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Preservation theorems for algebraic and relational models of logic

Wilmari Morton

A thesis submitted to the School of Computer Science,
Faculty of Science,
University of the Witwatersrand, Johannesburg
in fulfilment of the requirements for the degree of Doctor of Philosophy.

Supervisor: Prof. Clint J. van Alten,
School of Computer Science,
University of the Witwatersrand.

Co-supervisor: Dr. Willem Conradie,
Department of Mathematics,
University of Johannesburg.

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DECLARATION

I declare that this thesis entitled *"Preservation theorems for algebraic and relational models of logic"* is the result of my own research except as cited in the references. It is being submitted for the degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. The thesis has not been accepted for any degree and is not concurrently submitted in candidature of any other degree.



Wilmari Morton
Johannesburg, 15 May 2013

ABSTRACT

In this thesis a number of different constructions on ordered algebraic structures are studied. In particular, two types of constructions are considered: completions and finite embeddability property constructions.

A main theme of this thesis is to determine, for each construction under consideration, whether or not a class of ordered algebraic structures is closed under the construction. Another main focus of this thesis is, for a particular construction, to give a syntactical description of properties preserved by the construction. A property is said to be preserved by a construction if, whenever an ordered algebraic structure satisfies it, then the structure obtained through the construction also satisfies the property.

The first four constructions investigated in this thesis are types of completions. A completion of an ordered algebraic structure consists of a completely lattice ordered algebraic structure and an embedding that embeds the former into the latter. Firstly, different types of filters (dually, ideals) of partially ordered sets are investigated. These are then used to form the filter (dually, ideal) completions of partially ordered sets. The other completions of ordered algebraic structures studied here include the MacNeille completion, the canonical extension (also called the completion with respect to a polarization) and finally a prime filter completion.

A class of algebras has the finite embeddability property if every finite partial subalgebra of some algebra in the class can be embedded into some finite algebra in the class. Firstly, two constructions that establish the finite embeddability property for residuated ordered structures are investigated. Both of these constructions are based on completion constructions: the first on the MacNeille completion and the second on the canonical extension. Finally, algebraic filtrations on modal algebras are considered and a duality between algebraic and relational versions of filtrations is established.

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1. GENERAL INTRODUCTION

Non-classical logics, such as modal or substructural logics, often have both natural algebraic and relational (Kripke-style) semantics. One of the primary concerns of algebraic logic is the identification of classes of algebras that are suitable for the study of various logics. If a class of algebras can be found that algebraizes a given logic in a natural way, then algebraic methods may be used to better understand the logic. A very well-known example of such a class of algebras is the class of Boolean algebras that algebraizes classical propositional logic.

The properties of such a class of algebras often correspond to properties of the logic and the relational semantics — thus introducing new ways of establishing results through duality and correspondence theories. An example relevant to this thesis is that decidability of the logic can often be obtained by showing that the class of algebras is generated by its finite members.

Once a class of algebras has been identified there are, broadly speaking, two courses of investigation to follow. Firstly, we can undertake a thorough investigation of the class of algebras. In doing so, we seek to obtain favourable results that will be applicable in the logic setting. This then introduces the second course of investigation, namely establishing links between properties of the logic and properties of the class of algebras. That is, translating logic problems into their algebraic counterparts and then translating algebraic results back into logic terms.

In this thesis we will focus, mostly, on algebraic models of non-classical logics — usually without explicit mention of the possible logics that may be interpreted on these algebraic structures. Moreover, for the most part we will pursue the first course of investigation by focussing our attention on the development of the algebraic theory. In particular, we will study a number of constructions on classes of ordered algebraic structures that are algebraic models for various non-classical logics. We do, however, also consider one problem where our focus will be on the translation of relational methods into algebraic ones and vice

versa.

By *preservation theorems* we mean the following types of results. We say that *a class of algebras is closed under a construction* if, given an algebra of the class, the algebraic structure obtained via the construction also belongs to the class. For example, substructural logics are logics whose algebraic models are residuated structures [GJKO07]. We are therefore interested in constructions that, when performed on residuated structures, yield residuated structures.

Furthermore, we say that an identity (an expression of the form $(\forall\vec{x})(s(\vec{x}) = t(\vec{x}))$, for terms s, t in the language) is *preserved by a construction* if, whenever the original structure satisfies the identity, then the structure obtained through the construction also satisfies the identity. We similarly define what it means for an inequality (an expression of the form $(\forall\vec{x})(s(\vec{x}) \leq t(\vec{x}))$ for terms s, t in the language), quasi-identity or any other property to be preserved by a construction. Our research is partly motivated by considerations on modal algebras. An important problem in modal logic is that of canonicity — the preservation of identities by the canonical extension. A classical result there is that the class of Sahlqvist identities [Sah75], a syntactically defined class, is preserved by the canonical extension [Jón94]. For a number of the constructions considered in this thesis our aim has been to prove Sahlqvist-like results by giving syntactic descriptions of classes of identities preserved by the respective constructions. Our algebraic approach is motivated by the algebraic approach for modal algebras in [Jón94] and [GV99].

This thesis is divided into two parts. In the first part we focus on completions. A *completion* of an ordered algebraic structure is a pair consisting of a complete ordered structure (see Chapter 2) and an embedding that maps the original structure into the complete one (see Chapter 3 for a precise definition). We will explore the motivations for completions in Chapter 3.

In most cases the properties of completions do not depend on the algebraic structure, but only on the underlying partial order of the algebra we wish to complete. For this reason we focus, firstly, on completions of partially ordered sets into complete lattices. We investigate ways of extending operations defined on the partially ordered set, to operations defined on the completions. We investigate properties of the operations preserved by the completions, for example order-preservation, residuation and distribution properties.

To start with we consider the *filter and ideal completions* of partially ordered

sets in Chapter 4. In the literature a wide range of different up-sets and down-sets have been called the 'filters' and 'ideals', respectively, of a partially ordered set. We survey these possible definitions. In particular, we will focus on four different families of up-sets and down-sets — different types of filters and ideals of a partially ordered set — that we believe are representative. Three of these families of filters (respectively, ideals) form complete lattices into which the original partially ordered set can be embedded, i.e., completions of the partially ordered set. The four types of filters and ideals introduced in this chapter will also be used in a number of the other constructions studied in this thesis (see Chapters 6 and 7). Next we investigate some of the properties of these filter and ideal completions. In the course of our investigations we consider the possible definitions of 'prime filters' (respectively, 'prime ideals') of a partially ordered set and relate them to the join-irreducible (respectively, meet-irreducible) elements of the completions. Prime filters and ideals of a partially ordered set will be used again in the completions considered in Chapter 7. Finally we consider the extension of order-preserving operations to these completions.

Next, in Chapter 5, we turn our attention to the *MacNeille completion*. The MacNeille completion of partially ordered sets and lattices has been studied in great depth and is well understood, see for instance [Mac37, TV07]. Furthermore, we can use the MacNeille completion to complete MTL-algebras [vA11] — the algebraic models of *monoidal t-norm logic* (see Chapter 5.2). We consider the expansion of an MTL-algebra with a single unary, order-preserving operation that we will call a modality. Such algebras will be called *modal MTL-algebras*. We begin by axiomatizing the class of modal MTL-algebras. We then use the MacNeille completion of modal MTL-algebras to obtain a Sahlqvist-like result for modal MTL, i.e., we give a syntactic description of properties (involving the newly added modality) that are preserved by the MacNeille completion of a modal MTL-algebra.

The third type of completion we study in Chapter 6 is the *canonical extension* or *completion with respect to polarizations* of partially ordered sets. We note that different completions have been called 'the canonical extension' of a partially ordered set. This is due to the fact that, depending on the type of filters and ideals of a partially ordered set used in the construction, one may obtain distinct completions. We investigate the construction in general, but also consider specific instances of completions obtained through this construction.

More specifically, the four types of filters and ideals introduced in Chapter 4 are used to obtain four, generally different, completions of a partially ordered set. We consider the extension of additional operations to each of these completions — focusing on distribution and residuation properties. An alternative construction that makes use of an intermediate structure is described. Finally we give some results toward a syntactical description of a class of properties preserved by these completions.

In Chapter 7 we characterize the partially ordered sets that can be embedded into completely distributive complete lattices and describe the construction of such a completely distributive complete lattice. The construction makes use of the 'prime filters' and 'prime ideals' defined in Chapter 4. We characterize the partially ordered sets for which the completion obtained in this chapter is isomorphic to one of the completions obtained in Chapter 6. We also consider the extension of order-preserving and order-reversing operations to the completion.

In the second part of the thesis we focus on constructions that produce finite models. These constructions have been used to prove the *finite embeddability property* for many varieties of algebras. A class of algebras has the finite embeddability property if every finite partial subalgebra of some algebra in the class can be embedded into a finite algebra in the class. Once the finite embeddability property has been established for a variety of algebras, the decidability of its universal (and hence, equational) theory and of the associated logic (if it is finitely axiomatized) may follow via algebraization results. See Chapter 8 for more details on the motivation behind such constructions.

In Chapter 9 we use the *standard construction* [vA09] for obtaining the finite embeddability property for a class of residuated (ordered) algebras, to obtain the finite embeddability property for the class of modal MTL-algebras. We also establish the finite embeddability property for a number of its subclasses by investigating properties preserved by the construction. The standard construction is based on the MacNeille completion (studied in Chapter 5). This then introduces the following question: Can a construction be devised that is based on the canonical extension (studied in Chapter 6)? In fact, it was this question that led us to investigate the canonical extensions of partially ordered sets. The answer to this question is 'yes' for decreasing lattice-ordered residuated structures. We describe this alternative construction, called the *canonical FEP construction*, and show that it may also be used to establish the finite em-

beddability property of some classes of algebras. We consider some additional properties preserved by this construction.

Finally, in Chapter 10 we study finite embeddability constructions for modal algebras. A *modal algebra* is a Boolean algebra equipped with a unary operator. We show that the algebraic constructions considered in this chapter are algebraic versions of model-theoretic filtrations. A *filtration* of a (Kripke) model is a finite (Kripke) model obtained with respect to a subformula closed set of formulas. We use the methods developed in this chapter to obtain the algebraic versions of a number of well-known model-theoretic filtrations.

2. PRELIMINARIES

In this chapter we give some basic definitions and fix the notations. The reader may consult [Bir67], [DP02] or [BS81] for more on the definitions and results given here.

For a set Q, let $\mathcal{P}(Q)$ denote the *powerset* of Q. We write $M \subseteq^{fin} Q$ to denote that M is a *finite subset* of Q. For $n \in \mathbb{N}$, let $M \subseteq^{n} Q$ denote that $M \subseteq Q$ *and* M *has* n *or fewer elements*. If $S \subseteq Q$, then $Q - S$ will denote the *set complement* of S in Q, i.e., $Q - S = \{a \in Q : a \notin S\}$.

2.1 Partially ordered sets

One of the main themes of this thesis will be to investigate various 'completions' (see Definition 3.0.1 in Chapter 3) of partially ordered sets. In this section we recall the definitions of partially ordered sets and related notions.

Definition 2.1.1. *A* quasi-ordered set *is a pair* $\mathbf{Q} = \langle Q, \leq \rangle$ *such that* Q *is a set and* $\leq$ *is a binary relation on* Q *such that, for all* $x, y, z \in Q$*,*

(i) $x \leq x$*, i.e.,* $\leq$ *is reflexive, and*

(ii) $x \leq y$ *and* $y \leq z$ *imply* $x \leq z$*, i.e.,* $\leq$ *is transitive.*

Then $\leq$ *is called a* quasi-order *on* Q *and* Q *is called the* universe *of* $\mathbf{Q}$*.*

Definition 2.1.2. *A* partially ordered set*, or* poset *for short, is a quasi-ordered set* $\mathbf{P} = \langle P, \leq \rangle$ *such that, in addition to (i) and (ii) in Definition 2.1.1 above,* $\leq$ *satisfies, for all* $x, y \in P$*,*

(iii) $x \leq y$ *and* $y \leq x$ *imply* $x = y$*, i.e.,* $\leq$ *is antisymmetric.*

Then $\leq$ *is called a* partial order *on* P*.*

We will sometimes write $\leq^{\mathbf{P}}$ to indicate that we are working with the order defined on the universe P of a poset $\mathbf{P} = \langle P, \leq^{\mathbf{P}} \rangle$. If there exists an element $y \in P$ such that $y \geq x$ for all $x \in P$, then y is called the *top element* and is denoted by $\top$ or 1. On the other hand, if P contains an element z such that $z \leq x$ for all $x \in P$, then z is called the *bottom element* and is denoted by $\bot$ or 0. We sometimes write $\top^{\mathbf{P}}$ and $\bot^{\mathbf{P}}$ to avoid ambiguity. A poset $\mathbf{P}$ is called *bounded* if it has both a top element and a bottom element.

Definition 2.1.3. *The* dual of a poset $\mathbf{P} = \langle P, \leq^{\mathbf{P}} \rangle$, *is the poset* $\mathbf{P}^{\partial} = \left\langle P, \leq^{\mathbf{P}^{\partial}} \right\rangle$ *such that* $\mathbf{P}^{\partial}$ *has the same universe as* $\mathbf{P}$ *but where* $\leq^{\mathbf{P}^{\partial}} \subseteq P \times P$ *is defined by:*

$$x \leq^{\mathbf{P}^{\partial}} y \iff y \leq^{\mathbf{P}} x$$

for all $x, y \in P$.

In general, given any statement about posets, the *dual statement* can be obtained by replacing $\leq$ with $\geq$ and vice versa.

A poset $\mathbf{P}$ is said to be *linearly ordered* if, for all $x, y \in P$ either $x \leq y$ or $y \leq x$. That is, any two elements of P are comparable. A linearly ordered poset is also called a *chain*.

Let $\mathbf{P} = \langle P, \leq \rangle$ be a poset and let $S \subseteq P$. An element $x \in P$ is an *upper bound of* S if $x \geq y$ for all $y \in S$. Dually, an element $z \in P$ is called a *lower bound of* S if $z \leq y$ for all $y \in S$. Let S^u and S^ℓ denote the sets of all upper and lower bounds of S, respectively. That is, define ${}^\ell : \mathcal{P}(P) \to \mathcal{P}(P)$ and ${}^u : \mathcal{P}(P) \to \mathcal{P}(P)$ by

$$S^\ell = \{a \in P : a \leq b \text{ for all } b \in S\}$$

and

$$S^u = \{a \in P : a \geq b \text{ for all } b \in S\}.$$

If S^u has a least element, then that least element is called the *supremum of* S. Dually, if S^ℓ has a greatest element, then that greatest element is called the *infimum of* S.

(i) If the *supremum of* S exists in $\mathbf{P}$, then we denote it by $\bigvee S$ and call it *the join of* S.

(ii) If the *infimum of* S exists in $\mathbf{P}$, then we denote it by $\bigwedge S$ and call it *the meet of* S.

The join and meet of a set $\{x, y\}$ are denoted by $x \vee y$ and $x \wedge y$, respectively. We will use a superscript $\mathbf{P}$ if it is necessary to indicate that a join or a meet is being found in a particular poset $\mathbf{P}$, i.e., we write $\vee^{\mathbf{P}}$ or $\wedge^{\mathbf{P}}$. It should be clear that the partial operations $\vee$ and $\wedge$ are induced by the ordering $\leq$ and implicitly defined for any poset $\mathbf{P}$.

Definition 2.1.4. *A* subposet *of a poset* $\mathbf{P} = \langle P, \leq \rangle$ *is any poset* $\mathbf{Q} = \langle Q, \leq^{\mathbf{Q}} \rangle$ *such that* $Q \subseteq P$ *and* $\leq^{\mathbf{Q}} \subseteq Q \times Q$ *is defined by*

$$x \leq^{\mathbf{Q}} y \iff x \leq^{\mathbf{P}} y$$

for all $x, y \in Q$.

Definition 2.1.5. *The* (direct or Cartesian) product *of the posets* $\mathbf{P}_1, \ldots, \mathbf{P}_n$ *for some* $n \in \mathbb{N}$, *where* $\mathbf{P}_i = \langle P_i, \leq^{\mathbf{P}_i} \rangle$ *for* $i = 1, \ldots, n$, *is the poset* $\prod_{i=1}^{n} \mathbf{P}_i$ *such that* $\prod_{i=1}^{n} P_i$ *is its universe and its ordering* $\leq$ *is the coordinate-wise ordering defined by:*

$$(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n) \iff x_i \leq^{\mathbf{P}_i} y_i \text{ for } i = 1, \ldots, n$$

for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \prod_{i=1}^{n} P_i$. *If* $n = 2$, *then we write* $\mathbf{P}_1 \times \mathbf{P}_2$ *to denote the product of* $\mathbf{P}_1$ *and* $\mathbf{P}_2$.

2.2 Algebras, lattices and Boolean algebras

A *type* (or *language*) of algebras is a set $\mathbb{T}$ of function symbols such that a nonnegative integer n, called the *arity*, is assigned to each $f \in \mathbb{T}$. If the arity of f is n, then f is said to be an n-ary function symbol.

An *algebra of type* $\mathbb{T}$ is a structure $\mathbf{A} = \langle A, \mathbb{T}^{\mathbf{A}} \rangle$ such that A is a set called the *universe* (or *underlying set*) of the algebra $\mathbf{A}$ and for each n-ary function symbol $f \in \mathbb{T}$ there is a function $f^{\mathbf{A}} : A^n \to A$ in $\mathbb{T}^{\mathbf{A}}$. Each operation $f^{\mathbf{A}} \in \mathbb{T}^{\mathbf{A}}$ is called a *fundamental operation* of the algebra. We will often omit the superscript $\mathbf{A}$ (and write f instead of $f^{\mathbf{A}}$) when it is clear from the context.

Definition 2.2.1. *A* join-semilattice *is a poset* $\mathbf{L} = \langle L, \leq \rangle$ *such that* L *is a non-empty set and the supremum of each finite subset of* L *exists.*

The order $\leq$ now induces a (fully defined) binary operation $\vee^{\mathbf{L}}$ on L such that $x \vee^{\mathbf{L}} y$ equals the supremum of $\{x, y\}$ for all $x, y \in L$.

We will sometimes write $\mathbf{L} = \langle L, \vee^{\mathbf{L}} \rangle$, where $\vee^{\mathbf{L}}$ is idempotent, commutative and associative, when we refer to a join-semilattice to emphasize that $\vee^{\mathbf{L}}$ is defined on the entire $L \times L$ and that $\vee^{\mathbf{L}}$ forms part of the language. Then the associated partial order of $\mathbf{L}$ is defined by, for all $x, y \in L$,

$$x \leq y \iff x \vee^{\mathbf{L}} y = y.$$

A *meet-semilattice* can now be defined dually.

Definition 2.2.2. *A* meet-semilattice *is a poset* $\mathbf{L} = \langle L, \leq \rangle$ *such that* L *is a non-empty set and the infimum of each finite subset of* L *exists.*

The order $\leq$ now induces a (fully defined) binary operation $\wedge^{\mathbf{L}}$ on L such that $x \wedge^{\mathbf{L}} y$ equals the infimum of $\{x, y\}$ for all $x, y \in L$.

As with join-semilattices we will sometimes write $\mathbf{L} = \langle L, \wedge^{\mathbf{L}} \rangle$, where $\wedge^{\mathbf{L}}$ is idempotent, commutative and associative, to indicate that $\wedge^{\mathbf{L}}$ is fully defined and that $\wedge^{\mathbf{L}}$ forms part of the language. The associated partial order of $\mathbf{L}$ is defined by, for all $x, y \in L$,

$$x \leq y \iff x \wedge^{\mathbf{L}} y = x.$$

Definition 2.2.3. *A* lattice *is an algebra* $\mathbf{L} = \langle L, \vee, \wedge \rangle$ *such that* L *is a non-empty set equipped with two binary operations* $\vee : L \times L \to L$ *and* $\wedge : L \times L \to L$ *that satisfies*

$$\begin{aligned} (x \vee y) \vee z = x \vee (y \vee z) \quad &\text{and} \quad (x \wedge y) \wedge z = x \wedge (y \wedge z) \\ x \vee y = y \vee x \quad &\text{and} \quad x \wedge y = y \wedge x \\ x \vee x = x \quad &\text{and} \quad x \wedge x = x \\ x \vee (x \wedge y) = x \quad &\text{and} \quad x \wedge (x \vee y) = x \end{aligned}$$

for all $x, y, z \in L$.

We can now define a partial order $\leq^{\mathbf{L}}$ on L in terms of $\vee$ and $\wedge$ as follows:

$$x \leq^{\mathbf{L}} y \iff x \vee y = y \iff x \wedge y = x$$

for all $x, y \in L$. Then $\leq^{\mathbf{L}}$ is called *the associated lattice order of* $\mathbf{L}$ and $\langle L, \leq^{\mathbf{L}} \rangle$ is a poset. Furthermore, the operations $\vee$ and $\wedge$ correspond with the induced join and meet operations of $\leq^{\mathbf{L}}$, respectively, i.e., the processes of obtaining

supremums and infimums in $\langle L, \leq^{\mathbf{L}} \rangle$, as defined in the previous section. That is, $x \vee y$ is the least element of $\{x, y\}^u$ and $x \wedge y$ is the greatest element of $\{x, y\}^\ell$, for all $x, y \in L$. We may therefore view a lattice as a poset such that the supremum and infimum exist for all finite subsets of L (even though the languages are technically not the same). Hence, a lattice can be seen as both a join-semilattice and a meet-semilattice. Depending on the context, we will sometimes view a lattice as an algebraic structure and at other times view it as a poset.

If $\mathbf{L} = \langle L, \vee, \wedge \rangle$ is a lattice such that $\langle L, \leq^{\mathbf{L}} \rangle$ is bounded, then we often denote the top element of $\mathbf{L}$ by 1 and the bottom element by 0. Moreover, $x \wedge 1 = x$ and $x \vee 0 = x$ for all $x \in L$. Sometimes $1, 0$ are included in the language in which case we write $\mathbf{L} = \langle L, \vee, \wedge, 0, 1 \rangle$.

Given the above we can now make the following definition.

Definition 2.2.4. *A* complete lattice *is a lattice* $\mathbf{L} = \langle L, \vee, \wedge \rangle$ *such that the sumpremum and the infimum (with respect to the associated lattice order* $\leq^{\mathbf{L}}$*) exist for all subsets of* L*.*

Definition 2.2.5. *A* sublattice *of a lattice* $\mathbf{L} = \langle L, \vee^{\mathbf{L}}, \wedge^{\mathbf{L}} \rangle$ *is a lattice* $\mathbf{L}' = \langle L', \vee^{\mathbf{L}'}, \wedge^{\mathbf{L}'} \rangle$ *such that* $L' \subseteq L$ *and if* $x, y \in L'$*, then* $x \vee^{\mathbf{L}'} y = x \vee^{\mathbf{L}} y \in L'$ *and* $x \wedge^{\mathbf{L}'} y = x \wedge^{\mathbf{L}} y \in L'$*.*

Thus, $\vee^{\mathbf{L}'}$ is the restriction of $\vee^{\mathbf{L}}$ to L' and $\wedge^{\mathbf{L}'}$ is the restriction of $\wedge^{\mathbf{L}}$ to L'.

Definition 2.2.6. *Let* $\mathbf{L} = \langle L, \vee, \wedge \rangle$ *be a lattice. Then* $\mathbf{L}$ *is said to be*

(i) distributive *if it satisfies the* distributive law*: for all* $x, y, z \in L$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

(ii) modular *if it satisfies the* modular law*: for all* $x, y, z \in L$

$$x \geq z \text{ implies } x \wedge (y \vee z) = (x \wedge y) \vee z.$$

For complete lattices we have the following stronger condition.

Definition 2.2.7. *If a lattice* $\mathbf{L}$ *is complete, then* $\mathbf{L}$ *is called* completely distributive *if, for any doubly indexed subset* $\{x_{ij}\}_{i \in \Psi, j \in \Phi}$ *of* L*, we have*

$$\bigwedge_{i \in \Psi} \left(\bigvee_{j \in \Phi} x_{ij} \right) = \bigvee_{\gamma : \Psi \to \Phi} \left(\bigwedge_{i \in \Psi} x_{i\gamma(i)} \right),$$

where $\gamma : \Psi \to \Phi$ *is a choice function, i.e.,* $\gamma(i) = j$ *for some* $j \in \Phi$.

Let S be any set. Then $\mathbf{L} = \langle \mathcal{P}(S), \cup, \cap \rangle$ is a completely distributive complete lattice. In fact, any complete lattice $\langle L, \vee, \wedge \rangle$ such that L is a set of sets, $\vee$ is $\cup$ and $\wedge$ is $\cap$, is completely distributive.

Definition 2.2.8. *Let* $\mathbf{L} = \langle L, \vee, \wedge \rangle$ *be a lattice. An element* $x \in L$ *is* join-irreducible *if*

(i) x *is not the bottom element (if it exists in* $\mathbf{L}$*),*

(ii) $x = y \vee z$ *implies* $x = y$ *or* $x = z$ *for all* $y, z \in L$.

The following condition is equivalent to condition (ii):

(ii)' If $y < x$ *and* $z < x$*, then* $y \vee z < x$ *for all* $y, z \in L$.

Dually, an element $x \in L$ *is* meet-irreducible *if*

(a) x *is not the top element (if it exists in* $\mathbf{L}$*),*

(b) $x = y \wedge z$ *implies* $x = y$ *or* $x = z$ *for all* $y, z \in L$.

The following condition is equivalent to condition (b):

(b)' If $y > x$ *and* $z > x$*, then* $y \wedge z > x$ *for all* $y, z \in L$.

Equivalently, an element $x \in L$ is join-irreducible if, whenever $x = \bigvee X$ and X is finite, then $x \in X$. Dually, an element $x \in L$ is meet-irreducible if, whenever $x = \bigwedge X$ and X is finite, then $x \in X$. That is, an element is join-irreducible (respectively, meet-irreducible) if it cannot be written as a finite join (respectively, meet) unless it forms part of the finite join (respectively, meet).

Definition 2.2.9. *A* Boolean algebra *is an algebra* $\mathbf{A} = \langle A, \vee, \wedge, \neg, 0, 1 \rangle$ *such that*

(i) $\langle A, \vee, \wedge, 0, 1 \rangle$ *is a distributive lattice with greatest element* 1 *and least element* 0*,*

(ii) $\neg$ *is a unary operation on* A *such that* $x \vee \neg x = 1$ *and* $x \wedge \neg x = 0$ *for all* $x \in A$.

2.3 Varieties

We give some background on varieties that will be required.

Definition 2.3.1. *A* subalgebra $\mathbf{B}$ *of an algebra* $\mathbf{A}$ *is an algebra of the same type as* $\mathbf{A}$ *with* $B \subseteq A$ *such that every fundamental operation of* $\mathbf{B}$ *is the restriction of the corresponding operation of* $\mathbf{A}$ *to* B*, the universe of* $\mathbf{B}$*.*

Definition 2.3.2. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are two algebras of the same type* $\mathbb{T}$*, then a map* $\varphi : A \to B$ *is called a* homomorphism *from* $\mathbf{A}$ *to* $\mathbf{B}$ *if* $\varphi(f^{\mathbf{A}}(a_1, \ldots, a_n)) = f^{\mathbf{B}}(\varphi(a_1), \ldots \varphi(a_n))$*, for all* $a_1, \ldots, a_n \in A$ *and each* n*-ary fundamental operation* $f^{\mathbf{A}} \in \mathbb{T}^{\mathbf{A}}$*. If, in addition,* φ *is onto, then* $\mathbf{B}$ *is called a* homomorphic image *of* $\mathbf{A}$*.*

Definition 2.3.3. *Let* $\mathbf{A}_i$ *be an algebra of type* $\mathbb{T}$ *for each* $i \in \Psi$*. Then the* (direct) product $\prod_{i\in\Psi} \mathbf{A}_i$ *is defined to be the algebra of type* $\mathbb{T}$ *with* $\prod_{i\in\Psi} A_i$ *as the universe such that for each* n*-ary* $f \in \mathbb{T}$ *and each* $(a_i)_{i\in\Psi} \in \prod_{i\in\Psi} A_i$*, we have that* $f^{\prod_{i\in\Psi} \mathbf{A}_i}((a_i)_{i\in\Psi}) = \left(f^{\mathbf{A}_i}(a_i)\right)_{i\in\Psi}$*.*

For a class of algebras $\mathbb{K}$, we define $\mathrm{S}(\mathbb{K})$, $\mathrm{H}(\mathbb{K})$ and $\mathrm{P}(\mathbb{K})$ to be, respectively, the class of all subalgebras of algebras from $\mathbb{K}$, the class of all homomorphic images of algebras from $\mathbb{K}$ and the class of all direct products of algebras from $\mathbb{K}$. Then S, H and P are called *class operators.*

Lemma 2.3.4. *The class operators* S*,* H *and* P *preserve identities, i.e., if an identity is valid in a class of algebras* $\mathbb{K}$*, then it is valid in* $\mathrm{S}(\mathbb{K})$*,* $\mathrm{H}(\mathbb{K})$ *and* $\mathrm{P}(\mathbb{K})$*.*

We note that in this thesis we will take the universal quantification over the variables occurring in an identity, inequality or quasi-identity as implicit. For example, we will write $s = t$ rather than $(\forall \vec{x})(s(\vec{x}) = t(\vec{x}))$.

We can now define the notions of a *variety* and a *subvariety.*

Definition 2.3.5. *A non-empty class of algebras* $\mathbb{K}$ *of type* $\mathbb{T}$ *is called a* variety *if it is closed under* S*,* H *and* P*.*

The smallest variety containing a class of algebras $\mathbb{K}$ of the same type, is called *the variety generated by* $\mathbb{K}$. A variety is *finitely generated* if it is generated by a finite set of finite algebras.

Definition 2.3.6. *A subclass of a variety that is itself also a variety is called a* subvariety *of the variety.*

2.4 Operations and operators on ordered sets

We introduce the types of operations on ordered sets that will be considered.

Let $\mathbf{P}, \mathbf{Q}$ and $\mathbf{P}_i$ be posets, for $i = 1, \ldots, n$ for some $n \in \mathbb{N}$. For a unary map $f : P \to Q$ and $S \subseteq P$, let $f(S) = \{f(a) : a \in S\}$. Similarly, for an n-ary map $f : \prod_{i=1}^{n} P_i \to Q$ and $S_i \subseteq P_i$, let

$$f(S_1, \ldots, S_2) = \{f(a_1, \ldots, a_n) : a_i \in S_i, i = 1, \ldots, n\}.$$

The maps $f_1, f_2 : P \to Q$ can be ordered using the point-wise ordering: $f_1 \leq f_2$ if, and only if, $f_1(x) \leq f_2(x)$ for every $x \in P$. Let $f_1 : P \to Q$ and $f_2 : R \to P$; then we will write $f_1 \cdot f_2$ for the composition of f_1 and f_2.

Definition 2.4.1. *Let* $\mathbf{P} = \langle P, \leq^{\mathbf{P}} \rangle$ *and* $\mathbf{Q} = \langle Q, \leq^{\mathbf{Q}} \rangle$ *be posets. A map* $f : P \to Q$ *is called*

(i) one-to-one *if:* $f(x) = f(y)$ *implies* $x = y$ *for all* $x, y \in P$*;*

(ii) onto *if: for every* $y \in Q$ *there exists* $x \in P$ *such that* $f(x) = y$*;*

(iii) order-preserving *(or* monotone*) if:* $x \leq^{\mathbf{P}} y$ *implies that* $f(x) \leq^{\mathbf{Q}} f(y)$ *for all* $x, y \in P$*;*

(iv) an order-embedding *if:* $x \leq^{\mathbf{P}} y$ *if, and only if,* $f(x) \leq^{\mathbf{Q}} f(y)$ *for all* $x, y \in P$*;*

(v) an order-isomorphism *if:* f *is an order-embedding that maps* P *onto* Q*.*

If $\mathbf{P}$ and $\mathbf{Q}$ are posets such that there exists an order-isomorphism $f : P \to Q$ from P onto Q, then $\mathbf{P}$ and $\mathbf{Q}$ are said to be *order isomorphic*. If $\mathbf{P}$ and $\mathbf{Q}^{\partial}$ are order isomorphic, then $\mathbf{P}$ and $\mathbf{Q}$ are said to be *reverse order-isomorphic*.

Let $\mathbf{P} = \langle P, \leq^{\mathbf{P}} \rangle$ and $\mathbf{Q} = \langle Q, \leq^{\mathbf{Q}} \rangle$ be posets. A map $f : P \to Q$ *distributes over finite joins* if $\bigvee f(M)$ exists and $\bigvee f(M) = f(\bigvee M)$ for all $M \subseteq^{fin} P$ such that $\bigvee M$ exists. If $\bigvee f(S)$ exists and $\bigvee f(S) = f(\bigvee S)$ for all $S \subseteq P$ such that $\bigvee S$ exists, then f *distributes over arbitrary joins*.

Distribution of a map over finite and arbitrary meets can be defined dually. That is, $f : P \to Q$ *distributes over finite meets* if $\bigwedge f(M)$ exists and $\bigwedge f(M) = f(\bigwedge M)$ for all $M \subseteq^{fin} P$ such that $\bigwedge M$ exists. If $\bigwedge f(S)$ exists and $\bigwedge f(S) = f(\bigwedge S)$ for all $S \subseteq P$ such that $\bigwedge S$ exists, then f *distributes over arbitrary meets*.

The distribution of a map over joins is often called *join-preservation* in the literature (see, for example, [GH01], [GJKO07] and [Suz11]). Observe that if f distributes over finite joins or meets, then f is order-preserving.

Definition 2.4.2. *A map f between posets $\mathbf{P}$ and $\mathbf{Q}$ is called*

(i) *an* operator *if it distributes over finite joins.*

(ii) *a* complete operator *if it distributes over arbitrary joins.*

(iii) *a* dual operator *if it distributes over finite meets.*

(iv) *a* complete dual operator *if it distributes over arbitrary meets.*

An *operator* on a Boolean algebra $\mathbf{A} = \langle A, \vee, \wedge, \neg, 0, 1\rangle$ is an operation $f : A \to A$ that distributes over finite joins and satisfies $f(0) = 0$.

We can now generalise these notions to n-ary maps.

Definition 2.4.3. *Let $\mathbf{P}_i$, for $i = 1, \ldots, n$ and $\mathbf{Q}$ be posets. An n-ary map $f : \prod_{i=1}^{n} P_i \to Q$ is called*

(i) *an* operator *if it distributes over finite joins in each coordinate.*

(ii) *a* complete operator *if it distributes over arbitrary joins in each coordinate.*

(iii) *a* dual operator *if it distributes over finite meets in each coordinate.*

(iv) *a* complete dual operator *if it distributes over arbitrary meets in each coordinate.*

Let $\mathbf{L}_1 = \left\langle L_1, \vee^{\mathbf{L}_1}, \wedge^{\mathbf{L}_1}\right\rangle$ and $\mathbf{L}_2 = \left\langle L_2, \vee^{\mathbf{L}_2}, \wedge^{\mathbf{L}_2}\right\rangle$ be lattices. If $f : L_1 \to L_2$ is one-to-one, onto and both an operator and a dual operator, then f is called a *lattice isomorphism* and the lattices $\mathbf{L}_1$ and $\mathbf{L}_2$ are *isomorphic*. Lattices $\mathbf{L}_1$ and $\mathbf{L}_2$ are (lattice) isomorphic if, and only if, $\left\langle L_1, \leq^{\mathbf{L}_1}\right\rangle$ and $\left\langle L_2, \leq^{\mathbf{L}_2}\right\rangle$ are order-isomorphic.

Definition 2.4.4. *A* closure operator *$f : P \to P$ is a map that satisfies, for all $x, y \in P$,*

(i) *$x \leq f(x)$, i.e., f is increasing,*

(ii) *$x \leq y$ implies $f(x) \leq f(y)$, i.e., f is order-preserving, and*

(iii) *$f(f(x)) = f(x)$, i.e., f is idempotent.*

2.5 Residuated operators

In this section we recall the definitions of residuated operators. We also give some standard results concerning residuated operators that we will use when proving preservation theorems involving residuated operators.

Let $\mathbf{P} = \langle P, \leq^{\mathbf{P}} \rangle$, $\mathbf{Q} = \langle Q, \leq^{\mathbf{Q}} \rangle$ and $\mathbf{R} = \langle R, \leq^{\mathbf{R}} \rangle$ be posets.

Definition 2.5.1. *A map $f : P \to Q$ is called* residuated *if there exists a corresponding map $g : Q \to P$, called the* residual *of f, such that, for all $x \in P$ and all $y \in Q$*

$$f(x) \leq^{\mathbf{Q}} y \iff x \leq^{\mathbf{P}} g(y).$$

The following holds for unary residuated operators.

Lemma 2.5.2. *Let $f : P \to Q$ be residuated with residual $g : Q \to P$ and let $S \subseteq P$ and $T \subseteq Q$. If $\bigvee S$ exists in $\mathbf{P}$, then $\bigvee f(S)$ exists in $\mathbf{Q}$ and $\bigvee f(S) = f(\bigvee S)$. Similarly, if $\bigwedge T$ exists in $\mathbf{Q}$, then $\bigwedge g(T)$ exists in $\mathbf{P}$ and $\bigwedge g(T) = g(\bigwedge T)$.*

Hence, a unary residuated map is a complete operator while its residual is a complete dual operator. Let $f : P \to Q$ be a residuated operator with residual $g : Q \to P$. Then,

(i) g is uniquely determined by f.

(ii) f and g are both order-preserving.

(iii) $g \cdot f$ is a closure operator.

(iv) $x \leq g(f(x))$ for all $x \in P$.

(v) $f(g(y)) \leq y$ for all $y \in Q$.

Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be complete lattices. If $f : L_1 \to L_2$ is a complete operator, then f is residuated. In this case, the residual $g : L_2 \to L_1$ is definable by $g(y) = \bigvee\{x \in L_1 : f(x) \leq y\}$ for all $y \in L_2$. We can also define f in terms of g by $f(x) = \bigwedge\{y \in L_2 : x \leq g(y)\}$ for all $x \in L_1$.

Definition 2.5.3. *A binary map $\circ : P \times Q \to R$ is called* residuated *if there exist maps $\backslash : P \times R \to Q$ and $/ : R \times Q \to P$ such that for all $x \in P$, $y \in Q$ and $z \in R$*

$$x \circ y \leq^{\mathbf{R}} z \iff y \leq^{\mathbf{Q}} x \backslash z \iff x \leq^{\mathbf{P}} z/y.$$

The maps $\backslash$ *and* $/$ *are called the* left and right residuals of f*, respectively.*

A binary residuated map, $\circ : P \times Q \to R$, is order-preserving in both arguments. If $\backslash : P \times R \to Q$ and $/ : Q \times R \to P$ are the left and right residuals of $\circ$, respectively, then $\backslash$ is order-preserving in its second argument and order-reversing in its first, while $/$ is order-preserving in its first argument and order-reversing in its second. For all $x \in P$ and $z \in R$ we have that $x \circ (x\backslash z) \leq^{\mathbf{R}} z$; and for all $y \in Q$ and $z \in R$ we have that $(z/y) \circ y \leq^{\mathbf{R}} z$.

Lemma 2.5.4. *Let* $\circ : P \times Q \to R$ *be a binary residuated map with left and right residuals* $\backslash : P \times R \to Q$ *and* $/ : R \times Q \to P$*, respectively. Let* $S \subseteq P$*,* $T \subseteq Q$ *and* $U \subseteq R$*.*

(i) If $\bigvee S$ *exists in* $\mathbf{P}$*, then* $\bigvee_{a \in S}(a \circ b)$ *exists in* $\mathbf{R}$*, for any* $b \in Q$*, and* $(\bigvee S) \circ b = \bigvee_{a \in S}(a \circ b)$*.*

(ii) If $\bigvee T$ *exists in* $\mathbf{Q}$*, then* $\bigvee_{b \in T}(a \circ b)$ *exists in* $\mathbf{R}$*, for any* $a \in P$*, and* $a \circ (\bigvee T) = \bigvee_{b \in T}(a \circ b)$*.*

(iii) If $\bigvee S$ *exists in* $\mathbf{P}$*, then* $\bigwedge_{a \in S}(a\backslash c)$ *exists in* $\mathbf{Q}$*, for any* $c \in R$*, and* $(\bigvee S)\backslash c = \bigwedge_{a \in S}(a\backslash c)$*.*

(iv) If $\bigwedge U$ *exists in* $\mathbf{R}$*, then* $\bigwedge_{c \in U}(a\backslash c)$ *exists in* $\mathbf{Q}$*, for any* $a \in P$*, and* $a\backslash (\bigwedge U) = \bigwedge_{c \in U}(a\backslash c)$*.*

(v) If $\bigwedge U$ *exists in* $\mathbf{R}$*, then* $\bigwedge_{c \in U}(c/b)$ *exists in* $\mathbf{P}$*, for any* $b \in Q$*, and* $(\bigwedge U)/b = \bigwedge_{c \in U}(c/b)$*.*

(vi) If $\bigvee T$ *exists in* $\mathbf{Q}$*, then* $\bigwedge_{b \in T}(c/b)$ *exists in* $\mathbf{P}$*, for any* $c \in R$*, and* $c/(\bigvee T) = \bigwedge_{b \in T}(c/b)$*.*

Hence, a binary residuated map $\circ$ is a complete operator. If $\mathbf{P}, \mathbf{Q}$ and $\mathbf{R}$ are complete lattices, then a binary map $\circ : P \times Q \to R$ is residuated if, and only if, it is a complete operator. If this is the case then the left and right residuals of $\circ$ are definable as $a\backslash c = \bigvee\{b \in Q : a \circ b \leq c\}$ and $c/b = \bigvee\{a \in P : a \circ b \leq c\}$.

If a binary residuated operator $\circ : P \times P \to P$ is commutative, then its left and right residuals coincide, i.e., $x\backslash y = y/x$ for all $x, y \in P$ and the symbol $\to$ is usually used to denote *the residual*. That is, $x \to y = x\backslash y = y/x$ for all $x, y \in P$.

2.6 Galois connections

The constructions studied in Chapters 5, 6 and 9 all make use of Galois connections as defined in this section.

Let $\mathbf{P} = \langle P, \leq^{\mathbf{P}} \rangle$ and $\mathbf{Q} = \langle Q, \leq^{\mathbf{Q}} \rangle$ be posets.

Definition 2.6.1. *Maps* $^{\triangleright} : P \rightleftarrows Q :^{\triangleleft}$ *form a* Galois connection*, if, for all* $x \in P$ *and* $y \in Q$ *we have* $y \leq^{\mathbf{Q}} x^{\triangleright}$ *if, and only if,* $x \leq^{\mathbf{P}} y^{\triangleleft}$*. The maps* $^{\triangleright}$ *and* $^{\triangleleft}$ *are called the* polarities *of the Galois connection.*

Lemma 2.6.2. *Let* $^{\triangleright} : P \rightleftarrows Q :^{\triangleleft}$ *be maps that form a Galois connection. Then, for* $x, x_1, x_2 \in P$ *and* $y, y_1, y_2 \in Q$*:*

(i) If $x_1 \leq^{\mathbf{P}} x_2$*, then* $x_2^{\triangleright} \leq^{\mathbf{Q}} x_1^{\triangleright}$*. That is,* $^{\triangleright}$ *is order-reversing.*

(ii) If $y_1 \leq^{\mathbf{Q}} y_2$*, then* $y_2^{\triangleleft} \leq^{\mathbf{P}} y_1^{\triangleleft}$*. That is,* $^{\triangleleft}$ *is order-reversing.*

(iii) The maps $^{\triangleright\triangleleft} : P \to P$ *and* $^{\triangleleft\triangleright} : Q \to Q$ *are closure operators. Therefore,* $x \leq^{\mathbf{P}} x^{\triangleright\triangleleft}$ *and* $y \leq^{\mathbf{Q}} y^{\triangleleft\triangleright}$*.*

(iv) We have $^{\triangleright\triangleleft\triangleright} = ^{\triangleright}$ *and* $^{\triangleleft\triangleright\triangleleft} = ^{\triangleleft}$*, i.e.,* $x^{\triangleright\triangleleft\triangleright} = x^{\triangleright}$ *and* $y^{\triangleleft\triangleright\triangleleft} = y^{\triangleleft}$*.*

Lemma 2.6.3. *Let* $^{\triangleright} : P \rightleftarrows Q :^{\triangleleft}$ *be maps that form a Galois connection. Then both maps convert existing joins into meets, i.e., for* $S \subseteq P$ *and* $T \subseteq Q$*:*

(i) If $\bigvee S$ *exists in* $\mathbf{P}$*, then* $\bigwedge(S^{\triangleright})$ *exists in* $\mathbf{Q}$ *and* $(\bigvee S)^{\triangleright} = \bigwedge(S^{\triangleright})$*.*

(ii) If $\bigvee T$ *exists in* $\mathbf{Q}$*, then* $\bigwedge(T^{\triangleleft})$ *exists in* $\mathbf{P}$ *and* $(\bigvee T)^{\triangleleft} = \bigwedge(T^{\triangleleft})$*.*

Let P, Q and R be sets. If $R \subseteq P \times Q$, then R induces a Galois connection between $\langle \mathcal{P}(P), \subseteq \rangle$ and $\langle \mathcal{P}(Q), \subseteq \rangle$. The maps $^{\triangleright} : \mathcal{P}(P) \rightleftarrows \mathcal{P}(Q) :^{\triangleleft}$, defined by, for $S \subseteq P$ and $T \subseteq Q$

$$S^{\triangleright} = \{y \in Q : x \in S \text{ implies } (x, y) \in R\}$$

and

$$T^{\triangleleft} = \{x \in P : y \in T \text{ implies } (x, y) \in R\}$$

are called the *polarities* of R and form the Galois connection induced by R.

2.7 Up-sets, down-sets, filters and ideals

Different families of up-sets and down-sets of posets are central to this thesis. A special family of up-sets (respectively, down-sets) is the filters (respectively, ideals) of a lattice. In Chapter 4 we investigate possible generalizations of these notions to posets. We then use these different generalizations in the constructions studied in Chapters 4, 6 and 7. One of the constructions considered in Chapter 9 employs the filters (respectively, ideals) of meet-semilattices (respectively, join-semilattices).

Let $\mathbf{P} = \langle P, \leq \rangle$ be a poset.

Definition 2.7.1. *A subset $F \subseteq P$ is called an* up-set *(or* order-filter*) of $\mathbf{P}$ if F satisfies:*

$$\text{if } x \in F \text{ and } y \in P \text{ such that } y \geq x, \text{ then } y \in F, \tag{2.1}$$

and whenever $\mathbf{P}$ has a top element $F \neq \varnothing$.

Dually, a subset $I \subseteq P$ is called a down-set *(or* order-ideal*) of $\mathbf{P}$ if I satisfies:*

$$\text{if } x \in I \text{ and } y \in P \text{ such that } y \leq x, \text{ then } y \in I, \tag{2.2}$$

and whenever $\mathbf{P}$ has a bottom element $I \neq \varnothing$.

For $S \subseteq P$, let $[S)$ and $(S]$ denote the up-set and the down-set of $\mathbf{P}$, respectively, generated by S, i.e., $[S) = \{a \in P : a \geq b \text{ for some } b \in S\}$ and $(S] = \{a \in P : a \leq b \text{ for some } b \in S\}$. If $S = \{x\}$, we write $[x)$ and $(x]$ for $[\{x\})$ and $(\{x\}]$, respectively. Up-sets (down-sets) of the form $[x)$ ($(x]$) are called *principal*. We note that our definition includes the empty set in the family of up-sets of a poset, but only for posets that do not have a top element. Dually, the empty set is a down-set of a poset if it does not have a bottom element. In some instances in the literature the empty set is always excluded (see for instance [DP02]), while in others it is always included (see for instance [Sch72]).

Definition 2.7.2. *Let $\mathbf{L}$ be a lattice (respectively, meet-semilattice). A non-empty subset F of L is called a* filter *of $\mathbf{L}$ if it satisfies (2.1) and, for $x, y \in L$,*

$$x, y \in F \text{ implies } x \wedge y \in F.$$

Definition 2.7.3. *Let $\mathbf{L}$ be a lattice (respectively, join-semilattice). A non-empty subset I of L is called an* ideal *of $\mathbf{L}$ if it satisfies (2.2) and, for $x, y \in L$,*

$$x, y \in I \text{ implies } x \vee y \in F.$$

The family of filters (respectively, ideals) of a lattice $\mathbf{L}$ will be denoted by $\mathcal{F}(\mathbf{L})$ (respectively, $\mathcal{I}(\mathbf{L})$), or just $\mathcal{F}$ (respectively, $\mathcal{I}$) if $\mathbf{L}$ is understood. Both $\mathcal{F}$ and $\mathcal{I}$ are closed under arbitrary intersection.

A filter or an ideal of $\mathbf{L}$ is called *proper* if it does not coincide with L. If $\mathbf{L}$ has a bottom element, then $F \in \mathcal{F}$ is proper if, and only if, $\bot(= 0) \notin F$. Dually, if $\mathbf{L}$ has a top element, then $I \in \mathcal{I}$ is proper if, and only if, $\top(= 1) \notin I$.

Let $\mathbf{L}$ be a lattice (respectively, meet-semilattice) and let $S \subseteq L$. Then there exists a smallest filter containing S, denoted by $[S\rangle$, namely $[S\rangle = \bigcap\{F \in \mathcal{F} : S \subseteq F\}$. Dually, if $\mathbf{L}$ is a lattice (respectively, join-semilattice), then there exists a smallest ideal containing S which we will denote by $\langle S]$, namely $\langle S] = \bigcap\{I \in \mathcal{I} : S \subseteq I\}$. We call $[S\rangle$ the *filter generated by* S and $\langle S]$ the *ideal generated by* S. If $S = \{a\}$ for some $a \in L$, then $[\{a\}\rangle = [a)$ and it is called a *principal filter* of $\mathbf{L}$. Similarly, $\langle\{a\}] = (a]$ and it is called a *principal ideal* of $\mathbf{L}$. Let $S \subseteq L$; observe that $b \in [S\rangle$ if, and only if, $b \geq \bigwedge M$ for some $M \subseteq^{fin} S$. Similarly, $c \in \langle S]$ if, and only if, $c \leq \bigvee N$ for some $N \subseteq^{fin} S$.

Definition 2.7.4. *A* prime filter F *of a lattice* $\mathbf{L}$ *is a filter of* $\mathbf{L}$ *that satisfies, for* $x, y \in L$,

$$x \vee y \in F \text{ implies } x \in F \text{ or } y \in F.$$

A prime ideal *of a lattice* $\mathbf{L}$ *is an ideal of* $\mathbf{L}$ *that satisfies, for* $x, y \in L$,

$$x \wedge y \in I \text{ implies } x \in I \text{ or } y \in I.$$

We will denote the family of prime filters (respectively, prime ideals) of a lattice $\mathbf{L}$ by $\mathscr{F}(\mathbf{L})$ (respectively, $\mathscr{I}(\mathbf{L})$), or simply $\mathscr{F}$ (respectively, $\mathscr{I}$) if $\mathbf{L}$ is understood. A filter F of $\mathbf{L}$ is prime if, and only if, $L - F$ is a prime ideal of $\mathbf{L}$.

Definition 2.7.5. *A proper filter* F *of a lattice* $\mathbf{L}$ *is called an* ultrafilter *or a* maximal filter *if the only filter that properly contains* F *is the set* L *itself.*

Dually, a proper ideal I *of a lattice* $\mathbf{L}$ *is called* maximal *if the only ideal that properly contains* I *is* L.

If $\mathbf{L}$ is a distributive lattice with a bottom element, then every ultrafilter of $\mathbf{L}$ is a prime filter. If $\mathbf{L}$ is a distributive lattice with a top element, then every maximal ideal of $\mathbf{L}$ is a prime ideal (see, for example, [DP02]).

Definition 2.7.6. *A subset* F *of a lattice* $\mathbf{L}$ *is called a* complete filter *of* $\mathbf{L}$, *if* $\bigwedge S \in F$ *for every* $S \subseteq F$ *such that* $\bigwedge S$ *exists in* $\mathbf{L}$.

Dually, a subset I of a lattice $\mathbf{L}$ is called a complete ideal *of $\mathbf{L}$, if $\bigvee S \in I$ for every $S \subseteq I$ such that $\bigvee S$ exists in $\mathbf{L}$.*

Let $\mathcal{F}^c$ and $\mathcal{I}^c$ denote the families of complete filters and ideals of a lattice $\mathbf{L}$, respectively.

Part I

COMPLETIONS

3. INTRODUCTION TO COMPLETIONS

We are often interested in algebraic structures of which the underlying set is (partially) ordered. Such *ordered algebraic structures* occur naturally in many areas of mathematics. Examples include ordered groups, ordered rings, fields, ordered vector spaces, the sets of open or closed elements of a topology and the algebraic models of logics. Given an ordered algebraic structure, we are often interested in the supremums (joins) and infimums (meets) of its (arbitrary) subsets. If these do not exist, then one way to get around this non-existence is to embed the partial structure into a complete structure for which the necessary supremums and infimums do exist. We will call a pair consisting of a complete structure and an embedding a *completion*. The following definition makes this precise.

Definition 3.0.1. *A* completion *of a poset* $\mathbf{P}$ *is a pair* $(\mathbf{L}, \gamma)$ *where* $\mathbf{L}$ *is a complete lattice (viewed as a poset) and* $\gamma : P \to L$ *is an order-embedding.*

There are various reasons why one might wish to embed ordered algebraic structures into complete ones. A complete lattice is representable, both as a complete lattice of sets and as the image of a closure operator on a powerset lattice. Therefore, if it is important that the algebraic structures under consideration are representable, then completing the algebras would be one way of obtaining exactly what is needed.

From a logician's perspective, we may wish to model predicate logics. However, since (bounded) universal quantification corresponds with infinite meets and (bounded) existential quantification corresponds with infinite joins, we need to complete the algebra under consideration first to ensure that the necessary infinite joins and meets exist.

Furthermore, the proofs of many completeness theorems of various propositional and predicate logics have made use of results on completions. For example, in [MO02] the MacNeille completion of a residuated lattice is used to prove

that the predicate logic MTL∀ is standard complete. In [Ono03b, Theorems 5 and 6] the MacNeille completion of a (integral weakly idempotent) commutative residuated lattice (with exponentials) is shown to be a commutative residuated lattice (with exponentials). These results are then used to show that intuitionistic linear predicate logic (with exponentials) is complete with respect to the class of complete commutative residuated lattices (with exponentials). Similarly, it can be used to show that intuitionistic predicate logic without the contraction rule is complete with respect to the class of complete integral (weakly idempotent) commutative residuated lattices [Ono03b, Corollary 13].

Another application of completions that can be found in the literature, is the use of the canonical extension to obtain relational semantics for non-classical logics — including some substructural logics. It is often the case that a logic is closely related to a corresponding class of algebraic structures. These algebraic structures then provide algebraic semantics for the logic. Relational semantics for the logic may then be obtained by taking the canonical extensions of the algebraic structures and then using discrete duality theory to obtain relational structures. See [DGP05] and [CGvR] for examples of where this has been done in the literature.

In the following four chapters we will investigate the construction of various completions of posets and other algebraic structures. Depending on our purposes, we may need different completions. Each completion has its advantages and disadvantages. For example, the MacNeille completion of a lattice preserves the existing infinite structure while the canonical extension destroys it. Having various completions at our disposal makes it more likely that we will have a construction that does what we need it to do.

4. FILTER AND IDEAL COMPLETIONS

It is well known that if $\mathbf{L}$ is a lattice with a top element, then $\mathbf{F} = \langle \mathcal{F}(\mathbf{L}), \vee^{\mathbf{F}}, \wedge^{\mathbf{F}} \rangle$ forms a complete lattice where $\mathcal{F}(\mathbf{L})$ is the set of filters of $\mathbf{L}$, $\bigvee^{\mathbf{F}}_{i\in\Psi} F_i = \left[\bigcup_{i\in\Psi} F_i\right\rangle$ and $\bigwedge^{\mathbf{F}}_{i\in\Psi} F_i = \bigcap_{i\in\Psi} F_i$ for $F_i \in \mathcal{F}(\mathbf{L})$, $i \in \Psi$. Then $\subseteq$ is the associated lattice order. If $\mathbf{L}$ does not have a top element, then $\mathcal{F}(\mathbf{L}) \cup \{\varnothing\}$, is the universe of a complete lattice. Let $\mathbf{F}_\top(\mathbf{L})$ denote the complete lattice obtained from the set of filters of $\mathbf{L}$ with the possible inclusion of $\varnothing$. Similarly, if $\mathbf{L}$ has a bottom element, then $\mathbf{I} = \langle \mathcal{I}(\mathbf{L}), \vee^{\mathbf{I}}, \wedge^{\mathbf{I}} \rangle$, is a complete lattice where $\mathcal{I}(\mathbf{L})$ is the set of all ideals of $\mathbf{L}$, $\bigvee^{\mathbf{I}}_{j\in\Phi} = \left[\bigcup_{j\in\Phi} I_j\right\rangle$ and $\bigwedge^{\mathbf{I}}_{i\in\varphi} = \bigcap_{j\in\Phi} I_j$ for I_j, $j \in \Phi$. Then $\leq^{\mathbf{I}}$ is $\subseteq$. If $\mathbf{L}$ does not have a bottom element, then we can include $\varnothing$ to form a complete lattice. Let $\mathbf{I}_\perp(\mathbf{L})$ denote the complete lattice obtained from the set of ideals of $\mathbf{L}$ with the possible inclusion of $\varnothing$. Then, $\nu : L \to \mathcal{F}_\top(\mathbf{L})$ defined by $\nu(a) = [a)$, for all $a \in L$, is a lattice embedding of $\mathbf{L}$ into $\mathbf{F}_\top(\mathbf{L})^\partial$, that is, $\nu(L)$ is the universe of a sublattice of $\mathbf{F}_\top(\mathbf{L})^\partial$. Dually, $\omega : L \to \mathcal{I}_\perp(\mathbf{L})$ defined by $\omega(a) = (a]$, for all $a \in L$, is a lattice embedding of $\mathbf{L}$ into $\mathbf{I}_\perp(\mathbf{L})$, that is, $\omega(L)$ is the universe of a sublattice of $\mathbf{I}_\perp(\mathbf{L})$. Thus, $(\mathbf{F}_\perp(\mathbf{L})^\partial, \nu)$ and $(\mathbf{I}_\top(\mathbf{L}), \omega)$ are completions of $\mathbf{L}$. For more on the filter and ideal lattices of a lattice $\mathbf{L}$ the reader is referred to [Bir67, Chapter V.2].

In this chapter we would like to generalise 'filter' and 'ideal' completions to the poset setting. This has been done in various, decidedly distinct, ways in the literature. We begin by recalling the definition of four different families of up-sets and down-sets of a poset $\mathbf{P}$. Many more have been defined in the literature and we give a quick survey of the remaining families that do not form a part of this thesis. The different types of filters and ideals defined in this chapter will be used again in some of the completions studied in later chapters.

Next we investigate the complete lattices formed by three of these families. At this point the notions of a 'prime ideal' and a 'prime filter' of a poset become of interest. However, the literature does not agree on what the correct definitions of these notions are either. We recall the definitions found in the literature and

compare them. We will use the family of *prime filters* defined here in the completions studied in Chapter 7. Furthermore, we show that *strictly prime filters* (see Definition 4.2.18) are meet-irreducible elements in the associated lattice of filters. Similarly, *strictly prime ideals* are meet-irreducible elements in the associated lattice of ideals.

Finally, we consider the extension of operations defined on the poset to operations defined on the completions. In particular, we show that if $f : P \to P$ is an operator, then its extension (to one of the completions) is a complete operator. Dually, the extension (to one of the completions) of a dual operator is a complete dual operator.

4.1 Filters and ideals of posets

In the literature one may encounter various families of up-sets (respectively, down-sets) of a poset that have been called the 'filters' (respectively, 'ideals') of the poset.

Let $\mathbf{P}$ be a poset and suppose $\mathcal{F}'$ is defined to be the set of 'filters of $\mathbf{P}$', then, as stated in [Fri54], it would be desirable for $\mathcal{F}'$ to satisfy the following conditions:

(i) If $a \in P$, then $[a) \in \mathcal{F}'$.

(ii) If $\mathcal{T} \subseteq \mathcal{F}'$, then $\bigcap \mathcal{T} \in \mathcal{F}'$.

(iii) If P is a lattice, then $\mathcal{F}'$ is exactly the family of filters of P.

In [GJKO07] a so-called 'rich enough' family of up-sets, $\mathcal{F}'$, (that is, 'rich enough' to be used in a construction studied in [GJKO07, Chapter 6]) is required to satisfy:

(a) If $a \in P$, then $[a) \in \mathcal{F}'$.

(b) If $F \in \mathcal{F}'$, then F is closed under existing finite meets.

(c) $\varnothing \in \mathcal{F}'$ if, and only if, P does not have a top element.

The family of filters, $\mathcal{F}(\mathbf{L})$, of a lattice $\mathbf{L}$ satisfies conditions (i)-(iii), (a) and (b) above. On the other hand, $\varnothing$ is never a filter of $\mathbf{L}$ and the closure of $\mathcal{F}(\mathbf{L})$ under intersection is not affected by this exclusion. For a poset $\mathbf{P}$ the satisfaction of condition (ii) depends on the satisfaction of condition (c).

It would therefore seem natural to require that the eventual definition of a 'filter of a poset' satisfies all of the conditions above, since we are looking for a generalisation of the notion of a filter on a lattice. However, upon closer inspection it would appear that such a requirement would be expecting too much. Consider, for instance, condition (ii): "If $\mathcal{T} \subseteq \mathcal{F}'$, then $\bigcap \mathcal{T} \in \mathcal{F}'$". As stated above, if $\mathbf{P}$ does not have a top element, then condition (ii) will be satisfied only if condition (c) is satisfied, i.e., $\varnothing \in \mathcal{F}'$. However, if condition (c) is satisfied, then $\mathcal{F}'$ need not equal the set of filters when $\mathbf{P}$ is a lattice, which is condition (iii). It should be apparent that defining filters (and ideals) on a poset is not straightforward.

Let $\mathbf{P} = \langle P, \leq \rangle$ be a poset. Recall that $F \subseteq P$ is an *up-set* of $\mathbf{P}$ if, whenever $a \in F$ and $b \in P$ such that $a \leq b$, then $b \in F$; and $F \neq \varnothing$ whenever $\mathbf{P}$ has a top element. Dually, $I \subseteq P$ is a *down-set* of $\mathbf{P}$ if, whenever $a \in I$ and $b \in P$ such that $a \geq b$, then $b \in I$; and $I \neq \varnothing$ whenever $\mathbf{P}$ has a bottom element.

We now recall the definitions of some of the various families of up-sets and down-sets that have been called the 'filters' and 'ideals' of a poset, respectively, in the literature.

Definition 4.1.1 ([AA90])**.** *A subset $F \subseteq P$ is called a* pseudo filter *of $\mathbf{P}$ if F is an up-set that satisfies*

$$\text{if } x, y \in F \text{ and } x \wedge y \text{ exists in } \mathbf{P}, \text{ then } x \wedge y \in F \tag{4.1}$$

and if $\mathbf{P}$ has a top element then $F \neq \varnothing$. Pseudo ideals *can be defined dually.*

Pseudo ideals were defined in [AA90] where, in addition to satisfying the above properties, they were defined to be non-empty proper subsets of P. However, we define P to be both a pseudo filter and a pseudo ideal, while $\varnothing$ is a pseudo filter (respectively, ideal) when $\mathbf{P}$ does not have a top element (respectively, a bottom element).

In [Doy50] the notion of an 'ideal' of a poset was defined. This notion actually corresponds to the notion of a filter on a bounded lattice.

Definition 4.1.2 ([Doy50])**.** *A subset $F \subseteq P$ is called a* Doyle-pseudo filter *of $\mathbf{P}$ if F is an up-set that satisfies*

$$\text{if } M \subseteq^{fin} F \text{ such that } \bigwedge M \text{ exists in } \mathbf{P}, \text{ then } \bigwedge M \in F \tag{4.2}$$

and if $\mathbf{P}$ has a top element then $F \neq \varnothing$. Doyle-pseudo ideals *can be defined dually.*

In [Ven71] Doyle-pseudo ideals were defined to be non-empty, and were called the 'ideals' of a poset. Doyle-pseudo ideals were also simply called the 'ideals' of a poset in [Tun74] — though we note that it was not explicitly said that ideals are only closed under *existing* finite joins.

Next we consider a family of up-sets first introduced in [Fri54].

Definition 4.1.3 ([Fri54]). *A subset $F \subseteq P$ is called a* Frink filter *of* $\mathbf{P}$ *if F satisfies*

$$\text{if } M \subseteq^{fin} F, \text{ then } M^{\ell u} \subseteq F \tag{4.3}$$

and if $\mathbf{P}$ *has a top element then $F \neq \varnothing$.* Frink ideals *can be defined dually.*

Note that all Frink filters are up-sets. Frink ideals were called the 'ideals' of a poset in [Fri54] and [War55]. This family of down-sets were used in [War55] in the study of relations between topologies in posets, one of which was the Frink ideal topology.

The definitions of Doyle-pseudo and Frink filters given here, differ from the original definitions obtained in [Doy50] and [Fri54], respectively, in that $\varnothing$ is excluded for posets with a top element.

For more on pseudo, Doyle-pseudo and Frink filters and ideals the reader may consult [Nie06].

In [Hof79] it was suggested that the following family of up-sets may be viewed as the 'filters' of a poset.

Definition 4.1.4 ([Hof79]). *A non-empty subset $F \subseteq P$ is called a* directed filter *of* $\mathbf{P}$ *if it is an up-set that satisfies*

$$\text{if } x, y \in F, \text{ then there exists } z \in F \text{ such that } z \leq x \text{ and } z \leq y. \tag{4.4}$$

Directed ideals *can be defined dually.*

If $S \subseteq P$ satisfies (4.4), then S is called a *down-directed* subset of $\mathbf{P}$. *Up-directed subsets* can be defined dually.

The directed filters and ideals defined above have also been called the 'filters' and the 'ideals' of a poset in the literature (see for instance [DGP05], [Por12] or [DP02]).

Tables 4.1 and 4.2 respectively summarize the different types of filters and ideals of a poset considered in this thesis.

An up-set F of a poset $\mathbf{P}$ is called a	
pseudo filter if:	$x, y \in F$ and $x \wedge y$ exists in $\mathbf{P}$ implies $x \wedge y \in F$.
Doyle-pseudo filter if:	$M \subseteq^{fin} F$ and $\bigwedge M$ exists in $\mathbf{P}$ implies $\bigwedge M \in F$.
Frink filter if:	$M \subseteq^{fin} F$ implies $M^{\ell u} \subseteq F$.
directed filter if:	$x, y \in F$ implies there exists $z \in F$ such that $z \in \{x, y\}^{\ell}$.

Tab. 4.1: A summary of the various types of filters under consideration.

A down-set I of a poset $\mathbf{P}$ is called a	
pseudo ideal if:	$x, y \in I$ and $x \vee y$ exists in $\mathbf{P}$ implies $x \vee y \in I$.
Doyle-pseudo ideal if:	$M \subseteq^{fin} I$ and $\bigvee M$ exists in $\mathbf{P}$ implies $\bigvee M \in I$.
Frink ideal if:	$M \subseteq^{fin} I$ implies $M^{u\ell} \subseteq I$.
directed ideal if:	$x, y \in I$ implies there exists $z \in I$ such that $z \in \{x, y\}^{u}$.

Tab. 4.2: A summary of the various types of filters under consideration.

Let $\mathcal{F}^p$, $\mathcal{F}^{dp}$, $\mathcal{F}^f$ and $\mathcal{F}^d$ denote the families of pseudo, Doyle-pseudo, Frink and directed filters of $\mathbf{P}$, respectively. The families of pseudo, Doyle-pseudo, Frink and directed ideals of $\mathbf{P}$ will be denoted by $\mathcal{I}^p$, $\mathcal{I}^{dp}$, $\mathcal{I}^f$ and $\mathcal{I}^d$, respectively. We write $\mathcal{F}^*(\mathbf{P})$ and $\mathcal{I}^*(\mathbf{P})$, for $* \in \{p, dp, f, d\}$, if it is necessary to indicate which poset is used. For $* \in \{p, dp, f, d\}$, we will sometimes refer to the elements of $\mathcal{F}^*$ as $*$-filters and to the elements of $\mathcal{I}^*$ as $*$-ideals.

Remark 4.1.5. *The members of $\mathcal{F}_{dp}$, $\mathcal{F}_f$ and $\mathcal{F}_d$ are closed under existing finite meets. To see this, let $M \subseteq^{fin} P$ such that $\bigwedge M$ exists in $\mathbf{P}$. Then, by definition, if $F \in \mathcal{F}_{dp}$ such that $M \subseteq F$, it follows that $\bigwedge M \in F$. If $F \in \mathcal{F}_f$ such that $M \subseteq F$, then $\bigwedge M \in M^{\ell u} \subseteq F$. Finally, if $F \in \mathcal{F}_d$ such that $M \subseteq F$, then there exists $z \in M^{\ell}$ such that $z \in F$. But then $\bigwedge M \in F$.*

Dually, the members of $\mathcal{I}_{dp}$, $\mathcal{I}_f$ and $\mathcal{I}_d$ are closed under existing finite joins.

Lemma 4.1.6. *The following inclusions hold: $\mathcal{F}^d \subseteq \mathcal{F}^f \subseteq \mathcal{F}^{dp} \subseteq \mathcal{F}^p$ and $\mathcal{I}^d \subseteq \mathcal{I}^f \subseteq \mathcal{I}^{dp} \subseteq \mathcal{I}^p$. In general, these inclusions are strict.*

Proof. We prove the claim for the families of filters. The proof of the claim for the families of ideals follows dually.

Let $F \in \mathcal{F}^d$ and $M = \{a_1, a_2, \ldots, a_n\} \subseteq^{fin} F$. Since F is directed there exists a $z_1 \in F$ such that $z_1 \leq a_1$ and $z_1 \leq a_2$. Furthermore, there exists a

$z_2 \in F$ such that $z_2 \leq z_1$ (hence also $z_2 \leq a_1, a_2$) and $z_2 \leq a_3$. Continuing in this way we can find a $z_{n-1} \in F$ such that $z_{n-1} \leq z_{n-2}$ (and hence also $z_{n-1} \leq a_i$ for $i = 1, \ldots, n-1$) and $z_{n-1} \leq a_n$. That is, $z_{n-1} \in F$ and $z_{n-1} \leq a_i$ for $i = 1, \ldots, n$. Then $z_{n-1} \in M^\ell$ which implies that $M^{\ell u} \subseteq [z_{n-1}) \subseteq F$. Thus $F \in \mathcal{F}^f$.

Next let $F \in \mathcal{F}^f$ and let $M \subseteq^{fin} F$ such that $\bigwedge M$ exists in P. Then $\bigwedge M \in M^{\ell u} \subseteq F$, i.e., $\bigwedge M \in F$ and hence $F \in \mathcal{F}^{dp}$.

Since any subset containing two or fewer elements is still a finite subset, it follows that $\mathcal{F}^{dp} \subseteq \mathcal{F}^p$.

Consider the following poset to see that these inclusions generally are strict:

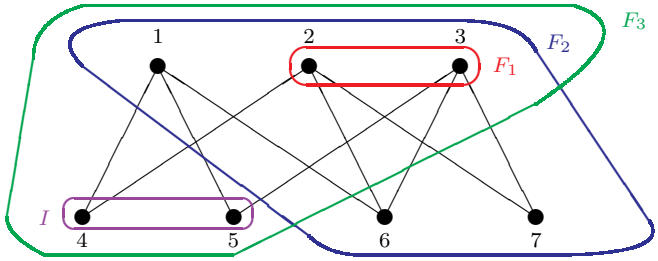


Fig. 4.1: The inclusions $\mathcal{F}^d \subseteq \mathcal{F}^f \subseteq \mathcal{F}^{dp} \subseteq \mathcal{F}^p$ may be strict.

Let $\mathbf{P}'$ be the poset in Figure 4.1. Then $F_1 = \{2, 3\} \in \mathcal{F}^f$, but $F_1 \notin \mathcal{F}^d$ since it does not contain a common lower bound of 2 and 3. Furthermore, $F_2 = \{1, 2, 3, 6, 7\} \in \mathcal{F}^{dp}$, but $F_2 \notin \mathcal{F}^f$. To see why, observe that $\{6, 7\}^\ell = \varnothing$ and therefore $\{6, 7\}^{\ell u} = P \not\subseteq F_2$. Finally, $F_3 = \{1, 2, 3, 4, 5\} \in \mathcal{F}^p$ but $F_3 \notin \mathcal{F}^{dp}$. Here $\bigwedge\{1, 2, 3\} = 6 \notin F_3$. Also note that, in general, $\mathcal{F}^p$ is strictly included in the family of all up-sets. In this particular example all up-sets are also pseudo filters. However, $I = \{4, 5\}$ is a down-set that is not a pseudo ideal since $4 \vee 5 = 1 \notin I$. □

If $\mathbf{P}$ is bounded then the inclusions $\mathcal{F}^d \subseteq \mathcal{F}^f$ and $\mathcal{I}^d \subseteq \mathcal{F}^f$ may still be strict. We note that every principal up-set (respectively, down-set) of a poset $\mathbf{P}$ is a directed filter (respectively, directed ideal) of $\mathbf{P}$, and therefore included in all four families of filters (respectively, ideals). In fact, if $\mathbf{P}$ is finite, then $\mathcal{F}^d = \{[a) : a \in P\}$. Furthermore, observe that if $\mathbf{L}$ is a bounded lattice, then $\mathcal{F}^p(\mathbf{L}) = \mathcal{F}^{dp}(\mathbf{L}) = \mathcal{F}^f(\mathbf{L}) = \mathcal{F}^d(\mathbf{L}) = \mathcal{F}(\mathbf{L})$ and $\mathcal{I}^p(\mathbf{L}) = \mathcal{I}^{dp}(\mathbf{L}) = \mathcal{I}^f(\mathbf{L}) = \mathcal{I}^d(\mathbf{L}) = \mathcal{I}(\mathbf{L})$.

In this thesis we will focus our attention on the families of up-sets and down-sets defined above. However, more (generally distinct) families of up-sets and down-sets, that have been used in various completions of posets, have been defined in the literature. Among these is the family $\{S^{u\ell} : S \subseteq P\}$ used by MacNeille in [Mac37]. If $\mathbf{P}$ is finite, then the family of Frink filters of $\mathbf{P}$ correspond exactly with this family of down-sets. However, this need not be the case if $\mathbf{P}$ is infinite. For example, if $F \in \mathcal{F}^f$ is infinite, then $F^{u\ell}$ need not be a subset of F — this may be the case when $\bigwedge F$ exists in $\mathbf{P}$, but is not included in F. See Chapter 5 for more on the MacNeille completion.

In [BS66] the collection of all *non-empty down-sets of a poset* $\mathbf{P}$ *that are bounded above*, i.e.,

$$\{I : I \text{ is a down-set and there exists } a \in P \text{ such that } I \subseteq (a]\},$$

was used to complete $\mathbf{P}$. In general, this family of down-sets does not correspond with any of the families of down-sets under consideration in this thesis. Let $\mathbf{P}'$ be the poset depicted in Figure 4.1. Then $\mathbf{P}'$ is a Frink ideal, but it is not bounded above. On the other hand, $\{5, 7\} \subseteq P'$ is bounded above since $\{5, 7\} \subseteq (3] = \{3, 5, 6, 7\}$, but since $5 \vee 7 = 3 \notin \{5, 7\}$ it is not a pseudo ideal (and hence none of the other three types of ideals under consideration).

In [Abi68] Abian defined the *initial cuts* of a poset and showed that the family of initial cuts generally differs from the family of lower cuts used by MacNeille. For $a \in P$, an *initial segment* with respect to a was defined to be the set $\{x \in P : x \not\geq a\}$ and an *initial cut* the union of any family of initial segments. An initial cut need not be closed under existing joins. To see why we consider the poset $\mathbf{P}'$ in Figure 4.1 again. The sets $\{3, 5, 6, 7\}$ and $\{2, 4, 6, 7\}$ are the initial segments with respect to 4 and 5, respectively. Their union, $\{2, 3, 4, 5, 6, 7\}$ then forms an initial cut, but not a pseudo ideal since $4 \vee 5 = 1$ is not included. On the other hand, $F_1 \in \mathcal{F}^f$, but F_1 is not an initial cut.

In [Sch72] a down-set I is called *k-small generated*, for $k \geq 2$, if there exists $S \subseteq I$ such that $|S| \leq k$ and $a \in I$ if, and only if, $a \leq s$ for some $s \in S$. Since $k \geq 2$ it follows that all principal down-sets are k-small generated. It should be clear that k-small generated down-sets need not be closed under existing joins. These types of down-sets are essentially generalizations of the principal down-sets. In [WWT78] generated down-sets, with a variety of restrictions on the generating subsets, were used. For example, the generating subsets were

assumed to have cardinality less than or equal to some n; or, assumed to have an upper bound in the poset for every pair of elements from the generating subset; or, assumed to have an upper bound in the poset for every finite subset of the generating subset; or, assumed to be bounded, directed or linearly ordered.

In [Hal00] an 'ideal' of an ordered set is defined to be a subset I for which $\{a, b\}^{u\ell} \subseteq I$ whenever $a, b \in I$. Consider the poset $\mathbf{P}'$ from Figure 4.1 once more. If we were to define the notion of a 'filter' dually to the definition above, then $\{1, 2, 3\}$ would be a 'filter' of $\mathbf{P}'$, but not a Frink filter since $\{1, 2, 3\}^{\ell u} = \{1, 2, 3, 6\} \not\subseteq \{1, 2, 3\}$.

Another collection of down-sets that should be mentioned here, though not called ideals in the literature, is the family of *Scott-closed* sets [Sco72]. A Scott-closed set is a down-set that is closed under the existing joins of its directed subsets. If $\mathbf{P}$ is finite, then the Scott-closed sets correspond with the directed ideals of $\mathbf{P}$. For more on Scott-closed sets and the Scott topology the reader is referred to [Ern81], [GHK+80] and [Ros84].

In [Doc67] and [Sch74] ideals closed under selected joins were defined. Let $\mathcal{T} \subseteq \mathcal{P}(P)$. Then an $\mathcal{T}$-ideal of $\mathbf{P}$, I, is a down-set satisfying: if $S \in \mathcal{T}$, $S \subseteq I$ and $\bigvee S$ exists in $\mathbf{P}$, then $\bigvee S \in I$. Clearly, if $\mathcal{T}$ is all finite subsets of P, then the $\mathcal{T}$-ideals of $\mathbf{P}$ are exactly the Doyle-pseudo ideals of $\mathbf{P}$. Similarly, if $\mathcal{T}$ is all binary subsets of P, then the $\mathcal{T}$-ideals of $\mathbf{P}$ are exactly the pseudo ideals of $\mathbf{P}$.

In [MN65] (see also [Ros72]) the following definition of $\mathfrak{m}$-ideals was given: a subset $F \subseteq P$ is called an $\mathfrak{m}$-ideal if $S^{u\ell} \subseteq F$ for all $S \subseteq F$ such that $|S| < \mathfrak{m}$. If $\mathfrak{m} = \aleph_0$, then the $\mathfrak{m}$-ideals are exactly the Frink ideals.

4.1.1 Complete filters and ideals

Next we generalise the notion of a 'complete filter' to the poset setting. Again the generalisation is not straightforward, since a number of different families of up-sets may be identified as candidates for the generalisation.

Definition 4.1.7 ([Tun74, Jan78])**.** *A subset $F \subseteq P$ is called a* complete Doyle-pseudo filter *of $\mathbf{P}$ if F is an up-set that satisfies*

$$\text{if } S \subseteq F \text{ such that } \bigwedge S \text{ exists in } \mathbf{P}, \text{ then } \bigwedge S \in F \tag{4.5}$$

and if $\mathbf{P}$ has a top element then $F \neq \varnothing$. Complete Doyle-pseudo ideals *can be defined dually.*

In [Jan78] complete Doyle-pseudo filters are called *conditionally complete filters* and in [Tun74] simply the 'complete filters' of a poset.

Following the above we make the following analogous definitions.

Definition 4.1.8. *A subset $F \subseteq P$ will be called a* complete Frink filter *of P if F satisfies*

$$\text{if } S \subseteq F, \text{ then } S^{\ell u} \subseteq F \tag{4.6}$$

and if $\mathbf{P}$ has a top element then $F \neq \varnothing$. Complete Frink ideals *can be defined dually.*

Definition 4.1.9. *A non-empty subset $F \subseteq P$ will be called a* complete directed filter *of $\mathbf{P}$ if F is an up-set that satisfies*

$$\text{if } S \subseteq F, \text{ then there exists } z \in F \text{ such that } z \leq x \text{ for all } x \in S. \tag{4.7}$$

Complete directed ideals *can be defined dually.*

Let $\mathcal{F}^{cdp}$, $\mathcal{F}^{cf}$ and $\mathcal{F}^{cd}$ denote the families of complete Doyle-pseudo, complete Frink and complete directed filters, respectively. The families of complete Doyle-pseudo, complete Frink and complete directed ideals will be denoted by $\mathcal{I}^{cdp}$, $\mathcal{I}^{cf}$ and $\mathcal{I}^{cd}$, respectively.

If P is finite, then $\mathcal{F}^{cdp} = \mathcal{F}^{dp}$, $\mathcal{F}^{cf} = \mathcal{F}^{f}$, $\mathcal{F}^{cd} = \mathcal{F}^{d}$, $\mathcal{I}^{cdp} = \mathcal{I}^{dp}$, $\mathcal{I}^{cf} = \mathcal{I}^{f}$ and $\mathcal{I}^{cd} = \mathcal{I}^{d}$. If $\mathbf{L}$ is a lattice and $\mathbf{L}$ has a top element, then the families of complete Doyle-pseudo, complete Frink and complete directed filters all coincide with the family of complete filters of $\mathbf{L}$, i.e., $\mathcal{F}^{cdp}(\mathbf{L}) = \mathcal{F}^{cf}(\mathbf{L}) = \mathcal{F}^{cd}(\mathbf{L}) = \mathcal{F}^{c}(\mathbf{L})$. Similarly, $\mathcal{I}^{cdp}(\mathbf{L}) = \mathcal{I}^{cf}(\mathbf{L}) = \mathcal{I}^{cd}(\mathbf{L}) = \mathcal{I}^{c}(\mathbf{L})$ if $\mathbf{L}$ has a bottom element.

Lemma 4.1.10. *The following inclusions hold: $\mathcal{F}^{cd} \subseteq \mathcal{F}^{cf} \subseteq \mathcal{F}^{cdp}$ and $\mathcal{I}^{cd} \subseteq \mathcal{I}^{cf} \subseteq \mathcal{F}^{cdp}$. In general, these inclusions are strict.*

Proof. The proof of Lemma 4.1.6 may be suitably modified to prove the inclusions. The example given in Figure 4.1 suffices to show that these inclusions are strict, since the various types of complete filters coincide with the corresponding types of filters on finite posets. In Figure 4.2 we provide infinite posets demonstrating that these inclusions are strict. If $\mathbf{P}'$ is the infinite anti-chain, depicted in Figure 4.2, then any proper subset of $\mathbf{P}'$ will be a complete Doyle-pseudo filter, but not a complete Frink filter. In particular, if $F_1 = P' - \{1\}$, then $F_1 \in \mathcal{F}^{cdp}(\mathbf{P}')$, but $F_1 \notin \mathcal{F}^{cf}(\mathbf{P}')$. Next let $\mathbf{Q}'$ be the poset depicted in Figure 4.2. Then $F_2 = Q' - \{1, 2\} \in \mathcal{F}^{cf}(\mathbf{Q})$, but $F_2 \notin \mathcal{F}^{cd}(\mathbf{Q})$. □

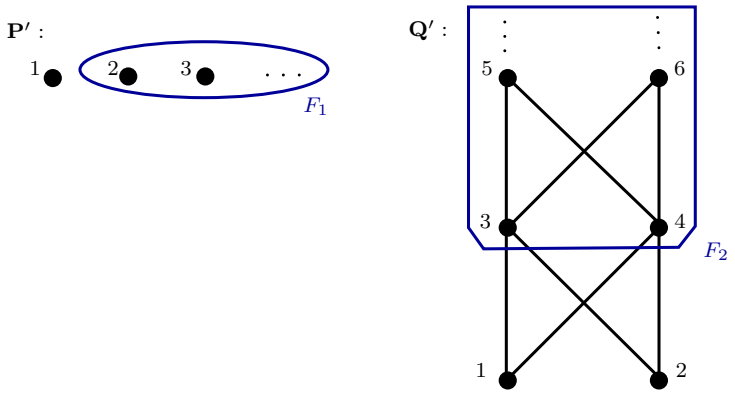


Fig. 4.2: The inclusions $\mathcal{F}^{cd} \subseteq \mathcal{F}^{cf} \subseteq \mathcal{F}^{cdp}$ may be strict.

4.2 Filter and ideal lattices

The different types of filters and ideals ordered by inclusion form posets, i.e., $\langle \mathcal{F}^*, \subseteq \rangle$ and $\langle \mathcal{I}^*, \subseteq \rangle$ are posets for $* \in \{p, dp, f, d\}$. If $* \in \{p, dp, f\}$, then we will be able to say more. We will show that $\mathcal{F}^*$ and $\mathcal{I}^*$ are closed under intersection. Furthermore, we will show that an arbitrary subset of P generates both a $*$-filter and a $*$-ideal of $\mathbf{P}$. Then the sets $\mathcal{F}^*$ and $\mathcal{I}^*$ are the universes of complete lattices.

We first have a closer look at closure under intersection. The fact that $\mathcal{F}^{dp}$ and $\mathcal{I}^{dp}$ are closed under intersection was shown in, for instance, [Sch72] and [GJKO07]. In [Fri54] it was stated that $\mathcal{F}^f$ and $\mathcal{I}^f$ are closed under intersection. We include a proof here.

Lemma 4.2.1. *The families $\mathcal{F}^p$, $\mathcal{F}^{dp}$, $\mathcal{F}^f$, $\mathcal{I}^p$, $\mathcal{I}^{dp}$ and $\mathcal{I}^f$ are closed under arbitrary intersections.*

Proof. We only show the closure under arbitrary intersection for the families of filters. It can be shown similarly for the families of ideals.

The family of all up-sets is closed under intersection: let $\mathcal{G}$ be an arbitrary set of up-sets and let $F = \bigcap \mathcal{G}$. If $F = \varnothing$, then P does not have a top element and F is an up-set by definition. If $a \in F$, $b \in P$ and $b \geq a$, then $a \in G$ for every $G \in \mathcal{G}$. But every $G \in \mathcal{G}$ is an up-set which implies that $b \in G$ for every $G \in \mathcal{G}$. Therefore, $b \in \bigcap \mathcal{G} = F$.

Now suppose $\mathcal{G} \subseteq \mathcal{F}^{dp}$ (respectively, $\mathcal{G} \subseteq \mathcal{F}^{p}$). If $F = \varnothing$, then P does

not have a top element and $F \in \mathcal{F}^{dp}$ (respectively, $F \in \mathcal{F}^{p}$) by definition. Otherwise, let $M \subseteq^{fin}$ (respectively, $\subseteq^2$) F such that $\bigwedge M$ exists in P. Then $M \subseteq^{fin}$ (respectively, $\subseteq^2$) G for every $G \in \mathcal{G}$. By definition $\bigwedge M \in G$ for every $G \in \mathcal{G}$. Therefore, $\bigwedge M \in \bigcap \mathcal{G} = F$ and hence, $F \in \mathcal{F}^{dp}$ (respectively, $F \in \mathcal{F}^{p}$).

Let $\mathcal{G} \subseteq \mathcal{F}^{f}$. If $F = \varnothing$, then P does not have a top element and $F \in \mathcal{F}^{f}$. If $F \neq \varnothing$ and $M \subseteq^{fin} F$, then $M \subseteq^{fin} G$ for every $G \in \mathcal{G}$. Then $M^{\ell u} \subseteq G$ for every $G \in \mathcal{G}$, which implies that that $M^{\ell u} \subseteq \bigcap \mathcal{G} = F$. Thus, $F \in \mathcal{F}^{f}$. □

Remark 4.2.2. *We note that the inclusion of $\varnothing$ in $\mathcal{F}^{p}$, $\mathcal{F}^{dp}$ and $\mathcal{F}^{f}$ when* $\mathbf{P}$ *does not have a top element is necessary for the closure of these families of up-sets under intersection. Similarly, the inclusion of $\varnothing$ in $\mathcal{I}^{p}$, $\mathcal{I}^{dp}$ and $\mathcal{I}^{f}$ when* $\mathbf{P}$ *does not have a bottom element, ensures that each of these families of down-sets is closed under intersection.*

Next we investigate the generation of 'filters' and 'ideals' by arbitrary subsets. Since these families are closed under intersection, the 'filter' and 'ideal' generated by a subset of the poset can be defined from above. Indeed, in the literature the definitions from above of generated 'filters' and 'ideals' have often been used. However, in the sequel we provide definitions from below and show that they are equivalent to the definitions from above.

Lemma 4.2.3. *Let $S \subseteq P$ be arbitrary. Then there exists a Doyle-pseudo (respectively, pseudo) filter, denoted by $[S\rangle_{dp}$ (respectively, $[S\rangle_{p}$), that is the intersection of all Doyle-pseudo (respectively, pseudo) filters containing S. The set $[S\rangle_{dp}$ (respectively, $[S\rangle_{p}$) is called* the Doyle-pseudo (respectively, pseudo) filter generated by S. *Moreover, define the sequence S_i, $i \in \mathbb{N}$, of subsets of P as follows:*

$$S_0 = S$$

$$S_{i+1} = \left[\left\{\bigwedge M : \varnothing \neq M \subseteq^{fin} (respectively, \subseteq^2) S_i \text{ and } \bigwedge M \text{ exists } \right\}\right)$$

Then, $[S\rangle_{dp}$ (respectively, $[S\rangle_{p}$) $= \bigcup_{i\in\mathbb{N}} S_i$.

Proof. Observe that $\{F \in \mathcal{F}^{dp} : S \subseteq F\} \neq \varnothing$ since $S \subseteq P \in \mathcal{F}^{dp}$. By Lemma 4.2.1, $\bigcap\{F \in \mathcal{F}^{dp} : S \subseteq F\} \in \mathcal{F}^{dp}$.

For the second part of the claim, let $F = \bigcup_{i\in\mathbb{N}} S_i$. If $a \in S_i$, then $\{a\} \subseteq^{fin} S_i$ with $a = \bigwedge\{a\}$. Thus $a \in S_{i+1}$ and $S_i \subseteq S_{i+1}$. That is, the sequence S_i, $i \in \mathbb{N}$, is increasing. In particular, $S \subseteq S_i$ for each $i \in \mathbb{N}$ and hence $S \subseteq F$. Furthermore, since S_i is an up-set for $i \geq 1$, it follows that F is also an up-set.

Suppose $M \subseteq^{fin} F$ such that $\bigwedge M$ exists and $M = \{a_1, \ldots, a_n\}$. For $j = 1, \ldots, n$, let T_j be the first set in the sequence S_i, $i \in \mathbb{N}$, such that $a_j \in T_j$. Since M is finite and S_i, $i \in \mathbb{N}$, is increasing, there is a largest element, S_k, in $\{T_1, \ldots, T_n\}$. Then, $M \subseteq^{fin} S_k$ and $\bigwedge M \in S_{k+1}$. Hence, $\bigwedge M \in F$ and $F \in \mathcal{F}^{dp}$.

Let $G \in \mathcal{F}^{dp}$ such that $S \subseteq G$. We show by induction that each $S_i \subseteq G$ for $i \in \mathbb{N}$. If $a \in S_0$, then $a \in G$ by hypothesis. Suppose $S_i \subseteq G$ and say $a \in S_{i+1}$. Then $a \geq \bigwedge M$ for some $M \subseteq^{fin} S_i \subseteq G$. But $G \in \mathcal{F}^{dp}$ implies that $\bigwedge M \in G$ and therefore also $a \in G$. Hence, $S_{i+1} \subseteq G$. That is, $S_i \subseteq G$ for all $i \in \mathbb{N}$. Now let $a \in F$; then $a \in S_j \subseteq G$ for some $j \in \mathbb{N}$. Therefore, $F \subseteq G$.

The proof of the claim for generated pseudo filters is similar. □

Example 4.2.4. *One may wonder whether or not the process of finding the Doyle-pseudo (pseudo) filter generated by an arbitrary subset S can be described in finitely many steps. In general it is not possible. The sequence S_i, $i \in \mathbb{N}$, may be a strictly increasing sequence as illustrated in Figure 4.3.*

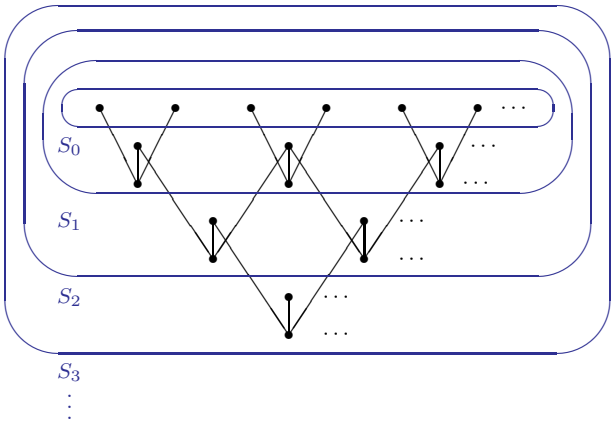


Fig. 4.3: The generation process of $[S\rangle_{pd}$ need not be finite.

Lemma 4.2.5. *Let $S \subseteq P$ be arbitrary. Then there exists a Frink filter, denoted by $[S\rangle_f$, that is the intersection of all Frink filters containing S. The set $[S\rangle_f$ is called* the Frink filter generated by S. *Moreover, $[S\rangle_f = \bigcup \{M^{\ell u} : M \subseteq^{fin} S\}$.*

Proof. Note that $\{F \in \mathcal{F}^f : S \subseteq F\} \neq \varnothing$ since $S \subseteq P \in \mathcal{F}^f$. Then, by Lemma 4.2.1, $\bigcap\{F \in \mathcal{F}^f : S \subseteq F\} \in \mathcal{F}^f$.

We will show that $S \subseteq F$ and that $F \in \mathcal{F}^f$. Then we will show that F is the smallest, set theoretically speaking, Frink filter for which this is the case.

Let $F = \bigcup \{M^{\ell u} : M \subseteq^{fin} S\}$. If $a \in S$, then $\{a\} \subseteq^{fin} S$ and $\{a\} \subseteq \{a\}^{\ell u} \subseteq F$. Therefore, $S \subseteq F$.

Next we show that $F \in \mathcal{F}^f$. If $M \subseteq^{fin} S$ such that $M^\ell = \varnothing$, then $M^{\ell u} = P$ and $F = P \in \mathcal{F}^f$.

Now suppose that $M^\ell \neq \varnothing$ for every $M \subseteq^{fin} S$. Let $N \subseteq^{fin} F$ such that $N = \{a_1, \ldots, a_n\}$. Then there exists $N_i \subseteq^{fin} S$ such that $a_i \in N_i^{\ell u}$ for $i = 1, \ldots, n$. Furthermore, $\bigcup_{i=1}^n N_i \subseteq^{fin} S$. Let $b \in (\bigcup_{i=1}^n N_i)^\ell$, then $b \in N_i^\ell$ for each $i = 1, \ldots, n$. That is, $b \leq a_i$ for each $i = 1, \ldots, n$. But then $b \in N^\ell$ and $(\bigcup_{i=1}^n N_i)^\ell \subseteq N^\ell$. Therefore, $N^{\ell u} \subseteq (\bigcup_{i=1}^n N_i)^{\ell u} \subseteq F$. Hence, $F \in \mathcal{F}^f$.

Finally, let $G \in \mathcal{F}^f$ such that $S \subseteq G$. If $M \subseteq^{fin} S$, then $M \subseteq^{fin} G$ and $M^{\ell u} \subseteq G$. Hence, $\bigcup \{M^{\ell u} : M \subseteq^{fin} S\} = F \subseteq G$. □

Notice that if $S \subseteq^{fin} P$, then $[S\rangle_f = S^{\ell u}$. Clearly $S \subseteq^{fin} S$ and $S^{\ell u} \subseteq [S\rangle_f$. Let $M \subseteq^{fin} S$; then $S^\ell \subseteq M^\ell$ and $M^{\ell u} \subseteq S^{\ell u}$.

If $S = \{a\}$ for some $a \in P$, then $[\{a\}\rangle_p = [\{a\}\rangle_{dp} = [\{a\}\rangle_f = [a)$.

The pseudo, Doyle-pseudo and Frink ideals generated by an arbitrary set $S \subseteq P$ can be defined dually and will be denoted by $\langle S]_p$,$\langle S]_{dp}$ and $\langle S]_f$, respectively. Furthermore, if $S = \{a\}$ for some $a \in P$, then $\langle\{a\}]_p = \langle\{a\}]_{dp} = \langle\{a\}]_f = (a]$.

Since the families $\mathcal{F}^*$ and $\mathcal{I}^*$ are closed under intersection and since arbitrary subsets of P generate elements in $\mathcal{F}^*$ and $\mathcal{I}^*$, these sets form the universes of complete lattices. If $* \in \{p, dp, f\}$, then $\mathbf{F}^* = \langle \mathcal{F}^*, \vee^{\mathbf{F}^*}, \wedge^{\mathbf{F}^*} \rangle$ is a complete lattice, where

$$\bigvee_{i\in\Psi}^{\mathbf{F}^*} F_i = \left[\bigcup_{i\in\Psi} F_i\right\rangle_* \quad \text{and} \quad \bigwedge_{i\in\Psi}^{\mathbf{F}^*} F_i = \bigcap_{i\in\Psi} F_i$$

for $F_i \in \mathcal{F}^*$, $i \in \Psi$. Then $\subseteq$ is the associated lattice order $\leq^{\mathbf{F}^*}$. Similarly, $\mathbf{I}^* = \langle \mathcal{I}^*, \vee^{\mathbf{I}^*}, \wedge^{\mathbf{I}^*} \rangle$ is a complete lattice if $* \in \{p, dp, f\}$, with

$$\bigvee_{j\in\Phi}^{\mathbf{I}^*} F_i = \left\langle\bigcup_{j\in\Phi} I_j\right]_* \quad \text{and} \quad \bigwedge_{j\in\Phi}^{\mathbf{I}^*} F_i = \bigcap_{j\in\Phi} I_j$$

for $I_j \in \mathcal{I}^*$, $j \in \Phi$. Then the associated lattice order $\leq^{\mathbf{I}^*}$ is $\subseteq$.

The following corollary is a consequence of these facts. The claim for the pseudo and Doyle-pseudo cases follows from results in [Doc67] and [Sch72]. The claim for the Frink case follows from the results in [Fri54].

Corollary 4.2.6. *Let* $* \in \{p, dp, f\}$ *and let* $\mathbf{P} = \langle P, \leq \rangle$ *be a poset. Define* $\nu_* : P \to \mathcal{F}^*$ *by* $\nu_*(a) = [a)$*, for all* $a \in P$*. Then* $((\mathbf{F}^*)^\partial, \nu_*)$ *is a completion of* $\mathbf{P}$ *and, furthermore,* ν_* *is an order-embedding of* $\mathbf{P}$ *into* $(\mathbf{F}^*)^\partial$ *that preserves all existing finite meets and joins in* $\mathbf{P}$*.*

Define $\omega_* : P \to \mathcal{I}^*$ *by* $\omega_*(a) = (a]$*, for all* $a \in P$*. Then* $(\mathbf{I}^*, \omega_*)$ *is a completion of* $\mathbf{P}$*. Moreover,* ω_* *is an order-embedding of* $\mathbf{P}$ *into* $\mathbf{I}^*$ *that preserves all existing finite meets and joins in* $\mathbf{P}$*.*

It is interesting to note that the embeddings ω_* (respectively, ν_*) that map elements of a poset onto the principal ideal (respectively, principal filter) generated by it, is called the *canonical embedding* in the literature — see for instance [Ern83] and [Sch72].

Example 4.2.7. *Let* $\mathbf{P}'$ *be the poset depicted in Figure 4.1. Then the complete lattices* $(\mathbf{F}^*)^\partial$ *and* $\mathbf{I}^*$*, for* $* \in \{p, dp, f\}$ *are depicted in Figures 4.4 and 4.5. The image of* P' *is shaded in each of these. See Example A.1.1 in Appendix A.1 for more details.*

The following notions will be explored for most of the completions studied in this thesis and will turn out to be very useful when we investigate extensions of additional operators.

Definition 4.2.8. *Let* $\mathbf{L} = \langle L, \vee, \wedge \rangle$ *be a complete lattice. Then,* $S \subseteq L$ *is said to be* join-dense in $\mathbf{L}$ *if every element in* L *is the join of elements in* S*. Dually,* $T \subseteq L$ *is said to be* meet-dense in $\mathbf{L}$ *if every element in* L *is the meet of elements in* T*.*

If $\mathbf{P}$ *is a poset and* $(\mathbf{L}, \gamma)$ *is a completion of* $\mathbf{P}$*, then* $(\mathbf{L}, \gamma)$ *is called a* join-completion of $\mathbf{P}$ *if* $\gamma(P)$ *is join-dense in* $\mathbf{L}$*. Dually,* $(\mathbf{L}, \gamma)$ *is called a* meet-completion of $\mathbf{P}$ *if* $\gamma(P)$ *is meet-dense in* $\mathbf{P}$*.*

If $\gamma(P)$ *is both meet-dense and join-dense in* $\mathbf{L}$*, then the completion* $(\mathbf{L}, \gamma)$ *is called* doubly dense*.*

Join-completions have also been called *upper completions* [Sch72] or *superior completions* [BS66] in the literature, while meet-completions have also been called *inferior completions* [BS66].

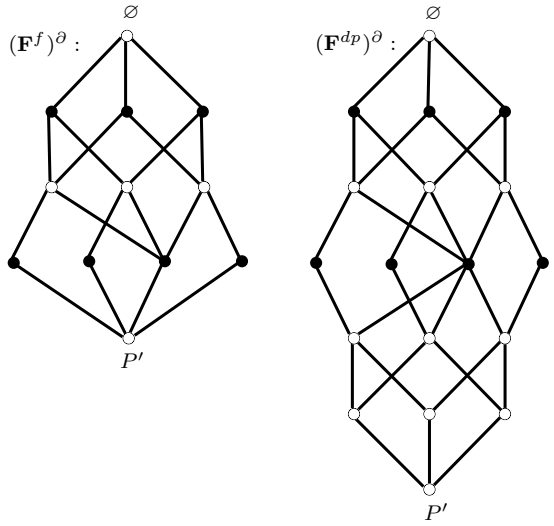


Fig. 4.4: The complete lattices $(\mathbf{F}^f)^\partial$ and $(\mathbf{F}^{dp})^\partial$.

In [Ern83] and [Sch74] the completions $(\mathbf{I}^*, \omega_*)$, for $* \in \{p, dp, f\}$, of a poset $\mathbf{P}$ were shown to be join-completions of $\mathbf{P}$ since $\omega_*(P)$ is join-dense in $\mathbf{I}^*$. Recall that $\bigvee \varnothing = \bot$. Dually, each completion $((\mathbf{F}^*)^\partial, \nu_*)$, for $* \in \{p, dp, f\}$, is a meet-completion of $\mathbf{P}$ since $\nu_*(P)$ is meet-dense in $(\mathcal{F}^*)^\partial$. Here we use the fact that $\bigwedge \varnothing = \top$. See [Sch74] for more on join-completions.

Furthermore, for $* \in \{p, dp, f\}$, the completions $\mathbf{I}^*$ are instances of a *standard completion*. A standard completion of $\mathbf{P}$ is a collection of down-sets of $\mathbf{P}$ that includes all principal down-sets. Standard completions of posets have been studied extensively in, for example, [BN82], [Ern83], [EW83] and [ER87].

We now turn our attention to the families of directed filters and ideals. In general, $\mathcal{F}^d$ and $\mathcal{I}^d$ are not closed under intersection as was noted in [Hof79]. Consider the following counterexample to see why.

Example 4.2.9. *Let* $\mathbf{P}'$ *be the poset depicted in Figure 4.6. Then* $\{1, 2, 3\}$, $\{1, 2, 4\} \in \mathcal{F}^d$, *but* $\{1, 2, 3\} \cap \{1, 2, 4\} = \{1, 2\} \notin \mathcal{F}^d$ *since it does not contain a common lower bound of* 1 *and* 2.

Furthermore, an arbitrary subset of a poset need not generate a unique directed filter. Since $\mathcal{F}^d$ is not closed under intersection a 'generated directed filter' cannot be defined from above. On the other hand, a definition from

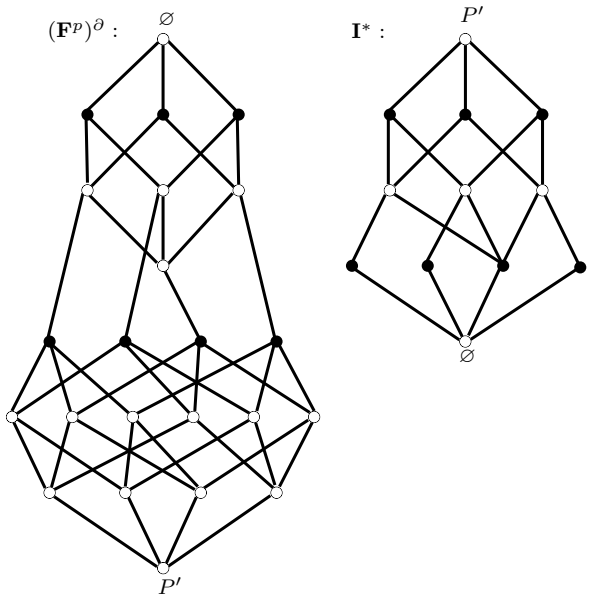


Fig. 4.5: The complete lattices $(\mathbf{F}^p)^\partial$ and $\mathbf{I}^*$.

below would not produce a unique directed filter — if it even exists. Consider the following counterexample to see why.

Example 4.2.10. *Let* $\mathbf{P}'$ *be the poset depicted in Figure 4.7. If* $S = \{3, 4\}$*, then there does not exist a directed filter containing* S*, since* $\mathbf{P}'$ *does not contain a common lower bound for* 3 *and* 4*. Furthermore, if* $T = \{1, 2\}$*, then* $\{1, 2, 3\}$ *and* $\{1, 2, 4\}$ *are two directed filters both containing* T*, but there does not exist a least directed filter containing* T*.*

In fact, it will only make sense to refer to a 'directed filter generated by a set', if we start off with a set that is already directed.

Lemma 4.2.11. *If* $D \subseteq P$ *is down-directed, then* $[D)$ *is the least directed filter containing* D*. Dually, if* $U \subseteq P$ *is up-directed, then* $(U]$ *is the least directed ideal containing* U*.*

The proof is straightforward and is omitted.

The families of directed filters and ideals therefore do not, in general, form complete lattices. In [DP02, Definition 8.1] *pre-complete partially ordered sets* (*pre-CPO* for short) are defined to be posets for which the join $\bigvee D$ of each

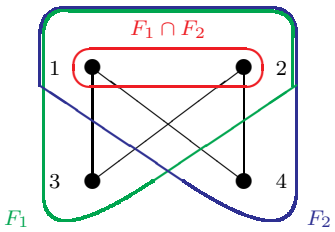


Fig. 4.6: The poset $\mathbf{P}'$ with F_1, F_2 and $F_1 \cap F_2$.

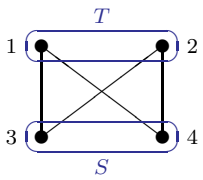


Fig. 4.7: In general, 'generated directed filters' are not well-defined.

directed subset D of $\mathbf{P}$ exists. Then $(\mathbf{F}^d)^\partial$ and $\mathbf{I}^d$ are both pre-CPO's and $\mathcal{I}^d$ is called the *ideal completion* of a poset $\mathbf{P}$ in [DP02, Exercise 9.6].

We now give a quick summary of the properties of the various types of filters and ideals discussed in this chapter. Recall the list of conditions that one might expect the 'filters' of a poset to satisfy from Section 4.1. In Table 4.3 we list these conditions and indicate which of the families of up-sets under consideration in this thesis satisfy the respective conditions. Table 4.4 contains a similar summary for the families of down-sets under consideration in this thesis.

Property	$\mathcal{F}^p$	$\mathcal{F}^{dp}$	$\mathcal{F}^f$	$\mathcal{F}^d$
Includes the principal filters.	Y	Y	Y	Y
Closed under arbitrary intersection.	Y	Y	Y	N
Each member is closed under existing finite meets.	N	Y	Y	Y
Contains $\varnothing$ if, and only if, the poset has no top element.	Y	Y	Y	N
Corresponds with the family of filters on a bounded lattice.	Y	Y	Y	Y

Tab. 4.3: A summary of the properties of the various types of filters.

Property	$\mathcal{I}^p$	$\mathcal{I}^{dp}$	$\mathcal{I}^f$	$\mathcal{I}^d$
Includes the principal ideals.	Y	Y	Y	Y
Closed under arbitrary intersection.	Y	Y	Y	N
Each member is closed under existing finite joins.	N	Y	Y	Y
Contains $\varnothing$ if, and only if, the poset has no bottom element.	Y	Y	Y	N
Corresponds with the family of ideals on a bounded lattice.	Y	Y	Y	Y

Tab. 4.4: A summary of the properties of the various types of ideals.

4.2.1 Prime filters and ideals

In this section we deviate from the main theme of this thesis by considering some properties of the filter and ideal completions that are not directly related to property preservation.

Recall that a prime filter F of a lattice $\mathbf{L} = \langle L, \vee, \wedge \rangle$ is a filter of $\mathbf{L}$ satisfying: if $a \vee b \in F$, then $a \in F$ or $b \in F$. Dually, a prime ideal I of $\mathbf{L}$ is an ideal of $\mathbf{L}$ that satisfies: if $a \wedge b \in I$, then $a \in I$ or $b \in I$. Equivalently, a filter F of $\mathbf{L}$ (respectively, an ideal I of $\mathbf{L}$) is prime if, and only if, $L - F$ (respectively, $L - I$) is a prime ideal (respectively, prime filter).

In [Mac37] the meet of two or more elements, in a poset, is called their product. In analogy with prime numbers, it is then natural to assume that a prime element should be one that cannot be expressed as the meet (product) of two other elements. That is, we would expect a prime filter not to be the intersection of two or more strictly greater filters. In fact, we have the following for lattices.

Lemma 4.2.12. *If* $\mathbf{L}$ *is a lattice and* F *is a prime filter of* $\mathbf{L}$*, then* F *is a meet-irreducible element of* $\mathbf{F}$*.*

Proof. Suppose F is a prime filter and let G_1 and G_2 be two filters of $\mathbf{L}$ such that $F \subset G_1$ and $F \subset G_2$. Then there exists $a \in G_1$ such that $a \notin F$ and there exists $b \in G_2$ such that $b \notin F$. Furthermore, $a \vee b \in G_1 \cap G_2$, but $a \vee b \notin F$ since F is prime. Therefore, $F \subset G_1 \cap G_2$. Hence, F is a meet-irreducible element of $\mathcal{F}$. □

We will now explore possible definitions of 'prime filters' and 'prime ideals' on posets. The first possible definition of a 'prime filter' we consider will not

suffice for meet-irreducibility, but will prove useful in the construction of another completion of the poset, studied in Chapter 7.

In [Jan78] a prime complete Doyle-pseudo filter of a poset $\mathbf{P}$ (called a 'prime conditionally complete filter' in [Jan78]) was defined to be a complete Doyle-pseudo filter, F, that also satisfies: if $S \subseteq P$ such that $\bigvee S$ exists and $\bigvee S \in F$, then $S \cap F \neq \varnothing$. This condition was then shown to be equivalent to requiring that $P - F$ be a prime complete Doyle-pseudo ideal. In line with this we give the following definition of a prime Doyle-pseudo filter.

Definition 4.2.13. *A proper (complete) Doyle-pseudo filter F of a poset $\mathbf{P}$ is said to be* prime *if, and only if, $P - F$ is a (complete) Doyle-pseudo ideal.*

A proper (complete) Doyle-pseudo ideal I is said to be prime *if, and only if, $P - I$ is a (complete) Doyle-pseudo filter.*

Observe that if $F \in \mathcal{F}^{dp}$ is prime, then $P - F \in \mathcal{I}^{dp}$ is prime. Similarly, if $I \in \mathcal{I}^{dp}$ is prime, then $P - I \in \mathcal{F}^{dp}$ is also prime.

The definition implies that $\varnothing$ is not a prime filter or ideal, since $P - \varnothing = P$ which is not proper. We denote the set of prime Doyle-pseudo filters (respectively, prime complete Doyle-pseudo filters) by $\mathscr{F}^{dp}$ (respectively, $\mathscr{F}^{cdp}$) and the set of prime Doyle-pseudo ideals (respectively, prime complete Doyle-pseudo ideals) by $\mathscr{I}^{dp}$ (respectively, $\mathscr{I}^{cdp}$). Prime pseudo filters and ideals can be defined similarly.

Lemma 4.2.14. *A Doyle-pseudo filter F of a poset $\mathbf{P}$ is prime if, and only if, whenever $\bigvee M$ exists and $\bigvee M \in F$ for some $M \subseteq^{fin} P$, then $F \cap M \neq \varnothing$.*

Proof. Suppose F is prime and let $M \subseteq^{fin} P$ such that $\bigvee M$ exists and $\bigvee M \in F$. Assume $M \cap F = \varnothing$. Then $M \subseteq^{fin} P - F$. But $P - F$ is a Doyle-pseudo ideal since F is prime. Therefore, $\bigvee M \in P - F$ — contradicting the initial assumption that $\bigvee M \in F$. Hence, $M \cap F \neq \varnothing$.

Next suppose F satisfies: whenever $\bigvee M$ exists and $\bigvee M \in F$ for some $M \subseteq^{fin} P$, then $F \cap M \neq \varnothing$. Now let $N \subseteq^{fin} P - F$. If $\bigvee N$ exists, then $\bigvee N \in P - F$ by the contrapositive of the assumption, since $F \cap N = \varnothing$. Thus, $P - F$ is a Doyle-pseudo ideal and F is prime. □

We now define the prime Frink filters and prime directed filters in the same way.

Definition 4.2.15. *A proper (complete) Frink filter F of a poset $\mathbf{P}$ is said to be* prime *if, and only if, $P - F$ is a (complete) Frink ideal. A proper (complete) Frink ideal I is said to be* prime *if, and only if, $P - I$ is a (complete) Frink filter.*

Similarly, a proper (complete) directed filter F is said to be prime *if, and only if, $P - F$ is a (complete) directed ideal. A proper (complete) directed ideal I is said to be* prime *if, and only if, $P - I$ is a (complete) directed filter.*

Lemma 4.2.16. *A Frink (respectively, directed) filter F of a poset $\mathbf{P}$ is prime if, and only if, whenever $\bigvee M$ exists and $\bigvee M \in F$ for some $M \subseteq^{fin} P$, then $F \cap M \neq \varnothing$.*

We note that the dual of the above statement holds for Frink (respectively, directed) ideals. The proof is similar to the proof of Lemma 4.2.14 and follows from the fact that Frink and directed filters are closed under existing finite meets while Frink and directed ideals are closed under existing finite joins. See Remark 4.1.5.

In [GJP10] a prime directed ideal, I, of a meet-semilattice is defined to be a directed ideal satisfying: for any $a, b \in P$ if $a \wedge b \in I$, then $a \in I$ or $b \in I$. The above result ensures that the definition of a prime directed ideal given here is a generalisation of the definition given in [GJP10].

Let $\mathscr{F}^f$ and $\mathscr{F}^d$ denote the sets of prime Frink and prime directed filters, respectively, while $\mathscr{I}^f$ and $\mathscr{I}^d$ denote the sets of prime Frink and prime directed ideals, respectively.

Since $\langle \mathcal{F}^d, \subseteq \rangle$ is not complete, we do not consider the directed filters or ideals in the discussion below.

In general the prime pseudo, Doyle-pseudo and Frink filters defined above need not be meet-irreducible elements in $\mathbf{F}^*$, $* \in \{p, dp, f\}$. Consider the following example to see why.

Example 4.2.17. *Let $\mathbf{P}'$ be the poset depicted in Figure 4.8. Then $\mathcal{F}^p = \mathcal{F}^{dp} = \mathcal{F}^f$. Moreover, $\{1, 2\} \in \mathscr{F}^*$ since $\{3, 4\} \in \mathcal{I}^*$. However, $\{1, 2\}$ is not meet-irreducible in $\mathbf{F}^*$ since $\{1, 2, 3\} \cap \{1, 2, 4\} = \{1, 2\}$.*

The definitions of prime $*$-filters given above therefore seem to be insufficient. In [Hal00] 'prime Doyle-pseudo ideals' of a poset (simply called the 'prime ideals' of a poset in [Hal00]) were defined as follows: $I \in \mathcal{I}^{dp}$ of a poset $\mathbf{P}$ is called

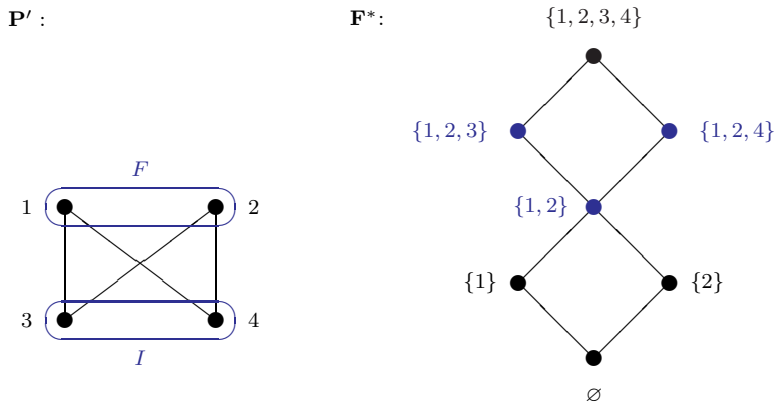


Fig. 4.8: A prime filter need not be meet-irreducible in $\mathbf{F}^*$.

prime if I is proper and non-empty and

$$\text{if } a, b \in P \text{ such that } \{a, b\}^\ell \subseteq I, \text{ then } a \in I \text{ or } b \in I. \tag{4.8}$$

In light of the above we make the following definitions.

Definition 4.2.18. *A proper, non-empty pseudo filter F of a poset $\mathbf{P}$ is called a* strictly prime *pseudo filter of $\mathbf{P}$ if it satisfies*

$$\text{If } \varnothing \neq M \subseteq^2 P \text{ such that } M^u \subseteq F, \text{ then } M \cap F \neq \varnothing. \tag{4.9}$$

Strictly prime *pseudo ideals can be defined dually.*

Definition 4.2.19. *A proper, non-empty Doyle-pseudo filter F of a poset $\mathbf{P}$ is called a* strictly prime *Doyle-pseudo filter of $\mathbf{P}$ if it satisfies*

$$\text{If } \varnothing \neq M \subseteq^{fin} P \text{ such that } M^u \subseteq F, \text{ then } M \cap F \neq \varnothing. \tag{4.10}$$

Similarly, a proper, non-empty Frink filter F of $\mathbf{P}$ is called a strictly prime *Frink filter of $\mathbf{P}$ if it satisfies (4.10).*

Strictly prime *Doyle-pseudo and Frink ideals can be defined dually.*

Let $\mathscr{F}_s^*$ (respectively, $\mathscr{I}_s^*$) denote the sets of all strictly prime $*$-filters (respectively, strictly prime $*$-ideals). It follows from Example 4.2.17 that, in general, $\mathscr{F}^* \nsubseteq \mathscr{F}_s^*$ for $* \in \{p, dp, f\}$. However, the inclusion in the other direction holds.

Lemma 4.2.20. *Let $* \in \{p, dp, f\}$. Then $\mathscr{F}_s^* \subseteq \mathscr{F}^*$.*

Proof. Let $F \in \mathscr{F}_s^{dp}$ and let $M \subseteq^{fin} P - F$ such that $\bigvee M$ exists. Then $M^u \neq \varnothing$ as $\bigvee M \in M^u$. If $M^u \subseteq F$, then by (4.10) $M \cap F \neq \varnothing$ — a contradiction. Therefore, there exists an element $b \in M^u$ such that $b \in P - F$. But then $\bigvee M \in P - F$: if $\bigvee M \in F$, then $b \in F$ since $\bigvee M \leq b$ and F is an upset — again a contradiction. Thus, $P - F \in \mathscr{I}^{dp}$ and $F \in \mathscr{F}^{dp}$.

The proof of $\mathscr{F}_s^p \subseteq \mathscr{F}^p$ is similar.

Let $F \in \mathscr{F}_s^f$ and let $M \subseteq^{fin} P - F$. If $M^u \subseteq F$, then $M \cap F \neq \varnothing$ since F satisfies (4.10) — contradicting our choice of M. Hence $M^u \cap (P - F) \neq \varnothing$. Let $b \in M^u \cap (P - F)$, then $b \geq c$ for every $c \in M^{u\ell}$. If there exists $c \in M^{u\ell}$ such that $c \in F$, then $b \in F$ since F is an upset — contradicting the fact that $b \in P - F$. Therefore, $c \in P - F$ for all $c \in M^{u\ell}$, i.e., $M^{u\ell} \subseteq P - F$ and $P - F \in \mathcal{F}^f$. Hence, $F \in \mathscr{F}^f$. □

Similarly we can show that $\mathscr{I}_s^* \subseteq \mathscr{I}^*$ for $* \in \{p, dp, f\}$.

Next we show that the strictly prime filters are meet-irreducible in the lattice of filters.

Lemma 4.2.21. *Let $\mathbf{P}$ be a poset and $* \in \{p, dp, f\}$. If $F \in \mathscr{F}_s^*(\mathbf{P})$, then F is meet-irreducible in $\mathbf{F}^*(\mathbf{P})$ and hence a join-irreducible element in $(\mathbf{F}^*(\mathbf{P}))^\partial$.*

Proof. Let $F \in \mathscr{F}_s^*(\mathbf{P})$ and $G_1, G_2 \in \mathcal{F}^*(\mathbf{P})$ such that $F \subset G_1$ and $F \subset G_2$. Then $F \subseteq G_1 \cap G_2$. If $G_1 \subseteq G_2$, then $G_1 \cap G_2 = G_1$ and $F \subset G_1 \cap G_2$. Similarly, if $G_2 \subseteq G_1$, then $F \subset G_1 \cap G_2$. Now suppose $G_1 \nsubseteq G_2$ and $G_2 \nsubseteq G_1$. Then there exist elements $a \in G_1$ and $b \in G_2$ such that $a \notin G_2$ and $b \notin G_1$. Then $\{a, b\}^u \subseteq G_1 \cap G_2$: if $\{a, b\}^u = \varnothing$, then $\{a, b\}^u \subseteq G_1 \cap G_2$. On the other hand, suppose $\{a, b\}^u \neq \varnothing$ and let $c \in \{a, b\}^u$. Then $c \in G_1$ since $c \geq a$ and $c \in G_2$ since $c \geq b$, i.e., $c \in G_1 \cap G_2$. Hence, $\{a, b\}^u \subseteq G_1 \cap G_2$. Now suppose $\{a, b\}^u \subseteq F$. Then $a \in F$ or $b \in F$ since F is strictly prime. But since $F \subseteq G_1 \cap G_2$, it follows that $a \in G_1 \cap G_2$ or $b \in G_1 \cap G_2$ — contradicting our choice of a and b. Therefore, $\{a, b\}^u \nsubseteq F$. We now choose $G = G_1 \cap G_2$. By Lemma 4.2.1 $G \in \mathcal{F}^*$. Then, since $a, b \notin G$ we have $G \subset G_1$ and $G \subset G_2$. Moreover, since $\{a, b\}^u \subseteq G$ we know that $F \subset G$.

We have shown that if $F \subset G_1$ and $F \subset G_2$, then $F \subset G_1 \cap G_2$ for all $G_1, G_2 \in \mathcal{F}^*$. Thus, F is meet-irreducible in $\mathbf{F}^*$ and therefore also join-irreducible in $(\mathbf{F}^*)^\partial$. □

Similarly, for $* \in \{p, dp, f\}$, if $I \in \mathscr{I}_s^*(\mathbf{P})$, then I is meet-irreducible in $\mathbf{I}^*(\mathbf{P})$.

The converse of Lemma 4.2.21 need not be true.

Example 4.2.22. *Let $* \in \{p, dp, f\}$ and let $\mathbf{P}'$ be the poset depicted in Figure 4.9. Let $F = \{1\} \in \mathcal{F}^*$ and $M = \{2, 3\} \subseteq P'$. Then $M^u = \varnothing \subseteq F$ but $M \cap F = \varnothing$. Therefore, $F \notin \mathscr{F}_s^*$. However, as can be seen in the depiction of $\mathbf{F}^*$ in Figure 4.9, F is meet-irreducible in $\mathbf{F}^*$.*

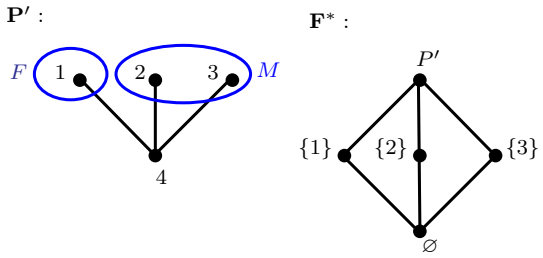


Fig. 4.9: Not all meet-irreducible elements in $\mathbf{F}^*$ are strictly prime.

If $\mathbf{P}$ is a lattice, then a filter of $\mathbf{P}$ is strictly prime if, and only if, it is prime. We note not every meet-irreducible element in $\mathbf{F}(\mathbf{L})$ of a lattice $\mathbf{L}$ is a prime filter of $\mathbf{L}$. To see why, consider the meet-irreducible elements in the lattice of filters of the complete lattice $\mathbf{F}^*$ in the previous example.

In [LR88] generalised notions of distributivity and modularity of posets are given. A poset $\mathbf{P}$ said to be *distributive* if $(\{a, b\}^u \cup \{c\})^\ell = (\{a, c\}^\ell \cup \{b, c\}^\ell)^{u\ell}$ for all $a, b, c \in P$. Furthermore, a poset $\mathbf{P}$ is called *ideal distributive* if the lattice of ideals (as defined in [Hal00]) is distributive. In [HR95] it was then shown that if $\mathbf{P}$ is ideal distributive, then $I \subseteq P$ is a strictly prime ideal if, and only if, it is meet-irreducible. A consequence of this result is that an ideal of a distributive lattice is prime if, and only if, it is meet-irreducible in its lattice of ideals. We give a direct proof here.

Lemma 4.2.23. *If $\mathbf{L}$ is a distributive lattice then $F \in \mathcal{F}$ is prime if, and only if, F is meet-irreducible in $\mathbf{F}$.*

Proof. The forward implication follows from Lemma 4.2.21. We must therefore only prove the backward implication. We prove the contra-positive.

Let $\mathbf{L}$ be a distributive lattice and let F be a filter of $\mathbf{L}$ that is not prime. Then there exist elements $a, b \in L$ such that $a \vee b \in F$ but $a \notin F$ and $b \notin F$. Suppose $a \leq b$, then $b = a \vee b \in F$ — contradicting the choice of a and b. Similarly, if $b \leq a$. Thus $a \not\leq b$ and $b \not\leq a$. Now let $G_1 = [\{c \wedge a : c \in F\})$ and $G_2 = [\{c \wedge b : c \in F\})$. Then G_1 and G_2 are filters of $\mathbf{L}$ such that $F \subseteq G_1 \cap G_2$.

Let $d \in G_1 \cap G_2$. Then there exist elements $c_1, c_2 \in F$ such that $d \geq c_1 \wedge a$ and $d \geq c_2 \wedge b$. Then $d \geq (c_1 \wedge a) \vee (c_2 \wedge b) \geq (c_1 \wedge c_2 \wedge a) \vee (c_1 \wedge c_2 \wedge b)$. Let $c = c_1 \wedge c_2$. Then $c \in F$ and $d \geq (c \wedge a) \vee (c \wedge b) = c \wedge (a \vee b)$ since $\mathbf{L}$ is distributive. But $c \in F$ and $a \vee b \in F$ imply that $c \wedge (a \vee b) \in F$. Since F is an upset we have $d \in F$. Hence, $F = G_1 \cap G_2$.

Now suppose $a \in G_2$. Then there exists $c \in F$ such that $a \geq c \wedge b$. For any $c' \in F$ we have $c' \wedge a \geq c' \wedge c \wedge b$. But $c' \wedge c \in F$ which implies that $(c' \wedge c) \wedge b \in G_2$ and therefore so is $c' \wedge a$. Then $G_1 \subseteq G_2$ and $G_1 \cap G_2 = G_1 = F$. But then $a \in F$ which contradicts our choice of a and b. Thus $a \notin G_2$. Similarly, $b \notin G_1$. Then $F \subset G_1$ and $F \subset G_2$ but $F = G_1 \cap G_2$, i.e., F is not meet-irreducible. □

4.3 Extensions of maps

It is often the case that a poset $\mathbf{P}$ is the underlying ordered structure of an algebra. If this is the case, then there will usually be some additional operations defined on P. The process of completing an algebra then includes finding extensions of these additional operations to the complete algebra. We now consider the extensions of operations to the completions studied in this chapter.

If $f : P \to Q$ is a map defined between two posets, then we would like to define extensions of f on the filter and ideal completions of the posets. That is, for $* \in \{p, dp, f\}$, we want to define maps $f^{(\mathbf{F}^*)^\partial} : \mathcal{F}^*(P) \to \mathcal{F}^*(Q)$ and $f^{\mathbf{I}^*} : \mathcal{I}^*(P) \to \mathcal{I}^*(Q)$ in such a way that $f^{(\mathbf{F}^*)^\partial}(\nu_*(a)) = \nu_*(f(a))$ and $f^{\mathbf{I}^*}(\omega_*(a)) = \omega_*(f(a))$. Similarly, we would also like to define extensions of n-ary maps. In [BS66] completions of partially ordered algebras were considered. In particular the authors of [BS66] considered join-completions, meet-completions and doubly dense completions of partially ordered algebras. Their definition of the extension of an order-preserving operation of an algebra heavily relied on the join-denseness of the poset universe in its completion. In this section we will employ similar methods for the extension of maps.

We begin by considering the extension of unary maps.

For the remainder of this section let $* \in \{p, dp, f\}$ and posets $\mathbf{P} = \langle P, \leq^{\mathbf{P}} \rangle$ and $\mathbf{Q} = \langle Q, \leq^{\mathbf{Q}} \rangle$ be fixed. Recall that $\nu_*(P)$ and $\nu_*(Q)$ are meet-dense in $(\mathbf{F}^*(\mathbf{P}))^\partial$ and $(\mathbf{F}^*(\mathbf{Q}))^\partial$, respectively; and $\omega_*(P)$ and $\omega_*(Q)$ are join-dense in $\mathcal{I}^*(\mathbf{P})$ and $\mathcal{I}^*(\mathbf{Q})$, respectively. That is, for $F \in \mathcal{F}^*(\mathbf{P})$ we have that $F = \bigwedge^{(\mathbf{F}^*)^\partial} \{\nu_*(a) : a \in P \text{ such that } \nu_*(a) \geq^{(\mathbf{F}^*)^\partial} F\}$ and for $I \in \mathcal{I}^*(\mathbf{P})$ we have that $I = \bigvee^{\mathbf{I}^*} \{\omega_*(a) : a \in P \text{ such that } \omega_*(a) \leq^{\mathbf{I}^*} I\}$. If $f : P \to Q$ is an order-preserving unary map, then we have the following natural extensions of f to $(\mathbf{F}^*(\mathbf{P}))^\partial$ and $\mathbf{I}^*(\mathbf{P})$, respectively.

Definition 4.3.1. *Let $f : P \to Q$ be order-preserving. Define $f_*^\wedge : \mathcal{F}^*(\mathbf{P}) \to \mathcal{F}^*(\mathbf{Q})$ by, for $F \in \mathcal{F}^*(\mathbf{P})$,*

$$\begin{aligned} f_*^\wedge(F) &= \bigwedge^{(\mathbf{F}^*(\mathbf{Q}))^\partial} \{[f(a)) : a \in P \text{ such that } \nu_*(a) \subseteq F\} \\ &= \bigwedge^{(\mathbf{F}^*(\mathbf{Q}))^\partial} \{[f(a)) : a \in P \text{ such that } [a) \subseteq F\} \end{aligned}$$

Define $f_*^\vee : \mathcal{I}^*(\mathbf{P}) \to \mathcal{I}^*(\mathbf{Q})$ by, for $I \in \mathcal{I}^*(\mathbf{P})$,

$$\begin{aligned} f_*^\vee(I) &= \bigvee^{\mathcal{I}^*(\mathbf{Q})} \{(f(a)] : a \in P \text{ such that } \omega_*(a) \subseteq I\} \\ &= \bigvee^{\mathcal{I}^*(\mathbf{Q})} \{(f(a)] : a \in P \text{ such that } (a] \subseteq I\} \end{aligned}$$

For $F \in \mathcal{F}^*(\mathbf{P})$, let $f(F) = \{f(a) : a \in F\}$. Similarly, for $I \in \mathcal{I}^*(\mathbf{P})$ let $f(I) = \{f(a) : a \in I\}$. The maps $f_*^\wedge$ and $f_*^\vee$ can now be simplified in the following way.

Lemma 4.3.2. *Let $f : P \to Q$ be order-preserving. Then,*

$$f_*^\wedge(F) = [f(F)\rangle_* \quad \text{and} \quad f_*^\vee(I) = \langle f(I)]_* .$$

Furthermore, $f_*^\wedge$ and $f_*^\vee$ are both order-preserving and are extensions of f. That is, for $a \in P$,

$$f_*^\wedge(\nu_*(a)) = \nu_*(f(a)) \quad \text{and} \quad f_*^\vee(\omega_*(a)) = \omega_*(f(a)).$$

Proof. We prove the claims for $f_*^\wedge$. The claims for $f_*^\vee$ can be proved similarly.

Let $a \in F$. Then $\nu_*(a) = [a) \subseteq F$ and it follows that $f(F) \subseteq \bigcup\{[f(a)) : a \in P \text{ such that } \nu_*(a) \subseteq F\}$. Therefore, $[f(F)\rangle_* \subseteq f_*^\wedge(F)$.

For the inclusion in the other direction, let $a \in P$ such that $\nu_*(a) = [a) \subseteq F$. If $b \in [f(a))$, then $b \geq f(a) \in [f(F)\rangle_*$. Since $[f(F)\rangle_*$ is an up-set we have that $b \in [f(F)\rangle_*$. Thus, $[f(a)) \subseteq [f(F)\rangle_*$ for all $a \in P$ such that $\nu_*(a) = [a) \subseteq F$. Then $\bigcup\{[f(a)) : a \in P \text{ such that } \nu_*(a) \subseteq F\} \subseteq [f(F)\rangle_*$. Since the filter generated by a set is the intersection of all filters that include it, we have that $f_*^\wedge(F) \subseteq [f(F)\rangle_*$.

Let $F, G \in \mathcal{F}^*(P)$ such that $F \leq^{(\mathbf{F}^*(\mathbf{P}))^\partial} G$, i.e., $G \subseteq F$. Then $f(G) \subseteq f(F)$. This implies that $[f(G)\rangle_* \subseteq [f(F)\rangle_*$, i.e., $f_*^\wedge(G) \leq^{(\mathbf{F}^*(\mathbf{Q}))^\partial} f_*^\wedge(F)$.

Finally we show that $f_*^\wedge$ extends f. By the above $f_*^\wedge(\nu_*(a)) = [f([a))\rangle_*$. Since $f(a) \subseteq f([a))$ it follows that $[f(a)) \subseteq [f([a))\rangle_*$. Now let $b \in [a)$. Then $b \geq a$ and since f is order-preserving $f(b) \geq f(a)$. Then, $f(b) \in [f(a))$ and $f([a)) \subseteq [f(a))$. Then, by the definition of a generated filter, $[f([a))\rangle_* \subseteq [f(a)\rangle_*$. Therefore, $f_*^\wedge(\nu_*(a)) = [f(a)\rangle_* = \nu_*(f(a))$. □

Lemma 4.3.3. *Let $f : P \to P$ be a unary order-preserving operation with extensions $f_*^\wedge$ and $f_*^\vee$ to $(\mathbf{F}^*)^\partial$ and $\mathbf{I}^*$, respectively. Then,*

(i) if f is increasing (also known as extensive), then so are $f_*^\wedge$ and $f_*^\vee$,

(ii) if f is decreasing, then so are $f_*^\wedge$ and $f_*^\vee$,

(iii) if f is the identity map on P, then $f_*^\wedge$ is the identity map on $\mathcal{F}^*$ and $f_*^\vee$ is the identity map on $\mathcal{I}^*$.

Proof. We prove the claims for $f_*^\wedge$. The claims for $f_*^\vee$ follow similarly.

(i) Suppose $a \leq f(a)$ for all $a \in P$. Let $F \in \mathcal{F}^*$ and $a \in F$. Then, $a \leq f(a)$ by assumption and $f(a) \in F$ since F is an up-set. Then, $f(F) \subseteq F$ and hence $[f(F)\rangle_* \subseteq F$, i.e., $F \leq^{(\mathbf{F}^*)^\partial} f_*^\wedge(F)$.

(ii) Suppose $f(a) \leq a$ for all $a \in P$. Let $F \in \mathcal{F}^*$ and $a \in F$. Then, $f(a) \leq a$ which implies that $a \in [f(F)\rangle_*$. Therefore, $F \subseteq [f(F)\rangle_*$, i.e., $f_*^\wedge(F) \leq^{(\mathbf{F}^*)^\partial} F$.

(iii) Suppose $f(a) = a$ for all $a \in P$ and let $F \in \mathcal{F}^*$. Then, $[f(F)\rangle_* = [\{f(a) : a \in F\}\rangle^* = [\{a : a \in F\}\rangle_* = F$. That is, $f_*^\wedge(F) = F$.

□

Example 4.3.4. *Let* $\mathbf{P'}$ *be the poset depicted in Figure 4.10. If* $h : P' \to P'$ *is defined by* $h(1) = h(2) = 2$ *and* $h(3) = 3$*, then* h *is an operator (since no non-trivial joins exist in* $\mathbf{P'}$*). However,* $h^\wedge_f$ *is not an operator: let* $F = \{1\}$ *and* $G = \{2\}$*. Then,* $[h(F)\rangle_f = [\{2\}\rangle_f = \{2\}$ *and* $[h(G)\rangle_f = [\{2\}\rangle_f = \{2\}$*. Therefore,* $h^\wedge_f(F) \vee^{(\mathbf{F}^f(\mathbf{P'}))^\partial} h^\wedge_f(G) = \{2\}$*. On the other hand,* $F \vee^{(\mathbf{F}^f(\mathbf{P'}))^\partial} G = F \cap G = \varnothing$*. Then,* $h^\wedge_f(F \vee^{(\mathbf{F}^f(\mathbf{P'}))^\partial} G) = [h(F \cap G)\rangle_f = [h(\varnothing)\rangle_f = [\varnothing\rangle_f = \varnothing$*.*

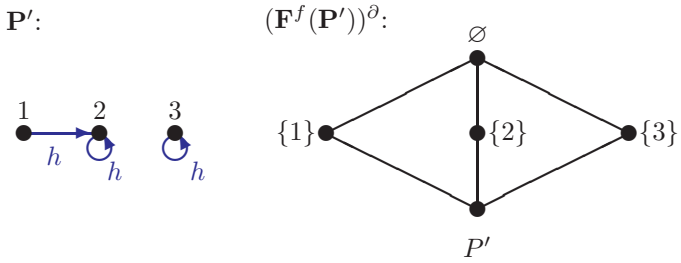


Fig. 4.10: $h^\wedge_f$ need not be an operator when h is.

Remark 4.3.5. *If we examine Example 4.3.4 further, we observe that there does not exist an extension of* h *to some* h' *defined on* $\mathcal{F}^f(\mathbf{P'})$ *such that* h' *will be an operator. Suppose to the contrary that some extension* h' *of* h *is an operator. Then* $h'(\varnothing) \geq h'(\nu_f(2)) = \nu_f(h(2)) = \nu_f(2)$ *and* $h'(\varnothing) \geq h'(\nu_f(3)) = \nu_f(h(3)) = \nu_f(3)$ *implies that* $h'(\varnothing) = \varnothing$*. But* $h'(\varnothing) = h'(\nu_f(1) \vee \nu_f(2)) = h'(\nu_f(1)) \vee h'(\nu_f(2)) = \nu_f(2)$ *and we have reached a contradiction. However, this does not mean that it is impossible to find some completion of* $\mathbf{P'}$ *for which* h *can be extended to an operator — for an example see Remark 6.3.9.*

One can also use the poset $\mathbf{P'}$ in Example 4.3.4 to see that $h^\wedge_f$ and $h^\vee_f$ need not be dual operators when h is a dual operator and that $h^\vee_f$ need not be an operator when h is an operator. See Example A.1.2 in Appendix A.1 for the details.

On the other hand, if f is a dual operator, then $f^\wedge_{dp}$ is a complete dual operator. Similarly, if f is an operator, then $f^\vee_{dp}$ is a complete operator. We prove the latter statement here. The proof of the former follows dually.

Lemma 4.3.6. *If* $f : P \to Q$ *is an operator, then* $f^\vee_{dp}$ *is a complete operator.*

Proof. Let $I_i \in \mathcal{I}^{dp}(\mathbf{P})$ for $i \in \Psi$. We need to compare

$$f^{\vee}_{dp}\left(\bigvee_{i\in\Psi} I_i\right) = \left\langle f\left(\left\langle \bigcup_{i\in\Psi} I_i\right]_{dp}\right)\right]_{dp} \text{ and } \bigvee_{i\in\Psi} f^{\vee}_{dp}(I_i) = \left\langle \bigcup_{i\in\Psi} \langle f\,(I_i)]_{dp}\right]_{dp}.$$

We first show by induction that $f\left(\left\langle \bigcup_{i\in\Psi} I_i\right]_{dp}\right) \subseteq \left\langle \bigcup_{i\in\Psi} \langle f\,(I_i)]_{dp}\right]_{dp}$. If $a \in S_0 = \bigcup_{i\in\Psi} I_i$, then $a \in I_{i_0}$ for some $i_0 \in \Psi$. Then $f(a) \in \langle f(I_{i_0})]_{dp} \subseteq \left\langle \bigcup_{i\in\Psi} \langle f\,(I_i)]_{dp}\right]_{dp}$ and $f(S_0) \subseteq \left\langle \bigcup_{i\in\Psi} \langle f\,(I_i)]_{dp}\right]_{dp}$. Now suppose $f(S_j) \subseteq \left\langle \bigcup_{i\in\Psi} \langle f\,(I_i)]_{dp}\right]_{dp}$ and let $a \in S_{j+1}$. Then $a \leq \bigvee M$ for some $M \subseteq^{fin} S_j$ such that $\bigvee M$ exists. By the inductive hypothesis $f(M) \subseteq \left\langle \bigcup_{i\in\Psi} \langle f\,(I_i)]_{dp}\right]_{dp}$. Furthermore, since $\bigvee M$ exists and f is an operator, we have that $\bigvee f(M)$ exists and $\bigvee f(M) = f(\bigvee M)$. Then $f(\bigvee M) \in \left\langle \bigcup_{i\in\Psi} \langle f\,(I_i)]_{dp}\right]_{dp}$ since it is closed under existing joins. Since f is order-preserving, $f(a) \leq f(\bigvee M)$ and it follows that $f(a) \in \left\langle \bigcup_{i\in\Psi} \langle f\,(I_i)]_{dp}\right]_{dp}$. Thus, $f(S_{j+1}) \subseteq \left\langle \bigcup_{i\in\Psi} \langle f\,(I_i)]_{dp}\right]_{dp}$. This proves that $f\left(\left\langle \bigcup_{i\in\Psi} I_i\right]_{dp}\right) \subseteq \left\langle \bigcup_{i\in\Psi} \langle f\,(I_i)]_{dp}\right]_{dp}$. Hence, $\left\langle f\left(\left\langle \bigcup_{i\in\Psi} I_i\right]_{dp}\right)\right]_{dp} \subseteq \left\langle \bigcup_{i\in\Psi} \langle f\,(I_i)]_{dp}\right]_{dp}$.

For the inclusion in the other direction:

$$\begin{aligned}
& I_i \subseteq \left\langle \bigcup_{i\in\Psi} I_i\right]_{dp} \text{ for all } i \in \Psi \\
\Rightarrow\ & f(I_i) \subseteq f\left(\left\langle \bigcup_{i\in\Psi} I_i\right]_{dp}\right) \text{ for all } i \in \Psi \\
\Rightarrow\ & \langle f(I_i)]_{dp} \subseteq \left\langle f\left(\left\langle \bigcup_{i\in\Psi} I_i\right]_{dp}\right)\right]_{dp} \text{ for all } i \in \Psi \\
\Rightarrow\ & \bigcup_{i\in\Psi} \langle f(I_i)]_{dp} \subseteq \left\langle f\left(\left\langle \bigcup_{i\in\Psi} I_i\right]_{dp}\right)\right]_{dp} \\
\Rightarrow\ & \left\langle \bigcup_{i\in\Psi} \langle f(I_i)]_{dp}\right]_{dp} \subseteq \left\langle f\left(\left\langle \bigcup_{i\in\Psi} I_i\right]_{dp}\right)\right]_{dp}.
\end{aligned}$$

□

A consequence of the above is that if $f : P \to Q$ is residuated, then $f^{\vee}_{dp}$ is residuated. Moreover, if $f : P \to Q$ is residuated with residual $g : Q \to P$, then

$f^\vee_{dp} : \mathcal{I}^{dp}(\mathbf{P}) \to \mathcal{I}^{dp}(\mathbf{Q})$ is residuated and $f^\vee_{dp}$'s residual, $g^r_{dp} : \mathcal{I}^{dp}(\mathbf{Q}) \to \mathcal{I}^{dp}(\mathbf{P})$, is defined by, for $J \in \mathcal{I}^{dp}(\mathbf{Q})$,

$$g^r_{dp}(J) = \left\langle \bigcup \{I \in \mathcal{I}^{dp}(\mathbf{P}) : f^\vee_{dp}(I) \subseteq J\} \right]_{dp}.$$

Lemma 4.3.7. *If $f : P \to Q$ is residuated with residual $g : Q \to P$ and $f^\vee_{dp} : \mathcal{I}^{dp}(\mathbf{P}) \to \mathcal{F}^{dp}(\mathbf{Q})$ is its extension with residual $g^r_{dp} : \mathcal{I}^{dp}(\mathbf{Q}) \to \mathcal{I}^{dp}(\mathbf{P})$, then g^r_{dp} extends g.*

Proof. Let $a \in P$. We need to show that $g^r_{dp}(\omega_{dp}(a)) = (g(a)] = \omega_{dp}(g(a))$.

We first show that $f(I) \subseteq f^\vee_{dp}(I)$ for all $I \in \mathcal{I}^{dp}(\mathbf{P})$. Let $I \in \mathcal{I}^{dp}(\mathbf{P})$ and $a \in I$. Then $(a] \subseteq I$ which, by Lemma 4.3.2, implies that $(f((a])] = (f(a)] \subseteq f^\vee_{dp}(I)$. Thus, $f(a) \in f^\vee_{dp}(I)$ and hence $f(I) \subseteq f^\vee_{dp}(I)$.

Let $I \in \mathcal{I}^{dp}(\mathbf{P})$ such that $f^\vee_{dp}(I) \subseteq (a]$. Let $b \in I$. Then by the above,

$$\begin{aligned} & f(b) \in f^\vee_{dp}(I) \subseteq (a] \\ \Rightarrow\ & f(b) \leq a \\ \Rightarrow\ & b \leq g(a) \quad \text{by residuation} \\ \Rightarrow\ & b \in (g(a)]. \end{aligned}$$

Thus, $I \subseteq (g(a)]$. But then $\bigcup\{I \in \mathcal{I}^{dp}(\mathbf{P}) : f^\vee_{dp}(I) \subseteq J\} \subseteq (g(a)]$ and since the dp-ideal generated by the set is the intersection of all dp-ideals that include it, we have that $g^r_{dp}(\omega_{dp}(a)) \subseteq (g(a)]$.

For the inclusion in the other direction let $b \in (g(a)]$. Then,

$$\begin{aligned} & b \leq g(a) \\ \Rightarrow\ & f(b) \leq a \quad \text{by residuation} \\ \Rightarrow\ & (f(b)] = f^\vee_{dp}((b]) \subseteq (a] \\ \Rightarrow\ & (b] \in \{I \in \mathcal{I}^{dp}(\mathbf{P}) : f^\vee_{dp}(I) \subseteq \omega_{dp}(a)\} \\ \Rightarrow\ & b \in g^r_{dp}(\omega_{dp}(a)). \end{aligned}$$

Thus, $(g(a)] \subseteq g^r_{dp}(\omega_{dp}(a))$.

□

Lastly we show how to define the extensions of n-ary maps in a similar way.

Let $n \in \mathbb{N}$ and let $\mathbf{P}_i = \langle P_i, \leq^{\mathbf{P}_i} \rangle$, for $i = 1, \ldots, n$, and $\mathbf{Q} = \langle Q, \leq^{\mathbf{Q}} \rangle$ be posets. Let $f : \prod_{j=1}^n P_j \to Q$ be an order-preserving n-ary map, i.e.,

order-preserving in each coordinate. For $F_i \in \mathcal{F}^*(\mathbf{P}_i)$ let $f(F_1, \ldots, F_n) = \{f(a_1, \ldots, a_n) : a_i \in F_i, i = 1, \ldots, n\}$. Similarly, for $I_i \in \mathcal{F}^*(\mathbf{P}_i)$ let $f(I_1, \ldots, I_n) = \{f(a_1, \ldots, a_n) : a_i \in F_i, i = 1, \ldots, n\}$.

Lemma 4.3.8. *Let $f : \prod_{i=1}^n P_i \to Q$ be an order-preserving n-ary map. Define $f_*^\wedge : \prod_{i=1}^n \mathcal{F}^*(\mathbf{P}_i) \to \mathcal{F}^*(\mathbf{Q})$ and $f_*^\vee : \prod_{i=1}^n \mathcal{I}^*(\mathbf{P}_i) \to \mathcal{I}^*(\mathbf{Q})$ as follows, for $F_i \in \mathcal{F}^*(\mathbf{P}_i)$*

$$f_*^\wedge(F_1, \ldots, F_n) = [f(F_1, \ldots, F_n)\rangle_*$$

and for $I_i \in \mathcal{I}^*(\mathbf{P}_i)$

$$f_*^\vee(I_1, \ldots, I_n) = \langle f(I_1, \ldots, I_n)]_* .$$

Then $f_*^\wedge$ and $f_*^\vee$ are both order-preserving and they both extend f.

The proof is similar to the proof of Lemma 4.3.2.

5. THE MACNEILLE COMPLETION

The construction considered in this chapter is also called the *Dedekind-MacNeille completion*, the *completion by cuts* or the *normal completion* of a poset $\mathbf{P}$. In [Mac37] MacNeille generalised Dedekind's construction of the real numbers from the rational numbers, to yield a completion for any poset. We begin by describing this completion for posets in general. Next we focus our attention on the MacNeille completion of lattices. In particular, we are interested in the MacNeille completion of a subclass of the class residuated lattices, namely the MTL-algebras. We summarize some of the results obtained in [vA09] and [vA11]. Next we study the MacNeille completion of modal MTL-chains, where a modal MTL-chain is a residuated lattice equipped with an additional order-preserving unary map. We begin by axiomatizing the class of modal MTL-algebras. Next we consider a possible extension of a 'modality', defined on a MTL-chain, to the MacNeille completion of the underlying lattice. Given this extension, we focus our attention on the preservation of properties.

5.1 The MacNeille completion of a poset

In [Mac37] MacNeille proved that any poset $\mathbf{P} = \langle P, \leq \rangle$ can be embedded into a complete lattice $\mathbf{L}$ in such a way that the embedding preserves all joins and meets existing in $\mathbf{P}$. He described the construction of such a completion of a poset $\mathbf{P}$, i.e., he constructed a complete lattice $\mathbf{L} = \langle \mathcal{L}, \vee^{\mathbf{L}}, \wedge^{\mathbf{L}} \rangle$ and described the order-embedding ι that maps P into $\mathcal{L}$. In [Ban56], [Sch56] and [Bru62] the MacNeille completion of a poset $\mathbf{P}$ is characterized as the completion $(\mathbf{L}, \iota)$, unique up to isomorphism, that fixes P and in which $\iota(P)$ is doubly dense (see Definition 4.2.8). That is, if $\mathbf{L}' = \left\langle \mathcal{L}', \vee^{\mathbf{L}'}, \wedge^{\mathbf{L}'} \right\rangle$ is a complete lattice and P is a subset of $\mathcal{L}'$ that is both join-dense and meet-dense in $\mathbf{L}'$, then $\mathbf{L}'$ is isomorphic to the MacNeille completion $(\mathbf{L}, \iota)$ of $\mathbf{P}$ via an order-isomorphism that agrees with ι on P. The uniqueness, up to isomorphism, allows us to speak of *the*

MacNeille completion of a poset. Furthermore, the MacNeille completion $(\mathbf{L}, \iota)$ of a poset $\mathbf{P}$ is minimal in the sense that if $(\mathbf{C}, \varphi)$ is any completion of $\mathbf{P}$ with $\mathbf{C} = \langle \mathcal{C}, \vee^{\mathbf{C}}, \wedge^{\mathbf{C}} \rangle$, then there exists an order-embedding $\psi : \mathcal{L} \to \mathcal{C}$ such that $\psi(\iota(P)) = \varphi(P)$, i.e., $\psi \cdot \iota = \varphi$ [Mac37]. The abstract characterization above has been used as the definition of the MacNeille completion of a poset in the literature (see for instance [TV07]). We, however, will use a concrete definition of the completion. To this end, consider the following construction.

Let $\mathbf{P} = \langle P, \leq \rangle$ be a poset. A set $S \subseteq P$ is called *stable* if $S^{\ell u} = S$. Note that a stable set is upward closed in $\mathbf{P}$ and closed under existing arbitrary meets.

Definition 5.1.1. *Let* $\mathbf{L} = \langle \mathcal{L}, \vee^{\mathbf{L}}, \wedge^{\mathbf{L}} \rangle$ *where* $\mathcal{L}$ *is the set of all stable sets and for* $S_i \in \mathcal{L}$, $i \in \Psi$,

$$\bigvee_{i \in \Psi}^{\mathbf{L}} S_i = \bigcap_{i \in \Psi} S_i \text{ and } \bigwedge_{i \in \Psi}^{\mathbf{L}} S_i = \bigcap \{T \in \mathcal{L} : S_i \subseteq T \text{ for all } i \in \Psi\}.$$

The associated complete lattice order $\leq^{\mathbf{L}}$ *on* $\mathbf{L}$ *is* $\supseteq$.

Let $\iota : P \to \mathcal{L}$ *be defined by* $\iota(a) = \{a\}^u$ *for* $a \in P$. *Then* ι *is an order-embedding of* $\mathbf{P}$ *into* $\mathbf{L}$ *that preserves all existing meets and joins in* $\mathbf{P}$.

The MacNeille completion *of the poset* $\mathbf{P}$ *is the pair* $(\mathbf{L}, \iota)$.

If $\mathbf{P}$ is a chain, then its MacNeille completion is also a chain and, for $S, T \in \mathcal{L}$, $S \wedge^{\mathbf{L}} T = S \cup T$.

We note that $\iota(P)$ is doubly dense in $\langle \mathcal{L}, \vee^{\mathbf{L}}, \wedge^{\mathbf{L}} \rangle$ (see Definition 4.2.8). Since $\iota(P)$ is join-dense in its MacNeille completion, the MacNeille completion of a poset is a so-called 'standard completion' of the poset. Standard completions of posets have been studied in [Sch74, Ern81, EW83, ER87].

It is well known that the pair of maps $({}^{\ell}, {}^{u})$ used in the MacNeille completion of a poset $\mathbf{Q} = \langle Q, \leq \rangle$ form a Galois connection between $\langle \mathcal{P}(Q), \subseteq \rangle$ and $\langle \mathcal{P}(Q), \supseteq \rangle$. The stable sets are just the closed elements of the closure operator ${}^{\ell u}$, and are also called the *Galois closed* sets.

Remark 5.1.2. *It is interesting to note that in [Mac37] the MacNeille completion of a poset is called a 'canonical extension' of the poset. The definition of a 'canonical extension' of a poset given in [Mac37] ensures that it is minimal in a sense. The term 'canonical extension' has since been used for a generally different completion of lattices and posets (see for instance [GJ94, GH01, GJKO07, DGP05]). Since this completion is generally different from the MacNielle completion, it is also in general not minimal in the required sense and hence is not a*

'canonical extension' in the sense of [Mac37]. We investigate these completions in Chapter 6. In [Kri47] some minor modifications to the theory of 'canonical extensions', as defined in [Mac37], were suggested. This modified theory of 'canonical extensions' was developed further in [Kri47] and in [Kri48] explicit constructions of 'canonical extensions' were considered.

5.2 The MacNeille completion of MTL-chains

When studying a construction it is natural to ask whether or not a class of algebras is closed under the construction. Some equationally defined classes of algebras turn out to be closed under the MacNeille completion. In [Mac37] it was shown that the class of Boolean Algebras is closed under the MacNeille completion, i.e., the MacNeille completion (of the lattice reduct) of a Boolean algebra, is again a Boolean algebra. Similarly, it was shown in [BD75] that the class of Heyting Algebras is closed under the MacNeille completion (also see [BH04]). On the other hand, some prominent equational properties are not preserved by the MacNeille completion. For example, in [Fun44] a counterexample was given to show that the MacNeille completion of a distributive lattice need not be distributive. In fact, in [Har93a] it was shown that any lattice can be embedded into the MacNeille completion of a distributive lattice in such a way that all existing joins and meets are preserved by the embedding. Completion-invariant properties of posets, i.e., properties that are satisfied by a poset if, and only if, it is satisfied by its MacNeille completion, were considered in [Ern91].

The discussion thusfar has not included classes of algebras expanded with additional operations. We note that the algebraic structure of Boolean algebras and Heyting algebras are completely determined by their lattice reducts. The extension of additional operations defined on lattices to operations defined on their MacNeille completions has been studied for a wide variety of algebras. In [Mon70] and [GV99] the MacNeille completion of Boolean algebras with operators was studied. The extension of maps to the various ideal completions, considered in Chapter 4.3, was done similarly to the extension of the operators in [Mon70] — utilising the join-denseness of the image of P in the completion. In [TV07] this was called the 'lower completion' of a Boolean algebra with operators. An 'upper completion' of an algebra would uniformly utilise the meet-denseness of the image of the poset in its MacNeille completion, when ex-

tending additional operations. The question of "Which equational properties of lattice with additional operations are preserved under the upper and lower completions?", was addressed in [TV07]. The MacNeille completion of more specific classes of lattice expansions have also been the subject of many research projects. The MacNeille completion of ortholattices have been studied in [Mac64], of orthomodular lattices in [Ada69, Har91, Har93b], of modal algebras in [BH07] and of modal algebras extended with fixpoint operators in [San08].

We now turn our attention to the following class of algebras.

Definition 5.2.1. *An* (integral, bounded, commutative) residuated lattice *is an algebra* $\mathbf{A} = \langle A, \circ, \to, \vee, \wedge, 0, 1\rangle$ *such that*

(i) $\langle A, \vee, \wedge, 0, 1\rangle$ *is a bounded lattice with* 1 *and* 0 *as greatest and least elements, respectively, and*

(ii) $\circ$ *is a binary operation that is associative, commutative, has identity* 1 *and is residuated with residual* $\to$*, i.e., for all* $x, y, z \in A$

$$x \circ y \leq z \iff y \leq x \to z.$$

If, in addition, the residuated lattice $\mathbf{A}$ is linearly ordered, then $\mathbf{A}$ will be called a *residuated chain*. The following hold for residuated lattices:

$$1 \to x = x, \quad x \to 1 = 1, \quad x \to x = 1$$
$$x \circ (x \to y) \leq y$$
$$(x \circ y) \to z = x \to (y \to z)$$
$$x \circ y \leq x \wedge y$$
$$x \leq y \iff x \to y = 1.$$

The residual operation $\to$ satisfies:

$$x \to y = \bigvee\{z : x \circ z \leq y\}.$$

The MacNeille completion of residuated lattices has been studied in [Ono03a] and [Ono03b]. Therein it was shown that many classes of residuated lattices are closed under the MacNeille completion. One such subclass of residuated lattices is the class of FL-algebras studied in [CGT11] and [CGT12]. Another is the class of *MTL-algebras*.

In [EG01], monoidal t-norm logic, MTL for short, was introduced as the logic of left-continuous t-norms and an algebraic semantics for the logic, namely the variety of 'MTL-algebras', was defined.

Definition 5.2.2. *An* MTL-algebra $\mathbf{A} = \langle A, \circ, \to, \vee, \wedge, 0, 1\rangle$ *is a residuated lattice that satisfies the prelinearity identity: for all* $x, y \in A$

$$(x \to y) \vee (y \to x) = 1.$$

The following hold in all MTL-algebras:

$$\begin{aligned} x \to (y \vee z) &= (x \to y) \vee (x \to z) \\ x \to (y \wedge z) &= (x \to y) \wedge (x \to z) \\ (x \vee y) \to z &= (x \to z) \wedge (y \to z) \\ (x \wedge y) \to z &= (x \to z) \vee (y \to z). \end{aligned}$$

We note that the middle two of the equations above also hold in residuated lattices in general. We will use the abbreviation $\neg x := x \to 0$, which defines a negation operation. From the properties listed above it follows that the negation is order-reversing, $\neg 0 = 1$ and $\neg 1 = 0$. We inductively define the terms x^n, for $n \in \mathbb{N}$, as follows: $x^0 = 1$ and $x^{n+1} = x \circ x^n$.

An MTL-algebra whose underlying lattice order is linear is called an *MTL-chain*. A main result concerning the variety of MTL-algebras is that it is generated by the class of MTL-chains (see, for example, [EG01]).

The MacNeille completion of MTL-chains has been studied in [vA09, vA11]. In the following section we will consider expansions of MTL-algebras with (order-preserving, unary) operations. We will restrict our attention to the MacNeille completion of MTL-chains. Therefore, we now give a brief summary of the results from [vA09] and [vA11].

Throughout the rest of this section let $\mathbf{A} = \langle A, \circ, \to, \wedge \vee 0, 1\rangle$ be a fixed MTL-chain.

We shall use the MacNeille completion to construct a complete lattice into which the underlying ordering on $\mathbf{A}$ embeds. Next we extend $\circ$ and $\to$ to binary operations on the complete lattice. The definitions of $\circ^{\mathbf{L}}$ and $\to^{\mathbf{L}}$ given below, were used in [vA11].

Let $\langle \mathcal{L}, \vee^{\mathbf{L}}, \wedge^{\mathbf{L}}\rangle$ be the MacNeille completion of the underlying ordering on $\mathbf{A}$.

For $H_1, H_2 \subseteq A$, define

$$H_1 \circ H_2 = \{a \circ b : a \in H_1 \text{ and } b \in H_2\}$$

and for $S, T \in \mathcal{L}$ define

$$S \circ^{\mathbf{L}} T = (S^\ell \circ T^\ell)^u.$$

Theorem 5.2.3. *[vA11]*

(i) Let $S,T \in \mathcal{L}$. If $H_1, H_2 \subseteq A$ is such that $S = H_1^u$ and $T = H_2^u$, then $S \circ^{\mathbf{L}} T = H_1^u \circ^{\mathbf{L}} H_2^u = (H_1 \circ H_2)^u$.

(ii) The operation $\circ^{\mathbf{L}}$ on $\mathcal{L}$ is residuated with respect to $\supseteq$ and the residual of $\circ^{\mathbf{L}}$ is, for $S,T \in \mathcal{L}$,

$$S \to^{\mathbf{L}} T = \{a \in A : (S^\ell \circ \{a\})^u \supseteq T\}^u.$$

(iii) The algebra $\mathbf{L} = \langle \mathcal{L}, \circ^{\mathbf{L}}, \to^{\mathbf{L}}, \vee^{\mathbf{L}}, \wedge^{\mathbf{L}}, 0^{\mathbf{L}}, 1^{\mathbf{L}} \rangle$, where $0^{\mathbf{L}} = A$ and $1^{\mathbf{L}} = \{1\}$, is a complete MTL-chain and the map $\iota : A \to \mathcal{L}$ defined by $\iota(a) = \{a\}^u$ for all $a \in A$, is an embedding of $\mathbf{A}$ into $\mathbf{L}$ that preserves all existing meets and joins in $\mathbf{A}$. Hence, $(\mathbf{L}, \iota)$ is the MacNeille completion *of $\mathbf{A}$.*

Lemma 5.2.4. *[MvAb] Let $S,T \in \mathcal{L}$. If $H_1, H_2 \subseteq A$ such that $S = H_1^u$ and $T = H_2^u$, then*

$$S \wedge^{\mathbf{L}} T = H_1^u \wedge^{\mathbf{L}} H_2^u = (H_1 \wedge H_2)^u,$$

where $H_1 \wedge H_2 = \{a \wedge b : a \in H_1 \text{ and } b \in H_2\}$.

Proof. Observe that for each $a \in T^\ell$, either $T = [a)$ or there exists $b \in H_2$ such that $a \leq b$. To see this, suppose that $b < a$ for all $b \in H_2$; so $a \in H_2^u = T$. Then $a \in T \cap T^\ell$, which is only possible if $T = [a)$.

Since $\mathbf{L}$ is a chain we may assume, without loss of generality, that $S \leq^{\mathbf{L}} T$, i.e., $S \supseteq T$, so $S \wedge^{\mathbf{L}} T = S$. Note that $H_1 \subseteq S^\ell$ hence $H_1 \wedge H_2 \subseteq S^\ell$, and therefore $S = S^{\ell u} \subseteq (H_1 \wedge H_2)^u$.

For the reverse inclusion, if $H_1 \subseteq H_1 \wedge H_2$, then $(H_1 \wedge H_2)^u \subseteq H_1^u = S$. If $H_1 \not\subseteq H_1 \wedge H_2$, then there exists $a \in H_1$ such that for all $b \in H_2$, $a \not\leq b$, i.e., $b < a$. But then $a \in H_2^u = T \subseteq S$. Since $a \in H_1 \subseteq S^\ell$, it follows that $a \in S \cap S^\ell$, so $S = [a)$. Now $T = S = [a)$ since $a \in T$. That is, a is the least upper bound of H_2. Then $b = a \wedge b \in H_1 \wedge H_2$ for any $b \in H_2$ since $a \in H_1$. Thus, $H_2 \subseteq H_1 \wedge H_2$ and hence $(H_1 \wedge H_2)^u \subseteq H_2^u = [a) = S$. □

We now consider the preservation of properties by the construction. Note that for MTL-chains, an inequality $s \leq t$ is equivalent to an identity $s = s \wedge t$ or $t = s \vee t$ or $s \to t = 1$. (Recall that the universal quantification over the variables occurring in s and t is implicit.) In the sequel we consider the preservation of inequalities by the completion and therefore, implicitly, also the preservation

of identities. In [vA11] 'approximation terms' were used to obtain preservation results. A general scheme of inequalities, whose preservation by the completion is determined by the form of the terms s and t, was described. This is related to the methods used in [Jón94] and [GV99] to obtain preservation results in completions of modal algebras. (See also [TV07] for similar results on ordered algebras.)

Let t be an MTL-term. If the variables occurring in t are in the sequence $\vec{x} = x_1, \ldots, x_n$, then we denote this by $t(x_1, \ldots, x_n)$ or $t(\vec{x})$. If $\vec{a} = a_1, \ldots, a_n$ is a sequence of elements of A, then we write $t(\vec{a})$ to denote the evaluation of the term t in $\mathbf{A}$ under the assignment $x_i \mapsto a_i$. If $\vec{S} = S_1, \ldots, S_n$ is a sequence of elements of $\mathcal{L}$, then we write $t^{\mathbf{L}}(\vec{S})$ to denote the evaluation of the term t in $\mathbf{L}$ under the assignment $x_i \mapsto S_i$. We write $\vec{S}^\ell$ to denote the sequence $S_1^\ell, \ldots, S_n^\ell$ and $\vec{a} \in \vec{S}^\ell$ means that $a_i \in S_i^\ell$ for each $i = 1, \ldots, n$. Where a term $t(\vec{x})$ and either $\vec{a} \in A$ or $\vec{S} \in \mathcal{L}$ are given, it is assumed that $\vec{x}$ and $\vec{a}$ or $\vec{S}$ are sequences of the same length.

Given a term $t(\vec{x})$ and $\vec{S} \in \mathcal{L}$, the evaluation of $t^{\mathbf{L}}(\vec{S})$ can be approximated by the set of $t^{\mathbf{A}}(\vec{a})$'s where each $a_i \in S_i^\ell$, which we write as $\{t^{\mathbf{A}}(\vec{a}) : \vec{a} \in \vec{S}^\ell\}$. Since this set is not stable it is necessary to close it in $\mathcal{L}$, which can be done in two ways, namely:

$$t^\exists(\vec{S}) = \{t^{\mathbf{A}}(\vec{a}) : \vec{a} \in \vec{S}^\ell\}^u,$$
$$t^\forall(\vec{S}) = \{t^{\mathbf{A}}(\vec{a}) : \vec{a} \in \vec{S}^\ell\}^{\ell u},$$

which are then our approximations to $t^{\mathbf{L}}(\vec{S})$. We say that $t(\vec{x})$ is:

$$\begin{aligned}
\exists\textit{-stable} \text{ if } \quad & t^{\mathbf{L}}(\vec{S}) = t^\exists(\vec{S}) \\
\exists\textit{-expanding} \text{ if } \quad & t^{\mathbf{L}}(\vec{S}) \subseteq t^\exists(\vec{S}) \\
\exists\textit{-contracting} \text{ if } \quad & t^{\mathbf{L}}(\vec{S}) \supseteq t^\exists(\vec{S}) \\
\forall\textit{-stable} \text{ if } \quad & t^{\mathbf{L}}(\vec{S}) = t^\forall(\vec{S}) \\
\forall\textit{-expanding} \text{ if } \quad & t^{\mathbf{L}}(\vec{S}) \subseteq t^\forall(\vec{S}) \\
\forall\textit{-contracting} \text{ if } \quad & t^{\mathbf{L}}(\vec{S}) \supseteq t^\forall(\vec{S}) \qquad \text{for all } \vec{S} \in \mathcal{L}.
\end{aligned}$$

If $\mathbf{A}$ satisfies the inequality $s(\vec{x}) \leq t(\vec{x})$, then $s^\exists(\vec{S}) \supseteq t^\exists(\vec{S})$ and $s^\forall(\vec{S}) \supseteq t^\forall(\vec{S})$ for all $\vec{S} \in \mathcal{L}$. Thus, for example, if $\mathbf{A}$ satisfies $s(\vec{x}) \leq t(\vec{x})$ and s is $\exists$-contracting and t is $\exists$-expanding, then for any $\vec{S} \in \mathcal{L}$, $s^{\mathbf{L}}(\vec{S}) \supseteq s^\exists(\vec{S}) \supseteq t^\exists(\vec{S}) \supseteq t^{\mathbf{L}}(\vec{S})$, i.e., $s^{\mathbf{L}}(\vec{S}) \leq^{\mathbf{L}} t^{\mathbf{L}}(\vec{S})$. More generally we have the following results.

Theorem 5.2.5. *[vA09]*

(i) If both s and t are $\exists$-stable terms, or both s and t are $\forall$-stable terms, then $s = t$ is preserved by the MacNeille completion.

(ii) If s is an $\exists$-contracting term and t is an $\exists$-expanding term, or s is a $\forall$-contracting term and t is a $\forall$-expanding term, then $s \leq t$ is preserved by the MacNeille completion.

Lemma 5.2.6. *[vA11]*

(i) The constants 1 and 0, every variable x, every $\{\circ, \vee\}$-term and every $\{\wedge, \vee\}$-term is $\exists$-stable.

(ii) If s is $\exists$-stable, then $\neg s$ is $\forall$-stable.

(iii) If s_1 and s_2 are $\exists$-stable (resp., $\exists$-contracting, $\exists$-expanding) terms, then $s_1 \vee s_2$ is $\exists$-stable (resp., $\exists$-contracting, $\exists$-expanding).

(iv) If s_1 and s_2 are $\exists$-stable (resp., $\exists$-contracting) terms that have no variables in common, then $s_1 \circ s_2$ is $\exists$-stable (resp., $\exists$-contracting).

(v) If $t(\vec{x})$ is an $\exists$-expanding term and y is a variable not in $\vec{x}$, then $t(\vec{x}) \to y$ is $\exists$-contracting.

Since $S^u \subseteq S^{\ell u}$ for any $S \subseteq A$, we have that $t^{\exists}(\vec{S}) \subseteq t^{\forall}(\vec{S})$ for any term $t(\vec{x})$ in the language and any $\vec{S} \in \mathcal{L}$. Hence, if a term is $\forall$-contracting, then it is also $\exists$-contracting. Similarly, if a term is $\exists$-expanding, then it is also $\forall$-expanding.

Definition 5.2.7. *The sets of* positive *and* negative *terms are the smallest sets of terms closed under the following rules:*

(i) 0 and 1 are both positive and negative;

(ii) the term $t(x) = x$ is positive for each variable x;

(iii) if s is negative and t is positive, then $s \to t$ is positive and $t \to s$ is negative;

(iv) if $s(x_1, \ldots, x_n)$ is a $\{\circ, \wedge, \vee\}$-term and each t_i is positive (respectively, negative) terms, then $s(t_1, \ldots, t_n)$ is positive (respectively, negative).

It was shown in [vA09] that every positive term in the language $\{\circ, \to, \vee, \wedge, 1, 0\}$ is $\exists$-expanding (hence also $\forall$-expanding), while every negative term is $\forall$-contracting (hence also $\exists$-contracting).

For example, if $\mathbf{A}$ is 'involutive', i.e., satisfies $\neg\neg x = x$, then so is $\mathbf{L}$. If $\mathbf{A}$ is 'strict', i.e., satisfies $x \wedge (\neg x) = 0$, then so is $\mathbf{L}$. The following identities are also preserved [vA11]: $\neg(x \circ y) \vee ((x \wedge y) \to (x \circ y)) = 1$ (weak nilpotent minimum), $x \vee \neg x^n$ (weak excluded middle), $x^{n+1} = x^n$ (n-contraction) and $\neg x^{n+1} = \neg x^n$ (weak n-contraction).

5.3 Modal MTL-chains

Substructural logics are logics with structure sensitive consequence relations, for example, logics without structural rules like contraction, weakening, commutativity or associativity that form part of intuitionistic and classical logic. It the literature, *modalities* have then been added to the substructural logics as a way to reintroduce limited structural rules. This was done in [Gir87]: the exponentials ! and ? of linear logic can be viewed as modal operators, since they have some similarities with the modalities $\Diamond$ and $\Box$. The addition of modal operators to various non-classical logics has since been studied increasingly. Another example is the Baaz Delta Δ, intended to mean complete (classical) truth, added to fuzzy logic [Baa96]. In [Mon04] and [CMM10] storage operators and truth stresser modalities are added to many-valued logics and, in particular, to MTL. For more examples of the addition of modalities to various (substructural) logics the reader can consult, for example, [Res93, Ven95, Buc94, DGR97, Kam03, Ono05].

It is therefore natural to consider the expansion of MTL-algebras with a 'modality'. Before we consider the MacNeille completion of 'modal MTL-algebras', we must make the notion of a 'modal MTL-algebra' precise.

The results in this section have been obtained in collaboration with Prof. Clint van Alten and have been published in [MvAb].

5.3.1 Axiomatization of (reverse) modal MTL-algebras

Motivated by the fact that the variety of MTL-algebras is generated by the class of MTL-chains, we define a 'modal MTL-chain' to be an MTL-chain equipped with an additional order-preserving (unary) operation f, and a 'modal MTL-

algebra' as any algebra in the variety generated by the class of modal MTL-chains. We will show that this class is strictly smaller than the class of all MTL-algebras that have an additional order-preserving operation. In particular, we show that modal MTL-algebras are axiomatized by the axioms of MTL-algebras together with $f(x \vee y) = f(x) \vee f(y)$ and $(f(y) \to f((x \to z) \circ y)) \vee (z \to x) = 1$.

We note that the less general notion (in the sense that there are additional constraints) of a 'modal residuated lattice' has been considered by Ono in [Ono05], where such algebras are defined to be residuated lattices equipped with an operation f that satisfies $f(x) \leq x$, $f(x) \leq f(f(x))$, $1 \leq f(1)$ and $f(x) \circ f(y) \leq f(x \circ y)$ in addition to being order-preserving.

We also consider 'reverse modal MTL-chains' that are algebras in which the modality is order-reversing rather than order-preserving. A natural motivating example is the operation $h(x) = 1 - x$ on any standard MTL-algebra, that is, an MTL-algebra whose universe is the real interval $[0, 1]$. We show that an axiomatization for the class of reverse modal MTL-algebras consists of the axioms for MTL together with $h(x \vee y) = h(x) \wedge h(y)$ and $(h((x \to z) \circ y) \to h(y)) \vee (z \to x) = 1$.

Definition 5.3.1.

(i) *A* modal residuated lattice *is an algebra* $\langle A, \circ, \to, \vee, \wedge, f, 0, 1\rangle$*, where* $\langle A, \circ, \to, \vee, \wedge, 0, 1\rangle$ *is a residuated lattice and* f *is a unary operation that is order-preserving, i.e.,* $x \leq y$ *implies* $f(x) \leq f(y)$*.*

(ii) *A* reverse modal residuated lattice *is an algebra* $\langle A, \circ, \to, \vee, \wedge, h, 0, 1\rangle$*, where* $\langle A, \circ, \to, \vee, \wedge, 0, 1\rangle$ *is a residuated lattice and* h *is a unary operation that is order-reversing, i.e.,* $x \leq y$ *implies* $h(y) \leq h(x)$*.*

Let $\mathbf{A} = \langle A, \circ, \to, \vee, \wedge, f, 0, 1\rangle$ be a fixed modal residuated lattice.

Definition 5.3.2. *A subset* F *of* A *is a* congruence filter *(or* c-filter *for short)*[1] *of* $\mathbf{A}$ *if:* $1 \in F$*,* F *is upward closed, closed under* $\circ$*, and* $f(d) \to f(d \circ a) \in F$ *whenever* $a \in F$ *and* $d \in A$*.*

We note that c-filters have also been called *implicative filters* [Ono10], *deductive filters* [GJKO07] or *normal filters*.

[1] sometimes called an 'ideal', as it satisfies the notion of ideal from [GU84].

The set of all c-filters of $\mathbf{A}$, ordered by inclusion, forms a complete lattice, denoted $\mathrm{Fil}^{\mathrm{c}}\,\mathbf{A}$. If θ is a congruence on $\mathbf{A}$ and $a \in A$, we use $[a]_\theta$ to denote the congruence class of a with respect to θ.

Proposition 5.3.3. *The congruence lattice of* $\mathbf{A}$, $\mathrm{Con}\,\mathbf{A}$, *is isomorphic to the c-filter lattice,* $\mathrm{Fil}^{c}\,\mathbf{A}$*; the isomorphisms are given by:* $\theta \mapsto [1]_\theta$ *and* $F \mapsto \theta_F = \{(a,b) : a \to b, b \to a \in F\}$.

Proof. Since the result is known for residuated lattices [GJKO07], we need only check that the result extends to the operation f. Let F be a c-filter of $\mathbf{A}$. To see that θ_F is compatible with f, suppose $(a,b) \in \theta_F$, i.e., $a \to b, b \to a \in F$. Since $a \circ (a \to b) \leq b$ and f is order preserving it follows that $f(a \circ (a \to b)) \leq f(b)$. By the definition of a c-filter, $f(a) \to f(a \circ (a \to b)) \in F$, and $f(a) \to f(a \circ (a \to b)) \leq f(a) \to f(b)$, so $f(a) \to f(b) \in F$. Similarly, we can show that $f(b) \to f(a) \in F$, hence $(f(a), f(b)) \in \theta_F$. Next, let θ be a congruence on $\mathbf{A}$. To see that $[1]_\theta$ is a c-filter, suppose $a \in [1]_\theta$, i.e., $(a,1) \in \theta$. Then for any $d \in A$ we have that $(f(d) \to f(d \circ a), f(d) \to f(d \circ 1)) \in \theta$, i.e., $(f(d) \to f(d \circ a), 1) \in \theta$. Thus, $f(d) \to f(d \circ a) \in [1]_\theta$. □

We would like to extend the fact that the MTL-chains generate the variety of MTL-algebras to the modal case. In order to do so, we make the following definitions.

Definition 5.3.4.

(i) *A* modal MTL-chain *is a modal residuated lattice whose underlying lattice order is linear.*

(ii) *A* modal MTL-algebra *is any algebra in the variety generated by modal MTL-chains.*

We note that since the underlying lattice order is linear it follows that modal MTL-chains satisfy the prelinearity identity. Furthermore, since modal MTL-algebras are in the variety generated by modal MTL-chains, it follows that modal MTL-algebras also satisfy the prelinearity identity.

Since f is order-preserving, the following identity holds in all modal MTL-chains, and hence also modal MTL-algebras:

$$f(x \vee y) = f(x) \vee f(y). \tag{5.1}$$

The identity $f(x \wedge y) = f(x) \wedge f(y)$ also holds. Note that the order-preserving property of f can be inferred from (5.1).

We now introduce some notions from Universal Algebra that we will need going forward. The reader is referred to [BS81] for more details on the notions defined here as well as the proofs of the results listed here. Also see Chapter 2.3 for the definition of a varieties.

An algebra $\mathbf{B}$ is called *congruence-distributive* if $\operatorname{Con}\mathbf{B}$ is a distributive lattice. We call a class of algebras *congruence-distributive* if, and only if, every algebra in the class is congruence-distributive. We now have the following result that follows from Mal'cev conditions [Mal54].

Theorem 5.3.5. *The variety of lattices is congruence distributive.*

See [GJKO07] for a direct proof.

Definition 5.3.6. *An algebra* $\mathbf{B}$ *is a* subdirect product *of an indexed family* $(\mathbf{B}_i)_{i\in\Psi}$ *of algebras if,*

(i) $\mathbf{B}$ *is a subalgebra of* $\prod_{i=1}^{n}\mathbf{B}_i$*, the direct product of the algebras in* $(\mathbf{B}_i)_{i\in\Psi}$*, and*

(ii) all the coordinate projections restricted to $\mathbf{B}$ *are onto, i.e., each* $\mathbf{B}_i$ *is a homomorphic image of* $\mathbf{B}$*.*

The family of algebras $(\mathbf{B}_i)_{i\in\Psi}$ *is called a* subdirect representation *of* $\mathbf{B}$*.*

We now call an algebra $\mathbf{B}$ *subdirectly irreducible* if every subdirect representation $(\mathbf{B}_i)_{i\in\Psi}$ of $\mathbf{B}$ contains (an isomorphic copy of) $\mathbf{B}$ as a factor.

Let $(\mathbf{B}_i)_{i\in\Psi}$ be an indexed family of algebras of the same type. We will call an ultrafilter on $\mathcal{P}(\Psi)$ (viewed as a Boolean algebra) an *ultrafilter on* Ψ. Let U be an ultrafilter on Ψ. Then, for $\vec{a}$ and $\vec{b}$ in the direct product $\prod_{i\in\Psi}\mathbf{B}_i$, define

$$||\vec{a} = \vec{b}|| = \{i \in \Psi : a_i = b_i\}.$$

Furthermore, let $\equiv_U \subseteq \prod_{i\in\Psi} B_i \times \prod_{i\in\Psi} B_i$ be defined by

$$\vec{a} \equiv_U \vec{b} \iff ||\vec{a} = \vec{b}|| \in U.$$

Then $\equiv_U$ is a congruence on $\prod_{i\in\Psi}\mathbf{B}_i$.

Definition 5.3.7. *The* ultraproduct *of an indexed family of algebras* $(\mathbf{B}_i)_{i\in\Psi}$ *with respect to an ultrafilter* U *on* Ψ *is the quotient algebra* $(\prod_{i\in\Psi}\mathbf{B}_i)/\equiv_U$*.*

Finally we introduce the notion of a *quasivariety*.

Definition 5.3.8. *A class of algebras* $\mathbb{K}$ *of the same type is called a* quasivariety *if it is closed under isomorphism, subalgebras, direct products and ultraproducts.*

A class of algebras is a quasivariety if, and only if, it can be axiomatized by quasi-identities.

In [Jón67] Jónsson gave a Mal'cev condition for congruence-distributive varieties. He also proved the following result.

Theorem 5.3.9. *[Jón67] Let* $\mathbb{K}$ *be a congruence-distributive variety generated by a subclass* $\mathbb{K}'$*. If* $\mathbf{B}$ *is a subdirectly irreducible algebra in* $\mathbb{K}$*, then* $\mathbf{B}$ *is the homomorphic image of a subalgebra of an ultraproduct of members of* $\mathbb{K}'$*.*

Observe that the variety of modal MTL-algebras is congruence-distributive since its algebras contain lattice reducts and since congruence distributivity is a Mal'cev condition. Since the variety of modal MTL-algebras is congruence-distributive, it follows that Jónsson's theorem applies. Furthermore, since the variety of modal MTL-algebras is generated by the modal MTL-chains, it follows that every subdirectly irreducible modal MTL-algebra is a homomorphic image of a subalgebra of an ultraproduct of modal MTL-chains. But the class of modal MTL-chains is closed under ultraproducts, subalgebras and homomorphic images since its algebras are linearly ordered. Thus, the variety generated by modal MTL-chains may be obtained by taking subdirect products only. In particular, this means that the variety coincides with the quasivariety generated by modal MTL-chains.

In order to axiomatize the variety of modal MTL-algebras it is sufficient, therefore, to determine identities that a modal residuated lattice must satisfy to be embeddable into a product of modal MTL-chains, and hence a subdirect product of modal MTL-chains. From the theory of universal algebra (see, for instance, [BS81, Lemma 8.2]) we know that if the intersection of a set of congruences of an algebra is the trivial congruence, then the algebra is a subdirect product of the associated quotient algebras. Thus, we shall characterize the congruences of a modal MTL-algebra for which the quotient algebra is a modal MTL-chain. Since the congruence lattice of a modal MTL-algebra is isomorphic to the c-filter lattice, we characterize the c-filters F for which $\mathbf{A}/\theta_F$ is a modal MTL-chain where $\mathbf{A}$ is a modal residuated lattice (see, for example, [EG01]).

The methods used here make use of ideas from the theory of ℓ-groups — see, for example, [AF88].

In the sequel, let $\mathbf{A} = \langle A, \circ, \to, \vee, \wedge, f, 0, 1\rangle$ be a fixed modal residuated lattice as before.

As with a lattice filter, a c-filter of $\mathbf{A}$ is called *prime* if, for all $a, b \in A$, $a \vee b \in F$ implies that at least one of $a \in F$ or $b \in F$.

The following result can be obtained for modal MTL-algebras in the same way that it is obtained for MTL-algebras.

Lemma 5.3.10. *If $\mathbf{A}$ satisfies the prelinearity identity, then a c-filter F of $\mathbf{A}$ is prime if, and only if, $\mathbf{A}/\theta_F$ is linearly ordered.*

Lemma 5.3.11. *The variety of modal MTL-algebras satisfies:*

$$x \vee z = 1 \text{ implies } (f(y) \to f(x \circ y)) \vee z = 1. \tag{5.2}$$

Proof. If $a \vee c = 1$ in a chain, then either $a = 1$, in which case $f(b) \to f(a \circ b) = 1$, or $c = 1$, in which case $(f(b) \to f(a \circ b)) \vee c = 1$. Thus, by Jónsson's theorem every modal MTL-algebra satisfies (5.2). □

Definition 5.3.12. *For every ideal I of the lattice reduct of $\mathbf{A}$ define:*

$$F_I = \{a \in A : \text{there exists } c \in I \text{ such that } a \vee c = 1\}.$$

Lemma 5.3.13. *Suppose $\mathbf{A}$ satisfies the quasi-identity (5.2).*

(i) If I is an ideal of the lattice reduct of $\mathbf{A}$, then F_I is a c-filter of $\mathbf{A}$.

(ii) If I is a maximal (proper) ideal of the lattice reduct of $\mathbf{A}$, then F_I is prime.

Proof. (i) It is clear that $1 \in F_I$. Let $a, b \in A$. If $a, b \in F_I$, then there exist $c_1, c_2 \in I$ such that $a \vee c_1 = 1$ and $b \vee c_2 = 1$. To see that $a \circ b \in F_I$, observe that $c_1 \vee c_2 \in I$ and $(a \vee c_1) \circ (b \vee c_2) = 1$. After distributing the left-hand side, we obtain

$$\begin{aligned}
&(a \circ b) \vee (a \circ c_2) \vee (c_1 \circ b) \vee (c_1 \circ c_2) = 1\\
\Rightarrow\ &(a \circ b) \vee ((c_1 \circ b) \vee (c_1 \circ c_2)) \vee ((a \circ c_2) \vee (c_1 \circ c_2)) = 1\\
\Rightarrow\ &(a \circ b) \vee (c_1 \circ (b \vee c_2)) \vee ((a \vee c_1) \circ c_2) = 1\\
\Rightarrow\ &(a \circ b) \vee (c_1 \circ 1) \vee (1 \circ c_2) = 1\\
\Rightarrow\ &(a \circ b) \vee (c_1 \vee c_2) = 1,
\end{aligned}$$

so $a \circ b \in F_I$. If $a \in F_I$ and $a \leq b$, then there exists $d \in I$ such that $a \vee d = 1$, hence also $b \vee d = 1$, so $b \in F_I$. If $a \in F_I$, say $a \vee d = 1$ for some $d \in I$, then $(f(b) \to f(a \circ b)) \vee d = 1$, by (5.2), so $f(b) \to f(a \circ b) \in F_I$.

(ii) Suppose I is a maximal ideal of the lattice reduct of $\mathbf{A}$ and $a \vee b \in F_I$. Then there exists $c \in I$ such that $(a \vee b) \vee c = 1$. Suppose that $a \notin F_I$. Then $a \vee c \neq 1$ for every $c \in I$ so the ideal of the lattice reduct of $\mathbf{A}$ generated by $I \cup \{a\}$ is a proper ideal containing I. Since I is maximal, we must have $a \in I$. Thus, $a \vee c \in I$ and hence $b \in F_I$ since $b \vee (a \vee c) = 1$. A similar argument shows that if $b \notin F_I$, then $a \in F_I$. Therefore, at least one of $a \in F_I$ or $b \in F_I$. □

Theorem 5.3.14. *The variety of modal MTL-algebras is axiomatized by the axioms of MTL-algebras together with* (5.1) *and* (5.2).

Proof. Let $\mathbf{A}$ be a modal residuated lattice that satisfies the prelinearity identity, (5.1) and (5.2). Note that the identity (5.1) implies the quasi-identity for the order-preserving property of f. For each $a \in A\backslash\{1\}$ there exists, by Zorn's Lemma, a maximal ideal I_a of the lattice reduct of $\mathbf{A}$ with $a \in I_a$. Note that $a \notin F_{I_a}$ or else there exists $c \in I_a$ such that $a \vee c = 1$, which implies that $1 \in I_a$, but I_a is proper. It follows that:

$$\bigcap \{F_{I_a} : a \in A\backslash\{1\}\} = \{1\}.$$

By the isomorphism between the congruence lattice and c-filter lattice of $\mathbf{A}$, it follows that $\mathbf{A}$ is a subdirect product of $\{\mathbf{A}/\theta_{F_{I_a}} : a \in A\backslash\{1\}\}$. Furthermore, from Lemma 5.3.13 and Lemma 5.3.10 it follows that $\mathbf{A}$ is a subdirect product of modal MTL-chains. □

As we shall show, the quasi-identity (5.2) in the above results may be replaced by the following identity:

$$(f(y) \to f((x \to z) \circ y)) \vee (z \to x) = 1. \tag{5.3}$$

Corollary 5.3.15. *The variety of modal MTL-algebras is axiomatized by the axioms of MTL-algebras together with* (5.1) *and* (5.3).

Proof. Since every chain satisfies: $x \leq z$ or $z \leq x$, it follows easily that every modal MTL-chain, and hence every modal MTL-algebra, satisfies (5.3). Suppose $\mathbf{A}$ is a modal residuated lattice that satisfies the prelinearity condition, (5.1) and (5.3); we show that it also satisfies (5.2), from which the result follows. If

$a, b, c \in A$ and $a \vee c = 1$, then $c = 1 \to c = (a \vee c) \to c = (a \to c) \wedge (c \to c) = a \to c$. Similarly, $a = c \to a$. By (5.3), $(f(b) \to f(a \circ b)) \vee c = 1$. □

We show, by example, that the quasi-identity (5.2) cannot be dropped from our axiomatization of modal MTL-algebras.

Example 5.3.16. *Let* $\mathbf{A}$ *be modal residuated lattice described as follows. The universe* $A = \{0, a, b, 1\}$*, the lattice order is given by* $0 \leq a, b \leq 1$ *with* a *and* b *incomparable (see Figure 5.1). For* $x, y \in A$*, let* $x \circ^{\mathbf{A}} y = x \wedge y$ *and* $x \to^{\mathbf{A}} y = \bigvee\{z : x \wedge z \leq y\}$*. Let* $f^{\mathbf{A}}$ *be the operation defined by* $f^{\mathbf{A}}(0) = 0$*,* $f^{\mathbf{A}}(a) = b$*,* $f^{\mathbf{A}}(b) = a$ *and* $f^{\mathbf{A}}(1) = 1$*. Then the* $f^{\mathbf{A}}$*-free reduct of* $\mathbf{A}$ *is an MTL-algebra and hence a subdirect product of MTL-chains. In addition* f *distributes over joins, i.e., (5.1) holds, however,* $\mathbf{A}$ *does not satisfy (5.2). To see this, observe that* $a \vee b = 1$ *but* $f^{\mathbf{A}}(1) \to^{\mathbf{A}} f^{\mathbf{A}}(1 \circ^{\mathbf{A}} a) = 1 \to^{\mathbf{A}} b = b$*.* $\mathbf{A}$ *has only two c-filters, namely* $\{1\}$ *and* $\{0, a, b, 1\}$*, and therefore only two congruences. Thus,* $\mathbf{A}$ *is subdirectly irreducible and hence cannot be represented as a subdirect product of modal MTL-chains.*

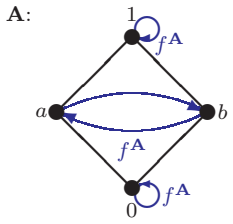


Fig. 5.1: The operation $f^{\mathbf{A}}$ on $\mathbf{A}$.

The above results can be adapted to reverse modal MTL-algebras as follows. We omit the proofs as they are similar to those for modal MTL-algebras. Let $\mathbf{B} = \langle B, \circ, \to, \vee, \wedge, h, 0, 1\rangle$ be a fixed reverse modal residuated lattice.

Definition 5.3.17. *A subset* F *of* B *is a* c-filter *of* $\mathbf{B}$ *if:* $1 \in F$*,* F *is upward closed, closed under* $\circ$*, and* $h(d \circ a) \to h(d) \in F$ *whenever* $a \in F$ *and* $d \in B$*.*

Corollary 5.3.18. *The congruence lattice,* $\operatorname{Con}\mathbf{B}$*, is isomorphic to the c-filter lattice,* $\operatorname{Fil}^c \mathbf{B}$*; the isomorphisms are given by:* $\theta \mapsto [1]_\theta$ *and* $F \mapsto \theta_F$*.*

Definition 5.3.19.

(i) *A* reverse modal MTL-chain *is a reverse modal residuated lattice whose underlying lattice order is linear.*

(ii) *A* reverse modal MTL-algebra *is any algebra in the variety generated by reverse modal MTL-chains.*

Since h is order-reversing, the identities

$$h(x \vee y) = h(x) \wedge h(y) \tag{5.4}$$

and

$$h(x \wedge y) = h(x) \vee h(y)$$

hold in all reverse modal MTL-chains and MTL-algebras; either identity implies that h is order-reversing. Consider the following quasi-identity and identity:

$$x \vee z = 1 \text{ implies } (h(x \circ y) \to h(y)) \vee z = 1, \tag{5.5}$$

$$(h((x \to z) \circ y) \to h(y)) \vee (z \to x) = 1. \tag{5.6}$$

Theorem 5.3.20. *The variety of reverse modal MTL-algebras is axiomatized by the axioms for MTL-algebras together with* (5.4)*, and* (5.5) *or* (5.6)*.*

We show, by example, that the quasi-identity (5.5) cannot be dropped from our axiomatization of reverse modal MTL-algebras.

Example 5.3.21. *Let* $\mathbf{B}$ *be the reverse modal residuated lattice defined as follows. The* h*-free reduct of* $\mathbf{B}$ *is the same as the* f*-free-reduct in the previous example, so it is an MTL-algebra. Let* $h^{\mathbf{B}}$ *be defined by* $h^{\mathbf{B}}(0) = 1$*,* $h^{\mathbf{B}}(a) = a$*,* $h^{\mathbf{B}}(b) = b$ *and* $h^{\mathbf{B}}(1) = 0$*. See Figure 5.2. Then* $\mathbf{B}$ *satisfies (5.4) but not (5.5) since* $a \vee^{\mathbf{B}} b = 1$ *but,* $(h^{\mathbf{B}}(1 \circ^{\mathbf{B}} a) \to^{\mathbf{B}} h^{\mathbf{B}}(1)) \vee^{\mathbf{B}} b = b$*. Again,* $\mathbf{B}$ *has only two c-filters, namely* $\{1\}$ *and* $\{0, a, b, 1\}$*, and therefore only two congruences. Thus,* $\mathbf{B}$ *is subdirectly irreducible and hence cannot be represented as a subdirect product of reverse modal MTL-chains.*

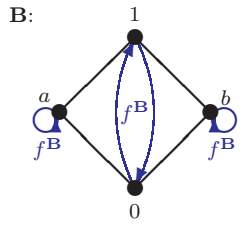


Fig. 5.2: The operation $f^{\mathbf{B}}$ on $\mathbf{B}$.

5.3.2 The MacNeille completion of modal MTL-chains

Throughout this section $\mathbf{A} = \langle A, \circ, \to, \vee, \wedge, f, 0, 1\rangle$ will be a fixed modal MTL-chain.

Let $\mathbf{A}'$ be the f-free reduct of $\mathbf{A}$. Then $\mathbf{A}'$ is an MTL-chain and we can obtain its MacNeille completion $\mathbf{L}' = \langle \mathcal{L}, \circ^{\mathbf{L}}, \to^{\mathbf{L}}, \vee^{\mathbf{L}}, \wedge^{\mathbf{L}}, 0^{\mathbf{L}}, 1^{\mathbf{L}}\rangle$ as described in Section 5.2. We then extend the operation f on $\mathbf{A}$ to an operation $f^{\mathbf{L}}$ on $\mathcal{L}$ so that the resulting algebra $\mathbf{L}$ is a modal MTL-chain into which $\mathbf{A}$ embeds, called the MacNeille completion of $\mathbf{A}$. Thereafter, various preservation properties of the completion of $\mathbf{A}$ into $\mathbf{L}$ are considered, that is, properties of $\mathbf{A}$ that are also satisfied by $\mathbf{L}$. The results obtained here build on the results obtained in [vA09] and [vA11] (summarised in Section 5.2); in particular, the general scheme of inequalities given there is extended to include terms built up with an additional modal operator f.

Define a unary operation (the modal operator) on $\mathbf{L}$ as follows: for $S \in \mathcal{L}$,

$$f^{\mathbf{L}}(S) = \{f(a) : a \in S^{\ell}\}^{u}.$$

Lemma 5.3.22. *The operation $f^{\mathbf{L}}$ is order-preserving on $\mathbf{L}$.*

Proof. Let $S, T \in \mathcal{L}$ such that $S \leq^{\mathbf{L}} T$, i.e., $S \supseteq T$. Then $S^{\ell} \subseteq T^{\ell}$, so $\{f(a) : a \in S^{\ell}\} \subseteq \{f(a) : a \in T^{\ell}\}$, hence $\{f(a) : a \in S^{\ell}\}^{u} \supseteq \{f(a) : a \in T^{\ell}\}^{u}$, i.e., $f^{\mathbf{L}}(S) \leq^{\mathbf{L}} f^{\mathbf{L}}(T)$. □

Recall that $\iota : A \to \mathcal{L}$ defined by $\iota(a) = \{a\}^{u}$ is embedding of $\mathbf{A}$ into $\mathbf{L}$.

Lemma 5.3.23. *The embedding ι of $\mathbf{A}$ into $\mathbf{L}$ preserves f, i.e., for $a \in A$, we have $f^{\mathbf{L}}(\{a\}^{u}) = \{f(a)\}^{u}$.*

Proof. We have $f^{\mathbf{L}}(\{a\}^{u}) = \{f(b) : b \in \{a\}^{u\ell}\}^{u} = \{f(b) : b \leq a\}^{u}$, which is equal to $\{f(a)\}^{u}$ since f is order-preserving. □

Theorem 5.3.24. *The algebra* $\mathbf{L} = \langle F, \circ^{\mathbf{L}}, \to^{\mathbf{L}}, \vee^{\mathbf{L}}, \wedge^{\mathbf{L}}, f^{\mathbf{L}}, 0^{\mathbf{L}}, 1^{\mathbf{L}} \rangle$ *is a complete modal MTL-chain and* ι *is an embedding of* $\mathbf{A}$ *into* $\mathbf{L}$ *that preserves all existing meets and joins in* $\mathbf{A}$.

Then $(\mathbf{L}, \iota)$ *is the* MacNeille completion *of* $\mathbf{A}$.

In the remainder of this section we describe some classes of (∃-, ∀-) stable, expanding and contracting terms, which may be used in conjunction with the above theorem to obtain a class of identities preserved by the MacNeille completion. We define positive and negative terms in the language $\{\circ, \to, \vee, \wedge, f, 0, 1\}$ by modifying condition (iv) in Definition 5.2.7 as follows:

(iv) if $s(x_1, \ldots, x_n)$ is a $\{\circ, \vee, \wedge, f\}$-term and each t_i is a positive (respectively, negative) term, then $s(t_1, \ldots, t_n)$ is positive (respectively, negative).

We show that every positive term is ∃-expanding, while every negative term is ∃-contracting. However, in order to classify positive and negative terms in this way, it is useful to show the stronger result that every negative term is ∀-contracting, which implies that it is ∃-contracting. Note also that a (∃-, ∀-) stable term is both contracting and expanding. Recall that every positive term in the language $\{\circ, \to, \vee, \wedge, 0, 1\}$ is ∃-expanding and every such negative term is ∀-contracting (hence also ∃-contracting) [vA09]. We shall extend these results to include the modal operator.

Lemma 5.3.25. *If* t *is an* ∃*-expanding term, then* $f(t)$ *is* ∃*-expanding.*

Proof. Let $s(\vec{x}) = f(t(\vec{x}))$ and $\vec{S} \in \mathcal{L}$. Then,

$$\begin{aligned} t^{\mathbf{L}}(\vec{S}) \subseteq t^{\exists}(\vec{S}) \quad &\Rightarrow \quad (t^{\mathbf{L}}(\vec{S}))^{\ell} \supseteq (t^{\exists}(\vec{S}))^{\ell} \\ &\Rightarrow \quad \{f(\vec{a}) : \vec{a} \in t^{\mathbf{L}}(\vec{S})^{\ell}\} \supseteq \{f(\vec{a}) : \vec{a} \in t^{\exists}(\vec{S})^{\ell}\} \\ &\Rightarrow \quad \{f(\vec{a}) : \vec{a} \in t^{\mathbf{L}}(\vec{S})^{\ell}\}^{u} \subseteq \{f(\vec{a}) : \vec{a} \in t^{\exists}(\vec{S})^{\ell}\}^{u} \\ &\Rightarrow \quad f^{\mathbf{L}}(t^{\mathbf{L}}(\vec{S})) \subseteq \{f(\vec{a}) : a \in \{t(\vec{b}) : \vec{b} \in \vec{S}^{\ell}\}^{u\ell}\}^{u}. \end{aligned}$$

Since $\{t(\vec{b}) : \vec{b} \in \vec{S}^{\ell}\} \subseteq \{t(\vec{b}) : b \in \vec{S}^{\ell}\}^{u\ell}$ we have

$$\{f(\vec{a}) : a \in \{t(\vec{b}) : \vec{b} \in \vec{S}^{\ell}\}\}^{u} \supseteq \{f(\vec{a}) : a \in \{t(\vec{b}) : \vec{b} \in \vec{S}^{\ell}\}^{u\ell}\}^{u}$$

and hence

$$s^{\mathbf{L}}(\vec{S}) = f^{\mathbf{L}}(t^{\mathbf{L}}(\vec{S})) \subseteq \{f(\vec{a}) : \vec{a} \in t^{\exists}(\vec{S})^{\ell}\}^{u} \subseteq \{f(t(\vec{a})) : \vec{a} \in \vec{S}^{\ell}\}^{u} = s^{\exists}(\vec{S}).$$

□

Lemma 5.3.26. *If t is a $\forall$-contracting term, then $f(t)$ is $\forall$-contracting.*

Proof. Let $s(\vec{x}) = f(t(\vec{x}))$ and $\vec{S} \in \mathcal{L}$. Then,

$$\begin{aligned} s^{\mathbf{L}}(\vec{S}) &= f^{\mathbf{L}}(t^{\mathbf{L}}(\vec{S})) \\ &\supseteq f^{\mathbf{L}}(t^{\forall}(\vec{S})) \\ &= \{f(b) : b \in \{t(\vec{a}) : \vec{a} \in \vec{S}^{\ell}\}^{\ell u \ell}\}^{u} \\ &= \{f(b) : b \in \{t(\vec{a}) : \vec{a} \in \vec{S}^{\ell}\}^{\ell}\}^{u}, \end{aligned}$$

which we must show to include $s^{\forall}(\vec{S}) = \{f(t(\vec{a})) : \vec{a} \in \vec{S}^{\ell}\}^{\ell u}$. Let $b \in \{t(\vec{a}) : \vec{a} \in \vec{S}^{\ell}\}^{\ell}$, so $b \leq t(\vec{a})$ for all $\vec{a} \in \vec{S}^{\ell}$, hence also $f(b) \leq f(t(\vec{a}))$ for all $\vec{a} \in \vec{S}^{\ell}$. Thus, $f(b) \in \{f(t(\vec{a})) : \vec{a} \in \vec{S}^{\ell}\}^{\ell}$, from which we deduce $s^{\forall}(\vec{S}) \subseteq s^{\mathbf{L}}(\vec{S})$ after taking upper bounds. □

Combining Proposition 13.27 in [vA09] with Lemmas 5.3.25 and 5.3.26 gives the following result.

Proposition 5.3.27. *Every positive $\{\circ, \to, \vee, \wedge, f, 0, 1\}$-term is $\exists$-expanding, hence also $\forall$-expanding; and every negative $\{\circ, \to, \vee, \wedge, f, 0, 1\}$-term is $\forall$-contracting, hence also $\exists$-contracting.*

Consequently, any inequality $s \leq t$ in which s is negative and t is positive is preserved by the completion. Observe that such an inequality is equivalent to $1 = s \to t$, and $s \to t$ is positive. We next consider $\exists$-stable terms; since such terms are both $\exists$-contracting and $\exists$-expanding, they may appear on either side of the inequality.

Lemma 5.3.28. *If s_1 and s_2 are $\exists$-stable (respectively, $\exists$-contracting) terms that have no variables in common, then $s_1 \wedge s_2$ is $\exists$-stable (respectively., $\exists$-contracting).*

Proof. Let $t(\vec{x}, \vec{y}) = s_1(\vec{x}) \wedge s_2(\vec{y})$, where $\vec{x}$ and $\vec{y}$ have no variables in common, and $\vec{S}, \vec{T} \in \mathcal{L}$. Then, using Lemma 5.2.4,

$$\begin{aligned} t^{\mathbf{L}}(\vec{S}, \vec{T}) &= s_1^{\mathbf{L}}(\vec{S}) \wedge^{\mathbf{L}} s_2^{\mathbf{L}}(\vec{T}) \\ &= \text{ (respectively, } \supseteq) \; s_1^{\exists}(\vec{S}) \wedge^{\mathbf{L}} s_2^{\exists}(\vec{T}) \\ &= \{s_1(\vec{a}) : \vec{a} \in \vec{S}^{\ell}\}^{u} \wedge^{\mathbf{L}} \{s_2(\vec{b}) : \vec{b} \in \vec{T}^{\ell}\}^{u} \\ &= \{s_1(\vec{a}) \wedge s_2(\vec{b}) : \vec{a}, \vec{b} \in \vec{S}^{\ell}, \vec{T}^{\ell}\}^{u} \\ &= t^{\exists}(\vec{S}, \vec{T}). \end{aligned}$$

□

Lemma 5.3.29. *Let t_1 and t_2 be terms that are order-preserving in each co-ordinate and have exactly one variable in common, say x, and let $\vec{y_1}$ and $\vec{y_2}$ be the remainder of the variables occurring in t_1 and t_2, respectively. If $s(x, \vec{y_1}, \vec{y_2}) = t_1(x, \vec{y_1}) \circ t_2(x, \vec{y_2})$ or $t_1(x, \vec{y_1}) \wedge t_2(x, \vec{y_2})$ and both t_1 and t_2 are $\exists$-stable (respectively, $\exists$-contracting), then s is $\exists$-stable (respectively, $\exists$-contracting).*

Proof. For $S, \vec{T_1}, \vec{T_2} \in \mathcal{L}$ and $s = t_1 \circ t_2$, we have

$$\begin{aligned} s^{\mathbf{L}}(S, \vec{T_1}, \vec{T_2}) &= t_1^{\mathbf{L}}(S, \vec{T_1}) \circ^{\mathbf{L}} t_2^{\mathbf{L}}(S, \vec{T_2}) \\ &= (\text{respectively, } \supseteq)\ t_1^{\exists}(S, \vec{T_1}) \circ^{\mathbf{L}} t_2^{\exists}(S, \vec{T_2}). \end{aligned}$$

Using Theorem 5.2.3(i),

$$\begin{aligned} & t_1^{\exists}(S, \vec{T_1}) \circ^{\mathbf{L}} t_2^{\exists}(S, \vec{T_2}) \\ = \ & \{t_1(a, \vec{b}) : a \in S^\ell, \vec{b} \in \vec{T_1}^{\,\ell}\}^u \circ^{\mathbf{L}} \{t_2(c, \vec{d}) : c \in S^\ell, \vec{d} \in \vec{T_2}^{\,\ell}\}^u \\ = \ & \{t_1(a, \vec{b}) \circ t_2(c, \vec{d}) : a, c \in S^\ell, \vec{b} \in \vec{T_1}^{\,\ell}, \vec{d} \in \vec{T_2}^{\,\ell}\}^u \\ \subseteq \ & \{t_1(a, \vec{b}) \circ t_2(a, \vec{d}) : a \in S^\ell, \vec{b} \in \vec{T_1}^{\,\ell}, \vec{d} \in \vec{T_2}^{\,\ell}\}^u \\ = \ & s^{\exists}(S, \vec{T_1}, \vec{T_2}). \end{aligned}$$

For the inclusion in the other direction, let $e \in s^{\exists}(S, \vec{T_1}, \vec{T_2})$, i.e., $t_1(a, \vec{b}) \circ t_2(a, \vec{d}) \leq e$ for all $a \in S^\ell, \vec{b} \in \vec{T_1}^{\,\ell}$ and $\vec{d} \in \vec{T_2}^{\,\ell}$ and let $a, c \in S^\ell$. Since $\mathbf{A}$ is a chain either $a \leq c$ or $c \leq a$. Suppose $c \leq a$; then $t_2(c, \vec{d}) \leq t_2(a, \vec{d})$ for all $\vec{b} \in \vec{T_1}^{\,\ell}$ and $\vec{d} \in \vec{T_2}^{\,\ell}$ since t_2 is order-preserving in each co-ordinate. Then $t_1(a, \vec{b}) \circ t_2(c, \vec{d}) \leq t_1(a, \vec{b}) \circ t_2(a, \vec{d}) \leq e$, so $e \in \{t_1(a, \vec{b}) \circ t_2(c, \vec{d}) : a, c \in S^\ell, \vec{b} \in \vec{T_1}, \vec{d} \in \vec{T_2}\}^u$, as required. If $a \leq c$ the proof is similar. The proof for $s = t_1 \wedge t_2$ follows similarly, using Lemma 5.2.4. □

By the definition of $f^{\mathbf{L}}$ it is immediate that $f(x)$ is $\exists$-stable. Thus, a particular consequence of the above lemma is that $(f(x))^n$ is $\exists$-stable for each $n \geq 1$, as are $(f(x_1))^{n_1} \circ \cdots \circ (f(x_k))^{n_k}$ and $(f(x_1))^{n_1} \wedge \cdots \wedge (f(x_k))^{n_k}$, where each x_i is a variable, 0 or 1 and each $n_i \geq 1$. More generally, we have the following:

Lemma 5.3.30. *The following terms are $\exists$-stable: any term built up inductively from 0, 1, x, $f(x)$, for any variable x, by taking $\circ$ or $\wedge$ of terms that share at most one variable, or any $\vee$.*

Combining the above results we obtain the following theorem.

Theorem 5.3.31. *An inequality $s \leq t$ is preserved by the completion if t is any positive term and s is:*

(i) a negative term,

(ii) an $\exists$-stable term (as in Lemma 5.3.30),

(iii) a term $t(\vec{x}) \to y$, where $t(\vec{x})$ is an $\exists$-expanding term and y is a variable not in $\vec{x}$,

(iv) any term built up inductively from terms in (i-iii) by taking $\circ$'s or $\wedge$'s of any terms that have no variables in common or any $\vee$'s.

Special subclasses of modal MTL-chains

In [CMM10], Ciabattoni et al. study the addition of truth stresser modalities to MTL and its extensions. When considering the semantics of these logics a number of classes of algebras are studied, all of which are subclasses of modal MTL-algebras as considered here. The various logics studied in [CMM10] are the monoidal t-norm logic (MTL), the involutive monoidal t-norm logic (IMTL) and the strict monoidal t-norm logic (SMTL). The last two of the aforementioned logics axiomatize t-norm logics whose negations are, respectively, involutive and strict. Furthermore, t-norm logics satisfying an n-contraction property were also studied — with involutive negations (C_nIMTL) and without (C_nMTL).

Let $\mathit{Logics} = \{\mathrm{MTL}, \mathrm{IMTL}, \mathrm{SMTL}\} \cup \{\mathrm{C}_n\mathrm{MTL} : n \geq 2\} \cup \{\mathrm{C}_n\mathrm{IMTL} : n \geq 2\}$. For $\mathrm{L} \in \mathit{Logics}$, an L-algebra is an MTL-algebra such that all the MTL-axioms as well as the additional axioms of the logic L are all valid. That is,

- An IMTL-algebra is an MTL-algebra satisfying: $\neg\neg x = x$.
- An SMTL-algebra is an MTL-algebra satisfying: $x \wedge (\neg x) \leq 0$.
- A C_nMTL-algebra, $n \geq 2$, is an MTL-algebra satisfying: $x^n \leq x^{n-1}$.
- A C_nIMTL-algebra, $n \geq 2$, is an IMTL-algebra that is also a C_nMTL-algebra.

Then, an

- LK^r-algebra is a modal L-algebra satisfying:
  (i) $f(x \to y) \leq f(x) \to f(y)$
  (ii) $f(x \vee y) = f(x) \vee f(y)$
  (iii) $f(1) = 1$.
- LKT^r-algebra is an LK^r-algebra additionally satisfying: $f(x) \leq x$.
- $\mathrm{LS4}^r$-algebra is an LKT^r-algebra additionally satisfying: $f(f(x)) \geq f(x)$ (hence also $f(f(x)) = f(x)$).
- $\mathrm{L}!^r$-algebra is an $\mathrm{LS4}^r$-algebra additionally satisfying: $f(x) \circ f(x) = f(x)$.
- L^r_Δ-algebra is an $\mathrm{LS4}^r$-algebra additionally satisfying: $f(x) \vee (f(x) \to 0) = 1$.

It is shown in [CMM10] that an LK^r-algebra is a subdirect product of linearly ordered algebras, hence the quasi-identity (5.2) holds in such algebras.

Observe that in a modal MTL-algebra, the identity

$$f(x) \circ f(y) \leq f(x \circ y) \tag{5.7}$$

is equivalent to the identity

$$f(x \to y) \leq f(x) \to f(y). \tag{5.8}$$

To see that (5.7) implies (5.8) we recall that $x \circ (x \to y) \leq y$. Since f is order-preserving and by (5.7), $f(x) \circ f(x \to y) \leq f(x \circ (x \to y)) \leq f(y)$, hence $f(x \to y) \leq f(x) \to f(y)$. Conversely, to see that (5.8) implies (5.7), recall that $x \leq y \to (x \circ y)$. Since f is order-preserving and by (5.8), $f(x) \leq f(y \to (x \circ y)) \leq f(y) \to f(x \circ y)$, hence $f(x) \circ f(y) \leq f(x \circ y)$.

The following corollary is now a straightforward consequence of Theorem 5.3.31.

Corollary 5.3.32. *If* $\mathbf{A}$ *is a linearly ordered* LK^r-, LKT^r-, $\mathrm{LS4}^r$-, $\mathrm{L}!^r$- *or* L^r_Δ-*algebra, then so is* $\mathbf{L}$.

Complete operators

Recall that an operation f is called a *complete operator* if $f(\bigvee a_i) = \bigvee f(a_i)$ whenever $\bigvee a_i$ exists. Complete operators are often also called *left-continuous*.

If, in addition to being order-preserving, we assume that the operation f is a complete operator, a wider class of properties is preserved by the completion. Firstly, if f is a complete operator, then $f^{\mathbf{L}}$ is a complete operator.

Lemma 5.3.33. *Let $H \subseteq A$. If $d \in H^{u\ell}$ such that $d \not\leq e$ for any $e \in H$, i.e., $e < d$ for all $e \in H$, then $d = \bigvee H$.*

Proof. Suppose $d \in H^{u\ell}$ such that $e < d$ for every $e \in H$. Then d is an upper bound for H. Let $a \in H^u$, then $d \leq a$ since $d \in H^{u\ell}$. Thus, d is the least upper bound for H. □

Lemma 5.3.34. *If f is a complete operator, then for all $H \subseteq A$,*

(i) $f^{\mathbf{L}}(H^u) = (f(H))^u$, where $f(H) = \{f(a) : a \in H\}$,

(ii) if $S = H^u$ is stable, then $f^{\mathbf{L}}(S) = (f(H))^u$.

Proof. We shall prove part (i); part (ii) then follows directly. Since $H \subseteq H^{u\ell}$ we have $f(H) \subseteq f(H^{u\ell})$ and also $(f(H^{u\ell}))^u \subseteq (f(H))^u$, i.e., $f^{\mathbf{L}}(H^u) \subseteq (f(H))^u$. Conversely, let $a \in (f(H))^u$, i.e., $a \geq f(b)$ for every $b \in H$, and let $c \in H^{u\ell}$. If $c \leq d$ for some $d \in H$, then $f(c) \leq f(d) \leq a$ since $a \in (f(H))^u$. If not, then $d < c$ for every $d \in K$, so $c = \bigvee H$, by Lemma 5.3.33. Thus, $f(c) = f(\bigvee H) = \bigvee f(H)$ since f is a complete operator. But a is an upper bound for $f(H)$, so $\bigvee f(H) \leq a$ and, in particular, $f(c) \leq a$. In either case we find that $f(c) \leq a$ and therefore $a \in (f(H^{u\ell}))^u$. We conclude that $(f(H))^u \subseteq (f(H^{u\ell}))^u$. □

Lemma 5.3.35. *If f is a complete operator, then $f^{\mathbf{L}}$ is also a complete operator.*

Proof. Let $S_i \in \mathcal{L}$. By Lemma 2.6.3, $f^{\mathbf{L}}(\bigvee_i^{\mathbf{L}} S_i) = f^{\mathbf{L}}(\bigcap_i S_i) = f^{\mathbf{L}}(\bigcap_i S_i^{\ell u}) = f^{\mathbf{L}}((\bigcup_i S_i^{\ell})^u)$. Then, by Lemma 5.3.34, $f^{\mathbf{L}}(\bigvee_i^{\mathbf{L}} S_i) = f(\bigcup_i S_i^{\ell})^u$. Finally, by Lemma 2.6.3, $f^{\mathbf{L}}(\bigvee_i^{\mathbf{L}} S_i) = (\bigcup_i f(S_i^{\ell}))^u = \bigcap_i f(S_i^{\ell})^u = \bigvee_i^{\mathbf{L}} f^{\mathbf{L}}(S_i)$. □

It is well known that residuated unary operations distribute over all existing joins. Examples of complete operators therefore include all residuated operators. However, only a partial converse holds: if the underlying lattice of an algebra $\mathbf{A}$ is complete, then a unary operation f on A that distributes over all joins is

residuated. Hence, a consequence of f being a complete operator is that $f^{\mathbf{L}}$ is residuated.

Lemma 5.3.36. *If f is a complete operator and s is an $\exists$-contracting term, then $f(s)$ is also $\exists$-contracting.*

Proof. Let $t = f(s(\vec{x}))$ and $\vec{S} \in \mathcal{L}$. Then

$$\begin{aligned} t^{\mathbf{L}}(\vec{S}) = f^{\mathbf{L}}(s^{\mathbf{L}}(\vec{S})) &\supseteq f^{\mathbf{L}}(s^{\exists}(\vec{S})) = f^{\mathbf{L}}(\{s(\vec{a}) : \vec{a} \in \vec{S}^{\ell}\}^{u}) \\ &= \{f(s(\vec{a})) : \vec{a} \in \vec{S}^{\ell}\}^{u} \quad \text{by Lemma 5.3.34} \\ &= t^{\exists}(\vec{S}). \end{aligned}$$

□

Corollary 5.3.37. *If f is a complete operator and s is an $\exists$-stable term, then $f(s)$ is also $\exists$-stable.*

Proof. A term that is both $\exists$-expanding and $\exists$-contracting is an $\exists$-stable term. The result follows from Lemmas 5.3.25 and 5.3.36. □

Lemma 5.3.38. *If f is a complete operator, then the following terms are $\exists$-stable: any term built up inductively from 0, 1, x, $f(x)$, for any variable x, by taking $\circ$ or $\wedge$ of terms that share at most one variable, taking f of any term or $\vee$ of any two terms.*

Combining the above results we obtain the theorem below.

Theorem 5.3.39. *If f is a complete operator, then an inequality $s \leq t$ is preserved by the completion if t is any positive term and s is:*

(i) a negative term,

(ii) an $\exists$-stable term (as in Lemma 5.3.38)

(iii) a term $t(\vec{x}) \to y$, where $t(\vec{x})$ is an $\exists$-expanding term and y is a variable not in $\vec{x}$,

(iv) a term built up inductively from terms in (i-iii) by taking $\circ$ or $\wedge$ of any terms have no variables in common, any f or any $\vee$.

Order-reversing modalities

If $\mathbf{A} = \langle A, \vee, \wedge, \circ, \to, h, 0, 1\rangle$ is a reverse modal MTL-algebra, then we define the unary operation $h^{\mathbf{L}}$ on $\mathbf{L}$ by

$$h^{\mathbf{L}}(S) = \{h(a) : a \in S^{\ell}\}^{\ell u}$$

and let $\mathbf{L} = \langle F, \circ^{\mathbf{L}}, \vee^{\mathbf{L}}, \wedge^{\mathbf{L}}, h^{\mathbf{L}}, 0^{\mathbf{L}}, 1^{\mathbf{L}}\rangle$, where the other operations are as described in Section 5.2. Note that $h^{\mathbf{L}}(S)$ is the $\forall$-approximation.

Recall that $\iota : A \to \mathcal{L}$ is the embedding of $\mathbf{A}$ into $\mathbf{L}$ defined by $\iota(a) = \{a\}^u$ for all $a \in A$.

Lemma 5.3.40. *The embedding $\iota : A \to \mathcal{L}$ of $\mathbf{A}$ into $\mathbf{L}$ preserves h, i.e., for $a \in A$, $h^{\mathbf{L}}(\{a\}^u) = \{h(a)\}^u$.*

Lemma 5.3.41. *The operation $h^{\mathbf{L}}$ is order-reversing.*

The proofs of Lemmas 5.3.40 and 5.3.41 are similar to the proofs of Lemmas 5.3.23 and 5.3.22, respectively.

Proposition 5.3.42. *The algebra $\mathbf{L} = \left\langle F, \circ^{\mathbf{L}}, \to^{\mathbf{L}}, \vee^{\mathbf{L}}, \wedge^{\mathbf{L}}, h^{\mathbf{L}}, 0^{\mathbf{L}}, 1^{\mathbf{L}}\right\rangle$ is a complete reverse modal MTL-algebra and $\iota : A \to \mathcal{L}$, the embedding of $\mathbf{A}$ into $\mathbf{L}$, preserves all existing meets and joins in $\mathbf{A}$.*

Then $(\mathbf{L}, \iota)$ is the MacNeille completion *of $\mathbf{A}$.*

We now turn our attention to properties preserved by this construction. Clearly, h gives the impression of a negative term and, indeed, we show that this is the case.

Lemma 5.3.43. *If s is an $\exists$-expanding term, then $h(s)$ is $\forall$-contracting.*

Proof. Let $t = h(s(\vec{x}))$ and $\vec{S} \in \mathcal{L}$. Since s is $\exists$-expanding and $h^{\mathbf{L}}$ is order-reversing,

$$t^{\mathbf{L}}(\vec{S}) = h^{\mathbf{L}}(s^{\mathbf{L}}(\vec{S})) \supseteq h^{\mathbf{L}}(s^{\exists}(\vec{S})) = \{h(b) : b \in \{s(\vec{a}) : \vec{a} \in \vec{S}^{\ell}\}^{u\ell}\}^{\ell u},$$

which we must show to be a superset of $t^{\forall}(\vec{S}) = \{h(s(\vec{a})) : \vec{a} \in \vec{S}^{\ell}\}^{\ell u}$. Since $\{s(\vec{a}) : \vec{a} \in \vec{S}^{\ell}\} \subseteq \{s(\vec{a}) : \vec{a} \in \vec{S}^{\ell}\}^{u\ell}$, $\{h(s(\vec{a})) : \vec{a} \in \vec{S}^{\ell}\} \subseteq \{h(b) : b \in \{s(\vec{a}) : \vec{a} \in \vec{S}^{\ell}\}^{u\ell}\}$ and the result follows after taking the ${}^{\ell u}$-closures. □

Lemma 5.3.44. *If s is a $\forall$-contracting term, then $h(s)$ is $\exists$-expanding.*

Proof. Let $t = h(s(\vec{x}))$ and $\vec{S} \in \mathcal{L}$. Since s is $\forall$-contracting and $h^{\mathbf{L}}$ is order-reversing,

$$\begin{aligned} t^{\mathbf{L}}(\vec{S}) = h^{\mathbf{L}}(s^{\mathbf{L}}(\vec{S})) \subseteq h^{\mathbf{L}}(s^{\forall}(\vec{S})) &= \{h(b) : b \in \{s(\vec{a}) : \vec{a} \in \vec{S}^{\ell}\}^{\ell u\ell}\}^{\ell u} \\ &= \{h(b) : b \in \{s(\vec{a}) : \vec{a} \in \vec{S}^{\ell}\}^{\ell}\}^{\ell u}, \end{aligned}$$

which we must show to be a subset of $t^{\exists}(\vec{S}) = \{h(s(\vec{a})) : \vec{a} \in \vec{S}^{\ell}\}^{u}$. Let $\vec{a} \in \vec{S}^{\ell}$ and $b \in \{s(\vec{a}) : \vec{a} \in \vec{S}^{\ell}\}^{\ell}$. Then $b \leq s(\vec{a})$ so $h(s(\vec{a})) \leq h(b)$, hence $\{h(s(\vec{a})) : \vec{a} \in X^{\ell}\}^{\ell} \subseteq \{h(b) : b \in \{s(\vec{a}) : \vec{a} \in \vec{S}^{\ell}\}^{\ell}\}^{\ell}$ and the result follows after taking ^{u}'s. □

Extend the notion of positive and negative terms to the language of reverse modal MTL-algebras by defining $h(s)$ negative whenever s is positive and $h(s)$ positive whenever s is negative. The above two results give the following.

Proposition 5.3.45. *Every positive term is $\exists$-expanding and every negative term is $\forall$-contracting, hence also $\exists$-contracting.*

Theorem 5.3.46. *If h is a reverse modality, an inequality $s \leq t$ is preserved by the completion if t is any positive term and s is:*

(i) a negative term,

(ii) an $\exists$-stable term on the language $\{\circ, \vee, \wedge, 0, 1\}$ (as in Lemma 5.3.30),

(iii) a term $t(\vec{x}) \to y$, where $t(\vec{x})$ is an $\exists$-expanding term and y is a variable not in $\vec{x}$,

(iv) a term built up inductively from terms in (i-iii) by taking $\circ$ or $\wedge$ of any terms have no variables in common, or any $\vee$.

In addition, we have the following preservation result.

Proposition 5.3.47. *If $\mathbf{A}$ satisfies $x \leq h(h(x))$ or $h(h(x)) \leq x$ then $\mathbf{L}$ satisfies the same. Thus, if h is involutive, then so is $h^{\mathbf{L}}$.*

Proof. The inequality $x \leq h(h(x))$ is preserved by Theorem 5.3.46 since x is $\exists$-stable and $h(h(x))$ is positive.

For $S \in \mathcal{L}$,

$$h^{\mathbf{L}}(h^{\mathbf{L}}(S)) = \{h(b) : b \in \{h(a) : a \in S^{\ell}\}^{\ell}\}^{\ell u},$$

where we have used the fact that $H^{\ell u \ell} = H^{\ell}$ for any $H \subseteq A$. Suppose $\mathbf{A}$ satisfies $h(h(x)) \leq x$ and let $S \in \mathcal{L}$. Let $c \in \{h(b) : b \in \{h(a) : a \in S^{\ell}\}^{\ell}\}^{\ell}$, i.e., $c \leq h(b)$ for all $b \in \{h(a) : a \in S^{\ell}\}^{\ell}$. If $d \in S$, then $a \leq d$ for all $a \in S^{\ell}$ hence $h(d) \leq h(a)$ so $h(d) \in \{h(a) : a \in S^{\ell}\}^{\ell}$. Thus, $c \leq h(h(d)) \leq d$, so $c \in S^{\ell}$, hence $\{h(b) : b \in \{h(a) : a \in S^{\ell}\}^{\ell u \ell}\}^{\ell} \subseteq S^{\ell}$ and the result follows after taking superscript ^{u}'s. □

6. CANONICAL EXTENSIONS

Completions of algebraic structures that have received substantial attention are so-called *canonical extensions*. Canonical extensions were first introduced in [JT51, JT52] for Boolean algebras with operators. Canonical extensions of (bounded) distributive lattices were studied in [GJ94, GJ00, CP12] and canonical extensions of (bounded) lattice expansions were first studied in [GH01] and later also in [Har06]. We note that in this chapter we will not use the term 'canonical extension' for the notion defined in [Mac37] (see Remark 5.1.2), which is generally different from the notion of 'canonical extension' discussed here.

In [GH01] both a concrete description and an abstract characterization of the canonical extension of a (bounded) lattice were given. It turns out that the construction given in [GH01] is a special case of the construction described in [Tun74] wherein Tunnicliffe described the completion of posets with respect to a polarization (see Definition 6.1.1). Subsequently, 'canonical extensions' of posets have been explored in [DGP05, GJKO07, GJP].

In [DGP05] the construction described in [GH01] (first appearing in [Tun74]) was modified for posets. An alternative construction, closely related to the construction of the canonical extension of Boolean algebras with operators given in [GM97], was also given in [DGP05]. The authors then focussed their attention on 'canonical extensions' of additional operations and relational completeness of some substructural logics.

In [GJKO07] the construction from [Tun74] (and [GH01]) was (again) described generally for the poset setting and properties of 'canonical extensions' were investigated. The 'canonical extensions' of additional operations as well as residuated groupoids were also considered.

However, upon closer inspection it becomes apparent that the completions used in [DGP05] and [GJKO07] are generally different. The difference between these completions is due to a choice of 'filters' and 'ideals' (families of up-sets

and down-sets of posets, respectively). To use the terminology from [Tun74], different 'polarizations of posets' were used in [DGP05] and [GJKO07]. We note that in [DGP05, Remark 2.3] the authors acknowledged that a choice had to be made and subsequently explained their choice. Thus, it is clear that a choice must be made, but it is not clear which choice would give one 'the correct' definition of the 'canonical extension' of a poset, or if there even is 'a correct' definition. Further investigation is therefore warranted.

In [GJP] a completion obtained through the construction in [Tun74] is called an $(\mathcal{F}, \mathcal{I})$-construction — after its polarization. The authors of [GJP] investigate which properties a polarization should satisfy in order for the completion obtained from it to satisfy certain desirable properties, for example restrictive distributive laws [GH01] or commuting with products.

In the next section we study the construction from [Tun74, GH01] for the posets setting. We then investigate some of the properties of completions obtained through this construction. Perhaps unsurprisingly not all the results that hold for canonical extensions of (bounded) lattices are true for these completions. Crucially, these completions do not, in general, commute with (Cartesian) products — see Example 6.2.14.

We then investigate four specific completions that may be obtained through the construction. We use the four different types of filters and ideals of a poset, defined in Chapter 4, to construct the four completions under consideration. Among these are the 'canonical extensions' of posets studied in [DGP05] and [GJKO07], respectively. We take a closer look at some of the properties of each of the individual completions.

Next we focus our attention on extensions of additional operations. We first consider extensions of unary operations. Again the results are not always favourable. For example, the extensions (used here) of operators on posets are not necessarily operators on the completions of the posets — see Examples 6.3.8 and 6.3.10. On the other hand, for three of the completions considered here (the completions using Doyle-pseudo, Frink and directed filters and ideals, respectively) the extensions of unary residuated operators are again residuated — see Propositions 6.3.13 and 6.3.14.

We also explore extensions of n-ary operations. Since the construction does not commute with products, extensions of n-ary operations are not straightforward. For three of the completions under consideration the extensions of

arbitrary unary operations can be generalized to extensions for arbitrary n-ary operations. We investigate some properties of these extensions of n-ary operations. In particular, we are interested in order-preserving n-ary operations and binary residuated operators.

As stated earlier, an alternative construction of the canonical extension of a Boolean Algebra with operators was described in [GM97]. In this construction the canonical extension is the MacNeille completion of an intermediate structure. This construction was generalised in [DGP05, Suz11] to the poset setting for a particular choice of families of up-sets and down-sets. We show here that the construction can be generalized to use a number of different families of up-sets and down-sets.

Finally we focus our attention on the preservation of properties through the construction studied in this chapter. As in the previous chapters we follow the approach used in [Jón94]. Using the denseness of the completion, we can approximate terms in the completion from below and from above. We combine the use of these approximations in order to give a syntactical description of inequalities preserved by the construction. We note that in [GM97, Suz11, Suz10] an alternative approach was followed to investigate property preservation by the completion.

A part of this chapter has been submitted for publication in the form of [Mor].

6.1 The general case

Throughout this section let $\mathbf{P} = \langle P, \leq \rangle$ be a fixed poset.

6.1.1 The construction

The construction described in this section corresponds with the construction of completions of posets with respect to polarizations [Tun74]. It also corresponds with the construction of canonical extensions of bounded lattices [GH01]. Throughout this section let $\mathcal{F}$ and $\mathcal{I}$ be fixed families of *non-empty* subsets of P. Let $R \subseteq \mathcal{F} \times \mathcal{I}$ be the relation defined by $(F, I) \in R$ if, and only if, $F \cap I \neq \varnothing$.

The polarities of R yield the Galois connection [Bir67], ${}^{\triangleright} : \mathcal{P}(\mathcal{F}) \rightleftarrows \mathcal{P}(\mathcal{I}) : {}^{\triangleleft}$, where, for $X \in \mathcal{P}(\mathcal{F})$ and $\Lambda \in \mathcal{P}(\mathcal{I})$

$$X^{\triangleright} = \{I \in \mathcal{I} : F \in X \text{ implies } I \cap F \neq \varnothing\}$$

$$\Lambda^{\triangleleft} = \{F \in \mathcal{F} : I \in \Lambda \text{ implies } F \cap I \neq \varnothing\}.$$

Then $X \in \mathcal{P}(\mathcal{F})$ is *Galois closed* if $X = X^{\triangleright\triangleleft}$ and $\Lambda \in \mathcal{P}(\mathcal{I})$ is *Galois closed* if $\Lambda = \Lambda^{\triangleleft\triangleright}$. In [Tun74] the term 'regular' is used for Galois closed sets.

Let $\mathcal{C} = \{X \in \mathcal{P}(\mathcal{F}) : X = X^{\triangleright\triangleleft}\}$. For $\mathcal{T} \subseteq \mathcal{C}$ let

$$\bigwedge^{\mathbf{C}} \mathcal{T} = \bigcap \mathcal{T} \text{ and } \bigvee^{\mathbf{C}} \mathcal{T} = \Big(\bigcup \mathcal{T}\Big)^{\triangleright\triangleleft},$$

i.e., meet is intersection and join is the Galois closure of the union. Then $\mathbf{C} = \langle \mathcal{C}, \vee^{\mathbf{C}}, \wedge^{\mathbf{C}} \rangle$ is a complete lattice where $\subseteq$ is the associated lattice ordering $\leq^{\mathbf{C}}$.

Definition 6.1.1. *[Tun74, Definition 3] A pair $(\mathcal{F}, \mathcal{I})$ of sets of non-empty subsets of P is called a* polarization of $\mathbf{P}$ *if:*

(i) If $x, y \in P$ such that $x \neq y$, then there exists $S \in \mathcal{F} \cup \mathcal{I}$ such that $x \in S$ and $y \notin S$.

(ii) If $F \in \mathcal{F}$ and $x \notin F$, then there exists $I \in \mathcal{I}$ such that $x \in I$ and $F \cap I = \varnothing$ and, dually, if $x \notin I \in \mathcal{I}$ then there exists $F \in \mathcal{F}$ such that $x \in F$ and $I \cap F = \varnothing$.

For the remainder of this section we assume that $(\mathcal{F}, \mathcal{I})$ forms a polarization of $\mathbf{P}$ and that $\mathbf{C} = \langle \mathcal{C}, \vee^{\mathbf{C}}, \wedge^{\mathbf{C}} \rangle$ is the complete lattice of Galois closed sets with respect to $(\mathcal{F}, \mathcal{I})$.

Define the map $\alpha : P \to \mathcal{C}$ by $\alpha(a) = \{F \in \mathcal{F} : a \in F\}$. Then, for each $a \in P$ the set $\alpha(a)$ is Galois closed, i.e., $\alpha(a) \in \mathcal{C}$. Furthermore, α is one-to-one. For $S \subseteq P$ let $\alpha(S) = \{\alpha(a) : a \in S\}$.

Lemma 6.1.2. *[Tun74, Proposition 4] Let $S \subseteq P$. Then*

(i) $\bigwedge^{\mathbf{C}} \alpha(S) = \{F \in \mathcal{F} : S \subseteq F\}$.

(ii) $\bigvee^{\mathbf{C}} \alpha(S) = \{F \in \mathcal{F} : S \subseteq I \in \mathcal{I} \text{ implies } F \cap I \neq \varnothing\}$. In particular, if $S \in \mathcal{I}$ then $\bigvee^{\mathbf{C}} \alpha(S) = \{F \in \mathcal{F} : F \cap S \neq \varnothing\}$.

Analogous claims were made for bounded lattices in [GH01, Proposition 2.6]. In the sequel we omit the superscript $\mathbf{C}$ when denoting $\vee$'s and $\wedge$'s in $\mathbf{C}$ and only use it when we need to indicate which lattice is under consideration.

Definition 6.1.3. *[Tun74, Definition 5] A polarization $(\mathcal{F}, \mathcal{I})$ of $\mathbf{P}$ is* consistent *if $\mathcal{F}$ is a set of non-empty up-sets of $\mathbf{P}$ such that each principal up-set of $\mathbf{P}$ is an intersection of sets in $\mathcal{F}$, and, dually, $\mathcal{I}$ is a set of non-empty down-sets of $\mathbf{P}$ such that each principal down-set of $\mathbf{P}$ is an intersection of sets in $\mathcal{I}$.*

Then we have the following.

Theorem 6.1.4. *[Tun74, Theorem 1]* $(\mathbf{C}, \alpha)$ *is a completion of* $\mathbf{P}$ *if, and only if, the polarization used to construct* $\mathbf{C}$ *is consistent.*

Hence, if $(\mathcal{F}, \mathcal{I})$ is a consistent polarization then α is an order-embedding of $\mathbf{P}$ into $\mathbf{C}$. If it is necessary to specify $\mathbf{P}$, we will write $(\mathbf{C}(\mathbf{P}), \alpha^{\mathbf{P}})$ and $\mathcal{C}(\mathbf{P})$.

Let $\mathcal{B} = \{\Lambda \in \mathcal{P}(\mathcal{I}) : \Lambda = \Lambda^{\triangleleft\triangleright}\}$. We note that $\mathbf{B} = \langle \mathcal{B}, \vee^{\mathbf{B}}, \wedge^{\mathbf{B}} \rangle$, such that $\bigvee^{\mathbf{B}} \mathcal{T} = \bigcap \mathcal{T}$ and $\bigwedge^{\mathbf{B}} \mathcal{T} = (\bigcup \mathcal{T})^{\triangleleft\triangleright}$ for $\mathcal{T} \subseteq \mathcal{B}$ where $\leq^{\mathbf{B}}$ is $\supseteq$, also forms a complete lattice. If $(\mathcal{F}, \mathcal{I})$ is consistent, then $\gamma : P \to \mathcal{B}$ defined by $\gamma(a) = \{I \in \mathcal{I} : a \in I\}$ is an order-embedding of $\mathbf{P}$ into $\mathbf{B}$ and therefore $(\mathbf{B}, \gamma)$ is also a completion of $\mathbf{P}$. Moreover, $\mathbf{C}$ is order-isomorphic to $\mathbf{B}$ with isomorphism $\psi : X \mapsto X^{\triangleright}$. As a matter of fact, in [Tun74] the construction of $(\mathbf{B}, \gamma)$ is described. However, we prefer working with $(\mathbf{C}, \alpha)$.

In [GJKO07] a slightly more restrictive condition was placed on the sets $\mathcal{F}$ and $\mathcal{I}$ for their investigation of the above construction for the poset setting: the families $\mathcal{F}$ and $\mathcal{I}$ of up-sets and down-sets of $\mathbf{P}$, respectively, are called *rich enough* if

(i) each member of $\mathcal{F}$ (respectively, $\mathcal{I}$) is closed under existing finite meets (respectively, joins).

(ii) $\mathcal{F}$ (respectively, $\mathcal{I}$) contains all principal up-sets (respectively, down-sets).

In [GJKO07] the empty set is allowed to be a member of $\mathcal{F}$ (respectively, $\mathcal{I}$) if, and only if, $\mathbf{P}$ does not have a top element (respectively, bottom element). If we assume that rich enough families of up-sets and down-sets may not include the empty set, then any pair consisting of a rich enough family of up-sets and a rich enough family of down-sets clearly forms a consistent polarization.

In [GJP] families of up-sets and down-sets satisfying condition (ii) above (from [GJKO07]) are called 'standard collections of filters' and 'standard collections of ideals', respectively.

If $(\mathbf{C}, \alpha)$ is a completion obtained from a consistent polarization, then α need not preserve existing meets and joins in $\mathbf{P}$. In [Tun74] a polarization $(\mathcal{F}, \mathcal{I})$ is called *lattice-consistent* if $\mathcal{F} = \mathcal{F}^{dp}$ and $\mathcal{I} = \mathcal{I}^{dp}$, i.e., the families of Doyle-pseudo filters and ideals, respectively (see Definition 4.1.2 in Chapter 4.1); and *completely consistent* if $\mathcal{F} = \mathcal{F}^{cdp}$ and $\mathcal{I} = \mathcal{I}^{cdp}$ (see Definition 4.1.7 in Chapter 4.1). The author then states in [Tun74, Proposition 6] that α preserves existing finite joins and meets if, and only if, the polarization is lattice-consistent;

and α preserves existing joins and meets if, and only if, the polarization is completely consistent. However, from the proof it becomes clear that a correct definition of *lattice-consistent* polarization should be relaxed to only require condition (i) from [GJKO07] stated above. Similarly, the definition of a completely consistent polarization should be altered to include more families of up-sets and down-sets. Therefore we make the following definitions.

Definition 6.1.5. *A polarization $(\mathcal{F}, \mathcal{I})$ of $\mathbf{P}$ will be called* lattice-consistent *if each member of $\mathcal{F}$ is closed under existing finite meets and each member of $\mathcal{I}$ is closed under existing finite joins. Similarly, $(\mathcal{F}, \mathcal{I})$ will be called* completely consistent *if each member of $\mathcal{F}$ is closed under existing arbitrary meets and each member of $\mathcal{I}$ is closed under existing arbitrary joins.*

Recall from Definition 4.2.8 that a set S is meet-dense in a complete lattice $\mathbf{A}$ if every element in A is the meet of elements in S. Dually, a set T is said to be join-dense in a complete lattice $\mathbf{A}$ if every element in A is the join of elements in T.

Theorem 6.1.6. *[Tun74, Theorem 2] Let $(\mathbf{A}, \varphi)$ be a completion of $\mathbf{P}$ where $\mathbf{A} = \langle \mathcal{A}, \vee^{\mathbf{A}}, \wedge^{\mathbf{A}} \rangle$. Then there exists a consistent polarization $(\mathcal{F}, \mathcal{I})$ of $\mathbf{P}$ such that $(\mathbf{C}, \alpha)$, obtained from $(\mathcal{F}, \mathcal{I})$, is isomorphic to $(\mathbf{A}, \varphi)$ (in such a way that the image of P is fixed by the isomorphism) if, and only if, there exist $S, T \subseteq \mathcal{A}$ such that*

(i) S is meet-dense in $\mathbf{A}$ and T is join-dense in $\mathbf{A}$; and

(ii) if $a \in S$ and $b \in T$ such that $a \geq b$, then there exists $c \in P$ such that $a \geq \varphi(c) \geq b$.

We will call a completion $(\mathbf{A}, \varphi)$ *dense* with respect to a pair of subsets (S, T) of $\mathcal{A}$, if S is meet-dense in $\mathbf{A}$ and T is join-dense in $\mathbf{A}$. The first condition in the theorem above can then be restated as: there exist $S, T \subseteq \mathcal{A}$ such that $\mathbf{A}$ is dense with respect to (S, T).

A consequence of the preceding theorem is that the completions studied in Chapters 4 and 5 are obtainable, up to isomorphism fixing the image of P, using the construction described in this section.

For example, let $(\mathbf{L}, \iota)$ be the MacNeille completion of $\mathbf{P}$, as described in Chapter 5.1. Recall that $\iota(P)$ is both join-dense and meet-dense in $\mathbf{L}$. Hence, in the notation of Theorem 6.1.6, we may set $S = \iota(P) = T$. Then by [Tun74,

Corollary 1] S and T also satisfy the second condition. Moreover, if $\mathcal{F}$ is the set of all principal up-sets of $\mathbf{P}$ and $\mathcal{I}$ the set of all principal down-sets of $\mathbf{P}$, then $(\mathbf{C}, \alpha)$, the completion obtained from the polarization $(\mathcal{F}, \mathcal{I})$, is (isomorphic to) the MacNeille completion of $\mathbf{P}$.

For more examples the reader is referred to [Tun74, pg. 23].

Remark 6.1.7. *One may now wonder whether or not every completion of a poset is obtainable from a polarization. In [Tun74, Example 3] the author attempted to address this problem by providing a counterexample. However, the example does not appear to disprove what it was intended to disprove. For more details on why this example does not work the reader may consult Example A.2.1 in Appendix A.2. The problem remains open.*

6.1.2 Properties of the completion

Throughout this section let $\mathcal{F}$ be a fixed family of *non-empty* up-sets of $\mathbf{P}$ that includes all the principal up-sets, and $\mathcal{I}$ a fixed family of *non-empty* down-sets of $\mathbf{P}$ that includes all principal down-sets. Then $(\mathcal{F}, \mathcal{I})$ forms a consistent polarization as defined above. Let $(\mathbf{C}, \alpha)$ be the completion obtained from the polarization $(\mathcal{F}, \mathcal{I})$, as described in the previous section. We write $\mathcal{F}(\mathbf{P})$ and $\mathcal{I}(\mathbf{P})$ when it is necessary to specify $\mathbf{P}$.

Lemma 6.1.8. *The following holds for* $(\mathbf{C}, \alpha)$.

(i) $\top = \mathcal{F}$.

(ii) $P \in \mathcal{F}$ *if, and only if,* $\bot = \{P\}$. *Moreover,* $P \notin \mathcal{F}$ *if, and only if,* $\bot = \varnothing$.

(iii) $\bigvee \alpha(P) = \top$ *and* $\bigwedge \alpha(P) = \bot$.

Proof. (i) Recall that $\varnothing \notin \mathcal{F}$. If $P \in \mathcal{I}$, then $\mathcal{F}^{\triangleright\triangleleft} = \{P\}^{\triangleleft} = \mathcal{F}$. If $P \notin \mathcal{I}$, then for all $J \in \mathcal{I}$ there exists $F \in \mathcal{F}$ such that $J \cap F = \varnothing$. Therefore, $\mathcal{F}^{\triangleright\triangleleft} = \varnothing^{\triangleleft} = \mathcal{F}$. The last equality follows since the implication "$I \in \varnothing$ implies $F \cap I \neq \varnothing$" is trivially true for all $F \in \mathcal{F}$.

In either case $\mathcal{F}$ is Galois closed and hence $\top = \mathcal{F}$.

(ii) If $P \in \mathcal{F}$, then $\{P\}^{\triangleright\triangleleft} = \mathcal{I}^{\triangleleft} = \{P\}$, i.e., $\{P\}$ is Galois closed. The only proper subset of $\{P\}$ is $\varnothing$, but $\varnothing^{\triangleright\triangleleft} = \mathcal{I}^{\triangleleft} = \{P\}$ and therefore does not belong to $\mathcal{C}$. Hence, $\bot = \{P\}$. The implication in the other direction is immediate.

Now suppose $P \notin \mathcal{F}$. Then $\varnothing^{\triangleright\triangleleft} = \mathcal{I}^{\triangleleft} = \varnothing$ and $\bot = \varnothing$ since $\varnothing$ is the least subset of $\mathcal{F}$ and it is Galois closed. Again, the implication in the other direction is immediate.

(iii) If $P \in \mathcal{I}$, then $\bigvee \alpha(P) = \{F \in \mathcal{F} : F \cap P \neq \varnothing\} = \mathcal{F} = \top$ by part (i) and Lemma 6.1.2 (ii). If $P \notin \mathcal{I}$, then $\bigvee \alpha(P) = \{F \in \mathcal{F} : P \subseteq I \in \mathcal{I} \text{ implies } F \cap I \neq \varnothing\} = \mathcal{F} = \top$ since there is no $I \in \mathcal{I}$ such that $P \subseteq I$ and therefore the implication is trivially true for all $F \in \mathcal{F}$.

If $P \in \mathcal{F}$, then $\bigwedge \alpha(P) = \{F \in \mathcal{F} : P \subseteq F\} = \{P\} = \bot$ by part (ii) and Lemma 6.1.2 (i). On the other hand, if $P \notin \mathcal{F}$, then $\bigwedge \alpha(P) = \{F \in \mathcal{F} : P \subseteq \mathcal{F}\} = \varnothing = \bot$ again by part (ii) and Lemma 6.1.2 (i).

□

Following [DGP05] and [GJP] we define the closed and open elements of the completion in terms of the elements of $\mathcal{F}$ and $\mathcal{I}$, respectively.

Definition 6.1.9. *An element $Y \in \mathcal{C}$ is called* closed *if $Y = \bigwedge \alpha(F)$ for some $F \in \mathcal{F}$. On the other hand, an element $Z \in \mathcal{C}$ is called* open *if $Z = \bigvee \alpha(I)$ for some $I \in \mathcal{I}$.*

Let K, O and KO denote the sets of closed, open and clopen elements of $\mathbf{C}$, respectively. When necessary we will use $K(\mathbf{P})$, $O(\mathbf{P})$ and $KO(\mathbf{P})$ to denote the closed, open and clopen elements of $\mathbf{C}(\mathbf{P})$.

For a bounded lattice the closed and open elements of its canonical extension are defined as the meets and joins, respectively, of the images of arbitrary subsets of the lattice [GH01]. Since an arbitrary subset of a lattice generates both a filter and an ideal of the lattice, on lattices our definitions of the sets of closed and open elements will be the same as the definitions given in [GH01]. This will be explored further in Section 6.2.

The following notion of a parametrised compactness, called *internal compactness* in [GJKO07], was also mentioned in [GH01].

Proposition 6.1.10. *The completion $(\mathbf{C}, \alpha)$ is* internally compact with respect to $(\mathcal{F}, \mathcal{I})$ *in the sense that it satisfies: for any $S, T \subseteq P$,*

$$\bigwedge \alpha(S) \leq \bigvee \alpha(T) \text{ if, and only if, } F \cap I \neq \varnothing \text{ for any } F \in \mathcal{F} \text{ and any } I \in \mathcal{I} \text{ such that } S \subseteq F \text{ and } T \subseteq I.$$

Proof. Let $S, T \subseteq P$.

Suppose $\bigwedge \alpha(S) \leq \bigvee \alpha(T)$ for $S, T \subseteq P$ and let $G \in \mathcal{F}$ and $J \in \mathcal{I}$ such that $S \subseteq G$ and $T \subseteq J$. Then $\bigwedge \alpha(G) \leq \bigwedge \alpha(S) \leq \bigvee \alpha(T) \leq \bigvee \alpha(J)$. That is, $\{F \in \mathcal{F} : G \subseteq F\} \subseteq \{F \in \mathcal{F} : J \subseteq I \in \mathcal{I} \text{ implies } F \cap I \neq \varnothing\}$. In particular, $G \cap J \neq \varnothing$.

Next suppose $F \cap I \neq \varnothing$ for every $F \in \mathcal{F}$ and $I \in \mathcal{I}$ such that $S \subseteq F$ and $T \subseteq I$. Then $F \in \{F \in \mathcal{F} : T \subseteq I \in \mathcal{I} \text{ implies } F \cap I \neq \varnothing\} = \bigvee \alpha(T)$ and hence $\bigwedge \alpha(S) \leq \bigvee \alpha(T)$. □

In [GJP] a slightly weaker parametrised compactness is used in considering Δ_1-extensions: a completion $(\mathbf{C}, \alpha)$ obtained from $(\mathcal{F}, \mathcal{I})$ will be called $(\mathcal{F}, \mathcal{I})$-*compact* or *parametrically compact* if it satisfies: $\bigwedge \alpha(F) \leq \bigvee \alpha(I)$ if, and only if, $F \cap I \neq \varnothing$. Clearly $(\mathbf{C}, \alpha)$ is also parametrically compact.

Lemma 6.1.11. *If $Y \in K$, then $G = \{a \in P : \alpha(a) \geq Y\} \in \mathcal{F}$ and $Y = \bigwedge \alpha(G)$. Similarly, if $Z \in O$, then $J = \{a \in P : \alpha(a) \leq Z\} \in \mathcal{I}$ and $Z = \bigvee \alpha(J)$.*

Proof. If $Y \in K$, then $Y = \bigwedge \alpha(G')$ for some $G' \in \mathcal{F}$. It is immediate that $G' \subseteq G$. Let $a \in G$, then $Y \leq \alpha(a)$, i.e., $\{F \in \mathcal{F} : G' \subseteq F\} \subseteq \{F \in \mathcal{F} : a \in F\}$. Then $G' \in \alpha(a)$ and $a \in G'$. Thus, $G \subseteq G'$. Then $G' = G$ and $Y = \bigwedge \alpha(G)$.

For $Z \in O$ there is some $J' \in \mathcal{I}$ such that $Z = \bigvee \alpha(J')$. Clearly $J' \subseteq J$. Now let $a \in J$. Then $\alpha(a) \leq Z = \bigvee \alpha(J')$, i.e., $\{F \in \mathcal{F} : a \in F\} \subseteq \{F \in \mathcal{F} : F \cap J' \neq \varnothing\}$. Therefore, if $a \in F$, then $F \cap J' \neq \varnothing$. In particular $[a) \cap J' \neq \varnothing$. Thus $a \in J'$ and $J \subseteq J'$. We conclude that $J' = J$ and $Z = \bigvee \alpha(J)$. □

Lemma 6.1.12. *The set KO of clopen elements of $\mathbf{C}$ is exactly the set $\alpha(P)$.*

Proof. Recall that $[a) \in \mathcal{F}$ and $(a] \in \mathcal{I}$ for every $a \in P$. If $a \in P$, then $\bigvee \alpha((a]) = \alpha(a) = \bigwedge \alpha([a))$ and $\alpha(a) \in KO$. Hence, $\alpha(P) \subseteq KO$.

Next let $X \in KO$. Then there exists $G \in \mathcal{F}$ and $J \in \mathcal{I}$ such that $\bigwedge \alpha(G) = X = \bigvee \alpha(J)$. By the internal compactness, Proposition 6.1.10, this is the case if, and only if, $G \cap J \neq \varnothing$. Let $a \in G \cap J$, then $\alpha(a) \leq \bigvee \alpha(J) = X = \bigwedge \alpha(G) \leq \alpha(a)$. Consequently, $G = [a)$, $J = (a]$ and $X = \alpha(a) \in \alpha(P)$. Therefore, $KO \subseteq \alpha(P)$. □

This result further motivates the choice to define the closed and open elements of $(\mathbf{C}, \alpha)$ in terms of $\mathcal{F}$ and $\mathcal{I}$, respectively. If we defined the closed

and open elements of $(\mathbf{C}, \alpha)$ in terms of arbitrary subsets of P instead, then Lemma 6.1.12 would not necessarily be true — see Example 6.2.2. In Section 6.3 we use Lemma 6.1.12 when we consider possible extensions of additional operations.

Proposition 6.1.13. *The completion $(\mathbf{C}, \alpha)$ is dense with respect to the sets of closed and open elements, i.e., every element of $\mathcal{C}$ is both the join of all the closed elements below it and the meet of all the open elements above it.*

Proof. We first show that $X \in \mathcal{C}$ is an up-set in $\mathcal{F}$: let $F \in X$, $G \in \mathcal{F}$ such that $F \subseteq G$ and $I \in X^{\triangleright}$. Then $I \cap F \subseteq I \cap G$, so $I \cap G \neq \varnothing$. Hence, $G \in X^{\triangleright\triangleleft} = X$.

It is immediate that $\bigvee\{Y \in K : Y \leq X\} \leq X$ since $Y \leq X$ for every $Y \in \{Y \in K : Y \leq X\}$. If $X = \varnothing$, then $X = \bot$ by Lemma 6.1.8 (ii) and $X \leq \bigvee\{Y \in K : Y \leq X\}$. Now suppose $X \neq \varnothing$ and let $G \in X$. Then $\bigwedge \alpha(G) \leq X$ since X is an up-set in $\mathcal{F}$. Furthermore, $G \in \bigwedge \alpha(G) \subseteq (\bigcup\{\bigwedge \alpha(F) : \bigwedge \alpha(F) \leq X\})^{\triangleright\triangleleft} = \bigvee\{Y \in K : Y \leq X\}$. Hence, $X \subseteq \bigvee\{Y \in K : Y \leq X\}$, i.e., $X \leq \bigvee\{Y \in K : Y \leq X\}$.

Since $Z \geq X$ for every $Z \in \{Z \in O : Z \geq X\}$ it follows that $\bigwedge\{Z \in O : Z \geq X\} \geq X$. If $I \in X^{\triangleright}$, then $X \leq \bigvee \alpha(I)$, i.e., $X \subseteq \{F \in \mathcal{F} : F \cap I \neq \varnothing\}$ by Lemma 6.1.2 (ii) and therefore $X^{\triangleright} \subseteq \{I \in \mathcal{I} : \bigvee \alpha(I) \geq X\}$. Then, $X = X^{\triangleright\triangleleft} \supseteq \{I \in \mathcal{I} : \bigvee \alpha(I) \geq X\}^{\triangleleft} = \{F \in \mathcal{F} : I \in \mathcal{I} \text{ such that } \bigvee \alpha(I) \geq X \text{ implies } F \cap I \neq \varnothing\} = \bigcap\{\{F \in \mathcal{F} : F \cap I \neq \varnothing\} : \bigvee \alpha(I) \geq X\} = \bigwedge\{Z \in O : Z \geq X\}$. □

Even though the terms 'closed' and 'open' elements were not used in [Tun74], the above result was also shown in [Tun74, Proposition 4 (b)].

In [GH01] the term 'canonical extension of a lattice $\mathbf{L}$' is used for a completion of $\mathbf{L}$ that is both dense and compact (in the sense of Definition 6.2.10). The uniqueness of the canonical extension of a lattice, up to isomorphism, is then proved [GH01, Proposition 2.7]. In [GJKO07, Theorem 6.2], the uniqueness, up to isomorphism, of $(\mathbf{C}, \alpha)$ is shown. In [DGP05, Theorem 2.5] the uniqueness, up to isomorphism, of $(\mathbf{C}, \alpha)$ is shown for a specific choice of $\mathcal{F}$ and $\mathcal{I}$. We note that the significance of this uniqueness is lessened by the fact that the notions of compactness and denseness depend entirely on the sets $\mathcal{F}$ and $\mathcal{I}$. That is, $(\mathbf{C}, \alpha)$ is the unique completion of $\mathbf{P}$ that is internally compact and dense with respect to $\mathcal{F}$ and $\mathcal{I}$. But, other (distinct) completions may be obtained, through the construction described in Section 6.1.1, for different choices of $\mathcal{F}$ and $\mathcal{I}$. These

completions will also be internally compact and dense, but now with respect to the new choices of $\mathcal{F}$ and $\mathcal{I}$.

The following result was first noted in [GH01]. See [GJKO07, Theorem 6.2] for a proof of the statement.

Proposition 6.1.14. *Let $S \subseteq P$ such that $\bigwedge S$ exists in $\mathbf{P}$ and all $F \in \mathcal{F}$ are closed under $\bigwedge S$, i.e., $S \subseteq F$ implies $\bigwedge S \in F$. Then $\bigwedge S$ is preserved by the extension, i.e., $\alpha(\bigwedge S) = \bigwedge \alpha(S)$.*

Similarly, let $T \subseteq P$ such that $\bigvee T$ exists in $\mathbf{P}$ and every $I \in \mathcal{I}$ is closed under $\bigvee T$, i.e., $T \subseteq I$ implies $\bigvee T \in I$. Then $\bigvee T$ is preserved by the extension, i.e., $\alpha(\bigvee T) = \bigvee \alpha(T)$.

This result motivates the alteration of the notions of lattice-consistent and completely consistent polarizations to that given in Definition 6.1.5.

If $\mathcal{F}$ is a family of non-empty up-sets of $\mathbf{P} = \langle P, \leq \rangle$, then $\mathcal{F}$ is a family of non-empty down-sets of $\mathbf{P}^\partial = \langle P, \geq \rangle$. Similarly, if $\mathcal{I}$ is a family of non-empty down-sets of $\mathbf{P} = \langle P, \leq \rangle$, then $\mathcal{I}$ is a family of non-empty up-sets of $\mathbf{P}^\partial = \langle P, \geq \rangle$. Furthermore, if $(\mathcal{F}, \mathcal{I})$ is a lattice-consistent polarization of $\mathbf{P}$, then $(\mathcal{I}, \mathcal{F})$ is a lattice-consistent polarization of $\mathbf{P}^\partial$. Then, the set of sets that are Galois closed with respect to $(\mathcal{I}, \mathcal{F})$ and $\geq$ is exactly the set $\mathcal{B} = \{\Lambda \in \mathcal{P}(\mathcal{I}) : \Lambda = \Lambda^{\triangleleft\triangleright}\}$ and $\mathbf{C}(\mathbf{P}^\partial) = \langle \mathcal{B}, \vee^{\mathbf{C}}, \wedge^{\mathbf{C}} \rangle$ is the completion of $\mathbf{P}^\partial$ obtained from the polarizaiton $(\mathcal{I}, \mathcal{F})$ (see Section 6.1.1), where $\mathcal{I}$ is used as the up-sets and $\mathcal{F}$ as the down-sets. As we showed earlier, $\mathbf{C}(\mathbf{P})$ is isomorphic to $\mathbf{B}(\mathbf{P})$. But, from the above, it follows that $\mathbf{B}(\mathbf{P}) = \mathbf{C}(\mathbf{P}^\partial)^\partial$. Hence, $\mathbf{C}(\mathbf{P})^\partial$ is isomorphic to $\mathbf{C}(\mathbf{P}^\partial)$, i.e., the dual of the completion of a poset is isomorphic to the completion of its dual. (However, one can easily find an example to show that it need not be the case that $\mathbf{C}(\mathbf{P})^\partial$ is equal to $\mathbf{C}(\mathbf{P}^\partial)$ for some sets $\mathcal{F}$ and $\mathcal{I}$.)

Finally we consider the product of completions. This will play a major role in Section 6.3 where we will investigate extensions of operations defined on posets. For example, an n-ary operation on a poset $\mathbf{P}$ is a map $f : P^n \to P$.

Lemma 6.1.15. *For $n \in \mathbb{N}$ and $i = 1, \ldots, n$, let each $\mathbf{P}_i$ be a poset with completion $(\mathbf{C}(\mathbf{P}_i), \alpha^{\mathbf{P}_i})$. Then $\beta : \prod_{i=1}^n P_i \to \prod_{i=1}^n \mathcal{C}(\mathbf{P}_i)$ defined by $\beta((a_1, \ldots, a_n)) = (\alpha^{\mathbf{P}_1}(a_1), \ldots, \alpha^{\mathbf{P}_n}(a_n))$ is an order-embedding of $\prod_{i=1}^n \mathbf{P}_i$ into $\prod_{i=1}^n \mathbf{C}(\mathbf{P}_i)$.*

Proof. Let $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \prod_{i=1}^n P_i$. Then by the definition of the coordinate-wise ordering defined on the product and since α is an order embedding we have $(a_1, \ldots, a_n) \leq (b_1 \ldots, b_n)$ if, and only if, $a_i \leq^{\mathbf{P}_i} b_i$ for $i = 1, \ldots, n$

if, and only if, $\alpha^{\mathbf{P}_i}(a_i) \leq \alpha^{\mathbf{P}_i}(b_i)$ for $i = 1, \ldots, n$. That is, if, and only if, $\beta((a_1, \ldots, a_n)) \leq \beta((b_1, \ldots, b_n))$. □

6.2 Some specific cases

The reader is referred to Chapter 4.1 for the definitions of pseudo, Doyle-pseudo, Frink and directed filters and ideals (see Definitions 4.1.1, 4.1.2, 4.1.3 and 4.1.4, respectively).

Let $* \in \{p, dp, f, d\}$ and set $\mathcal{F}_* = \mathcal{F}^* \backslash \{\varnothing\}$ and $\mathcal{I}_* = \mathcal{I}^* \backslash \{\varnothing\}$. Then each $\mathcal{F}_*$ is a family of non-empty up-sets of $\mathbf{P}$ that includes the principal up-sets and each $\mathcal{I}_*$ is a family of non-empty down-sets of $\mathbf{P}$ that includes the principal down-sets. Let $(\mathbf{C}_*, \alpha_*)$ be the completion obtained from $(\mathcal{F}_*, \mathcal{I}_*)$ as described in Section 6.1.1. Then results from Section 6.1.2 apply. Let $\top^*$ and $\bot_*$ denote the top and bottom elements of $\mathbf{C}_*$, respectively. In general the four extensions obtained in this way are distinct, as the following example illustrates.

Example 6.2.1. *Let $\mathbf{P}'$ be the poset depicted in Figure 6.1. (Note that $\mathbf{P}'$ was also considered in Example 4.2.7.) Then $\mathbf{C}_d$, $\mathbf{C}_f$ and $\mathbf{C}_{pd}$ are depicted in Figure 6.1 with $\alpha_*(P')$ shaded. The completion $\mathbf{C}_p$ contains 48 elements and is not depicted here due to its size. See Example A.2.2 in Appendix A.2 for more details.*

We note that $\mathbf{C}_d$ was referred to as the 'canonical extension' of $\mathbf{P}$ in [DGP05], while in [GJKO07] the term the 'canonical extension' of $\mathbf{P}$ was used for $\mathbf{C}_{dp}$.

Let $* \in \{p, dp, f, d\}$ and let K_*, O_* and KO_* denote the sets of closed, open and clopen elements of $(\mathbf{C}_*, \alpha_*)$, respectively. By Definition 6.1.9, $K_* = \{\bigwedge \alpha_*(F) : F \in \mathcal{F}_*\}$ and $O_* = \{\bigvee \alpha_*(I) : I \in \mathcal{I}_*\}$. Furthermore, by Propositions 6.1.10 and 6.1.13 $\mathbf{C}_*$ is internally compact and dense with respect to (K_*, O_*).

Let $\mathbf{L} = \langle L, \vee, \wedge \rangle$ be a bounded lattice. Recall that, for $* \in \{p, dp, f, d\}$, $\mathcal{F}^*(\mathbf{L}) = \mathcal{F}_*(\mathbf{L}) = \mathcal{F}(\mathbf{L})$ and $\mathcal{I}^*(\mathbf{L}) = \mathcal{I}_*(\mathbf{L}) = \mathcal{I}(\mathbf{L})$. We will denote the completion of $\mathbf{L}$ obtained from the polarization $(\mathcal{F}(\mathbf{L}), \mathcal{I}(\mathbf{L}))$ by $(\mathbf{C}(\mathbf{L}), \alpha^{\mathbf{L}})$ (or simply $(\mathbf{C}, \alpha)$, if $\mathbf{L}$ is understood). Then $(\mathbf{C}(\mathbf{L}), \alpha^{\mathbf{L}})$ is the canonical extension of $\mathbf{L}$ [GH01]. In the literature the sets of closed and open elements of $(\mathbf{C}(\mathbf{L}), \alpha^{\mathbf{L}})$ are often defined as the meet and join, respectively, of the image of arbitrary subsets of L (see for instance [Jón94] and [GHV05]). Furthermore, in [GH01] it was shown that the set of closed elements of the canonical extension of a lattice

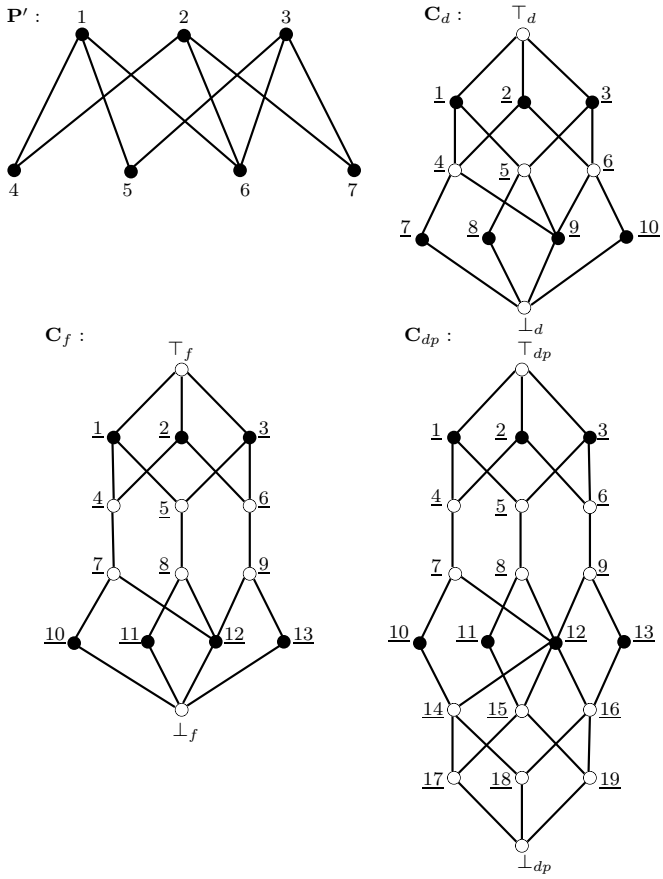


Fig. 6.1: The poset $\mathbf{P}'$ with the complete lattices $\mathbf{C}_d, \mathbf{C}_f$ and $\mathbf{C}_{dp}$.

$\mathbf{L}$ forms a sublattice of $\mathbf{C}(\mathbf{L})$ and is reverse isomorphic to the lattice of its filters $\mathbf{F}(\mathbf{L})$. Similarly, the set of open elements of $\mathbf{C}(\mathbf{L})$ forms a sublattice of $\mathbf{C}(\mathbf{L})$ and is isomorphic to the lattice of its ideals $\mathbf{I}(\mathbf{L})$.

In [GJKO07] it was suggested that closed and open elements of the completion $(\mathbf{C}, \alpha)$ of a poset, as described in Section 6.1.1, may also be defined in terms of arbitrary subsets of the poset, regardless of the choice of $\mathcal{F}$ and $\mathcal{I}$ used. However, if we choose to alter Definition 6.1.9 accordingly, then the set of clopen elements of $(\mathbf{C}, \alpha)$ need not be exactly $\alpha(P)$. We will need $KO(\mathbf{P}) = \alpha(P)$ for the extensions of maps considered in Section 6.3. If Definition 6.1.9 is left unchanged, the set of closed elements is order-isomorphic to $\mathbf{F}^\partial$, but in general does not form a sublattice of $\mathbf{C}$. Similarly, the set of open elements is order-

isomorphic to $\mathbf{I}$, but in general does not form a sublattice of $\mathbf{C}$. Consider the following example to see why.

Example 6.2.2. *Let $\mathbf{P}'$ be the poset depicted in Figure 6.2 with the complete lattices $\mathbf{C}_p$, $\mathbf{C}_{dp}$, $\mathbf{C}_f$ and $\mathbf{C}_d$ also depicted. Closed elements of $(\mathbf{C}_*, \alpha_*)$, $* \in \{p, dp, f, d\}$, are depicted by $\bullet$, open elements by $\circ$ and elements that are neither open nor closed by $\circledast$.*

If arbitrary subsets of P' were used in the definition of closed and open elements, then $\underline{3} \in \mathcal{C}_d$ would be clopen, but $\underline{3} \notin \alpha_d(P')$.

For $* \in \{p, dp, f, d\}$, the set of open elements of $(\mathbf{C}_*, \alpha_*)$ is isomorphic to the poset $\mathbf{I}_*$, but does not form a sublattice of $\mathbf{C}_*$: for all the completions $\underline{1}, \underline{2} \in O_*$ and $\underline{1} \wedge \underline{2} = \underline{3}$, but $\underline{3} \notin O_*$. Similarly, the closed elements of $(\mathbf{C}_*, \alpha_*)$ are isomorphic to the poset $\mathbf{F}_*^\partial$, but the join of closed elements need not be closed again: $\underline{4}, \underline{5} \in K_d$ but $\underline{4} \vee \underline{5} = \underline{3} \notin K_d$ in $\mathbf{C}_d$ and $\underline{5}, \underline{6} \in K_*$ but $\underline{5} \vee \underline{6} = \underline{4} \notin K_*$ in $\mathbf{C}_*$, $* \in \{p, dp, f\}$. Hence, the closed elements do not form a sublattice of $\mathbf{C}_*$.

For more details on the completion in this example the reader is referred to Example A.2.3 in Appendix A.2.

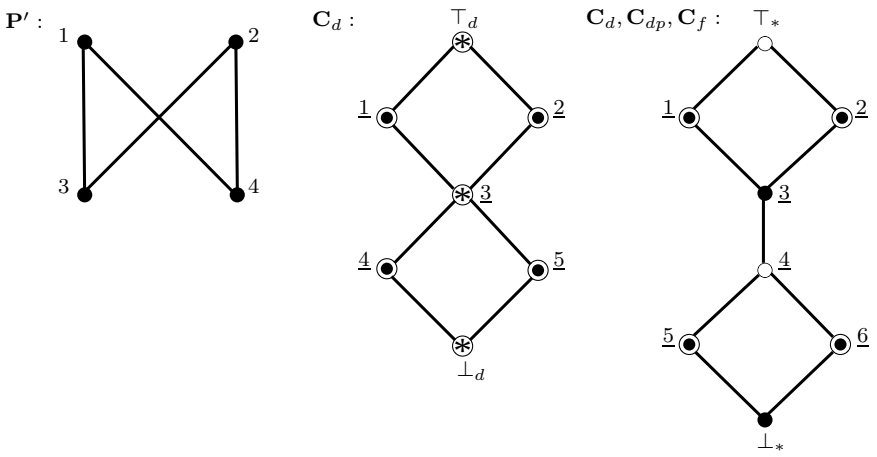


Fig. 6.2: Closed and open elements of $\mathbf{C}_*$ for $* \in \{p, pd, f, d\}$.

Lemma 6.2.3. *Let $\varepsilon_K : \mathcal{F} \to K$ be defined by $\varepsilon_K(F) = \bigwedge \alpha(F)$. Then ε_K is an order-isomorphism between $\mathbf{F}^\partial$ and $\langle K, \subseteq \rangle$. Similarly, let $\varepsilon_O : \mathcal{I} \to O$ be defined by $\varepsilon_O(I) = \bigvee \alpha(I)$. Then ε_O is an order-isomorphism between $\mathbf{I}$ and $\langle O, \subseteq \rangle$.*

Proof. We prove the claim for ε_K. The claim for ε_O follows similarly.

Observe that ε_K is clearly onto by the definition of closed elements. Now suppose $\varepsilon_K(F) = \bigwedge \alpha(F) = \bigwedge \alpha(G) = \varepsilon_K(G)$ for some $F, G \in \mathcal{F}$. Let $a \in F$. Then $F \cap (a] \neq \varnothing$ and by the internal compactness $\bigwedge \alpha(F) \leq \bigvee \alpha((a])$. Then $\bigwedge \alpha(G) \leq \bigvee \alpha((a])$ and, again by the internal compactness, $G \cap (a] \neq \varnothing$. Thus, $a \in G$ and $F \subseteq G$. Similarly we can show that $G \subseteq F$ by using the internal compactness and the fact that $\bigwedge \alpha(F) \leq \bigwedge \alpha(G)$. Then $F = G$ and ε_K is one-to-one.

The above also shows that $F \leq^{\mathbf{F}^\partial} G$ if $\varepsilon_K(F) \leq \varepsilon_K(G)$. Next suppose $F \leq^{\mathbf{F}^\partial} G$ for $F, G \in \mathcal{F}$. That is, $F \supseteq G$. Let $F' \in \mathcal{F}$ such that $F \subseteq F'$. Then $G \subseteq F'$ and $\varepsilon_K(F) = \bigwedge \alpha(F) = \{F' \in \mathcal{F} : F \subseteq F'\} \subseteq \{F' \in \mathcal{F} : G \subseteq F'\} = \bigwedge \alpha(G) = \varepsilon_K(G)$. □

We conclude that it is natural for K and O to depend on $\mathcal{F}$ and $\mathcal{I}$, respectively.

Let $* \in \{p, dp, f\}$. Recall from Lemmas 4.2.3 and 4.2.5 that an arbitrary $S \subseteq P$ generates both a $*$-filter and a $*$-ideal, denoted $[S\rangle_*$ and $\langle S]_*$. We have the following closures for K_* and O_*.

Lemma 6.2.4. *Let $* \in \{p, dp, f\}$. Then K_* is closed under meets, i.e., the meet of closed elements is again a closed element; and O_* is closed under joins, i.e., the join of open elements is again an open element.*

Proof. Let $F_i \in \mathcal{F}$ for $i \in \Psi$. Then,

$$\begin{aligned}
\bigwedge_{i\in\Psi} \left(\bigwedge \alpha(F_i)\right) &= \bigcap_{i\in\Psi} \{F \in \mathcal{F} : F_i \subseteq F\} \\
&= \{F \in \mathcal{F} : F_i \subseteq F \text{ for all } i \in \Psi\} \\
&= \left\{F \in \mathcal{F} : \left(\bigcup_{i\in\Psi} F_i\right) \subseteq F\right\} \\
&= \left\{F \in \mathcal{F} : \left[\bigcup_{i\in\Psi} F_i\right\rangle_* \subseteq F\right\} \\
&= \bigwedge \alpha\left(\left[\bigcup_{i\in\Psi} F_i\right\rangle_*\right) \in K.
\end{aligned}$$

Similarly we can show that for $I_j \in \mathcal{I}$, $j \in \Phi$, we have $\bigvee_{j\in\Phi} (\bigvee \alpha(I_j)) = \bigvee \alpha\left(\left\langle \bigcup_{j\in\Phi} I_j\right]_*\right) \in O$. □

Proposition 6.2.5. *Let $* \in \{p, dp, f\}$. If $S \subseteq P$, then $\bigvee \alpha_*(S) = \bigvee \alpha_*(\langle S]_*)$ and $\bigwedge \alpha_*(S) = \bigwedge \alpha_*([S\rangle_*)$.*

Proof. By Lemma 6.1.2 (ii) $\bigvee \alpha_*(S) = \{F \in \mathcal{F}_* : S \subseteq I \in \mathcal{I}_* \text{ implies } F \cap I \neq \varnothing\}$ and $\bigvee \alpha_*(\langle S]_*) = \{F \in \mathcal{F}_* : F \cap \langle S]_* \neq \varnothing\}$. Then, $F \in \bigvee \alpha_*(\langle S]_*)$ if, and only if, $F \cap \bigcap\{I \in \mathcal{I}_* : S \subseteq I\} \neq \varnothing$ if, and only if, $S \subseteq I \in \mathcal{I}_*$ implies $F \cap I \neq \varnothing$ if, and only if, $F \in \bigvee \alpha_*(S)$. Hence, $\bigvee \alpha_*(S) = \bigvee \alpha_*(\langle S]_*)$.

By Lemma 6.1.2 (i), $\bigwedge \alpha_*(S) = \{F \in \mathcal{F}_* : S \subseteq F\}$ and $\bigwedge \alpha_*([S\rangle_*) = \{F \in \mathcal{F}_* : [S\rangle_* \subseteq F\}$. Let $F \in \mathcal{F}_*$ such that $[S\rangle_* \subseteq F$. Then $S \subseteq [S\rangle_* \subseteq F$ and $F \in \bigwedge \alpha_*(S)$. On the other hand, since $[S\rangle_* = \bigcap\{F \in \mathcal{F}_* : S \subseteq F\}$ we have that $[S\rangle_* \subseteq F$ for each $F \in \bigwedge \alpha_*(S)$ and $\bigwedge \alpha_*([S\rangle_*) \subseteq \bigwedge \alpha_*(S)$. □

A consequence of the above is that the sets of closed and open elements of $(\mathbf{C}_*, \alpha_*)$, $* \in \{p, dp, f\}$, can indeed be defined in terms of arbitrary subsets of the poset.

Corollary 6.2.6. *Let $* \in \{p, dp, f\}$, then $K_* = \{\bigwedge \alpha_*(S) : S \subseteq P\}$ and $O_* = \{\bigvee \alpha_*(T) : T \subseteq P\}$.*

On the other hand, recall from Example 4.2.10 that an arbitrary subset of P need not generate a directed filter or a directed ideal. In fact, the notion of a directed filter or ideal being generated only makes sense if we begin with a directed set (see Lemma 4.2.11). Then, by Lemmas 4.2.11 and 6.1.2 (i) and (ii), we have the following for $\mathbf{C}_d$.

Proposition 6.2.7. *Let $D \subseteq P$ be down-directed and $U \subseteq P$ up-directed, then $\bigwedge \alpha_d(D) = \bigwedge \alpha_d([D))$ and $\bigvee \alpha_d(U) = \bigvee \alpha_d((U])$.*

The elements of K_d and O_d can therefore be described in terms of arbitrary directed sets.

Corollary 6.2.8. *Let $X, Y \in \mathcal{C}_d$. Then $X \in K_d$ if, and only if, $X = \bigwedge \alpha_d(D)$ for some down-directed $D \subseteq P$. Also, $Y \in O_d$ if, and only if, $Y = \bigvee \alpha_d(U)$ for some up-directed $U \subseteq P$.*

In [DGP05] 'compactness' of $(\mathbf{C}_d, \alpha_d)$ is defined as follows: $(\mathbf{C}_d, \alpha_d)$ *is* compact *provided that whenever $D \subseteq P$ is non-empty and down-directed, $U \subseteq P$ is non-empty and up-directed and $\bigwedge \alpha_d(D) \leq \bigvee \alpha_d(U)$, then there exist $a \in D$ and $b \in U$ with $a \leq^{\mathbf{P}} b$.* This notion of compactness is implied by the internal compactness of $(\mathbf{C}_d, \alpha_d)$.

Lemma 6.2.9. *If $D \subseteq P$ is non-empty and down-directed, $U \subseteq P$ is non-empty and up-directed and $\bigwedge \alpha_d(D) \leq \bigvee \alpha_d(U)$, then there exist $a \in D$ and $b \in U$ with $a \leq^{\mathbf{P}} b$.*

Proof. Suppose $D, U \subseteq P$ are both non-empty with D down-directed, U up-directed and $\bigwedge \alpha_d(D) \leq \bigvee \alpha_d(U)$. Then, $F \cap I \neq \varnothing$ for all $F \in \mathcal{F}_d$ with $D \subseteq F$ and all $I \in \mathcal{I}_d$ with $U \subseteq I$ by the internal compactness. By Lemma 4.2.11 it then follows that $[D) \in \mathcal{F}_d$, $(U] \in \mathcal{I}_d$ and $[D) \cap (U] \neq \varnothing$. Let $c \in [D) \cap (U]$, then $a = c \leq^{\mathbf{P}} c = b$. □

In [GH01] the following stronger notion of compactness is defined.

Definition 6.2.10. *A completion $(\mathbf{C}, \alpha)$ is called* compact *if for any $\mathcal{Y} \subseteq K$ and any $\mathcal{Z} \subseteq O$ it satisfies:*

$$\bigwedge \mathcal{Y} \leq \bigvee \mathcal{Z} \text{ if, and only if,}$$
$$\text{there exist } \mathcal{Y}_0 \subseteq^{fin} \mathcal{Y} \text{ and } \mathcal{Z}_0 \subseteq^{fin} \mathcal{Z} \text{ such that } \bigwedge \mathcal{Y}_0 \leq \bigvee \mathcal{Z}_0.$$

In general, it is not the case that $(\mathbf{C}, \alpha)$ is compact. In [GH01] it was observed that this stronger notion of compactness is not a property of the complete lattice $\mathbf{C}$, but rather of $(\mathbf{C}, \alpha)$ since the sets K and O depend on α, $\mathcal{F}$ and $\mathcal{I}$. Hence, this form of compactness is also indirectly parametrised by $\mathcal{F}$ and $\mathcal{I}$. That is to say, the sets $\mathcal{Y}$ and $\mathcal{Z}$ of closed and open elements cannot be replaced by arbitrary subsets of $\mathcal{C}$.

For certain choices of $\mathcal{F}$ and $\mathcal{I}$ the completion $(\mathbf{C}, \alpha)$ will be compact. In particular, we will show that $(\mathbf{C}_*, \alpha_*)$ is compact for $* \in \{p, dp, f\}$.

We will need the following to prove compactness.

Lemma 6.2.11. *Let $* \in \{p, dp, f\}$ and $S \subseteq P$. If $a \in [S\rangle_*$, then there exists $M \subseteq^{fin} S$ such that $a \in [M\rangle_*$.*

Proof. Recall from Lemma 4.2.3 that $[S\rangle_{dp} = \bigcup_{i \in \mathbb{N}} S_i$ where $S_0 = S$ and $S_{i+1} = [\{\bigwedge M : \varnothing \neq M \subseteq^{fin} S \text{ and } \bigwedge M \text{ exists}\})$. If $a \in S_0 = S$, then $\{a\} \subseteq^{fin} S$ and $a \in [a)$. For $1 \leq j \in \mathbb{N}$, suppose that if $b \in S_j$, then there exists $M \subseteq^{fin} S$ such that $b \in [M\rangle_{dp}$. Let $a \in S_{j+1}$. Then $a \geq \bigwedge N$ for some $N \subseteq^{fin} S_j$ such that $\bigwedge N$ exists. Suppose $N = \{b_1, \ldots, b_n\}$. Then, by the inductive hypothesis, there exists $M_i \subseteq^{fin} S$ such that $b_i \in [M_i\rangle_{dp}$ for $i = 1, \ldots, n$. Now let $M = \bigcup_{i=1}^{n} M_i \subseteq^{fin} S$. Then $N \subseteq^{fin} [M\rangle_{dp}$ and therefore $a \in [M\rangle_{dp}$.

The proof of the statement for pseudo filters is similar.

Next recall from Lemma 4.2.5 that $[S\rangle_f = \bigcup\{M^{\ell u} : M \subseteq^{fin} S\}$. Let $a \in [S\rangle_f$. Then there exists $M \subseteq^{fin} S$ such that $a \in M^{\ell u}$ and therefore $a \in [M\rangle_f$. □

The dual statements for generated $*$-ideals also hold.

Proposition 6.2.12. *Let $* \in \{p, dp, f\}$, then $(\mathbf{C}_*, \alpha_*)$ is compact.*

Proof. Let $\mathcal{Y} \subseteq K_*$ and $\mathcal{Z} \subseteq O_*$.

The backward implication is immediate. If there exist $\mathcal{Y}_0 \subseteq^{fin} \mathcal{Y}$ and $\mathcal{Z}_0 \subseteq^{fin} \mathcal{Z}$ such that $\bigwedge \mathcal{Y}_0 \leq \bigvee \mathcal{Z}_0$, then $\bigwedge \mathcal{Y} \leq \bigwedge \mathcal{Y}_0 \leq \bigvee \mathcal{Z}_0 \leq \bigvee \mathcal{Z}$.

To prove the implication in the other direction, suppose that $\bigwedge \mathcal{Y} \leq \bigvee \mathcal{Z}$. Furthermore, let $\mathcal{G} \subseteq \mathcal{F}_*$ such that $\mathcal{Y} = \{\bigwedge \alpha_*(G) : G \in \mathcal{G}\}$ and $\mathcal{J} \subseteq \mathcal{I}_*$ such that $\mathcal{Z} = \{\bigvee \alpha_*(J) : J \in \mathcal{J}\}$. Then, by Lemma 6.2.4,

$$\bigwedge \mathcal{Y} = \bigwedge \Big\{ \bigwedge \alpha_*(G) : G \in \mathcal{G} \Big\} = \bigwedge \alpha_* \left(\Big[\bigcup \mathcal{G} \Big\rangle_* \right).$$

Similarly,

$$\bigvee \mathcal{Z} = \bigvee \Big\{ \bigvee \alpha_*(J) : J \in \mathcal{J} \Big\} = \bigvee \alpha_* \left(\Big\langle \bigcup \mathcal{J} \Big]_* \right).$$

By the internal compactness $[\bigcup \mathcal{G}\rangle_* \cap \langle \bigcup \mathcal{J}]_* \neq \varnothing$. Let $c \in [\bigcup \mathcal{G}\rangle_* \cap \langle \bigcup \mathcal{J}]_*$, then by Lemma 6.2.11 there exist sets $M \subseteq^{fin} \bigcup \mathcal{G}$ and $N \subseteq^{fin} \bigcup \mathcal{J}$ such that $c \in [M\rangle_*$ and $c \in \langle N]_*$, i.e., $c \in [M\rangle_* \cap \langle N]_*$ and $\bigwedge \alpha_*([M\rangle_*) \leq \alpha_*(c) \leq \bigvee \alpha_*(\langle N]_*)$.

For each $a \in M$, let $G_a \in \mathcal{G}$ such that $a \in G_a$. Then, $\bigwedge \alpha_*(G_a) \leq \alpha_*(a)$ for each $a \in M$ and

$$\bigwedge \Big\{ \bigwedge \alpha_*(G_a) : a \in M \Big\} \leq \bigwedge \alpha_*(M) = \bigwedge \alpha_*([M\rangle_*).$$

Similarly, for each $b \in N$, let $J_b \in \mathcal{J}$ such that $b \in J_b$. Then, $\bigvee \alpha_*(J_b) \geq \alpha_*(b)$ for each $b \in N$ and

$$\bigvee \alpha_*(\langle N]_*) = \bigvee \alpha_*(N) \leq \bigvee \Big\{ \bigvee \alpha_*(J_b) : b \in N \Big\}.$$

Let $\mathcal{Y}_0 = \{\bigwedge \alpha_*(G_a) : a \in M\}$ and $\mathcal{Z}_0 = \{\bigvee e_*(J_b) : b \in N\}$. Then $\bigwedge \mathcal{Y}_0 \leq \bigvee \mathcal{Z}_0$. □

In [GJKO07, Lemma 6.3] it is stated that $(\mathbf{C}_{dp}, \alpha_{dp})$ is compact and in the paragraph preceding Lemma 6.3 it is claimed that $(\mathbf{C}_{dp}, \alpha_{dp})$ is the only completion for which internal compactness implies compactness. From the above

it can be seen that $(\mathbf{C}_{dp}, \alpha_{dp})$ is not the only compact completion of $\mathbf{P}$, nor is it the only one for which internal compactness implies compactness.[1]

In [GJP] it is shown that certain Δ_1-completions of a poset are compact by giving a number of sufficient conditions for compactness. If $* \in \{p, dp, f\}$, then $(\mathbf{C}_*, \alpha_*)$ satisfies those sufficient conditions and compactness of $(\mathbf{C}_*, \alpha_*)$ can also be established via their result.

Let $* \in \{dp, f, d\}$. Recall from Remark 4.1.5 that the members of $\mathcal{F}^*$ are closed under existing finite meets, while the members of $\mathcal{I}^*$ are closed under existing finite joins. The polarization $(\mathcal{F}_*, \mathcal{I}_*)$ is therefore lattice-consistent as per Definition 6.1.5 and rich enough in the sense of [GJKO07] (though always excluding the empty set). The members of $\mathcal{F}_p$ and $\mathcal{I}_p$ are closed under existing binary meets and joins, respectively. Combining this with Proposition 6.1.14 gives the following.

Corollary 6.2.13. *If $* \in \{dp, f, d\}$, then $(\mathbf{C}_*, \alpha_*)$ preserves all existing finite meets and joins, while $(\mathbf{C}_p, \alpha_p)$ preserves all existing binary meets and joins.*

Finally, for $n \in \mathbb{N}$ and $i = 1, \ldots, n$, let each $\mathbf{P}_i$ be a poset and let $* \in \{p, dp, f, d\}$. Then $\beta_* : \prod_{i=1}^n P_i \to \prod_{i=1}^n \mathcal{C}_*(\mathbf{P}_i)$, defined by $\beta_*((a_1, \ldots, a_n)) = (\alpha_*^{\mathbf{P}_1}(a_1), \ldots, \alpha_*^{\mathbf{P}_n}(a_n))$, is the order-embedding of $\prod_{i=1}^n \mathbf{P}_i$ into $\prod_{i=1}^n \mathbf{C}_*(\mathbf{P}_i)$.

In [DGP05, Theorem 2.8] it is claimed that the completion commutes with products, i.e., $\mathbf{C}_d(\prod_{i=1}^n \mathbf{P}_i) = \prod_{i=1}^n \mathbf{C}_d(\mathbf{P}_i)$, up to isomorphism. Similarly, it is claimed in [GJKO07, Corollary 6.9] that $\mathbf{C}_{dp}(\prod_{i=1}^n \mathbf{P}_i)$ is isomorphic to $\prod_{i=1}^n \mathbf{C}_{dp}(\mathbf{P}_i)$. However, the following example serves as a counterexample to both these claims. In general it is not necessarily the case that $\mathbf{C}(\prod_{i=1}^n \mathbf{P}_i)$ is isomorphic to $\prod_{i=1}^n \mathbf{C}(\mathbf{P}_i)$.

Example 6.2.14. *Let $\mathbf{P}'$ be the 2-element anti-chain, then $\mathbf{P}' \times \mathbf{P}'$ is the 4-element anti-chain. For $* \in \{p, pd, f, d\}$, the completion $\mathbf{C}_*(\mathbf{P}')$ has 4 elements, as depicted in Figure 6.3, and hence $\mathbf{C}_*(\mathbf{P}') \times \mathbf{C}_*(\mathbf{P}')$ has 16 elements. On the other hand, the completion $\mathbf{C}_*(\mathbf{P}' \times \mathbf{P}')$, for $* \in \{f, d\}$, has only 6 elements.*

[1] We note that [GJKO07, Lemma 6.3] states the equivalence of the compactness and the internal compactness of $(\mathbf{C}_{dp}, \alpha_{dp})$ with a third statement. However, in the poset setting, this third statement need not be equivalent to the compactness nor the internal compactness of $(\mathbf{C}_{dp}, \alpha_{dp})$ — it relies on the existence of meets and joins that in fact need not exist in the poset. On the other hand, in [GH01, Lemma 2.4] the claim was proved for the canonical extension of a lattice.

Moreover, if $* \in \{p, dp\}$ *then* $\mathbf{C}_*(\mathbf{P}' \times \mathbf{P}')$ *has far more than* 16 *elements. See Example A.2.4 in Appendix A.2 for more details.*

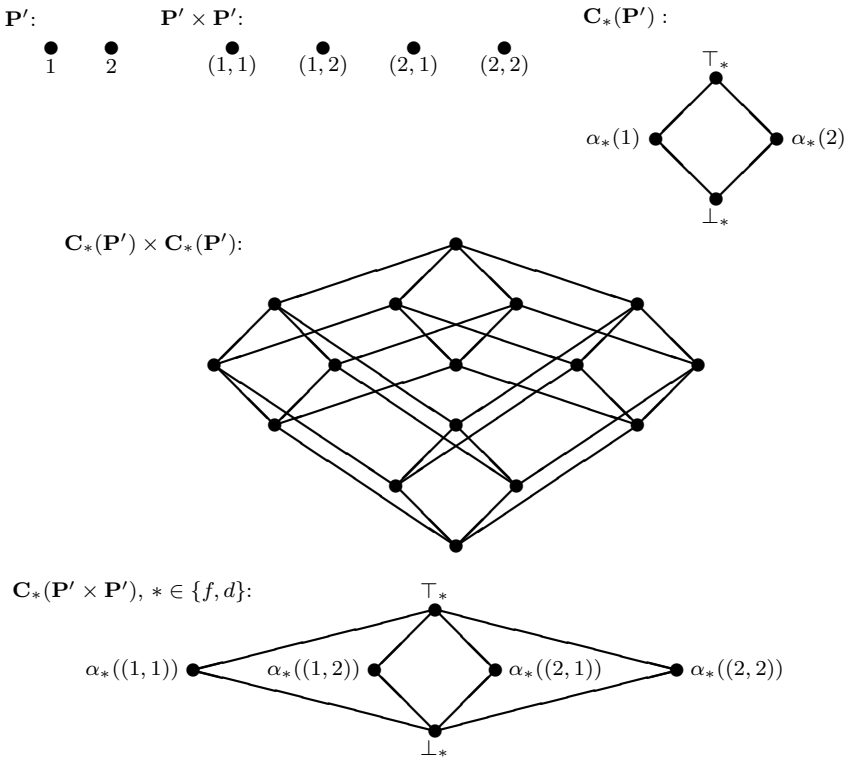


Fig. 6.3: The construction generally does not commute with products.

Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be bounded posets. We note that in [GJP] it is shown that if $\mathcal{F}_*(\mathbf{P}_1 \times \mathbf{P}_2) = \mathcal{F}_*(\mathbf{P}_1) \times \mathcal{F}_*(\mathbf{P}_2)$, then $\mathbf{C}_*(\mathbf{P}_1 \times \mathbf{P}_2)$ is isomorphic to $\mathbf{C}_*(\mathbf{P}_1) \times \mathbf{C}_*(\mathbf{P}_2)$. However, the boundedness is crucial for the implication to be true. In [GJP] the authors also provide the example above to show that even though $\mathcal{F}_d(\mathbf{P} \times \mathbf{P}) = \mathcal{F}_d(\mathbf{P}) \times \mathcal{F}_d(\mathbf{P})$, the construction does not commute with products. We note that in the case of bounded lattices $\mathbf{C}(\prod_{i=1}^n \mathbf{L}_i)$ is isomorphic to $\prod_{i=1}^n \mathbf{C}(\mathbf{L}_i)$, [GH01]. This lack of commutativity of the construction of the completion and products of posets will have an impact on the extension of n-ary operations.

For the remainder of this section we examine the choice we made at the start of this chapter to exclude the empty set from the sets that form polarizations.

We will show that nothing is gained by allowing the empty set.

In [GJKO07] the empty set is included in a rich enough family of up-sets (respectively, down-sets), $\mathcal{F}$ (respectively, $\mathcal{I}$), if $\mathbf{P}$ does not have a top (respectively, bottom) element. However, following [Tun74] we require that the members of both $\mathcal{F}$ and $\mathcal{I}$ be non-empty. By definition $\varnothing \notin \mathcal{F}^d$. However, recall if $\mathbf{P}$ does not have a top element, then $\varnothing \in \mathcal{F}^*$ for $* \in \{p, dp, f\}$; and if $\mathbf{P}$ does not have a bottom element, then $\varnothing \in \mathcal{I}^*$ for $* \in \{p, dp, f\}$. One may now wonder why we choose to exclude the empty set and what would happen if we included it. We will show that the complete lattice obtained from $(\mathcal{F}^*, \mathcal{I}^*)$ has up to two more elements than the complete lattice obtained from $(\mathcal{F}_*, \mathcal{I}_*)$. There will be a new top element (above the top element of $\mathbf{C}_*$) if $\mathbf{P}$ does not have a top element and a new bottom element (below the bottom element of $\mathbf{C}_*$) if $\mathbf{P}$ does not have a bottom element.

We introduce the following notation. Let $* \in \{p, dp, f\}$, $R^* \subseteq \mathcal{F}^* \times \mathcal{I}^*$ be defined by $(F, I) \in R^*$ if and only if $F \cap I \neq \varnothing$ and let the polarities of R^* be denoted by

$$^{\blacktriangleright} : \mathcal{P}(\mathcal{F}^*) \rightleftarrows \mathcal{P}(\mathcal{I}^*) : ^{\blacktriangleleft}$$

where, for $X \in \mathcal{P}(\mathcal{F}^*)$ and $\Lambda \in \mathcal{P}(\mathcal{I}^*)$ we have

$$X^{\blacktriangleright} = \{I \in \mathcal{I}^* : F \in X \text{ implies } I \cap F \neq \varnothing\}$$

$$\Lambda^{\blacktriangleleft} = \{F \in \mathcal{F}^* : I \in \Lambda \text{ implies } F \cap I \neq \varnothing\}.$$

Let $\mathcal{C}^* = \{X \in \mathcal{P}(\mathcal{F}^*) : X = X^{\blacktriangleright\blacktriangleleft}\}$ and $\mathbf{C}^* = \langle \mathcal{C}^*, \vee, \wedge \rangle$ where meet is intersection, join the Galois closure of the union and $\subseteq$ the associated lattice ordering. Define the map $\alpha^* : P \to \mathcal{C}^*$ by $\alpha^*(a) = \{F \in \mathcal{F}^* : a \in F\}$ for all $a \in P$. Then the map α^* is an order-embedding of $\mathbf{P}$ into $\mathbf{C}^*$.

Lemma 6.2.15. *Let $* \in \{p, dp, f\}$ and $X \in \mathcal{P}(\mathcal{F}_*)$. Then, $X \in \mathcal{C}_*$, i.e., $X = X^{\triangleright\triangleleft}$ if, and only if, $X = X^{\blacktriangleright\blacktriangleleft}$. In particular, $\{P\}^{\blacktriangleright\blacktriangleleft} = \{P\}^{\triangleright\triangleleft} = \{P\}$ and $\mathcal{F}_*^{\blacktriangleright\blacktriangleleft} = \mathcal{F}_*^{\triangleright\triangleleft} = \mathcal{F}_*$.*

Next let $X \in \mathcal{P}(\mathcal{F}^*)$. If $X = X^{\blacktriangleright\blacktriangleleft}$ such that $\varnothing \in X$, then P does not have a top element and $X = \mathcal{F}^*$. If $\varnothing = \varnothing^{\blacktriangleright\blacktriangleleft}$, then P does not have a bottom element and $\varnothing \in \mathcal{I}^*$.

Proof. Let $X \in \mathcal{P}(\mathcal{F}_*)$ such that $X \neq \varnothing$. Notice that $\varnothing \notin X^{\blacktriangleright}$ since $X \neq \varnothing$ and $\varnothing \cap F = \varnothing$ for all $F \in X$. Then $I \in X^{\triangleright}$ if, and only if, $I \cap F \neq \varnothing$ for all $F \in X$ if, and only if, $I \in X^{\blacktriangleright}$. Thus, $X^{\triangleright} = X^{\blacktriangleright}$. Furthermore, $X^{\triangleright} = X^{\blacktriangleright} \neq \varnothing$

since $P \in \mathcal{I}_*$ such that $P \cap F \neq \varnothing$ for all $F \in X$. Now $\varnothing \notin X^{\blacktriangleright\blacktriangleleft}$ since $X^{\blacktriangleright} \neq \varnothing$ and $\varnothing \cap I = \varnothing$ for all $I \in X^{\blacktriangleright}$. Then, $F \in X^{\triangleright\triangleleft}$ if, and only if, $F \cap I \neq \varnothing$ for all $I \in X^{\triangleright} = X^{\blacktriangleright}$ if, and only if, $F \in X^{\blacktriangleright\blacktriangleleft}$. Therefore, $X^{\triangleright\triangleleft} = X^{\blacktriangleright\blacktriangleleft}$.

By Lemma 6.1.8 (ii) we have that $\bot_* = \{P\}$, since $P \in \mathcal{F}_*$. Therefore, if $X \in \mathcal{C}_*$, then $X \neq \varnothing$. Thus we may conclude that $X \in \mathcal{C}_*$ if, and only if, $X = X^{\blacktriangleright\blacktriangleleft}$.

Now let $X \in \mathcal{P}(\mathcal{F}^*)$ such that $X = X^{\blacktriangleright\blacktriangleleft}$ and $\varnothing \in X$. Then $\varnothing \in \mathcal{F}^*$ and it follows from the definition of $*$-filters that P does not have a top element. Furthermore, $X^{\blacktriangleright} = \varnothing$ since no ideal can have a non-empty intersection with $\varnothing$. Then, $X^{\blacktriangleright\blacktriangleleft} = \varnothing^{\blacktriangleleft} = \mathcal{F}^*$.

Lastly suppose $\varnothing = \varnothing^{\blacktriangleright\blacktriangleleft}$. But $\varnothing^{\blacktriangleright} = \mathcal{I}^*$ which implies that $(\mathcal{I}^*)^{\blacktriangleleft} = \varnothing$. This is only the case if $P \cap I = \varnothing$ for some $I \in \mathcal{I}^*$. That can only be true for $I = \varnothing$. Then $\varnothing \in \mathcal{I}^*$ and P has no bottom element by the definition of $*$-ideals. □

The following example now illustrates the difference between $\mathbf{C}_*$ and $\mathbf{C}^*$.

Example 6.2.16. *Let* $\mathbf{P}'$ *be the* 2*-element anti-chain depicted in Figure 6.4. Then* $\mathcal{F}^* = \mathcal{I}^* = \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$ *and* $\mathcal{F}_* = \mathcal{I}_* = \{\{1\}, \{2\}, \{1,2\}\}$ *for* $* \in \{p, dp, f\}$*. Then* $\mathbf{C}_*$ *is the completion obtained from* $(\mathcal{F}_*, \mathcal{I}_*)$*, while* $\mathbf{C}^*$ *is the completion obtained from* $(\mathcal{F}^*, \mathcal{I}^*)$*.*

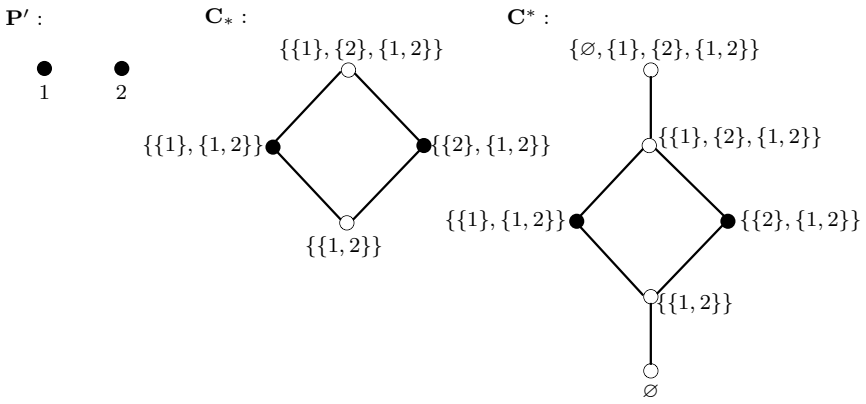


Fig. 6.4: Using $\mathcal{F}^*$ and $\mathcal{I}^*$ in the construction.

It should be clear that nothing is really gained by using $\mathcal{F}^*$ and $\mathcal{I}^*$ instead

of $\mathcal{F}_*$ and $\mathcal{I}_*$. This further justifies our choice to exclude the empty set from families of up-sets and down-sets used to form polarizations.

6.3 Extensions of maps

6.3.1 Unary maps

Throughout this section let the posets $\mathbf{P} = \langle P, \leq^{\mathbf{P}} \rangle$ and $\mathbf{Q} = \langle Q, \leq^{\mathbf{Q}} \rangle$ be fixed. Let $f : P \to Q$ be an arbitrary map between $\mathbf{P}$ and $\mathbf{Q}$. We wish to extend f to a map f_* from $\mathbf{C}_*(\mathbf{P})$ to $\mathbf{C}_*(\mathbf{Q})$. Following [GJ00] and [GH01], we have the following two ways of naturally extending a unary map f, since $(\mathbf{C}_*(\mathbf{P}), \alpha_*^{\mathbf{P}})$ is dense.

Definition 6.3.1. *For $f : P \to Q$, define $f_*^\sigma, f_*^\pi : \mathcal{C}_*(\mathbf{P}) \to \mathcal{C}_*(\mathbf{Q})$ by*

$$f_*^\sigma(X) = \bigvee \Big\{ \bigwedge \{\alpha_*^{\mathbf{Q}}(f(a)) : a \in P, Y \leq \alpha_*^{\mathbf{P}}(a) \leq Z\} : Y \in K_*(\mathbf{P}), Z \in O_*(\mathbf{P}), Y \leq X \leq Z \Big\}$$

$$f_*^\pi(X) = \bigwedge \Big\{ \bigvee \{\alpha_*^{\mathbf{Q}}(f(a)) : a \in P, Y \leq \alpha_*^{\mathbf{P}}(a) \leq Z\} : Y \in K_*(\mathbf{P}), Z \in O_*(\mathbf{P}), Y \leq X \leq Z \Big\}.$$

The results in this section are closely related to the results obtained in [GH01] for bounded lattices. However, some statements that are true for bounded lattices do not hold in the more general poset setting.

Lemma 6.3.2. *Let $f : P \to Q$. Then f_*^σ and f_*^π both extend f, i.e., $f_*^\sigma(\alpha_*^{\mathbf{P}}(b)) = \alpha_*^{\mathbf{Q}}(f(b)) = f_*^\pi(\alpha_*^{\mathbf{P}}(b))$, for $b \in P$.*

The proof is straightforward and relies on Lemma 6.1.12, i.e., the fact that $KO_*(\mathbf{P}) = \alpha_*^{\mathbf{P}}(P)$.

In the case of bounded lattices $f^\sigma \leq f^\pi$ under the point-wise order [GH01, Lemma 4.2]. However, as illustrated by the following example, this need not be the case in the poset setting if $f : P \to Q$ is not order-preserving. This contradicts [GJKO07, Lemma 6.7].

Example 6.3.3. *Let $\mathbf{P}'$ be the poset depicted in Figure 6.5, and let $f : P' \to P'$ be defined by $f(1) = f(3) = 3$ and $f(2) = f(4) = 4$. Then, $f_d^\sigma(X) = \bigvee\{\underline{3}, \underline{4}, \bot_d\} = X$, while $f_d^\pi(X) = \bigwedge\{X, \underline{3}, \underline{4}\} = \bot_d$. Furthermore, $f_*^\sigma(X_1) =$*

$\bigvee\{\underline{3}, \underline{4}, \bot_*\} = X_2$, *while* $f^\pi_*(X_1) = \bigwedge\{X_2, \underline{3}, \underline{4}\} = \bot_*$, *for* $* \in \{p, dp, f\}$. *The reader is referred to Example A.2.3 in Appendix A.2 for details on the completions.*

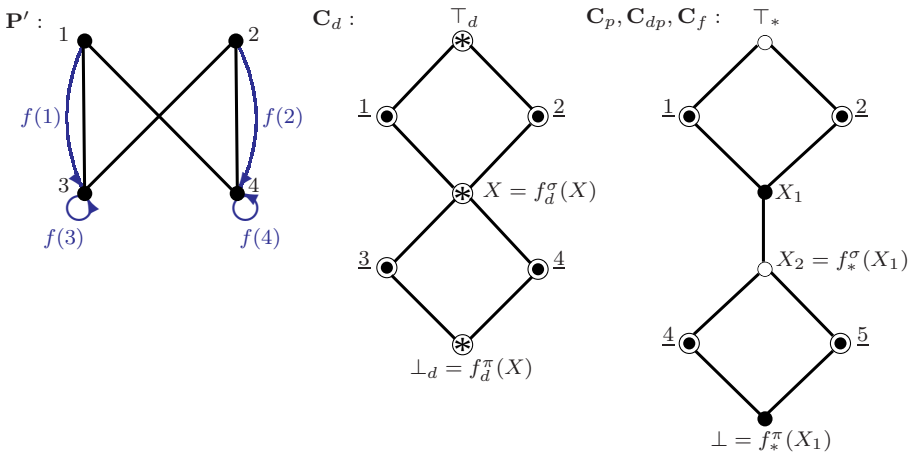


Fig. 6.5: f^σ_* need not be less than f^π_*.

If $\mathbf{P}$ does not have a top element, then it may be the case that $P \notin \mathcal{I}_d$. If $P \notin \mathcal{I}_d$, then it follows from Lemmas 6.1.8 (iii) and 6.1.11 that $\top_d \notin O_d$. Then $f^\sigma_d(\top_d) = \bigvee \varnothing = \bot_d$ and $f^\pi_d(\top_d) = \bigwedge \varnothing = \top_d$, regardless of the definition of f. If $\mathbf{P}$ does not have a bottom element, then we may have that $P \notin \mathcal{F}_d$. Again by Lemmas 6.1.8 (iii) and 6.1.11, $P \notin \mathcal{F}_d$ implies that $\bot_d \notin K_d$ and consequently $f^\sigma_d(\bot_d) = \bigvee \varnothing = \bot_d$, while $f^\pi_d(\bot_d) = \bigwedge \varnothing = \top_d$, regardless of f's definition. Therefore, unless $\mathbf{P}$ has a top and a bottom element, f^σ_d and f^π_d need not be order-preserving when f is. Since we would prefer an extension of an order-preserving map to be order-preserving, we redefine f^σ_d and f^π_d when f is order-preserving. The definition below was used in [DGP05] for the extension of order-preserving maps.

Definition 6.3.4. *Let* $f : P \to Q$ *be order-preserving. Then define* $f^\sigma_d, f^\pi_d : \mathcal{C}_d(\mathbf{P}) \to \mathcal{C}_d(\mathbf{Q})$ *by*

$$f^\sigma_d(X) = \bigvee \left\{ \bigwedge \{\alpha^{\mathbf{Q}}_d(f(a)) : a \in P, Y \leq \alpha^{\mathbf{P}}_d(a)\} : X \geq Y \in K_d(\mathbf{P}) \right\}$$

$$f^\pi_d(X) = \bigwedge \left\{ \bigvee \{\alpha^{\mathbf{Q}}_d(f(a)) : a \in P, \alpha^{\mathbf{P}}_d(a) \leq Z\} : X \leq Z \in O_d(\mathbf{P}) \right\}.$$

In [DGP05] it was shown that both f_d^σ and f_*^π are extensions of f and are order-preserving when f is.

For $* \in \{p, dp, f\}$ we need not redefine f_*^σ and f_*^π, since we have the following simplifications.

Lemma 6.3.5. *If $* \in \{p, dp, f\}$ and $f : P \to Q$ is order-preserving, then*

$$f_*^\sigma(X) = \bigvee \left\{ \bigwedge \{\alpha_*^{\mathbf{Q}}(f(a)) : a \in P, Y \leq \alpha_*^{\mathbf{P}}(a)\} : X \geq Y \in K_*(\mathbf{P}) \right\}$$

$$f_*^\pi(X) = \bigwedge \left\{ \bigvee \{\alpha_*^{\mathbf{Q}}(f(a)) : a \in P, \alpha_*^{\mathbf{P}}(a) \leq Z\} : X \leq Z \in O_*(\mathbf{P}) \right\}.$$

These simplifications are straightforward and the proofs are omitted.

We will need the following to prove Lemma 6.3.7 (iv) for f_d^σ and f_d^π. It was noted in [DGP05].

Lemma 6.3.6. *Let $f : P \to Q$ be order-preserving, $F \in \mathcal{F}_d$ and $I \in \mathcal{I}_d$. Then, $f(F)$ is down-directed and $f(I)$ is up-directed.*

Lemma 6.3.7. *Let $* \in \{p, dp, f, d\}$. Let $f : P \to Q$ be order-preserving. Then:*

(i) f_*^σ *and* f_*^π *are order-preserving.*

(ii) $f_*^\sigma \leq f_*^\pi$ *under the point-wise ordering.*

(iii) We have the following simplifications

(a) $f_*^\sigma(Y) = \bigwedge \{\alpha_*^{\mathbf{Q}}(f(a)) : a \in P, Y \leq \alpha_*^{\mathbf{P}}(a)\}$ *for all* $Y \in K_*(\mathbf{P})$.

(b) $f_*^\sigma(X) = \bigvee \{f_*^\sigma(Y) : X \geq Y \in K_*(\mathbf{P})\}$ *for all* $X \in \mathcal{C}_*(\mathbf{P})$.

(c) $f_*^\pi(Z) = \bigvee \{\alpha_*^{\mathbf{Q}}(f(a)) : a \in P, Z \geq \alpha_*^{\mathbf{P}}(a)\}$ *for all* $Z \in O_*(\mathbf{P})$.

(d) $f_*^\pi(X) = \bigwedge \{f_*^\sigma(Z) : X \leq Z \in O_*(\mathbf{P})\}$ *for all* $X \in \mathcal{C}_*(\mathbf{P})$.

(iv) $f_*^\sigma = f_*^\pi$ *on* $K_*(\mathbf{P}) \cup O_*(\mathbf{P})$. *Moreover,* $f_*^\sigma(K_*(\mathbf{P})) \subseteq K_*(\mathbf{Q})$, $f_*^\pi(K_*(\mathbf{P})) \subseteq K_*(\mathbf{Q})$, $f_*^\sigma(O_*(\mathbf{P})) \subseteq O_*(\mathbf{Q})$ *and* $f_*^\pi(O_*(\mathbf{P})) \subseteq O_*(\mathbf{Q})$.

Proof. The proofs of parts (i) to (iii) are similar to those of the analogous claims for bounded lattices [GH01]. We prove (iv) since we need to consider the case where $*$ is d separately.

(iv) Let $Y \in K_*(\mathbf{P})$. Then $f_*^\sigma(Y) \leq f_*^\pi(Y)$ by part (ii). Let $a \in P$ such that $Y \leq \alpha_*^{\mathbf{P}}(a)$. Then, by Lemma 6.1.12, $\alpha_*^{\mathbf{P}}(a) \in O_*(\mathbf{P})$ and therefore

$\{\alpha_*^{\mathbf{Q}}(f(a)) : a \in P, Y \leq \alpha_*^{\mathbf{P}}(a)\} \subseteq \{f_*^\pi(Z) : Y \leq Z \in O_*(\mathbf{P})\}$. Then, by part (iii) (d), $f_*^\pi(Y) \leq f_*^\sigma(Y)$.

The proof that $f_*^\sigma = f_*^\pi$ on $O_*(\mathbf{P})$ is similar.

Let $* \in \{p, dp, f\}$. Then it follows immediately from Corollary 6.2.6 that $f_*^\pi(Y) = f_*^\sigma(Y) \in K_*(\mathbf{Q})$ for $Y \in K_*(\mathbf{P})$ and $f_*^\sigma(Z) = f_*^\pi(Z) \in O_*(\mathbf{Q})$ for $Z \in O_*(\mathbf{P})$.

If $*$ is d, then, by Lemma 6.1.11, $G = \{a \in P : Y \leq \alpha_d^{\mathbf{P}}(a)\} \in \mathcal{F}_d(\mathbf{P})$ for $Y \in K_d(\mathbf{P})$. Then $f(G)$ is down-directed by Lemma 6.3.6. Finally, by Corollary 6.2.8 it follows that $\bigwedge \alpha_d^{\mathbf{Q}}(f(G)) \in K_d(\mathbf{Q})$. But $\bigwedge \alpha_d^{\mathbf{Q}}(f(G)) = f_*^\sigma(Y)$ $(= f_*^\pi(Y)$ by the above). Similarly, one can show that $f_*^\pi(Z) (= f_*^\sigma(Z)$ by the above) $\in O_d(\mathbf{Q})$ for $Z \in O_d(\mathbf{P})$. □

We now consider the extension of operators defined on $\mathbf{P}$ to $\mathbf{C}_*$. Observe that if f is an operator, then f is order-preserving. The simplifications from Lemma 6.3.7 therefore apply.

In the case of bounded lattices, f^σ and f^π are complete operators when f is an operator. Furthermore, if f is a dual operator, then f^σ and f^π are complete dual operators [GH01, Corollary 4.7]. In the next two examples we illustrate that, for $* \in \{dp, f, d\}$, f_*^σ need not be a (complete) operator when f is an operator. Dually, f_*^π need not be a (complete) dual operator when f is a dual operator. We no longer consider the completion $(\mathbf{C}_p, \alpha_p)$ since it does not preserve all existing finite meets and joins (Corollary 6.2.13).

Example 6.3.8. *Let $\mathbf{P}'$ be the 3-element anti-chain, depicted in Figure 6.6, with $f : P' \to P'$ defined by $f(1) = 2$, $f(2) = 2$ and $f(3) = 3$. Then, f distributes over all existing joins since no non-trivial joins exist in $\mathbf{P}'$. However, $f_*^\sigma(\alpha_*(1) \vee \alpha_*(2)) = \top_* \neq \alpha_*(2) = f_*^\sigma(\alpha_*(1)) \vee f_*^\sigma(\alpha_*(2))$, for $* \in \{f, d\}$. Therefore, f_*^σ does not distribute over finite joins. See Example A.2.5 in Appendix A.2 for more details.*

We note that $\mathbf{C}_*(\mathbf{P}')$, $* \in \{f, d\}$, from Example 6.3.8 is isomorphic to $\mathbf{F}_*(\mathbf{P}')$ in Example 4.3.4. Therefore, by the argument in Remark 4.3.5, there is no way to extend f to an operator on $\mathbf{C}_*(\mathbf{P}')$. However, f may be extended to an operator on other completions of $\mathbf{P}'$.

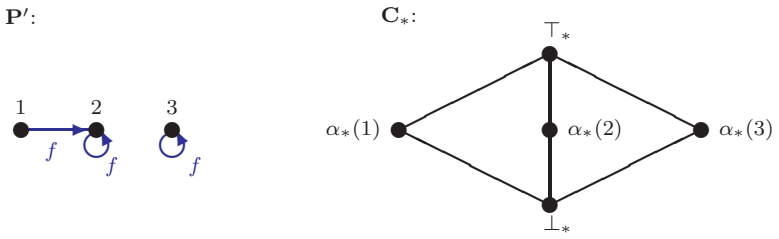


Fig. 6.6: f^σ_*, $* \in \{f, d\}$, need not be an operator when f is.

Remark 6.3.9. *Let* $\mathbf{P}'$ *be the poset from Example 6.3.8 with operator* f *defined on* P'. *Then* f^σ_{dp} *is an operator on* $\mathbf{C}_{dp}(\mathbf{P}')$. *See Example A.2.5 in Appendix A.2 for more details.*

In [GJKO07, Lemma 6.12] it is claimed that if f is an operator, then f^σ_{dp} is a complete operator. Furthermore, it is also claimed that if f is a dual operator, then f^π_{dp} is a complete dual operator. The following example contradicts these claims.

Example 6.3.10. *Let* $\mathbf{P}'$ *and* $\mathbf{Q}'$ *be the posets depicted in Figure 6.7 and let* $f : P' \to Q'$ *be the map defined by* $f(a) = \underline{a}$ *for* $a = 1, \ldots, 7$. *Then* f *distributes over all existing joins. Due to their sizes, diagrams for* $\mathbf{C}_{dp}(\mathbf{P}')$ *and* $\mathbf{C}_{dp}(\mathbf{Q}')$ *are not depicted here.*

Now let $X_1 = \bigwedge \alpha^{\mathbf{P}'}_{dp}(\{1,2\})$ *and* $X_2 = \alpha^{\mathbf{P}'}_{dp}(4)$. *Then* $X_1 \vee X_2 = (X_1 \cup X_2)^{\rhd\lhd} = \{I \in \mathcal{I}_{dp}(\mathbf{P}') : 4 \in I \text{ and } (1 \in I \text{ or } 2 \in I)\}^{\lhd} = \mathcal{F}_{dp}(\mathbf{P}') - \{\{1\}, \{2\}\}$. *But then* $\alpha^{\mathbf{P}'}_{dp}(3) \subseteq X_1 \vee X_2$ *and since* f^σ_{dp} *is order-preserving,* $f^\sigma_{dp}(\alpha^{\mathbf{P}'}_{dp}(3)) \leq f^\sigma_{dp}(X_1 \vee X_2)$. *That is,* $f^\sigma_{dp}(X_1 \vee X_2) \geq \alpha^{\mathbf{Q}'}_{dp}(\underline{3}) \in K_{dp}(\mathbf{Q}')$.

On the other hand, since $\{1,2\} \in \mathcal{F}_{dp}(\mathbf{P}')$, *we have* $X_1 \in K_{dp}(\mathbf{P}')$ *and* $f^\sigma_{dp}(X_1) = \alpha^{\mathbf{Q}'}_{dp}(f(1)) \wedge \alpha^{\mathbf{Q}'}_{dp}(f(2)) = \alpha^{\mathbf{Q}'}_{dp}(\underline{1}) \wedge \alpha^{\mathbf{Q}'}_{dp}(\underline{2}) = \alpha^{\mathbf{Q}'}_{dp}(\underline{8})$. *Also,* $f^\sigma_{dp}(X_2) = \alpha^{\mathbf{Q}'}_{dp}(\underline{4})$. *Then,* $f^\sigma_{dp}(X_1) \vee f^\sigma_{dp}(X_2) = \alpha^{\mathbf{Q}'}_{dp}(\underline{8}) \vee \alpha^{\mathbf{Q}'}_{dp}(\underline{4}) = \{F \in \mathcal{F}_{dp}(\mathbf{Q}') : \underline{8} \in F \text{ or } \underline{4} \in F\}^{\rhd\lhd}$. *In particular,* $\{\underline{4}, \underline{7}, \underline{8}\} \in \{F \in \mathcal{F}_{dp(\mathbf{Q}')} : \underline{8} \in F \text{ or } \underline{4} \in F\}^{\rhd}$, *but* $\{\underline{3}\} \cap \{\underline{4}, \underline{7}, \underline{8}\} = \varnothing$ *and therefore* $\{\underline{3}\} \notin \alpha^{\mathbf{Q}'}_{dp}(\underline{8}) \vee \alpha^{\mathbf{Q}'}_{dp}(\underline{4})$. *Thus,* $\alpha^{\mathbf{Q}'}_{dp}(\underline{3}) \not\leq f^\sigma_{dp}(X_1) \vee f^\sigma_{dp}(X_2)$.

By the denseness of $\mathbf{C}_{dp}(\mathbf{Q}')$ *it then follows that* $f^\sigma_{dp}(X_1) \vee f^\sigma_{dp}(X_2) \neq f^\sigma_{dp}(X_1 \vee X_2)$. *That is, the extension* f^σ_{dp} *does not distribute over finite joins.*

In [Suz11] a more restrictive notion of distribution over joins was defined. A

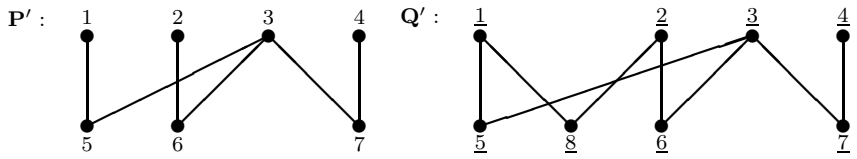


Fig. 6.7: f^{σ}_{dp} need not be an operator when f is.

map $f : P \to Q$ will be called 'join-preserving' if it satisfies: for all $a_1, a_2 \in P$ and each $c \in Q$ satisfying $f(a_1) \leq c$ and $f(a_2) \leq c$, there exists $b \in P$ such that $a_1 \leq b$, $a_2 \leq b$ and $f(b) \leq c$. We note that the map f defined in Example 6.3.10 satisfies this definition of 'join-preservation'. Hence, f^{σ}_{dp} need not be an operator even if f satisfies the more restrictive join-preservation property.

Next we focus our attention on residuated operators. We ask the questions: Is f^{σ}_{*} residuated when f is? If so, can we describe its residual? For $* \in \{dp, f, d\}$, we will show that if g is f's residual then f^{σ}_{*} is residuated when f is and that g^{π}_{*} is its residual.

Lemma 6.3.11. *Let $f : P \to Q$ be residuated with residual $g : Q \to P$. Let $G \in \mathcal{F}_{dp}(\mathbf{P})$ and $J \in \mathcal{I}_{dp}(\mathbf{Q})$. Then*

$$[f(G)\rangle_{dp} \cap J \neq \varnothing \iff G \cap \langle g(J)]_{dp} \neq \varnothing.$$

Proof. Let $c \in [f(G)\rangle_{dp} \cap J$. Clearly $g(c) \in \langle g(J)]_{dp}$. We show by induction that $g(c) \in G$. Recall that $[f(G)\rangle_{dp} = \bigcup_{i\in\mathbb{N}} S_i$ where $S_0 = f(G)$ and $S_{i+1} = \big[\{\bigwedge M : \varnothing \neq M \subseteq^{fin} S_i \text{ such that } \bigwedge M \text{ exists}\}\big)$.

If $c \in S_0$ then $c = f(a)$ for some $a \in G$. Then, by residuation, $f(a) \leq c$ implies that $a \leq g(c)$. Since $G \in \mathcal{F}_{dp}(\mathbf{P})$, it follows that $g(c) \in G$.

Suppose that $d \in S_i$ implies that $g(d) \in G$. Let $c \in S_{i+1}$, i.e., $\bigwedge M \leq c$ for some $M \subseteq^{fin} S_i$ such that $\bigwedge M$ exists. Then $g(\bigwedge M) \leq g(c)$. By Lemma 2.5.2 it follows that $\bigwedge g(M)$ exists and $\bigwedge g(M) = g(\bigwedge M)$. Thus, $\bigwedge g(M) \leq g(c)$. By the inductive hypothesis $g(d) \in G$ for every $d \in M$. But $G \in \mathcal{F}_{dp}(P)$, therefore $\bigwedge g(M) \in G$ and hence $g(c) \in G$. Then, $g(c) \in G \cap \langle g(J)]_{dp}$.

The implication in the other direction follows similarly. □

Lemma 6.3.12. *Let $f : P \to Q$ be residuated with residual $g : Q \to P$. Let $G \in \mathcal{F}_{f}(\mathbf{P})$ and $J \in \mathcal{I}_{f}(\mathbf{Q})$. Then*

$$[f(G)\rangle_{f} \cap J \neq \varnothing \iff G \cap \langle g(J)]_{f} \neq \varnothing.$$

Proof. We begin by showing that $g(f(M)^\ell)^u \subseteq g(f(M^\ell))^u$ for any $M \subseteq^{fin} G$. If $M^\ell = \varnothing$, then $g(f(\varnothing))^u = g(\varnothing)^u = \varnothing^u = P$. Hence, $g(f(M)^\ell)^u \subseteq g(f(M^\ell))^u$. Now suppose $M^\ell \neq \varnothing$ and let $b \in M^\ell$. Then for every $a \in M$, $b \leq a$ and therefore $f(b) \leq f(a)$. Thus, $f(b) \in f(M)^\ell$ and $f(M^\ell) \subseteq f(M)^\ell$. But then $g(f(M^\ell)) \subseteq g(f(M)^\ell)$ and hence $g(f(M)^\ell)^u \subseteq g(f(M^\ell))^u$.

Next we show that $g(f(M^\ell))^u \subseteq G$ for any $M \subseteq^{fin} G$. Let $d \in g(f(M^\ell))^u$, then $d \geq g(f(a))$ for every $a \in M^\ell$. By residuation, $g(f(a)) \geq a$ for every $a \in M^\ell$. Then, $d \geq a$ for every $a \in M^\ell$ and $d \in M^{\ell u}$. Hence, $g(f(M^\ell))^u \subseteq M^{\ell u} \subseteq G$ since $G \in \mathcal{F}_f(\mathbf{P})$.

Now let $c \in [f(G)\rangle_f \cap J$. Then, $g(c) \in \langle g(J)]_f$ is immediate. It remains to show that $g(c) \in G$. Since $c \in [f(G)\rangle_f$, we know that $c \in N^{\ell u}$ for some $N \subseteq^{fin} f(G)$. Let $M \subseteq^{fin} G$ such that $N = f(M)$. Then $c \in f(M)^{\ell u}$, i.e., $c \geq d$ for every $d \in f(M)^\ell$. But then $g(c) \geq g(d)$ for every $d \in f(M)^\ell$ and $g(c) \in g(f(M)^\ell)^u$. By the claims above it follows that $g(c) \in g(f(M^\ell))^u \subseteq G$. That is, $g(c) \in G \cap \langle g(J)]_f$.

Similarly we can show that $f(g(M)^u)^\ell \subseteq f(g(M^u))^\ell$ and $f(g(M^u))^\ell \subseteq J$ for any $M \subseteq^{fin} J$. We can then prove, in a similar way, that $f(c) \in [f(g)\rangle_f \cap J$ when $c \in G \cap \langle g(J)]_f$. □

Proposition 6.3.13. *Let $* \in \{dp, f\}$. If $f : P \to Q$ is residuated with residual $g : Q \to P$, then $f_*^\sigma : \mathcal{C}_*(\mathbf{P}) \to \mathcal{C}_*(\mathbf{Q})$ is residuated and $g_*^\pi : \mathcal{C}_*(\mathbf{Q}) \to \mathcal{C}_*(\mathbf{P})$ is its residual, i.e., for all $X \in \mathcal{C}_*(\mathbf{P})$ and all $X' \in \mathcal{C}_*(\mathbf{Q})$*

$$f_*^\sigma(X) \leq X' \iff X \leq g_*^\pi(X').$$

Proof. By the denseness of $\mathbf{C}_*(\mathbf{P})$ and $\mathbf{C}_*(\mathbf{Q})$ we have $X = \bigvee\{Y \in K_*(\mathbf{P}) : Y \leq X\}$ for all $X \in \mathcal{C}_*(\mathbf{P})$ and $X' = \bigwedge\{Z \in O_*(\mathbf{Q}) : Z \geq X'\}$ for all $X' \in \mathcal{C}_*(\mathbf{Q})$. Furthermore, since $P \in \mathcal{F}_*(\mathbf{P})$ we have $\bigwedge \alpha_*^{\mathbf{P}}(P) = \bot_*^{\mathbf{C}_*(\mathbf{P})} \in \{Y \in K_*(\mathbf{P}) : Y \leq X\}$ for all $X \in \mathcal{C}_*(\mathbf{P})$. Similarly, $Q \in \mathcal{I}_*(\mathbf{Q})$ implies that $\bigvee \alpha_*^{\mathbf{Q}}(Q) = \top_*^{\mathbf{C}_*(\mathbf{Q})} \in \{Z \in O_*(\mathbf{Q}) : Z \geq X'\}$ for all $X' \in \mathcal{C}_*(\mathbf{Q})$. Since f is residuated, it is order-preserving and by Lemma 6.3.7 (i) so is f_*^σ. Therefore, $f_*^\sigma(X) \leq X'$ if, and only if, $f_*^\sigma(Y) \leq Z$ for every $Y \in K_*(\mathbf{P})$ such that $Y \leq X$ and every $Z \in O_*(\mathbf{Q})$ such that $X' \leq Z$. Similarly, $X \leq g_*^\pi(X')$ if, and only if, $Y \leq g_*^\pi(Z)$ for every $Y \in K_*(\mathbf{P})$ such that $Y \leq X$ and every $Z \in O_*(\mathbf{Q})$ such that $X' \leq Z$. It is therefore sufficient to prove that $f_*^\sigma(Y) \leq Z$ if, and only if, $Y \leq g_*^\pi(Z)$ for $Y \in K_*(\mathbf{P})$ and $Z \in O_*(\mathbf{P})$.

Let $Y \in K_*(\mathbf{P})$ and $Z \in O_*(\mathbf{P})$. Then there exists $G \in \mathcal{F}_*(\mathbf{P})$ such that $Y = \bigwedge \alpha_*^{\mathbf{P}}(G)$ and there exists $J \in \mathcal{I}_*(\mathbf{Q})$ such that $Z = \bigvee \alpha_*^{\mathbf{Q}}(J)$. Furthermore, $f_*^\sigma(Y) = \bigwedge\{\alpha_*^{\mathbf{Q}}(f(a)) : a \in P, Y \leq \alpha_*^{\mathbf{P}}(a)\}$ and $g_*^\pi(Z) = \bigvee\{\alpha_*^{\mathbf{P}}(g(b)) : b \in Q, Z \geq \alpha_*^{\mathbf{Q}}(b)\}$.

Suppose $f_*^\sigma(Y) \leq Z$, i.e., $\bigwedge\{\alpha_*^{\mathbf{Q}}(f(a)) : a \in P, Y \leq \alpha_*^{\mathbf{P}}(a)\} \leq \bigvee \alpha_*^{\mathbf{Q}}(J)$. By the internal compactness of $\mathbf{C}_*(\mathbf{Q})$, the above holds if, and only if, $F \cap I \neq \varnothing$ for every $F \in \mathcal{F}_*(\mathbf{Q})$ such that $\{f(a) : Y \subseteq \alpha_*^{\mathbf{P}}(a)\} \subseteq F$ and every $I \in \mathcal{I}_*(\mathbf{Q})$ such that $J \subseteq I$. That is, if, and only if, $\big[\{f(a) : Y \subseteq \alpha_*^{\mathbf{P}}(a)\}\big\rangle_* \cap J \neq \varnothing$. But $Y \subseteq \alpha_*^{\mathbf{P}}(a)$ if, and only if, $\{F \in \mathcal{F}_*(\mathbf{P}) : G \subseteq F\} \subseteq \{F \in \mathcal{F}_*(\mathbf{P}) : a \in F\}$ if, and only if, $a \in G$. Therefore, $\{a \in P : Y \subseteq \alpha_*^{\mathbf{P}}(a)\} = G$ and

$$f_*^\sigma(Y) \leq Z \iff [f(G)\rangle_* \cap J \neq \varnothing.$$

On the other hand, suppose $Y \leq g_*^\pi(Z)$, i.e., $\bigwedge \alpha_*^{\mathbf{P}}(G) \leq \bigvee\{\alpha_*^{\mathbf{P}}(g(b)) : b \in Q, Z \geq \alpha_*^{\mathbf{Q}}(b)\}$. By the internal compactness of $\mathbf{C}_*(\mathbf{P})$ the above holds if, and only if, $F \cap I \neq \varnothing$ for every $F \in \mathcal{F}_*(\mathbf{P})$ such that $G \subseteq F$ and every $I \in \mathcal{I}_*(\mathbf{P})$ such that $\{g(b) : Z \supseteq \alpha_*^{\mathbf{Q}}(b)\} \subseteq I$. In particular, if, and only if, $G \cap \Big\langle\{g(b) : Z \supseteq \alpha_*^{\mathbf{Q}}(b)\}\Big]_* \neq \varnothing$. But $Z \supseteq \alpha_*^{\mathbf{Q}}(b)$ if, and only if, $\{F \in \mathcal{F}_*(\mathbf{Q}) : b \in F\} \subseteq \{F \in \mathcal{F}_*(\mathbf{Q}) : F \cap J \neq \varnothing\}$ if, and only if, $[b) \cap J \neq \varnothing$ if, and only if, $b \in J$. Therefore, $\{b \in Q : Z \supseteq \alpha_*^{\mathbf{Q}}(b)\} = J$ and

$$Y \leq g_*^\pi(Z) \iff G \cap \langle g(J)]_* \neq \varnothing.$$

Hence, we need to prove $[f(G)\rangle_* \cap J \neq \varnothing$ if, and only if, $G \cap \langle g(J)]_* \neq \varnothing$ to prove the claim.

Lemmas 6.3.11 and 6.3.12 prove the equivalence for the Doyle-pseudo and Frink cases, respectively. □

We note that a similar claim was made in [GJKO07, Lemma 6.15] for binary residuated operators. However, in Example 6.3.30 we provide a counter-example to that claim.

To prove that f_d^σ has residual g_d^π, we use the same argument that was used in [DGP05, Proposition 3.6] for a similar claim on binary residuated operators.

Proposition 6.3.14. *If $f : P \to Q$ is residuated with residual $g : Q \to P$, then $f_d^\sigma : \mathcal{C}_d(\mathbf{P}) \to \mathcal{C}_d(\mathbf{Q})$ is residuated and $g_d^\pi : \mathcal{C}_d(\mathbf{Q}) \to \mathcal{C}_d(\mathbf{P})$ is its residual, i.e., for all $X \in \mathcal{C}_d(\mathbf{P})$ and all $X' \in \mathcal{C}_d(\mathbf{Q})$*

$$f_d^\sigma(X) \leq X' \iff X \leq g_d^\pi(X').$$

Proof. Let $X \in \mathcal{C}_d(\mathbf{P})$ and $X' \in \mathcal{C}_d(\mathbf{Q})$.

To prove that $f_d^\sigma(X) \leq X' \iff X \leq g_d^\pi(X')$ it is sufficient to show that for $Y \in K_d(\mathbf{P})$ and $Z \in O_d(\mathbf{Q})$, $f_d^\sigma(Y) \leq Z \iff Y \leq g_d^\pi(Z)$ by the denseness of $\mathbf{C}_d(\mathbf{P})$ and $\mathbf{C}_d(\mathbf{Q})$.

Note that if $\{Y \in K_d(\mathbf{P}) : Y \leq X\} = \varnothing$, then $X = \bigvee \varnothing = \bot_d^{\mathbf{C}_d(\mathbf{P})}$, $f_d^\sigma(X) = \bigvee \varnothing = \bot_d^{\mathbf{C}_d(\mathbf{Q})}$ and $f_d^\sigma(X) \leq X' \iff X \leq g_d^\pi(X')$ is true for all $X' \in \mathcal{C}_d(\mathbf{Q})$. Similarly, if $\{Z \in O_d(\mathbf{Q}) : Z \geq X'\} = \varnothing$, then $X' = \bigwedge \varnothing = \top_d^{\mathbf{C}_d(\mathbf{Q})}$, $g_d^\pi(X') = \bigwedge \varnothing = \top_d^{\mathbf{C}_d(\mathbf{P})}$ and $f_d^\sigma(X) \leq X' \iff X \leq g_d^\pi(X')$ is true for all $X \in \mathcal{C}_d(\mathbf{P})$.

Suppose $\{Y \in K_d(\mathbf{P}) : Y \leq X\} \neq \varnothing$ and $\{Z \in O_d(\mathbf{Q}) : Z \geq X'\} \neq \varnothing$. Let $Y \in K_d(\mathbf{P})$ and $Z \in O_d(\mathbf{Q})$. Then there exists a filter $G \in \mathcal{F}_d(\mathbf{P})$ such that $Y = \bigwedge \alpha_d^{\mathbf{P}}(G)$ and there exists an ideal $J \in \mathcal{I}_d(\mathbf{Q})$ such that $Z = \bigvee \alpha_d^{\mathbf{Q}}(J)$.

Suppose $f_d^\sigma(Y) \leq Z$. Now, $f_d^\sigma(Y) = \bigwedge\{\alpha_d^{\mathbf{Q}}(f(a)) : a \in P, Y \leq \alpha_d^{\mathbf{P}}(a)\} = \bigwedge \alpha_d^{\mathbf{Q}}(f(G))$. That is, $\bigwedge \alpha_d^{\mathbf{Q}}(f(G)) \leq \bigvee \alpha_d^{\mathbf{Q}}(J)$. By Lemma 6.3.6, $f(G)$ is down-directed and therefore, by Corollary 6.2.8, $\bigwedge \alpha_d^{\mathbf{Q}}(f(G)) = \bigwedge \alpha_d^{\mathbf{Q}}([f(G))$. Then, by Lemma 6.2.9, there exist elements $d \in [f(G))$ and $b \in J$ such that $d \leq b$. But $f(a) \leq d$ for some $a \in G$ since $d \in [f(G))$. That is, there exists an $a \in G$ such that $f(a) \leq b$. By the residuation we have $a \leq g(b)$. Hence, $\bigwedge \alpha_d^{\mathbf{P}}(G) \leq \alpha_d^{\mathbf{P}}(a) \leq \alpha_d^{\mathbf{P}}(g(b)) \leq \bigvee \alpha_d^{\mathbf{P}}(g(J))$, i.e., $Y \leq g_d^\pi(Z)$.

The implication in the other direction follows similarly. □

6.3.2 *n-ary maps*

Given an n-ary map $f : \prod_{i=1}^n P_i \to Q$, for $n \in \mathbb{N}$, and posets $\mathbf{P}_1, \ldots, \mathbf{P}_n, \mathbf{Q}$, we would like to define an extension of f from $\prod_{i=1}^n \mathcal{C}_*(\mathbf{P}_i)$ to $\mathcal{C}_*(\mathbf{Q})$. On lattices the canonical extension commutes with finite products [GH01]. Therefore, in the lattice setting the extension of any n-ary map may be viewed as the extension of a unary map. That is, for $f : \prod_{i=1}^n L_i \to L$ where $\mathbf{L}_i$, $1 \leq i \leq n$, and $\mathbf{L}$ are lattices, an extension of f, say $f^{\mathbf{C}} : \prod_{i=1}^n \mathcal{C}(\mathbf{L}_i) \to \mathcal{C}(\mathbf{L})$, can be viewed as the unary map $f^{\mathbf{C}} : \mathcal{C}(\prod_{i=1}^n \mathbf{L}_i) \to \mathcal{C}(\mathbf{L})$. However, since the construction of completions of posets described in Section 6.1.1 does not commute with products, see Example 6.2.14, the extension of an n-ary map must be treated as an n-ary map in the poset setting.

For $* \in \{p, dp, f\}$, the sets of closed and open elements in $\prod_{i=1}^n \mathbf{C}_*(\mathbf{P}_i)$ are defined as follows.

Definition 6.3.15. *Let $* \in \{p, dp, f\}$. An element $(Y_1, \ldots, Y_n) \in \prod_{i=1}^n \mathcal{C}_*(\mathbf{P}_i)$*

is called closed *if* $Y_i \in K_*(\mathbf{P}_i)$, *for* $i = 1, \ldots, n$. *Furthermore, an element* $(Z_1, \ldots, Z_n) \in \prod_{i=1}^n \mathcal{C}_*(\mathbf{P}_i)$ *is called* open *if* $Z_i \in O_*(\mathbf{P}_i)$, *for* $i = 1, \ldots n$.

Let $\mathcal{K}_*$, $\mathcal{O}_*$ *and* $\mathcal{KO}_*$ *denote the sets of closed, open and clopen elements of* $\prod_{i=1}^n \mathbf{C}_*(\mathbf{P}_i)$, *respectively. Then,* $\mathcal{K}_* = \prod_{i=1}^n K_*(\mathbf{P}_i)$ *and* $\mathcal{O}_* = \prod_{i=1}^n O_*(\mathbf{P}_i)$.

Recall that the order-embedding $\beta_* : \prod_{i=1}^n P_i \to \prod_{i=1}^n \mathcal{C}_*(\mathbf{P}_i)$ is defined by $\beta_*((a_1, \ldots, a_n)) = (\alpha_*^{\mathbf{P}_1}(a_1), \ldots, \alpha_*^{\mathbf{P}_n}(a_n))$ (see Lemma 6.1.15). Then the pair $(\prod_{i=1}^n \mathbf{C}_*(\mathbf{P}_i), \beta_*)$ is a completion of $\prod_{i=1}^n \mathbf{P}_i$.

Let $\vec{a} \in \prod_{i=1}^n P_i$, $\vec{X} \in \prod_{i=1}^n \mathcal{C}_*(\mathbf{P}_i)$, $\vec{Y} \in \mathcal{K}_*$ and $\vec{Z} \in \mathcal{O}_*$ denote the n-tuples $(a_1, \ldots, a_n)$, $(X_1, \ldots, X_n)$, $(Y_1, \ldots, Y_n)$ and $(Z_1, \ldots, Z_n)$, respectively, where $a_i \in P_i$, $X_i \in \mathcal{C}_*(\mathbf{P}_i)$, $Y_i \in K_*(\mathbf{P}_i)$ and $Z_i \in O_*(\mathbf{P}_i)$ for $i = 1, \ldots, n$.

Proposition 6.3.16. *Let* $* \in \{p, dp, f\}$. *The completion* $(\prod_{i=1}^n \mathbf{C}_*(\mathbf{P}_i), \beta_*)$ *is dense with respect to the sets* $\mathcal{K}_*$ *and* $\mathcal{O}_*$, *i.e., for every* $\vec{X} \in \prod_{i=1}^n \mathcal{C}_*(\mathbf{P}_i)$ *we have that* $\vec{X} = \bigvee\{\vec{Y} \in \mathcal{K}_* : \vec{Y} \leq \vec{X}\} = \bigwedge\{\vec{Z} \in \mathcal{O}_* : \vec{Z} \geq \vec{X}\}$.

The denseness of $(\prod_{i=1}^n \mathbf{C}_*(\mathbf{P}_i), \beta_*)$ follows directly from the definitions of $\mathcal{K}_*$ and $\mathcal{O}_*$ and the denseness of each $(\mathbf{C}_*(\mathbf{P}_i), \alpha_*)$, established in Proposition 6.1.13.

Lemma 6.3.17. *Let* $* \in \{p, dp, f\}$. *Then,* $\mathcal{KO}_* = \beta_*\left(\prod_{i=1}^n P_i\right)$.

The proof follows from the fact that $KO_*(\mathbf{P}_i) = \alpha_*^{\mathbf{P}_i}(P_i)$ for $i = 1, \ldots, n$ (see Lemma 6.1.12).

There are now two natural extensions for an n-ary map, as was the case for unary maps.

Definition 6.3.18. *For* $* \in \{p, dp, f\}$ *and an* n*-ary map* $f : \prod_{i=1}^n P_i \to Q$, *define* $f_*^\sigma : \prod_{i=1}^n \mathcal{C}_*(\mathbf{P}_i) \to \mathcal{C}_*(\mathbf{Q})$ *and* $f_*^\pi : \prod_{i=1}^n \mathcal{C}_*(\mathbf{P}_i) \to \mathcal{C}_*(\mathbf{Q})$ *by: for all* $\vec{X} \in \prod_{i=1}^n \mathcal{C}_*(\mathbf{P}_i)$,

$$f_*^\sigma(\vec{X}) = \bigvee \left\{ \bigwedge \left\{ \alpha_*^{\mathbf{Q}}(f(\vec{a})) : \vec{a} \in \prod_{i=1}^n P_i, \vec{Y} \leq \beta_*(\vec{a}) \leq \vec{Z} \right\} : \vec{Y} \in \mathcal{K}_*, \vec{Z} \in \mathcal{O}_*, \vec{Y} \leq \vec{X} \leq \vec{Z} \right\},$$

$$f_*^\pi(\vec{X}) = \bigwedge \left\{ \bigvee \left\{ \alpha_*^{\mathbf{Q}}(f(\vec{a})) : \vec{a} \in \prod_{i=1}^n P_i, \vec{Y} \leq \beta_*(\vec{a}) \leq \vec{Z} \right\} : \vec{Y} \in \mathcal{K}_*, \vec{Z} \in \mathcal{O}_*, \vec{Y} \leq \vec{X} \leq \vec{Z} \right\}.$$

We note that since $\prod_{i=1}^n \mathbf{C}_*(\mathbf{P}_i)$ and $\mathbf{C}_*(\prod_{i=1}^n \mathbf{P}_i)$ are isomorphic for bounded lattices, $f_*^\sigma = f^\sigma$ and $f_*^\pi = f^\pi$ if $\mathbf{P}_i$, $1 \leq i \leq n$, and $\mathbf{Q}$ are bounded lattices. Therefore, the definitions of f_*^σ and f_*^π given here are generalizations of the definitions on lattices.

Lemma 6.3.19. *Let $* \in \{p, dp, f\}$. If $f : \prod_{i=1}^n P_i \to Q$, then f_*^σ and f_*^π extend f, i.e., for $\vec{a} \in \prod_{i=1}^n P_i$ we have $f_*^\sigma(\beta_*(\vec{a})) = \alpha_*^{\mathbf{Q}}(f(\vec{a})) = f_*^\pi(\beta_*(\vec{a}))$.*

The proof is similar to the proof of Lemma 6.3.2 where the extension of unary maps are considered.

Lemma 6.3.20. *If $f : \prod_{i=1}^n P_i \to Q$ is order-preserving, then*

$$f_*^\sigma(\vec{X}) = \bigvee \left\{ \bigwedge \left\{ \alpha_*^{\mathbf{Q}}(f(\vec{a})) : \vec{a} \in \prod_{i=1}^n P_i, \vec{Y} \leq \beta_*(\vec{a}) \right\} : \vec{X} \geq \vec{Y} \in \mathcal{K}_* \right\},$$

$$f_*^\pi(\vec{X}) = \bigwedge \left\{ \bigvee \left\{ \alpha_*^{\mathbf{Q}}(f(\vec{a})) : \vec{a} \in \prod_{i=1}^n P_i, \beta_*(\vec{a}) \leq \vec{Z} \right\} : \vec{X} \leq \vec{Z} \in \mathcal{O}_* \right\}.$$

The proofs of these simplifications are similar to the proofs of the simplifications for the unary cases, done in Lemma 6.3.5.

If $*$ is d, then $\prod_{i=1}^n \mathbf{C}_d(\mathbf{P}_i)$ need not be dense with respect to $\prod_{i=1}^n K_d(\mathbf{P}_i)$ and $\prod_{i=1}^n O_d(\mathbf{P}_i)$, as illustrated in the example below.

Example 6.3.21. *Let $\mathbf{P}'$ be the 2-element anti-chain. Then $\mathbf{C}_d$ and $\mathbf{C}_d \times \mathbf{C}_d$ are depicted in Figure 6.8. The elements from K_d and $K_d \times K_d$ are depicted by $\bullet$, the elements from O_d and $O_d \times O_d$ are depicted by $\circ$ and all other elements are depicted by $\circledast$.*

Now consider, for example, the element $(\top_d, \bot_d)$. Firstly, $(\top_d, \bot_d)$ has no elements from $K_d \times K_d$ below it and can therefore not be expressed as a join of such elements. Furthermore, $(\top_d, \bot_d)$ has no elements from $O_d \times O_d$ above it and cannot be expressed as a meet of these elements either.

In [DGP05] order-preserving n-ary maps are extended in terms of the sets $\prod_{i=1}^n K_d(\mathbf{P}_i)$ and $\prod_{i=1}^n O_d(\mathbf{P}_i)$ in the following two ways:

$$f_d^1(\vec{X}) = \bigvee \left\{ \bigwedge \left\{ \alpha_d^{\mathbf{Q}}(f(\vec{a})) : \vec{a} \in \prod_{i=1}^n P_i, \vec{Y} \leq \beta_d(\vec{a}) \right\} : \vec{X} \geq \vec{Y} \in \prod_{i=1}^n K_d(\mathbf{P}_i) \right\},$$

$$f_d^2(\vec{X}) = \bigwedge \left\{ \bigvee \left\{ \alpha_d^{\mathbf{Q}}(f(\vec{a})) : \vec{a} \in \prod_{i=1}^n P_i, \beta_d(\vec{a}) \leq \vec{Z} \right\} : \vec{X} \leq \vec{Z} \in \prod_{i=1}^n O_d(\mathbf{P}_i) \right\}.$$

Then f_d^1 and f_d^2 are both order-preserving and extend f.

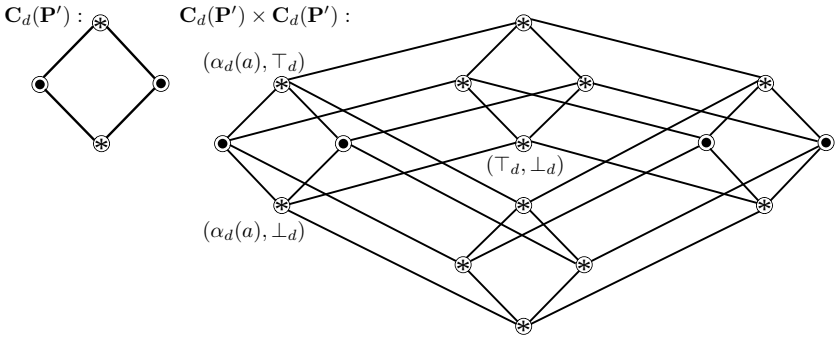


Fig. 6.8: $\mathbf{C}_d \times \mathbf{C}_d$ need not be dense w.r.t. $K_d \times K_d$ and $O_d \times O_d$.

Example 6.3.22. *Let* $\mathbf{P}' = \langle \{1, 2\}, \leq \rangle$ *be the* 2*-element anti-chain from Example 6.3.21 with* $\mathbf{C}_d \times \mathbf{C}_d$ *depicted in Figure 6.8. Define* $\rho_1 : P' \times P' \to P'$ *by* $\rho_1(1, 1) = 1 = \rho_1(1, 2)$ *and* $\rho_1(2, 1) = 2 = \rho_1(2, 2)$*. Then* ρ_1 *is the projection map on the first coordinate of* $\mathbf{P}' \times \mathbf{P}'$ *and it is order-preserving. However, neither of the extensions* $(\rho_1)^1_d$ *or* $(\rho_1)^2_d$ *are the projection map on the first coordinate of* $\mathbf{C}_d \times \mathbf{C}_d$*. To see this observe that since there are no elements from* $K_d \times K_d$ *less than or equal to* $(\alpha_d(a), \bot_d)$*, we have* $(\rho_1)^1_d((\alpha_d(a), \bot_d)) = \bigvee \varnothing = \bot_d \neq \alpha_d(a)$*. Furthermore, since there are no elements from* $O_d \times O_d$ *greater than or equal to* $(\alpha_d(a), \top_d)$*, we have* $(\rho_1)^2_d((\alpha_d(a), \top_d)) = \bigwedge \varnothing = \top_d \neq \alpha_d(a)$*.*

The previous example could be altered slightly to show that the extensions f^1_d and f^2_d of a constant n-ary map f need not be constant. There are therefore some natural and simple properties that are not preserved by the extensions considered in [DGP05]. It would appear that the main reason why the extensions from [DGP05] do not preserve these properties, is the lack of denseness of $\prod_{i=1}^n \mathbf{C}_d(\mathbf{P}_i)$ with respect to $\prod_{i=1}^n K_d(\mathbf{P}_i)$ and $\prod_{i=1}^n O_d(\mathbf{P}_i)$.

Next we define a pair of sets such that $\prod_{i=1} \mathbf{C}_d(\mathbf{P}_i)$ is dense with respect to it.

Definition 6.3.23. *Define the sets of* closed *and* open *elements of* $\prod_{i=1}^n \mathbf{C}_d(\mathbf{P}_i)$ *by*

$$\mathcal{K}_d = \prod_{i=1}^n (K_d(\mathbf{P}_i) \cup \{\bot_d^{\mathbf{P}_i}\}) \quad \text{and} \quad \mathcal{O}_d = \prod_{i=1}^n (O_d(\mathbf{P}_i) \cup \{\top_d^{\mathbf{P}_i}\}),$$

where $\bot_d^{\mathbf{P}_i}$ *and* $\top_d^{\mathbf{P}_i}$ *denote the bottom and top elements of* $\mathbf{C}_d(\mathbf{P}_i)$*, respectively. Let* $\mathcal{KO}_d$ *be the set of* clopen *elements of* $\prod_{i=1}^n \mathbf{C}_d(\mathbf{P}_i)$*.*

Proposition 6.3.24. *The completion* $(\prod_{i=1}^n \mathbf{C}_d(\mathbf{P}_i), \beta_d)$ *is dense with respect to the sets* $\mathcal{K}_d$ *and* $\mathcal{O}_d$*, i.e., for every* $\vec{X} \in \prod_{i=1}^n \mathcal{C}_d(\mathbf{P}_i)$ *we have* $\vec{X} = \bigvee\{\vec{Y} \in \mathcal{K}_d : \vec{Y} \leq \vec{X}\} = \bigwedge\{\vec{Z} \in \mathcal{O}_d : \vec{Z} \geq \vec{X}\}$.

Proof. Clearly $\bigvee\{\vec{Y} \in \mathcal{K}_d : \vec{Y} \leq \vec{X}\} \leq \vec{X}$ and $\vec{X} \leq \bigwedge\{\vec{Z} \in \mathcal{O}_d : \vec{Z} \geq \vec{X}\}$.

Let $\mathcal{T} = \{\vec{Y} \in \mathcal{K}_d : \vec{Y} \leq \vec{X}\}$ and $T_i = \{Y \in \mathcal{C}_d(\mathbf{P}_i) : Y = Y_i \text{ for some } \vec{Y} \in \mathcal{T}\}$ for $i = 1, \ldots, n$. Then $\bigvee \mathcal{T} = (\bigvee T_1, \ldots, \bigvee T_n)$. Furthermore, let $S_i = \{Y \in K_d(\mathbf{P}_i) : Y \leq X_i\}$. Then $\vec{X} = (\bigvee S_1, \ldots, \bigvee S_n)$. Let $Y \in S_i$, then $(\bot_d^{\mathbf{P}_1}, \ldots, Y, \ldots, \bot_d^{\mathbf{P}_n}) \in \mathcal{T}$, which implies that $Y \in T_i$. Thus, $S_i \subseteq T_i$ and $\bigvee S_i \leq \bigvee T_i$. Therefore, $\vec{X} = (\bigvee S_1, \ldots, \bigvee S_n) \leq (\bigvee T_1, \ldots, \bigvee T_n) = \bigvee \mathcal{T}$.

The proof that $\vec{X} \geq \bigwedge\{\vec{Z} \in \mathcal{O}_d : \vec{Z} \geq \vec{X}\}$ is similar. □

Lemma 6.3.25. *In* $\prod_{i=1}^n \mathcal{C}_d(\mathbf{P}_i)$*, we have* $\mathcal{KO}_d = \beta_d\left(\prod_{i=1}^n P_i\right)$.

Proof. By Lemma 6.1.12 we have $\alpha_d(P_i) = KO_d(\mathbf{P}_i)$ for $i = 1, \ldots, n$. Therefore $\beta_d(\prod_{i=1}^n P_i) \subseteq \mathcal{KO}_d$.

Let $\vec{X} \in \mathcal{KO}_d$. For $i = 1, \ldots, n$, if $X_i = \bot_d^{\mathbf{P}_i}$, then $\bot_d^{\mathbf{P}_i} \in O_d(\mathbf{P}_i)$ or $\bot_d^{\mathbf{P}_i} = \top_d^{\mathbf{P}_i}$. If $\bot_d^{\mathbf{P}_i} \in O_d(\mathbf{P}_i)$, then there exists $I \in \mathcal{I}_d(\mathbf{P}_i)$ such that $\bot_d^{\mathbf{P}_i} = \bigvee \alpha_d^{\mathbf{P}_i}(I)$. But $I \neq \varnothing$ implies that P_i is a singleton and $X_i \in \alpha_d^{\mathbf{P}_i}(P_i)$. If $\bot_d^{\mathbf{P}_i} = \top_d^{\mathbf{P}_i} = \mathcal{F}_d(\mathbf{P}_i) \neq \varnothing$, then P_i is a singleton and $X_i \in \alpha_d^{\mathbf{P}_i}(P_i)$.

In the same way we can show that $X_i \in \alpha_d^{\mathbf{P}_i}(P_i)$ if $X_i = \top_d^{\mathbf{P}_i}$ for some $i = 1, \ldots, n$.

If $X_i \neq \bot_d^{\mathbf{P}_i}$ and $X_i \neq \top_d^{\mathbf{P}_i}$, then $X_i \in KO_d(\mathbf{P}_i)$ and $X_i \in \alpha_d^{\mathbf{P}_i}(P_i)$ by Lemma 6.1.12.

Therefore, $\vec{X} \in \mathcal{KO}_d$ implies that $X_i \in \alpha_d^{\mathbf{P}_i}(P_i)$ for each $i = 1, \ldots, n$. It then follows that $\vec{X} \in \beta_d(\prod_{i=1}^n P_i)$ and $\mathcal{KO}_d \subseteq \beta_d(\prod_{i=1}^n P_i)$. □

Definition 6.3.26. *For an order-preserving* n*-ary map* $f : \prod_{i=1}^n P_i \to Q$*, define the extensions* $f_d^\sigma : \prod_{i=1}^n \mathcal{C}_d(\mathbf{P}_i) \to \mathcal{C}_d(\mathbf{Q})$ *and* $f_d^\pi : \prod_{i=1}^n \mathcal{C}_d(\mathbf{P}_i) \to \mathcal{C}_d(\mathbf{Q})$ *by, for all* $\vec{X} \in \prod_{i=1}^n \mathcal{C}_d(\mathbf{P}_i)$*,*

$$f_d^\sigma(\vec{X}) = \bigvee \left\{ \bigwedge \left\{ \alpha_d^{\mathbf{Q}}(f(\vec{a})) : \vec{a} \in \prod_{i=1}^n P_i, \vec{Y} \leq \beta_d(\vec{a}) \right\} : \vec{X} \geq \vec{Y} \in \mathcal{K}_d \right\},$$

$$f_d^\pi(\vec{X}) = \bigwedge \left\{ \bigvee \left\{ \alpha_d^{\mathbf{Q}}(f(\vec{a})) : \vec{a} \in \prod_{i=1}^n P_i, \beta_d(\vec{a}) \leq \vec{Z} \right\} : \vec{X} \leq \vec{Z} \in \mathcal{O}_d \right\}.$$

Lemma 6.3.27. *If* $f : \prod_{i=1}^n P_i \to Q$*, then* f_d^σ *and* f_d^π *extend* f*, i.e., for* $\vec{a} \in \prod_{i=1}^n P_i$ *we have* $f_d^\sigma(\beta_d(\vec{a})) = \alpha_d^{\mathbf{Q}}(f(\vec{a})) = f_d^\pi(\beta_d(\vec{a}))$.

Using Lemma 6.3.25 we can show the above similarly to the way in which the analogous claims are proved for the other cases.

Example 6.3.28. *Consider the poset* $\mathbf{P}'$ *and binary map* $\rho_1 : P' \times P' \to P'$ *from Example 6.3.22. Recall that* $\mathbf{P}'$ *is the* 2*-element anti-chain and* ρ_1 *the projection map on the first coordinate of* $\mathbf{P}' \times \mathbf{P}'$*. Then both* $(\rho_1)^\sigma_d$ *and* $(\rho_1)^\pi_d$ *are the projection map on the first coordinate of* $\mathbf{C}_d \times \mathbf{C}_d$*.*

If each $\mathbf{P}_i$ is bounded then our extensions of an order-preserving n-ary map f correspond with the extensions f^1_d and f^2_d defined in [DGP05]. Consider the following to see why. Let $\rho_i : \mathcal{F}_d(\prod_{i=1}^n \mathbf{P}_i) \to \mathcal{P}(P_i)$ be defined by, for $F \in \mathcal{F}_d(\prod_{i=1}^n \mathbf{P}_i)$

$$\rho_i(F) = \{a \in P_i : \text{there exists } \vec{a} \in F \text{ such that } a = a_i\}.$$

In [GJP, Proposition 6.13] it was shown that if $\rho_i(F) \in \mathcal{F}_d(\mathbf{P}_i)$ for all $F \in \mathcal{F}(\prod_{i=1}^n \mathbf{P}_i)$, then $\mathbf{F}_d(\prod_{i=1}^n \mathbf{P}_i) = \prod_{i=1}^n \mathbf{F}_d(\mathbf{P}_i)$. Furthermore, as stated earlier, in [GJP, Proposition 6.12] it was shown that if each $\mathbf{P}_i$ is bounded and $\mathcal{F}_d(\prod_{i=1}^n \mathbf{P}_i) = \prod_{i=1}^n \mathcal{F}_d(\mathbf{P}_i)$, then $\mathbf{C}_d(\prod_{i=1}^n \mathbf{P}_i) = \prod_{i=1} \mathbf{C}_d(\mathbf{P}_i)$. Though not shown in [GJP], it is an easy exercise to show that $\rho_i(F) \in \mathcal{F}_d(\mathbf{P}_i)$ for all $F \in \mathcal{F}(\prod_{i=1}^n \mathbf{P}_i)$. Hence, if each $\mathbf{P}_i$, $i = 1, \ldots, n$, is bounded, then $\mathcal{K}_d(\prod_{i=1}^n \mathbf{P}_i) = \prod_{i=1}^n K_d(\mathbf{P}_i)$ and it follows that $f^\sigma_d = f^1_d$ and $f^\sigma_d = f^2_d$.

We have the following for the extensions of order-preserving n-ary maps.

Lemma 6.3.29. *Let* $* \in \{p, dp, f, d\}$ *and let* $f : \prod_{i=1}^n P_i \to Q$ *be order-preserving. Then,*

(i) f^σ_* *and* f^π_* *are order-preserving.*

(ii) $f^\sigma_* \leq f^\pi_*$ *under the point-wise ordering.*

(iii) we have the following simplifications:

a) $f^\sigma_*(\vec{Y}) = \bigwedge\{\alpha^{\mathbf{Q}}_*(f(\vec{a})) : \vec{a} \in \prod_{i=1}^n P_i, \vec{Y} \leq \beta_*(\vec{a})\}$ *for all* $\vec{Y} \in \mathcal{K}_*$.

b) $f^\sigma_*(\vec{X}) = \bigvee\{f^\sigma_*(\vec{Y}) : \vec{X} \geq \vec{Y} \in \mathcal{K}_*\}$ *for all* $\vec{X} \in \prod_{i=1}^n \mathcal{C}_*(P_i)$.

c) $f^\pi_*(\vec{Z}) = \bigvee\{\alpha^{\mathbf{Q}}_*(f(\vec{a})) : \vec{a} \in \prod_{i=1}^n P_i, \vec{Z} \geq \beta_*(\vec{a})\}$ *for all* $\vec{Z} \in \mathcal{O}_*$.

d) $f^\pi_*(\vec{X}) = \bigwedge\{f^\pi_*(\vec{Z}) : \vec{X} \leq \vec{Z} \in \mathcal{O}_*\}$ *for all* $\vec{X} \in \prod_{i=1}^n \mathcal{C}_*(\mathbf{P}_i)$.

(iv) $f_*^\sigma = f_*^\pi$ *on* $\mathcal{K}_* \cup \mathcal{O}_*$. *Moreover, if* $* \in \{p, dp, f\}$, *then* f_*^σ *and* f_*^π *map elements in* $\mathcal{K}_*$ *to elements in* $K_*(\mathbf{Q})$*; and elements in* $\mathcal{O}_*$ *to elements in* $O_*(\mathbf{Q})$.

Proof. If $* \in \{p, dp, f\}$, then the proofs of (i) to (iv) are similar to the proofs of (i) to (iv) of Lemma 6.3.7. Similarly for (i), (iii) and (iv) if $*$ is d.

Let $*$ be d, then the following proves (ii):

Recall that in a (complete) lattice we have that $\bigvee S \leq \bigwedge T$ if, and only if, $a \leq b$ for all $a \in S$ and all $b \in T$ where $S, T \subseteq P$. To prove that $f_d^\sigma \leq f_d^\pi$, we need to show that, for every $\vec{Y} \in \mathcal{K}_d$ and every $\vec{Z} \in \mathcal{O}_d$, such that $\vec{Y} \leq \vec{X} \leq \vec{Z}$,

$$\bigwedge\{\alpha_d^{\mathbf{Q}}(f(\vec{a})) : \vec{a} \in \prod_{i=1}^n P_i, \vec{Y} \leq \beta_d(\vec{a})\} \leq \bigvee\{\alpha_d^{\mathbf{Q}}(f(\vec{a})) : \vec{a} \in \prod_{i=1}^n P_i, \vec{Z} \geq \beta_d(\vec{a})\}$$

Let $\vec{Y} \in \mathcal{K}_d$ and $\vec{Z} \in \mathcal{O}_d$. For $i = 1, \ldots, n$, we consider the various possible combinations of Y_i's and Z_i's and construct an element $\vec{c} = (c_1, \ldots, c_2) \in \prod_{i=1}^n P_i$ such that $\vec{Y} \leq \beta_d(\vec{c}) \leq \vec{Z}$.

- If $Y_i = \bot_d^{\mathbf{P}_i}$ and $Z_i = \top_d^{\mathbf{P}_i}$, let c_i be any element in P_i. Then $\bot_d^{\mathbf{P}_i} \leq \alpha_d^{\mathbf{P}_i}(c_i) \leq \top_d^{\mathbf{P}_i}$.
- If $Y_i = \bot_d^{\mathbf{P}_i}$ and $Z_i \in O_d(\mathbf{P}_i)$, then there exists a $J \in \mathcal{I}_d(\mathbf{P}_i)$ such that $Z_i = \bigvee \alpha_d^{\mathbf{P}_i}(J)$. Let $c_i \in J$, then $\bot_d^{\mathbf{P}_i} \leq \alpha_d^{\mathbf{P}_i}(c_i) \leq \bigvee \alpha_d^{\mathbf{P}_i}(J) = Z_i$.
- If $Y_i \in K_d(\mathbf{P}_i)$ and $Z_i = \top_d^{\mathbf{P}_i}$, then there exists a $G \in \mathcal{F}_d(\mathbf{P}_i)$ such that $Y_i = \bigwedge \alpha_d^{\mathbf{P}_i}(G)$. Let $c_i \in G$, then $Y_i = \bigwedge \alpha_d^{\mathbf{P}_i}(G) \leq \alpha_d^{\mathbf{P}_i}(c_i) \leq \top_d^{\mathbf{P}_i}$.
- If $Y_i \in K_d(\mathbf{P}_i)$ and $Z_i \in O_d(\mathbf{P}_i)$, then there exist $G \in \mathcal{F}_d(\mathbf{P}_i)$ and $J \in \mathcal{I}_d(\mathbf{P}_i)$ such that $Y_i = \bigwedge \alpha_d^{\mathbf{P}_i}(G)$ and $Z_i = \bigvee \alpha_d^{\mathbf{P}_i}(J)$. But then $\bigwedge \alpha_d^{\mathbf{P}_i}(G) \leq X_i \leq \bigvee \alpha_d^{\mathbf{P}_i}(J)$. By the internal compactness of $\mathbf{C}_d(\mathbf{P}_i)$ it follows that $G \cap J \neq \varnothing$. Let $c_i \in G \cap J$, then $\bigwedge \alpha_d^{\mathbf{P}_i}(G) \leq \alpha_d^{\mathbf{P}_i}(c_i) \leq \bigvee \alpha_d^{\mathbf{P}_i}(J)$.

Then,

$$\begin{aligned}
&\bigwedge\{\alpha_d^{\mathbf{Q}}(f(\vec{a})) : \vec{a} \in \prod_{i=1}^n P_i, \vec{Y} \leq \beta_d(\vec{a})\} \\
&\leq \alpha_d^{\mathbf{Q}}(f(\vec{c})) \\
&\leq \bigvee\{\alpha_d^{\mathbf{Q}}(f(\vec{a})) : \vec{a} \in \prod_{i=1}^n P_i, \vec{Z} \geq \beta_d(\vec{a})\}
\end{aligned}$$

and the result follows. □

If we modify Examples 6.3.8 and 6.3.10 suitably, we can construct counterexamples showing that f^σ_* need not be an operator if f is, for $* \in \{dp, f, d\}$.

We now focus on binary residuated operators. Let $\mathbf{P} = \langle P, \leq^{\mathbf{P}} \rangle$, $\mathbf{Q} = \langle Q, \leq^{\mathbf{Q}} \rangle$ and $\mathbf{R} = \langle R, \leq^{\mathbf{R}} \rangle$. Let $\circ : P \times Q \to R$ be a binary residuated operator with left residual $\backslash : P \times R \to Q$ and right residual $/ : Q \times R \to P$. Recall that the dual of the completion of a poset is isomorphic to the completion of its dual. We can therefore view $\backslash$ and $/$ as maps on $\mathbf{P}^\partial \times \mathbf{R}$ and $\mathbf{Q} \times \mathbf{R}^\partial$, respectively. When viewed in this way, the π-extensions, for $* \in \{f, dp\}$, of $\backslash$ and $/$ are described in the following way:

$$Y \backslash^\pi_* Z = \bigvee \{\alpha^{\mathbf{Q}}_*(a \backslash b) : a \in P, b \in R, Y \leq \alpha^{\mathbf{P}}_*(a), Z \geq \alpha^{\mathbf{R}}_*(b)\}$$
$$\text{for all } Y \in K_*(\mathbf{P}) \text{ and } Z \in O_*(\mathbf{R}).$$

$$X_1 \backslash^\pi_* X_2 = \bigwedge \{Y \backslash^\pi_* Z : X_1 \geq Y \in K_*(\mathbf{P}), X_2 \leq Z \in O_*(\mathbf{R})\}$$
$$\text{for all } X_1 \in P \text{ and } X_2 \in R.$$

In [GJKO07, Lemma 6.15] it was claimed that the σ-extension, $\circ^\sigma_{dp}$, of a binary residuated operator $\circ$ with left and right residuals $\backslash$ and $/$, respectively, has left and right residuals $\backslash^\pi_{dp}$ and $/^\pi_{dp}$, respectively. However, if $\mathbf{P}, \mathbf{Q}$ and $\mathbf{R}$ are posets, then the following example shows that $\backslash^\pi_*$ need not be the left residual of $\circ^\sigma_*$ for $* \in \{dp, f\}$. We note that if $\mathbf{P}$ and $\mathbf{Q}$ are meet-semilattices and $\mathbf{R}$ is a join-semilattice, then the proof of [GJKO07, Lemma 6.15] proves that $\circ^\sigma_{dp}$ has residuals $\backslash^\pi_{dp}$ and $/^\pi_{dp}$.

Example 6.3.30. *Let $* \in \{dp, f\}$. Let $\mathbf{P}'$ be the poset depicted in Figure 6.9 and define $\circ, \backslash, / : P' \times P' \to P'$ as in Table 6.1. Then $\circ$ is a binary residuated operator with left and right residuals $\backslash$ and $/$.*

Observe that $F = \{1, 2, 3\} \in \mathcal{F}_*$ and $Y = \bigwedge \alpha_*(F) \in \mathcal{K}_*$. Furthermore, $I = \{4, 5, 6\} \in \mathcal{I}_*$ and $Z = \bigvee \alpha_*(I) \in \mathcal{O}_*$. Then $\alpha_*(2) \wedge \alpha_*(3) = Y > Z = \alpha_*(4) \vee \alpha_*(5)$.

Now, $Y \circ^\sigma_* Y = \alpha_*(6)$ since $2 \circ 3 = 4$, $3 \circ 2 = 5$ and $4 \wedge 5 = 6$. If $\backslash^\pi_*$ is the left-residual of $\circ^\sigma_*$, then $Y \leq Y \backslash^\pi_* \alpha_*(6)$ by the residuation. However, $Y \backslash^\pi_* \alpha_*(6) = \bigvee \{\alpha_*(1 \backslash 6), \alpha_*(2 \backslash 6), \alpha_*(3 \backslash 6)\} = \bigvee \{\alpha_*(6), \alpha_*(4), \alpha_*(5)\} = Z < Y$, which violates the residuation condition in the above. Hence, $\backslash^\pi_*$ is not the left-residual of $\circ^\sigma_*$.

Note that since $\circ^\sigma_*$ is a complete operator it must be residuated. See Example A.2.6 in Appendix A.2 for more details on the completion.

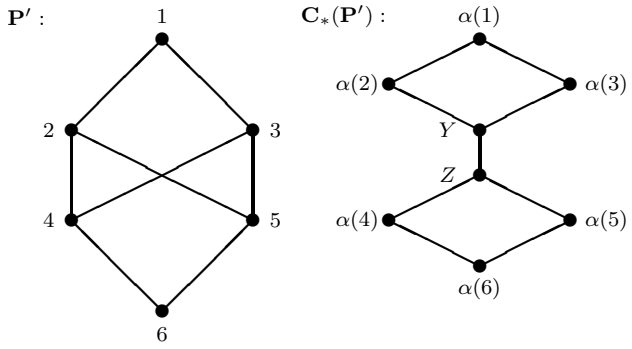


Fig. 6.9: $\backslash_*^\pi$ need not be the left residual of $\circ_*^\sigma$.

The question of whether or not $\circ_*^\sigma$ is residuated when $\circ$ is, for $* \in \{dp, f, d\}$, is still open. If one could prove that $\circ_*^\sigma$ is always a complete operator when $\circ$ is residuated, then it would follow that $\circ_*^\sigma$ is residuated.

In [DGP05, Proposition 3.6] it was shown that if $\circ$ is a binary residuated operator with left and right residuals $\backslash$ and $/$, respectively, then $\circ_d^1$ has left and right residuals $\backslash_d^2$ and $/_d^2$, respectively. Recall that if $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$ are bounded, then $\circ_d^\sigma = \circ_d^1$, $\backslash_d^\pi = \backslash_d^2$ and $/_d^\pi = /_d^2$. If $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$ are not bounded, then $\circ_d^1$ is essentially an operator on $\mathbf{C}_d(\mathbf{P} \times \mathbf{Q})$ rather than on $\mathbf{C}_d(\mathbf{P}) \times \mathbf{C}_d(\mathbf{Q})$.

6.4 An alternative construction of $\mathbf{C}_*$

In [DGP05] and [Suz11] an alternative construction of $\mathbf{C}_d$ (up to isomorphism) was described. This construction is a generalization of the construction of the canonical extension of Boolean algebras with operators described in [GM97]. Here we give a brief description and overview of the construction, but now also using polarizations different from $(\mathcal{F}_d, \mathcal{I}_d)$ to illustrate that, up to isomorphism, the other completions obtained from lattice-consistent polarizations may also be obtained through this construction.

Throughout this section let $\mathbf{P}$ be a fixed poset and $\mathcal{F}$ a fixed family of *non-empty* up-sets of $\mathbf{P}$ that includes all the principal up-sets and such that each member of $\mathcal{F}$ is closed under existing finite meets. Furthermore let $\mathcal{I}$ be a fixed family of *non-empty* down-sets of $\mathbf{P}$ that includes all principal down-sets and such that each member of $\mathcal{I}$ is closed under existing finite joins. That is, $\mathcal{F}$ and $\mathcal{I}$ are rich enough in the sense of [GJKO07] (excluding the empty set) and

$1 \circ 1 = 1$	$2 \circ 1 = 2$	$3 \circ 1 = 3$	$4 \circ 1 = 4$	$5 \circ 1 = 5$	$6 \circ 1 = 6$
$1 \circ 2 = 2$	$2 \circ 2 = 2$	$3 \circ 2 = 5$	$4 \circ 2 = 6$	$5 \circ 2 = 5$	$6 \circ 2 = 6$
$1 \circ 3 = 3$	$2 \circ 3 = 4$	$3 \circ 3 = 3$	$4 \circ 3 = 4$	$5 \circ 3 = 6$	$6 \circ 3 = 6$
$1 \circ 4 = 5$	$2 \circ 4 = 6$	$3 \circ 4 = 5$	$4 \circ 4 = 6$	$5 \circ 4 = 6$	$6 \circ 4 = 6$
$1 \circ 5 = 4$	$2 \circ 5 = 4$	$3 \circ 5 = 6$	$4 \circ 5 = 6$	$5 \circ 5 = 6$	$6 \circ 5 = 6$
$1 \circ 6 = 6$	$2 \circ 6 = 6$	$3 \circ 6 = 6$	$4 \circ 6 = 6$	$5 \circ 6 = 6$	$6 \circ 6 = 6$

$1 \backslash 1 = 1$	$2 \backslash 1 = 1$	$3 \backslash 1 = 1$	$4 \backslash 1 = 1$	$5 \backslash 1 = 1$	$6 \backslash 1 = 1$
$1 \backslash 2 = 2$	$2 \backslash 2 = 1$	$3 \backslash 2 = 2$	$4 \backslash 2 = 1$	$5 \backslash 2 = 1$	$6 \backslash 2 = 1$
$1 \backslash 3 = 3$	$2 \backslash 3 = 3$	$3 \backslash 3 = 1$	$4 \backslash 3 = 1$	$5 \backslash 3 = 1$	$6 \backslash 3 = 1$
$1 \backslash 4 = 5$	$2 \backslash 4 = 3$	$3 \backslash 4 = 5$	$4 \backslash 4 = 1$	$5 \backslash 4 = 3$	$6 \backslash 4 = 1$
$1 \backslash 5 = 4$	$2 \backslash 5 = 4$	$3 \backslash 5 = 2$	$4 \backslash 5 = 2$	$5 \backslash 5 = 1$	$6 \backslash 5 = 1$
$1 \backslash 6 = 6$	$2 \backslash 6 = 4$	$3 \backslash 6 = 5$	$4 \backslash 6 = 2$	$5 \backslash 6 = 3$	$6 \backslash 6 = 1$

$1/1 = 1$	$2/1 = 2$	$3/1 = 3$	$4/1 = 4$	$5/1 = 5$	$6/1 = 6$
$1/2 = 1$	$2/2 = 1$	$3/2 = 3$	$4/2 = 4$	$5/2 = 3$	$6/2 = 4$
$1/3 = 1$	$2/3 = 2$	$3/3 = 1$	$4/3 = 2$	$5/3 = 5$	$6/3 = 5$
$1/4 = 1$	$2/4 = 1$	$3/4 = 1$	$4/4 = 2$	$5/4 = 1$	$6/4 = 2$
$1/5 = 1$	$2/5 = 1$	$3/5 = 1$	$4/5 = 1$	$5/5 = 3$	$6/5 = 3$
$1/6 = 1$	$2/6 = 1$	$3/6 = 1$	$4/6 = 1$	$5/6 = 1$	$6/6 = 1$

Tab. 6.1: The definitions of $\circ$, $\backslash$ and $/$.

therefore the polarization $(\mathcal{F}, \mathcal{I})$ is lattice-consistent.

Now define a binary relation $\sqsubseteq$ on the union $\mathcal{F} \cup \mathcal{I}$ as follows: for all $F, G \in \mathcal{F}$ and all $I, J \in \mathcal{I}$

(i) $F \sqsubseteq G$ if, and only if, $F \supseteq G$,

(ii) $I \sqsubseteq J$ if, and only if, $I \subseteq J$,

(iii) $F \sqsubseteq I$ if, and only if, $F \cap I \neq \varnothing$,

(iv) $I \sqsubseteq F$ if, and only if, for all $a \in I$ and all $b \in F$, $a \leq b$.

The relation $\sqsubseteq$ is a quasi-order on $\mathcal{F} \cup \mathcal{I}$ since it is reflexive and transitive, but not a partial order since it is not antisymmetric: if $F = [a)$ and $I = (a]$ for some

$a \in P$, then $F \sqsubseteq I$ and $I \sqsubseteq F$, but $F \neq I$. Now define the following equivalence relation, $\sim$, on $\mathcal{F} \cup \mathcal{I}$. For $F \in \mathcal{F}$ and $I \in \mathcal{I}$ we have $F \sim F$, $I \sim I$ and

$$F \sim I \text{ if, and only if, } F \sqsubseteq I \text{ and } I \sqsubseteq F.$$

Then $\sim$ identifies the principal filters and ideals generated by the same element, i.e., $[a) \sim (a]$ for all $a \in P$. For $S \in \mathcal{F} \cup \mathcal{I}$ let $[S]_\sim$ denote the equivalence class of S with respect to $\sim$ and let $\mathbb{D} = \{[S]_\sim : S \in \mathcal{F} \cup \mathcal{I}\}$. Let $\sqsubseteq_\mathbb{D}$ be the binary relation on $\mathbb{D}$ defined by:

$$[S]_\sim \sqsubseteq_\mathbb{D} [T]_\sim \text{ if, and only if, } S \sqsubseteq T \text{ for all } S, T \in \mathcal{F} \cup \mathcal{I}.$$

Then $\sqsubseteq_\mathbb{D}$ is reflexive, transitive and antisymmetric and $\mathbf{D} = (\mathbb{D}, \sqsubseteq_\mathbb{D})$ is a poset. The poset $\mathbf{D} = (\mathbb{D}, \sqsubseteq_\mathbb{D})$ is called the *intermediate structure* or *intermediate level* (see for instance [DGP05, Suz11]). Now let $(\mathbf{L}(\mathbf{D}), \iota_\mathbf{D})$, with $\mathbf{L}(\mathbf{D}) = \langle \mathcal{L}(\mathbb{D}), \vee^{\mathbf{L}(\mathbf{D})}, \wedge^{\mathbf{L}(\mathbf{D})} \rangle$, be the MacNeille completion of $\mathbf{D}$. See Chapter 5 for more on the MacNeille completion. Recall that the MacNeille completion can abstractly be defined as the unique (up to isomorphism) completion of $\mathbf{D}$ such that $\mathbf{D}$ is doubly dense in it. It should be clear, from the definition of closed and open elements, that there is a one-to-one correspondence between $K \cup O$ and $\mathbb{D}$. Moreover, since $K \cup O$ is doubly dense in $\mathbf{C}$, it follows that $\mathbf{L}(\mathbf{D})$ is isomorphic to $\mathbf{C}$, the completion obtained from the polarization $(\mathcal{F}, \mathcal{I})$.

Though only dealt with abstractly in the literature, we will now explicitly define the order-embedding of $\mathbf{P}$ into $\mathbf{L}(\mathbf{D})$. Define $\iota_\mathbf{P} : P \to \mathbb{D}$ by $\iota_P(a) = [(a]]_\sim = [[a)]_\sim$ for all $a \in P$. Then $\iota_\mathbf{P}$ is one-to-one. Furthermore, recall that $\iota_\mathbf{D} : \mathbb{D} \to \mathcal{L}(\mathbb{D})$ is defined by $\iota_\mathbf{D}([S]_\sim) = [S]^u_\sim$. Then the composition of $\iota_\mathbf{P}$ with $\iota_\mathbf{D}$, i.e., $\iota_\mathbf{D} \cdot \iota_\mathbf{P}$, is an order-embedding of $\mathbf{P}$ into $\mathbf{L}(\mathbf{D})$. For $a \in P$ we have $\iota_\mathbf{D}(\iota_\mathbf{P}(a)) = \iota_\mathbf{D}([[a)]_\sim) = [[a)]^u_\sim$.

Lemma 6.4.1. *Let $\mathbf{L}(\mathbf{D})$ be the complete lattice obtained through the construction described above. Then $\iota_\mathbf{D} \cdot \iota_\mathbf{P}$ preserves existing finite joins and meets in $\mathbf{P}$, i.e., if $M, N \subseteq^{fin} P$ such that $\bigvee M$ and $\bigwedge N$ exist in $\mathbf{P}$, then $\iota_\mathbf{D}(\iota_\mathbf{P}(\bigvee M)) = \bigvee^{\mathbf{L}(\mathbf{D})} \iota_\mathbf{D}(\iota_\mathbf{P}(M))$ and $\iota_\mathbf{D}(\iota_\mathbf{P}(\bigwedge N)) = \bigwedge^{\mathbf{L}(\mathbf{D})} \iota_\mathbf{D}(\iota_\mathbf{P}(N))$.*

Proof. (i) Let $M \subseteq^{fin} P$ be the set $M = \{a_1, \ldots, a_n\}$ for some $n \in \mathbb{N}$. Then,

for $i = 1, \ldots, n$,

$$\begin{aligned} a_i \leq \bigvee M &\Rightarrow [a_i) \supseteq \left[\bigvee M\right) \\ &\Rightarrow [a_i) \sqsubseteq \left[\bigvee M\right) \\ &\Rightarrow [[a_i)]_\sim \sqsubseteq_{\mathbb{D}} \left[\left[\bigvee M\right)\right]_\sim \\ &\Rightarrow \left[\left[\bigvee M\right)\right]_\sim \in [[a_i)]^u_\sim. \end{aligned}$$

Then, $[[\bigvee M)]_\sim \in \bigcap [[a_i)]^u_\sim$. Hence, $[[\bigvee M)]^u_\sim \subseteq \bigcap_{i=1}^n [[a_i)]^u_\sim$, i.e.,

$$\bigvee^{\mathbf{L(D)}} \iota_{\mathbf{D}}\left(\iota_{\mathbf{P}}\left(M\right)\right) \leq^{\mathbf{L(D)}} \iota_{\mathbf{D}}\left(\iota_{\mathbf{P}}\left(\bigvee M\right)\right).$$

On the other hand, suppose $[S]_\sim \in \bigcap_{i=1}^n [[a_i)]^u_\sim$ for some $S \in \mathcal{F} \cup \mathcal{L}$. Then $[S]_\sim \in [[a_i)]^u_\sim$ for $i = 1, \ldots, n$, i.e., $[[a_i)]_\sim \sqsubseteq_{\mathbb{D}} [S]_\sim$ for $i = 1, \ldots, n$.

- If $S \in \mathcal{F}$, then $S \subseteq [a_i)$ for $i = 1, \ldots, n$. Therefore, $S \subseteq \bigcap_{i=1}^n [a_i) = [\bigvee M)$ which implies that $[\bigvee M) \sqsubseteq S$. Then $[[\bigvee M)]_\sim \sqsubseteq_{\mathbb{D}} [S]_\sim$ and $[S]_\sim \in [[\bigvee M)]^u_\sim$.
- If $S \in \mathcal{I}$, then $[a_i) \cap S \neq \varnothing$ for $i = 1, \ldots, n$. Then $a_i \in S$ for $i = 1, \ldots, n$. But by assumption each member of S is closed under existing joins. Therefore, $\bigvee M \in S$ and $[\bigvee M) \cap S \neq \varnothing$. Then $[[\bigvee M)]_\sim \sqsubseteq_{\mathbb{D}} [S]_\sim$ and $[S]_\sim \in [[\bigvee M)]^u_\sim$.

Thus we have shown that $\bigcap_{i=1}^n [[a_i)]^u_\sim \subseteq [[\bigvee M)]^u_\sim$, i.e.,

$$\iota_{\mathbf{D}}\left(\iota_{\mathbf{P}}\left(\bigvee M\right)\right) \leq^{\mathbf{L(D)}} \bigvee^{\mathbf{L(D)}} \iota_{\mathbf{D}}(\iota_{\mathbf{P}}(M)).$$

Hence, $\iota_{\mathbf{D}} \cdot \iota_{\mathbf{P}}$ preserves existing joins.

(ii) Let $N \subseteq^{fin} P$ be the set $N = \{b_1, \ldots, b_m\}$ for some $m \in \mathbb{N}$. Then, for $i = 1, \ldots, m$

$$\begin{aligned} \bigwedge N \leq b_i &\Rightarrow \left(\bigwedge N\right] \subseteq (b_i] \\ &\Rightarrow \left(\bigwedge N\right] \sqsubseteq (b_i] \\ &\Rightarrow \left[\left(\bigwedge N\right]\right]_\sim \sqsubseteq_{\mathbb{D}} [(b_i]]_\sim \\ &\Rightarrow [(b_i]]^u_\sim \subseteq \left[\left(\bigwedge N\right]\right]^u_\sim. \end{aligned}$$

Then $[(\bigwedge N]]^u_\sim \in \{\mathcal{T} \in \mathcal{L}(\mathbb{D}) : [(b_i]]^u_\sim \subseteq \mathcal{T} \text{ for } i = 1, \ldots, n\}$. Therefore, $\bigcap\{\mathcal{T} \in \mathcal{L}(\mathbb{D}) : [(b_i]]^u_\sim \subseteq \mathcal{T} \text{ for } i = 1, \ldots, n\} \subseteq [(\bigwedge N]]^u_\sim$, i.e.,

$$\iota_{\mathbf{D}}\left(\iota_{\mathbf{P}}\left(\bigwedge N\right)\right) \leq^{\mathbf{L(D)}} \bigwedge^{\mathbf{L(D)}} \iota_{\mathbf{D}}(\iota_{\mathbf{P}}(N)).$$

On the other hand, let $\mathcal{T} \in \mathcal{L}(\mathbb{D})$ such that $[(b_i]]^u_\sim \subseteq \mathcal{T}$ for $i = 1, \ldots, n$. This is equivalent to requiring that $\bigcup_{i=1}^n [(b_i]]^u_\sim \subseteq \mathcal{T}$. Then $(\bigcup_{i=1}^n [(b_i]]^u_\sim)^{\ell u} \subseteq \mathcal{T}^{\ell u} = \mathcal{T}$ since $\mathcal{T} \in \mathcal{L}(\mathbb{D})$. By the properties of Galois connection we have that $(\bigcap_{i=1}^n [(b_i]]^{u\ell}_\sim)^u \subseteq \mathcal{T}$. Now let $S \in \mathcal{F} \cup \mathcal{I}$ such that $[S]_\sim \in \bigcap_{i=1}^n [(b_i]]^{u\ell}_\sim$. Then, $[S]_\sim \in [(b_i]]^{u\ell}_\sim$ for $i = 1, \ldots, n$. Since $[(b_i]]^{u\ell}_\sim = [(b_i]]^{\ell}_\sim$ we have that $[S]_\sim \sqsubseteq_{\mathbb{D}} [(b_i]]_\sim$ for $i = 1, \ldots, n$. Then $S \sqsubseteq (b_i]$ for $i = 1, \ldots, n$.

- If $S \in \mathcal{F}$, then $S \cap (b_i] \neq \varnothing$ for $i = 1, \ldots, n$ and it follows that $N \subseteq^{fin} S$. Since each member of $\mathcal{F}$ is closed under existing meets we have $\bigwedge N \in S$. Then $[\bigwedge N) \subseteq S$ and therefore $S \sqsubseteq [\bigwedge N)$. Now $[S]_\sim \sqsubseteq_{\mathbb{D}} [[\bigwedge N)]_\sim = [(\bigwedge N]]_\sim$ and $[S]_\sim \in [(\bigwedge N]]^{\ell}_\sim$.
- If $S \in \mathcal{I}$, then $S \subseteq (b_i]$ for $i = 1, \ldots, n$ and therefore $S \subseteq \bigcup_{i=1}^n (b_i] = (\bigwedge N]$. Then $S \sqsubseteq (\bigwedge N]$ and $[S]_\sim \sqsubseteq_{\mathbb{D}} [(\bigwedge N]]_\sim$. Hence, $[S]_\sim \in [(\bigwedge N]]^{\ell}_\sim$

We may therefore conclude that $\bigcap_{i=1}^n [(b_i]]^{u\ell}_\sim \subseteq [(\bigwedge N]]^{\ell}_\sim$.

Finally we have that $[(\bigwedge N]]^u_\sim = [(\bigwedge N]]^{\ell u}_\sim \subseteq (\bigcup_{i=1}^n [(b_i]]^{u\ell}_\sim)^u \subseteq \mathcal{T}$ which implies that

$$\bigwedge^{\mathbf{L(D)}} \iota_{\mathbf{D}}(\iota_{\mathbf{P}}(N)) \leq^{\mathbf{L(D)}} \iota_{\mathbf{D}}\left(\iota_{\mathbf{P}}\left(\bigwedge N\right)\right).$$

□

6.5 *Preservation theorems*

Following the methods used in [Jón94] and [GV99] we use approximation terms to obtain preservation results. In particular, we will give a syntactic description of terms s and t for which $s \leq t$ is preserved by the completion. (Recall that we take universal quantification over such expressions as implicit.)

Throughout this section let $* \in \{dp, f, d\}$. In the sequel we will identify $\mathbf{P}$ with the sub-poset of $\mathbf{C}_*$ that $\mathbf{P}$ is isomorphic to. That is, we consider $P \subseteq \mathcal{C}_*$. From here on we will use the letters a, b or c to denote element of $\mathcal{C}_*$; elements of K_* will be denoted by y or y' and elements of O_* will be denoted by z or z'.

Let $\langle P, \{f_i : i \in \Psi\}, \leq\rangle$ be an ordered algebra such that each f_i is order-preserving. Then both f_i^σ and f_i^π are order-preserving and we can take $f_i^{\mathbf{C}_*}$ to mean either of the two. Let t be a term in the language $\{\vee, \wedge, \{f_i : i \in \Psi\}\}$. If the variables occurring in t are in the sequence $\vec{x} = x_1, \ldots, x_n$, then we denote this by $t(\vec{x})$. If $\vec{a} = a_1, \ldots, a_n$ is a sequence of elements of $\mathcal{C}_*$, then $t^{\mathbf{C}_*}(\vec{a})$ denotes the evaluation of t in $\mathbf{C}_*$ under the assignment $x_i \mapsto a_i$.

For each term $t(\vec{x})$ and $\vec{a} \in \mathcal{C}_*^n$ define

$$t_*^\sigma(\vec{a}) = \bigvee \left\{ \bigwedge \left\{ t(\vec{b}) : \vec{b} \in P^n, \vec{y} \leq \vec{b} \leq \vec{z} \right\} : \vec{a} \geq \vec{y} \in \mathcal{K}_*, \vec{a} \leq \vec{z} \in \mathcal{O}_* \right\},$$

$$t_*^\pi(\vec{a}) = \bigwedge \left\{ \bigvee \left\{ t(\vec{b}) : \vec{b} \in P^n, \vec{y} \leq \vec{b} \leq \vec{z} \right\} : \vec{a} \geq \vec{y} \in \mathcal{K}_*, \vec{a} \leq \vec{z} \in \mathcal{O}_* \right\}.$$

For each f_i, $i \in \Psi$, assume that $f_i^{\mathbf{C}_*}$ is a fixed extension of f_i, either f_i^σ or f_i^π, on $\mathbf{C}_*$.

Definition 6.5.1. *A term $t(\vec{x})$ is called*

- σ-stable *if* $t^{\mathbf{C}_*}(\vec{a}) = t_*^\sigma(\vec{a})$,
- σ-expanding *if* $t^{\mathbf{C}_*}(\vec{a}) \geq t_*^\sigma(\vec{a})$,
- σ-contracting *if* $t^{\mathbf{C}_*}(\vec{a}) \leq t_*^\sigma(\vec{a})$,
- π-stable *if* $t^{\mathbf{C}_*}(\vec{a}) = t_*^\pi(\vec{a})$,
- π-expanding *if* $t^{\mathbf{C}_*}(\vec{a}) \geq t_*^\pi(\vec{a})$,
- π-contracting *if* $t^{\mathbf{C}_*}(\vec{a}) \leq t_*^\pi(\vec{a})$,

for all $\vec{a} \in \mathcal{C}_*^n$.

Lemma 6.5.2. *If $\langle P, \{f_i : i \in \Psi\}, \leq\rangle$ satisfies $s(\vec{x}) \leq t(\vec{x})$, then $s_*^\sigma(\vec{a}) \subseteq t_*^\sigma(\vec{a})$ and $s_*^\pi(\vec{a}) \subseteq t_*^\pi(\vec{a})$ for all $\vec{a} \in \mathcal{C}_*^n$.*

Proof. Let $\vec{a} \in \mathcal{C}_*^n$, $\vec{y} \in \mathcal{K}_*$ and $\vec{z} \in \mathcal{O}_*$ such that $\vec{y} \leq \vec{a} \leq \vec{z}$. If $\vec{b} \in P^n$ such that $\vec{y} \leq \vec{b} \leq \vec{z}$, then $s(\vec{b}) \leq t(\vec{b})$ since $\langle P, \{f_i : i \in \Psi\}, \leq\rangle$ satisfies $s(\vec{x}) \leq t(\vec{x})$. Then,

$$\bigwedge \left\{ s(\vec{b}) : \vec{b} \in P^n, \vec{y} \leq \vec{b} \leq \vec{z} \right\} \leq \bigwedge \left\{ t(\vec{b}) : \vec{b} \in P^n, \vec{y} \leq \vec{b} \leq \vec{z} \right\}$$

and

$$\bigvee \left\{ s(\vec{b}) : \vec{b} \in P^n, \vec{y} \leq \vec{b} \leq \vec{z} \right\} \leq \bigvee \left\{ t(\vec{b}) : \vec{b} \in P^n, \vec{y} \leq \vec{b} \leq \vec{z} \right\}.$$

Therefore, $s_*^\sigma(\vec{a}) \leq t_*^\sigma(\vec{a})$ and $s_*^\pi(\vec{a}) \leq t_*^\pi(\vec{a})$. □

Corollary 6.5.3. *If s is a σ-contracting term and t is a σ-expanding term, or s is a π-contracting term and t is a π-expanding term, then $s \leq t$ is preserved by the completion.*

Proof. Suppose $\langle P, \{f_i : i \in \Psi, \leq\}\rangle$ satisfies $s \leq t$, where s is a σ-contracting term and t is a σ-expanding term. Then by Lemma 6.5.2, $s^{\mathbf{C}_*}(\vec{a}) \leq s^{\sigma}_*(\vec{a}) \leq t^{\sigma}_*(\vec{a}) \leq t^{\mathbf{C}_*}(\vec{a})$. The proof is similar if s is π-contracting and t is π-expanding. □

We note that if s is σ-stable, then s is σ-contracting and if t is σ-stable, then t is σ-expanding. Thus, if s is σ-stable and t is σ-stable, then $s \leq t$ is preserved by the completion. Similarly, if s is π-stable and t is π-stable, then $s \leq t$ is preserved by the completion.

Lemma 6.5.4. *Let s_1 and s_2 be σ-contracting terms, i.e., $s_1^{\mathbf{C}_*}(\vec{a}) \leq (s_1)^{\sigma}_*(\vec{a})$ and $s_2^{\mathbf{C}_*}(\vec{a}) \leq (s_2)^{\sigma}_*(\vec{a})$ and for all $\vec{a} \in \mathcal{C}^n_*$. Let $t(\vec{x}) = s_1(\vec{x}) \vee s_2(\vec{x})$. Then t is a σ-contracting term.*

Proof. Let $\vec{a} \in \mathcal{C}^n_*$. Then,

$$\begin{aligned}
& t^{\mathbf{C}_*}(\vec{a}) \\
&= s_1^{\mathbf{C}_*}(\vec{a}) \vee s_2^{\mathbf{C}_*}(\vec{a}) \\
&\leq (s_1)^{\sigma}_*(\vec{a}) \vee (s_2)^{\sigma}_*(\vec{a}) \\
&= \bigvee \left\{ \bigwedge \left\{ s_1(\vec{b}) : \vec{b} \in P^n, \vec{y} \leq \vec{b} \leq \vec{z} \right\} : \vec{a} \geq \vec{y} \in \mathcal{K}, \vec{a} \leq \vec{z} \in \mathcal{O} \right\} \vee \\
&\quad \bigvee \left\{ \bigwedge \left\{ s_2(\vec{b}) : \vec{b} \in P^n, \vec{y} \leq \vec{b} \leq \vec{z} \right\} : \vec{a} \geq \vec{y} \in \mathcal{K}, \vec{a} \leq \vec{z} \in \mathcal{O} \right\} \\
&= \bigvee \left\{ \bigwedge \left\{ s_1(\vec{b}) : \vec{b} \in P^n, \vec{y} \leq \vec{b} \leq \vec{z} \right\} \vee \right. \\
&\qquad \left. \bigwedge \left\{ s_2(\vec{b}) : \vec{b} \in P^n, \vec{y} \leq \vec{b} \leq \vec{z} \right\} : \vec{a} \geq \vec{y} \in \mathcal{K}, \vec{a} \leq \vec{z} \in \mathcal{O} \right\}.
\end{aligned}$$

Let $\vec{y} \in \mathcal{K}$ and $\vec{z} \in \mathcal{O}$ such that $\vec{y} \leq \vec{a} \leq \vec{z}$. If $\vec{b} \in P^n$ such that $\vec{y} \leq \vec{b} \leq \vec{z}$, then $s_1(\vec{b}) \leq s_1(\vec{b}) \vee s_2(\vec{b})$ and $s_2(\vec{b}) \leq s_1(\vec{b}) \vee s_2(\vec{b})$. Therefore,

$$\bigwedge \{ s_1(\vec{b}) : \vec{b} \in P^n, \vec{y} \leq \vec{b} \leq \vec{z} \} \leq \bigwedge \{ s_1(\vec{b}) \vee s_2(\vec{b}) : \vec{b} \in P^n, \vec{y} \leq \vec{b} \leq \vec{z} \}$$

and

$$\bigwedge \{ s_2(\vec{b}) : \vec{b} \in P^n, \vec{y} \leq \vec{b} \leq \vec{z} \} \leq \bigwedge \{ s_1(\vec{b}) \vee s_2(\vec{b}) : \vec{b} \in P^n, \vec{y} \leq \vec{b} \leq \vec{z} \}.$$

Then,

$$\bigwedge\{s_1(\vec{b}) : \vec{b} \in P^n, \vec{y} \leq \vec{b} \leq \vec{z}\} \vee \bigwedge\{s_2(\vec{b}) : \vec{b} \in P^n, \vec{y} \leq \vec{b} \leq \vec{z}\}$$
$$\leq \bigwedge\{s_1(\vec{b}) \vee s_2(\vec{b}) : \vec{b} \in P^n, \vec{y} \leq \vec{b} \leq \vec{z}\}$$

so $t^{\mathbf{C}_*}(\vec{a}) \leq t^{\sigma}_*(\vec{a})$. Hence, t is σ-contracting. □

Lemma 6.5.5. *Let s_1 and s_2 be π-expanding terms, i.e., $s_1^{\mathbf{C}_*}(\vec{a}) \geq (s_1)^{\pi}_*(\vec{a})$ and $s_2^{\mathbf{C}_*}(\vec{a}) \geq (s_2)^{\pi}_*(\vec{a})$ for all $\vec{a} \in \mathcal{C}^n_*$. Let $t(\vec{x}) = s_1(\vec{x}) \wedge s_2(\vec{x})$. Then t is a π-expanding term.*

The proof follows the dual argument to that used in the proof of Lemma 6.5.4.

We now consider terms involving additional operations. By Lemmas 6.3.7 and 6.3.29 we have that $f^{\sigma}_* \leq f^{\pi}_*$ under the point-wise ordering if f is order-preserving. If every operation occurring in a term t is order-preserving, then the term (function) t is order-preserving. Moreover, $t^{\sigma}_* \leq t^{\pi}_*$. Let $f : P \to P$ be a residuated operator with residual $g : P \to P$. Then f^{σ}_* and g^{π}_* form a residuated pair on $\mathbf{C}_*$ by Lemmas 6.3.13 and 6.3.14. For the remainder of this section we consider only terms from the language $\{\vee, \wedge, f, g\}$ and assume that $f^{\mathbf{C}_*}$ is the extension f^{σ}_* of f; while $g^{\mathbf{C}_*}$ will be the extension g^{π}_* of g. Then all terms under consideration from now on are order-preserving. Furthermore, by Lemma 6.3.7, we have the following simplification of our approximations:

$$t^{\sigma}_*(\vec{a}) = \bigvee\left\{\bigwedge\left\{t(\vec{b}) : \vec{y} \leq \vec{b} \in P^n\right\} : \vec{a} \geq \vec{y} \in \mathcal{K}_*\right\},$$

$$t^{\pi}_*(\vec{a}) = \bigwedge\left\{\bigvee\left\{t(\vec{b}) : \vec{z} \geq \vec{b} \in P^n\right\} : \vec{a} \leq \vec{z} \in \mathcal{O}_*\right\}.$$

Now suppose a term t is σ-contracting, i.e., $t^{\mathbf{C}_*}(\vec{a}) \leq t^{\sigma}_*(\vec{a})$ for all $\vec{a} \in \mathcal{C}^n_*$. Then $t^{\sigma}_*(\vec{a}) \leq t^{\pi}_*(\vec{a})$ for all $\vec{a} \in \mathcal{C}^n_*$ and t is π-contracting. On the other hand, suppose t is π-expanding, i.e., $t^{\mathbf{C}_*}(\vec{a}) \geq t^{\pi}_*(\vec{a})$. Then $t^{\pi}_*(\vec{a}) \geq t^{\sigma}_*(\vec{a})$ for all $\vec{a} \in \mathcal{C}^n_*$ and t is σ-expanding.

By definition $f(x)$ is σ-stable and therefore also π-contracting. Similarly, $g(x)$ is π-stable and therefore also σ-expanding.

We will call a term $t(\vec{x})$ *totally defined* if $t(\vec{a})$ is defined in P, i.e., $t(\vec{a})$ exists and $t(\vec{a}) \in P$, for all $\vec{a} \in P^n$. We note that since not all finite joins and meets exist in P, only $\{f, g\}$-terms will be totally defined terms.

Lemma 6.5.6. *Let s be a totally defined σ-contracting term, i.e., $s^{\mathbf{C}_*}(\vec{a}) \leq s^{\sigma}_*(\vec{a})$ for all $\vec{a} \in \mathcal{C}^n_*$. Let $t(\vec{x}) = f(s(\vec{x}))$. Then t is a σ-contracting term.*

Proof. Let $\vec{a} \in \mathcal{C}_*^n$. Then,

$$\begin{aligned}
& t^{\mathbf{C}_*}(\vec{a}) \\
=& f_*^\sigma(s^{\mathbf{C}_*}(\vec{a})) \\
\leq& f_*^\sigma(s_*^\sigma(\vec{a})) \\
=& f_*^\sigma\left(\bigvee\left\{\bigwedge\{s(\vec{c}) : \vec{c} \in P^n, \vec{y} \leq \vec{c}\} : \vec{a} \geq \vec{y} \in \mathcal{K}_*\right\}\right) \\
=& \bigvee\left\{f_*^\sigma\left(\bigwedge\{s(\vec{c}) : \vec{c} \in P^n, \vec{y} \leq \vec{c}\}\right) : \vec{a} \geq \vec{y} \in \mathcal{K}_*\right\}
\end{aligned}$$

where the final equality follows from the fact that f_*^σ is residuated on $\mathbf{C}_*$ and therefore a complete operator.

Let $\vec{y} \in \mathcal{K}_*$ such that $\vec{y} \leq \vec{a}$. Then,

$$\begin{aligned}
& f_*^\sigma\left(\bigwedge\{s(\vec{c}) : \vec{c} \in P^n, \vec{y} \leq \vec{c}\}\right) \\
=& \bigvee\left\{\bigwedge\{f(b) : b \in P, y' \leq b\} : \bigwedge\{s(\vec{c}) : \vec{c} \in P^n, \vec{y} \leq \vec{c}\} \geq y' \in K_*\right\}.
\end{aligned}$$

Let $y' \in K_*$ such that $y' \leq \bigwedge\{s(\vec{c}) : \vec{c} \in P^n, \vec{y} \leq \vec{c}\}$. Then, $y' \leq s(\vec{c})$ for all $\vec{c} \in P^n$ such that $\vec{y} \leq \vec{c}$. But $s(\vec{c}) \in P$, since s is a totally defined term. Then,

$$\begin{aligned}
& \{f(b) : b \in P, y' \leq b\} \supseteq \{f(s(\vec{c})) : \vec{c} \in P^n, \vec{y} \leq \vec{c}\} \\
\Rightarrow & \bigwedge\{f(b) : b \in P, y' \leq b\} \leq \bigwedge\{f(s(\vec{c})) : \vec{c} \in P^n, \vec{y} \leq \vec{c}.\}
\end{aligned}$$

By the above we have that

$$\begin{aligned}
& \bigvee\left\{\bigwedge\{f(b) : b \in P, y' \leq b\} : \bigwedge\{s(\vec{c}) : \vec{c} \in P^n, \vec{y} \leq \vec{c}\} \geq y' \in K_*\right\} \\
\leq & \bigwedge\{f(s(\vec{c})) : \vec{c} \in P^n, \vec{y} \leq \vec{c}\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
t^{\mathbf{C}_*}(\vec{a}) \leq & \bigvee\left\{\bigwedge\{f(s(\vec{c})) : \vec{c} \in P^n, \vec{y} \leq \vec{c}\} : \vec{a} \geq \vec{y} \in \mathcal{K}_*\right\} \\
=& t_*^\sigma(\vec{a}).
\end{aligned}$$

Hence, t is σ-contracting. □

Lemma 6.5.7. *Let s be a totally defined π-expanding term, i.e., $s^{\mathbf{C}_*}(\vec{a}) \geq s_*^\pi(\vec{a})$ for all $\vec{a} \in \mathcal{C}_*^n$. Let $t(\vec{x}) = g(s(\vec{x}))$. Then t is a π-expanding term.*

The proof is similar to the proof of Lemma 6.5.6, but makes use of the fact that g_*^π is a complete dual operator on $\mathbf{C}_*$.

We summarize the above results in the following theorem.

Theorem 6.5.8. *An inequality $s \leq t$ is preserved by the completion $(\mathbf{C}_*, \alpha_*)$, where $\mathbf{C}_* = \langle \mathcal{C}_*, \vee^{\mathbf{C}}, \wedge^{\mathbf{C}}, f_*^{\sigma}, g_*^{\pi} \rangle$, if s is any term built up from variables using $\vee$ and f; and t is any term built up from variables using $\wedge$ and g.*

7. PRIME FILTER COMPLETION

A number of representation theorems for (completely) distributive (complete) lattices can be found in the literature. In [Ran52, Theorem 1] it was shown that a lattice is completely distributive if, and only if, it is a complete homomorphic image of a complete ring of sets. A year later, in [Ran53, Theorem A] the same author showed that every completely distributive complete lattice can be embedded isomorphically into the direct union of a family of complete chains.

Another seven years later he improved the proof of the above representation result in [Ran60, Theorem 7], greatly reducing the number of chains required in the representation, by making use of the following notions. Let $\mathbf{L} = \langle L, \vee, \wedge \rangle$ be a complete lattice with associated ordering relation $\leq$. An ordered pair of elements (a, b) of L is called a *blanket* if, and only if, for every $c \in L$, either $c \geq a$ or $c \leq b$. If, in addition, $a_0 \in L$ such that $a_0 > a$ implies that (a_0, b) is not a blanket and $b_0 \in L$ such that $b_0 < b$ implies that (a, b_0) is not a blanket, then (a, b) is called a *minimax blanket*. Finally, a blanket (a, b) *separates* the elements $c, d \in L$ if, and only if, $c \geq a$ and $d \leq b$.

A combination of [Ran60, Theorems 5, 6 and 7] then gives the following representation result. A complete lattice $\mathbf{L} = \langle L, \vee, \wedge \rangle$ can be embedded isomorphically into the direct union of a family of complete chains if, and only if, for any $c, d \in L$ with $c \not\leq d$, there exists a minimax blanket that separates c and d. The embedding preserves all meets and joins existing in L.

The representation results above are all closely related to Priestley's representation theorem for bounded distributive lattices [Pri70]. A *topology* $\mathscr{T}$ on a set P is a family of subsets of P that contains P and $\varnothing$ and that is closed under arbitrary unions and finite intersections. Let $\mathbf{L} = \langle L, \vee, \wedge \rangle$ be a bounded distributive lattice and let $S_a = \{I \in \mathscr{I}(\mathbf{L}) : a \notin I\}$ and $\mathcal{B} = \{S_a \cap (\mathscr{I}(\mathbf{L}) - S_b) : a, b \in L\}$. Define $\mathscr{T}$ by: $U \in \mathscr{T}$ if, and only if, U is a union of members of $\mathcal{B}$. Then $\mathscr{T}$ is a topology on $\mathscr{I}(\mathbf{L})$ and $\langle \mathscr{I}(\mathbf{L}), \subseteq, \mathscr{T} \rangle$ is called the *dual space of* $\mathbf{L}$ or *the prime ideal space of* $\mathbf{L}$. The sets S_a, for $a \in L$, then form the clopen down-

sets of $\mathscr{I}(\mathbf{L})$. Priestley's representation theorem now states: If $\mathbf{L} = \langle L, \vee, \wedge \rangle$ is a bounded distributive lattice, then the map $a \mapsto S_a$ is an isomorphism of $\mathbf{L}$ onto the lattice of clopen down-sets of the dual space $\langle \mathscr{I}(\mathbf{L}), \subseteq, \mathscr{T} \rangle$ of $\mathbf{L}$. The above representation theorem relied on the following being satisfied by the dual space of a bounded distributive lattice: For any points $x, y \in P$, if $x \not\leq y$, then there exists a clopen upset U of $\mathbf{P}$ such that $x \in U$ and $y \notin U$. This is known as Priestley's separation axiom and is clearly closely related to the notion of separation by a blanket.

In [Jan78] the combined representation result from [Ran60], as stated above, was generalised for the poset setting. We summarize the results from [Jan78] in Section 7.1 and then give a similar result, but with a much simpler construction. We then investigate a possible connection between this construction and the construction of a complete lattice obtained from a polarization, as studied in Chapter 6. Finally we consider possible extensions of maps to the completely distributive complete lattice obtained through the construction.

The work done in this chapter is part of an on-going collaboration with Prof. Clint van Alten [MvAc].

7.1 The construction

The reader is referred to Chapter 4 for the definitions of pseudo and Doyle-pseudo filters (Definitions 4.1.1 and 4.1.2, respectively), complete Doyle-pseudo filters (Definition 4.1.7) and (complete) prime pseudo and Doyle-pseudo filters (Definition 4.2.13).

Firstly, we summarise the results from [Jan78] wherein the authors proved a sub-direct representation of certain posets.

Let $\mathbf{P} = \langle P, \leq \rangle$ be a poset. A pair (F, I), with $F \in \mathcal{F}^{cdp}$ and $I \in \mathcal{I}^{cdp}$, is called a *blanket* if, and only if, $F \cup I = P$. If $x, y \in P$ such that $x \not\leq y$, then a blanket (F, I) is said to *separate* x *and* y if, and only if, $x \in F$ and $y \in I$.

Define the relations $R_{\mathcal{F}} \subseteq \mathcal{F}^{cdp} \times \mathcal{F}^{cdp}$ and $R_{\mathcal{I}} \subseteq \mathcal{I}^{cdp} \times \mathcal{I}^{cdp}$ by

$$(F, G) \in R_{\mathcal{F}} \iff F \cap \langle P - G]_{cdp} = \varnothing$$

and

$$(I, J) \in R_{\mathcal{I}} \iff [P - I\rangle_{cdp} \cap J = \varnothing.$$

A subset S of $\mathcal{F}$ is said to be a $R_{\mathcal{F}}$*-chain* if, and only if, for all $F, G \in S$ we have that $(F, G) \in R_{\mathcal{F}}$, $F = G$ or $(G, F) \in R_{\mathcal{F}}$. Furthermore, a $R_{\mathcal{F}}$-chain, S,

will be called *weakly dense* if, and only if, whenever $F, G \in S$ and $(F, G) \in R_{\mathcal{F}}$, there exists $F' \in S$ such that $(F, F') \in R_{\mathcal{F}}$ and $(F', G) \in R_{\mathcal{F}}$. *$R_{\mathcal{I}}$-chains* and *weakly dense $R_{\mathcal{I}}$-chains* in $\mathcal{I}$ are defined similarly.

A filter $F \in \mathcal{F}^{cdp}$ will be called *accessible* if, and only if, it is either prime or there exists a weakly dense $R_{\mathcal{F}}$-chain S in $\mathcal{F}$ such that $F \notin S$, but $F = \bigcap S$. An ideal $I \in \mathcal{I}^{cdp}$ will be called *accessible* if, and only if, it is either prime or there exists a weakly dense $R_{\mathcal{I}}$-chain T in $\mathcal{I}$ such that $I \notin T$, but $I = \bigcap T$. Finally a blanket (F, I) will be called an *accessible blanket* if, and only if, both F and I are accessible.

Then we have the following representation result.

Theorem 7.1.1 ([Jan78])**.** *A poset* $\mathbf{P}$ *can be embedded isomorphically into the direct union of a family of complete chains if, and only if, for any* $a, b \in P$ *with* $a \not\leq b$ *there exists an accessible blanket which separates* a *and* b*. The embedding preserves all meets and joins existing in* $\mathbf{P}$*.*

The construction of the direct union of a family of complete chains in the above is fairly involved and rather cumbersome to work with. We propose weakening the condition required of $\mathbf{P}$ and embedding $\mathbf{P}$ into a completely distributive complete lattice instead. Since the variety of distributive lattices is generated by the 2-element chain, nothing is really sacrificed.

For $* \in \{p, dp\}$, recall that $\mathscr{F}^*$ and $\mathscr{I}^*$ denote the families of prime $*$-filters and prime $*$-ideals of $\mathbf{P}$, respectively (see Definition 4.2.13). In the sequel we will be interested in posets that satisfy one of the following:

$$\begin{gathered}\text{For any } x, y \in P, \text{ if } x \not\leq y, \text{ then there} \\ \text{exists } F \in \mathscr{F}^{dp}(\mathbf{P}) \text{ such that } x \in F, \text{ but } y \notin F.\end{gathered} \tag{7.1}$$

$$\begin{gathered}\text{For any } x, y \in P, \text{ if } x \not\leq y, \text{ then there} \\ \text{exists } F \in \mathscr{F}^{p}(\mathbf{P}) \text{ such that } x \in F, \text{ but } y \notin F.\end{gathered} \tag{7.2}$$

By the definition of prime filters it then follows that $y \in P - F \in \mathscr{I}^*$ for $* \in \{p, dp\}$. All distributive lattices satisfy the above.

Theorem 7.1.2. *A poset* $\mathbf{P}$ *can be embedded into a completely distributive complete lattice such that all existing finite (respectively, binary) meets and joins*

in $\mathbf{P}$ *are preserved by the embedding if, and only if,* $\mathbf{P}$ *satisfies (7.1) (respectively,(7.2)).*

Proof. We prove the claim for the case where $\mathbf{P}$ satisfies (7.1). The proof of the case where $\mathbf{P}$ satisfies (7.2) is similar.

Let $\mathbf{L} = \langle L, \vee, \wedge \rangle$ be a completely distributive complete lattice and let ξ be an order-embedding of $\mathbf{P}$ into $\mathbf{L}$ such that ξ preserves all existing finite meets and joins in $\mathbf{P}$. Let $a, b \in P$ such that $a \not\leq b$. Then $\xi(a) \not\leq \xi(b)$ since ξ is an order-embedding. Since $\mathbf{L}$ is distributive, it follows that there exists a prime filter $F \in \mathscr{F}(\mathbf{L})$ such that $\xi(a) \in F$, but $\xi(b) \notin F$. Let $G = \{c \in P : \xi(c) \in F\}$. Then $a \in G$, but $b \notin G$. Moreover, $G \in \mathscr{F}^{pd}(\mathbf{P})$: Let $c_1 \in G$ and $c_2 \in P$ such that $c_1 \leq c_2$. Then $\xi(c_1) \leq \xi(c_2)$ and $\xi(c_1) \in F$. Since F is an up-set we have $\xi(c_2) \in F$ and consequently $c_2 \in G$. Hence, G is an up-set in $\mathbf{P}$. Next let $M \subseteq^{fin} G$ such that $\bigwedge M$ exists in $\mathbf{P}$. Then $\xi(M) \subseteq^{fin} F$ and $\bigwedge \xi(M) \in F$. Since ξ preserves all existing finite meets in $\mathbf{P}$, we have that $\bigwedge \xi(M) = \xi(\bigwedge M) \in F$. Therefore, $\bigwedge M \in G$ and we conclude that G is closed under existing finite meets. Finally, suppose $N \subseteq^{fin} P$ such that $\bigvee N$ exists and $\bigvee N \in G$. Then $\xi(\bigvee N) \in F$, i.e., $\bigvee \xi(N) \in F$ since ξ preserves existing finite joins. But $F \in \mathscr{F}(\mathbf{L})$ implies that $F \cap \xi(N) \neq \varnothing$. Let $c \in N$ such that $\xi(c) \in F \cap \xi(N)$. Then $c \in G$ by definition and $G \cap N \neq \varnothing$. Hence, $G \in \mathscr{F}^{dp}(\mathbf{P})$.

Now suppose $\mathbf{P}$ satisfies (7.1). To prove the backward implication we construct a completely distributive complete lattice into which $\mathbf{P}$ can be embedded and we describe the embedding.

Let $\mathcal{E}_{dp} = \{U \in \mathcal{P}(\mathscr{F}^{dp}) : U \text{ is an up-set in } \langle \mathscr{F}^{dp}, \subseteq \rangle\}$. We note that since $\varnothing$ and $\mathscr{F}^{dp}$ are up-sets in $\langle \mathscr{F}^{dp}, \subseteq \rangle$, it follows that $\mathcal{E}_{dp} \neq \varnothing$. Then we will show that $\mathbf{E}_{dp} = \langle \mathcal{E}_{dp}, \cup, \cap \rangle$ is a completely distributive complete lattice where $\subseteq$ is the associated lattice order $\leq^{\mathbf{E}}$. Let $\mathcal{T} \subseteq \mathcal{E}_{dp}$. Then $\bigcup \mathcal{T} = \{F \in \mathscr{F}^{dp} : F \in U \text{ for some } U \in \mathcal{T}\}$. Let $F \in \bigcup \mathcal{T}$ and $G \in \mathscr{F}^{dp}$ such that $F \subseteq G$. Then $F \in U$ for some $U \in \mathcal{T}$. Since U is an up-set in $\langle \mathscr{F}^{dp}, \subseteq \rangle$, it follows that $G \in U$. Hence, $G \in \bigcup \mathcal{T}$ and $\bigcup \mathcal{T}$ is an up-set in $\langle \mathscr{F}^{dp}, \subseteq \rangle$. Next we consider $\bigcap \mathcal{T} = \{F \in \mathscr{F}^{dp} : U \in \mathcal{T} \text{ implies } F \in U\}$. Let $F' \in \bigcap \mathcal{T}$ and $G' \in \mathscr{F}^{dp}$ such that $F' \subseteq G'$. Then $F' \in U$ for all $U \in \mathcal{T}$. But each $U \in \mathcal{T}$ is an up-set in $\langle \mathscr{F}^{dp}, \subseteq \rangle$. Therefore, $G' \in U$ for all $U \in \mathcal{T}$ and hence $G' \in \bigcap \mathcal{T}$. Thus, $\bigcap \mathcal{T}$ is an up-set in $\langle \mathscr{F}^{dp}, \subseteq \rangle$. This proves that $\mathbf{E}_{dp}$ is a complete lattice. It is well known that any complete lattice of sets is completely distributive [DP02].

Now define $\xi_{dp} : P \to \mathcal{E}_{dp}$ as follows: for $a \in P$

$$\xi_{dp}(a) = \{F \in \mathscr{F}^{dp} : a \in F\}.$$

Then ξ_{dp} is an order-embedding of $\mathbf{P}$ into $\mathbf{E}_{dp}$ that preserves the finite meets and join that exist in $\mathbf{P}$: Let $a, b \in P$. If $a \leq b$, then $a \in F \in \mathscr{F}^{dp}$ implies that $b \in F$ since F is an up-set. Hence $\xi_{dp}(a) \subseteq \xi_{dp}(b)$. If $a \not\leq b$, then by assumption there exists $F' \in \mathscr{F}^{dp}$ such that $a \in F'$, but $b \notin F'$. Then $F' \in \xi_{dp}(a)$ but $F' \notin \xi_{dp}(b)$. Therefore, $\xi_{dp}(a) \not\subseteq \xi_{dp}(b)$.

Next let $M \subseteq^{fin} P$ such that $\bigwedge M$ exists in $\mathbf{P}$. Then,

$$\begin{aligned}
\xi_{dp}\left(\bigwedge M\right) &= \left\{F \in \mathscr{F}^{dp} : \bigwedge M \in F\right\} \\
&= \{F \in \mathscr{F}^{dp} : M \subseteq F\} \\
&= \bigcap_{a \in M} \{F \in \mathscr{F}^{dp} : a \in F\} \\
&= \bigcap \xi_{dp}(M),
\end{aligned}$$

where the second equality follows from the closure of Doyle-pseudo filters under existing finite meets. Now let $N \subseteq^{fin} P$ such that $\bigvee N$ exists in $\mathbf{P}$. Then,

$$\begin{aligned}
\xi_{dp}\left(\bigvee N\right) &= \left\{F \in \mathscr{F}^{dp} : \bigvee N \in F\right\} \\
&= \{F \in \mathscr{F}^{dp} : N \cap F \neq \varnothing\} \\
&= \bigcup_{a \in N} \{F \in \mathscr{F}^{dp} : a \in F\} \\
&= \bigcup \xi_{dp}(N),
\end{aligned}$$

where the second equality follows from the fact that the filters are prime.

Thus, $(\mathbf{E}_{dp}, \xi_{dp})$ is a completion of $\mathbf{P}$. □

In the sequel let $(\mathbf{E}_*(\mathbf{P}), \xi_*^{\mathbf{P}})$, $* \in \{dp, p\}$, denote the completion of a poset $\mathbf{P}$ constructed as in the proof of Theorem 7.1.2. If $\mathbf{P}$ is understood we write $(\mathbf{E}_*, \xi_*)$.

Example 7.1.3. *Let $\mathbf{P}'$ be the poset depicted in Figure 7.1. Then, $\mathbf{P}'$ satisfies (7.1) and (7.2). By Theorem 7.1.2 it follows that $\mathbf{P}'$ can be embedded into the completely distributive complete lattice $\mathbf{E}_*$, also depicted in Figure 7.1. The image of $\mathbf{P}$ under ξ_* is shaded in the depiction of $\mathbf{E}_*$.*

One may wonder whether or not one of the smaller families of up-sets of a poset $\mathbf{P}$ would suffice. For example, would a poset $\mathbf{P}$ be embeddable into a

completely distributive complete lattice if, and only if, for any $x, y \in P$ there exists a prime Frink filter, $F \in \mathscr{F}^f$, such that $x \in F$, but $y \notin F$? The answer to this question is no. The poset $\mathbf{P}'$ considered in this example is clearly embeddable into a completely distributive complete lattice, but, for example, there does not exist a prime Frink filter F such that $2 \in F$, but $3 \notin F$.

For the full details, see Example A.3.1 in Appendix A.3.

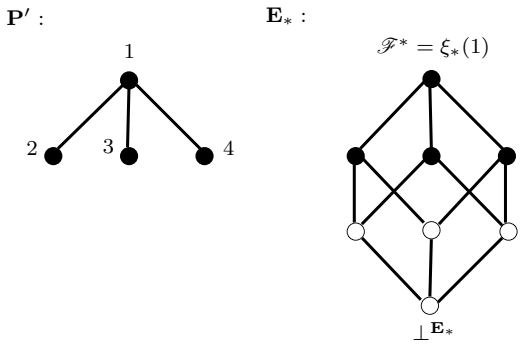


Fig. 7.1: The poset $\mathbf{P}'$ and the complete lattice $\mathbf{E}_*$, for $* \in \{d, dp\}$.

7.2 Relation to the canonical extension

If we assume the axiom of choice, then we have the following result (see for instance [DP02, Theorem 10.18]).

Theorem 7.2.1. *Let $\mathbf{L} = \langle L, \vee, \wedge \rangle$ be a distributive lattice, $F \in \mathcal{F}(\mathbf{L})$ and $I \in \mathcal{I}(\mathbf{L})$ such that $F \cap I = \varnothing$, then there exist $G \in \mathscr{F}(\mathbf{L})$ and $J = L - G \in \mathscr{I}(\mathbf{L})$ such that $F \subseteq G$ and $I \subseteq J$.*

Recall that if $\mathbf{L}$ is a bounded lattice, then the families of pseudo and Doyle-pseudo filters and ideals correspond with the families of filters and ideals of $\mathbf{L}$. We will therefore drop the subscript "$_*$" when we refer to the completion $(\mathbf{E}, \xi)$ of $\mathbf{L}$. Moreover, (7.1) now becomes:

$$\begin{gathered} \text{For any } x, y \in L, \text{ if } x \not\leq y, \text{ then there} \\ \text{exists } F \in \mathscr{F}(\mathbf{L}) \text{ such that } x \in F, \text{ but } y \notin F. \end{gathered} \tag{7.3}$$

It is well known that a lattice $\mathbf{L}$ is distributive if, and only if, it satisfies (7.3).

It is also known that if $\mathbf{L}$ is a bounded distributive lattice, then $\mathbf{E}$ is (isomorphic to) the canonical extension of $\mathbf{L}$ [GJ94]. We now give an explicit correspondence between $\mathbf{E}$ and $\mathbf{C}$, as described in Chapter 6.

Lemma 7.2.2. *Let $\mathbf{L}$ be a bounded distributive lattice. Let $\mathbf{E}$ be the completely distributive complete lattice obtained through the construction described in Section 7.1 and let $\mathbf{C}$ be the complete lattice obtained from the polarization $(\mathcal{F}(\mathbf{L}), \mathcal{I}(\mathbf{L}))$, as described in Chapter 6.1.1. Define $\eta : \mathcal{C} \to \mathcal{E}$ by, for $X \in \mathcal{C}$*

$$\eta(X) = X \cap \mathscr{F}(\mathbf{L})$$

Then η is a lattice isomorphism between $\mathbf{C}$ and $\mathbf{E}$ that fixes L.

Proof. (i) η is one-to-one: Let $X_1, X_2 \in \mathcal{C}$ such that $\eta(X_1) = \eta(X_2)$, i.e., $X_1 \cap \mathscr{F}(\mathbf{L}) = X_2 \cap \mathscr{F}(\mathbf{L})$. Let $F \in X_1$ and suppose $F \notin X_2 = X_2^{\triangleright\triangleleft}$. Then there exists $I \in X_2^{\triangleright}$ such that $F \cap I = \varnothing$. By Theorem 7.2.1, there exists $G \in \mathscr{F}(\mathbf{L})$ such that $F \subseteq G$ and $G \cap I = \varnothing$. Therefore, $G \notin X_2^{\triangleright\triangleleft} = X_2$. But, since X_1 is an up-set in $\mathcal{F}$ and $F \subseteq G$, we have that $G \in X_1$. This contradicts our assumption that $\eta(X_1) = \eta(X_2)$. Thus, $F \in X_2$ and $X_1 \subseteq X_2$. Similarly, we can show that $X_2 \subseteq X_1$. Hence, $X_1 = X_2$.

(ii) η is onto: Let $U \in \mathcal{E}$. We will show that $\eta(U^{\triangleright\triangleleft}) = U$. It follows from the properties of Galois connections that $U^{\triangleright\triangleleft} \in \mathcal{C}$. Since $U \subseteq U^{\triangleright\triangleleft}$ and $U \subseteq \mathscr{F}(\mathbf{L})$, the inclusion $U \subseteq U^{\triangleright\triangleleft} \cap \mathscr{F}(\mathbf{L})$ is immediate. To prove the inclusion in the other direction, let $F \in U^{\triangleright\triangleleft} \cap \mathscr{F}(\mathbf{L})$. Then $F \cap I \neq \varnothing$ for all $I \in U^{\triangleright}$. Now $J = L - F \in \mathscr{I}(\mathbf{L})$, since $F \in \mathscr{F}(\mathbf{L})$, and $F \cap J = \varnothing$. Therefore, $J \notin U^{\triangleright}$. This implies that there exists $G \in U$ such that $G \cap J = \varnothing$. Then $G \subseteq L - J = F$ and $F \in U$, since U is an up-set of prime filters. Hence, $U^{\triangleright\triangleleft} \cap \mathscr{F}(\mathbf{L}) \subseteq U$.

(iii) η distributes over meets and joins: Let $X_i \in \mathcal{C}$ for $i \in \Psi$. Then,

$$\begin{aligned}\eta\left(\bigwedge_{i\in\Psi}^{\mathbf{C}} X_i\right) &= \eta\left(\bigcap_{i\in\Psi} X_i\right)\\ &= \left(\bigcap_{i\in\Psi} X_i\right) \cap \mathscr{F}(\mathbf{L})\\ &= \bigcap_{i\in\Psi} (X_i \cap \mathscr{F}(\mathbf{L}))\\ &= \bigcap_{i\in\Psi} \eta(X_i)\\ &= \bigwedge_{i\in\Psi}^{\mathbf{E}} \eta(X_i).\end{aligned}$$

Furthermore, recall that $\bigvee_{i\in\Psi}^{\mathbf{C}} X_i = (\bigcup_{i\in\Psi} X_i)^{\triangleright\triangleleft}$ which equals $(\bigcap_{i\in\Psi} X_i^{\triangleright})^{\triangleleft}$ by Lemma 2.6.3. Let $F \in \eta(\bigvee_{i\in\Psi}^{\mathbf{C}} X_i) = (\bigcap_{i\in\Psi} X_i^{\triangleright})^{\triangleleft} \cap \mathscr{F}(\mathbf{L})$. This is the case if, and only if, $F \cap I \neq \varnothing$ for all $I \in \bigcap_{i\in\Psi} X_i^{\triangleright}$ if, and only if, $J \notin \bigcap_{i\in\Psi} X_i^{\triangleright}$ for $J = L - F \in \mathscr{I}(\mathbf{L})$ (since F is prime). Moreover,

$$\begin{aligned}& J \notin \bigcap_{i\in\Psi} X_i^{\triangleright}\\ \iff\ & J \notin X_j^{\triangleright} \text{ for some } j \in \Psi\\ \iff\ & \text{there exists } G_j \in X_j \text{ such that } G_j \cap J = \varnothing \text{ for some } j \in \Psi\\ \iff\ & \text{there exists } G_j \in X_j \text{ such that } G_j \subseteq F \text{ for some } j \in \Psi\\ \iff\ & F \in X_j \text{ for some } j \in \Psi\\ \iff\ & F \in \eta(X_j) \text{ for some } j \in \Psi\\ \iff\ & F \in \bigcup_{i\in\Psi} \eta(X_i) = \bigvee_{i\in\Psi}^{\mathbf{E}} \eta(X_i).\end{aligned}$$

Hence, $\eta(\bigvee_{i\in\Psi}^{\mathbf{C}} X_i) = \bigvee_{i\in\Psi}^{\mathbf{E}} \eta(X_i)$.

(iv) η fixes L: Let $a \in L$. Then,

$$\begin{aligned}\eta(\alpha(a)) &= \eta(\{F \in \mathcal{F} : a \in F\})\\ &= \{F \in \mathcal{F} : a \in F\} \cap \mathscr{F}(\mathbf{L})\\ &= \{F \in \mathscr{F}(\mathbf{L}) : a \in F\}\\ &= \xi(a).\end{aligned}$$

□

Implicit from the above is that the map $\zeta : \mathcal{E} \to \mathcal{C}$ defined by $\zeta(U) = U^{\triangleright\triangleleft}$, for $U \in \mathcal{E}$, is the inverse of η and therefore a lattice isomorphism from $\mathbf{E}$ to $\mathbf{C}$ that fixes L.

Consider the proof of Lemma 7.2.2. The fact that $\mathbf{L}$ is distributive is only called upon when we prove that η, the isomorphism between $\mathbf{C}$ and $\mathbf{E}$, is one-to-one, since we appeal to Theorem 7.2.1. One may now wonder whether or not there exists a larger class of posets for which the completions discussed in this chapter correspond to the completions studied in Chapter 6. We will show that this is indeed the case and give a characterization of a larger class of posets for which $\mathbf{E}_{dp}$ and $\mathbf{C}_{dp}$ are isomorphic.

We begin by recalling a result obtained in [Tun74].

Following [Tun74] a polarization $(\mathcal{F}, \mathcal{I})$ of a poset $\mathbf{P}$ is called *normal* if, whenever $F \in \mathcal{F}$ and $I \in \mathcal{I}$ such that $F \cap I = \varnothing$, then there exist $G \in \mathcal{F}$ and $J \in \mathcal{I}$ such that $G \cap I = \varnothing = F \cap J$ and $G \cup J = P$. Then we have the following result.

Theorem 7.2.3. *[Tun74, Theorem 3] The completion $\mathbf{C}$ obtained from a polarization $(\mathcal{F}, \mathcal{I})$ of a poset $\mathbf{P}$ is completely distributive if, and only if, $(\mathcal{F}, \mathcal{I})$ is a normal polarization of $\mathbf{P}$.*

Observe that a normal polarization does not require that the larger sets G and J be disjoint. Let $\mathbf{P}$ be a poset for which $(\mathcal{F}_{dp}(\mathbf{P}), \mathcal{I}_{dp}(\mathbf{P}))$ form a normal polarization. Define $\eta_{dp} : \mathcal{C}_{dp} \to \mathcal{E}_{dp}$ by $\eta_{dp}(X) = X \cap \mathscr{F}^{dp}(\mathbf{P})$. To prove that η_{dp} is one-to-one for a poset, we need the following stronger condition:

$$\begin{aligned} &\text{If } F \in \mathcal{F}_{dp}(\mathbf{P}) \text{ and } I \in \mathcal{I}_{dp}(\mathbf{P}) \text{ such that } F \cap I = \varnothing, \text{ then there exist} \\ &G \in \mathscr{F}^{dp}(\mathbf{P}) \text{ and } J = L - G \in \mathscr{I}^{dp}(\mathbf{P}) \text{ such that } F \subseteq G \text{ and } I \subseteq J. \end{aligned} \tag{7.4}$$

We will now show that if $(\mathcal{F}_{dp}(\mathbf{P}), \mathcal{I}_{dp}(\mathbf{P}))$ is a normal polarization for a poset $\mathbf{P}$, then $\mathbf{P}$ satisfies (7.4).

Lemma 7.2.4. *Let $\mathbf{P}$ be a poset, $F \in \mathcal{F}_{dp}(\mathbf{P})$ and $I \in \mathcal{I}_{dp}(\mathbf{P})$. Then $F \cap I = \varnothing$ if, and only if, $[\alpha(F)\rangle \cap \langle\alpha(I)] = \varnothing$.*

Proof. Let $F \in \mathcal{F}_{dp}(\mathbf{P})$ and $I \in \mathcal{I}_{dp}(\mathbf{P})$. Observe that $[\alpha(F)\rangle = [\bigwedge \alpha(F))$ and $\langle\alpha(I)] = (\bigvee \alpha(I)]$. Then, $[\alpha(F)\rangle \cap \langle\alpha(I)] \neq \varnothing$ if, and only if, $[\bigwedge \alpha(F)) \cap (\bigvee \alpha(I)] \neq \varnothing$. This is the case if, and only if, $\bigwedge \alpha(F) \leq \bigvee \alpha(I)$ if, and only if, $F \cap I \neq \varnothing$ by the internal compactness of $\mathbf{C}_{dp}$. □

Corollary 7.2.5. *If $\mathbf{C}_{dp}$ of a poset $\mathbf{P}$ is distributive and $F \in \mathcal{F}_{dp}(\mathbf{P})$ and $I \in \mathcal{I}_{dp}(\mathbf{P})$ such that $F \cap I = \varnothing$, then there exist $G \in \mathscr{F}(\mathbf{C}_{dp})$ and $J = \mathcal{C}_{dp} - G \in \mathscr{I}(\mathbf{C}_{dp})$ such that $\alpha(F) \subseteq G$ and $\alpha(I) \subseteq J$.*

This is a direct consequence of Theorem 7.2.1 and Lemma 7.2.4.

Lemma 7.2.6. *Let $G \in \mathscr{F}(\mathbf{C}_{dp})$ of a poset $\mathbf{P}$. Then $\alpha^{-1}(G \cap \alpha(P)) \in \mathscr{F}^{dp}(\mathbf{P})$.*

Proof. We first show that $\alpha^{-1}(G \cap \alpha(P)) \in \mathcal{F}^{dp}(\mathbf{P})$. Clearly $\alpha^{-1}(G \cap \alpha(P))$ is an up-set in $\mathbf{P}$, since G is an up-set and α and α^{-1} are order-preserving. Let $M \subseteq^{fin} \alpha^{-1}(G \cap \alpha(P))$ such that $\bigwedge M$ exists in $\mathbf{P}$. Then, since α preserves finite meets that exist in $\mathbf{P}$,

$$\begin{aligned} & M \subseteq^{fin} \alpha^{-1}(G \cap \alpha(P)) \\ \Rightarrow\ & \alpha(M) \subseteq^{fin} G \cap \alpha(P) \\ \Rightarrow\ & \bigwedge \alpha(M) \in G \\ \Rightarrow\ & \alpha\left(\bigwedge M\right) \in G \\ \Rightarrow\ & \alpha\left(\bigwedge M\right) \in G \cap \alpha(P) \\ \Rightarrow\ & \bigwedge M \in \alpha^{-1}(G \cap \alpha(P)). \end{aligned}$$

Therefore, $\alpha^{-1}(G \cap \alpha(P))$ is closed under finite meets that exist in $\mathbf{P}$. Hence, $\alpha^{-1}(G \cap \alpha(P)) \in \mathcal{F}_{dp}(\mathbf{P})$.

Now let $N \subseteq^{fin} P - \alpha^{-1}(G \cap \alpha(P))$ such that $\bigvee N$ exists in $\mathbf{P}$. Suppose $\bigvee N \in \alpha^{-1}(G \cap \alpha(P))$. Then, since α preserves finite joins that exist in $\mathbf{P}$ and since G is prime,

$$\begin{aligned} & \bigvee N \in \alpha^{-1}(G \cap \alpha(P)) \\ \Rightarrow\ & \alpha\left(\bigvee N\right) \in G \cap \alpha(P) \\ \Rightarrow\ & \bigvee \alpha(N) \in G \cap \alpha(P) \\ \Rightarrow\ & G \cap \alpha(N) \neq \varnothing \\ \Rightarrow\ & N \cap \alpha^{-1}(G \cap \alpha(P)) \neq \varnothing, \end{aligned}$$

which contradicts our choice of N. Thus, $\bigvee N \notin \alpha^{-1}(G \cap \alpha(P))$ and therefore $P - \alpha^{-1}(G \cap \alpha(P)) \in \mathcal{I}_{dp}(\mathbf{P})$. But then, $\alpha^{-1}(G \cap \alpha(P)) \in \mathscr{F}^{dp}(\mathbf{P})$. □

Combining Corollary 7.2.5 and Lemma 7.2.6 now gives the following.

Corollary 7.2.7. *If $\mathbf{C}_{dp}$ of a poset $\mathbf{P}$ is distributive, then $\mathbf{P}$ satisfies (7.4).*

If we now combine Corollary 7.2.7 with the backward implication of Theorem 7.2.3, we get the following result.

Corollary 7.2.8. *If $\mathbf{P}$ is poset such that $(\mathcal{F}_{dp}(\mathbf{P}), \mathcal{I}_{dp}(\mathbf{P}))$ is a normal polarization, then $\mathbf{P}$ satisfies (7.4).*

We also have the following.

Lemma 7.2.9. *If a poset $\mathbf{P}$ satisfies (7.4), then $\mathbf{P}$ satisfies (7.1).*

Proof. Let $\mathbf{P}$ be a poset that satisfies (7.4) and let $a, b \in P$ such that $a \not\leq b$. Then $[a) \in \mathcal{F}_{dp}(\mathbf{P})$, $(b] \in \mathcal{I}_{dp}(\mathbf{P})$ and $[a) \cap (b] = \varnothing$. By assumption there exist $G \in \mathscr{F}^{dp}(\mathbf{P})$ and $J = P - G \in \mathscr{I}^{dp}(\mathbf{P})$ such that $[a) \subseteq G$ and $(b] \subseteq J$. Then, $a \in G$, but $b \notin G$. □

A consequence of the above is that if $(\mathcal{F}_{dp}(\mathbf{P}), \mathcal{I}_{dp}(\mathbf{P}))$ is a normal polarization of a poset $\mathbf{P}$, then we can embed $\mathbf{P}$ into a completely distributive complete lattice, $\mathbf{E}_{dp}$, constructed as in the proof of Theorem 7.1.2. Furthermore, recall from our earlier discussion that the only part of the proof of Lemma 7.2.2 that would not hold for all posets satisfying (7.1), is the proof that η is one-to-one. It should be clear that if a poset $\mathbf{P}$ satisfies (7.4) (the poset analogue of the property described in Theorem 7.2.1), then the map η_{dp} will be one-to-one. Therefore, we have the following result.

Corollary 7.2.10. *Let $\mathbf{P}$ be a poset such that $(\mathcal{F}_{dp}(\mathbf{P}), \mathcal{I}_{dp}(\mathbf{P}))$ is a normal polarization. Let $\mathbf{E}_{dp}$ be the completely distributive complete lattice obtained through the construction described in Section 7.1 and let $\mathbf{C}_{dp}$ be the complete lattice obtained from the polarization $(\mathcal{F}_{dp}(\mathbf{P}), \mathcal{I}_{dp}(\mathbf{P}))$, as described in Chapter 6.1.1. Let $\eta_{dp} : \mathcal{C}_{dp} \to \mathcal{E}_{dp}$ be defined by, for $X \in \mathcal{C}_{dp}$,*

$$\eta_{dp}(X) = X \cap \mathscr{F}^{dp}(P).$$

Then η_{dp} is a lattice isomorphism between $\mathbf{C}_{dp}$ and $\mathbf{E}_{dp}$ that fixes P.

Remark 7.2.11. *Recall the following result for lattices. Let $\mathbf{L}$ be a lattice. Then the following are equivalent:*

(i) $\mathbf{L}$ is distributive.

(ii) *Given $J \in \mathcal{I}(\mathbf{L})$ and $G \in \mathcal{F}(\mathbf{L})$ such that $J \cap G = \varnothing$, there exist $I \in \mathscr{I}(\mathbf{L})$ and $F = L - I \in \mathscr{F}(\mathbf{L})$ such that $J \subseteq I$ and $G \subseteq F$.*

(iii) *Given $a, b \in L$ with $a \not\leq b$, there exists $F \in \mathscr{F}(\mathbf{L})$ such that $a \in F$, but $b \notin F$.*

One may now wonder whether or not an analogous claim would be true in the poset setting. By Lemma 7.2.9 we have that (ii) implies (iii) for posets. However, the question of whether or not the other implications hold is still open.

7.3 Extension of maps

For the remainder of this section let $* \in \{p, dp\}$ and let $\mathbf{P} = \langle P, \leq^{\mathbf{P}} \rangle$ and $\mathbf{Q} = \langle Q, \leq^{\mathbf{Q}} \rangle$ be fixed posets that satisfy (7.1) if $*$ is dp and (7.2) if $*$ is p. Furthermore, let $(\mathbf{E}_*(\mathbf{P}), \xi_*^{\mathbf{P}})$ and $(\mathbf{E}_*(\mathbf{Q}), \xi_*^{\mathbf{Q}})$ be the completions of $\mathbf{P}$ and $\mathbf{Q}$ obtained through the construction described in Section 7.1.

We will treat order-preserving and order-reversing maps separately.

Lemma 7.3.1. *If $U \in \mathcal{E}_*(\mathbf{P})$, then $\bigcap U$ is an up-set in $\mathbf{P}$. In particular, if $U = \xi_*^{\mathbf{P}}(a)$, then $\bigcap U = [a)$.*

Dually, if Υ is an up-set of ideals, then $\bigcap \Upsilon$ is a down-set in $\mathbf{P}$. In particular, $\bigcap\{I \in \mathscr{I}^*(\mathbf{P}) : a \in I\} = (a]$

Proof. Let $b \in \bigcap U$ and $c \in P$ such that $b \leq c$. Then $b \in F$ for all $F \in U$. But each $F \in U$ is an up-set in P. Hence, $c \in F$ for all $F \in U$ and consequently $c \in \bigcap U$. Thus, $\bigcap U$ is an up-set in $\mathbf{P}$.

Suppose $U = \xi_*(a)$ for some $a \in U$. Clearly $[a) \subseteq \bigcap U$. If $b \in P$ such that $a \not\leq b$, then, by assumption, there exists $F \in \mathscr{F}^*(\mathbf{P})$ such that $a \in F$, but $b \notin F$. Then $F \in \xi_*(a) = U$ and $b \notin \bigcap U$. Therefore, $\bigcap U = [a)$.

The proof of the second claim is similar. □

Lemma 7.3.2. *Let $f : P \to Q$ be an order-preserving map. Then $f^{\mathbf{E}_*} : \mathcal{E}_*(\mathbf{P}) \to \mathcal{E}_*(\mathbf{Q})$ defined by, for $U \in \mathcal{E}_*(\mathbf{P})$,*

$$f^{\mathbf{E}_*}(U) = \left\{F \in \mathscr{F}^*(\mathbf{Q}) : f\left(\bigcap U\right) \subseteq F\right\}$$

is order-preserving and extends f, i.e., for all $a \in P$ we have $f^{\mathbf{E}_*}(\xi_*^{\mathbf{P}}(a)) = \xi_*^{\mathbf{Q}}(f(a))$.

Proof. Let $U, V \in \mathcal{E}_*(\mathbf{P})$ such that $U \subseteq V$ and $F \in f^{\mathbf{E}_*}(U)$. Then,

$$\begin{aligned} & U \subseteq V \text{ and } f\left(\bigcap U\right) \subseteq F \\ \Rightarrow & \bigcap U \supseteq \bigcap V \\ \Rightarrow & f\left(\bigcap U\right) \supseteq f\left(\bigcap V\right) \\ \Rightarrow & f\left(\bigcap V\right) \subseteq F \\ \Rightarrow & F \in f^{\mathbf{E}_*}(V). \end{aligned}$$

Hence, $f^{\mathbf{E}_*}(U) \subseteq f^{\mathbf{E}_*}(V)$ and we conclude that $f^{\mathbf{E}_*}$ is order-preserving.

Now let $a \in P$ and let $F \in f^{\mathbf{E}_*}(\xi_*^{\mathbf{P}}(a))$. Since $a \in \bigcap \xi_*^{\mathbf{P}}(a)$, we have that $f(a) \in F$ and $F \in \xi_*^{\mathbf{Q}}(f(a))$. Hence, $f^{\mathbf{E}_*}(\xi_*^{\mathbf{P}}(a)) \subseteq \xi_*^{\mathbf{Q}}(f(a))$. Next let $G \in \xi_*^{\mathbf{Q}}(f(a))$ and let $b \in \bigcap \xi_*^{\mathbf{P}}(a)$. By Lemma 7.3.1, $b \geq a$, and since f is order-preserving, it follows that $f(b) \geq f(a)$. Then $f(b) \in G$, since G is an up-set and $f(a) \in G$. Therefore, $f(\bigcap \xi_*^{\mathbf{P}}(a)) \subseteq G$ and $G \in f^{\mathbf{E}_*}(\xi_*^{\mathbf{P}}(a))$. Thus, $\xi_*^{\mathbf{Q}}(f(a)) \subseteq f^{\mathbf{E}_*}(\xi_*^{\mathbf{P}}(a))$. □

Lemma 7.3.3. *Let $h : P \to Q$ be an order-reversing map. Then $h^{\mathbf{E}_*} : \mathcal{E}_*(\mathbf{P}) \to \mathcal{E}_*(\mathbf{Q})$ defined by, for $U \in \mathcal{E}_*(\mathbf{P})$,*

$$h^{\mathbf{E}_*}(U) = \left\{F \in \mathscr{F}^*(\mathbf{Q}) : h\left(\bigcap \left\{I \in \mathscr{I}^*(\mathbf{P}) : \bigcap U \nsubseteq (P - I)\right\}\right) \subseteq F\right\}$$

is order-reversing and extends h, i.e., for all $a \in P$ we have $h^{\mathbf{E}_*}(\xi_*^{\mathbf{P}}(a)) = \xi_*^{\mathbf{Q}}(h(a))$.

Proof. Let $U, V \in \mathcal{E}_*(\mathbf{P})$ such that $U \subseteq V$. Then,

$$\begin{aligned} & \bigcap V \subseteq \bigcap U \\ \Rightarrow & \bigcap U \nsubseteq (P - J) \text{ for all } J \in \left\{I \in \mathscr{I}^*(\mathbf{P}) : \bigcap V \nsubseteq (P - I)\right\} \\ \Rightarrow & J \in \left\{I \in \mathscr{I}^*(\mathbf{P}) : \bigcap U \nsubseteq (P - I)\right\} \\ & \text{for all } J \in \left\{I \in \mathscr{I}^*(\mathbf{P}) : \bigcap V \nsubseteq (P - I)\right\} \\ \Rightarrow & \left\{I \in \mathscr{I}^*(\mathbf{P}) : \bigcap V \nsubseteq (P - I)\right\} \subseteq \left\{I \in \mathscr{I}^*(\mathbf{P}) : \bigcap U \nsubseteq (P - I)\right\} \\ \Rightarrow & \bigcap \left\{I \in \mathscr{I}^*(\mathbf{P}) : \bigcap V \nsubseteq (P - I)\right\} \supseteq \bigcap \left\{I \in \mathscr{I}^*(\mathbf{P}) : \bigcap U \nsubseteq (P - I)\right\} \\ \Rightarrow & h\left(\bigcap \left\{I \in \mathscr{I}^*(\mathbf{P}) : \bigcap V \nsubseteq (P - I)\right\}\right) \\ & \supseteq h\left(\bigcap \left\{I \in \mathscr{I}^*(\mathbf{P}) : \bigcap U \nsubseteq (P - I)\right\}\right). \end{aligned}$$

Let $F \in h^{\mathbf{E}_*}(V)$. Then, $h\left(\bigcap\{I \in \mathscr{I}^*(\mathbf{P}) : \bigcap V \nsubseteq P - I\}\right) \subseteq F$ implies that $h\left(\bigcap\{I \in \mathscr{I}^*(\mathbf{P}) : \bigcap U \nsubseteq P - I\}\right) \subseteq F$ and therefore, $F \in h^{\mathbf{E}_*}(U)$. Hence, $h^{\mathbf{E}_*}(V) \subseteq h^{\mathbf{E}_*}(U)$ and we may conclude that $h^{\mathbf{E}_*}$ is order-reversing.

Let $a \in P$. Then $\bigcap \xi_*^{\mathbf{P}}(a) = [a)$ by Lemma 7.3.1. Now,

$$\begin{aligned} J \in \{I \in \mathscr{I}^*(\mathbf{P}) : [a) \nsubseteq P - I\} &\iff \text{there exists } b \in P \text{ such that} \\ &\qquad b \geq a \text{ and } b \in J \\ &\iff a \in J \\ &\iff J \in \{I \in \mathscr{I}^*(\mathbf{P}) : a \in I\}. \end{aligned}$$

That is, $\{I \in \mathscr{I}^*(\mathbf{P}) : \bigcap \xi_*^{\mathbf{P}}(a) \nsubseteq P - I\} = \{I \in \mathscr{I}^*(\mathbf{P}) : a \in I\}$. Then, by Lemma 7.3.1, it now follows that $\bigcap\{I \in \mathscr{I}^*(\mathbf{P}) : \bigcap \xi_*^{\mathbf{P}}(a) \nsubseteq P - I\} = \bigcap\{I \in \mathscr{I}^*(\mathbf{P}) : a \in I\} = (a]$. Then $h^{\mathbf{E}_*}(\xi_*^{\mathbf{P}}(a)) = \{F \in \mathscr{F}^*(\mathbf{Q}) : h((a]) \subseteq F\}$. Let $c \in (a]$. Then, $c \leq a$ implies $h(a) \leq h(c)$, since h is order-reversing. Consequently, $h(a) \in F \in \mathscr{F}^*(\mathbf{Q})$ if, and only if, $h((a]) \subseteq F$ since F is an up-set. Hence, $h^{\mathbf{E}_*}(\xi_*^{\mathbf{P}}(a)) = \{F \in \mathscr{F}^*(\mathbf{Q}) : h(a) \in F\} = \xi_*^{\mathbf{Q}}(h(a))$. □

In the following example we show that $f^{\mathbf{E}_*}$ need not be an operator when f is one. Similarly, $h^{\mathbf{E}_*}$ need not be a dual operator when h is one.

Example 7.3.4. *Let $\mathbf{P}'$ be the poset depicted in Figure 7.2. Note that no non-trivial joins or meets exist in $\mathbf{P}'$. Let $f : P' \to P'$ be the identity map. Then f is an operator on $\mathbf{P}'$. However, the extension $f^{\mathbf{E}_*}$ of f to $\mathbf{E}_*$ (also depicted in Figure 7.2 with $\xi_*(P')$ shaded) is not an operator on $\mathbf{E}_*$. In particular, $f^{\mathbf{E}_*}(\xi_*(3) \cup \xi_*(4)) \neq f^{\mathbf{E}_*}(\xi_*(3)) \cup f^{\mathbf{E}_*}(\xi_*(4))$.*

Now let $h : P' \to P'$ be the unary operation defined by $h(1) = 3$, $h(2) = 4$, $h(3) = 1$ and $h(4) = 2$. Then h is order-reversing and a dual operator on $\mathbf{P}'$. However, the extension $h^{\mathbf{E}_*}$ of h to $\mathbf{E}_*$ is not a dual operator on $\mathbf{E}_*$. In particular, $h^{\mathbf{E}_*}(\xi_*(3) \cap \xi_*(4)) \neq h^{\mathbf{E}_*}(\xi_*(3)) \cap h^{\mathbf{E}_*}(\xi_*(4))$.

For more details the reader may consult Example A.3.2 in Appendix A.3.

Lemma 7.3.5. *Let $f : P \to P$ be order-preserving.*

(i) If f is increasing, then $f^{\mathbf{E}_*}$ is increasing.

(ii) If f is such that $f(f(x)) \leq f(x)$ for all $x \in P$, the $f^{\mathbf{E}_*}(f^{\mathbf{E}_*}(U)) \subseteq f^{\mathbf{E}_*}(U)$ for all $U \in \mathcal{E}_*$.

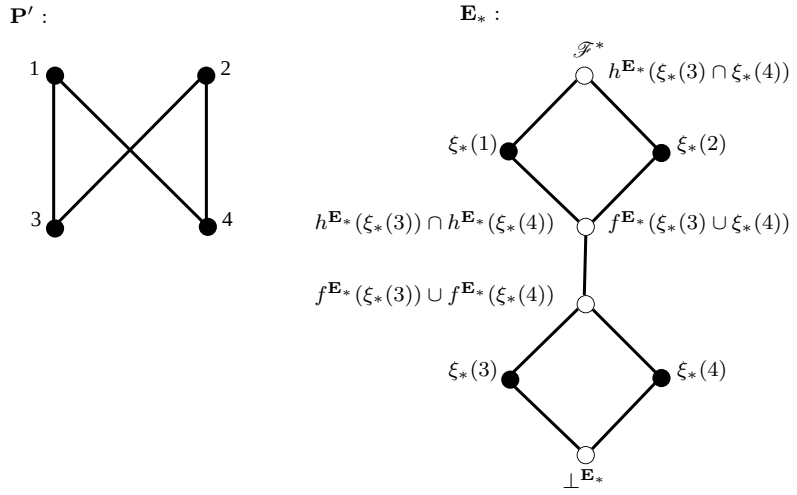


Fig. 7.2: Extensions of (dual) operators need not be (dual) operators.

Proof. (i) Suppose $f(x) \geq x$ for all $x \in P$. Let $U \in \mathcal{E}_*$ and $F \in U$. If $a \in \bigcap U$, then $f(a) \geq a$ and since $\bigcap U$ is an upset (by Lemma 7.3.1) we have $f(a) \in \bigcap U$. Therefore, $f(\bigcap U) \subseteq \bigcap U \subseteq F$. Then $F \in f^{\mathbf{E}_*}(U)$ and hence $U \subseteq f^{\mathbf{E}_*}(U)$.

(ii) Suppose $f(f(x)) \leq f(x)$ for all $x \in P$. Let $U \in \mathcal{E}_*$. Then, by definition, $f(\bigcap U) \subseteq F$ for all $F \in f^{\mathbf{E}_*}(U)$. Thus, $f(\bigcap U) \subseteq \bigcap f^{\mathbf{E}_*}(U)$ and $f(f(\bigcap U)) \subseteq f(\bigcap f^{\mathbf{E}_*}(U))$. Let $F \in f^{\mathbf{E}_*}(f^{\mathbf{E}_*}(U))$. Then,

$$\begin{aligned} & f\left(\bigcap f^{\mathbf{E}_*}(U)\right) \subseteq F \\ \Rightarrow\ & f\left(f\left(\bigcap U\right)\right) \subseteq F \\ \Rightarrow\ & f\left(\bigcap U\right) \subseteq F \\ \Rightarrow\ & F \in f^{\mathbf{E}_*}(U), \end{aligned}$$

where the second implication follows from our assumption and the fact that F is an up-set. Hence, $f^{\mathbf{E}_*}(f^{\mathbf{E}_*}(U)) \subseteq f^{\mathbf{E}_*}(U)$.

□

Finally, we can also define extensions of n-ary maps. Let $\mathbf{P}_1, \ldots, \mathbf{P}_n$ and $\mathbf{Q}$ be posets, for some $n \in \mathbb{N}$. Let $f : \prod_{i=1}^{n} P_i \to Q$ be an n-ary map that is

order-preserving in each coordinate. Define $f^{\mathbf{E}_*} : \prod_{i=1}^n (\mathcal{E}_*(\mathbf{P}_i)) \to \mathcal{E}_*(\mathbf{Q})$ by, for $U_i \in \mathcal{E}_*(\mathbf{P}_i)$, $i = 1, \ldots, n$,

$$f^{\mathbf{E}_*}(U_1, \ldots, U_2) = \left\{ F \in \mathscr{F}^*(\mathbf{Q}) : f\left(\bigcap U_1, \ldots, \bigcap U_n\right) \subseteq F \right\}.$$

The proofs that $f^{\mathbf{E}_*}$ extends f and is order-preserving in each coordinate are similar to the proofs of the analogous claims for unary maps.

On the other hand, let $h : \prod_{i=1}^n P_i \to Q$ be an n-ary map that is order-reversing in each coordinate. For $U_i \in \mathcal{E}_*(\mathbf{P}_i)$ let $\underline{U_i} = \bigcap \{ I \in \mathscr{I}^*(\mathbf{P}_i) : \bigcap U_i \nsubseteq (P - I) \}$. Then $h^{\mathbf{E}_*} : \prod_{i=1}^n (\mathcal{E}_*(\mathbf{P}_i)) \to \mathcal{E}_*(\mathbf{Q})$ defined by, for $U_i \in \mathcal{E}_*(P_i)$, $i = 1, \ldots, n$,

$$h^{\mathbf{E}_*}(U_1, \ldots, U_2) = \left\{ F \in \mathscr{F}^*(\mathbf{Q}) : h\left(\underline{U_1}, \ldots, \underline{U_n}\right) \subseteq F \right\},$$

extends h and is order-reversing in each coordinate.

Part II

THE FINITE EMBEDDABILITY PROPERTY

8. INTRODUCTION TO THE FINITE EMBEDDABILITY PROPERTY

An important question in Mathematical logic is determining whether or not a given logic is decidable. A logic is *decidable* if there exists an algorithm that decides whether or not a formula is a theorem of the logic. One way to prove the decidability of a logic is to reduce the decidability problem to determining satisfaction on finite models. A logic is said to have the *finite model property* (FMP) if every formula that is not a theorem of the logic can be refuted in a finite model of the logic. Furthermore, a logic has the *strong finite model property* (SFMP) if, for every finite set of premises Σ, and every formula φ, if φ does not follow from Σ, then all the formulas of Σ are satisfied by some interpretation in a finite model of the logic that makes φ false. If a finitely axiomatized logic has the SFMP, then it is decidable.

Due to algebraization results for logics (algebraization in the sense of [BP89]), the above is directly related to the identification of classes of algebras with decidable theories.

If $\mathbb{K}$ is a class of algebras, then the *universal theory* (respectively, *equational theory*) of $\mathbb{K}$ is the set of universal sentences (respectively, universally quantified identities) that are valid in all members of $\mathbb{K}$. A class of algebras has a *decidable* universal (respectively, equational) theory if there exists an algorithm that decides whether or not a universal sentence (respectively, identity) is a member of the theory, i.e., is valid in all members of the class.

We will need the following notions in the sequel.

Definition 8.0.1. *Let* $\mathbf{A} = \langle A, \{f_i^{\mathbf{A}} : i \in \Psi\}, \leq^{\mathbf{A}} \rangle$ *be an ordered algebra (of any type) and let* B *be any subset of* A*. The* partial subalgebra $\mathbf{B}$ *of* $\mathbf{A}$ *with domain* B *is the partial ordered algebra* $\langle B, \{f_i^{\mathbf{B}} : i \in \Psi\}, \leq^{\mathbf{B}} \rangle$*, where for* $i \in \Psi$*,*

f_i k-ary, $b_1, \ldots, b_k \in B$

$$f_i^{\mathbf{B}}(b_1, \ldots, b_k) = \begin{cases} f_i^{\mathbf{A}}(b_1, \ldots, b_k) & : \quad \text{if } f_i^{\mathbf{A}}(b_1, \ldots, b_k) \in B \\ \text{undefined} & : \quad \text{if } f_i^{\mathbf{A}}(b_1, \ldots, b_k) \notin B, \end{cases}$$

and $\leq^{\mathbf{B}}$ is the restriction of $\leq^{\mathbf{A}}$ to B, i.e., for $b_1, b_2 \in B$ we have that

$$b_1 \leq^{\mathbf{B}} b_2 \iff b_1 \leq^{\mathbf{A}} b_2.$$

Definition 8.0.2. *An* embedding *of a partial subalgebra $\mathbf{B}$ into an ordered algebra $\mathbf{C}$ is a one-to-one map $\gamma : B \to C$ that preserves and reflects the partial order and all existing operations; i.e., for $b_1, b_2 \in B$ we have that $b_1 \leq^{\mathbf{B}} b_2$ if, and only if, $\gamma(b_1) \leq^{\mathbf{C}} \gamma(b_2)$; and if f_i is some k-ary operation such that $f_i^{\mathbf{B}}(b_1, \ldots, b_k)$ is defined for $b_1, \ldots, b_k \in B$, then $\gamma\left(f_i^{\mathbf{B}}(b_1, \ldots, b_k)\right) = f_i^{\mathbf{C}}\left(\gamma(b_1), \ldots, \gamma(b_k)\right)$.*

A class $\mathbb{K}$ of (ordered) algebras has the *finite embeddability property* (FEP, for short) if every finite partial subalgebra of some member of $\mathbb{K}$ can be embedded into some finite member of $\mathbb{K}$.

Suppose we are interested in whether or not a given identity $(\forall \vec{x})(s(\vec{x}) = t(\vec{x}))$ is in the equational theory of some variety $\mathbb{V}$ of algebras. If we start with the assumption that it is not in the equational theory of $\mathbb{V}$, then there exists an algebra $\mathbf{A} \in \mathbb{V}$, and some assignment $\vec{x} \mapsto \vec{a}$ of elements of A to the variables such that the evaluations of s and t are different, i.e., $s^{\mathbf{A}}(\vec{a}) \neq t^{\mathbf{A}}(\vec{a})$. The set of elements of A used in the evaluation of s and t form a finite subset, say $M \subseteq^{fin} A$. Let $\mathbf{B}$ be the finite partial subalgebra of $\mathbf{A}$ with universe M. Now, if $\mathbb{V}$ has the FEP, then $\mathbf{B}$ can be embedded into a finite member $\mathbf{C}$ of $\mathbb{V}$ in such a way that all existing operations in $\mathbf{B}$ are preserved and therefore $s^{\mathbf{C}}(\vec{a}) \neq t^{\mathbf{C}}(\vec{a})$. That is, if $\mathbb{V}$ has the FEP, then the following result holds: an identity holds in $\mathbb{V}$ if, and only if, it holds in all finite members of $\mathbb{V}$. The same method can be used to find a finite countermodel for a given quasi-identity or universal sentence that is not valid in $\mathbb{V}$. As a consequence, $\mathbb{V}$ is generated, as a quasivariety, by its finite members. If, in addition, $\mathbb{V}$ is finitely axiomatized, then the (quasi-)equational theory of $\mathbb{V}$ is decidable.

The FEP has been used to prove the decidability of the universal theories of various varieties — usually associated with logic. Examples of varieties that have the FEP include the variety of closure algebras [McK41, MT44], Heyting algebras [MT46], integral residuated lattices [BvA02] and integral residuated ordered groupoids [BvA05]. In [GJ13, vA09] it was shown that a large selection

of subvarieties of (integral) residuated lattice-ordered unital groupoids, have the FEP.

For a more extensive background on the FEP the reader is referred to [BvA02] or [Eva69].

9. THE FEP FOR RESIDUATED STRUCTURES

We would like to identify varieties of residuated (ordered) structures that have the FEP. Such varieties correspond to substructural logics and for finitely axiomatized logics the FEP implies decidability thereof. In this chapter we consider two different constructions that may be used to establish the FEP for a variety of residuated structures. Each of these constructions is based on a completion construction. The first construction we study is the *standard construction* for obtaining the FEP for residuated structures (see, for example [vA09]). The standard construction is based on the MacNeille completion of a poset (see Remark 9.1.3). The second construction we investigate in this chapter is based on the canonical extension of a bounded lattice (see Remark 9.2.19). We therefore call it the *canonical FEP construction*.

In Section 9.1.1 we describe the standard construction for residuated (partially) ordered algebras [vA09]. We include this description here in order to highlight the similarities and differences between the standard construction and the construction considered in Section 9.2. Then, in Section 9.1.2 we describe the standard construction for MTL-chains [vA11]. The reader is referred to Chapter 5.2 for more on MTL-algebras and MTL-chains. Since the algebras under consideration are linearly ordered, the construction simplifies significantly.

Recall that if it is the case that the finite algebra obtained through the construction satisfies an inequality $s \leq t$ whenever the original algebra does, then we say that the inequality is preserved by the FEP construction. (Also recall that the universal quantification over the variables occurring in s and t is implicit.) In [vA09] and [vA11] a general description of inequalities $s \leq t$ that are preserved by the construction for residuated ordered algebras and MTL-chains, respectively, was given. As in the case of various completion constructions, an approximation term was used to establish the preservation of properties. In Section 9.1.2 we recall the definition of the approximation term that was used to prove the preservation of properties by the standard FEP construction for

MTL-chains. We then give a summary of the results from [vA11].

In Section 9.1.3 we extend the construction to modal MTL-chains (see Chapter 5.3). Thereafter, we consider the preservation of various identities under the FEP construction, building on the results in [vA11]. In doing so we obtain the FEP for various classes of modal MTL-chains. That is, if an inequality $s \leq t$ is preserved by the construction, then the subclass of modal MTL-chains characterized by $s \leq t$ has the FEP, and hence the corresponding subvariety of modal MTL-algebras has the FEP. Since any variety of modal MTL-algebras is generated by its subclass of modal MTL-chains, the FEP for such a variety follows from the FEP for its subclass of modal MTL-chains [vA11] (see also [BF00]). Thus, we only need to consider modal MTL-chains.

In Section 9.2 we describe an alternative construction for obtaining the FEP for residuated lattice ordered algebras. The construction in this section is based on the canonical extension of a lattice, studied in Chapter 6 — hence the title *canonical FEP construction*. We show (again) that the class of decreasing residuated (lattice) ordered algebras has the FEP through this construction. Finally we investigate some additional properties preserved by the construction.

9.1 The standard FEP construction

9.1.1 The FEP for residuated ordered algebras

For the full details of the construction described in this section the reader may consult [vA09]. The reader is also referred to [Bus11] for more on the FEP for residuated ordered algebras.

By a *residuated ordered algebra* (of type $\mathbb{T}$) we shall mean a structure $\mathbf{A} = \langle A, \mathbb{T}^{\mathbf{A}}, \leq \rangle$, where $\langle A, \leq \rangle$ is a poset and $\langle A, \mathbb{T}^{\mathbf{A}} \rangle$ is an algebra whose set of operations $\mathbb{T}^{\mathbf{A}}$ is a finite set consisting of constants, unary and binary residuated operators and their residuals. Now, let

(i) $\mathbb{T}_0^{\mathbf{A}}$ denote the set of constants in $\mathbb{T}^{\mathbf{A}}$;

(ii) $\mathbb{T}_1^{\mathbf{A}}$ denote the set of residuated unary operators in $\mathbb{T}^{\mathbf{A}}$;

(iii) $(\mathbb{T}_1^{\bullet})^{\mathbf{A}}$ denote the set of residuals of the operators in $\mathbb{T}_1^{\mathbf{A}}$;

(iv) $\mathbb{T}_2^{\mathbf{A}}$ denote the set of residuated binary operators in $\mathbb{T}^{\mathbf{A}}$; and

(v) $(\mathbb{T}_2^{\bullet})^{\mathbf{A}}$ denote the set of left and right residuals of the operators in $\mathbb{T}_2^{\mathbf{A}}$.

If $\mathbf{B}$ is a partial subalgebra of a residuated ordered algebra $\mathbf{A}$ (see Definition 8.0.1), then we introduce the following notion.

Definition 9.1.1 ([vA09]). *Let $\mathbf{A} = \langle A, \mathbb{T}^{\mathbf{A}}, \leq \rangle$ be a residuated ordered algebra of type $\mathbb{T}$ and $\mathbf{B}$ a partial subalgebra of $\mathbf{A}$. A pair $\mathbf{W} = \langle W, W^{\bullet} \rangle$ of subsets of A is called a* $\mathbf{B}$-residual pair *if the following conditions are satisfied:*

(i) W contains $B \cup \mathbb{T}_0^{\mathbf{A}}$ and is closed under the operations in $\mathbb{T}_1^{\mathbf{A}} \cup \mathbb{T}_2^{\mathbf{A}}$;

(ii) $W^{\bullet}$ contains $B \cup \mathbb{T}_0^{\mathbf{A}}$ and is closed under the operations in $(\mathbb{T}_1^{\bullet})^{\mathbf{A}}$ and closed under $a \backslash_k x$ and $x /_k a$ for all $a \in W$ and $\backslash_k, /_k \in (\mathbb{T}_2^{\bullet})^{\mathbf{A}}$.

For the remainder of this section let $\mathbf{A} = \langle A, \mathbb{T}^{\mathbf{A}}, \leq \rangle$ be a fixed residuated ordered algebra and let $\mathbf{B} = \langle B, \mathbb{T}^{\mathbf{B}}, \leq^{\mathbf{B}} \rangle$ be a fixed partial ordered subalgebra of $\mathbf{A}$. Let $\mathbf{W} = \langle W, W^{\bullet} \rangle$ be a $\mathbf{B}$-residual pair. Now defined ${}^{\mathfrak{l}}$ and ${}^{\mathfrak{u}}$ as follows. For $S \subseteq A$, let

$$S^{\mathfrak{l}} = \{a \in W : a \leq c \text{ for all } c \in S\}$$

$$S^{\mathfrak{u}} = \{a \in W^{\bullet} : a \geq c \text{ for all } c \in S\}$$

That is, $S^{\mathfrak{l}}$ is the set of all lower bounds of S in W and $S^{\mathfrak{u}}$ is the set of all upper bounds of S in $W^{\bullet}$. We note that if $W = A = W^{\bullet}$, then $S^{\mathfrak{l}} = S^{\ell}$ and $S^{\mathfrak{u}} = S^{u}$. Moreover, the pair of maps $({}^{\mathfrak{l}}, {}^{\mathfrak{u}})$ forms a Galois connection between $\langle \mathcal{P}(W), \subseteq \rangle$ and $\langle \mathcal{P}(W^{\bullet}), \supseteq \rangle$.

A set $S \subseteq W^{\bullet}$ will be called *stable* if $S = S^{\mathfrak{l}\mathfrak{u}}$. Let C denote the set of all stable sets. Then $\supseteq$ is a complete lattice order on C, and for $S_i \in C$ for $i \in \Psi$

$$\bigvee_{i \in \Psi}^{\mathbf{C}} S_i = \bigcap_{i \in \Psi} S_i \quad \text{and} \quad \bigwedge_{i \in \Psi}^{\mathbf{C}} S_i = \bigcap \{T \in C : S_i \subseteq T \text{ for all } i \in \Psi\}.$$

Furthermore, for each $f \in \mathbb{T}_1^{\mathbf{A}}$ and $\circ \in \mathbb{T}_2^{\mathbf{A}}$, define the operations $f^{\mathbf{C}}$ and $\circ^{\mathbf{C}}$ on C as follows [vA09, Definition 5.8]. For $L_1, L_2 \subseteq W$, let $f(L_1) = \{f(a) : a \in L_1\}$ and $L_1 \circ L_2 = \{a \circ b : a \in L_1 \text{ and } b \in L_2\}$. For $S_1, S_2 \in C$, define:

$$f^{\mathbf{C}}(S_1) = (f(S_1^{\mathfrak{l}}))^{\mathfrak{u}} \quad \text{and} \quad S_1 \circ^{\mathbf{C}} S_2 = (S_1^{\mathfrak{l}} \circ S_2^{\mathfrak{l}})^{\mathfrak{u}}.$$

Then for each $f \in \mathbb{T}_1^{\mathbf{A}}$ and $\circ \in \mathbb{T}_2^{\mathbf{A}}$, the operations $f^{\mathbf{C}}$ and $\circ^{\mathbf{C}}$ on C are residuated with respect to the order $\supseteq$ [vA09, Lemma 5.10]. If $f \in \mathbb{T}_1^{\mathbf{A}}$ has residual $g \in (\mathbb{T}_1^{\bullet})^{\mathbf{A}}$, then denote the residual of $f^{\mathbf{C}}$ by $g^{\mathbf{C}}$. Similarly, if $\circ \in \mathbb{T}_2^{\mathbf{A}}$ has left and right residuals $\backslash, / \in (\mathbb{T}_2^{\bullet})^{\mathbf{A}}$, then denote the left and right residuals of $\circ^{\mathbf{C}}$ by $\backslash^{\mathbf{C}}$ and $/^{\mathbf{C}}$, respectively. Finally, for each $k \in \mathbb{T}_0^{\mathbf{A}}$, let $k^{\mathbf{C}} = \{k\}^{\mathfrak{u}}$. Let

$\mathbb{T}^{\mathbf{C}} = \{t^{\mathbf{C}} : t \in \mathbb{T}\}$, $\leq^{\mathbf{C}} = \supseteq$ and $\mathbf{C} = \langle C, \mathbb{T}^{\mathbf{C}}, \leq^{\mathbf{C}} \rangle$. Then we have the following result.

Theorem 9.1.2. *[vA09, Theorem 5.11] The structure* $\mathbf{C}$ *is a complete residuated ordered algebra of the same type as* $\mathbf{A}$*, and there exists an embedding of* $\mathbf{B}$ *into* $\mathbf{C}$ *that preserves all existing meets and joins in* $\mathbf{B}$*.*

The map $\zeta : B \to C$ defined by $\zeta(b) = \{b\}^{\mathrm{u}}$ for $b \in B$ is an order embedding of $\mathbf{B}$ into $\mathbf{C}$ (see Definition 8.0.2) that preserves all existing meets and joins in $\mathbf{B}$.

Remark 9.1.3. *We note that if* $\mathbf{B} = \mathbf{A}$*, then* $\mathbf{W} = \langle A, A \rangle$ *is the only possible* $\mathbf{B}$*-residual pair, and* $\mathbf{C}$ *is a completion of* $\mathbf{A}$*. In fact, we obtain the MacNeille completion of the lattice-reduct of* $\mathbf{A}$*. For more on the MacNeille completion the reader may consult Chapter 5.*

An infinite sequence $a_1, a_2, \ldots$ of elements of a quasi-ordered set $\langle Q, \leq \rangle$ will be called *good* if there exist $i, j \in \mathbb{N}$ such that $i < j$ and $a_i \leq a_j$. If no such indices exist, i.e., if $a_i \not\leq a_j$ whenever $i < j$, then the sequence is called *bad*. A quasi-ordered set $\langle Q, \leq \rangle$ is *well-quasi-ordered* if every infinite sequence of elements of Q is good. That is, $\langle Q, \leq \rangle$ is well-quasi-ordered if it does not contain an infinite descending chain nor does it contain an infinite anti-chain. A quasi-ordered set $\langle Q, \leq \rangle$ is *reverse well-quasi-ordered* if, for every infinite sequence of elements $a_1, a_2, \ldots$, there exist $i, j \in \mathbb{N}$ with $i < j$ and $a_j \leq a_i$. That is, $\langle Q, \leq \rangle$ is reverse well-quasi-ordered if it does not contain an infinite ascending chain nor does it contain an infinite anti-chain.

Theorem 9.1.4. *[vA09, Theorem 7.1] Let* $\mathbf{A}$ *be a residuated ordered algebra,* $\mathbf{B}$ *a finite partial subalgebra of* $\mathbf{A}$*, and* $\mathbf{W} = \langle W, W^{\bullet} \rangle$ *a* $\mathbf{B}$*-residual pair.*

(i) If $\langle W, \leq \rangle$ *is reverse well-quasi-ordered and* $\langle W^{\bullet}, \leq \rangle$ *is well-quasi-ordered, then* $\mathbf{C}$ *is finite.*

(ii) If $\langle W, \leq \rangle$ *is well-quasi-ordered and* $\langle W^{\bullet}, \leq \rangle$ *is reverse well-quasi-ordered, then* $\mathbf{C}$ *is finite.*

9.1.2 The FEP for MTL-chains

Recall from Definition 5.2.2 that an MTL-algebra $\mathbf{A} = \langle A, \circ, \to, \vee, \wedge, 0, 1 \rangle$ is a residuated lattice that satisfies the prelinearity identity: for $x, y \in A$

$$(x \to y) \vee (y \to x) = 1.$$

An MTL-chain is a linearly ordered MTL-algebra. The reader is referred to Definitions 5.2.1 and 5.2.2, and the discussions that follow these definitions, for more on residuated lattices and MTL-algebras.

Throughout this section let $\mathbf{A}$ be a fixed MTL-chain, B a finite subset of A containing 1 and 0, and $\mathbf{B}$ the partial subagebra of $\mathbf{A}$ with domain B.

Let W and $W^\bullet$ be two sets satisfying:

(W1) $B \subseteq W \subseteq A$ and $B \subseteq W^\bullet \subseteq A$,

(W2) W is closed under $\circ$,

(W3) if $a \in W$ and $b \in W^\bullet$, then $a \to b \in W^\bullet$,

(W4) $\langle W, \leq \rangle$ is reverse well-ordered and $\langle W^\bullet, \leq \rangle$ is well-ordered.

If we use the terminology from the previous subsection, then (W1-W3) ensures that $\langle W, W^\bullet \rangle$ is a $\mathbf{B}$-residual pair, while (W4) ensures that the algebra obtained through the construction described in this section will be finite by Theorem 9.1.4.

Since we are only considering linearly ordered algebras, the standard construction simplifies to the following.

For W and $W^\bullet$ satisfying (W1 - W4) define, for each $a \in A$,

$$a^{\mathfrak{l}} = \bigvee \{b \in W : b \leq a\}, \qquad a^{\mathfrak{u}} = \bigwedge \{c \in W^\bullet : a \leq c\}.$$

The well-ordering and reverse well-ordering assumptions ensure that the relevant supremums and infimums of the above sets exist. The maps $^{\mathfrak{u}}$ and $^{\mathfrak{l}}$ are both order-preserving. In addition, the following properties are easily derived:

Lemma 9.1.5. *For any $a \in W$, $c, d \in W^\bullet$ and $e \in A$,*

(i) $a \leq a^{\mathfrak{u}\mathfrak{l}}$ *and* $c^{\mathfrak{l}\mathfrak{u}} \leq c$,

(ii) $a \leq c$ *iff* $a^{\mathfrak{u}} \leq c$,

(iii) $a \leq c$ *iff* $a \leq c^{\mathfrak{l}}$,

(iv) $a^{\mathfrak{u}\mathfrak{l}\mathfrak{u}} = a^{\mathfrak{u}}$, $c^{\mathfrak{l}\mathfrak{u}\mathfrak{l}} = c^{\mathfrak{l}}$,

(v) $e^{\mathfrak{l}\mathfrak{u}\mathfrak{l}\mathfrak{u}} = e^{\mathfrak{l}\mathfrak{u}}$.

In fact, the pair of maps $({}^{\mathsf{l}},{}^{\mathsf{u}})$ (considered as maps between W and $W^\bullet$) forms a Galois connection between $\langle W,\leq\rangle$ and $\langle W^\bullet,\leq\rangle$.

An element $c\in W^\bullet$ is said to be *stable* if $c=c^{\mathsf{lu}}$. Let C denote the set of stable elements. By Lemma 9.1.5 (iv) and (v), a^{u} is stable for $a\in W$ and e^{lu} is stable for $e\in A$. In [vA11] is was shown that the well-ordering and reverse well-ordering assumptions in $(W4)$ imply that C is a finite set.

We define an MTL-chain with universe C. Since $C\subseteq A$, the order on A, restricted to C, is linear and defines lattice operations $\wedge^{\mathbf{C}}$ and $\vee^{\mathbf{C}}$ which coincide with the corresponding operations on A. The product operation is defined, for $c,d\in C$, by:

$$c\circ^{\mathbf{C}} d=(c^{\mathsf{l}}\circ d^{\mathsf{l}})^{\mathsf{u}}.$$

The following property holds: If $c,d\in C$ and $a,b\in W$ for which $c=a^{\mathsf{u}}$ and $d=b^{\mathsf{u}}$, then $c\circ^{\mathbf{C}} d=(a\circ b)^{\mathsf{u}}$. Using this property one can show that $\circ^{\mathbf{C}}$ is associative, commutative, has identity 1 and is residuated with respect to $\leq$; for $c,d\in C$, the residual is:

$$c\to^{\mathbf{C}} d=(c^{\mathsf{l}}\to d)^{\mathsf{lu}}.$$

The algebra $\mathbf{C}=\langle C,\circ^{\mathbf{C}},\to^{\mathbf{C}},\wedge^{\mathbf{C}},\vee^{\mathbf{C}},0,1\rangle$ is therefore a finite MTL-chain and the identity map is an embedding of $\mathbf{B}$ into $\mathbf{C}$; that is, if $a\circ^{\mathbf{B}} b$ is defined in B, then $a\circ^{\mathbf{B}} b=a\circ^{\mathbf{C}} b$, and, similarly, for $\to^{\mathbf{B}}$.

Let $t(\vec{x})=t(x_1,\ldots,x_n)$ be any $\{\circ,\to,\vee,\wedge,0,1\}$-term. If $\vec{c}=c_1,\ldots,c_n$ is a sequence of elements of C, then $t^{\mathbf{C}}(\vec{c})$ denotes the evaluation of t in $\mathbf{C}$ under the assignment $x_i\mapsto c_i$. Where a term $t(\vec{x})$ and $\vec{c}\in C$ are given, $\vec{x}$ and $\vec{c}$ are assumed to be sequences of the same length. If $\vec{c}=c_1,\ldots,c_n$ is a sequence of elements in C, then $\vec{c}^{\,\mathsf{l}}$ denotes the sequence $c_1^{\mathsf{l}},\ldots,c_n^{\mathsf{l}}$ of elements of W.

For each term $s(\vec{x})$ and $\vec{c}\in C$, define

$$s^\star(\vec{c})=s^{\mathbf{A}}(\vec{c}^{\,\mathsf{l}})^{\mathsf{lu}}.$$

Note that $s^\star(\vec{c})\in C$ by Lemma 9.1.5 (v). A term $s(\vec{x})$ is called:

$$\begin{aligned}
&\star\text{-}\textit{stable} && \text{if} \quad s^{\mathbf{C}}(\vec{c})=s^\star(\vec{c})\\
&\star\text{-}\textit{expanding} && \text{if} \quad s^{\mathbf{C}}(\vec{c})\geq s^\star(\vec{c})\\
&\star\text{-}\textit{contracting} && \text{if} \quad s^{\mathbf{C}}(\vec{c})\leq s^\star(\vec{c}) \qquad \text{for all } \vec{c}\in C.
\end{aligned}$$

If $\mathbf{A}$ satisfies an inequality $s\leq t$, then $s^\star(\vec{c})\leq t^\star(\vec{c})$ for all $\vec{c}\in C$. Thus, if s is $\star$-contracting and t is $\star$-expanding, then $\mathbf{C}$ satisfies $s\leq t$ and the inequality

is preserved. Observe that $\star$-stable implies both $\star$-contracting and $\star$-expanding. This gives the following results.

Theorem 9.1.6. *[vA11] The following hold for all terms s and t:*

(i) If s and t are both $\star$-stable, then $s = t$ is preserved by the FEP construction.

(ii) If s is $\star$-contracting and t is $\star$-expanding, then $s \leq t$ is preserved by the FEP construction.

The following proposition summarizes the results regarding MTL-terms.

Proposition 9.1.7. *[vA11]*

(i) If $s(\vec{x})$ is a $\{\circ, \vee, \wedge, 0, 1\}$-term and $\vec{c} \in C$, then $s^{\mathbf{C}}(\vec{c}) = s(\vec{c}^{\mathrm{l}})^{\mathrm{u}}$ and s is $\star$-stable.

(ii) If $t(\vec{x}) = \neg s(\vec{x})$, where $s(\vec{x})$ is a $\{\circ, \vee, \wedge, 0, 1\}$-term and $\vec{c} \in C$, then $t^{\mathbf{C}}(\vec{c}) = t(\vec{c}^{\mathrm{l}})^{\mathrm{lu}}$, i.e., t is $\star$-stable.

(iii) For all variables $x_1, \ldots, x_n, y$, the term $(x_1 \circ \cdots \circ x_n) \to y$ is $\star$-contracting.

(iv) If $t_1, \ldots, t_m$ are $\star$-stable (resp., $\star$-expanding, $\star$-contracting) terms and $s(y_1, \ldots, y_m)$ is a $\{\wedge, \vee\}$-term, then $s(t_1, \ldots, t_m)$ is $\star$-stable (resp., $\star$-expanding, $\star$-contracting).

(v) If $t_1, \ldots, t_m$ are $\star$-contracting terms and $s(y_1, \ldots, y_m)$ is a $\{\circ, \wedge, \vee\}$-term, then $s(t_1, \ldots, t_m)$ is $\star$-contracting.

(vi) If s is a $\star$-contracting term and t is a $\star$-expanding term, then $s \to t$ is $\star$-expanding.

9.1.3 The FEP for modal MTL-chains

The results from this section were obtained in collaboration with Prof. Clint van Alten and have been published in [MvAb].

Recall from Definitions 5.3.1 and 5.3.4 that a modal MTL-chain $\mathbf{A} = \langle A, \circ, \to, \wedge, \vee, f, 0, 1\rangle$ is a linearly ordered residuated lattice such that f is an order-preserving unary operation.

Throughout this section let $\mathbf{A} = \langle A, \circ, \to, \wedge, \vee, f, 0, 1\rangle$ be a fixed modal MTL-chain, let B be a finite subset of A containing 1 and 0, let $\mathbf{B}$ be the partial

subagebra of $\mathbf{A}$ with domain B and let $\mathbf{C}$ be the finite MTL-chain obtained by the construction described in the previous subsection, from the modality-free reduct of $\mathbf{B}$.

In order to extend the construction to modal MTL-chains, we define the operation $f^{\mathbf{C}}$ on C by:

$$f^{\mathbf{C}}(c) = f(c^{\mathfrak{l}})^{\mathfrak{l}\mathfrak{u}}.$$

For ease of notation, we assume that f binds more strongly than $^{\mathfrak{u}}$ and $^{\mathfrak{l}}$.

Lemma 9.1.8. *The identity embedding of* $\mathbf{B}$ *into* $\mathbf{C}$ *preserves the operation* f*, i.e., if* $f(b) \in B$ *for some* $b \in B$*, then* $f^{\mathbf{C}}(b) = f^{\mathbf{B}}(b) = f(b)$*.*

Proof. Note that, by the definitions of $^{\mathfrak{u}}$ and $^{\mathfrak{l}}$ and the fact that $B \subseteq W \cap W^{\bullet}$, if $b \in B$, then $b^{\mathfrak{u}} = b = b^{\mathfrak{l}}$. Thus, if $f(b) \in B$ as well, then we have: $f^{\mathbf{C}}(b) = f(b^{\mathfrak{l}})^{\mathfrak{l}\mathfrak{u}} = f(b)^{\mathfrak{l}\mathfrak{u}} = f(b)$. □

Lemma 9.1.9. $f^{\mathbf{C}}$ *is order-preserving, hence* $f^{\mathbf{C}}$ *distributes over* $\wedge^{\mathbf{C}}$ *and* $\vee^{\mathbf{C}}$*.*

Proof. If $a, b \in C$ such that $a \leq b$, then $a^{\mathfrak{l}} \leq b^{\mathfrak{l}}$ and also $f(a^{\mathfrak{l}}) \leq f(b^{\mathfrak{l}})$. Thus, $f^{\mathbf{C}}(a) = f(a^{\mathfrak{l}})^{\mathfrak{l}\mathfrak{u}} \leq f(b^{\mathfrak{l}})^{\mathfrak{l}\mathfrak{u}} = f^{\mathbf{C}}(b)$. □

Theorem 9.1.10. *For* $W, W^{\bullet}$ *satisfying (W1 - W4) the algebra* $\mathbf{C} = \langle C, \circ^{\mathbf{C}}, \to^{\mathbf{C}}, \wedge^{\mathbf{C}}, \vee^{\mathbf{C}}, f^{\mathbf{C}}, 0, 1\rangle$ *is a finite modal MTL-chain and the identity map is an embedding of* $\mathbf{B}$ *into* $\mathbf{C}$*.*

Since a choice of W and $W^{\bullet}$ exists that satisfies (W1 - W4), namely, W the $\{\circ\}$-closure of B in A and $W^{\bullet} = \{a \to b : a \in W, b \in B\}$, we have the following result.

Theorem 9.1.11. *The class of modal MTL-chains has the FEP, hence the variety of modal MTL-algebras has the FEP.*

We now extend the results summarized in Section 9.1.2 to include the modal operator. We obtain different preservation results depending on the choice of W and $W^{\bullet}$. Generally, the larger W is, the stronger the results, but we must always ensure that $\langle W, \leq\rangle$ is reverse well-ordered and $\langle W^{\bullet}, \leq\rangle$ is well-ordered for $\mathbf{C}$ to be finite. In the subsections below, we consider variations of the above construction by making different choices for W and $W^{\bullet}$. We then describe $\star$-stable, $\star$-contracting and $\star$-expanding terms involving the modality f. Larger classes of $\star$-stable, $\star$-contracting and $\star$-expanding terms can then be inferred

from Proposition 9.1.7. The preservation results and FEP for each class are then obtained directly from Theorem 9.1.6.

The first choice of W and $W^\bullet$ we shall consider is W the $\{\circ\}$-closure of B in A and $W^\bullet = \{a \to b : a \in W, b \in B\}$. By definition, $B \subseteq W$, and $B \subseteq W^\bullet$ since $b \in B$ can be written as $1 \to b$. Note that if $c = a \to b \in W^\bullet$ for $a \in W$ and $b \in B$, then for any $d \in W$, $d \to c = (a \circ d) \to b \in W^\bullet$. That $\langle W, \leq \rangle$ is reverse well-ordered and $\langle W^\bullet, \leq \rangle$ is well-ordered is proved in [vA11]. Note that W is also closed under $\wedge$ and $\vee$ and contains 0 and 1. From the definition of $f^{\mathbf{C}}$, we immediately get that $f(x)$, $f(1)$ and $f(0)$ are $\star$-stable terms. Thus, by Proposition 9.1.7, we have the following results for this particular choice of W and $W^\bullet$.

Lemma 9.1.12. *If s is a $\{\circ, \vee, \wedge, 0, 1\}$-term, then $f(s)$ is $\star$-expanding.*

Proof. Let $t(\vec{x}) = f(s(\vec{x}))$ and $\vec{c} \in C$. By Proposition 9.1.7 (i), we have $s(\vec{c}^{\mathfrak{l}}) \in W$ and $s^{\mathbf{C}}(\vec{c}) = s(\vec{c}^{\mathfrak{l}})^{\mathfrak{u}}$, so $t^{\mathbf{C}}(\vec{c}) = f(s(\vec{c}^{\mathfrak{l}})^{\mathfrak{u}\mathfrak{l}})^{\mathfrak{l}\mathfrak{u}} \geq f(s(\vec{c}^{\mathfrak{l}}))^{\mathfrak{l}\mathfrak{u}} = t^\star(\vec{c})$. □

Lemma 9.1.13. *If s is a $\{\circ, \wedge, \vee, 0, 1\}$-term, then $f(\neg s)$ is $\star$-contracting.*

Proof. Let $t(\vec{x}) = f(\neg s(\vec{x}))$ and $\vec{c} \in C$. Then $t^{\mathbf{C}}(\vec{c}) = f^{\mathbf{C}}(\neg^{\mathbf{C}} s^{\mathbf{C}}(\vec{c})) = f^{\mathbf{C}}(\neg^{\mathbf{C}}(s(\vec{c}^{\mathfrak{l}}))^{\mathfrak{u}}) = f^{\mathbf{C}}(((s(\vec{c}^{\mathfrak{l}}))^{\mathfrak{u}\mathfrak{l}} \to 0)^{\mathfrak{l}\mathfrak{u}})$, where the second equality follows from Proposition 9.1.7 (i). But $(s(\vec{c}^{\mathfrak{l}}))^{\mathfrak{u}\mathfrak{l}} \to 0 \in W^\bullet$, and by Lemma 9.1.5 (i), we have that $((s(\vec{c}^{\mathfrak{l}}))^{\mathfrak{u}\mathfrak{l}} \to 0)^{\mathfrak{l}\mathfrak{u}} \leq ((s(\vec{c}^{\mathfrak{l}}))^{\mathfrak{u}\mathfrak{l}} \to 0)$. Since $f^{\mathbf{C}}$ is order-preserving, $t^{\mathbf{C}}(\vec{c}) \leq f^{\mathbf{C}}((s(\vec{c}^{\mathfrak{l}}))^{\mathfrak{u}\mathfrak{l}} \to 0) = f(((s(\vec{c}^{\mathfrak{l}}))^{\mathfrak{u}\mathfrak{l}} \to 0)^{\mathfrak{l}})^{\mathfrak{l}\mathfrak{u}}$. Since $\vec{c}^{\mathfrak{l}} \in W$ and W is closed under $\{\circ, \wedge, \vee, 0, 1\}$ it follows that $s(\vec{c}^{\mathfrak{l}}) \in W$ and $s(\vec{c}^{\mathfrak{l}}) \leq (s(\vec{c}^{\mathfrak{l}}))^{\mathfrak{u}\mathfrak{l}}$. But $\to$ is order-reversing in the first coordinate and ${}^{\mathfrak{l}}$ is order-preserving, so $((s(\vec{c}^{\mathfrak{l}}))^{\mathfrak{u}\mathfrak{l}} \to 0)^{\mathfrak{l}} \leq (s(\vec{c}^{\mathfrak{l}}) \to 0)^{\mathfrak{l}}$. Therefore, $t^{\mathbf{C}}(\vec{c}) \leq f(s(\vec{c}^{\mathfrak{l}}) \to 0)^{\mathfrak{l}\mathfrak{u}} = t^\star(\vec{c})$. □

We note that a number of the special classes of modal MTL-chains considered in Chapter 5.3.2 are closed under the standard FEP construction and hence have the FEP. To see that this is the case observe that, by the above and Proposition 9.1.7, the following inequalities and identities (that form the additional axioms of these classes) are preserved by the construction: $f(x) \circ f(y) \leq f(x \circ y)$ (which is equivalent to $f(x \to y) \leq f(x) \to f(y)$), $f(1) = 1$ and $f(x) \leq x$. In addition, the strict condition $x \wedge (\neg x) \leq 0$ and n-contraction $x^n \leq x^{n+1}$ are preserved, as is the involution $\neg\neg x = x$, although for this case it is necessary to first close B under $\neg$ (see [vA11]). Thus, the varieties of LK^r-algebras and LKT^r-

algebras have the FEP for each $\mathrm{L} \in \mathit{Logics} = \{\mathrm{MTL}, \mathrm{IMTL}, \mathrm{SMTL}\} \cup \{\mathrm{C}_n\mathrm{MTL} : n \geq 2\} \cup \{\mathrm{C}_n\mathrm{IMTL} : n \geq 2\}$.

When W is closed under f

Throughout this subsection we assume that $\mathbf{A}$ satisfies $f(x) \leq x$ and take W to be the $\{\circ, f\}$-closure of B and $W^\bullet = \{a \to b : a \in W, b \in B\}$. It is immediate that (W1 - W3) hold; for (W4), the reverse well-ordering of W follows directly from Higman's theorem [Hig52]: we may consider W as an ordered algebra generated by a finite set B, with operations ($\circ$ and f) compatible with the order and decreasing in all arguments. The reverse well-ordering of $W^\bullet$ then follows. Note that $f(x) \leq x$ is preserved by the FEP construction since both $f(x)$ and x are $\star$-stable terms; that is, $f^{\mathbf{C}}$ is decreasing.

Lemma 9.1.14. *If $t(x) = (f(x))^n$, for any $n \geq 1$, and $c \in C$, then $t^{\mathbf{C}}(c) = (t(c^{\mathfrak{l}}))^{\mathfrak{u}}$ and t is $\star$-stable. In addition, $(f(1))^n$ and $(f(0))^n$ are $\star$-stable for any $n \geq 1$.*

Proof. This is shown by induction on n: if $n = 1$ then $t(x) = f(x)$ and t is $\star$-stable by definition. Moreover, since $c^{\mathfrak{l}} \in W$ and W is closed under f we have $(f(c^{\mathfrak{l}}))^{\mathfrak{l}} = f(c^{\mathfrak{l}})$ and $t^\star(c) = (f(c^{\mathfrak{l}}))^{\mathfrak{u}}$. Now suppose that for some $n \geq 1$, the term $s(x) = (f(x))^n$ is $\star$-stable and $s^{\mathbf{C}}(c) = (s(c^{\mathfrak{l}}))^{\mathfrak{u}}$. Let $t(x) = f(x) \circ s(x)$. Then:

$$\begin{aligned} t^{\mathbf{C}}(c) &= f^{\mathbf{C}}(c) \circ^{\mathbf{C}} s^{\mathbf{C}}(c) \\ &= f(c^{\mathfrak{l}})^{\mathfrak{u}} \circ^{\mathbf{C}} (s(c^{\mathfrak{l}}))^{\mathfrak{u}} \\ &= (f(c^{\mathfrak{l}}) \circ s(c^{\mathfrak{l}}))^{\mathfrak{u}} \\ &= (t(c^{\mathfrak{l}}))^{\mathfrak{u}} = t^\star(c). \end{aligned}$$

The final equality follows since $c^{\mathfrak{l}} \in W$ and W's closure under $\circ$ and f implies that $t(c^{\mathfrak{l}}) \in W$, i.e., $(t(c^{\mathfrak{l}}))^{\mathfrak{l}} = t(c^{\mathfrak{l}})$. The proofs for $(f(1))^n$ and $(f(0))^n$ are similar. □

Lemma 9.1.15. *Any $\{\circ, \wedge, \vee, f, 0, 1\}$-term is $\star$-expanding.*

Proof. Let $s(\vec{x})$ be a $\{\circ, \wedge, \vee, f, 0, 1\}$-term and $\vec{c} \in C$. If s does not contain f then, by Proposition 9.1.7 (i), s is $\star$-stable, and therefore also $\star$-expanding for the standard construction and hence also for the modified construction. We

just need to consider the case where s contains f, so let $s(\vec{x}) = f(t(\vec{x}))$ where $t(\vec{x})$ is a $\{\circ, \wedge, \vee, f, 0, 1\}$-term and assume, inductively, that t is $\star$-expanding. We have $s^{\mathbf{C}}(\vec{c}) = f^{\mathbf{C}}(t^{\mathbf{C}}(\vec{c})) \geq f^{\mathbf{C}}(t^{\star}(\vec{c})) = f^{\mathbf{C}}((t(\vec{c}^{\mathfrak{l}}))^{\mathfrak{lu}})$. Since $\vec{c}^{\mathfrak{l}} \in W$ and W is closed under $\{\circ, \wedge, \vee, f, 0, 1\}$ we have $(t(\vec{c}^{\mathfrak{l}}))^{\mathfrak{lu}} = (t(\vec{c}^{\mathfrak{l}}))^{\mathfrak{u}}$. Then $s^{\mathbf{C}}(\vec{c}) \geq f^{\mathbf{C}}((t(\vec{c}^{\mathfrak{l}}))^{\mathfrak{u}}) = f((t(\vec{c}^{\mathfrak{l}}))^{\mathfrak{ul}})^{\mathfrak{lu}}$. Moreover, since $t(\vec{c}^{\mathfrak{l}}) \in W$ we have $(t(\vec{c}^{\mathfrak{l}}))^{\mathfrak{ul}} \geq t(\vec{c}^{\mathfrak{l}})$, by Proposition 9.1.5 (i), and so $s^{\mathbf{C}}(\vec{c}) \geq f(t(\vec{c}^{\mathfrak{l}}))^{\mathfrak{lu}} = s^{\star}(\vec{c})$. □

Lemma 9.1.16. *If $t = \neg s$ where s is a $\{\circ, \wedge, \vee, f, 0, 1\}$-term, then t is $\star$-contracting.*

Proof. Let $t(\vec{x}) = \neg s(\vec{x})$ and $\vec{c} \in C$. By Lemma 9.1.15, $s^{\mathbf{C}}(\vec{c}) \geq s^{\star}(\vec{c})$ and, since $\neg^{\mathbf{C}}$ is order-reversing, $\neg^{\mathbf{C}} s^{\mathbf{C}}(\vec{c}) \leq \neg^{\mathbf{C}} s^{\star}(\vec{c})$. Furthermore, since $\vec{c} \in W$ and W is closed under $\{\circ, \wedge, \vee, f, 0, 1\}$, it follows that $s(c^{\mathfrak{l}}) \in W$, $s^{\star}(\vec{c}) = (s(\vec{c}^{\mathfrak{l}}))^{\mathfrak{u}}$ and $s(\vec{c}^{\mathfrak{l}}) \leq (s(\vec{c}^{\mathfrak{l}}))^{\mathfrak{ul}}$. Thus, $t^{\mathbf{C}}(\vec{c}) = \neg^{\mathbf{C}} s^{\mathbf{C}}(\vec{c}) \leq \neg^{\mathbf{C}} s^{\star}(\vec{c}) = ((s(\vec{c}^{\mathfrak{l}}))^{\mathfrak{ul}} \to 0)^{\mathfrak{lu}} \leq (s(\vec{c}^{\mathfrak{l}}) \to 0)^{\mathfrak{lu}} = t^{\star}(\vec{c})$. □

By the above results, we have that $f(x) \leq f(f(x))$ and $f(x) \circ f(x) = f(x)$ are preserved by the above FEP construction. Recall that $\mathit{Logics} = \{\mathrm{MTL}, \mathrm{IMTL}, \mathrm{SMTL}\} \cup \{\mathrm{C}_n\mathrm{MTL} : n \geq 2\} \cup \{\mathrm{C}_n\mathrm{IMTL} : n \geq 2\}$. Let $\mathrm{L} \in \mathit{Logics}$; then the varieties of $\mathrm{LS4}^r$- and $\mathrm{L}!^r$-algebras (see Chapter 5.3.2) have the FEP since they have decreasing operators. In addition, L^r_{Δ}-algebras have the FEP since the identity $f(x) \vee (f(x) \to 0) = 1$ is easily seen to be preserved, as in the proof of Corollary 5.3.32.

In [CMM10], Ciabattoni et al. investigated the FEP for MTL-algebras and IMTL-algebras. They showed that the subvarieties of IMTL-,SMTL-, MTL!^{r}- and IMTL!^{r}-algebras have the FEP. Using our construction, we show that all of the subvarieties of algebras considered in [CMM10] have the FEP - thus extending their results. Moreover, using our FEP construction any subvariety obtained by adding identities preserved by our FEP construction have the FEP.

Residuated Operators

In the previous subsections, the sets of $\star$-stable terms excluded those with iterated f's, such as $f(f(x))$. One case in which such terms are $\star$-stable is if f is *residuated* with residual g. The reader is referred to Chapter 2.5 for more on residuated operators. Residuated operators are a special case of the complete operators considered in Chapter 5.3.2.

Results regarding the FEP for residuated lattices with additional residuated operators were obtained in [vA09]. These results specialise to the case of modal MTL-algebras, as we show here. The FEP construction is modified as follows: Set W to be the $\{\circ, f\}$-closure of B and $W^\bullet$ the least set containing B and closed under the operations $a \to x$, $a \in W$, and g.

It is immediate that this choice of W and $W^\bullet$ satisfy conditions (W1 - W3), however (W4) is not generally true. If f is decreasing, then (W4) is true [vA09]. Thus, we assume that $\mathbf{A}$ satisfies $f(x) \leq x$. Observe that since W is closed under f and $\circ$, the preservation results of the previous subsection hold.

Lemma 9.1.17. *The operation $f^{\mathbf{C}}$ is residuated and its residual is $g^{\mathbf{C}}(c) = \bigvee\{d \in C : f^{\mathbf{C}}(d) \leq c\} = g(c)^{\mathfrak{l}\mathfrak{u}}$ for all $c \in C$. Thus, if $b, g(b) \in B$, then $g^{\mathbf{C}}(b) = g(b)$.*

Proof. Let $c \in C$. We begin by showing that $g(c)^{\mathfrak{l}\mathfrak{u}}$ belongs to $\{d \in C : f^{\mathbf{C}}(d) \leq c\}$. Since $g(c) \in W^\bullet$,

$$f^{\mathbf{C}}(g(c)^{\mathfrak{l}\mathfrak{u}}) = f(g(c)^{\mathfrak{l}\mathfrak{u}\mathfrak{l}})^{\mathfrak{u}} = f(g(c)^{\mathfrak{l}})^{\mathfrak{u}} \leq f(g(c))^{\mathfrak{u}} \leq c^{\mathfrak{u}} = c.$$

Next, suppose $d \in C$ such that $f^{\mathbf{C}}(d) \leq c$, i.e., $f(d^{\mathfrak{l}})^{\mathfrak{u}} \leq c$. Then $f(d^{\mathfrak{l}}) \leq c$, hence $d^{\mathfrak{l}} \leq g(c)$. By Lemma 9.1.5, since $d^{\mathfrak{l}} \in W$, we have $d^{\mathfrak{l}} \leq g(c)^{\mathfrak{l}}$ hence $d = d^{\mathfrak{l}\mathfrak{u}} \leq g(c)^{\mathfrak{l}\mathfrak{u}}$, which completes the proof of the first statement. Thus, if $b, g(b) \in B$, then $g^{\mathbf{C}}(b) = g(b)^{\mathfrak{l}\mathfrak{u}} = g(b)$. □

The usefulness of the residuation property for f comes from the following result, which is then used to describe a large set of $\star$-stable terms.

Lemma 9.1.18. *If $a \in W$, then $f^{\mathbf{C}}(a^{\mathfrak{u}}) = f(a)^{\mathfrak{u}}$.*

Proof. Let $a \in W$. Recall that $f^{\mathbf{C}}(a^{\mathfrak{u}}) = f(a^{\mathfrak{u}\mathfrak{l}})^{\mathfrak{u}}$. By Lemma 9.1.5, $a \leq a^{\mathfrak{u}\mathfrak{l}}$, hence $f(a)^{\mathfrak{u}} \leq f(a^{\mathfrak{u}\mathfrak{l}})^{\mathfrak{u}}$. Let $e = f(a)^{\mathfrak{u}} \in W^\bullet$. Then $f(a) \leq e$ hence $a \leq g(e)$ by residuation. Then $a^{\mathfrak{u}} \leq g(e)$ by Lemma 9.1.5, since $g(e) \in W^\bullet$, so $a^{\mathfrak{u}\mathfrak{l}} \leq a^{\mathfrak{u}} \leq g(e)$. Thus, $f(a^{\mathfrak{u}\mathfrak{l}}) \leq f(g(e)) \leq e$ so $f^{\mathbf{C}}(a^{\mathfrak{u}}) = f(a^{\mathfrak{u}\mathfrak{l}})^{\mathfrak{u}} \leq e = f(a)^{\mathfrak{u}}$. □

Lemma 9.1.19. *If $s(\vec{x})$ is a $\{\circ, \vee, \wedge, f, 0, 1\}$-term and $\vec{c} \in C$, then $s^{\mathbf{C}}(\vec{c}) = s(\vec{c}^{\mathfrak{l}})^{\mathfrak{u}}$ and s is $\star$-stable.*

Proof. If s is a $\{\circ, \vee, \wedge, 0, 1\}$-term, the results follow from Lemma 9.1.7 (i). Inductively, suppose $s = f(\vec{x})$, where $t(\vec{x})$ is a $\{\circ, \vee, \wedge, f, 0, 1\}$-term and $t^{\mathbf{C}}(\vec{c}) =$

$t(\vec{c}^{\,\mathfrak{l}})^{\mathfrak{u}}$ for all $\vec{c} \in C$. For each $c \in C$, $c^{\mathfrak{l}} \in W$ and W is closed under the operations in $\{\circ, \vee, \wedge, f, 0, 1\}$, so $t(\vec{c}^{\,\mathfrak{l}}) \in W$ and $f(t(\vec{c}^{\,\mathfrak{l}})) \in W$. Thus,

$$\begin{aligned} s^{\mathbf{C}}(\vec{c}) &= f^{\mathbf{C}}(t^{\mathbf{C}}(\vec{c})) = f^{\mathbf{C}}(t(\vec{c}^{\,\mathfrak{l}})^{\mathfrak{u}}) \\ &= f(t(\vec{c}^{\,\mathfrak{l}}))^{\mathfrak{u}} \qquad \text{(by Lemma 9.1.18)} \\ &= f(t(\vec{c}^{\,\mathfrak{l}}))^{\mathfrak{l}\mathfrak{u}} = s^{\star}(\vec{c}). \end{aligned}$$

□

Order-reversing modalities

The standard construction can also be extended to reverse modal MTL-chains and the results obtained in the case of modal MTL-chains can be adapted for order-reversing modalities.

Recall from Definitions 5.3.1 and 5.3.19 that a reverse modal MTL-chain $\mathbf{A} = \langle A, \circ, \to, \vee, \wedge, h, 0, 1\rangle$ is a linearly ordered residuated lattice with an additional order-reversing unary operation h.

For the remainder of this section let $\mathbf{A} = \langle A, \circ, \to, \vee, \wedge, h, 0, 1\rangle$ be a fixed reverse modal MTL-chain. Assume that W and $W^{\bullet}$ are sets satisfying (W1-W4), in particular, we may take W to be the $\{\circ\}$-closure of B in A and $W^{\bullet} = \{a \to b : a \in W, b \in B\}$. Let $\mathbf{C}$ be the finite MTL-algebra obtained by the construction in Section 9.1.2. Extend $\mathbf{C}$ with the operation $h^{\mathbf{C}}$ on C defined by: for $c \in C$,

$$h^{\mathbf{C}}(c) = h(c^{\mathfrak{l}})^{\mathfrak{l}\mathfrak{u}}.$$

Lemma 9.1.20. *$h^{\mathbf{C}}$ is order-reversing.*

Proof. For $c, d \in C$ with $c \leq d$, we have $c^{\mathfrak{l}} \leq d^{\mathfrak{l}}$ hence $h(d^{\mathfrak{l}}) \leq h(c^{\mathfrak{l}})$, and so $h^{\mathbf{C}}(d) \leq h^{\mathbf{C}}(d)$. □

The proofs of the following results are straightforward.

Lemma 9.1.21. *The identity embedding of $\mathbf{B}$ into $\mathbf{C}$ preserves the operation h, i.e., if $b, h(b) \in B$, then $h^{\mathbf{C}}(b) = h(b)$.*

Theorem 9.1.22. *For $W, W^{\bullet}$ satisfying (W1-W4) the algebra $\mathbf{C} = \langle C, \circ^{\mathbf{C}}, \to^{\mathbf{C}}, \wedge^{\mathbf{C}}, \vee^{\mathbf{C}}, h^{\mathbf{C}}, 0, 1\rangle$ is a finite reverse modal MTL-chain, and the identity map is an embedding of $\mathbf{B}$ into $\mathbf{C}$.*

Corollary 9.1.23. *The class of reverse modal MTL-chains has the FEP, hence the variety of reverse modal MTL-algebras has the FEP.*

We now investigate preservation theorems that include the order-reversing operation. From the definition of $h^{\mathbf{C}}$ it follows that $h(x), h(1)$ and $h(0)$ are $\star$-stable terms. Moreover, by Proposition 9.1.7 we have the following results.

Lemma 9.1.24. *If s is a $\{\circ, \wedge, \vee, 0, 1\}$-term, then $h(s)$ is $\star$-contracting and $h(\neg s)$ is $\star$-expanding.*

Proof. If $s(\vec{x})$ is a $\{\circ, \wedge, \vee, 0, 1\}$-term and $\vec{c} \in C$, then, by Proposition 9.1.7 (i),

$$h^{\mathbf{C}}(s^{\mathbf{C}}(\vec{c}) = h^{\mathbf{C}}(s(\vec{c}^{\,\mathfrak{l}})^{\mathfrak{u}}) = h(s(\vec{c}^{\,\mathfrak{l}})^{\mathfrak{u}\mathfrak{l}})^{\mathfrak{l}\mathfrak{u}} \leq h(s(\vec{c}^{\,\mathfrak{l}}))^{\mathfrak{l}\mathfrak{u}},$$

so $h(s)$ is $\star$-contacting. The remaining statement follows similarly from Proposition 9.1.7 (ii). □

Using the above results together with Proposition 9.1.7 larger classes of $\star$-stable, $\star$-contracting and $\star$-expanding terms can be inferred. From Theorem 9.1.6 we then obtain preservation results and the FEP for subvarieties of reverse modal MTL-algebras whose corresponding characteristic properties are preserved.

9.2 The canonical FEP construction

The results obtained in this section form part of an on-going collaboration with Prof. Clint van Alten [MvAa].

The standard construction for obtaining the FEP for a variety of residuated ordered algebras is based on the MacNeille completion of lattices (see Remark 9.1.3). However, a lattice can be completed in many different ways as can be seen from Part I of this thesis. In particular, we now describe an alternative construction for obtaining the FEP for residuated lattice ordered algebras that is based on the construction of a completion of a lattice with respect to a polarization, i.e., the canonical extension. See Chapter 6 for more on this construction.

9.2.1 The construction

Throughout this section $\mathbf{A} = \langle A, \vee, \wedge, \mathbb{T}^{\mathbf{A}}, \leq \rangle$ will be a fixed residuated lattice ordered algebra (of type $\mathbb{T}$). The set of operations $\mathbb{T}^{\mathbf{A}}$ of $\mathbf{A}$ is a finite set con-

sisting of constants, unary and binary residuated operations and their residuals as in Section 9.1.1. Moreover, let $\mathbf{B} = \langle B, \vee^{\mathbf{B}}, \wedge^{\mathbf{B}}, \mathbb{T}^{\mathbf{B}}, \leq^{\mathbf{B}} \rangle$ be a fixed partial (ordered) subalgebra of $\mathbf{A}$.

We modify the definition of a $\mathbf{B}$-residual pair as follows.

Definition 9.2.1. *A pair* $\mathbf{W} = \langle W, W^\bullet \rangle$ *of subsets of* A *is called a* $\mathbf{B}$-residual pair *if the following conditions are satisfied:*

(i) W *contains* $B \cup \mathbb{T}_0^{\mathbf{A}}$ *and is closed under* $\wedge$ *and the operations in* $\mathbb{T}_1^{\mathbf{A}} \cup \mathbb{T}_2^{\mathbf{A}}$*;*

(ii) $W^\bullet$ *contains* $B \cup \mathbb{T}_0^{\mathbf{A}}$ *and is closed under* $\vee$ *and the operations in* $(\mathbb{T}_1^\bullet)^{\mathbf{A}}$ *and* $a \backslash_k x$ *and* $x /_k a$ *for all* $a \in W$ *and* $\backslash_k, /_k \in (\mathbb{T}_2^\bullet)^{\mathbf{A}}$.

Observe that $\langle W, \leq \rangle$ (i.e., $\langle W, \wedge \rangle$) is a meet-semilattice and $\langle W^\bullet, \leq \rangle$ (i.e., $\langle W^\bullet, \vee \rangle$) is a join-semilattice.

For the remainder of this section let $\mathbf{W} = \langle W, W^\bullet \rangle$ be a fixed $\mathbf{B}$-residual pair.

Let $\mathcal{F}(W)$ denote the set of all filters of W and $\mathcal{I}(W^\bullet)$ set of all ideals of $W^\bullet$ (see Definitions 2.7.2 and 2.7.3). Let $R \subseteq \mathcal{F}(W) \times \mathcal{I}(W^\bullet)$ be the binary relation defined by: $(F, I) \in R$ if, and only if, there exists $a \in F$ and there exists $b \in I$ such that $a \leq^{\mathbf{A}} b$. Then the polarities of R yield a Galois connection, ${}^\rhd : \mathcal{P}(\mathcal{F}(W)) \leftrightarrows \mathcal{P}(\mathcal{I}(W^\bullet)) : {}^\lhd$ where, for $X \in \mathcal{P}(\mathcal{F}(W))$ and $\Lambda \in \mathcal{P}(\mathcal{I}(W^\bullet))$

$$X^\rhd = \{I \in \mathcal{I}(W^\bullet) : F \in X \text{ implies } (F, I) \in R\},$$

$$\Lambda^\lhd = \{F \in \mathcal{F}(W) : I \in \Lambda \text{ implies } (F, I) \in R\}.$$

Then $\Lambda \in \mathcal{P}(\mathcal{I}(W^\bullet))$ is *Galois closed* if $\Lambda^{\lhd\rhd} = \Lambda$ and $X \in \mathcal{P}(\mathcal{F}(W))$ is *Galois closed* if $X^{\rhd\lhd} = X$. Let $\mathcal{S} = \{\Lambda \in \mathcal{P}(\mathcal{I}(W^\bullet)) : \Lambda = \Lambda^{\lhd\rhd}\}$.

Lemma 9.2.2. *If* Λ_i, $i \in \Psi$, *are Galois closed, then* $\bigcap_{i \in \Psi} \Lambda_\Psi$ *is Galois closed.*

Proof. Using the properties of Galois connections described in Lemmas 2.6.2, we have the following: for $\Lambda_i \in \mathcal{S}$, $i \in \Psi$,

$$\left(\bigcap_{i\in\Psi} \Lambda_i\right)^{\lhd\rhd} = \left(\bigcap_{i\in\Psi} \Lambda_i^{\lhd\rhd}\right)^{\lhd\rhd} = \left(\bigcup_{i\in\Psi} \Lambda_i^{\lhd}\right)^{\rhd\lhd\rhd} = \left(\bigcup_{i\in\Psi} \Lambda_i^{\lhd}\right)^{\rhd} = \bigcap_{i\in\Psi} \Lambda_i^{\lhd\rhd} = \bigcap_{i\in\Psi} \Lambda_i.$$

□

For $\Lambda_i \in \mathcal{S}$, $i \in \Psi$, let

$$\bigvee_{i\in\Psi}^{\mathbf{S}} \Lambda_i = \bigcap_{i\in\Psi} \Lambda_i \quad \text{and} \quad \bigwedge_{i\in\Psi}^{\mathbf{S}} \Lambda_i = \left(\bigcup_{i\in\Psi} \Lambda_i\right)^{\triangleleft\triangleright}.$$

Then $\mathbf{S} = \langle \mathcal{S}, \vee^{\mathbf{S}}, \wedge^{\mathbf{S}} \rangle$ is a complete lattice such that the associated complete lattice order $\leq^{\mathbf{S}}$ is $\supseteq$. Let $\mu : B \to \mathcal{S}$ be the map defined by $\mu(b) = \{I \in \mathcal{I}(W^\bullet) : b \in I\}$.

Remark 9.2.3. *Recall from Lemma 2.6.3 that $^\triangleright$ and $^\triangleleft$ convert existing joins into meets. That is, for $\Lambda_i \in \mathcal{S}$, $i \in \Psi$*

$$\left(\bigcap_{i\in\Psi} \Lambda_i\right)^{\triangleleft} = \left(\bigvee_{i\in\Psi}^{\mathbf{S}} \Lambda_i\right)^{\triangleleft} = \left(\bigwedge_{i\in\Psi}^{\mathbf{S}} \Lambda_i^{\triangleleft}\right) = \left(\bigcup_{i\in\Psi} \Lambda_i^{\triangleleft}\right)^{\triangleleft\triangleright}.$$

If meet is intersection and join is the Galois closure of the union on the set of Galois closed elements in $\mathcal{P}(\mathcal{I}(W^\bullet))$, then for $X_j \in \mathcal{P}(\mathcal{I}(W^\bullet))$, $j \in \Phi$,

$$\left(\bigcup_{j\in\Phi} X_j\right)^{\triangleright} = \left(\bigcup_{j\in\Phi} X_j\right)^{\triangleright\triangleleft\triangleright} = \left(\bigvee_{j\in\Phi} X_j\right)^{\triangleright} = \bigwedge_{j\in\Phi} X_j^{\triangleright} = \bigcap_{j\in\Phi} X_j^{\triangleright}.$$

We observe that $\{F\}^{\triangleright} = \{I \in \mathcal{I}(W^\bullet) : (F, I) \in R\} \in \mathcal{S}$ for any $F \in \mathcal{F}(W)$. Let $\mathcal{S}(W) = \{\{F\}^{\triangleright} : F \in \mathcal{F}(W)\}$.

Lemma 9.2.4. *If $\Lambda \in \mathcal{S}$, then Λ is an intersection of elements of $\mathcal{S}(W)$.*

Proof. Let $\Lambda \in \mathcal{S}$, i.e., $\Lambda = \Lambda^{\triangleleft\triangleright}$. Then,

$$\begin{aligned}
\Lambda^{\triangleleft\triangleright} &= \left\{I \in \mathcal{I}(W^\bullet) : F \in \Lambda^{\triangleleft} \text{ implies } (F, I) \in R\right\} \\
&= \bigcap \left\{\{I \in \mathcal{I}(W^\bullet) : (F, I) \in R\} : F \in \Lambda^{\triangleleft}\right\} \\
&= \bigcap \{\{F\}^{\triangleright} : F \in \Lambda^{\triangleleft}\}.
\end{aligned}$$

□

Let $b \in B$. Now let $\Lambda_b = \{I \in \mathcal{I}(W^\bullet) : b \in I\}$ and $X_b = \{F \in \mathcal{F}(W) : b \in F\}$. Furthermore, let $\langle T]^{W^\bullet}$ denote the ideal generated by $T \subseteq W^\bullet$ in $W^\bullet$ and $(b]^{W^\bullet}$ be the principal ideal generated by b in $W^\bullet$. Dually, we denote the filter generated by $T' \subseteq W$ in W by $[T'\rangle^W$ and the principal filter generated by c in W by $[c)^W$.

Lemma 9.2.5. *If $b \in B$, then*

(i) $\Lambda_b^{\triangleleft} = X_b$, *and*

(ii) $X_b^{\triangleright} = \Lambda_b$.

Proof. We only prove the first statement. The proof of the second follows dually.

Let $F \in \Lambda_b^{\triangleleft} = \{F \in \mathcal{F}(W) : I \in \Lambda_b \text{ implies } (F, I) \in R\}$. In particular $(b]^{W^\bullet} \in \Lambda_b$ and therefore $(F, (b]^{W^\bullet}) \in R$. Then there exist $a \in F$ and $c \in (b]^{W^\bullet}$ such that $a \leq c$. But $c \in (b]^{W^\bullet}$ implies $c \leq b$. Hence, $a \leq b$ and $b \in F$ since $b \in B \subseteq W \cap W^\bullet$. Therefore, $F \in X_b$ and $\Lambda_b^{\triangleleft} \subseteq X_b$. On the other hand, let $F \in X_b$. Since $b \in F$, $b \in I$ and $b \leq b$ for each $I \in \Lambda_b$, we have that $(F, I) \in R$ for each $I \in \Lambda_b$. Thus, $F \in \Lambda_b^{\triangleleft}$ and $X_b \subseteq \Lambda_b^{\triangleleft}$. □

Corollary 9.2.6. *Let $b \in B$. Then $\mu(b) \in \mathcal{S}$.*

Proof. By Lemmas 9.2.5 and 2.6.2 we have $\mu(b) = \Lambda_b = X_b^{\triangleright} \in \mathcal{S}$. □

Lemma 9.2.7. *The map μ preserves the ordering in* $\mathbf{B}$.

Proof. Let $a, b \in B$ be such that $a \leq b$ and let $I \in \mu(b)$. Then $b \in I$. Since $a \leq b$ and I is a down-set, we have $a \in I$. Thus, $I \in \mu(a)$. Hence, $\mu(b) \subseteq \mu(a)$, i.e., $\mu(a) \leq^{\mathbf{S}} \mu(b)$. □

Lemma 9.2.8. *The map μ is one-to-one and preserves the existing finite meets and existing finite joins in* $\mathbf{B}$.

Proof. The map μ is one-to-one since the principal ideal, $(b]^{W^\bullet} \in \Lambda_b = \mu(b)$ for all $b \in B$: If $a \neq b$, then at least one of $a \not\leq b$ or $b \not\leq a$. Suppose $a \not\leq b$, then $a \notin (b]^{W^\bullet}$. Therefore, $(b]^{W^\bullet} \in \mu(b)$ but $(b]^{W^\bullet} \notin \mu(a)$ and $\mu(b) \neq \mu(a)$. Similarly, $\mu(b) \neq \mu(a)$ if $b \not\leq a$.

Let $b_i \in B$ for $i = 1, \ldots, n$, and suppose $\bigvee_{i=1}^n b_i$ exists in B. Then $\bigvee_{i=1}^n b_i \in W^\bullet$ since $B \subseteq W^\bullet$ and $W^\bullet$ is closed under $\vee$. Furthermore,

$$\begin{aligned}
\bigvee_{i=1}^n {}^{\mathbf{S}}\mu(b_i) &= \bigcap_{i=1}^n \Lambda_{b_i} = \bigcap_{i=1}^n \{I \in \mathcal{I}(W^\bullet) : b_i \in I\} \\
&= \{I \in \mathcal{I}(W^\bullet) : b_i \in I \text{ for all } i = 1, \ldots, n\} \\
&= \left\{ I \in \mathcal{I}(W^\bullet) : \bigvee_{i=1}^n b_i \in I \right\} \\
&= \Lambda_{(\bigvee_{i=1}^n b_i)} = \mu\left(\bigvee_{i=1}^n b_i \right).
\end{aligned}$$

Next suppose $\bigwedge_{i\in 1}^{n} b_i$ exists in B; then $\bigwedge_{i=1}^{n} b_i \in W$ since $B \subseteq W$ and W is closed under $\wedge$. By Lemma 9.2.5 and Remark 9.2.3 and the properties of filters, we have that:

$$\begin{aligned}
\bigwedge_{i=1}^{n}{}^{\mathbf{S}}\mu(b_i) &= \left(\bigcup_{i=1}^{n}\mu(b_i)\right)^{\triangleleft\triangleright} = \left(\bigcap_{i=1}^{n}\Lambda_{b_i}^{\triangleleft}\right)^{\triangleright} = \left(\bigcap_{i=1}^{n}X_{b_i}\right)^{\triangleright} \\
&= \left(\bigcap_{i=1}^{n}\{F\in\mathcal{F}(W): b_i\in F\}\right)^{\triangleright} \\
&= \{F\in\mathcal{F}(W): b_i\in F \text{ for all } i=1,\dots,n\}^{\triangleright} \\
&= \left\{F\in\mathcal{F}(W): \bigwedge_{i=1}^{n} b_i\in F\right\}^{\triangleright} \\
&= X^{\triangleright}_{(\bigwedge_{i=1}^{n} b_i)} = \Lambda_{(\bigwedge_{i=1}^{n} b_i)} = \mu\left(\bigwedge_{i=1}^{n} b_i\right).
\end{aligned}$$

□

Definition 9.2.9. *For each $f \in \mathbb{T}_1^{\mathbf{A}}$ and $\circ \in \mathbb{T}_2^{\mathbf{A}}$ we define the operations $f^{\mathbf{S}} : \mathcal{S} \to \mathcal{S}$ and $\circ^{\mathbf{S}} : \mathcal{S}\times\mathcal{S} \to \mathcal{S}$ as follows. For $F, G \in \mathcal{F}(W)$ define*

$$\hat{f}(F) = [\{f(a) : a \in F\}\rangle^{W} \quad \text{and} \quad F\hat{\circ}\,G = [\{a\circ b : a\in F, b\in G\}\rangle^{W}.$$

Next, for $X, Y \in \mathcal{P}(\mathcal{F}(W))$, define

$$f(X) = \{\hat{f}(F) : F\in X\} \quad \text{and} \quad X\circ Y = \{F\hat{\circ}\,G : F\in X,\, G\in Y\}.$$

Then, for $\Lambda, \Upsilon \in S$, we define

$$f^{\mathbf{S}}(\Lambda) = f(\Lambda^{\triangleleft})^{\triangleright} \quad \text{and} \quad \Lambda\circ^{\mathbf{S}}\Upsilon = (\Lambda^{\triangleleft}\circ\Upsilon^{\triangleleft})^{\triangleright}.$$

Let $f \in \mathbb{T}_1^{\mathbf{A}}$ such that $g \in (\mathbb{T}_1^{\bullet})^{\mathbf{A}}$ is its residual and let $\circ \in \mathbb{T}_2^{\mathbf{A}}$ such that $\backslash, / \in (\mathbb{T}_2^{\bullet})^{\mathbf{A}}$ are the left and right residuals of $\circ$, respectively. For $I \in \mathcal{I}(W^{\bullet})$ and $F \in \mathcal{F}(W)$, define

$$\hat{g}(I) = \langle\{g(a) : a\in I\}]^{W^{\bullet}},$$

$$F\hat{\backslash}I = \langle\{a\backslash b : a\in F, b\in I\}]^{W^{\bullet}} \text{ and}$$

$$I\hat{/}F = \langle\{a/b : a\in I, b\in F\}]^{W^{\bullet}}.$$

Furthermore, for $\Lambda \in \mathcal{S}$ and $X \in \mathcal{P}(\mathcal{F}(W))$ define

$$g(\Lambda) = \{\hat{g}(I) : I \in \Lambda\},$$

$$X\backslash\Lambda = \{F\hat{\backslash}I : F \in X, I \in \Lambda\} \text{ and}$$

$$\Lambda/X = \{I\hat{/}F : I \in \Lambda, F \in X\}.$$

Recall from Chapter 2.7 that since W is a meet-semilattice, $a \in [T\rangle^W$ for some $T \subseteq W$ if, and only if, there exist elements $b_1, \ldots, b_n \in T$ such that $a \geq \bigwedge_{i=1}^n b_i$. Similarly, since $W^\bullet$ is a join-semilattice, $a \in \langle T']^{W^\bullet}$ for some $T' \subseteq W^\bullet$ if, and only if, there exist $b_1, \ldots, b_m \in T'$ such that $a \leq \bigvee_{i=1}^n b_i$.

Lemma 9.2.10. *Let $F, G \in \mathcal{F}(W)$ and $I \in \mathcal{I}(W^\bullet)$. Then*

(i) $(\hat{f}(F), I) \in R$ *if, and only if, there exist* $a \in F$ *and* $b \in I$ *such that* $f(a) \leq b$.

(ii) $(F, \hat{g}(I)) \in R$ *if, and only if, there exist* $a \in F$ *and* $b \in I$ *such that* $a \leq g(b)$.

(iii) $(F\hat{\circ}G, I) \in R$ *if, and only if, there exist* $a \in F$, $b \in G$ *and* $c \in I$ *such that* $a \circ b \leq c$.

(iv) $(G, F\hat{\backslash}I) \in R$ *if, and only if, there exist* $a \in F$, $b \in G$ *and* $c \in I$ *such that* $b \leq a\backslash c$.

(v) $(F, I\hat{/}G) \in R$ *if, and only if, there exist* $a \in F$, $b \in G$ *and* $c \in I$ *such that* $a \leq c/b$.

Proof. We prove the third and the fourth statements. The other statements can be proved similarly.

(iii) If there exist $a \in F$, $b \in G$ and $c \in I$ such that $a \circ b \leq c$, then $(F\hat{\circ}G, I) \in R$ since $a \circ b \in F \circ G$. Next suppose $(F\hat{\circ}G, I) \in R$. Then there exist $a' \in F\hat{\circ}G$ and $c \in I$ such that $a' \leq c$. Since $a' \in F\hat{\circ}G$ there exist $a_1, \ldots, a_n \in F$ and $b_1, \ldots, b_n \in G$, for some $n \in \mathbb{N}$, such that $\bigwedge_{i=1}^n (a_i \circ b_i) \leq a'$. Then $a = \bigwedge_{i=1}^n a_i \in F$ and $b = \bigwedge_{i=1}^n b_i \in G$ since W is closed under finite meets. Furthermore, $a \circ b \leq \bigwedge_{i=1}^n (a_i \circ b_i) \leq a' \leq c$.

(iv) If there exist $a \in F$, $b \in G$ and $c \in I$ such that $b \leq a\backslash c$, then $(G, F\hat{\backslash}I) \in R$ since $a\backslash c \in F\hat{\backslash}I$. Now suppose $(G, F\hat{\backslash}I) \in R$. Then there exist $b \in G$ and

$c' \in F\hat{\backslash}I$ such that $b \leq c'$. But $c' \in F\hat{\backslash}I$ implies there exist $a_1, \ldots, a_n \in F$ and $c_1, \ldots, c_n \in I$, for some $n \in \mathbb{N}$, such that $c' \leq \bigvee_{i=1}^{n}(a_i \backslash c_i)$. Then $a = \bigwedge_{i=1}^{n} a_i \in F$ since W is closed under finite meets and $c = \bigvee_{i=1}^{n} c_i \in I$ since $W^\bullet$ is closed under finite joins. Moreover, $b \leq c' \leq \bigvee_{i=1}^{n}(a_i \backslash c_i) \leq a \backslash c$.

□

Lemma 9.2.11. *Let $f \in \mathbb{T}_1^{\mathbf{A}}$, $\circ \in \mathbb{T}_2^{\mathbf{A}}$ and $\Lambda, \Upsilon \in \mathcal{S}$. If $X, Y \in \mathcal{P}(\mathcal{F}(W))$ such that $\Lambda = X^{\triangleright}$ and $\Upsilon = Y^{\triangleright}$, then*

(i) $f^{\mathbf{S}}(\Lambda) = f^{\mathbf{S}}(X^{\triangleright}) = (f(X))^{\triangleright}$, *and*

(ii) $(\Lambda \circ^{\mathbf{S}} \Upsilon) = X^{\triangleright} \circ^{\mathbf{S}} Y^{\triangleright} = (X \circ Y)^{\triangleright}$.

Proof. We prove the second statement. The proof of the first follows a similar argument and relies on Lemma 9.2.10 parts (i) and (ii).

The inclusion from left to right, $(X^{\triangleright\triangleleft} \circ Y^{\triangleright\triangleleft})^{\triangleright} \subseteq (X \circ Y)^{\triangleright}$, is immediate from the properties of Galois connections.

For the inclusion in the other direction observe that, for $I \in \mathcal{I}(W^\bullet)$

$$
\begin{aligned}
& I \in (X \circ Y)^{\triangleright} \\
\iff & (F' \hat{\circ} G', I) \in R \text{ for all } F' \in X \text{ and all } G' \in Y \\
\iff & \text{there exist } a \in F', b \in G' \text{ and } c \in I \text{ such that} && \text{by Lemma 9.2.10 (iii)} \\
& a \circ b \leq c \text{ for all } F' \in X \text{ and all } G' \in Y \\
\iff & \text{there exist } a \in F', b \in G' \text{ and } c \in I \text{ such that} && \text{by residuation} \\
& b \leq a \backslash c \text{ for all } F' \in X \text{ and all } G' \in Y \\
\iff & (G', F'\hat{\backslash}I) \in R \text{ for all } F' \in X \text{ and all } G' \in Y && \text{by Lemma 9.2.10 (iv)} \\
\iff & F'\hat{\backslash}I \in Y^{\triangleright} \text{ for all } F' \in X \\
\iff & (G, F'\hat{\backslash}I) \in R \text{ for all } F' \in X \text{ and all } G \in Y^{\triangleright\triangleleft} \\
\iff & \text{there exist } a \in F', b \in G \text{ and } c \in I \text{ such that} && \text{by Lemma 9.2.10 (iv)} \\
& b \leq a \backslash c \text{ for all } F' \in X \text{ and all } G \in Y^{\triangleright\triangleleft} \\
\iff & \text{there exist } a \in F', b \in G \text{ and } c \in I \text{ such that} && \text{by residuation} \\
& a \leq c / b \text{ for all } F' \in X \text{ and all } G \in Y^{\triangleright\triangleleft} \\
\iff & (F', I\hat{/}G) \in R \text{ for all } F' \in X \text{ and all } G \in Y^{\triangleright\triangleleft} && \text{by Lemma 9.2.10 (v)} \\
\iff & I\hat{/}G \in X^{\triangleright} \text{ for all } G \in Y^{\triangleright\triangleleft}
\end{aligned}
$$

$\iff (F, I\hat{/}G) \in R$ for all $F \in X^{\triangleright\triangleleft}$ and all $G \in Y^{\triangleright\triangleleft}$

$\iff$ there exist $a \in F, b \in G$ and $c \in I$ such that by Lemma 9.2.10 (v)
$a \leq c/b$ for all $F \in X^{\triangleright\triangleleft}$ and all $G \in Y^{\triangleright\triangleleft}$

$\iff$ there exist $a \in F, b \in G$ and $c \in I$ such that by residuation
$a \circ b \leq c$ for all $F \in X^{\triangleright\triangleleft}$ and all $G \in Y^{\triangleright\triangleleft}$

$\iff (F\hat{\circ}G, I) \in R$ for all $F \in X^{\triangleright\triangleleft}$ and all $G \in Y^{\triangleright\triangleleft}$ by Lemma 9.2.10 (iii)

$\iff I \in (X^{\triangleright\triangleleft} \circ Y^{\triangleright\triangleleft})^{\triangleright}$.

□

Lemma 9.2.12. *For each $f \in \mathbb{T}_1^{\mathbf{A}}$ and $\circ \in \mathbb{T}_2^{\mathbf{A}}$, the operations $f^{\mathbf{S}}$ and $\circ^{\mathbf{S}}$ on $\mathcal{S}$ are residuated with respect to the order $\supseteq$.*

Proof. Since $\mathbf{S}$ is a complete lattice it suffices to show that $f^{\mathbf{S}}$ and $\circ^{\mathbf{S}}$ distribute over all joins. We prove the claim for $\circ^{\mathbf{S}}$. The claim for $f^{\mathbf{S}}$ can be shown similarly. Let $\Upsilon, \Lambda_i \in \mathcal{S}$, $i \in \Psi$. To show that $\circ^{\mathbf{S}}$ distribute over all joins, we must show that

$$\Upsilon \circ^{\mathbf{S}} \bigvee_{i\in\Psi}^{\mathbf{S}} \Lambda_i = \bigvee_{i\in\Psi}^{\mathbf{S}} (\Upsilon \circ^{\mathbf{S}} \Lambda_i) \text{ and } \bigvee_{i\in\Psi}^{\mathbf{S}} \Lambda_i \circ^{\mathbf{S}} \Upsilon = \bigvee_{i\in\Psi}^{\mathbf{S}} (\Lambda_i \circ^{\mathbf{S}} \Upsilon).$$

Let us consider the first condition. By Remark 9.2.3 and Lemma 9.2.11 (ii),

$$\begin{aligned}
\Upsilon \circ^{\mathbf{S}} \bigvee_{i\in\Psi}^{\mathbf{S}} \Lambda_i &= \Upsilon \circ^{\mathbf{S}} \bigcap_{i\in\Psi} \Lambda_i \\
&= \Upsilon^{\triangleleft\triangleright} \circ^{\mathbf{S}} \bigcap_{i\in\Psi} (\Lambda_i^{\triangleleft\triangleright}) \\
&= \Upsilon^{\triangleleft\triangleright} \circ^{\mathbf{S}} \left(\bigcup_{i\in\Psi} \Lambda_i^{\triangleleft} \right)^{\triangleright} \\
&= \left(\Upsilon^{\triangleleft} \circ \bigcup_{i\in\Psi} \Lambda_i^{\triangleleft} \right)^{\triangleright}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\bigvee_{i\in\Psi}^{\mathbf{S}} (\Upsilon \circ^{\mathbf{S}} \Lambda_i) &= \bigcap_{i\in\Psi} (\Upsilon \circ^{\mathbf{S}} \Lambda_i) \\
&= \bigcap_{i\in\Psi} (\Upsilon^{\triangleleft} \circ \Lambda_i^{\triangleleft})^{\triangleright} \\
&= \left(\bigcup_{i\in\Psi} (\Upsilon^{\triangleleft} \circ \Lambda_i^{\triangleleft}) \right)^{\triangleright} \\
&= \left(\bigcup_{i\in\Psi} \{F \hat{\circ} G : F \in \Upsilon^{\triangleleft}, G \in \Lambda_i^{\triangleleft}\} \right)^{\triangleright} \\
&= \left\{ F \hat{\circ} G : F \in \Upsilon^{\triangleleft}, G \in \bigcup_{i\in\Psi} \Lambda_i^{\triangleleft} \right\}^{\triangleright} \\
&= \left(\Upsilon^{\triangleleft} \circ \bigcup_{i\in\Psi} \Lambda_i^{\triangleleft} \right)^{\triangleright}.
\end{aligned}$$

Similarly, the second condition holds. Hence, $\circ^{\mathbf{S}}$ is residuated. □

We will use the following auxiliary result to describe the residuals of $f^{\mathbf{S}}$ and $\circ^{\mathbf{S}}$.

Lemma 9.2.13. *Let $X \in \mathcal{P}(\mathcal{F}(W))$ and $\Lambda \in \mathcal{P}(\mathcal{I}(W^{\bullet}))$. Then,*

$$X^{\triangleright} \supseteq \Lambda^{\triangleleft\triangleright} \iff X \subseteq \Lambda^{\triangleleft}$$

Proof. If $X^{\triangleright} \supseteq \Lambda^{\triangleleft\triangleright}$, then by the properties of Galois connections $X \subseteq X^{\triangleright\triangleleft} \subseteq \Lambda^{\triangleleft\triangleright\triangleleft} = \Lambda^{\triangleleft}$. The inclusion in the other direction follows immediately from the properties of Galois connections. □

Lemma 9.2.14. *For each $f \in \mathbb{T}_1^{\mathbf{A}}$ with residual $g \in (\mathbb{T}_1^{\bullet})^{\mathbf{A}}$, define $g^{\mathbf{S}} : \mathcal{S} \to \mathcal{S}$ by, for $\Lambda \in \mathcal{S}$,*

$$g^{\mathbf{S}}(\Lambda) = g(\Lambda)^{\triangleleft\triangleright}.$$

Then $g^{\mathbf{S}}$ is the residual of $f^{\mathbf{S}}$.

Proof. For all $\Lambda, \Upsilon \in \mathcal{S}$ we must show that $f^{\mathbf{S}}(\Lambda) \leq^{\mathbf{S}} \Upsilon$ if, and only if, $\Lambda \leq^{\mathbf{S}} g^{\mathbf{S}}(\Upsilon)$. First note that by Lemma 9.2.13,

$$\begin{aligned}
f^{\mathbf{S}}(\Lambda) \leq^{\mathbf{S}} \Upsilon &\iff f(\Lambda^{\triangleleft})^{\triangleright} \supseteq \Upsilon^{\triangleleft\triangleright} \\
&\iff f(\Lambda^{\triangleleft}) \subseteq \Upsilon^{\triangleleft}.
\end{aligned}$$

Also, by Lemma 9.2.13,

$$\begin{aligned}\Lambda \leq^{\mathbf{S}} g^{\mathbf{S}}(\Upsilon) &\iff \Lambda^{\triangleleft\triangleright} \supseteq g(\Upsilon)^{\triangleleft\triangleright}\\ &\iff \Lambda^{\triangleleft} \subseteq g(\Upsilon)^{\triangleleft}.\end{aligned}$$

Therefore, to prove the claim we must show that $f(\Lambda^{\triangleleft}) \subseteq \Upsilon^{\triangleleft}$ if, and only if, $\Lambda^{\triangleleft} \subseteq g(\Upsilon)^{\triangleleft}$. We prove the forward implication. The implication in the other direction follows similarly.

Suppose $f(\Lambda^{\triangleleft}) \subseteq \Upsilon^{\triangleleft}$. Then,

$F \in \Lambda^{\triangleleft}$

$\Rightarrow \hat{f}(F) \in f(\Lambda^{\triangleleft})$

$\Rightarrow \hat{f}(F) \in \Upsilon^{\triangleleft}$ by assumption

$\Rightarrow (\hat{f}(F), I) \in R$ for all $I \in \Upsilon$

$\Rightarrow$ there exist $a \in F, b \in I$ such that $f(a) \leq b$ for all $I \in \Upsilon$ by Lemma 9.2.10 (i)

$\Rightarrow$ there exist $a \in F, b \in I$ such that $a \leq g(b)$ for all $I \in \Upsilon$ by residuation

$\Rightarrow (F, \hat{g}(I)) \in R$ for all $I \in \Upsilon$ by Lemma 9.2.10 (ii)

$\Rightarrow F \in g(\Upsilon)^{\triangleleft}$.

Hence, $\Lambda^{\triangleleft} \subseteq g(\Upsilon)^{\triangleleft}$. □

Lemma 9.2.15. *For each $\circ \in \mathbb{T}_2^{\mathbf{A}}$ with left and right residuals $\backslash, / \in (\mathbb{T}_2^{\bullet})^{\mathbf{A}}$, define $\backslash^{\mathbf{S}}, /^{\mathbf{S}} : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ by, for $\Lambda, \Upsilon \in \mathcal{S}$,*

$$\Lambda \backslash^{\mathbf{S}} \Upsilon = (\Lambda^{\triangleleft} \backslash \Upsilon)^{\triangleleft\triangleright} \quad \text{and} \quad \Lambda /^{\mathbf{S}} \Upsilon = (\Lambda / \Upsilon^{\triangleleft})^{\triangleleft\triangleright}.$$

Then $\backslash^{\mathbf{S}}$ and $/^{\mathbf{S}}$ are the left and right residuals, respectively, of $\circ^{\mathbf{S}}$.

Proof. Let $\Lambda, \Upsilon, \Gamma \in \mathcal{S}$. We must show that $\Lambda \circ^{\mathbf{S}} \Upsilon \leq^{\mathbf{S}} \Gamma$ if, and only if, $\Upsilon \leq^{\mathbf{S}} \Lambda \backslash^{\mathbf{S}} \Gamma$ if, and only if, $\Lambda \leq^{\mathbf{S}} \Gamma /^{\mathbf{S}} \Upsilon$. We show that $\Lambda \circ^{\mathbf{S}} \Upsilon \leq^{\mathbf{S}} \Gamma$ implies $\Upsilon \leq^{\mathbf{S}} \Lambda \backslash^{\mathbf{S}} \Gamma$. The other implications follow similarly.

Observe that, by Lemma 9.2.13,

$$\begin{aligned}\Lambda \circ^{\mathbf{S}} \Upsilon \leq^{\mathbf{S}} \Gamma &\iff (\Lambda^{\triangleleft} \circ \Upsilon^{\triangleleft})^{\triangleright} \supseteq \Gamma^{\triangleleft\triangleright}\\ &\iff \Lambda^{\triangleleft} \circ \Upsilon^{\triangleleft} \subseteq \Gamma^{\triangleleft}.\end{aligned}$$

Furthermore, again by Lemma 9.2.13,

$$\begin{aligned}\Upsilon \leq^{\mathbf{S}} \Lambda \backslash^{\mathbf{S}} \Gamma &\iff \Upsilon^{\triangleleft\triangleright} \supseteq (\Lambda^{\triangleleft} \backslash \Gamma)^{\triangleleft\triangleright}\\ &\iff \Upsilon^{\triangleleft} \subseteq (\Lambda^{\triangleleft} \backslash \Gamma)^{\triangleleft}.\end{aligned}$$

Thus we must show that $\Lambda^{\triangleleft} \circ \Upsilon^{\triangleleft} \subseteq \Gamma^{\triangleleft}$ implies $\Upsilon^{\triangleleft} \subseteq (\Lambda^{\triangleleft} \backslash \Gamma)^{\triangleleft}$.

Suppose $\Lambda^{\triangleleft} \circ \Upsilon^{\triangleleft} \subseteq \Gamma^{\triangleleft}$. Then,

$G \in \Upsilon^{\triangleleft}$

$\Rightarrow F \hat{\circ} G \in \Lambda^{\triangleleft} \circ \Upsilon^{\triangleleft}$ for all $F \in \Lambda^{\triangleleft}$

$\Rightarrow F \hat{\circ} G \in \Gamma^{\triangleleft}$ for all $F \in \Lambda^{\triangleleft}$ by assumption

$\Rightarrow (F \hat{\circ} G, I) \in R$ for all $F \in \Lambda^{\triangleleft}$ and all $I \in \Gamma$

$\Rightarrow$ there exist $a \in F, b \in G$ and $c \in I$ such that by Lemma 9.2.10 (iii)
$a \circ b \leq c$ for all $F \in \Lambda^{\triangleleft}$ and all $I \in \Gamma$

$\Rightarrow$ there exist $a \in F, b \in G$ and $c \in I$ such that by residuation
$b \leq a \backslash c$ for all $F \in \Lambda^{\triangleleft}$ and all $I \in \Gamma$

$\Rightarrow (G, F \hat{\backslash} I) \in R$ for all $F \in \Lambda^{\triangleleft}$ and all $I \in \Gamma$ by Lemma 9.2.10 (iv)

$\Rightarrow G \in (\Lambda^{\triangleleft} \backslash \Gamma)^{\triangleleft}$.

Hence, $\Upsilon^{\triangleleft} \subseteq (\Lambda^{\triangleleft} \backslash \Gamma)^{\triangleleft}$. □

Lemma 9.2.16. *Let $f \in \mathbb{T}_1^{\mathbf{A}}$ and $\circ \in \mathbb{T}_2^{\mathbf{A}}$.*

(i) If $a \in B$ such that $f(a) \in B$, then $\hat{f}([a]^W) = [f(a)]^W$.

(ii) If $a \in B$ such that $g(a) \in B$, then $\hat{g}((a]^{W^\bullet}) = (g(a)]^{W^\bullet}$.

(iii) If $a, b \in B$ such that $a \circ b \in B$, then $[a]^W \hat{\circ} [b]^W = [a \circ b)^W$.

(iv) If $a, b \in B$ such that $a \backslash b \in B$, then $[a]^W \hat{\backslash} (b]^{W^\bullet} = (a \backslash b]^{W^\bullet}$.

(v) If $a, b \in B$ such that $a / b \in B$, then $(a]^{W^\bullet} \hat{/} [b)^W = (a / b]^{W^\bullet}$.

Proof. We prove the third and the fourth statements. The other three statements can be proved similarly.

(iii) Suppose $a, b \in B$ such that $a \circ b \in B$. Let $c \in [a \circ b)^W$, then $a \circ b \leq c \in W$. But $a \circ b \in \{e \circ d : e \in [a)^W, d \in [b)^W\} = \{e \circ d : a \leq e \in W, b \leq d \in W\}$. Therefore, $c \in [\{e \circ d : a \leq e \in M, b \leq d \in M\}\rangle^W = [a)^W \hat{\circ} [b)^W$ by the upward closure of filters.

For the inclusion in the other direction, let $c \in [a)^W \hat{\circ} [b)^W = [\{e \circ d : \leq e \in W, b \leq d \in W\}\rangle^W$. Then $c \geq \bigwedge_{i=1}^n (e_i \circ d_i)$ for some $n \in \mathbb{N}$ and $a \leq e_i \in W$, $b \leq d_i \in W$ for $i = 1, \ldots, n$. Since $e_i \circ d_i \geq a \circ b$ for $i = 1, \ldots, n$, we have that $c \geq \bigwedge_{i=1}^n (e_i \circ d_i) \geq a \circ b$ and $c \in [a \circ b)^W$.

(iv) Suppose $a, b \in B$ such that $a\backslash b \in B$. Let $c \in (a\backslash b]^{W^\bullet}$. Then $c \leq a\backslash b \in \{e\backslash d : e \in [a)^W, d \in (b]^{W^\bullet}\} = \{e\backslash d : a \leq e \in M, b \geq d \in W^\bullet\}$. Thus, $c \in \langle\{e\backslash d : a \leq e \in W, b \geq d \in W^\bullet\}\rangle^{W^\bullet} = [a)^W \hat{\backslash} (b]^{W^\bullet}$ by the downward closure of ideals.

On the other hand, let $c \in [a)^W \hat{\backslash} (b]^{W^\bullet} = \langle\{e\backslash d : a \leq e \in W, b \geq d \in W^\bullet\}\rangle^{W^\bullet}$. Then $c \leq \bigvee_{i=1}^n (e_i\backslash d_i)$ for some $n \in \mathbb{N}$ and $a \leq e_i \in W$, $b \geq d_i \in W^\bullet$ for $i = 1, \ldots, n$. Since $e_i\backslash d_i \leq a\backslash b$ for $i = 1, \ldots n$ we have that $c \leq \bigvee_{i=1}^n (e_i\backslash d_i) \leq a\backslash b$ and $c \in (a\backslash b]^{W^\bullet}$.

□

Lemma 9.2.17. *The embedding $\mu : a \mapsto \{I \in \mathcal{I}(W^\bullet) : a \in I\}$ preserves each $f \in \mathbb{T}_1^{\mathbf{A}}$, each $g \in (\mathbb{T}_1^\bullet)^{\mathbf{A}}$, each $\circ \in \mathbb{T}_2^{\mathbf{A}}$ and each pair $\backslash, / \in (\mathbb{T}_2^\bullet)^{\mathbf{A}}$. That is:*

(i) If $a \in B$ such that $f(a) \in B$, then $f^{\mathbf{S}}(\mu(a)) = \mu(f(a))$.

(ii) If $a \in B$ such that $g(a) \in B$, then $g^{\mathbf{S}}(\mu(a)) = \mu(g(a))$.

(iii) If $a, b \in B$ such that $a \circ b \in B$, then $\mu(a) \circ^{\mathbf{S}} \mu(b) = \mu(a \circ b)$.

(iv) If $a, b \in B$ such that $a\backslash b \in b$, then $\mu(a)\backslash^{\mathbf{S}}\mu(b) = \mu(a\backslash b)$.

(v) If $a, b \in B$ such that $a/b \in b$, then $\mu(a)/^{\mathbf{S}}\mu(b) = \mu(a/b)$.

Proof. We only prove the third and fourth statements. The proofs of other statements follow similarly. Recall that $\mu(a) = \Lambda_a = \{I \in \mathcal{I}(W^\bullet) : a \in I\}$ and $X_a = \{F \in \mathcal{F}(W) : a \in F\}$.

(iii) Let $a, b \in B$ such that $a \circ b \in B$. By Lemmas 9.2.5 (i) and 9.2.11 (ii),

$$\begin{aligned}\mu(a) \circ^{\mathbf{S}} \mu(b) = \Lambda_a^{\triangleleft\triangleright} \circ^{\mathbf{S}} \Lambda_b^{\triangleleft\triangleright} = (\Lambda_a^{\triangleleft} \circ \Lambda_b^{\triangleleft})^{\triangleright} = (X_a \circ X_b)^{\triangleright}\\ = \{F \hat{\circ} G : F, G \in \mathcal{F}(W) \text{ such that } a \in F,\ b \in G\}^{\triangleright}.\end{aligned}$$

Then, $I \in \{F \hat{\circ} G : F, G \in \mathcal{F}(W) \text{ such that } a \in F,\ b \in G\}^{\triangleright}$ if, and only if, $([a)^W \hat{\circ} [b)^W, I) \in R$: The forward implication follows from the definition of $^{\triangleright}$ since $[a)^W \hat{\circ} [b)^W \in \{F \hat{\circ} G : F, G \in \mathcal{F}(W) \text{ such that } a \in F,\ b \in G\}$. For the implication in the other direction, suppose $([a)^W \hat{\circ} [b)^W, I) \in R$. Then, Lemma 9.2.10 (iii), there exist $c \in [a)^W$, $d \in [b)^W$ and $e \in I$ such that $a \circ b \leq c \circ d \leq e$. Now let $F, G \in \mathcal{F}(W)$ such that $a \in F$ and $b \in G$. Then $a \circ b \in F \hat{\circ} G$ and by Lemma 9.2.10 (iii) we have that $(F \hat{\circ} G, I) \in R$.

Since this is the case for all $F, G \in \mathcal{F}(W)$ such that $a \in F$ and $b \in G$, it follows that $I \in \{F \hat{\circ} G : F, G \in \mathcal{F}(W) \text{ such that } a \in F,\ b \in G\}^{\triangleright}$.

Furthermore,

$$\begin{aligned}
&([a)^W \hat{\circ} [b)^W, I) \in R \\
\iff &([a \circ b)^W, I) \in R \qquad \text{(by Lemma 9.2.16 (iii))} \\
\iff &\text{there exist } c \in [a \circ b)^W,\ d \in I \text{ such that } c \leq d \\
\iff &a \circ b \in I \qquad (a \circ b \leq c \leq d \text{ and } a \circ b \in B) \\
\iff &I \in \mu(a \circ b).
\end{aligned}$$

(iv) Let $a, b \in B$ such that $a \backslash b \in B$. Then,

$$\begin{aligned}
&\mu(a) \backslash^{\mathbf{S}} \mu(b) = (\Lambda_a^{\triangleleft} \backslash \Lambda_b)^{\triangleleft\triangleright} = (X_a \backslash \Lambda_b)^{\triangleleft\triangleright} \\
&= \{G \hat{\backslash} J : F \in \mathcal{F}(W), J \in \mathcal{I}(W^\bullet) \text{ such that } a \in F,\ b \in I\}^{\triangleleft\triangleright}.
\end{aligned}$$

Then $F \in \{G \hat{\backslash} J : G \in \mathcal{F}(W), J \in \mathcal{I}(W^\bullet) \text{ such that } a \in G,\ b \in I\}^{\triangleleft}$ if, and only if, $(F, [a)^W \hat{\backslash} (b]^{W^\bullet}) \in R$: Firstly $[a)^W \hat{\backslash} (b]^{W^\bullet} \in \{G \hat{\backslash} J : G \in \mathcal{F}(W), J \in \mathcal{I}(W^\bullet) \text{ such that } a \in G,\ b \in I\}$ since $a \in [a)$ and $b \in (b]$. Therefore, $(F, [a)^W \hat{\backslash} (b]^{W^\bullet}) \in R$ for all $F \in \{G \hat{\backslash} J : G \in \mathcal{F}(W), J \in \mathcal{I}(W^\bullet) \text{ such that } a \in G,\ b \in I\}^{\triangleleft}$. Next, suppose $(F, [a)^W \hat{\backslash} (b]^{W^\bullet}) \in R$. Then, by Lemma 9.2.10 (iv), there exist $c \in F$, $d \in [a)^W$ and $e \in (b]^{W^\bullet}$ such that $c \leq d \backslash e \leq a \backslash b$. Now let $G \in \mathcal{F}(W)$ and $J \in \mathcal{I}(W^\bullet)$ such that $a \in G$ and $b \in I$. Then, again by Lemma 9.2.10 (iv), $(F, G \hat{\backslash} J) \in R$. Hence, $F \in \{G \hat{\backslash} J : G \in \mathcal{F}(W), J \in \mathcal{I}(W^\bullet) \text{ such that } a \in G,\ b \in I\}^{\triangleleft}$.

Moreover,

$$\begin{aligned}
&(F, [a)^W \backslash (b]^{W^\bullet}) \in R \\
\iff &(F, (a \backslash b]^{W^\bullet}) \in R \qquad \text{(by Lemma 9.2.16 (iv))} \\
\iff &\text{there exist } c \in F,\ d \in (a \backslash b]^{W^\bullet} \text{ such that } c \leq d \\
\iff &a \backslash b \in F. \qquad (c \leq d \leq a \backslash b \text{ and } a \backslash b \in B)
\end{aligned}$$

Hence,

$$\begin{aligned}
&\{G \backslash J : G \in \mathcal{F}(W), J \in \mathcal{I}(W^\bullet) \text{ such that } a \in G,\ b \in I\}^{\triangleleft} \\
&= \{F \in \mathcal{F}(W) : a \backslash b \in F\}.
\end{aligned}$$

We can now show that $I \in \{F \in \mathcal{F}(W) : a\backslash b \in F\}^{\triangleright}$ if, and only if, $([a\backslash b)^W, I) \in R$: Suppose $I \in \{F \in \mathcal{F}(W) : a\backslash b \in F\}^{\triangleright}$. Since $[a\backslash b)^W \in \{F \in \mathcal{F}(W) : a\backslash b \in F\}$ it follows directly that $([a\backslash b)^W, I) \in R$. On the other hand, suppose $([a\backslash b)^W, I) \in R$. Then there exist $c \in [a\backslash b)^W$ and $d \in I$ such that $a\backslash b \leq c \leq d$. Let $G \in \{F \in \mathcal{F}(W) : a\backslash b \in F\}$; then $(G, I) \in R$ since $a\backslash b \in G$. Therefore, $I \in \{F \in \mathcal{F}(W) : a\backslash b \in F\}^{\triangleright}$.

Finally,

$$\begin{aligned} & ([a\backslash b)^W, I) \in R \\ \iff & \text{there exist } c \in [a\backslash b)^W,\ d \in I \text{ such that } c \leq d \\ \iff & a\backslash b \in I \qquad (a\backslash b \leq c \leq d \text{ and } a\backslash b \in B) \\ \iff & I \in \mu(a\backslash b). \end{aligned}$$

Thus, $\mu(a)\backslash^{\mathbf{S}}\mu(b) = \mu(a\backslash b)$.

□

Theorem 9.2.18. *The structure* $\mathbf{S}$ *is a complete residuated ordered algebra of the same type as* $\mathbf{A}$ *and there exists an embedding of* $\mathbf{B}$ *into* $\mathbf{S}$ *that preserves all existing meets and joins in* $\mathbf{B}$.

Remark 9.2.19. *Let* $\mathbf{L} = \langle L, \vee, \wedge \rangle$ *be a bounded lattice and let* $\mathbf{C}$ *be the completion of* $\mathbf{L}$ *obtained from the polarization* $(\mathcal{F}(\mathbf{L}), \mathcal{I}(\mathbf{L}))$ *as described in Chapter 6. Then* $\mathbf{C}$ *is the canonical extension of* $\mathbf{L}$ *[GH01]. Now, if* $B = A$, *then the only possible* $\mathbf{B}$*-residual pair is* $\langle A, A \rangle$. *Then* $(F, I) \in R$ *if, and only if,* $F \cap I \neq \varnothing$ *and the lattice reduct of* $\mathbf{S}$ *is just the canonical extension of the lattice reduct of* $\mathbf{A}$.

9.2.2 Finiteness

Recall that $\mathbf{A} = \langle A, \vee, \wedge, \mathbb{T}^{\mathbf{A}}, \leq \rangle$ is a residuated lattice ordered algebra (of type $\mathbb{T}$) and $\mathbf{B} = \langle B, \vee^{\mathbf{B}}, \wedge^{\mathbf{B}}, \mathbb{T}^{\mathbf{B}}, \leq^{\mathbf{B}} \rangle$ is a partial (ordered) subalgebra of $\mathbf{A}$. Also recall that $\mathbf{W} = \langle W, W^{\bullet} \rangle$ is a $\mathbf{B}$-residual pair, as per Definition 9.2.1.

Let $\mathcal{P}^{fin}(W)$ denote the set of all finite subsets of W. For $M, N \in \mathcal{P}^{fin}(W)$ define the ordering $\leqq$ on $\mathcal{P}^{fin}(W)$ by: $M \leqq N$ if, and only if, there exists a one-to-one function $\psi : M \to N$ such that $a \leq \psi(a)$ for every $a \in M$.

We will make use of the following result that was obtained in [Nas63].

Lemma 9.2.20. *[Nas63] If $\langle W, \geq \rangle$ is well-quasi-ordered, then so is $\langle \mathcal{P}^{fin}(W), \geqq \rangle$.*

Corollary 9.2.21. *If $\langle W, \leq \rangle$ is reverse well-quasi-ordered, then so is $\langle \mathcal{P}^{fin}(W), \leqq \rangle$.*

Proof. We that have that $\langle W, \leq \rangle$ is reverse well-quasi-ordered if, and only if, $\langle W, \geq \rangle$ is well-quasi-ordered; which implies that $\langle \mathcal{P}^{fin}(W), \geqq \rangle$ is well-quasi-ordered; which is the case if, and only if, $\langle \mathcal{P}^{fin}(W), \leqq \rangle$ is reverse well-quasi-ordered. □

Let $F \in \mathcal{F}(W)$. We note that $W - F$ is a downset in W. If $\langle W, \leq \rangle$ is reverse well-quasi-ordered, then $W - F$ contains only finitely many maximal elements (if not, the maximal elements would form a bad sequence in $\langle W, \leq \rangle$). Let D_F denote the set of maximal elements in $W - F$.

Lemma 9.2.22. *Suppose $\langle W, \leq \rangle$ is reverse well-quasi-ordered and let $F, G \in \mathcal{F}(W)$. If $D_F \geqq D_G$, then $F \subseteq G$.*

Proof. Since $\langle W, \leq \rangle$ is reverse well-quasi-ordered, D_F and D_G are both finite and $\geqq$ is defined for D_F and D_G.

By definition of $\geqq$ there exists a one-to-one function $\psi : D_G \to D_F$ such that $a \leq \psi(a)$ for every $a \in D_G$.

For any $x \in A$ it now follows that:

$$\begin{array}{lll} & x \notin G & \\ \Rightarrow & x \in W - G & \\ \Rightarrow & x \leq a & \text{for some } a \in D_G \\ \Rightarrow & x \leq \psi(a) & \text{since } D_F \geqq D_G \\ \Rightarrow & x \leq b & \text{for some } b \in D_F \text{ since } \psi(a) \in D_F \\ \Rightarrow & x \notin F. & \end{array}$$

Hence, $F \subseteq G$. □

For $F \in \mathcal{F}(W)$, let $F^{\triangleright}$ be an abbreviation for $\{F\}^{\triangleright} = \{I \in \mathcal{I}(W^{\bullet}) : (F, I) \in R\}$ and recall that $(F, I) \in R$ if, and only if, there exist $a \in F$ and $b \in I$ such that $a \leq b$. Furthermore, recall that $\mathbf{S} = \langle \mathcal{S}, \vee^{\mathbf{S}}, \wedge^{\mathbf{S}} \rangle$ is a complete lattice with $\mathcal{S} = \{\Lambda \in \mathcal{P}(\mathcal{I}(W^{\bullet})) : \Lambda = \Lambda^{\triangleleft\triangleright}\}$ and

$$\bigvee_{i \in \Psi}^{\mathbf{S}} \Lambda_i = \bigcap_{i \in \Psi} \Lambda_i \quad \text{and} \quad \bigwedge_{i \in \Psi}^{\mathbf{S}} \Lambda_i = \left(\bigcup_{i \in \Psi} \Lambda_i \right)^{\triangleleft\triangleright}$$

for $\Lambda_i \in \mathcal{S}$, $i \in \Psi$. The associated lattice order $\leq^{\mathbf{S}}$ is $\supseteq$.

Theorem 9.2.23. *If $\langle W, \leq \rangle$ is reverse well-quasi-ordered and $\langle W^\bullet, \leq \rangle$ is well-quasi-ordered, then* **S** *is finite.*

Proof. Recall that $S(W) = \{F^\rhd : F \in \mathcal{F}(W)\}$. Although $\mathcal{F}(W)$ is an infinite set, we claim that the set $S(W)$ is finite.

We begin by showing that $\langle S(W), \subseteq \rangle$ is well-quasi-ordered: Assume to the contrary that $\langle S(W), \subseteq \rangle$ is not well-quasi-ordered, i.e., there is a bad sequence $F_1^\rhd, F_2^\rhd, \ldots$ in $\langle S(W), \subseteq \rangle$. Then, whenever $i < j$, we have $F_i^\rhd \not\subseteq F_j^\rhd$. Now, $F_i^\rhd \not\subseteq F_j^\rhd$ implies that there exists an $I \in \mathcal{I}(W^\bullet)$ such that $(F_i, I) \in R$, but $(F_j, I) \notin R$. Moreover, $(F_i, I) \in R$ if, and only if, there exist $a \in F_i$ and $b \in I$ such that $a \leq b$. Then $a \notin F_j$, since $(F_j, I) \notin R$. Thus, $F_i \not\subseteq F_j$ whenever $i < j$. Hence, $F_1, F_2, \ldots$ is a bad sequence in $\langle \mathcal{F}(W), \subseteq \rangle$.

From the contrapositive of Lemma 9.2.22, it now follows that $D_{F_1}, D_{F_2}, \ldots$ is a sequence of finite subsets of W such that, whenever $i < j$, we have $D_{F_i} \not\geqq D_{F_j}$. That is, $D_{F_1}, D_{F_2}, \ldots$ is a bad sequence in $\langle \mathcal{P}^{fin}(W), \geqq \rangle$. Hence, $\langle \mathcal{P}^{fin}(W), \geqq \rangle$ is not well-quasi-ordered. But then $\langle \mathcal{P}^{fin}(W), \leqq \rangle$ is not reverse well-quasi-ordered and neither is $\langle W, \leq \rangle$, by Corollary 9.2.21. This, however contradicts our assumption that $\langle W, \leq \rangle$ is reverse well-quasi-ordered. We may therefore conclude that $\langle S(W), \subseteq \rangle$ is well-quasi-ordered. That is, it has no infinite anti-chains nor does it have any infinite descending chains.

Next, we show that $\langle S(W), \subseteq \rangle$ has no infinite ascending chains. Suppose, to the contrary, that there exists an infinite chain $F_1^\rhd \subset F_2^\rhd \subset \cdots$ in $S(W)$. For each $n \in \mathbb{N}$, choose $I_n \in F_{n+1}^\rhd - F_n^\rhd$. Note that $(F_{n+1}, I_n) \in R$, but $(F_n, I_n) \notin R$. Therefore, there exist $c_n \in F_{n+1}$ and $d_n \in I_n$ such that $c_n \leq d_n$, for each $n \in \mathbb{N}$. Then, $(F_{i+1}, I_j) \notin R$ whenever $i < j$: suppose to the contrary that $(F_{i+1}, I_j) \in R$ for some $i < j$. Then there exist $a \in F_{i+1}$ and $b \in F_j$ such that $a \leq b$. But then $a \in F_{i+1} \subseteq F_j$, which implies that $(F_j, I_j) \in R$ — contradicting our choice of I_j. Furthermore, $(F_{i+1}, I_j) \notin R$ whenever $i < j$ implies that, whenever $i < j$, we have $c \not\leq d$ for all $c \in F_{i+1}$ and all $d \in I_j$. In particular, $c_i \not\leq d_j$ whenever $i < j$. Since $c_i \leq d_i$, it follows that $d_i \not\leq d_j$ whenever $i < j$. Thus (d_n) is a bad sequence in $\langle W^\bullet, \leq \rangle$, contradicting our assumption. Therefore, $\langle S(W), \subseteq \rangle$ is finite.

Consequently, $S(W)$ is finite. Recall that every stable set $\Lambda \in \mathcal{S}$ is an intersection of elements of $S(W)$, by Lemma 9.2.4. Hence, $\mathcal{S}$ is finite. □

Thus, the problem of determining if **S** is finite is reduced to identifying **B**-

residual pairs, $\mathbf{W} = \langle W, W^\bullet \rangle$ for which $\langle W, \leq \rangle$ is reverse well-quasi-ordered and $\langle W^\bullet, \leq \rangle$ is well-quasi-ordered. We briefly summarise some results from [Hig52] that we will use to find reverse well-quasi-ordered W's and well-quasi-ordered $W^\bullet$'s.

An algebra $\mathbf{A} = \langle A, \mathbb{O}^{\mathbf{A}}, \preceq \rangle$ such that $\preceq$ is a quasi-order is called a *quasi-ordered algebra* if:

$$\text{Each operation in } \mathbb{O}^{\mathbf{A}} \text{ preserves } \preceq \text{ in each of its arguments.} \tag{9.1}$$

The quasi-order $\preceq$ is called a *divisibility order* if, in addition to (9.1), it satisfies:

$$\begin{aligned} &\text{For each } n\text{-ary operation } f^{\mathbf{A}} \text{ of } \mathbf{A} \text{ and all } a_1, \ldots, a_n \in A, \\ &a_i \preceq f^{\mathbf{A}}(a_1, \ldots, a_n) \text{ for each } i = 1, \ldots, n. \end{aligned}$$

For each $k \in \mathbb{N}$, let $\mathbb{O}_k^{\mathbf{A}}$ denote the set of all k-ary operations in $\mathbb{O}^{\mathbf{A}}$. Suppose $\leq_k$ is a quasi-order on $\mathbb{O}_k^{\mathbf{A}}$ for each $k \in \mathbb{N}$. Then $\preceq$ is called *compatible* with $\leq_k$ if, for all $f^{\mathbf{A}}, h^{\mathbf{A}} \in \mathbb{O}_k^{\mathbf{A}}$:

$$\text{If } f^{\mathbf{A}} \leq_k h^{\mathbf{A}} \text{ and } a_1, \ldots, a_k \in A, \text{ then } f^{\mathbf{A}}(a_1, \ldots, a_k) \preceq h^{\mathbf{A}}(a_1, \ldots, a_k).$$

If $\langle A, \mathbb{O}^{\mathbf{A}}, \preceq \rangle$ has no proper subalgebras, then it is said to be *minimal*. For example, an algebra generated by its set of constants is a minimal algebra.

Theorem 9.2.24. *[Hig52, Theorem 1.1] Suppose that $\langle A, \mathbb{O}^{\mathbf{A}}, \preceq \rangle$ is a minimal algebra endowed with a divisibility order $\preceq$. If $\langle \mathbb{O}_k^{\mathbf{A}}, \leq_k \rangle$ is a well-quasi-ordered set such that $\preceq$ is compatible with $\leq_k$ for each $k = 0, \ldots, n$, and $\mathbb{O}_k^{\mathbf{A}}$ is empty for $k > n$, then $\langle A, \preceq \rangle$ is well-quasi-ordered.*

Theorem 9.2.25. *[Hig52, Theorem 1.2] Suppose that $\langle A, \mathbb{O}^{\mathbf{A}}, \preceq \rangle$ is an algebra of finite type endowed with a divisibility order $\preceq$. If $\langle A, \mathbb{O}^{\mathbf{A}} \rangle$ is generated by a subset B and $\langle B, \preceq |_B \rangle$ is well-quasi-ordered, then $\langle A, \preceq \rangle$ is well-quasi-ordered. In particular, if $\langle A, \mathbb{O}^{\mathbf{A}}, \preceq \rangle$ is generated by a finite set, then $\langle A, \preceq \rangle$ is well-quasi-ordered.*

Finally we note that a finite direct product of well-quasi-ordered sets is again well-quasi-ordered; and the union of a finite number of well-quasi-ordered subsets of some partially ordered set $\langle A, \leq \rangle$ is also well-quasi-ordered.

Decreasing residuated ordered algebras

Recall from Section 9.1.1 that for a residuated lattice ordered algebra, $\mathbf{A} = \langle A, \vee, \wedge, \mathbb{T}^{\mathbf{A}}, \leq \rangle$, the set of operations $\mathbb{T}^{\mathbf{A}}$ is finite and consists of constants and unary and binary residuated operators. If, in addition, each operator in $\mathbb{T}_1^{\mathbf{A}} \cup \mathbb{T}_2^{\mathbf{A}}$ is decreasing, then $\mathbf{A} = \langle A, \vee, \wedge, \mathbb{T}^{\mathbf{A}}, \leq \rangle$ is called a *decreasing residuated lattice ordered algebra*. Then, for each $f \in \mathbb{T}_1^{\mathbf{A}}$, we have $f(x) \leq x$ which implies that $x \leq g(x)$, where $g \in (\mathbb{T}_1^{\bullet})^{\mathbf{A}}$ is f's residual. Hence, each $g \in (\mathbb{T}_1^{\bullet})^{\mathbf{A}}$ is increasing. Furthermore, if $\circ \in \mathbb{T}_2^{\mathbf{A}}$, then $y \circ x \leq x$ and $x \circ y \leq x$ implies that $x \leq y \backslash x$ and $x \leq x / y$.

Now let $\mathbf{A}$ be a decreasing residuated ordered algebra and $\mathbf{B}$ a partial subalgebra of $\mathbf{A}$. Let W be the closure of $B \cup \mathbb{T}_0^{\mathbf{A}}$ under the operations in $\mathbb{T}_1^{\mathbf{A}} \cup \mathbb{T}_2^{\mathbf{A}}$ and $\wedge$. Then $\geq$ is a divisibility order on W and W is generated by a finite set. Thus, by Theorem 9.2.25, $\langle W, \geq \rangle$ is well-quasi-ordered and hence $\langle W, \leq \rangle$ is reverse well-quasi-ordered.

Next let $W^{\bullet}$ be the closure of $B \cup \mathbb{T}_0^{\mathbf{A}}$ under $\vee$, each $g_i \in (\mathbb{T}_1^{\bullet})^{\mathbf{A}}$, $i \in \Psi$, and under $a \backslash_j x$ and $x /_j a$ for all $a \in W$ and $j \in \Phi$. Then,

(i) $\leq$ is preserved by $\vee$ and g_i, $a \backslash_j x$ and $a /_j x$ for all $i \in \Psi$, all $j \in \Phi$ and all $a \in W$.

(ii) As explained in the discussion above, each $g \in (\mathbb{T}_1^{\bullet})^{\mathbf{A}}$, each $a \backslash_j x$ and each $x /_j a$ is increasing, i.e., $c \leq g(c)$, $c \leq a \backslash_j c$ and $c \leq a /_j c$ for all $c \in W^{\bullet}$. Hence, $\leq$ is a divisibility order on $W^{\bullet}$.

(iii) $W^{\bullet}$ is closed under infinitely many unary operations and one binary operation. The set $\mathbb{O}_2$ is just $\{\vee\}$ which is trivially well-quasi-ordered and the ordering $\leq$ on $W^{\bullet}$ is (trivially) compatible with its trivial ordering.

Furthermore, for each $j \in \Phi$, let $L_j = \{a \backslash_j x : a \in M\}$ and $R_j = \{x /_j a : a \in W\}$. Define the relation $\leq_j^L$ on each L_j as follows: $a_1 \backslash_j x \leq_j^L a_2 \backslash_j x$ if, and only if, $a_1 \geq a_2$. Similarly, define $\leq_j^R$ on each R_j by: $x /_j a_1 \leq_j^R x /_j a_2$ if, and only if, $a_1 \geq a_2$. Then each $\langle L_j, \leq_j^L \rangle$ and each $\langle R_j, \leq_j^R \rangle$ is well-quasi-ordered: $\langle W, \leq \rangle$ is reverse well-quasi-ordered and $a_1 \geq a_2$ implies $a_1 \backslash_j c \leq a_2 \backslash_j c$ and $c /_j a_1 \leq c /_j a_2$ for all $c \in W^{\bullet}$. Now let $\mathbb{O}_1$ be the union of all L_j, R_j and $\{g_i \in (\mathbb{T}_1^{\bullet})^{\mathbf{A}} : i \in \Psi\}$ and let $\leq_1$ be the union of all $\leq_j^L$'s, $\leq_j^R$'s and the trivial order on $\{g_i \in (\mathbb{T}_1^{\bullet})^{\mathbf{A}} : i \in \Psi\}$. Then $\langle \mathbb{O}_1, \leq_1 \rangle$ is well-quasi-ordered. Moreover, the ordering $\leq$ on $W^{\bullet}$ is compatible with

$\leq_1$.

Hence, $\langle W^\bullet, \leq\rangle$ is well-quasi-ordered by Theorem 9.2.24.

Then $\langle W, W^\bullet\rangle$ is a $\mathbf{B}$-residual pair such that W is reverse well-quasi-ordered and $W^\bullet$ is well-quasi-ordered. Moreover, $\mathbf{S}$ obtained from $\langle W, W^\bullet\rangle$ by the construction described in Section 9.2.1, is finite by Theorem 9.2.23.

9.2.3 Additional properties preserved by the construction

Though not explored in full, we show some initial preservation results here. In particular we show how Lemma 9.2.11 can be used to great effect to prove the preservation of some important properties of the operations.

Lemma 9.2.26. *Let $f \in \mathbb{T}_1^{\mathbf{A}}$. Then:*

(i) If f is decreasing, then so is $f^{\mathbf{S}}$.

(ii) If f is increasing, then so is $f^{\mathbf{S}}$.

(iii) If f is idempotent, then so is $f^{\mathbf{S}}$.

Proof. (i) Suppose f is decreasing, i.e., $f(a) \leq a$ for all $a \in A$. Let $\Lambda \in \mathcal{S}$ and $I \in \Lambda$. Then,

$$\begin{aligned}
&(F, I) \in R \text{ for all } F \in \Lambda^\triangleleft \\
\Rightarrow\ &\text{there exist } a \in F,\ b \in I \text{ such that } a \leq b \text{ for all } F \in \Lambda^\triangleleft \\
\Rightarrow\ &\text{there exist } a \in F,\ b \in I \text{ such that } f(a) \leq a \leq b \text{ for all } F \in \Lambda^\triangleleft \\
\Rightarrow\ &(\hat{f}(F), I) \in R \text{ by Lemma 9.2.10 (i) for all } F \in \Lambda^\triangleleft \\
\Rightarrow\ &I \in f(\Lambda^\triangleleft)^\triangleright.
\end{aligned}$$

Hence, $\Lambda \subseteq f(\Lambda^\triangleleft)^\triangleright$, i.e., $f^{\mathbf{S}}(\Lambda) \leq^{\mathbf{S}} \Lambda$.

(ii) Suppose f is increasing, i.e., $a \leq f(a)$ for all $a \in A$. Let $\Lambda \in \mathcal{S}$ and $I \in f(\Lambda^\triangleleft)^\triangleright$. Then,

$$\begin{aligned}
&(\hat{f}(F), I) \in R \text{ for all } F \in \Lambda^\triangleleft \\
\Rightarrow\ &\text{there exist } a \in \hat{f}(F),\ b \in I \text{ such that } a \leq b \text{ for all } F \in \Lambda^\triangleleft \\
&\text{by Lemma 9.2.10 (i)}
\end{aligned}$$

$\Rightarrow$ there exist $c_1, \ldots, c_n \in F,\ n \in \mathbb{N},\ a \in \hat{f}(F)$ and $b \in I$ such that

$\bigwedge_{i=1}^{n} f(c_i) \leq a \leq b$ for all $F \in \Lambda^{\triangleleft}$

$\Rightarrow$ there exist $c_1, \ldots, c_n \in F,\ n \in \mathbb{N},\ a \in \hat{f}(F)$ and $b \in I$ such that

$f\left(\bigwedge_{i=1}^{n} c_i\right) \leq \bigwedge_{i=1}^{n} f(c_i) \leq a \leq b$ for all $F \in \Lambda^{\triangleleft}$

$\Rightarrow$ there exist $c \in F$(set, $c = \bigwedge_{i=1}^{n} c_i$), $b \in I$ such that $c \leq f(c) \leq b$ for all $F \in \Lambda^{\triangleleft}$

since W and F are closed under finite meets and f is increasing

$\Rightarrow (F, I) \in R$ for all $F \in \Lambda^{\triangleleft}$

$\Rightarrow I \in \Lambda^{\triangleleft\triangleright} = \Lambda$.

Therefore, $f(\Lambda^{\triangleleft})^{\triangleright} \subseteq \Lambda$, i.e., $\Lambda \leq^{\mathbf{S}} f^{\mathbf{S}}(\Lambda)$.

(iii) Suppose f is idempotent, i.e., $f(f(a)) = f(a)$ for all $a \in A$. We first show that $\hat{f}(\hat{f}(F)) = \hat{f}(F)$: Let $a \in f(F)$. Then $a \geq \bigwedge_{i=1}^{n} f(c_i) \geq f(\bigwedge_{i=1}^{n} c_i)$ for some $c_1, \ldots, c_n \in F$, $n \in \mathbb{N}$. But $c = \bigwedge_{i=1}^{n} c_i \in F$ since W and F are closed under finite meets. Thus, $a \geq f(c) = f(f(c))$ for some $c \in F$, since f is idempotent. This implies that $a \in \hat{f}(\hat{f}(F))$. Hence, $\hat{f}(F) \subseteq \hat{f}(\hat{f}(F))$.

On the other hand, let $b \in \hat{f}(\hat{f}(F))$. Using the fact that W and F are closed under finite meets, we can follow a similar argument to the one above to prove the following: there exists $a \in \hat{f}(F)$ such that $b \geq f(a)$. But $a \in \hat{f}(F)$ implies that $a \geq f(c)$ for some $c \in F$. Then $b \geq f(a) \geq f(f(c)) = f(c)$ which implies that $b \in \hat{f}(F)$. Hence, $\hat{f}(\hat{f}(F)) \subseteq \hat{f}(F)$.

Next let $X \in \mathcal{P}(\mathcal{F}(W))$. Then:

$$\begin{aligned} f(f(X)) &= f(\{\hat{f}(F) : F \in X\}) \\ &= \{\hat{f}(\hat{f}(F)) : F \in X\} \\ &= \{\hat{f}(F) : F \in X\} \\ &= f(X). \end{aligned}$$

Now, by the above and Lemma 9.2.11 (i) we have: for $\Lambda \in \mathcal{S}$,

$$\begin{aligned} f^{\mathbf{S}}(f^{\mathbf{S}}(\Lambda)) &= f^{\mathbf{S}}(f(\Lambda^{\triangleleft})^{\triangleright}) \\ &= f(f(\Lambda^{\triangleleft}))^{\triangleright} \\ &= f(\Lambda^{\triangleleft})^{\triangleright} \\ &= f^{\mathbf{S}}(\Lambda). \end{aligned}$$

□

Lemma 9.2.27. *Let $\circ \in \mathbb{T}_2^{\mathbf{A}}$. Then,*

(i) If $\circ$ is decreasing (in each coordinate), then so is $\circ^{\mathbf{S}}$.

(ii) If $\circ$ is associative, then so is $\circ^{\mathbf{S}}$.

(iii) If $\circ$ is commutative, then so is $\circ^{\mathbf{S}}$.

(iv) If $1 \in \mathbb{T}_0^{\mathbf{A}}$ such that 1 is a (left- or right-) identity of $\circ$, then $\mu(1)$ is a (left- or right-) identity of $\circ^{\mathbf{S}}$.

Proof. (i) Suppose $\circ$ is decreasing, i.e., $a \circ b \leq a$ and $a \circ b \leq b$ for all $a, b \in A$. We must show that $\Lambda \circ^{\mathbf{S}} \Upsilon \leq^{\mathbf{S}} \Lambda$ and $\Lambda \circ^{\mathbf{S}} \Upsilon \leq^{\mathbf{S}} \Upsilon$ for all $\Lambda, \Upsilon \in \mathcal{S}$. We will prove the first inequality. The second follows similarly. Let $\Lambda, \Upsilon \in \mathcal{S}$ and let $I \in \Lambda$. Then,

$$\begin{aligned} &(F, I) \in R \text{ for all } F \in \Lambda^{\triangleleft} \\ \Rightarrow\ &\text{there exist } a \in F,\ b \in I \text{ such that } a \leq b \text{ for all } F \in \Lambda^{\triangleleft} \\ \Rightarrow\ &\text{there exist } a \in F,\ b \in I \text{ such that } a \circ c \leq a \leq b \\ &\text{for all } c \in G,\ \text{for all } G \in \Upsilon^{\triangleleft} \text{ and all } F \in \Lambda^{\triangleleft} \\ \Rightarrow\ &(F \hat{\circ}\, G, I) \in R \text{ for all } F \in \Lambda^{\triangleleft} \text{ and all } G \in \Upsilon^{\triangleleft}, \text{ by Lemma 9.2.10 (iii)} \\ \Rightarrow\ &I \in (\Lambda^{\triangleleft} \circ \Upsilon^{\triangleleft})^{\triangleright}. \end{aligned}$$

Hence, $\Lambda \subseteq (\Lambda^{\triangleleft} \circ \Upsilon^{\triangleleft})^{\triangleright}$, i.e., $\Lambda \circ^{\mathbf{S}} \Upsilon \leq^{\mathbf{S}} \Lambda$.

(ii) Suppose $\circ$ is associative. Let $F_1, F_2, F_3 \in \mathcal{F}(W)$. We will show that $F_1 \hat{\circ}\, (F_2 \hat{\circ}\, F_3) = (F_1 \hat{\circ}\, F_2) \hat{\circ}\, F_3$. Let

$$e \in F_1 \hat{\circ}\, (F_2 \hat{\circ}\, F_3) = [\{a \circ b : a \in F_1, b \in F_2 \hat{\circ}\, F_3\}\rangle\,.$$

Then $e \geq \bigwedge_{i=1}^{n}(a_i \circ b_i) \geq \bigwedge_{i=1}^{n} a_i \circ \bigwedge_{i=1}^{n} b_i$ where $a_i \in F_1$ and $b_i \in F_2 \hat{\circ} F_3$ for $i = 1, \ldots, n$. But $a = \bigwedge_{i=1}^{n} a_i \in F_1$ and $b = \bigwedge_{i=1}^{n} b_i \in F_2 \hat{\circ} F_3$. Thus, $e \geq a \circ b$ for $a \in F_1$ and $b \in F_2 \hat{\circ} F_3$. Using a similar argument we can now show that $b \in F_2 \circ F_3$ implies that $b \geq c \circ d$ for some $c \in F_2$ and $d \in F_3$. Then, $e \geq a \circ (c \circ d) = (a \circ c) \circ d$ by the associativity and $e \in (F_1 \hat{\circ} F_2) \hat{\circ} F_3$. Therefore, $F_1 \hat{\circ} (F_2 \hat{\circ} F_3) \subseteq (F_1 \hat{\circ} F_2) \hat{\circ} F_3$. The inclusion in the other direction follows similarly.

A consequence of the above is that $X_1 \circ (X_2 \circ X_3) = (X_1 \circ X_2) \circ X_3$ for $X_1, X_1, X_3 \in \mathcal{P}(\mathcal{F}(W))$:

$$\begin{aligned} X_1 \circ (X_2 \circ X_3) &= X_1 \circ \{F_2 \hat{\circ} F_3 : F_i \in X_i, i = 1, 2\} \\ &= \{F_1 \hat{\circ} (F_2 \hat{\circ} F_3) : F_i \in X_i, i = 1, 2, 3\} \\ &= \{(F_1 \hat{\circ} F_2) \hat{\circ} F_3 : F_i \in X_i, i = 1, 2, 3\} \\ &= (X_1 \circ X_2) \circ X_3. \end{aligned}$$

Then, by the above and Lemma 9.2.11 (ii) we have, for $\Lambda, \Upsilon, \Gamma \in \mathcal{S}$ (and therefore $\Lambda^{\triangleleft}, \Upsilon^{\triangleleft}, \Gamma^{\triangleleft} \in \mathcal{P}(\mathcal{F}(W))$):

$$\begin{aligned} \Lambda \circ^{\mathbf{S}} (\Upsilon \circ^{\mathbf{S}} \Gamma) &= \Lambda^{\triangleleft\triangleright} \circ^{\mathbf{S}} (\Upsilon^{\triangleleft} \circ \Gamma^{\triangleleft})^{\triangleright} \\ &= (\Lambda^{\triangleleft} \circ (\Upsilon^{\triangleleft} \circ \Gamma^{\triangleleft}))^{\triangleright} \\ &= ((\Lambda^{\triangleleft} \circ \Upsilon^{\triangleleft}) \circ \Gamma^{\triangleleft})^{\triangleright} \\ &= (\Lambda^{\triangleleft} \circ \Upsilon^{\triangleleft})^{\triangleright} \circ^{\mathbf{S}} \Gamma^{\triangleleft\triangleright} \\ &= (\Lambda \circ^{\mathbf{S}} \Upsilon) \circ^{\mathbf{S}} \Gamma. \end{aligned}$$

(iii) Suppose $\circ$ is commutative. We first show that $\hat{\circ}$ is commutative. Let $F_1, F_2 \in \mathcal{F}(W)$. Then,

$$\begin{aligned} F_1 \hat{\circ} F_2 &= [\{a \circ b : a \in F_1, b \in F_2\}\rangle \\ &= [\{b \circ a : b \in F_2, a \in F_1\}\rangle \\ &= F_2 \hat{\circ} F_1. \end{aligned}$$

Now we have that, for $X_1, X_2 \in \mathcal{P}(\mathcal{F}(W))$:

$$\begin{aligned} X_1 \circ X_2 &= \{F_1 \hat{\circ} F_2 : F_1 \in X_1, F_2 \in X_2\} \\ &= \{F_2 \hat{\circ} F_1 : F_2 \in X_2, F_1 \in X_1\} \\ &= X_2 \circ X_1. \end{aligned}$$

Finally, for $\Lambda, \Upsilon \in \mathcal{S}$:

$$\begin{aligned}\Lambda \circ^{\mathbf{S}} \Upsilon &= \left(\Lambda^{\triangleleft} \circ \Upsilon^{\triangleleft}\right)^{\triangleright} \\ &= \left(\Upsilon^{\triangleleft} \circ \Lambda^{\triangleleft}\right)^{\triangleright} \\ &= \Upsilon \circ^{\mathbf{S}} \Lambda.\end{aligned}$$

(iv) We show that if 1 is a left-identity of $\circ$, then $\mu(1)$ is a left-identity of $\circ^{\mathbf{S}}$. The proofs of the other cases are similar.

Let $\Lambda \in \mathcal{S}$. Then, by Lemma 9.2.5,

$$\begin{aligned}&\mu(1) \circ^{\mathbf{S}} \Lambda \\ &= (\mu(1)^{\triangleleft} \circ \Lambda^{\triangleleft})^{\triangleright} \\ &= (\{F \in \mathcal{F}(W) : 1 \in F\} \circ \{G \in \mathcal{F}(W) : I \in \Lambda \text{ implies } (G, I) \in R\})^{\triangleright} \\ &= \{F \hat{\circ} G : 1 \in F, I \in \Lambda \text{ implies } (G, I) \in R\}^{\triangleright}.\end{aligned}$$

We first show that $\Lambda \subseteq \mu(1) \circ^{\mathbf{S}} \Lambda$. Suppose $F = G_1 \hat{\circ} G_2$ such that $1 \in G_1$ and $I \in \Lambda$ implies that $(G_2, I) \in R$ and let $J \in \Lambda$. Then $(G_2, J) \in R$ by assumption. By the definition of R there then exist $a \in G_2$ and $b \in J$ such that $a \leq b$. But $1 \in G_1$ implies that $1 \circ a = a \in F$. Therefore, $(F, J) \in R$. Since $(F, J) \in R$ for all $J \in \Lambda$, it follows that $F \in \Lambda^{\triangleleft}$. Hence, $\{F \hat{\circ} G : 1 \in F, I \in \Lambda \text{ implies } (G, I) \in R\} \subseteq \Lambda$. By the properties of Galois connections we then have that

$$\Lambda = \Lambda^{\triangleleft\triangleright} \subseteq \{F \hat{\circ} G : 1 \in F, I \in \Lambda \text{ implies } (G, I) \in R\}^{\triangleright} = \mu(1) \circ^{\mathbf{S}} \Lambda.$$

For the inclusion in the other direction let $G \in \Lambda^{\triangleleft}$, i.e., $G \in \mathcal{F}(W)$ such that $I \in \Lambda$ implies that $(G, I) \in R$. Observe that since $1 \in \mathbb{T}_0^{\mathbf{A}} \subseteq W$, we have that $[1)^W \in \mathcal{F}(W)$ such that $1 \in [1)^W$. We will now show that $[1)^W \hat{\circ} G = G$.

Let $a \in G$; then $1 \circ a = a \in [1)^W \hat{\circ} G$. Hence, $G \subseteq [1)^W \hat{\circ} G$. Next consider $[1)^W \hat{\circ} G = \left[\{b \circ c : b \in [1)^W, c \in G\}\right\rangle$. If $b \in [1)^W$, then $b \geq 1$ which implies that $b \circ c \geq 1 \circ c = c$ for all $c \in G$. Thus, $b \circ c \in G$ since G is an up-set. Then $[1)^W \hat{\circ} G \subseteq G$. Therefore, $[1)^W \hat{\circ} G = G$.

From the above it now follows that

$$\Lambda^{\triangleleft} \subseteq \{F \hat{\circ} G : 1 \in F, I \in \Lambda \text{ implies } (G, I) \in R\}.$$

Then, by the properties of Galois connections,

$$\{F \mathbin{\hat{\circ}} G : 1 \in F, I \in \Lambda \text{ implies } (G, I) \in R\}^{\triangleright} = \mu(1) \circ^{\mathbf{S}} \Lambda \subseteq \Lambda^{\triangleleft\triangleright} = \Lambda.$$

□

10. ALGEBRAIC FILTRATIONS IN MODAL LOGIC

Many well-known propositional modal logics are algebraizable and classes of Boolean algebras with operators (or BAOs for short) are the equivalent algebraic semantics of such logics. Since Boolean algebras form the algebraic semantics for classical propositional logic, the task of the additional operators in BAOs is to represent the modalities of the logic. Following the discussion in Chapter 8 and since modal logics are algebraizable, it is natural to seek classes of BAOs that have the FEP. Recall that the algebraizability ensures that if a class of BAOs has the FEP, then the associated logic is decidable if it is finitely axiomatized.

In this chapter we will use the method of *algebraic filtration* to prove the FEP for classes of modal algebras (BAOs with a single unary operator). The method of filtration has been used to prove finite model properties in modal logic. Although this method is usually associated with relational (Kripke) models, it was originally an algebraic one. In [McK41] filtrations were used to prove the finite model property for the modal logics S2 and S4 (see also [MT44]). The (Kripke) model-theoretic version of filtration first appeared in [Lem66a, Lem66b, LS77], where the algebraic and model-theoretic methods were connected for some particular cases. The filtration method was further developed in [Seg68, Seg71] (where the term 'filtration' was apparently first used). Algebraic filtrations have also been applied in the settings of, for example, cylindric algebras [HMT85] and relation algebras [Ném87]. For an extensive history of modal logics and filtrations, we refer the interested reader to [Gol03].

We investigate connections between the algebraic and model-theoretic versions of filtrations and develop a duality between the two methods. In the next section we recall the definitions of the notions we will use: modal algebras, Kripke frames and models and the basic modal language.

We begin our investigation by describing algebraic constructions that produce finite modal algebras, called *algebraic filtrations*, in Section 10.2. In Section 10.3 we recall the method of filtration for Kripke models and then adjust

the method for Kripke frames. That is, we introduce the notion of a *set filtration*, which was already implicit in [Lem66a]. Set filtrations are equivalent to ordinary model-theoretic filtrations, but operate on frames rather than models. They are better suited for the duality theory developed in Section 10.4.

Finally, in Section 10.5 we use the correspondence developed in Section 10.3 to translate well known model-theoretic filtrations into set filtrations. In particular, we will consider the *largest, smallest, transitive* and *symmetric* filtrations. We use the duality theory developed in Section 10.4 to find the algebraic versions of each of these filtrations.

For the sake of readability we will restrict ourselves to modal algebras and to frames with only one binary relation. The definitions of algebraic and set filtrations as well as all other definitions and results in the rest of the chapter can, however, be generalized in a natural way to the settings of arbitrary Boolean algebras with operators (BAOs) and frames of different modal similarity types, respectively.

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10.1 Modal algebras, modal logic and Kripke semantics

In this section we give the definitions of the objects of study for this chapter. See [BdRV01] for more on the notions defined here.

Recall that an operator $f : A \to A$ on a Boolean algebra $\mathbf{A}$ distributes over finite joins, i.e., $f(x \vee y) = f(x) \vee f(y)$ for all $x, y \in A$, and satisfies $f(0) = 0$.

Definition 10.1.1. *A* (normal) modal algebra *is an algebra* $\mathbf{A} = \langle A, \vee, \wedge, \neg, 0, 1, f\rangle$ *is such that* $\langle A, \vee, \wedge, \neg, 0, 1\rangle$ *is a Boolean algebra and* f *is a unary operator.*

An element $a \in A$ is an *atom of* $\mathbf{A}$ if $0 < a$ and there is no element $b \in A$ such that $0 < b < a$. Let $At\,\mathbf{A}$ denote the set of all atoms of $\mathbf{A}$. A modal algebra $\mathbf{A}$ is called *atomic* if every non-0 element of A has an atom less than or equal to it. An element $a \in A$ is a *co-atom of* $\mathbf{A}$ if $a < 1$ and there is no element $b \in A$ such that $a < b < 1$. Let $Ca\,\mathbf{A}$ denote the set of all co-atoms of $\mathbf{A}$.

If $S \subseteq A$, then *the Boolean subalgebra* of $\mathbf{A}$ generated by S, say $\mathbf{B}_S = \langle B, \vee^{\mathbf{B}}, \wedge^{\mathbf{B}}, \neg^{\mathbf{B}}, 0, 1\rangle$, is the intersection of all subalgebras of the f-free reduct of $\mathbf{A}$ containing S. We note that the unary operator f is partially defined on B. Furthermore, if S is finite, then $\langle \mathbf{B}, f\rangle$ is a finite partial subalgebra of $\mathbf{A}$.

The formulas of the *basic modal language* over a denumerably infinite set of proposition letters Φ are given by the following recursive definition:

$$\varphi := \bot \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \Diamond\varphi$$

where $p \in \Phi$. This language is interpreted in models $\mathcal{M} = \langle W, R, V\rangle$ in the usual way, i.e., given $w \in W$:

- it is never the case that $\mathcal{M}, w \Vdash \bot$;
- $\mathcal{M}, w \Vdash p$ if, and only if, $w \in V(p)$;
- $\mathcal{M}, w \Vdash \neg\varphi$ if, and only if, $\mathcal{M}, w \nVdash \varphi$;
- $\mathcal{M}, w \Vdash \varphi \vee \psi$ if, and only if, $\mathcal{M}, w \Vdash \varphi$ or $\mathcal{M}, w \Vdash \psi$;
- $\mathcal{M}, w \Vdash \Diamond\varphi$ if, and only if, there exists $v \in W$ such that $(w, v) \in R$ and $\mathcal{M}, v \Vdash \varphi$.

Given the definition of the semantics, it is possible to extend the valuation $V : \Phi \to \mathcal{P}(W)$ to a map from the set of all formulas to $\mathcal{P}(W)$ by letting $V(\varphi) = \{w \in W : \mathcal{M}, w \Vdash \varphi\}$. We shall make use of the fact that $V(\Diamond\varphi) = f_R(V(\varphi))$, which can be seen as follows:

$$\begin{aligned} V(\Diamond\varphi) &= \{w \in W : \mathcal{M}, w \Vdash \Diamond\varphi\} \\ &= \{w \in W : \text{there exists } x \in W \text{such that } (w, x) \in R \text{ and } \mathcal{M}, x \Vdash \varphi\} \\ &= \{w \in W : \text{there exists } x \in W \text{ such that } (w, x) \in R \text{ and } x \in V(\varphi)\} \\ &= f_R(V(\varphi)). \end{aligned}$$

Definition 10.1.2. *A* (Kripke) frame *is a pair $\mathfrak{F} = \langle W, R\rangle$ where W is a non-empty set and R a binary relation on W.*

A valuation on $\mathfrak{F}$ *is a function $V : \Phi \to \mathcal{P}(W)$ that assigns a subset of W to every proposition letter.*

A (Kripke) model $\mathcal{M} = \langle \mathfrak{F}, V\rangle$ *is a relational structure where $\mathfrak{F}$ is a frame and V is a valuation on $\mathfrak{F}$. If $\mathfrak{F} = \langle W, R\rangle$, then we also write $\mathcal{M}$ as $\langle W, R, V\rangle$.*

It is well known that the duality between frames and modal algebras rests on the following notions.

Definition 10.1.3. *The* complex algebra *of* $\mathfrak{F} = \langle W, R\rangle$ *is the modal algebra* $\mathfrak{F}^+ = \langle \mathcal{P}(W), \cap, \cup, -, \varnothing, W, f_R\rangle$, *where the operator* f_R *on* $\mathcal{P}(W)$ *is given by:*

$$f_R(X) = \{w \in W : \text{there exists } x \in X \text{ such that } (w, x) \in R\}.$$

Recall that an ultrafilter F of an Boolean algebra $\mathbf{A}$ is a maximal proper filter of the lattice reduct of $\mathbf{A}$, i.e., a proper filter of the lattice reduct of $\mathbf{A}$ that satisfies: for all $a, b \in A$

$$\text{if } a \vee b \in F, \text{ then } a \in F \text{ or } b \in F.$$

Since $a \vee \neg a = 1 \in F$ for all $a \in A$ and all filters F of a Boolean algebra $\mathbf{A}$, the above is equivalent to: for all $a \in A$ either $a \in F$ or $\neg a \in F$.

Theorem 10.1.4 (Ultrafilter theorem)**.** *Let* $\mathbf{A}$ *be a Boolean algebra,* $a \in A$ *and* G *a proper filter of* $\mathbf{A}$ *such that* $a \notin G$. *Then there is an ultrafilter* F *of* $\mathbf{A}$ *such that* $G \subseteq F$ *and* $a \notin F$.

Definition 10.1.5. *The* ultrafilter frame *of* $\mathbf{A}$ *is the frame* $\mathbf{A}_\bullet = \langle Uf\mathbf{A}, R_f\rangle$ *where* $Uf\mathbf{A}$ *is the set of all ultrafilters of* $\mathbf{A}$ *and* R_f *is a binary relation on* $Uf\mathbf{A}$ *such that*

$$(u, v) \in R_f \iff f(a) \in u \text{ whenever } a \in v.$$

For each modal algebra $\mathbf{A}$, define a binary relation $R^{\mathbf{A}}$ on A such that

$$(a, b) \in R^{\mathbf{A}} \iff a \leq f(b).$$

Definition 10.1.6. *Suppose* $\mathbf{A}$ *is an atomic modal algebra. The* atom structure *of* $\mathbf{A}$ *is the frame* $\mathbf{A}_+ = \langle At\mathbf{A}, R^{\mathbf{A}}{\restriction} At\mathbf{A}\rangle$ *where* $R^{\mathbf{A}}{\restriction} At\mathbf{A}$ *denotes the restriction of* $R^{\mathbf{A}}$ *to* $At\mathbf{A}$.

The duality theory developed in Section 10.4 will rely on the following two theorems.

Theorem 10.1.7. *Let* $\mathfrak{F}$ *be a frame. The* $\mathfrak{F}$ *is isomorphic to* $(\mathfrak{F}^+)_+$, *i.e., the atom structure of the complex algebra of* $\mathfrak{F}$.

Theorem 10.1.8 (Jónsson-Tarski theorem [JT51])**.** *Let* $\mathbf{A} = \langle A, \vee, \wedge, \neg, 0, 1, f\rangle$ *be a modal algebra. Then the representation function* $\varpi : A \to \mathcal{P}(Uf\mathbf{A})$ *given by*

$$\varpi(a) = \{u \in Uf\mathbf{A} : a \in u\}$$

is an embedding of $\mathbf{A}$ *into* $(\mathbf{A}_\bullet)^+$.

That is, every modal algebra is (isomorphic to) a subalgebra of the complex algebra of its ultrafilter frame. The above theorem can be restated more generally for BAOs.

10.2 Algebraic Filtrators

Let $\mathbf{A}$ be a modal algebra. Recall from the discussion on the FEP in Chapter 8 that if we assume that an identity $(\forall \vec{x})(s(\vec{x}) = t(\vec{x}))$ fails in $\mathbf{A}$, then there exist some assignment $\vec{x} \mapsto \vec{a}$ of elements of A to the variables such that $s^{\mathbf{A}}(\vec{a}) \neq t^{\mathbf{A}}(\vec{a})$. Then the set of elements of A used in the evaluation of s and t forms a finite subset, say $M \subseteq^{fin} A$. Recall that the aim is now to construct a finite modal algebra in which the identity fails. In general the subalgebra of $\mathbf{A}$ generated by M is infinite since the operator f may force the inclusion of infinitely many elements. Hence we cannot expect to produce a finite modal algebra in this way. However, the partial subalgebra $\langle \mathbf{B}_M, f \rangle$ where $\mathbf{B}_M = \langle B, \vee^{\mathbf{B}}, \wedge^{\mathbf{B}}, \neg^{\mathbf{B}}, 0, 1 \rangle$ is the Boolean subalgebra of the f-free reduct of $\mathbf{A}$ generated by M and f the modal operator partially defined on B, is finite and is closed under all existing Boolean operations in M. For $\langle \mathbf{B}_M, f \rangle$ to be a modal algebra, we must now extend the partial modal operator f to an operator f' defined on the entire B. We note that $\langle \mathbf{B}_M, f \rangle$ is not the only choice of finite partial subalgebra that can be used in such a construction — any finite partial subalgebra of $\mathbf{A}$ that contains M and is closed under the existing Boolean operations in M may be used. We will, however, focus on $\langle \mathbf{B}_M, f \rangle$ in this thesis.

In [McK41] the extension f' of f was defined on the universe B of $\mathbf{B}_M$ in the following way:

$$f'(b) = \bigwedge \{a \in B : \text{there exists } c \in B \text{ such that } a = f(c) \text{ and } b \leq c\}.$$

Given some restrictions, it is easy to show that f' as defined above is an operator that extends f. We can now generalize this approach to describe a way to define an operation f^Q on B in terms of an arbitrary binary relation $Q \subseteq B \times B$. For any $Q \subseteq B \times B$ define f^Q on B by:

$$f^Q(b) = \bigwedge \{a \in B : (a, b) \in Q\}.$$

In order to ensure that f^Q extends f and is an operator, we will require that Q satisfy certain conditions. This approach will be explored further in Section 10.2.1 where we will also state the conditions required of Q.

The suggested extension of a partial operation given above uses a conjunction of elements of B to give an approximation of $f(b)$ from above. Alternatively, we may also use a disjunction of elements of B to give an approximation $f(b)$ from below. We will develop this approach in the remainder of this section since it is better suited for the duality theory developed in Section 10.4. As one might expect, there is a close connection between the approximations from below and from above and we will investigate it in more detail in Section 10.2.1.

Throughout this section let $\mathbf{A}$ be a modal algebra with operator f, $M \subseteq^{fin} A$ and $\mathbf{B}_M$ the finite Boolean subalgebra of the f-free reduct of $\mathbf{A}$ generated by M. Furthermore, let $\underline{M} \subseteq M$ such that $f(a) \in M$ whenever $a \in \underline{M}$. If f' is an operator on $\mathbf{B}_M$ we use $\langle \mathbf{B}_M, f' \rangle$ to denote the modal algebra with $\mathbf{B}_M$ as its Boolean part and f' as its operator.

It is well known that an operator f' on the finite Boolean algebra $\mathbf{B}_M$ is uniquely determined by its operation on $At\mathbf{B}_M$, the set of atoms of $\mathbf{B}_M$. To see why, note that if $b = x_1 \vee \cdots \vee x_n$ where $\{x_1, \ldots, x_n\}$ is the set of all atoms of $\mathbf{B}_M$ below b, then

$$f'(b) = f'(x_1 \vee \cdots \vee x_n) = f'(x_1) \vee \cdots \vee f'(x_n) = \bigvee \{f'(y) : y \in At\mathbf{B}_M \text{ and } y \leq b\}.$$

Moreover, for each function $g : At\mathbf{B}_M \to B_M$ (and hence each operator on $\mathbf{B}_M$) there is an associated binary relation R^g on $At\mathbf{B}_M$ defined by:

$$(x, y) \in R^g \iff x \leq g(y). \tag{10.1}$$

That is, $R^g = \leq \circ g$, where $\circ$ denotes relational composition.

Conversely, any binary relation R on $At\mathbf{B}_M$ has an associated function $g^R : At\mathbf{B}_M \to B_M$ defined by:

$$g^R(y) = \bigvee \{x \in At\mathbf{B}_M : (x, y) \in R\}. \tag{10.2}$$

We now have a one-to-one correspondence between the functions $g : At\mathbf{B}_M \to B_M$ and relations $R \subseteq At\mathbf{B}_M \times At\mathbf{B}_M$.

Lemma 10.2.1. *For any function $g : At\mathbf{B}_M \to B_M$, $g^{R^g} = g$, and, for any binary relation R on $At\mathbf{B}_M$, $R^{g^R} = R$.*

Proof. For $y, z \in At\mathbf{B}_M$,

$$g^{R^g}(y) = \bigvee \{x \in At\mathbf{B}_M : (x, y) \in R^g\} = \bigvee \{x \in At\mathbf{B}_M : x \leq g(y)\} = g(y),$$

and

$$(z,y) \in R^{g^R} \iff z \leq g^R(y) \iff z \leq \bigvee\{x \in At\mathbf{B}_M : (x,y) \in R\}.$$

Since z is an atom, the last condition holds if, and only if,

$$z \in \{x \in At\mathbf{B}_M : (x,y) \in R\} \iff (z,y) \in R.$$

□

Given any binary relation R on $At\mathbf{B}_M$, the function $g^R : At\mathbf{B}_M \to B_M$ defined in (10.2) can be extended to an operator on B_M, denoted by f^R, in the following *unique* way:

$$\begin{aligned} f^R(b) &= \bigvee\{g^R(y) : y \in At\mathbf{B}_M \text{ and } y \leq b\} \\ &= \bigvee\Big\{\bigvee\{x \in At\mathbf{B}_M : (x,y) \in R\} : y \in At\mathbf{B}_M \text{ and } y \leq b\Big\} \\ &= \bigvee\{x \in At\mathbf{B}_M : \text{there exists } y \in At\mathbf{B}_M \text{ such that } y \leq b \text{ and } (x,y) \in R\}. \end{aligned}$$

Note that $f^R(0) = \bigvee \varnothing = 0$ and, by the definition of f^R on B_M, it is immediate that f^R distributes over finite joins, i.e., f^R is an operator on $\mathbf{B}_M$. To ensure that f^R extends f, we require the following condition:

$$\begin{aligned} &\text{For all } b \in \underline{M} \text{ and all } x \in At\mathbf{B}_M \text{ we have } x \leq f(b) \text{ if, and only if,} \\ &\qquad \text{there exists } y \in At\mathbf{B}_M \text{ such that } y \leq b \text{ and } (x,y) \in R. \end{aligned} \tag{R}$$

We note that condition (R) states that $\leq \circ f \restriction_{\underline{M}} \subseteq R\circ \leq$ where $\circ$ denotes relational composition.

Lemma 10.2.2. *If R is a binary relation on $At\mathbf{B}_M$ such that (R) holds, then $f^R(b) = f(b)$ for all $b \in \underline{M}$.*

Proof. By (R), we have

$$\begin{aligned} f^R(b) &= \bigvee\{x \in At\mathbf{B}_M : \text{there exists } y \in At\mathbf{B}_M \text{ such that } y \leq b \text{ and } (x,y) \in R\} \\ &= \bigvee\{x \in At\mathbf{B}_M : x \leq f(b)\} \\ &= f(b) \quad (\text{since } f(b) \in B_M). \end{aligned}$$

□

The next lemma provides a converse.

Lemma 10.2.3. *If f' is an operator on $\mathbf{B}_M$ that extends f, then $R^{f'}$ satisfies (R).*

Proof. Let $b \in \underline{M}$, and $x \in At\mathbf{B}_M$. Suppose $x \leq f(b) = f'(b)$. But then $x \leq f'(\bigvee\{y \in At\mathbf{B}_M : y \leq b\}) = \bigvee\{f'(y) : y \in At\mathbf{B}_M \text{ and } y \leq b\}$, and hence $x \leq f'(y_0)$ for some atom $y_0 \leq b$. But $x \leq f'(y_0)$ means $(x, y_0) \in R^{f'}$. This proves the implication from left to right in (R). Conversely, suppose $y_0 \leq b$ and $(x, y_0) \in R^{f'}$. Then $x \leq f'(y_0) \leq f'(b) = f(b)$. □

Thus, we have established a one-to-one relationship between operators on $\mathbf{B}_M$ that extend f and binary relations on $At\mathbf{B}_M$ that satisfy condition (R). We now make the following definition.

Definition 10.2.4. *An* algebraic filtrator of $\mathbf{A}$ through $(M, \underline{M})$ *is a binary relation R on $At\mathbf{B}_M$ that satisfies (R). In that case, the modal algebra $\langle \mathbf{B}_M, f^R \rangle$, where f^R is defined by:*

$$f^R(b) = \bigvee\{x \in At\mathbf{B}_M : \text{there exists } y \in At\mathbf{B}_M \text{ such that } y \leq b \text{ and } (x, y) \in R\} \tag{10.3}$$

is called the algebraic filtration of $\mathbf{A}$ through $(M, \underline{M})$ with R.

An algebraic filtrator R is called rigid *if, in addition to (R), it also satisfies:*

$$\begin{aligned} &\text{For all } x, y \in At\mathbf{B}_M \text{ and for all } c, d \in A \text{ we have that} \\ &0 \neq c \leq x \text{ and } d \leq y \text{ and } c \leq f(d) \text{ implies } (x, y) \in R. \end{aligned} \tag{$R1$}$$

Observe that since M is finite, so is $\mathbf{B}_M$. Hence, $\mathbf{B}_M$ is indeed atomic, as was assumed in the definition. Moreover, the atoms of $\mathbf{B}_M$ are maximal non-0 meets of elements of M and their negations. That is, if $M = \{a_1, \ldots, a_n\}$ and a^0 and a^1 denote $\neg a$ and a, respectively, then $At\mathbf{B}_M = \{a_1^{h(1)} \wedge \cdots \wedge a_n^{h(n)} : h : \{1, \ldots, n\} \to \{0, 1\}\} - \{0\}$.

We will show that the additional rigidity condition ($R1$) is necessary if we want the algebraic filtrations to correspond exactly to the standard filtrations of models found in the literature. A detailed comparison between filtrations of algebras and of models is made in Section 10.4.

The discussion in this section can be summarized in the following theorem:

Theorem 10.2.5 (Algebraic Filtration Theorem)**.** *Suppose that $\mathbf{A} \not\models s = t$. Then there exist subsets $\underline{M} \subseteq M \subseteq A$ such that for any algebraic filtration*

$\langle \mathbf{B}_M, f^R \rangle$ *of* $\mathbf{A}$ *through* $(M, \underline{M})$ *with* R *we have* $\langle \mathbf{B}_M, f^R \rangle \not\models s = t$. *Moreover,* $|\mathbf{B}_M| \leq 2^{2^n}$, *where* n *is the number of subterms of* s *and* t. *Specifically, one can take* $M = \{v(u) : u \text{ a subterm of } s \text{ or } t\}$, *and* $\underline{M} = \{a \in S : f(a) \in M\}$, *where* v *is any assignment on* $\mathbf{A}$ *falsifying* $s = t$.

10.2.1 Extending operators on Boolean subalgebras

Recall from the opening discussion of this section that a conjunction of elements may be used to define an extension of a partial operator, as opposed to a disjunction of elements as was used in the above. We will now show that there is a natural connection between the two approaches and we will give an explicit translation between them.

As before, let $\mathbf{A} = \langle A, \vee, \wedge, \neg, 0, 1, f \rangle$ be a modal algebra and $\underline{M} \subseteq M \subseteq^{fin} A$ such that $f(a) \in M$ whenever $a \in \underline{M}$. Also, let $\mathbf{B}_M$ be the Boolean subalgebra of $\mathbf{A}$ generated by M. The following operation defined on B_M was used by McKinsey in [McK41] to prove the finite model properties for $S2$ and $S4$ (see also [MT44]):

$$f'(b) = \bigwedge \{a \in B_M : \text{there exists } c \in \underline{M} \text{ such that } a = f(c) \text{ and } b \leq c\}. \tag{10.4}$$

If we assume that $\underline{M}$ is closed under $\vee$, then it can be shown that f' distributes over finite joins in $\mathbf{B}_M$ and extends f. This assumption is not restrictive since we may first close M under $\vee$ in $\mathbf{B}_M$. If we do so, then for all $a, b \in \underline{M}$ we have that $f(a \vee b) = f(a) \vee f(b) \in M$ and therefore $a \vee b \in \underline{M}$. To ensure that $f'(0) = 0$ for operations defined in this way, we include 0 in the set $\underline{M}$ (and hence also in M). Then f' is an operator that extends f.

Another example of an operation defined on B_M in this way is:

$$f'(b) = \bigwedge \{a \in B_M : f(b) \leq a\}. \tag{10.5}$$

which can easily be shown to be an operator that extends f.

Throughout this section we assume that $0 \in \underline{M}$.

In general, for any binary relation Q on B_M, we may define an operation $f^Q : B_M \to B_M$ by:

$$f^Q(b) = \bigwedge \{a \in B_M : (a, b) \in Q\}. \tag{10.6}$$

The relation Q does not uniquely determine f^Q, that is, different relations may induce the same operation. This is illustrated by the following results.

Lemma 10.2.6. *Suppose Q and Q' are binary relations on B_M such that Q is the upward closure in the first co-ordinate of Q', i.e., $(a,b) \in Q$ if, and only if, $(c,b) \in Q'$ for some $c \leq a$. Then $f^{Q'} = f^Q$.*

Proof. Since $Q' \subseteq Q$, it is clear that $f^Q(b) \leq f^{Q'}(b)$ for each $b \in B_M$. To prove the inequality in the other direction, note that

$$\begin{aligned} f^Q(b) &= \bigwedge\{a \in B_M : (a,b) \in Q\} \\ &= \bigwedge\{a \in B_M : \text{ there exists } c \in B_M \text{ such that } a \geq c \text{ and } (c,b) \in Q'\}. \end{aligned}$$

Suppose $a \in \{a \in B_M : (a,b) \in Q\}$, i.e., there exists $c \in B_M$ such that $a \geq c$ and $(c,b) \in Q'$. Then $a \geq c \geq \bigwedge\{d \in B_M : (d,b) \in Q'\}$ since $c \in \{d \in B_M : (d,b) \in Q'\}$. Hence $a \geq f^{Q'}(b)$ and it follows that $f^Q(b) \geq f^{Q'}(b)$. □

Lemma 10.2.7. *Suppose Q and Q' are binary relations on B_M such that Q is the meet closure in the first co-ordinate of Q', i.e., $(a,b) \in Q$ if, and only if, $a = a_1 \wedge a_2$ where $(a_1,b) \in Q'$ and $(a_2,b) \in Q'$. Then $f^{Q'} = f^Q$.*

Proof. Observe that $f^Q(b) \leq f^{Q'}(b)$ for each $b \in B_M$ since $Q' \subseteq Q$. We now prove that the inequality in the other direction holds. Note that

$$\begin{aligned} f^Q(b) &= \bigwedge\{a \in B_M : (a,b) \in Q\} \\ &= \bigwedge\{a \in B_M : \text{ there exist } a_1, a_2 \in B_M \text{ such that} \\ &\qquad a = a_1 \wedge a_2 \text{ and } (a_1,b) \in Q' \text{ and } (a_2,b) \in Q'\}. \end{aligned}$$

Let $a \in \{a \in B_M : (a,b) \in Q\}$; then there exist $a_1, a_2 \in B_M$ such that $a = a_1 \wedge a_2$, $(a_1,b) \in Q'$ and $(a_2,b) \in Q'$. Then $a_1, a_2 \geq f^{Q'}(b)$ since $a_1, a_2 \in \{a \in B_M : (a,b) \in Q'\}$ and therefore $a = a_1 \wedge a_2 \geq f^{Q'}(b)$. Hence $f^Q(b) \geq f^{Q'}(b)$. □

A binary relation Q on B_M will be called *f^Q-maximal* if, whenever $Q \subseteq Q' \subseteq B_M \times B_M$ such that $f^Q = f^{Q'}$, we have that $Q' = Q$.

Observe that, if $Q \subseteq B_M \times B_M$ is meet closed in its first co-ordinate, then $(f^Q(b), b) \in Q$ for all $b \in B_M$. This follows immediately from the definition of f^Q and the fact that B_M is finite.

Lemma 10.2.8. *A relation $Q \subseteq B_M \times B_M$ is f^Q-maximal if, and only if, it is meet and upward closed in its first co-ordinate.*

Proof. In light of Lemmas 10.2.6 and 10.2.7, it is clear than an f^Q-maximal relation is meet and upward closed in its first co-ordinate. Conversely, suppose $Q \subseteq B_M \times B_M$ is meet and upward closed in its first co-ordinate and there exists $Q' \subseteq B_M \times B_M$ such that $Q \subseteq Q'$ and $f^Q = f^{Q'}$. For $a, b \in B_M$ such that $(a, b) \in Q'$, we have that $a \geq f^{Q'}(b) = f^Q(b) = \bigwedge\{c \in B_M : (c, b) \in Q\}$. Then $(a, b) \in Q$ since $(f^Q(b), b) \in Q$ and Q is upward closed in the first co-ordinate. Thus $Q' \subseteq Q$ and hence $Q' = Q$. □

We are only interested in relations Q such that f^Q is an operator on $\mathbf{B}_M$ that extends f. In order to characterize such relations we define the following conditions.

- For all $b_1, b_2, a_1, a_2 \in B_M$, if $(a_1, b_1) \in Q$ and $(a_2, b_2) \in Q$, then $(a_1 \vee a_2, b_1 \vee b_2) \in Q$. ($Q1$)
- For all $b_1, b_2, a \in B_M$, if $(a, b_1) \in Q$ and $b_2 \leq b_1$, then $(a, b_2) \in Q$. ($Q2$)
- For all $a \in B_M$ and all $b \in \underline{M}$, if $(a, b) \in Q$, then $f(b) \leq a$. ($Q3$)
- For all $b \in \underline{M}$ we have $(f(b), b) \in Q$. ($Q4$)

Lemma 10.2.9. *Let $Q \subseteq B_M \times B_M$.*

(i) If ($Q1$) and ($Q2$) hold, then f^Q is distributes over finite joins in $\mathbf{B}_M$.

(ii) If ($Q3$) and ($Q4$) hold, then $f^Q(b) = f(b)$ for all $b \in \underline{M}$.

Proof. (i) Let $b_1, b_2 \in B_M$. Then,

$$f^Q(b_1 \vee b_2) = \bigwedge\{a \in B_M : (a, b_1 \vee b_2) \in Q\}.$$

We will show that the above is equal to:

$$\begin{aligned} & f^Q(b_1) \vee f^Q(b_2) \\ = & \bigwedge\{a_1 \in B_M : (a_1, b_1) \in Q\} \vee \bigwedge\{a_2 \in B_M : (a_2, b_2) \in Q\} \\ = & \bigwedge\{a_1 \vee a_2 : (a_1, b_1) \in Q \text{ and } (a_2, b_2) \in Q\}. \end{aligned}$$

If $(a, b_1 \vee b_2) \in Q$, then $(a, b_1) \in Q$ and $(a, b_2) \in Q$ by ($Q2$). Hence $f^Q(b_1 \vee b_2) \geq f^Q(b_1) \vee f^Q(b_2)$. If $(a_1, b_1) \in Q$ and $(a_2, b_2) \in Q$, then $(a_1 \vee a_2, b_1 \vee b_2) \in Q$ by ($Q1$). Therefore, $f^Q(b_1 \vee b_2) \leq f^Q(b_1) \vee f^Q(b_2)$ and the equality follows.

(ii) Let $b \in \underline{M}$. Then $(f(b), b) \in Q$ so $f(b) \in \{a \in B_M : (a, b) \in Q\}$ by $(Q4)$. Furthermore, if $a \in B_M$ such that $(a, b) \in Q$, then $f(b) \leq a$ by $(Q3)$. Hence $f^Q(b) = f(b)$. □

By our assumption that $0 \in \underline{M}$, it follows by the above lemma that if $(Q1) - (Q4)$ hold, then f^Q is an operator that extends f.

Lemma 10.2.10. *Let $Q \subseteq B_M \times B_M$ be f^Q-maximal. Then:*

(i) f^Q is distributes over finite joins on $\mathbf{B}_M$ if, and only if, $(Q1)$ and $(Q2)$ hold.

(ii) $f^Q(b) = f(b)$ for all $b \in \underline{M}$ if, and only if, $(Q3)$ and $(Q4)$ hold.

Proof. (i) The backward implication follows from Lemma 10.2.9. We prove the forward implication. Assume that f^Q distributes over finite joins. To show that $(Q1)$ holds, suppose that $(a_1, b_1) \in Q$ and $(a_2, b_2) \in Q$ for $b_1, b_2, a_1, a_2 \in B_M$. Then,

$$\begin{aligned} & f^Q(b_1 \vee b_2) \\ =& f^Q(b_1) \vee f^Q(b_2) \\ =& \bigwedge\{a_1 \in B_M : (a_1, b_1) \in Q\} \vee \bigwedge\{a_2 \in B : (a_2, b_2) \in Q\} \\ =& \bigwedge\{a_1 \vee a_2 \in B_M : (a_1, b_1) \in Q \text{ and } (a_2, b_2) \in Q\}. \end{aligned}$$

Thus, $a_1 \vee a_2 \geq f^Q(b_1 \vee b_2)$. Furthermore, since Q is finite and meet closed in the first co-ordinate we have that $(f^Q(b_1 \vee b_2), b_1 \vee b_2) \in Q$. Finally, the fact that Q is upward closed in the first co-ordinate implies that $(a_1 \vee a_2, b_1 \vee b_2) \in Q$.

To show that $(Q2)$ holds, suppose we have that $(a, b_1) \in Q$ and $b_2 \leq b_1$ for $a, b_1, b_2 \in B_M$. Then $f^Q(b_1) = f^Q(b_1 \vee b_2) = f^Q(b_1) \vee f^Q(b_2)$. Therefore $f^Q(b_2) \leq f^Q(b_1) \leq a$. As observed earlier, $(f^Q(b_2), b_2) \in Q$. It then follows that $(a, b_2) \in Q$, since Q is upward closed in the first co-ordinate.

(ii) As in part (i) we need only prove the forward implication. Suppose $f^Q(b) = f(b)$ for all $b \in \underline{M}$. To see that $(Q3)$ holds, suppose $b \in \underline{M}$ and $(a, b) \in Q$. Then, $a \geq f^Q(b) = f(b)$.

Finally, from $f^Q(b) = f(b)$ and the fact that Q is meet closed in its first co-ordinate, it follows that $(f(b), b) \in Q$, i.e., $(Q4)$ holds. □

Suppose $Q \subseteq B_M \times B_M$ satisfies (Q1) – (Q4). Then we can restrict Q to $At\mathbf{B}_M$ in the second co-ordinate since f^Q is an operator. Furthermore, we can also restrict Q to $Ca\mathbf{B}_M$, the set of co-atoms of $\mathbf{B}_M$, in the first co-ordinate since the upward closure of Q in the first co-ordinate does not change f^Q. Let P^Q denote the restriction of Q to $Ca\mathbf{B}_M \times At\mathbf{B}_N$, i.e., P^Q satisfies:

$$\begin{aligned}(c, y) \in P \text{ if, and only if, there exists } a \in B_M \\ \text{such that } a \leq c \text{ and } (a, y) \in Q.\end{aligned} \tag{10.7}$$

Observe that if Q and Q' are different relations on B_M that define the same operator, the relations on $Ca\mathbf{B}_M \times At\mathbf{B}_M$ obtained from Q and Q' by (10.7) are the same.

Starting with an arbitrary relation $P \subseteq Ca\mathbf{B}_M \times At\mathbf{B}_M$, we may define an operation f^P on B_M as follows: for $y \in At\mathbf{B}_M$,

$$f^P(y) = \bigwedge\{c \in Ca\mathbf{B}_M : (c, y) \in P\}$$

and extend the operation to arbitrary $b \in B_M$ by:

$$\begin{aligned}f^P(b) &= \bigvee\{f^P(y) : y \in At\mathbf{B}_M \text{ and } y \leq b\} \\ &= \bigwedge\{c \in Ca\mathbf{B}_M : \text{for all } y \in At\mathbf{B}_M, \text{ if } y \leq b, \text{ then } (c, y) \in P\}.\end{aligned} \tag{10.8}$$

We now show that the operation f^P as defined above is an operator.

Lemma 10.2.11. *For every $P \subseteq Ca\mathbf{B}_M \times At\mathbf{B}_M$, f^P distributes over finite joins in $\mathbf{B}_M$.*

Proof. Let $b_1, b_2 \in B_M$. Then

$$f^P(b_1 \vee b_2) = \bigwedge\{c \in Ca\mathbf{B}_M : \text{for all } y \in At\mathbf{B}_M, \text{ if } y \leq b_1 \vee b_2, \text{ then } (c, y) \in P\}.$$

On the other hand,

$$\begin{aligned}&f^P(b_1) \vee f^P(b_2) \\ &= \bigwedge\{c_1 \vee c_2 : \text{ for all } y \in At\mathbf{B}_M, \text{ if } y \leq b_1, \text{ then } (c_1, y) \in P, \text{ and,} \\ &\qquad \text{for all } y \in At\mathbf{B}_M, \text{ if } y \leq b_2, \text{ then } (c_2, y) \in P\} \\ &= \bigwedge\{c \in Ca\mathbf{B}_M : \text{for all } y \in At\mathbf{B}_M, \text{ if } y \leq b_1, \text{ then } (c, y) \in P, \text{ and,} \\ &\qquad \text{for all } y \in At\mathbf{B}_M, \text{ if } y \leq b_2, \text{ then } (c, y) \in P\}.\end{aligned}$$

The second equality follows from the fact that if c_1 and c_2 are distinct co-atoms, then $c_1 \vee c_2 = 1$.

Let $c \in Ca\mathbf{B}_M$ such that $(c, y') \in P$ whenever $y' \leq b_1$ for $y' \in At\mathbf{B}_M$ and $(c, y') \in P$ whenever $y' \leq b_2$ for $y' \in At\mathbf{B}_M$. Now let $y \in At\mathbf{B}_M$ such that $y \leq b_1 \vee b_2$. Since $y \in At\mathbf{B}_M$, it follows that $y \leq b_1$ or $y \leq b_2$. In either case, it follows that $(c, y) \in P$. Thus,

$$c \in \{c \in Ca\mathbf{B}_M : \text{for all } y \in At\mathbf{B}_M, \text{ if } y \leq b_1 \vee b_2, \text{ then } (c, y) \in P\},$$

and $f^P(b_1 \vee b_2) \leq f^P(b_1) \vee f^P(b_2)$.

The inclusion in the other direction (and hence the inequality in the other direction) is straightforward. □

To ensure that f^P extends f, we require the following condition:

$$\begin{aligned} &\text{For all } b \in \underline{M} \text{ and all } c \in Ca\mathbf{B}_M \text{ we have that } f(b) \leq c \\ &\text{if, and only if, for all } y \in At\mathbf{B}_M, y \leq b \text{ implies } (c, y) \in P. \end{aligned} \tag{P}$$

Lemma 10.2.12. *Let $P \subseteq Ca\mathbf{B}_M \times At\mathbf{B}_M$. Then $f^P(b) = f(b)$ for all $b \in \underline{M}$ if, and only if, P satisfies (P).*

Proof. Suppose that P satisfies (P). We must show that

$$f(b) = \bigwedge\{c \in Ca\mathbf{B}_M : \text{for all } y \in At\mathbf{B}_M, \text{ if } y \leq b, \text{ then } (c, y) \in P\}.$$

Since $f(b)$ is in the Boolean algebra $\mathbf{B}_M$, it is equal to the meet of all the co-atoms greater than it. Thus, it is sufficient to show that, for any $c \in Ca\mathbf{B}_M$,

$$f(b) \leq c \iff \text{for all } y \in At\mathbf{B}_M, \text{ if } y \leq b, \text{ then } (c, y) \in P,$$

which is just condition (P).

To prove the implication in the other direction, suppose that $f^P(b) = f(b)$ for all $b \in \underline{M}$ and let $b \in \underline{M}$ and $d \in Ca\mathbf{B}_M$. Then,

$$\begin{aligned} & f(b) \leq d \\ \iff & f^P(b) \leq d \\ \iff & d \in \{c \in Ca\mathbf{B}_M : \text{for all } y \in At\mathbf{A}_S, \text{ if } y \leq b, \text{ then } (c, y) \in P\} \\ \iff & y \leq b \text{ implies } (d, y) \in P \text{ for all } y \in At\mathbf{B}_M, \end{aligned}$$

as required. □

If $Q \subseteq A_S \times A_S$ is f^Q-maximal and satisfies (Q1) – (Q4), then $P^Q \subseteq Ca\,\mathbf{A}_S \times At\,\mathbf{A}_S$ is just the restriction of Q to $Ca\,\mathbf{A}_S \times At\,\mathbf{A}_S$. Thus, in this case the operations f^{P^Q} (as defined in (10.8)) and f^Q (as defined in (10.6)) are the same.

Lemma 10.2.13. *If $Q \subseteq B_M \times B_M$ is f^Q-maximal and satisfies (Q1) – (Q4), then $f^Q(b) = f^{P^Q}(b)$ for all $b \in B_M$. Thus, $f^{P^Q}(b) = f(b)$ for all $b \in \underline{M}$.*

Proof. It follows from Lemmas 10.2.9 (i) and 10.2.11 that f^{P^Q} and f^Q are operators. Hence, it will be sufficient show that $f^{P^Q}(y) = f^Q(y)$ for all $y \in At\mathbf{B}_M$, since that will imply that $f^{P^Q}(b) = f^Q(b)$ for all $b \in B_M$. Recall that for Boolean algebras we have that

$$\bigwedge S = \bigwedge\{c \in Ca\,\mathbf{B}_M : \text{ there exists } d \in S \text{ such that } d \leq c\}.$$

for any $S \subseteq B$. Thus, for $y \in At\mathbf{B}_M$,

$$\begin{aligned} f^Q(y) &= \bigwedge\{a \in B_M : (a, y) \in Q\} \\ &= \bigwedge\{c \in Ca\,\mathbf{B}_M : \text{there exists } a \in B \text{ such that } a \leq c \text{ and } (a, y) \in Q\} \\ &= \bigwedge\{c \in Ca\,\mathbf{B}_M : (c, y) \in P^Q\} \\ &= f^{P^Q}(y). \end{aligned}$$

Thus, f^{P^Q} and f^Q agree on B_M.

By Lemma 10.2.9 (ii) we have that $f^{P^Q}(b) = f^Q(b) = f(b)$ for all $b \in \underline{M}$. □

Then the following result is a consequence of Lemmas 10.2.13 and 10.2.12.

Corollary 10.2.14. *If $Q \subseteq B_M \times B_M$ is f^Q-maximal and satisfies (Q1)–(Q4), then P^Q satisfies (P).*

Note that if Q is f^Q-maximal, then Q can be recovered from P^Q by taking the meet closure in the first co-ordinate and the join-closure in the second co-ordinate. More generally, suppose that $P \subseteq Ca\,\mathbf{B}_M \times At\,\mathbf{B}_M$ satisfies (P). Define a relation $Q^P \subseteq B_M \times B_M$ by taking the meet and upward closure in the first coordinate and the join and downward closure in the second coordinate, i.e.,

$$\begin{aligned} &(a, b) \in Q^P \\ \iff &\text{for all } c \in Ca\,\mathbf{B}_M \text{ and all } y \in At\,\mathbf{B}_M, \text{ if } a \leq c \text{ and } y \leq b, \text{ then } (c, y) \in P. \end{aligned}$$

It should be clear that the restriction of Q^P to $Ca\mathbf{B}_M \times At\mathbf{B}_M$ is just P, i.e., $P^{Q^P} = P$. Furthermore, we also have that $Q^{P^Q} = Q$ for any $Q \subseteq B_M \times B_M$ that is f^Q-maximal and satisfies $(Q1) - (Q4)$.

Finally we wish to establish connections between relations $P \subseteq Ca\mathbf{B}_M \times At\mathbf{B}_M$ satisfying (P) and relations $R \subseteq At\mathbf{B}_M \times At\mathbf{B}_M$ satisfying (R). For $P \subseteq Ca\mathbf{B}_M \times At\mathbf{B}_M$, define $R^P \subseteq At\mathbf{B}_M \times At\mathbf{B}_M$ as the relation that satisfies:

$$(x, y) \in R \iff (\neg x, y) \notin P \tag{10.9}$$

Lemma 10.2.15. *If P satisfies (P), then R^P satisfies (R) and $f^{R^P} = f^P$.*

Proof. Let $b \in \underline{M}$ and $x \in At\mathbf{B}_M$. Then $\neg x \in Ca\mathbf{B}_M$ and therefore, by (P),

$$\begin{aligned}
& f(b) \leq \neg x \\
\iff & \text{for all } y \in At\mathbf{B}_M, \text{ if } y \leq b, \text{ then } (\neg x, y) \in P \\
\iff & \text{there does not exist } y \in At\mathbf{B}_M \text{ such that } y \leq b \text{ and } (\neg x, y) \notin P \\
\iff & \text{there does not exist } y \in At\mathbf{A}_S \text{ such that } y \leq b \text{ and } (x, y) \in R^P.
\end{aligned}$$

Equivalently,

$$f(b) \not\leq \neg x \iff \text{there exists } y \in At\mathbf{B}_M \text{ such that } y \leq b \text{ and } (x, y) \in R^P.$$

Since x is an atom, we have that $x \leq f(b)$ if, and only if, $f(b) \not\leq \neg x$, as required.

Thus, R^P satisfies (R), so f^{R^P} is an operator. Since f^P is also an operator, to show that $f^{R^P} = f^P$ we need only show that $f^{R^P}(y) = f^P(y)$ for all atoms y. Recall that in a finite Boolean algebra with atoms $x_1, \ldots, x_n$, if $a = x_1 \vee \cdots \vee x_k$, then $\neg a = x_{k+1} \vee \cdots \vee x_n$. Now, for $y \in At\mathbf{B}_M$,

$$\begin{aligned}
f^P(y) = & \bigwedge \{c : c \in Ca\,\mathbf{A}_S \text{ and } (c, y) \in P\} \\
= & \neg \bigvee \{\neg c : c \in Ca\,\mathbf{A}_S \text{ and } (c, y) \in P\} \\
= & \neg \bigvee \{x : x \in At\,\mathbf{A}_S \text{ and } (\neg x, y) \in P\} \\
= & \bigvee \{x : x \in At\,\mathbf{A}_S \text{ and } (\neg x, y) \notin P\} \\
= & f^{R^P}(y).
\end{aligned}$$

□

On the other hand, for $R \subseteq At\mathbf{B}_M \times At\mathbf{B}_M$, define $P^R \subseteq Ca\mathbf{B}_M \times At\mathbf{B}_M$ to be the relation that satisfies:

$$(c, y) \in P \iff (\neg c, y) \notin R \tag{10.10}$$

Lemma 10.2.16. *If R satisfies (R), then P^R satisfies (P) and $f^R = f^{P^R}$.*

Proof. Let $b \in \underline{S}$ and $c \in Ca\mathbf{B}_M$. Then $\neg c \in At\mathbf{B}_M$ and by (R),

$$\begin{aligned}
&\neg c \leq f(b)\\
\iff &\text{there exists } y \in At\mathbf{B}_M \text{ such that } y \leq b \text{ and } (\neg c, y) \in R\\
\iff &\text{it is not the case that } y \leq b \text{ implies } (\neg c, y) \notin R \text{ for all } y \in At\mathbf{B}_M\\
\iff &\text{it is not the case that } y \leq b \text{ implies } (c, y) \in P^R \text{ for all } y \in At\mathbf{B}_M.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
&\neg c \not\leq f(b)\\
&\text{for all } y \in At\mathbf{B}_M, \text{ if } y \leq b, \text{ then } (c, y) \in P^R.
\end{aligned}$$

Since c is a co-atom, we have $f(b) \leq c$ if, and only if, $\neg c \not\leq f(b)$, as required for P^R to satisfy (P).

Since f^R and f^{P^R} are operators, we need only show that $f^R(y) = f^{P^R}(y)$ for all atoms y to show that $f^R = f^{P^R}$. Indeed, for $y \in At\mathbf{B}_M$,

$$\begin{aligned}
f^{P^R}(y) =& \bigwedge\{c : c \in Ca\mathbf{B}_M \text{ and } (c, y) \in P^R\}\\
=& \bigwedge\{c : c \in Ca\mathbf{B}_M \text{ and } (\neg c, y) \notin R\}\\
=& \neg \bigvee\{\neg c : c \in Ca\mathbf{B}_M \text{ and } (\neg c, y) \notin R\}\\
=& \neg \bigvee\{x : x \in At\mathbf{B}_M \text{ and } (x, y) \notin R\}\\
=& \bigvee\{x : x \in At\mathbf{B}_M \text{ and } (x, y) \in R\}\\
=& f^R(y).
\end{aligned}$$

□

For $P \subseteq Ca\mathbf{B}_M \times At\mathbf{B}_M$ and $x, y \in At\mathbf{B}_M$, we have:

$$(x, y) \in R^{P^R} \iff (\neg x, y) \notin P^R \iff (\neg\neg x, y) \in R \iff (x, y) \in R.$$

For $R \subseteq At\mathbf{B}_M \times At\mathbf{B}_M$, $c \in Ca\mathbf{B}_M$ and $y \in At\mathbf{B}_M$, we have:

$$(c, y) \in P^{R^P} \iff (\neg c, y) \notin R^P \iff (\neg\neg c, y) \in P \iff (c, y) \in P.$$

Thus there is a one-to-one correspondence between relations $R \subseteq At\mathbf{B}_M \times At\mathbf{B}_M$ satisfying (R), relations $P \subseteq Ca\mathbf{B}_M \times At\mathbf{B}_M$ satisfying (P) and relations

$Q \subseteq B_M \times B_M$ that are f^Q-maximal and satisfy (Q1) – (Q4). Indeed, for $R \subseteq At\mathbf{B}_M \times At\mathbf{B}_M$ satisfying (R) and $a, b \in B_M$,

$$\begin{aligned} &(a,b) \in Q^{P^R} \\ \iff &\text{for all } c \in Ca\mathbf{B}_M \text{ and all } y \in At\mathbf{B}_M, \text{ if } a \leq c \text{ and } y \leq b, \text{ then } (\neg c, y) \notin R. \end{aligned}$$

Conversely, for $Q \subseteq B_M \times B_M$ that is f^Q-maximal and satisfies (Q1) – (Q4), and $x, y \in At\mathbf{B}_M$,

$$(x,y) \in R^{P^Q} \iff (\neg x, y) \notin Q.$$

In addition, for relations P, Q, R that are related as above, the operators f^P, f^Q and f^R are the same.

In summary then: if Q is any binary relation on B_M that satisfies (Q1) – (Q4), define P as in (10.7) and R by: $(x,y) \in R$ if, and only if, $(\neg x, y) \notin P$, which is equivalent to:

$$(x,y) \in R \text{ if, and only if, for all } a \in B_M, \text{ if } (a,y) \in Q, \text{ then } x \leq a. \quad (10.11)$$

Then R is the unique algebraic filtrator of $\mathbf{A}$ through $(M, \underline{M})$ such that $f^R = f^Q$. Conversely, starting with an algebraic filtrator R of $\mathbf{A}$ through $(M, \underline{M})$, there is no unique relation $Q \subseteq B_M \times B_M$ corresponding to R, however there is a unique relation $P \subseteq Ca\mathbf{B}_M \times At\mathbf{B}_M$, namely $(c,y) \in P$ if, and only if, $(\neg c, y) \notin R$, such that $f^P = f^R$. A suitable, but not unique, relation $Q \subseteq B_M \times B_M$ corresponding to P and R may be obtained from (10.8):

$$\begin{aligned} &(a,b) \in Q \text{ if, and only if,} \\ &a \in Ca\mathbf{B}_M \text{ and for all } y \in At\mathbf{B}_M, \text{ if } y \leq b, \text{ then } (c,y) \in P. \end{aligned}$$

10.3 Frames and Filtrations

We now recall the standard definition of a filtration for a (Kripke) model. The process of finding a filtration of a model can be described as follows: Given an equivalence relation of finite index on a model, construct a finite model by defining a relation on the equivalence classes that preserves truth for all formulas in a given finite subformula-closed set of formulas. This method relies heavily on the subformula-closed set of formulas and the valuation on the model. However, we do not have dual notions for these in the algebraic setting. For this reason we reformulate the notion of a filtration for a frame instead — introducing

set filtrations that operate on frames. Finally, we give a comparison between filtrations for frames and models.

Definition 10.3.1. *Let Σ be a finite, subformula-closed set of formulas (in the basic modal language) and $\mathcal{M} = \langle W, R, V\rangle$ a model. For $u, v \in W$, let $u \sim_\Sigma v$ if, and only if, for all $\varphi \in \Sigma$, $\mathcal{M}, u \Vdash \varphi$ if, and only if, $\mathcal{M}, v \Vdash \varphi$. Then $\sim_\Sigma$ is an equivalence relation on W. Let $[u]_\Sigma$ denote the equivalence class of u with respect to $\sim_\Sigma$, and let $W_\Sigma = \{[u]_\Sigma : u \in W\}$. A* filtration of $\mathcal{M} = \langle W, R, V\rangle$ through Σ *is then any model $\mathcal{M}' = \langle W', R', V'\rangle$ such that:*

- $W' = W_\Sigma$, $(F1)$
- $V'(p) = \{[u]_\Sigma : u \in V(p)\}$, $(F2)$
- $(u, v) \in R$ *implies* $([u]_\Sigma, [v]_\Sigma) \in R'$, $(F3)$
- *if* $([u]_\Sigma, [v]_\Sigma) \in R'$, *then* $\mathcal{M}, u \Vdash \Diamond\varphi$ *whenever* $\mathcal{M}, v \Vdash \varphi$ *and* $\Diamond\varphi \in \Sigma$. $(F4)$

The following theorem demonstrates how filtrations can be applied.

Theorem 10.3.2 (Filtration theorem)**.** *If $\mathcal{M}'$ is a filtration of $\mathcal{M}$ through a subformula closed set of formulas Σ, then for all $w \in W$ and all $\varphi \in \Sigma$, $\mathcal{M}, w \Vdash \varphi$ iff $\mathcal{M}', [w]_\Sigma \Vdash \varphi$. Moreover, $|W_\Sigma| \leq 2^{|\Sigma|}$.*

It should be clear that the method of filtration described above relies on a subformula-closed set Σ and a valuation V. In the sequel we generalise this method to frames.

Recall that the *complex algebra* of a frame $\mathfrak{F} = \langle W, R\rangle$ is the modal algebra $\mathfrak{F}^+ = \langle \mathcal{P}(W), \cap, \cup, -, \varnothing, W, f_R\rangle$, where f_R is defined on $\mathcal{P}(W)$ by:

$$f_R(X) = \{w \in W : \text{there exists } x \in X \text{ such that } (w, x) \in R\}.$$

The atoms of $\mathfrak{F}^+$ are exactly the singleton subsets $\{w\}$ of W.

In the following definition of a *set filtrator*, we replace Σ and V that were used in the filtration method for models, by two subsets $\underline{M} \subseteq M \subseteq \mathcal{P}(W)$ where $f_R(X) \in M$ for each $X \in \underline{M}$.

Definition 10.3.3. *Let $\mathfrak{F} = \langle W, R\rangle$ be a frame and $\underline{M} \subseteq M \subseteq^{fin} \mathcal{P}(W)$ such that $f_R(X) \in M$ whenever $X \in \underline{M}$. Let $\sim_M$ be the equivalence relation on W defined by:*

$$u \sim_M v \iff \text{for all } X \in M \text{ we have that } u \in X \text{ if, and only if, } v \in X.$$

For each $u \in W$, denote by $[u]_M$ the equivalence class of u with respect to $\sim_M$, and let $W_M = \{[u]_M : u \in W\}$.

A set filtrator of $\mathfrak{F}$ through $(M, \underline{M})$ *is a binary relation R' on W_M satisfying:*

For all $X \in \underline{M}$ and for all $u \in W$ we have that $u \in f_R(X)$ if, and only if, there exists $v \in W$ such that $v \in X$ and $([u]_M, [v]_M) \in R'$. (*SF*)

The frame $\langle W_M, R' \rangle$ is then called a set filtration of $\mathfrak{F}$ *through* $(M, \underline{M})$.

A rigid *set filtrator R' is a set filtrator that additionally satisfies, for all $u, v \in W$,*

$$(u, v) \in R \text{ implies } ([u]_S, [v]_S) \in R'. \tag{SF1}$$

Next we show that every filtration of a model corresponds to a set filtration.

Proposition 10.3.4. *Let $\mathcal{M}' = \langle W_\Sigma, R', V' \rangle$ be filtration of a model $\mathcal{M} = \langle W, R, V \rangle$ through a subformula closed set of formulas Σ. Let $M = \{V(\varphi) : \varphi \in \Sigma\}$ and $\underline{M} = \{V(\varphi) : \Diamond\varphi \in \Sigma\}$. Then $\langle W_M, R' \rangle$ is a rigid set filtration of $\langle W, R \rangle$ through $(M, \underline{M})$ and $\langle W_M, R' \rangle = \langle W_\Sigma, R' \rangle$.*

Proof. Recall that, for $\psi \in \Sigma$, $V(\psi) = \{x \in W : \mathcal{M}, x \Vdash \psi\}$. If $X \in \underline{M}$, then $X = V(\varphi)$ for some $\varphi \in \Sigma$ such that $\Diamond\varphi \in \Sigma$. Hence $V(\Diamond\varphi) = f_R(V(\varphi)) = f_R(X)$. Thus $f_R(X) \in M$ whenever $X \in \underline{M}$. Note that,

$$\begin{aligned} & X \in M \\ \iff & \text{there exist } \varphi \in \Sigma \text{ such that } X = V(\varphi) \\ \iff & \text{there exist } \varphi \in \Sigma \text{ such that } X = \{w \in W : \mathcal{M}, w \Vdash \varphi\}. \end{aligned}$$

Therefore, it should be clear that $u \sim_M v$ if, and only if, $u \sim_\Sigma v$. Thus $W_\Sigma = W_M$. By the above we may now rewrite (*F*4) as:

$$\text{If } ([u]_M, [v]_M) \in R', \text{ then } u \in f_R(X) \text{ whenever } v \in X \text{ and } X \in \underline{M},$$

or, equivalently,

$$\begin{aligned} & \text{for all } X \in \underline{M} \text{ there exists } v \in X \text{ such that} \\ & ([u]_M, [v]_M) \in R' \text{ implies that } u \in f_R(X). \end{aligned} \tag{10.12}$$

The above is just the forward implication in (*SF*).

To prove the backward implication, note that, if $X \in \underline{M}$ and $u \in f_R(X)$, then there exists $v \in X$ such that $(u, v) \in R$. By (*F*3) we then have that, if $X \in \underline{M}$ and $u \in f_R(X)$, then there exists $v \in X$ such that $([u]_M, [v]_M) \in R'$. Thus, (*SF*) holds. The rigidity condition (*SF*1) follows from (*F*3). □

The converse is not true. The following example illustrates that not every set filtration corresponds to a filtration of a model.

Example 10.3.5. *Let $\mathfrak{F} = \langle W, R\rangle$ be the frame with $W = \{a, b, c\}$ and $R = \{(a, b), (a, c), (b, a)\}$ depicted in Figure 10.1.*

Then $\{\{b, c\}\} = \underline{M} \subseteq M = \{\{a\}, \{b, c\}\}$ satisfy $f_R(X) \in M$ whenever $X \in \underline{M}$. The sets in M form the equivalence classes with respect to $\sim_M$, i.e., $W_M = \{[a]_M, [b]_M\}$. Now let $R' = \{([a]_M, [b]_M)\}$, then R' satisfies (SF) and $\langle W_M, R'\rangle$, also depicted in Figure 10.1, is a set filtration of $\mathfrak{F}$ through $(M, \underline{M})$.

Note, however, that since $(b, a) \in R$ but $([b]_M, [a]_M) \notin R'$, this frame cannot be obtained as the result of filtering a model based on $\mathfrak{F}$, since (F3) will never hold.

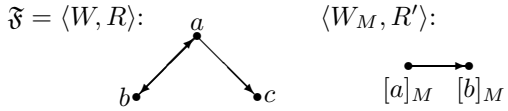


Fig. 10.1: The frame $\mathfrak{F} = \langle W, R\rangle$ and its set filtration $\langle W_M, R'\rangle$

We now show that even though the rigidity is not necessary to find a set filtrator of a frame, it is necesarry to obtain a filtration of a model. That is, as a partial converse for Lemma 10.3.4 we show that every rigid set filtration of a frame corresponds to a filtration of a model.

Proposition 10.3.6. *Let $\langle W, R\rangle$ be a frame and $\langle W_M, R'\rangle$ a rigid set filtration of $\langle W, R\rangle$ through $(M, \underline{M})$. Then there exists a subformula-closed set of formulas Σ and a valuation V such that $\langle W_\Sigma, R', V'\rangle$ is a filtration of $\langle W, R, V\rangle$ through Σ and $\langle W_\Sigma, R'\rangle = \langle W_M, R'\rangle$.*

Proof. Assume, without loss of generality, that $M = \{X_1, \ldots, X_n\}$ and $\underline{M} = \{X_1, \ldots, X_m\}$, $m \leq n$. By the definition of set filtration, for each $i = 1, \ldots, m$, there is some $j_i \in \{1, \ldots, n\}$, such that $f_R(X_i) = X_{j_i}$. Let $\Sigma = \{p_1, \ldots, p_n, \Diamond p_1,$

$\ldots, \Diamond p_m\}$ and $V(p_i) = X_i$ for $i = 1, \ldots, n$. Note that Σ is a subformula-closed set of formulas.

As before, $V(\Diamond p_i) = f_R(X_i) = X_{j_i} = V(p_{j_i})$ for all $i = 1, \ldots, m$. Then, for $\varphi \in \Sigma$,

$$\begin{aligned} & \mathcal{M}, u \Vdash \varphi \\ \iff & u \in V(\varphi) \\ \iff & u \in X_i \text{ for some } i = 1, \ldots, n. \end{aligned}$$

Thus, $u \sim_\Sigma v$ if, and only if, $u \sim_M v$. Thus $W_\Sigma = W_M$ and it follows that ($F1$) holds. Note that (SF) clearly implies (10.12), which is equivalent to ($F4$), while ($SF1$) clearly implies ($F3$). □

One may now wonder whether or not a weakened notion of filtration in modal logic would suffice. The answer is 'yes'. In the following definition we introduce the notion of a *weak filtration* which corresponds with the definition of a set filtrator.

Definition 10.3.7. *Let Σ be a finite, subformula-closed set of formulas and $\mathcal{M} = \langle W, R, V\rangle$ a model. A* weak filtration *of $\mathcal{M} = \langle W, R, V\rangle$ through Σ is any model $\mathcal{M}' = \langle W', R', V'\rangle$ such that:*

- $W' = W_\Sigma$, ($F1$)
- $V'(p) = \{[u]_\Sigma : u \in V(p)\}$, ($F2$)
- *for all $\Diamond\varphi \in \Sigma$ we have $\mathcal{M}, u \Vdash \Diamond\varphi \iff ([u]_\Sigma, [v]_\Sigma) \in R'$ for some $v \in W$ such that $\mathcal{M}, v \Vdash \varphi$.* ($WF$)

Suppose $\langle W, R\rangle$ is a frame and let $\langle W_M, R'\rangle$ be a set filtration of $\langle W, R\rangle$ through $(M, \underline{M})$. Then there exist a subformula-closed set of formulas Σ and a valuation V such that $\langle W_M, R', V'\rangle$ is a *weak filtration* of $\langle W, R, V\rangle$ through Σ and $\langle W_\Sigma, R'\rangle = \langle W_M, R'\rangle$. We can obtain Σ and V in exactly the same way as was described in the proof of Proposition 10.3.6.

Example 10.3.8. *The set filtration described in Example 10.3.5 now corresponds to a weak filtration of a model. Let $\Sigma = \{p_1, p_2, \Diamond p_1\}$, $V(p_1) = \{\{b, c\}\}$ and $V(p_2) = \{\{a\}\}$. Then it can easily be shown that R' satisfies (WF). Thus, $\langle W_M, R', V'\rangle$ is a weak filtration of $\langle W, R, V\rangle$ through Σ.*

Observe that, if a relation R' satisfies both (*F*3) and (*F*4), then it also satisfies (*WF*). On the other hand, from Examples 10.3.5 and 10.3.8 it follows that (*WF*) is a strictly weaker condition than conditions (*F*3) and (*F*4) together. However, the weakened condition (*WF*) suffices to prove a modified version of the Filtration Theorem.

Theorem 10.3.9 (Weak Filtration Theorem)**.** *If $\mathcal{M}'$ is a weak filtration of $\mathcal{M}$ through a subformula-closed set of formulas Σ, then for all $u \in W$ and all $\varphi \in \Sigma$, $\mathcal{M}, u \Vdash \varphi$ if, and only if, $\mathcal{M}', [u]_\Sigma \Vdash \varphi$. Moreover, $|W_\Sigma| \leq 2^{|\Sigma|}$.*

Proof. As in the standard proof of the Filtration Theorem (see, for example, [BdRV01]), we prove the claim by induction on the complexity of formulas in the basic modal logic. The base case as well as the Boolean cases remain unchanged since they do not involve the relation R'. We therefore only need to check the case where $\varphi := \Diamond\psi$.

Suppose $\mathcal{M}, u \Vdash \Diamond\psi$. By (*WF*) we have that $([u]_\Sigma, [v]_\Sigma) \in R'$ for some $v \in W$ such that $\mathcal{M}, v \Vdash \psi$. Since Σ is subformula-closed, we have that $\psi \in \Sigma$. Then, by the inductive hypothesis, $\mathcal{M}', [v]_\Sigma \Vdash \psi$. It then follows that $\mathcal{M}', [u]_\Sigma \Vdash \Diamond\psi$ since $([u]_\Sigma, [v]_\Sigma) \in R'$.

For the implication in the other direction, suppose $\mathcal{M}', [u]_\Sigma \Vdash \Diamond\psi$. Then there exists a $[v]_\Sigma$ such that $([u]_\Sigma, [v]_\Sigma) \in R'$ and $\mathcal{M}', [v]_\Sigma \Vdash \psi$. Then $\psi \in \Sigma$ since Σ is subformula-closed and by the inductive hypothesis we have that $\mathcal{M}, v \Vdash \psi$. Finally, since R' satisfies (*WF*), it follows from the backward implication of (*WF*) that $\mathcal{M}, u \Vdash \Diamond\psi$. □

10.4 Duality

Recall from Section 10.1 that every frame is (isomorphic to) the atom structure of some modal algebra and every modal algebra is (isomorphic to) a subalgebra of the complex algebra of some frame. Furthermore, operations on frames such as taking generated subframes, bounded morphic images, and disjoint unions correspond naturally with operations on algebras, namely, taking homomorphic images, subalgebras, and products. In this section we will establish a similar duality between set filtrations (operating on frames) and algebraic filtrations (operating on algebras). This correspondence motivates the use of the term "algebraic filtration" for the construction defined in Section 10.2.

10.4.1 Starting from frames

Let $\mathfrak{F} = \langle W, R\rangle$ be a frame with $\underline{M} \subseteq M \subseteq^{fin} \mathcal{P}(W)$. On the one hand, we can obtain the set filtration $\langle W_M, R'\rangle$ for some set filtrator R' (as in Definition 10.3.3). On the other hand, we may choose to consider the complex algebra of the frame, i.e., $\mathfrak{F}^+ = \langle \mathcal{P}(W), \cap, \cup, -, \varnothing, W, f_R\rangle$. Then (the same) R' is an algebraic filtrator of $\mathfrak{F}^+$ through $(M, \underline{M})$. Hence, $\langle \mathfrak{F}^+_M, f^{R'}_R\rangle$, where $\mathfrak{F}^+_M$ is the Boolean subalgebra of $\mathfrak{F}^+$ generated by M and $f^{R'}_R$ is as defined in (10.3), is the algebraic filtration of $\mathfrak{F}^+$ through $(M, \underline{M})$ with R'. In the following proposition we show that the atom structure of $\langle \mathfrak{F}^+_M, f^{R'}_R\rangle$, i.e., $\langle \mathfrak{F}^+_M, f^{R'}_R\rangle_+$ is precisely $\langle W_M, R'\rangle$. This is illustrated in the diagram in Figure 10.2.

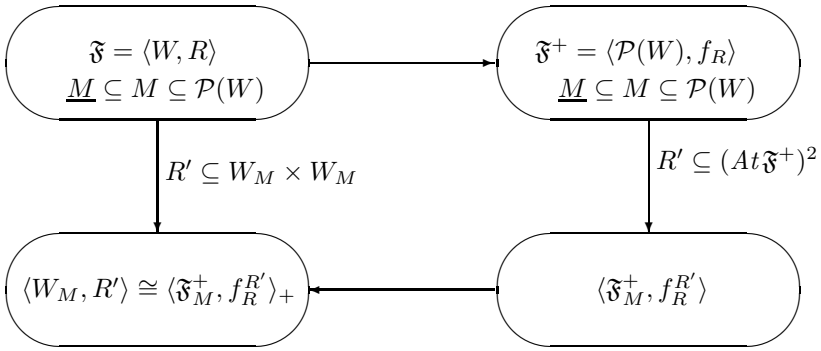


Fig. 10.2: Starting from frames.

Proposition 10.4.1. *Let $\mathfrak{F} = \langle W, R\rangle$ be a frame, let $\underline{M} \subseteq M \subseteq^{fin} \mathcal{P}(W)$ and let R' be a (rigid) set filtrator of $\mathfrak{F}$ through $(M, \underline{M})$. If $\mathfrak{F}^+_M$ is the Boolean subalgebra of $\mathfrak{F}^+$ generated by M and $\langle \mathfrak{F}^+_M, f^{R'}_R\rangle$ the algebraic filtration of $\mathfrak{F}^+$ through $(M, \underline{M})$ with R', then*

(i) R' is a (rigid) algebraic filtrator of $\mathfrak{F}^+$ through $(M, \underline{M})$,

(ii) $\langle W_M, R'\rangle \cong \langle \mathfrak{F}^+_M, f^{R'}_R\rangle_+$.

Proof. (i) Recall that $\mathfrak{F}^+ = \langle \mathcal{P}(W), \cap, \cup, -, \varnothing, W, f_R\rangle$ where f_R is given by

$$f_R(X) = \{w \in W : \text{there exists } x \in X \text{ such that } (w, x) \in R\}.$$

We first show that R' is an algebraic filtrator of $\mathfrak{F}^+$ through $(M, \underline{M})$. In order to do so we must show that (a) R' is an relation of $At\mathfrak{F}^+_M$; and (b) R' satisfies (R).

(a) Recall that since $\mathfrak{F}^+_M$ is the Boolean subalgebra generated by M, $At\mathfrak{F}^+_M$ consists of the maximal non-empty meets of elements of M. That is, if $M = \{X_1, \ldots, X_n\}$, then $At\mathfrak{F}^+_M$ consists of all sets of the form $X'_1 \cap X'_2 \cap \cdots \cap X'_n \neq \varnothing$, where X'_i is either X_i or $-X_i$. Now let $u, v \in W$. If $u, v \in X_1 \cap \cdots \cap X_k \cap (-X_{k+1}) \cap \cdots \cap (-X_n)$, then $u \in X_i$ if, and only if, $v \in X_i$ for all $X_i \in M$, $i = 1, \ldots, n$. Thus, the atoms of $\mathfrak{F}^+_M$ are just the equivalence classes $[u]_M$. Hence, $W_M = At\mathfrak{F}^+_M$ and R' is a relation on $W_M = At\mathfrak{F}^+_M$.

(b) Secondly, since R' is a set filtrator, it satisfies (SF). But (SF) is equivalent to:

For all $X \in \underline{M}$ *and all* $[u]_M \in W_M$ *we have that* $[u]_M \subseteq f_R(X)$
if, and only if,
there exists $[v]_M \in W_M$ *such that* $[v]_M \subseteq X$ *and* $([u]_S, [v]_S) \in R'$.

Clearly the above is equivalent to (R) for $\mathfrak{F}^+$ and it follows that R' satisfies (R).

Next we show that R' is a rigid algebraic filtrator when it is a rigid set filtrator. Suppose R' satisfies ($SF1$), i.e., for all $u, v \in W$, $(u, v) \in R$ implies that $([u]_M, [v]_M) \in R'$. We must show that ($R1$) holds. For $\mathfrak{F}^+$, ($R1$) can be written as: For all $[u]_M, [v]_M \in W_M$ and $c, d \subseteq W$,

$$\text{if } \varnothing \neq c \subseteq [u]_M, d \subseteq [v]_M \text{ and } c \subseteq f_R(d), \text{ then } ([u]_M, [v]_M) \in R'.$$

But if $\varnothing \neq c \subseteq [u]_M$, $d \subseteq [v]_M$ and $c \subseteq f_R(d)$, then every element in c has an R-successor in d. Since $c \neq \varnothing$, it then follows that there are elements $u' \in c \subseteq [u]_M$ and $v' \in d \subseteq [v]_M$ such that $(u', v') \in R$. Hence, $([u]_M, [v]_M) \in R'$ by ($SF1$) and R' is a rigid algebraic filtrator.

(ii) As noted in Section 10.2, the atom structure of $\langle \mathfrak{F}^+_M, f^{R'}_R \rangle$, i.e., $\langle \mathfrak{F}^+_M, f^{R'}_R \rangle_+$, is the structure $\langle At\mathfrak{F}^+_M, R' \rangle$. Since $At\mathfrak{F}^+_M = W_M$, the result follows.

□

10.4.2 Starting from algebras

Let $\mathbf{A} = \langle A, \vee, \wedge, \neg, 0, 1, f\rangle$ be a modal algebra with $\underline{M} \subseteq M \subseteq^{fin} A$. On the one hand, we may obtain the algebraic filtration $\langle \mathbf{B}_M, f^{R'}\rangle$ of $\mathbf{A}$ through $(M, \underline{M})$ with some algebraic filtrator R'. Recall that $\mathbf{B}_M$ is the Boolean subalgebra of $\mathbf{A}$ generated by M and $f^{R'}$ is defined by

$$f^R(b) = \bigvee\{x \in At\mathbf{B}_M : \text{there exists } y \in At\mathbf{B}_M \text{ such that } y \leq b \text{ and } (x, y) \in R\}.$$

On the other hand, we can consider the ultrafilter frame of $\mathbf{A}$, $\mathbf{A}_\bullet = \langle Uf\mathbf{A}, R'_f\rangle$ where $(u, v) \in R'_f$ if, and only if, $f(a) \in u$ whenever $a \in v$ for all $u, v \in Uf\mathbf{A}$. By the Jónsson-Tarski theorem, Theorem 10.1.8, $\mathbf{A}$ may be embedded into the complex algebra of its ultrafilter frame, i.e., $(\mathbf{A}_\bullet)^+$, via the map $\varpi : A \to Uf\mathbf{A}$ given by $\varpi(a) = \{u \in Uf\mathbf{A} : a \in u\}$.

Now let

$$M_\bullet = \{\varpi(a) : a \in M\}, \tag{10.13}$$

$$\underline{M}_\bullet = \{\varpi(a) : a \in \underline{M}\}, \tag{10.14}$$

$$R'_\bullet \subseteq \{\varpi(x) : x \in At\mathbf{A}\}^2 \text{ such that } (\varpi(x), \varpi(y)) \in R'_\bullet \iff (x, y) \in R'. \tag{10.15}$$

Then $\underline{M}_\bullet \subseteq M_\bullet \subseteq \mathcal{P}(Uf\mathbf{A})$. We will show that $R'_\bullet$ is a set filtrator of $\mathbf{A}_\bullet$ through $(\underline{M}_\bullet, M_\bullet)$ with resulting set filtration $\langle (Uf\mathbf{A})_{M_\bullet}, R'_\bullet\rangle$. Moreover, the complex algebra of $\langle (Uf\mathbf{A})_{M_\bullet}, R'_\bullet\rangle$, i.e., $\langle (Uf\mathbf{A})_{M_\bullet}, R'_\bullet\rangle^+$ is isomorphic to the algebraic filtration $\langle \mathbf{B}_M, f^{R'}\rangle$. This is illustrated in the diagram in Figure 10.3.

Lemma 10.4.2. *Let $\mathbf{A}$ be a modal algebra, R' an algebraic filtrator of $\mathbf{A}$ through $(M, \underline{M})$ and $\mathbf{A}_\bullet$ the ultrafilter frame of $\mathbf{A}$. If $M_\bullet$, $\underline{M}_\bullet$ and $R'_\bullet$ are defined as in (10.13), (10.14) and (10.15), respectively, then R' is a set filtrator of $\mathbf{A}_\bullet$ through $(M_\bullet, \underline{M}_\bullet)$. If R' is rigid, then so is $R'_\bullet$.*

Proof. We first show that $M_\bullet$, $\underline{M}_\bullet$ and $R'_\bullet$ are correctly defined for it to be possible that $R'_\bullet$ is a set filtrator of $\mathbf{A}_\bullet$ through $(M_\bullet, \underline{M}_\bullet)$. Suppose that $M = \{a_1, \ldots, a_n\}$ and $\underline{M} = \{a_1, \ldots, a_m\}$, $m \leq n$. Then $\underline{M}_\bullet \subseteq M_\bullet \subseteq \mathcal{P}(Uf\mathbf{A})$. Furthermore, the quotient structure $\langle (Uf\mathbf{A})_{M_\bullet}, R'_\bullet\rangle$ has universe:

$$(Uf\mathbf{A})_{M_\bullet} = \{\varpi(a_1)^{h(1)} \cap \cdots \cap \varpi(a_n)^{h(n)} : h : \{1, \ldots, n\} \to \{0, 1\}\} - \{\varnothing\},$$

where $\varpi(a_i)^0 = -\varpi(a_i)$ and $\varpi(a_i)^1 = \varpi(a_i)$. Since ϖ is an embedding of $\mathbf{A}$

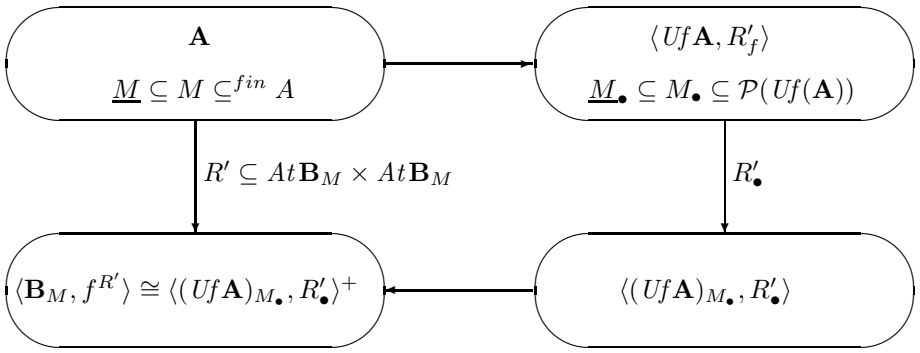


Fig. 10.3: Starting from algebras.

into $(\mathbf{A}_\bullet)^+$, we have that $\varpi(a)^i = \varpi(a^i)$, $i \in \{0,1\}$. Hence

$$\begin{aligned}\varpi(a_1)^{h(1)} \cap \cdots \cap \varpi(a_n)^{h(n)} &= \varpi(a_1^{h(1)}) \cap \cdots \cap \varpi(a_n^{h(n)}) \\ &= \varpi(a_1^{h(1)} \wedge \cdots \wedge a_n^{h(n)}).\end{aligned}$$

It follows that $(Uf\mathbf{A})_{M_\bullet} = \{\varpi(x) : x \in At\mathbf{B}_M\}$. Hence $R'_\bullet$ is a relation on the equivalence classes in $(Uf\mathbf{A})_{M_\bullet}$. Thus, $M_\bullet$, $\underline{M}_\bullet$ and $R'_\bullet$ are correctly defined.

Now, to show that $R'_\bullet$ is a set filtrator we must show that it satisfies (*SF*). In this context (*SF*) can be rewritten as: For all $a \in \underline{M}$ and all $u \in Uf\mathbf{A}$,

$$\begin{aligned}u \in & f_{R_f}(\varpi(a)) \text{ if, and only if, there exists } v \in Uf\mathbf{A} \\ & \text{such that } v \in \varpi(a) \text{ and } ([u]_{S_\bullet}, [v]_{S_\bullet}) \in R'_\bullet. \end{aligned} \tag{10.16}$$

Since ϖ is an embedding, we have that $f_{R_f}(\varpi(a)) = \varpi(f(a))$. Moreover $u \in \varpi(f(a))$ is equivalent to $[u]_{M_\bullet} \subseteq \varpi(f(a))$ and $v \in \varpi(a)$ is equivalent to $[v]_{M_\bullet} \subseteq \varpi(a)$ since an element of $(Uf\mathbf{A})_{M_\bullet}$ is the intersection of all members of $M_\bullet$ that contain it. Thus (10.16) is equivalent to: For all $a \in \underline{M}$ and all $u \in Uf\mathbf{A}$

$$\begin{aligned}[u]_{M_\bullet} \subseteq \varpi(f(a))) \text{ if, and only if, } & \text{ there exists } v \in Uf\mathbf{A} \\ \text{such that } [v]_{M_\bullet} \subseteq \varpi(a) \text{ and } & ([u]_{M_\bullet}, [v]_{M_\bullet}) \in R'_\bullet. \end{aligned} \tag{10.17}$$

Since the equivalence classes with respect to $M_\bullet$ are of the form $\varpi(x)$ for $x \in At\mathbf{B}_M$, we may quantify over $At\mathbf{B}_M$ instead of over $Uf\mathbf{A}$. Thus (10.17) is

equivalent to: For all $a \in \underline{M}$ and all $x \in At\mathbf{B}_M$

$$\varpi(x) \subseteq \varpi(f(a)) \text{ if, and only if, there exists } y \in At\mathbf{B}_M \text{ such that } \varpi(y) \subseteq \varpi(a) \text{ and } (\varpi(x), \varpi(y)) \in R'_\bullet. \tag{10.18}$$

Now, $\varpi(a) \subseteq \varpi(b)$ if, and only if, $a \leq b$, for all $a, b \in A$. From this and the definition of $R'_\bullet$, it follows that (10.18) is equivalent to: For all $a \in \underline{M}$ and $x \in At\mathbf{B}_M$,

$$x \leq f(a) \text{ if, and only if, there exists } y \in At\mathbf{B}_M \text{ such that } y \leq a \text{ and } (x, y) \in R'$$

which is just (*R*). Thus, R' satisfies (*SF*) if, and only if, it satisfies (*R*). But R' satisfies (*R*) by assumption. Hence, R' satisfies (*SF*).

Next we show that $R'_\bullet$ is a rigid set filtrator when R' is rigid. Suppose that R' is rigid, i.e., it satisfies (*R*1). We have to show that $R'_\bullet$ satisfies (*SF*1). Let $x, y \in At\mathbf{B}_M$, $u \in \varpi(x)$, $v \in \varpi(y)$ such that $(u, v) \in R'_f$. By the definition of R'_f (see Definition 10.1.5) it follows that $f(a) \in u$ for all $a \in v$. In particular, $f(y) \in u$. Then, $x \wedge f(y) \in u$, since u is a filter and $x \wedge f(y) \neq 0$ since u is proper. Moreover, since $0 \neq x \wedge f(y) \leq x$ and $y \leq y$ and $x \wedge f(y) \leq f(y)$, we have that $(x, y) \in R'$ by (*R*1). Hence, by definition, $(\varpi(x), \varpi(y)) \in R'_\bullet$. □

Define $\delta : At\mathbf{B}_M \to (Uf\mathbf{A})_{M_\bullet}$ to be the restriction of ϖ to $At\mathbf{B}_M$, i.e., for all $x \in At\mathbf{B}_M$,

$$\delta(x) = \{u \in Uf\mathbf{A} : x \in u\}.$$

Proposition 10.4.3. *Let* $\mathbf{A}$ *be a modal algebra and* $\langle \mathbf{B}_M, f^{R'} \rangle$ *the algebraic filtration of* $\mathbf{A}$ *through* $(M, \underline{M})$ *with some algebraic filtrator* R'. *Furthermore, let* $\mathbf{A}_\bullet$ *the ultrafilter frame of* $\mathbf{A}$ *and let* $M_\bullet$, $\underline{M}_\bullet$ *and* $R'_\bullet$ *be defined as in the equations (10.13), (10.14) and (10.15), respectively, such that* $\langle (Uf\mathbf{A})_{M_\bullet}, R'_\bullet \rangle$ *is the set filtration of* $\mathbf{A}_\bullet$ *through* $(\underline{M}_\bullet, M_\bullet)$ *with* $R'_\bullet$. *Then* δ *is an isomorphism between the atom structure* $\langle \mathbf{B}_M, f^{R'} \rangle_+$ *and* $\langle (Uf\mathbf{A})_{M_\bullet}, R'_\bullet \rangle$. *Consequently,* $\langle \mathbf{B}_M, f^{R'} \rangle$ *is isomorphic to the complex algebra* $\langle (Uf\mathbf{A})_{M_\bullet}, R'_\bullet \rangle^+$.

Proof. Recall from Definition 10.1.6 that $\langle \mathbf{B}_M, f^{R'} \rangle_+ = \langle At\mathbf{B}_M, R' \rangle$. Since $\delta(x) = \varpi(x)$ for all $x \in At\mathbf{B}_M$, it follows from the definition of $R'_\bullet$ that $(x, y) \in R'$ if, and only if, $(\delta(x), \delta(y)) \in R'_\bullet$. Thus, δ is a homomorphism. In the proof of Lemma 10.4.2 it was shown that $(Uf\mathbf{A})_{M_\bullet} = \{\varpi(x) : x \in At\mathbf{B}_M\} = \{\delta(x) :$

$x \in At\mathbf{B}_M\}$. Hence, δ is onto. To see that δ is one-to-one, observe that if $x, y \in At\mathbf{B}_B$ such that $x \neq y$, then $x \wedge y = 0$. But ultrafilters are proper, so no ultrafilter of $\mathbf{A}$ contains both x and y. Then $\delta(x) \cap \delta(y) = \varnothing$ and it follows that $\delta(x) \neq \delta(y)$

Since $\mathbf{B}_M$ is finite, $\langle \mathbf{B}_M, f^{R'} \rangle$ is isomorphic to $(\langle \mathbf{B}_M, f^{R'} \rangle_+)^+$ which in turn is isomorphic to $\langle (Uf\mathbf{A})_{M_\bullet}, R'_\bullet \rangle^+$. □

10.5 Analogues of model-theoretic filtrations

In this section we translate a number of well-known filtrations from the literature into their corresponding set filtrations and algebraic filtrations. In addition, we will use the correspondences of Section 10.2.1 to give equivalent descriptions of the algebraic filtrations. In particular, we will consider the largest, smallest, transitive and symmetric filtrations.

To start with we will make use of the correspondence developed in Section 10.3 between filtrations operating on models and set filtrations operating on frames to find a definition of the corresponding set filtration of each of the four filtrations we will consider. We will then make use of the duality developed in Section 10.4.1 to obtain the algebraic version of the filtration in terms of a relation R on the atoms. Finally, we also give the definition of the algebraic filtrator in terms of an arbitrary binary relation Q and use the correspondence developed in Section 10.2.1 to show that it is equivalent to the algebraic filtrator obtained through the duality.

We now consider some well-known (model-theoretic) filtrations used in modal logic. The 'largest' (respectively, 'smallest', 'transitive', 'symmetric') filtration of a model, as referred to in the literature, is a description of how a filtration of any given model with respect to any given subformula closed set of formulas can be defined. We use the correspondence theory to give a description of how a set filtration of any given frame through an appropriate pair of sets $(M, \underline{M})$ with a set filtrator can be defined. Using the duality theory we give a description of how an algebraic filtration of any given modal algebra through an appropriate pair of sets $(M, \underline{M})$ with an algebraic filtrator can be defined.

We introduce the following notions to assist us with the translation of a set filtrator into an algebraic filtrator.

Definition 10.5.1. *An* augmented modal algebra (AMA for short) *is a struc-*

ture $\mathfrak{A} = \langle A, \vee, \wedge, \neg, 0, 1, f, \underline{M}, M, R' \rangle$ *where*

- $\mathbf{A} = \langle A, \vee, \wedge, \neg, 0, 1, f \rangle$ *is a modal algebra,*
- $\underline{M} \subseteq M \subseteq A$ *such that* $f(a) \in S$ *whenever* $a \in \underline{M}$*,*
- R' *is binary relation on the* $At\mathbf{B}_M$*.*

We let $\mathscr{L}$ be the first-order language of AMAs, but where only restricted quantification over $\underline{M}$, M, B_M, and $At\mathbf{B}_M$ is allowed. The following definition makes this precise:

Definition 10.5.2. *Let* $\mathscr{L}$ *be the first-order language with*

- *function symbols* $\wedge$ *and* $\vee$ *(binary),* $\neg$ *and* f *(unary),*
- *constant symbols* 0 *and* 1*,*
- *unary predicates symbols* $\underline{M}$*,* M*,* A_M*, and* AtB_M*, and*
- *binary predicate symbol* R'*.*

The usual Boolean connectives will be denoted by $\&, \sqcup, \sim$*, and* $\Rightarrow$ *to avoid confusion with the operations of the Boolean algebra. We will often write* $x \in M$ *instead of* $M(x)$*, and similarly for the other unary predicate symbols.*

The only quantification allowed in $\mathscr{L}$ *is bounded quantification over the extensions of the unary predicates, i.e., quantification of the form* $\forall x(x \in M \Rightarrow \varphi)$ *and* $\exists x(x \in M \text{ and } \varphi)$ *(abbreviated as usual as* $(\forall x \in M)\varphi$ *and* $(\exists x \in M)\varphi$ *and similarly for the other unary predicates).*

The language $\mathscr{L}$ is interpreted in AMAs in the obvious way. Notice that, even though the signature of AMAs does not explicitly accommodate the predicate symbols B_M and $At\mathbf{B}_M$, the interpretations of these are entirely determined by the interpretation of M.

We are now ready to consider the filtration constructions mentioned above. In addition to the notions of AMAs and the language $\mathscr{L}$ we will need some further technical results. We include these with the investigation of the largest filtration to make their motivation and significance clearer.

Throughout the following subsections, $\mathbf{A} = \langle A, \vee, \wedge, \neg, 0, 1, f \rangle$ will be a fixed modal algebra and $\underline{M} \subseteq M \subseteq^{fin} A$ such that $f(a) \in M$ whenever $a \in \underline{M}$.

10.5.1 The largest filtration

We first consider the largest (or coarsest) filtration (see, for example, [BdRV01]). Recall that the *largest filtration* of a model $\mathcal{M} = \langle W, R, V\rangle$ through a finite subformula-closed set of formulas Σ is given by

$$([u]_\Sigma, [v]_\Sigma) \in R^\ell \text{ if, and only if, for all } \Diamond\varphi \in \Sigma, \text{if } \mathcal{M}, v \Vdash \varphi, \text{ then } \mathcal{M}, u \Vdash \Diamond\varphi,$$

or, equivalently,

$$([u]_\Sigma, [v]_\Sigma) \in R^\ell \text{ if, and only if, for all } \Diamond\varphi \in \Sigma, \text{if } v \in V(\varphi), \text{ then } u \in V(\Diamond\varphi).$$

We now formulate an equivalent version in terms of a set filtrator acting on a frame. Thus, instead of $\mathcal{M}$ and Σ, we have a frame $\mathfrak{F} = \langle W, R\rangle$ and $\underline{M} \subseteq M \subseteq \mathcal{P}(W)$ where, for each $X \in \underline{M}$, $f_R(X) \in M$. From the correspondence developed in Section 10.3 it follows that $X \in \underline{M}$ corresponds to $V(\varphi)$, for some $\varphi \in \Sigma$ with $\Diamond\varphi \in \Sigma$, and that $f_R(X) = V(\Diamond\varphi)$. Thus, the *largest set filtrator* of $\mathfrak{F}$ through $(M, \underline{M})$ is given by

$$\begin{gathered}([u]_M, [v]_M) \in R^\ell \text{ if, and only if,}\\ \text{for all } X \in \underline{M}, \text{if } v \in X, \text{ then } u \in f_R(X).\end{gathered} \tag{10.19}$$

Then R^ℓ satisfies (*SF*) by Proposition 10.3.4.

Recall from Section 10.4 that R^ℓ may also be viewed as an algebraic filtrator through $(M, \underline{M})$ of the complex algebra $\mathfrak{F}^+$. In particular, if $\mathfrak{F}^+_M$ is the Boolean subalgebra of $\langle W, R\rangle^+$ generated by M, then the atoms of $\mathfrak{F}^+_M$ are just the equivalence classes $[u]_M$, where $u \in W$. Furthermore, for $v \in W$ and $X \in S$, we have that $v \in X$ if, and only if, $[v]_M \subseteq X$. Thus, (10.19) is equivalently to:

$$\begin{gathered}([u]_M, [v]_M) \in R^\ell \text{ if, and only if,}\\ \text{for all } X \in \underline{M}, \text{ if } [v]_M \subseteq X, \text{ then } [u]_M \subseteq f_R(X).\end{gathered}$$

Thus the duality theory gives us the following definition. In abuse of notation, we will use R^ℓ for different relations, but it should be clear from the context which relation we are referring to.

Definition 10.5.3. *The* largest algebraic filtrator of $\mathbf{A}$ through $(M, \underline{M})$, *denoted* R^ℓ, *is defined by, for all* $x, y \in At\mathbf{B}_M$,

$$(x, y) \in R^\ell \text{ if, and only if, for all } a \in \underline{M}, \text{ if } y \leq a, \text{ then } x \leq f(a).$$

The filtration of $\mathbf{A}$ *through* $(M, \underline{M})$ *with* R^ℓ*, namely* $\langle \mathbf{B}_M, f^{R^\ell} \rangle$*, is called the* largest algebraic filtration of $\mathbf{A}$ through $(M, \underline{M})$.

Recall that (*R*) is the following condition:

For all $b \in \underline{M}$ and all $x \in At\mathbf{B}_M$ we have $x \leq f(b)$ if, and only if,
there exists $y \in At\mathbf{B}_M$ such that $y \leq b$ and $(x, y) \in R$.

Remark 10.5.4. *We must now confirm that* R^ℓ *is an algebraic filtrator, i.e., that* R^ℓ *satisfies* (*R*)*. This can be done by a direct computation. However, we would like to show that whenever we translate the definition of a set filtrator in the above way, we get the definition of an algebraic filtrator. As a result we avoid tedious computations for each future translation.*

If $\mathbf{A}$ is the complex algebra of some frame, then the fact that R^ℓ is an algebraic filtrator follows directly from Proposition 10.4.1 and the fact that R^ℓ satisfies (*SF*). However, we need to prove that this is the case for arbitrary modal algebras $\mathbf{A}$. To do so we will make use of AMAs and $\mathscr{L}$.

In [JT51] it was shown that every modal algebra $\mathbf{A}$ is (isomorphic to) a subalgebra of a complete and atomic modal algebra $\mathbf{A}^\sigma$, called its canonical extension. See Chapter 6 for more on the canonical extension. The canonical extension of an AMA $\mathfrak{A} = \langle A, \vee, \wedge, \neg, 0, 1, f, \underline{M}, M, R' \rangle$ is the AMA $\mathfrak{A}^\sigma = \langle A^\sigma, \vee, \wedge, \neg, 0, 1, f, \underline{M}, M, R' \rangle$ where $\langle A^\sigma, \vee, \wedge, \neg, 0, 1, f \rangle$ is the canonical extension $\mathbf{A}^\sigma$ of $\mathbf{A}$, and $\underline{M}$, M, and R' are unchanged. This definition makes sense since $\mathbf{B}_M$ is finite and therefore isomorphic to its canonical extention, and in fact we may identify the two, i.e., $\mathbf{B}_M = \mathbf{B}_M^\sigma$.

The following lemma can be established by a straightforward induction, using the fact that the bounded quantification of $\mathscr{L}$ restricts all considerations to the substructure $\mathfrak{A}$ of $\mathfrak{A}^\sigma$.

Lemma 10.5.5. *For any* AMA $\mathfrak{A}$ *and any* $\mathscr{L}$*-sentence* φ*, it holds that* $\mathfrak{A} \models \varphi$ *if, and only if,* $\mathfrak{A}^\sigma \models \varphi$.

Futhermore, from [JT51] we know that the complete and atomic modal algebras are, up to isomorphism, the complex algebras of Kripke frames. Let CAMA be the class of all AMAs with complete and atomic modal algebra reducts. We now show that if we can define a relation with an $\mathscr{L}$-formula, then it will be an algebraic filtrator on all modal algebras whenever it is an algebraic filtrator on complex algebras.

Observe that (R) can be rewritten as follows:

$$(\forall b \in \underline{M})(\forall x \in At B_M)(x \leq f(b) \Leftrightarrow (\exists y \in At B_M)(y \leq b \,\&\, Rxy)).$$

It should be clear from the above that (R) is an $\mathscr{L}$-sentence.

Proposition 10.5.6. *Let ψ an $\mathscr{L}$-sentence, such that $\psi \models_{\mathrm{CAMA}} (R)$. Then $\psi \models_{\mathrm{AMA}} (R)$.*

Proof. Let $\mathfrak{A} \in$ AMA, and suppose $\mathfrak{A} \models \psi$. By the forward implication of Lemma 10.5.5 we have that $\mathfrak{A}^\sigma \models \psi$, and hence, by assumption, $\mathfrak{A}^\sigma \models (R)$. Since (R) is an $\mathscr{L}$-sentence, it follows that $\mathfrak{A} \models (R)$ by the backward implication of Lemma 10.5.5. □

We can now rewrite the condition in Definition 10.5.3 as follows:

$$(\forall x, y \in At B_M)(R'xy \Leftrightarrow (\forall a \in \underline{M})(y \leq a \Rightarrow x \leq f(a))) \tag{10.20}$$

Clearly the above is an $\mathscr{L}$-sentence. Then, if ψ is the $\mathscr{L}$-sentence (10.20), then R^ℓ satisfies (R) by Proposition 10.5.6. That is, we have accomplished what we set out to do in Remark 10.5.4.

The set filtrator R^ℓ as defined in (10.19) is rigid (this is immediate from the definition and (*SF*1)). Then it follows that the algebraic filtrator R^ℓ is also rigid by Proposition 10.4.1.

Lemma 10.5.7. *The largest binary relation on $At\mathbf{B}_M$ satisfying (R) is R^ℓ.*

Proof. Let $R \subseteq At\mathbf{B}_M \times At\mathbf{B}_M$ such that R satisfies (R) and suppose $(x, y) \in R$. If $y \leq a$, then $x \leq f(a)$ by (R). Thus $(x, y) \in R^\ell$ and it follows that $R \subseteq R^\ell$. □

Observe that R^ℓ assigns the largest value (in terms of the ordering on $\mathbf{B}_M$) to $f^R(b)$ when compared to all the binary relations on $At\mathbf{B}_M$ satisfying (R) (or then, algebraic filtrators of $\mathbf{A}$ through $(M, \underline{M})$). This follows from the fact that R^ℓ is the largest binary relation on $At\mathbf{B}_M$ (set theoretically) to satisfy (R), Lemma 10.5.7, and from the definition of f^{R^ℓ} — recall that for $b \in B_M$,

$$f^{R^\ell}(b) = \bigvee\{x \in At\mathbf{B}_M : \text{there exists } y \in At\mathbf{B}_M \text{ such that } y \leq b \text{ and } (x, y) \in R^\ell\}.$$

Next we use the correspondence developed in Section 10.2.1 to show that the operator f^{R^ℓ} coincides with the operator given in (10.4) used in [McK41] to prove finite model properties for $S2$ and $S4$. Let $Q^\ell \subseteq B_M \times B_M$ be defined by:

$$(a, b) \in Q^\ell \text{ if, and only if, there exists } d \in \underline{M} \text{ such that } a = f(d) \text{ and } b \leq d,$$

so that

$$\begin{aligned} f^{Q^\ell}(b) &= \bigwedge\{a \in B_M : \text{there exists } d \in \underline{M} \text{ such that } a = f(d) \text{ and } b \leq d\} \\ &= \bigwedge\{f(d) : d \in \underline{M} \text{ and } b \leq d\}. \end{aligned}$$

It can easily be shown that Q^ℓ satisfies $(Q2) - (Q4)$; to ensure that Q^ℓ satisfies $(Q1)$ we require that $\underline{M}$ be closed under $\vee$ and $0 \in \underline{M}$. Thus, f^{Q^ℓ} is an operator that extends f. The algebraic filtrator on $\mathbf{A}$ through $(M, \underline{M})$ corresponding to Q^ℓ, as given by (10.11), is:

$$\begin{aligned} & (x, y) \in R \\ \iff & \text{for all } a \in B_M, \text{ if } (a, y) \in Q^\ell, \text{ then } x \leq a \\ \iff & \text{for all } a \in B_M, \text{ if there exists } d \in \underline{M} \text{ such that } a = f(d) \text{ and } y \leq d, \\ & \text{then } x \leq a \\ \iff & \text{for all } d \in \underline{M}, \text{ if } y \leq d, \text{ then } x \leq f(d)). \end{aligned}$$

But this is just R^ℓ. Hence we have the following.

Corollary 10.5.8. *If $\underline{M}$ is closed under $\vee$ and $0 \in \underline{M}$, then the modal algebra $\langle \mathbf{B}_M, f^{Q^\ell} \rangle$ is the largest algebraic filtration through $(M, \underline{M})$, i.e., $f^{Q^\ell} = f^{R^\ell}$.*

10.5.2 The smallest filtration

In this section we turn our attention to the smallest (or finest) filtration is of a model. If $\mathcal{M} = \langle W, R, V \rangle$ is a model and Σ a finite, subformula-closed set of formulas, then the *smallest filtration* (see, for example, [BdRV01]) of $\mathcal{M}$ is given by the relation:

$$\begin{aligned} ([u]_\Sigma, [v]_\Sigma) \in R^s \text{ if, and only if, there exists } u' \in [u]_\Sigma \\ \text{and there exists } v' \in [v]_\Sigma \text{ such that } (u,' v') \in R. \end{aligned}$$

As with the largest filtration, suppose we have a frame $\mathfrak{F} = \langle W, R \rangle$ and $\underline{M} \subseteq M \subseteq \mathcal{P}(W)$ where $f_R(X) \in M$ for each $X \in \underline{M}$, instead of $\mathcal{M}$ and Σ. Then, by the correspondence developed in Section 10.3, the *smallest set filtrator* of $\mathfrak{F}$ through $(M, \underline{M})$ is:

$$\begin{aligned} ([u]_M, [v]_M) \in R^s \text{ if, and only if, there exists } u' \in [u]_M \\ \text{and there exists } v' \in [v]_M \text{ such that } (u', v') \in R. \end{aligned} \tag{10.21}$$

The filtration $\langle W_M, R^s\rangle$, of $\mathfrak{F}$ obtained through $(M, \underline{M})$ with R^s is called the *smallest set filtration* of $\mathfrak{F}$ through $(M, \underline{M})$.

Recall that R^s may also be viewed as an algebraic filtrator through $(M, \underline{M})$ of the complex algebra $\mathfrak{F}^+$ (see Section 10.4). Then (10.21) equivalent to:

$$([u]_M, [v]_M) \in R^s \text{ if, and only if, } [u]_M \cap f_R([v]_M) \neq \varnothing.$$

On modal algebras in general this becomes Definition 10.5.9, below. Again we will abuse notation and use R^s to also denote the relation on $At\mathbf{B}_M$.

Definition 10.5.9. *The* smallest algebraic filtrator of $\mathbf{A}$ through $(M, \underline{M})$ *is defined by, for all* $x, y \in At\mathbf{B}_M$,

$$(x, y) \in R^s \text{ if, and only if, } x \wedge f(y) \neq 0,$$

The filtration of $\mathbf{A}$ *through* $(M, \underline{M})$ *with* R^s, *namely* $\langle \mathbf{B}_M, f^{R^s}\rangle$, *is called the* smallest algebraic filtration of $\mathbf{A}$ through $(M, \underline{M})$

Now let ψ be the $\mathscr{L}$-sentence:

$$(\forall x, y \in At\mathbf{B}_M)(R'xy \Leftrightarrow (x \wedge f(y) \neq 0)).$$

Then, since all CAMAs satisfying ψ also satisfy (R), it follows from Proposition 10.5.6 that R^s will always satisfy (R).

The relation R^s defined above, commonly known as the *smallest*, *finest*, or *least* filtration, is *not* the smallest relation, set theoretically speaking, which satisfies the property (R) nor does it produce the smallest value for $f^R(b)$ when compared to other binary relations on $At\mathbf{B}_M$ satisfying (R).

Example 10.5.10. *Consider the complex algebra* $\mathbf{A}$ *of a frame consisting of three points* u, v, *and* w, *with accessibility relation* $R = \{(u, v), (v, v), (w, w)\}$ *(depicted in Figure 10.4). Let* $x = \{u\}$, $y = \{v\}$ *and* $z = \{w\}$ *denote the atoms of* $\mathbf{A}$. *Then* $f(x) = 0$, $f(y) = x \vee y$, $f(z) = z$ *and* $f(x \vee y) = f(x) \vee f(y) = x \vee y$. *Now let* $M = A$; *then* $\mathbf{B}_M = \mathbf{A}$ *and* $\underline{M} = \{x \vee y\}$.

The three relations $R_1 = \{(x, y), (y, x)\}$, $R_2 = \{(x, x), (y, y)\}$, *and* $R_3 = \{(x, y), (y, y)\}$ *all satisfy condition* (R). *However, their intersection does not contain a relation that satisfies* (R). *Hence, in this instance, no least filtration exists. Furthermore, the smallest filtration* R^s *as defined above, would be* $R^s = \{(x, y), (y, y), (z, z)\}$ *which is strictly includes* R_3. *These relations are illustrated in Figure 10.4.*

Note that on frames, the smallest set filtrator R^s is (set theoretically) the smallest rigid *set filtrator. Recall from Section 10.3 that (model-theoretic) filtrations (Definition 10.3.1), in the usual sense of the term, correspond to rigid set filtrators. Then the smallest filtration is, per definition, the least relation satisfying the conditions of Definition 10.3.1. From Proposition 10.4.1 it follows that R^s is the smallest rigid algebraic filtrator.*

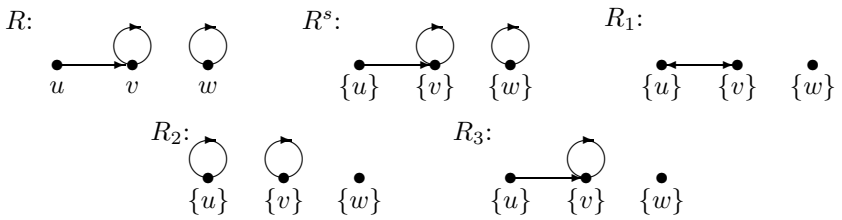


Fig. 10.4: In some instances no least filtration exists.

Next, we use the correspondence developed in Section 10.2.1 to show that the operator obtained from the smallest algebraic filtrator is equivalent to the operator obtained from the relation $Q^s \subseteq B_M \times B_M$ defined by:

$$(a, b) \in Q^s \text{ if, and only if, } f(b) \leq a.$$

It is easy to verify that Q^s satisfies $(Q1) - (Q4)$. Thus, f^{Q^s} defined by, for all $b \in B_M$

$$f^{Q^s}(b) = \bigwedge \{a \in B_M : f(b) \leq a\}$$

is an operator that extends f. To obtain the algebraic filtrator on $\mathbf{A}$ through $(M, \underline{M})$ corresponding to Q^s, we make use of the intermediate relation $P^s \subseteq Ca\mathbf{B}_M \times At\mathbf{B}_M$ defined as in (10.7):

$$\begin{aligned}
&(c, y) \in P^s \\
\iff &\text{there exists } a \in B_M \text{ such that } a \leq c \text{ and } (a, y) \in Q^s \\
\iff &\text{there exists } a \in B_M \text{ such that } f(y) \leq a \leq c \\
\iff &f(y) \leq c.
\end{aligned}$$

Thus, the algebraic filtrator R corresponding to Q^s is defined by:

$$\begin{aligned} & (x,y) \in R \\ \iff & (\neg x, y) \notin P^s \\ \iff & f(y) \not\leq \neg x \\ \iff & x \wedge f(y) \neq 0 \\ \iff & (x,y) \in R^s. \end{aligned}$$

Thus, we have the following result.

Corollary 10.5.11. *The modal algebra* $\langle \mathbf{B}_M, f^{Q^s} \rangle$ *is the smallest algebraic filtration through* $(M, \underline{M})$*, i.e.,* $f^{Q^s} = f^{R^s}$.

10.5.3 The transitive filtration

Filtrations are often designed to preserve specific frame properties of the models they are applied to. In this section we will consider filtrations designed to preserve *transitivity*. By Sahlqvist's theorem we know that a frame is transitive if, and only if, the modal formula $\Diamond\Diamond p \to \Diamond p$ is valid on the frame. We will say that a modal algebra is *transitive* if it validates $f(f(x)) \leq f(x)$. Let Tr denote the class of transitive modal algebras.

Given a class $\mathbb{K}$ of modal algebras, let AMA($\mathbb{K}$) (respectively, CAMA($\mathbb{K}$)) be the class of all AMAs (respcetively, CAMAs) $\mathbf{A}$ such that the modal algebra reduct of $\mathbf{A}$ is a (complete and atomic) member of $\mathbb{K}$. Then AMA(Tr) (respectively, CAMA(Tr)) are the (complete and atomic) augmented transitive modal algebras.

If $\mathcal{M} = \langle W, R, V \rangle$ is a model and Σ a finite, subformula-closed set of formulas, the *transitive filtration* of $\mathcal{M}$ (see, for example, [BdRV01]) is given by the relation:

$$\begin{aligned} ([u]_\Sigma, [v]_\Sigma) \in R^t \text{ if, and only if, for all } \Diamond\varphi \in \Sigma, \\ \text{if } v \in V(\varphi \vee \Diamond\varphi), \text{ then } u \in V(\Diamond\varphi). \end{aligned}$$

When applied to transitive models this is a filtration, and produces a transitive model. (We note that the resulting model will be transitive even if the original was not.)

If we now translate the above to sets and frames, we see that the *transitive set filtrator* of $\mathfrak{F}$ through $(M, \underline{M})$ is given by:

$$([u]_M,[v]_M) \in R^t \text{ if, and only if, for all } X \in \underline{M},$$
$$\text{if } v \in X \cup f_R(X), \text{ then } u \in f_R(X),$$

or, equivalently,

$$([u]_M, [v]_M) \in R^t \text{ if, and only if, for all } X \in \underline{M},$$
$$\text{if } [v]_M \subseteq X \cup f_R(X), \text{ then } [u]_M \subseteq f_R(X)).$$

The relation R^t satisfies (*SF*) by the correspondence developed in Section 10.3 and, moreover, preserves transitivity. From the duality theory of Section 10.4 we know that R^t also defines an algebraic filtrator on complex algebras of transitive frames. If we generalize to modal algebras in general, then we have the following.

Definition 10.5.12. *The* transitive algebraic filtrator *of* $\mathbf{A}$ *through* $(M, \underline{M})$ *is defined by, for all* $x, y \in At\mathbf{B}_M$,

$$(x, y) \in R^t \text{ if, and only if, for all } a \in \underline{M}, \text{ if } y \leq a \vee f(a), \text{ then } x \leq f(a).$$

The filtration of $\mathbf{A}$ *obtained through* $(M, \underline{M})$ *with* R^t, *namely* $\langle \mathbf{B}_M, f^{R^t} \rangle$, *is called the* transitive algebraic filtration *of* $\mathbf{A}$ *through* $(M, \underline{M})$.

As in the previous examples we need to show that R^t satisfies (*R*). In addition to that, we need to show that the transitive algebraic filtration $\langle \mathbf{B}_M, f^{R^t} \rangle$, as defined above, is again a transitive modal algebra. The following proposition now generalizes Proposition 10.5.6 for classes of AMAs.

Proposition 10.5.13. *Let* $\mathbb{K}$ *be a class of modal algebras closed under canonical extensions, and* ψ *an* $\mathscr{L}$*-sentence, such that*

(1) $\psi \models_{\mathrm{CAMA(K)}} (R)$, *and*

(2) $\left\langle B_M, f^{R'} \right\rangle \in \mathsf{K}$ *whenever* $\mathfrak{A} = \langle A, \vee, \wedge, \neg, 0, 1, f, \underline{M}, M, R' \rangle \in \mathrm{CAMA(K)}$ *and* $\mathfrak{A} \models \psi$.

Then

(3) $\psi \models_{\mathrm{AMA(K)}} (R)$, *and*

(4) $\left\langle B_M, f^{R'} \right\rangle \in \mathsf{K}$ *whenever* $\mathfrak{A} = \langle A, \vee, \wedge, \neg, 0, 1, f, \underline{M}, M, R' \rangle \in \mathrm{AMA}(\mathsf{K})$ *and* $\mathfrak{A} \models \psi$.

Proof. The proof of (3) is similar to the proof of Proposition 10.5.6

To prove (4), suppose $\mathfrak{A} = \langle A, \vee, \wedge, \neg, 0, 1, f, \underline{M}, M, R' \rangle \in \mathrm{AMA}(\mathsf{K})$ and $\mathfrak{A} \models \psi$. Then $\mathfrak{A}^\sigma \in \mathrm{CAMA}(\mathsf{K})$ by the assumption that $\mathbb{K}$ is closed under canonical extensions, and it therefore follows that $\mathfrak{A}^\sigma \models \psi$ by Lemma 10.5.5. Thus $\left\langle \mathbf{B}^\sigma_M, f^{R'} \right\rangle \in \mathbb{K}$. But then the claim follows, since $\left\langle \mathbf{B}^\sigma_M, f^{R'} \right\rangle = \left\langle \mathbf{B}_M, f^{R'} \right\rangle$. □

Proposition 10.5.14. *If* $\mathbf{A}$ *is transitive, then* R^t *defines an algebraic filtrator of* $\mathbf{A}$ *through* $(M, \underline{M})$. *Moreover* $\left\langle \mathbf{B}_M, f^{R^t} \right\rangle$ *is transitive.*

Proof. In order to be able to apply Proposition 10.5.13 the class Tr of transitive modal algebras must be closed under taking canonical extensions. Since the inequality $f(f(x)) \leq f(x)$ falls within the Sahlqvist class, it follows from the canonicity of Sahlqvist identities studied in [Jón94] that Tr is indeed closed under taking canonical extensions.

Furthermore, let ψ be the $\mathscr{L}$-sentence:

$$(\forall x, y \in At\, B_M)(R'xy \Leftrightarrow (\forall a \in \underline{M})(y \leq a \vee f(a) \Rightarrow x \leq f(a))).$$

From the discussion above we know that R^t is an algebraic filtration on all complex algebras of transitive frames. Thus we have that $\psi \models_{\mathrm{CAMA}(\mathsf{Tr})} (R)$. Moreover, the filtration $\langle \mathbf{B}_M, R^t \rangle$ is transitive whenever $\mathbf{A}$ is a transitive complex algebra. That is, both conditions of Proposition 10.5.13 are met. Thus, by Proposition 10.5.13 we have that R^t satisfies (R) and $\left\langle \mathbf{B}_M, f^{R^t} \right\rangle$ is transitive. □

If we consider the correspondence of Section 10.2.1 again, we show that the operator obtained from the transitive algebraic filtrator is equivalent to the operator obtained from the relation $Q^t \subseteq B_M \times B_M$ defined by:

$(a, b) \in Q^t$ if, and only if, there exists $d \in \underline{M}$ such that $a = f(d)$ and $b \leq d \vee a$.

It is easy to show that Q^t satisfies conditions $(Q2)$ and $(Q4)$. As with the largest filtration, Q satisfies condition $(Q1)$ if $\underline{M}$ be closed under $\vee$. If $\mathbf{A}$ is transitive and that $0 \in \underline{M}$, then Q^t satisfies $(Q3)$. Under these condition f^{Q^t}, defined by

$$f^{Q^t}(b) = \bigwedge \{a \in B_M : (a, b) \in Q^t\},$$

is an operator that extends f. The algebraic filtration of $\mathbf{A}$ through $(M, \underline{M})$ corresponding to Q^t is given by (10.11):

$$\begin{aligned}
&(x,y) \in R^{Q^t} \\
\iff &\text{for all } a \in B_M, \text{ if } (a,y) \in Q^t, \text{ then } x \leq a \\
\iff &\text{for all } a \in B_M, \text{ if there exists } d \in \underline{M} \text{ such that} \\
&a = f(d) \text{ and } y \leq d \vee a, \text{ then } x \leq a \\
\iff &\text{for all } d \in \underline{M}, \text{ if } y \leq d \vee f(d), \text{ then } x \leq f(d).
\end{aligned}$$

Hence, R^{Q^t} is just R^t and we have the following result.

Corollary 10.5.15. *If* $\mathbf{A}$ *is transitive,* $\underline{M}$ *is closed under* $\vee$ *and* $0 \in \underline{M}$*, then* $\langle \mathbf{B}_M, f^{Q^t} \rangle$ *is the transitive algebraic filtration through* $(M, \underline{M})$*, i.e.,* $f^{Q^t} = f^{R^t}$.

10.5.4 The symmetric filtration

It is well-known (again by Sahlqvist's Theorem) that a frame is *symmetric* if, and only if, the modal formula $p \to \Box\Diamond p$ is valid on it. A modal algebra is called *symmetric* if it validates $x \leq \neg f(\neg f(x))$. The class of all symmetric modal algebras will be denoted by Sym.

If $\mathcal{M} = \langle W, R, V \rangle$ is a model and Σ is a finite, subformula-closed set of formulas, then the *symmetric filtration* of $\mathcal{M}$ (see [LS77]) is given by the relation:

$$\begin{aligned}
&([u]_\Sigma, [v]_\Sigma) \in R^{sym} \text{ if, and only if, for all } \Diamond\varphi \in \Sigma \text{ we have that} \\
&\mathcal{M}, v \Vdash \varphi \text{ implies } \mathcal{M}, u \Vdash \Diamond\varphi \text{ and } \mathcal{M}, u \Vdash \varphi \text{ implies } \mathcal{M}, v \Vdash \Diamond\varphi.
\end{aligned}$$

When applied to symmetric models this is a filtration and produces symmetric models. (As in the transitive case, the resulting model will be symmetric even if the original model was not.)

Translating the above to sets and frames, we define the *symmetric set filtrator* of $\mathfrak{F}$ through $(M, \underline{M})$ by:

$$\begin{aligned}
&([u]_M, [v]_M) \in R^{sym} \text{ if, and only if, for all } X \in \underline{M} \text{ we have that} \\
&v \in X \text{ implies } u \in f_R(X) \text{ and } u \in X \text{ implies } v \in f_R(X).
\end{aligned}$$

As in the previous examples, we define the algebraic filtrator corresponding to the frame filtrator given above as follows.

Definition 10.5.16. *The* symmetric algebraic filtrator *of* $\mathbf{A}$ *through* $(M, \underline{M})$ *is defined by, for all* $x, y \in At\mathbf{B}_M$,

$$(x,y) \in R^{sym} \text{ if, and only if, for all } a \in \underline{M} \text{ we have that}$$
$$y \leq a \text{ implies } x \leq f(a) \text{ and } x \leq a \text{ implies } y \leq f(a).$$

The filtration of $\mathbf{A}$ *through* $(M, \underline{M})$ *with* R^t, *namely* $\langle \mathbf{B}_M, f^{R^t} \rangle$, *is called the* symmetric algebraic filtration *of* $\mathbf{A}$ *through* $(M, \underline{M})$.

Now let ψ be the $\mathscr{L}$-sentence:

$$(\forall x, y \in At\mathbf{B}_M)(\forall a \in M)((y \leq a \Rightarrow x \leq f(a)) \,\&\, (x \leq a \Rightarrow y \leq f(a)))$$

Then R^{sym} is defined by ψ. Furthermore, the class Sym of symmetric modal algebras is closed under canonical extensions. Thus, following a similar argument to the one used in the proof of Proposition 10.5.14, we can prove the following result.

Proposition 10.5.17. *If* $\mathbf{A}$ *is symmetric, then the relation* R^{sym} *is an algebraic filtrator, i.e.,* R^{sym} *satisfies* (R). *Moreover,* $\langle \mathbf{B}_M, f^{R^{sym}} \rangle$ *is symmetric.*

Finally, we obtain a relation on B_M that induces the same operator as R^{sym}, by applying the correspondence developed in Section 10.2.1. In this case, we derive a suitable relation Q^{sym} from R^{sym}. Let $P^{sym} \subseteq Ca\mathbf{B}_M \times At\mathbf{B}_M$ be the relation given by:

$$\begin{aligned}
&(c, y) \in P^{sym} \\
\iff &(\neg c, y) \notin R^{sym} \\
\iff &\text{there exists } d \in \underline{M} \text{ such that } y \leq d \text{ and } \neg c \not\leq f(d), \text{ or, } \neg c \leq d \text{ and } y \not\leq f(d) \\
\iff &\text{there exists } d \in \underline{M} \text{ such that } y \leq d \text{ and } f(d) \leq c, \text{ or, } \neg c \leq d \text{ and } f(d) \leq \neg y.
\end{aligned}$$

Then, by Lemma 10.2.16, the relation P^{sym} satisfies (P). Furthermore, $f^{P^{sym}}$ defined by:

$$f^{P^{sym}}(b) = \bigwedge \{c \in Ca\mathbf{B}_M : \text{for all } y \in At\mathbf{B}_M, \text{ if } y \leq b, \text{ then } (c, y) \in P^{sym}\}$$

is an operator on $\mathbf{B}_M$ that extends f.

The relation P^{sym} can now be extended to a relation on $B_M \times At\mathbf{B}_M$ by allowing any $a \in B_M$ in its first co-ordinate. To see why, observe that if $a \in B_M$

and $c \in Ca\mathbf{B}_M$ such that $c \geq a$, then $(a, y) \in P^{sym}$ implies that $(c, y) \in P^{sym}$. Therefore,

$$f^{P^{sym}}(b) = \bigwedge\{a \in B_M : \text{for all } y \in At\mathbf{B}_M, \text{ if } y \leq b, \text{ then } (a, y) \in P^{sym}\}.$$

We now define a relation Q^{sym} as follows:

$$(a, b) \in Q^{sym} \text{ if, and only if, for all } y \in At\mathbf{B}_M, \text{ if } y \leq b \text{ then } (a, y) \in P^{sym}$$

Then Q^{sym} satisfies (*Q*1), (*Q*2) and (*Q*4). If $\mathbf{A}$ is symmetric, then Q^{sym} satisfies (*Q*3) . Hence we have the following result.

Corollary 10.5.18. *If* $\mathbf{A}$ *is symmetric, then* $\langle \mathbf{B}_M, f^{Q^{sym}} \rangle$ *is the symmetric algebraic filtration through* $(M, \underline{M})$*, i.e.,* $f^{Q^{sym}} = f^{R^{sym}}$.

11. CONCLUSIONS AND FUTURE WORK

In Part I of this thesis we studied four different constructions for completing partially ordered sets. Generally these constructions produce different completions of the same poset. For different applications one may choose to employ different completions, depending on which properties one needs the completion to preserve. For example, the Doyle-pseudo ideal (respectively, filter) completion of a poset is the only completion (of those considered in this thesis) for which the extension of an operator (respectively, dual operator) is a complete operator (respectively, complete dual operator). Thus, if the distribution over joins is of importance in a particular problem, then one would choose to perform the ideal completion.

Unary residuation maps are preserved by both the MacNeille completion and completions with respect to polarizations. In order to decide which completion would be more advantageous, a thorough comparison of properties preserved by the respective completions still needs to be done. On the other hand, whether or not completions obtained via polarizations preserve binary residuation maps is still unknown. In particular, we would like to determine whether or not the σ-extension of a binary residuated map is residuated on the completion and, if it is, we would like to describe its residual. It is known that binary residuated maps are preserved by the MacNeille completion and it can therefore be used for problems requiring such preservation results.

The methods employed in Chapters 5 and 6 to obtain syntactical descriptions of properties preserved by the completions, may also be used to obtain preservation results for the other completions. That is, if appropriate approximation terms for the filter, ideal and prime filter completions are identified, then the approximation terms may be used to determine inequalities preserved by these completions.

Future work includes further development of the canonical FEP construction. We would like to answer questions like: Can we use the canonical FEP

construction to prove the FEP for classes of algebras for which the standard construction could not be used? Does the finite lattice obtained through the canonical FEP construction have denseness properties since it is related to completions obtained via polarizations?

Finally, a further question to consider regarding filtrations is: what properties of a modal algebra are preserved in the finite modal algebra constructed by a filtration? In particular, does the finite modal algebra belong to the same varieties as the original algebra?

APPENDICES

A. DETAILS OF SELECTED EXAMPLES

A.1 Examples from Chapter 4

Example A.1.1. *Let* $\mathbf{P}'$ *be the poset depicted in Figure 4.1 considered in Example 4.2.7. Then,*

(i) $\mathcal{F}^d = \{\{1\},\{2\},\{3\},\{1,2,4\},\{1,3,5\},\{2,3,7\},\{1,2,3,6\}\}$
and $\mathcal{I}^d = \{\{4\},\{5\},\{6\},\{7\},\{1,4,5,6\},\{2,4,6,7\},\{3,5,6,7\}\}$.

(ii) $\mathcal{F}^f = \mathcal{F}^d \cup \{\varnothing,\{1,2\},\{1,3\},\{2,3\},P'\}$
and $\mathcal{I}^f = \mathcal{F}^d \cup \{\varnothing,\{4,6\},\{5,6\},\{6,7\},P'\}$.

(iii) $\mathcal{F}^{dp} = \mathcal{F}^f \cup \{\{1,2,3,4,6\},\{1,2,3,5,6\},\{1,2,3,6,7\},\{1,2,3,4,5,6\},$
$\{1,2,3,4,6,7\},\{1,2,3,5,6,7\}\}$ *and* $\mathcal{I}^{dp} = \mathcal{I}^f$.

(iv) $\mathcal{F}^p = \mathcal{F}^{dp} \cup \{\{1,2,3\},\{1,2,3,4,5\},\{1,2,3,4,7\},\{1,2,3,5,7\},\{1,2,3,4,5,7\}$
and $\mathcal{I}^p = \mathcal{I}^{dp}$.

Example A.1.2. *Let* $\mathbf{P}' = \langle P', \leq \rangle$ *be the poset depicted in Figure 4.10 with* $h : P' \to P'$ *defined by* $h(1) = h(2) = 2$ *and* $h(3) = 3$ *as in Example 4.3.4. Then* h *is both an operator and a dual operator since no non-trivial joins or meets exist. Let* $S = \{1\}$ *and* $T = \{2\}$. *Note that* $S, T \in \mathcal{F}^f(\mathbf{P}')$ *and* $S, T \in \mathcal{I}^f(\mathbf{P}')$.

(i)

$$[h(S)\rangle_f = [\{2\}\rangle_f = \{2\}, \quad [h(T)\rangle_f = [\{2\}\rangle_f = \{2\},$$
$$[S \cup T\rangle_f = [\{1,2\}\rangle_f = \{1,2,3\}.$$

Then,

$$h_f^{\wedge}(S) \wedge^{(\mathbf{F}^f(\mathbf{P}'))^{\partial}} h_f^{\wedge}(T) = [\{2\}\rangle_f = \{2\},$$

but

$$h_f^{\wedge}(S \wedge^{(\mathbf{F}^f(\mathbf{P}'))^{\partial}} T) = \Big[h([S \cup T\rangle_f)\Big\rangle_f = [\{2,3\}\rangle_f = \{1,2,3\}.$$

Hence, $h_f^{\wedge}$ *is not a dual operator.*

(ii)

$$\langle h(S)]_f = \langle\{2\}]_f = \{2\}, \quad \langle h(T)]_f = \langle\{2\}]_f = \{2\},$$
$$\langle h(S \cap T)]_f = \langle h(\varnothing)]_f = \langle\varnothing]_f = \varnothing, \quad \langle S \cup T]_f = \langle\{1,2\}]_f = \{1,2,3\}.$$

Then,

$$h_f^\vee(S) \wedge^{\mathbf{I}^f(\mathbf{P}')} h_f^\vee(T) = \langle\{2\}]_f = \{2\},$$

but

$$h_f^\vee(S \wedge^{\mathbf{I}^f(\mathbf{P}')} T) = \langle h(S \cap T)]_f = \varnothing.$$

Hence, $h_f^\vee$ is not a dual operator.

Furthermore,

$$h_f^\vee(S) \vee^{\mathbf{I}^f(\mathbf{P}')} h_f^\vee(T) = \langle\{2\}]_f = \{2\},$$

but

$$h_f^\vee(S \vee^{\mathbf{I}^f(\mathbf{P}')} T) = \Big\langle h(\langle S \cup T]_f)\Big]_f = \langle\{2,3\}]_f = \{1,2,3\}.$$

Hence, $h_f^\vee$ is not an operator.

A.2 Examples from Chapter 6

Example A.2.1. *The reader is referred to Remark 6.1.7 for the context of this example.*

Let $\mathbf{P}' = \left\langle P', \leq^{\mathbf{P}'} \right\rangle$ be the poset depicted in Figure A.1 and let $\mathbf{Q}' = \left\langle Q', \vee^{\mathbf{Q}'}, \wedge^{\mathbf{Q}'} \right\rangle$ the complete lattice depicted in the same figure with associated lattice order $\leq^{\mathbf{Q}'}$. Let $\alpha : P' \to Q'$ be defined by $\alpha(1) = \underline{1}, \alpha(2) = \underline{2}, \alpha(3) = \underline{4}, \alpha(4) = \underline{6}$ and $\alpha(5) = \underline{7}$. Then $(\mathbf{Q}', \alpha)$ is a completion of $\mathbf{P}'$. The subposet of $\mathbf{Q}'$ that is order-isomorphic to $\mathbf{P}'$ is shaded in the depiction of $\mathbf{Q}'$ in Figure A.1.

The author of [Tun74] wanted to use Theorem 6.1.6 to argue that $(\mathbf{Q}', \alpha)$ is not a completion of $\mathbf{P}'$ that can be obtained from some polarization. See Chapter 6.1.1 for the construction referred to here.

Suppose $\mathbf{Q}'$ can be obtained form a polarization. Then there must exist $S, T \subseteq Q'$ that satisfy the conditions of Theorem 6.1.6. Recall that the first condition in Theorem 6.1.6 states that S is meet-dense in $\mathbf{Q}'$ and T is join-dense in $\mathbf{Q}'$. Since $\underline{5}$ is a completely meet-irreducible element in $\mathbf{Q}'$, it must be the case that $\underline{5} \in S$. Similarly, since $\underline{3}$ is a completely join-irreducible in $\mathbf{Q}'$, it must be the case that $\underline{3} \in T$. The author of [Tun74] now claimed that $\underline{5} \in S$ and $\underline{3} \in T$ implies that S and T must violate the second condition in Theorem 6.1.6.

Thus, reaching a contradiction. He makes this claim since there does not exist an element in the image of P' in between $\underline{3}$ and $\underline{5}$. However, $\underline{5} \leq^{\mathbf{Q}'} \underline{3}$ and not $\underline{5} \geq^{\mathbf{Q}'} \underline{3}$. Hence, S and T need not violate the second condition in Theorem 6.1.6.

In fact, $\mathbf{Q}'$ is isomorphic to $\mathbf{C}_f(\mathbf{P}')$:

$$\mathcal{F}^f(\mathbf{P}') = \{\varnothing, \{1\}, \{2\}, \{1,2\}, \{1,3\}, \{1,2,5\}, \{1,2,3,4\}, P'\},$$

$$\mathcal{I}^f(\mathbf{P}') = \{\varnothing, \{4\}, \{5\}, \{3,4\}, \{4,5\}, \{2,4,5\}, \{1,3,4,5\}, P'\}$$

and

$$\begin{aligned}\mathcal{C}_f(\mathbf{P}') = \{ &\{P'\}, \{\{1,2,3,4\}, P'\}, \{\{1,2,5\}, P'\}, \{\{1,3\}, \{1,2,3,4\}, P'\},\\ &\{\{1,2,5\}, \{1,2,3,4\}, P'\}, \{\{1,2\}, \{1,2,5\}, \{1,2,3,4\}, P'\},\\ &\mathcal{F}_f - \{\{1\}, \{1,3\}\}, \mathcal{F}_f - \{\{2\}\}, \mathcal{F}_f \ \}\end{aligned}$$

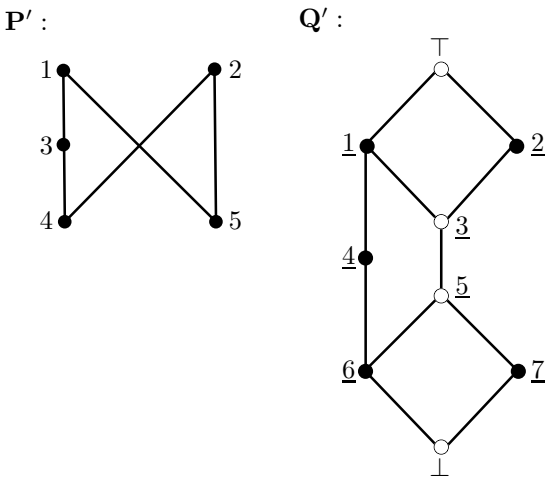


Fig. A.1: The poset $\mathbf{P}'$ and the complete lattice $\mathbf{Q}'$.

Example A.2.2. *In this example we give more details on the completions in Example 6.2.1. Let $\mathbf{P}'$ be the poset depicted in Figure 6.1. Then $\mathbf{P}'$ was also considered in Example 4.2.7. See Example A.1.1 for the set $\mathcal{F}^*$ and $\mathcal{I}^*$, $* \in$*

$\{p, dp, f, d\}$. *Then,*

$$\begin{aligned}\mathcal{C}_d = \{ &\varnothing, \{\{1,2,4\}\}, \{\{1,3,5\}\}, \{\{2,3,7\}\}, \{\{1,2,3,6\}\}, \{\{1,2,4\},\{1,2,3,6\}\},\\ &\{\{1,3,5\},\{1,2,3,6\}\}, \{\{2,3,7\},\{1,2,3,6\}\}, \{\{1\},\{1,2,4\},\{1,3,5\},\{1,2,3,6\}\},\\ &\{\{2\},\{1,2,4\},\{2,3,7\},\{1,2,3,6\}\}, \{\{3\},\{1,3,5\},\{2,3,7\},\{1,2,3,6\}\}, \mathcal{F}_d \}.\end{aligned}$$

$$\begin{aligned}\mathcal{C}_f = \{ &\{P'\}, \{\{1,2,4\},P'\}, \{\{1,3,5\},P'\}, \{\{2,3,7\},P'\}, \{\{1,2,3,6\},P'\},\\ &\{\{1,2,4\},\{1,2,3,6\},P'\}, \{\{1,3,5\},\{1,2,3,6\},P'\}, \{\{2,3,7\},\{1,2,3,6\},P'\},\\ &\{\{1,2\},\{1,2,4\},\{1,2,3,6\},P'\}, \{\{1,3\},\{1,3,5\},\{1,2,3,6\},P'\},\\ &\{\{2,3\},\{2,3,7\},\{1,2,3,6\},P'\},\\ &\{\{1\},\{1,2\},\{1,3\},\{1,2,4\},\{1,3,5\},\{1,2,3,6\},\\ &\{\{2\},\{1,2\},\{2,3\},\{1,2,4\},\{2,3,7\},\{1,2,3,6\},P'\},\\ &\{\{3\},\{1,3\},\{2,3\},\{1,3,5\},\{2,3,7\},\{1,2,3,6\},P'\},P'\},\mathcal{F}_f \}.\end{aligned}$$

$$\begin{aligned}\mathcal{C}_{dp} = \{ &\{P'\}, \{\{1,2,3,4,5,6\},P'\}, \{\{1,2,3,4,6,7\},P'\}, \{\{1,2,3,5,6,7\},P'\},\\ &\{\{1,2,3,4,6\},\{1,2,3,4,5,6\},\{1,2,3,4,6,7\},P'\},\\ &\{\{1,2,3,5,6\},\{1,2,3,4,5,6\},\{1,2,3,5,6,7\},P'\},\\ &\{\{1,2,3,6,7\},\{1,2,3,4,6,7\},\{1,2,3,5,6,7\},P'\},\\ &\{\{1,2,4\},\{1,2,3,4,6\},\{1,2,3,4,5,6\},\{1,2,3,4,6,7\},P'\},\\ &\{\{1,3,5\},\{1,2,3,5,6\},\{1,2,3,4,5,6\},\{1,2,3,5,6,7\},P'\}\\ &\{\{2,3,7\},\{1,2,3,6,7\},\{1,2,3,4,6,7\},\{1,2,3,5,6,7\},P'\}\\ &\{\{1,2,3,6\},\{1,2,3,4,6\},\{1,2,3,5,6\},\{1,2,3,6,7\},\{1,2,3,4,5,6\},\\ &\quad\{1,2,3,4,6,7\},\{1,2,3,5,6,7\},P'\},\\ &\{\{1,2,4\},\{1,2,3,6\},\{1,2,3,4,6\},\{1,2,3,5,6\},\{1,2,3,6,7\},\\ &\quad\{1,2,3,4,5,6\},\{1,2,3,4,6,7\},\{1,2,3,5,6,7\},P'\},\\ &\{\{1,3,5\},\{1,2,3,6\},\{1,2,3,4,6\},\{1,2,3,5,6\},\{1,2,3,6,7\},\\ &\quad\{1,2,3,4,5,6\},\{1,2,3,4,6,7\},\{1,2,3,5,6,7\},P'\},\\ &\{\{2,3,7\},\{1,2,3,6\},\{1,2,3,4,6\},\{1,2,3,5,6\},\{1,2,3,6,7\},\\ &\quad\{1,2,3,4,5,6\},\{1,2,3,4,6,7\},\{1,2,3,5,6,7\},P'\},\end{aligned}$$

$$\mathcal{F}_{dp} - \{\{1\},\{2\},\{3\},\{1,2\},\{2,3\},\{1,2,4\},\{2,3,7\}\},$$
$$\mathcal{F}_{dp} - \{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{1,2,4\},\{1,3,5\}\},$$
$$\mathcal{F}_{dp} - \{\{1\},\{2\},\{3\},\{1,3\},\{2,3\},\{1,3,5\},\{2,3,7\}\},$$
$$\mathcal{F}_{dp} - \{\{2\},\{3\},\{2,3\},\{2,3,7\}\}, \mathcal{F}_{dp} - \{\{1\},\{3\},\{1,3\},\{1,3,5\}\},$$
$$\mathcal{F}_{dp} - \{\{1\},\{2\},\{1,2\},\{1,2,4\}\}, \mathcal{F}_{dp}\}.$$

Example A.2.3. *Here we provide more details on the completions considered in Examples 6.2.2 and 6.3.3. Let* $\mathbf{P}'$ *be the poset depicted in Figures 6.2 and 6.5. Then,*

$$\mathcal{F}^d = \{\{1\},\{2\},\{1,2,3\},\{1,2,4\}\} \quad \text{and} \quad \mathcal{I}^d = \{\{3\},\{4\},\{1,3,4\},\{2,3,4\}\}.$$

For $* \in \{p, dp, f\}$,

$$\mathcal{F}^* = \mathcal{F}^d \cup \{\varnothing, \{1,2\}, P'\} \quad \text{and} \quad \mathcal{I}^* = \mathcal{I}^d \cup \{\varnothing, \{3,4\}, P'\}.$$

Furthermore,

$$\mathcal{C}_d = \{\varnothing, \{\{1,2,3\}\}, \{\{1,2,4\}\}, \{\{1,2,3\},\{1,2,4\}\}, \mathcal{F}_d - \{\{2\}\}, \mathcal{F}_d - \{\{1\}\}, \mathcal{F}_d\}$$

and, for $* \in \{p, dp, f\}$,

$$\mathcal{C}_* = \{\{P'\}, \{\{1,2,3\}, P'\}, \{\{1,2,4\}, P'\}, \{\{1,2,3\},\{1,2,4\}, P'\}$$
$$\{\{1,2\},\{1,2,3\},\{1,2,4\}, P'\}, \mathcal{F}_* - \{\{1\}\}, \mathcal{F}_* - \{\{2\}\}, \mathcal{F}_*\}$$

Clearly then $\bot_d = \varnothing$, $\underline{3} = \{\{1,2,3\},\{1,2,4\}\}$ *and* $\top_d = \mathcal{F}_d$ *are neither open nor closed in* $\mathbf{C}_d$.

On the other hand, there are elements that are not in $\alpha_*(P')$ *that are either closed or open in* $\mathbf{C}_*$ *for* $* \in \{p, dp, f\}$. *For example,* $\underline{3} = \{\{1,2\},\{1,2,3\},\{1,2,4\}, P'\} = \bigwedge \alpha_*(\{1,2\}) \in K_*$ *and* $\underline{4} = \{\{1,2,3\},\{1,2,4\}, P'\} = \bigvee \alpha_*(\{3,4\}) \in O_*$.

Example A.2.4. *Let* $\mathbf{P}' = \langle P', \leq \rangle$ *be the poset from Example 6.2.14 depicted in Figure 6.3. Then,* $\mathcal{F}^d(\mathbf{P}') = \{\{1\},\{2\}\} = \mathcal{I}^d(\mathbf{P}')$ *and, for* $* \in \{p, dp, f\}$, $\mathcal{F}^*(\mathbf{P}') = \{\varnothing, \{1\},\{2\}, P'\} = \mathcal{I}^*(\mathbf{P}')$. *We then have that,* $\mathcal{C}_d(\mathbf{P}') = \{\varnothing, \{\{1\}\}, \{\{2\}\}, \mathcal{F}_d(\mathbf{P}')\}$ *and* $\mathcal{C}_*(\mathbf{P}') = \{\{P'\}, \{\{1\}, P'\}, \{\{2\}, P'\}, \mathcal{F}_*(\mathbf{P}')\}$. *Clearly* $\mathbf{C}_d(\mathbf{P}')$ *is isomorphic to* $\mathbf{C}_*(\mathbf{P}')$ *and is the complete lattice depicted in Figure 6.3.*

Let $\mathbf{Q}' = \mathbf{P}' \times \mathbf{P}'$; *then* $\mathbf{Q}'$ *is also depicted in Figure 6.3. Label the elements of* Q' *with* a, b, c, d *form left to right. Then,*

$$\mathcal{F}^d(\mathbf{Q}') = \{\{a\},\{b\},\{c\},\{d\}\} = \mathcal{I}^d(\mathbf{Q}'),$$

$$\mathcal{F}^f(\mathbf{Q}') = \mathcal{F}^d(\mathbf{Q}') \cup \{\varnothing, \{a,b,c,d\}\} = \mathcal{I}^f(\mathbf{Q}')$$

and, for $* \in \{p, dp\}$,

$$\begin{aligned}&\mathcal{F}^*(\mathbf{Q}')\\ &= \mathcal{F}^f(\mathbf{Q}') \cup \{\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\},\{a,b,c\},\{a,b,d\},\{b,c,d\}\}\\ &= \mathcal{I}^*(\mathbf{Q}').\end{aligned}$$

We now have that,

$$\mathcal{C}_d(\mathbf{Q}') = \{\varnothing, \{\{a\}\},\{\{b\}\},\{\{c\}\},\{\{d\}\},\mathcal{F}_d(\mathbf{Q}')\},$$

$$\mathcal{C}_f(\mathbf{Q}') = \{Q', \{\{a\},Q'\},\{\{b\},Q'\},\{\{c\},Q'\},\{\{d\},Q'\},\mathcal{F}_f(\mathbf{Q}')\}$$

and $\mathcal{C}_*(\mathbf{Q}')$ *contains* 129 *elements for* $* \in \{p, dp\}$. *See Figure 6.3 for a depiction of* $\mathbf{C}_*(\mathbf{Q}')$, $* \in \{f, d\}$.

Example A.2.5. *Let* $\mathbf{P}'$ *be the* 3*-element anti-chain considered in Example 6.3.8 and depicted in Figure 6.6. Then,*

$$\mathcal{F}_d = \{\{1\},\{2\},\{3\}\} = \mathcal{I}^d \quad \textit{and} \quad \mathcal{F}^f = \mathcal{F}^d \cup \{\varnothing, P'\} = \mathcal{I}^f.$$

Furthermore,

$$\mathcal{C}_d = \{\varnothing, \{\{1\}\},\{\{2\}\},\{\{3\}\},\mathcal{F}_d\}$$

and

$$\mathcal{C}_f = \{\{P'\},\{\{1\},P'\},\{\{2\},P'\},\{\{3\},P'\},\mathcal{F}_f\}.$$

Clearly $\mathbf{C}_d$ *and* $\mathbf{C}_f$ *are isomorphic. Then* $\mathbf{C}_*$, $* \in \{f, d\}$, *is the complete lattice depicted in Figure 6.6.*

On the other hand,

$$\mathcal{F}^{dp} = \mathcal{F}^f \cup \{\{1,2\},\{1,3\},\{2,3\}\} = \mathcal{I}^{dp}$$

and

$$\begin{aligned}\mathcal{C}_{dp} = \{&\{P'\},\{\{1,2\},P'\},\{\{1,3\},P'\},\{\{2,3\},P'\},\\ &\{\{1,2\},\{1,3\},P'\},\{\{1,2\},\{2,3\},P'\},\{\{1,3\},\{2,3\},P'\},\\ &\{\{1\},\{1,2\},\{1,3\},P'\},\{\{2\},\{1,2\},\{2,3\},P'\},\\ &\{\{3\},\{1,3\},\{2,3\},P'\},\mathcal{F}_{dp} - \{\{1\},\{2\},\{3\}\},\\ &\mathcal{F}_{dp} - \{\{1\},\{2\}\},\mathcal{F}_{dp} - \{\{1\},\{3\}\},\mathcal{F}_{dp} - \{\{2\},\{3\}\},\\ &\mathcal{F}_{dp} - \{\{1\}\},\mathcal{F}_{dp} - \{\{2\}\},\mathcal{F}_{dp} - \{\{3\}\},\mathcal{F}_{dp}\ \}\end{aligned}$$

The complete lattice $\mathbf{C}_{dp}$ *is depicted in Figure A.2. Now label the element in* $\mathcal{C}_{dp}$ *with 'a' to 'r' from top to bottom and from left to right. Let* $f : P \to P$ *be the operator defined in Example 6.3.8 by* $f(1) = f(2) = 2$ *and* $f(3) = 3$*. Then* $f^{\sigma}_{dp} : \mathcal{C}_{dp} \to \mathcal{C}_{dp}$ *is defined as in Table A.1. It is easy to check that* f^{σ}_{dp} *is an operator.*

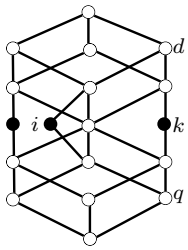


Fig. A.2: $\mathbf{C}_{dp}$ of the the 3-element anti-chain.

$f^{\sigma}_{dp}(a) = d$	$f^{\sigma}_{dp}(g) = d$	$f^{\sigma}_{dp}(m) = i$
$f^{\sigma}_{dp}(b) = i$	$f^{\sigma}_{dp}(h) = i$	$f^{\sigma}_{dp}(n) = q$
$f^{\sigma}_{dp}(c) = d$	$f^{\sigma}_{dp}(i) = i$	$f^{\sigma}_{dp}(o) = i$
$f^{\sigma}_{dp}(d) = d$	$f^{\sigma}_{dp}(j) = i$	$f^{\sigma}_{dp}(p) = q$
$f^{\sigma}_{dp}(e) = i$	$f^{\sigma}_{dp}(k) = k$	$f^{\sigma}_{dp}(q) = q$
$f^{\sigma}_{dp}(f) = i$	$f^{\sigma}_{dp}(l) = i$	$f^{\sigma}_{dp}(r) = q$

Tab. A.1: The definition of $f^{\sigma}_{dp} : \mathcal{C}_{dp} \to \mathcal{C}_{dp}$.

Example A.2.6. *Let* $\mathbf{P}'$ *be the poset depicted in Figure 6.9 considered in Example 6.3.30. Then, for* $* \in \{dp, f\}$*,*

$$\mathcal{F}^* = \{\varnothing, \{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}, \{1,2,3,4\}, \{1,2,3,5\}, P'\}$$

and

$$\mathcal{I}^* = \{\varnothing, \{6\}, \{4,6\}, \{5,6\}, \{4,5,6\}, \{2,4,5,6\}, \{3,4,5,6\}, P'\}.$$

Now,

$$\begin{aligned}\mathcal{C}_* = \{&\{P'\}, \{\{1,2,3,4\}, P'\}, \{\{1,2,3,5\}, P'\}, \{\{1,2,3,4\}, \{1,2,3,5\}, P'\},\\ &\{\{1,2,3\}, \{1,2,3,4\}, \{1,2,3,5\}, P'\}, \mathcal{F}_* - \{\{1\}, \{1,3\}\},\\ &\mathcal{F}_* - \{\{1\}, \{1,2\}\}, \mathcal{F}_*\}\end{aligned}$$

and $\mathbf{C}_*$ *is the complete lattice depicted in Figure 6.9.*

A.3 Examples from Chapter 7

Example A.3.1. *This example provides the full details of Example 7.1.3. Let* $\mathbf{P}'$ *be the poset depicted in Figure 7.1. Then,*

$$\mathcal{F}^* = \{\{1\}, \{1,2\}, \{1,3\}, \{1,4\}, \{\mathbf{1},\mathbf{2},\mathbf{3}\}, \{\mathbf{1},\mathbf{2},\mathbf{4}\}, \{\mathbf{1},\mathbf{3},\mathbf{4}\}, P'\}$$

and

$$\mathcal{I}^* = \{\varnothing, \{\mathbf{2}\}, \{\mathbf{3}\}, \{\mathbf{4}\}, P'\}$$

for $* \in \{p, dp\}$. *Furthermore,* $\mathscr{F}^*$ *and* $\mathscr{I}^*$ *contain the elements in* $\mathcal{F}^*$ *and* $\mathcal{I}^*$, *respectively, printed in bold. Then,*

$1 \not\leq 2,\ 1 \in \{1,3,4\}$ *and* $\{2\} \in \mathscr{I}^*$, $\quad 1 \not\leq 3,\ 3 \in \{1,2,4\}$ *and* $\{3\} \in \mathscr{I}^*$,
$1 \not\leq 4,\ 2 \in \{1,2,3\}$ *and* $\{4\} \in \mathscr{I}^*$, $\quad 2 \not\leq 3,\ 2 \in \{1,2,4\}$ *and* $\{3\} \in \mathscr{I}^*$,
$3 \not\leq 2,\ 3 \in \{1,3,4\}$ *and* $\{2\} \in \mathscr{I}^*$, $\quad 2 \not\leq 4,\ 2 \in \{1,2,3\}$ *and* $\{4\} \in \mathscr{I}^*$,
$4 \not\leq 2,\ 4 \in \{1,3,4\}$ *and* $\{2\} \in \mathscr{I}^*$, $\quad 3 \not\leq 4,\ 3 \in \{1,2,3\}$ *and* $\{4\} \in \mathscr{I}^*$,
$4 \not\leq 3,\ 4 \in \{1,2,4\}$ *and* $\{3\} \in \mathscr{I}^*$.

Therefore, $\mathbf{P}'$ *satisfies (7.1) and (7.2). Then,*

$$\begin{aligned}\mathcal{E}_* = \{&\varnothing, \{\{1,2,3\}\}, \{\{1,2,4\}\}, \{\{1,3,4\}\}, \{\{1,2,3\}, \{1,2,4\}\}, \{\{1,2,3\}, \{1,3,4\}\},\\ &\{\{1,2,4\}, \{1,3,4\}\}, \mathscr{F}^*\}\end{aligned}$$

and $\mathbf{E}_* = \langle \mathcal{E}_*, \cup, \cap \rangle$ *can be depicted as in Figure 7.1.*

On the other hand, $\mathcal{F}^f = \{\{1\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,2,3,4\}\}$ *and* $\mathcal{I}^f = \{\varnothing, \{2\}, \{3\}, \{4\}, \{1,2,3,4\}\}$. *Then,* $\mathscr{F}^f = \varnothing$ *and* $\mathscr{I}^f = \varnothing$ *and clearly a similar construction using prime Frink filters instead of prime Doyle-pseudo filters would not yield the required result.*

Example A.3.2. *Let $* \in \{p, dp\}$ and let $\mathbf{P}'$ be the poset depicted in Figure 7.2 considered in Example 7.3.4. Compare with Example A.2.3. Then,*

$$\mathcal{F}^* = \{\varnothing, \{\mathbf{1}\}, \{\mathbf{2}\}, \{\mathbf{1}, \mathbf{2}\}, \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}, \{\mathbf{1}, \mathbf{2}, \mathbf{4}\}, P'\}$$

$$\mathcal{I}^* = \{\varnothing, \{\mathbf{3}\}, \{\mathbf{4}\}, \{\mathbf{3}, \mathbf{4}\}, \{\mathbf{1}, \mathbf{3}, \mathbf{4}\}, \{\mathbf{2}, \mathbf{3}, \mathbf{4}\}, P'\}.$$

Moreover, $\mathscr{F}^*$ and $\mathscr{I}^*$ contain the elements in $\mathcal{F}^{dp}$ and $\mathcal{I}^{dp}$, respectively, printed in bold. Then,

$1 \not\leq 2,\ \{1\} \in \mathscr{F}^*$ *and* $\{2,3,4\} \in \mathscr{I}^*$, $\quad 1 \not\leq 3,\ \{1\} \in \mathscr{F}^*$ *and* $\{2,3,4\} \in \mathscr{I}^*$,
$1 \not\leq 4,\ \{1\} \in \mathscr{F}^*$ *and* $\{2,3,4\} \in \mathscr{I}^*$, $\quad 2 \not\leq 1,\ \{2\} \in \mathscr{F}^*$ *and* $\{1,3,4\} \in \mathscr{I}^*$,
$2 \not\leq 3,\ \{2\} \in \mathscr{F}^*$ *and* $\{1,3,4\} \in \mathscr{I}^*$, $\quad 2 \not\leq 4,\ \{2\} \in \mathscr{F}^*$ *and* $\{1,3,4\} \in \mathscr{I}^*$,
$3 \not\leq 4,\ \{1,2,3\} \in \mathscr{F}^*$ *and* $\{4\} \in \mathscr{I}^*$, $\quad 4 \not\leq 3,\ \{1,2,4\} \in \mathscr{F}^*$ *and* $\{3\} \in \mathscr{I}^*$.

Thus, $\mathbf{P}'$ satisfies (7.1) and (7.2). Therefore, we can construct a completely distributive complete lattice $\mathbf{E}_*$ by Theorem 7.1.2. Then,

$$\mathcal{E}_* = \{\varnothing, \{\{1,2,3\}\}, \{\{1,2,4\}\}, \{\{1,2,3\}, \{1,2,4\}\}, \{\{1,2\}, \{1,2,3\}, \{1,2,4\}\},$$
$$\{\{1\}, \{1,2\}, \{1,2,3\}, \{1,2,4\}\}, \{\{2\}, \{1,2\}, \{1,2,3\}, \{1,2,4\}\}, \mathscr{F}^*\}$$

and $\mathbf{E}_*$ is the complete lattice depicted in Figure 7.2.

Let $f : P' \to P'$ be the identity map. Then,

$$f^{\mathbf{E}_*}(\xi_*(3)) = \{F \in \mathscr{F}^* : [3) \subseteq F\} = \xi_*(3)$$

and

$$f^{\mathbf{E}_*}(\xi_*(4)) = \{F \in \mathscr{F}^* : [4) \subseteq F\} = \xi_*(4).$$

Therefore,

$$f^{\mathbf{E}_*}(\xi_*(3)) \cup f^{\mathbf{E}_*}(\xi_*(4)) = \xi_*(3) \cup \xi_*(4) = \{\{1,2,3\}, \{1,2,4\}\}.$$

On the other hand, $\bigcap\{\{1,2,3\}, \{1,2,4\}\} = \{1,2\}$. Thus,

$$\begin{aligned} f^{\mathbf{E}_*}(\xi_*(3) \cup \xi_*(4)) &= \{F \in \mathscr{F}^* : \{1,2\} \subseteq F\} \\ &= \{\{1,2\}, \{1,2,3\}, \{1,2,4\}\} \\ &\neq f^{\mathbf{E}_*}(\xi_*(3)) \cup f^{\mathbf{E}_*}(\xi_*(4)). \end{aligned}$$

Next let $h : P' \to P'$ be defined by $h(1) = 3$, $h(2) = 4$, $h(3) = 1$ and $h(4) = 2$. Then, as established in the proof of Lemma 7.3.3,

$$h^{\mathbf{E}_*}(\xi_*(3)) = \{F \in \mathscr{F}^* : h([3]) \subseteq F\} = \{F \in \mathscr{F}^* : 1 \in F\} = \xi_*(1)$$

and

$$h^{\mathbf{E}_*}(\xi_*(4)) = \{F \in \mathscr{F}^* : h((4]) \subseteq F\} = \{F \in \mathscr{F}^* : 2 \in F\} = \xi_*(2).$$

Therefore,

$$h^{\mathbf{E}_*}(\xi_*(3)) \cap h^{\mathbf{E}_*}(\xi_*(4)) = \xi_*(1) \cap \xi_*(2) = \{\{1,2\},\{1,2,3\},\{1,2,4\}\}.$$

On the other hand, since $\xi_*(3) \cap \xi_*(4) = \varnothing$*, we have that*

$$\begin{aligned} h^{\mathbf{E}_*}(\xi_*(3) \cap \xi_*(4)) &= \{F \in \mathscr{F}^* : \varnothing \subseteq F\} \\ &= \mathscr{F}^* \\ &\neq h^{\mathbf{E}_*}(\xi_*(3)) \cap h^{\mathbf{E}_*}(\xi_*(4)). \end{aligned}$$

B. IMPLEMENTATION OF ALGORITHM TO COMPUTE $\mathcal{C}_*$

Though the construction described in Chapter 6.1.1 of the complete lattice $\mathcal{C}$, from a polarization (S_1, S_2), is not particularly complicated, it may prove to be a time consuming process. This is essentially due to the fact that, for each set in $\mathcal{P}(S_1)$, we must check whether or not it is Galois closed. Hence, we have $2^{|S_1|}$ sets to consider. Even for a small poset, computing $\mathcal{C}$ may turn out to be a big undertaking. For instance, the 3-element anti-chain considered in Example A.2.5 has 7 non-empty Doyle-pseudo filters and ideals. Hence, there are 128 sets in $\mathcal{P}(\mathcal{F}_{dp})$ to test for Galois closure.

On the up side, the process can easily be implemented since it requires no human intervention. Since we relied upon numerous examples and counterexamples during our study of these completions, we found it necessary to implement the algorithm to generate our examples.

Our implementation was done in Java and we made use of a pre-defined set object. The input file contains the two sets that form the polarization with respect to which the complete lattice is constructed. The program then computes the set of Galois closed subsets of the first set, and generates a file containing this set as output. In this Appendix we provide the source code of our implementation in order for the reader to easily verify the correctness of the examples generated through this. We also provide a sample input and output file.

B.1 Source code

```
package CompletionWrtPolarization;

import java.util.*;
import java.nio.file.Files;
import java.nio.file.Path;
```

```
import java.nio.file.Paths;
import java.nio.charset.Charset;
import java.nio.charset.StandardCharsets;
import java.io.IOException;
/*
 * @author Wilmari Morton
 * Note: No effort has been made to optimise this code.
 * The object was to create a working algorithm.
 */
public class CompletionWrtPolarization {
    final static Charset ENCODING = StandardCharsets.UTF_8;
    /*
     * @param args the command line arguments
     * The first argument is the path to the input file
     * containing the polarization.
     * The second argument is the path of the output file.
     */
    public static void main(String[] args) {
        try {
            //Read the input file into a List<String>.
            List<String> fullFileContents =
            readInputFile(args[0]);

            //Extract Set 1 from the input file contents.
            Set<Object> set1 =
            GetSetsFromStringList(ExtractSet1(fullFileContents));

            //Extract Set 2 from the input file contents.
            Set<Object> set2 =
            GetSetsFromStringList(ExtractSet2(fullFileContents));

            //Perform the completion.
            Set<Object> resultSet =
            PerformCompletionWrtPolarization(set1, set2);
```

```
            /*
            * Write the resulting Galois closed sets
            * out to file.
            */
            WriteOutput(args[1], resultSet);
        } catch (IOException ex) {
            System.out.println(ex.getMessage());
        }
    }

    //Algorithm to compute the Galois closed elements.
    public static Set<Object>
PerformCompletionWrtPolarization
(Set<Object> set1, Set<Object> set2) {
        Set<Object> galoisClosedSets = new HashSet<Object>();
        Set powerSetOfSet1 = powerSet(set1);
        boolean nonEmptyIntersection;
        /*
         * Iterator through the power set of Set 1.
         * In our construction of C,
         * Set 1 is the set of *-filters.
         */
        for (Iterator powerSetOfSet1Iterator =
        powerSetOfSet1.iterator();
        powerSetOfSet1Iterator.hasNext();) {
            Set powerSetElement =
            (Set) powerSetOfSet1Iterator.next();
            Set<Object> nonEmptyIntersectionSet2 =
            new HashSet<Object>();
            Set<Object> nonEmptyIntersectionSet1 =
            new HashSet<Object>();
            /*
             * Find the elements of Set 2 that
             * have a non-empty intersection with
             * all the elements of the current set.
```

```
 */
for (Iterator set2Iterator =
set2.iterator(); set2Iterator.hasNext();) {
    Set set2Element =
    (Set) set2Iterator.next();
    nonEmptyIntersection = true;
    for (Iterator powerSetElementIterator =
    powerSetElement.iterator();
    powerSetElementIterator.hasNext();) {
        Set set1Element =
        (Set) powerSetElementIterator.next();
        boolean elementFound = false;
        for (Iterator set1ElementIterator =
        set1Element.iterator();
        set1ElementIterator.hasNext();) {
            String element =
            (String) set1ElementIterator.next();
            if (set2Element.contains(element)) {
                elementFound = true;
                break;
            }
        }
        if (!elementFound) {
            nonEmptyIntersection = false;
            break;
        }
    }
    if (nonEmptyIntersection) {
        nonEmptyIntersectionSet2.add(set2Element);
    }
}
for (Iterator set1Iterator =
set1.iterator(); set1Iterator.hasNext();) {
    Set set1Element = (Set) set1Iterator.next();
    nonEmptyIntersection = true;
```

```
                for (Iterator nonEmptyIntersectionSet2Iterator =
                nonEmptyIntersectionSet2.iterator();
                nonEmptyIntersectionSet2Iterator.hasNext();) {
                    Set set2Element =
                    (Set) nonEmptyIntersectionSet2Iterator.next();
                    boolean elementFound2 = false;
                    for (Iterator set2ElementIterator =
                    set2Element.iterator();
                    set2ElementIterator.hasNext();) {
                        String el =
                        (String) set2ElementIterator.next();
                        if (set1Element.contains(el)) {
                            elementFound2 = true;
                            break;
                        }
                    }
                    if (!elementFound2) {
                        nonEmptyIntersection = false;
                        break;
                    }
                }
                if (nonEmptyIntersection) {
                    nonEmptyIntersectionSet1.add(set1Element);
                }
            }
            if ((powerSetElement.size() ==
            nonEmptyIntersectionSet1.size())) {
                galoisClosedSets.add(powerSetElement);
            }
        }
        return galoisClosedSets;
    }

    /*
     * Returns the power set of a set.
```

```
 */
public static <T> Set<Set<T>> powerSet(Set<T> originalSet) {
    Set<Set<T>> sets = new HashSet<Set<T>>();
    if (originalSet.isEmpty()) {
        sets.add(new HashSet<T>());
        return sets;
    }
    List<T> list = new ArrayList<T>(originalSet);
    T head = list.get(0);
    Set<T> rest = new HashSet<T>(list.subList(1, list.size()));
    for (Set<T> set : powerSet(rest)) {
        Set<T> newSet = new HashSet<T>();
        newSet.add(head);
        newSet.addAll(set);
        sets.add(newSet);
        sets.add(set);
    }
    return sets;
}

/*
 * Reads in the file at the specified path
 * and returns a List<String>.
 * Each list item is a line in the file.
 */
public static List<String>
readInputFile(String aFileName) throws IOException {
    Path path = Paths.get(aFileName);
    return Files.readAllLines(path, ENCODING);
}

/*
 * Iterates through the List<String> returned from
 * reading the input file and returns a List<String>
 * which only contains the lines from the input file
```

```
 * pertaining to Set 1.
 */
public static List<String>
ExtractSet1(List<String> fullFileContents)
{
    ArrayList<String> returnList =
    new ArrayList<String>();
    Iterator<String> iter = fullFileContents.iterator();
    boolean done = false;
    boolean started = false;
    while (iter.hasNext()&&!done)
    {
        String currentLine = iter.next();
        if (currentLine.equalsIgnoreCase("--Set 1--"))
            started = true;
        if ((started) &&
        (currentLine.equalsIgnoreCase("--Set 2--")))
            done = true;
        if ((started) && (!done) &&
        (!currentLine.equalsIgnoreCase("--Set 1--")))
            returnList.add(currentLine);
    }
    return returnList;
}

/*
 * Iterates through the List<String> returned from
 * reading the input file and returns a List<String>
 * which only contains the lines from the input file
 * pertaining to Set 2.
 */
public static List<String>
ExtractSet2(List<String> fullFileContents)
{
    ArrayList<String> returnList = new ArrayList<String>();
```

```
    Iterator<String> iter = fullFileContents.iterator();
    boolean done = false;
    boolean started = false;
    while (iter.hasNext()&&!done)
    {
        String currentLine = iter.next();
        if (currentLine.equalsIgnoreCase("--Set 2--"))
            started = true;
        if ((started) &&
        (currentLine.equalsIgnoreCase("--Set 1--")))
            done = true;
        if ((started) && (!done) &&
        (!currentLine.equalsIgnoreCase("--Set 2--")))
            returnList.add(currentLine);
    }
    return returnList;
}

/*
 * Converts a List of strings of the for a,b,c etc
 * into a Set of Set<Strings>.
 */
public static Set<Object>
GetSetsFromStringList(List<String> inputList) {
    Set<Object> returnSet = new HashSet<Object>();
    for (String s : inputList) {
        Set<String> addSet = new HashSet<String>();
        String[] innerString = s.split(",");
        for (String element : innerString) {
            if ((!element.equalsIgnoreCase("empty")) &&
            (!element.isEmpty())) {
                addSet.add(element);
            }
        }
        returnSet.add(addSet);
```

```
    }
    return returnSet;
}

/*
 * Takes in the string representation of a Set object,
 * e.g. [b,c,a], orders the elements alphabetically
 * and returns a string in the format:
 * {a,b,c}
 */
public static String OrderedSet(String inputString) {

    inputString = inputString.replaceAll("\\[", "");
    inputString = inputString.replaceAll("\\]", "");
    inputString = inputString.replaceAll(" ", "");
    String[] strArray = inputString.split(",");
    java.util.Arrays.sort(strArray);
    String setString = "{";
    for (String element : strArray) {
        setString = setString.concat(element);
        setString = setString.concat(",");
    }
    setString = setString.substring(0,
    setString.length() - 1);
    setString = setString.concat("}");
    return setString;
}

/*
 * Writes the sets out to the output file
 */
public static void WriteOutput(String filePath,
Set<Object> galoisClosedSets) throws IOException {
    /*
     * This loop gets the maximum size of the sets.
```

```
 * This is used to output the sets in order of size.
 */
int maxElements = 0;
for (Iterator itera = galoisClosedSets.iterator();
itera.hasNext();) {
    Set stb = (Set) itera.next();
    if (stb.size() > maxElements) {
        maxElements = stb.size();
    }
}
ArrayList<String> aLines = new ArrayList<String>();
aLines.add("C={");
/*
 * Iterate from 0 to the max set size.
 * At each iteration write out the sets that have
 * that number of sub-sets.
 */
for (int count = 0; count <= maxElements; count++) {
    for (Iterator itera = galoisClosedSets.iterator();
    itera.hasNext();) {
        Set stb = (Set) itera.next();
        if (stb.size() == count) {
            String printString = "{";
            for (Iterator inner = stb.iterator();
            inner.hasNext();) {
                if (!printString.endsWith("{")) {
                    printString =
                    printString.concat(",");
                }
                Set inset = (Set) inner.next();
                printString =
                printString.concat
                (OrderedSet(inset.toString()));
            }
            printString = printString.concat("}");
```

```
                    aLines.add(printString);
                }
            }
        }
        aLines.add("}");
        Path path = Paths.get(filePath);
        Files.write(path, aLines, ENCODING);
    }
}
```

B.2 Sample input file

Let P' be the 3-element anti-chain from Example A.2.5. To compute $\mathcal{C}_{dp}$, the completion with respect to $(\mathcal{F}_{dp}, \mathcal{I}_{dp})$ the following is received as input.

input.txt:

– – Set 1 – –
1
2
3
1, 2
1, 3
2, 3
1, 2, 3
– – Set 2 – –
1
2
3
1, 2
1, 3
2, 3
1, 2, 3

B.3 Sample output file

The following file is received as output, given the input $(\mathcal{F}_{dp}, \mathcal{I}_{dp})$ from the sample above.

output.txt:

$C = \{$
$\{\{1,2,3\}\}$
$\{\{1,3\},\{1,2,3\}\}$
$\{\{2,3\},\{1,2,3\}\}$
$\{\{1,2\},\{1,2,3\}\}$
$\{\{1,3\},\{2,3\},\{1,2,3\}\}$
$\{\{2,3\},\{1,2\},\{1,2,3\}\}$
$\{\{1,3\},\{1,2\},\{1,2,3\}\}$
$\{\{3\},\{1,3\},\{2,3\},\{1,2,3\}\}$
$\{\{2\},\{2,3\},\{1,2\},\{1,2,3\}\}$
$\{\{1\},\{1,3\},\{1,2\},\{1,2,3\}\}$
$\{\{1,3\},\{2,3\},\{1,2\},\{1,2,3\}\}$
$\{\{2\},\{1,3\},\{2,3\},\{1,2\},\{1,2,3\}\}$
$\{\{3\},\{1,3\},\{2,3\},\{1,2\},\{1,2,3\}\}$
$\{\{1\},\{1,3\},\{2,3\},\{1,2\},\{1,2,3\}\}$
$\{\{3\},\{1,3\},\{1\},\{2,3\},\{1,2\},\{1,2,3\}\}$
$\{\{3\},\{2\},\{1,3\},\{2,3\},\{1,2\},\{1,2,3\}\}$
$\{\{2\},\{1,3\},\{1\},\{2,3\},\{1,2\},\{1,2,3\}\}$
$\{\{3\},\{2\},\{1\},\{1,3\},\{2,3\},\{1,2\},\{1,2,3\}\}$
$\}$

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