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Ogden's Lemma for Random Permitting and Forbidding Context Picture Languages and Table-Driven Context-Free Picture Languages

A Dissertation submitted to the Faculty of Science, University of the Witwatersrand, in partial fulfilment of the requirements for the degree of Master of Science

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Johannesburg,
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Declaration of Authorship

I, Joy Oghogho Idahosa, declare that the research dissertation on Ogden's Lemma for Random Permitting and Forbidding Context Picture Languages and Table-Driven Context-Free Picture Languages and all research done were done on my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly, while in candidature for a MSc degree at the University of the Witwatersrand;
- where any part of this research report has previously been submitted for a degree or any other qualification at the University of the Witwatersrand or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this dissertation is entirely my own work;
- I have acknowledged all main sources of help;
- where the research is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;

Signed: _____

Abstract

Random context picture grammars are used to generate pictures through successive refinement. There are three important subclasses of random context picture grammars, namely random permitting context picture grammars, random forbidding context picture grammars and table-driven context-free picture grammars. These grammars generate the random permitting context picture languages, random forbidding context picture languages and table-driven context-free picture languages, respectively. Theorems exist which provide necessary conditions that have to be satisfied by a language before it can be classified under a particular subclass. Some of these theorems include the pumping and shrinking lemmas, which have been developed for random permitting context picture languages and random forbidding context picture languages respectively. Two characterization theorems were developed for the table-driven context-free picture languages.

This dissertation examines these existing theorems for picture languages, i.e., the pumping and shrinking lemmas and the two characterisation theorems, and attempts to prove theorems, which will provide an alternative to the existing theorems and thus provide new tools for identifying languages that do not belong to the various classes. This will be done by adapting Ogden's idea of marking parts of a word which was done for the string case. Our theorems essentially involve marking parts of a picture such that the pumping operation increases the number of marked symbols and the shrinking operation reduces it.

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Contents

Declaration of Authorship i
Abstract ii
Acknowledgment iii

1 Introduction **1**
1.1 General Introduction 1
1.2 Random Context Picture Grammars 2
1.3 Necessary Conditions 3
1.4 Roadmap 5

2 Background **7**
2.1 Introduction 7
2.2 General Definitions 7
2.3 String Languages 8
2.3.1 Definitions 9
2.3.2 Existing Necessary Conditions for String Languages 10
2.4 Picture Languages 13
2.4.1 Definitions 13
2.4.2 Existing Necessary Conditions for Picture Languages 19
2.4.2.1 Other Necessary Conditions 23
2.4.2.2 More Background 26
2.4.2.3 Established Relationships 27
2.5 Conclusion 27

3 Ogden's Lemma for Context-Free Galleries **28**
3.1 Introduction 28

3.2 Definitions 28
3.3 Results 29
3.4 Conclusion 35

4 Ogden's Lemma for Random Permitting and Forbidding Context Galleries 36
4.1 Introduction 36
4.2 Definitions 37
4.3 Ogden's Lemma for Random Permitting Context Galleries 39
4.4 Ogden's Lemma for Random Forbidding Context Galleries 41
4.5 Conclusion 44

5 Table-Driven Context-free Picture Grammars 45
5.1 Introduction 45
5.2 Ogden's Lemma for Table-Driven Context-Free Galleries 46
5.3 Conclusion 53

6 Conclusion and Future Work 54
References 56

List of Figures

1.1 A diagram showing the relationship between classes of picture grammars . . . 4

2.1 Productions of a random context picture grammar . . . 14
2.2 Hollow isosceles right-angle triangle: second refinement . . . 16
2.3 Hollow isosceles right-angle triangle: third refinement . . . 16
2.4 Hollow isosceles right-angle triangle: fourth refinement . . . 17
2.5 Φ_1 of $\mathcal{G}_{\mathsf{xyzoh}}$ . . . 21
2.6 Φ_2 of $\mathcal{G}_{\mathsf{xyzoh}}$ . . . 21
2.7 Φ_3 of $\mathcal{G}_{\mathsf{xyzoh}}$ . . . 22
2.8 A production of a collage grammar . . . 23
2.9 A derivation using the rule in Figure 2.8 . . . 24

3.1 Derivation tree showing branch points (branch points are indicated by ovals and marked nodes by asterisks) . . . 31
3.2 Pictures from the gallery $\mathcal{G}_{\mathsf{triangleandcrosses}}$ . . . 34

4.1 Pictures from the gallery $\mathcal{G}_{\mathsf{triangles}}$ . . . 42
4.2 The shrinking operation performed on Γ_1 . . . 43

5.1 The picture $\Phi_{1,2}$ from the gallery $\mathcal{G}_{\mathsf{FalseTCFPG}}$ . . . 51
5.2 The picture $\Phi_{2,2}$ from the gallery $\mathcal{G}_{\mathsf{FalseTCFPG}}$ . . . 51

List of Abbrevations

CFG	context-free grammar
CSG	context-sensitive grammar
RCG	random context grammar
RPCG	random permitting context grammar
RFCG	random forbidding context grammar
CFL	context-free language
CSL	context-sensitive language
RCL	random context language
RPCL	random permitting context language
RFCL	random forbidding context language
CFPG	context-free picture grammar
CSPG	context-sensitive picture grammar
RCPG	random context picture grammar
RPCPG	random permitting context picture grammar
RFCPG	random forbidding context picture grammar
TCFPG	table-driven context-free picture grammar
CFPL	context-free picture language
CSPL	context-sensitive picture language
RCPL	random context picture language
RPCPL	random permitting context picture language
RFCPL	random forbidding context picture language
TCFPL	table-driven context-free picture language

Chapter 1

Introduction

1.1 General Introduction

Various types of grammars exist in formal language theory and their rewriting rules differ in complexity. These grammars generate their respective languages; for example, picture grammars generate picture languages. A picture language is defined as a set of pictures which share a certain unique characteristic. Picture languages are also called *galleries*. In the rest of this dissertation, picture languages will often be referred to as galleries to make the text more readable.

Some picture grammar types include the context-free picture grammars (CFPGs), random context picture grammars (RCPGs) and the collage grammars. The CFPGs make use of rewriting rules and these rules are made up of a single symbol being replaced by other symbols. The CFPGs generate the context-free picture languages (CFPLs) [Matz, 1997] and are known for their elegance and simplicity, but are not powerful enough to generate certain languages. One set of these context-free grammars is the class of collage grammars [Drewes and Kreowski, 1999].

In both the string and the picture case, some grammars are able to sense context, unlike the context-free grammars. In the string case, these are called context-sensitive grammars (CSGs). The CSGs are known to be too powerful, restrictive and too complex to use. In the picture case, some context-free picture grammars are able to sense context to some degree, making them context-sensitive to an extent. They are called context-free but use context to regulate the ap-

plication of context-free productions. For example, in so-called puzzle grammars, a production can only be applied if the right-hand side of the production fits into the developing picture, just as in jigsaw puzzles.

Due to the limited generative power of context-free grammars and the fact that context-sensitive grammars are restrictive, researchers were motivated to find an intermediate generative model as a trade-off between the simplicity of context-free grammars and the power of context-sensitive grammars. This led to the development of the random context picture grammars. The RCPGs are a variation of CFPGs and make use of regulated rewriting, which involves the use of context present in any part of the picture, such that restrictions are placed on the application of these rewriting rules to a given picture based on this context, hence the name "random context". In RCPGs, context is not structurally connected. It is globally distributed, unlike in the case of context-free array [Yamamoto *et al.*, 1989] and puzzle grammars [Laroche *et al.*, 1992], which use local context.

Chain-code picture grammars [Maurer *et al.*, 1983], Lindenmayer systems [Rozenberg and Salomaa, 1976] [Prusinkiewicz and Hanan, 1976], array grammars [Yamamoto *et al.*, 1989] and puzzle grammars [Laroche *et al.*, 1992], amongst others, generate pictures using syntactic methods, which is a popular form of generating pictures. These grammars are variations of the CFPGs where some are context-free and others context-sensitive to some extent, e.g, the context-free array grammars.

1.2 Random Context Picture Grammars

Random context picture grammars are an adaptation of the random context (string) grammars [van der Walt, 1972] (RCGs). The word "random" in random context only means that the position of each of these context symbols in the picture is not considered: as long as they exist in the picture, the production rule can either be applied or not. RCPGs use successive refinement to generate pictures, which involves subdividing a shape until a picture is generated. Basically, a pictorial form, which is made up of a square divided into smaller squares containing variables, generates a picture, which contains only terminals, by rewriting variables using the productions of the given grammar. Every terminal is associated with a colour and the square in which it is

contained is filled with that color, making up a picture. For the purpose of this research, squares divided into four equal squares were used, which helped to simplify the process, without affecting the final results.

RCPGs are grammars which use regulated rewriting. They are similar to context-free picture grammars (context-free picture grammars share the same properties as the context free grammars, but are adapted for pictures), but differ in that the application of their production rules is controlled by context symbols which are distributed randomly in the pictorial form, and thus can generate a larger class of languages than grammars without context, as shown in Ewert [1999, Examples 2.5–2.9]. With context being globally distributed, these rewriting restrictions control the development of a picture to some degree. This context is either permitting or forbidding where the permitting context enables the use of a production and the latter prevents it.

RCPGs have three natural subclasses; context-free picture grammars, random permitting context picture grammars (RPCPGs) and random forbidding context picture grammars (RFCPGs). They generate the context-free picture languages (CFPLs), random permitting context picture languages (RPCPLs) and random forbidding context picture languages (RFCPLs), respectively. A special case of the RFCPGs exists called the table-driven context-free picture grammars (TCFPGs). They generate the table-driven context-free picture languages (TCFPLs). In TCFPGs, productions are chosen from a table and they are applied in parallel to all the variables in the pictorial form [Ewert, 2009]. The relationship between these subclasses is shown in Figure 1.1.

1.3 Necessary Conditions

An important aspect in formal language theory is finding necessary conditions that must be satisfied by a language before it can be generated by a grammar of the respective class [Ewert and van der Walt, 2013], thus providing an understanding of the power and limitations of these grammars. Various attempts have been made to develop theorems that can be used to determine whether a picture language belongs to a certain class. The pumping lemma is a good example on how this can be done. It is also called an iteration theorem because it generates a series of pictures in a language by "iterating some repetition operation" [Rabkin, 2012].

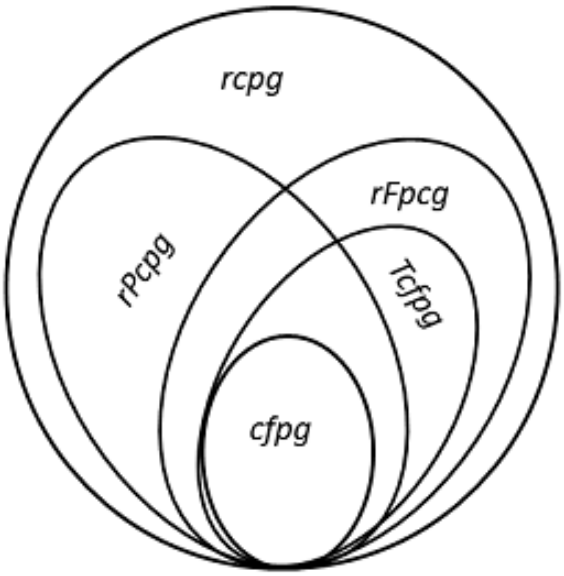


Figure 1.1: A diagram showing the relationship between classes of picture grammars

Some of the attempts at developing these theorems include the pumping-shrinking lemma for CFPLs by Ewert [1999], the pumping lemma for RPCPLs by Ewert and van der Walt [1999], the shrinking lemma for RFCPLs by Ewert and van der Walt [1998], two characterisation lemmas for TCFPLs by Bhika *et al.* [2007], necessary conditions for random context galleries by Tkachova [2013] and criteria for context-freeness of collage languages by Drewes *et al.* [1997]. These necessary conditions have been successful at classifying a number of languages into their specific classes and they have also been used in determining the generative power of these grammars. The pumping lemma for RPCPLs was used to show that RPCPGs are strictly weaker than RCPGs. The shrinking lemma for RFCPLs showed that RFCPGs are strictly weaker than RCPGs, and the two characterisation theorems for TCFPLs were used to show that TCFPGs are strictly weaker than RFCPGs. These are discussed in detail in Chapter 2.

The purpose of this research is to investigate these existing necessary conditions and to develop new theorems as an alternative or even an improvement on the existing ones. We will make use of an idea called marking, proposed by Ogden [1968] for Ogden's lemma. This lemma is a generalisation of the pumping lemma for CFLs by Bar-Hillel *et al.* [1961]. Basically, Ogden's lemma states that any two segments of a word can be pumped, ensuring that certain parts of the word are chosen as marked, such that the pumping operation is guaranteed to increase the

number of marked symbols in the word. In this research, this idea was used and adapted for the picture case.

The aim of this research is to achieve results similar to those obtained by Rabkin [2012], which introduced Ogden's lemma for subclasses of random context (string) languages. Random context picture grammars are a two-dimensional adaptation of the random context (string) grammars (RCGs), which also have subclasses, just as the RCPGs, which include context-free grammars (CFGs), random permitting context grammars (RPCGs), random forbidding context grammars (RFCGs) and a special case of the RFCGs, Extended Table-driven 0-Context Lindenmayer-system (ET0L) grammars. They generate the context-free languages (CFLs), random permitting context languages (RPCLs), random forbidding context languages (RFCLs) and the Extended Table-driven 0-context Lindenmayer-system (ET0L) languages, respectively. Ogden's lemma for CFLs was given by Ogden [1968] and Rabkin [2012] used this approach to develop Ogden's lemma for RPCLs, RFCLs and ET0L languages.

This research aims to develop Ogden-like lemmas for three subclasses of random context picture languages beginning with a simpler class, Ogden's lemma for CFPLs, which will assist our understanding on how to proceed with the three subclasses. To our knowledge, there has been no attempt to apply Ogden's idea of marking to picture languages. Therefore, we use this idea to proffer new tools for analysing languages.

1.4 Roadmap

A brief introduction to picture languages and string languages was provided, including a discussion of the usefulness of necessary conditions. In Chapter 2, the existing necessary conditions for string languages are discussed, with a specific focus on Ogden's lemma for subclasses of random context (string) languages, on which this research is based. We then discuss in detail, the existing conditions for the subclasses of random context picture languages, which we intend to provide an alternative to, and other necessary conditions that exist for picture languages. In Chapter 3, the first result of this research is given; we provide Ogden's lemma for context-free picture languages, the simplest of the subclasses. We also provide an example to show the usefulness of the new theorem. We then give Ogden's lemma for RPCPLs and RFCPLs in Chapter

4 and use these new theorems to show that certain galleries are not random permitting or random forbidding context galleries. An introduction to TCFPGs is given in Chapter 5 and an analogue of Ogden's lemma for TCFPLs is also given; the usefulness of the new theorem is shown by proving that a certain gallery is not a TCFPL. In Chapter 6, a conclusion is given, showing that the aim of this research has been achieved and we provide possible future work that can be done, based on the results found here.

Chapter 2

Background

2.1 Introduction

This chapter presents and discusses the existing necessary conditions for subclasses of random context string languages which serve as a background for this research. We also discuss the existing conditions for subclasses of random context picture languages and show how we intend to improve on them. Necessary conditions which exist for other classes of picture languages, aside from the random context picture languages are also discussed briefly in this chapter.
In Section 2.2 we give general definitions which will be useful in understanding these necessary conditions. In Section 2.3 a detailed discussion of string languages is provided, followed by an equally detailed treatment of picture languages in Section 2.4. The chapter is concluded in Section 2.5.

2.2 General Definitions

Definition 1. $\mathbb{N}$ denotes the set $\{0, 1, 2, ...\}$ of natural numbers.
Definition 2. $\mathbb{N}^+$ denotes the set $\{1, 2, ...\}$ of positive natural numbers.
Definition 3. $l \in [m]$ denotes that the value of l is a positive natural number between and including 1 and m.

Definition 4. $A \subseteq B$ denotes that every element of A is an element of B, where A and B are sets.

Definition 5. We denote the set of words (or strings) over an alphabet (finite set) Σ by Σ^*. The empty word is denoted by λ. The length of a word w is denoted by $|w|$ and can be defined as: if $|w| = 1$ for all $w \in \Sigma$, then $|vw| = |v| + 1$ for all $w \in \Sigma$, $v \in \Sigma^*$.

2.3 String Languages

This section focusses primarily on random context (string) grammars. Just like the RCPGs, RCGs have three subclasses: the random permitting context grammars, random forbidding context grammars and a special case of the RFCGs, ET0L (Extended Table-driven 0-context Lindenmayer) grammars. These subclasses generate respectively, the random permitting context languages (RPCLs), random forbidding context languages (RFCLs) and the ET0L languages. The RPCGs use context that allows the application of a production, while the RFCGs use context that inhibits the application of a production. ET0L grammars use productions which are applied in parallel to the variables in a string and the choice of these productions is table-driven.

This work focuses on Ogden's lemma by Ogden [1968], which is a generalization of the pumping lemma for context-free languages [Bar-Hillel *et al.*, 1961]. The pumping lemma generally states that for a context-free language, any string in the language which is adequately long contains two sections which can either be removed or repeated any number of times, with the resulting string remaining in the language.

Ogden [1968] then generalized this pumping lemma by introducing the concept of marking and thus ensuring that the repeatable segments that can be selected are partially controlled. It ensures that at least one of the repeatable segments contains a marked position, provided enough positions in the string are chosen as marked. The major aim of Ogden [1968]'s research was to show that certain context-free languages are naturally ambiguous, i.e., it is possible to derive two or more derivation trees for a word in a language for every grammar for that language. Ogden's lemma ensures that the pumping operation increases the number of marked symbols (i.e., the resulting word is larger than the initial one).

Various necessary conditions have been developed in the past. Rabkin [2012]'s research

examined existing theorems on string languages and proved variations of Ogden's lemma for subclasses of random context languages. Rabkin [2012] developed Ogden's lemma for random permitting and random forbidding context languages and ET0L languages, thus providing new tools for classifying which languages belong to each subclass. Rabkin [2012]'s new lemmas basically state that substituting larger (or smaller) subwords for smaller (or larger) subwords which are contained in a word will give a resulting word that is still in the language and if parts of these words are chosen as marked, the resulting larger (or smaller) word must contain more (or less) marked symbols than the initial word. Rabkin [2012] introduced the concept of densely-marked words, stating that there are no large unmarked words or subwords, i.e., every subword of a word with a minimum size, contains at least one marked symbol. These lemmas were shown by Rabkin [2012] to strengthen the known pumping for RPCLs [Ewert and van der Walt, 2002], provide an alternative for the shrinking [van der Walt and Ewert, 2000] lemma for RFCLs and also provide a pumping lemma and Ogden-like lemma for ET0L languages.

This research is based on Rabkin [2012]'s work. The aim is to develop necessary conditions for subclasses of *picture* languages, analogous to those developed by Rabkin [2012] for *string* languages.

In Section 2.3.1 we provide definitions relevant to these classes of languages. In Section 2.3.2 we discuss the three necessary conditions by Rabkin [2012].

2.3.1 Definitions

We now give some definitions obtained from Rabkin [2012].

Definition 6. (Random Context Grammar) A random context grammar (RCG) is a tuple $G = (V, \Sigma, S, P)$ where V and Σ are disjoint finite sets (the alphabets of variables and terminals, respectively), $S \in V$, and $P \subseteq V \times (V \cup \Sigma)^+ \times 2^V \times 2^V$ is a finite set of rules or productions. An element $(A, x, \mathcal{P}, \mathcal{F})$ of P is written as $A \to x(\mathcal{P}, \mathcal{F})$. $\mathcal{P}$ and $\mathcal{F}$ are respectively called the permitting and forbidding sets of the production, while A and x are respectively called the left-hand side and right-hand side.
If $\mathcal{F} = \emptyset$ for every production in P, then G is called a *random permitting context grammar*; if $\mathcal{P} = \emptyset$ for every production in P, then G is a *random forbidding context grammar*; if $\mathcal{P} = \mathcal{F} = \emptyset$

for every production rule, then G is a *context-free grammar*.

Definition 7. ($\Rightarrow$, $\Rightarrow^*$) If $G = (V, \Sigma, S, P)$ is a random context grammar, then we define $\Rightarrow$ such that $x \Rightarrow y$ if $x = x_1 A x_2$ and $y = x_1 w x_2$, where $A \to w(\mathcal{P}; \mathcal{F})$ is in P, no variable in $\mathcal{F}$ appears in $x_1 x_2$, and every variable in $\mathcal{P}$ appears in $x_1 x_2$. We define $\Rightarrow^*$ such that $x \Rightarrow^* y$ denotes that x derives y in zero or more steps.

Definition 8. (Element of $\mathcal{L}(G)$) The *random context language* $\mathcal{L}(G)$ generated by an RCG $G = (V, \Sigma, S, P)$ is the set $\{z \in \Sigma^* \mid S \Rightarrow^* z\}$. An element of $\mathcal{L}(G)$ is called a word.

Definition 9. (Sentential Form) If $G = (V, \Sigma, S, P)$ is a random context grammar, and $x \in (V \cup \Sigma)^*$, then we say x is a *sentential form* of G if $S \Rightarrow^* x$.

Definition 10. (Strict Subword) If $x = uvw$ is a word, then v is called a *subword* of x or symbolically, $v \sqsubseteq x$. If $v \sqsubseteq x$ and $v \neq x$ then we say v is a strict subword of x and write $v \overset{\sqsubset}{\neq} x$.

Definition 11. (Marked Word) A word w with marked symbols can be defined formally as a pair (w, M), where $M \subseteq [|w|]$: an instance of a symbol in w is called *marked* if its position is in M.

Definition 12. (k-Densely marked word) A marked word w is *k-densely marked* if every subword u of w with length at least k contains at least one marked symbol.

Definition 13. (ET0L System) An *ET0L system* is a tuple $G = (V, \Sigma, \mathcal{T}, S)$ where Σ, the terminal alphabet, is a non-empty subset of the alphabet V ($\Sigma \subseteq V$), $S \in V$ and $\mathcal{T}$ is a finite collection $\{T_1, ..., T_n\}$ of tables. Each table is a finite set of productions $A \to u$, such that for every $A \in V$ there is a production $A \to u$ in T_i for every $i \in [n]$.

2.3.2 Existing Necessary Conditions for String Languages

In this section, we give the known pumping lemma for context-free languages and the variations of Ogden's lemma for the subclasses of random context languages.

Theorem 1. ***Pumping lemma for CFLs by Bar-Hillel et al. [1961]:*** *If a language L is context-free, then there exists some integer $p \geq 1$ (the pumping threshold) such that every word $w \in L$ with $|w| \geq p$ can be written as $w = uvxyz$, such that:*

1. $|vxy| \leq p$,
2. $|vy| \geq 1$, *and*

3. uv^nxy^nz *is in* L *for all* $n \geq 0$.

Theorem 2. ***Ogden's lemma for CFLs by Ogden [1968]:*** *If* L *is a context-free language, then there exists an integer* m *such that for any* $w \in L$ *with at least* m *marked positions,* w *can be written as* $w = uvxyz$ *such that:*

1. x *and at least one of* v *or* y *both contain a marked position;*
2. vxy *contain at most* m *marked positions;*
3. $uv^nxy^nz \in L$ *for all* $n \in \mathbb{N}$.

Theorem 3. ***Ogden's lemma for RPCLs by Rabkin [2012]:*** *For any RPCL* L *and* $k \in \mathbb{N}^+$, *there is an* $m \in \mathbb{N}^+$ *(the pumping threshold) such that for any* k*-densely marked word* $w \in L$ *with* $|w| \geq m$ *there is a number* $l \in [m]$ *such that:*

1. w *contains* l *mutually disjoint non-empty subwords* $u_1, u_2, ..., u_l$ *and* l *mutually disjoint non-empty subwords* $v_1, v_2, ..., v_l$, *such that for each* $i \in [l]$ *there exists a* $j \in [l]$ *such that* $v_i \sqsubseteq u_j$;
2. *if each* v_i *is replaced with* u_i, *then the resulting word is still in* L, *and this process can be applied iteratively to always yield a word in* L;
3. *if* v_i *contains a marked symbol, then so does* u_i;
4. *there are strictly more marked symbols in* $u_1u_2...u_l$ *than in* $v_1v_2...v_l$.

This lemma was used to show that the language in Rabkin [2012, Example 4.14] is not random permitting. The lemma showed improvement on the pumping lemma for RPCLs [van der Walt and Ewert, 2000] by showing that this example, which satisfies the pumping lemma for RPCLs, does not satisfy the Ogden's lemma and is therefore not random permitting. An example of a language, [Rabkin, 2012, Example 4.15], which is not an RPCL but satisfies Ogden's lemma for RPCLs was given, showing that the theorem is a necessary and not sufficient condition for RPCLs.

Theorem 4. ***Ogden's lemma for RFCLs by Rabkin [2012]:*** *For any RFCL* L *and* k, $t \in \mathbb{N}^+$, *there is an* $m \in \mathbb{N}^+$ *(the shrinking threshold) such that for any* k*-densely marked word* $w \in L$ *with* $|w| \geq m$ *there are* t *words* $w^{(1)}, w^{(2)}, ..., w^{(t)} = w$ *such that for every* $j \in \{2, 3, ..., t\}$:

1. *there exists a number* $l \in [m]$*;*
2. $w^{(j)}$ *contains* l *mutually disjoint non-empty subwords* $u_1, u_2, ..., u_l$ *and* l *mutually disjoint non-empty subwords* $v_1, v_2, ..., v_l$*, such that for each* $i \in [l]$*, there exists a* $p \in [l]$ *such that* $v_i \sqsubseteq u_p$*;*
3. *if each* u_i *is replaced with* v_i*, then the resulting word is* $w^{(j-1)}$*;*
4. *if* v_i *contains a marked symbol, then so does* u_i*;*
5. *there are strictly more marked symbols in* $u_1u_2...u_l$ *than in* $v_1v_2...v_l$*.*

This theorem was not able to improve on the previous shrinking lemma for RFCLs [Ewert and van der Walt, 2002] and this is explained in Section 4.2 of Rabkin [2012]. An example of a non-rFc language was given, which satisfies this theorem, concluding that it is also a necessary but not sufficient condition.

Theorem 5. ***Ogden's lemma for ET0L languages by Rabkin [2012]****: If* L *is an ET0L language, then there exists an* $l \in \mathbb{N}$ *(which we will call the threshold for* L*) such that for any word* $w \in L$ *with at least* l *marked positions,*

1. w *can be written as* $w = u_1u_2...u_n$ *and each* u_i *can be written* $u_i = v_{(i,1)}v_{(1,2)}...v_{(i,n_i)}$ *(we will denote the set of subscripts of* v*, i.e.,* $(i, j) : i \in [n], j \in [n_i]$ *by* I*);*
2. *there is a map* $\phi : I \to [n]$ *such that if each* $v_{(i,j)}$ *is replaced with* $u_{\phi(i,j)}$*, then the resulting word is still in* L*, and this process can be applied iteratively to always yield a word in* L*;*
3. *if* $v_{(i,j)}$ *contains a marked position, then so does* $u_{\phi(i,j)}$*;*
4. *there is an* $(i, j) \in I$ *such that* $\phi(i, j) = i$*, and there are at least two marked positions in* $v_{(i,j)}$ *and at least one in* u_i*, but outside of* $v_{(i,j)}$*.*

The language in Rabkin [2012, Example 6.5], which is an ET0L language was shown to satisfy the theorem and the language in Rabkin [2012, Example 6.6], which is not an ET0L language but satisfies the theorem was given, thus showing that the theorem is a necessary but not sufficient condition for ET0L languages.

2.4 Picture Languages

Picture grammars are a two-dimensional form of the string grammars. They differ from string grammars in that, for the languages they generate, some details about the shape of a derivation tree for a picture are preserved in the positions and sizes of squares in the language, while the structure of a derivation tree of a word is almost completely concealed in a word and preserves only the ordering of the leaves [Rabkin, 2012].

A number of necessary conditions also exist for picture languages; Ewert [1999] developed the shrinking-pumping lemma for CFPLs, Ewert and van der Walt [1999] developed the pumping lemma for RPCPLs, a shrinking lemma was developed by Ewert and van der Walt [1998] for RFCPLs and for TCFPLs, two characterization theorems were developed in Bhika *et al.* [2007]. Tkachova [2013] developed new necessary conditions for subclasses of random context picture languages, Drewes *et al.* [1997] developed the criteria for the context-freeness of collage languages, a pumping lemma for array languages was developed in Shen-Pei and Lin [1986] and Kim [1990] studied two properties of picture languages which are picture iteration and picture ambiguity. We discuss these existing necessary conditions in detail in the next sections and provide the theorems and results for those relevant to this research. We also later discuss other existing necessary conditions for other classes of picture languages.

2.4.1 Definitions

We now provide some definitions specific to picture languages, from Ewert [1999] and Bhika *et al.* [2007].

Random context picture grammars generate pictures using productions of the form shown in Figure 2.1, where:

- A is a variable,
- $x_{11}, x_{12}, x_{21}, x_{22}$ are variables or terminals,
- $\mathcal{P}$ and $\mathcal{F}$ are sets of variables.

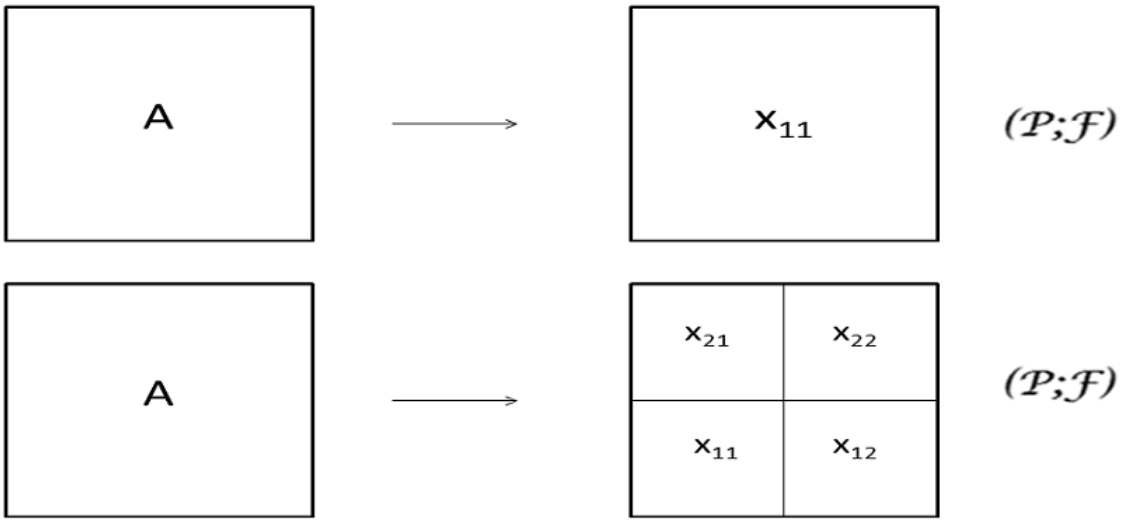


Figure 2.1: Productions of a random context picture grammar

If a picture which is being developed has a square labelled A and if the developing picture contains none of the variables in $\mathcal{F}$ and all the variables of $\mathcal{P}$, then the square labelled A can be replaced by a square labelled x_{11} or by equal-sized squares with labels $x_{11}, x_{12}, x_{21}, x_{22}$. We denote the square with sides parallel to the axes, lower left corner at (u, v) and upper right corner at (x, y) by coordinates $((u, v), (x, y))$, using lower-case Greek letters, e.g, the square (A, α) is the square labelled A in the coordinates $((u, v), (x, y))$, where α denotes $((u, v), (x, y))$.

Definition 14. (Random Context Picture Grammar) *A random context picture grammar* $G = (V_N, V_T, P, (S, \sigma))$ is defined as consisting of a finite alphabet V of *labels*, consisting of disjoint subsets V_N of *variables* and V_T of *terminals*, a finite set of *productions* P of the form $A \to [x_{11}, x_{12}, x_{21}, x_{22}](\mathcal{P}; \mathcal{F})$ or, $A \to x_{11}(\mathcal{P}; \mathcal{F})$, where $A \in V_N, x_{11}, x_{12}, x_{21}, x_{22} \in V$ and $\mathcal{P}, \mathcal{F} \subseteq V_N$ and an *initial labelled square* (S, σ) with $S \in V_N$.

We now illustrate these concepts with an example from Ewert [1999]:

Example 1. We want to generate the gallery of hollow isosceles right-angle triangles with side length 2^i, $i \geq 1$. This we achieve with the RCPG:

$G_{\text{triangle}} = \{S, T_{lb}, T_{rb}, T_t, T_{ls}, T_b, T_h, T'_{lb}, T'_{rb}, T'_t, T'_{ls}, T'_b, T'_h, F\}, \{w, b\}, P, (S, ((0, 0), (1, 1)))),$

where P is the set:

$$
\begin{aligned}
S &\rightarrow [T_{lb}, T_{rb}, T_t, w] && (2.1)\\
T_{lb} &\rightarrow [T'_{lb}, T'_b, T'_{ls}, w](\{\}; \{T'_{rb}, T'_t, T'_{ls}, T'_b, T'_h\}) \mid && (2.2)\\
&\quad F(\{\}; \{T'_{rb}, T'_t, T'_{ls}, T'_b, T'_h\}) && (2.3)\\
T_{rb} &\rightarrow [T'_b, T'_{rb}, T'_h, w](\{T'_{lb}\}; \{\}) \mid && (2.4)\\
&\quad b(\{F\}; \{\}) && (2.5)\\
T_t &\rightarrow [T'_{ls}, T'_h, T'_t, w](\{T'_{lb}\}; \{\}) \mid && (2.6)\\
&\quad b(\{F\}; \{\}) && (2.7)\\
T_{ls} &\rightarrow [T'_{ls}, w, T'_{ls}, w](\{T'_{lb}\}; \{\}) \mid && (2.8)\\
&\quad b(\{F\}; \{\}) && (2.9)\\
T_b &\rightarrow [T'_b, T'_b, w, w](\{T'_{lb}\}; \{\}) \mid && (2.10)\\
&\quad b(\{F\}; \{\}) && (2.11)\\
T_h &\rightarrow [w, T'_h, T'_h, w](\{T'_{lb}\}; \{\}) \mid && (2.12)\\
&\quad b(\{F\}; \{\}) && (2.13)\\
T'_{lb} &\rightarrow T_{lb}(\{\}; \{T_{rb}, T_t, T_{ls}, T_b, T_h\}) && (2.14)\\
T'_{rb} &\rightarrow T_{rb}(\{T_{lb}\}; \{\}) && (2.15)\\
T'_t &\rightarrow T_t(\{T_{lb}\}; \{\}) && (2.16)\\
T'_{ls} &\rightarrow T_{ls}(\{T_{lb}\}; \{\}) && (2.17)\\
T'_b &\rightarrow T_b(\{T_{lb}\}; \{\}) && (2.18)\\
T'_h &\rightarrow T_h(\{T_{lb}\}; \{\}) && (2.19)\\
F &\rightarrow b(\{\}; \{T_{rb}, T_t, T_{ls}, T_b, T_h\}) && (2.20)
\end{aligned}
$$

The terminals w and b represent the colors *light* and *dark* respectively.

The pictorial form α_i contains 2^i equally big squares on each side of the triangle at the beginning of the ith iteration of the sequence of rules 2.2–2.19. Of these squares, the lower left corner is labelled T_{lb}, the lower right corner is labelled T_{rb} and the square at the top labelled T_t. The other squares on the left-hand side are labelled T_{ls}, T_h on the hypotenuse, and T_b at the bottom. The remaining squares are labelled w.

One pictorial form α_i can derive another α_{i+1} depending on whether T_{lb} is present in α_i or not. T_{lb} produces an F if it decides to terminate (2.3) and the other variables produce b (2.5, 2.7, 2.9, 2.11 and 2.13) on sensing F and then F produces a b (2.20) and the picture is complete.

Otherwise, α_{i+1} can be derived by dividing each square into four equal squares (2.2, 2.4, 2.6, 2.8, 2.10, and 2.12) such that the length of the sides of the pictorial form increases to twice the previous length, each square labelled with variables. The original variables can then be replaced (2.14–2.19) and the main loop terminated.

Pictures in this gallery are shown in Figures 2.2, 2.3 and 2.4.

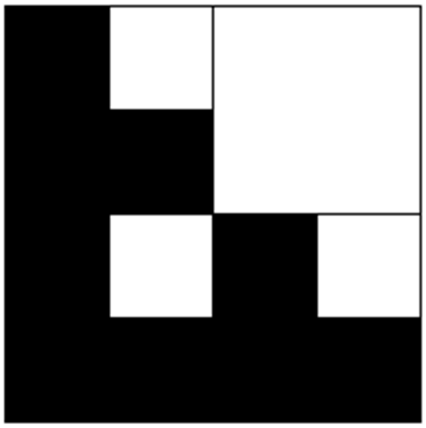


Figure 2.2: Hollow isosceles right-angle triangle: second refinement

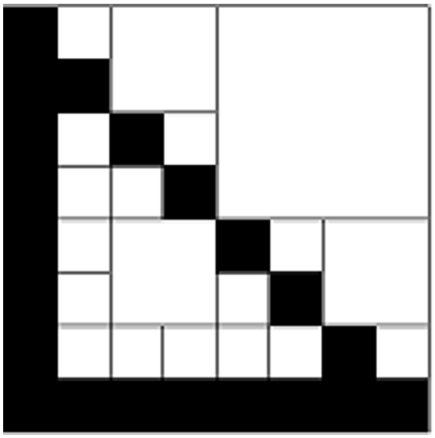


Figure 2.3: Hollow isosceles right-angle triangle: third refinement

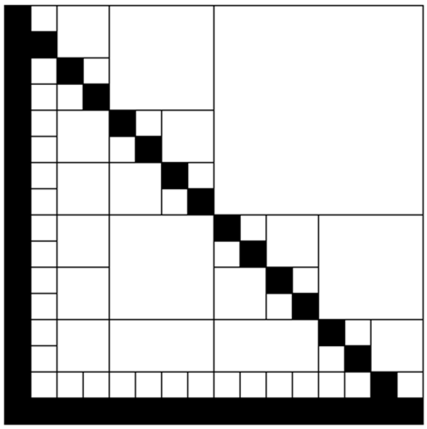


Figure 2.4: Hollow isosceles right-angle triangle: fourth refinement

Definition 15. (PICTORIAL FORM) A *pictorial form* is any finite set of non-overlapping labelled squares in the plane. If Π is a pictorial form, we denote by $l(\Pi)$ the set of labels used in Π. The size of a pictorial form Π is the number of squares contained in it, i.e., $|\Pi|$.

Definition 16. ($\Rightarrow$, $\Rightarrow^*$) For an RCPG G and pictorial forms Π and Γ we write $\Pi \Rightarrow \Gamma$ if there is a production $A \rightarrow [x_{11}, x_{12}, x_{21}, x_{22}](\mathcal{P}, \mathcal{F})$ or $A \rightarrow x_{11}(\mathcal{P}, \mathcal{F})$ in G, Π contains a labelled square $(A, \alpha), l(\Pi \backslash \{(A, \alpha)\}) \supseteq \mathcal{P}$ and $l(\Pi \backslash \{(A, \alpha)\}) \cap \mathcal{F} = \emptyset$, and $\Gamma = (\Pi \backslash \{(A, \alpha)\}) \cup \{(x_{11}, \alpha_{11}), (x_{12}, \alpha_{12}), (x_{21}, \alpha_{21}), (x_{22}, \alpha_{22})\}$ or $\Gamma = (\Pi \backslash \{(A, \alpha)\}) \cup \{(x_{11}, \alpha_{11})\}$. As usual, $\Rightarrow^*$ denotes the reflexive transitive closure of $\Rightarrow$.

Definition 17. (PICTURE, GALLERY) A *picture* is a pictorial form Π with $l(\Pi) \subseteq V_T$, where V_T is a set of terminals. A *subpicture* Ω of a picture Π is any subset of Π that fills a square, i.e., the union of all the squares in Ω is a square. If $\Omega \sqsubseteq \Pi$ and $\Omega \neq \Pi$ then we say Ω is a strict subpicture of Π and write $\Omega \sqsubsetneq \Pi$. The gallery $\mathcal{G}(G)$ generated by a grammar $G = (V_N, V_T, P, (S, \sigma))$ is $\{\Phi \mid \{(S, \sigma)\} \Rightarrow^* \Phi \text{ and } l(\Phi) \subseteq V_T\}$. An element of $\mathcal{G}(G)$ is called a picture. We call the gallery generated by an RCPG a *random context gallery*.

Definition 18. ($\rightarrow$) $\Pi \rightarrow \beta$ denotes the pictorial form obtained from Π by uniformly scaling (up or down) and translating all the labelled squares in Π to fill the square β, retaining all the labels.

Definition 19. ($\#_v(\Phi)$) For a picture Φ, $\#_v(\Phi)$ denotes the number of appearances of the symbol v in Φ.

Definition 20. (TABLE-DRIVEN CONTEXT-FREE PICTURE GRAMMARS) A *table-driven context-*

free picture grammar is a system $G = (V_N, V_T, \mathcal{T}, (S, \sigma))$, where $V_N, V_T, V = V_N \cup V_T$ and (S, σ) are as defined in Definition 14. $\mathcal{T}$ is a finite set of tables, each table $R \in \mathcal{T}$ satisfying the following conditions:

1. R is a finite set of productions of the form $A \to [x_{11}]$ or $A \to [x_{11}, x_{12}, x_{21}, x_{22}]$, where $A \in V_N$, and $x_{11}, x_{12}, x_{21}, x_{22} \in V$.
2. R is complete, i.e., for each $A \in V_N$, there exist $x_{11}, x_{12}, x_{21}, x_{22} \in V$ such that $A \to [x_{11}]$ or $A \to [x_{11}, x_{12}, x_{21}, x_{22}]$ is in R.

In TCFPGs, as in RCPGs, terminals are never rewritten, but the squares containing variables are replaced. The completeness condition ensures that every direct derivation replaces all the variables in the pictorial form.

Definition 21. ($\mathsf{lhs}(p)$, $\mathsf{rhs}(p)$) For any production p, say $A \to [x_{11}, x_{12}, x_{21}, x_{22}]$, A is called the left hand side of p, and $[x_{11}, x_{12}, x_{21}, x_{22}]$ the right hand side of p, denoted by $\mathsf{lhs}(p)$ and $\mathsf{rhs}(p)$, respectively.

Definition 22. ($\mathsf{repl}((A, \alpha), p)$) For a labelled square (A, α) and a production p with $A = \mathsf{lhs}(p)$, say $A \to [x_{11}, x_{12}, x_{21}, x_{22}]$, we denote $\{(x_{11}, \alpha_{11}), (x_{12}, \alpha_{12}), (x_{21}, \alpha_{21}), (x_{22}, \alpha_{22})\}$ by $\mathsf{repl}((A, \alpha), p)$.

Definition 23. ($\mathsf{var}(\Pi)$, $base$) For pictorial form Π, we define $\mathsf{var}(\Pi) = \{(A, \alpha) \in \Pi | A \in V_N\}$. For pictorial form Π and table R, we call $b : \mathsf{var}(\Pi) \to R$ a $base$ on Π if for each $(A, \alpha) \in \mathsf{var}(\Pi)$, $\mathsf{lhs}(b((A, \alpha))) = A$.

Definition 24. (TCFPGs, $\Rightarrow$) Let Π and Γ be pictorial forms. We say that $\Pi \Rightarrow \Gamma$, if there exists a base b on Π such that

$$\Gamma = \Pi \backslash \mathsf{var}(\Pi) \cup \bigcup_{(A,\alpha) \in \mathsf{var}(\Pi)} \mathsf{repl}((A, \alpha), b((A, \alpha))).$$

Note: It is not a contradiction that we use the same symbol, $\Rightarrow$, in Definition 16 and Definition 24 since TCFPGs are a subclass of RCPGs with certain restrictions on their production rules.

Definition 25. (NONFREQUENT, RARE) Let $\mathcal{G}$ be a set of pictures with labels from the alphabet V_T, and B a nonempty subset of V_T. Then:

1. B is called *nonfrequent* in $\mathcal{G}$ if there exists a constant k, which may depend on both $\mathcal{G}$ and B, such that for every $\Phi \in \mathcal{G}$, $\#_B(\Phi) < k$; otherwise B is called *frequent* in $\mathcal{G}$.

2. B is *rare* in $\mathcal{G}$ if for every $k \in \mathbb{N}_+$ there exists an $n_k > 0$ such that for every $n \in \mathbb{N}$ with $n > n_k$, if a picture $\Phi \in \mathcal{G}$ contains n occurrences of letters from B, then for each two such occurrences, the smallest subpicture containing those occurrences has size at least k; otherwise B is called *nonrare* in $\mathcal{G}$.

2.4.2 Existing Necessary Conditions for Picture Languages

We now present existing necessary conditions that have been developed for picture languages. This research hopes to improve on these conditions, using Ogden's idea of marking parts of a word and applying this to the picture case.

Theorem 6. ***Shrinking-Pumping lemma for context-free picture languages by Ewert [1999]:*** *For any infinite CFPL $\mathcal{G}$ there is a positive integer m such that if any picture $\Phi \in \mathcal{G}$ contains at least m squares, then:*

1. *Φ contains two nonempty subpictures (Ω, α) and (Ψ, β) with $\beta \sqsubseteq_{\neq} \alpha$;*
2. *the picture obtained from Φ by substituting $(\Psi \to \alpha)$ for (Ω, α) is in $\mathcal{G}$;*
3. *the picture obtained from Φ by substituting $(\Omega \to \beta)$ for (Ψ, β) is in $\mathcal{G}$;*
4. *recursively carrying out the operation in (3) always results in a picture in $\mathcal{G}$.*

This essentially states that if a part of a picture is pumped, (i.e., a larger subpicture is put in place of a smaller subpicture) or shrunk, (i.e., a smaller subpicture replaces a larger subpicture), the resulting picture will still be in the gallery and if these operations are carried out repeatedly, the result will still be in the gallery. This shrinking-pumping lemma for CFPLs was used to prove that certain galleries cannot be generated by a CFPG. A gallery, [Ewert, 1999, Example 2.3], which can be generated by an RPCPG, but not by a CFPG was given, thus showing that CFPGs are strictly weaker than RPCPGs. The same was done for the RFCPGs, where a gallery, [Ewert, 1999, Example 2.4], which can be generated by an RFCPG and not by a CFPG was given, thus showing that CFPGs are also strictly weaker than RFCPGs.

Theorem 7. ***Pumping lemma for RPCPLs by Ewert and van der Walt [1999]:*** *For any RPCPL $\mathcal{G}$ there is an $m \in \mathbb{N}^+$ such that for any picture $\Phi \in \mathcal{G}$ with $|\Phi| \geq m$ there is a number l, $l \in [m]$,*

such that:

1. *Φ contains l mutually disjoint nonempty subpictures $(\Omega_1, \alpha_1), ..., (\Omega_l, \alpha_l)$ and l mutually disjoint nonempty subpictures $(\Psi_1, \beta_1), ..., (\Psi_l, \beta_l)$, these being related by a function $\vartheta : \{1, ..., l\} \to \{1, ..., l\}$ such that for each i, $i \in [l], \beta_i \sqsubseteq \alpha_{\vartheta_{(i)}}$ and for at least one i, $i \in [l], \beta_i \overset{\sqsubseteq}{\neq} \alpha_{\vartheta_{(i)}}$;*

2. *the picture obtained from Φ by substituting $(\Omega_i \to \beta_i)$ for (Ψ_i, β_i) for all i, $i \in [l]$, is in $\mathcal{G}$.*

3. *recursively carrying out the operation in (2) always results in a picture in $\mathcal{G}$.*

This essentially states that if the pumping operation is performed on a large enough picture in a gallery (i.e., l smaller subpictures are replaced by l larger subpictures) and this is carried out any number of times, the resulting picture will also be in that gallery. The lemma showed that an actual effect of the pumping property is that the set of sizes of the pictures in an infinite gallery generated by an RPCPG contains an infinite arithmetic progression [Ewert and van der Walt, 1999]. A gallery, [Ewert, 1999, Example 2.5] which can be generated by an RCPG and not by an RPCPG was given, showing that RPCPGs are strictly weaker than RCPGs. This pumping lemma for RPCPLs was also used to show that there exists a gallery, [Ewert, 1999, Example 2.4], which can be generated by an RFCPG, but not by an RPCPG, concluding that RFCPLs are not included in the class of RPCPLs. In this case, they were unable to find a gallery which can be generated by an RPCPG and not by an RFCPG and this was posed as an open problem which, to our knowledge, has not been solved as yet.

We now give an example of the pumping operation on a gallery $\mathcal{G}_{xyzoh}$ which can be generated by an RPCPG.

Example 2. We use an example from Ewert [2009, Example 4.3]: Consider Φ_1 in Figure 2.5. Let (Ω_1, α_1) be the lower left hand quarter, (Ω_2, α_2) the lower right hand quarter, (Ω_3, α_3) the upper left hand quarter and (Ω_4, α_4) the upper right hand quarter of Φ_1. Furthermore, let (Ψ_1, β_1) be equal to (Ω_2, α_2), (Ψ_2, β_2) the letter Y, (Ψ_3, β_3) the letter Z and (Ψ_4, β_4) the letter H. Then $\vartheta(1) = 2, \vartheta(2) = \vartheta(3) = 1$ and $\vartheta(4) = 4$.

Φ_2 in Figure 2.6 is obtained by substituting $(\Omega_i \to \beta_i)$ for (Ψ_i, β_i), $i \in [4]$, in Φ_1. Then Φ_3 in Figure 2.7 is obtained by carrying out this same operation on Φ_2, and so on.

Theorem 8. ***Shrinking lemma for RFCPLs by Ewert and van der Walt [1998]**: Let $\mathcal{G}$ be an RFCPL. For any integer $t \geq 2$ there exists an integer $m = m(t)$ such that for any picture $\Phi \in \mathcal{G}$*

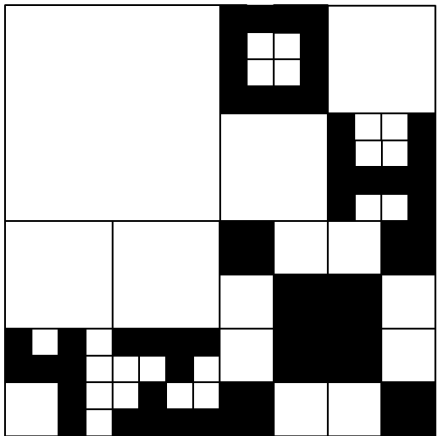


Figure 2.5: Φ_1 of $\mathcal{G}_{\mathsf{xyzoh}}$

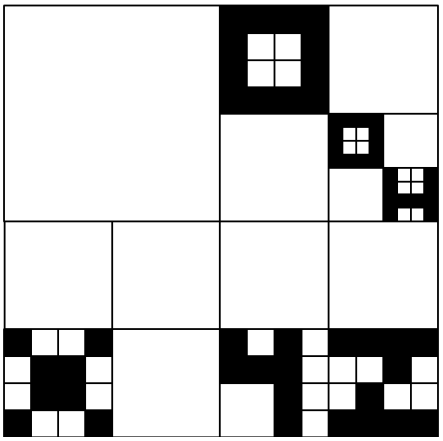


Figure 2.6: Φ_2 of $\mathcal{G}_{\mathsf{xyzoh}}$

with at least m squares there are t pictures $\Phi^1, ..., \Phi^t = \Phi$ in $\mathcal{G}$ and $t-1$ numbers $l_2, ..., l_t$ such that for each j, $2 \leq j \leq t$,

1. *Φ^j contains l_j mutually disjoint nonempty square subpictures $(\Omega_{j1}, \alpha_{j1}), ..., (\Omega_{jl_j}, \alpha_{jl_j})$ and l_j mutually disjoint non-empty square subpictures $(\Psi_{j1}, \beta_{j1}), ..., (\Psi_{jl_j}, \beta_{jl_j})$, these being related by a function $\vartheta_j : \{1, ..., l_j\} \rightarrow \{1, ..., l_j\}$ such that for each i, $i \in [l_j], \beta_{ji} \sqsubseteq \alpha_{j\vartheta_j(i)}$ and for at least one i, $i \in [l_j], \beta_{ji} \stackrel{\sqsubseteq}{\neq} \alpha_{j\vartheta_j(i)}$;*
2. *the picture Φ_{j-1} is obtained by substituting $(\Psi_{ji} \rightarrow \alpha_{ji})$ for $(\Omega_{ji}, \alpha_{ji})$ for all i, $i \in [l_j]$, in Φ^j.*

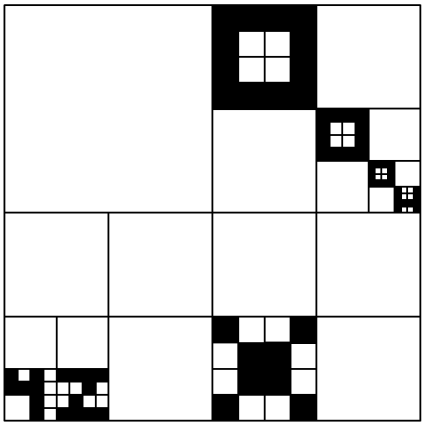


Figure 2.7: Φ_3 of $\mathcal{G}_{\text{xyzoh}}$

This essentially states that if the shrinking operation is performed on a large enough picture in a gallery (i.e., l larger subpictures are replaced by l smaller subpictures), and this is carried out a number of times, the resulting pictures will also be in that gallery. Using the shrinking lemma, a gallery, [Ewert, 1999, Lemma 5.2], which can be generated by an RCPG, but not by an RFCPG was given and this showed that RFCPGs are strictly weaker than RCPGs.

Two conditions necessary for a gallery to be generated by a TCFPG are listed below.

Theorem 9. ***Characterisation lemmas for TCFPLs by Bhika et al. [2007]:***

1. *Let $\mathcal{G}$ be a gallery generated by a TCFPG with terminal alphabet V_T. Then for every $V_1 \subseteq V_T$, $V_1 \neq \emptyset$, there exists a positive integer k such that for every picture Φ in $\mathcal{G}$ either:*
   - $\#_{V_1}(\Phi) \leq 1$, *or*
   - Φ *contains a subpicture* Π *such that* $|\Pi| \leq k$ *and* $\#_{V_1}(\Pi) \geq 2$, *or*
   - *there exists an infinite subset* $\mathcal{H} \subseteq \mathcal{G}$ *such that, for every* $\Upsilon \in \mathcal{H}, \#_{V_1}(\Upsilon) = \#_{V_1}(\Phi)$.
2. *Given a TCFPG $G = (N, T, \mathcal{T}, (S, \sigma))$ and $B \subseteq T, B \neq \emptyset$; if B is rare in $\mathcal{G}$, then B is nonfrequent in $\mathcal{G}$.*

The gallery of the Sierpiński carpet in Bhika *et al.* [2007, Example 2.1] was used to show the power of TCFPGs and it was shown that no RPCPG or CFPG can generate this gallery. Also, it was shown that every gallery that can be generated by a TCFPG can also be generated

by an RFCPG. The first necessary condition was used to show that a gallery, [Ewert, 1999, Theorem 4.3], cannot be generated by any TCFPG. The second condition was used to show that a certain gallery, [Bhika *et al.*, 2007, Example 5.9], cannot be generated by any TCFPG but can be generated by an RFCPG, concluding that TCFPGs are strictly weaker than RFCPGs.

2.4.2.1 Other Necessary Conditions

We now discuss other attempts to develop necessary conditions for certain picture languages. Due to little work being done in this area, we were only able to find research work done on four classes of picture language, which we discuss in the following paragraphs.

Tkachova [2013] developed alternative necessary conditions for subclasses of random context galleries, based on existing necessary conditions for random context (string) languages [Ewert and van der Walt, 2013]. Tkachova [2013] used a different approach from that used in Ewert and van der Walt [1999], Ewert and van der Walt [1998] and Bhika *et al.* [2007]. In Ewert and van der Walt [1999], Ewert and van der Walt [1998] and Bhika *et al.* [2007], no restriction is put on the size of the subpicture used for pumping while Tkachova [2013]'s work gives a minimum size for the subpicture. Tkachova [2013] states that for any large enough picture Φ in a gallery, there exists a subpicture α of Φ of a defined minimum size such that α is also a subpicture of a larger (or smaller) picture β in that gallery. These new necessary conditions were developed for RPCPLs, RFCPLs, CFPLs and TCFPLs and a new gallery, [Tkachova, 2013, Example 4.3.1], was created to show the usefulness of the new necessary conditions.

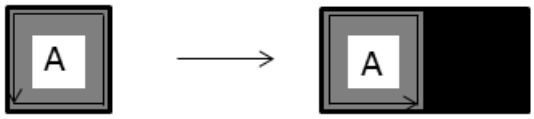


Figure 2.8: A production of a collage grammar

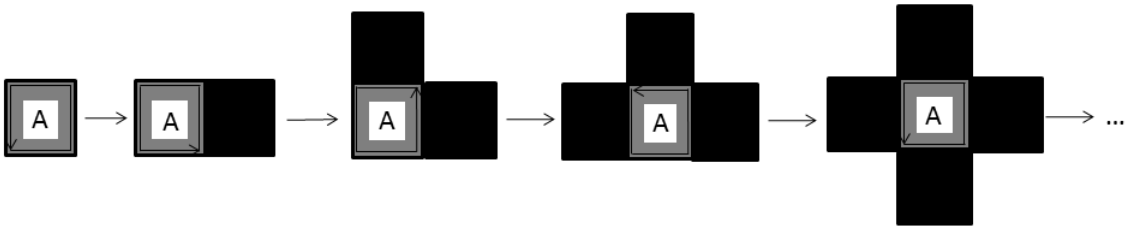


Figure 2.9: A derivation using the rule in Figure 2.8

Drewes *et al.* [1997] developed criteria for the context-freeness of collage languages. Collage grammars [Drewes and Kreowski, 1999] generate pictures that are made up of collages, where each collage is a set of parts, and a part is "a set of points in a Euclidean space of some dimension" [Drewes *et al.*, 1997]. A picture is obtained by overlaying all of these parts. Collages are generated by a replacement operation involving atomic variables and the replacement of one variable by another variable or by parts is done by applying affine transformations. Figures 2.8 and 2.9 show some derivations in a collage grammar. In Figure 2.8, the first A-labelled square is a variable and the right-hand side contains the same variable as the left-hand side but it is turned at $90°$ counterclockwise, and on the right, a square, which is a part. If the left-hand side is used as the start collage, the right-hand side is derived as shown in the first step of Figure 2.9. For one to apply the production again, it must be turned by $90°$ again so the left-hand side matches the variable which is being replaced which obtains the upper square which is a new part. Iterating this process only yields two more collages since the same parts are reproduced repeatedly after four steps. A pumping lemma for collage languages was developed based on the iteration occurring in the generation of pictures, but it was not sufficient as "the most essential property was missing" [Drewes *et al.*, 1997], i.e., the ability to yield infinitely many results. In Figure 2.9 we see that iterating productions only yields the same result eventually, therefore it is not possible to yield infinitely many results and since the pumping lemma works on the iteration of productions, it is obvious that the pumping lemma is useless in this case. The pumping lemma cannot show whether a collage language can be generated by a collage grammar. To solve this problem, Drewes *et al.* [1997] developed a theorem which was able to show that certain collage languages are not context-free. The first criterion of the theorem involves shrinking a collage in a language such that there is, at most, a constant difference in the number of parts in subsequent

collages; this leads on to show that the number of parts of collages in a context-free collage language increases at most linearly. The second criterion ensures that the volume of parts over all collages in a given language increases or decreases exponentially. It was previously known that the Sierpiński gasket, [Drewes *et al.*, 1997, Fig. 1], could not be generated by a collage grammar because the refinement is too uniform but there was no theorem to prove this. The new lemma by Drewes *et al.* [1997] was able to show that since the number of parts in the square pattern of the Sierpiński gasket grows quadratically, it cannot be generated by a context-free collage grammar, thus showing the usefulness of the new theorem.

A certain model of two-dimensional languages [Rosenfeld and Siromoney, 1993], array languages, which are also a general form of Chomsky grammars, are studied in Shen-Pei and Lin [1986]; these languages are also discussed in Yamamoto *et al.* [1989] and Subramanian *et al.* [2013]. Array grammars are accurate and flexible which makes them better, to a certain degree, than other two-dimensional models. A pumping lemma for these languages was developed in Shen-Pei and Lin [1986] using the idea of a tree structure. A definition for the derivation trees of two-dimensional array grammars was given. This was used to prove the pumping lemma, [Shen-Pei and Lin, 1986, Theorem 2–3], which determined whether an array language is context-free, depending on some properties that exist in the language. These properties are dependent on the length and width of the array. The new pumping lemma was derived using the existing pumping lemma for the string case. This new lemma led on to show that, due to the possible space constraints, the pumping lemma cannot be utilized efficiently and thus left this open for future work. To our knowledge, no work has been done to improve on this. We refer the reader to the article, Drewes *et al.* [1997] for a clearer understanding of how the lemma works since it is too lengthy to be summarised here.

Kim [1990] studied two properties of regular, linear and context-free picture languages: picture iteration and picture ambiguity. Iteration theorems similar to those given for the string case in Hopcroft and Ullman [1979] were given for the regular, linear and context-free picture languages. These iteration theorems were used to show that certain picture sets do not belong to these classes of picture languages. These iteration theorems use the same method as the pumping lemma for CFPLs. This lemma states that for any large enough picture in a regular, linear or context-free picture language, if one "pumps" a picture in the picture set to generate a larger picture, the larger picture must also be in that picture set. The iteration theorem for context-

free picture languages, [Hopcroft and Ullman, 1979, Theorem 3.3], was used to show that the picture language D, derived from $L = \{u^i d^i r^i | i \geq 1\}$, cannot be generated by a context-free picture grammar since the pumping operation does not yield a picture that is still in D. Examples ([Hopcroft and Ullman, 1979, Example 3.1 and 3.2]) were also given to show the usefulness of the iteration theorems for regular and linear picture languages, respectively.

2.4.2.2 More Background

Ewert [1999] investigated the power of random context and gave examples of five galleries that can be generated using context. In van der Walt and Ewert [2003], a property shared by all random context galleries was presented and it was shown that pictures which are composed of squares of equal sizes share a certain commutativity. Two pictures are said to be commutative if it is possible to convert one picture to another by alternating two of its subsquares (squares in the picture), which may involve uniformly increasing the size of one subsquare and also decreasing the size of the other [van der Walt and Ewert, 2003]. A picture set which cannot be generated using only random context was then obtained using this notion, thus presenting a limitation of random context.

Other work has been done to show the power of the permitting feature. Cooperating context-free array grammar systems with permitting features were investigated in Subramanian *et al.* [2013]. Context-free arrays [Yamamoto *et al.*, 1989] are called context-free but have the ability to sense context by using the blank symbol, $(\#)$, to regulate the application of productions. They are known to be able to produce geometric figures like solid rectangles because of their context-sensing ability. However, the ability to sense context prevents the development of a two-dimensional equivalent of the pumping lemma for context-free string languages. Regulated rewriting was then included in the grammar system by using permitting symbols with the rules of the grammar. This helps in reducing the number of components needed for generating a set of picture arrays in a cooperating distributed array grammar system. From this research, it was shown that the number of components in a permitting cooperating distributed context-free array grammar system reduces when compared to a context-free array grammar system, thus showing the power of restricted rewriting using permitting symbols.

Some of the work done on random context string languages has been discussed, which provides the background on which this research is based. Some of the previous work done on picture languages was also introduced. In the next section, the relationship between the subclasses of RCPLs is discussed.

2.4.2.3 Established Relationships

The subclasses of random context galleries have been grouped into different categories using developed necessary conditions. A class of grammars is said to be weaker than another if every language that the first class of grammars can generate, the second class can also generate. A class of grammars is *strictly* weaker than another if the first class of grammars is weaker and there is at least one language that can be generated by the second class which cannot be generated by the first.

The different theorems that were given in Subsection 2.4.2 have provided a categorization of these classes. CFPGS are known to be strictly weaker than RPCPGs and RFCPGs [Ewert, 1999, Examples 2.5 and 2.6], and RPCPGs are strictly weaker than RCPGs, as shown in Ewert [1999]. In Ewert and van der Walt [1998], it was shown that RFCPGs are also strictly weaker than RCPGs. TCFPGs are a special case of the RFCPGs and are more powerful than CFPGs and strictly weaker than RFCPGs [Bhika *et al.*, 2007]. The relationship is shown in Figure 1.1. No definition exists for the class of context-sensitive picture grammars but it is known that there are galleries that cannot be generated using RCPGs [Ewert, 1999].

2.5 Conclusion

We have discussed the existing necessary conditions for random context picture languages. We now extend these existing conditions using marking in the next chapters.

In Chapter 3, we provide Ogden's lemma for context-free galleries which will provide us with information on how to proceed with the more complicated classes.

Chapter 3

Ogden's Lemma for Context-Free Galleries

3.1 Introduction

This chapter provides the proof of Ogden's lemma for CFPLs. We first define some terms relevant to this theorem, then we prove some necessary lemmas which will be used for the proof. We finally prove Ogden's lemma for CFPLs and provide an example of a gallery that cannot be generated by this class of grammars.

3.2 Definitions

Some of the definitions in this section are based on previous work done for the string case and are adapted for the picture case, which is the main focus of this research.

Definition 26. (Derivation Tree) A *derivation tree* is an ordered, rooted tree that represents the syntactic structure of a picture according to a grammar.

Definition 27. (Root Node) A root node is a variable from which a derivation begins.

Definition 28. (Parent, Child) If there is an edge from node v to w, then v is said to be the

parent of w and w the child of v.

Definition 29. (Interior Node) An interior node corresponds to a variable that appears within the derivation and every node is labelled with a variable. A leaf node is a node with no children.

Definition 30. (Descendant) For a derivation tree T, a node n is called a descendant of node m in T if there is a sequence of nodes $n_1, n_2, ..., n_k$, such that $n_k = n$, $n = m$ and for each i, $i \in [k-1]$, n_{i+1} is a direct descendant of n_i.

Definition 31. (Subtree) A subtree of a tree T is a tree consisting of a node in T and all of its descendants in T.

Definition 32. (Proper Subtree) A proper subtree is a subtree corresponding to any node that is not the root node.

Definition 33. (Path) A *path* in a nonempty derivation tree consists either of a single node or of a node, one of its descendants, and all the nodes in between.

Definition 34. (Length of a Path) The *length* of a path is the number of nodes it contains.

Definition 35. (Height of a Tree) The *height* of a derivation tree is the length of the longest path.

Definition 36. (Marked Picture) Let Φ be a picture. A marked picture Φ is a pair (Φ, M), where $M \neq \emptyset$ and $M \subseteq \Phi$.

Definition 37. (Contribution of a Variable) Let $G = (V_N, V_T, P, (S, \sigma))$ be a CFPG and $\mathcal{G}$ be the gallery generated by G. Let $\{(S, \sigma)\} \Rightarrow^* \Pi \Rightarrow^* \Phi$ where $\Phi \in \mathcal{G}$. Then the contribution of a variable in Π is the subpicture of Φ formed by the terminals which descend from the variable. A variable is said to be marked if there is a marked terminal in its contribution.

Definition 38. (Branch Point) A node is called a *branch point* if it has more than one marked symbol in its contribution.

3.3 Results

First, we prove Lemma 1. It is adapted from Martin [2003, Lemma 8.1].

Lemma 1. *For any $h \geq 1$, a derivation tree having more than 4^{h-1} leaf nodes must have height greater than h.*

Proof. We will prove this by mathematical induction on h, using the contrapositive statement: If the height is less than or equal to h, the number of leaf nodes is less than or equal to 4^{h-1}.

Basis Step: A derivation tree with height less than or equal to 1 has no more than one node and therefore no more than one leaf node.

Induction Step: For $k \geq 1$, a derivation tree of height less than or equal to k has no more than 4^{k-1} leaf nodes.

Proof of Induction Step: Let T be a derivation tree with height less than or equal to $k+1$. The statement holds for T with no more than one node. Otherwise, each proper subtree of T has height less than or equal to k, and thus each has 4^{k-1} or fewer leaf nodes, by the induction hypothesis. The number of leaf nodes in T is the sum of the numbers in all subtrees and therefore no greater than $4^{k-1} + 4^{k-1} + 4^{k-1} + 4^{k-1} = 4^k$. □

Lemma 2. *If the derivation tree consisting of a node on a path and its descendants has more than 4^h marked leaf nodes (Definition 36), it has more than h branch points.*

Now we prove Lemma 2, adapted from Martin [2003, Lemma 8.2]. The derivation tree in Figure 3.1 serves as an aid to understanding this proof.

Consider the following situation: Starting at the top of the derivation tree, we choose a path containing the root node and a leaf node. Starting at the root node, we repeatedly select a node N in the path and the child of N having the largest number of marked positions among its descendants. Following the path down the tree, starting at a branch point, the number of marked descendants of the current node decreases at the node right below it and then remains constant until another branch point is reached. Thus, we can see that from the way the path is chosen, every branch point below the top one has at least half as many marked descendants as the branch point above it.

Proof. We will now prove the lemma by mathematical induction on h using the contrapositive statement:
If the number of branch points is no more than h, it has no more than 4^h marked leaf nodes.

Basis Step: A derivation tree with no more than one branch point has no more than four marked

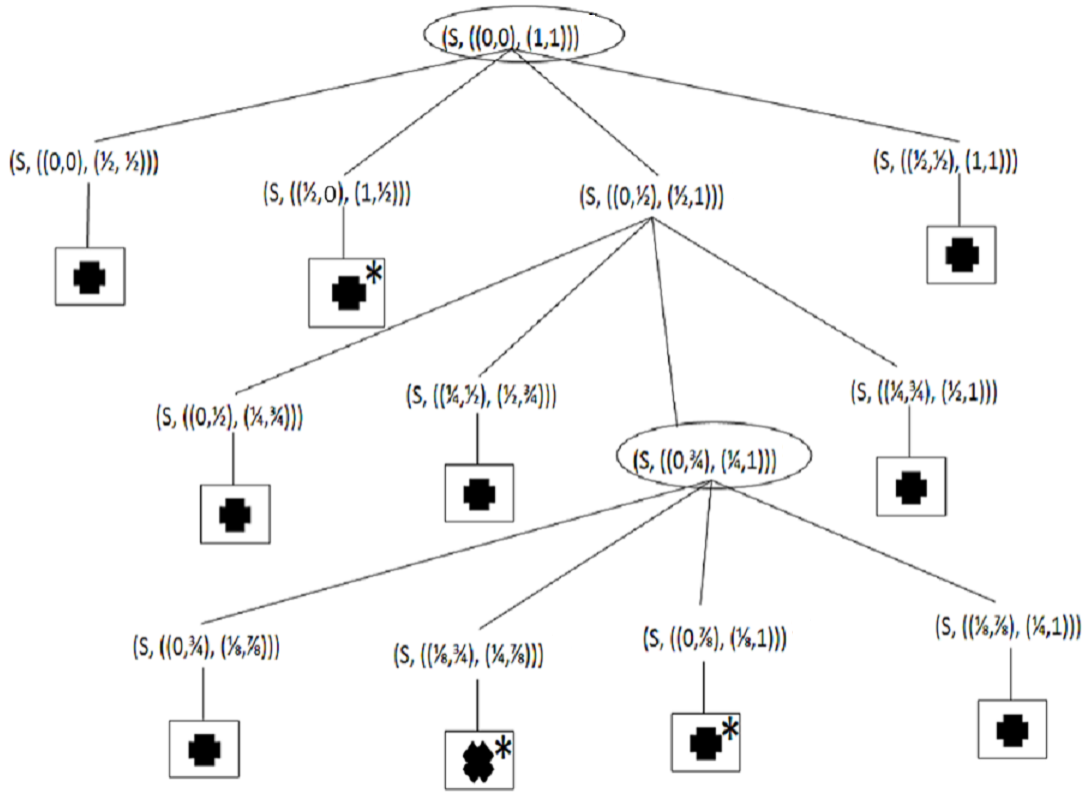


Figure 3.1: Derivation tree showing branch points (branch points are indicated by ovals and marked nodes by asterisks)

leaf nodes.

Induction Step: For $k \geq 1$, a derivation tree with no more than k branch points has no more than 4^k marked leaf nodes.

Proof of Induction Step: Let T be a derivation tree with no more than $k+1$ branch points. Then every proper subtree of T has no more than k branch points and thus has no more than 4^k marked leaf nodes, from the induction hypothesis. The number of marked leaf nodes in T is the sum of the numbers in each subtree and therefore no more than $4^k + 4^k + 4^k + 4^k = 4 \times 4^k = 4^{k+1}$. $\square$

We now prove Ogden's lemma for context-free galleries:

Theorem 10. *Suppose $\mathcal{G}$ is a CFPL. Then there is an integer m, such that if Φ is any picture in $\mathcal{G}$ with $|\Phi| \geq m$, and any m or more positions of Φ are designated as marked, then:*

1. *Φ contains nonempty subpictures (Ω, α) and (Ψ, β) with $\beta \overset{\sqsubset}{\neq} \alpha$,*
2. *the subpicture (Ω, α) contains no more than m marked positions,*
3. *$\Gamma = \Omega \backslash \Psi$ contains at least one marked position,*
4. *the subpicture (Ψ, β) contains at least one marked position,*
5. *the picture obtained from Φ by substituting $(\Psi \rightarrow \alpha)$ for (Ω, α) is in $\mathcal{G}$,*
6. *the picture obtained from Φ by substituting $(\Omega \rightarrow \beta)$ for (Ψ, β) is in $\mathcal{G}$, and*
7. *recursively carrying out the operation described in Condition 6 always results in a picture in $\mathcal{G}$.*

Proof. The first section of this proof in the first four paragraphs is similar to the proof of the pumping lemma for context-free galleries given in Ewert [1999, Theorem 3.1].

Let $G = (V_N, V_T, P, (S, \sigma))$ be a CFPG with no productions of the form $A \rightarrow x_{11}, x_{11} \in V_N$, that generates the gallery, $\mathcal{G}$; it is easily seen that such a CFPG must exist. Let V_N have h elements.

Let Φ be a picture in $\mathcal{G}$ with $|\Phi| \geq 4^{h+1}$. Consider any derivation tree for Φ. It has more than 4^h leaf nodes, and therefore, by Lemma 1, its height is greater than $h+1$. Then any derivation tree for Φ must have a path which contains at least $h+1$ nodes labelled with variables. Among these $h+1$ selected nodes, some variable, say A, must appear twice. Assume (A, α) and (A, β) are the nodes with (A, α) closer to the root. Let (Ω, α) and (Ψ, β) be the subpictures of Φ generated by (A, α) and (A, β) respectively. Since (A, α) and (A, β) are on the same path, (A, β) is derived from (A, α). Moreover, since (A, α) has exactly four descendants, β is properly contained in α, i.e., $\beta \overset{\sqsubset}{\neq} \alpha$. This shows that Condition 1 holds.

If we start at (A, α) and copy the derivation sequence that led from (A, β) to (Ψ, β), then (A, α) will generate the subpicture $(\Psi \rightarrow \alpha)$. Thus Condition 5 holds.

Otherwise, if we start at (A, β) and copy the derivation sequence that led from (A, α) to (Ω, α), then (A, β) will generate the subpicture $(\Omega \rightarrow \beta)$. Thus Condition 6 holds. Carrying this out repeatedly will always give a picture in $\mathcal{G}$, i.e., $\{(A, \alpha)\} \Rightarrow^* (\Omega, \alpha)$ can be copied

arbitrarily often, showing that Condition 7 holds. We still need to prove Conditions 2, 3 and 4.

Let Φ have more than 4^h marked positions. We choose a path containing the root node and a leaf node. Starting at the root node S, we repeatedly select an interior node N in the path and the child having the largest number of marked positions among its descendants. Lemma 2 implies that there must be more than h branch points in our path. Consider the $h+1$ branch points farthest down in the path, and the subtree whose root is the topmost such node. Since there are only h variables in the grammar, at least two of these branch points are labelled with the same variable, say A. Condition 3 must hold from the definition of a branch point, since Γ is derived from the first A which is a branch point. Since the tree has at most $h+1$ branch points, the tree has at most 4^{h+1} marked leaf nodes (Lemma 2). If we choose $m = 4^{h+1}$, Condition 2 holds. Condition 4 follows because the bottom A is a branch point. Thus the theorem holds. □

The operation in Condition 5 is referred to as *shrinking* because the resulting picture has fewer squares than the initial picture. Operations in Conditions 6 and 7 are called *pumping* operations since the resulting pictures are larger than the initial picture.

Corollary 1 is similar to that in Rabkin [2012], but adapted for the picture case.

Corollary 1. *Assume $\mathcal{G} = \{\Phi_1, \Phi_2, ...\}$ is a CFPL. Let $\Phi \in \mathcal{G}$, let $\Phi_{i_1}, \Phi_{i_2}, ...$, be the sequence of pictures pumped from Φ. Then:*

(a) the number of appearances of any terminal, say b, in $\Phi_{i_1}, \Phi_{i_2}, ...$ is a non-decreasing arithmetic progression.

(b) the number of marked positions in $\Phi_{i_1}, \Phi_{i_2}, ...$ form an increasing arithmetic progression.

Proof. In the replacement operation in Condition 6, the number of the terminals b increases by the difference between the number of b's in Φ_{i_1} and Φ_{i_2}. Also, the count of marked positions increases by the difference between the number of marked positions in Φ_{i_1} and Φ_{i_2}. □

We now give an example of how Theorem 10 is useful in proving that a language is not context-free, using Corollary 1a and 1b.

Consider the gallery $\mathcal{G}_{\mathsf{triangleandcrosses}} = \{\Phi_1, \Phi_2, ...\}$ where each picture consists of crosses and one hollow isosceles right-angled triangle which are dark on a light background. A picture has the following structure: The lower left and upper right quarters each contain one cross. The

upper left quarter contains a hollow isosceles right-angled triangle with side lengths 2^i, $i \geq 1$. The lower right quarter is divided into four and a cross is placed in each quarter. Each of these crosses can be divided into four repeatedly, independently from each other. Some pictures in this gallery are shown in Figure 3.2.

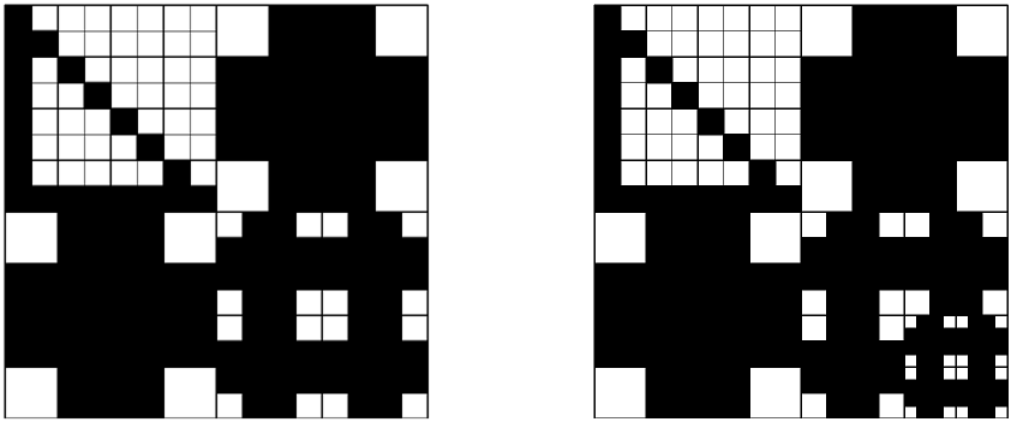


Figure 3.2: Pictures from the gallery $\mathcal{G}_{\mathsf{triangleandcrosses}}$

We now prove that $\mathcal{G}_{\mathsf{triangleandcrosses}}$ is not context-free.

Theorem 11. *$\mathcal{G}_{\mathsf{triangleandcrosses}}$ is not context-free.*

Proof. Suppose $\mathcal{G}_{\mathsf{triangleandcrosses}}$ is context-free. Let m be the integer of Theorem 10. Let $\Phi \in \mathcal{G}_{\mathsf{triangleandcrosses}}$ be such that its triangle has side length $2^i > m$, $i \in \mathbb{N}^+$. Then $|\Phi| \geq m$.

Mark any m of the dark squares that make up the sides of the triangle. Let (Ω, α) and (Ψ, β) be two subpictures of Φ with $\beta \sqsubseteq_{\neq} \alpha$. By Conditions 3 and 4, each of the subpictures must contain a marked position. If we substitute $(\Omega \to \beta)$ for (Ψ, β), then all the black squares that form the sides of the triangle do not have the same refinement anymore. Thus, the resulting picture is not in the gallery. This contradicts Theorem 10. Therefore $\mathcal{G}_{\mathsf{triangleandcrosses}}$ is not context-free. □

However, using the pumping property from the shrinking-pumping lemma for context-free picture languages, we cannot prove that this gallery is not context-free. The reason is as follows: in the bottom right quarter of any picture, we can find a small subpicture (one cross) which can be replaced by a large subpicture (four crosses) with the resulting picture still being in the gallery. An example is shown in Figure 3.2 with the result of this operation on the right-hand

side. Otherwise, if we decide to shrink the picture on the left-hand side instead and replace the large subpicture by the small subpicture, we derive a picture which is not in the gallery. This leads to a contradiction, it shows that the gallery is not a CFPL. Thus the new lemma does not improve on the shrinking-pumping lemma for CFPLs (Theorem 6). Instead, it can be used as an alternative to prove that a certain gallery is not a CFPL.

3.4 Conclusion

In this chapter, we have proven a generalisation of Ogden's lemma for context-free galleries and we were able to show that the gallery $\mathcal{G}_{\mathsf{triangleandcrosses}}$ is not context-free, using the newly developed theorem. It is also now clear that this new lemma does not necessarily improve on the old lemma since they both can show that the gallery in our example is not a context-free gallery. However, it may be possible to find a gallery which is not a CFPL but satisfies the old lemma and does not satisfy the new lemma; we leave this open for future work.

In the next chapter, we prove generalisations of Ogden's lemma for random permitting context galleries and random forbidding context galleries.

Chapter 4

Ogden's Lemma for Random Permitting and Forbidding Context Galleries

4.1 Introduction

This chapter provides the proof of Ogden's lemma for random permitting and random forbidding context galleries (RPCPLs and RFCPLs respectively). We first define terms relevant to this proof, give the necessary lemmas and then prove the theorem. Examples are then given to show the usefulness of the new theorem.
In the next section, we explain the concept of k-density.

4.1.1 Concept of k-density

The k-density concept ensures that there are no large unmarked subpictures. Every subpicture of a particular size has at least one marked position. This concept ensures that the number of marked symbols is dependent on the size of the pictorial form so that the marked symbols in the developing picture are not introduced at a later stage when they become irrelevant. k-density is hereditary in that if a pictorial form, say Φ_3, is k-densely marked and $\Phi_0 \Rightarrow^* \Phi_3$, then Φ_0 must also be k-densely marked. Marking also tends to increase in the developing picture in an

RPCPL, i.e., if $\Gamma \Rightarrow^* \Phi$, then Φ has more marked symbols than Γ, and vice-versa in an RFCPL. First we give some definitions.

4.2 Definitions

The definitions in this section are based on previous work done in Rabkin [2012], but adapted for the picture case.

Definition 39. (k-Densely marked picture) For $k \geq 0$, a marked picture Φ is k-densely marked if every subpicture Γ of Φ with size 4^k, contains at least one marked position.

Definition 40. ($\#_b^{\mathsf{m}}(\Phi)$) Let $G = (V_N, V_T, P, (S, \sigma))$ be an RCPG generating the gallery $\mathcal{G}$. Let Φ be a picture in $\mathcal{G}$. For a terminal, say b, if $\Phi \in \mathcal{G}$, then $\#_b^{\mathsf{m}}(\Phi)$ is the number of marked appearances of b in Φ. Let Φ be a pictorial form derived from (S, σ). For a variable, say A, $\#_A^{\mathsf{m}}(\Phi)$ is the number of marked appearances of A in Φ.

Definition 41. Let $G = (V_N, V_T, P, (S, \sigma))$ be an RCPG generating the gallery $\mathcal{G}$. Let Φ be a pictorial form and $V_N = \{X_1, X_2, ..., X_n\}$. Then $\mathsf{cnt}^{\mathsf{m}}(\Phi) = (\#_{X_1}^{\mathsf{m}}(\Phi), \#_{X_1}(\Phi) - \#_{X_1}^{\mathsf{m}}(\Phi), \#_{X_2}^{\mathsf{m}}(\Phi), \#_{X_2}(\Phi) - \#_{X_2}^{\mathsf{m}}(\Phi), ..., \#_{X_n}^{m}(\Phi), \#_{X_n}(\Phi) - \#_{X_n}^{\mathsf{m}}(\Phi))$, i.e., the vector of counts of marked and unmarked variables. Here, cnt means "count of non-terminals (variables)".

Definition 42. If v is a vector, we write $|v|$ to denote the sum of all the entries in v.

We now prove the following lemmas which are based on work done in Rabkin [2012, Lemma 4.7–4.11] and adapted for the picture case which is relevant for this research.

Lemma 3. *Let $G = (V_N, V_T, P, (S, \sigma))$ be an RCPG generating the gallery $\mathcal{G}$. If $\{(S, \sigma)\} = \Phi_1 \Rightarrow \Phi_2 \Rightarrow ... \Rightarrow \Phi_n \in \mathcal{G}$, and Φ_n is k-densely marked, then so also is Φ_i for all $i \in [n]$.*

Proof. Let Γ be a subpicture of Φ_i such that $|\Gamma| \geq 4^k$ and it contains no marked symbols. If Γ appears in Φ_{i+1} exactly as it is without getting changed, then Φ_{i+1} is not k-densely marked. On the other hand, if an unmarked symbol in Γ is rewritten, Φ_{i+1} contains an unmarked subpicture of size at least $|\Gamma|$ since an unmarked symbol can only also derive unmarked symbols; so it is not k-densely marked. Thus, by downward induction, if Φ_n is k-densely marked, so are Φ_i for all $i \in [n]$. □

Lemma 4. *Let $G = (V_N, V_T, P, (S, \sigma))$ be an RCPG with $\{(S, \sigma)\} \Rightarrow^n \Phi$, $n \geq 0$, Φ a pictorial form. Then, $|\Phi| \leq 3n + 1$.*

Proof. Using induction on n; if $n = 0$, then $\Phi = \{(S, \sigma)\}$, so the statement follows. Assume it is true for n, and $\{(S, \sigma)\} \Rightarrow^n \Phi' \Rightarrow \Phi$. Then it means Φ' generates Φ by rewriting a single variable, thus $|\Phi| \leq |\Phi'| - 1 + 4$ (since the maximum length of the right hand side of a rule in P is 4, as stated in Definition 14). Thus, $|\Phi| \leq (3n + 1) + 3 = 3n + 3 + 1 = 3(n + 1) + 1$ and thus it is true for all $n \in \mathbb{N}^+$. □

Lemma 5. *Let $G = (V_N, V_T, P, (S, \sigma))$ be an RCPG generating the gallery $\mathcal{G}$ with $\{(S, \sigma)\} = \Phi_1 \Rightarrow \Phi_2 \Rightarrow ... \Rightarrow \Phi_n$, where the Φ_i's are pictorial forms, such that at most n' branch variables are rewritten in the derivation. Then $\#^m(\Phi_n) \leq 3n' + 1$.*

Proof. Using induction on n', if $n' = 0$, then $\#^{\mathsf{m}}(\Phi_n) \leq 1$. Assume it is true for n' and if a single branch symbol in Φ_{n-1} is rewritten to generate Φ_n, then $\#^{\mathsf{m}}(\Phi_n) \leq \#^{\mathsf{m}}(\Phi_{n-1}) - 1 + 4$. Thus $\#^{\mathsf{m}}(\Phi_n) \leq (3n' + 1) + 3 = 3n' + 3 + 1 = 3(n' + 1) + 1$ and thus it is true for all $n' \in \mathbb{N}^+$. □

Lemma 6. *Let $G = (V_N, V_T, P, (S, \sigma))$ be an RCPG generating the gallery $\mathcal{G}$ with $\{(S, \sigma)\} = \Phi_1 \Rightarrow \Phi_2 \Rightarrow ... \Rightarrow \Phi_n \in \mathcal{G}$, where Φ_is are pictorial forms, with Φ_n k-densely marked, such that at most n' branch variables are rewritten in the derivation. Then $|\Phi_{n'}| < 4^k(3n' + 2)$.*

Proof. A k-densely marked picture with m marked positions has size less than $4^k(m + 1)$; this lemma then follows from Lemma 5. □

Lemma 7. *Suppose $p_1, p_2, ... \in \mathbb{N}$, and $n, t \in \mathbb{N}^+$. Then there exists a number $b = b(t) \in \mathbb{N}$ such that if $v_1, v_2, ...$ is a sequence of n-vectors of non-negative integers satisfying $|v_i| \leq p_i$, $i \in \mathbb{N}^+$, then there exist t indices $i_1 < i_2 < ... < i_t \in [b]$ such that $v_{i_1} \leq v_{i_2} \leq ... \leq v_{i_t}$.*

Proof. Proof given in Ewert and van der Walt [1998, Lemma 3.1]. □

We now prove Ogden's lemma for random permitting context galleries.

4.3 Ogden's Lemma for Random Permitting Context Galleries

Theorem 12. *For any RPCPL $\mathcal{G}$ and $k \in \mathbb{N}^+$, there is an $m \in \mathbb{N}^+$ (the pumping threshold) such that for any k-densely marked picture $\Phi \in \mathcal{G}$ with $|\Phi| > m$, there is a number $l \in [m]$ such that:*

1. *Φ contains l mutually disjoint non-empty subpictures $(\Omega_1, \alpha_1), (\Omega_2, \alpha_2), ..., (\Omega_l, \alpha_l)$ and l mutually disjoint non-empty subpictures $(\Psi_1, \beta_1), (\Psi_2, \beta_2), ..., (\Psi_l, \beta_l)$, such that for each $i \in [l]$, there exists a $j \in [l]$ such that $\beta_i \sqsubseteq \neq \alpha_j$;*
2. *the picture obtained from Φ by subsituting $(\Omega_i \rightarrow \beta_i)$ for (Ψ_i, β_i) for all $i, i \in [l]$, is in $\mathcal{G}$;*
3. *repeatedly carrying out the operation in (2) will always yield a picture in $\mathcal{G}$;*
4. *if (Ψ_i, β_i) contains a marked symbol, so does (Ω_i, α_i);*
5. *there are strictly more marked symbols in $\bigcup_{i=1}^{l}(\Omega_i, \alpha_i)$ than in $\bigcup_{i=1}^{l}(\Psi_i, \beta_i)$.*

Proof. Let $G = (V_N, V_T, P, (S, \sigma))$ be an RPCPG and $\mathcal{G}$ be the gallery generated by G. Let $\{(S, \sigma)\} = \Phi_1 \Rightarrow \Phi_2 \Rightarrow ... \Rightarrow \Phi_n = \Phi$ be a derivation of Φ with n' branch points.

Let $i_1, i_2, ..., i_{n'}$ be the indices of the pictorial forms where a branch symbol is rewritten. By Lemma 6, $|\Phi_{i_j}| < 4^k(3(j-1)+2)$, where $j \in [n']$. Let $b = b(2)$ be the integer of Lemma 7 for the sequence $4^k(3(j-1)+2)$ for all j; and it is dependent only on G and k. If $n^{'} \geq b$, there are g and h such that $g, h \in [b]$ with $g < h$ and $\mathsf{cnt}^{\mathsf{m}}(\Phi_{i_g}) \leq \mathsf{cnt}^{\mathsf{m}}(\Phi_{i_h})$. Let us denote Φ_{i_g} by Π and Φ_{i_h} by Γ. Since a branch variable is rewritten in Π, there are strictly more marked symbols in Γ than in Π.

Let $l = |\mathsf{cnt}^{\mathsf{m}}(\Pi)|$. We can repeat the derivation $\Pi \Rightarrow^* \Gamma$ arbitrarily many times since $\mathsf{cnt}^{\mathsf{m}}(\Pi) \leq \mathsf{cnt}^{\mathsf{m}}(\Gamma)$ and any needed context is available. Align the l variables in Π with variables in Γ having a function $\delta : \{1, ..., l\} \rightarrow \{1, ..., l\}$ such that each instance of a symbol is associated with another instance of the same symbol, and marked symbols are associated with other marked symbols. Let $\Pi_1, \Pi_2, ..., \Pi_l$ be the contribution of the variables in Π and $\Gamma_1, \Gamma_2, ..., \Gamma_l$, the contributions of the variables in Γ. If the derivation $\Pi \Rightarrow^* \Gamma$ is repeated, mapping the productions applied to symbols in Π to the corresponding symbols in Γ, the result will

be that every Γ_i will replace Π_i for all $i \in [l]$.

By setting m to $4^k(3b+2)$, we ensure $n' > b$ (by Lemma 6) and since $l = |\mathsf{cnt}^{\mathsf{m}}(\Pi)| \leq |\Pi| \leq m$, we obtain the theorem. □

A special case of the above theorem where every picture is 0-densely marked will give the same result as the pumping lemma for RPCPLs (Theorem 7).

Using Theorem 12, we can show that the gallery in Section 3.3, $\mathcal{G}_{\mathsf{triangleandcrosses}}$, cannot be generated using permitting context only.

Theorem 13. *The gallery $\mathcal{G}_{\mathsf{triangleandcrosses}}$ is not a random permitting context gallery.*

Proof. Suppose $\mathcal{G}_{\mathsf{triangleandcrosses}}$ is a random permitting context gallery. Let l and m be the integer of Theorem 12. Let $\Phi \in \mathcal{G}_{\mathsf{triangleandcrosses}}$ be such that its triangle has side length $2^i \geq m$, $i \in \mathbb{N}^+$. Then $|\Phi| \geq m$.

Mark any m positions with a dark background that make up the sides of the triangles. Let $(\Omega_1, \alpha_1), (\Omega_2, \alpha_2), ..., (\Omega_l, \alpha_l)$ and $(\Psi_1, \beta_1), (\Psi_2, \beta_2), ..., (\Psi_l, \beta_l)$ be the large and small subpictures respectively such that for each $i \in [l]$, there exists a $j \in [l]$ such that $\beta_i \sqsubseteq \neq \alpha_j$. By Condition 5, there are more marked symbols in the union of the larger subpictures (Ω_i, α_i) than in the union of the smaller subpictures (Ψ_i, β_i). If we substitute $(\Omega_i \to \beta_i)$ for (Ψ_i, β_i), then all the black squares that form the sides of the triangle do not have the same refinement anymore. Thus the resulting picture is not in the gallery. This leads to a contradiction of Theorem 12. Therefore $\mathcal{G}_{\mathsf{triangleandcrosses}}$ is not a random permitting context gallery. □

However, using the pumping lemma for RPCPLs (Theorem 7), we cannot prove that this gallery is not context-free. It is possible to find l smaller subpictures which can be replaced by l larger subpictures with the resulting picture still being in the gallery. This is because, in the bottom right quarter of any picture, we can find a small subpicture which can be replaced by a large subpicture with the resulting picture still being in the gallery. Examples are shown in Figure 3.2. We can thus see that Theorem 12 improves on the pumping lemma for RPCPLs.

Note that if all positions are chosen as marked in this example, using the new lemma, it will yield the same result as in the pumping lemma for RPCPLs. Therefore, the ability of Theorem 12 to focus on marked symbols is necessary in this example.

In the next section, we prove Ogden's lemma for random forbidding context galleries and use the theorem to show that a particular gallery is not random forbidding.

4.4 Ogden's Lemma for Random Forbidding Context Galleries

We now prove the corresponding result to Theorem 12 for RFCPLs.

Theorem 14. *For any RFCPL $\mathcal{G}$ and $k, t \in \mathbb{N}^+$, there is an $m \in \mathbb{N}^+$ (the shrinking threshold) such that for any k-densely marked picture $\Phi \in \mathcal{G}$ with $|\Phi| > m$, there are t pictures $\Phi^{(1)}, \Phi^{(2)}, ..., \Phi^{(t)} = \Phi$ such that for every $j \in \{2, 3, ..., t\}$:*

1. *there exists a number $l \in [m]$;*
2. *$\Phi^{(j)}$ contains l mutually disjoint non-empty subpictures $(\Omega_1, \alpha_1), (\Omega_2, \alpha_2), ..., (\Omega_l, \alpha_l)$ and l mutually disjoint non-empty subpictures $(\Psi_1, \beta_1), (\Psi_2, \beta_2), ..., (\Psi_l, \beta_l)$, such that for each $i \in [l]$, there exists a $p \in [l]$ such that $\beta_i \sqsubseteq \alpha_p$;*
3. *the picture $\Phi^{(j-1)}$ obtained by substituting $(\Psi_i \to \alpha_i)$ for (Ω_i, α_i) for all $i, i \in [l]$, is in $\mathcal{G}$;*
4. *if (Ψ_i, β_i) contains a marked position, then so does (Ω_i, α_i);*
5. *there are strictly more marked positions in $\bigcup_{i=1}^{l}(\Omega_i, \alpha_i)$ than in $\bigcup_{i=1}^{l}(\Psi_i, \beta_i)$.*

Proof. Let $G = (V_N, V_T, P, (S, \sigma))$ be an RFCPG generating $\mathcal{G}$. Let $\{(S, \sigma)\} = \Phi_1 \Rightarrow^* \Phi_2 \Rightarrow^* ... \Rightarrow^* \Phi_{n'} = \Phi$ be a derivation in G with n' branch points, where $\Phi_1, \Phi_2, ..., \Phi_{n'}$ are the pictorial forms in which a branch symbol is rewritten.

By Lemma 6, $|\Phi_j| < 4^k(3(j-1)+2)$, where $j \in [n']$. Let $b = b(t)$ be the number in Lemma 7 for the sequence $4^k(3(j-1)+2)$ for all j, and it is dependent only on G, k and t. If $n' \geq b$, then there exists $r_1 < r_2 < ... < r_t \in [b]$ such that $\mathsf{cnt}^{\mathsf{m}}(\Phi_{r_1}) \leq \mathsf{cnt}^{\mathsf{m}}(\Phi_{r_2}) \leq ... \leq \mathsf{cnt}^{\mathsf{m}}(\Phi_{r_t})$.

Since this is the forbidding case where less context permits the use of more productions, we use the opposite procedure to that used in the proof of Theorem 12. To obtain $\Phi^{(j-1)}$ from $\Phi^{(j)}$,

let $l = |\mathsf{cnt}^{\mathsf{m}}(\Phi_{r_{(j-1)}})|$, let $(\Omega_1, \alpha_1), (\Omega_2, \alpha_2), ..., (\Omega_l, \alpha_l)$ be the contributions of the corresponding variables in $\Phi_{r_{(j)}}$ and $(\Psi_1, \beta_1), (\Psi_2, \beta_2), ..., (\Psi_l, \beta_l)$ be the contributions of the corresponding variables in $\Phi_{r_{(j-1)}}$. The replacement operation then results from applying the same rules to the variables in $\Phi_{r_{(j-1)}}$ the same way they were applied to $\Phi_{r_{(j)}}$.

By setting m to $4^k(3b+2)$, we ensure $n' > b$ (by Lemma 6) and since $l = |\mathsf{cnt}^{\mathsf{m}}(\Phi_{r_{(j-1)}})| \leq (\Phi_{r_{(j-1)}}) \leq m$, we obtain the theorem.

□

We now give an example of how Theorem 14 can be used to prove that a language is not random forbidding.

Consider the gallery $\mathcal{G}_{\mathsf{triangles}} = \{\Phi_1, \Phi_2, ...\}$ where each picture in the gallery consists of 4^i, $i \geq 1$, isosceles right-angle triangles and each triangle has side length 2^k, $k \geq 1$. The triangles $(\Gamma_1, \Gamma_2, ..., \Gamma_l)$, $l \in [4^i]$, in a picture have side lengths $(2^k, 2^{k+1}, ...)$. Pictures in this gallery are shown in Figure 4.1.

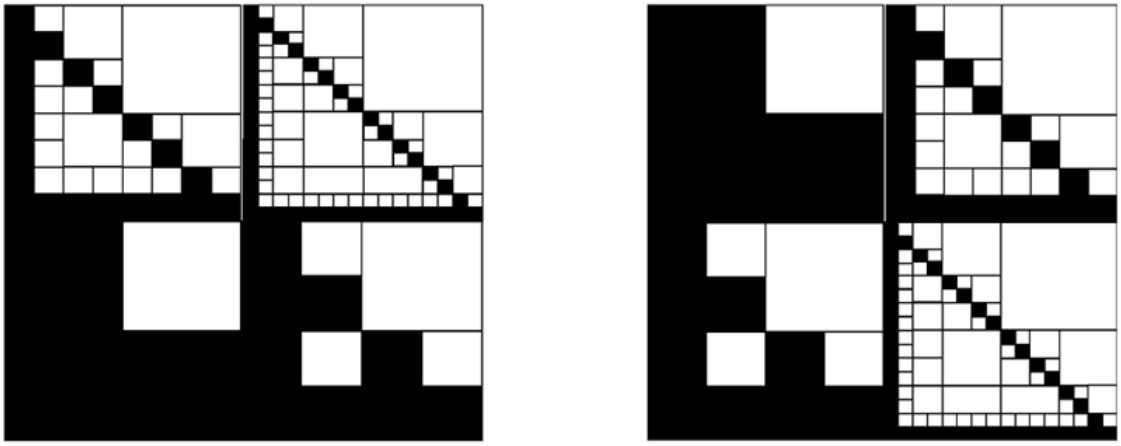


Figure 4.1: Pictures from the gallery $\mathcal{G}_{\mathsf{triangles}}$

We now prove that this gallery is not a random forbidding context gallery.

Theorem 15. *$\mathcal{G}_{\mathsf{triangles}}$ is not a random forbidding context gallery.*

Proof. Suppose $\mathcal{G}_{\mathsf{triangles}}$ is a random forbidding context gallery. Let k be the integer of the lemma, we choose $k = 4$. Let t be the integer of the lemma, we choose $t = 2$. Let m be the

integer of the lemma. Let $\Phi \in \mathcal{G}_{\mathsf{triangles}}$ be such that Φ has at least $\min(m, 4)$ triangles. Moreover, let Φ be such that its four smallest triangles are $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 with side lengths $2^3, 2^4, 2^5$ and 2^6 respectively.

We choose to mark all black squares. Because of Condition 5, the operation in Condition 3 cannot be performed on white squares only, thus we cannot just swop one white square with another. The operation of Condition 3 has to involve the sides of the triangles. The operation can only be performed in l places where $l \in [m]$. In order to shrink the smallest triangle, i.e., the triangle with side length 2^3, we need to shrink at least 9 places simultaneously (as shown in Figure 4.2, therefore we cannot change all m triangles in the picture such that the new picture is in the gallery again. Thus $\mathcal{G}_{\mathsf{triangles}}$ is not an RFCPL. □

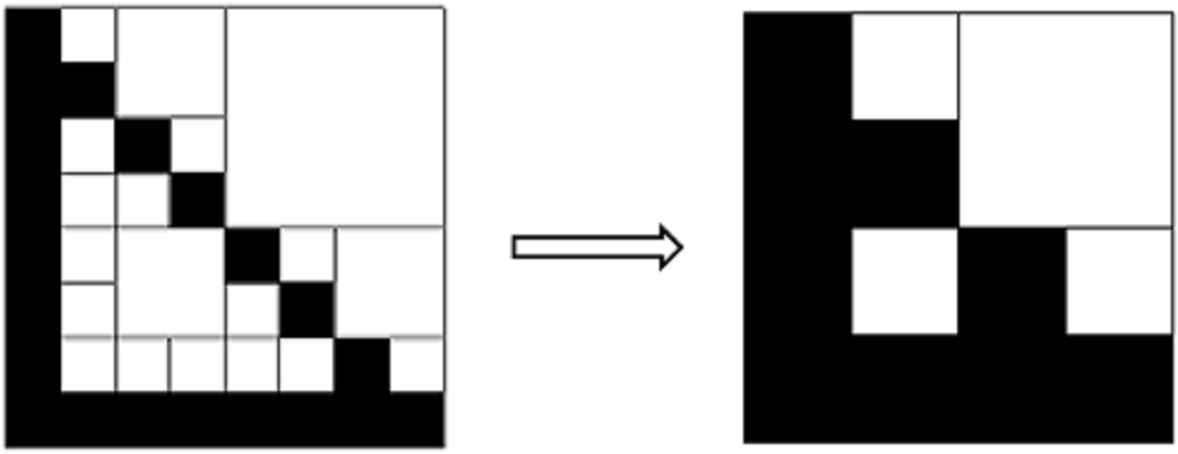


Figure 4.2: The shrinking operation performed on Γ_1

Note that if all positions are chosen as marked in this example, it will give the same result as the shrinking lemma for RFCPLs. Therefore, the ability of Theorem 14 to focus on marked symbols helped to only shrink the sides of the triangles and not the white squares. It is possible that the marking gave us a more elegant proof than without the marking. We can then say this new lemma is an alternative and not necessarily an improvement on the shrinking lemma for RFCPLs.

4.5 Conclusion

An analogue of Ogden's lemma for the class of random permitting and random forbidding picture languages was developed. The usefulness of the new lemmas was demonstrated by using them to prove that certain galleries do not belong to the specific class. In the case of the random permitting context, we were able to improve on the pumping lemma for random permitting context picture languages with the new lemma. For the example used, the new lemma does not improve on the shrinking lemma for RFCPLs (Theorem 8) since they both show that the gallery in our example is not an RFCPL. In the time available for the dissertation, we could not find an example that shows the improvement of the new lemma on the old one, therefore we leave it as an open question.

In the next chapter, we discuss the TCFPGs and provide Ogden's lemma for this class of picture languages.

Chapter 5

Table-Driven Context-free Picture Grammars

5.1 Introduction

In this chapter, table-driven context-free picture grammars are introduced. In this class of grammars, the productions of the grammar are applied in parallel to all the variables in the pictorial form, and the choice of production is table driven. That is, at each step, a table of productions is chosen from a fixed set and the production applied to each variable is obtained from the table.

There are other models of the TCFPGs. These include the table-driven 0-context Lindenmayer (T0L) collage grammars [Klempien-Hinrichs *et al.*, 1999], and the Extended T0L collage grammars [Drewes *et al.*, 2003]. Collage grammars can generate pictures which consist of geometric objects that overlap. TCFPGs are a restricted form of ET0L collage grammars because their productions cannot generate pictures with squares that overlap.

A normal form for TCFPGs was given in Bhika *et al.* [2007]. It states that TCFPGs can be written in such a way that the right hand side of each production consists of either terminals only or nonterminals only. A lemma was given to show this in Bhika *et al.* [2007, Lemma 3.1].Two characterisation theorems were developed which need to be satisfied by a gallery for it to be generated by a TCFPG. They are discussed in Chapter 2. It was also shown in Bhika *et al.* [2007,

Lemma 6.1] that every gallery that can be generated by a TCFPG, can also be generated by an RFCPG, and thus TCFPGs are strictly weaker than RFCPGs. A gallery [Bhika *et al.*, 2007, Lemma 6.2] was given to show the power of RFCPGs over TCFPGs. Therefore it can be concluded that the shrinking lemma for RFCPLs (Theorem 8) holds for all galleries generated by any TCFPG.

Having discussed the TCFPGs, in the next section the proof of Ogden's lemma for table-driven context-free galleries is given.

5.2 Ogden's Lemma for Table-Driven Context-Free Galleries

In this section a version of Ogden's lemma for table-driven context-free galleries is provided. We first define some terms that are relevant to the proof, give the necessary lemmas and prove the theorem. We provide a pumping lemma from the new theorem as a corollary and give an example to show the usefulness of the new theorem and then use this new theorem to prove Bhika *et al.* [2007]'s theorem about rare and non-frequent symbols.

We now give some definitions adapted from Rabkin [2012] for the picture case.

Definition 43. (Level, Root) A level refers to the set of nodes at the same depth which corresponds to one of the pictorial forms in a derivation.
The root is called the first level, its children the second level, and so on.

Definition 44. (Out-degree) The maximum out-degree of a tree is the maximum number of children of any node in the tree.
For the purpose of this research, the maximum out-degree is four, since the maximum length of the right hand side of a rule is 4, as stated in Definition 14.

Definition 45. (Marked Leaf) A leaf is marked if it corresponds to a marked symbol in the picture.
A non-leaf is marked if any of its child nodes are marked.

5.2.1 Results

We now prove our main result.

The following lemma, which is similar to Lemma 1, is obtained from Rabkin [2012] and adapted for the picture case.

Lemma 8. *For any $h \in \mathbb{N}$, if a tree with maximum out-degree of four has more than 4^h marked leaves, it must have a path from the root to a leaf with more than h branch nodes.*

Proof. We prove this by structural induction on the tree.

A derivation tree consisting of a single leaf has at most $1 = 4^0$ marked leaf, thus no branch nodes, which satisfies the lemma.

Let $\mathbb{T}$ be a (non-leaf) derivation tree with more than 4^h marked leaves, for some $h > 0$. Assume its children, $\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3, \mathbb{T}_4$, satisfy the lemma. If, of all the children, only one $\mathbb{T}_i$ has a marked symbol in its contribution, then $\mathbb{T}$ contains the same number of marked leaves and a path with the same number of branch nodes as $\mathbb{T}_i$. Therefore, $\mathbb{T}$ satisfies the lemma.

Otherwise, the root of $\mathbb{T}$ is a branch node and this means that one of the $\mathbb{T}_i$ must have more than $4^h/4 \geq 4^{h-1}$ marked leaves. Therefore, that $\mathbb{T}_i$ has a path with more than $(h-1)$ branch nodes and because the root of $\mathbb{T}$ is also a branch node, $\mathbb{T}$ must have a path with more than h branch nodes. □

We now prove Ogden's lemma for TCFPLs as adapted from Rabkin [2012].

Theorem 16. *If $\mathcal{G}$ is a TCFPL, then there exists an $l \in \mathbb{N}$ (which we call the threshold for $\mathcal{G}$) such that for any picture $\Phi \in \mathcal{G}$ with at least l marked positions,*

1. *Φ is composed of n subpictures, $(\Omega_1, \alpha_1), (\Omega_2, \alpha_2), ..., (\Omega_n, \alpha_n)$ and each (Ω_i, α_i) is composed of n_i subpictures, $(\Psi_{(i,1)}, \beta_{(i,1)}), (\Psi_{(i,2)}, \beta_{(i,2)}), ..., (\Psi_{(i,n_i)}, \beta_{(i,n_i)})$ (we will denote the set of subscripts of (Ψ, β), i.e., $\{(i,j) : i \in [n], j \in [n_i]\}$, by I);*
2. *there is a map $\phi : I \to [n]$ such that the picture obtained by substituting $(\Omega_{\phi(i,j)} \to \beta_{(i,j)})$ for $(\Psi_{(i,j)}, \beta_{(i,j)})$ is in $\mathcal{G}$;*
3. *recursively carrying out the operation described in (2) always results in a picture in $\mathcal{G}$;*
4. *if $(\Psi_{(i,j)}, \beta_{(i,j)})$ contains a marked position, then so does $(\Omega_{\phi(i,j)}, \alpha_{\phi(i,j)})$;*

5. *there is an* $(i,j) \in I$ *such that* $\phi(i,j) = i$*, and there are at least two marked positions in* $(\Psi_{(i,j)}, \beta_{(i,j)})$ *and at least one in* (Ω_i, α_i)*, but outside of* $(\Psi_{(i,j)}, \beta_{(i,j)})$.

Proof. Let $G = (V_N, V_T, \mathcal{T}, (S, \sigma))$ be a TCFPG that generates $\mathcal{G}$. Let $V = V_N \cup V_T$. Let $h = |V|4^{|V|}$. Let $\Phi \in \mathcal{G}$ with at least $4^h + 1$ marked positions.

Consider a derivation tree of Φ, say $\mathbb{T}$. Since $\mathbb{T}$ has more than 4^h marked leaves, then it must have a path with more than h branch nodes (Lemma 8).

On this path, there must be at least one symbol, $A \in V$, which appears more than $h/|V| = 4^{|V|}$ times as a label of a branch node on this path. For each of the appearances of the symbol A, there are two sets of symbols where one is the set of symbols which appear on the same level as A and the other is the set of marked symbols which appear on the same level. Given that there exist only $4^{|V|}$ distinct such pairs of sets (the set of all the subsets of V for the two sets), there must be two branch nodes having the label A, one being the parent of the other, with the same pair of sets. Let A_1 and A_2 be the parent and child node respectively and let Φ_1 and Φ_2 the pictorial forms in which they appear respectively.

The sequence of tables of productions which were applied to Φ_1 to derive Φ can also be applied to Φ_2. There may be several possible results from this as a consequence of non-determinism, but one result can be obtained by replacing each subtree corresponding to an instance of a symbol in Φ_2 with a subtree corresponding to an instance of the same symbol in Φ_1. In order to achieve the result we need, it is necessary to choose A_1 when an instance of A is sought, and if there is a marked instance of a symbol, we must choose it. Carrying this replacement any number of times will still give the desired result. Thus Conditions 2, 3 and 4 are given.

Condition 5 follows from the fact that A_2 is replaced with A_1 and both are branch nodes. By setting l to $4^h + 1$, we obtain the theorem.

□

The pumping operation in Condition 2 is described precisely below, and was taken from Rabkin [2012] and adapted for the picture case. The pumping operation generates the picture

$\Phi^{(t)}$ (the result of applying the pumping operation t times) for all $t \in \mathbb{N}$, where:

$$(\Psi_{(i,j)}, \beta_{(i,j)})^{(0)} = (\Psi_{(i,j)}, \beta_{(i,j)}) \tag{5.1}$$

$$(\Omega_i, \alpha_i)^{(t)} = (\Psi_{(i,1)}, \beta_{(i,1)})^{(t)}, (\Psi_{(i,2)}, \beta_{(i,2)})^{(t)}, ..., (\Psi_{(i,n_i)}, \beta_{(i,n_i)})^{(t)} \tag{5.2}$$

$$(\Psi_{(i,j)}, \beta_{(i,j)})^{(t+1)} = (\Omega_{\phi(i,j)}, \alpha_{\phi(i,j)})^{(t)} \tag{5.3}$$

$$\Phi^{(t)} = (\Omega_1, \alpha_1)^{(t)} \cup (\Omega_2, \alpha_2)^{(t)} \cup ... \cup (\Omega_n, \alpha_n)^{(t)} \tag{5.4}$$

Note that $\Phi^{(0)} = \Phi$.

Theorem 16 is now used to prove that a gallery is not a TCFPL. For this purpose, the gallery $\mathcal{G}_{\mathsf{triangles}}$ will be used. Examples of pictures in this gallery are shown in Figure 4.1.

Theorem 17. *The gallery $\mathcal{G}_{\mathsf{triangles}}$ is not a TCFPL.*

Proof. Suppose $\mathcal{G}_{\mathsf{triangles}}$ is a TCFPL. Let l be the pumping threshold for $\mathcal{G}_{\mathsf{triangles}}$. Let Φ be a picture in the gallery with at least l dark squares.

We choose to mark all dark squares in the picture. Following from Condition 5 of Theorem 16, it must be possible to find a smaller subpicture with two marked positions (black squares), within a bigger subpicture, containing at least one marked position outside of the smaller subpicture, such that substituting the larger subpicture for the smaller subpicture will yield a triangle with the proper refinement on all sides. However, this operation will yield a picture that is no longer a proper hollow-isosceles right-angle triangle. Therefore the resulting picture is not in the gallery. This contradicts Theorem 16, thus $\mathcal{G}_{\mathsf{triangles}}$ is not a TCFPL. □

We will now prove two facts adapted from Rabkin [2012] about the pumping operation which will help when using Theorem 16 to prove that a certain gallery is not a TCFPL.

As seen in Theorem 12, the pumping operation increases the number of marked symbols in the case of RPCPLs, but in the case of Theorem 16 the pumping operation can initially reduce the number of marked symbols by replacing subpictures containing many marked symbols by subpictures containing a few. Eventually, the number of marked symbols will have to increase and this will be shown in Corollary 2:

Corollary 2. *If $\mathcal{G}$ is a TCFPL with threshold l, and $\Phi \in \mathcal{G}$ has at least l marked symbols, then the number of marked symbols in $\Phi^{(t)}$ tends to infinity. Specifically, $\Phi^{(t)}$ contains at least $t + 2$ marked symbols for all $t \in \mathbb{N}$.*

Proof. Let $(\Omega_i, \alpha_i)^{(t)}$ and $(\Psi_{(i,j)}, \beta_{(i,j)})^{(t)}$ be as defined previously (5.1–5.4), and i and j as in Condition 5 of Theorem 16.

By Condition 5 of Theorem 16, $(\Psi_{(i,j)}, \beta_{(i,j)})^{(0)}$ contains at least two marked symbols.

Assume $(\Psi_{(i,j)}, \beta_{(i,j)})^{(t)}$ contains at least $t+2$ marked symbols. Then $(\Psi_{(i,j)}, \beta_{(i,j)})^{(t+1)} = (\Omega_i, \alpha_i)^{(t)}$. Since $(\Psi_{(i,j)}, \beta_{(i,j)})^{(t)}$ is a subpicture of $(\Omega_i, \alpha_i)^{(t)}$, then $(\Psi_{(i,j)}, \beta_{(i,j)})^{(t)}$ contributes $t+2$ marked symbols to $(\Omega_i, \alpha_i)^{(t)}$. However, from Theorem 16 (Ω_i, α_i) contains a marked symbol which is not inside of $(\Psi_{(i,j)}, \beta_{(i,j)})$ and since marked subpictures are only replaced with other marked subpictures, this property is inherited: $(\Omega_i, \alpha_i)^{(t)}$ has a marked symbol outside of $(\Psi_{(i,j)}, \beta_{(i,j)})^{(t)}$. This other marked symbol then adds to the $t+2$ marked symbols in $(\Psi_{(i,j)}, \beta_{(i,j)})^{(t)}$, giving a total of $t+3$ marked symbols.

By induction, it is clear that $\Phi^{(t)}$ contains at least $t+2$ marked symbols for all $t \in \mathbb{N}$. □

On the other hand, the number of symbols cannot grow superexponentially by the pumping operation.

Corollary 3. *If $\mathcal{G}$ is a TCFPL with threshold l, and $\Phi \in \mathcal{G}$ has size at least l, then $|\Phi^{(t)}| \leq |I|^t \times |\Phi|$, where $\Phi^{(t)}$ is the result of applying the pumping operation t times and I is the set defined in Condition 1 of Theorem 16.*

Proof. Let $(\Omega_i, \alpha_i)^{(t)}$ and $(\Psi_{(i,j)}, \beta_{(i,j)})^{(t)}$ be as defined previously (5.1–5.4).
For all $(i,j) \in I$ and all t, we have $|(\Psi_{(i,j)}, \beta_{(i,j)})^{(t+1)}| = |(\Omega_{\phi(i,j)}, \alpha_{\phi(i,j)})^{(t)}| \leq |\Phi^{(t)}|$.
Since $|\Phi^{(t+1)}| = \sum_{(i,j)\in I} |(\Psi_{(i,j)}, \beta_{(i,j)})^{(t+1)}|$, we get $|\Phi^{(t+1)}| \leq |I| \times |\Phi^{(t)}|$.

□

We now show that Theorem 16, combined with Corollaries 2 and 3, is useful in proving that a certain gallery $\mathcal{G}_{\mathsf{FalseTCFPG}}$, which is similar to the string example in Rabkin [2012, Example 6.9], is not a TCFPL.

Consider the gallery $\mathcal{G}_{\mathsf{FalseTCFPG}} = \{\Phi_{m,n} \mid m < n, m > 0\}$. Let the terminals b, g and w represent the squares with the colors black, grey and white respectively. Then $\Phi_{m,n}$ is such that the terminals on its diagonal, read from bottom left to top right, form the string $b^m g^{n^k}$, where $k = 2^m$, while the rest of the picture is white. Pictures from this gallery are shown in Figures 5.1 and 5.2.

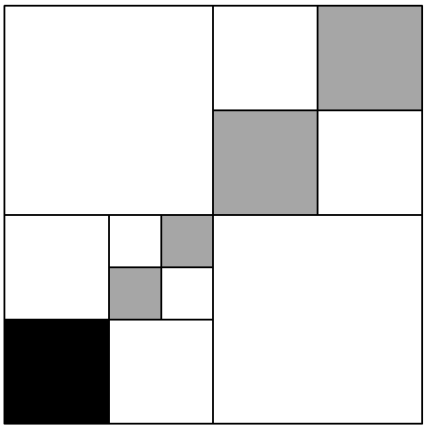


Figure 5.1: The picture $\Phi_{1,2}$ from the gallery $\mathcal{G}_{\mathsf{FalseTCFPG}}$

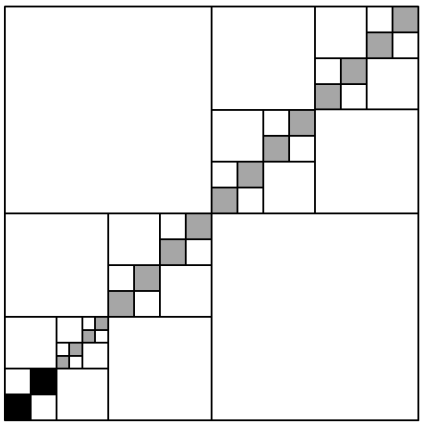


Figure 5.2: The picture $\Phi_{2,2}$ from the gallery $\mathcal{G}_{\mathsf{FalseTCFPG}}$

Theorem 18. *$\mathcal{G}_{\mathsf{FalseTCFPG}}$ cannot be generated by any TCFPG.*

Proof. Suppose $\mathcal{G}_{\mathsf{FalseTCFPG}}$ is a TCFPL. Let l be the pumping threshold for $\mathcal{G}_{\mathsf{FalseTCFPG}}$ and let Φ be a picture $b^l g^{n^k}$, $k = 2^l$, on its diagonal with all the black squares marked.

By Corollary 2, $\#_b(\Phi^{(t)}) \geq t + 2$. From Corollary 3, $\#_g(\Phi^{(t)}) \leq |\Phi^{(t)}| \leq |I|^t|\Phi|$. It is known that any constant, raised to the power of an exponential function (i.e., $g^{n^{2^l}}$) will eventually get larger than one with a single exponential function, and thus $|I|^t|\Phi| < |g^{n^{2^{t+2}}}|$ for a large enough t. It is easily seen that for a large enough t, Φ^t is not in the gallery and thus contradicts Theorem 16; thus $\mathcal{G}_{\mathsf{FalseTCFPG}}$ is not a TCFPL. □

Also note that if all positions in Φ are marked, the pumping operation could yield a picture which is still in the gallery by increasing the number of g's while the number of b's remain the same. Thus, it is clear that using marking in Theorem 16 is necessary for this example.

There is a simpler form of Theorem 16, which we will call the pumping lemma for TCFPLs. The idea was obtained from Rabkin [2012] and adapted for the picture case. It has the same relationship with Theorem 16 as between Ogden's lemma (Theorem 10) and the pumping lemma (Theorem 6) for context-free galleries.

Corollary 4. *If $\mathcal{G}$ is a TCFPL, then there exists an $l \in \mathbb{N}$ such that for any picture $\Phi \in \mathcal{G}$ with $|\Phi| \geq l$,*

1. *Φ is composed of n subpictures $(\Omega_1, \alpha_1), (\Omega_2, \alpha_2), ..., (\Omega_n, \alpha_n)$ subpictures and each (Ω_i, α_i) is composed of $(\Psi_{(i,1)}, \beta_{(i,1)}), (\Psi_{(i,2)}, \beta_{(i,2)}), ..., (\Psi_{(i,n_i)}, \beta_{(i,n_i)})$ (we will denote the set of subscripts of (Ψ, β), i.e., $\{(i,j) : i \in [n], j \in [n_i]\}$, by I);*
2. *there is a map $\phi : I \to [n]$ such that the picture obtained by substituting $(\Omega_{\phi(i,j)} \to \beta_{(i,j)})$ for $(\Psi_{(i,j)}, \beta_{(i,j)})$ is in $\mathcal{G}$;*
3. *recursively carrying out the operation described in (2) always results in a picture in $\mathcal{G}$;*
4. *there is an $(i,j) \in I$ such that $\phi(i,j) = i$, and $(\Psi_{(i,j)}, \beta_{(i,j)})$ is a proper subpicture of (Ω_i, α_i).*

Proof. This follows directly from Theorem 16 if we mark all squares in Φ. □

A theorem by Bhika *et al.* [2007] on rare and nonfrequent symbols in TCFPLs can be deduced from Theorem 16 without making additional reference to the structure of TCFPL derivations.

Theorem 19. *Let $G = (V_N, V_T, \mathcal{T}, (S, \sigma))$ be a TCFPG generating the gallery $\mathcal{G}$ and $B \subseteq V_T$, $B \neq \emptyset$. If B is rare in $\mathcal{G}$, then B is nonfrequent in $\mathcal{G}$.*

Proof. Let l be the number from Theorem 16. Assume that B is frequent in $\mathcal{G}$, then there must be a picture Φ with more than l symbols from B. If we choose to mark all of these symbols and only them, Theorem 16 applies.

Using Corollary 2, $\Phi^{(t)}$ (the result of pumping Φ t times) contains at least $t+2$ symbols from

B. However, by Condition 5 of Theorem 16, the subpicture $(\Psi_{(i,j)}, \beta_{(i,j)})$ contains two marked symbols, and this subpicture appears in all $\Phi^{(t)}$, so these two symbols are a fixed distance apart. So B is not rare in $\mathcal{G}$. □

Using Theorem 19, it is possible to show that the gallery $\mathcal{G}_{\mathsf{FalseTCFPG}}$, defined above, cannot be generated by any TCFPG. Theorem 19 cannot show that the gallery $\mathcal{G}_{\mathsf{triangles}}$ is not a TCFPL since it has no rare set of symbols, thus we see that Theorem 16 can be used in cases where Theorem 19 cannot.

We can also conclude from Theorem 15 and other known results that the gallery $\mathcal{G}_{\mathsf{triangles}}$ cannot be generated by any TCFPG. It is known that TCFPGs are strictly weaker than RFCPGs and since we proved that the gallery $\mathcal{G}_{\mathsf{triangles}}$ cannot be generated by any RFCPG in Theorem 15, it is right to conclude that it cannot be generated by any TCFPG. Following this, we can also say that Ogden's lemma for RFCPLs also holds for all TCFPLs.

5.3 Conclusion

In this Chapter, we discussed the TCFPGs briefly and developed an Ogden-like lemma for TCFPLs. A gallery was given and this was used to show that the new lemma is useful by using it to prove that that gallery is not a TCFPL. Another gallery was developed and two corollaries were given which were used together with the new lemma to prove that this gallery is not a TCFPL, showing that the use of marking is necessary. Using the corollaries given, a different proof was developed for the theorem of *rare* and *nonfrequent* symbols [Bhika *et al.*, 2007], and this was used to show that the gallery in Theorem 17 is not a TCFPL.

There already exist necessary conditions for TCFPLs given in Bhika *et al.* [2007], but we provided a pumping lemma for TCFPLs that adds to the work done by Bhika *et al.* [2007].

Chapter 6

Conclusion and Future Work

The aim of this research was achieved since it was possible to develop Ogden-like lemmas for all the three subclasses of random context picture languages discussed here. Ogden's lemma for context-free picture languages was also provided, which was mainly done to give us more understanding on how to proceed with the other 'more complicated' subclasses. However, it is a result in itself.

Ogden-like lemmas for the context-free picture languages were developed in Chapter 3 and for the random permitting and forbidding context picture languages in Chapter 4. For the table-driven context-free picture languages the lemma was developed in Chapter 5. For each of these subclasses, an example was given showing the usefulness of the theorem, i.e., galleries which do not satisfy the theorem, thus showing that they do not belong to the respective classes. It also was shown that the new Ogden's lemma for context-free picture languages can be used as an alternative to the shrinking-pumping lemma for context-free languages. In the random permitting context case, we were able to show that the new theorem improves on the previous pumping lemma by giving an example of a non-rPc gallery which satisfies the pumping lemma, but does not satisfy the new lemma. In the random forbidding case, we were unable to find a non-rFc gallery which satisfies the shrinking lemma, but not the new lemma, thus we leave it as an open problem to find a gallery which shows that the new lemma improves on the previous shrinking lemma. For the table-driven case, an example of a gallery was given and the newly developed lemma for TCFPLs was used to show that it is not a TCFPL. A pumping lemma for TCFPLs

was also developed, which did not exist before, to the best of our knowledge. Using the newly developed Ogden's lemma for TCFPLs, we were able to devise another way to prove the theorem on *nonfrequent* and *rare* symbols without making additional reference to the structure of TCFPL derivations.

Future work can also be done to determine if these new necessary conditions are necessary, but not sufficient conditions. That is, finding one or more galleries which do not belong to the respective class, but still satisfy the new necessary conditions for each class of languages.

It will also be useful for future work to be done on other Ogden-like lemmas for these subclasses of random context picture languages, which does not focus on the density of markings in the picture, i.e., not enforcing that every subpicture of a certain size must have a marked position.

Bibliography

[Bar-Hillel *et al.* 1961] Y. Bar-Hillel, M. Perles, and E. Shamir. On formal properties of simple phrase structure grammars. *Zeitschrift f́ur Phonetik, Sprachwissenschaft und Kommunikationsforschung*, (14):143–172, 1961.

[Bhika *et al.* 2007] Charita Bhika, Sigrid Ewert, Ryan Schwartz, and Mutahi Waruhiu. Table-driven context-free picture grammars. *International Journal of Foundations of Computer Science*, (18):1151, 2007.

[Drewes and Kreowski 1999] Frank Drewes and Hans-Jörg Kreowski. Picture generation by collage grammars. In Hartmut Ehrig, Gregor Engels, Hans-Jörg Kreowski, and Grzegorg Rozenberg, editors, *Handbook of Graph Grammars and Computing by Graph Transformation*, volume 2, pages 397–457, Singapore, 1999. Applications, Languages, and Tools, World Scientific.

[Drewes *et al.* 1997] Frank Drewes, Hans-Jörg Kreowski, and Denis Lapoire. Criteria to disprove context-freeness of collage languages. Krakow, Poland, 1-3 September 1997. Proceedings of the 11th International Symposium on Fundamentals of Computation Theory.

[Drewes *et al.* 2003] Frank Drewes, Renate Klempien-Hinrichs, and Hans-Jörg Kreowski. Table-driven and context-sensitive collage languages. *Journal of Automata, Languages and Combinatorics*, 8(1):5–24, 2003.

[Ewert and van der Walt 1998] Sigrid Ewert and Andries van der Walt. Generating pictures using random forbidding context. *International Journal of Pattern Recognition and Artificial Intelligence*, 12(7):939–950, 1998.

[Ewert and van der Walt 1999] Sigrid Ewert and Andries van der Walt. Generating pictures using random permitting context. *International Journal of Pattern Recognition and Artificial Intelligence*, 13(3):339–355, 1999.

[Ewert and van der Walt 2002] Sigrid Ewert and Andries van der Walt. A pumping lemma for random permitting context languages. *Theoretical Computer Science*, 270:959–967, 2002.

[Ewert and van der Walt 2013] Sigrid Ewert and Andries van der Walt. Necessary conditions for subclasses of random context languages. *Theoretical Computer Science*, 475:66–72, 2013.

[Ewert 1999] Sigrid Ewert. *Random Context Picture Grammars*. Technical report, Department of Computer Science, University of Stellenbosh, South Africa, March 1999. PhD Thesis.

[Ewert 2009] Sigrid Ewert. Random context picture grammars: The state of the art. In Frank Drewes, A. Habel, B. Hoffman, and D. Plump, editors, *Manipulation of Graphs, Algebras and Pictures*, pages 1–148. 2009.

[Hopcroft and Ullman 1979] John E. Hopcroft and Jeffery D. Ullman. *Introduction to Automata Theory, Languages and Computation*. Addison-Wesley, Reading, Massachusetts, 1st edition, 1979.

[Kim 1990] Changwook Kim. Picture iteration and picture ambiguity. *Journal of Computer and System Sciences*, 40(3):289–306, 1990.

[Klempien-Hinrichs *et al.* 1999] Renate Klempien-Hinrichs, Hans-Jörg Kreowski, and Stefan Taubenberger. Correct translation of mutually recursive function systems into TOL collage grammars. In *Lecture Notes in Computer Science*, volume 1684, pages 350–361, Iaşi, Romania, August 30–September 2 1999. Proceedings of the 12th International Symposium on Fundamentals of Computation Theory (FCT).

[Laroche *et al.* 1992] Patrick Laroche, Maurice Nivat, and Ahmed Saoudi. Context-sensitivity of puzzle grammars. In A. Nakamura, M. Nivat, A. Saoudi, P. S. P. Wang, and K. Inoue, editors, *Lecture Notes in Computer Science*, volume 654, pages 195–212, Springer, Berlin, December 1992. Proceedings Second International Conference Parallel Image Analysis, ICPIA, Ube, Japan.

[Martin 2003] John C. Martin. *Introduction to Languages and the Theory of Computation*. McGraw-Hill, 3rd edition, 2003.

[Matz 1997] Oliver Matz. Regular expressions and context-free grammars for picture languages. In *14th Annual Symposium on Theoretical Aspects of Computer Science*, pages 283–294. Springer-Verlag, 1997.

[Maurer *et al.* 1983] Hermann A. Maurer, Grzegorz Rozenberg, and Emo Welzl. Chain code picture languages. In Hartmut Ehrig, Manfred Nagl, and Grzegorg Rozenberg, editors, *Graph-Grammars and Their Application to Computer Science*, volume 153, pages 232–244, Springer, Berlin, 1983. Lecture Notes in Computer Science.

[Ogden 1968] William Ogden. A helpful result for proving inherent ambiguity. *Mathematical Systems Theory*, (2):191–194, 1968.

[Prusinkiewicz and Hanan 1976] Przemyslaw Prusinkiewicz and James Hanan. Lindenmayer systems, fractals, and plants. In *Lecture Notes in Biomathematics*, volume 79. Springer New York, 1976.

[Rabkin 2012] Max Rabkin. *Ogden's Lemma for Random Permitting and Forbidding Context and ETOL Languages*. MSc Dissertation, School of Computer Science, University of the Witwatersrand, Johannesburg, South Africa, October 2012.

[Rosenfeld and Siromoney 1993] A. Rosenfeld and R. Siromoney. Picture language - a survey. *Academic Press*, (1):229–245, 1993.

[Rozenberg and Salomaa 1976] Grzegorz Rozenberg and Arto Salomaa. The mathematical theory of L systems. In Julius T. Tou, editor, *Advances in Information Systems Science*, pages 161–206. Springer US, 1976.

[Shen-Pei and Lin 1986] Patrick Shen-Pei and Hwei-Jen Lin. A pumping lemma for two-dimensional array languages. *International Conference on Pattern Recognition*, 8, 27-31 October 1986.

[Subramanian *et al.* 2013] K.G. Subramanian, Ibrahim Venkat, and Erzsébet Csuhaj-Varjú. On the power of permitting features in cooperating context-free array grammar systems. *Discrete Applied Mathematics*, (161):2328–2335, 2013.

[Tkachova 2013] Daria Sergeevna Tkachova. *Necessary Conditions for Random Context Galleries*. Technical report, Department of Computer Science, University of the Witwatersrand, Johannesburg, South Africa, November 2013. Honors Research Report.

[van der Walt and Ewert 2000] Andries van der Walt and Sigrid Ewert. A shrinking lemma for random forbidding context languages. *Theoretical Computer Science*, 237(1–2):149–158, 2000.

[van der Walt and Ewert 2003] Andries van der Walt and Sigrid Ewert. A property of random

context picture grammars. *Theoretical Computer Science*, (301):313–320, 2003.

[van der Walt 1972] Andries van der Walt. Random context languages. *Information Processing*, (71):66–68, 1972.

[Yamamoto *et al.* 1989] Y. Yamamoto, K. Morita, and K. Sugata. Context-sensitivity of two-dimensional regular array grammars. *International Journal of Pattern Recognition and Artificial Intelligence 3*, 13(3 & 4):295–319, 1989.