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2011-07-07
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Symmetries and Conservation Laws of Higher-order Systems of Partial Differential Equations

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A dissertation submitted to the Faculty of Science, University of the Witwatersrand, in fulfilment of the requirements for the degree of Master of Science.

Johannesburg, 2011.

DECLARATION

I declare that the contents of this dissertation are original except where due references have been made. It is being submitted for the degree of Master of Science at the University of the Witwatersrand in Johannesburg. It has not been submitted before for any degree to any other institution.

_____

R.M. Morris

This _____ day of _____ 2011.

Acknowledgements

I gratefully acknowledge the continued support and advice of my supervisor, Professor A.H. Kara. I really appreciate the time he has taken to share his guidance and technical assistance.

This work has been made possible in part by the University of the Witwatersrand as well as the National Research Foundation (NRF) of South Africa (grant number FA - 2007041200006).

I would also like to say a special thanks to my family and friends for their moral support, advice and encouragement while compiling my dissertation.

Abstract

Conservation laws for nonlinear partial differential equations (pdes) have been determined through different approaches. In this dissertation, we study conservation laws for some third-order systems of pdes, viz., some versions of the Boussinesq equations, as well as a version of the BBM equation and the well-known Ito equation. It is shown that new and interesting conserved quantities arise from 'multipliers' that are of order greater than one in derivatives of the dependent variables. Furthermore, the invariance properties of the conserved flows with respect to the Lie point symmetry generators are investigated via the symmetry action on the multipliers.

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Introduction

The Lie symmetry approach is now an established route for the reduction of differential equations and its advantages in the analysis of nonlinear partial differential equations (pdes) is vast. The method centres around the algebra of one parameter Lie groups of transformations that are admitted by the pde; when these have been determined, the reduction of the pde is standard and may lead to exact (symmetry invariant) solutions [7, 17]. There are a number of reasons to compute conserved densities and fluxes of pdes. Some conservation laws are fundamental laws of physics (e.g., conservation of momentum, mass and energy) and others facilitate the analysis of the pde. The existence of a large number of conservation laws is a predictor of complete integrability [5]. Other reasons are related to the numerical solution of pdes. For example, one should check whether the conserved quantities are in fact constant [4].

The role of 'multipliers' has been shown to play a significant role in the construction of conserved densities and fluxes [3] and in determining hierarchies. In short, knowledge of a multiplier, by formula, leads to a conserved flow. Furthermore, the use of symmetry properties of a given system of pdes to construct or generate new conservation laws from known conservation laws has been investigated [9, 10].

This dissertation is structured as follows.

In the first chapter, we state the definitions of the fundamental concepts that will be required to perform the calculations and provide an illustrative example.

In the second chapter, we calculate the conserved quantities via the multiplier approach as discussed above and then we analyse the symmetry action of the Lie point symmetry generators on the multipliers for a version of the BBM equation (section 2.2), the classical Boussinesq system of equations (section 2.3) and the bi-Hamiltonian Boussinesq system of equations (section 2.4).

In the third and final chapter, we apply the same technique on the fifth-order Ito equation.

Chapter 1

Preliminaries

1.1 Introduction

In this chapter, we state the necessary definitions of the fundamental concepts that will be used throughout this dissertation and then we provide an illustrative example based on a version of the nonlinear Schrödinger equation.

1.2 Fundamental Concepts

The definitions and notations that will be used are presented below.

A function $f(x, u, u_{(1)}, \ldots, u_{(r)})$ of a finite number of variables is called a differential function of order r.

$u_{(1)}, u_{(2)}, \ldots, u_{(r)}$ denotes the collections of all first, second, ..., rth-order partial derivatives, that is, $u_i^\alpha = D_i(u^\alpha), u_{ij}^\alpha = D_j D_i(u^\alpha), \ldots$ respectively, with the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \ldots, \quad i = 1, \ldots, n, \tag{1.1}$$

where the summation convention is used whenever appropriate.

We denote $\mathcal{A}$ to be the universal vector space of differential functions.

Consider an rth-order system of pdes of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$

$$G^\mu(x, u, u_{(1)}, \ldots, u_{(r)}) = 0, \quad \mu = 1, \ldots, \tilde{m}. \tag{1.2}$$

A current $\Phi = (\Phi^1, \ldots, \Phi^n)$ is conserved if it satisfies

$$D_i \Phi^i = 0 \tag{1.3}$$

along the solutions of (1.2). It can be shown that every admitted conservation law arises from *multipliers* $Q_\mu(x, u, u_{(1)}, \ldots)$ such that

$$Q_\mu G^\mu = D_i \Phi^i \tag{1.4}$$

holds identically (i.e., off the solution space) everywhere, not just on solutions for some current Φ^i.

Definition 1. The *Euler* operator, for each α, is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 \cdots i_s}^\alpha}, \quad \alpha = 1, \ldots, m. \tag{1.5}$$

Definition 2. The Lie-Bäcklund or generalised operator is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \qquad \xi^i, \ \eta^\alpha \in \mathcal{A}. \tag{1.6}$$

This operator is an abbreviated form of the following infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} \zeta^\alpha_{i_1\ldots i_s} \frac{\partial}{\partial u^\alpha_{i_1\ldots i_s}}, \tag{1.7}$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} \zeta^\alpha_i &= D_i(W^\alpha) + \xi^j u^\alpha_{ij}\ , \\ \zeta^\alpha_{i_1\ldots i_s} &= D_{i_1}\ldots D_{i_s}(W^\alpha) + \xi^j u^\alpha_{ji_1\ldots i_s}, \quad s>1. \end{aligned} \tag{1.8}$$

In (1.8), W^α is the Lie characteristic function given by

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j. \tag{1.9}$$

One can write the Lie-Bäcklund or generalised operator (1.7) in the characteristic form

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} D_{i_1}\ldots D_{i_s}(W^\alpha) \frac{\partial}{\partial u^\alpha_{i_1\ldots i_s}}. \tag{1.10}$$

We determine the conserved flows by first constructing the multipliers Q_μ which are obtained by noting that the Euler operator annihilates total divergences, i.e., the defining equation would be

$$\frac{\delta}{\delta u^\alpha}[Q_\mu G^\mu] = 0. \tag{1.11}$$

To calculate the conserved quantities, we need to invert the total divergence operator. This requires the integration (by parts) of an expression in multi-dimensions involving arbitrary functions and its derivatives, which is a difficult and cumbersome task. The homotopy operator [1, 5, 17] is a powerful algorithmic tool (explicit formula) that originates from homological algebra and variational bi-complexes. It is used in integrability testing and inversion problems involving pdes. It reduces the inversion of the total divergence operator to a standard integration of one auxiliary

variable and is calculated via calculus based formulas that involve higher-order Euler operators [5].

After calculating the conserved quantities, we observe the relationship between the Lie point symmetry generators and the multipliers. If the action of a Lie point symmetry generator on a multiplier is equal to 0, this implies strict invariance, while if this action results in a scalar multiple of the original multiplier, this implies ray invariance. In the case where neither strict invariance nor ray invariance occurs, this action results in new multipliers in terms of the old multipliers, so that the space of the multipliers is closed under the action of the Lie point symmetry generators. Thus new conservation laws can be calculated.

1.3 Illustrative Example

We first consider an illustrative example via a version of the nonlinear Schrödinger equation. This version is called the Gross-Pitaevskii equation, which has been discussed and analysed in many texts, e.g., [17]. It arises in quantum physics where it describes a zero temperature Bose-Einstein condensate (BEC) for which the scattering length between atoms is smaller than the spacing between atoms [20]. It is of the form

$$iq_t - q_{xx} - w(x)q - g|q|^2 q = 0, \tag{1.12}$$

where q is the condensate wave function of a complex order parameter.

If we write $q = u + iv$ and choose $w(x) = x^2$ and $g = 1$, then (1.12) becomes

$$i(u_t + iv_t) - u_{xx} - iv_{xx} - x^2(u + iv) - (u^2 + v^2)(u + iv) = 0. \tag{1.13}$$

Separating (1.13) into real and imaginary parts leads to the following second-order

system of pdes

$$\begin{aligned} G^1 &= u_t - v_{xx} - x^2 v - (u^2 + v^2)v = 0, \\ G^2 &= v_t + u_{xx} + x^2 u + (u^2 + v^2)u = 0, \end{aligned} \tag{1.14}$$

where G^1 and G^2 are functions satisfying (1.2).

The multiplier (f, g) will satisfy the 'joint' Euler operator

$$\frac{\delta}{\delta(u, v)}(f * G^1 + g * G^2) = 0. \tag{1.15}$$

Equation (1.15) is a consequence of two dependent variables and is equivalent to the action of the Euler operator on each dependent variable, given by

$$\begin{aligned} \frac{\delta}{\delta u}(f * G^1 + g * G^2) = 0, \\ \frac{\delta}{\delta v}(f * G^1 + g * G^2) = 0. \end{aligned} \tag{1.16}$$

Thus by (1.4), we have

$$f * G^1 + g * G^2 = D_x \Phi^x + D_t \Phi^t.$$

It turns out that, inter alia, we get the following pairs of multipliers

$$\begin{aligned} (f_1, g_1) &= (u, v), \\ (f_2, g_2) &= (-v_t, u_t). \end{aligned} \tag{1.17}$$

To calculate the conservation laws, the densities and fluxes respectively are given by the homotopy formula [5] or by expanding the following

$$D_x \Phi_i^x + D_t \Phi_i^t = f_i * G^1 + g_i * G^2. \tag{1.18}$$

This results in the components of the conserved vectors

$$\begin{aligned} \Phi_1^x &= v u_x - u v_x, \\ \Phi_1^t &= \tfrac{1}{2}(u^2 + v^2) \end{aligned}$$

and

$$\Phi_2^x = \tfrac{1}{2}(u_t u_x + v_t v_x - u u_{xt} - v v_{xt}),$$
$$\Phi_2^t = \tfrac{1}{4}[u^4 + v^4 + 2u^2v^2 + 2x^2(u^2 + v^2) + 2u u_{xx} + 2v v_{xx}].$$

In complex form, the densities are given as

$$\Phi_1^t = \tfrac{1}{2}|q|^2,$$
$$\Phi_2^t = |q|^4 + 2x^2|q|^2 + q\bar{q_{xx}} + q_{xx}\bar{q},$$

respectively.

To conclude this section, we apply the total divergence on (Φ_1^x, Φ_1^t) and (Φ_2^x, Φ_2^t). That is,

$$\begin{aligned}
&D_x\Phi_1^x + D_t\Phi_1^t \\
&= D_x(vu_x - uv_x) + D_t[\frac{1}{2}(u^2 + v^2)] \\
&= v_x u_x + v u_{xx} - u_x v_x - u v_{xx} + u u_t + v v_t \\
&= u(u_t - v_{xx}) + v(v_t + u_{xx}) \\
&= u[u_t - v_{xx} - x^2 v - (u^2 + v^2)v] + v[v_t + u_{xx} + x^2 u + (u^2 + v^2)u] \\
&= f_1 * G^1 + g_1 * G^2
\end{aligned}$$

and

$$\begin{aligned}
&D_x\Phi_2^x + D_t\Phi_2^t \\
&= D_x[\frac{1}{2}(u_t u_x + v_t v_x - u u_{xt} - v v_{xt})] + D_t[\frac{1}{4}(u^4 + v^4 + 2u^2v^2 + 2x^2u^2 \\
&\quad + 2x^2v^2 + 2u u_{xx} + 2v v_{xx})] \\
&= \frac{1}{2}u_{xt}u_x + \frac{1}{2}u_t u_{xx} + \frac{1}{2}v_{xt}v_x + \frac{1}{2}v_t v_{xx} - \frac{1}{2}u_x u_{xt} - \frac{1}{2}u u_{xxt} - \frac{1}{2}v_x v_{xt} \\
&\quad - \frac{1}{2}v v_{xxt} + u^3 u_t + x u u_t + u v^2 u_t + u^2 v v_t + \frac{1}{2}u_t u_{xx} + \frac{1}{2}u u_{xxt}
\end{aligned}$$

$$
\begin{aligned}
& + xvv_t + v^3 v_t + \frac{1}{2} v_t v_{xx} + \frac{1}{2} v v_{xxt} \\
= \; & -v_t[-v_{xx} - xv - (u^2 + v^2)v] + u_t[u_{xx} + xu + (u^2 + v^2)u] \\
= \; & -v_t[-v_{xx} - xv - (u^2 + v^2)v + u_t] + u_t[u_{xx} + xu + (u^2 + v^2)u + v_t] \\
= \; & f_2 * G^1 + g_2 * G^2.
\end{aligned}
$$

This shows that the conserved vector $(\Phi_i^1, \Phi_i^2) = (\Phi_i^x, \Phi_i^t)$, as described above, satisfies (1.3) along the solutions of (1.14).

Chapter 2

New Conservation Laws of some Third-order Systems of PDE's arising from Higher-order Multipliers

2.1 Introduction and Background

The evolutionary third-order classical Boussinesq system of equations

$$u_t + v_x + uu_x - \frac{1}{3}u_{xxt} = 0, \quad v_t + uv_x + vu_x + u_x = 0 \tag{2.1}$$

arises in the study of dissipative mechanics which models two-dimensional small amplitude long wavelength water waves (see [2, 15]). Generally, finding the con-

served quantities of (2.1) from first principles is not a trivial task. Determining a Lagrangian, if it exists, would in itself be a nontrivial undertaking. Moreover, obtaining higher-order conservation laws can be an even more cumbersome undertaking. We resort to an approach that produces a larger class of nontrivial conserved vectors from multipliers that are higher than first-order in derivatives (of the dependent variables) [18, 19].

Another third-order system arises from rewriting the Boussinesq equation in the model for unidirectional propagation of long waves in shallow water (this will be elaborated as a final example in this chapter) [17].

Before analysing the above cases, for illustrative purposes, we study a preliminary third-order scalar example [8, 12, 15], viz., the Benjamin-Bona-Mahony (BBM) equation, of which we consider the following version

$$u_t + 2uu_x + nu_x - u_{xxt} = 0. \tag{2.2}$$

This equation was originally derived as an approximation for surface water waves in a uniform channel. We will investigate the possible existence of higher-order derivative dependent multipliers giving rise to conservation laws.

The results of this chapter appear in [16].

2.2 The BBM Equation

We first consider a slight variation of the scalar BBM equation with a dispersion term, viz.,

$$u_t + 2uu_x + nu_x - u_{xxt} = 0 \tag{2.3}$$

where n is a real constant.

By (1.6), we define $X = \xi\partial_x + \tau\partial_t + \phi\partial_u$ to be the Lie-Bäcklund operator, where $\xi = \xi(x,t,u),\ \tau = \tau(x,t,u),\ \phi = \phi(x,t,u)$ that leaves invariant (2.3), i.e.,

$$X^{[3]}(u_t + 2uu_x + nu_x - u_{xxt}) = 0 \tag{2.4}$$

subject to

$$u_t = u_{xxt} - nu_x - 2uu_x.$$

To expand (2.4) further, we use the prolongation formulae from (1.8) and the Lie characteristic function from (1.9). This results in the following equation

$$\begin{aligned} &2\phi u_x + \phi_t - \xi_t u_x + (\phi_u - \tau_t)u_t - \xi_u u_x u_t - \tau_u u_t^2 + (2u+n)[\phi_x + (\phi_u - \xi_x)u_x \\ &- \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t] - \phi^{xxt} = 0. \end{aligned} \tag{2.5}$$

The governing equations are obtained from (2.5) by the separation of the monomials,

viz., coefficients of the derivatives in the dependent variable u. That is,

$$\begin{gathered}
\xi_u = 0, \\
\tau_u = 0, \\
\phi_{uu} = 0, \\
\xi_t = 0, \\
\phi_{ut} = 0, \\
\tau_x = 0, \\
-\phi_t - n\phi_x - 2u\phi_x + \phi_{xxt} = 0, \\
-2\phi + n\xi_x + 2u\xi_x - n\tau_t - 2u\tau_t + 2\phi_{xtu} - n\phi_{xxu} - 2u\phi_{xxu} = 0, \\
-2\xi_x + \phi_{xxu} = 0, \\
-\xi_{xx} + 2\phi_{xu} = 0.
\end{gathered} \tag{2.6}$$

The Lie point symmetry generators of (2.3) by a solution of the over determined system (2.6) are given by

$$X = \partial_t, \quad Y = \partial_x, \quad Z = -\frac{2}{n}t\partial_t + \frac{1}{n}(n + 2u)\partial_u.$$

We now solve

$$\frac{\delta}{\delta u}(Q(u_t + 2uu_x + nu_x - u_{xxt})) = 0, \tag{2.7}$$

where we suppose second-order derivative dependence of the multiplier

$$Q = f(x, t, u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}).$$

Equation (2.7) has to be satisfied for all functions $u(x, t)$ and not only the solutions of (2.3).

When we expand the left hand side of equation (2.7), we first use Leibniz's rule.

This results in

$$\begin{aligned} & Q_u(u_t + 2uu_x + nu_x - u_{xxt}) + 2Qu_x \\ & \quad - D_x[Q_{u_x}(u_t + 2uu_x + nu_x - u_{xxt}) + Q(2u + n)] \\ & \quad - D_t[Q_{u_t}(u_t + 2uu_x + nu_x - u_{xxt}) + Q] \\ & \quad + D_x^2 D_t(Q) = 0. \end{aligned} \tag{2.8}$$

The elaborated and expanded form of (2.8) is calculated using the computer algebra system (CAS) Maple.

The tedious calculations after expansion and separation by monomials of (2.8) reveals that $Q = f$ is of order zero or two in derivatives with respect to u. We list these as

$$\begin{aligned} Q_1 = &\ 1, \\ Q_2 = &\ u, \\ Q_3 = &\ u_{xt} - u^2. \end{aligned} \tag{2.9}$$

A symmetry study of the multipliers in (2.9) is done using Z, i.e.,

$$ZQ_1 = 0, \quad ZQ_2 = Q_1 + \frac{2}{n}Q_2, \quad ZQ_3 = \frac{4}{n}Q_3 - 2Q_2 \tag{2.10}$$

so that all the Q_i's are strictly invariant under the translations X and Y, and Q_1 is also strictly invariant under Z. The action of Z on Q_2 gives rise to Q_1 and since the action of Z on Q_3 is a linear combination of Q_3 and Q_2, we may suppose that Q_3 forms a basis of the space of multipliers and so too would the conserved vector following Q_3. The components of the conserved vectors (Φ^x, Φ^t) (via the homotopy

formula) are, respectively,

$$\begin{aligned}
&\Phi_1^x = u^2 + nu - u_{xt},\\
&\Phi_1^t = u,\\
&\\
&\Phi_2^x = \tfrac{1}{6}(3nu^2 + 4u^3 + 2u_t u_x - 4uu_{xt}),\\
&\Phi_2^t = \tfrac{1}{6}(3u^2 + {u_x}^2 - 2uu_{xx})
\end{aligned}$$

and

$$\begin{aligned}
\Phi_3^x &= \frac{1}{12}[-4nu^3 - 6u^4 + 3{u_t}^2 + 12u^2u_{xt} - 4{u_{xt}}^2 + 2u_x u_{xtt} + u_t(3nu_x - u_{xxt})\\
&\quad - u(3u_{tt} - 3nu_{xt} + u_{xxtt})],\\
\Phi_3^t &= \frac{1}{12}[-4u^3 + 3u_t u_x + 3n{u_x}^2 - 2u_{xt}u_{xx} - u_x u_{xxt} + u(3u_{xt} - 3nu_{xx} + u_{xxxt})].
\end{aligned}$$

The total divergence on (Φ_1^x, Φ_1^t) and (Φ_2^x, Φ_2^t) is

$$\begin{aligned}
&D_x\Phi_1^x + D_t\Phi_1^t\\
&\quad = \quad D_x(u^2 + nu - u_{xt}) + D_t(u)\\
&\quad = \quad Q_1(u_t + 2uu_x + nu_x - u_{xxt}),
\end{aligned}$$

and

$$\begin{aligned}
&D_x\Phi_2^x + D_t\Phi_2^t\\
&\quad = \quad D_x[\frac{1}{6}(3nu^2 + 4u^3 + 2u_t u_x - 4uu_{xt})] + D_t[\frac{1}{6}(3u^2 + {u_x}^2 - 2uu_{xx})]\\
&\quad = \quad \frac{1}{6}(6nuu_x + 12u^2u_x + 2u_t u_{xx} + 2u_{xt}u_x - 4uu_{xxt} - 4u_x u_{xt})\\
&\qquad + \frac{1}{6}(6uu_t + 2u_x u_{xt} - 2uu_{xxt} - 2u_t u_{xx})\\
&\quad = \quad Q_2(u_t + 2uu_x + nu_x - u_{xxt}).
\end{aligned}$$

Similarly, the total divergence on the conserved vector (Φ_3^x, Φ_3^t) is

$$D_x\Phi_3^x + D_t\Phi_3^t = Q_3(u_t + 2uu_x + nu_x - u_{xxt}),$$

so that the above conserved vectors satisfy (1.3) along the solutions of (2.3).

2.3 The Classical Boussinesq System of Equations

Our main example is the classical Boussinesq system of equations, wherein we emphasize the role of the Euler operator in two variables and its complexity as the order function on which it operates increases (we choose possible second-order multipliers).

This system is given by

$$\begin{aligned} G^1 &= u_t + v_x + uu_x - \tfrac{1}{3}u_{xxt} = 0, \\ G^2 &= v_t + uv_x + vu_x + u_x = 0, \end{aligned} \tag{2.11}$$

where G^1 and G^2 are functions satisfying (1.2).

By (1.6), we define $X = \xi\partial_x + \tau\partial_t + \phi\partial_u + \eta\partial_v$ to be the Lie-Bäcklund operator, where $\xi = \xi(x,t,u,v),\ \tau = \tau(x,t,u,v),\ \phi = \phi(x,t,u,v),\ \eta = \eta(x,t,u,v)$ that leaves invariant (2.11), i.e.,

$$\begin{aligned} X^{[3]}(u_t + v_x + uu_x - \tfrac{1}{3}u_{xxt}) = 0, \\ X^{[1]}(v_t + uv_x + vu_x + u_x) = 0 \end{aligned} \tag{2.12}$$

subject to

$$u_t = \frac{1}{3}u_{xxt} - v_x - uu_x, \quad v_t = -u_x - uv_x - vu_x.$$

When we apply the prolongation formulae on (2.12), we obtain

$$\begin{aligned} \phi u_x + \phi^x u + \phi^t + \eta^x - \tfrac{1}{3}\phi^{xxt} = 0, \\ \phi v_x + \eta u_x + \phi^x(v+1) + \eta^x u + \eta^t = 0. \end{aligned} \tag{2.13}$$

Separation by monomials of (2.13) leads to the over determined system

$$\begin{gathered}
\phi_v = 0, \\
\xi_v = 0, \\
\tau_v = 0, \\
\eta_u = 0, \\
\xi_u = 0, \\
\tau_u = 0, \\
\phi_{uu} = 0, \\
\xi_t = 0, \\
\phi_{tu} = 0, \\
\tau_x = 0, \\
3\eta_x + 3\phi_t + 3u\phi_x - \phi_{xxt} = 0, \\
3\phi - 3u\xi_x + 3u\tau_t - 2\phi_{xtu} + u\phi_{xxu} = 0, \\
6\xi_x - \phi_{xxu} = 0, \\
3\eta_v - 3\xi_x + 3\tau_t - 3\phi_u + \phi_{xxu} = 0, \\
-\xi_{xx} + 2\phi_{xu} = 0, \\
\eta - \eta_v - v\eta_v - \xi_x - v\xi_x + \tau_t + v\tau_t + \phi_u + v\phi_u = 0, \\
\phi - u\xi_x + u\tau_t = 0, \\
\eta_t + u\eta_x + \phi_x + v\phi_x = 0.
\end{gathered} \tag{2.14}$$

We can conclude that the Lie point symmetry generators of (2.11) are

$$X = \partial_t, \quad Y = \partial_x, \quad Z = -\frac{1}{2}t\partial_t + \frac{1}{2}u\partial_u + (1+v)\partial_v.$$

Now, we apply the 'joint' Euler operator, which annihilates the total divergence

$$\frac{\delta}{\delta(u,v)}(f * G^1 + g * G^2) = 0, \tag{2.15}$$

where

$$Q = f(x, t, u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xt}, v_{xt})$$

and

$$P = g(x, t, u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xt}, v_{xt}).$$

Equation (2.15) has to be satisfied for all functions $u(x,t)$ and $v(x,t)$ and not only the solutions of (2.11).

The expansion of the left hand side of (2.15) is extensive and requires the use of Maple to enumerate, particularly in the separation of the monomials and solving the over determined system of pdes. In summarized form, the 'more interesting' multipliers are given by

$$\begin{aligned} f_1 &= k, \quad g_1 = F(t)[(1+v)u_{xx} + uv_{xx} + v_{xt} + 2u_x v_x] \\ f_2 &= u, \quad g_2 = \ln(1+v) \end{aligned} \tag{2.16}$$

where F is an arbitrary function of t and k is a constant.

We now observe a symmetry analysis of the multipliers f_1 and g_1 under the generators X, Y and Z as follows

$$\begin{aligned} Xf_1 &= 0, \quad Xg_1 = g_1^*, \\ Yf_1 &= 0, \quad Yg_1 = 0, \\ Zf_1 &= 0, \quad Zg_1 = \tfrac{1}{2}g_1, \end{aligned} \tag{2.17}$$

where we chose $F(t) = t^2$.

This implies partial strict invariance under the generators X and Z, since the action of these symmetries on the multiplier f_1 only is equal to 0, while we have strict invariance under the generator Y.

We note that $g_1^* = H(t)[(1+v)u_{xx} + uv_{xx} + v_{xt} + 2u_x v_x]$ for some arbitrary function $H(t)$ different from $F(t)$, so that the action of the generator X on g_1 results in a new multiplier in terms of the old multiplier and thus would result in a new conservation law.

The components of the conserved vector corresponding to the multipliers f_1 and g_1 is given by

$$\begin{aligned}
\Phi_1^x &= \frac{1}{36}[9t^2{v_t}^2 + 27t^2 v_t u_x + 18t^2{u_x}^2 + 18u^2(k + t^2{v_x}^2) - 8ku_{xt} \\
&\quad + 6tv^2(-2u_x + 3t{u_x}^2 - tu_{xt}) + 3v(12k - 3t^2 v_{tt} - 6tu_x + 12t^2{u_x}^2 \\
&\quad + 2tv_t(-3 + 4tu_x) - 2t^2 u_t v_x - 3t^2 u_{xt}) + 6tu(t(5v_t + 6u_x)v_x \\
&\quad + v((-2 + 6tu_x)v_x - tv_{xt}))], \\
\Phi_1^t &= \frac{1}{36}[9t^2 v_t v_x + 9t^2 u_x v_x + 18t^2 v u_x v_x + 9t^2 v v_{xt} - 4ku_{xx} + 9t^2 v u_{xx} \\
&\quad + 6t^2 v^2 u_{xx} + 6u(6k + t^2{v_x}^2 + t^2 v v_{xx})].
\end{aligned}$$

The symmetry analysis of the multipliers f_2 and g_2 under the generators X and Y reveals that

$$\begin{aligned}
&Xf_2 = 0, \quad Xg_2 = 0, \quad \text{strict invariance} \\
&Yf_2 = 0, \quad Yg_2 = 0, \quad \text{strict invariance.}
\end{aligned} \tag{2.18}$$

However, the action of the generator Z shows that $Zf_2 = \frac{1}{2}f_2$ and $Zg_2 = 1$. This implies partial ray invariance, since the action of the generator Z on f_2 only is a scalar multiple of f_2.

The components of the conserved vector corresponding to these multipliers is given by

$$\Phi_2^x = \frac{1}{9}[3u^3 + u_t u_x + u(9(1+v)\ln(1+v) - 2u_{xt})],$$

$$\Phi_2^t = \ln(1+v) + \frac{1}{2}u^2 + [-1+\ln(1+v)]v + \frac{{u_x}^2}{18} - \frac{1}{9}uu_{xx}.$$

It suffices to illustrate the total divergence on the conserved flow (Φ_2^x, Φ_2^t). That is,

$$\begin{aligned}
&D_x\Phi_2^x + D_t\Phi_2^t \\
&= D_x[\frac{1}{9}(3u^3 + u_tu_x + u(9(1+v)\ln(1+v) - 2u_{xt})] + D_t[\ln(1+v) + \frac{1}{2}u^2 \\
&\quad + (-1+\ln(1+v))v + \frac{{u_x}^2}{18} - \frac{1}{9}uu_{xx}] \\
&= u^2u_x + u_x(1+v)\ln(1+v) + uv_x\ln(1+v) + uv_x + \frac{v_t}{(1+v)} \\
&\quad + uu_t - v_t + v_t\ln(1+v) + \frac{vv_t}{(1+v)} - \frac{1}{3}uu_{xxt} \\
&= u(u_t + v_x + uu_x - \frac{1}{3}u_{xxt}) + \ln(1+v)(v_t + uv_x + vu_x + u_x) \\
&= f_2 * G^1 + g_2 * G^2.
\end{aligned}$$

Thus, the conserved flow (Φ_2^x, Φ_2^t) satisfies (1.3) along the solutions of (2.11).

2.4 The bi-Hamiltonian Boussinesq System of Equations

The version of the Boussinesq equation that arises in the model for unidirectional propagation of long waves in shallow water is given by

$$u_{tt} = \frac{4}{3}(u^2)_{xx} + \frac{1}{3}u_{xxxx} \tag{2.19}$$

which, as a (bi-Hamiltonian) system, can be written in an evolutionary form

$$u_t - v_x = 0, \quad v_t = \frac{1}{3}u_{xxx} + \frac{8}{3}uu_x. \tag{2.20}$$

The conservation laws of (2.19) and a systems version of the Boussinesq equation have been discussed and determined in [11, 13] using 'partial Lagrangians'. Only the conservation laws corresponding to multipliers of up to first-order, at best, were obtainable from Noether's Theorem via the *Noether-type symmetries*. For example, in the former case, the multipliers were 1, t, x and xt with the corresponding Noether-type generators ∂_u, $t\partial_u$, $x\partial_u$ and $xt\partial_u$. The procedure used above (in section 2.3) on (2.20) gives rise to the known multipliers $(Q, P) = (v, u)$ and the previously unknown second-order case

$$(Q, P) = (u_{xx} + 4u^2, 3v).$$

The corresponding conserved flow is then calculated to be

$$(\Phi^x, \Phi^t) = (\frac{1}{2}(-8u^2v + u_tu_x - uu_{xt} - 2vu_{xx}), \frac{1}{6}(8u^3 + 9v^2 + 3uu_{xx})).$$

The total divergence relating to this conserved flow (Φ^x, Φ^t) is given by

$$\begin{aligned} & D_x\Phi^x + D_t\Phi^t \\ &= D_x[\frac{1}{2}(-8u^2v + u_tu_x - uu_{xt} - 2vu_{xx})] + D_t[\frac{1}{6}(8u^3 + 9v^2 + 3uu_{xx})] \\ &= Q(u_t - v_x) + P[(v_t - \frac{1}{3})u_{xxx} - \frac{8}{3}uu_x], \end{aligned}$$

so that the conserved flow (Φ^x, Φ^t) satisfies (1.3) along the solutions of (2.20).

As in section 2.3, we define $X = \xi\partial_x + \tau\partial_t + \phi\partial_u + \eta\partial_v$ to be the Lie-Bäcklund operator, that leaves invariant (2.20), i.e.,

$$\begin{aligned} & X^{[1]}(u_t - v_x) = 0, \\ & X^{[3]}(v_t - \tfrac{1}{3}u_{xxx} - \tfrac{8}{3}uu_x) = 0 \end{aligned} \tag{2.21}$$

subject to

$$u_t = v_x, \quad v_t = \frac{1}{3}u_{xxx} + \frac{8}{3}uu_x.$$

After applying the prolongation formulae on (2.20), we obtain

$$\begin{aligned} &\phi^t - \eta^x = 0, \\ &-\tfrac{8}{3}\phi u_x - \tfrac{8}{3}\phi^x u + \eta^t - \tfrac{1}{3}\phi^{xxx} = 0. \end{aligned} \tag{2.22}$$

The separation of monomials of (2.22) leads to the over determined system

$$\begin{gathered} \phi_v = 0, \\ \xi_v = 0, \\ \tau_v = 0, \\ \xi_u = 0, \\ \tau_u = 0, \\ \phi_{uu} = 0, \\ \tau_x = 0, \\ 8\phi - 8u\eta_v - 8u\xi_x + 8u\tau_t + 8u\phi_u - \xi_{xxx} + 3\phi_{xxu} = 0, \\ 3\eta_t - 8u\phi_x - \phi_{xxx} = 0, \\ -\xi_{xx} + \phi_{xu} = 0, \\ \eta_v - \xi_x + \tau_t - \phi_u = 0, \\ \eta_v + 3\xi_x - \tau_t - \phi_u = 0, \\ -\eta_x + \phi_t = 0, \\ \eta_u - \xi_t = 0, \\ \eta_u + \xi_t = 0. \end{gathered} \tag{2.23}$$

Consequently, the Lie point symmetry generators of (2.20) are

$$W = \partial_t, \quad X = \partial_x, \quad Y = \partial_v, \quad Z = -x\partial_x - 2t\partial_t + 2u\partial_u + 3v\partial_v.$$

We observe the symmetry analysis of the multipliers $(Q, P) = (u_{xx} + 4u^2, 3v)$ under the generator Z as

$$ZQ = 4Q, \quad ZP = 3P, \quad \text{ray invariance.} \tag{2.24}$$

2.5 Discussion and Conclusion

We have obtained new conservation laws for some well known classes of third-order systems, viz., some versions of the Boussinesq and BBM equations by studying the possible existence of higher-order multipliers - each of which gives rise to a unique conserved flow.

In the following chapter, we will apply the same approach to a fifth-order pde.

Chapter 3

Analysis of the Fifth-order Ito Equation

3.1 Introduction and Background

The fifth-order Ito equation has been studied extensively. It was first proposed by Ito and he showed that this equation has multi-soliton solutions, derives a Lax representation and a bilinear Bäcklund transformation.

It has been observed that this equation is closely associated to a Kac-Moody algebra and a nonlinear superposition formula has been derived [6]. The Ito equation reads

$$D_t(D_t + D_x^3)f \cdot f = 0 \tag{3.1}$$

so that via the dependent variable transform $u = 2(\ln f)_{xx}$, it becomes

$$u_{tt} + u_{xxxt} + 3(2u_x u_t + u u_{xt}) + 3u_{xx} \int_{-\infty}^{x} u_t(x', t)dx' = 0. \tag{3.2}$$

We will assume the well known version ([14, 21])

$$u_{xtt} + u_{xxxxt} + 6u_{xx}u_{xt} + 3u_x u_{xxt} + 3u_{xxx}u_t = 0, \tag{3.3}$$

which plays a very important role in mathematical physics and engineering sciences.

3.2 Application of the Ito Equation

As in chapter 2 (section 2.2), we define $X = \xi\partial_x + \tau\partial_t + \phi\partial_u$ to be the Lie-Bäcklund operator, where $\xi = \xi(x, t, u)$, $\tau = \tau(x, t, u)$, $\phi = \phi(x, t, u)$ that leaves invariant (3.3), i.e.,

$$X^{[5]}(u_{xtt} + u_{xxxxt} + 6u_{xx}u_{xt} + 3u_x u_{xxt} + 3u_{xxx}u_t) = 0. \tag{3.4}$$

After applying the prolongation formulae on (3.4), we obtain

$$\phi^{xtt} + \phi^{xxxxt} + 6\phi^{xx}u_{xt} + 6\phi^{xt}u_{xx} + 3\phi^{x}u_{xxt} + 3\phi^{xxt}u_x + 3\phi^{xxx}u_t + 3\phi^{t}u_{xxx} = 0. \tag{3.5}$$

The separation of monomials of (3.5) results in the over determined system

$$\begin{gathered}
\xi_u = 0, \\
\tau_u = 0, \\
\phi_{uu} = 0, \\
\phi_t = 0, \\
\xi_t = 0, \\
\phi_{tu} = 0, \\
\phi_x = 0, \\
\tau_x = 0, \\
\phi_{xu} = 0, \\
\phi_{xtu} = 0, \\
\xi_{xx} = 0, \\
\phi_{xxu} = 0, \\
\phi_{xtt} + \phi_{xxxxt} = 0, \\
3\phi_{xxx} + \phi_{xxxxu} = 0, \\
\phi_{ttu} + 3\phi_{xxt} + 4\phi_{xxxtu} = 0, \\
-\tau_{tt} + 6\phi_{xx} + 4\phi_{xxxu} = 0, \\
\phi_{xt} + \phi_{xxtu} = 0, \\
-3\xi_x + \tau_t = 0, \\
-2\xi_x + \tau_t + \phi_u = 0, \\
-2\xi_x + \tau_t + \phi_u = 0.
\end{gathered} \tag{3.6}$$

Consequently, the Lie point symmetry generators of (3.3) are

$$W = \partial_x, \quad X = \partial_t, \quad Y = \partial_u, \quad Z = x\partial_x + 3t\partial_t - u\partial_u.$$

We assume the multiplier $Q = f$ to be of third-order derivative dependence (of the dependent variable u), i.e., $Q = f(x, t, u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, u_{xxt}, u_{xtt}, u_{ttt})$.

When we apply the Euler operator (1.11) on (3.3), it annihilates the total divergence

$$\frac{\delta}{\delta u}(Q(u_{xtt} + u_{xxxxt} + 6u_{xx}u_{xt} + 3u_x u_{xxt} + 3u_{xxx}u_t)) = 0. \tag{3.7}$$

As before, we first use Leibniz's rule to expand the left hand side of (3.7).
This results in

$$\begin{aligned}
&Q_u(u_{xtt} + u_{xxxxt} + 6u_{xx}u_{xt} + 3u_x u_{xxt} + 3u_{xxx}u_t) \\
&\quad - D_x[Q_{u_x}(u_{xtt} + u_{xxxxt} + 6u_{xx}u_{xt} + 3u_x u_{xxt} + 3u_{xxx}u_t) + 3Qu_{xxt}] \\
&\quad - D_t[Q_{u_t}(u_{xtt} + u_{xxxxt} + 6u_{xx}u_{xt} + 3u_x u_{xxt} + 3u_{xxx}u_t) + 3Qu_{xxx}] \\
&\quad + 6D_x^2(Qu_{xt}) + 6D_x D_t(Qu_{xx}) - 3D_x^3(Qu_t) - 3D_x^2 D_t(Qu_x) \\
&\quad - D_x D_t^2(Q) - D_x^4 D_t(Q) \;=\; 0.
\end{aligned} \tag{3.8}$$

To expand (3.8) further, it similarly requires some kind of software package to enumerate.

The tedious calculations after expansion and separation by monomials of (3.8) reveals that there are no first, second or third order multipliers. The other nontrivial ones independent of derivatives are, with the corresponding conserved densities,

$$\begin{aligned}
Q_1 &= xt, \\
\Phi_1^t &= \frac{1}{30}[-10tu_t - 10xu_x - 15t{u_x}^2 + 20txu_{xt} + 45txu_x u_{xx} - 6tu_{xxx} \\
&\qquad - 5u(-4 + 3tu_{xx} - 3txu_{xxx}) + 6txu_{xxxx}],
\end{aligned}$$

$$\begin{aligned}
Q_2 &= x, \\
\Phi_2^t &= \frac{1}{30}(-10u_t - 15{u_x}^2 + 20xu_{xt} - 15uu_{xx} + 45xu_x u_{xx} - 6u_{xxx} + 15xuu_{xxx} \\
&\qquad + 6xu_{xxxx}),
\end{aligned}$$

$$
\begin{aligned}
Q_3 &= u, \\
\Phi_3^t &= \frac{1}{30}[-10u_t u_x - 10{u_x}^3 + 20uu_{xt} + 3{u_{xx}}^2 + 6u_x(5uu_{xx} - u_{xxx}) \\
&\qquad + 10u^2 u_{xxx} + 6uu_{xxxx}]
\end{aligned}
$$

and

$$
\begin{aligned}
Q_\infty &= F(t), \\
\Phi_4^t &= \frac{1}{30}[-10F'u_x + F(20u_{xt} + 45u_x u_{xx} + 15uu_{xxx} + 6u_{xxxx})],
\end{aligned}
$$

where F is an arbitrary function of t.

The action of all the symmetries on the above multipliers are listed as

$$
\begin{array}{llll}
WQ_1 = t, & WQ_2 = 1, & WQ_3 = 0, & WQ_\infty = 0, \\
XQ_1 = 4Q_1, & XQ_2 = Q_2, & XQ_3 = -Q_3, & XQ_\infty = 3tF', \\
YQ_1 = 0, & YQ_2 = 0, & YQ_3 = 1, & YQ_\infty = 0, \\
ZQ_1 = Q_2, & ZQ_2 = 0, & ZQ_3 = 0, & ZQ_\infty = F'.
\end{array}
\tag{3.9}
$$

In particular, the action of W on Q_1 and Q_2, X on Q_∞, Y on Q_3 and Z on Q_∞ results in new multipliers that are functions of $Q_\infty = F(t)$. The action of W on Q_3 and Q_∞, Y on Q_1, Q_2 and Q_∞, Z on Q_2 and Q_3 implies strict invariance. The action of X on Q_1, Q_2 and Q_3 implies ray invariance.

We observe that the space of the multipliers is closed under the action of the Lie point symmetry generators, where $\{Q_1, Q_3, Q_\infty\}$ forms a basis of the space of multipliers. Q_2 is not included in the set since the action of the generator Z on Q_1 resulted in Q_2.

3.3 Discussion and Conclusion

In this chapter, we obtained new conserved densities of the well known fifth-order Ito equation, by assuming the possible existence of higher-order multipliers. Four multipliers were determined using the Euler operator and they were all derivative independent. Lastly, the action of the Lie point symmetry generators on the multipliers in some cases lead to some alternative multipliers and thus would result in alternative conservation laws.

Conclusion

In this dissertation, our main objective was to calculate Lie symmetries and conservation laws of some higher-order systems of pdes and investigate the relationship between these. To calculate the conserved quantities, we first calculated the multipliers via the Euler operator acting on a total divergence. The corresponding conserved quantities were then determined via the homotopy formula. From calculating the Lie point symmetry generators, we observed that in some cases the symmetry properties of the multipliers gave rise to alternative multipliers. This implies that alternative conservation laws can be calculated.

In the second chapter, we analysed a different version of the scalar BBM equation and some third-order Boussinesq systems of equations. Some of the multipliers were calculated to be of order greater than one in derivatives of the dependent variables. We could then construct new and interesting conserved quantities.

In the third chapter, we applied the same approach on the fifth-order Ito equation. All of the multipliers were calculated to be derivative independent and each of these gave rise to a unique conserved quantity.

It was necessary to exclude many of the tedious calculations, as they were very involved.

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