P}^{\rm a}_k$ into two perpendicular transverse polarisations, so that

$$\mathcal{H} = \frac{1}{2}\sum_{{\rm a},k\alpha}\left(P^{{\rm a}2}_{k\alpha} + \omega_k^2 Q^{{\rm a}2}_{k\alpha}\right). \tag{A.16}$$

Quantization is now achieved by requiring that

$$[\hat{P}^{\rm a}_{k\alpha}, \hat{Q}^{\rm b}_{k'\beta}] = -i\,\delta^3(\vec{k}-\vec{k}')\,\delta_{\alpha\beta}\,\delta_{\rm ab}. \tag{A.17}$$

The Hamiltonian may be generalised directly from the classical form, as there are no difficult product terms, such as $\hat{Q}^{\rm a}_{k\alpha}\hat{P}^{\rm a}_{k\alpha}$, present, due to their integrand which goes as $\sin(\vec{k}.\vec{r})\cos(\vec{k}'.\vec{r})$ and thus yields zero upon integration. Thus

$$\hat{\mathcal{H}} = \frac{1}{2}\sum_{{\rm a},k\alpha}\left(\hat{P}^{{\rm a}2}_{k\alpha} + \omega_k^2 \hat{Q}^{{\rm a}2}_{k\alpha}\right). \tag{A.18}$$

The eigenvalues of this Hamiltonian are easily found, as the Hamiltonian is identical to that of a system of independent linear oscillators, and thus by analogy we know that

$$E = \sum_{{\rm a},k\alpha}\left(N^{\rm a}_{k\alpha} + \frac{1}{2}\right)\omega_k, \tag{A.19}$$

where the $N^{\rm a}_{k\alpha}$ are integers. We are also led by this analogy to the familiar raising and lowering operators

$$\begin{aligned} {\rm c}^\dagger_{k\alpha{\rm a}} &= \frac{1}{\sqrt{2\omega_k}}\left(\omega_k\hat{Q}^{\rm a}_{k\alpha} - i\hat{P}^{\rm a}_{k\alpha}\right) \\ {\rm c}_{k\alpha{\rm a}} &= \frac{1}{\sqrt{2\omega_k}}\left(\omega_k\hat{Q}^{\rm a}_{k\alpha} + i\hat{P}^{\rm a}_{k\alpha}\right), \end{aligned} \tag{A.20}$$

with commutator

$$[c_{k\alpha a}, c^\dagger_{k'\beta b}] = \frac{1}{\sqrt{2\omega_k 2\omega_{k'}}}\left\{\omega_k\omega_{k'}[\hat{Q}^a_{k\alpha}, \hat{Q}^b_{k'\beta}] + i\omega_{k'}[\hat{P}^a_{k\alpha}, \hat{Q}^b_{k'\beta}] - i\omega_k[\hat{Q}^a_{k\alpha}, \hat{P}^b_{k'\beta}] + [\hat{P}^a_{k\alpha}, \hat{P}^b_{k'\beta}]\right\}, \tag{A.21}$$

which, using the original quantization requirement (A.17), gives

$$[c_{k\alpha a}, c^\dagger_{k'\beta b}] = \frac{1}{\sqrt{2\omega_k 2\omega_{k'}}}\{\omega_k + \omega_{k'}\}\,\delta^3(\vec{k}-\vec{k}')\,\delta_{\alpha\beta}\,\delta_{ab} = \delta^3(\vec{k}-\vec{k}')\,\delta_{\alpha\beta}\,\delta_{ab}. \tag{A.22}$$

The non-zero matrix elements of these operators are

$$\langle N^a_{k\alpha}-1 \mid c_{k\alpha a} \mid N^a_{k\alpha}\rangle = \langle N^a_{k\alpha} \mid c^\dagger_{k\alpha a} \mid N^a_{k\alpha}-1\rangle = \sqrt{N^a_{k\alpha}}. \tag{A.23}$$

Expressing these operators in terms of our original expansion, we find

$$c_{k\alpha a} \sim \frac{1}{\sqrt{2\omega_k}}\left(\omega_k\,[a^a_k + a^{*a}_k] + i(-i\omega_k)\,[a^a_k - a^{*a}_k]\right), \tag{A.24}$$

i.e.

$$c_{k\alpha a} \sim \sqrt{2\omega_k}\,a^a_k \quad \text{and} \quad c^\dagger_{k\alpha a} \sim \sqrt{2\omega_k}\,a^{*a}_k, \tag{A.25}$$

which leads us to the quantized expression for $\hat{A}(\vec{r},t)$

$$\hat{A}(\vec{r},t) = \sum_{a,k\alpha}\vec{A}^*_{k\alpha}(\vec{r})\,c^\dagger_{k\alpha a} + \sum_{a,k\alpha}\vec{A}_{k\alpha}(\vec{r})\,c_{k\alpha a}, \tag{A.26}$$

where

$$\vec{A}_{k\alpha}(\vec{r}) = \frac{1}{\sqrt{2\omega_k}}\hat{n}_\alpha\,e^{i\vec{k}\cdot\vec{r}}, \tag{A.27}$$

so that the orthonormality condition on the eigenfunctions takes the form

$$\int d^3r\,\vec{A}^*_{k\alpha}(\vec{r}).\,\vec{A}_{k'\beta}(\vec{r}) = \frac{1}{2\omega_k}\,\delta^3(\vec{k}-\vec{k}')\,\delta_{\alpha\beta}, \tag{A.28}$$

and the completeness relation is

$$\sum_\alpha \int d^3k 2\omega_k A^i_{k\alpha}(\vec{r})A^{i*}_{k\alpha}(\vec{r}') = \delta^3(\vec{r}-\vec{r}')\,\delta^{ij}. \tag{A.29}$$

A.3 Quantization within an M.I.T. Bag

Due to the spherical nature of the M.I.T. bag we are led to solve the isotropic Helmholtz Equation

$$\left(\vec{\nabla}^2 + k^2\right)\begin{pmatrix}\varphi(\vec{r})\\ \vec{A}(\vec{r})\end{pmatrix} = 0, \tag{A.30}$$

in spherical co-ordinates. Assuming that the functions are separable in these co-ordinates

$$\begin{aligned}\varphi(\vec{r}) &= f(r)g(\hat{r}),\\ \vec{A}(\vec{r}) &= \vec{\mathcal{O}}\,f'(r)g'(\hat{r}),\end{aligned} \tag{A.31}$$

where $\vec{\mathcal{O}}$ is any vector operator such that

$$[\,\vec{\mathcal{O}}, \vec{\nabla}^2\,] = 0, \tag{A.32}$$

we arrive at radial and angular component equations that go as

$$\left\{\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \left[r^2k^2 - j(j+1)\right]\right\}\begin{pmatrix}\varphi(\vec{r})\\ \vec{A}(\vec{r})\end{pmatrix} = 0, \tag{A.33}$$

$$\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + j(j+1)\right\}\begin{pmatrix}\varphi(\vec{r})\\ \vec{A}(\vec{r})\end{pmatrix} = 0. \tag{A.34}$$

We recognise the radial equation (A.33) as that of a spherical Bessel function, and hence the radial dependence takes the form [1]

$$f(r) = \mathrm{j}_j(kr),$$

as we need a function that is finite everywhere within the bag. The angular dependence becomes obvious when we write equation (A.34) in terms of the angular momentum operator $\vec{L}$

$$\left\{\vec{L}^2 - j(j+1)\right\}\begin{pmatrix}\varphi(\vec{r})\\ \vec{A}(\vec{r})\end{pmatrix} = 0, \tag{A.35}$$

which clearly is solved by spherical harmonics and vector spherical harmonics. As we will work in the spatial representation the basic functions are thus

$$g(\hat{r}) = Y_{jm}(\hat{r}).$$

The most general representation of our eigenfunctions is thus

$$\begin{aligned} \varphi(\vec{r}) &= \mathcal{N} \mathrm{j}_j(kr) Y_{jm}(\hat{r}), \\ \vec{A}(\vec{r}) &= \mathcal{N}' \vec{\mathcal{O}} \mathrm{j}_j(kr) Y_{jm}(\hat{r}), \end{aligned} \tag{A.36}$$

with the further restriction that $\varphi(\vec{r})$ and $\vec{A}(\vec{r})$ must be related by the Lorentz gauge condition

$$\frac{\partial \varphi(\vec{r})}{\partial t} + \vec{\nabla}.\vec{A}(\vec{r}) = 0. \tag{A.37}$$

In order to determine three independent solutions, we start from the principle that any vector field $\vec{A}(\vec{r})$ may be split into two pieces

$$\vec{A}(\vec{r}) = \vec{A}_1(\vec{r}) + \vec{A}_2(\vec{r}), \tag{A.38}$$

where $\vec{A}_1(\vec{r})$ is the transverse part of the field characterised by

$$\vec{\nabla}.\vec{A}_1(\vec{r}) = 0, \tag{A.39}$$

and $\vec{A}_2(\vec{r})$ is the longitudinal part of the field which may always be written in the form

$$\vec{A}_2(\vec{r}) = \vec{\nabla} f(\vec{r}), \tag{A.40}$$

and where clearly $\vec{\nabla} \times \vec{A}_2(\vec{r}) = 0$. This follows from the guaranteed solubility of the linear second order differential equation [3]

$$\nabla^2 f(\vec{r}) = \vec{\nabla}.\vec{A}(\vec{r}). \tag{A.41}$$

Since $[\vec{\nabla}, \vec{\nabla}^2] = 0$, $f(\vec{r})$ must also obey the isotropic Helmholtz equation

$$\left(\vec{\nabla}^2 + k^2\right) f(\vec{r}) = 0, \tag{A.42}$$

and, using the Lorentz condition (A.37), it is clear that we must take

$$\vec{f}(\vec{r}) = \frac{1}{ik}\varphi(\vec{r}), \tag{A.43}$$

where $\varphi(\vec{r})$ is the corresponding scalar potential.

Thus we have derived the so called *Longitudinal-Scalar*, or L, mode (where we introduce the $R^{\frac{3}{2}}$ factor explicitly, for convenience)

$$\begin{aligned} \varphi_n^{\mathrm{L}}(\vec{r}) &= \frac{\mathcal{N}_j^{\mathrm{L}}}{R^{\frac{3}{2}}} \mathrm{j}_j(k_j^{\mathrm{L}} r) Y_{jm}(\hat{r}), \\ \vec{A}_n^{\mathrm{L}}(\vec{r}) &= \frac{\mathcal{N}_j^{\mathrm{L}}}{R^{\frac{3}{2}}} \frac{1}{ik_j^{\mathrm{L}}} \vec{\nabla} \mathrm{j}_j(k_j^{\mathrm{L}} r) Y_{jm}(\hat{r}), \end{aligned} \tag{A.44}$$

for $j \geq 0$, with parity $(-)^j$.

For the transverse part, we will need a further two independent modes. Since the angular momentum operator $\vec{L}$ is by definition transverse, we are led to the *Transverse Magnetic*, or M, mode

$$\vec{A}_n^{\mathrm{M}}(\vec{r}) = \frac{\mathcal{N}_j^{\mathrm{M}}}{R^{\frac{3}{2}}} \frac{\vec{L}}{\sqrt{j(j+1)}} \mathrm{j}_j(k_j^{\mathrm{M}} r) Y_{jm}(\hat{r}), \tag{A.45}$$

for $j > 0$, with parity $(-)^{j+1}$. The other transverse mode, found to be the curl of this expression, are the so called *Transverse Electric*, or E, mode

$$\vec{A}_n^{\mathrm{E}}(\vec{r}) = \frac{\mathcal{N}_j^{\mathrm{E}}}{R^{\frac{3}{2}}} \frac{1}{ik_j^{\mathrm{E}}} \vec{\nabla} \times \frac{\vec{L}}{\sqrt{j(j+1)}} \mathrm{j}_j(k_j^{\mathrm{E}} r) Y_{jm}(\hat{r}), \tag{A.46}$$

again for $j > 0$, but with parity $(-)^j$. Note that since both the Magnetic and Electric modes are transverse, the corresponding scalar potentials $\varphi_j^{\mathrm{M}}(\vec{r})$ and $\varphi_j^{\mathrm{E}}(\vec{r})$ are zero.

The M.I.T. bag boundary conditions correspond to the requirement that no flux pass through a spherical surface located at $r = R$. The constraints on

	Longitudinal		*Magnetic*		*Electric*	
$j=0$	4.493	7.725				
$j=1$	2.082	5.940	2.744	6.117	4.493	7.725
$j=2$	3.342	7.290	3.870	7.443	5.763	9.095
$j=3$	4.514	8.584	4.973	8.722	6.988	10.417
$j=4$	5.647	9.840	6.062	9.968	8.182	11.705
	All energies specified in units $1/R$					

Table A.1: The Low-Lying Gluon Eigenstates of the M.I.T. Bag Model
The first two radial excitations for each angular momentum j are shown for the three possible polarisations L,M and E. Note how only the Longitudinal mode has an angular momentum $j=0$ mode.

the potentials are thus [6, 7]

$$\begin{aligned} \hat{r}.\vec{\nabla}\varphi(\vec{r})\Big|_{r=R} &= 0, \\ \hat{r}\times\left(\vec{\nabla}\times\vec{A}(\vec{r})\right)\Big|_{r=R} &= 0. \end{aligned} \tag{A.47}$$

This gives rise to eigenvalue equations for each of the modes as follows

$$\frac{d}{dr}\mathrm{j}_j(k_j^{\mathrm{L}}r)\Big|_{r=R} = 0, \tag{A.48}$$

$$\frac{d}{dr}\left[r\mathrm{j}_j(k_j^{\mathrm{M}}r)\right]\Big|_{r=R} = 0, \tag{A.49}$$

$$\mathrm{j}_j(k_j^{\mathrm{E}}r)\Big|_{r=R} = 0, \tag{A.50}$$

each of which, due to the periodic nature of the spherical Bessel functions, gives rise to an infinite series of radial excitations for each j and m. The first few low-lying states are shown in Table A.1, along with the first radial excitation of each. The states are now normalised by requiring that

$$\begin{aligned} \int d^3r\,|\,\varphi(\vec{r})\,|^2 &= \frac{1}{2k}, \\ \int d^3r\vec{A}^*(\vec{r}).\vec{A}(\vec{r}) &= \frac{1}{2k}. \end{aligned} \tag{A.51}$$

We find that the required normalisation constants are then

$$
\begin{aligned}
\mathcal{N}_j^{\mathrm{L}} &= \left\{ k_j^{\mathrm{L}} \mathrm{j}_j^2(k_j^{\mathrm{L}} R) \left[1 - \frac{j(j+1)}{(k_j^{\mathrm{L}} R)^2} \right] \right\}^{-\frac{1}{2}}, \\
\mathcal{N}_j^{\mathrm{M}} &= \left\{ k_j^{\mathrm{M}} \mathrm{j}_j^2(k_j^{\mathrm{M}} R) \left[1 - \frac{j(j+1)}{(k_j^{\mathrm{M}} R)^2} \right] \right\}^{-\frac{1}{2}}, \\
\mathcal{N}_j^{\mathrm{E}} &= \left\{ k_j^{\mathrm{E}} \mathrm{j}_{j+1}^2(k_j^{\mathrm{E}} R) \right\}^{-\frac{1}{2}}.
\end{aligned}
\tag{A.52}
$$

A.4 Second Quantization in the M.I.T. Bag

A.4.1 Notation

The gauge particle of QCD is both massless and equal to its antiparticle, thus the fields describing it must be hermitian. Expanding the field operator in terms of the eigenstates of the M.I.T. bag of section A.3 above, using the second quantisation of section A.2, we find

$$
\hat{A}_{\mathrm{a}}^{\mu}(x^{\kappa}) = \sum_n A_n^{*\mu}(x^{\kappa})\, \mathrm{c}_{n\mathrm{a}}^{\dagger} + \sum_n A_n^{\mu}(x^{\kappa})\, \mathrm{c}_{n\mathrm{a}}. \tag{A.53}
$$

in terms of analogous creation and annihilation operators respectively, with commutator

$$
[\mathrm{c}_{m\mathrm{a}}, \mathrm{c}_{n\mathrm{b}}^{\dagger}] = \delta_{mn}\, \delta_{\mathrm{ab}}. \tag{A.54}
$$

This expansion is somewhat cumbersome, and it is easier to define a negative index $-n$ typified as follows :

$$
\begin{aligned}
\omega_{-n} &= -\omega_n, \\
A_{-n}^{\mu}(x^{\kappa}) &= A_n^{*\mu}(x^{\kappa}), \\
\mathrm{c}_{-n\mathrm{a}} &= \mathrm{c}_{n\mathrm{a}}^{\dagger},
\end{aligned}
$$

etc. That is to say, that $-n$ will be used to denote the negative energy equivalent of the n'th particle state, i.e. the n'th anti-particle state. In this notation

the field operator is simply expanded as

$$\begin{aligned}\hat{A}^{\mu}_{\mathrm{a}}(x^{\kappa}) &= \sum_{\pm n} A^{\mu}_{n}(x^{\kappa})\, \mathrm{c}_{\mathrm{na}}, \\ &= \sum_{\pm n} A^{*\mu}_{n}(x^{\kappa})\, \mathrm{c}^{\dagger}_{\mathrm{na}},\end{aligned} \tag{A.55}$$

and the commutator becomes

$$\begin{aligned}[\mathrm{c}_{\mathrm{ma}}, \mathrm{c}_{\mathrm{nb}}] &= \mathrm{sgn}(m)\, \delta_{m,-n}\, \delta_{\mathrm{ab}}, \\ [\mathrm{c}^{\dagger}_{\mathrm{ma}}, \mathrm{c}^{\dagger}_{\mathrm{nb}}] &= -\mathrm{sgn}(m)\, \delta_{m,-n}\, \delta_{\mathrm{ab}}, \\ [\mathrm{c}_{\mathrm{ma}}, \mathrm{c}^{\dagger}_{\mathrm{nb}}] &= \mathrm{sgn}(m)\, \delta_{m,n}\, \delta_{\mathrm{ab}}.\end{aligned} \tag{A.56}$$

A.4.2 The QCD Hamiltonian

Given the Lagrange density for non-interacting QCD gauge fields

$$\hat{\mathcal{L}}(x^{\kappa}) = -\frac{1}{4}\sum_{\mathrm{a}} \hat{F}^{\mu\nu}_{\mathrm{a}}(x^{\kappa})\hat{F}_{\mathrm{a}\mu\nu}(x^{\kappa}) - \frac{\lambda}{2}\sum_{\mathrm{a}} \left(\partial_{\mu}\hat{A}^{\mu}_{\mathrm{a}}(x^{\kappa})\right)^{2}, \tag{A.57}$$

one may derive from the definition of the energy-momentum tensor [18]

$$\hat{\Theta}^{\mu\nu}(x^{\kappa}) = \sum_{\rho\mathrm{a}} \frac{\partial\hat{\mathcal{L}}(x^{\kappa})}{\partial(\partial_{\mu}\hat{A}_{\mathrm{a}\rho}(x^{\kappa}))}\, \partial^{\nu}\hat{A}_{\mathrm{a}\rho}(x^{\kappa}) - g^{\mu\nu}\hat{\mathcal{L}}(x^{\kappa}) \tag{A.58}$$

$$\begin{aligned}&= -\sum_{\rho\mathrm{a}} \hat{F}^{\mu\rho}_{\mathrm{a}}(x^{\kappa})\partial^{\nu}\hat{A}_{\mathrm{a}\rho}(x^{\kappa}) \\ &\quad - \lambda\sum_{\rho\mathrm{a}} (\partial_{\rho}\hat{A}^{\rho}_{\mathrm{a}}(x^{\kappa}))\partial^{\nu}\hat{A}^{\mu}_{\mathrm{a}}(x^{\kappa}) \\ &\quad - g^{\mu\nu}\hat{\mathcal{L}}(x^{\kappa}).\end{aligned} \tag{A.59}$$

Consider now the first term of this last expression, which we would expect to be gauge invariant in free space. Under the transformation $\hat{A}^{\mu}_{\mathrm{a}}(x^{\kappa}) \mapsto \hat{A}^{\mu}_{\mathrm{a}}(x^{\kappa}) + \partial^{\mu}\hat{\psi}_{\mathrm{a}}(x^{\kappa})$, ignoring the gauge fixing term,

$$\hat{\Theta}^{\mu\nu}(x^{\kappa}) \mapsto \hat{\Theta}^{\cdot\nu}(x^{\kappa}) - \sum_{\rho\mathrm{a}} \hat{F}^{\mu\rho}_{\mathrm{a}}(x^{\kappa})\partial^{\nu}\partial_{\rho}\hat{\psi}_{\mathrm{a}}(x^{\kappa}), \tag{A.60}$$

and thus even this part of $\hat{\Theta}^{\mu\nu}(x^\kappa)$ is manifestly not gauge-invariant. However following [18], we make use of a freedom of choice in $\hat{\Theta}^{\mu\nu}(x^\kappa)$, defined by

$$\Delta\hat{\Theta}^{\mu\nu}(x^\kappa) = \sum_\rho \partial_\rho \hat{\Theta}^{\mu\rho,\nu}(x^\kappa), \tag{A.61}$$

where $\hat{\Theta}^{\mu\rho,\nu}(x^\kappa)$ is a function that is antisymmetric in μ, ρ, and local in the fields. The gauge dependence found above suggests

$$\Delta\hat{\Theta}^{\mu\nu}(x^\kappa) = \sum_\rho \partial_\rho \left(\sum_a \hat{F}_a^{\mu\rho}(x^\kappa)\hat{A}_a^\nu(x^\kappa) \right), \tag{A.62}$$

whence the full tensor becomes

$$\begin{aligned}
\hat{\Theta}^{\mu\nu}(x^\kappa) &= -\sum_{\rho a} \hat{F}_a^{\mu\rho}(x^\kappa)\left(\partial^\nu \hat{A}_{a\rho}(x^\kappa) - \partial_\rho \hat{A}_a^\nu(x^\kappa)\right) \\
&\quad - \lambda \sum_{\rho a} (\partial_\rho \hat{A}_a^\rho(x^\kappa))\left(\partial^\mu \hat{A}_a^\nu(x^\kappa) + \partial^\nu \hat{A}_a^\mu(x^\kappa)\right) \\
&\quad + \lambda \partial^\mu \sum_{\rho a} (\partial_\rho \hat{A}_a^\rho(x^\kappa))\hat{A}_a^\nu(x^\kappa) - g^{\mu\nu}\hat{\mathcal{L}}(x^\kappa),
\end{aligned} \tag{A.63}$$

since using the equation of motion (A.6),

$$\begin{aligned}
\partial_\rho \hat{F}_a^{\mu\rho}(x^\kappa) &= \partial_\rho \partial^\mu \hat{A}_a^\rho(x^\kappa) - \partial_\rho \partial^\rho \hat{A}_a^\mu(x^\kappa) \\
&= \lambda \partial^\mu \partial_\rho \hat{A}_a^\rho(x^\kappa).
\end{aligned}$$

Now the Hamiltonian $\hat{\mathcal{H}}$ is defined by

$$\hat{\mathcal{H}} = \int d^3x\, \hat{\Theta}^{00}(x^\kappa), \tag{A.64}$$

which gives

$$\begin{aligned}
\hat{\mathcal{H}}_0 &= \sum_a \int d^3x \hat{F}_a^{0\rho}(x^\kappa)\left(\partial_\rho \hat{A}_a^0(x^\kappa) - \partial^0 \hat{A}_{a\rho}(x^\kappa)\right) \\
&\quad + \frac{1}{4}\sum_a \int d^3x \hat{F}_a^{\mu\nu}(x^\kappa)\hat{F}_{a\mu\nu}(x^\kappa) \\
&\quad + \text{appropriate gauge fixing terms.}
\end{aligned} \tag{A.65}$$

A.4.3 The Second Quantized Hamiltonian

Consider first the second term of the above Hamiltonian for the case of non-interacting fields. Using the field expansions of section A.4.1, we find

$$\frac{1}{4}\sum_{\mathbf{a}}\int d^3x\left(\partial^\mu \hat{A}^\nu_{\mathbf{a}}(x^\kappa)-\partial^\nu \hat{A}^\mu_{\mathbf{a}}(x^\kappa)\right)\left(\partial_\mu \hat{A}_{\mathbf{a}\nu}(x^\kappa)-\partial_\nu \hat{A}_{\mathbf{a}\mu}(x^\kappa)\right)$$
$$=\frac{1}{4}\sum_{\pm m,n}\left\{\int d^3x\left(\partial^\mu A^{*\nu}_m(x^\kappa)-\partial^\nu A^{*\mu}_m(x^\kappa)\right)\left(\partial_\mu A_{n\nu}(x^\kappa)-\partial_\nu A_{n\mu}(x^\kappa)\right)\right\}$$
$$\times\sum_{\mathbf{a}} c^\dagger_{m\mathbf{a}}c_{n\mathbf{a}}. \tag{A.66}$$

Now

$$\int d^3x\left(\partial^\mu A^{*\nu}_m(x^\kappa)-\partial^\nu A^{*\mu}_m(x^\kappa)\right)\left(\partial_\mu A_{n\nu}(x^\kappa)-\partial_\nu A_{n\mu}(x^\kappa)\right)$$
$$=2\int d^3x\left(\partial^\mu A^{*\nu}_m(x^\kappa)\partial_\mu A_{n\nu}(x^\kappa)-\partial^\mu A^{*\nu}_m(x^\kappa)\partial_\nu A_{n\mu}(x^\kappa)\right). \tag{A.67}$$

By partial integration we have

$$\int d^3x\,\partial^\mu A^{*\nu}_m(x^\kappa)\partial_\mu A_{n\nu}(x^\kappa)$$
$$=\int d^3x\left(-\partial_i\partial^i A^{*\nu}_m(x^\kappa)A_{n\nu}(x^\kappa)+\partial^0 A^{*\nu}_m(x^\kappa)\partial_0 A_{n\nu}(x^\kappa)\right)$$
$$=(-\omega_m^2+\omega_m\omega_n)\int d^3x A^{*\nu}_m(x^\kappa)A_{n\nu}(x^\kappa), \tag{A.68}$$

since, from the equation of motion (A.7), we know that

$$\partial_i\partial^i A^\mu_n(x^\kappa)=-\nabla^2 A^\mu_n(x^\kappa)=-\partial_0^2 A^\mu_n(x^\kappa)=\omega_n^2 A^\mu_n(x^\kappa). \tag{A.69}$$

Also by partial integration and the commutativity of first order differentiation, we have

$$\int d^3x\,\partial^\mu A^{*\nu}_m(x^\kappa)\partial_\nu A_{n\mu}(x^\kappa)$$

$$= \int d^3x \left(\partial^0 A^{*0}_m(x^\kappa) \partial_0 A_{n0}(x^\kappa) - \partial^0 \partial_i A^{*i}_m(x^\kappa) A_{n0}(x^\kappa) \right.$$
$$\left. - A^{*0}_m(x^\kappa) \partial^i \partial_0 A_{ni}(x^\kappa) + \partial_i A^{*i}_m(x^\kappa) \partial^j A_{nj}(x^\kappa) \right)$$
$$= (-\omega_m^2 - \omega_n^2 + 2\omega_m\omega_n) \int d^3x A^{*0}_m(x^\kappa) A_{n0}(x^\kappa), \qquad \text{(A.70)}$$

using the Lorentz condition

$$\partial_i A^i_n(x^\kappa) = -\partial_0 A^0_n(x^\kappa) = -i\omega_n A^0_n(x^\kappa). \qquad \text{(A.71)}$$

However, from the orthonormalisation (A.28), we know that

$$\int d^3x A^{*i}_m(x^\kappa) A_{ni}(x^\kappa) = -\frac{\delta_{mn}}{2\omega_m},$$
$$\int d^3x A^{*0}_m(x^\kappa) A_{n0}(x^\kappa) = \begin{cases} \frac{\delta_{mn}}{2\omega_m} & \text{for L,} \\ 0 & \text{for M,E,} \end{cases} \qquad \text{(A.72)}$$

where **L**,**M** and **E** denote the Longitudinal, Transverse Magnetic and Transverse Electric modes, respectively, as before. Thus it is clear that this term of the Hamiltonian

$$\frac{1}{4} \sum_a \int d^3x \left(\partial^\mu \hat{A}^\nu_a(x^\kappa) - \partial^\nu \hat{A}^\mu_a(x^\kappa) \right) \left(\partial_\mu \hat{A}_{a\nu}(x^\kappa) - \partial_\nu \hat{A}_{a\mu}(x^\kappa) \right) = 0. \qquad \text{(A.73)}$$

We now consider the first term of the Hamiltonian, again for the case of non-interacting fields,

$$\sum_a \int d^3x \left(\partial^0 \hat{A}^\rho_a(x^\kappa) - \partial^\rho \hat{A}^0_a(x^\kappa) \right) \left(\partial_\rho \hat{A}^0_a(x^\kappa) - \partial^0 \hat{A}_{a\rho}(x^\kappa) \right)$$
$$= \sum_{\pm m,n} \left\{ \int d^3x \left(\partial^0 A^{*\rho}_m(x^\kappa) - \partial^\rho A^{*0}_m(x^\kappa) \right) \left(\partial_\rho A^0_n(x^\kappa) - \partial^0 A_{n\rho}(x^\kappa) \right) \right\}$$
$$\times \sum_a c^\dagger_{ma} c_{na}. \qquad \text{(A.74)}$$

If we now proceed by using the Lorentz condition and the orthonormalisation in an analogous manner to the method above, then this term yields

$$\int d^3x \left(\partial^0 A^{*\rho}_m(x^\kappa) - \partial^\rho A^{*0}_m(x^\kappa) \right) \left(\partial_\rho A^0_n(x^\kappa) - \partial^0 A_{n\rho}(x^\kappa) \right)$$

$$= -\omega_m^2 \int d^3x A_m^{*0}(x^\kappa) A_{n0}(x^\kappa) - \omega_m \omega_n \int d^3x A_m^{*i}(x^\kappa) A_{ni}(x^\kappa). \tag{A.75}$$

Furthermore, since all the gauge fixing terms go as $\delta_\mu \hat{A}_a^\mu(x^\kappa)$, they all evaluate to zero, and so we are left with the non-interacting Hamiltonian

$$\begin{aligned}
\hat{\mathcal{H}}_0 &= \sum_a \sum_{\pm m,n} \left\{ -\omega_m^2 \begin{pmatrix} \frac{\delta_{mn}}{2\omega_m} \\ 0 \end{pmatrix} - \omega_m \omega_n \frac{(-\delta_{mn})}{2\omega_m} \right\} c_{ma}^\dagger c_{na} \begin{cases} \text{for L,} \\ \text{for M,E} \end{cases} \\
&= \sum_a \sum_{\pm n(\text{TRANSVERSE})} \frac{1}{2} \omega_n c_{na}^\dagger c_{na} \\
&= \sum_a \sum_{n(\text{TRANSVERSE})} \omega_n c_{na}^\dagger c_{na},
\end{aligned} \tag{A.76}$$

after performing the normal ordering necessary to remove infinite negative sea contributions. Putting in the full form of $\hat{F}_a^{\mu\nu}(x^\kappa)$ and evaluating the interaction terms, we find for the full Hamiltonian

$$\begin{aligned}
\hat{\mathcal{H}} &= \sum_a \sum_{n(\text{TRANSVERSE})} \omega_n c_{na}^\dagger c_{na} \\
&\quad + \frac{g^2}{4} \sum_{\pm mnpq} \left(\int d^3x A_m^{*\mu}(x^\kappa) A_n^{*\nu}(x^\kappa) A_{p\mu}(x^\kappa) A_{q\nu}(x^\kappa) \right) \\
&\qquad \times \sum_{abcde} f_{abc} f_{ade} c_{mb}^\dagger c_{ne}^\dagger c_{pd} c_{qe} \\
&\quad + \cdots,
\end{aligned} \tag{A.77}$$

where the other terms are trilinear in the fields, and do not interest us at this stage. It is interesting to note that it is only the Transverse Magnetic and Electric modes that contribute to the Hamiltonian, the non-physical Longitudinal mode having dropped out. This is why we only consider the transverse modes when we calculate the Gluonium spectrum in Chapter 4.

A.5 The Non-Interacting Propagator in the M.I.T. Bag

We consider the propagator

$$i\tilde{D}'^{(0)\mu\nu}_{ab}(x,x') = \langle \mathrm{T}\{\hat{A}^{\mu}_{a}(x)\hat{A}^{\nu}_{b}(x')\}\rangle^{(0)}. \tag{A.78}$$

Expanding now in terms of the eigenstates of the M.I.T. bag as in equation (A.53), we find

$$\begin{aligned} i\tilde{D}'^{(0)\mu\nu}_{ab}(x,x') &= \langle \sum_{mn} A^{\mu}_{m}(x)A^{*\nu}_{n}(x')\, c_{ma}c^{\dagger}_{nb}\rangle^{(0)}\theta(t-t') \\ &\quad + \langle \sum_{mn} A^{*\mu}_{m}(x)A^{\nu}_{n}(x')\, c_{nb}c^{\dagger}_{ma}\rangle^{(0)}\theta(t'-t). \end{aligned} \tag{A.79}$$

Transforming to energy co-ordinates, we have

$$\begin{aligned} i\tilde{D}'^{(0)\mu\nu}_{ab}(\vec{x},\vec{x}',\omega) &= \int_{-\infty}^{\infty} d(t-t')i\tilde{D}'^{(0)\mu\nu}_{ab}(x,x')\, e^{i\omega(t-t')} \\ &= \delta_{ab}\sum_{m} A^{*\mu}_{m}(\vec{x})A^{\nu}_{m}(\vec{x}')\int_{-\infty}^{0} d(t-t')e^{-i(\omega^{0}_{m}-\omega)(t-t')} \\ &\quad + \delta_{ab}\sum_{m} A^{\mu}_{m}(\vec{x})A^{*\nu}_{m}(\vec{x}')\int_{0}^{\infty} d(t-t')e^{i(\omega^{0}_{m}+\omega)(t-t')} \\ &= \delta_{ab}\sum_{m}\left\{\frac{A^{*\mu}_{m}(\vec{x})A^{\nu}_{m}(\vec{x}')}{-i(\omega^{0}_{m}-\omega-i0^{+})} - \frac{A^{\mu}_{m}(\vec{x})A^{*\nu}_{m}(\vec{x}')}{i(\omega^{0}_{m}+\omega-i0^{+})}\right\} \\ &= -i\delta_{ab}\sum_{m}\left\{\frac{(\omega^{0}_{m}-\omega)A^{\mu}_{m}(\vec{x})A^{*\nu}_{m}(\vec{x}')}{\omega^{2}-{\omega^{0}_{m}}^{2}+i0^{+}}\right. \\ &\quad \left. + \frac{(\omega^{0}_{m}+\omega)A^{*\mu}_{m}(\vec{x})A^{\nu}_{m}(\vec{x}')}{\omega^{2}-{\omega^{0}_{m}}^{2}+i0^{+}}\right\}. \end{aligned} \tag{A.80}$$

Relabeling $Y^{*}_{j,m} = (-)^{j-m}Y_{j,-m}$, the two sums may be re-ordered into one

$$i\tilde{D}'^{(0)\mu\nu}_{ab}(\vec{x},\vec{x}',\omega) = -i\delta_{ab}\sum_{m}\frac{2\omega^{0}_{m}A^{\mu}_{m}(\vec{x})A^{*\nu}_{m}(\vec{x}')}{\omega^{2}-{\omega^{0}_{m}}^{2}+i0^{+}}. \tag{A.81}$$

Appendix B

The Chemical Potential μ_0

B.1 Relating μ_0 to n_0

If one examines more closely [2] the approximative method of Section 2.4 in Chapter 2, defining

$$\langle \hat{S} \rangle' = e^{\sigma} \tag{B.1}$$

where σ is a finite functional in pairs of $\hat{A}^{0\mu}_{\rm In}(x^\kappa)$ operators, then one finds that

$$\frac{dK(t)}{dt} = \frac{\delta\sigma(K^\mu K^{*\nu})}{\delta K^*(t)} = -i\mu_0\sqrt{n_0}\, e^{-i\mu_0 t}, \tag{B.2}$$

in terms of the functions defined by

$$\begin{aligned} K^\mu(x) &= A^\mu_0(\vec{x})\, K(t) = \sqrt{n_0}\, A^\mu_0(\vec{x})\, e^{-i\mu_0 t} \\ K^{*\mu}(x) &= A^{*\mu}_0(\vec{x})\, K^*(t) = \sqrt{n_0}\, A^{*\mu}_0(\vec{x})\, e^{+i\mu_0 t}. \end{aligned} \tag{B.3}$$

Note that the finite nature of our system is reflected in the spatial dependence of these functions, approximated here by the lowest state wavefunction.

Now one may define σ in terms of a sum over vacuum-loops with m-pairs of incomplete vertices :

$$\begin{aligned} \sigma(K^\mu K^{*\nu}) \;=\; -i\sum_{(W)} \int W^{\mu_1 \ldots \mu_m, \nu_1 \ldots \nu_m}_m (x'_1 \ldots x'_m; x_1 \ldots x_m) K^*_{\mu_1}(x'_1) \ldots \\ \ldots K^*_{\mu_m}(x'_m) K_{\nu_1}(\qquad K_{\nu_m}(x_m) (d^4x)(d^4x'), \end{aligned} \tag{B.4}$$

where the sum runs over all such loops. Now the vacuum-loop expectation value W_m will depend only on $(2m-1)$ time differences. There are $2m$ exponentials (from the K's and K^*'s) and $2m$ time integrals so that each term has a divergent form. Variation with respect to one $K^*(t)$ will remove one integral, and thus give us a finite result for each term and also for the sum. Performing the functional derivative, we find that

$$\begin{aligned}\frac{\delta\sigma(K^\mu K^{*\nu})}{\delta j_\lambda(t)} &= -i\sum_{n=1}^{m}\sum_{(W)}\int W_m^{\mu_1\ldots\mu_m,\nu_1\ldots\nu_m}(x';x)K^*_{\mu_1}(x'_1)\ldots \\ &\quad \ldots K^*_{\mu_{n-1}}(x'_{n-1})A^*_{0\mu_n}(\vec{x}'_n)K^*_{\mu_{n+1}}(x'_{n+1})\ldots K^*_{\mu_m}(x'_m) \\ &\quad \times K_{\nu_1}(x_1)\ldots K_{\nu_m}(x_m)\,\delta(t'_n-t)(d^4x)(d^4x').\end{aligned} \tag{B.5}$$

If we define a generalised Fourier-like transform

$$\begin{aligned}&W_m(\omega'_1\ldots\omega'_m;\omega_1\ldots\omega_m)\,2\pi\delta(\Sigma\omega-\Sigma\omega') \\ &\quad = \int W_m^{\mu_1\ldots\mu_m,\nu_1\ldots\nu_m}(x';x)A^*_{0\mu_1}(\vec{x}'_1)\ldots A^*_{0\mu_m}(\vec{x}'_m) \\ &\qquad \times A_{0\nu_1}(\vec{x}_1)\ldots A_{0\nu_m}(\vec{x}_m)e^{i\Sigma_n(\omega'_nt'_n-\omega_nt_n)}(d^4x)(d^4x'),\end{aligned} \tag{B.6}$$

and use that

$$\delta(t'_n-t)=\int\frac{d\omega}{2\pi}e^{i\omega(t'_n-t)}, \tag{B.7}$$

we find, after substituting for $K^{*\mu}(x)$ and $K^\mu(x)$, that

$$\begin{aligned}\frac{\delta\sigma(K^\mu K^{*\nu})}{\delta K^*(t)} &= -i\sqrt{n_0}\int d\omega\,\delta(\omega-\mu_0)\,e^{-i\omega t}\sum_{(W)}n_0^{m-1} \\ &\quad \times\{W_m(\omega\ldots\mu_0;\mu_0\ldots\mu_0)+\cdots+W_m(\mu_0\ldots\omega;\mu_0\ldots\mu_0)\}.\end{aligned} \tag{B.8}$$

Comparing this expression to that of equation (B.2), we see that

$$\mu_0=\sum_{(W)'}n_0^{m-1}W_m(\mu_0\ldots\mu_0;\mu_0\ldots\mu_0), \tag{B.9}$$

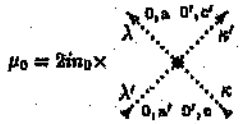


Figure B.1: Lowest Order Contributions to the Chemical Potential μ_0

where now the sum runs not only over geometric structure, but also over the position of the special vertex at which the derivative occurs.

B.2 Lowest Order Calculation of μ_0

If we consider the contributions to μ_0 as given in equation (B.9) above then we see that the lowest order contribution must be that of figure B.1, where the factor 2 comes from the two possible positions of the special vertex, and the factor in_0 from the definitions above. Using expression (3.10) of Chapter 3 to evaluate the diagram, we find

$$\begin{aligned}\mu_0 &= 2n_0 g^2 \int d^3x \frac{1}{4n_0^2} i\tilde{D}^{0}_{aa'}{}^{\lambda\lambda'}(\vec{x},\vec{x})\, i\tilde{D}^{0'}_{cc'}{}^{\kappa\kappa'}(\vec{x},\vec{x}) \\ &\times \sum_{e} \{f_{ace}f_{a'c'e}(g_{\lambda\lambda'}g_{\kappa\kappa'} - g_{\lambda\kappa'}g_{\lambda'\kappa}) + f_{ace}f_{c'ae}(g_{\lambda\kappa}g_{\lambda'\kappa'} - g_{\lambda\lambda'}g_{\kappa\kappa'})\},\end{aligned} \tag{B.10}$$

where the $1/4n_0^2$ factor reduces the condensate propagators to the appropriate lowest state functions as they appear in the definition of $W_m(\mu_0 \ldots \mu_0)$, each represented by a vectored missing line in the diagram above. However their propagator source yields the two residual m-sums denoted by the 0 and 0' indices. Finally then, filling in the condensate propagator, we obtain

$$\mu_0 = 2n_0 g^2 C_A N_C \Big\{ 2 \int d^3x A^a_0(\vec{\ }) A^a_{0\mu}(\vec{x}) A^\nu_{0'}(\vec{x}) A^a_{0'\nu}(\vec{x})$$

$$- \int d^3x A_0^{\mu}(\vec{x}) A^{*}_{0'\mu}(\vec{x}) A_0^{*\nu}(\vec{x}) A_{0'\nu}(\vec{x})$$

$$- \int d^3x A_0^{*\mu}(\vec{x}) A^{*}_{0'\mu}(\vec{x}) A_0^{\nu}(\vec{x}) A_{0'\nu}(\vec{x}) \Big\}, \tag{B.11}$$

where $N_C = 8$ is the number of colours, and where C_A is the SU(3) constant defined by

$$\sum_{bc} f_{abc} f_{a'bc} = C_A \, \delta_{aa'}. \tag{B.12}$$

Note that μ_0 takes a definite non-zero value. The evaluation of this value, in particular the residual sums denoted by 0 and 0′, is handled in Appendix D.

Appendix C

Colour Sums and Colour Factors

C.1 Anomalous Self-Energies

We consider colour sums such as

$$m_{ab} = \sum_{cde} f_{ace} c_{cd} f_{dbe}, \tag{C.1}$$

where f_{abc} denotes the structure constants of the SU(3) group, defined by

$$[\lambda_a, \lambda_b] = 2i \sum_c f_{abc} \lambda_c, \tag{C.2}$$

in terms of the Gellmann matrices λ_a of SU(3), and where c_{ab} denotes the coupling matrix of two octets to a singlet in SU(3) colour space. In the equation for $i\Sigma^{02}$ (see Chapter 3)

$$\begin{aligned}\Sigma^{02}_{mnab}(\omega + \mu_0) = g^2 \sum_{cde} \{ & f_{cae} f_{bde} \int d^3x A_{n\nu}(\vec{x}) i\overline{D^0}^{\nu\mu}_{cd}(\vec{x}, \vec{x}) A_{m\mu}(\vec{x}) \\ & - f_{cae} f_{bde} \int d^3x i\overline{D^0}^{\lambda\lambda'}_{cd}(\vec{x}, \vec{x}) g_{\lambda\lambda'} A^\mu_m(\vec{x}) A_{n\mu}(\vec{x}) \\ & + f_{cbe} f_{ade} \int d^3x A_{m\mu}(\vec{x}) i\overline{D^0}^{\mu\nu}_{cd}(\vec{x}, \vec{x}) A_{n\nu}(\vec{x}) \\ & - f_{cbe} f_{ade} \int d^3x i\overline{D^0}^{\lambda\lambda'}_{cd}(\vec{x}, \vec{x}) g_{\lambda\lambda'} A^\mu_m(\vec{x}) A_{n\mu}(\vec{x})\end{aligned}$$

$$+ f_{cde} f_{bae} \int d^3x A_{n\nu}(\vec{x}) i\overline{D^0_{cd}}^{\nu\mu}(\vec{x},\vec{x}) A_{m\mu}(\vec{x})$$
$$- f_{cde} f_{bae} \int d^3x A_{m\mu}(\vec{x}) i\overline{D^0_{cd}}^{\mu\nu}(\vec{x},\vec{x}) A_{n\nu}(\vec{x}) \},$$
(C.3)

we have six terms, the first two going as

$$\sum_{cde} f_{cde} f_{bde} c_{cd} = \sum_{cde} f_{ace} c_{cd} f_{dbe} = m_{ab}, \tag{C.4}$$

using that the f_{abc} are antisymmetric in the indices 'a','b' and 'c'. Similarly the second two terms go as

$$\sum_{cde} f_{cbe} f_{ade} c_{cd} = \sum_{cde} f_{ace} c_{cd} f_{dbe} = m_{ab}, \tag{C.5}$$

assuming that c_{ab} is *symmetric* in its indices, which we will show below. With this assumption it is clear that the last two terms go as

$$\sum_{cde} f_{cde} f_{bae} c_{cd} = 0. \tag{C.6}$$

Thus it is clear that the colour dependence of $i\Sigma^{02}$ is indeed of the form m_{ab} and it remains for us to find a closed form for this quantity.

C.2 Coupling of Two Octets to a Singlet in SU(3)

The coupling of two octets to a singlet in SU(3) is best described in the notation

$$\mathrm{T}^i_j = x_j \bar{x}^i - \frac{1}{3}\sum_k (x_k \bar{x}^k)\delta^i_j, \tag{C.7}$$

where the singlet is given by

$$\mathrm{S} = \sum_{ij} \mathrm{T}^i_j \mathrm{T}^j_i, \tag{C.8}$$

that is, it goes as $\mathbf{T}^{\mathrm{T}}\mathbf{T}'$. Comparing this notation with that of the Gellmann matrices λ_a we have been using up till now, we find

$$\begin{aligned}
\mathrm{T}_1^1 &= \frac{1}{2\sqrt{3}}\lambda_8 + \frac{1}{2}\lambda_3, \\
\mathrm{T}_2^1 &= \frac{1}{2}\lambda_1 - \frac{i}{2}\lambda_2, \\
\mathrm{T}_3^1 &= \frac{1}{2}\lambda_4 - \frac{i}{2}\lambda_5, \\
\mathrm{T}_1^2 &= \frac{1}{2}\lambda_1 + \frac{i}{2}\lambda_2, \\
\mathrm{T}_2^2 &= \frac{1}{2\sqrt{3}}\lambda_8 - \frac{1}{2}\lambda_3, \\
\mathrm{T}_3^2 &= \frac{1}{2}\lambda_6 - \frac{i}{2}\lambda_7, \\
\mathrm{T}_1^3 &= \frac{1}{2}\lambda_4 + \frac{i}{2}\lambda_5, \\
\mathrm{T}_2^3 &= \frac{1}{2}\lambda_6 + \frac{i}{2}\lambda_7, \\
\mathrm{T}_3^3 &= -\frac{1}{\sqrt{3}}\lambda_8.
\end{aligned} \tag{C.9}$$

Defining an *orthogonal* transformation from the one notation to the other by

$$\mathrm{T}_\alpha = \sum_a \mathcal{U}_{\alpha a}\lambda_a, \tag{C.10}$$

we find the transformation matrix is

$$\mathcal{U}_{\alpha a} = \frac{1}{2}\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} \\
1 & i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & i & 0 & 0 & 0 \\
1 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} \\
0 & 0 & 0 & 0 & 0 & 1 & i & 0 \\
0 & 0 & 0 & 1 & -i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -i & 0
\end{bmatrix}. \tag{C.11}$$

Here the α index denotes the eight independent components of the tensor, which we choose to be the first eight of those given above, the other component being given in terms of them by

$$\mathbf{T}_3^3 = -\mathbf{T}_1^1 - \mathbf{T}_2^2. \tag{C.12}$$

The 'a' index refers to the Gellmann matrices as usual. Writing the singlet in the form

$$\mathbf{S} = \sum_{\alpha\beta} \mathbf{T}_\alpha^{\mathrm{T}} c_{\alpha\beta} \mathbf{T}'_\beta, \tag{C.13}$$

and rewriting equation (C.8) in terms of the first eight components of the tensor, we identify, using (C.12),

$$c_{\alpha\beta} = \begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{C.14}$$

Now we wish to determine c_{ab} such that

$$\mathbf{S} = \sum_{ab} \lambda_a^{\mathrm{T}} c_{ab} \lambda_b = \sum_{ab\alpha\beta} \lambda_a^{\mathrm{T}} \mathcal{U}_{a\alpha}^{\mathrm{T}} c_{\alpha\beta} \mathcal{U}_{\beta b} \lambda_b. \tag{C.15}$$

Thus we find, after some matrix manipulation, that

$$c_{ab} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \frac{1}{2} \delta_{ab}, \tag{C.16}$$

which is symmetric as required. We may now calculate directly

$$m_{ab} = \frac{1}{2}\sum_{ce} f_{ace}f_{cbe} = -\frac{1}{2}C_A\delta_{ab}, \tag{C.17}$$

in terms of the SU(3) constant $C_A = 3$.

C.3 Colour Dependence of Full Propagator

We have obtained, in Chapter 3, an explicit form for the full propagator

$$i\tilde{D}'(+) = i\frac{\left[\Sigma_{02}^{-1}(+)\tilde{D}'^{-1}_{(0)}(-)\Sigma_{11}^{-1}(-)\Sigma_{02}(+) - 1\right]\Sigma_{11}(-)}{\text{Denominator}}, \tag{C.18}$$

where the denominator is

$$\left[\tilde{D}'^{-1}_{(0)}(+) - \Sigma_{11}(+)\right]\left[\Sigma_{02}^{-1}(+)\tilde{D}'^{-1}_{(0)}(-)\Sigma_{11}^{-1}(-)\Sigma_{02}(+) - 1\right]\Sigma_{11}(-)$$
$$- \Sigma_{20}(+)\left[\Sigma_{11}^{-1}(-)\Sigma_{02}(+)\Sigma_{11}(-)\right], \tag{C.19}$$

and where $(+)$ denotes an argument of $\omega+\mu_0$, and $(-)$ an argument of $-\omega+\mu_0$. Now all the quantities in this equation are proportional to unit matrices in colour space, and thus so too are all their inverses. Since the product of any number of unit matrices yields just another unit matrix, it is clear that the colour dependence of the full propagator is trivial, and we are left with a straightforward matrix equation in the eigenstate indices alone, with the appropriate colour factors introduced for each quantity.

Appendix D

Evaluation of Matrix Element Integrals

D.1 The General Form of the Integrals

We need to calculate matrix elements of the form

$$\begin{aligned} I^{(1)}_{mn} &= \int d^3x A^{\lambda}_0(\vec{x}) A^*_{0\lambda}(\vec{x}) A^{\mu}_m(\vec{x}) A^*_{n\mu}(\vec{x}), \\ I^{(2)}_{mn} &= \int d^3x A^{\mu}_m(\vec{x}) A^*_{0\mu}(\vec{x}) A^{\nu}_0(\vec{x}) A^*_{n\nu}(\vec{x}). \end{aligned} \tag{D.1}$$

Now in general, the wavefunctions in these matrix elements are

$$A^{\lambda}_n(\vec{x}) = \frac{\mathcal{N}_n}{R^{\frac{3}{2}}} \begin{pmatrix} a_0 \\ \vec{a} \end{pmatrix}^{\lambda} j_{l_n}(k_n x) Y_{l_n m_n}(\hat{x}), \tag{D.2}$$

where a_0 is some constant (1, 0, 0 for L, M, E states respectively), and $\vec{a}$ some vector operator ($\frac{1}{ik_n}\vec{\nabla}$, $\frac{\vec{L}}{\sqrt{l_n(l_n+1)}}$, $\frac{1}{ik_n}\vec{\nabla} \times \frac{\vec{L}}{\sqrt{l_n(l_n+1)}}$). Thus we have integrals that go as

$$\begin{aligned} I_{1234} &= \int d^3x A^{\mu}_1(\vec{x}) A^*_{2\mu}(\vec{x}) A^{\nu}_3(\vec{x}) A^*_{4\nu}(\vec{x}) \\ &= a^{(1)}_0 a^{*(2)}_0 a^{(3)}_0 a^{*(4)}_0 \int d^3x A_1 A^*_2 A_3 A^*_4 \\ &- a^{(1)}_0 a^{*(2)}_0 \int d^3x A_1 A^*_2 \vec{a}^{(3)} A_3 . \vec{a}^{*(4)} A^*_4 - a^{(3)}_0 a^{*(4)}_0 \int d^3x \vec{a}^{(1)} A_1 . \vec{a}^{*(2)} A^*_2 A_3 A^*_4 \end{aligned}$$

$$+ \int d^3x\, \vec{a}^{\,(1)} A_1 . \vec{a}^{\,*(2)} A_2^* \, \vec{a}^{\,(3)} A_3 . \vec{a}^{\,*(4)} A_4^*, \tag{D.3}$$

in terms of the basic functions

$$A_n(\vec{x}) = \frac{\mathcal{N}_n}{R^{\frac{3}{2}}} j_{l_n}(k_n x)\, Y_{l_n m_n}(\hat{x}) = \varphi_n(x)\, Y_{l_n m_n}(\hat{x}). \tag{D.4}$$

In order to proceed, we expand the vector operators as follows

$$\vec{a}\,|lm\rangle = \sum_{l'm'} |l'm'\rangle\langle l'm'|\,\vec{a}\,|lm\rangle, \tag{D.5}$$

or rather

$$\vec{a}\,\varphi(x)\, Y_{lm}(\hat{x}) = \sum_{l'm'} Y_{l'm'}(\hat{x})\,\langle l'm'|\,\vec{a}\,(x)\,|lm\rangle, \tag{D.6}$$

where the $\varphi(x)$ is implicit in the function $\vec{a}\,(x)$. In particular, we now shift to spherical tensor co-ordinates a_μ, where

$$\vec{a}^{\,(1)} . \vec{a}^{\,*(2)} = \sum_{\mu=-1}^{+1} (-)^\mu a_\mu^{(1)} a_{-\mu}^{*(2)}. \tag{D.7}$$

In this notation, the most general term, that containing the double dot-product, becomes

$$\int d^3x\, \vec{a}^{\,(1)} A_1 . \vec{a}^{\,*(2)} A_2^* \, \vec{a}^{\,(3)} A_3 . \vec{a}^{\,*(4)} A_4^* \quad =$$

$$\sum_{\mu\mu'} \sum_{l_i' m_i'} (-)^{\mu+\mu'} \int d^3x \langle l_1' m_1'|\, a_\mu^{(1)}\,|l_1 m_1\rangle \langle l_2' m_2'|\, a_{-\mu}^{*(2)}\,|l_2 m_2\rangle \langle l_3' m_3'|\, a_{\mu'}^{(3)}\,|l_3 m_3\rangle$$

$$\times \langle l_4' m_4'|\, a_{-\mu'}^{*(4)}\,|l_4 m_4\rangle\, (-)^{m_2+m_4} Y_{l_1' m_1'}(\hat{x})\, Y_{l_2' m_2'}(\hat{x})\, Y_{l_3' m_3'}(\hat{x})\, Y_{l_4' m_4'}(\hat{x}),$$

where we use that

$$Y_{lm}^*(\hat{x}) = (-)^m Y_{l-m}(\hat{x}). \tag{D.8}$$

The angular integral may now be evaluated, using the relation [11, page 63]

$$\sum_{lm} \left[\frac{(2l_1+1)(2l_2+1)(2l+1)}{4\pi}\right]^{\frac{1}{2}} \begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix} Y_{lm}^*(\hat{x}) \begin{pmatrix} l_1 & l_2 & l \\ 0 & 0 & 0 \end{pmatrix}$$

$$= Y_{l_1 m_1}(\hat{x})\, Y_{l_2 m_2}(\hat{x}), \tag{D.9}$$

and the orthonormality of the Spherical Harmonics [11, page 21]

$$\int Y^*_{lm}(\hat{x})\, Y_{l'm'}(\hat{x})\, d\Omega = \delta_{ll'}\,\delta_{mm'}. \tag{D.10}$$

Thus we find for the angular integral in the expression above

$$\begin{aligned}
&\int Y_{l_1'm_1'}(\hat{x})\, Y_{l_2'm_2'}(\hat{x})\, Y_{l_3'm_3'}(\hat{x})\, Y_{l_4'm_4'}(\hat{x})\, d\Omega \\
&\quad = \sum_{lm}(-)^m \frac{(2l+1)}{4\pi}\left[(2l_1'+1)(2l_2'+1)(2l_3'+1)(2l_4'+1)\right]^{\frac{1}{2}} \\
&\quad \times \begin{pmatrix} l_1' & l_2' & l \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l_3' & l_4' & l \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l_1' & l_2' & l \\ m_1' & m_2' & m \end{pmatrix}\begin{pmatrix} l_3' & l_4' & l \\ m_3' & m_4' & -m \end{pmatrix}.
\end{aligned} \tag{D.11}$$

Furthermore, expanding in terms of reduced matrix elements [11, page

$$\begin{aligned}
\langle l'm' |\, a_\mu \,| lm\rangle &= \frac{(-)^{l-m}}{\sqrt{3}}(l'm'l-m \mid 1\mu)\langle l' \| a \| l\rangle \\
&= (-)^{l'-m'}\begin{pmatrix} l' & l & 1 \\ m' & -m & -\mu \end{pmatrix}\langle l' \| a \| l\rangle,
\end{aligned} \tag{D.12}$$

we end up with a sum in m_1', m_2' and μ that goes as

$$\sum_{m_1'm_2'\mu}(-)^{l_1'+l_2'-m_1'-m_2'+\mu}\begin{pmatrix} l_1' & l_2' & l \\ m_1' & m_2' & m \end{pmatrix}\begin{pmatrix} l_1' & l_1 & 1 \\ m_1' & -m_1 & -\mu \end{pmatrix}\begin{pmatrix} l_2' & l_2 & 1 \\ m_2' & m_2 & \mu \end{pmatrix}, \tag{D.13}$$

which simplifies to [11, page 95]

$$(-)^{l_1+l_2'}\begin{pmatrix} l_1 & l_2 & l \\ m_1 & -m_2 & m \end{pmatrix}\begin{Bmatrix} l_1 & l_2 & l \\ l_2' & l_1' & 1 \end{Bmatrix}. \tag{D.14}$$

The sum in m_3', m_1' and μ' may be performed in an analogous manner and we find the double dot-product term finally to be

$$\int d^3x\, \vec{a}^{(1)}A_1 . \vec{a}^{*(2)}A_2^* \, \vec{a}^{(3)}A_3 . \vec{a}^{*(4)}A_4^* \quad =$$

$$\sum_{l_1' l m} (-)^{m_2+m_4-m} \frac{(2l+1)}{4\pi} [(2l_1'+1)(2l_2'+1)(2l_3'+1)(2l_4'+1)]^{\frac{1}{2}}$$

$$\times (-)^{l_1+l_1'} \begin{pmatrix} l_1' & l_2' & l \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l \\ m_1 & -m_2 & m \end{pmatrix} \begin{Bmatrix} l_1 & l_2 & l \\ l_2' & l_1' & 1 \end{Bmatrix}$$

$$\times (-)^{l_3+l_3'} \begin{pmatrix} l_3' & l_4' & l \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_3 & l_4 & l \\ m_3 & -m_4 & -m \end{pmatrix} \begin{Bmatrix} l_3 & l_4 & l \\ l_4' & l_3' & 1 \end{Bmatrix}$$

$$\times \int x^2 dx \, (l_1' \| a^{(1)}(x) \| l_1)(l_2' \| a^{*(2)}(x) \| l_2)(l_3' \| a^{(3)}(x) \| l_3)(l_4' \| a^{*(4)}(x) \| l_4). \tag{D.15}$$

The other terms of our original integral I_{1234} may be found analogously. For example

$$-a_0^{(1)}a_0^{*(2)} \int d^3x A_1 A_2^* \, \vec{a}^{(3)}A_3 . \vec{a}^{*(4)}A_4^* \quad =$$

$$-\sum_{l_1' l m} (-)^{m_3+m_4-m} \frac{(2l+1)}{4\pi} [(2l_1+1)(2l_2+1)(2l_3'+1)(2l_4'+1)]^{\frac{1}{2}}$$

$$\times \begin{pmatrix} l_1 & l_2 & l \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l \\ m_1 & -m_2 & m \end{pmatrix}$$

$$\times (-)^{l_3+l_3'} \begin{pmatrix} l_3' & l_4' & l \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_3 & l_4 & l \\ m_3 & -m_4 & -m \end{pmatrix} \begin{Bmatrix} l_3 & l_4 & l \\ l_4' & l_3' & 1 \end{Bmatrix}$$

$$\times a_0^{(1)}a_0^{*(2)} \int x^2 dx \, \varphi_1(x)\varphi_2^*(x) \, (l_3' \| a^{(3)}(x) \| l_3)(l_4' \| a^{*(4)}(x) \| l_4), \tag{D.16}$$

and the other dot-product term is found by exchanging indices 1 with 3, and 2 with 4. The pure spherical harmonic term may be evaluated directly, using

the above relations, and turns out to be

$$a_0^{(1)}a_0^{*(2)}a_0^{(3)}a_0^{*(4)}\int d^3\mathbf{x}A_1A_2^*A_3A_4^* \;=\; \sum_{lm}(-)^{m_2+m_4-m}\frac{(2l+1)}{4\pi}[(2l_1+1)(2l_2+1)(2l_3+1)(2l_4+1)]^{\frac{1}{2}} \times \begin{pmatrix} l_1 & l_2 & l \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l_1 & l_2 & l \\ m_1 & -m_2 & m \end{pmatrix}\begin{pmatrix} l_3 & l_4 & l \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l_3 & l_4 & l \\ m_3 & -m_4 & -m \end{pmatrix} \times a_0^{(1)}a_0^{*(2)}a_0^{(3)}a_0^{*(4)}\int x^2dx\,\varphi_1(x)\varphi_2^*(x)\varphi_3(x)\varphi_4^*(x). \tag{D.17}$$

Further residual m-sums remain, corresponding to the m-projection of the condensate propagator, and these will be handled explicitly in Section D.3.

D.2 Reduced Matrix Elements for the Vector Operators

We will require the reduced matrix elements of the $\vec{\nabla}$, $\vec{L}$ and $\vec{\nabla}\times\vec{L}$ operators. Now the first of these is [11, page 80]

$$(l'\|\,\vec{\nabla}\varphi(x)\,\|l) \;=\; (-)^{l'}\begin{pmatrix} l' & 1 & l \\ 0 & 0 & 0 \end{pmatrix}^{-1} \times \begin{cases} \dfrac{l+1}{[(2l+1)(2l+3)]^{\frac{1}{2}}}\left(\dfrac{\partial}{\partial x}-\dfrac{l}{x}\right)\varphi(x), & l'=l+1, \\ \dfrac{l}{[(2l-1)(2l+1)]^{\frac{1}{2}}}\left(\dfrac{\partial}{\partial x}+\dfrac{l+1}{x}\right)\varphi(x), & l'=l-1, \\ 0, & \text{otherwise.} \end{cases} \tag{D.18}$$

The two 3-j symbols may be evaluated directly [11, page 50]

$$\begin{pmatrix} l+1 & 1 & l \\ 0 & 0 & 0 \end{pmatrix} \;=\; (-)^{l+1}\left[\frac{l+1}{(2l+1)(2l+3)}\right]^{\frac{1}{2}},$$

$$\begin{pmatrix} l-1 & 1 & l \\ 0 & 0 & 0 \end{pmatrix} = (-)^l \left[\frac{l}{(2l-1)(2l+1)}\right]^{\frac{1}{2}},$$

and so

$$\langle l' \| \vec{\nabla}\varphi(x) \| l \rangle = \begin{cases} (l+1)^{\frac{1}{2}} \left(\dfrac{\partial}{\partial x} - \dfrac{l}{x}\right)\varphi(x), & l' = l+1, \\ -l^{\frac{1}{2}} \left(\dfrac{\partial}{\partial x} + \dfrac{l+1}{x}\right)\varphi(x), & l' = l-1, \\ 0, & \text{otherwise.} \end{cases} \tag{D.19}$$

The second reduced matrix element is [11, page 76]

$$\langle l' \| \vec{L}\varphi(x) \| l \rangle = [(2l+1)l(l+1)]^{\frac{1}{2}} \delta_{ll'} \varphi(x). \tag{D.20}$$

In order to determine the third reduced matrix element, we make use of the decomposition reation [11, page 109] :

$$\begin{aligned} \langle l' \| \vec{\nabla} \times \vec{L}\varphi(x) \| l \rangle &= \sqrt{3}(-)^{l+l'+1} \sum_{l''} \begin{Bmatrix} 1 & 1 & 1 \\ l & l' & l'' \end{Bmatrix} \\ &\quad \times \langle l' \| \vec{\nabla}\varphi(x) \| l'' \rangle \langle l'' \| \vec{L} \| l \rangle, \end{aligned} \tag{D.21}$$

and thus

$$\begin{aligned} \langle l' \| \vec{\nabla} \times \vec{L}\varphi(x) \| l \rangle &= \sqrt{3}(-)^{l+l'+1} \begin{Bmatrix} 1 & 1 & 1 \\ l & l' & l \end{Bmatrix} (2l+1)^{\frac{1}{2}} \\ &\quad \times \begin{cases} (l+1)l^{\frac{1}{2}} \left(\dfrac{\partial}{\partial x} - \dfrac{l}{x}\right)\varphi(x), & l' = l+1, \\ -(l+1)^{\frac{1}{2}} l \left(\dfrac{\partial}{\partial x} + \dfrac{l+1}{x}\right)\varphi(x), & l' = l-1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{D.22}$$

Thus for the full vector operators between our states we find for the L, M and E cases

$$\begin{aligned}\langle l_m \| \vec{a}_n^{\mathrm{L}} \varphi_n(x) \| l_n \rangle &= \langle l_m \| \frac{1}{ik_n^{\mathrm{L}}} \vec{\nabla} \varphi_n(x) \| l_n \rangle \\ &= \begin{cases} i(l_n+1)^{\frac{1}{2}} \varphi_{n(l_m)}(x), & l_m = l_n + 1, \\ i(l_n)^{\frac{1}{2}} \varphi_{n(l_m)}(x), & l_m = l_n - 1, \\ 0, & \text{otherwise}, \end{cases}\end{aligned} \tag{D.23}$$

$$\begin{aligned}\langle l_m \| \vec{a}_n^{\mathrm{M}} \varphi_n(x) \| l_n \rangle &= \langle l_m \| \frac{\vec{L}}{\sqrt{l_n(l_n+1)}} \varphi_n(x) \| l_n \rangle \\ &= (2l_n+1)^{\frac{1}{2}} \varphi_{n(l_m)}(x) \delta_{l_m l_n},\end{aligned} \tag{D.24}$$

$$\begin{aligned}\langle l_m \| \vec{a}_n^{\mathrm{E}} \varphi_n(x) \| l_n \rangle &= \langle l_m \| \frac{1}{ik_n^{\mathrm{E}}} \vec{\nabla} \times \frac{\vec{L}}{\sqrt{l_n(l_n+1)}} \varphi_n(x) \| l_n \rangle \\ &= \sqrt{3} \begin{Bmatrix} 1 & 1 & 1 \\ l_n & l_m & l_n \end{Bmatrix} (2l_n+1)^{\frac{1}{2}} \\ &\times \begin{cases} i(l_n+1)^{\frac{1}{2}} \varphi_{n(l_m)}(x), & l_m = l_n + 1, \\ i(l_n)^{\frac{1}{2}} \varphi_{n(l_m)}(x), & l_m = l_n - 1, \\ 0, & \text{otherwise}, \end{cases}\end{aligned} \tag{D.25}$$

respectively, where we have used that [1]

$$\begin{aligned}\frac{1}{k_n}\left(\frac{\partial}{\partial x} - \frac{l_n}{x}\right) \mathrm{j}_{l_n}(k_n x) &= -\mathrm{j}_{l_n+1}(k_n x), \\ \frac{1}{k_n}\left(\frac{\partial}{\partial x} + \frac{l_n+1}{x}\right) \mathrm{j}_{l_n}(k_n x) &= \mathrm{j}_{l_n-1}(k_n x),\end{aligned} \tag{D.26}$$

and defined

$$\varphi_{n(\mu)}(x) = \frac{\mathcal{N}_n}{R^{\frac{3}{2}}} j_\mu(k_n x). \tag{D.27}$$

We also make use of the fact that both the vector operator $\vec{a}_n$, and the radial wavefunction $\varphi_n(x)$ originate out of the ket of these matrix elements.

D.3 Residual m-sums in Matrix Elements

If we compare the four terms of the integral

$$I_{1234} = \int d^3x A^\mu_1(\vec{x}) A^*_{2\mu}(\vec{x}) A^\nu_3(\vec{x}) A^*_{4\nu}(\vec{x}) \tag{D.28}$$

in equations (D.17), (D.16) and (D.15) we find that the m-dependence of this integral goes as

$$\frac{(2l+1)}{4\pi} \sum_m (-)^{m_2+m_4-m} \begin{pmatrix} l_1 & l_2 & l \\ m_1 & -m_2 & m \end{pmatrix} \begin{pmatrix} l_3 & l_4 & l \\ m_3 & -m_4 & -m \end{pmatrix}. \tag{D.29}$$

D.3.1 m-sums in the Full Propagator

Consider now the form of the self energy (as derived in Chapter 3)

$$\begin{aligned} \Sigma^{11}_{mnab}(\omega+\mu_0) = 2n_0 g^2 C_\Lambda \delta_{ab} \Big\{ & 2 \int d^3x A^\mu_0(\vec{x}) A^*_{0\mu}(\vec{x}) A^\nu_m(\vec{x}) A^*_{n\nu}(\vec{x}) \\ & - \int d^3x A^\mu_m(\vec{x}) A_{0\mu}(\vec{x}) A^{*\nu}_0(\vec{x}) A^*_{n\nu}(\vec{x}) \\ & - \int d^3x A^\mu_m(\vec{x}) A^*_{0\mu}(\vec{x}) A^\nu_0(\vec{x}) A^*_{n\nu}(\vec{x}) \Big\}. \end{aligned} \tag{D.30}$$

In the first term we have an m-dependence that goes as (remembering that there is also a sum over the m-projection of the condensate propagator, i.e. over m_0)

$$\frac{(2l+1)}{4\pi} \sum_{mm_0} (-)^{m_0+m_n-m} \begin{pmatrix} l_0 & l_0 & l \\ m_0 & -m_0 & m \end{pmatrix} \begin{pmatrix} l_m & l_n & l \\ m_m & -m_n & -m \end{pmatrix}$$

$$= \frac{(2l+1)}{4\pi}\left\{\sum_{m_0}(-)^{m_0}\begin{pmatrix} l_0 & l_0 & l \\ m_0 & -m_0 & 0\end{pmatrix}\right\}(-)^{m_n}\begin{pmatrix} l_m & l_n & l \\ m_n & -m_n & 0\end{pmatrix}\delta_{m_m m_n}$$

$$= \frac{(-)^{l_0}}{4\pi}(2l_0+1)^{\frac{1}{2}}(-)^{l_n}(2l_n+1)^{-\frac{1}{2}}\,\delta_{l_m l_n}\,\delta_{m_m m_n}\,\delta_{l0}, \qquad \text{(D.31)}$$

since

$$\sum_{m_0}(-)^{m_0}\begin{pmatrix} l_0 & l_0 & l \\ m_0 & -m_0 & 0\end{pmatrix} = \sum_{m_0}(-)^{l_0}(2l_0+1)^{\frac{1}{2}} \times \begin{pmatrix} l_0 & l_0 & 0 \\ m_0 & -m_0 & 0\end{pmatrix}\begin{pmatrix} l_0 & l_0 & l \\ m_0 & -m_0 & 0\end{pmatrix} = (-)^{l_0}(2l_0+1)^{\frac{1}{2}}\,\delta_{l0} \qquad \text{(D.32)}$$

(see [11, pages 48–50]). For the second term the m-dependence is

$$\frac{(2l+1)}{4\pi}\sum_{m m_0}(-)^{m_0+m_n-m}\begin{pmatrix} l_m & l_0 & l \\ m_m & m_0 & m\end{pmatrix}\begin{pmatrix} l_0 & l_n & l \\ -m_0 & -m_n & -m\end{pmatrix}$$

$$= \frac{(2l+1)}{4\pi}\sum_{m m_0}\begin{pmatrix} l_0 & l & l_m \\ m_0 & m & m_m\end{pmatrix}\begin{pmatrix} l_0 & l & l_n \\ m_0 & m & m_n\end{pmatrix}$$

$$= \frac{(2l+1)}{4\pi}(2l_n+1)^{-1}\,\delta_{l_m l_n}\,\delta_{m_m m_n}\,\Delta(l_0, l_n, l), \qquad \text{(D.33)}$$

and the third term, by analogy has the same dependence. Hence we see that the m-dependence is trivial, being independent of the m-projection as, we would have expected. The self-energy is also diagonal in the angular momentum index l_n, but does exhibit an l-dependence.

We must also consider the anomalous self-energies derived in Chapter 3

$$\Sigma^{02}_{mnab}(\omega+\mu_0) = 2n_0\,g^2 m_{ab}\Big\{\int d^3x A_{m\mu}(\vec{x})A_0^{*\mu}(\vec{x})A_0^{*\nu}(\vec{x})A_{n\nu}(\vec{x}) - \int d^3x A_0^{*\mu}(\vec{x})A^*_{0\mu}(\vec{x})A^{\nu}_{m}(\vec{x})A_{n\nu}(\vec{x})\Big\}. \qquad \text{(D.34)}$$

In this case, we wish to add no angular momentum to the condensate, so we create the two $A_0^n(\vec{x})$ particles in the anomalous condensate propagator with opposite m-projections, coupled to angular momentum zero. The m-dependence of the first term is thus

$$\begin{aligned}
&\frac{(2l+1)}{4\pi}\sum_{mm_0}(-)^m\begin{pmatrix} l_m & l_0 & l \\ m_m & -m_0 & m\end{pmatrix}\begin{pmatrix} l_0 & l_n & l \\ m_0 & m_n & -m\end{pmatrix}\begin{pmatrix} l_0 & l_0 & 0 \\ m_0 & -m_0 & 0\end{pmatrix} \\
&= \frac{(2l+1)}{4\pi}(-)^{l_0-m_m}(2l_0+1)^{-\frac{1}{2}} \\
&\times \sum_{mm_0}\begin{pmatrix} l_0 & l & l_m \\ -m_0 & m & m_m\end{pmatrix}\begin{pmatrix} l_0 & l & l_n \\ -m_0 & m & -m_n\end{pmatrix} \\
&= \frac{(2l+1)}{4\pi}(-)^{l_0+m_n}(2l_0+1)^{-\frac{1}{2}}(2l_n+1)^{-1}\,\delta_{l_m l_n}\,\delta_{m_m-m_n}\,\Delta(l_0,l_n,l).
\end{aligned} \tag{D.35}$$

The second term we handle analogously :

$$\begin{aligned}
&\frac{(2l+1)}{4\pi}\sum_{mm_0}(-)^m\begin{pmatrix} l_0 & l_0 & l \\ m_0 & -m_0 & m\end{pmatrix}\begin{pmatrix} l_m & l_n & l \\ m_m & m_n & -m\end{pmatrix}\begin{pmatrix} l_0 & l_0 & 0 \\ m_0 & -m_0 & 0\end{pmatrix} \\
&= \frac{(2l+1)}{4\pi}(-)^{l_0}(2l_0+1)^{-\frac{1}{2}} \\
&\times \left\{\sum_{m_0}(-)^{m_0}\begin{pmatrix} l_0 & l_0 & l \\ m_0 & -m_0 & 0\end{pmatrix}\right\}\begin{pmatrix} l_m & l_n & l \\ -m_n & m_n & 0\end{pmatrix}\delta_{m_m-m_n} \\
&= \frac{(-)^{l_n+m_n}}{4\pi}(2l_n+1)^{-\frac{1}{2}}\,\delta_{l_m l_n}\,\delta_{m_m-m_n}\,\delta_{l0}.
\end{aligned} \tag{D.36}$$

Thus the matrix element is diagonal in l, but off-diagonal in m with dependence

$$(-)^{m_n} f(l_n)\,\delta_{l_m l_n}\,\delta_{m_m-m_n}. \tag{D.37}$$

However this anomalous self-energy only enters our propagator in a product term, where it is multiplied by the Σ^{20} self-energy, which is given by

$$\Sigma^{02}_{mnab}(\omega+\mu_0) = \left[\Sigma^{20}_{mnab}(\omega+\mu_0)\right]^*, \tag{D.38}$$

so that the product term goes as

$$f^2(l_n)\,\delta_{l_m l_n}\,\delta_{m_m m_n} \tag{D.39}$$

i.e. it is diagonal. Thus all the quantities contributing to the full propagator have a trivial m-dependence and thus so does the propagator itself. We need only consider the $m_m = m_n = 0$ subspace, all the m-values being degenerate.

D.3.2 m-sums in the Chemical Potential μ_0

In Appendix B, we found that the chemical potential was given by

$$\begin{aligned}\mu_0 = 2n_0 g^2 C_A N_C \Big\{ 2 \int d^3x A_0^\mu(\vec{x}) A_{0\prime}^*(\vec{x}) A_{0\prime}^\nu(\vec{x}) A_{0\prime\nu}^*(\vec{x}) \\ - \int d^3x A_0^\mu(\vec{x}) A_{0\prime\mu}^*(\vec{x}) A_{0\prime}^\nu(\vec{x}) A_{0\prime\nu}(\vec{x}) \\ - \int d^3x A_0^{*\mu}(\vec{x}) A_{0\prime\mu}^*(\vec{x}) A_0^\nu(\vec{x}) A_{0\prime\nu}(\vec{x}) \Big\},\end{aligned} \tag{D.40}$$

where the 0 and 0′ indices denoted the two residual m_0-sums associated with the two independent condensate propagators found in the diagrams. Thus in the first term, we have m-dependence

$$\begin{aligned}&\frac{(2l+1)}{4\pi} \sum_{m m_0 m_0'} (-)^{m_0+m_0'-m} \begin{pmatrix} l_0 & l_0 & l \\ m_0 & -m_0 & m \end{pmatrix} \begin{pmatrix} l_0 & l_0 & l \\ m_0' & -m_0' & -m \end{pmatrix} \\ &= \frac{(2l+1)}{4\pi} \left\{ \sum_{m_0} (-)^{m_0} \begin{pmatrix} l_0 & l_0 & l \\ m_0 & -m_0 & 0 \end{pmatrix} \right\}^2 \\ &= \frac{1}{4\pi}(2l_0+1)\,\delta_{l0},\end{aligned} \tag{D.41}$$

using the result of (D.32). The second term is handled similarly and goes as

$$\frac{(2l+1)}{4\pi} \sum_{m m_0 m_0'} (-)^{m_0+m_0'-m} \begin{pmatrix} l_0 & l_0 & l \\ m_0 & -m_0' & m \end{pmatrix} \begin{pmatrix} l_0 & l_0 & l \\ -m_0 & m_0' & -m \end{pmatrix}$$

$$= \frac{(2l+1)}{4\pi}(-)^l \sum_m \left\{ \sum_{m_0 m_0'} \begin{pmatrix} l_0 & l_0 & l \\ -m_0 & m_0' & -m \end{pmatrix} \begin{pmatrix} l_0 & l_0 & l \\ -m_0 & m_0' & -m \end{pmatrix} \right\}$$

$$= \frac{(2l+1)}{4\pi}(-)^l \sum_m \left\{ (2l+1)^{-1} \Delta(l_0, l_0, l) \right\}$$

$$= \frac{(2l+1)}{4\pi}(-)^l \Delta(l_0, l_0, l). \tag{D.42}$$

The third term goes as

$$\frac{(2l+1)}{4\pi} \sum_{m m_0 m_0'} (-)^{m_0+m_0'-m} \begin{pmatrix} l_0 & l_0 & l \\ -m_0 & -m_0' & m \end{pmatrix} \begin{pmatrix} l_0 & l_0 & l \\ m_0 & m_0' & -m \end{pmatrix}$$

$$= \frac{(2l+1)}{4\pi}(-)^l \sum_m \left\{ \sum_{m_0 m_0'} \begin{pmatrix} l_0 & l_0 & l \\ m_0 & m_0' & -m \end{pmatrix} \begin{pmatrix} l_0 & l_0 & l \\ m_0 & m_0' & -m \end{pmatrix} \right\}$$

$$= \frac{(2l+1)}{4\pi}(-)^l \Delta(l_0, l_0, l). \tag{D.43}$$

The integrals are then evaluated using the general method detailed above.

D.4 Selection Rules of the Interaction

As shown above, all the self-energies are diagonal in the angular-momentum index l, and thus we know that angular momentum must be conserved and only states of the same angular momentum can interact. Closer examination of the forms of the terms of the interaction given by equations (D.17), (D.16) and (D.15), reveals that for states with a scalar nature, where the term (D.15) is non-zero, there is no further selection rule. Hence all Longitudinal states of a given angular momentum will interact. However when one of the two external lines in the interaction denotes a transverse state, only the term given by (D.15) is non-zero, and this has a dependence

$$\begin{pmatrix} l_m' & l_n' & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_0 & l_0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{D.44}$$

in one case, and

$$\begin{pmatrix} l'_m & l_0 & l \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l'_n & l_0 & l \\ 0 & 0 & 0 \end{pmatrix}, \tag{D.45}$$

in the other. Since we know that

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \tag{D.46}$$

is only non zero if $l_1 + l_2 + l_3$ is even, it is clear from the above that this term of the interaction will only be zero if l'_m and l'_n are both even or both odd. However, we already know that $l_m = l_n$ and the reduced matrix elements in Section D.2 above tell us that the allowed values of l' for a given l are

$$\begin{array}{ll} l' = l \pm 1, & \text{for L, E} \\ l' = l, & \text{for M.} \end{array} \tag{D.47}$$

Hence we see that Longitudinal and Transverse Electric states will form one sub-block together, and the Transverse Magnetic states another, for each angular momentum value l. We make use of this fact, in Chapter 4, to consider the convergence of results as a function of the size of the model space in which they are calculated.

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