

COVER PAGE

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God bless you all

Thanks

DECLARATION

I, the undersigned, declare that the thesis which I hereby submit for the degree of Doctor of Philosophy to the University of Witwatersrand contains my own independent work and has not previously been submitted by me for any degree at this or any other tertiary institution.



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RESEARCH TITLE

**TECHNICAL AND VOCATIONAL EDUCATION AND TRAINING (TVET)
MATHEMATICS TEACHERS' PROFESSIONAL NOTICING OF STUDENTS'
PRODUCTIVE STRUGGLE IN CALCULUS TASKS**

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LIST OF ABBREVIATIONS AND ACRONYMS

APOS:	Action, Process, Object and Schema
CCK:	Common Content Knowledge
KCT:	Knowledge of Content and Teaching

MKT:	Mathematical Knowledge for Teaching
NCV:	National Curriculum Vocational
PCK:	Pedagogical Content Knowledge
SCK:	Specialized Common Knowledge
TIMMS:	Third International Mathematics and Science Study
TVET:	Technical Vocational and Education Training
ZPD:	Zone of Proximal Development

ABSTRACT

Research has shown that mathematics students have difficulties understanding calculus concepts. This situation is mainly attributed to the poor and ineffective traditional methods of teaching and learning mathematics in schools and institutions of higher learning. In attempting to improve this situation, mathematics education researchers are calling for research on innovative constructivist methods of mathematics teaching and learning. This study explored productive struggle and professional noticing as new pedagogical strategies of teaching and learning of Technical and Vocational Education and Training (TVET) College mathematics, calculus in particular. Three constructivist theories of APOS, Concept Image and Concept Definition and Self-directed learning are discussed, merged and used in the study. Two groups of TVET College lecturers were formed with one as the experimental group and the other as a control group. Experimental group lecturers underwent training on productive struggle and professional noticing new strategies of TVET College mathematics teaching and learning. The control group did not get training on these new teaching strategies and was kept as the benchmark point of comparison. These participating lecturers from the two groups were purposefully sampled as lecturers offering TVET College mathematics in exiting levels of N5 and N6 mathematics classes. Data collection was done through a triangulation approach of using observations, interviews and document analysis in the form of term test scripts. This study found that productive struggle and professional noticing strategies of teaching and learning mathematics were superior to traditional methods of mathematics teaching and learning and were also found effective in enhancing calculus understanding in students. The study has put up a pedagogical framework that can be used by TVET College mathematics lecturers when teaching complex calculus topics. This study recommended in-service training for TVET College lecturers on productive struggle and professional noticing new teaching strategies. Again, the department of higher education and training should make bursaries available for these TVET College lecturers to further their studies in mathematics education.

Keywords: *Productive struggle and Professional noticing, Strategies, College Calculus, Mathematics knowledge for teaching*

CHAPTER ONE: INTRODUCTION AND BACKGROUND OF STUDY

Chapter One introduces and gives the reader an understanding of the background and context of the study. It highlights the role of mathematics and its importance in life and its relation to this study. The chapter also discusses its place in higher education institutions like Technical and Vocational Education and Training (TVET) colleges which are contributing to the country's workforce. The chapter further sheds some light on the challenges encountered in teaching and learning mathematics and current issues in teaching and learning calculus.

1.1 Background and Context of Study

Mathematics education researchers, internationally and locally, are faced with the problem of improving the teaching and learning of mathematics in schools and institutions of higher learning. Modern life demands the requirement to have good mathematical knowledge that supports all round personal development (Hodaňová & Nocar, 2016). Many of these researchers allude to the fact that traditional methods of teaching and learning mathematics are inferior and ineffective in improving students' mathematics understanding (Kapur, 2010; Luneta & Makonye, 2010; Azlan & Zanzali, 2011). According to Crain (1992), mathematics traditional teaching methods promote superficial understanding in students. These methods fail to address deep rooted mathematical errors and misconceptions in students (Smith, DiSessa & Roschelle, 1994). Battista (1999) argues that to date, teachers are still using traditional methods in their mathematics teaching. Matlala (2015) affirms that educators continue to be hesitant to use problem-solving techniques in their instruction. Similarly, teachers still face some problems in teaching mathematical problem-solving (Ishak, Osman, Wei, Kurniati, Ismail & Nanna, 2021). Matlala (2015) further argues that students resort to rote learning and memorization of facts and procedures which make it impossible to reason, validate and argue their mathematical viewpoints. In the quest to improve the situation stated above, more research must be done to find suitable methods of mathematics teaching and learning. Mabena, Mokgosi and Ramapela (2021) put it clear that in South Africa, there is generally an underachievement in mathematics and this include the TVET Colleges, the focus of this study. These researchers call for a shift from traditional methods to strategies that allow for problem solving that provoke higher order thinking skills.

Worldwide, mathematics is regarded as a critical curriculum subject (McKnight, 1987). To Rimkus (2017), mathematics and science are essential to the everyday applications of technological discoveries and inventions which improve the quality of life by making the world a better place for all to live in. Again, Howie (1997) indicates that for South Africa to succeed as a country in a rapidly changing and competitive world, it must address the science, engineering and technology issues to develop and prosper economically. This is where the conception of study came in as this can be achieved through TVET college programs with Mathematics as an integral component of this initiative.

Furthermore, Tanujawa, Prahmana and Mumu (2017) indicate that mathematics is a tool for scientific thinking which is responsible for developing the ability to think logically, systematically and critically. On the other hand, Ali (2013) sees mathematics as the mother of all learning, with other subjects deriving their concepts from it. Also, Glazer and McConnell (2002) argue that all fields of study use mathematical tools.

Notwithstanding the importance of mathematics, many teachers find teaching of mathematics difficult (William, 2018). The way mathematics is generally taught through behaviourist methods of chalk and talk, knowledge transmission as well as telling, particularly in South Africa, continues to fail to transform learning outcomes (Reddy, 2005; Reddy et al., 2012; Reddy et al., 2016; Reddy & Hannan, 2018). Do teachers know how to teach mathematics effectively? Kilpatrick and Silver (2000) argue that school mathematics is superficial, boring and repetitious and needs to prepare students enough to use mathematics daily. This means that South African teachers need good strategies to teach mathematics effectively.

Reddy and Hannan (2018), writing on the results of the Third International Mathematics and Science Study (TIMSS), argue that 'learner performance in school mathematics and science are proxy indicators of the health of our educational system' (p.1). This International comparative study shows that South Africa needs a better record of mathematics performance. Teaching and learning mathematics in South Africa is difficult, resulting in most students continuing to experience many challenges in the subject. As evidenced in this comparative study of about fifty (50) countries of the world in the (TIMSS) of 2003, South Africa continues to perform poorly in mathematics (Fleisch, 2008). According to Fleisch, in about 65 problem-solving questions, only 2.1 percent (2.1%) of the students from South Africa solved these problems correctly. This brings to the fore: the importance of teaching mathematics through problem-solving methods. Mogari (2014) indicates that South Africa can develop if it pays much attention to the quality of mathematics and its teaching and learning. Particularly for this study, problem-solving is explored through students' productive struggle and lecturers' professional noticing.

Productive struggle is students' effort to figure out solutions that are not immediately obvious from challenging mathematical questions that cannot be solved by relying on formulas (Hiebert & Grouws, 2007). This is a situation where students are engaged in high cognitive mathematical problems that are beyond their current ability, given for them to solve with 'think time' allocated. As students struggle through problem solving, opportunities to grapple with important mathematical ideas develop which ultimately yield desired results (Warshauer, 2015). On the other hand, Van Es and Sherin (2002) and Thomas, Jong, Fisher and Schack (2017) define professional noticing as a skill that enables teachers to pay particular attention to the ideas raised by students and then act accordingly on such ideas. This means that when students are busy figuring out mathematical solutions, lecturers should be around to notice their mathematical thinking.

It follows that productive struggle and professional noticing together form a problem-solving technique that allows students to seek any appropriate approach to solving mathematical tasks

without much help from the teacher. This happens when solutions are not immediately available for students. This problem-solving technique is a student-centered approach. It further allows teachers to notice students' mathematical thinking and assist where necessary. In the researcher's view, these two ideas (productive struggle and professional noticing) belong together because when students are struggling with a task, the teacher should notice, interpret and decide on how to assist the poor student. If students are given simple problems to solve, for which they get solutions easily, then there is absolutely no place for these two ideas. This study explored the impact of these two new strategies in the teaching and learning of TVET College mathematics.

1.2 Decolonization and Decoloniality issues of TVET colleges in South Africa

Von Bismarck (2012) opines that the traditionalist definition of decolonization as a process by which legally dependent territories obtained their constitutional independence and entered the world stage of international relations as sovereign states is quite narrow. According to Von Bismarck (2012), decolonization refers to the process of undoing the political, economic, social, cultural, and linguistic consequences of European colonial expansion. This researcher puts forward the argument that the latter definition provides methodological advantages by making it possible to investigate topics and developments that the strictly constitutional viewpoint ignores. In agreement, Wildcat (2022) understands decolonization as the global decolonization movement where countries in Asia and Africa overthrew imperial powers, as well as the struggles of Indigenous peoples in settler contexts. Mhango (2018) indicates that it is worth noting that in Africa, since the inception of colonialism, a wrong and misleading education system has been operating.

Hall and Tandon (2017) indicate that there are elements of decolonizing practice emerging in most universities, at least in some small ways. For example, the voices of students in South Africa, the UK, India, and Canada calling for decolonizing the curriculum of higher education have been

gaining more attention.

Minga (2021) asserts that South African students, together with a small number of academics, began a campaign in 2015 to decolonize the curriculum at universities to tackle and dismantle the epistemic violence and hegemony of Eurocentrism (According to Shohat and Stam (2009), Eurocentrism is the belief that Europe, in the broadest meaning of western hegemony, is the centre of the universe, the ontological reality that casts a shadow over the rest of the globe, and the source from which all good things originate).

The aim was to completely rethink, reframe, and reconstruct the curriculum and place South Africa, Southern Africa, and Africa at the centre of teaching, learning, and research (Heleta, 2016). Ramoupi and Ntongwe (2017) downplay the colonial and apartheid worldviews and ask why we should follow the curriculum if it does not improve our thinking or say something about us and our lived experience. These researchers seem to align themselves with South African students in calling for the end to white domination and European views in higher education by claiming the insertion of a curriculum that is capable of bringing in a multiplicity of worldviews in teaching and learning practices. Western education in Africa is designed to evangelize blacks, and Mbembe (2016) argues that something is seriously wrong and profoundly demeaning when we are asked to bow in deference before the statues of those who did not consider us as humans and who deployed every single means in their power to remind us of our supposed worthlessness. There is something perverse about engaging in this ritual of self-humiliation and self-debasement every time we happen to find ourselves in such an environment. Mbembe (2016) outcries that the current situation with most of our universities, if not all, is that they are Westernized. According to Mbembe (2016), being Westernized means that they are local instances of a dominant academic model based on a Eurocentric epistemic standard. To make it worse, African students received more memorization than originality from their Western schooling (Mbembe, 2016).

Given the ideas presented above, it is necessary to recover and reorganize formerly marginalized teaching methods and information to decolonize the curriculum. The goal is to construct a new educational system that is sensitive to the unique needs of each student and fulfils the demands of

all teachers, staff, and students. Analyzing these, entails looking at the shortcomings and biases in the current curriculum, the shortcomings of programs for preparing teachers, and the socio-political effects of colonialism that have influenced educational practices. To achieve decolonization, colonial practices that have historically influenced education and are still doing so must be addressed and opposed.

On the other hand, decoloniality is a movement that seeks to liberate those who have been colonized from worldwide colonialism (Ndlovu-Gatsheni, 2015). This researcher claims that the battles against the inequality, apartheid, colonialism, imperialism, and slavery of Euro-North American-centric modernity gave rise to decoloniality. According to Maldonado-Torres (2006), decoloniality is the destruction of knowledge and power structures that lead to the replication of geopolitical, racial, and gender inequalities that emerged or found new, more potent ways to be expressed in the modern/colonial world.

Taking Curriculum Studies into account, Decoloniality seeks to answer questions about who is in charge of the field both globally and domestically, as well as who runs the organizations and institutions in the field and generates knowledge inside and for it (Le Grange, 2018). Furthermore, as Reyes (2019) demonstrates, decoloniality directs attention toward pedagogies that ask why things are done a certain way. Decoloniality calls for a pedagogy in education that visibilizes, critiques, and dismantles colonial processes (Reyes, 2019). This researcher opines that decoloniality encourages educators to use their creative capacities to change oppressive societies, rejecting Western values and cultural variation. It acknowledges that children have boundless potential to learn, think, create, and love in ways consistent with their social and cultural histories. According to Reyes (2019), pedagogy acknowledges the impact of past experiences on students' relationships with material, teachers, and the school, emphasizing the prevalence of colonial conditions and the need for active, personal, and thoughtful engagements for successful teaching (Reyes, 2019).

Heleta (2016) indicates that decoloniality involves reorganizing curriculums, placing South Africa at the forefront of education, research, and teaching, and incorporating indigenous knowledge into

histories, cultures, and textbooks. This researcher strongly argues that disrupting colonial knowledge and respecting indigenous culture is crucial, as it has been used to undermine indigenous people. Universities should prioritize African cultural values by promoting diversity and incorporating African values and cultures into science, which can enhance society's perception of universities (Heleta, 2016). It makes sense to think that by promoting multilingualism, resolving power conflicts, and Africanizing the curriculum, institutions of higher learning can contribute significantly to the decolonization movement.

Ndlovu-Gatsheni (2013) asserts that decoloniality challenges the dominant curriculum notion and calls for the disruption of curriculum studies. It critiques Western universities and schools for privileging and demeaning knowledge, resulting in the decimation of the knowledge of colonized and racialized peoples, questioning what knowledge is of most worth (Ndlovu-Gatsheni, 2013).

The literature on decolonization and decoloniality explained in this section explores the importance of decolonization in education, focusing on reconstructing knowledge systems and incorporating diverse pedagogies and resources. It examines the experiences of indigenous people and how global education methods may support colonialism (Wa Thiong'o, 1998). Frantz Fanon's (1963) ideas promote cultural identity and critical thinking, while Professor Wiredu emphasizes the role of education in promoting political, economic, and social change in Africa (Osha, 1999). All three philosophers mentioned have one goal: to investigate power structures, question colonial ideologies, and empower students.

1.2.1 Mbembe's teaching approaches

Mbembe (2016) came up with powerful theories that prioritize interactive and collaborative teaching approaches on how to remodel and reinvent the purpose of education to enhance the development of students' senses of capability, identity, and understanding, as well as their overall well-being. This researcher indicates that to tackle societal, technological, economic, and cultural

difficulties in the ever-evolving global village of today's classroom, Active learning is a successful teaching strategy that develops students' critical, analytical, and evaluative skills while requiring them to participate in meaningful activities (Mbembe, 2016).

A student-centered approach to learning is more successful in fostering deeper comprehension of the material, the development of critical thinking abilities, and the transfer of knowledge. This method has affected the creation of curricula and pedagogy, especially in the fields of medicine and vocational education. Nonetheless, opposition from educators and learners could lead to a two-tiered educational structure. Teachers should concentrate on creating difficult tasks and keeping track of students' progress to successfully adopt student-centered learning. For the development of realistic activities and hands-on experiences in computer science education, Mbempe's real-world application principle is essential. To excel in their education and future employment, engineers, project managers, technology developers, and researchers apply this idea.

1.2.2 Positioning Decolonization and Decoloniality of education in the study

Proponents of decolonization of education, for example, Minga (2021); Wildcat (2022; Mhango (2018) underpinning their ideas on the famous works of Waredu (1997), Fanon (1963), and Wa Thiong'o (1988) in line with Mbempe's (2016) theories, advocate for curriculum change in institutions of higher learning.

Mbempe's (2016) constructivist approach to active learning is seen as a successful teaching strategy that develops students' critical, analytical, and evaluative skills while requiring them to participate in meaningful activities. It fits well with the well-thought-out theoretical framework reinforced with relevant literature in this study. The constructivist's APOS theory, Concept Image and Concept Definition theory, and self-directed learning theory rhyme well with Mbempe's (2016) ideas of a student-centered approach to learning, creating difficult tasks, keeping track of students' progress, and developing realistic activities and hands-on experiences. The three theories explained in Chapter 2 all find application in an active classroom where students actively construct their

meaning.

On the point of decoloniality, on the point of why things are done in a particular manner, this study responded to that by changing how calculus topics should be taught to TVET college students. Lecturers had to change the culture of instruction and introduce newly researched and effective teaching and learning methods. The Western ways of engaging in teaching and learning were changed to new interactive and active approaches to teaching and learning.

This study changed the way calculus courses should be taught to TVET college students in response to the decoloniality movement, which questions why things are done a certain way. Lecturers needed to alter the educational environment and implement recently developed, very successful teaching and learning strategies. New interactive and active techniques for teaching and learning have replaced the old methods of instruction. Researchers like Nchindila (2018), indicates that in Africa, English as a foreign language, is recipe for ineffective learners' knowledge, skills, and understanding of the subject.

1.3 TVET Colleges and the training of a skilled workforce in South Africa

The South African government's position is that TVET colleges train and produce artisans to address the skills shortage in the country (Alexander & Masoabi, 2017). The National Planning Commission of 2013, in its National Development Plan Vision 2030 – Chapter 13, indicates that the country's skills shortage is blamed on the education system, which is failing to address imbalances of the past. The commission has also found that the decline in apprenticeship systems brings about a shortage in artisan trades. Van Rooyen, Du Toit, Botha, and Rothmann (2010) indicate that South Africa faces a critical shortage of artisans and therefore suffers from infrastructure development, economic growth and wealth creation. TVET sector has been earmarked to address this shortage by training artisans. A successful TVET artisan training depends to a greater extent on understanding mathematics and calculus. This research study focused on a TVET college where students study to be artisans. Artisans are engineers in the making.

Observation have shown that, some of those who passed as graduands, both best and poorly performing learners, come to register at a TVET College and continue their struggle to understand mathematics courses. They find TVET mathematics too challenging, so their dreams of qualifying as artisans diminish when they are not assisted to pass their grades.

Ronald (1990) indicates that mathematics is the language of engineering which is so important to artisans and engineers because the laws and patterns of nature are elegantly coded mathematical expressions. Hodanova and Nocar (2016) argue that mathematical knowledge and the study of technical fields enable students to find better jobs and, by so doing, increase the quality of their life and professional orientation.

This study focused on exploring TVET College lecturers' new teaching strategies in productive struggle and professional noticing, which aimed at improving the teaching and learning of mathematics, which is needed in undergoing successful artisan training. Technical mathematics forms part of the subject matrix in all engineering courses offered at TVET Colleges. As already mentioned, a successful TVET College artisan training program depends largely on the understanding of mathematics (calculus in particular). It is because calculus is a pre-requisite course for students keen to study engineering (Makonye, 2012). Furthermore, Rahman (2017) argues that engineering cannot be without calculus. It then follows that the importance of calculus in TVET College artisanship programs cannot be overemphasized.

1.4 What is calculus, and why is it an important topic in Mathematics?

What exactly is mathematics? According to Orton and Wain (1994), mathematics is an organized collection of knowledge, an abstract system of ideas, a practical tool, a key to understanding the universe, a way of thinking, a deductive system, an intellectual challenge, a language, and the purest form of logic, an aesthetic experience, and a product of the human mind. The complexity of this definition points to the complexity of mathematics. On the same breadth, Smith (2004) indicates that mathematics is widely recognized as the underpinning language for science and

engineering. Rahman (2017) argues that mathematics is the brain of all engineering disciplines, and no engineering study is possible without it. This means that to become a good artisan or engineer; one must have sound knowledge of mathematics (calculus). Flegg, Mallet and Lupton (2012) indicate that the calculus section of mathematics plays an important role in engineering training courses.

Domondon, Pardo and Rin (2022) emphasizes the importance of mathematical learning in that it equips learners with powerful tools to understand and change the world. According to this researcher, those tools are logical reasoning, problem-solving skills and the ability to think in abstract ways. Domondon et al., (2022) argue that mathematics is one of the subjects that is taken very seriously in the school system, irrespective of country or level of education.

Again, what is calculus? The birth of calculus is credited to Sir Isaac Newton and Gottfried Leibniz, who independently conceived the idea (Kollerstrom, 2012; Rosenthal, 1951). Calculus is composed of two distinct but intimately related components. These are differential calculus and integral calculus. The fundamental theorem of calculus captures this. This means that differentiation and integration are inverse operations, symbolically captured in Fig. 1.

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$
$$\int_a^b f(x) dx = F(b) - F(a)$$

Figure 1: Fundamental Theorem of Calculus

Differential calculus deals with differentiation and determination of derivatives (rates of change and slopes of graphs/curves). Integral calculus focuses on the accumulation of quantities and the area under graphs. Tall (1996) indicates that differentiation deals with the rate of change, while integration concerns cumulative growth. These two calculus sections undo each other as they are opposite in operation.

As was already said, Rahman (2017) contends that calculus is essential and fundamental to all engineering disciplines, making the idea of engineering without it nearly unthinkable. Calculus is widely used in science, engineering, medicine, business, industry, and many other fields that apply the concept of rate of change and motion, according to Mokhtar, Tarmizi, Ayub, and Tarmizi (2010). As a result, calculus is regarded as the most important subject for science and engineering students in universities. According to these authors, understanding calculus is essential for success in engineering courses like mechanics, fluid mechanics, and thermodynamics. According to Tall, Smith, and Piez (2008), calculus has attracted the most attention and investment in the use of technology. Most students who take calculus as their final mathematics course realize the beauty and power of solving mathematical problems in engineering applications.

Johor, Sabri and Hamzah (2014) see calculus as a major and useful subject in engineering courses and argue that this cannot be denied when considering topics such as exponents and surds, graphs, functions, complex numbers, limits of function, domain and range, integration, differentiation, integration by parts integration of trigonometric substitutions, chain rule, implicit function, partial derivatives and L'Hopital's rule and their applications. According to Flegg, Mallet, and Lupton (2012), the great structural designs and stunning buildings that surround us are the outcome of the usage of calculus in civil engineering. Tall, Smith, and Piez (2008) assert that calculus has established itself as the most distinct section among mathematics courses due to the beauty of its applications. Calculus is crucial because it helps students develop their technical and analytical skills while being applied across a wide range of disciplines (Bajracharya, 2012; Standler, 1990). This is also applicable TVET College students whose skills should develop make them realize their dreams of qualifying as artisans.

1.5 Research Problem

Indicated earlier, Mathematics education researchers are overwhelmed with research that aims to

improve the quality of mathematics, including calculus teaching and learning (Zanzali, 2011). A growing body of research show that students have difficulties understanding mathematics and calculus concepts (Zakaria & Salleh, 2015). Researchers' findings indicate that traditional mathematics teaching and learning methods are responsible for poor calculus understanding. Similarly, Tall (1993) blames traditional mathematics teaching and learning methods and proclaims that they make students memorize procedures for solving calculus problems. These traditional methods, for example, telling method, have proven unproductive in realizing desired mathematics outcomes (Kapur, 2010; Reddy et al., 2018). Charmaz (2006) indicates that traditional mathematics teaching methods start with the teacher introducing a new concept or procedure by showing students one or more examples or definitions and explaining how the new idea is connected to previously learned ideas. Battista (1999) indicates that in traditional mathematics instruction, teachers show students few examples of how to solve a certain type of problem and then has them rehearse this method in class and homework. According to Battista (1999), students imitate what they have seen and heard from their teacher than understanding what they are doing. The study is totally against this kind of practice.

Research has also reported different reasons for poor student understanding of calculus (Engelbrecht et al., 2005). Sometimes, students use calculus procedures for differentiation and integration without knowing their conceptual basis. Clement, Lochhead and Monk (1981) also found that most students needed a deeper conceptual understanding of the underlying theorems and clarification about calculus tasks. Johor, Sabri and Hamzah (2014) indicate that learning calculus is not easy as counting from number one to hundred; but indeed a complex and dynamic process.

Other researchers, for example, Luneta and Makonye (2010) and Kiat (2005), found that students needed a stronger background in mathematics in pre-calculus concepts such as algebra, limit values, factorization, functions, solving of equations, and simplification of indices. This is of great concern because algebra and functions are the language in which calculus is mediated. The teaching

and practice of instructional approach (Kapur, 2010), quite common in South African schools, has proven ineffective in bringing about good results in mathematics (Reddy et al., 2018). This situation calls for research in innovative pedagogies in the teaching and learning of mathematics that will transform the whole space. This was the gap the study intended to close.

Calculus is a pre-requisite course for students keen to study engineering, science, and technology because calculus is a language of higher mathematics (Makonye, 2012). Makonye's experience is that teachers and learners face daunting challenges in teaching and learning calculus, respectively.

Some researchers, for example (Mabena et al., 2021), call for constructive, innovative mathematics teaching and learning methods. This study found it imperative to respond to the call by engaging in pedagogical innovative research that focused on the productive struggle and professional noticing as new strategies for teaching and learning mathematics. Kapur (2010) found that students taught via productive struggle methods outperformed those taught through traditional methods. Again, Edward and Beattie (2016) indicate that productive struggle promotes mathematical meaning-making in students and further enhances collaborative learning settings in which students explore rich mathematical tasks together. Productive struggle occurs when students are given ill-defined and cognitively challenging mathematics tasks (Stein et al., 2000), with students needing help finding the solutions immediately. This kind of task leads to learning gains for students as knowledge gained this way is not easily forgotten.

When students work independently through struggles, they ultimately develop the capacity to think deeper, reason and solve more problems (Stein et al., 2000). Students' struggles allow the teacher to attend to their strategies, interpret their understandings and decide how to respond to their understanding (Jacobs, Lamb & Phillip, 2010). Indeed, it is imperative to research calculus teaching and learning from new perspectives, such as productive struggles (Van Es & Sherin, 2002, Thomas, Jong, Fisher & Schack, 2017) and professional noticing (Hiebert & Grouws, 2007, Jacobs et al., 2010; Ding & Domínguez, 2016).

Stein et al. (2000) argue that for students to develop the capacity to think, reason, and solve problems, teachers must prepare for their high-level cognitively complex mathematical tasks. The argument put forward by Stein et al. (2000) is that high-level cognitively designed tasks are rich enough to support fruitful classroom discussions. This kind of task leads to learning gains for students.

Lecturers need to design and give challenging calculus tasks, which must throw students deep into problem-solving struggles. The study subscribed to Polya's approach and Schoenfeld's approach on how to solve difficult mathematical problems. For Polya (1954), students must first understand the problem, develop a plan, carry it out, and finally look back at the result to check if it makes sense. On the other side, Schoenfeld (2014) argues that when students are faced with a difficult task, they must try to solve a simpler form of the question and use the results to solve the original problem. Polya's (1954) and Schoenfeld's (2014) approaches throw students deep into positions of struggling with the solutions. This study accommodated these approaches as it found them relevant. During the students' struggles, lecturers must carefully notice and practice unbiased hearing of what students do and say respectively. Wallach and Even (2005) argue that hearing is about teachers understanding what their students say, show, feel and do during instruction. This means that teachers should interpret students' talk and actions professionally.

Teachers should notice by picking up, identifying, pointing out, and detecting all errors and misconceptions occurring during class interactions. Teachers should possess the skill of attending to students' strategies, interpreting their understandings and deciding how to respond to their understanding (Jacobs et al., 2010). Students' strategies provide an idea of the extent of their understanding.

One way to enhance professional noticing is to resource learner errors and misconceptions. Moru, Qhobela, Wetsi and Nchejane (2014) argue that analyzing errors empowers the teacher to effectively support learners in engaging with new mathematics concepts. Teachers should know

about analyzing errors and detecting misconceptions to interpret students' errors and misconceptions appropriately. This builds upon the work of Ball, Thames and Phelps (2008) about mathematics knowledge for teaching (MKT), in particular - knowledge of content and students (KC&S), together with specialized content knowledge (SCK) (Ball et al., 2008), these unpack and make new mathematical knowledge understandable to learners. Ball et al.'s model helps teachers to channel their professional noticing skills. This helps teachers to develop professionally situated knowledge in teaching calculus at TVET colleges in this study.

Teaching in a TVET college and offering mathematics classes made the researcher realize how students battle with understanding calculus topics (William, 2018). The researcher has argued in his master's work that students bring superficial calculus knowledge upon enrolling at TVET colleges. Sam (2014) argues that various efforts have been made - to improve students' performance in calculus, but most of such efforts concentrated mainly on weak students than on teachers. This calls for more efforts towards innovative pedagogical practices to teach calculus at TVET colleges.

What is the TVET College lecturers' way of teaching deep calculus topics so that students understand better? Oleson and Hora (2014) argue that teachers teach how they were taught by simply mimicking and replicating their mentors. The study wants to challenge such practices and take a different approach to students' poor understanding of calculus topics through professional noticing of learner thinking while solving calculus tasks with little teacher support or structure.

In capturing the problem statement: TVET college mathematics students experience many challenges in learning calculus because they have learnt mathematics through traditional methods of teaching. One wonders if students are taught through innovative teaching methods such as new productive struggle and professional noticing can help improve mathematics students' achievement.

In light of the above dilemmas in mathematics teaching and learning in school and higher education institutions, the study explored productive struggle and professional noticing that could help improve TVET college mathematics students' achievement.

1.6 Purpose of the study

TVET college mathematics lecturers play a crucial role in preparing students for success in the workforce. One of the key skills that is essential for students to develop in order to succeed in the field of mathematics is the ability to engage in productive struggle. Productive struggle refers to the process of students grappling with challenging mathematical problems and persevering through difficulties in order to arrive at a solution. In the context of complex calculus tasks, productive struggle is particularly important as it helps students develop a deep understanding of mathematical concepts and prepares them for real-world problem-solving scenarios.

The aim of this research is to investigate TVET college mathematics lecturers' professional noticing of students' productive struggle in calculus tasks. Professional noticing refers to teachers' ability to interpret and make sense of what students are doing mathematically, and to use this information to inform their teaching practices. By studying teachers' professional noticing of students' productive struggle in calculus tasks, we can gain insights into how teachers support and facilitate students' learning experiences in the classroom.

One of the key questions that this research seeks to answer is how can TVET College lecturers professionally notice students' errors and misunderstandings during interactive class activities and while marking students' term assessments. Do teachers recognize the signs of productive struggle in their students, and do they have the knowledge and skills to effectively support students through this process? By examining teachers' professional noticing practices, we can better understand the ways in which teachers can foster a productive struggle environment in the classroom and help students develop their problem-solving skills.

Furthermore, this research aims to explore the impact of TVET college lecturers' professional noticing of students' productive struggle on students' learning outcomes in complex calculus tasks. Do students who experience productive struggle in calculus tasks with the support of their teachers achieve better results than those who do not? By examining the relationship between teachers' professional noticing practices and students' learning outcomes, we can identify strategies that effectively support students' development of mathematical proficiency.

This research on TVET mathematics teachers' professional noticing of students' productive struggle in calculus tasks has the potential to inform teaching practices and improve student learning outcomes. By understanding how teachers identify and respond to productive struggle in the classroom, we can better support students in developing their mathematical problem-solving abilities and prepare them for success in the workforce. Ultimately, this research aims to contribute to the development of effective teaching practices in TVET college mathematics education and enhance students' mathematical experiences in the classroom.

1.7 Research questions

TVET college mathematics students experience many challenges in learning calculus, a key course in engineering. It is proposed that TVET College lecturers embark on professionally noticing their students' mathematical thinking during students' productive struggle activities given in class. In light of this statement, the research articulates the following questions:

- a) What is the impact of TVET college lecturers' professional noticing of students' productive struggle on students' learning outcomes in complex calculus tasks?
- b) How can lecturers professionally notice students' errors and misunderstandings during

interactive activities and while marking students' term assessments? What errors do TVET College students do and what misconceptions do they display in their calculus tasks?

- c) What kind of pedagogical framework can be recommended for the enhanced teaching and learning of challenging calculus topics?

1.8 Significance of the Study

As already indicated earlier, traditional methods of talk and chalk, knowledge transmission, and teaching by telling, particularly in South Africa, continues to fail to transform learning outcomes (Reddy & Hannan, 2018). Traditional methods are superficial, boring and repetitious and, for a long time, failed to prepare students enough to use mathematics daily (Kilpatrick & Silver, 2000). Teaching and learning mathematics, particularly calculus, has reported many challenges for students and teachers (Makonye, 2013). Clement, Lochhead and Monk (1981) also found that most students needed a deeper conceptual understanding of deep calculus topics and showed errors and misconceptions about calculus tasks. Calculus is used in various disciplines by providing logical paths for solving complex problems and equips students with technical and analytic skills (Bajracharya, 2012 & Standler, 1990).

Identifying students' productive struggles in calculus assignments is a crucial aspect of effective doctorate-level teaching, as noted by Robinson (2017). Teachers may foster a learning environment that encourages academic achievement, critical thinking, and intellectual development by being aware of their students' difficulties and offering considerate, encouraging responses.

A supportive and inclusive learning environment where students feel valued and encouraged to take intellectual risks is created when teachers identify and address students' productive struggles in calculus problems (Brown, 2020). Increased student enthusiasm, self-assurance, and perseverance in the face of academic difficulties can result from this strategy.

The significance of this study is contributing to knowledge base of the teaching and learning of deep calculus topics of the TVET sector in South Africa and globally. The findings of this study provide information to all stake holders in the TVET college sector in assisting to come up with the appropriate mathematics teaching and learning strategies which will ultimately improve students' understanding of complex calculus topics. The study findings have the potential to contribute to the existing literature on the teaching of Calculus by teachers of mathematics globally and on mathematics teacher education. Other mathematics education researchers can use the results of this study to build on many more studies that might be carried out on the same study area and may also provide a rich ground for further research based on the gaps left out by this study. The potential for this study to serve as a useful blueprint for future educational improvements means that it has the ability to revolutionize educational practice.

1.9 Conclusions

The chapter gave the background of the study – the problem statement and the purpose of the study, amongst other things. The challenges of teaching and learning Mathematics/calculus are experienced worldwide. There seems to be an accepted overwhelming consensus that traditional methods are to be blamed for students' poor calculus understanding. The traditional methods are so ineffective that many students are prone to make errors and display misconceptions in the study of TVET College Mathematics.

1.10 The lay-out of the research report

Chapter 1 – Introduction and Background of the Study

Chapter 2 – Theoretical Framework and Literature Review

Chapter 3 – Analytical Framework

Chapter 4 - Research Design and Methodology

Chapter 5 - Implementation of Data Collection

Chapter 6 – Data Analysis and Discussions

Chapter 7 - Discussions of collected data

Chapter 8– Findings Conclusion and Recommendations

References

Appendices

CHAPTER TWO: THEORETICAL FRAMEWORK AND LITERATURE REVIEW

This chapter is about the theoretical framework as well as the literature review of the study. It highlights three things as the main: the grand theory underpinning the research, middle-order and substantial theories found relevant to the study. It goes on to look at a few other ideas: errors and misconceptions in the teaching and learning of calculus, Ball et al., (2008) 's Mathematics Knowledge for Teaching (MKT) and teaching interventions of the experimental group lecturers.

Varpio, Paradis, Uijtdehaage and Young (2020) define a theory as a set of logically related suggestions and constructs and their relationships described to help understand the world in which we live. In addition to theories' illustrative power, they are descriptive, explanatory, emancipatory, disruptive and predictive (Varpio et al., 2020). According to these researchers, a theoretical framework is a logically developed and connected set of concepts developed from the theory. Similarly, Lester (2005) defines a framework as a set of ideas, principles, agreements or rules that provides the basis or the outline of something to be studied. Therefore, a theory is a guide for research activities (Johnson, 1980).

On the other hand, Olivier (1989) indicates that theory serves as a lens through which one views the facts. In this way, it means that facts stem from theory. Grand, middle-range, and substantial theories depend on generality and specificity (Creswell, 1994). According to these researchers, Grand theories explain the main categories of phenomena, while Middle-ranged theories find application between working hypotheses of everyday life. Substantial theories offer explanations in constrained settings and limited scopes. This study pinned on Constructivism's grand theory of learning. In the space between middle-ranged and substantial theories lie APOS theory, Self-directed learning theory and Concept image and Concept definition theory. These theories were chosen and argued for their merger in relevance to the study. Ball et al. (2008) 's

Mathematics Knowledge for Teaching framework reinforced with relevant literature were included to give more meaning to the merger of these chosen theories.

2.1 The constructivist theories of the study

Constructivism is a theory of learning. English (1995) argues that the constructivist way of learning is a dynamic process that allows learners to construct, modify, and integrate ideas by interacting with the physical world, materials, and other learners. In agreement with this, Fosnot (2005) adds that Constructivism is about deep conceptual learning, which is self-regulated at moments of criticality. Piaget (1970) asserts that knowledge cannot be packaged and transferred intact from lecturers to students because students actively construct their learning.

In light of the above, classrooms must become workshops with teachers as facilitators, not as knowledge transmitters. This further means that the new imagined classroom should be characterized by: questioning, learners paraphrasing each other and discussing ideas together. In addition to these, the classroom should uphold the importance of thinking time, pair talk and problem solving and inquiry approaches.

Crain (1992) argues that in Constructivism, the students are explorers who make their discoveries and formulate their understanding. Constructivists like Crain (1992) are against traditional teaching for its `teaching by telling method` because it does not create opportunities for students to reason, justify, and argue their standpoints.

Constructivists believe that knowledge is fallible and subjective as it is constructed by students (Von Glassersfeld, 1989). Like most other constructivists, Von Glassersfeld (1989) posits that knowledge can never be copied as is to students, but instead, students only actively build their knowledge. Since lecturers cannot transmit knowledge to their students, it makes sense that teachers should assist students by providing mental models on which students can impose abstract

mathematical meaning on their own (Makonye, 2014). Interestingly, Smith, diSessa, and Roschelle (1993) talk of `tabula rasa`, meaning that learners can never come to class as empty slates or vessels bare of knowledge. This means that teachers must be mindfully considerate of this situation by nurturing this knowledge.

On the other hand, Vygotsky's theory of learning points out that the social sphere and not the individual mind are the genesis of learning (Vygotsky, 1930). This theory is rooted in the belief that the world is socially constructed, with knowledge being constructed through interactions with the environment and other people. Simons (2000) indicates that learning is a process through which people construct collective meaning and develop their perspectives on situations. In agreement, Morris (2019) views learning as an individual, interpretive, and active meaning-making process. This researcher posits that for teachers to assist learners in learning, they should engage these learners in solving authentic, real-world problems so that they attain their true potential.

It is of utmost importance that constructivism allows lecturers to interpret and examine students' mathematical thinking and enable them to pinpoint erroneous learning (Crain, 1992). This researcher argues that the student's true potential manifests through the Zone of Proximal Development (ZPD). This means what students can do with little help from the lecturer that which they could not do earlier - this ties well with the new strategies of productive struggle and professional noticing explored in this study.

Diagnosing students' errors and misconceptions during struggles with calculus tasks helps pinpoint aspects of their ZPD. This helps to provide a pedagogical model for which this research hopes to achieve and contribute to mathematics education.

This research study adopted a constructivist view of learning instead of traditional ways of teaching and learning mathematics. As indicated earlier, the traditional methods of mathematics teaching and learning have been found to promote superficial understanding, and learners become more reliant and dependent on their lecturers

2.1.1 APOS theory

The word APOS stands for Action, Process, Object, and Schema and it is a theory of both teaching and learning and a research methodology in mathematics education (Tziritas, 2011). Each letter in APOS represents a stage of the process of students' understanding during learning.

In understanding a concept and as well as constructing meaning, these stages do not necessarily follow one another in a linear order (Luneta & Makonye, 2010). These researchers indicate that when students learn a mathematical concept, it first occurs as an *activity* to which little meaning is attached. Students then view it as a *process* or procedure; thirdly, they view it as an *object* internalized in the mind; and lastly, this object is incorporated into the student's *schema*. According to these researchers, doing mathematics should be dominated by activities and processes before forming mathematical concepts. Doing mathematical tasks involves conceptualizing mathematical ideas, which must be taught in context first as it helps learners develop mathematical meanings before learning procedures. Luneta and Makonye (2010) further indicate that doing mathematics provides learners with physical and mental objects upon which mathematical meanings and concepts are imposed. These researchers indicate that once mathematical concepts are abstracted from reality through actions and processes, learners can hold them as generic mental objects that can be applied to other practical and theoretical positions differently from concrete situations that bore them.

Tzur and Simon (2004) explain APOS theory's 'A' for *action* as any repeatable physical or mental operation that transforms objects. According to these researchers, action requires a definite procedure, which the subject is aware of following. Tzur and Simon (2004) argue that the steps of the process and the objects at all action phases must be present, physically or in the subject's mind. Tzur and Simon (2004) indicate further that when the need for an explicit process becomes less necessary and when the total action can take place entirely in the mind of the subject or can be imagined as happening without following all of the specific steps, it means the action has been

interiorized (internalized) and can now become a *process*. These researchers further indicate that when it becomes possible for a *process* to be changed by *action*, it has been encapsulated to become an *object*. According to this theoretical view, there is one way to make mathematical objects, and that is by encapsulating a process. Encapsulation occurs when a student considers the steps that are used in a specific process, becomes aware of the process as a whole, understands that transformations (whether they be actions or processes) can have an impact on it, is able to create such transformations, and then thinks of the process as an object (Asiala, Brown, De Vries, Dubinsky, Mathews, & Thomas, 1997).

All the points explained above are only possible when the individual learner has made a conscious decision to learn. This scenario takes us to the second theory used in the study.

2.1.2 Self-directed learning theory

Knowles (1975) describes self-directed learning as a practice in which individuals take the initiative, with or without the help of others, in establishing their learning needs, formulating goals, identifying human and material resources, and assessing learning outcomes. Peters (1989) explains self-directed learning as a process in which the learner decides to learn to achieve a goal, formulates a plan to reach the goal, and takes action toward the set goal. This means that self-direction happens when the learner takes control of her learning by planning, setting goals, and then actioning the plan. In other words, it means that self-directed learning is motivated by the desire to learn and does not require any external motivational force. After the student has decided to self-direct, it becomes easier for the teacher to assist in imparting knowledge.

Self-directed learning places students at the heart of learning and has the potential to make them successful in their learning process (du Toit-Brits, 2018). Self-directed learning is seen as an active process of meaning-making and makes learners not to become passive receivers of information. According to du Toit-Brits (2018), the classroom atmosphere must create opportunities for learners

to take responsibility and ownership of their learning in support of the development and progress of students' self-directedness within the learning process.

Boyatzis (2002) opines that self-directed learning becomes possible when students discover who they want to be. When intrinsically motivated, their ideal selves develop from their egos, dreams, and aspirations. According to Boyatzis (2002), our aspirations, dreams, and desires are shaped by our values, philosophy, life and career, motives, and role models. To attain this, Brockett and Hiemstra (1985) indicate that teachers should use various teaching and learning techniques to arouse and maintain learner interest as they promote discussions, questioning, and self-directed inquiry skills. Kerka (1994) argues that for self-directedness, the learner is neither self-regulating nor reliant on but interdependent, forming new understandings through discourse, opinion, and consideration with fellow students and the teacher as facilitator.

Students, who are more reliant and dependent on their lecturers, are more likely not to realize and identify errors and mistakes in their work (Zimmerman, 2002). For such students to develop an understanding of their learning process, they should be supported to become self-directed so that they can develop and implement strategies and modify their study habits and learning behaviors to address difficulties and, by so doing, become more successful learners. When students are self-motivated, they go the extra mile to do more work by solving more problems that further enhance their mathematical understanding.

2.1.3 Concept Image and Concept Definition

In mathematics education, 'concept image and concept definition theory' is regarded as another strategy used to understand mathematical concepts. Concept image is the total mental configuration related with the concept, including all the mental pictures and associated properties and processes, while concept definition refers to the own words used to specify a concept (Tall & Vinner, 1981). According to these researchers, concept images and concept definitions are built

over time from different experiences. A distinction is made between mathematical concepts as formally defined and the cognitive processes by which they are conceived. It is argued that during the mental processes of recalling and manipulating a concept, many associated processes are brought into play, consciously and unconsciously affecting its meaning and usage. From this point, a personal concept definition can differ from a formal concept definition. Tall and Vinner (1981) advise that students should always strive to give formal concept definitions. These experts indicate that the development of a concept image is only sometimes coherent because the brain works differently. This means that different concept images are activated at different times (evoked concept images). Concept definition may be learned through rote, and the student may use their explanation of evoked concept image. Concept definition may vary from time to time and may differ from formal concept definition as accepted by the mathematical community.

One will argue that concept images and concept definitions stem from mental imagery. Mental imagery refers to anything and everything that happens inside us when we think, plan, consider, and reflect (Mason, 2002). This researcher asserts that the power to imagine is the most fundamental of all the powers we are born with and further argues that imagery and imagination must be used effectively in the mathematics classroom and in preparation for lessons. Weber (2014) indicates that imagining (mentally forming and manipulating images) is personal; therefore, students experience mathematics as beginning in their minds. During this process, teachers are supposed to professionally notice and gain insight into their students' thought processes.

Concept images and concept definitions assist in shedding more light on students' errors and misconceptions. Makonye (2014) defines a concept image as an internal exemplary of reality constructed by a learner due to involvement with a particular concept. According to this researcher, the concept image is similar to Piaget's schema and is adjustable over time. Important to note is the fact that the concept image may differ from the formal mathematical knowledge held by the

mathematical community (Makonye, 2014). Errors and misconceptions are likely to build up if a learner has an inadequate concept image.

In an ideal setup, the concept image and concept definition should be harmonious and without conflict. Otherwise, there will be weak constructions when students are aware of the concept but have yet to construct it and cannot tell what it is (Makonye, 2014). This researcher indicates that solving mathematical problems involves conceptualizing mathematical ideas by forming mathematical concepts, which are mental objects that later form part of students' schema.

2.1.4 The interrelatedness and overlapping of chosen theories

The paragraphs below, explain a merger of the three theories that were found relevant for this research study.

In APOS theory, when students repeat and reflect on the actions they perform, they tend to interiorize (internalize) these actions into a mental process. When they become aware of such mental processes, they can construct transformations and encapsulate (encase or envelop) the process into a cognitive object (Arnon et al., 2014). According to these researchers, APOS theory brings about exteriorization and encapsulation as the only mental mechanisms needed to enhance mathematics understanding.

Most researchers indicate that the idea of schema is very similar to the concept image in Tall and Vinner's Concept image and Concept Definition for mathematics understanding (Dubinsky & McDonald, 2001; Connolly, Murphy & Moore, 2006). Students should be able to define and explain the mathematical concepts formed. Looking at Tall and Vinner's Concept Image, students should say to some extent, their mental pictures and properties of their formed concepts. The Concept Image is an inner model of reality constructed by learners as a result of experience with a particular concept (Luneta and Makonye, 2013).

The above processes are likely to yield the desired results, only when the students must have made an individual decision to learn mathematics (being self-directed). When students are not coerced in any manner to learn mathematics, when they are motivated from within themselves, when they take control of their learning, then the learning of mathematics becomes beneficial to them. These kinds of students are motivated to try new mathematical problems and even ill-defined problems because they are always ready to go and do an extra mile.

2.1.5 Mathematics Knowledge for Teaching Framework

Hill, Rowan, and Ball (2005) describe mathematical knowledge for teaching (MKT) as the mathematical knowledge used to teach mathematics. It is the mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students (Ball et al., 2008). According to these authors, teaching mathematics includes explaining terms and concepts to students, interpreting students' statements and solutions, judging and correcting textbook statements of particular topics, using representations accurately in the classroom, and providing students with examples of mathematical concepts, algorithms, or proofs. Quite interesting is the argument put forward by Bansilal (2014) that a teacher with only a three-year teaching diploma is considered to be underqualified and may not be aware of how mathematical topics are related throughout mathematics included in the curriculum. Coming to the study, lecturers offering Mathematics N4 knew and understood Mathematics N5 and Mathematics N6 and emphasized sections that overlapped and continued to the next level of study.

In the following paragraphs, there are discussions of some pertinent aspects of Ball et al. (2008) 's CCK, SCK, KCS and KCT as determinants of MKT.

Common Content Knowledge (CCK) is defined by Ball et al. (2008) as the mathematical knowledge and skill that teachers possess to know the material they teach, recognize incorrect

answers from students, and recognize inaccurate textbook definitions. This knowledge is not unique to teaching and is used in a wide variety of settings; for example, teachers need to write mathematical notations correctly on the board. Ball et al. (2008) argue that the lack of CCK may result in teachers incorrectly pronouncing mathematical terms, making calculation errors, and being unable to solve mathematical problems correctly. A classroom characterized by all these will surely suffer from valuable time loss. CCK requires teachers to have basic competence with the content, and in this sense, it is not likely to be unique to teachers.

Specialized Content Knowledge (SCK) refers to the mathematical knowledge and skills unique to teaching that are not needed in other settings (Ball et al., 2008). This mathematical knowledge is only needed for teaching, for example, in analyzing student errors or checking whether an alternative solution approach is correct. SCK calls for unique mathematical understanding and reasoning, such as how to choose, make, and use mathematical representations effectively, and explain and justify mathematical ideas. The mathematical demands of teaching require specialized mathematical knowledge in situations like knowing which statement best explains why we should find common denominators when adding fractions, for example.

Knowledge of Content and Students (KCS) is the type of knowledge that combines knowing about students and knowing about mathematics. During teaching and learning mathematics, lecturers must anticipate what students are likely to think and what they are likely to find confusing. Mathematical examples chosen by teachers should assist them in predicting students' thinking and, at the same time, arouse interest and motivate them to learn. Lecturers must also be able to notice, hear, and interpret students' mathematical thinking.

Knowledge of Content and Teaching (KCT) refers to knowing about teaching and knowing about mathematics. Mathematical knowledge of the design of instruction is pivotal in sequencing particular content for instruction. Teachers need to know which examples to start with and which examples to use to take students deeper into the content. Teachers should know which teaching

methods to use for desirable results. They should also know how to conduct themselves during classroom discussions, for example, when to pause for more clarification, paraphrase, and probe.

Teachers possessing all four domains make their students understand mathematics far better. They are likely to skillfully notice when students give wrong answers or when textbooks give inaccurate definitions. It suffices to believe that the extent to which teachers recognize errors and mistakes in the students' work depends largely on the amount of mathematical knowledge in them (Ball et al., 2008). For this study, examples are replaced by good tasks that allow for productive struggles.

Similarly, Walshaw and Antony (2008) argue that a teacher's mathematical subject knowledge enhances the quality of class discussions. How mathematics teachers listen to and probe students greatly increases the teacher's understanding of students' mathematical thinking. In responding to students' mathematical ideas, teachers should delve into the minds of these students (Walshaw & Antony, 2008). This allows teachers to notice mathematical students thinking and respond appropriately and professionally.

In concluding this part, three constructivist theories, APOS, Self-directed theory, and Concept Definition and Concept Image theory, explained above, have been used in the study. The merger and rationale behind choosing these theories about their interrelatedness and overlapping have been explained.

In a scenario where students are self-directed and motivated to try any mathematical problem, lecturers should meet these initiatives somewhere halfway by designing tasks that are suitable to throw students deep into productive struggles. The Ball et al., (2008) 's MKT assist in planning for good lesson tasks. These tasks must always be within the student's Zone of Proximal Development (ZPD). Lecturers should be able to notice errors and misconceptions professionally, attend to what they notice, interpret it, and decide on how to respond to it. This study follows a knitting thread that brings together all the chosen theories in this section. Calculus should be fun for TVET college lecturers to teach though it is only possible if they have a solid

command of the subject content.

2.2 Mathematical challenges in the teaching and learning of Calculus

The main causes of problems with calculus studies are either the lecturers or the students. Simovwe (2020) reports that the majority of students found the calculus portions of the introduction to be difficult, and that their issues grew more difficult when they studied functions, graphs, and other related algebraic topics (Judson & Nishimori, 2005). Many students, Judson and Nishimori (2005) found, lacked a developed knowledge of functions as a result, which might have led to errors when differentiation and integration were needed.

Usiskin (2003) proposed that early instruction in inequality, summation, and other algebraic concepts could improve students' calculus performance. The majority of students do not acquire the requisite knowledge of the defined integral notion because of the arrangement of the teaching materials (Anatoli, 2008; Orton, 1983; Thomas & Hong, 1996). Tall and Vinner (1981) claim that the reason why students misinterpret and find it difficult to learn calculus concepts is because there is a difference between the concept definition and the concept image of the limit concept.

According to research, a lot of students find it difficult to understand the fundamentals of calculus (Artigue, Batanero & Kent, 2007). Concerns about students memorizing facts due to an excessive emphasis on algebraic drill and practice have been expressed by many academics (Gordon, 2004). According to Gordon (2004), students who do not have a firm grasp of calculus principles often approach differentiation or integration problems by using the procedures they must memorize. Calculus learning is rendered useless, according to Bezuidenhout, 2001; Davis and Vinner, 1986, when pupils only take into account the computational aspect of the subject, ignoring its conceptual component. Students complete their calculus studies with minimal conceptual comprehension because, according to Aspinwell and Miller (1997), they view computation as the primary goal of the subject.

On the contrary, Tol (2005) suggests that the difficulties teachers face with calculus concepts may have originated during their time in secondary education because calculus training in schools frequently emphasizes algebraically based procedural knowledge (Morris, 1999). The question that needs to be addressed is whether or not instructors are able to adequately communicate Calculus concepts to students.

Mastorides and Zachariades (2004) discovered that lecturers struggled to explain concepts like limits and continuity of functions conceptually and were unable to give conceptual justifications for the procedural tasks they carried out. One recurrent finding throughout these research, for instance, is that pre-service teachers tend to depend solely on a sharing interpretation of division and do not comprehend measurement division (Simon, 1993).

Gordon's (2004) research found that calculus students typically demand algebraic answers to issues and lengthy formula derivations. Gordon also mentioned the belief held by several calculus instructors that they were teaching algebra instead of calculus topics. This effect was also seen by Axtell (2006), who believed that pupils who receive traditional calculus instruction are not able to comprehend the subject's foundations. According to Zachariades et al. (2007), when teaching calculus, teachers appear to place more emphasis on processes than on students' conceptual comprehension.

2.3 Strategies/Methods of teaching calculus in institutions of higher education.

Research by Musonda (2009) discovered that the way lecturers teach mathematics might have a positive or negative impact on students' performance. According to (Chifwa, 2015), lecturers are free to choose how they want to teach calculus, but they must first be aware of all the methods that are accessible, as well as their benefits and drawbacks, goal, and application. This is because there are both quick learners and slow learners in the classroom. Nyaumwe, Bappoo, Buzuzi, and Kasiyandima (2004) discovered that traditional, lecturer-centered methods are ineffective in fostering conceptual comprehension of mathematics. According to these researchers, these

conventional approaches have drawn criticism for failing to help pupils develop their problem-solving abilities. Nyaumwe et al. (2004) contend that traditional methods of instruction do not incorporate discussion and questioning.

Teaching strategies can be either student-centered or lecturer-centered, according to Chifwa (2015). The researcher claims that while lecturer-centered instruction focuses on providing students with an explanation of the material for passive assimilation with little opportunity for clarifying questions, student-centered instruction helps students develop their mathematical reasoning skills (Petty, 2009 & Mohan, 2012). These scholars contend that under lecturer-centered techniques, the lecturers are the information source and the students are like empty canisters that need to be filled.

According to Zhang (2003), the reason why student-centered teaching methodologies are favoured is because they allow students to actively participate in the teaching and learning of calculus by providing them with time to reflect and think. According to Zhang (2003), students learn best when they actively participate in the process of experiencing, forming ideas, and addressing issues (Muzumara, 2008). Muzumara (2008) asserts that by participating in class discussions, voicing their perspectives, providing evidence for them, and accepting the viewpoints of their peers, students taught using student-centered techniques develop into critical thinkers and problem solvers. Students that are driven to study and take an active role in the process might create valuable information that will last a lifetime.

Students can acquire a variety of skills through class discussions, according to Muzumara (2008) and Petty (2009). These skills include listening intently, speaking clearly, taking notes, being open to new ideas, focusing, being attentive, arranging their thoughts, and respecting the opinions of others. According to Muzumara (2008), students are given the chance to put together opinions, assess ideas, look critically at their own thinking, and ask questions. Another point brought up by Denhere, Chinyoka, and Mambeu (2013) is that educators are expected to be specialists who can plan difficult mathematics assignments that push students without making them feel frustrated or less motivated.

2.4 Constructivist Approach to the Teaching of Calculus

Mathematics education research found that traditional mathematics teaching and learning methods have failed to improve students' understanding, as indicated earlier. The constructivist approaches above support the kind of mathematics teaching and learning where students are put at the centre of learning - a student-centered approach. Under these merged theories, as explained, students are no longer passively receiving readymade information but building their knowledge actively. Students are given ill-defined problems in which they are supposed to construct meaning and solve them with minimal teacher support.

Students are expected to be independent and become self-directed learners by taking full responsibility of their learning (Makonye, 2016). This means that students take control over their learning as they search for knowledge, concepts and skills required to solve mathematical problems. Makonye (2016) argues that self-directed learners tend to obtain higher grades in mathematics. This researcher, urges South African mathematics educators to learn how self-directed learning can be incorporated to empower learning. This study accepted the challenge and explored the productive struggle and professional noticing interventions in promoting self-directed learning. For Makonye (2016), students should believe that the success of their learning is a result of their efforts. The issues of struggle and self-directed learning subsume the idea that learners construct their knowledge to make any headway in their work. Ernest (1996) asserts that knowledge construction is a cognitive process driven by self-mental regulation, meaning that the individual must be actively willing to learn. Crain (1992) argues that teachers do not intend to teach errors and misconceptions, but students make sense of the concepts taught and, in doing so, develop errors and display misconceptions in the process. Nesher (1987) regards error making as the expertise of learners. This may mean that error making may be the only contribution that

students make. According to Nesher (1987), teachers should welcome errors and misconceptions because the road to a state of expertise is paved with these errors and misconceptions. Nesher (1987) further argues that each error has the potential to become a significant milestone in learning. Learners hold errors and misconceptions as they try to assimilate and accommodate new knowledge in prior acquired knowledge structures (Piaget, 1964). Unfortunately, learners' errors and misconceptions sometimes resist instruction because they are the psychological states created by themselves (Smith et al., 1993). Smith et al. (1993) argue that traditional teaching methods will not help address the problem in areas where errors and misconceptions are deep-rooted. Smith et al. (1993) call for new teaching strategies in teaching and learning mathematics. This study found it important to explore innovative pedagogical approaches such as productive struggle and professional noticing in which students are deeply involved in problem-solving situations. In such approaches, students correct themselves instead of the teacher. This helps to overcome the resistance to new knowledge referred to above.

2.5 Teacher responsibilities towards a constructivist classroom

Livy, Muir and Sullivan (2018) indicate that teachers should create a classroom culture that values and promotes learning struggles by providing students with challenging tasks. These researchers argue that teachers should allow students to explain and justify their responses of mathematical tasks. Roble (2017) strongly opines that current classrooms need to shift from traditional teaching methods to strategies that allow for problem-solving that provoke higher-order thinking skills.

According to Roble (2017), students, who learn from their struggling and mistakes, ultimately improve their ability to attempt and complete tasks they did not know how to solve. To encourage students to stay with complex tasks, lecturers must allow for clarity-seeking questions and motivation, allow thinking time, and acknowledge performance. During teaching and learning of mathematics, lecturers must provide non-routine problems that cannot be solved with formulas. These call for lecturers to become experts at creating questions requiring students to think outside

the box. Unlike traditional methods of teaching and learning mathematics, lecturers must call on students who may not know the correct answers and probe them. This follows that lecturers must develop a habit of praising students' efforts for both correct and incorrect responses and adopt a saying that 'No question or response is foolish'. Lecturers must give students reasonable time to work through given problems. At the end of it, lecturers must give students informative feedback.

Unlike what happens in a traditional teaching and learning atmosphere, the lecturer's role is mainly to try and avoid giving answers and instead ask questions that throw students into deep thinking. The lecturer is an investigator of how students think through a task. The new method of productive struggle and professional noticing empowered students to become problem-solvers and building their confidence. Likely, students work together and become a community of learners who support each other.

Lecturers are responsible for creating tasks that engage students in thinking with some perplexity, confusion, or doubt (Dewey, 1933) and yet were within students' zone of proximal development (Vygotsky, 1978). This was seen happening in this study. Dewey (1933) argues that lecturers must not look at students' struggles when taking too long to arrive at correct answers and quickly think of assisting them and showing them an effective procedure to solve the problem. This will discourage self-directedness in the classroom and lead to the detriment of students' learning and conceptual understanding of new mathematical ideas.

Sengupta-Irving (2015) indicates that struggling to solve problems allows students first to fail, make mistakes and debate ideas and possible solutions. This exercise leads to making students learn to reason mathematically. In agreement with this, Warshauer (2015) found that when students were given problems to solve, they had difficulties getting started on the problem, did not know what processes to carry out and could not explain their work and mathematics thinking. Lecturers should not see these students struggling and quickly resort to giving them solutions to problems because the cognitive demand of the task will be lowered (Stein et al., 2000). Instead, the lecturer must

support these students' struggles by probing their thinking and asking questions that will make students ultimately find ways to solve given problems. Warshauer (2015) advises that lecturers must place student-centered activities at the centre of instruction to create and allow for opportunities for students to grapple with challenging mathematical problems regularly.

Researchers (Sengupta-Irving, 2015; Warshauer, 2015)) in this section are making a few important points to consider. Students' struggles are an important part of learning mathematics. They help students learn how to solve problems and develop a deeper understanding of mathematical concepts. Teachers can support these students' struggles in their classrooms by providing opportunities for students to wrestle with problems, questioning them about their thinking, and helping them find their mistakes. Struggles in mathematics ultimately lead to better understanding and mastery of concepts. Mathematics should not be taught in a way that spoon-feeds students all of the steps needed to arrive at an answer. When this happens, students do not have the opportunity to wrestle with the material independently and may become dependent on teachers for every step along the way. This type of instruction also robs students of the satisfaction of conquering a problem through independent effort. This approach was totally avoided in the study.

Learning struggles allow for complex problems where students are encouraged to take risks, struggle and fail but still feel good about working on hard problems (Boaler, 2016). According to Boaler (2016), lecturers must adopt a classroom culture that values and promotes learning struggles at all times. The study subscribed to this advice.

2.6 Teaching Interventions of the experimental group of Lecturers

The study was interventional and had its main focus on making a change of behaviour in lecturers' practice towards the teaching and learning of mathematics in a TVET college. Its focus was on teaching interventions by giving specialized training to one of the two participant groups of lecturers who were compared towards the end of the study. The study examined the effects of the

intervention on the experimental group by comparing the results with the control group which was kept as a benchmark point of comparison. The primary purpose of this intervention research was to engender a desirable outcome of results towards the new teaching strategies of productive struggle and professional noticing against the old traditional mathematics teaching and learning methods.

Cusack, Del Mar, Chalmers, Gibson, and Hoffmann (2018, 2016) argue that an intervention must improve the understanding of key relevant concepts for evaluating the interpretation of research results. On the other hand, Kumar (1999) indicates that the control group need not undergo training because it has to serve as a benchmarking point of comparison (Levy & Ellis, 2011; White & Sabarwal, 2014). This was done in the study.

The two groups were comparable in every respect except for the intervention. For this study, the intervention was a focused teaching session that deviated from the existing teaching methodology to improve the teaching and learning of TVET mathematics. The intervention focused on key concepts: productive struggle and professional noticing. Petursdottir, Haraldsdottir and Svavarsdottir (2019) assert that intervention should impact and enhance positive research outcomes. The details on how the interventions were done have been detailed in the data collection chapter.

2.7 Errors and Misconceptions in the Teaching and Learning of Mathematics

Olivier (1989) defines an error as a wrong answer due to systematic planning in that it is applied regularly in the same circumstances and defines misconception as an underlying conceptual structure that gives rise to errors. According to Olivier (1989), a slip is a wrong answer that happens carelessly and is quickly detected and corrected spontaneously. This researcher argues that slips are unavoidable and can be done by both novices and professors. In agreement with this,

Voon, Julaihi and Tang (2017) indicate that a mathematics error due to carelessness is less serious. On the other side, Luneta and Makonye (2010) indicate that an error is a mistake, slip, blunder or inaccuracy and a deviation from accuracy. Researchers assert that errors are symptomatic of misconceptions (Riccomini, 2005; Nesher, 1987). According to Makonye (2010), errors and misconceptions are related but different. The definitions above mean that errors are indicators of the existence of misconceptions.

Over the past decades, mathematics education researchers have grappled with the idea of trying to understand the errors and misconceptions in teaching and learning mathematics. Most of these researchers agree that errors and misconceptions are significant and should be addressed in the teaching and learning of mathematics (Luneta & Makonye, 2010; Nesher, 1987; Olivier, 1989).

Olivier (1989) acknowledges that errors in teaching and learning mathematics are functions of cognitive variables, teachers, the curriculum, social factors, affective factors, emotional factors, motivation and attitudes. This researcher argues that errors and misconceptions are students' efforts to construct their knowledge intelligently. Misconceptions should, therefore, not be uprooted and avoided because they are part of the learning process and should be welcomed by both teachers and students. Li (2006) indicates that student errors are a symptom of misunderstanding.

Smith, DiSessa and Roschelle (1993) indicate that errors are visible in learners' artefacts, such as written text or speech, and further assert that misconceptions can even be hidden in correct answers. It means that educators need to listen carefully to determine why learners give the answers they give so that they can correctly follow learners' mathematical reasoning and attend to it.

Sarwadi and Shahrill (2014) indicate that if a learner cannot assimilate and accommodate, a gap is created in the learning of the concept, which leads to errors and misconceptions. These researchers advise that errors and misconceptions must not be seen as obstacles or 'dead ends' but must be

regarded as opportunities to reflect and learn. Teachers should therefore prescribe appropriate instructional strategies to be more diagnostically oriented to avoid any subsequent major conceptual problems.

Donaldson (1963) distinguishes between three types of errors in teaching and learning mathematics: that is - structural, arbitrary and executive. According to Donaldson (1963), structural errors arise from failure to grasp some principle or certain rule which is essential to finding the solution; arbitrary errors are said to be those in which the subject behaves arbitrarily and fails to take account of the constraints laid down in what is given, for example - arbitrary errors could be caused by learners wanting questions to fit and suit what they know or are familiar with and lastly executive errors involve failure to carry out manipulations or procedures even though the required concepts have been understood.

Eisenhart, Borko, Underhill, Brown, Jones and Agard (1993) describe two types of errors: procedural and conceptual. According to these researchers, procedural errors are associated with procedural knowledge and refer to mastery of computational skills and knowledge of procedures for identifying mathematical components, algorithms and definitions. In contrast, conceptual errors are associated with conceptual knowledge and refer to knowledge of the underlying structure of mathematics, which is about the relationships and interconnections of ideas that explain and give meaning to mathematical procedures.

Radatz (1979) classifies three types of errors in teaching and learning mathematics: errors arising from a lack of mastery of the language used, difficulties in obtaining visual information and lastly, those arising from a lack of prerequisite knowledge and skills. The language issue has also been raised by Mofolo-Mbokane, Engelbrecht and Harding (2013), who discovered that students need a sufficient mathematics register due to their poor mathematics background.

It then follows that teachers should know their students` error types for them to be able to attend, interpret and decide how to respond to them.

2.8 Schematic Representation of the Theoretical Framework

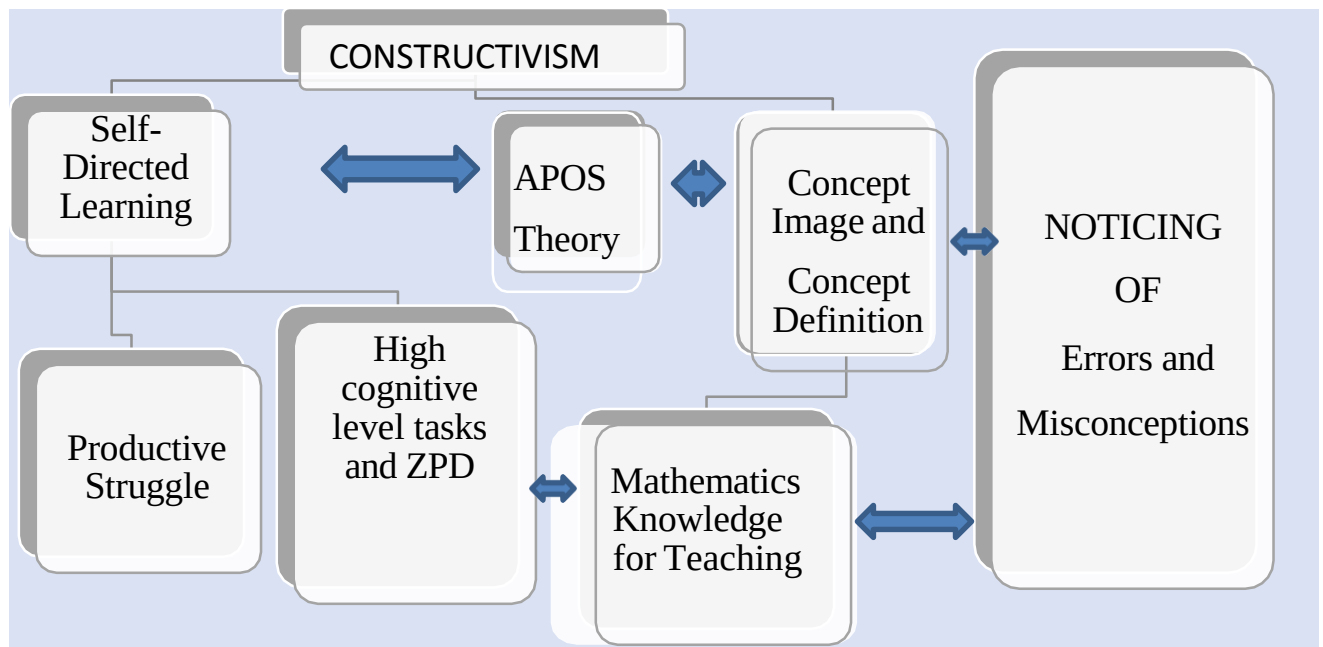


Figure 2: Schematic representation of Theoretical Framework

The theories selected for this study have been explained above. Although they are stand-alone theories, each focusing on a particular area of emphasis, they are all pulling together to confirm their interconnectedness. The above is a schematic representation of the theoretical framework showing three theories reinforced by teacher mathematics content knowledge. The three theories are APOS theory, Concept Image and Concept Definition, as well as Self-directed theory.

Mathematics knowledge for teachers has been added through literature to give meaning to the merger of these theories.

2.9 Conclusions

This chapter has outlined the constructivist grand theory underpinning the study. The middle-order theories have been included to put more meaning to the grand theory. The `Mathematics Knowledge for Teaching` (MKT) framework has been added to support the implementation of the theories in the teaching and learning space. The chapter also explained how all the theories, including the `MKT`, should affect mathematics teaching and learning and further explained the teacher responsibilities that should go along during productive struggles and professional noticing in a mathematics classroom.

CHAPTER THREE: ANALYTICAL CONCEPTUAL FRAMEWORK

This chapter outlines the analytical, conceptual framework with well-thought constructs and their relationships and their relevance to the study. Peshkin (1993) defines conceptual framework as the researcher's explanation of how the research problem would be explored. Adom, Hussein and Adu-Agyem (2018) indicate that the conceptual framework narrates all actions to be taken by the researcher in carrying out the research study.

Two main concepts impacting the study are - students' productive struggle and teachers' professional noticing. Lobato's teaching by telling method, as well as active *listening* taking place in classrooms, were added to intensify meaning to these two concepts. It made sense to include marking tests and feedback-giving sessions as they form an integral part of teaching and learning. Lastly, a schematic representation of the analytical conceptual framework was given.

3.1 Students' Productive Struggle

The effort made by students to understand and solve mathematical problems in order to figure out something that is not immediately clear is referred to as productive struggle (Hiebert & Grouws, 2007). According to Sayster and Mhakure (2020), students' difficulties with mathematics are crucial to both their intellectual development and their ability to fully grasp and deepen their understanding of mathematical topics. These scholars contend that opportunities for strengthening one's conceptual comprehension of mathematical topics are presented by the confusion and doubt that are brought about by difficulties encountered, while solving problems. According to Hiebert and Grouws (2007), problem-solving challenges help pupils understand mathematics and are essential to learning the subject with comprehension. These researchers vehemently contend that if students are to acquire mathematics in depth, they must engage in difficult problem-solving.

Sayster and Mhakure (2020) indicate that teachers are expected to create a learning environment that values and promotes productive struggles among students by giving challenging learning tasks that are accessible to all students. This means that tasks must be within students' Zone of Proximal Development (ZPD). Nyikos and Hashimoto (1997) regard ZPD as the distance between the actual development level as determined by independent problem-solving and the level of potential development as determined through problem-solving under adult guidance or in collaboration with capable peers. Hiebert and Grouws (2007) argue that students' struggles will be productive when they are within their ZPDs. In agreement, Dickey, Ewing, Caruso and Fulmer (2018) assert that tasks given for productive struggle should be challenging and yet within students' zone of proximal development, Vygotsky's (1978).

Sayster and Mhakure (2020) argue that exposing students to complex problem-solving tasks beyond their cognitive levels, skills, and abilities can result in productive failures on the part of students. This is undesirable as it will discourage students and dampen their eagerness to try more problems. This scenario supports the idea that all struggles must be within students' ZPD's. On the same point, Warshauer (2015) indicates that during the productive struggle, students encounter difficulties while working on challenging mathematical tasks, creating opportunities to grapple with important mathematical ideas and processes. In agreement, Permatasari (2016) argues that struggles are a natural part of learning mathematics and improve understanding and retention of learned strategies in long-term memory. An important point raised by Warshauer (2015) is that lecturers must provide appropriate minimum guidance and support, maintain the mathematical cognitive demand of the tasks, and advance students productively towards understanding. The mathematical demand of the tasks must be maintained by giving as little direct support and structure as possible (Stein et al., 2000). These researchers found that it is very tempting for teachers to want to decline the demand for mathematics tasks to assist their learners by giving hints to learners. This must be avoided as much as possible because it defeats the same purpose for

which the struggles stand. For this study, declining the task demand will compromise the reliability and validity aspects.

The following table by Warshauer (2015) and Permatasari (2016) explains strategies and indicators of productive struggle in the teaching and learning of mathematics.

Action	Teaching Strategies	Student Indicators of a Productive Struggle
Question	Teachers ask questions that help students focus on their thinking and identify the source of their struggle, then encourage students to build on their thinking or look at other ways to approach the problem.	Students ask questions to identify the source of their struggle, write down their ideas, clarify ideas with others, and consider alternative strategies to address their struggle.
Encourage	Teachers encourage students to reflect on their work and support students who struggle in their effort and not just in getting the correct answers.	Students use their effort to solve problems and try to make sense of their work, not only satisfied with a correct answer or that they perceive themselves as smart.

Give Time	Teachers give time and support for students to manage their struggles through adversity and failure by not stepping in too soon or too much, thereby taking the intellectual work away from the students.	Students use their time to develop and get through their strategies, evaluate their progress, and understand what they can do and what remains to be done.
Acknowledge	Teachers acknowledge that struggle is an important part of learning and doing mathematics.	Students persist in their work to make sense of and solve their problems without giving up or getting discouraged easily.

Table 1: Warshauer (2015) and Permatasari (2016) explain strategies and indicators of productive struggle in the teaching and learning of mathematics

The information provided in Table 1 was used to check for productive struggle strategies and student indicators during the teaching and learning of calculus. This shows that teachers should ask good questions that encourage all students to find solutions on their own. Reasonable time should be given for students to respond to questions. Finally, teachers should acknowledge the effort put in by students in struggling to find solutions.

On the other hand, Edwards and Beattie (2016) argue that productive struggle activities promote mathematical meaning-making for students. According to Edwards and Beattie (2016), productive struggle enhances collaborative learning settings where students explore rich mathematical tasks together. Edwards and Beattie (2016) developed an instructional routine called 'Problem Cycle,' which, in their view, is responsible for effective instruction that makes students realize the true potential of learning opportunities. The cycle has the following four stages.

Stage	The description of what the teacher should do.
Problem Launch	This is about preparing students for productive struggle by creating a shared understanding of the problem to be solved. The teacher should ensure that students clearly understand the problem and what they are expected to do.
Working the Problem	Teachers must engage students in productive struggle and monitor their ways of thinking in preparation for later discussions. During this time, the teacher should recognize and note students' ways of thinking in preparation for the coming discussions.
Discussion	The teacher should announce students' ways of thinking, whether correct or incorrect, encourage students to learn new ways of thinking by understanding each other, and explicitly connect their thinking to the key concepts. At this time, students present alternative ways of thinking about the problem and see opportunities to analyse mistakes and misconceptions.
Conclusion	Teacher has to concisely highlight the key concepts drawn from students' thinking, express the concepts with appropriate notation and representations, and explicitly connect the lesson concepts with the course organizing concepts.

Table 2: The strategies of the `Problem Cycle` by Edward and Beaties (2016)

Table 2 shows stages that should be followed during the `Problem Cycle` of productive struggle. Clearly, lecturers should give calculus questions, give instructions on what is expected and then

sit back and monitor students' struggles. Later on, the lecturer is expected to come in and allow students to give their views and allow for discussions as well. Finally, the lecturer must wrap up and consolidate all indicated key areas.

3.2 Mathematics Professional Teacher Noticing

As a TVET College mathematics lecturer, teaching entails all activities, from planning lessons and presenting them to assessing students and giving feedback. During these activities, as teachers, we should notice all occurrences in class by picking, identifying, pointing out, and detecting all errors and misconceptions from students (Van Es & Sherin, 2002).

Van Es and Sherin (2002) and Thomas, Jong, Fisher and Schack (2017) define professional noticing as a skill that enables teachers to pay particular attention to the ideas raised by students and then act on such ideas. This is the teacher's capacity to attend to students' mathematical conceptions. The teacher has to interpret what students say and develop a productive instructional course of action. Van Es and Sherin (2002) argue that professional noticing leads teachers to identify what is important and make connections during classroom interactions.

Jacobs, Lamb and Phillip (2010) comprehend professional teacher noticing as a skill broken down into three parts - attending to students' strategies, interpreting their understandings and deciding how to respond to their understanding. Of utmost importance is that these researchers indicate that students' strategies provide a window through which teachers can gauge their students' mathematical understanding.

Choy (2015) asserts that professional noticing can only be done in a more interactive classroom. According to Choy (2015), professional noticing assists teachers in attending to students' ideas and making sense of them to make sound decisions about their teaching. Ding and Domínguez (2016) argue that professional noticing is mainly about how teachers and students communicate during a lesson.

Erickson (2011) argues that teachers who lack professional noticing skills normally focus on superficial features of classroom interactions to believe that their students understand the lesson. These actions include, amongst others – raising hands, staying on tasks and following classroom routines.

Professional teacher noticing is of vital importance to this study. One of the methods of collecting data was done in classrooms where students engaged in productive struggle activities. Lecturers were expected to notice almost all activities during lesson presentations professionally. The researcher observed and took notes of all occurrences in class. Observation tools were used to collect data; see appendices 10.5 to 10.7. The researcher recorded findings and later analyzed, discussed and interpreted them.

3.3 The Skill of Listening in mathematics classrooms

Saricoban (1999) explains listening as identifying and understanding what others say. According to this researcher, listening involves understanding a speaker's accent or pronunciation, grammar and vocabulary, and grasping meaning. In the researcher's opinion, listening and hearing form an integral part of the teaching process to the extent that teaching and learning will be incomplete without them. Again, listening and hearing is about – paying attention to what is said and appropriately responding to the call.

Who should listen, and when should it happen in a classroom setup? Is it a case that everyone should listen, including the teacher? Must listening take place throughout the lesson? The answer to both questions is most definitely `Yes`.

Davis (1997) indicates that listening is an activity of all senses through the ears, mind and body. According to Davis (1997), listening can be evaluative, interpretive or hermeneutic. Evaluative listening happens when the teacher has pre-determined responses that leave no room for deviation. This happens when the teacher is not interested in what the student says because of the pre-

determined responses he/she has. From this perspective, it suggests that mathematics is consistent, primarily a system of already established facts, formal truths that cannot be altered and challenged. Secondly, interpretative listening is about making sense of what students say. The teacher probes, paraphrases students' answering, creates opportunities for discussions and nurtures emerging understanding in students. Lastly, hermeneutic listening takes place when teachers create opportunities for interactions and come together in shared or joined actions amongst themselves and their students. This avoids subject matter transmission as well as teaching by telling. Instead, it allows for teacher participation during classroom activities. This suggests that a teacher should become a student in his/her classroom and learn as other students are learning. Davis (1997) opines that teachers should participate with head nodding and making "unh-hunh" expressions during their interactions with students. On the other side, Arcavi and Isoda (2007) argue that listening to students in productive ways is in line with the basic tenets of the constructivist approaches because it is not a passive undertaking. The argument put forward by Arcavi and Isoda (2007) is that careful listening sustains discussions and makes questioning and probing possible during the teaching and learning of mathematics.

The new productive struggle interventions proposed for the study are learner-centered and advocate for discussions during teaching and learning sessions. Listening makes discussions possible. For Yagang (1993), listening enables teachers to identify and understand what students say. For this study, listening was exercised during class discussions on calculus tasks. Lecturers listened to notice and hear out students' mathematical thinking with the ultimate goal of addressing faulty mathematical ideas in students. At the same time, students were also expected to actively listen to their teacher because otherwise, they would need to capture what is expected of them to do correctly.

Wallach and Even (2005) argue that listening carefully and listening to what students say leads to teachers understanding their students' mathematics conceptions and ways of thinking. These researchers also indicate that it is important for teachers to listen attentively to their students and

correctly hear through because, at times, what teachers hear could be different from what students are saying or doing. In addition, Wallach and Even (2005) argue further that evaluating students for understanding is an integral part of instruction. During class discussions, teachers need to assess students understanding by listening to their mathematical deliberations and the nature of their participation by being sensitive to their feelings (Wallach & Even, 2005).

In this study, the teacher and students exercised listening and hearing skills fully. Lecturers created conducive classroom atmospheres that allowed for discussions to take place. During discussions, the teacher deeply probed students leaving them with one option of answering the question or asking back for clarity. Teachers listened carefully to students talk and responded appropriately during the interactions. At the same time, students heard out their teachers. Most importantly, teacher professionally noticed errors and misconceptions in their students` mathematical ideas/thinking.

3. 4 Lobato's a constructivist way of telling as opposed to traditional ways of telling.

Traditional methods of mathematics teaching and learning could be better and more effective in bringing about understanding in students (Tall, 1993; Battista, 1999). It is found that the traditional teaching-by-telling method does not create opportunities for students to reason, justify and argue their mathematical thinking standpoints but instead promotes a superficial understanding of mathematics and learner dependency (Crain, 1992). Smith et al. (1993) found that traditional teaching methods by telling failed to address deep-rooted mathematical errors and misconceptions. Roble (2017) strongly opines that current classrooms need to shift from traditional teaching methods to strategies that allow for problem-solving that provoke higher-order thinking skills. This is the gap area that the research study filled.

The moment teachers believe that mathematics is a fixed set of rules, facts and procedures, surely their teaching will be centered on telling students how to carry out those procedures (Battista,

1999). Constructivist theories do not allow teachers to stand before their classes and impart facts and procedures to students (Richardson, 2001).

Lobato, Clarke and Ellis (2005) strongly believe that constructivism has unjustly downplayed and removed away `teaching by telling` without refining how it should be done in a mathematics teaching and learning environment. In support of Lobato et al. (2005), other renowned researchers (Clement, 1997; Cobb, 1994 & Ernest, 1995) argue that some researchers believe that constructivism is against teaching by telling students anything that students need to know. These researchers argue that students cannot construct all knowledge, yet these researchers agree that traditional telling has some drawbacks. According to these researchers, traditional telling methods focus on procedural knowledge at the expense of conceptual knowledge elevating the teachers as the last authority and ultimate arbiter of mathematical knowledge and truth, undermining the possible students` cognitive engagement and contribution, making mathematics problems look like they have one solution path, creates no opportunities for teachers to students to learn about their students' ideas and mathematical strategies.

Instead, Lobato et al. (2005) indicate that introducing new information at critical moments by telling students catalyzes the development of new ideas. It is argued that teachers should provide information if it is required for students to continue their problem-solving efforts and if they need help finding it themselves (Lobato et al., 2005). These researchers argue that facts and ideas should be presented as something to consider and not as a prescription to follow. This gives enough bases for teachers to tell without fear of suppressing students' mathematics. The point made is that the teacher should introduce some information. According to Lobato et al. (2005), teachers must become facilitators and resort to telling only when there is a need to do so. Teachers need not act as the sole source of knowledge but should still be allowed to introduce new knowledge into their classrooms at critical moments.

Teachers need to ask questions to their students in a manner that the questions actually tell, and again, some statements may sometimes question more than they tell (Lobato et al., 2005). Teachers should be allowed to tell students when they want to - describe new concepts, summarize students' work, give critical information and check students' mathematical thinking. According to Lobato et al. (2005), teachers need to know when to engage in teaching by telling and how to do it.

3.5 Marking of Assessments and providing feedback to Students

Marking is a process whereby teachers judge the standard of responses and decide how well students answered questions asked in a test or examination by awarding marks or points for correct answers (Easdown et al., 2018). This section on marking of assessments was included because it sheds some light on term test scripts analysis, one of the three methods of data collection explained later in the following chapters. These researchers assert that during the marking process, teachers use marking guidelines or marking schemes which show how marks should be awarded to students. De Moira, Massey, Baird and Morrissey (2002) indicate that marking should be free from bias to prevent students from being both unfairly advantaged and disadvantaged. Morphett, Giagos, Gunn and Reid (2018) describe the traditional and manual approach to marking, which involves teachers awarding marks to certain steps or parts of a problem's solution. These researchers indicate that marking may also be done with written comments and annotations on the students' work to serve as feedback.

Grove and Good (2020) describe feedback as an essential part of the assessment, which provides students with information on how they fared in the assessment. According to these researchers, feedback should enhance student performance by pointing out where errors and misconceptions occur. On the same breadth, Chalmers, Mowat and Chapman (2018) indicate that feedback is given mainly to assist students to improve their performance. Similarly, Dawson, Henderson, Mahoney, Phillips, Ryan, Boud and Molloy (2019) argue that the sole purpose of feedback is to improve performance on the part of students.

Dawson et al. (2019) argue that assessing without giving feedback is completely unproductive for students. It is because feedback provides more detailed guidance to students and helps them improve their knowledge, understanding, and skills. Crisp and Johnson (2007) indicate that annotation is a feedback component. In agreement with this, Ball (2010) views annotation as an effective device for student feedback. This means that annotation makes feedback more meaningful. These researchers argue that it is a common practice for teachers to annotate their students' scripts during marking by way of underlining, circling some parts of the answer, or even commenting on the scripts. Annotation is seen to act as a communicative device between the student and the marker. Crisp and Johnson (2007) opine that annotations serve two purposes communicating the marker's decisions and facilitating of marker's judgment. They also argue that feedback should be specific, accurate, given on time and focused and should encourage students too.

3.6 Graphical representation of the conceptual framework

Miles and Huberman (1994) indicate that the conceptual framework serves as an anchor for the study and is referred to at the data interpretation stage. According to Miles and Huberman (1994), the conceptual framework allows the researcher to gather general constructs into intellectual bins. Conceptual framework may be given graphically or in a narrative form showing key variables or constructs to be studied and the presumed relationships between them (Miles & Huberman, 1994). Table 3 shows the conceptual framework of the study. The main focus is on students' productive struggle and professional noticing, as shown in the representation. Lecturers are responsible for designing challenging calculus tasks that throw students into productive struggles. As shown in the diagrammatic representation, the lecturer's observation of students' productive struggles allows for professional noticing of these students' thinking about the task at hand. Listening and hearing skills come into play when lecturers and students ask questions to each other. Students could manifest their erroneous mathematical thinking when they engage in productive struggles. During this time, the lecturer should notice students' mathematical ideas, attend,

interpret and decide on how to respond to them. In the case of written work, lecturers notice students' mathematical ideas when they mark official students' term tests. When lecturers know error types and misconceptions from their students, they are likely to attend to such errors and leave students with improved understanding.

See in table the graphical representation of the conceptual framework for this study:

See in table the graphical representation of the conceptual framework for this study:

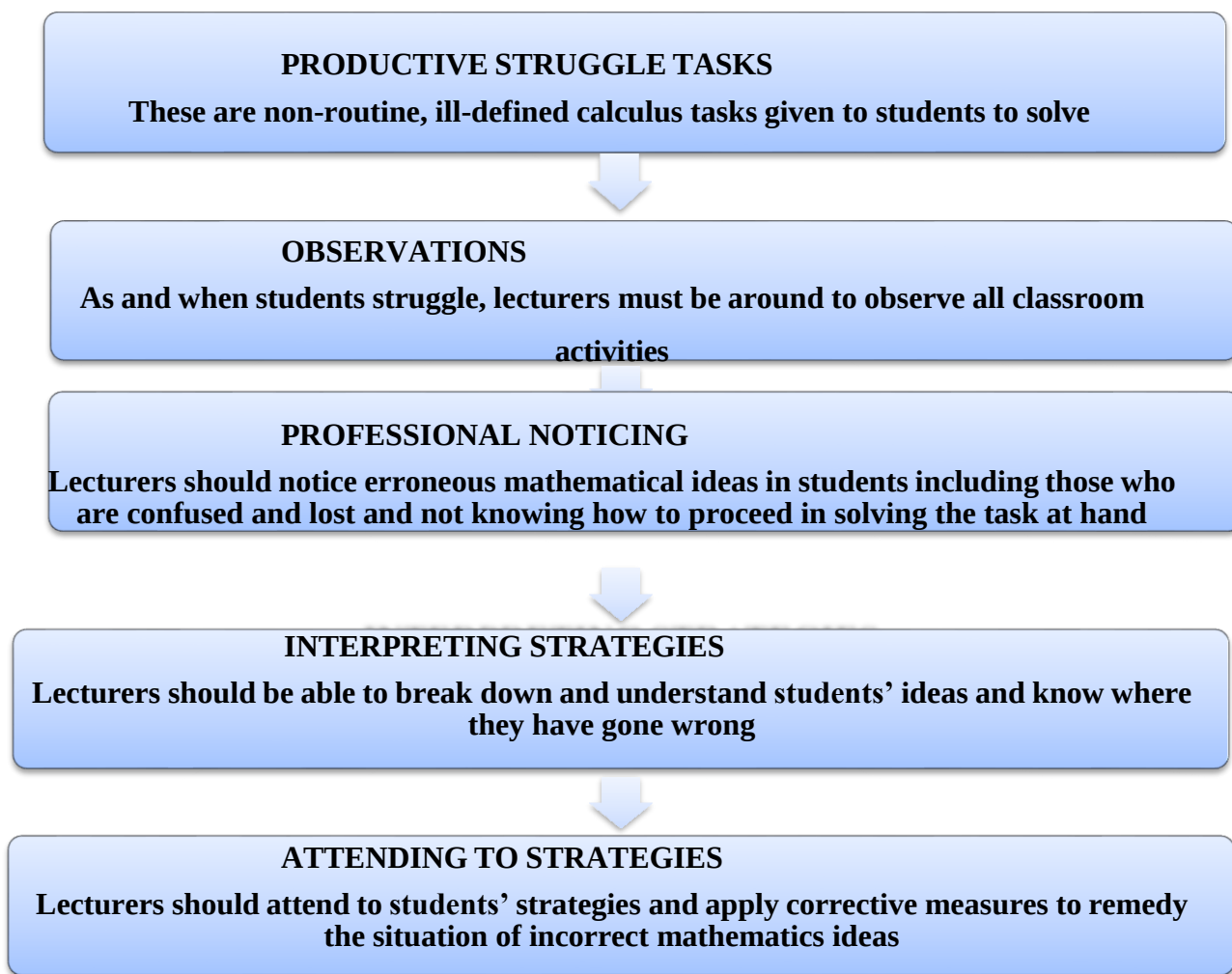


Table 3: Graphical representation of the Conceptual Framework

3.7 Conclusion

This chapter focused on new teaching and learning strategies for productive struggle and professional noticing. The students' productive struggles and the teacher's professional noticing, listening and hearing served as mechanisms to determine errors and misconceptions in students' thinking. These new teaching and learning strategies are capable of exposing weak areas in students which is platform suitable for teachers to professionally notice, interpret and decide on how they are going to attend the mathematics erroneous ideas.

CHAPTER FOUR: RESEARCH DESIGN AND METHODOLOGY

This chapter presents and explains the research design and methodology relevant to the study. Kuhn (1964) coined and brought the term `paradigm` into the academic community and defined it as a worldwide view that embodies the beliefs of scientists. Similarly, Willis, Jost and Nilakanta (2007) define a paradigm as the complete belief system, world view or framework that guides research and practice in a particular field of study. Anand, Larson and Mahoney (2020) indicate that a paradigm creates clear confines for a given field and defines which theory is relevant to the study. The research paradigm anchors the study in the applicable ontological and epistemological perspectives (Makonye & Luneta, 2014). Marsh and Furlong (2002) explain ontology as the very nature of being, namely, the theory of being. Simply put - ontology studies what can be known in the universe. It is about whether the real world is independent of our knowledge.

On the other hand, Marsh and Furlong (2002) describe epistemology as a viewpoint on the nature of the universe or the perspective on what we can know about the world and how we can know it. This indicates that epistemology is concerned with the theory of knowledge. It means that epistemology is the study of how we come to know what knowledge exists in the universe. According to Taylor and Settelaar (2003), philosophically, a paradigm consists of ontology, epistemology, and methodology.

Each paradigm has underlying ontological and epistemological presuppositions that are reflected in its methodology and methods (Scotland, 2012). According to Scotland (2012), methodology refers to a strategy or plan of action that involves the selection and application of specific methodologies. Methods are the specific procedures and strategies used to gather and analyze data in either a qualitative or quantitative way.

4.1 Research Paradigms

According to Taylor and Settelmaier (2003), several major paradigms govern educational practices. A few paradigms that befit the study will be discussed for this undertaking. Positivist Paradigm strives to investigate, confirm and predict law-like patterns of behaviour and is commonly used to test theories or hypotheses (Creswell, 2008). According to this researcher, generally, positivist paradigm has its focus on the objectivity of the research process. This researcher argues that positivism bases knowledge on recognizable facts and discards assumption about ultimate origins. According to Creswell (2008), the ontological position of positivists is that there is a single reality, while their epistemological position is that of the knower and the unknown being independent. Al-Saadi (2014) argues that in positivism; facts and values are very discrete, thus making it possible to conduct a neutral and value-free investigation. This means that the researcher detaches himself or herself from any impact on the research findings. Positivism epistemology holds that meaning and meaningful realities already exist in objects awaiting discovery (Al-Saadi, 2014). The above explained positivist research paradigm does not apply to this study.

Willis, Jost and Nilakanta (2007) describe Post-Positivist Paradigm as a slighter form of positivism that follows the same ideologies but allows more interaction between the researcher and research participants. (Creswell, 2008) indicates that post-positivism is an improved scientific method for the social sciences that aims to produce objective and generalizable knowledge about social patterns and seeks to affirm the presence of entire properties/laws in relationships amongst pre-defined variables.

4.2 Research Paradigm for the Study

This study falls within the interpretivist paradigm. The Interpretivist Paradigm rejects the notion that the world exists autonomously of our knowledge; but believes that the world is socially created (Marsh & Furlong, 2002).

Ontologically, the interpretivist paradigm embraces fallibilism in that absolute knowledge may not be possible. Scotland (2012) indicates that the ontological position of interpretivist is the view that reality is subjective and differs from person to person. This means that reality is individually constructed. One then realizes that there are as many realities as individuals. Makonye and Luneta (2014) argue that research is a shared process informed by participants, scrutinized and validated by others. In support of this statement, Powell (2001) also argues that from a fallibilism position, knowledge claims are inseparable from their claimants and instead tell us more about the claimants than the phenomenon in question.

This study is suited to the interpretive paradigm in its quest to harness students' mathematical thinking during their struggles by professionally noticing how they construct meaning while learning calculus. Taylor and Settelmaier (2003) argue that an interpretive paradigm enables researchers to build rich local understandings of the life-world experiences of teachers and students and the cultures of classrooms. An interpretive researcher will - interpret the thoughts and feelings of others, make sense of students' talk and know students better. A deeper understanding of all these helps improve the quality of mathematics education. Scotland (2012) argues that interpretive methods produce insight and understanding of behaviour, explain actions from the participant's perspective and do not take control over the participants.

4.3 Research Design

Kumar (1999) describes research design as a plan through which a researcher decides how to: collect information from respondents, select respondents, analyze collected data, and lastly, communicate the findings. Research design serves as a road map, a structure or strategy of investigation to be followed during the research journey to find answers to research questions (Kumar, 1999; Kerlinger, 1986). On the other hand, Bhattacharjee (2012) defines research design as a complete plan used for data collection in a research study. Bhattacharjee (2012) argues that research design must specify the data collection process, instruments to collect data and sampling process. According to Bhattacharjee (2012), quantitative and qualitative data can be collected using a combination technique such as - interviews, observations and documents. For this study, data collection techniques were done by observing lecturers in class, interviewing them, and sourcing data from official past-trimester students' answer scripts. Data collecting instruments, as well as sampling techniques, are fully explained in the next sections.

This research was underpinned by a quasi-experimental case study taking a qualitative research approach. The study also used a quantitative research approach to some little extent. Ultimately, the study took a near multi-method approach of using a mixed-method approach. Onwuegbuzie and Johnson (2006) indicate that mixed methods draw from the strengths of both qualitative and quantitative approaches and minimize their weaknesses in studies. These researchers argue that the mixed method approach gets deeper into the dataset to understand its meaning and further uses one method to verify findings from the other. Data was collected using a quasi-experimental approach of the experimental and control groups. The experimental group undertook the prescribed treatment, while the control group received no treatment and was used as the benchmarking point of comparison (Levy & Ellis, 2011; White & Sabarwal, 2014). Kumar (1999) indicates that both should be comparable as far as possible in every respect except for the intervention.

For this study, two groups of lecturers were purposely chosen from two campuses of a TVET College identified for the study. The two campuses were named `Campus A` and `Campus B`. Lecturers from `Campus A` formed the experimental group; meanwhile, those from `Campus B` were treated as the control group. These lecturers were comparable in all respect before the treatment of the experimental group lecturers.

4.4 Research Sample and Sampling Technique

A sample is a small subset of a population chosen to be studied (Lunsford, 1995). It therefore means that a sample represents a population. According to Lunsford (1995), a population is a group of people sharing some common characteristics.

Sampling is gathering information about the whole population instead of measuring every population unit (Khan, 2014). `Indeed, we cannot study everyone, everywhere doing everything` (Punch, 1998, p. 193). This means we can only study some cases of whatever situation that interests us and the ones we want to study. Neuman (2011) indicates that the size of sampling, whether small or big, does not matter; only the representativeness matters.

Shorten and Morley (2014) argue that sample selection is necessary and can determine whether research questions can be answered before the study begins. These researchers argue that most studies conducted with entire population census data could have used samples instead. They found that such studies achieved low response rates instead of using adequate samples with high-quality data collection efforts, which resulted in more reliable, valid and generalizable results.

This study adopted a purposeful sampling technique. Khan (2014) opines that purposeful sampling is nearly closer to theoretical sampling. Theoretical sampling is based on concepts emerging from data (Strauss & Corbin, 1998). According to Koerber and McMichael (2008), theoretical sampling involves a well-defined sample systematically aimed at representativeness. Marshall (1996) indicates that a purposeful sampling technique selects the most productive sample to answer the

research questions. Etikan, Musa and Alkassim (2016) indicate that the purposive sampling technique is the deliberate choice of participants due to the qualities the participants possess. Onwuegbuzie and Collins (2012) indicate that the researcher must decide on the number of participants to select and how to select these sample members.

The sample for this study was purposefully selected from lecturers working in a TVET College with six campuses. Lecturers from only two campuses participated in the study – `Campus A` and `Campus B`. These lecturers offered mathematics in higher levels of study, N5 and N6 classes. They were identified from the respective campuses` timetables. They all had at least a three-year teaching qualification with mathematics as a major subject. Lecturers from `Campus A` formed the experimental group, and those from `Campus B` formed the control group. The following section gives the demographics of these two groups of participant lecturers.

4.5 Demographics of the Experimental Group Sample

Three lecturers from `Campus A`, one female and two males, formed the experimental group. The female lecturer, looking just over 30 years, had just completed an Honours degree in mathematics studies. This lecturer has a National Diploma in Mathematics. She joined the campus to teach National Certificate and Vocational (NCV) but later moved to Report 191 programs. She taught NCV Mathematics Level 2 and Mathematics Level 3. When she moved over to Report 191, she was allocated N4 Mathematics classes. After doing N4 Mathematics for a few trimesters, she was allocated N5 Mathematics classes. This lecturer was observed during her teaching of mathematics N5 classes.

The two male lecturers have National Diplomas as their highest qualifications, with Mathematics as a major subject. They taught TVET mathematics, N5 and N6 higher classes for over 15 trimesters. Their ages ranged from 38 - 43 years.

When sampling for the study, these lecturers did not know whatsoever of productive struggle or professional noticing before the training. They used old traditional methods of mathematics teaching and learning.

They were trained on students' productive struggle and teachers' professional noticing and related activities. They were robustly engaged in these new teaching strategies. The detailed intervention process is explained later in the study.

4.6 Demographics of the Control Group Sample

Three lecturers, one male senior lecturer, and two females formed the control group. They came from 'Campus B'. The senior lecturer had three junior mathematics lecturers reporting under him. He taught higher classes of mathematics N5 and mathematics N6. He had a National Diploma with mathematics as a major. He taught TVET mathematics for over five trimesters. He was in his early thirties.

The two female lecturers qualified with National Diplomas with mathematics as a major subject. Their ages ranged from 35 - 42 years. One lady taught TVET mathematics for over 30 trimesters, and the other had been teaching the same for about six trimesters.

The lecturers in this group were not trained in the professional observation of teachers' and students' positive struggles. It served as a standard by which to compare the experimental group previously described.

A meeting was scheduled to go over the research protocol and all of the expectations for the participants, including the requirement that they provide written consent to the researcher. To explain the procedure and get their permission for the observation process, meetings with students were scheduled.

To make sure that the control group participants understood the goals of the study and their part in

it, they had to give their informed permission. Communicating in order to be transparent was essential to upholding moral principles and enhancing the reliability of the study. The purpose of the control group was explicitly stated in the study to guarantee that they receive the proper amount of intervention or treatment in order to reduce any possible ethical issues.

The meticulous selection of the control group played a vital role in mitigating biases and guaranteeing the validity and reliability of the research. In essence, the control group was utilized to establish a correlation between treatments or interventions and results.

Taber (2019) indicates that multiple published research suggests that the innovation under consideration has the potential to be more successful than present teaching approaches. These studies aim to demonstrate this by comparing a novel therapy against a typical procedure that is not regarded to be very successful. This researcher believes that consent from participants in educational research studies should always be voluntary and informed. Because participants cannot be expected to fully understand the purpose and background of the research in the same way that researchers do, researchers bear a significant amount of responsibility for the ethics of experiments.

Ruthven, Mercer, Taber, Guardia, Hofmann, and Riga (2016), as well as Moore, Graham, and Diamond (2003), agree that it is best practice to offer an innovation (treatment) to the control condition whenever possible, or to create a design in which all participants experience the experimental condition at some point during the study. These researchers claim that such a design offers scientific and ethical benefits, as well as the potential to reduce any uncontrolled discrepancies between the two groups. According to these researchers, it is important to consider what should happen to the control group at the end of the research, regarding ethical considerations, the scientific validity of the study, and the implications for future research. This means that the research that involves control group, must always take into account the ethical considerations.

The aforementioned researchers make it quite evident that, should the intervention prove to be successful, the control group will receive it at the conclusion of the study (Taber, 2019). This

viewpoint is undoubtedly predicated on the beneficence principle, which maintains that researchers have an obligation to maximize benefits and minimize harm to study participants. It might be morally acceptable to give the intervention to the control group if it has been demonstrated to be helpful in order to enhance their results.

4.7 Research Sites

The research study took place in a TVET College situated in the far East Rand, Gauteng Province, in South Africa. This identified TVET College has six campuses or delivery sites. Two campuses offer old Report 191, N4 to N6 engineering courses, which were primitively called Nated Courses. The study was conducted in the two campuses offering Nated engineering courses, namely `Campus A` and `Campus B`. These campuses are approximately 15 kilometers apart from each other.

4.8 `Campus A` Site

`Campus A` is situated on the outskirts of a black township residence, strategically to accommodate students from all neighboring communities. It is a big campus running two schools, Report 191 (Nated School) and National Certificate Vocational (NCV) School. Nated programs are post-matric qualifications, while the NCV is a pre-matric qualification.

The Nated School has two engineering courses - electrical and mechanical and offers only N4 to N6 courses. The admission criteria to register for any N4 course require students to pass both matric Mathematics and Physical Science with at least 40 percent. Conversely, students who complete NCV qualification are allowed admission to register for a relevant N4 course.

This campus has a staff complement of over 70 lecturers, 50 classrooms and 5 different and well-equipped engineering workshops, as well as seven fully-resourced computer laboratories. Fourteen classrooms are set aside for use by Nated School, where the study took place. This Nated School had 12 lecturers, three females and nine males. Only two male lecturers and one female lecturer

participated in the study. These lecturers taught mathematics in N5 and N6 higher classes.

4.9 `Campus B` Site

`Campus B` is also situated just on the borders of a black township residence, a few miles away from `Campus A`. The narrative of this campus is nearly the same in all respects as `Campus A`, with the only difference being that it also offers civil engineering courses in both Nated School and NCV School. As a result, this campus is relatively bigger than `Campus A`. It has more lecturers, classrooms, workshops, and computer laboratories than `Campus A`.

The Nated School has 17 lecturers, 7 females and 10 male lecturers. Three lecturers, two females and a male, participated in the study.

4.10 Data collection strategies

Three data collection strategies of observations, interviews and document analysis in the form of students` term test scripts were used in the study. In support of this, Leydens, Moskal and Pavelich (2004) indicate that qualitative data collection mainly happens through observations, interviews, and documents. These data collection strategies are explained below:

4.10.1 Observations during Teaching and Learning of Calculus

An observation protocol was designed to collect data during normal lesson presentations. It is attached in the appendices section. Watson and Evans (2012) indicate that an observation protocol is a system for observing and evaluating a lesson presented by the teacher and further allows for the analysis of a lesson. Martin, Daley, Hutchings, Jones, Eames & Whitaker (2010) argue that observation is fundamental to classroom research because it provides real-time behaviour measures without reliance on participant reports. Lesson observation protocol is a tool that can be used to develop teachers professionally and improve their practice (Evans et al., 2015). These

researchers argue that most information captured during lesson observations are almost mathematics-based suggestions that focus on the Curriculum Knowledge of the Student (CKS) domain (Ball et al., 2008). These researchers argue that teachers draw on their knowledge of students when noticing and evaluating how students are progressing with a task. Ball et al. (2008) argue that many demands of teaching require knowledge at the intersection of content and students. This leads to knowledge of common student conceptions and misconceptions about particular mathematical content.

Both experimental as well as control groups were observed. Experimental group lecturers prepared challenging mathematical tasks that exposed students to struggles. During students' struggles, lecturers noticed students' mathematical thinking. At the same time, both lecturers and students asked questions from each other. All activities relating to teaching and learning were observed. Lecturers noticed, picked, identified and pointed out erroneous responses and misconceptions in students' mathematical thinking. The researcher recorded all observations per the consent that the participating lecturers and students negotiated. Observation protocol is provided in the appendix section (See appendices 7.4, 7.6 & 7.7). This protocol accommodates all activities regarding students' productive struggles and the lecturer's professional noticing.

4.10.2 Documents analysis.

The main aim of using document analysis is to gather facts (Owen, 2014). This researcher argues that documents are usually analyzed with some general idea that acts as selection devices of what one is looking for. The documents referred to here are the official end-of-trimester student tests. According to the Department of higher education and Training (Dhet), there must be only two-term tests per trimester (Curriculum Instruction, 2020/2021/2022/2023). Prior (2002) argues that documents form a field of research in their own right. This means that documents have much information for analysis and interpretation purposes. Yanow (2007) asserts that document reading and analyzing can be part of observational study or an interview-based project. Documents can

provide background information before designing the research project and can be used to clarify or challenge what has been said earlier. Caulley (1983) argues that researchers must choose documents closer to the event being investigated. The argument put forward by Caulley (1983) is that the longer the interval between the incident described and the document's writing, the less reliable the document is. This study considered the point made by Caulley (1983) by taking the students' immediate next term's test scripts from the time of observations. Porter and Masingila (2000) indicate that by analyzing students' written work, one can gain information about students' understanding of mathematics.

Most of the lessons observed were mainly in the calculus section of mathematics study. Typical test questions were asked to determine the derivatives or integrals of functions using various techniques studied during the course. Most students found calculus topics so challenging. Tests given and written by students had a greater potential to expose students to mathematical thinking. This mathematical thinking embedded in students' written responses assisted both the teacher and researcher in engaging in the professional noticing of students' mathematical thinking.

4.10.3 Interviews of Lecturers.

Dilshad and Latif (2013) describe 'interview' as a specialized conversation between persons for a specific goal related with some agreed-upon subject matter. According to Qu and Dumay (2011), the interview method of data collection is the skill of questioning and evaluating responses.

There are three types of interviews: organized, unstructured, and semi-structured. Structured interviews are rigid and allow for pre-established questions with only a limited number of response categories; that is, they allow for little or no deviations from the pre-determined set of questions, whereas informal or unstructured interviews use open-ended questions, and semi-structured interviews involve prepared questioning guided by identified themes consistently and systematically interposed with probes designed to elicit more elaborate responses (Qu & Dumay,

2011).

Qu and Dumay (2011) assert that careful interviewing in addition to a well-thought-out strategy can result in a rich data set. Semi-structured interviews are popular, according to researchers like (Qu & Dumay, 2011; Dilshad & Latif, 2013), because they are adaptable, accessible, and understandable. More importantly, they are capable of revealing significant and frequently concealed aspects of organizational and human behavior. Semi-structured interviews were employed in this study for the same reasons given above. Since the goal of interviewing is to get an understanding of other people's experiences and the meaning they assign to them, Knox and Burkard (2009) contend that prior to conducting any interview, thought must be made to the questions that will be asked. According to Shensul, Schensul, and LeCompte (1999), the flow of the interviewee's story, maintaining an excellent connection with the interviewee, and avoiding interviewer bias can all contribute to an interview's quality. According to a study by Ruffell, Mason, and Allen (1998), teachers who do not know much about mathematics do not enjoy teaching it very much and lack confidence when doing so. Observing and talking to such teachers will not help.

TVET mathematics lecturers from both campuses were interviewed after having observed their lessons. An interview protocol with sample questions is provided. See Appendix 10.8.

4.11 Data Analysis

Data analysis is defined by Bogdan and Biklen (2007) as a methodical process of sorting and organizing the information gathered from the collected data. Data for this research study was gathered through observations in the classroom, interviews, and documentation in the form of term test items.

Hatch (2002) asserts once more that data analysis is a systematic quest for meaning and a strategy to analyze data so that what has been learnt can be shared with others. This researcher goes on to say that data analysis should be about structuring and probing data in ways that enable researchers

to find patterns, identify themes, uncover linkages, build explanations, make interpretations, issue critiques, and generate theories.

4.11. 1 Data analysis methodology

According to Leydens, Moskal, and Pavelich (2004), it is advisable to concentrate on observational techniques, interviews, and document analysis when analyzing qualitative data. That is what this study has accomplished. According to Leydens et al. (2004), there are two main ways to record the outcomes of an interview: one is to take thorough field notes throughout the interview and update them right away; the other is to record the interview on an audiotape or videotape and have the transcript of the conversation written later. The first alternative was adopted in this study, with the interviewer taking comprehensive notes as the interviews progressed. This study's data analysis focused mostly on the theoretical constructions described in chapter two, with a special emphasis on the conceptual framework constructs described in chapter three. Under the theoretical constructions of APOS, Concept Image and Concept Definition, and Self-directed theories, data analysis focused on how productive struggle and professional noticing impacted mathematics teaching and learning. Lobato's teaching-by-telling approach and Ball's MKT were also used in conjunction with literature to help solidify the explanations surrounding these constructions.

4.11.2 Grounded Theory, Constant Comparison Method and Unit of Analysis

Glaser and Strauss (1967) define grounded theory as systematic discovery of theory from social research data. It means that this discovered theory is grounded or embedded in the data. Bryant (2002) and Bitsch (2005) assert that grounded theory is a methodology for developing inductive theories grounded in systematically gathered and analyzed data. Bitsch (2005) argues that grounded theory intends to generate new theory rather than forcing data into an existing theory and has no intention of verifying any grand theory. This is the position of the study. Similarly, Jacelon and O'Dell (2005) agree that grounded theory is a qualitative research methodology where

grounded theory is derived through analyzing data and answering research questions. These researchers say that the grounded theory is closely related to data collection and analyses. A grounded theory project does not begin with a theory but rather with a question, and what is relevant in answering this question is allowed to emerge during the research process. This study has done what is said above.

Khan (2014) indicates that the grounded theory approach applies one or more techniques to collect empirical data, like - interviews, observations, and document analysis. The analytic procedures in data analysis are based on the constant comparison method. Constant comparison serves to uncover and explain patterns and variations. Theoretically, sampling maximizes opportunities to compare events, incidents or occurrences (Khan, 2014). This researcher argues that data collection and analysis are closely related and constantly alternating. Khan (2014) clarifies that theory generation is not based on the raw data but on concepts and categories being developed from the raw data. According to Khan (2014), theoretical sampling is, to some extent, purposeful sampling and forms an integral part of the grounded theory approach as it leads the researcher to what to collect next (Charmaz, 2006). Therefore, it plays an important role in getting appropriate data that is important in developing theory. Charmaz (2006) argues that theoretical sampling allows researchers to discover a theory that perfectly matches data. This is because the researcher chooses what is needed and important in developing the theory and, by so doing, avoids wasting time on all sorts of things that have nothing to do with the developing theory. It was on this basis that the study used purposeful/theoretical sampling.

Of equal importance is the unit of analysis in this research study. Neuman (2011) indicates that the unit of analysis refers to the units, cases, or parts of social life that are under consideration, and these could be individuals or groups. According to Neuman (2011), a unit of analysis is influenced by the conceptual framework, data collection techniques and sample and size.

Systematic data analysis starts as soon as data is made available after collection. Kolb (2012) indicates that the constant comparative method is better suited when used with theoretical sampling

in developing concepts from the data during data analysis. According to Kolb (2012), the constant comparative method begins with raw data and ultimately yields a theory that has to be reported in either a narrative or a set of propositions. The constant comparison method shows how the participants interact with the phenomena under study (Khan, 2014).

In this research study, two groups of lecturers were observed teaching deep calculus topics. Secondly, the data analysis exercise focused on interview transcribed data. Leydens et al. (2004) indicate that interviews capture less observable data, for example, the participants' thoughts and perspectives, which can confirm or contradict, complicate or complement observational data. Lastly, the researcher checked how lecturers marked students' tests and what errors and misconceptions were noticed.

This brings to the fore the importance of carefully marking students' test scripts. To the researcher, marking reveals if students understood the lessons presented to them. Lecturers with experience in marking external examinations will likely manage the marking process better. My experience with external examinations, as Mathematics N6 chief marker for over 15 trimesters, advanced my marking skills to a greater extent. This experience assisted in checking how lecturers marked students' term test scripts.

The framework for analyzing errors and misconceptions is indicated in Appendix 10.9 on term test items analysis protocol. The framework shows the question asked, what the lecturer noticed, and how the lecturer marked the question. Lastly, it provides the error or misconception classification by the researcher. The framework is designed to show errors and misconceptions noticed by the researcher, which the lecturer should have noticed. Researchers, to mention but a few (Tisi et al., 2013; Wang & Cai, 2006; Cox, 1967) – all grappled with the idea of the reliability and validity of the marking process. They strongly advise that objectivity and accuracy of marking can be improved by eliminating biases, having detailed and clear marking schemes, training markers, being honest and marking only aspects that match assessment objectives.

4.12 Teaching Interventions

Teaching interventions were only made for the experimental group. The control group did not undergo training on professional noticing and productive struggle on mathematics teaching and learning strategies and therefore served as a benchmarking point of comparison (Levy & Ellis, 2011; White & Sabarwal, 2014). The two groups were comparable in every respect except for the intervention.

4.12.1 Interventions on Productive Struggle

Training of experimental group lecturers took place over six sessions of one hour each. Lecturers were introduced to and trained on students' productive struggle and teachers' professional noticing strategies of mathematics teaching and learning. The first session was dominated by robust discussions and clarity-seeking questions regarding the use of productive struggle and professional noticing strategies as opposed to old traditional methods. An agreement to use these new teaching strategies was happily reached. It meant that teaching by telling method, talk and chalk, dominating lessons, and teacher-centered approach were not going to be used.

Lecturers were taken through the work of Hiebert and Grouws (2007) that:

They had to know that kind and purposeful greetings to students set the right tone for teaching and learning to take place. Greetings are a way of recognizing students present in class. Purposeful, kind greetings make students feel welcome in class. Greetings maximize the willingness to participate in class interactions (Aung & Tepsuriwong, 2017).

They were made aware that students do not come to class to listen to them dominate lessons and watch them solve problems for them, but instead, lecturers have to provide safe learning atmospheres which create opportunities for students to actively participate in class activities. Safe learning atmospheres are classrooms characterized by: promoting problem-solving and reasoning, allowing for focused and fruitful discussions, purposeful questioning, and finally, allowing students to argue their mathematical standpoints. Lecturers were made to understand that

productive struggle and professional noticing approaches allow students to welcome and love challenges, be intrigued by mistakes, enjoy effort, and keep on learning. It was emphasized that teaching and learning mathematics without struggles has no progress or growth for students. Lecturers were encouraged to welcome errors and misconceptions made by students. It was shared that making mistakes and correcting them builds a culture where students try more and then learn more as they learn from such mistakes.

Lecturers were made aware that the planning stage of a lesson must consider launching the lesson by activating students' prior knowledge to allow them to draw on their experiences during the learning process. They were told to use good productive struggle tasks in their introduction and encouraged to give non-routine problems that may not be solved with formulas to engage their students in productive struggles. These are the kinds of tasks whose solutions are not immediately available and therefore students had to struggle first. They were warned never to decline or lower the cognitive level of tasks as this will defeat the purpose of the productive struggles. Lecturers were told the art of asking telling questions and asking statements that actually tells as explained by Lobato (2005).

The importance of classroom talk was highlighted. Lecturers were told that students have the tendency to manifest their erroneous mathematics thinking/ideas during active classroom interactions. They were told to stage activities that have potential to kick start classroom discussions. Students talk amongst themselves bring opportunities of paraphrasing each other's mathematical responses. This assist in students learning from each other.

In the end, lecturers understood that students should have opportunities to struggle while solving mathematical tasks. They were made aware that students will still experience discomfort when faced with a challenge but enduring the discomfort and working through the problem results in breakthroughs which lead to the satisfaction of overcoming a challenging task.

The meeting finally agreed that the results of productive struggle and professional noticing are likely to promote retention and discourage memorizing solutions and are the right tools to solve

problems and lessen students' dependency on waiting for lecturers to give answers.

4.12.2 Intervention of professional noticing

The experimental group lecturers underwent four one-hour sessions of professional noticing training. Professional noticing mostly focuses on how students and lecturers interact during lessons (Ding & Domínguez, 2016). They were introduced to the works of Thomas, Jong, Fisher, and Schack (2017) and Van Es and Sherin (2002), which view professional noticing as a skill that enables teachers to pay close attention to the ideas generated by students and then act on those ideas. Jacobs, Lamb, and Philipp (2010) further clarified that professional noticing can best be understood in three stages: teachers' capacity to (a) attend to the student's mathematical conceptions and practices, (b) interpret these conceptions and practices, and (c) decide upon a productive instructional course of action based on this interpretation. This was explained in simple terms: lecturers had to hear out what students say and think, interpret students' mathematics thinking, and decide on what to do in response to what students say.

Lecturers were made aware that the effective way to enhance professional noticing was to resource learner errors and misconceptions (Moru et al., 2014). The argument was that analyzing students' errors empowers the teacher to effectively support them in engaging with new mathematics concepts. That meant teachers should know about analyzing errors and detecting misconceptions to interpret these students' errors and misconceptions appropriately.

It was clear that professional noticing assists lecturers in attending to students' ideas and making sense of them to make sound decisions about their teaching (Choy, 2015). Finally, it was emphasized that professional noticing can only be done in a more interactive classroom (Choy, 2015). That meant lecturers had the duty to create classroom platforms that trigger class discussions in which they had to notice erroneous responses and misconceptions and then attend to them. It was for this reason that professional noticing strategy had to be paired with productive struggle strategy. They are inseparable.

Lecturers also learned the importance of exercising good listening skills during teaching and learning. They were told to carefully listen to their students if they wanted to notice erroneous statements during classroom discussions. They were told that people listen to hear out and obtain information, understand, interpret and respond (Jacobs et al., 2010). They were warned to respond only after ensuring they correctly heard and understood what to respond to.

The work of Davis (1997) on professional listening assisted in making lecturers understand the value of listening to their students. Thus three types of listening were explained to lecturers, which are evaluative listening (for pre-determined responses), interpretative listening (making sense of what students say) and hermeneutic listening (students' teacher interactions). It was emphasized that lecturers and students must listen professionally during classroom interactions to have meaningful discussions.

Lecturers were also taken through the work of Arcavi and Isoda (2007), who argue that listening to students in productive ways is in line with basic tenets of the constructivist approaches because it is not a passive undertaking. The argument put forward by Arcavi and Isoda (2007) is that careful listening sustains discussions and makes questioning and probing possible during the teaching and learning of mathematics. They were further told that Yagang (1993) says listening enables teachers to identify and understand what students say. Wallach and Even (2005) argue that listening carefully and hearing through what students say leads to lecturers understanding their students' mathematics conceptions and ways of thinking. Wallach and Even (2005) argue that it is important for lecturers to listen attentively to their students and correctly hear through because, at times, what

lecturers hear could be different from what students are saying or doing. In addition, Wallach and Even (2005) argue that assessing students for understanding is an integral part of instruction. During class discussions, teachers need to assess students understanding by listening to their mathematical deliberations and the nature of their participation by being sensitive to their feelings.

Lecturers indicated their acceptance of new mathematics teaching strategies and agreed to use them.

4.13 Trustworthiness and Credibility of the Study

Trustworthiness means the ability to be relied on and be honest, truthful, accurate and correct at all times. On the other hand, credibility refers to - having the quality of being trusted and being believed, that is - accepted as truth or honest. Rolfe (2006) asserts that a study is trustworthy if and only if the reader of the research report judges it to be so. This means that determining the trustworthiness and credibility of a research study is highly subjective and relies on individual judgment.

Trustworthiness and credibility of the study are major factors in reflecting the integrity of the research project (Glesne & Peshkin, 1992). Kolb (2012) argues that even though it is difficult to obtain absolute exactness of research, a triangulation technique can be used to increase the reliability of the interpretation of data by using multiple methods of data collection. Shenton (2004) indicates that triangulation involves using different methods, like observations, documents and interviews. This study used the three methods above to attain its trustworthiness and credibility. Triangulation refers to using more than one data collection method, which aims to enhance the validity and trustworthiness of the study (Glesne & Peshkin, 1992). According to Kolb (2012), validity is achieved by adopting a theoretical sampling of the study. Parry (1998) asserts that using constant comparison methods and theoretical sampling increases the validity of the research. Another way of increasing the integrity of the study is to honestly share potential limitations with readers (Collet-Klingenberg & Kolb, 2011). According to these researchers, using different methods compensates for their limitations and exploits their respective benefits. This study adopted a purposeful sampling in its attempt to attain validity and trustworthiness.

One of the most common criticisms of qualitative research is the idea of business. Stahl and King (2020) argue that biases can be minimized when a researcher engages in self-description and self-

reflection by discussing their position within the study and how their personal beliefs and past training have influenced the research findings. In this study, the researcher has indicated his position on some ideas and shared his experience as a TVET College mathematics lecturer and chief marker.

Jacelon and O'Dell (2005) opine that trustworthiness in qualitative research is established by ensuring rigour in data collection and analysis and using multiple data sources, which maximize the range of data that might contribute to the researcher's understanding of the issues. Stake (1998) argues that multiple sources of evidence provide opportunities for comparison of data among and between participants and different types of data sources. According to Jacelon and O'Dell (2005), seeking feedback from the study participants is part of establishing credibility as it aims at checking the accuracy of the researcher's interpretations of the participant's experience. This study did what the above researchers recommended by getting feedback from participants during semi-structured question approach interviews.

Similarly, Matlala (2015) indicates that using multi-method strategies for collecting data, namely questionnaires, interviews and observations, increases the trustworthiness and credibility of the study. Cohen, Manion and Morrison (2007) further argue that the exclusive use of one method may be biased or may distort the researcher's picture of the particular slice of reality being investigated. This study used three data collection strategies, as advised by the researchers above.

4.14 Ethical Considerations

The ethics clearance for this study was successfully applied for, approved and issued by the hosting institution. Khan (2014) indicates that it is the moral and professional obligation of any researcher to be ethical even when research participants are unaware of or undisturbed by ethics. Amongst other things, Khan (2014) advises that - the researcher should disclose all benefits and losses to participants, not cause any harm, and should not humiliate, degrade, or release harmful information

about participants. This research study has considered these ethical issues. Pseudonyms were used instead of participants' real names, and that was done to protect them from any potential harm.

Lecturers were told that participation in the research was voluntary, and the researcher got informed consent statements signed by the participants that informed them of their rights and provided brief information about the research study (Neuman, 2011).

The researcher told participants they might withdraw from participating at any time during the research study should they feel like doing so. No participant was forced to participate in this research study. They all agreed to participate in the study by giving their written consent. In addition, participants were told that they could withdraw from participating in the study at any stage of the research without any penalty.

Furthermore, the researcher assured the participants that everything remained highly confidential. The researcher knew that all physical evidence had to be kept in a lockable filing cabinet in the researcher's office, and electronic data files would be stored in a password-protected computer. Participants were told that all information shared with the researcher would be kept confidential at all stages of the process and even after the completion of the study.

To come out clear, Cohen, Manion and Morrison (2000) indicate that ethics relate to the rights and the interests of the participants in the research. This exercise helps to prevent participants against the threat of marginalization and exploitation. This study adhered to these conformations. Participants of the study were given information sheet packages with consent forms for willingness to participate in the study. Research ethical principles like anonymity, confidentiality, and voluntary participation were observed.

4.15 Conclusions

This chapter outlined the research paradigm for the study. It also described the research

methodology and the methodology befitting the study. Data collecting strategies have also been explained. Finally, it explains the trustworthiness and credibility of the study. The ethical considerations which are relevant to this research study were explained.

CHAPTER FIVE: DATA COLLECTION AND IMPLEMENTATION

This chapter presents three different methods used in collecting data for the study. The first method used was classroom observations during normal mathematics lessons, followed by document analysis of students' term test scripts and lastly interviews with lecturers.

This chapter first deals with all aspects of the experimental group lecturers and then switches to control group lecturers. It deals with observations, interviews, and term test script items of experimental group lecturers and then moves to control group lecturers to address the same.

5.1 Experimental Group Data Collection Strategies

Three data collecting strategies explained above were used to collect data for the experimental group lecturers.

5.1.1 Classroom Observations

TVET College mathematics lecturers have lesson plans which they use during their lesson presentations. Milkova (2012) describes a lesson plan as an instructor's road map of what students need to learn and how it will be done effectively during class. On the same note, Stauffer (2019) defines a lesson plan as a teacher's guide for what students need to learn, how it will be taught, and how learning will be measured. This means that a lesson plan briefly outlines how the teacher will teach, what students will learn and how the teacher will measure students' understanding

Participating lecturers were divided into experimental group and a control group. Each group had three lecturers named – Lecturer 1, Lecturer 2 and Lecturer 3. The lecturer-student ratio in class

was 1:25 owing to covid-19 protocols for social distancing. Students are named and the researcher is referred to as the researcher.

Each experimental group lesson is divided into three parts/stages, namely: firstly - lesson introduction and minimal guidance, secondly - new learning and productive struggles and lastly - lesson conclusion and a way forward. Each lesson opens with brief information indicating the type of mathematical problems learned on the day. Five lessons were observed from each group.

5.1.1.1 First Experimental Lesson

The first lesson observed was on differentiation techniques for implicit functions. The kinds of functions dealt with were:

1. $x^2 + y^2 = r^2$, and $\sin(x+y) = 1$
2. $\ln\sqrt{x^2 - y^2} = x - y$
3. $\cos(xy) = e^y - xe^y$ and
4. $x^2 + \ln\left(\frac{y}{x}\right) - 5x + 4 = 0$.

Each problem, except the first one, is of higher order and an examination standard-type question. Following Lobato's teaching-by-telling method, the first problem was used to introduce the differentiation of implicit function types. Students were expected to use the implicit technique to differentiate these different implicit type functions.

5.1.1.1.1 Lesson Introduction and minimal guidance of Lesson 1

This was an N5 Mathematics lesson on implicit function differentiation technique. Lecturer1 walked in, politely greeted students and wrote the following on the whiteboard:

$$1. \ y = \frac{e^{-x}}{\tan x} \quad 2. \ y = x^4 \ln x \quad 3. \ y = \ln^2(1 - \cos x) \quad 4. \ x^2 + y^2 = r^2$$

He then instructed students to determine the derivatives of each problem given on the board and gave them about five to seven minutes to do so. After that time, lecturer1 called student1 to put his answer on the board. Student1 went to the board and wrote:

$$\frac{dy}{dx} = \frac{-e^{-x} \cdot \tan x - e^{-x} (\sec^2 x)}{(\tan x)^2}.$$

Lecturer 1 asked student 1 to explain to the class how she got the answer. Student 1 explained how she managed the computations.

The lecturer asked all students in the classroom if the explanations and answers were correct. All students said they were correct. Lecturer1 praised student1. The lecturer asked student2 how he solved number 2, and the response was that the product rule was used. They all agreed that problem

number 2 was easier and moved on. Student 3 was ordered to go to number 3 on the board. This student simplified and got: $y = 2l(1 - \cos x)$, ... and then gave the solution as:

$$\frac{dy}{dx} = 2 \cdot \frac{1}{(1 - \cos x)} \cdot (-\sin x).$$

The lecturer asked if the solution was correct. Almost all students agreed it was correct. The lecturer looked happy. He praised the student for the correct answer.

A student raised his/her hand and indicated that he/she had challenges with problem number 4. The lecturer nodded and asked the student to wait for others to finish the other questions. The student happily consented. After a couple of minutes, most students indicated that they were done. The lecturer picked students randomly to say their answers to the first three questions. The lecturer indicated that the first three questions focused mainly on the three rules of differentiation done in N4 Mathematics. The lecturer then asked if he/she could continue the lesson, and the students agreed.

5.1.1.1.2 New Learning and productive struggles of Lesson 1

The lecturer went to the board, removed all problems, and left number 4 and said it was going to be used for a new rule of differentiation called the implicit differentiation technique. He explained that in functions where ' x and y ' are related in a more complex manner, where it may not be desirable to express y in terms of x , then implicit differentiation was needed. He then gave: $\sin(x + y) = 1$, to show that expressing the function explicitly in terms of ' y ' is unnecessary. Lecturer1 emphasized that in determining the derivative, both ' x and y ' are differentiated and, most importantly, was to show the dy/dx next to the ' y ' derivative. The lecturer said anything not ' x ' or ' y ' should be treated as a constant.

The lecturer ordered students to do $x^2 + y^2 = r^2$. Students had to rely on introductory explanations

made by the lecturer. Reasonable time was given for students to engage with the problem. As he was walking between desks, he realized that a few students were attempting the given problem. He then asked student4 why she was not solving the problem, and she responded that she was lost and did not know where to start. The lecturer asked her to differentiate x^2 , and the student answered $2x$. The lecturer asked her again to differentiate y^2 , and she answered $2y$. The lecturer asked her what she must do to the ' y ' part. The student said she was lost. The lecturer said she was not lost and needed to do something for the ' y ' part. The student remembered to put dy/dx next to the ' y ' part. The lecturer was excited and praised the student for remembering to put the derivative dy/dx next to the ' y ' part. The lecturer asked the same student what should happen to the right-hand side of r^2 . The student said it was a constant and would be zero. The teacher requested the student4 to say the solution to him as he went to the board to capture it. Student4 gave the following:

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$$x^2 + y^2 = r^2$$

$$2x + 2y \frac{dy}{dx} = 0$$

$$2y \frac{dy}{dx} = -2x$$

$$\therefore \frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

Implicit Differentiation

Figure 3: Differentiation of a good example of an Implicit type function

The lecturer told students that all other rules of differentiation hold for this technique, and it meant that an implicit function in the form of a product or quotient requires the product rule and quotient rule, respectively. The lecturer emphasized $\frac{dy}{dx}$ for differentiating the 'y' part of the function. It was necessary to do so.

The lecturer invited clarity-seeking questions from students. Students were quiet, and the lecturer asked if they understood the example. Most students indicated they understood the example. Students pleaded for one more example, but the lecturer decided to give them work to do.

The lecturer asked students to solve the following:

$$\ln \sqrt{x^2 - y^2} = x - y.$$

Students were given time to work on the given problem for about five to seven minutes. The lecturer did minimal movements between students' desks, checking if students were working on the problem. The lecturer was mindful of the Covid 19 regulations. After a reasonable time, lecturer1 pointed Student5 to say out the answer. The student said she was only sure about the right-hand side:

$$\text{RHS} = 1 - dy/dx.$$

The lecturer asked the student5 - How to differentiate: $y = \ln(x)$? and the student said that the derivative is *'one over the (x)', multiplied by the first derivative of the f(x)*

$$\dots \text{that is } \frac{dy}{dx} = \frac{1}{f(x)} \cdot f'(x).$$

Other students agreed with Student5. The lecturer went to the board and started asking students randomly to loudly say the answer, step by step, for him to put it on the board. Student1 said the lecturer should put $\frac{1}{\sqrt{x^2-y^2}}$, which was the part: *one over f(x)*. Student6 said they should multiply by the derivative of this $f(x)$, and then the student gave the first derivative of the $f(x)$ as :

$$\frac{1}{2}(x^2 - y^2)^{-\frac{1}{2}} \left(2x - 2y \frac{dy}{dx} \right).$$

The lecturer was so thrilled at this, and students helped lecturer1 to rewrite the final solution:

$$\frac{1}{2}(x^2 - y^2)^{-\frac{1}{2}} \left(2x - 2y \frac{dy}{dx} \right) = 1 - dy/dx.$$

The lecturer emphasized that he was not interested in simplifying the answer but wanted students to master how the solution was done. He indicated that simplifying the answer would always be done after mastering the technique. The lecturer asked if the students were happy with the solution. Almost all students indicated they were happy except for Student7 who raised his hand and said:

‘Sir, I first started by simplifying ‘ $\ln(x)$ ’ to have:

$$\frac{1}{2} \ln(x^2 - y^2) \dots \text{instead of working with } \ln\sqrt{x^2 - y^2}.$$

The lecturer threw the question at students to ask if it was correct to do so. Other students agreed it was quite correct to do so. Lecturer1 indicated that alternative methods were welcomed.

Another problem was put on the board for students to do $\cos(xy) = e^y - x$. After five to seven minutes, lecturer 1 carefully looked at his students and asked volunteers to put the solution on the board. Student 8 rose quickly and wrote on the board $RHS = e^y \cdot dy/dx - 1$ and said she was not sure of LHS . The lecturer asked others to assist.

Student9 went and wrote: $-\sin(xy)$ [...].

Lecturer 1 asked student 8 if he was done with his answer, and the response was ‘No’ but indicated that the part he wrote was correct. Lecturer1 had to probe student8 further by asking – what should go into the bracket, and student8 responded by saying – it should be the derivative of (xy) . Lecturer1 told student8 to determine the derivative of (xy) . Finally, Student8 gave the derivative of (xy) as $[(x) \frac{d}{dx} + y]$. Lecturer1 thanked student8 for the answer and then wrote:

$$-\sin(xy) \left[(x) \frac{dy}{dx} + y \right] = \frac{dy}{dx} - 1.$$

Lecturer 1 told students to put together the final answer. Lecturer1 gave one more problem for students to do:

$$x^2 + \ln\left(\frac{y}{x}\right) - 5x + 4 = 0.$$

After some time, lecturer1 selected student10 to put the solution on the board and instructed student10 to involve other students should he need information regarding the solution. Student11 asked student10 if he could transpose the: $-5x + 4$ to the right-hand side. That question sparked heated discussions in the classroom. Other students said that the lecturer indicated it was not desirable to do so, and others insisted there was nothing wrong with that move. Lecturer1 had to come and assist them. Finally, they all agreed that it could be done though there was no point.

5.1.1.1.3 Lesson Conclusion and way forward of Lesson 1

Lecturer1 went to the board and wrote solution: $-si(xy) [(x) \frac{dy}{dx} + y] = dy/dx - 1$ and told students to look at the solution and determine the derivative. Student12 indicated that the dy/dx must be grouped to take out this common dy/dx as a factor. She indicated that she believed the dy/dx could then be given on the left-hand side, with all other terms on the right-hand side. The lecturer emphasized that it was unnecessary to simplify any further once the dy/dx was determined.

Towards the end of the lesson, lecturer 1 gave the last problem as homework: $e^{2x} \ln y = 4x$. He reminded students to do some more problems from the prescribed textbook.

5.1.1.2 Second Experimental Lesson

The second lesson was on the logarithmic technique type of differentiation. It was continued from the previous lesson on the 'implicit technique' type of differentiation. Chosen productive struggle tasks used were:

$$1. y = (\sin x)^{e^x} \quad 2. y^{x^2} = x^{y^2} \quad 3. y = \frac{e^{4x}\sqrt{x}}{\sin\sqrt{x}} \quad 4. y = \ln\left(\frac{3x^2 \cos 2x}{e^{4x}}\right).$$

These two techniques of `implicit differentiation` and `logarithmic differentiation` have many similarities.

5.1.1.2.1 Lesson Introduction and minimal guidance of Lesson 2

This was a continued lesson from the previous one on the N5 implicit differentiation technique and it was on the `Logarithmic type of differentiation`. These two topics are nearly same. Lecturer1 came to class and after polite greetings, ordered students to take out their homework books and open them on solutions done at home. During that time, lecturer1 moved between desks to check if students did the homework. Student1, who did the homework correctly, asked the lecturer about a problem she saw in one of the past examination question papers. Lecturer1 insisted on seeing the homework first.

When lecturer1 was contented with the homework, she ordered Student1 to put the past examination question on the board: $e^{\cos y} = 10y^2 + xy$. Lecturer1 asked student1 to give the derivative of $e^{f(x)}$. Student 1 answered that the derivative is $f'(x) \cdot e^{f(x)}$. Student 1 wrote on the board: $-\sin y \cdot e^{\cos y} \frac{dy}{dx} = 2y \cdot 10y^2 \frac{dy}{dx} + 1 + \frac{dy}{dx}$. The lecturer asked if it was correct, and the students agreed.

5.1.1.2.2 New Learning and productive struggles of Lesson 2

The lecturer indicated they would do a 'logarithmic type of differentiation' and explained that it is similar to an 'explicit type of differentiation' in instances where 'y' is differentiated. Lecturer1 told students they should take logs on both sides of the equal sign. The following was put on the board:

$y = (\sin x)^{e^x}$. Student2 was picked to go to the board and take logs on both sides, and student2 wrote on the board: $\ln y = \ln(\sin x)^{e^x}$. While on the board, student3 was asked to differentiate the LHS of ' $\ln y$ ', and student3 gave: $\frac{1}{y} \frac{d}{dx}$. Lecturer1 ordered student2 and student3 to sit down and picked student4 to do the RHS of $\ln(\sin x)^{e^x}$. Student4 asked the lecturer if he could simplify $e^x(\ln \sin x)$ so as to have a product. Lecturer1 told student4 to ask his classmates and get their opinions. Most students agreed it was correct to do so. Student4 continued to solve the problem and gave the following solution: $e^x \cdot \cot x + \ln(\sin x) \cdot e^x$.

Student4 asked the class if they agreed with his solution, and most students nodded in agreement. Student5 was ordered to put the full solution on the board, and she went and wrote on the board:

$$\frac{1}{y} \frac{dy}{dx} = e^x \cdot \cot x + \ln(\sin x) \cdot e^x.$$

Without erasing anything on the board, the lecturer put another problem for students to solve: $y^{x^2} = x^{y^2}$. Lecturer1 looked around and picked student6 to do the problem on the board. Student6 got up and wrote the following on the board: $\ln y^{x^2} = \ln x^{y^2}$. This student was told to simplify, and he gave: $x^2 \ln y = y^2 \ln x$. The lecturer stopped the student at this step and asked the class if the step was correct. Students agreed it was correct. The teacher instructed the student6 to sit down and asked the student7 to come and take off from where student6 left. Student7 confidently rose and wrote the following:

$$x^2 \cdot \frac{1}{y} \frac{dy}{dx} + \ln y \cdot 2x = y^2 \cdot \frac{1}{x} + \ln x \cdot 2y \frac{dy}{dx}.$$

The lecturer asked if this was correct. Students agreed it was correct. Lecturer1 gave two more problems for students to do:

$$1. y = \frac{e^{4x}\sqrt{x}}{\sin\sqrt{x}} \quad \text{and} \quad 2. y = \ln\left(\frac{3x^2\cos 2x}{e^{4x}}\right)$$

–

He told students to look carefully at the two problems and gave them reasonable time to analyze them. Lecturer1 asked students to share their thoughts together. Student1 indicated the possibility of using exponential laws to simplify the first problem. Students agreed to the approach. Student3 said that taking `ln` on both sides of the second problem will result in `ln (ln(x))` and looked worried at this situation. Student8 opined that problem2 could be simplified without taking `ln` on both sides by simplifying through exponential laws. Student8 asked the lecturer if it was correct to do so. Lecturer1 asked other students and they indicated there was nothing wrong with that simplification. Student9 indicated that it would be possible to do the two solutions with the response to student8's question.

5.1.1.2.3 Lesson Conclusion and way forward of Lesson 2

The lecturer explained that students needed to be very careful when dealing with the logarithmic type of differentiation technique because other problems might not allow them to take logs on both sides,

like: $y = \ln\left(\frac{3x^2 \cos 2x}{e^{4x}}\right)$. It is because there is already a log used in this problem. Lecturer1 ordered students to open their textbooks and indicated the exercise questions to be done as homework.

5.1.1.3 Third experimental lesson

The third lesson was on the `Partial Fractions` technique of integration. The types of functions dealt with were:

1. $\int \frac{f(x)}{(ax \pm b)^n} \cdot n \in \{1, 2, 3\}$
2. $\int \frac{f(x)}{(ax^2 + bx + c)(dx \pm e)^n} \cdot n \in \{1, 2, 3\}$.
3. $\int \frac{f(x)}{(ax \pm b)^n (cx \pm d)^m} \cdot n, m \in \{1, 2, 3\}$

The problem used to introduce this technique was: $\int \frac{x}{(x+1)(x-2)(x+3)} dx$. The objective was to make students aware of different techniques for different function types.

Lesson Introduction and minimal guidance of Lesson 3

This was an N6 Mathematics lesson on the `Partial Fractions` integration technique. Lecturer2 entered the classroom and politely greeted students. After a few minutes, Lecturer2 indicated that few integration techniques had been done and wanted to find out which of those did students find challenging. Student1 responded that the `integration by parts` technique seemed difficult and requested to revisit it during revision time. All agreed on this.

New Learning and productive struggles of Lesson 3

This was an N6 Mathematics lesson on the `Partial Fractions` integration technique. Lecturer2 entered the classroom and politely greeted students. After a few minutes, Lecturer2 indicated that few integration techniques had been done and wanted to find out which of those did students find challenging. Student1 responded that the `integration by parts` technique seemed difficult and requested to revisit it during revision time. All agreed on this.

5.1.1.3.2 New Learning and productive struggles of Lesson 3

Lecturer2 wrote on the board: $\int \frac{x}{(x+1)(x-2)(x+3)} dx$ and requested students to analyze it and try determining the integral for the next five to seven minutes. After this time, Lecturer2 realized that some students were not doing anything and went to them individually to find out what was happening. Two students who appeared like working together indicated they needed help on figuring out where and how to start solving the problem. Lecturer2 looked around and saw two other students working together on the solution. Lecturer2 asked them how they were managing, and they responded that they were solving for values of A, B and C. This was a new thing for the first two students. Lecturer2 asked that all four students work together. However, before that, Lecturer2 asked the two students to explain their approach to the whole classroom, and they did that and went to write on the board:

$$\frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x+3)} = \frac{x}{(x+1)(x-2)(x+3)}.$$

Other students got confused, and the two students told them to eliminate the fractions, solve for unknown values, substitute and finally integrate. Lecturer2 immediately interjected and asked if they all understood. A few students were still doubtful, and one of the students explained that in eliminating the fraction form, students had to multiply by the denominator. After some moments, Lecturer 2 picked one of the doubtful students to put on the board the non-fraction step. This student wrote on the board:

$$(x - 2)(x + 3) + B(x + 1)(x + 3) + C(x + 1)(x - 2) = x.$$

This was correct. The lecturer looked puzzled at how these two students rescued other students. The other student told students that in solving for values of A, B and C , the unknown ' x ' should be assigned a value that will reduce the bracket to zero; for example, if it is said: *Let $x = 2$* , then this will eliminate A and C values and result in having $15B = 2$ and leading to $B = 2/15$. After a few moments, Lecturer2 asked one student to put her findings on the board. The student put this:

$$\int \frac{\frac{1}{6}}{(x+1)} + \frac{\frac{2}{15}}{(x-2)} + \frac{\frac{-3}{10}}{(x+3)} dx$$

Lecturer2 asked students if they all agreed with the work on the board. They all agreed. Lecturer2 demanded that students finalize the solution. He gave them five minutes to do so. He later picked another student to put down the integral on the board. The student went to the board and wrote:

$$\frac{1}{6} \ln(x+1) + \frac{2}{15} \ln(x-2) - \frac{3}{10} \ln(x+3) + c$$

Lecturer2 told students that `other problems` may not easily afford a chance to find all unknown values and warned them that they might need to remember the techniques of solving simultaneous equations. He used the same example to clarify what he meant: He wrote on the board:

$(x - 2)(x + 3) + B(x + 1)(x + 3) + C(x + 1)(x - 2) = x$, and asked students to multiply out brackets. Students gave:

$$Ax^2 + Ax - 6A + Bx^2 + 4Bx + 3B + Cx^2 - Cx - 2C = x$$

He then told students to equate coefficients to solve for unknown values like:

$$x^2: A + B + C = 0$$

$$x: A + 4B - C = 1$$

$$c: -6A + 3B - 2C = 0$$

Students found it very interesting and managed to find the same values found earlier.

5.1.1.3.3 Lesson Conclusion and way forward of Lesson 3

Lecturer2 emphasized that any known method of solving systems of equations may be used. He told students that the technique used was called `Partial Fractions`. Lecturer2 explained possible function types for Partial Fractions, that is:

$$1. \int \frac{f(x)}{(ax \pm b)^n} \cdot n \in \{1, 2, 3\},$$

$$2. \quad \int \frac{f(x)}{(ax \pm b)^n (cx \pm d)^m} \cdot n, m \in \{1, 2, 3\},$$

$$3. \quad \int \frac{f(x)}{(ax^2 + bx + c)(dx \pm e)^n} \cdot n \in \{1, 2, 3\}$$

The lecturer emphasized the importance of checking first the function type as that would determine the technique to be used. He ordered students to open their prescribed textbooks on page 49, to take exercise number 3.1, and do Question 2 as classwork and Question 6 as homework.

Exercise 3.1, Question 2.

$$1. \int \frac{2x^2 - 2x + 1}{(2x + 1)^3} dx$$

Exercise 3.1, Question 6

$$2. \int \frac{x^3 + 3x^2 + 6x + 6}{x^3(x + 1)^2} dx$$

After some while, most students got correct answers to the classwork question. Lecturer2 wished all students well for the homework question.

5.1.1.4 Fourth experimental lesson

This was the fourth lesson on Mathematics N6 topic of `Depth of Centre of Fluid Pressure`. The desired outcome was to make students know how pressure affects objects immersed deeply into liquids, for example - water. The objects could be *structural walls* taking different design shapes like trapezoidal, parabolic, triangular, semi-circular and rectangular. Students needed to know how

to represent each *structural wall* on the Cartesian plane. They were told that the x-axis represents the water surface and the y-axis has to bisect the wall into two halves. The ultimate objective was calculating the centre point of fluid pressure developing on the wall.

5.1.1.4.1 Lesson Introduction and minimal guidance of Lesson 4

This was an N6 Mathematics lesson on the topic 'The depth of centre of fluid pressure'. It was preceded by lessons several lessons on complex solids of revolutions, for example: second moment of area and the moment of inertia. Lecturer3 walked into the classroom, and after politely greeting them, he asked if any of them had been to the river, swimming pool or sea to enjoy swimming. Some students responded that they had been to such places. Lecturer3 asked them about their experience when they were in the water. Students looked like they did not understand the question. Lecturer3 rephrased the question and asked them to explain how they felt when they got deeper into the water compared to when they were at shallow depths. One student indicated that it felt different when he was at a shallow depth than when he was at an increased depth. Lecturer3 wanted to know what was different. The student indicated that he knew it felt different and could not explain or tell what was different. Lecturer3 went straight to the board and wrote 'Depth of centre of fluid pressure'. Lecturer3 asked students to share what they knew about pressure. Student3 explained that pressure is a force applied to a unit area. Lecturer3 asked if it could be given as a formula, and Student4 stood up and wrote on the board: $Pressure = \frac{Force}{Area}$. Lecturer3 asked students what happens to an object when immersed in water. Students were quiet, not knowing the answer. Lecturer3 explained that when an object goes deeper into a body of water, there is more water above the object, and therefore a greater weight of the water pushing the object down. This is the reason why water pressure increases with depth. The pressure depends only upon the depth and is the same anywhere at a given depth and in every direction. Lecturer3 further explained that water, like all objects on Earth, is pulled downward the earth's centre by gravitational force. Every part of the body of water has a certain weight, and this weight pushes downward on whatever is below it. Water pressure results from the weight of all the water above pushing down on the object below. Lecturer3 emphasized that pressure depends only upon the depth and is the same anywhere at a given depth and in every direction.

5.1.1.4.2 New Learning and productive struggles of Lesson 4

Lecturer3 further emphasized that any object submerged vertically in water experience pressure developing on it. He indicated that the Pressure `P` on an object at a particular depth `h` in a liquid is given by: $P = \omega h$, where `ω` is the weight-density of a liquid per unit of volume. He further explained that in any vertically immersed object, there is an elementary force acting on each element of the area of this object and that this force increases with the depth of the point below the surface of the liquid. He indicated that the resultant of these forces is a single force equal to the total force acting at a particular point below the surface of the liquid, and this is called the `centre of fluid pressure` denoted: $\bar{y}$ mathematically. Carefully so, Lecturer3 indicated that:

$$\bar{y} = (\text{Second Moment of Area}) / (\text{First Moment of Area})$$

Lecturer3 reminded students that they already learned `the first moment of area` and `the second moment of area` calculations in previous lessons. Lecturer3 repeated that there are usually five types of *structural walls* dealing with the depth of fluid pressure: rectangular, trapezoidal, parabolic, semi-circular and triangular-shaped walls. He further indicated that the $x - axis$ should be treated as the water level in drawing these walls, and the y-axis must always divide the wall into two vertical parts.

Lecturer3 gave the following example on the board: An isosceles triangular weir/wall is immersed vertically into the water with its vertex being 1m below the water surface. The base of the triangular weir/wall is 8 m, and its height is 4 m. Lecturer3 instructed students to draw and show the situation on the system of axes as explained.

After a few minutes, Lecturer3 appointed Student4 to put the diagram on the board.

This was put on the board by the student4:

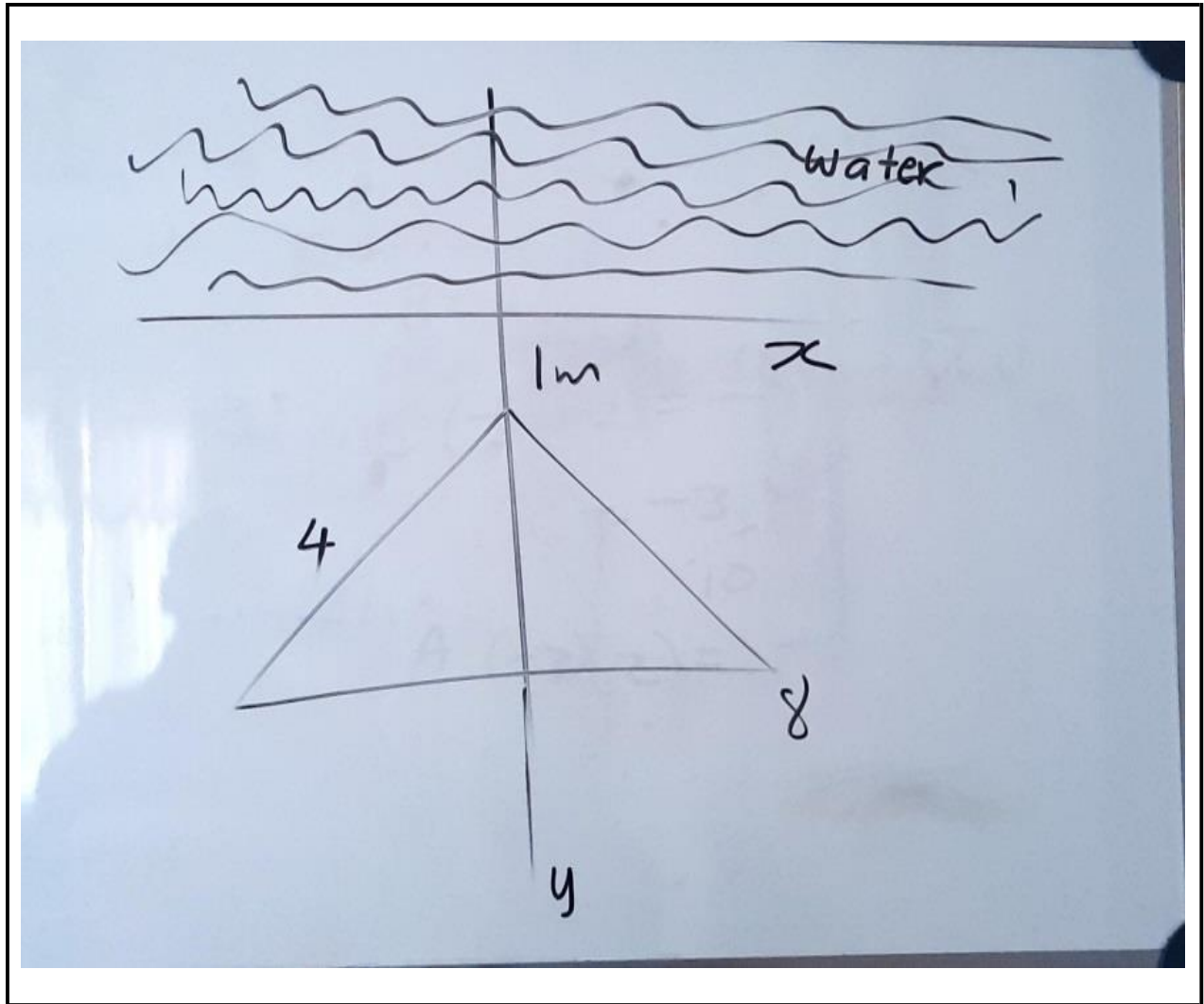


Figure 4: A student4's representation of a Triangular weir/wall

Lecturer3 asked if the sketch had been done correctly. Student5 agreed it was correct. Student6 disagreed and said it was incorrect. Lecturer3 asked Student6 to indicate the reason/s for his

disagreement. Student6 said the 8 m should be 4m because the base is 8m long. Student4 agreed that he made a mistake and should be 4 m.

Student7 said the 4 still needed to be corrected and should be 5. Other students asked why? Student7 said that the triangular wall is 4m high and, therefore, the base of the wall should be 5m below since the vertex is 1m below. Student8 indicated that student7 is confusing the base and the height. Student7 quickly agreed with Student8. Lecturer 3 asked students if they agreed. Lecturer3 saw a need to elaborate further on what Student8 said. Lecturer3 asked students to rectify student4 sketch below.

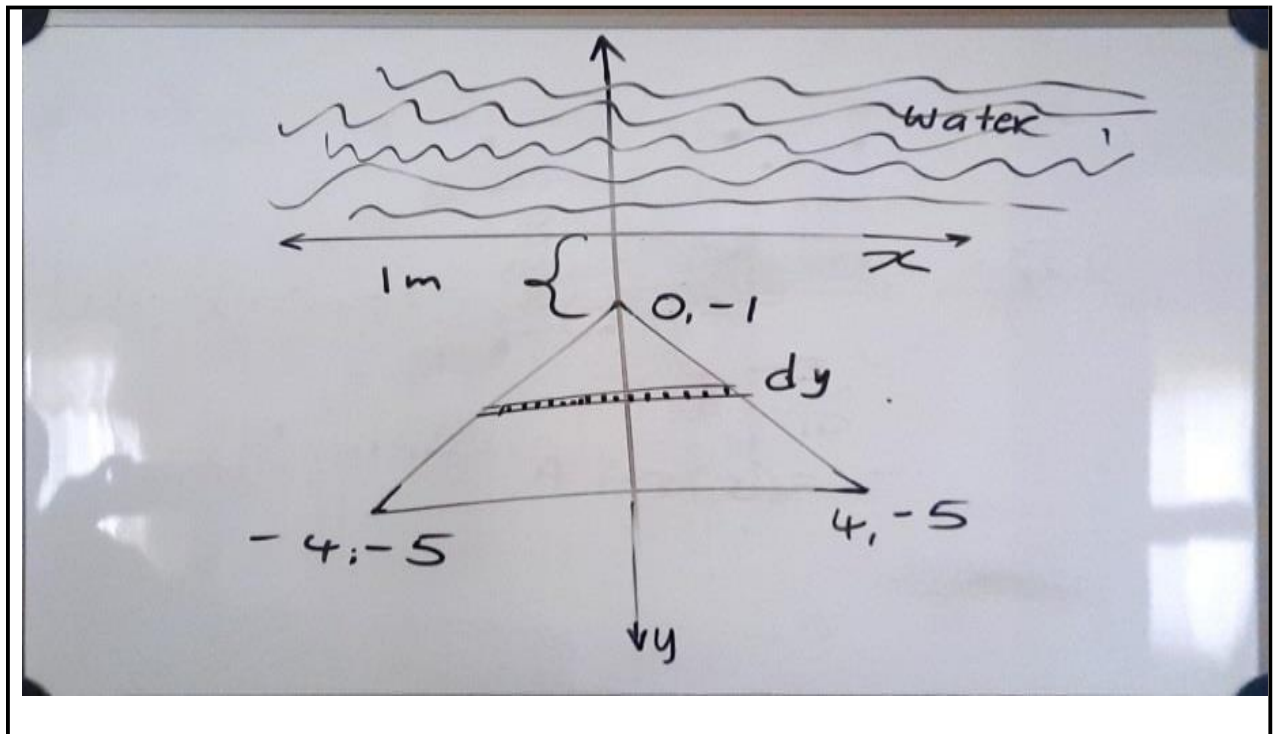


Figure 5: Lecturer's correct representation of a Triangular weir/wall

Lecturer8 explained that the base was 8 m, and therefore, the y -axis divided this base into 4m per side and further indicated that, with the vertex being 1m below the water surface, the base would be 5m below the water surface. Lecturer3 told students to note the points on the vertices. He explained that these points should be used to determine the equation of the straight line graph, which was the graph to be used for the first and the second moment of area.

Lecturer3 asked students – how they would determine the equation of a straight line when two points are given. Student 9 said she would first determine the gradient of the straight line. Lecturer3 asked student9 to give a formula for determining gradient. Student9 said the change in `y' over the change in `x'. Lecturer3 was ok with that response from student9 and went on to ask about the equation of the straight line. Student9 wrote on the board: $y - y_1 = (m)(x - x_1)$. Lecturer3 was happy and praised the student. Lecturer3 put another question on the board for students to do. It read: A water canal in the form of a trapezium is 8m wide at the top, 6m wide at the bottom, and 2 m high. The top of the weir is 1 m below the water surface. Determine the depth of the centre of pressure on the weir. After a few minutes, lecturer3 called student10 to sketch on the board.

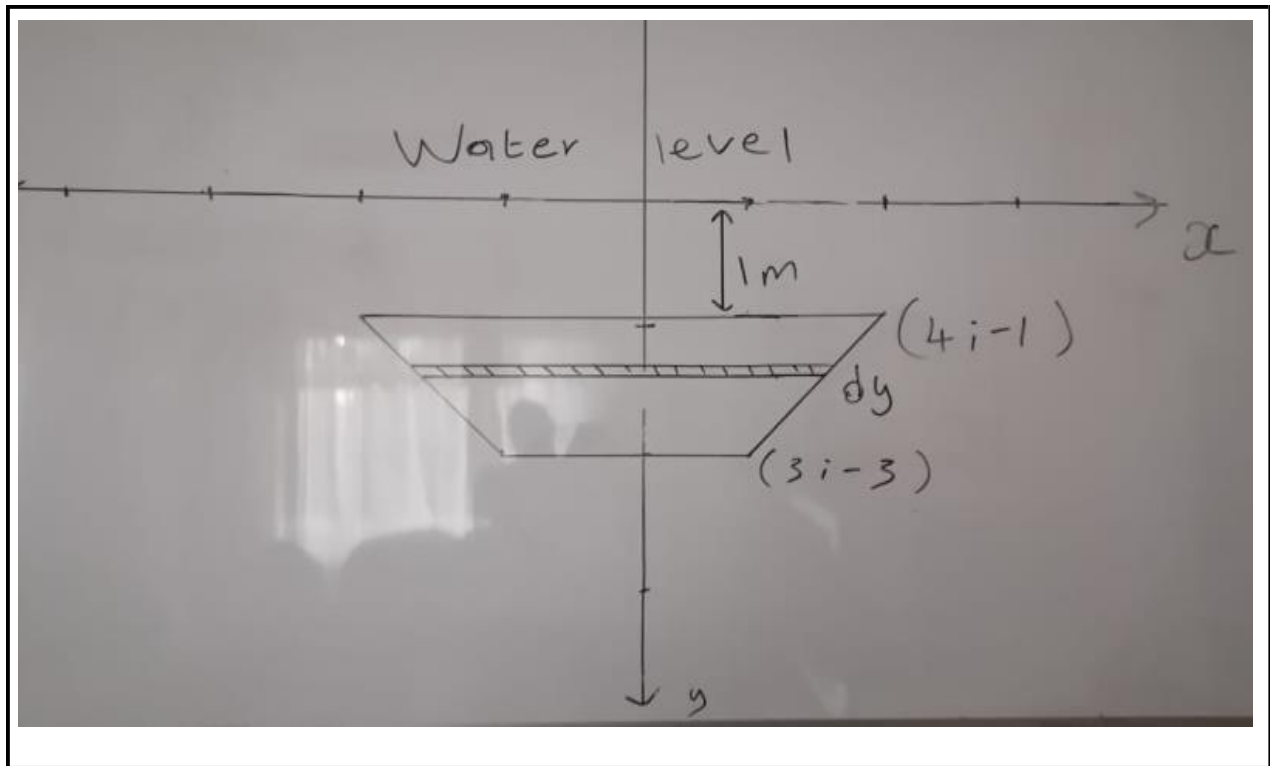


Figure 6: Student8` representation of Trapezoidal weir/wall

Lecturer3 looked at the sketch and asked students if it was correct. Most students nodded their heads in agreement to say it was done correctly.

Lecturer3 wrote another question on the board: This was the question: Determine the depth of the centre of pressure on a rectangular weir, 4 m by 8 m vertically immersed in water with its top 1m below the water surface.

He told student11 to sketch on the board.

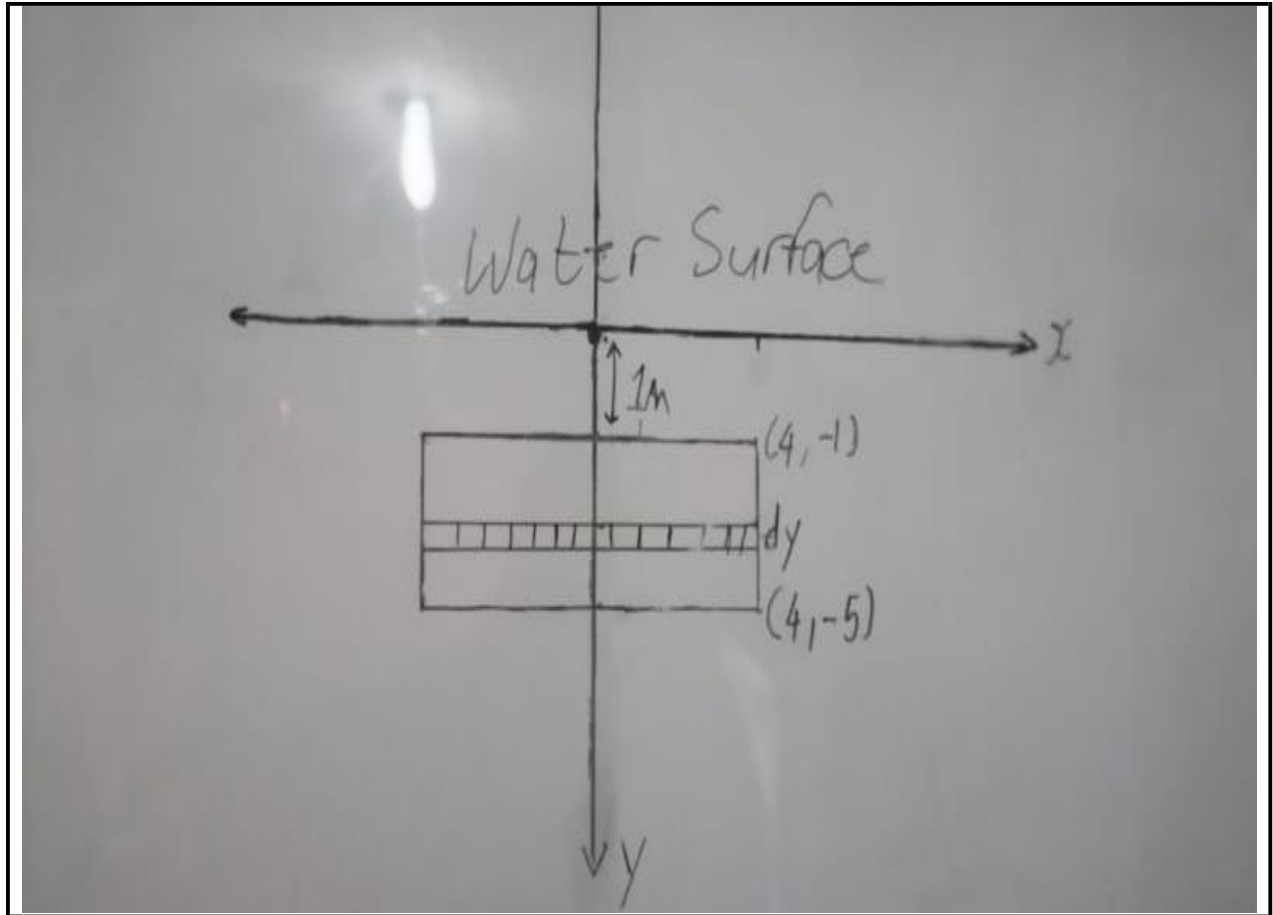


Figure 7: Student11's representation of Rectangular weir/wall

Lecturer3 again checked student11's work on the board and asked if it was done correctly. Students agreed it was correct.

5.1.1.4.3 Lesson Conclusion and way forward of Lesson 4

In closing the lesson, Lecturer3 ordered students to go to their textbook, Page number 151, Exercise 5.6 and do Question number 3 as homework.

5.1.1.5 Fifth experimental lesson

This lesson was about calculating the 'Depth of centre of fluid pressure'. Students needed to position the wall, correctly determine the equation of the relevant graph for the wall and calculate the centre of fluid pressure. If the wall was parabolic, they needed to determine the parabola equation for that situation.

5.1.15.1 Lesson Introduction and minimal guidance of Lesson 5

This was an N6 Mathematics lesson on the continued topic of 'Depth of centre of fluid pressure'. Lecturer3 came into the classroom and politely greeted students. He ordered students to take out their homework. The homework involved an isosceles triangular weir, shown in the picture below:

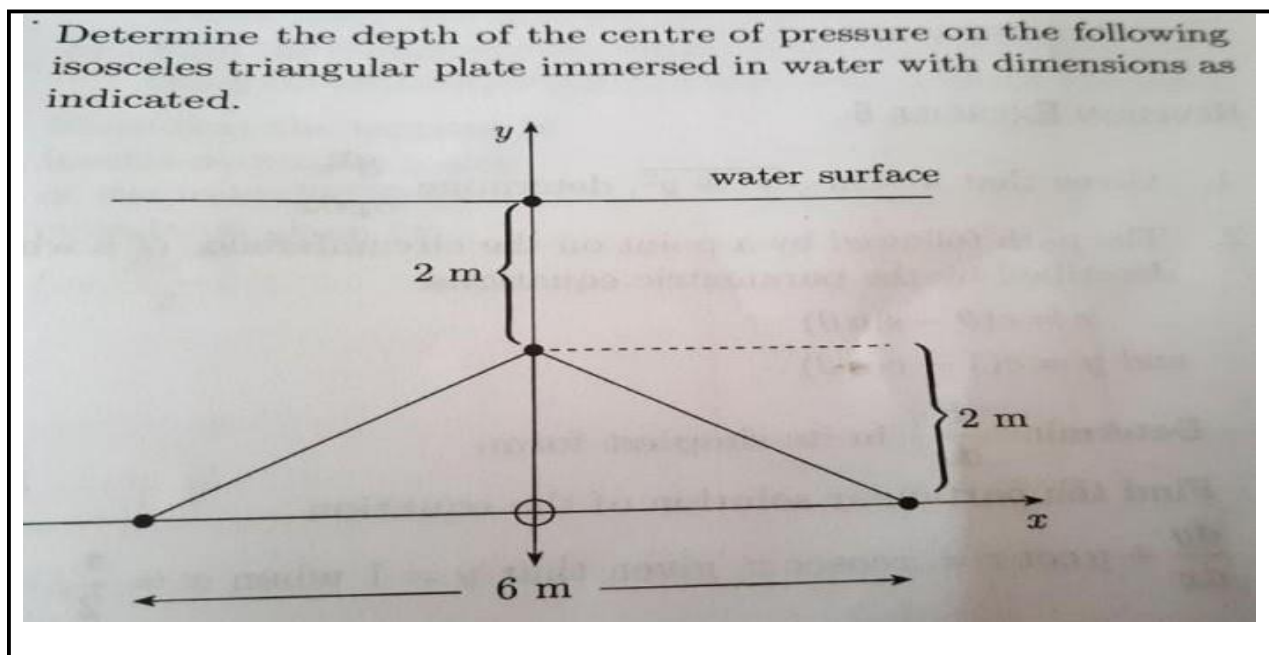


Figure 8: An Isosceles Triangular weir/wall

Lecturer3 picked Student1 to describe in his own words and share his understanding of Figure8. Student1 indicated that the vertex of the triangle is 2 m below the water level. Lecturer3 agreed and picked Student 2 to tell the triangle's height. Student2 answered: 2 m high. Lecturer3 picked Student3 to give the (x, y) points on the vertex, pointing up and on the vertex to the right- hand side of the $y - axis$. Student3 gave points for the vertex pointing up as: $(0, -2)$. Lecturer3 quickly stopped Student3 from saying the other point and picked Student4 to give the other point. Student4 gave the point as $(-3, -2)$. Other students looked unhappy and indicated their disagreement. Lecturer3 picked Student5 to give the correct point. Student5 said that he agreed with the $x - intercept$ of -3 but not with the y -intercept as -2 and that the $y - intercept$ should instead be -4 . That meant that the point was $(-3, -4)$. They all agreed on the coordinates of x as $(0, -2)$ and y as $(-3, -4)$. The whole class agreed.

5.1.1.5.2 New Learning and productive struggles of Lesson 5

Lecturer3 asked students to determine the equation of the straight line from the points agreed upon. After a few minutes, the lecturer picked Student6 to go to the board and asked other students to work with him to determine the straight-line equation. Student6 went to the board and started involving other students by writing the following on the board:

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1} \right) (x - x_1).$$

Student6 said that the points under consideration are $(0, -2)$ and $(-3, -4)$. Student6 asked Student7 to substitute the formula. Student7 gave: $y - (-2) = \left(\frac{(-2) - (-4)}{0 - (-3)} \right) (x - (0))$.

Student6 asked Student8 to give the simplified answer, and she gave: $y = \frac{4}{3}x - 2$. The lecturer checked if all students understood. They all looked fine.

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Lecturer3 instructed the first two rows of ten students, to calculate the first moment of area and the second two rows of ten students, to calculate the second moment of area. The remaining last row of five students was not given any work. Lecturer3 reminded students to refer to work already done in the first and second moments of the area. He gave them time to do so.

After some time, Lecturer3 picked a student from the first row to show how the first moment of the area was calculated. The student went to the board and wrote: *1st Moment of Area* = $\int_b^a y \, dA$. Lecturer3 asked the student what the ' dA ' stands for, and the student got confused about whether ' dA ' was for ydx or xdy . Lecturer3 asked other students, and only a few knew correctly that it was for ' xdy '. Lecturer3 asked another student from the same row about the ' x ' in the ' xdy '. A student from the last row of students who were idling and not assigned any work to do raised his hand and indicated that the ' x ' should be manipulated from the straight line equation and should be: $x = \frac{3}{4}y + \frac{3}{2}$ and therefore the whole thing should be given as:

$$\text{1st Moment of Area} = \int_b^a y \, x \, dy \text{ where } x = \frac{3}{4}y + \frac{3}{2}$$

This was correct. Lecturer3 asked students about the limiting values. He picked another student from the same row to give the limiting values. There was agreement that limiting values were -2 and -4 . Lecturer 3 went to the board and put: *1st Moment of Area* = $\int_{-4}^{-2} y \left(\frac{3}{4}y + \frac{3}{2} \right) dy$ and ordered the first two rows to finalize the answer quickly. A few moments later, Lecturer3 asked for the answer and was given 5. He asked if students from the same rows agreed with the 5. They all agreed.

Lecturer3 then turned to the second two rows and asked them how they calculated the second moment of area. A student from the second row stood up and went to the board to put the following:

2nd Moment of Area = $\int_{-4}^{-2} y^2 \left(\frac{3}{4}y + \frac{3}{2} \right) dy$. All students agreed on the formula substitution.

Lecturer3 instructed the second row of students to finalize the answer. After a couple of minutes, Lecturer3 asked for the answer. He was given -17 by a student from the second row of students. Lecturer3 asked all ten students from the second row if the answer was correct, and they all agreed it was correct. Lecturer3 asked the students from the last row of 5 students to calculate the final answer, and they gave the final answer as $-3,4\text{ m}$

5.1.1.5.3 Lesson Conclusion and way forward of Lesson 5

Lecturer3 called for students' attention as he made important explanations in summing up key aspects of the lesson. Firstly, he indicated that the unit for the second moment of area is m^4 and the unit for the first moment of area is m^3 , and therefore, the final answer should be in 'm' only. Secondly, he explained that the correct way of doing these calculations was to take twice the answer since there are two similar parts on both sides of the y-axis on the sketch. He indicated that the first moment should be: $5 \times 2 = 10m^3$ and the second moment should be: $-17 \times 2 = -34m^4$ to finally have: $\frac{-34m^4}{10m^3} = -3,4m$. Lastly, he indicated that the negative sign in the final answer only indicates that the point was under the water's surface. The lecturer picked a few questions from the textbook and told students to work on them at home.

5.1.2 Term test students' scripts

Twenty-three students' term test items were randomly gathered from students' scripts taught by both experimental and control group lecturers. All the test items were used for data collection, focusing on how lecturers marked students' mathematical responses. Marking focused on crediting correct responses and revealed errors and misconceptions in students' responses about questions

asked. Actual questions asked during term tests, students' responses and the lecturers' marking are well captured in the following figures.

5.1.2.1 Experimental Group Test Items

Six test items (about 26%) were randomly chosen from students' scripts marked by experimental group lecturers. These test items show the actual question, students' responses, how lecturers marked these responses, and a researcher's short description of the scenario under consideration.

1. Use 'Partial Fraction' to determine the integral of the function: $\int \frac{12x^2+x+27}{(2x^2+3)(x-2)} dx$

So $\int \frac{12x^2+x+27}{(2x^2+3)(x-2)} dx = \frac{Ax+B}{2x^2+3} + \frac{C}{x-2}$

$12x^2+x+27 = A(x-2) + C(2x^2+3)$

$12x^2+x+27 = Ax+Bx-2B+2x^2 \cdot C+3C$

Comparing:

$A = -2 \quad C = 7$
 $B = -3$

How did you get these values

$\therefore \int \frac{12x^2+x+27}{(2x^2+3)(x-2)} dx = \frac{-2x-3}{2x^2+3} + \frac{7}{x-2}$

$= -2x - 3 \ln(2x^2+3) + 7 \ln(x-2) + C$

Figure 9: Test - Item 1

The student gave values without showing how he got them and therefore was marked wrong by the lecturer. The marker annotated by writing – how did you get these values?

2. Use 'Integration by Parts' to determine the integral: $\int e^{3x} \cos 3x \, dx$

2.1 $y = e^{3x} \cos 3x$

$F(x) = e^{3x}$ $g'(x) = \cos 3x$
 $F'(x) = 3e^{3x}$ $g(x) = \frac{1}{3} \sin 3x$

$F(x)g(x) - \int F'(x)g(x)$

$e^{3x} \frac{1}{3} \sin 3x - \int 3e^{3x} \cdot \frac{1}{3} \sin 3x \, dx$

$e^{3x} \frac{1}{3} \sin 3x - \int e^{3x} \cdot \sin 3x \, dx$

$e^{3x} \frac{1}{3} \sin 3x - \frac{1}{3} \cos 3x \, dx$ whole

$\frac{1}{3} e^{3x} \sin 3x - \left[e^{3x} \cdot \frac{1}{3} \cos 3x - \int 3e^{3x} \cdot \frac{1}{3} \cos 3x \, dx \right]$

$\frac{1}{3} e^{3x} \sin 3x - \left[e^{3x} \cdot \frac{1}{3} \cos 3x - \int e^{3x} \cdot \cos 3x \, dx \right]$

$\frac{1}{3} e^{3x} \sin 3x - \frac{1}{3} e^{3x} \cos 3x - e^{3x} \frac{\cos 3x}{3}$

Figure 10: Test Item 2

The student should have remembered to carry through a part of the solution and therefore got the wrong marks for the mistake done. It is annotated by encircling and writing the correct value.

3. Determine the integral of $\int \tan^4 3x (\sec^2 3x - 1) dx$

$$2a) \int \tan^4 3x (\sec^2 3x + 1) dx$$

$$\int \tan^4 3x \sec^2 3x dx + \int \tan^4 3x dx$$

$$\int \tan^4 3x \sec^2 3x + (\sec^2 3x + 1) \tan^2 3x dx$$

$$\int \tan^4 3x \sec^2 3x + \sec^2 3x \tan^2 3x - \tan^2 3x dx$$

$$\int \tan^4 3x \sec^2 3x dx + \int \tan^2 3x \sec^2 3x dx - \int (\sec^2 3x - 1) dx$$

let $u = \tan 3x$
 $du = 3 \sec^2 3x dx$

$$\frac{1}{3} \int u^4 du + \frac{1}{3} \int u^2 du - \frac{1}{3} \tan 3x + x + C$$

$$= \frac{1}{3} \left[\frac{u^5}{5} \right] + \frac{1}{3} \left[\frac{u^3}{3} \right] - \frac{1}{3} \tan 3x + x + C$$

$$= \frac{1}{15} \tan^5 3x + \frac{1}{6} \tan^3 3x - \frac{1}{3} \tan 3x + x + C$$

Figure 11: Test Item 3

The student kept the principle correct but made mistakes when giving the final answer.

4. Given: $f(x) = \cos x$. Determine $f'(x)$ by using First Principles.

Q1 $F(x) = \cos x$

i $F(x+h)$

$\cos(x+h)$

$\cos x \cdot \cos h + \sin x \cdot \sin h$

ii $F(x+h) - F(x)$

$\cos x \cdot \cos h + \sin x \cdot \sin h - \cos x$

Figure 12: Test Item 4

The lecturer marked the student wrong for the incorrect compound angle identity used. The sign is wrong. The lecturer annotated by circling the wrong part of the student's response.

5. Given: $y = \pi^{\tan x} \cdot (\cos x)^x$, differentiate by using the appropriate technique

2.4. $y = \pi^{\tan x} (\cos x)^x$

$\ln y = \tan x \ln \pi + x \ln \cos x$

$\frac{1}{y} \cdot y' = \sec^2 x + x \cdot \frac{1}{\cos x} \cdot -\sin x + \ln \cos x$

$y' = (\sec^2 x + x \frac{\sin x}{\cos x} + \ln \cos x) \cdot y$

$y' = (\sec^2 x - x \tan x + \ln \cos x) \cdot \pi^{\tan x} (\cos x)^x$

Figure 13: Test Item 5

The student needed to have differentiated the first product part correctly. The teacher showed the student what he/she could have done. The lecturer annotated by writing the correct answer next to the wrong response

6. Determine the integral of the function: $\int \frac{x^2}{2x+1} dx$

1.1.4 $y = \frac{x^2}{2x+1}$

$\int \frac{x^2}{2x+1} dx$

$\frac{\frac{1}{2}x}{2x+1} \overline{) x^2 + 0x + 0}$

$x^2 + \frac{1}{2}x$

$0 - \frac{1}{2}x$

Figure 14: Test Item 6

The long division method remained a challenge as it was done incorrectly.

5.1.3 Consolidated experimental group lecturers` interviews

All lecturers who participated in the study were interviewed individually. Interviews took place after the observations and lasted about 40 - 45 minutes each. The following section brings the consolidated data collected from experimental group lecturers` interviews.

Lecturer1 was asked - `what is mathematics, and how does it help students?` The response was `Math is the Mother of all subjects`. He was asked to explain the response. He said that he rated and held technical Mathematics high in growing students' minds and further explained that technical mathematics/calculus is vital for students to become artisans, technicians and engineers. All experimental lecturers advocated that technical mathematics enhances critical and logical thinking and improves students' understanding of complex calculus concepts.

Lecturer1 was asked to share his thoughts on the lesson's introduction and the type of tasks used. The response was: `I do some research on the topic first and then select suitable productive struggle tasks to be used`. This lecturer indicated that the topic must always make students relate to real-life situations. This lecturer indicated that context is key in mathematics as it helps students to better understand the question by making mathematical connections.

Lecturer1 did not mention anything about greeting students, as they come into class. He was asked to share his views on whether greetings are necessary or not. Lecturer1 responded by saying - `greetings are like a key to start the lesson`. He said that `greetings set the good mood for learning` and again that `if students do not feel welcome in your classroom, forget about their contributions`. He further indicated his disappointment on some lecturers who downplay the importance of kindly welcoming and greeting students as they attend their lessons. He argued that a student`s positive attitude and relax state of mind go a long way in understanding mathematics during teaching and learning.

Lecturer2 was asked if he liked using productive struggle and professional noticing teaching strategies. He responded, 'I just like it and want to know more about it'. He indicated that the greatest highlight was that - 'students actively build their learning no matter how much a lecturer tells them what to do and how to do it, no matter how hard a lecturer says it'. This lecturer further indicated that he would never return to his old teaching methods. He said that he was going to search on google for more information about productive struggle approaches and professional noticing to teaching mathematics so that he could advance his skills in implementing it.

Experimental lecturers willingly allowed for productive struggle strategies and created a platform to be used. They explained topics/concepts by explaining their real-life applications and used well-chosen productive struggle tasks to do accompanying calculations. They then later gave work for students to do ill-defined calculus questions that had the potential to throw them deep into struggles. They always gave reasonable time for students to work on the given problems. They willingly allowed students to engage in productive struggles in trying to find solutions, during which period lecturers used to look out for those battling and those who did not know how and where to start attempting the solutions. These situations created perfect moments for students and lecturers to ask each other questions and afford further probing opportunities. They were asked if they were having any regrets for using these approaches. Lecturer3 answered that he instead had regrets for using the old ways of mathematics teaching and learning.

Lecturer3 asked if she has faith in giving students some 'think time' when a problem is given to them to solve. The response was 'think time allowed students to assemble their strategies and evaluate which would be relevant to solving the problem'. Lecturer3 argued that thinking is a process, and, therefore, students must not be hurried into giving answers to questions asked but must rather be given reasonable time to engage in it.

It was noticed that after lecturers gave students some work to do, they randomly called them to the front to put their solutions on the board. They always wanted some confirmation of whether the student's work on the board was correct or not, and that, in most cases, sparked off discussions. Lecturers were asked if calling students to the front to put their solutions on the board assisted students in some way. Lecturer3 responded that it was a way of starting classroom interactions where students learn from one another. Lecturer1 indicated that the interactions and discussions assist students in manifesting their mathematical ideas, a platform to professionally notice errors and misconceptions. When students discussed the solution on board, the lecturer sat back and listened but ultimately joined in redirecting and controlling the conversations. In the end, both students and the lecturer agreed about the solutions to the problems given on the board.

Lecturers were asked if they have confidence and belief in staging classroom discussions. Lecturer2 responded that students assisted one another with less pressure to give their views to their fellow students than when they would give their lecturers. They also mentioned that discussions have the potential to make students learn from each other, for example - paraphrasing each other's responses and at the same time building their knowledge actively.

Lecturer2 was asked how productive struggle and professional noticing assist in dealing with errors and misconceptions during the lesson. Lecturer2 acknowledged the difficulties students encountered in understanding calculus. This lecturer mentioned the inability to recognize and draw mathematical graphs, confusing differentiation with integration rules, confusing rotation about ' x and y ' axes on solids of revolutions, and making errors with certain techniques of integration, for example - integration by inspection, integration by parts, integration by completing the square and long division integrating technique. Most interestingly, lecturer 2 indicated that all these situations were noticed and attended to, and students left the classroom with error-free minds. This lecturer indicated that with the old way of teaching, students would forever have an unattended built of errors and misconceptions.

Lecturers were asked to share their views on how they went about noticing errors and misconceptions during teaching and learning mathematics. Lecturer1's response was that most errors and misconceptions manifest themselves during classroom discussions where students interact and exchange ideas and responses with each other. Again lecturer1 indicated that some errors and misconceptions might surface when students answered lecturer questions. They were asked how they responded to students' questions. Two lecturers said they always wait a bit and avoid giving quick answers but prefer throwing back the same questions to students.

Lecturer3 was asked: 'What do you do after marking students' test scripts? The long response was - 'like any other lecturer in our Maths department, we give feedback to our students'. This lecturer indicated his belief in giving feedback to students after an assessment is marked. Lecturer3 explained his approach to giving feedback on marked work - that he assigns questions to students and allows them to take turns putting them on the board. As they put their solutions on the board, they are expected to give explanations for their answers and also take questions from classmates as well as the lecturer and, by so doing, re-activating discussions in class. Towards the end, the lecturer will give additional information and hints on alternative methods that might be possible for the solutions.

The experimental group lecturers were asked if they were in a position to recommend for other mathematics lecturers and advocate for the new productive struggle strategies of mathematics teaching and learning. They all overwhelmingly supported the new productive struggle teaching strategies and vowed never to use old methods of teaching mathematics. Lecturer3 happily indicated that he was definitely going to share these powerful strategies with his fellow colleagues. Lecturer1 indicated that with the new strategies, he did not talk much as students engaged in discussions. She said that she found it very interesting to listen to students talk and quickly identify erroneous mathematical reasoning as they try argue their mathematical standpoints.

Lastly, experimental lecturers were asked to share their last thoughts about the new mathematics teaching and learning strategies. Lecturer2 indicated that he realized that the strategies are so powerful and effective that even the quietest and shyest students could positively contribute during classroom discussions.

5.2 Control Group Data Collection Strategies

Three strategies were used to collect data from control group lecturers. The strategies are: classroom observations, document analysis in the form of students' term scripts and lastly the interviews.

5.2.1 Classroom Observations

The lecturer student ratio was 1: 25. Lecturers were named Lecturer4, Lecturer5 and Lecturer6 and individual students as Student A, Student B, to Student Y. The researcher always remained the researcher. These lecturers taught mathematics using traditional methods. It was a challenge to distinguish different parts of control lessons as they were haphazardly planned and given to students.

5.2.1.1 First control lesson

This was an N5 lesson on the `Let Method` integration technique, as named by lecturer4. He entered the classroom and told students that they were going to do one of the `integration techniques` he called `Let Method`.

He wrote on the board: $\int (x\sqrt{x^2 - 2}) dx$

He immediately started showing the procedure for finding the solution and told students to say:

Let: $u = x^2 - 2$; from this, he determined: $\frac{dy}{dx} = 2x$, where $dx = \frac{du}{2x}$, by manipulation.

He substituted back and got: $\int x \cdot u^{\frac{1}{2}} \cdot \frac{du}{2x}$, and cancelled out the 'x' and remained with: $\int u^{\frac{1}{2}} \cdot \frac{du}{2}$.

Without involving the students, he continued explaining and integrating the function:

$$\frac{1}{2} \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c, \text{ and went on simplification as } \frac{1}{3} \cdot \frac{(x^2-2)^{\frac{3}{2}}}{1} + c.$$

The lecturer explained in detail all the steps involved in determining this integral. He asked students if they understood, and without paying attention to their responses, he continued to give another problem.

Lecturer4 gave students to solve: $\int \left(\frac{(\arctan x)^{\frac{1}{4}}}{1+x^2} \right) dx.$

He told students to say: Let $u = (\arctan x)$ and ordered students to take out the formula sheet and use it in solving this problem. He told students to follow exactly what he did with the first example done earlier. He gave students about five minutes to determine the integral. After five minutes, the lecturer went to the board and solved the problem without asking students for their answers or struggles. He only asked if they understood the solution on the board. Students responded in the majority that they understood. Some looked doubtful in their responses.

Lecturer 4 put another problem on the board: $\int (e^x \cdot \sec^2(e^x)) \cdot [1 + \tan(e^x)]^{-3} dx.$

He ordered students to solve the problem by telling them to say: Let: $u = [1 + \tan(e^x)]$ and follow the same approach in the abovementioned examples.

Lecturer 4 gave students three to five minutes to work on the problem. Without involving students, he gave the solution to students by putting it on the board. He asked students if they understood the work on the board. Students responded positively and said that they understood the solution.

Lecturer4 gave the following problems as homework:

1. $\int \frac{(\ln x)^{\frac{3}{2}}}{x} dx$

2. $\int \left(\sin^{\frac{1}{2}} x \right) \cos x \, dx.$

The lecturer called for questions regarding the lesson. Only one student raised her hand and requested the lecturer to come to her desk. The lecturer went to the student and softly addressed the student.

He dismissed the class and told students to hurry to their next lesson.

Interestingly when all students were leaving, two students went to the lecturer and started asking him questions about the first two examples done earlier. Lecturer 4 explained to them again and later sent them to their next lesson.

5.2.1.2 Second control lesson

This was an N5 lesson on the `Integration by Parts` technique. Lecturer 4 came into the classroom and greeted his students. He immediately told students they would do a new topic called `Integration by Parts`. He wrote on the board: $\int f(x) \cdot g'(x) dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x) dx$.

He told students that the above is a formula used to integrate parts. He told students to compare the rule with the product rule of differentiation. He said that they both deal with products. He explained the `Liate rule` and emphasized the order: *lnx, xⁿ, e^x and trig ratio* as found in most problems. He gave an example on the board: $\int x \sin x dx$.

He referred back to the example $\int x \sin x dx$ and explained that according to the `Liate rule`, the algebraic part of `x` should be the (x) , while $\sin x$ was $g'(x)$. He determined $f'(x) = 1$ and $g(x) = -\cos x$ and replaced them in the formula.

$$\int x \sin x dx = x(-\cos x) - \int (-\cos x) \cdot 1 dx$$

The lecturer gave the simplified final answer as $\int x \sin x dx = -x \cos x + \cos x + C$.

All steps were done to students with detailed step-by-step explanations.

Lecturer4 ordered students to do the following in the given order:

1. $\int x e^x dx$
2. $\int (x^2 + 1) \sin x dx$
3. $\int \ln x dx$

After about five minutes, he wrote $\int x e^x dx$ on the board and told students that `x` is (x) and e^x is $g'(x)$. He solved the problem for students. He ordered them to do number 2. He gave them

about three to five minutes to solve it. He then later put the solution on the board for students. He repeated the same approach with number 3.

He then asked students if they were happy with the new `Integration by Parts` rule. Only a few students responded softly, and he looked worried but did nothing about it. A student called him to his desk to indicate his problem areas. Lecturer4 attended to this student4. Afterwards, another student called him to her desk to explain some problem areas, and he went and explained to the student, and this took some more time.

When it was time up, Lecturer 4 dismissed the class. Three students approached him individually, each asking a clarity-seeking question about the lesson. The lecturer took his time to further explain to them. Afterwards, they all rushed to their next classes.

5.2.1.3 Third control lesson

The third lesson observed was an N6 Mathematics on `Second Order Differential Equations`. Lecturer 5 got into the lecture room, greeted students, and told them that they were going to do the `Second Order Differential Equations` without waste of time. A lesson on `First Differential Equations` had been completed the previous day. He then immediately put the following example on the board: $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 10$. He told students to take note of the degree of the highest power of the derivative in the example.

Lecturer5 told students that the general solution to this problem would be given by:

$y = \text{Complimentary Function (CF)} + \text{Particular Integral (PI)}$. [$y = CF + PI$]

He told students that the CF is determined from the Left Hand Side (LHS) and making Right Hand Side (RHS) = 0. He wrote on the board:

CF: $r^2 + 3r + 8 = 0$. He solved for `r` values as $r_1 = -2$ and $r_2 = -1$.

He said that because `r` values are unequal (-2 and -1), CF will be: $y_c = Ae^{-2x} + Be^{-x}$.

He further told students that if: $r_1 \neq r_2$, CF will be given by: $A = e^{mx}(A + Bx)$.

For complex roots where: $r_1 = a \pm bj$, the CF will be: $y = e^{mx}(A \cos bx + B \sin bx)$.

He told students that the CF always yields a quadratic equation with three possible solutions:

$r_1 = r_2$, $r_1 \neq r_2$ and $r = a \pm bj$. These are used to determine the CF nature.

He explained that the next step was to calculate the PI. He explained that in determining the PI, students should assume the general form of the function on the LHS of the given problem. He gave the following examples:

$y = 2$ (any constant), then for PI: $y = c$.

$y = 5x + 3$, then for PI: $y = Cx + D$

$y = 2e^{3x}$, then for PI: $y = Ce^{3x}$.

He pleaded with his students to always check the type of function on the LHS of the problem.

He said that for $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 10$, the assumed general form of the function would be:
 $y = C$.

Lecturer 5 rewrote the example on the board and ordered students to determine the CF and PI. He allocated about five minutes. After this time, he started solving the problem without involving students. He wrote:

$r^2 + 3r + 2 = 0$, and solved : $r_1 = -2$ and $r_2 = -1$ and gave CF as: $y = Ae^{-2x} + Be^{-x}$.

He took $y = C$ and determined the PI.

He explained that the first and second derivatives of $y = C$ had to be determined. The first and Second derivatives were determined as zero, leading to $2y = 10$ where $y = 5$. This meant that for PI: $y = 5$. Lecturer5 then put the full answer for students:

$$y = Ae^{-2x} + Be^{-x} = 5.$$

Lecturer 5 put another problem on the board: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = e^{-2x}$.

He ordered students to find CF and PI. After five to seven minutes, lecturer 5 solved the problem. He got the `r` values as $r_1 = 4$ and $r_2 = -2$ and CF: $y_c = Ae^{4x} + Be^{-2x}$.

He gave the PI as $y = Cxe^{-2x}$. He said that if the exponent of the function was the same as some part of the CF, then the PI takes an "x" as in $y = Cxe^{-2x}$. He determined the first as well as the second derivatives of $y = Cxe^{-2x}$. He emphasized the introduction of the `x` in the PI and said it would lead to using the product rule when determining the first and second derivatives.

Lecturer 5 then put another problem on the board: $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 18e^{-3x}$ and told students to solve it. He then walked between students' desks. After three to five minutes, he started solving the problem for the students. He then waited for students to call him to their desks when they had problems. The courageous ones called him to their desks and asked him questions softly. The lecturer gave another problem on the board, and the same approach was used again in doing the detailed solution to students without involving them. He then gave homework from the prescribed

textbook.

Towards the end of the lesson, Lecturer 5 issued students an information pack for the next topic. The pack summarizes all prescribed mathematical graphs and formulas for the different solids of revolutions.

5.2.1.4 Fourth control lesson

This was an N6 Mathematics lesson on 'Application of the definite integral'. Lecturer 5 greeted students as they walked into the lecturer's room and told them they were going to start a new chapter called, 'Application of the definite integral'. He wrote: $x^2 + y^2 = 6$ and $y = x$ and asked students to give the names of these graphs. Students mentioned circle and straight-line graphs, respectively. He drew these on the same system of axes and said that the points of intersection are calculated by equating the two graphs, that is $\sqrt{6 - x^2} = x$ and solving for x . He showed detailed steps on solving for 'x' and got: $x = \pm\sqrt{3}$

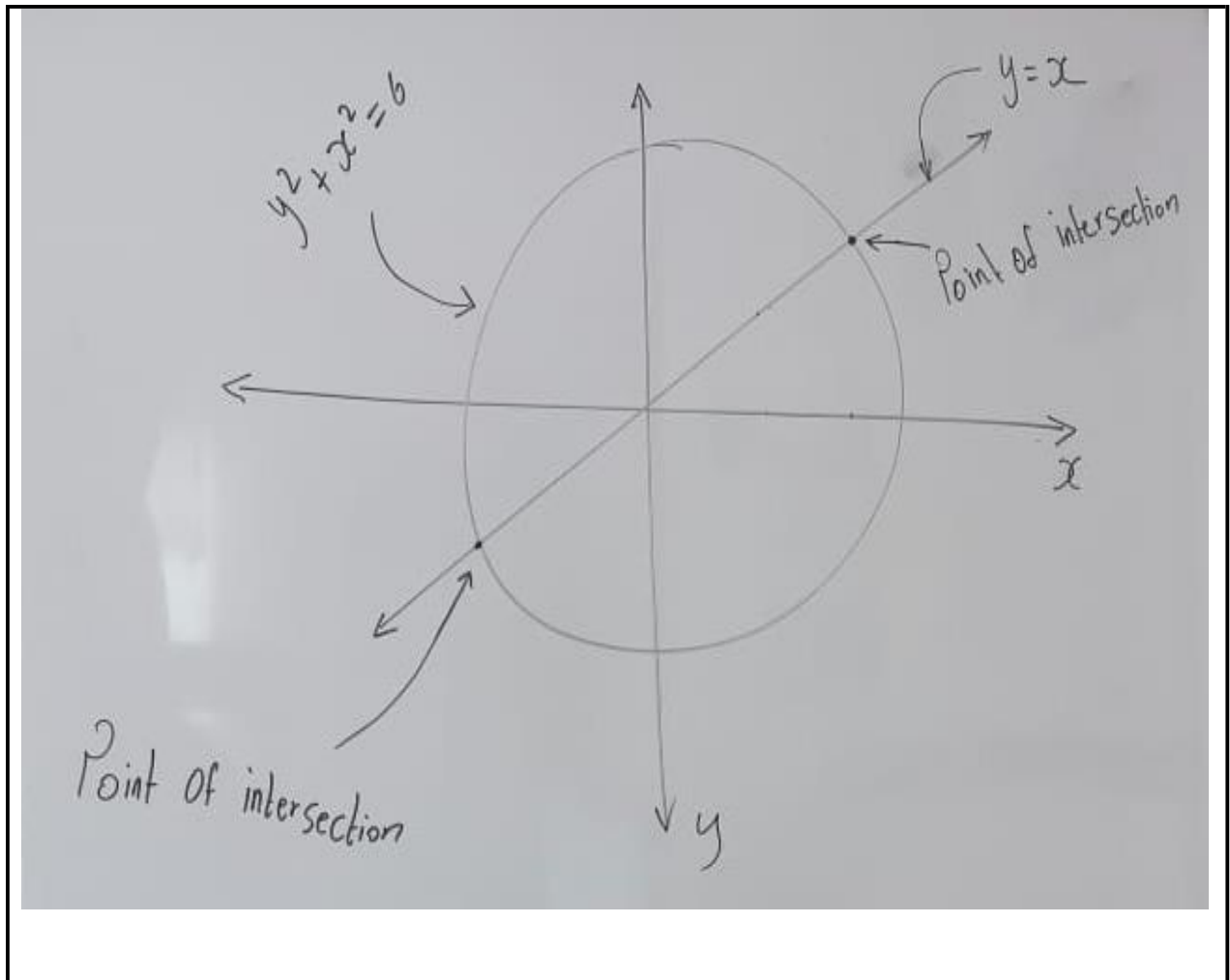


Figure 15: Points of intersection shown on Circle and Straight line graphs

Lecturer 5 gave students: $x = 4 - y^2$ and $x = y^2$ and ordered them to calculate the points of intersection. After some time, he calculated the points of intersection for students and drew a sketch that showed the enclosed area between the graphs. He explained that the enclosed area would be calculated from $(4 - y^2) - (y^2)$

He further told students that the first part of $(4 - y^2)$ will be integrated by identifying a formula on the formula sheet given on the hand-out during one of the recent lessons.

Lecturer5 wrote more pairs of equations and told students to calculate the points of intersections of each:

1. $y = 6 - x$ and $y = \frac{8}{x}$
2. $y = x^2 - 6x + 9$ and $y = 6x - x^2$
3. $\frac{x^2}{9} - \frac{y^2}{4} = 1$ and $y = 2$.

He later took $y = 6 - x$ and $y = \frac{8}{x}$ and drew the sketches on the board.

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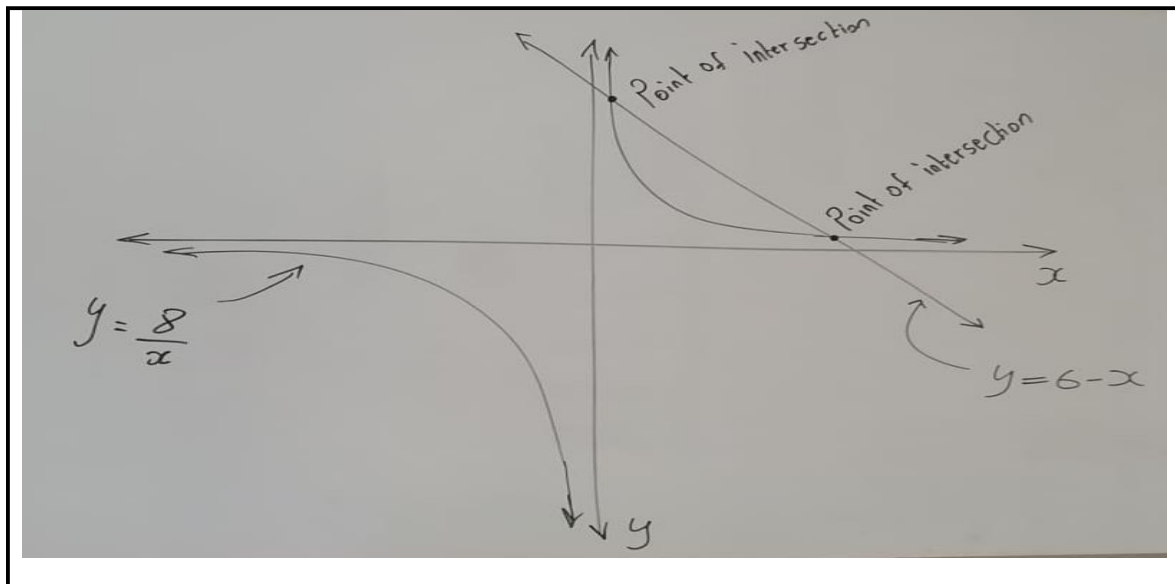


Figure 16: Points of intersection shown on Hyperbola and Straight line graphs

He calculated the points of intersection from $6 - x = \frac{8}{x}$ and found them to be (4,2) and (2,4).

He calculated from $Ax = \int_2^4 ((6 - x) - (\frac{8}{x})) dx$.

He integrated the functions as: $((6x - \frac{x^2}{2}) - (8\ln x)) \Big|_2^4$ and told students to substitute the upper

limit and subtract the substitution of the lower limit. Students looked busy calculating the answer. He calculated the answer, waited for students to calculate theirs, and later gave them the correct final answer. He said that students should check where they made mistakes and rectify them. The class was dismissed.

One male student went to the lecturer and showed him his work. The lecturer showed him the mistake in his calculation by explaining how the student should have done the problem

5.2.1.4 Fifth control lesson

This was an N4 Mathematics lesson on `Limits`. Lecturer 6 walked in and greeted students. Without wasting time, Lecturer 6 wrote on the board: Differential Calculus. She told students that from now onwards, they will be learning calculus. She pointed out that Mathematics N5 and Mathematics N6 syllabi are 100% calculus. She told students they were going to do `Limits`, and she went and put this topic on the board.

She explained that in determining the limit value of the function of x , *that is*, (x) , where `x` tends to `a`, then `x` must be substituted by `a`. She gave an example of $\lim_{x \rightarrow 2} (x^2 + 2)$. She explained that `x` tends to `2`; therefore, the `x` in the function must be substituted by `2` to give: $= (2^2 + 2) = 6$. She further explained that students must drop the limit part when substituting directly. She went on to explain a few rules of limits:

$$1. \lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

$$2. \lim_{x \rightarrow a} (f(x) \cdot g(x)) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$$

$$3. \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$$

She also gave and solved an example per each rule explained above. She also explained that the limit of a `constant` remains a `constant`.

She told students that there are two main methods of determining function limit values. That is the `direct substitution` method and the `simplification and factorization` method. She advised students to master areas of all types of factorization techniques, simplifications, trigonometric identities and other elements of algebra like exponents and logarithms. She wrote on the board and solved: $\lim_{x \rightarrow 2} \left(\frac{x^2 - 4}{x - 2} \right)$.

She substituted `x` directly and got the form: $\frac{0}{0}$ answer. She told students that the form `zero over zero` is not valid and can, therefore, not be accepted as an answer as it is indeterminable. She explained that the numerator should be factorized, and she then simplified the problem:

$$\lim_{x \rightarrow 2} \left(\frac{x^2 - 4}{x - 2} \right) = \left(\frac{(x - 2)(x + 2)}{x - 2} \right) = (x + 2) = 2 + 2 = 4.$$

She gave and solved another example: $\lim_{x \rightarrow 3} \left(\frac{x^3 - 27}{x - 3} \right)$.

Lecturer 6 checked where the challenge was but quickly realized that the student used her calculator in `degree mode` instead of `radian mode`. She addressed this with the student. Lecturer6 went on and explained the limits of the form: $\frac{\infty}{\infty}$, and gave an example:

$\lim_{x \rightarrow \infty} \left(\frac{x^2 - 2}{5x^2 - 3x + 1} \right)$. She then wrote on the board:

$\lim_{x \rightarrow \infty} \left(\frac{c}{x^n} \right) = 0$, and explained what this rule meant. She then solved the problem:

$\lim_{x \rightarrow \infty} \left(\frac{x^2 - 2}{5x^2 - 3x + 1} \right)$, and explained that direct substitution would give form: $\frac{\infty}{\infty}$.

She then explained that every term should be divided by the highest power of x in this way:

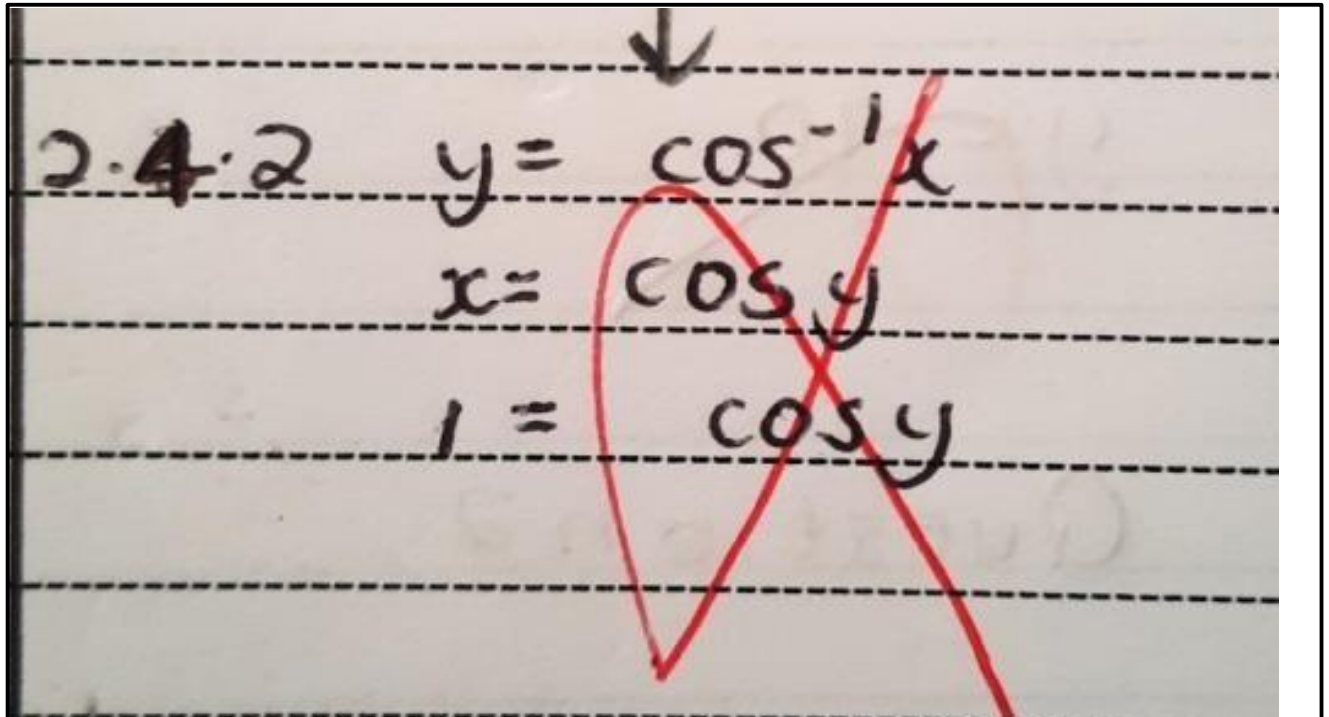
$$\lim_{x \rightarrow \infty} \left(\frac{x^2 - 2}{5x^2 - 3x + 1} \right) = \frac{\frac{x^2}{x^2} - \frac{2}{x^2}}{\frac{5x^2}{x^2} - \frac{3x}{x^2} + \frac{1}{x^2}} = \frac{1 - \frac{2}{x^2}}{5 - \frac{3}{x} + \frac{1}{x^2}} = \frac{1 - 0}{5 - 0 + 0} = \frac{1}{5}.$$

She asked students to ask questions, but none of them did. She gave homework from the prescribed textbook. She dismissed the class

5.2.2 Control group term test items from students' scripts

Seventeen test items (about 74%) were randomly chosen from students' scripts marked by control group lecturers. Test items show the questions, students' responses, how lecturers marked these responses and a researcher's short description of the scenario under consideration.

1. Differentiate: $y = \cos^{-1}x$



Handwritten student work on lined paper. The student has written the following equations:

$$2.4.2 \quad y = \cos^{-1}x$$
$$x = \cos y$$
$$1 = \cos y$$

A large red 'X' is drawn over the equations, indicating a wrong mark. An arrow points down to the first equation.

Figure 17: Test Item 7

The student got lost along the way but at least he/she knows what to do. This big wrong mark could somehow discourage the student. The student forgot to put the dy/dx next to the $\cos y$ and simplify further. This kind of marking undermines students' effort.

2. A follow-up to Q1.1.1: Solve for the value of y : $\ln y = 2$

1.1.2. $\ln y = 2$
 $y = 10^2$
 $= 100$

Figure 18: Test Item 8

The student got lost along the way but at least he/she knows what to do. This big wrong mark could somehow discourage the student. The student forgot to put the dy/dx next to the $\cos y$ and simplify further. This kind of marking undermines students effort.

3. Use L'Hopital's rule and determine: $\lim_{x \rightarrow \infty} \left(\frac{x}{\ln x} - \frac{1}{x} \right)$

1.1.1

$$\lim_{x \rightarrow \infty} \frac{x}{\ln x} - \frac{1}{\ln x}$$

$$\lim_{x \rightarrow \infty} \frac{x - 1}{\ln x}$$

$$\lim_{x \rightarrow \infty} \frac{\infty - 1}{\ln \infty}$$

$$\frac{\infty}{\infty} \quad [L'H]$$

$$\lim_{x \rightarrow \infty} \frac{1}{\frac{1}{x}}$$

$$\lim_{x \rightarrow \infty} \frac{1}{\frac{1}{\infty}}$$

$$= \frac{1}{0}$$

$$= 0$$

Figure 19: Test Item 9

The student has been penalized for making a fundamental mistake that: $\frac{1}{0} = 0$
This is a serious misconception displayed by the student.

1. Given: $f(x) = \frac{1}{2\sqrt{x^3}}$: Determine the derivative by using First Principles

$$\begin{aligned}
 2. \quad f(x) &= \frac{1}{2\sqrt{x^3}} \\
 &= \frac{1}{2}x^{-\frac{1}{3}} \\
 f(x+h) &= \frac{1}{2}(x+h)^{-\frac{1}{3}} \\
 &= \frac{1}{2} \left[\frac{x^{-\frac{1}{3}} \cdot h^0}{0!} - \frac{\frac{1}{3}x^{-\frac{4}{3}} \cdot h}{1!} + (-\frac{1}{3})(-\frac{4}{3}) \right. \\
 &= \frac{1}{2} \left[x^{-\frac{1}{3}} - \frac{1}{3}x^{-\frac{4}{3}}h + \frac{\frac{4}{9}x^{-\frac{7}{3}} \cdot h^2}{2!} \right]
 \end{aligned}$$

Figure 20: Test Item 10

The student did a few things correctly and should have been given marks for those steps. The student simplified correctly and correctly taken the bracket $(x+h)^{-\frac{1}{3}}$ and managed to correctly take out a common factor. All these efforts were not recognized by the marker.

3. Given: $f(x) = 2x^3 - x^2 + 5$, Determine the coordinates of the Turning Points.

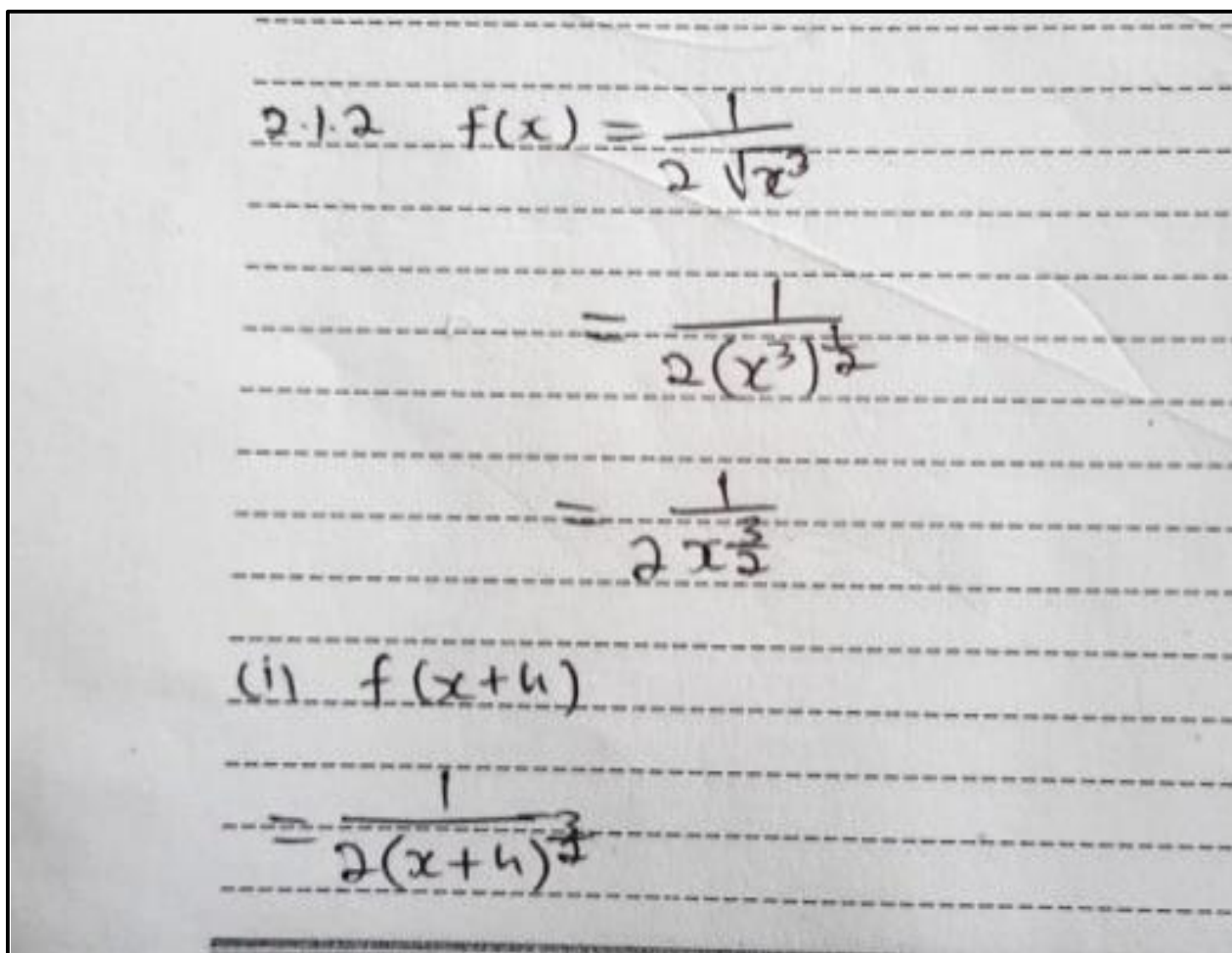
3.1 $f(x) = 2x^3 - x^2 + 5$
 $\frac{dx}{dy} = 6x^2 - 2x$
 $6x^2 - 2x = 0$
 $6x - 2 = 0$
 $6x = 2$
 $x = \frac{1}{3}$
 $(\frac{1}{3}, 0)$

3.1.2

Figure 21: Test Item 11

The student displays misconceptions. The function is $f(x)$ but he/she writes $\frac{dx}{dy}$ instead of $\frac{dy}{dx}$. A quadratic equation should yield two x -values but it is not the case here. The knowledge of factorization seems not to be well developed when taking out a common factor.

4. $f(x) = \frac{1}{2\sqrt{x^3}}$: Determine the derivative by using First Principles



2.1.2 $f(x) = \frac{1}{2\sqrt{x^3}}$

$= \frac{1}{2(x^3)^{\frac{1}{2}}}$

$= \frac{1}{2x^{\frac{3}{2}}}$

(i) $f(x+h)$

$= \frac{1}{2(x+h)^{\frac{3}{2}}}$

Figure 22: Test Item 12

The teacher did not mark this work as submitted by the student. The student partially simplified the given expression correctly. Was this left unmarked intentionally.

5. Determine $\int dx$ of $y = \tan^3 2x$

1.1.1) $y = -\tan^3 2x$
 $= \tan^3 2x \times \frac{1}{2}$
 $= \frac{1}{6} \tan^4 2x$

2.2) $y = \frac{\sin x}{e^{-x}}$

Figure 23: Test Item 13

The student seems to be taking chances and not knowing what to do. In the first step, is he exponent a 2 or a 4? This student does not know the integration technique to be used here.

Integrate:

$$\int \frac{x^3 - 2x - 1}{x - 3} dx$$

Handwritten student work for the integral of $\frac{x^3 - 2x - 1}{x - 3}$. The student incorrectly copied the denominator as $x - 1$ and used long division to get $x + 1$ with a remainder of 2. The work is crossed out with a red line.

$$\int \frac{x^3 - 2x - 1}{x - 1} dx$$

$$\begin{array}{r} x+1 \\ x-1 \overline{) x^3 - 2x - 1} \\ \underline{x^2 - x} \\ 0 + x - 1 \\ \underline{-x + 1} \\ 0 + 0 \end{array}$$

$$\frac{x^3 - 2x - 1}{x - 1} = \int x + 1 + \frac{2}{x - 1}$$

Figure 24: Test Item14

The student copied down the question correctly but rewrote it incorrectly, hence the wrong work. The student luckily recognizes the need for the long division method but, unfortunately, has taken down the question incorrectly. Could this be intentional by the student? This is an arbitrary error type. It must be marked wrong

6. Integrate: $\int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2} dx$

Question 1

1.1. $\int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2} dx$

$\frac{A}{(x-1)} + \frac{B}{(x+2)}$

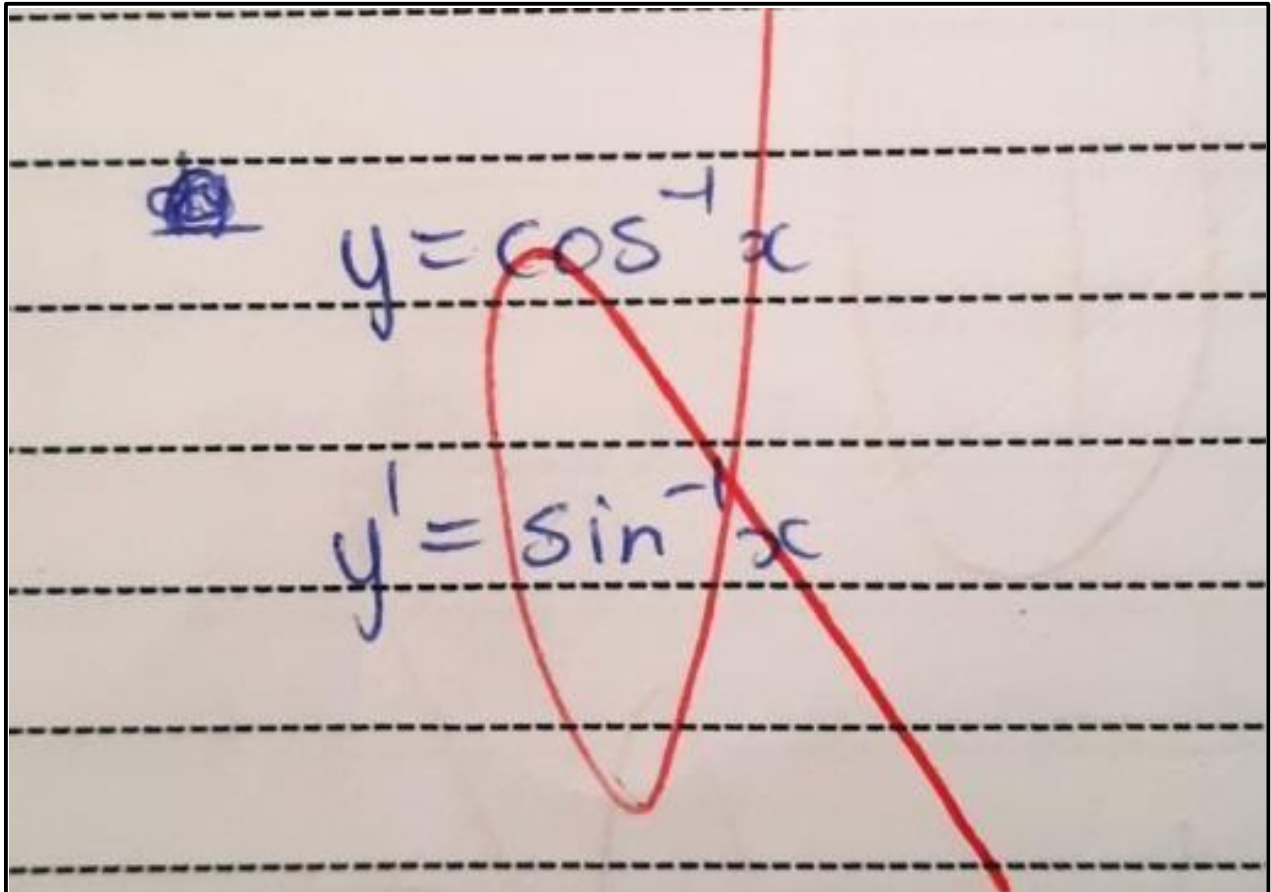
$\frac{A+B}{(x-1)(x+2)}$

$\frac{A}{(x-1)(x+2)} + \frac{B}{(x-1)(x+2)}$

Figure 25: Test Item15

The student made a principle error in his approach to solving the Partial Fraction problem. His cancellation suggests lot of confusion on what to do and how to go about solving the problem.

7. Differentiate: $y = \cos^{-1}x$



$y = \cos^{-1}x$

$y' = \sin^{-1}x$

Figure 26: Test Item 16

This student needs help to simplify the given function. Simplifying a given expression at times may be beneficial in opening up the next step. The student has no knowledge of inverse circular functions, a technique used in Mathematics N5.

8. By using L'Hopital's rule, determine $\lim_{x \rightarrow 0} \frac{\ln \sec x}{x \cdot e^{2-x}}$

$$\begin{aligned}
 &= \lim_{x \rightarrow 0} \frac{0}{0} \\
 &\quad \text{L'H} \\
 &= \lim_{x \rightarrow 0} \frac{\ln \sec x}{x \cdot e^{2-x}} \\
 &= \lim_{x \rightarrow 0} \frac{\frac{1}{\sec x}}{\frac{-1}{e^{-x}}} \\
 &= \lim_{x \rightarrow 0} \frac{\cos x}{e^{-x}} \\
 &= 1
 \end{aligned}$$

Figure 27: Test Item 17

The student failed to apply L'Hopital's rule correctly and also failed to differentiate the $\ln f(x)$ type. The student failed to recognize a product in the denominator.

9. If: $y = \pi^{\tan x} (\cos x)^x$, determine the derivative

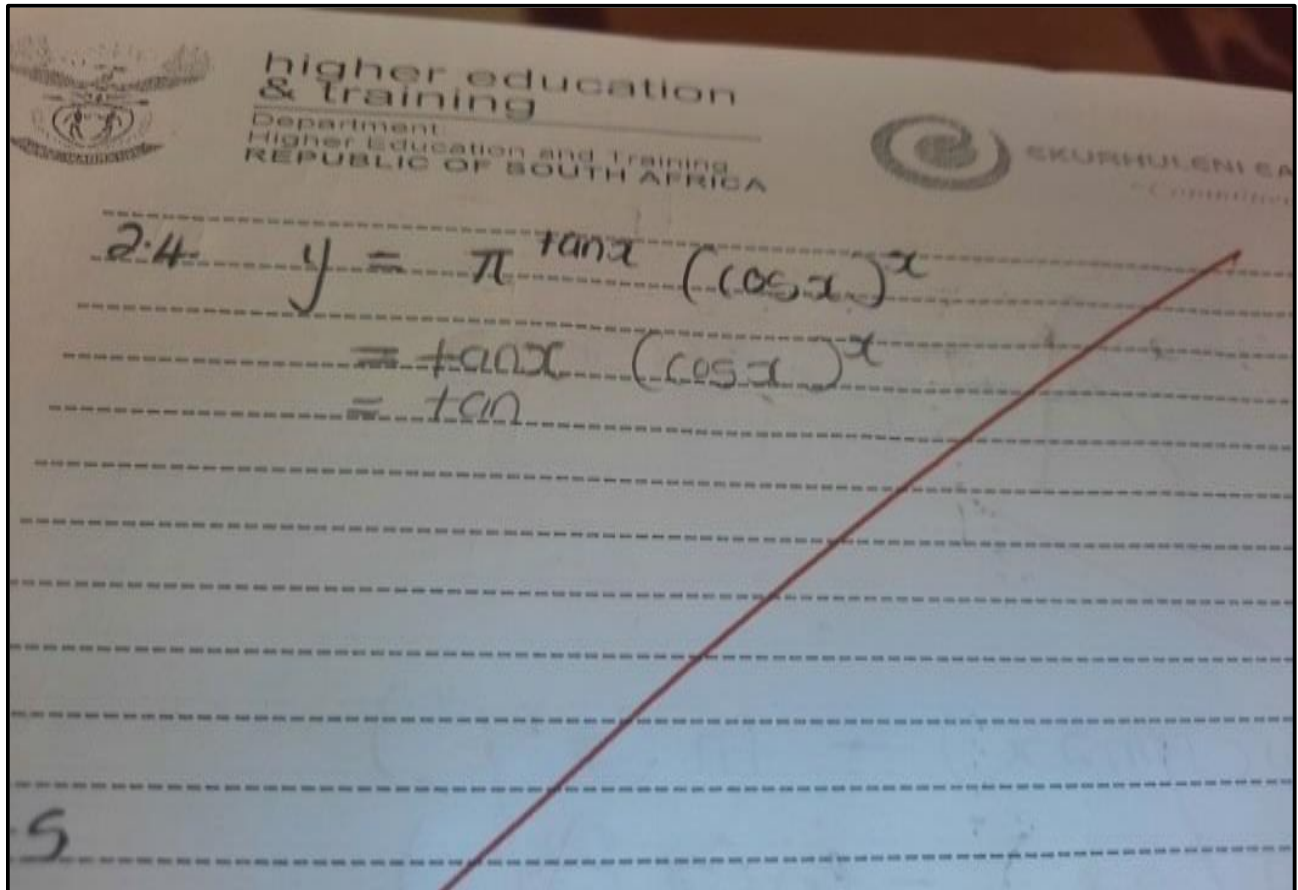


Figure 28: Test Item 18

The student did not at all have an idea of what to do and where to start solving this problem. This diagonal line means that there is nothing to work on and no mark will be given. This student has some misconception that: $\pi^{\tan x} = \tan x$ in the same way as: $e^{\tan x} = \tan x$. The lecturer should have noticed this instead of drawing a diagonal line which may be meaningless

10. Use L'Hospital's rule and determine:

$$: \lim_{x \rightarrow \infty} \left(\frac{x}{\ln x} - \frac{1}{x} \right)$$

l.l.l. $\lim_{x \rightarrow \infty} \frac{x}{\ln x} - \frac{1}{x}$

$\frac{d}{dx} = \frac{1}{x} - \frac{1}{x^2}$

$=$

Figure 29: Test Item 19

The lecturer marked the student wrong for not simplifying the question to, the form: infinity over infinity, so that L'Hospital's rule could be used.

13. Given: $f(x) = 1/(2\sqrt{x^3})$, differentiate from First Principles.

21.2. $f(x) = \frac{1}{2\sqrt{x^3}}$

$= \frac{1}{2} x^{-3}$

$\therefore f(x+h) = \frac{1}{2} (x+h)^{-3}$

$= \frac{1}{2} [x^{-3} - 3x^{-4}h]$

Figure 30: Test Item 20

The lecturer marked the response wrong because the student had incorrectly simplified the function. The incorrect opening step led to an incorrect derivative. The student failed to simplify correctly.

14. Integrate the function: $y = (8 - 5x)^3$

1.1.2. $y = (8 - 5x)^3$

$$y = \int (8 - 5x)^3 \cdot dx$$
$$= 8 \int (-5x)^3 \cdot dx$$
$$= -8 \cdot \frac{1}{5} \int (-5x)^3 \cdot dx$$
$$= -\frac{5}{8} \frac{(-5x)^4}{4} + C$$

Figure 31: Test Item 21

The student's pre-calculus knowledge is poor. Taking out the eight out of the bracket has been done illegally. The student needed to know which technique to use to find the integral.

15. Determine the derivative when: $y = \ln^2 \frac{2x}{e^x} - \ln e^{3x^3}$

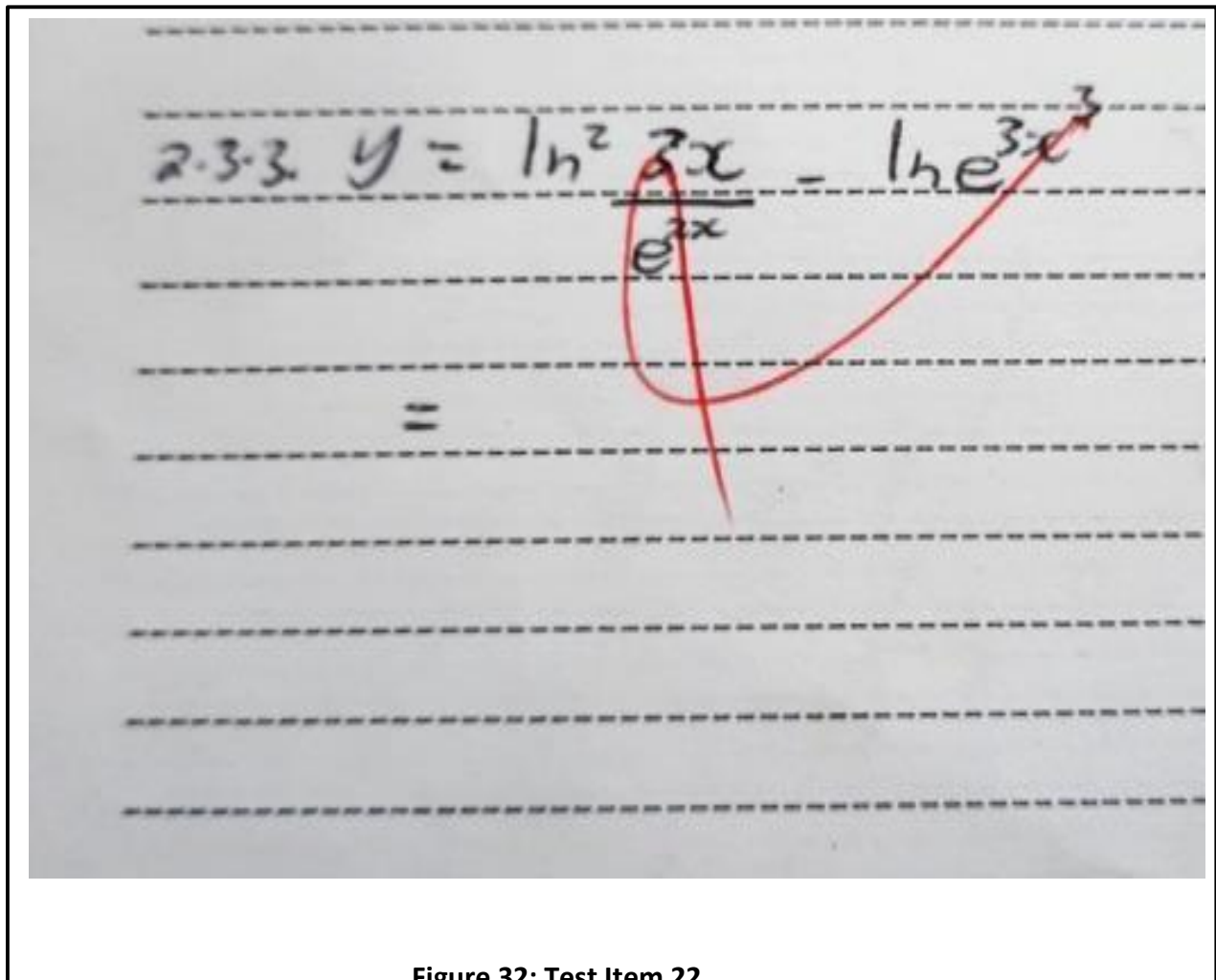


Figure 32: Test Item 22

The student could not move an inch; hence the wrong mark was given. There is too much work to be done here in order to make this student understand.

16. Determine the integral of the function: $\int \frac{x^2}{2x+1} dx$

1.1.4) $\int \frac{x^2}{2x+1} dx$ Let $x^2 = u$
 $\therefore x = u^2$

$\int \frac{u}{2u^2+1}$
 $4 \int (2u^2+1)'$

~~$4 \cdot \frac{2}{3} u^3$~~ $\frac{4(2u^2+1)^2}{2^2}$

$= \frac{4(2(x^4)+1)^2}{2}$

Figure 33: Test Item 23

The student could be right had he or she correctly used the long division method technique. For the student to say: *Let $x^2 = u$* , is a serious situation. He or she does not know what he or she is doing.

5.2.3 The consolidated control lecturers' interviews

Three lecturers from the control group were interviewed. The following information captured the composite and consolidated lecturers' responses and views to the interview questions.

These lecturers were not trained on productive struggle and professional noticing strategies of mathematics teaching and learning for the sole reason of using them as a benchmark point of comparison with experimental group lecturers. They used old traditional methods in their teaching. Their manner of teaching was student-centered, with little student involvement during lesson presentations.

These lecturers were asked what they liked about teaching TVET college mathematics. In the main, they responded that technical mathematics develops and improves students' way of thinking. Lecturer4, in particular, indicated that teaching technical mathematics is interesting because calculus topics are efficient and thought-provoking, for example - in projectiles when an object is thrown vertically up against gravity, minima and maxima situations, irrational roots by Newton/Taylor's method. This lecturer indicated that he views mathematics as a powerful subject that can be used to make students understand science and engineering.

These lecturers were asked to share their experiences and views on everyday greetings done to students. Lecturer6 indicated that as human beings we must greet each other when we meet. She registered her concern that at times students just keep quiet and look at her, while trying to greet them. She felt that greetings are not necessary since they are not part of the curriculum.

It was observed that there were no classroom discussions during mathematics teaching and learning. Lecturer4 was asked why he did not allow students to engage in active classroom discussions. He responded that 'students do not have the knowledge of the subject and therefore

their discussions will lack depth`. It was further observed that lecturers dominated and talked the most, with students relaxed and taking notes put on the board by lecturers.

Lecturer5 was asked if he used examples in his teaching and for what purpose did he use them. Lecturer5 said that he indeed used examples during lessons. He was asked why he used those examples. He replied that all mathematics lessons must have examples. Unfortunately, these examples could not achieve much as lecturers presented them with no good intentions or knowledge of how they could make them effective. Examples were presented as ordinary problems to solve and served no particular intended purpose of introducing lessons.

On how they noticed students who were lost and not understanding during the lesson, all lecturers responded that they could judge from the student's facial expressions and then make follow-ups with such students. Unfortunately, that was never seen happening.

They alluded to the idea that calculus topics are complex and difficult and indicated that students need more pre-knowledge to enhance their understanding of calculus. They gave examples of functions, mathematical graphs, logarithmic and exponential laws. Lecturer4 indicated that this problem could be attributed to the registration minimum entry requirements set for N4 courses. This lecturer opined that the minimum entry into N4-Course is too low (matric Mathematics and Physical Science both 40%). He suggested that 50% may bring better results. This lecturer was asked what assistance was given to struggling students and the response was `There is nothing I can do if students do not understand Maths`.

These lecturers indicated that students made many errors in their term tests. According to them, errors are attributed to a lack of basic and pre-knowledge of calculus. They indicated that most students confuse differential and integral calculus and cannot distinguish between the set of rules in each case. Lecturer5 was asked what he thought could improve calculus understanding and he

responded that maybe technology should be used to enhance the teaching and learning of mathematics by having well-resourced lecturer rooms with internet connections.

Control lecturers showed no respect to the process of giving feedback to students and they were asked if they believe in giving feedback to students. Lecturer4 responded that she did not have more confidence in feedback because the student could always check his/her understanding against the mark scored on the test. She said that the score should do the talking on her behalf whilst she concentrates on other things.

Lecturer6 was asked why she did not involve students in her teaching and did not allow them to dominate the lessons nor talk amongst themselves. She responded that she could not afford to waste time as she was always pressured to finish the syllabus. She confidently said, `Students do not know mathematics, and therefore they should be taught and told of the rules of the subject` and that `they must do as lecturers instruct them`.

Lecturer6 was asked why she gave solutions and never involved students after giving calculus problems to solve. She replied that students can always check their solutions against what she puts on the board. She said there is no point in working from students' wrong responses as these responses would waste time if she were to rely on those students' responses.

These control group lecturers were asked if they could think of any better teaching method that may enhance students understanding. Lecturer4 responded adamantly that the best method of teaching mathematics is through telling students what to do, doing examples for them, and solving mathematical problems for them. Lecturer6 responded that `classroom discussions are a waste of precious time` and argued that students need to learn mathematics and that their contributions about mathematics will lack depth. They do not realize that discussions are far more valuable and

potentially assist in leading to manifestations of errors and misconceptions, which can be used to improve students' mathematics understanding through professional noticing ways.

These lecturers hold a strong view that students should be given answers to questions given to them without subjecting them to struggles. They believe students should sit back, relax, and get spoon-fed with solutions. They were asked why they asked questions and shortly afterwards solved them without first checking how students attempted those problems. Lecturer5 responded that he was given a contract to teach students and not to subject students into frustrations. This approach relegated students to taking back sit and making them depend on their lecturers.

5.3 Conclusions

This chapter focused on data collection strategies in classroom observations, collecting data from term test items and lastly, doing interviews with lecturers. Lecturers were observed on their respective campuses, in their lecturer rooms and during their normal teaching and learning interaction with students as appearing on the campus's official timetable. Samples of actual term test scripts were used to see how the lecturers marked students' responses. The interviews were done with individual lecturers during their free periods.

CHAPTER SIX: DATA ANALYSIS

This chapter presents the analysis of data collected through classroom observations, interviews and students' marked scripts analysis. Data analysis touches base with the discussed theories and some constructs under the analytical and conceptual framework. Data analysis was also reinforced by considering other constructs like MKT and Lobato's teaching-by-telling method. Literature served as a cementing material in bringing together all constructs.

Constructivism posits that students construct knowledge through cognition driven by mental regulation (Piaget, 1968). In line with this idea, Sfard (1997) further argues that students acquire knowledge by constructing powerful mental structures, concepts or logic. Cobb, Yackel and Wood (1992) take it further that learning is an active process in which students solve problems that arise as they participate in the mathematical practices of the classroom. This act shows that teaching and learning mathematics is interactive and may involve negotiating mathematical meanings. Conversely, Vygotsky (1978) assumes that the same knowledge is fallible and subjective simply because it is constructed by individual students (Von Glassersfeld, 1989).

If students construct their learning, they are surely constructing their errors and misconceptions (Confrey & Kazak, 2006). No teacher, in his rightful state of mind, will go to class and teach errors and misconceptions to students. However, Confrey (1990) asserts that lecturers should not try to forcefully remove errors and misconceptions but rather negotiate them with students. These introductory paragraphs raise many ideas that necessitated the use of more than one theory in this study.

Regarding the layout, all areas of experimental group lecturers are analyzed first (observations, interviews and term test items) and then followed by the control group analyses of the same areas.

6.1 Analysis of experimental group Classroom Observations

The following work focuses on the consolidated data analysis of the three experimental group lecturers.

It is important to indicate that the study followed the chosen theories as they provided the basis of what was studied (Lester, 2005). Experimental group lecturers followed and upheld the constructivist principles of teaching and learning.

Experimental group lecturers changed their ways of doing business. They adopted the `student-centered` approach to teaching. They threw away old traditional methods of mathematics teaching and used the new productive struggle and professional noticing strategies of teaching mathematics as explored in the study. Their students seemed to have enjoyed the new strategies of mathematics teaching and learning. Lecturers avoided dominating their lessons and acted as facilitators by talking only when necessary (Lobato, 2005). They stopped their teaching-by-telling methods and avoided engaging in talk and chalk teaching methods. This new mathematics teaching and learning approach made students not to rely and depend heavily on their lecturers. They attempted and solved problems independently without much guidance from their lecturers. It means that students became self-directed learners as they did not wait to be told what to do at all times. Students were allowed to engage in productive struggle activities when solving challenging calculus tasks.

Lecturers chose and gave suitable productive struggle tasks that introduced new topics successfully and set students ready to venture into new learning content. Lecturers used introductory tasks to carefully explain real-world scenarios that allowed students to make mathematical connections and links from their known to unknown knowledge. Real life scenarios bring about correct context necessary to understand mathematical tasks (Luneta & Makonye, 2010). According to Morris (2019), engaging students in solving authentic, real-world problems, assist them to attain their true potential.

Lecturers created classroom atmospheres that allowed for active discussions and interactions among students. One lecturer told students that there are no wrong answers in his classroom, which is evident in a classroom that is tolerant of errors and misconceptions (Nesher, 1987). Students realized that making mistakes was not something sinister but was part of learning. Students knew they would not be embarrassed for their wrongful attempts when solving mathematical problems. This approach to teaching mathematics made students to always attend their classes in big numbers because of their high interest in learning mathematics, resulting in their willingness to actively participate in class discussions without fear.

Experimental group lecturers were able to engage their students in APOS theory exercises. In the introductory and minimal instruction part of all experimental lessons, lectures subjected their students to do some `Actions` through Lobato's rightful way of `teaching by telling` approach. For example: In the first experimental lesson, lecturer1 told students that in functions where `x` and `y` are related in a more complex manner, where it may not be desirable to express `y` in terms of `x`, implicit differentiation was needed. He further told students that in determining the derivative, both `x` and `y` must be differentiated, and the most important thing was to show the dy/dx next to the `y` derivative. Lastly, he told students that anything that was not `x` or `y` should be treated as a constant. This led students to internalize the concept to the point of being the objects (Tziritas, 2011) as these objects were incorporated into their schema. This object became part of their schema as we see them successfully solving more problems independently.

Experimental lesson number four illustrates the `Concept image and concept definition` as explained earlier. Students were asked to explain what the word `pressure` means, and they gave their concept images and their concept definitions. It is found that Student3's definition of `pressure as a force applied to a unit area` is more or less similar to that of the mathematics community. Student4 was also able to provide the formula: $Pressure = \frac{Force}{Area}$. Most interestingly was when the lecturer asked for the `concept image` when he asked about what happens to an object

submerged in water. One student indicated that it felt different when he was at a shallow depth compared to when he was at an increased depth but could not explain or tell what was different. This is evidence that `Concept images and concept definitions` will always be at play in a constructivist lesson design.

The students of the experimental group lecturers have shown much interest in their studies. Attending their classes in big numbers and actively participating in class discussions show elements of self-directedness. Most students who enroll at engineering TVET colleges want to qualify as artisans. They are focused on their studies and determined to achieve their dreams because they have discovered who they want to be in life (Boyatzis, 2002). Students were observed to be at the heart of their learning as they showed activeness in meaning-making by not becoming passive receivers of information. They took advantage of classroom atmospheres that their lecturers created and, by so doing, took responsibility and ownership of their learning.

All experimental group lessons have three main sections: introduction and minimal guidance of the lesson, new learning and productive struggles of the lesson and lastly, the lesson conclusion and way forward of the lesson.

The following data analysis confines what happened during normal mathematics teaching and learning in classrooms in the indicated sections.

6.1.1 Analysis of the `introduction and minimal guidance` of the experimental group lessons

Lecturers went to classrooms, and upon arrival, greeted their students in a kind and purposeful manner. They allowed students first to settle in because some students might still need to be ready, and chances were that if it was not done, the intended lesson would fail dismally long before it could start (Aung & Tepsuriwong, 2017).

The simple trick of greeting students politely and ensuring they are settled in has certain didactic functions. According to Aung and Tepsuriwong (2017), purposeful greetings to students ensure the balance of power between teachers and students and further detach students from what happened during the previous lesson by adjusting to the new lesson. These researchers point out that the kind greetings set a positive learning environment, motivate students, capture and focus students' attention and, most importantly, set the right tone for teaching and learning by foretelling a controlled atmosphere of respect for one another. It means that greetings create a space within which both students and lecturers function optimally in engaging and embarking on doing mathematics. The greetings were short but necessary (Aung & Tepsuriwong, 2017). Lesson introductions of experimental group lecturers followed this idea of making sure that their students got warm and purposeful greetings.

De Souza (2021) asserts that good lessons are determined by their introductions. Lecturers designed good introductory tasks to connect existing knowledge with the new knowledge they were going to teach. According to De Souza (2021), a well-chosen and well-articulated example helps build students' confidence and prepares them to tackle new problems independently. In the study, the good productive struggle tasks took the place of examples. Lecturers gave explanations relating to real-life situations that promoted understanding as students made mathematical connections with situations outside the classrooms. This was important because the psychology of education taught us that no student is a tabula rasa/empty vessel (Olivier, 1989).

These introductory explanations and tasks aroused interest, created an appetite of wanting to learn to know more in students, and prepared them for to be ready for new learning. All these resulted in students getting involved in lessons from the onset because otherwise, lecturers knew that if they did not do so, they could be running the risk of losing students until the end of the lesson. If students are not involved from the start of the lesson, it is doubtful that they will participate as it develops.

6.1.2 Analysis of `new learning and productive struggles` of the experimental lessons

The experimental group lecturers chose suitable productive struggle tasks that helped them to likely achieve their lesson objectives, as explained in the previous section. The chosen tasks prepared students to do more ill-defined problems that threw them into potential struggles. Lecturers kept the high task demand because they knew that declining the high task demand was defeating the same purpose for which difficult tasks were given.

In the first experimental lesson, when students did not know how to differentiate: $x^2 + y^2 = r^2$, the lecturer did not tell students what to do but instead probed for more. The problem was finally solved through probing and asking telling statements (Lobato et al., 2005). The lecturers left the task demand unchanged for students to struggle finding solutions. This was necessary as declining it would have brought easy moments to find solutions with no struggles.

Experimental group lessons were student-centered by design and implementation. One outstanding aspect was that students were not fully dependent on the lecturers to be fed and given solutions or information. Students got used to being on their own and attempted problems without expecting much assistance from their lecturers. As mentioned earlier, students were motivated to become independent (self-directed) and actively participated in classroom activities. We saw this when Lecturer1 told students how to differentiate implicit types of functions in the first experimental lesson. Lecturer1 told students only the most important aspects of the lesson and left all other things to be self-discovered. The lesson worked out to be a perfect productive struggle scenario when students were probed to ultimately find answers within themselves.

Classroom discussions created platforms for students to interact with classmates and their lecturers. The interactions in the classroom had the potential to make students realize their errors and misconceptions. In the fifth experimental lesson, lecturer3 asked Student4 to give the coordinates of a point, and Student4 gave the wrong coordinates. Student5 corrected student4, who

ultimately admitted he was wrong. The discussions further enhanced the growth of self-directedness on the part of students. At the same time, lecturers were afforded time to notice their students' mathematical thinking during these discussions. Lecturers used that time to check for erroneous responses from their students and interpret them to later attend to them.

Another important observation was that the experimental group students improved their listening and hearing skills. During my classroom presence, I could hardly hear any student asking the other student to say it louder or repeat the question. It was evident that their involvement in discussions improved big time. For people to talk and interact in communication there has to be effective listening and hearing taking place.

Lecturers randomly picked students to put their solutions on the board. This approach made students remain ready, knowing they may be picked anytime. It also ensured that as soon as a student encountered problems, he or she must quickly seek assistance because otherwise, he or she may be picked. No one wanted to be embarrassed in front of peers and so students did not want this to happen to them. They make sure that they were always ready to engage in mathematical problems in class.

Allowing students to discuss other students' work on the board contributed greatly to the teaching and learning atmosphere. It had many benefits. Firstly, when the lecturer drew back a bit, the shy and weak students rose to the occasion by participating in the discussions. These students could ask clarity-seeking questions to their fellow bright students, who explained to them in a manner and approach that made them understand better. This helped build confidence in weak students by ultimately growing confidence and understanding. It is because at times shy and weak students find it difficult to ask their lecturers on their challenges.

Sometimes students did not know how to solve given problems, not because they were blank in their minds but because they wanted a small piece of information to start them to solve. In such

cases, lecturers engaged in probing their students, and ultimately, these students were able to solve such problems. Vygotsky's ZPD came into play and worked perfectly, as students could correctly calculate the solutions after the probing exercise. Lecturers did not have to tell students what to do, but they probed deeper to let students think deeply. This exercise of probing students produced positive results.

Amiyani and Widjajanti (2018) indicate that students' self-created and self-taught knowledge improves the willingness to try new problems on their own, grow more resilient on challenging questions, and make deeper connections and renders students unafraid of making mistakes because they know they will learn from their attempts/mistakes. These authors argue that this kind of knowledge improves students' mathematical knowledge, mathematical skills, mathematics attitude, cognitive ability, conceptual understanding, and mathematical reasoning. According to these researchers, this knowledge builds self-confidence, interest in mathematics and the will to get better results. All of these characterized the experimental group's mathematics teaching and learning approaches in this study.

Students did not sit back and relax by listening and taking notes during the lessons; instead, they productively struggled to solve problems and build their knowledge. Lecturers allowed active participation during classroom discussions. Active talks by students encouraged them to somehow manifest their flawed mathematics ideas to be noticed by lecturers. This made the lessons to be interesting and student-centered. Students became actively involved in concept formulation and understanding and needed to know their meaning to focus on memorizing the formulae.

In response to Warshauer (2015) and Permatasari (2016), and Edwards and Beattie (2016) on strategies and indicators of productive struggle and the 'Problem Cycle' respectively, the following came out clear: Students were free to ask questions to lecturers, and lecturers developed the tendency to throw back the questions at the very students as a way of probing for more ideas around the questions. The responses to the probing questions gave the lecturers some ideas to measure

(notice) their students' mathematical understanding. Sometimes students asked questions not because they needed to know the answers but to check if their understanding was correct and how the lecturer and other students received their answers. When students were allowed reasonable time to do solutions on their own, they could explore and try possible methods that may come in handy in finding the solution. That allowed students to learn to be on their own and independent when solving mathematical problems.

The time given for students to productively struggle in attempting the solution to a given problem; enabled lecturers to exercise professional noticing skills, which assisted in spotting students who might be too reserved or lack the courage to seek advice when facing challenges and difficulties. It also created a platform for lecturers to have time to assist those who were needy. The assistance referred to here does not mean telling students answers to questions asked but probing them instead because the telling questions and the asking statements could unlock further understanding necessary for the solution.

Lecturers shifted away from traditional teaching methods to new strategies of productive struggle approach. These lecturers did not spoon-feed their students with ready-made answers; instead, they created a platform for students to actively construct their learning.

6.1.3 Analysis of the 'lesson conclusion and way forward' of experimental lessons.

Experimental group lecturers did not just leave things hanging after the classroom discussions; instead, they made sure that they resurfaced to summarize and emphasize important aspects of the lessons. They made sure that the learning objectives were consolidated; for example - in the third experimental lesson, the lecturer gave a summary by emphasizing possible function types for 'Partial Fractions'. This gave the lecturer an idea of whether the lesson objectives were met and allowed students to self-assess their mathematical understanding of what they learned. This did not happen with the control group of lecturers.

During the summary and conclusion of the lesson, the lecturers tied together all strategies and alternative methods used to solve the problems students initially struggled with. Lecturers used that time to reinforce important connections and helped students to wait for the next lessons in anticipation.

Experimental group lecturers gave their students homework at the end of each lesson presented. Sometimes, the homework questions needed to be more ill-defined than the ones in class. Students always returned with homework done, showing how committed and self-directed they were. In the second experimental lesson on the logarithmic type of differentiation technique, student1, who had done the homework correctly, asked the lecturer of a previous examination question that he needed to check if she did it correctly. The lecturer later threw that past exam question to the whole class. That was a way of avoiding giving and providing the solution to the student. The lecturer was too clever to know that other students would definitely learn from the very problem.

All experimental group lecturers' lessons successfully achieved their objectives as planned.

6.2 Analysis of experimental group's students' scripts.

Data collected from the indicated 23 test items show how lecturers marked their students' term test responses. The students' responses revealed their level of understanding in the questions asked. As Porter and Masingila (2000) argued, analyzing students' written work gives lecturers' information about students' understanding of mathematics. As shown on students' scripts, lecturers noticed students thinking when marking their written work. They have shown their reaction towards errors made by students in their written work by a way of annotating students responses.

It is true, as argued by Crisp and Johnson (2007), that it is common practice for markers to annotate students' scripts whilst marking - through underlining, circling, using abbreviations or making comments. According to Milanovic, Saville, Shuhong and Syndicate (2006), annotation supports

markers' thinking and marking decisions. Milanovic et al. (2006) argue that lecturers may use full words and sentences rather than standard annotations to communicate their reasoning. Arrowsmith and Chyriwsky (2009) indicate that annotation facilitates rapid communication of markers' reactions through comments and explanations by placing them precisely at the relevant point within the student's work.

I see annotation as a means of giving spontaneous feedback to students. A well-annotated student script may be given without going to class to do a feedback session.

Approximately 27% of students' term test items from experimental group lecturers have been analyzed for this research study. Marking was done fairly well. It was 'process oriented', considering all steps of the students' responses. This is seen in test item1, where the student gave unknown values for the partial fraction question without showing how he/she worked out those values. The marker did not get carried away by the correct values but instead questioned how the student got those values. The marker wrote: 'How did you get these values? '. This is an annotation. Annotation gives the marker an idea of the mistakes and error types and the frequency at which they occur, as opposed to just marking everything at once as either wrong or right without annotating.

Five responses out of a total of six have been well annotated. Annotation strengthens feedback sessions by making them sound and meaningful (Chyriwsky, 2009). The importance of annotation is that the minute students receive back scripts, they immediately realize their mistakes. This will likely encourage students to correct mistakes and improve their mathematical understanding by learning from them.

The marking of students' scripts distinguished between simple mistakes, errors and misconceptions. Test item2 shows the student's response with a mistake of forgetting to carry through a part of the solution to the final answer. Here the marker wrote: Where is: e^{3x} ? As soon

as the student receives his/her test back, immediately he/she realizes the mistake. In instances where errors were made, the marker may only know what was going on in the mind of the student if he/she calls the student to explain his or her response. It is because errors exist because of misconceptions (Olivier, 1989).

It was easy to see what experimental markers noticed in their students' responses by following their annotation remarks. In test item 4, the student made careless sign mistakes when giving the compound angle identity of $\cos(x+y)$. As soon as the student receives back the script, he or she will realize the mistake. That is the power of good annotation.

In a 'Partial Fraction' question (test item 1), the student gave answers only and was marked wrong. This was a long calculation with many steps, and the student might have done rough calculations somewhere else and transferred the calculated values to the script. The teacher yet asked where the student got those values. The student might have genuinely done the calculations somewhere and only gave the values without showing them. It could also be that the student had copied from a friend or even got the values mistakenly. It was proper for the marker to make such annotations.

Annotations made it easy to follow what lecturers noticed in students' written responses when answering calculus questions.

6.3 Analysis of experimental groups' interviews

All three experimental group lecturers possessed a national diploma during the time of data collection. They all did mathematics as a major subject in their qualifications. Two of these lecturers are furthering their studies with recognized institutions of higher learning in South Africa. The highest qualification of a lecturer determines the level of mathematics pedagogical content knowledge that this lecturer possesses (Bansilal, 2014). As argued by Ball et al. (2008), mathematical content knowledge and, in particular, Common Content Knowledge (CCC), Specialized Content Knowledge (SCK) well, as horizon knowledge, assist teachers in skillfully

noticing errors and mistakes in students thinking and also assist in preparing them for forthcoming topics or higher levels of learning. As argued further by Walshaw and Antony (2008), teacher knowledge, professional noticing and listening are at play during the lesson and enhance the quality of classroom discussions and further create a platform for the teacher to respond appropriately to students' erroneous thinking and ideas.

The response given by lecturer2 in appreciation and the love for productive struggle and professional noticing strategies of mathematics teaching and learning is something to ride home about. Comparatively speaking, the new teaching strategies are superior to old traditional methods of teaching. Students did not passively receive knowledge but instead they actively constructed their knowledge. This gave even more reasons why new productive struggle teaching strategies should be used.

Experimental group lecturers knew the power of setting up a positive classroom atmosphere through a warm welcome done with the art of kind greetings. They did not look down at greetings as just a formal exercise to engage in with their students. They were aware that making students feel at home, maximized their potential of understanding deep calculus topics. It is true as said by one lecturer that for students to participate actively in classroom activities, they should feel welcome and appreciated and the secret to achieving this is through positive welcome and greetings.

Experimental group lecturers knew that `calculus` is important for students who want a successful engineering career. This mindset and position surely affected and impacted their teaching. Their awareness that calculus promotes critical and logical thinking in students made them appropriately prepare their lessons and by so doing improving the quality of classroom interactions and mathematics understanding.

Experimental group lecturers always introduced topics by relating to and explaining real-life scenarios in line with the topic, followed by well-thought productive struggle tasks. After that, questions were put on the board for students to do. They always gave reasonable think time for students to work on the given problems. These lecturers would then draw back and let students struggle to find solutions. They later availed themselves to attend to all students' clarity-seeking questions during the struggles. They were asked why they did all of the above. Lecturer3 indicated that for students to answer mathematical questions correctly, 'context' plays a key role in analyzing and understanding the question. That is true. She said that giving students some time to work on a new problem creates a platform for them to actively engage with the question and check how they should solve it. It then showed that these lecturers were not only aware that students actively build their knowledge, but indeed they gave students that 'think time' to do so. The time given to students to be on their own and try to solve the problems was also indicative of the awareness on the lecturer's part that some students might get stuck and be lost in the process and not know where to start in doing the solution, and this needed them to intervene and assist such individual students who battled.

Lecturer3 was further asked - why she randomly picked students to put solutions on the board, and she responded by saying that – she was being strategic in using students' solutions to start classroom discussions. That sounds like a great strategy indeed. She indicated that whenever she asked other students if the work put on the board was correct, it always started fruitful engagements and robust discussions in class. That created a platform for the lecturer and students to discuss the solution on the board. During that time, the lecturer also had the opportunity to probe students on why they held particular views about their attempts and solutions. This scenario presented a perfect moment for lecturers to exercise their professional noticing skills.

These lecturers were asked to share their experiences about TVET college mathematics teaching and learning of calculus and how they see students' understanding and performance in the calculus section of the study. The response was that students do battle with calculus, and they were quick

to attribute that to poor mastery of foundational and pre-calculus knowledge as well as some poor background knowledge on mainly: graph functions, factorization, manipulation of technical formulae, logarithmic expressions as well as exponents. They indicated that during classroom discussions, some students were brave enough to open up about their problem areas and requested additional work to improve their understanding.

Lecturers argued that the true picture of whether students understood the lessons taught was looking at how they performed in term tests. They indicated that they picked mistakes and errors in the students' responses during the marking of term tests. They further said that they brought back to class the very mistakes and errors to re-look at them at the classroom level, where everyone got involved in the feedback discussions. That created a stage and platform for students to assist one another; those who gave correct solutions in particular questions, will lead the discussions in assisting those who got the wrong solutions. Technically speaking, this became yet another chance for students to refine and consolidate their overall understanding.

In conclusion, it was clear that these lecturers believed in productive struggle and professional noticing strategies because they saw that almost all students actively took part in classroom discussions and, by so doing, enhanced their mathematical understanding. They indicated that it was for the first time seeing such levels of students' confidence in their mathematics studies.

6.4 Analysis of Control group lecturers' Classroom Observations

As seen earlier, a lesson plan is a teacher's guide for what students need to learn, how it will be taught, and how learning will be measured (Stauffer, 2019). The experimental group lecturers' lessons were easily broken into three distinct parts of the introduction, the main part of the lesson and finally, the conclusion. Unfortunately, the same could not happen with control lessons as these lessons took haphazard lesson structures and did not show the three parts as experimental lessons.

All control lessons did not follow the constructivist teaching and learning mathematics approach. They instead followed the behaviourist approach. For the reader's benefit, though behaviourist theory did not form part of the study, it is necessary to explain it briefly. This is done to explain how constructivism theory differs from behaviourist theory.

Behaviourist is a theory of learning which views learning as a conditioned response to external stimuli (James, 2006). Staying with behaviourism, similarly, Pange and Pange (2011) define learning as an observable change in learner behaviour that originates from external conditions. This behaviourist theory has no concept of the mind and cognition in it. From this angle, learning is seen as accumulating skills and memorizing information or facts, which can be demonstrated in forming habits (Pange & Pange, 2011). Behaviorists' approach to learning believes that teaching should be teacher-centered, with teachers dominating their lessons (James, 2006). It also subscribes to rote learning. The implications for teaching include, amongst other things, training or drilling students to respond to instruction correctly and rapidly, while implications for assessment may be that performance is usually construed as either correct or incorrect.

The control lessons were teacher-centered, as students sat back and relaxed while lecturers did most of the talk and writing on the board. Lecturers dominated their lessons. In a way, lecturers did not allow students to construct their knowledge but instead tried to transfer knowledge to students by showing them how they should solve problems. Lecturers engaged in the 'old teaching by telling' approach and relegated their students to take back seat and wait to be spoon-fed.

Control group lecturers did not create platforms for classroom discussions. This made their students to be recipients of knowledge as they passively received this information from their lecturers. They were relegated to become spectators in their own game.

All control lecturers observed came to their lecturer rooms and immediately told students what they were going to do and immediately put the topic on the board. In introducing a topic, very

little, if no, explanations were given to give an idea of the real-life situations. That denied students an opportunity to make mathematical connections. Lecturers went straight to taking examples and then solving them fully with explanations without involving students. After that, lecturers tended to ask students if they understood the solutions. In most cases, students answered – ‘yes, Sir/Ma’am, ‘ we understand.

Like experimental group lecturers, the control group lecturers gave questions on the board for students to solve and also allowed for reasonable thinking time to find solutions, but surprisingly, control group lecturers always worked out and gave detailed solutions to students without involving them on how they solved the problems. These lecturers could have done those problems and solutions without giving students to solve as it did not serve any purpose to give time for students to find solutions to problems and later wards give detailed solutions on the board without checking with them on what they got in their solutions. It was a waste of precious time and unnecessary exercise. The approach of giving tasks and later doing detailed solutions without involving students did not allow lecturers to notice students’ mathematical thinking ideas, but instead, it promoted laziness in class. When lecturers gave students some work to do and afterwards solved the same work without seeking feedback, students got discouraged from attempting the next questions as they knew the solutions would be given freely. Students were not allowed to explore and use alternative methods that could be correct because no one would recognize and talk about such methods.

The control lecturers' habit of giving full explanations of concepts to their students somehow instilled a behaviour of total dependency on lecturers. There was a high likelihood that students would never verify the information given to them by the lecturer. The lecturers’ dominance of lessons with less student talk and engagement, with lecturers showing how to solve calculus tasks, somehow became boring to bright students and caused further damage to other students.

As for Warshauer (2015) and Permatasari (2016), and Edwards and Beattie (2016), the lecturers dominated the teaching and learning spaces with no area left for any discussions to take place. Students became passive and waited to be given information. They merely became knowledge recipients. Students were never given a platform to create their knowledge. Students became spectators in their games by sitting back and watching lecturers solving calculus tasks.

There were no productive struggles in class as the lecturers gave the problems and solved them for students. Nothing was left for students to work on. In most cases, problems arise when students are at home with no lecturer to offer assistance.

It could be safely concluded that the control group lessons did not achieve their objectives because students' errors and misconceptions were never noticed and attended to. The error and misconception built deepened unattended.

6.5 Analysis of control group` students' scripts.

In about 74% of the control group`s analyzed term test items, the marking of students' scripts is generally poor. Markers followed an `answer-oriented` approach by ignoring some correct steps in the students' responses. This means a lecturer looked at the student's solution and marked it wrong or drew diagonal line without checking the individual steps (test items 15, 16, 18 and 20). In a scenario where students did not write at all, then it is sound to draw a diagonal line or put a big wrong. Test items numbers (7, 10 and 11) tell a different story because one or two steps are correct and these were supposed to have been marked correct. This approach may discourage students because the few correct steps could motivate the student to want to do better next time. Another great concern is the unmarked question (test item 12). One wonders how many more questions were left unmarked in the whole batch of scripts. In this particular case, the student has managed to simplify correctly though he or she could not finish the calculation. This student has put a little effort into attempting the solution and should be marked correct for this part.

Almost all test items selected for the control group have no traces of annotation. Most term test items here presented a clear-cut opportunity for annotation, but the marker did not take advantage of it. It would be easy and efficient had the marker annotated these solutions, come feedback session.

The marked term test items revealed many errors made by students; for example - picking test item 9, a student made a fundamental error of saying: $\frac{1}{0} = 0$. This student did not know that division by zero is undefined. Students did not correctly apply L'Hospital's rule; First Principles differentiation is challenging. Students did not master the trigonometric identities, could not differentiate by using the product rule and quotient rule, and could not differentiate implicit functions and logarithmic type functions. The simplification of functions and the manipulation of formulae remains another major challenge for students.

Similarly, integral calculus remains equally challenging to students. They battle with integrating techniques such as Partial Fractions, long division method technique of integration, and the integration of trigonometric ratios of the form: $\int \sin^m x \cos^n x \, dx$. All these errors underlie many misconceptions in most sections of calculus.

6.6 Analysis of control group's interviews

Control group lecturers did not show much respect towards the power of kind greetings and its impact on mathematics teaching and learning. Lecturer6 mentioned that she stopped greeting her students because most times they do not greet back. This was a recipe towards traditional teaching methods because lecturers talk from start of lessons to the end without involving students.

When these lecturers were asked to explain how they approached the teaching and learning of mathematics to their students, it came out clear that they followed a student-centered approach. One of them was asked to explain what it means to have student-centered teaching, and the

response was that the teacher should involve students in his/her teaching by always asking them if they understood what the teacher explained. This needs to be corrected as it is wrong. As Mogari (2014) indicated, a student-centered approach allows teachers to hear out, notice students thinking and then assist when necessary. This means that students are taking centre stage in their learning.

These lecturers were asked why they gave students detailed explanations, detailed examples and detailed solutions for all problems done in class. To this, lecturer5 responded that mathematics is too challenging and, therefore, she was trying to simplify it for students. This lecturer believed that full explanations assisted and improved students' mathematics understanding. Asked whether she was aware of the danger of disadvantaging these students by relegating them to the back seat where they relax and get all solutions done for them, her response was clear she was given a contract to teach mathematics to students, and she was doing exactly that. She was asked to elaborate on her response, and she indicated that she prepared lessons to teach and not lessons to be discussed.

Control group lecturers should have engaged students in classroom discussions. As indicated, their sessions were mainly dominated by the lecturer's detailed explanations of examples on the board. Every time lecturers gave students some work to do, they rightfully allocated some reasonable thinking time for students to find solutions. Surprisingly, they solved those problems without first involving students and discovering how they attempted the work. They were asked why they were doing that, and Lecturer5 responded that she did not have time to waste. She adamantly indicated that students have nothing to offer mathematically as they do not know mathematics and, therefore, should be given mathematics without wasting time. That approach led to students copying work on the board without understanding it.

The control group lecturers' teaching method was similar to traditional ways of teaching, which are poor and ineffective in bringing about desirable results (Kapur, 2010; Kocsis, 2007; Kiat, 2005; Luneta & Makonye, 2010). As argued further by Reddy (2016) and Reddy and Hannan (2018), traditional teaching methods, for example, the 'chalk and talk method', continue to fail to

transform learning outcomes. The errors and misconceptions displayed in their written work serve as evidence to the above statements.

Students approached lecturers after the session to quickly ask for clarity on work done. Lecturers gave these students full attention. These lecturers were asked why they entertained these students because they had time to ask their questions during the lesson, and they decided to keep quiet and instead confirmed their understanding when asked if they understood the explanations. They responded that they were trying to accommodate the shy and weak students who lacked the courage to ask when all class members were present in class. That was surely a recipe for failure as there could be more students who needed help on same questions. This showed lack of self-confidence on the part of these students. Again, it can be argued that the teachers are to blame because they did not create classroom atmospheres that are friendly for such students to ask without being looked down by other students.

These lecturers indicated their frustrations with students' lack of understanding of calculus topics and also on more mistakes and errors made in written work. They indicated that students needed clarification on integration and differentiation techniques, battled with graph sketching, and struggled with differential and integral calculus applications. The continued built of errors and misconceptions deepened and became worse and worse.

6.7 Differences between experimental and control group lecturers teaching approaches

Experimental Group Lecturers (EGLs) methods	Control Group Lecturer (CGLs) methods

All EGLs were trained on productive struggle and professional noticing strategies of mathematics teaching and learning. They refrained from using traditional	CGLs followed traditional methods of teaching to serve as the benchmark point of comparison with EGLs` new strategies for mathematics teaching and learning. They
methods and successfully used these new strategies in their calculus teaching.	followed the teaching by telling, talk and chalk methods of calculus teaching.
Lessons were thoughtfully planned with distinct introduction parts, informative new subject matter with struggles and conclusions and were easy to follow through.	Lessons were haphazard with no recognizable stages and were not easy to follow through.
Greetings were purposeful and done to create conducive atmospheres for maximized teaching and learning benefits.	Greetings were done for formality with no particular purpose of creating enhanced classroom conditions for learning.
Lesson introductions done with well-chosen productive struggle tasks that helped students connect knowns to unknowns and further allowed them to make connections with real life situations to give correct context	The context was not considered to be important, and the examples used did not have much impact in unlocking an understanding of the new subject matter to follow
Lessons were student-centred, with students dominating classroom activities and lecturers acting as facilitators and knowing when to talk	Lessons were lecturer-centred, with lecturers dominating throughout the lessons and students sitting back, relaxed and waiting to be spoon-fed with detailed explanations and solutions

EGLs applied Lobato's telling method and talked only when necessary. Students engaged in active construction of their own knowledge by refraining from over relying on their lecturers.	CGLs applied traditional teaching by telling method and talked throughout the lessons. Students became knowledge recipients by heavily relying on being spoon-fed by their lecturers.
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Students engaged in fruitful classroom discussions with more interactions amongst themselves and their lecturer. Students learned from classmates. Students engaged in paraphrasing each other's responses.	There were no classroom discussions in all lessons observed. Students were denied the opportunity to learn from each other. Students had to sit back and relax as everything was done for them.
EGLs engaged in the type of marking that enhanced effective feedback to their students. Annotating written responses positively aided feedback.	CGLs marking did not show much respect for feedback because they seemed not to believe in it. No annotations were made to students' written responses.
EGLs lessons were successful, as seen at the end when students appeared fulfilled and content with lesson outcomes.	It was clear that CGL lessons were not successful, as students would still have questions at the end of the sessions.
Students became self-directed learners. They did home works correctly. They contributed and actively participated in classroom activities and learned from each other.	Students were fully reliant and dependent on lecturers. They never interacted at classroom level in engaging on subject matter.
Students errors and misconceptions got noticed, interpret by lecturers who at the end decided on how to address those areas. At the end of it all, students walked out of classrooms with error and misconception free minds.	Errors and misconceptions hardened as they never got noticed and attended to by lecturers. A continued built deepened and hardened.

EGLs did not decline high task demand, and they probed their students to think deeper and ultimately increased understanding and managed to solve problems.	Lecturers gave students solutions to all questions asked.
Students showed simple mistakes in their written work. It is easy to correct mistakes than dealing with errors and misconceptions. Students learned from their mistakes and improved their mathematics understanding.	Students committed many errors in their written work, which could be as a result of misconceptions.

Table 4: Tabulated differences between Experimental and Control groups methods

6.8 Conclusions

Chapter six focused more on the analysis of data collected through observations, interviews and term test items analysis for both groups of lecturers. It was found that productive struggle and professional noticing strategies of mathematics teaching and learning are superior to traditional mathematics teaching and learning methods.

CHAPTER SEVEN: DISCUSSIONS OF COLLECTED DATA

This chapter presents discussions of data collected through the three strategies of classroom observations, interviews, and trimester test items in the form of scripts. The chapter highlights the main themes critical in achieving the study's objectives and answering the research questions. The discussions revolve around theories explained in chapter two, conceptual constructs explained in chapter three with relevant literature cementing ideas together.

7.1 Experimental group lecturers' Classroom Observations

Data collected from different classrooms through experimental group lecturers' observations show how interventions positively impacted the teaching and learning of TVET College mathematics by increasing students' understanding of calculus tasks. Interventions took the form of training aimed at professionally developing experimental group lecturers in new productive struggle and professional noticing mathematics teaching strategies as explained by (Hiebert & Grouws, 2007) and (Van Es & Sherin, 2002), respectively. The ultimate goal of the training was to make lecturers refrain from following traditional teaching methods and use these new powerful mathematics teaching and learning methods explored in this study. Amongst other things, the purpose was to raise awareness that students actively construct their knowledge no matter how hard lecturers try to transfer knowledge to them (Piaget, 1970). It meant that lecturers had to change their teaching styles by adopting these new ways of teaching mathematics. The new productive struggle and professional noticing mathematics teaching strategies are only possible when lessons are student-centered, with lecturers withdrawing from the primitive 'teaching by telling' methods and throwing away the talk and chalk style of teaching. The experimental group lecturers understood these new teaching strategies during the training and further agreed to use them in their calculus teaching.

Their lessons had three distinct parts of introduction, new subject matter and conclusions. The introductory part consisted of kind greetings to students, amongst other things. The kind greetings action was strategic as it intended to put everyone at ease and set a good tone for the lesson. Shields-Lysiak, Boyd, Iorio and Vasquez (2020) opine that positive greetings set the correct tone that creates an engaging and responsive classroom with reduced disruptive student behavior. These authors argue that the art of greeting students translates into a more productive classroom. On the same point, Cook, Fiat, Larson, Daikos, Slemrod, Holland and Renshaw (2018) indicate that positive greetings assist teachers in getting their class period off to a good start by minimizing problem behavior and improving students' readiness to learn. It is sound to believe that kind and purposeful greetings to students make students feel welcome in class, set the right tone for teaching and learning to take place and thereby maximizing the willingness to participate in class interactions (Aung & Tepsuriwong, 2017). That was exactly what experimental lecturers did when they went to their respective classrooms. They also ensured that they introduced the day's topics by referring to real-life situations, which assisted students in making mathematical connections. This exercise fulfilled the notion that teaching must always start from known to unknown knowledge (Holland & Renshaw, 2018).

The use of good productive struggle tasks successfully introduced students to the intended new knowledge and of course this pulled off from known knowledge (Holland & Renshaw, 2018). Experimental group lecturers' choice and use of these tasks used in the teaching and learning of mathematics explained, gave and enhanced mathematical understanding to students (Binti Suffian, 2010). This author argues that a good mathematical task is pivotal in developing conceptual understanding. According to Rowland (2008), good tasks play a central role in mathematics pedagogy. Rowland (2008) opines that students should experience success with routine tasks before trying more challenging ones. Lecturers mainly used such tasks in different ways like - to introduce mathematical ideas or illustrate a mathematical concept (Rowland et al., 2005). For experimental group, good chosen tasks were used to stage students' productive struggles.

Interestingly, Rowland et al., (2005), argue that the choice and use of good productive struggle tasks depend largely on the teacher's knowledge of mathematics. All the experimental group lecturers had a fair knowledge of technical mathematics and have taught TVET College mathematics for a long time, and therefore, their experience assisted them in choosing good productive struggle tasks for their students. It is therefore safe to think they used their Pedagogical Content Knowledge of Ball et al, (2008) to select and use those mathematical tasks, which enhanced students' mathematical understanding.

All lesson introductions came with Lobato's minimal 'telling' of saying important information at critical moments during the lessons (Lobato, 2005). The lessons were student-centered, with students mainly engaging and taking part in classroom activities and relegating the lecturers to assume facilitator positions. Classroom activities in the form of calculus tasks emanated from ill-defined problems, which threw students deep into productive struggles. Lecturers ensured they kept the high tasks demand in all problems given to students and they did not lower the tasks cognitive levels (Stein et al., 2000). For every question asked, students were allocated reasonable 'think time' to search and figure out appropriate methods or techniques suitable and even alternative methods for finding solutions (Warshawer, 2015 & Permatasari, 2016). When students got lost and did not know what to do, lecturers probed them more and avoided giving solutions. This approach allowed students to actively get involved in their learning and by so doing, building their knowledge and understanding. The active classroom activities allowed for students to interact with each other. The interactions further allowed for questioning and paraphrasing each other's answers and discussing ideas together (English, 1995, Fosnot, 2005). In engaging in these activities, they became mathematical explorers and better their chances of understanding deep calculus topics (Crain, 1992). These class interactions and discussions created a platform for students to manifest their errors and misconceptions, which enabled lecturers to notice these errors and misconceptions professionally. Lecturers must know their students' error types and misconceptions in order to craft strategies to attend to them.

These lecturers knew they needed to be around during classroom discussions and acted as facilitators by talking only when necessary and further directing students skillfully (Lobato et al, 2005). Towards the end of the lesson, lecturers consolidated and wrapped up the lessons by highlighting and emphasizing important ideas. It was normal for experimental lecturers to give their students homework after a successful lesson. It deepened the understanding of the problems in the section at hand. Experimental lessons were successful, with students showing an improved understanding of the calculus concepts.

7.2 Experimental group lecturers' interviews

The most important accomplishment of experimental lecturers' interviews was that they managed so well to deal with productive struggle and professional noticing mathematics teaching strategies. For example, they planned and gave calculus tasks that threw students deep into struggles. They also managed to professionally notice their students' thinking/ideas during classroom discussions and were able to interpret and attend to such student ideas (Van Es & Sherin, 2002). They selected suitable productive struggle tasks to kick-start their lessons, gave real-life illustrations in their lesson introductions to give correct contexts, and enabled students to connect known knowledge to unknown knowledge by making mathematical connections. Their lessons were successful in achieving set objectives. Bansilal (2014) argues that a teacher with only a three-year diploma is considered underqualified and that those teachers are ineffective. This seemed not to be the case in this study. This claim needs revalidation because teachers with lots of experience teaching the subject might be more effective than a beginner with high qualification. This gap should be the next topic to research in the future. As it stands here, the experimental lecturers were more effective.

The other argument by Ball et al. (2008) is that Knowledge of Content and Students (KCS) assist teachers to notice, hear and interpret students' thinking. Again, Knowledge of Content and Teaching (KCT) helps in selecting good productive struggle tasks to start and move students

forward into the content. Teachers should know when and how to probe students for improved responses (Ball et al., 2008). Experimental lecturers did exceptionally well in all the areas raised above.

Given that experimental lecturers have been teaching TVET College mathematics for a long time, it was eye-catching to notice how they whole heartily welcomed the new productive struggle and professional noticing mathematics teaching and learning strategies. One would expect them to resist change and stick to their usual poor teaching methods. Change is painful. On the other hand, they looked ready to try any new method that might give good results. This gave the thought that they knew that their current methods were not yielding desirable results. The idea that students actively built their knowledge (Piaget, 1970), made these lecturers to completely change their way of teaching and adopted the new productive struggle and professional noticing teaching strategies learned during training. These lecturers got excited when they realized that their students' mathematics strategies were going to get noticed and attended to during the lesson and by so doing, maximizing students' chances of refining their mathematics understanding. They were happy to know that these strategies give spontaneous feedback on students' errors and misconceptions. They realized that with such feedback, students stood a better chance of improving their mathematics understanding, unlike in traditional methods where errors and misconceptions would be seen for the first time in written work/tests when the damage shall have already happened.

These lecturers expressed their dedication and love of teaching TVET College mathematics, and they also knew its importance to engineering students. They knew that the next best thing was to impart the same love and knowledge of mathematics/calculus to their students. Students' love of mathematics have potential to turn them into being self-directed learners by discovering and knowing what they want to become in future (Peters, 1989 & Boyatzis, 2002). TVET College students aspire to become artisans.

The classroom discussions were managed well, with lecturers knowing how to conduct themselves during interactions (Lobato et al., 2005). They did not dominate the class talks. They let students dominate the discussions. They became facilitators. They sat back and engaged professionally, noticing errors and misconceptions that students could be saying and displaying. The discussions became a perfect platform for manifesting their errors and misconceptions. They were aware that students had to contribute errors and misconceptions to discussions and welcomed it (Nesher, 1987). Their classrooms were tolerant of errors and misconceptions (Nesher, 1987). Their classrooms became productive, with most students understanding lessons.

7.3 Experimental group lecturers' students' test scripts

Experimental group lecturers have shown much respect for the process of marking students' test scripts. They followed a 'process-oriented' manner of marking. This approach means that in a many-stepped solution, they checked every step of the solution and awarded marks accordingly, as indicated in the marking guideline. Most of the scripts marked by experimental group lecturers depicted mistakes instead of serious errors. Unlike errors and misconceptions, mistakes are easily correctable and fixable (Olivier, 1989). Most certainly, these lecturers believe that, as human beings, we tend to learn from our mistakes.

When students made mistakes, lecturers commented and used annotations to give some remarks about students' thinking. Giving comments and annotating students' responses addresses the feedback issue (Morphett et al., 2018). These lecturers believed in feedback after giving and marking assessments. They were aware that feedback further enhanced understanding by providing a second chance to re-look at their work and correct all mistakes.

A general feeling regarding students' performance was that they fared fairly well except for careless mistakes. It is easier to rectify mistakes than to try and deal with errors because errors may be deep-rooted and resistant to change (Nesher, 1987).

7.4 Discussions of classroom observations of the control group lecturers

All lessons given by the control group followed the traditional mathematics teaching and learning approach. Lecturers dominated lessons using the `teaching by telling` approach, which relegated their students to passively receive information. According to (du Toit-Brits, 2018) these approach denies students a chance to engage in active meaning making process. Students sat back and relaxed, knowing their lecturers would do almost everything for them. These lecturers adopted a behaviourist approach to teaching and learning mathematics as explained in the third paragraph of section 6.4 on the analysis of control group lecturers` classroom observation. They believed that students understood all what they explained in their lessons. They further believed that they successfully managed to transfer knowledge from themselves to their students. Their teaching methodology emphasized procedures on how to solve particular mathematical problems. These lecturers needed to learn what it meant to have a student-centered approach to teaching and learning mathematics. The only time they involved students was when they asked them if they understood what was said, and to them, that was the student-centered approach. Mogari (2014) indicates that a student-centered approach allows teachers to hear and notice students' thinking when they manifest their mistakes. It means that control group lecturers did not engage their students in all these activities.

Control lecturers did not pay much attention to their lesson introductions. They hardly made illustrations for students to let them make connections with the real world. They did not care much about creating correct contexts for their students to make mathematical connections. They gave problems for students to solve and later solved them without checking and involving students on how they attempted solutions. This approach denied students opportunities to check their understanding of the tasks at hand. Students got used to the habit of lecturers giving them tasks and later solving those task problems without involving them. This practice encouraged students to heavily rely on and depend on their lecturers. According to (Zimmerman, 2002), students like

these ones, who are reliant and dependent on their lecturers, are more likely not to realize and identify errors and mistakes in their work. These happened to control group students.

Classrooms did not create discussion platforms, which subjected the less confident and shy students deeper into confusion and reserved mode. The aimless tendency of lecturers asking students whether they understood the work done did not assist in any form, as the students' responses were always superficial. These classrooms became boring and failed to ignite interest in students to learn mathematics with understanding (Kilpatrick & Silver, 2000).

Some students needed to gain self-confidence and ask clarity-seeking questions during the normal duration of the lesson. According to Amiyani and Widjajanti (2018), self-confidence leads to self-created and self-taught knowledge and these improve the willingness to try new tasks on their own. These control group students did not show self-confidence at all. We have seen students often go to the lecturer at the end of the period to ask for clarity on some work done. It could be perilous for lecturers to attend to such students who came to ask for clarity questions at the end of the period simply because there could be more other students who could be in the same situation.

7.5 Control group lecturers' interviews

The interviews of the control group lecturers revealed that they need to learn what student-centered teaching is, which was evident during observations. They thought asking students if they understood what they said was enough to be referred to as student-centered teaching. It was more of a formality to ask students such a question since it had no further intentions. Students always responded that they understood but did not mean it, as there was never a mechanism to check for understanding.

These lecturers somehow believed that their responsibility was to transfer knowledge to the students. Knowledge cannot be transferred as it is to students (Olivier (1989). They hold a strong position that giving students fully worked-out examples and solving problems improves

mathematics understanding. One of them affirmed that students do not know mathematics and that he/she had to tell them what to do and how to do it. On the point of throwing students into struggles, they indicated that mathematics is too difficult and such an exercise will be a waste of good time. On the same score, they opined that discussions will not yield desirable outcomes if there is no depth which is what students lacked. They showed no confidence in classroom discussions.

This position of control group lecturers showing less faith in classroom discussions and productive struggles as well as professional noticing made them remain captives of old ways of teaching mathematics, achieving less impact on the understanding of calculus topics. Old teaching methods have been proved to be poor and ineffective as already indicated (Reddy & Hannan, 2018).

On a different note, these lecturers did acknowledge errors and misconceptions by students but seemed to have hit a cul-de-sac (dead-end) situation in working out a strategy that could assist in improving the situation. They sounded to need more ideas on dealing with students' many errors and misconceptions. Could it be that they did not expect these errors and misconceptions simply because they believed knowledge could be transmitted intact? The answer will likely be 'Yes' because of their behaviourist view and approach to teaching. By now we know that knowledge can never be transmitted intact from lecturers to students (Olivier, 1989).

7.6 Control group lecturers' students' scripts

The kind of marking done by control group lecturers was 'answer oriented' in that it ignored all steps of a solution and focused on the final answer only. This marking method has some shortcomings: for example, a student might do all steps correctly and mistakenly give the wrong answer, or a student may have the wrong approach and steps but mistakenly give a correct answer. Looking at these shortcomings, lecturers might unjustly award marks to non-deserving students or even deny deserving students their rightful marks.

The unmarked test item 18 raises many questions, like: if a lecturer did not see it, what was wrong with his/her concentration? Firstly, could it be that he/she saw it and decided not to mark it because maybe he/she was unsure about the answer? Secondly, could it be that the student used a different/alternative method, and the lecturer did not know the method and decided not to get involved? All these are questions and concerns showing the poor quality marking done by the control group lecturers. These lecturers did not give any comments or annotations on students' scripts, and this action could be showing their attitude towards feedback. Crisp and Johnson (2007) indicate that annotation is a feedback component. In agreement with this, Ball (2010) views annotation as an effective device for student feedback. This means that annotation makes feedback more meaningful. The absence of comments and annotations in students' scripts leaves us with many unanswered questions, for example – did these lecturers ever notice students' errors and misconceptions? How did they give feedback to their students if they were not commenting on the scripts? In test item 9, the student said: $1 = 0$, and the lecturers did not comment or annotate on this. This situation made lecturers miss the critical opportunity to comment on the scripts to enhance the coming feedback-giving sessions.

7.7 Conclusions

Chapter Seven focused on the discussions of data collected through observations, interviews and student test scripts. Experimental group lecturers, who subscribed to constructivist approaches, outwitted their behaviourist control group counterparts who supported behaviourist ways of teaching with almost every area under consideration. This scenario means that behaviourist approaches to mathematics teaching and learning must be avoided at all costs, and in their place, constructivist methods must reign.

CHAPTER EIGHT: FINDINGS, CONCLUSIONS AND RECOMMENDATIONS

This chapter provides the study's findings under three sections: from classroom observations and from analyzing documents in the form of term test items and interviews. A framework for teaching and learning TVET College mathematics is given and explained. Research questions are answered, limitations of the study as well as the newly configured TVET College classroom are explained. Lastly, recommendations for the study are given. This chapter also touches the theories and all other constructs used in the study.

Experimental group lecturers are explained first and followed by control group lecturers.

8.1 Findings on experimental group lecturers' classroom observations

Experimental group lecturers were trained on new productive struggle (Hiebert & Grouws, 2007) and professional noticing (Van Es & Sherin, 2002) new strategies of mathematics teaching and learning strategies explored in the study. Definitions of these two terms are given on page three of this study. The main highlight and finding was that - lecturers were able to change and adapt to a new method/style of mathematics teaching and learning without any problems and without it causing any pain. Their positive attitude was vital in bringing about their change of mindset. These lecturers disclosed that the training about new productive struggle and professional noticing of mathematics teaching and learning had left them with no choice but to abandon the inferior traditional teaching methods and switch over to these new strategies. They indicated their happiness in welcoming the new strategies and using them as they are powerful in achieving sound mathematics understanding in students. Their responses aligned well with Kapur (2010) that students taught via productive struggle methods outperformed those taught through traditional methods. Their comments about the new mathematics teaching strategies, validate mathematics

education researchers' findings that traditional methods are indeed very poor and ineffective in bringing about desirable results (Kapur, 2010; Kocsis, 2007; Kiat, 2005)

It was found that the new mathematics teaching strategies explored in the study apply only to a student-centered method of teaching as indicated by Mogari (2014). The new teaching strategies do not find any place and application within the traditional mathematics teaching and learning methods. These productive struggle and professional noticing strategies may only be at play when the students actively build up their knowledge, as constructivists claim (Von Glassersfeld, 1989). As indicated by Lobato et al. (2005), minimal but clear instruction on what students should do in solving a mathematical problem sets off the platform for productive struggles in students with 'think time' given. This 'think time' allows students to interact and engage with the question and then contemplate of the best strategies to solve the problem (Warshauer, 2015 & Permatasari, 2016). Think time also allows for students to find alternative methods to be used. In this study, students did not operate like automated machines in working out answers, lecturers gave them 'think time' to solve the problems and, by so doing, engage in productive struggles. Giving reasonable think time for students to be on their own and solve problems maximized their performance more than if they were hurried.

Students, who got lost with no idea and direction and not knowing how and where to start finding the solutions, allowed lecturers to engage in probing exercises. Probing exercises allowed lecturers to know or have an idea of students' concept images and concept definitions (Tall and Vinner, 1981). In all instances where lecturers engaged in probing students, all students responded positively and accordingly and ultimately managed to solve initial problems correctly. It can be argued that these students were able to grow within their Zone of Proximal Development (ZPD), as argued by Vygotsky (1930), after some little assistance from their more knowledgeable lecturers.

Active classroom discussions during lessons did not only improve students' general mathematics understanding, but also brought other added advantages: Discussions in classrooms built confidence and self-belief in students. A confident student with the right mind will likely understand mathematics far better than a less confident student. Discussions are interactive in nature and reveal students' level of understanding of mathematics. Discussions allowed the lecturers to draw back a bit and observe students' mathematics thinking/ideas professionally. According to Stein et al. (2000), high cognitive tasks give birth to rich and fruitful classroom discussions. This happened in experimental group classrooms. Discussions were platforms for students to manifest their erroneous mathematical ideas and thereby creating opportunities for lecturers to notice, interpreted and attend to their students' mathematical thinking. Discussions offered everyone a chance to improve their listening skills. For everyone to participate actively in a discussion, they should listen carefully and capture the correct intended message before responding. A lecturer, who does not engage students in active classroom discussions, denies them an opportunity to learn from each other and misses the opportunity to engage in noticing students' mathematical thinking skills. In this scenario explained above, lecturers and students miss the opportunity to refine their listening skills. For this study, lecturers engaged students in active classroom discussions that saw them successfully noticing, interpreting and deciding on how to attend to their students' mathematical ideas (Van Es & Sherin, 2002).

Positive and kind greetings to students have been found to set the correct tone for teaching and learning to take place. Kind and polite greetings went a long way in bringing total preparedness and the desired state of mind for teaching and learning to take place in students (Aung & Tepsuriwong, 2017). It was not rushed for formality but was done appropriately to harness the benefits that came with it. This must be an eye opener to other teachers to take greeting their students seriously if they want to maximize participation in their classrooms.

Introducing new topics by first briefly explaining concepts and making illustrations about real-life situations assisted in bringing about correct context leading to students making quick mathematical

connections. Correct context assists in analyzing statements when looking for strategies of solving such tasks (Luneta & Makonye, 2010). Productive struggle and professional teaching strategies yielded positive classroom results by improving students' general mathematics understanding. Lesson introductions with good productive struggle tasks took students deeper into learning about new mathematical concepts. This process involved APOS theory as lecturers engaged in the actions during introductory stages the productive struggle tasks. These actions were interiorized and became objects. It was found that mathematical objects were successfully encapsulated, see definition of encapsulation process on APOS theory in chapter two. These mathematical objects finally got incorporated into their schemata.

8.2 Findings on experimental group lecturers' term test items

Twenty-three term test items were analyzed from both experimental and control group lecturers' marking. The focus of this section was how students responded to the questions asked (errors and misconceptions), how lecturers marked these students' responses (professional noticing), and what the researcher noticed from the lecturers' marking.

This process of marking, somehow dealt with Concept Image and Concept Definition theory, technically so, because students had to write down their ideas the way they would define them verbally. It is also because the same professional noticing skills that were applied to classroom interactions, were again applicable to the process of marking.

Marking students' responses is like a magistrate passing judgement on an offender in a court of law. The marker must be highly sober when executing the task of marking. Experimental group lecturers behaved like judges and have done a pretty good job marking their students' scripts using annotations and adopted a 'concept-oriented marking approach'. Annotations made feedback sessions more sound and effective (Crisp and Johnson (2007). A 'concept-oriented marking

approach` awarded marks to steps of the answer to show the recognition of students` efforts and struggles in attempting the solution by awarding credit accordingly.

In the main, students committed slips and mistakes in their written solutions with fewer errors and misconceptions because the errors were dealt with during teaching and learning sessions. Slips and mistakes due to carelessness are not too serious compared to errors and misconceptions (Voon, Julaihi & Tang, 2017). As lecturers we prefer to deal with slips and mistakes than errors and misconceptions. All lecturers probably agree with me. We all make mistakes; the good thing is that they can be rectified. On the other hand, it is not easy to rectify an error because an error occurs due to a misconception (Olivier, 1989). The few errors made by experimental group students suggest that productive struggle, as well as professional noticing, were effective in improving students` mathematics understanding.

The approach of marking students' written responses with annotations presented a continued second opportunity to notice any resistant errors that students might still have. Ultimately, students became ready and error-free to take their final external examinations with increased self-confidence and understanding.

8.3 Findings on experimental group lecturers` interviews

The following were the findings from interviews of experimental group lecturers:

The successful training of experimental group lecturers on new mathematics teaching and learning strategies made these lecturers look down on and undermine traditional approaches to mathematics teaching and learning. They were quick to believe that traditional methods are responsible for poor calculus understanding as they make students memorize procedures in solving calculus tasks (Tall, 1993) They strongly believed that old mathematics teaching and learning methods should be thrown away and be replaced by these new teaching strategies that fully support student-centered methods. They warm-heartedly welcomed productive struggle and professional noticing

approaches to teaching and learning of mathematics after witnessing the positive impact they brought on improving students' mathematical understanding. They understood and believed that knowledge cannot be transferred ready-made and intact from lecturers to students (Olivier, 1989), but rather, students actively built their knowledge. The active classroom discussions and interactions amongst students during calculus lessons served as evidence that experimental group lecturers believed in student-centered approach to mathematics teaching and learning. They created and gave productive struggle tasks that subjected their students into deep thinking with 'think time'. The importance of 'think time' cannot be overemphasized when students engage in complex calculus tasks. The classroom interactions created a platform for lecturers to notice, interpret and decide on how to respond and attend to students' strategies.

Picking students randomly to put their mathematical calculations (students thinking) on the board has improved their self-confidence. Students became self-directed learners, by not being too reliant and dependent on their lecturers to provide solutions to given calculus tasks. This exercise also motivated students to always be ready to contribute positively to classroom activities. Students appeared to have liked the idea of putting their solutions on the board as pointed out by lecturers. Their willingness was derived from their self-directedness of wanting to do more in contributing to classroom solutions of calculus tasks. These students modified their study habits and learning behaviour and became successful learners by refraining from being reliant on lecturers (Zimmerman, 2002). This means that when students are self-motivated, they go an extra mile to do more work by solving more problems that further enhance their mathematical understanding.

It was not immediately practicable to see and harness the quick results from the benefits of using productive struggles and professional noticing from examination results. However, already there was evidence that students understood lessons and their performance in the official term tests improved greatly. The argument put forward was that term tests are prepared to the same standard as external examinations and are administered and given under same conditions. However, this study would have loved to report on the examinations results, even though it would not make any

difference. The two official term tests serve as internal examinations as they contribute forty percent term mark towards the final promotion mark. Working with final examination results would not make a big difference.

8.4 Findings on control group lecturers' classroom observations

The teaching methods of this group resembled old traditional teaching methods in many ways: lessons were lecturer-centered; lecturers dominated the lessons by talking from start to finish, students passively received information, and lecturers were the final authority in their lessons. Their teaching methods prevented students from engaging in productive struggle activities. When lecturers talked from the first minute to the last minute of the period, it looked like students only remembered what was said in the last minutes of the session. Students seemed lazy and relaxed when lecturers gave detailed step-by-step solutions on the board. This was not assisting in growing students' knowledge as students were spectators and passively received information from their lecturers.

A classroom which is not characterized by active interactions and discussions becomes an unexciting and uninteresting teaching and learning space that elevates the lecturer to be the final authority in the subject matter taught and further relegates students to sit back and relax and become spectators. It was indeed observed that the control group lessons were somehow boring and did not arouse any interest in students (Kilpatrick & Silver, 2000). The lecturers refrained from developing interpretive, professional noticing skills, as these had no place in traditional mathematics teaching methods. This type of traditional classroom is fond of acquiring procedural knowledge as opposed to conceptual knowledge (Clement, 1997; Cobb, 1994 & Ernest, 1995). Procedural knowledge confines itself with the mastery of computational skills and knowledge of procedures for identifying mathematical components, algorithms and definitions (Eisenhart et al., 1993). It concerns itself mainly with rote learning (James, 2006) and mastery of rules and the

procedural application of such rules. All these above best describe and fit perfectly the control group lecturers' classrooms.

A traditional classroom in which the lecturer dominates the space by doing almost everything in explaining into details, and giving answers to mathematical problems, denies the student the chance to test themselves and check with themselves if they can handle the problems on their own. In this study, the control group approach gave weak and shy students reasons to remain in their hopeless dark spaces with no hope of rising above their struggles.

8.5 Findings on control group lecturers' term test items

These control lecturers did not annotate students' answers when marking their scripts. The students' written responses revealed many weaknesses, errors and displayed some misconceptions. Some students hold the mistaken belief that division by zero gives a zero answer. This is a serious concern. Students lacked calculus pre-knowledge. In some cases, students could not simplify surd expressions into simpler forms that will allow for easier handling. Most students did not know the conditions for 'L'Hospital'rule' and its correct application.

Students battled with product and quotient rules, which were introduced and done at the lower levels of study. The same confusion was seen with logarithmic and implicit differentiation techniques of differentiation, where students did not know which one to use and how to use it to determine derivatives. Differentiation from 'First Principles' is also on the list of sections that are problematic to control students. In some cases, students were unable to simplify given surd expressions to the form $y = ax^n$. Others failed to recognize that $(x) = \ln \pi$ is a constant and not a function like: $f(x) = \ln \pi x$.

Control group lecturers adopted an 'answer-oriented marking approach'. Mathematics answers and responses are always given in steps, and each step is a masterpiece of thinking supported by a

mathematical rule. For a lecturer to mark a five-stepped solution with one big tick or one big wrong is questionable because there could be few correct or incorrect steps that should be marked accordingly. Lecturers should mark steps rather than the overall work at once. Students gave correct answers with wrong steps and again gave wrong answers with correct steps, which had to be marked accordingly. Marking students' responses should be done with great care. These control lecturers failed to do all the above, so their marking was poor.

The absence of annotations prolonged the feedback-giving process as lecturers had to explain everything that students did wrong. Unlike when the annotation was given with experimental lecturers, students quickly saw their mistakes and adjusted their understanding quickly and accordingly.

Some student`s work had been left unmarked. This remained a big concern as this student missed the allocated marks on correctly answered steps. That situation could not be relegated to believing that the lecturer did not see the unmarked response as there could be possibilities that he or she did not know how to mark this work because maybe he or she did not know and understand the method used by the student. The other worrying factor was whether the moderation of marked scripts took place, as that could have solved that problem. The moderator could order the lecturer to mark the unmarked responses.

Control group lecturers showed less respect for the marking process; for example – a question was left unmarked, as indicated above. From the look of things, they did not care much because they undermined the importance of giving feedback to their students. It looked like they did not have the best interest of mathematics students understanding at heart.

8.6 Findings on control group lecturers` interviews

Control group lecturers supported and used old methods of mathematics teaching and learning. Their approach and way of teaching mathematics were mainly teacher-centered. They dominated lessons throughout the length of the period. Even though their approach was teacher-centered, they thought they were following a student-centered approach. They believed that by repeatedly asking students whether they understood the lesson meant a student-centered teaching approach. Their teaching style did not at all show elements of a student-centered teaching approach. This could be the case with many other teachers out there. There is a serious need to raise awareness.

There were no discussions in control classrooms during the teaching and learning of mathematics. Lecturers put up mathematical problems on the board for students to do and allocated few minutes for them to try, but surprisingly they came around and gave detailed solutions to the problems without checking with students on how they solved the problems. One would ask, why, in the first place did the lecturers give time to students to answer the questions if they would not check on how they tried finding those answers? This was a waste of time as it did not positively contribute to understanding the problem at hand. Instead, it relegated students to sit back and relax, knowing solutions would be done and given to them.

The lecturers did not take advantage of the opportunities provided for by productive struggle and professional noticing teaching strategies. After giving students some work to do, the lecturer could have used the time to exercise noticing skills as well as engage in probing students who may not know how to go about finding challenging calculus solutions. At the same time, both students and lecturers could have used the time to interact and discuss and attempt important aspects of the task

at hand. However, the control group teaching methodology does not accommodate the above ideas.

The observed practice of control group lecturers in failing to engage students in classroom discussions, not affording them time and space to share their views and opinions on the solutions to mathematical problems, allowing students to take back sit and relax by simply withdrawing from the lesson and adopting a passive mode, was evident of poor and ineffective traditional mathematics teaching and learning method. Students, who went and consulted with lecturers after class time, showed lack of self-confidence to seek clarity during allocated teaching times. The lack and absence of a culture that promote clarity-seeking questions subjected students to the fear of asking questions. These students could have asked their questions during the allocated time but decided to relegate them to the end, knowing that all other students would not be around to listen to their struggles. It was strange to see lecturers attending to those students. What if the questions asked were to assist other students who left already? Was it not necessary to tell these students to ask those question in class? However, the control teaching approach does not accommodate such exercises. These lecturers should realize that classroom discussions have the potential to motivate all students to partake and address individual challenges actively.

Control group lecturers' traditional mathematics teaching methods made students rely heavily on procedures as indicated already. Lecturer 5 gave students a handout of formulas and rules called a `formula sheet` to use when solving problems about the application of integration on solids of revolutions. That action supported procedures to a greater extent. That approach impacted little understanding because students relaxed and thought they had all the expertise needed to solve problems.

8.7 Grounded Theory and the Resulting Pedagogical Framework

Olivier (1989) indicates that a theory is a lens through which one views the facts. In this way, it means that facts stem from theory. The following text explains the findings that point at the embedded theory emanating from the data collected for the study. The data points out that the poor and ineffective traditional mathematics teaching and learning methods should be thrown away and get replaced by powerful productive struggle and professional noticing teaching strategies. Traditional methods subject students to struggling alone in their dark spaces without being noticed and assisted by the lecturers. On the other hand, new productive struggle and professional noticing methods create platforms and opportunities for students to come out of their struggling corners and spaces, get noticed and be assisted by their lecturers (Hiebert & Grouws, 2007; Van Es & Sherin, 2002). Lecturers need to plan their lessons in line with productive struggle and professional noticing strategies to achieve sound mathematics understanding in students. The introduction of new topics should be done in a more creative manner that connects the topic to everyday life situations. This is done to give correct context and allows for mathematical connections in students. Good and well-thought productive struggle tasks are effective in making understanding a lot easier.

Lecturers must always give high cognitive tasks to their students. They must be discouraged to lower the cognitive level of such tasks when students need help finding solutions (Stein et al., 2000). The least lecturers can do is probe students for deep thinking which is a strategy capable of bringing further understanding to solve problems. Lecturers must create classroom atmospheres that allow for class discussions and provide opportunities for all to ask questions freely, argue viewpoints and share all necessary information (Smith et al., 1993). Both the lecturer and students must positively contribute to the lesson during classroom interactions and discussions. Lecturers must be able to switch roles in being students in their classrooms and in being facilitators. They need to stop dominating their lessons from start to finish. They must know when to talk and fully mind their approach when interacting with students' mathematical ideas. They must refrain from

engaging in the old ways of teaching mathematics and instead adapt to Lobato's telling method approach.

Of importance is the marking of students' assessments. Lecturers must realize that all students' understanding displayed during lesson presentations gets written down when sitting for term assessments and final examinations. This means that marking is also an integral part of teaching and learning mathematics. Markers must adopt the 'concept-oriented marking' approach, not the 'answer-oriented marking' approach.

The 'concept oriented marking' is quite fair as compared to its counterpart 'answer-oriented marking' because it looks at each and every piece of thinking done in answering the questions. This kind of marking approach considers each step on its own as to whether the particular step is mathematically correct or not. For example: In Mathematics N6, there are questions that are asked for about 12 marks. This means that there are about 12 steps or about 12 potential areas for the mark allocation. Each step being a master piece of work on its own and supported by a valid mathematical rule. If this kind of answer was to be marked through the 'answer oriented approach', marking will not be fair as students' effort will not be credited. Justice will be done if each step is looked at individually as the 'concept oriented approach' suggests. The marking must be done through annotation as it aids and facilitates feedback sessions.

A framework is a set of ideas, guiding principles, or rules that provide the basis or outline of what is to be studied (Lester, 2005). On the other hand, Urquhart (2012) indicates that a narrative framework, a diagram or a set of propositions often represent ground theory. This research study wants to present the following framework on the teaching and learning of calculus in TVET colleges in line with Urquhart (2012)'s idea. The diagrammatic representation of the framework is presented in Table 5.

A suggested pedagogical framework for the study of calculus in TVET

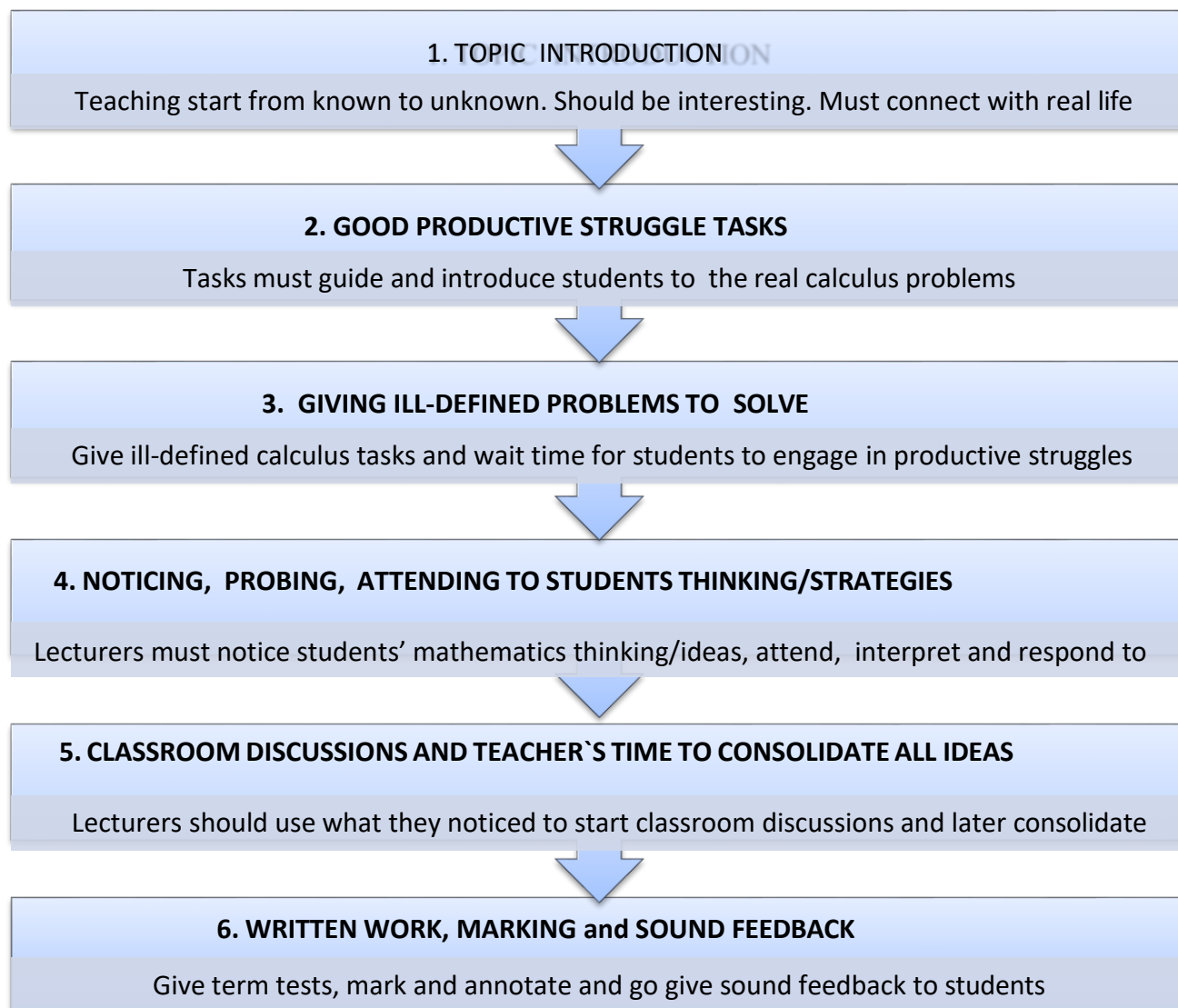


Table 5: A Pedagogical Framework for the study of calculus in TVET colleges

8.7.1 Explanation of the Pedagogical Framework

The following are the six key designs building up the framework for the teaching and learning of calculus in TVET Colleges.

Design 1: Topic introduction

Lecturers need to pay much attention to planning their lesson introductions. The introduction of a lesson must arouse interest in students for them to be ready to learn new knowledge. Lessons must always pull off from known to unknown information. Lecturers need to make students relate to everyday life situations by giving real-life illustrations. Differential calculus, as well as integral calculus, forms the basis of engineering, and therefore, lecturers must connect the new lesson with practical applications of the topic at hand. When students connect with the outside world, they develop a contextual understanding that promotes understanding and further enhances their problem-solving skills. The above processes make students either accommodate or assimilate knowledge, Piaget (1964), improving students' mathematical schemata. Well-thought introductions, assist students to accommodate and assimilate successfully without any challenges. The kind greetings and the warm welcome of students to the lecture-room cannot be overemphasized.

Design 2: The use of good productive struggle tasks

Lecturers have to design or search for good productive struggle tasks to use in introducing new topics. Good introductory productive struggle tasks enhance understanding. A well-chosen productive struggle task lights up the way by enhancing understanding between introduction and the transition into deep subject matter and thereby sustaining it. A well-thought productive struggle task completes the cycle of understanding and sets students ready to try and do new work on their own. This serves as motivation for students to grow into being self-directed learners who takes charge of their learning.

Design 3: Giving ill-defined problems to solve

Lecturers must give students high cognitive problems to solve but must be mindful of such tasks to be within students' ZPD. The instructions must be clear for students to know what is expected of them to do. To maximize the benefits of productive struggle, each student must work on his/her own to show his/her understanding. Lecturers must withdraw back a bit to afford students some time to work on the problems. Students are different; bright students will likely find solutions quicker than weaker students. Other students may need some help figuring out where and how to start attempting the questions. This presents a clear-cut opportunity for the lecturer to engage in probing exercises and later professionally notice students' mathematics thinking.

Design 4: Noticing, probing, and attending to students thinking/strategies

When students are busy attempting a given task, the lecturer must keep an attentive listening ear and a watchful eye to notice all activities taking place in the classroom. This includes students who will be stuck and not writing a thing in their books. Lost students who need to be told what to do will be probed with questions whose answers will lead them to open up understanding. Lecturers must walk in between students' desks to check students' work and avail himself/herself to students who will need individual attention.

Design 5: Classroom discussions and teacher's time to consolidate all ideas

In kick-starting classroom discussions, lecturers may order at least three students to put their solutions on the board. Lecturers must have seen these answers when busy walking between desks (noticing skill). The solutions on the board have a greater potential to start class discussions. Bright students will only accept correct answers with reasons. Weak students will rise and realize their mistakes. The lecturer may still draw back and allow students to engage in discussions about the solutions put on the board. During this time, the lecturer should notice students thinking and somehow note what mistakes, errors or misconceptions students make and display.

Towards the end of the lesson, the lecturer must wrap up the lesson by pulling together, refining every important aspect of the lesson and rectifying mistakes and errors with and in consultation with students. After this exercise, students are likely to be in a good space to do any questions under the section. Lecturers may now give an assessment to test understanding.

Design 6: Written work, marking and sound feedback

Assessment tasks should mainly consist of questions that require high cognitive thinking level. It should be questions that throw students deep into thinking.

Annotation impacts feedback sessions positively. Annotation makes feedback sessions more efficient and effective in making students quickly realize their mistakes and erroneous thinking. When students know their mistakes, they can always learn from them and grow.

8.8 Answering research questions

The study managed to successfully find answers to the research questions asked. The following section presents the replies to the three research questions asked by the study.

8.8.1 Answering research question

The research question - How can lecturers professionally notice students' errors and misunderstandings during interactive activities and while marking students' term assessments? What errors do TVET College students do and what misconceptions do they display in their calculus tasks?

Error-making and displaying of misconceptions by students, always characterize mathematics teaching and learning process (Nesher, 1993). This means that errors and misconceptions will always form part of learning regardless of the type of teaching method used. Dealing with errors and misconceptions depends largely on the type of teaching method used and how this method deals with errors and misconceptions and their sources. This study has already mentioned that errors and

misconceptions are resistant to traditional teaching methods. This is the case because traditional methods do not recognize errors and misconceptions or have no strategy to deal with them. However, current errors and misconceptions in students directly affect the next learning (Nesher, 1987). The point is that students taught through traditional methods will continue making errors and having misconceptions without being noticed by their lecturers, and this will further continue building up and affecting their next lessons in a bad manner (Nesher, 1987). These students will not do well in their official term tests and final examinations.

As expected, students taught by experimental group lecturers made errors and displayed some misconceptions, like any other. They confused differentiation and integration in the main. They lacked pre-calculus knowledge and had problems working with surds, exponential laws, manipulating technical formulae, working with indices, differentiating from first principles, confusing implicit differentiation with logarithmic differentiation, struggling with the long division technique of integration and integration of trigonometric ratios as well as other techniques of differentiation and integration. How then did the new mathematics teaching strategies assisted in this regard?

As indicated, experimental lecturers welcomed and allowed for errors and misconceptions in their classrooms. With the knowledge gained from the works of Olivier (1989) and Nesher (1987) that mathematics students will always make errors and develop misconceptions during mathematics learning, the experimental group lecturers knew beforehand that their students were going to commit some errors and display some misconceptions and most importantly they had a strategy to deal with such situations. Productive struggle and professional noticing strategies created opportunities for lecturers to notice and interpret students' mathematics thinking and then attend to such. These lecturers engaged in a process of noticing their students' errors, interpreting them and finally deciding how to attend to such errors and misconceptions. This process was able to set students free from these errors and misconceptions. Crain (1992) indicates that students' thinking is, in the main, faulty with errors and has misconceptions, while error-making is the students' expertise (Nesher, 1987). These new strategies of teaching and learning mathematics were capable of assisting in addressing resistant and deep-rooted errors and misconceptions held by students. Probing students on their errors and

misconceptions resulted in an increased understanding that saw students solving the same problems they could not solve earlier on. Students learned from their mistakes and were later ready to further explore, make discoveries, and formulate their own increased understanding. The processes explained above were possible because experimental group lecturers planned for them. Classroom interactions and discussions created opportunities for lecturers to notice and immediately correct erroneous students' thinking. It was found that teaching mathematics following these new strategies helped curb errors and misconceptions during class time. That scenario left students with a substantially reduced amount of errors and misconceptions. That was evident in students' term test items, where students made simple mistakes. Mistakes are easy to correct than errors.

It can be said confidently that with these new mathematics teaching and learning strategies, any error type and misconception, get noticed, interpret and attended to by lecturers. At the end, it does not matter the type of error and misconception as all kinds are noticed and attended to by lecturers. The students' written work taught by these lecturers, bear testimony to the ideas raised in this section.

On the other hand, students taught by control group lecturers remained with errors and misconceptions because these were never given attention. Errors and misconceptions hardened and negatively affected the next lessons (Nesher, 1987). Control lecturers saw their students' errors and misconceptions for the first time in their term test items. This makes sense because there was never a platform to notice and attend to these errors and misconceptions. Most errors and misconceptions were attributed to pre-knowledge of calculus, product and quotient rules of differentiation, logarithmic and implicit differentiation type, First principle differentiation and the section on limits (L'Hospital's rule).

This study does not want to claim that students taught by experimental group lecturers did not make errors and misconceptions. As indicated above, both groups of students taught by the two lecturers almost made similar mistakes. The difference is that students taught by experimental group lecturers had their errors and misconceptions noticed, interpreted and attended to during lesson time. The students realized their errors and misconceptions, learned from them, and refined their understanding

long before the term tests. This is why these students committed simple mistakes in their written work which were corrected with less effort (Nesher, 1987). On the other hand, students taught by control group lecturers remained with errors and misconceptions until they showed them for the first time in their term test. This was the reason why they committed many errors in their written work.

The new productive struggle and professional noticing mathematics teaching strategies are potent over old methods of teaching mathematics because:

They created platforms for students to actively engage in classroom interactions and discussions and in doing so manifest their errors and misconceptions. Lecturers noticed, interpreted, and developed strategies to address students' errors and misconceptions as manifested. Students had the opportunity to refine their understanding long before sitting for important assessments. These students were likely to have fewer errors in their written work than their counterparts.

8.8.2 Answering the second research question

The second question is - What is the impact of TVET college lecturers' professional noticing of students' productive struggle on students' learning outcomes in complex calculus tasks?

As mentioned earlier in the study, many mathematics researchers advocate for new strategies for teaching and learning mathematics (Jacobs et al., 2010; Ding & Domínguez, 2016; Makonye, 2012 & Kapur, 2006, 2008, 2010). Traditional ways of teaching and learning mathematics are poor and ineffective in achieving students' understanding and continuously fail to transform learning outcomes (Reddy, 2005; Reddy et al., 2012; 2016; & Hannan, 2018). All these researchers advocate for innovative pedagogies in mathematics teaching in order to transform this space. Kapur (2010) found that students taught via productive struggle methods outperformed those taught through traditional methods. It was, therefore, imperative to research calculus teaching and learning from new perspectives of productive struggle and professional noticing strategies.

The following are areas and situations impacted upon by the new productive struggle and professional noticing approaches to teaching and learning mathematics as observed from experimental group lecturers:

Experimental group lecturers created classroom atmospheres which changed them into being facilitators and no longer knowledge transmitters of knowledge to their students. They were made aware that knowledge cannot be packaged and transmitted to students as is and they welcomed this claim, believed it, and adopted it in their mathematics teaching (Piaget, 1970). They designed high cognitive level tasks that threw students deep into productive struggles. They maintained the tasks demand by not getting tempted to declining it (Stein et al., 2000). These lecturers did not assist students by giving them answers when they struggled but instead probed them to allow them to think deeper and, in so doing, discover and increase their understanding which ultimately enabled them to find correct answers on their own.

Experimental group lecturers' teaching method tolerated and welcomed errors and misconceptions (Nesher, 1987). They knew that some errors and misconceptions could be deep-rooted and that traditional methods were not going to help to address that problem. They strongly believed in the new teaching strategies explored in this study as the effective approach to dealing with errors and misconceptions (Smith et al., 1993). Experimental group lecturers accepted that traditional methods of teaching mathematics are passive and fail to address students' errors and misconceptions, which made them, welcome and use the new teaching strategies explored in the study. They believed that traditional methods promoted rote learning and memorization (Battista, 1999), which makes it almost impossible for students to reason, substantiate and argue their mathematical standpoints. Traditional methods of teaching brings /shallow understanding in students. Most importantly, unlike the new productive struggle teaching strategies, traditional teaching methods do not give teachers the capacity to notice erroneous students' mathematics thinking. Productive struggle, as well as professional noticing strategies of mathematics teaching and learning, further brought along with them the following:

It allowed students to take charge of their learning and become explorers who made their discoveries and actively formulated their understanding (Crain, 1992). It also created opportunities for students to grapple with important mathematical ideas and processes and, by so doing, increase their general mathematical understanding.

Productive struggle and professional noticing new strategies steered the lessons to revolve around students and, by so doing, creating opportunities for students to engage in activities and discussions taking place in their classrooms. Students manifested their errors and misconceptions, unaware that they were doing it. That got noticed and attended to by their lecturers and who immediate feedback to students' errors and misconceptions through exercising professional noticing skills. This improved the students' situation, leaving them with fewer errors and misconceptions.

Lecturers refrained from engaging in traditional teaching by telling methods and adopted Lobato's telling method, which enabled them to become facilitators (Lobato, 2005). They avoided unnecessary dominance of their lessons. Active classroom discussions somehow encouraged weak and shy students to rise to the occasion, get involved in the thick of things, and start participating in the meaningful classroom discussions. Nothing was as good as seeing all students in the classroom partaking in discussions. Classroom discussions motivated all students to prepare for the lessons thoroughly because they knew that lecturers randomly picked on them to share their viewpoints on the solutions they did. These platforms allowed students to give their answers with reasons and justifications. No student wanted to be embarrassed in front of peers, so they all got involved in finding solutions. They became self-directed learners in their learning.

The new teaching strategies allowed for more knowledgeable others to assist and guide the less knowledgeable others (Vygotsky, 1930). It also allowed students to grow within their ZPD upon deep probing by their lecturers. Productive struggle approaches allowed for increased conceptual knowledge instead of procedural knowledge which mostly emanates from rote learning. Conceptual knowledge allowed students to confidently become self-directed learners with increased self-confidence.

Experimental group lecturers realized the importance of giving feedback to students. They saw feedback as an essential exercise of the assessment as it revealed how students performed (Grove & Good, 2020). The practice of allowing lecturers to notice, to interpret, and to respond to students' mathematical thinking, allowed for instant feedback during lessons. This enhanced performance by pointing out errors and misconceptions and by so doing improving it (Chalmers, Mowat & Chapman, 2018). The marking of term tests provided yet another chance to further noticing students' mathematical thinking with an extended opportunity of giving another enhanced feedback on written work. The marking of scripts was done on a sound approach of annotating students responses and served as feedback component (Crisp & Johnson, 2007) by acting as an effective device for feedback (Ball, 2010).

The impact of productive struggle and professional noticing teaching strategies was of a positive nature in that they improved students' calculus understanding. They successfully transformed the mathematics teaching and learning space by doing away with poor and ineffective traditional teaching methods and replacing them with new powerful, effective teaching methods that improved students' calculus understanding. The new mathematics teaching strategies brought along new positive ways of doing business to lecturers and students.

8.8.3 Answering the third research question

- a) Lastly, the research aimed at suggesting a pedagogical framework for teaching and learning calculus through the use of productive struggle and professional noticing interventions. The research question is: What kind of pedagogical framework can be recommended for enhancing the teaching and learning of challenging calculus topics? Table 5 on page 191 gives the framework in a schematic representation followed by full explanations of key ideas raised in the framework.

The pedagogical framework was designed for new productive struggle (Hiebert & Grouws, 2007) and professional noticing (Van Es & Sherin, 2002) strategies of mathematics teaching and learning. These new mathematics teaching strategies can only be used with a student-centered teaching

approach. It will be impractical and a fruitless exercise to try using them with old traditional methods of teaching mathematics as they find no room for application with old teaching methods.

The framework suggests a sound path and guideline to be followed when teaching and learning complex calculus topics by TVET College students. The framework provides that students come to lecture rooms, engage in classroom activities and make errors in the process, with lecturers noticing these errors and attending to them, leaving students with fewer or no errors at the end of the lesson. This situation puts such students in a better position to perform better in their official term tests and external examinations.

8.9 New TVET College Classrooms

This study considered it necessary to indicate how the findings should affect the practice of mathematics teaching and learning in the current TVET College classrooms. The control group lecturers' classrooms best describe the current classroom. These are classrooms underpinned by behaviourist mathematics teaching and learning approaches. These are the classroom where lecturers serve as knowledge transmitters and students serve as knowledge recipients. They are teaching and learning spaces characterized by students loaded with hopeless errors and misconceptions. In short, the classrooms are boring teaching and learning spaces which promote laziness, reliance and dependence of students on lecturers.

Students stand a chance of not realizing their career dreams as most will not pass their grades. TVET Colleges will then not meet its mandate of producing artisans in the country as explained in the background section of the first chapter. It was on this basis that a need for a new reconfiguration of the current classroom was inevitable.

These above findings suggest a new classroom arrangement explained below.

The current classrooms are arranged in a manner that elevate teachers to take an authoritative position in front of the classrooms. This arrangement already suggests that teachers have the final say in every

activity taking place in the classrooms. All students look up to the teacher as the symbol of supremacy. Shor (1987) indicates that the apartheid era in South Africa emphasized rote learning within an authoritarian educational structure and to date classrooms are still suffering the results of this arrangement because.

Booyse and Chetty (2016) assert that a constructivist classroom is conducive to learning and makes learners become self-motivated and better able to master the next phase of the curriculum. This approach makes students become strategic and effective learners who take responsibility for their learning.

The first thing to be done in achieving such a classroom is to pair students in groups of four each, facing each other. This number can be increased depending on the available space. This kind of array promotes teamwork and makes students think and work together.

Teachers must change from using ineffective traditional methods and adopt new teaching strategies explained in the study. Their lesson preparations need good introductions that acknowledge students' everyday experiences by connecting known to unknown information. Freire (1991) regards learner experience as central to the construction of knowledge.

Lecturers must adopt Lobato's telling method by knowing when to talk during the lessons. They need to give students ill-defined tasks that formulas cannot assist to solve. These are the kind of tasks that throw students deep into struggles. Struggles lead to students actively constructing their knowledge. Lecturers need not decline the cognitive demand of the given mathematical tasks.

Lecturers need to have strategies for kick-starting classroom discussions. Classroom discussions have the potential to make students manifest their erroneous ideas and mathematics strategies. Active discussions bring about the opportunity for lecturers to notice students' mathematical thinking. This becomes the best moment for lecturers to attend, interpret and respond to students' errors and

misconceptions. Booyse and Chetty (2016) indicate that lecturers need to pinpoint the learner's strengths and weaknesses to determine how these might be built upon or addressed to improve the work. Once students' errors and misconceptions are attended to, mathematical understanding greatly improves. By attending to students' strategies, a stimulating environment that encourages learners is created and, by so doing, enables them to comprehend new concepts and content. When the understanding improves, students pass their grades. When grades are passed, TVET Colleges will be able to meet their mandate of producing artisans in the country.

8.10 Limitations of the Study

As alluded to earlier, sharing the study's potential limitations increases its integrity (Collet-Klingenberg & Kolb, 2011). Issues of trustworthiness and credibility have already been explained in chapter four.

One limitation might have affected the study. This study collected data from observing, interviewing and analyzing students' test scripts in determining the extent of the impact of lecturers' professional noticing and students' productive struggles on the teaching and learning of mathematics. The limitation could be that the test items were collected from official term tests and not final examination scripts. The term tests form part of the qualifying mark towards the final promotion mark. In other words, official term tests serve as the internal examination. The term test contributes 40% towards the promotion mark. Term tests are written under the same conditions as the final examinations. These term tests are set at the same standard as the external examinations. In other words, the results would still be the same had the external examination test items been used.

8.11 Recommendations

The research study made it very clear that to help students improve their understanding of deep calculus topics, lecturers must engage their students in new teaching strategies explored in this study. The proponents of productive struggle, Hiebert and Grouws (2007), Warshauer (2015) and

Permatasari (2016), indicate that this approach creates opportunities for students to grapple with important mathematical ideas and concepts, thereby improving mathematics understanding. These researchers further indicate that the mathematical task demand should be maintained for the productive struggle to yield positive results (Stein et al., 2000). Lobato (2005) supports the idea of giving as little direct support and structure as possible, though it is tempting for teachers to give full and detailed explanations. This sets off the scene for struggles. Olivier (1989) and Nesher (1987) assert that students always commit errors and show misconceptions in their learning/work. Therefore, lecturers should provide classrooms that promote students' thinking so they can notice errors made in the process and appropriately address them. These researchers argue that radical uprooting of students' errors and misconceptions may shake the student's confidence and cause confusion.

On the other hand, Porter and Masingila (2000) indicate that by marking students' test responses, lecturers gain information about students' understanding of mathematics. During this time, lecturers notice all errors and misconceptions in students' mathematics thinking. Ball et al. (2008) argue that common content knowledge lets teachers skillfully notice when students give wrong answers. On the same breadth, Walshaw and Antony (2008) argue that a teacher's mathematical subject knowledge enhances the quality of class discussions. It means that a teacher with vast mathematical knowledge is well equipped and better positioned to direct students during classroom discussions. How mathematics teachers listen, and probe students greatly increases the teacher's understanding of students' mathematical thinking.

In light of these statements made above by mathematics education researchers, this research presents some recommendations. The recommendations are as follows:

Recommendations for Practice and Professional Development for Lecturers:

Teaching calculus in TVET colleges requires a deep understanding of the subject matter as well as effective instructional strategies to support students' learning. A key recommendation for practice in promoting lecturers' professional noticing in calculus teaching is to provide ongoing professional

development and training. This could include workshops, seminars, and collaborative learning communities where teachers can engage in discussions and activities focused on improving their ability to professionally notice and interpret students' thinking. By enhancing their skills in this area, teachers can better understand their students' difficulties and misconceptions, leading to more effective instructional strategies.

Secondly, lecturers must continuously engage in implementing formative assessment practices in the classroom. A habit of regularly assessing students' understanding and providing feedback, lead to lecturers gaining insight into their students' thinking processes and thereby making informed instructional decisions. This ongoing feedback loop allows lecturers to adjust their teaching based on student needs and promote deeper learning.

Lecturers must always encourage students to grapple with complex calculus problems and persevere through challenges as this will help develop problem-solving skills and deepen their understanding of deep calculus concepts.

Lastly, lecturers must move away from conventional methods of instruction and gear toward a more successful strategy as outlined in the study that centres on professional noticing of students' productive struggles with challenging calculus assignments. This change may result in better learning outcomes, more student involvement, and a more thorough comprehension of the content learnt.

Recommendations on Support for Curriculum Design:

The researcher recommends that the policymakers should collaborate with curriculum designers to ensure that the new pedagogical framework for effective teaching and learning of deep calculus is integrated into the calculus curriculum in TVET colleges. The newly designed pedagogical framework provide lecturers with opportunities to approach teaching calculus differently, explore effective teaching strategies, and develop their skills in diagnosing and addressing students'

difficulties. On the side of policymakers, the adoption and the full utilization of the newly designed pedagogical framework in this study, will make sure that students receive high-quality instruction that supports their learning and growth in mathematics and calculus in particular. This will be possible only when the curriculum is designed in such a way that it provides opportunities for students to engage in productive struggle and for teachers to use their professional noticing skills to guide student learning, allow teachers to monitor student progress and make adjustments to their instruction as needed.

Recommendations for Future Research

The widespread perception seems to be that using technology enhances students' comprehension of mathematics. Productive struggles in math application sections can be fostered by using technology, such as YouTube videos and other educational related technologies. Getting more researchers to investigate technologies like gamification is one of the recommendations for future research. Since the 21st century generation appears to have microchips implanted in their brains, gamification is probably going to be more effective with them. It is anticipated that the teaching and learning of mathematics will be enhanced by the use of these innovative approaches. Educators can foster critical thinking, problem solving, and a deeper comprehension of difficult computational problems by investigating these concepts and creating a more student-centered teaching approach.

8.12 Conclusions

TVET college mathematics lecturers play a crucial role in shaping the future of students. One of their key responsibilities is the ability to professionally notice and respond to their students' productive struggle in complex calculus tasks. Students' productive struggle refers to the process of grappling with challenging mathematical problems in a way that leads to deep understanding and growth. In this thesis, the researcher explored the significance of lecturers' professional noticing of students' productive struggle in calculus tasks and discuss how this was cultivated and

enhanced through effective professional development.

Teachers' professional noticing of students' productive struggle in calculus tasks is essential for promoting student learning and engagement. When students are faced with challenging problems, they may experience frustration, confusion, and self-doubt. However, when lecturers are able to recognize and respond to these struggles in a supportive and constructive manner, they can help students develop perseverance, resilience, and problem-solving skills. By acknowledging and valuing students' efforts and mistakes, lecturers create a positive learning environment that encourages risk-taking and exploration.

The lecturers' professional noticing of students' productive struggle also helps identify misconceptions, gaps in understanding, and areas of difficulty. By observing students as they work through calculus tasks, lecturers gain valuable insights into their thought processes, problem-solving strategies, and reasoning skills. This information informs lecturers' instructional decisions, allowing them to tailor their teaching methods, provide targeted feedback, and offer timely interventions to support student learning. Additionally, lecturers can use their observations to differentiate instruction, scaffold learning, and design enrichment activities that cater to the diverse needs and abilities of their students.

An ongoing lecturer's professional development has the potential to enhance professional noticing of students' productive struggle in calculus tasks. Professional development arms lecturers with knowledge, skills, and tools needed to effectively observe, analyze, and interpret students' mathematical thinking and behaviors. Workshops, seminars, conferences, and collaborative learning communities, present opportunities for lecturers to learn about best practices in formative assessment, classroom observation, data analysis, and reflective practice.

In summary, by prioritizing and valuing students' productive struggle, teachers can create a learning environment that empowers students to persevere, overcome challenges, and achieve success in calculus and beyond.

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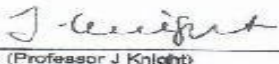
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10 APPENDICE

APPENDIX 1: Letter from TVET College Principal

 higher education & training Department: Higher Education and Training REPUBLIC OF SOUTH AFRICA	 EKURHULENI EAST TVET COLLEGE "Committed to Excellence"	HEAD OFFICE Sun Ngema Rd, Kwa-Thema, Springs Tel: (011) 730 6600 Fax: (011) 730 6608 E-mail: info@eec.edu.za Private Bag X12 SPRINGS 1560			
Enquiries: Mr M Mathe: 011 730 6600		EEC-DP-CS-Letter			
03 November 2021					
Mr Mmachuene William Hlako 2449 Masingo Street Tsakane 1550					
Dear Mr Hlako					
APPROVAL TO CONDUCT RESEARCH AT EKURHULENI EAST TVET COLLEGE					
This letter serves to acknowledge receipt of your request to conduct a research project titled: TVET Mathematics teachers professional noticing of student's productive struggle in calculus problem solving at Ekurhuleni East TVET College.					
We are pleased to inform you that your request is hereby approved. The approval is granted taking the following into account:					
1. The study is conducted by a Wits student researcher and remains the property of Wits. 2. The researcher can use our institution's name in his research project. 3. All data and information collected will be solely in the possession of the researcher. 4. The researcher may provide feedback of the research at his own discretion. 5. The research may be published in the public domain under the supervision of the supervisor. 6. The research is solely for academic purpose and may not be used for any other purpose other than for academic studies.					
Should you require further information, please do not hesitate to contact Mr M Mathe - Acting Deputy Principal: Corporate Services via email on: m.mathe@eec.edu.za					
Thank you for showing interest in our College. We are pleased to participate in this study and to contribute to this important research.					
Yours in Higher Education and Training,					
 Ms HM Sibanda Principal HMS/mm1	03/11/2021 Date				
E-mail: info@eec.edu.za Website: www.eec.edu.za					
ARTISAN AND SKILLS DEVELOPMENT CENTRE 16 Argon Street Purview, Springs Tel: (011) 730 6500 Fax: (011) 730 6505 E-mail: info@eec.edu.za Private Bag X52 SPRINGS 1560	BERONI CAMPUS 60 O'Reilly, Merry St Northmead, Bonaoni Tel: (011) 730 0600 Fax: (011) 426 3430 E-mail: info@eec.edu.za Private Bag X304 BERONI 1501	BRANDFAN CAMPUS 141 Victoria Avenue Brandfontein Tel: (011) 730 5900 Fax: (011) 740 8100 E-mail: info@eec.edu.za Private Bag X10 BRANDFONTEIN 1540	DAVEYTON CAMPUS 4444 St. Daveyton Tel: (011) 730 6600 Fax: (011) 438 4001 E-mail: info@eec.edu.za Private Bag X21 DAVEYTON 1507	KWA-THEMA CAMPUS Sun Ngema Rd. Kwa-Thema, Springs Tel: (011) 730 6600 Fax: (011) 730 6608 E-mail: info@eec.edu.za Private Bag X79 SPRINGS 1560	SPRINGS CAMPUS Flanckton Rd. Springs Tel: (011) 730 6600 Fax: (011) 382 8107 E-mail: info@eec.edu.za Private Bag X21 SPRINGS 1560

APPENDIX 2: Ethics Clearance Letter from Witwatersrand University

 UNIVERSITY OF THE WITWATERSRAND JOHANNESBURG	
Research Office	
HUMAN RESEARCH ETHICS COMMITTEE (NON-MEDICAL) R14/49 Hlako	
CLEARANCE CERTIFICATE	PROTOCOL NUMBER: H21/10/13
PROJECT TITLE	TVET mathematics teachers' professional noticing of students' productive struggle in calculus problem solving
INVESTIGATOR(S)	Mr M Hlako
SCHOOL/DEPARTMENT	School of Education
DATE CONSIDERED	22 October 2021
DECISION OF THE COMMITTEE	Approved Risk Level: Minimal
EXPIRY DATE	18 January 2025
DATE 19 January 2022	CHAIRPERSON  (Professor J Knight)
cc: Supervisor : Professor J Makonye	
DECLARATION OF INVESTIGATOR(S)	
To be completed in duplicate and ONE COPY returned to the Secretary at Room 10C04, 10th Floor, Senate House, University. Unreported changes to the application may invalidate the clearance given by the HREC (Non-Medical)	
I/We fully understand the conditions under which I am/we are authorized to carry out the abovementioned research and I/we guarantee to ensure compliance with these conditions. Should any departure to be contemplated from the research procedure as approved I/we undertake to submit an amendment of the protocol to the Committee. I agree to completion of a regular progress report. For Minimal and Low studies, this is due annually on 31 December. For Medium and High Risk studies, this is due twice annually on 30 June and 31 December.	
 Signature	<u>19.01.2022</u> Date
PLEASE QUOTE THE PROTOCOL NUMBER ON ALL ENQUIRIES	

APPENDIX 3: Amesa Long Paper Acceptance Letter



5 May 2023

Dear Veli and Paul

Thank you for submitting your Long Paper LP92 entitled:

**EXAMINING TVET MATHEMATICS LECTURERS' PROFESSIONAL NOTICING OF
STUDENTS' PRODUCTIVE STRUGGLE IN CALCULUS TASKS: FOSTERING GROWTH
AND DEVELOPMENT**

Hiako Mmachuene William and Makonye Paul Judah
University of the Witwatersrand

to the AMESA Congress 2023.

The Academic Coordinating Team (ACT) of the AMESA 2023 CONGRESS LOC would like to **congratulate** you on your long paper which has been **ACCEPTED** to be published in the AMESA 2023 CONGRESS proceedings.

Please go ahead and complete your AMESA 2023 CONGRESS registration.

The AMESA 2023 LOC is looking forward to your presentation at AMESA Congress 2023 at the University of Cape Town, 26 to 30 June 2023.

Kind regards,

A/Prof Duncan Mhakure (ACT – Coordinator)

APPENDIX 4: Students consent form


STUDENT'S CONSENT FORM

University of Witwatersrand, Wits School of Education, 27 Andrew Rd, Park town,
Johannesburg

DATE:

Title of research: TVET mathematics teachers' professional noticing of students' productive
struggle in calculus problem solving.

Name of Researcher: Hlako Mmachuene William

I (Name of the Student) agree
to participate in this research project as a student. The research has been explained to me and
I understand what my participation will involve.

I agree that my participation will remain confidential.

YES	NO
-----	----

I agree to be observed for the whole duration of the calculus lessons.

YES	NO
-----	----

I agree that the research has been explained to me and information sheet pack given to me by
the researcher

YES	NO
-----	----

I have received, read and understood the information sheet detailing the study.

YES	NO
-----	----

I agree that the research will be anonymous and confidential as explained in information
packs.

YES	NO
-----	----

I agree that the researcher will be audio recording the lessons that he will be observing.

YES	NO
-----	----

I agree that the researcher will look into my marked work for both official assessments given during the trimester.

YES NO

It is my responsibility to conduct research in an ethical manner according to Guidelines for Human Research Ethics Clearance Application/Non-medical. You are welcome to contact the University Research Ethics Committee (non-Medical), telephone +27(0)11 717 1408. Email: hrecnon-medical@wits.ac.za

Signature of Student: Date:

Signature of Researcher: Date:

NB: This was used to get consent on the part of students - to set/use/access their official scripts for purposes of analyzing how their lectures marked their mathematical thinking responses.

APPENDIX 5: Students information sheet



STUDENTS INFORMATION SHEET

Name of researcher: Hlako Mmachuene William

Title of research: TVET mathematics teachers' professional noticing of students' productive struggle in calculus problem solving.

Dear student of Ekurhuleni East TVET College – Kwa- Thema Campus

My name is Hlako Mmachuene William. I am a PhD student in the school of education at the University of Witwatersrand, Johannesburg. As part of my studies, I am doing research on TVET mathematics teachers' professional noticing of students' productive struggle in calculus problem solving. The main aim of the research is to explore how strategies of noticing students' productive struggle impact on teaching and learning of calculus as well as students' understanding of calculus topics

I would like to share with you important information regarding my proposed study.

Your role in the proposed study will involve only your presence in the classroom during normal teaching and learning of calculus topics while your lecturer will be observed on how he approaches the presentation of the subject matter. Your lecturer has been trained on new teaching strategies for which the researcher wants to observe their impact in students' understanding of challenging calculus topics. This includes basically all activities taking place in the classroom, ranging from topic introduction by lecturer, questions asked by lecturer and students, problems given to solve by lecturer as well as the kind of listening taking place in class. The researcher will therefore be coming into your classroom to observe these activities. The aim is to observe how your lecturer will be noticing students' mathematical thinking (which is a process that reveals students' errors and misconceptions) and how he/she deals with it. The observations will last for a period equivalent to the time allocated for calculus topics.

You are not going to be paid for participating in the research but you are going to benefit by being exposed to some strategies that may enhance your understanding of calculus. There are no disadvantages and penalties for not participating in the research. You may request to be

withdrawn from the research at any time if you feel uncomfortable with the way the research process is going on. You will be fully respected during the research. .

Reporting on the research will be confidential. The information given to me will be held securely and not disclosed to anyone. I will use pseudonym in reporting the research which means that I am not going to use real names of students. If you experience any distress or discomfort at any stage of this research process, please let me know and I will stop the data collection process and resume another time.

If you have any queries, concerns or complaints regarding ethical procedures of this study, you are welcome to contact the University Research Ethics Committee (Non-Medical), on the contact details given below.

It is my responsibility to conduct research in an ethical manner according to Guidelines for Human Research Ethics Clearance Application/Non-medical. You are welcome to contact the University Research Ethics Committee (non-Medical), telephone +27(0)11 717 1408. Email: hrecnon-medical@wits.ac.za

Should there be a need for me to clarify some part of this application; I am available on the contact details provided below. My supervisor details are also given below.

I am looking forward to hearing from you.

Yours faithfully

Hlako MW.

Phone Number: 073 155 1535

Signature Date

My supervisor Contact Details are as follows:

Prof. Makonye P.J

Office Number: 011 717 3206

Mobile Phone Number: 078 689 4572

Email Address: Judah.Makonye@wits.ac.za

B: This info sheet was issued to all students.

APPENDIX 6: Classroom activities observation tools

Classroom Activities Observation Tool

Title of research: TVET mathematics teachers` professional noticing of students` productive struggle in calculus tasks.

Lesson Design and Implementation	
Focus	Observation (A tick shows occurrence)
The lesson was designed to engage students as members of a learning community.	
In this lesson, students are allowed to explore ideas pertaining to the lesson.	
This lesson encouraged students to look for alternative approaches of investigation or problem solving.	
Students` ideas are encouraged and at times determine direction of the lesson.	
Classroom Culture and Communicative Interactions	
Students are given space to raise their ideas concerning the lesson.	
There was a high proportion of student talk and a significant amount of it occurred between and among students.	
Students were encouraged to generate conjectures, alternative solution strategies, and/or different ways of interpreting evidence.	
There was a climate of respect for what others had to say.	

The teacher's questions are well thought of questions that allow students to think deep about solutions.	
Student questions and comments often determined the focus and direction of classroom discourse.	
Encouraging learners to ask questions from each other	
Encourage learners to ask the teacher questions	
Uses cognitive terminology such as classify, analyze, predict or create when framing questions	
The teacher encourages learner inquiry by asking thoughtful, open-ended questions	
The teacher responds to learner contributions.	
Asks learners to elaborate on initial responses	

Allows for 'wait time' after posing questions	
The instructional strategies and activities respected students' prior knowledge and the preconceptions inherent therein.	
Students made predictions, estimations and/or hypotheses and devised means for testing them.	
There was a climate of respect for what others had to say.	
Active participation of students was encouraged and valued.	
Was student to student allowed during lesson presentation?	
Students are given space to raise their ideas concerning the lesson.	

There was a high proportion of student talk and a significant amount of it occurred between and among students.	
The focus and direction of the lesson was often determined by ideas originating with students.	
In this lesson, student exploration preceded formal presentation.	

APPENDIX 7: Teacher Professional Noticing and Listening Observation Tool

Teacher Professional Noticing and Listening Observation Tool

Title of research: TVET mathematics teachers` professional noticing of students` productive struggle in calculus tasks.

Professional Noticing	
Focus	Observation (A tick means occurrence)
Is there careful explanation of the target topic when introducing topic?	
Does teacher choose relevant instructional activities and provide various representations? E.g. What do students know already? What does he need to focus on? Are there other ways of solving the problem? Do the activities relate to areas of real-life experience? Why is it important for students to be able to do this part of mathematics?	
Working with pupils` answers, the ability to understand, interpret and react to pupils` responses and notions is highly dependent on PCK. Does the teacher notice students and tune into their learning processes? Does the teacher read and react adequately to pupils` responses and errors?	
Identifying errors, does the teacher identify errors? How?	
Adherence to textbook, does the teacher rely solely on textbook use? Is the teacher aware of other sources she / he can use?	

Use of terminology, does the teacher react to terminology dealt with in the topic of the day?	
Theoretical underpinning, is the teacher aware of any theoretical underpinning?	
Responding to children's ideas, does the teacher realize the importance of this? Or does he/she dismiss them?	
Recognition of conceptual appropriateness, what conceptual ideas does the teacher have?	
Choice of examples, is the teacher using relevant examples?	
Concentration on procedures, is the teacher heavily concentrating only on procedures in the expense of conceptual knowledge?	

Making connections, is there connections made with outside world?	
Creating platforms for discussions, does the teacher ever entertain any potential discussions?	
Group work, Does the teacher group the students?	
Teacher or student centered?	
Is the teaching focused on students or on teacher?	
Modern way of teaching / traditional ways, which teaching methods are practiced during teaching and learning.	
What is the teacher noticing during each of the stages below?	
Task introduction Mathematical discussions	
Small-group work	
Independent student work	
Post-lesson analysis	
What is the teacher noticing at each stage of the lesson presentation?	
Listening	

Listening is a two-way process, is the teacher ready to listen to students? Does the teacher have that patience?	
Listening type, which type of listening is the teacher doing during the lesson? Evaluative Interpretative Hermeneutically	
There different ways to listen. What kind of listening is exercised by the teacher?	

APPENDIX 8: Productive struggle observation

Productive struggle observation

Title of research: TVET mathematics teachers` professional noticing of students` productive struggle in calculus tasks.

Focus	Observation (A tick indicates occurrence)
The teacher has been trained on productive struggle strategies by researcher	
There is evidence/ pockets of productive struggle in the lesson presentation?	
The classroom activities meant for productive struggle, are they of higher order level?	
What was the teacher doing during productive struggle exercise?	
How does the teacher assist those who are unable to make a move in terms of finding the solution?	
Reasonable time given for students to try and solve the problem given to them? Noting the time given for productive struggle of each question.	
How does the teacher deal with different approaches to the solution?	
Does the teacher allow different approaches to find the solution?	
How does the teacher attend to individual students problems, during productive struggle?	
How does the teacher deal with students questions that seek clarity for the very problem given to them to solve?	
At the end of the productive struggle, how does the teacher consolidates for a meaningful lesson?	

Warshauer (2015) and Permatasari (2016) strategies and indicators of productive struggle

Action	Teaching Strategies	Student Indicators of a Productive Struggle	Observation
Question	Teachers ask questions that help students focus on their thinking and identify the source of their struggle, then encourage students to build on their thinking or look at other ways to approach the problem.	Students ask questions to identify the source of their struggle, write down their ideas, clarify ideas with others, and consider alternative strategies or representations to address their struggle.	
Encourage	Teachers encourage students to reflect on their work and support student struggle in their effort and not just in getting the correct answers	Students use their effort to solve problems and try to make sense of their work, not only satisfied with a correct answer or that they perceive themselves as smart or not.	
Give Time	Teachers give time and support for students to manage their struggles through adversity and failure by not stepping in too soon or too much, thereby taking the intellectual work a way from the students	Students use their time to develop and get through on their strategies, evaluate their progress, and understand what they can do and what still remains to be done.	
Acknowledge	Teachers acknowledge that struggle is an important part of learning and doing mathematics.	Students persist in their work to make sense of and to solve their problem and not give up or get discouraged easily.	

Table 4: Strategies and indicators of Productive Struggle adapted from Warshauer (2015) and Permatasari (2016).

Edwards and Beattie (2016) `Problem Cycle`

Stage	Description of what the teacher should do.	Observation
Problem Launch	This is about preparing students for productive struggle by creating a shared understanding of the problem to be solved. The teacher should make sure that students have a clear understanding of the problem and what they are expected to do.	
Working the Problem	Teachers must engage students in productive struggle and monitor their ways of thinking in preparation for later discussions. It is during this time that the teacher should recognize and note students' ways of thinking in preparation for the coming discussions.	
Conclusion	The teacher should announce students' ways of thinking whether correct or incorrect, encourage students to learn new ways of thinking by understanding each other, and explicitly connect their thinking to the key concepts. At this time, students present alternative ways of thinking about the problem and see opportunities to analyze mistakes and misconceptions.	
Conclusion	The teacher has to concisely highlight the key concepts drawn from students' thinking, express the concepts with appropriate notation and representations, and explicitly connect the lesson concepts with the course organizing concepts.	

Table 5: The stage of the `Problem Cycle` by Edward and Beatie (2016)

APPENDIX 9: Interview Protocol

Interview Protocol

Title of research: TVET mathematics teachers` professional noticing of students` productive struggle in calculus tasks.

Interviewer:	Hlako M.W	Interviewee:
Date:	Time:	
Personal Information will be asked, as a way of breaking any tension that might have built up.		
1. This will include finding the following:		
a. Post level and other responsibilities done by the particular lecturers in the campus.		
b. Number of years lecturing in a TVET College and in particular the Mathematics N5.		
c. Whether Mathematics was the lecturer`s favorite major subject at school and during training as lecturer.		
d. What the lecturer`s highest qualification is in Mathematics.		
The following questions relating to Mathematics teaching in TVET College will be asked:		
2. Whether the lecturer is offering Mathematics in any other level, e.g. Mathematics N4 to Mathematics N6.		
3. What/why do you dis/ like or enjoy most about offering TVET Mathematics?		
4. To you as a Mathematics lecturer, what is Mathematics and how do you see/perceive it?		
5. How can you explain your method/approach to teaching N5 mathematics?		
6. Do you ever allow discussions in your lecturer room? If yes, then how do you control it?		

7. We normally listen to hear. Do you know what it takes to listen? Who is doing most listening in your lecturer room?
8. Do you know of what is called `professional teacher noticing`? If yes, how normally do you go about exercising this skill?
9. Are there any challenges that you are encountering in the teaching and learning of calculus in Mathematics?
10. How do students respond to differential as well as integral calculus questions?
11. Maybe: do student makes errors in solving differentiation/integration/calculus problem? If yes, what kind of errors do they make?
12. Any misconceptions shown in learning about calculus topics?
13. It is known that there are two term tests to be written by TVET engineering students. After assessment is given, how do you give feedback? Do you believe in giving feedback to students? What role do you think feedback plays?
14. After how long do you give feedback to students, after having given them an assessment?
15. How do you deal with errors committed in the assessment tasks? How do you make sure that they do not occur again?
16. Are you happy with TVET Mathematics students` performance?
17. What do you think should be done to improve teaching and learning of mathematics?

APPENDIX 10: Test items analysis

Test items analysis

Title of research: TVET mathematics teachers` professional noticing of students` productive struggle in calculus tasks.

Test item	What the lecturer noticed and marked.	Similar or different error/misconception noticed by researcher

APPENDIX 11: MARKED STUDENTS' TERM TEST SCRIPTS SAMPLES

Cover page for ASSESSMENT TASKS

Surname and Initials of Student: ✖		
ID Number:		
Learning Programme: Report 191 Engineering Studies		
Subject: MATHEMATICS		
Level: N6	Group: A	Test : 1A
Topics covered: 1,2 AND 3		
Subject Outcomes & Learning Outcomes: DIFFERENTIATION AND INTEGRATION		
Marks allocated: 60	Duration: 109 MIN	Date: 25 /04/2022
Examiner: TSHIRANGWANA V		Moderator: MADASON S

Level of competency

RATING CODE	RATING	MARKS (%)
4	Excellent	90 – 100
3	Highly Competent	60 – 80
2	Competent	40 – 50
1	Not yet competent	0 – 30

Marks allocated

QUESTION NO	MARKS
1	
2	
3	
4	
5	
6	
7	
TOTAL MARKS	07

PERCENTAGE MARK

12 %

Feedback to student is:

SIGNATURES

Assessor: ✍	Student:	Moderator: ✍
Date: 11/03/2022	Date: 25 /04/2022	Date: 17/03/2022

ANSWER BOOK FOR FORMAL ASSESSMENTS

$$f(x) = e^{3x}$$

$$f'(x) = 3e^{3x}$$

$$g(x) = \sin 3x$$

$$g'(x) = \cos 3x$$

$$= -\frac{1}{3}e^{3x} \cdot \sin 3x + \int \frac{e^{3x} \cdot \cos 3x}{3} - \int \frac{3e^{3x} \cdot \cos 3x}{3} dx$$

$$= -\frac{1}{3}e^{3x} \cdot \sin 3x + \frac{1}{3}e^{3x} \cos 3x - \int e^{3x} \cdot \cos 3x dx$$

$$\int e^{3x} \cdot \cos 3x dx = -\frac{1}{3}e^{3x} \cdot \sin 3x + \frac{1}{3}e^{3x} \cos 3x - \int e^{3x} \cdot \cos 3x dx$$

$$\int e^{3x} \cdot \cos 3x dx + \int e^{3x} \cdot \cos 3x dx = -\frac{1}{3}e^{3x} \cdot \sin 3x + \frac{1}{3}e^{3x} \cos 3x$$

$$2 \int e^{3x} \cdot \cos 3x dx = -\frac{1}{3}e^{3x} \cdot \sin 3x + \frac{1}{3}e^{3x} \cos 3x$$

$$\int e^{3x} \cdot \cos 3x dx = -\frac{1}{6}e^{3x} \sin 3x + \frac{1}{6}e^{3x} \cos 3x + C$$

Q. y = tan⁴ 3x (sec² 3x + 1)

$$\int \tan^4 3x (\sec^2 3x + 1) dx$$

$$\int \tan^2 3x \cdot \tan^2 3x (\sec^2 3x + 1) dx$$

$$\int \tan^2 3x (\sec^2 3x - 1) \cdot (\sec^2 3x + 1) dx$$

$$\int \tan^2 3x (\sec^2 3x - 1) dx$$

ANSWER BOOK FOR FORMAL ASSESSMENTS

215. $y = \frac{\sin^2 x}{\cos^2 2x}$

$$\int \frac{\sin^2 x}{\cos^2 2x}$$

$$\int \sin^2 x \times \frac{1}{\cos^2 2x}$$

$$\int \sin^2 x \times \sec^2 2x$$

$$\int \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right)$$

$$\int \left[\frac{1}{2} - \frac{1}{2} (\cos 2x) \right] \times \sec^2 2x$$

$$\frac{1}{2} \int \sec^2 2x \, dx - \frac{1}{2} \int \cos 2x \, dx$$

$$\cos 2x = (1 - 2\sin^2 x)$$

$$f(x) = 1 - 2\sin^2 x$$

$$f'(x) = -4\sin x \cos x$$

$$g(x) = 1$$

$$g'(x) = 0$$

$$1 - 2\sin^2 x = \cos 2x$$

END OF REPORT