Algebraic Structures in Integer Partitions

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Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the Degree of Doctor of Philosophy at the University of Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

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Date: . 10th . . day of . . February . . 2022. at Johannesburg

Abstract

We study various algebraic structures based on the derived partitions of a partition λ of a positive integer n which are defined as partitions π of positive integers $m \leq n$ whose parts form a sub-multiset of the multiset of the parts of λ. We show that the set of derived partitions forms a group under addition and a monoid under multiplication. A ring isomorphism is established with the direct product of the rings of residue classes modulo a positive integer, where the number factors of the product is equal to the number of distinct parts of λ.

Using the algebraic tool provided by the ring of derived partitions we explore new properties of perfect partitions. We extend the latter to pseudo-perfect partitions, that is, partitions whose parts contain at most one partition of a smaller weight.

We also classify the types of additive cyclic groups of derived partitions and give a rule for the determination of subgroups.

As applications, we study k-complete partitions of n whose parts contain all the partitions of 1 to n. We found some properties of the class of k-complete partitions with least weight, and the class of strictly k-complete partitions, using generating functions and combinatorial methods.

We also give a brief discussion of perfect-partition-complete partitions which are single partitions that contain all the perfect partitions of a given number.

Dedication

In memory of my late mother

Phophi Takalani

1921 - 2013

Acknowledgment

I greatly acknowledge my supervisor, Professor AO Munagi, for his patience, assistance and guidance throughout my studies. I also thank the University of Limpopo, my employer, for the financial support. I thank my family for the moral support and special thanks to my sister, Tendani, for her encouraging words. Finally I thank God Almighty for giving me the strength to study.

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Chapter 1

Introduction

1.1 Essential Background

Partitions of integers have an old history. G.W. Leibniz asked Bernoulli in 1674 if he had investigated the number of ways a given integer or a number can be separated into two, three or more parts and remarked that the problem seemed difficult but important [12]. In modern terminology, Leibniz wanted Bernoulli to investigate the problem of determining the number of partitions of an integer n. He observed that:

For $n = 3$ there are three partitions: $3, 2 + 1, 1 + 1 + 1$.

For $n = 4$ there are five partitions: $4, 3 + 1, 2 + 2, 2 + 1 + 1, 1 + 1 + 1 + 1$.

For $n = 5$ there are seven partitions: $5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1$.

He also observed that the number of partitions of $n = 6$ is 11, which is prime. This suggested that the number of partitions of any n might always be prime, which turned out to be false after computing the partitions for $n = 7$, since they were 15 in number. This led to the beginning of the Theory of Partitions.

There are many mathematicians who have contributed tremendously to the development of the literature on partitions, like, Euler, Sylvester, MacMahon, Rogers, Ramanujan, Hardy, Rademacher, and at present, Andrews and many more. It was Leonhard Euler who did the first substantial systematic work on the study of partitions and partition identities. Srinivassa Ramanujan, who was an Indian mathematics genius, contributed much to the theory and discovered a lot of partition identities and special functions. Ramanujan in collaboration with G.H Hardy proved several results related to integer partitions, see for instance [11, 14, 15]. Ramanujan and Hardy derived an asymptotic estimate of the values of $p(n)$, where $p(n)$ is the number of partitions of n, given by

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}, n \to \infty,$$

and the formula was then perfected by Rademacher to

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{k} A_k(n) \frac{d}{dn}\left(\frac{sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}(n-\frac{1}{24})}\right)}{\sqrt{n-\frac{1}{24}}} \right),$$

see [6], [7] or [22]. Recently Bruinier and Ono derived an algebraic formula for partition function as explained by Murty, in [22], using the theory of harmonic weak Maass forms; see also [9].

In order to understand partitions,"George Andrews is a torch bearer of the theory of partitions and a medium to understand Ramanujan's work". Some of Ramanujan's work still remain inscrutable till today.

Definition 1.1.1. A partition λ of a positive integer n is a representation of n as a sum of positive integers without regard to order. The summands are called the parts or addends of the partition.

For example, the partitions of 6 are

$$\begin{aligned}&6, 5+1, 4+2, 4+1+1, 3+3, 3+2+1, 3+1+1+1, 2+2+2,\\ &2+2+1+1, 2+1+1+1+1, 1+1+1+1+1+1.\end{aligned}$$

The number of parts of λ is called the *length*, denoted by $l(\lambda)$. The sum of the parts is called the *weight*, denoted by $|\lambda|$. Generally the parts of the partition are arranged in exactly one way to suit the type of problem situation under discussion. Thus some authors define a partition using the 'weakly increasing' or 'nondecreasing' order, the 'weakly decreasing' or 'nonincreasing' order, etc. In this case a partition of n is defined to be a sequence of positive integers summing to n, where the sequence order is fixed in advance.

The corresponding sequence notation, for the foregoing example, is

$$\begin{aligned}&(6), (5,1), (4,2), (4,1,1), (3,3), (3,2,1), (3,1,1,1), (2,2,2),\\ &(2,2,1,1), (2,1,1,1,1), (1,1,1,1,1,1).\end{aligned}$$

If $|\lambda| = n$, this fact is also denoted by $\lambda \vdash n$, and both statements mean that λ is a partition of n.

Definition 1.1.2. A composition of an integer n is a partition of n in which the order of occurrence of the parts is taken into account.

Example 1.1.1. There are eight compositions of 4, namely,

$(4), (31), (13), (22), (211), (121), (112), (1111);$

and there are sixteen compositions of 5, namely,

$(5), (41), (14), (32), (23), (311), (131), (113), (221), (212), (122), (2111),$
$(1211), (1121), (1112), (11111).$

The number of compositions of an integer n is given by 2^{n-1}, see MacMahon [19].

The number of all partitions of n is called the *partition function*, and denoted by $p(n)$. Several questions are usually asked about $p(n)$, such as the following: How fast does $p(n)$ grow? Are there efficient ways to compute $p(n)$? What is its parity? What are its distinctive arithmetic properties? Is $p(n)$ multiplicative? Is $p(n)$ prime infinitely often?

The value of $p(n)$ increases as n increases. Conventionally, $p(0) = 1$ and $p(n) = 0$ for all $n < 0$. When $n = 1, 2, 3, \ldots$, the corresponding values of $p(n)$ are $1, 2, 3, 5, 7, 11, 15, 22, 30, 42, \ldots$, respectively.

Many authors have contributed to the study of restricted partition functions, i.e, proving identities on partitions with some restrictions. In this thesis, we will look at the algebraic structures of these integer partitions.

1.2 Methods of Proving Partition Identities

1.2.1 Ferrers Diagrams

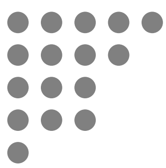
Partitions can be displayed using the Ferrer's diagram which was introduced by Norman Ferrers in 1853.

Ferrers diagrams are useful tools for visualizing partitions or sometimes for proving identities, see Wilf [30].

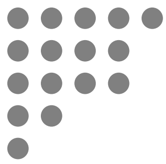
Definition 1.2.1. Let $\lambda \vdash n$ with $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\lambda_1 \geq \cdots \geq \lambda_k$. A Ferrers diagram of λ is a graphical representation of λ comprising of left-justified rows of dots (or unit squares) such that the jth row contains λ_j dots.

When 'unit squares' are emphasized in the definition, the resulting figure is commonly called a *Young diagram*.

Example 1.2.1. The partition $\lambda = (5, 4, 3, 3, 1) \vdash 16$ can be shown using the following Ferrers diagram

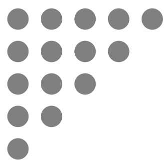


If we interchange the rows and columns of the Ferrers graph of a partition λ, we get a second Ferrers diagram belonging to the *conjugate* of the original partition denoted by λ'. For the partition λ above we have



Therefore $\lambda' = (5, 4, 4, 2, 1)$.

If the conjugate is the same as the original partition, then we have a *self-conjugate* partition. This happens when the Ferrers diagram is symmetric about its main diagonal. For example, the partition of $15 = 5 + 4 + 3 + 2 + 1$ is self-conjugate:



Since the conjugation operation transforms the largest part of a partition λ into the number of parts of λ', a simple theorem follows:

Theorem 1.2.1. *The number of partitions of an integer n whose largest part is k is equal to the number of partitions of n with k parts.*

This theorem has been proved, see [2],[7].

The algebraic or non-graphical computation of the conjugate of a partition is as follows, see, for instance, Munagi [21].

Let

$$\lambda = (\lambda_1^{a_1}, \lambda_2^{a_2}, \ldots, \lambda_k^{a_k}),\ \lambda_1 > \lambda_2 > \cdots > \lambda_k > 0.$$

Then

$$\lambda' = ((a_1 + \cdots + a_k)^{\lambda_k}, (a_1 + \cdots + a_{k-1})^{\lambda_{k-1}-\lambda_k}, \ldots, (a_1)^{\lambda_1-\lambda_2}). \tag{1.1}$$

We see that the largest part of λ, namely λ_1, is equal to the number of parts of λ' which is $\lambda_k + (\lambda_{k-1} - \lambda_k) + (\lambda_{k-2} - \lambda_{k-1}) + \cdots + (\lambda_1 - \lambda_2) = \lambda_k$. For example, if $\lambda = (5, 4, 3, 3, 1) = (5, 4, 3^2, 1)$, then

$$\lambda' = ((1 + 1 + 2 + 1)^1, (1 + 1 + 2)^{3-1}, (1 + 1)^{4-3}, (1)^{5-4}) = (5^1, 4^2, 2^1, 1^1)$$

which is $\lambda' = (5, 4, 4, 2, 1)$ as we found above.

Note that if λ is self-conjugate, that is, if $\lambda = \lambda'$, then the following equalities must hold:

$$\lambda_j = a_1 + a_2 + \cdots + a_{k-j+1},\ j = 1, 2, \ldots, k. \tag{1.2}$$

1.2.2 Generating Functions

According to Herbert Wilf [29], a generating function is a clothesline on which we hang up a sequence of numbers for display.

Partitions can be studied fruitfully using generating functions. A fundamental result is stated in the following theorem.

Theorem 1.2.2 (Euler). *For $|q| < 1$, we have*

$$\sum_{n=1}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

The proof of this can be found in [2], [7].

$$\prod_{n=1}^{\infty}(1-q^n)^{-1} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + \cdots$$

if and only if

$$p(0) = 1, p(1) = 1, p(2) = 2, p(3) = 3, p(4) = 5, p(5) = 7, p(6) = 11, \cdots$$

If the product extends only to $n = m$, the coefficient of q^n is the number of partitions of n into parts $\leq m$.

Euler found that using generating functions, most results on the theory of partitions can be proved. Generating functions can be manipulated more easily than the combinatorial objects in solving partition problems, especially proofs of partition identities. We consider the following examples.

Theorem 1.2.3 (Euler). *The number of partitions into unequal or distinct parts is equal to the number of its partitions into odd parts,*

Proof. Let $p(\mathcal{D}, n)$ be the number of partitions of n into distinct parts and let $p(\mathcal{O}, n)$ be the number of partitions of n into odd parts. We know that the generating functions for $p(n)$ is given by

$$\sum_{n=1}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

which is the same as

$$\begin{aligned}\prod_{n=1}^{\infty} \frac{1}{1-q^n} =&(1 + q + q^{1+1} + q^{1+1+1} + \cdots)\\ &(1 + q^2 + q^{2+2} + q^{2+2+2} + \cdots)\\ &(1 + q^3 + q^{3+3} + q^{3+3+3} + \cdots)\\ &\vdots\end{aligned}$$

So in order to obtain the generating function for $p(\mathcal{D}, n)$ we have to delete the terms $q^{i+i}, q^{i+i+i}, \cdots$, because the integer i is not allowed to appear more than once. Hence we obtain

$$\sum_{n=1}^{\infty} p(\mathcal{D}, n)q^n = (1+q)(1+q^2)(1+q^3)\cdots = \prod_{n=1}^{\infty}(1+q^n).$$

Then we have

$$\begin{aligned}
\sum_{n=1}^{\infty} p(\mathcal{D}, n)q^n &= (1+q)(1+q^2)(1+q^3)(1+q^4)(1+q^5)\cdots \\
&= \frac{1-q^2}{1-q} \cdot \frac{1-q^4}{1-q^2} \cdot \frac{1-q^6}{1-q^3} \cdot \frac{1-q^8}{1-q^4} \cdot \frac{1-q^{10}}{1-q^5} \cdots \\
&= \frac{\cancel{1-q^2}}{1-q} \cdot \frac{\cancel{1-q^4}}{\cancel{1-q^2}} \cdot \frac{\cancel{1-q^6}}{1-q^3} \cdot \frac{\cancel{1-q^8}}{\cancel{1-q^4}} \cdot \frac{\cancel{1-q^{10}}}{1-q^5} \cdots \\
&= \frac{1}{(1-q)(1-q^3)(1-q^5)(1-q^7)\cdots} \qquad (1.3) \\
&= \prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}} \\
&= \sum_{n=1}^{\infty} p(\mathcal{O}, n)q^n.
\end{aligned}$$

Thus $\quad p(\mathcal{D}, n) = p(\mathcal{O}, n).$

□

The following Theorem is found in [5], Exercise 78, p. 48 or [7], Example 1, p. 13, also [3].

Theorem 1.2.4. *The number of partitions of n which each part appears two, three or five times equals the number of partitions of n into parts congruent to 2,3,6,9,or 10 modulo 12.*

Proof. The generating function for the number of partition where each part appears 2, 3, or 5 times is

$$\prod_{n=1}^{\infty}(1+q^{2n}+q^{3n}+q^{5n})$$

So we have

$$\begin{aligned}
\prod_{n=1}^{\infty}(1+q^{2n}+q^{3n}+q^{5n}) &= \prod_{n=1}^{\infty}(1+q^{n})(1+q^{2n})(1-q^{n}+q^{2n}) \\
&= \prod_{n=1}^{\infty}(1+q^{n})(1+q^{2n})\left(\frac{1+q^{3n}}{1+q^{n}}\right) \\
&= \prod_{n=1}^{\infty}(1+q^{2n})(1+q^{3n}) \\
&= (1+q^{2})(1+q^{3})(1+q^{4})(1+q^{6})(1+q^{6})\cdots \\
&= (1+q^{2})(1+q^{4})(1+q^{6})\cdots(1+q^{3})(1+q^{6})\cdots \\
&= \frac{1-q^{4}}{1-q^{2}}\frac{1-q^{8}}{1-q^{4}}\frac{1-q^{12}}{1-q^{6}}\cdots\frac{1-q^{6}}{1-q^{3}}\frac{1-q^{12}}{1-q^{6}}\cdots \\
&= \frac{1}{(1-q^{2})(1-q^{3})(1-q^{6})(1-q^{9})(1-q^{10})}\cdots \\
&= \prod_{n=0}^{\infty}\frac{1}{(1-q^{4n+2})(1-q^{6n+3})}.
\end{aligned}$$

But $\prod_{n=0}^{\infty}\frac{1}{(1-q^{4n+2})(1-q^{6n+3})}$ is equal to

$$\prod_{n=0}^{\infty}\frac{1}{(1-q^{12n+2})(1-q^{12n+3})(1-q^{12n+6})(1-q^{12n+9})(1-q^{12n+10})}.$$

Thus the number of partitions of n where each part appears two, three or five times equals the number of partitions of n into parts congruent to 2, 3, 6, 9 or 10 modulo 12. $\square$

1.3 Thesis Overview

The main purpose of this thesis is to study various algebraic structures based on derived partitions λ of a partition of a positive integer n. By a partitions of n we mean a way of writing n as a sum of positive integers, and a derived partition is any partition π of integer $m \leq n$ whose parts form a sub-multiset of the multiset of parts of λ.

In Chapter 1 of this thesis, we introduce partitions, partition functions and some methods of solving partition identities. In Chapter 2, we give a brief review of relevant algebraic structures, including monoids, groups and rings.

Then in Chapter 3, we show that the set of a derived partitions of a fixed partition λ of n, $D(\lambda)$, forms a group under addition. We also establish a group isomorphism between $D(\lambda)$ and the group of residue classes of integers modulo n.

Chapter 4 is devoted to the study of a special case of the derived group of pseudo-perfect partitions, that is, when the group $D(\lambda)$ is generated by a perfect partition λ. Every element of such groups of weight $m \leq n$ contains at most one partition of each positive number less than m. Then in Chapter 5, we prove that the group of derived partitions also possesses a ring structure.

Lastly, in Chapter 6, we apply the structure of $D(\lambda)$ to the study of the class of partitions in which the parts contain all the partitions of 1 to n, called k-complete partitions. We also consider the analogous classes of perfect-partition complete partitions and full-perfect-partition complete partitions.

Chapter 2

Introduction to Ring Theory

In this chapter, our main focus is to provide a theoretical background of the theory of rings, and outline the concepts of groups and rings which will be useful in subsequent chapters. Since we will be dealing with algebraic structures, we need to understand the algebraic structures that will be used in integer partitions.

The following statements are mostly given without proofs since the proofs already exist in standard Algebra texts, see [4], [10], [13], [20] and [27].

A ring is an algebraic structure where we can add and multiply elements. We have to show that some sets of partitions form a ring.

2.1 Basic Definitions and Examples

In this section we shall review some basic definitions that need to be understood before we define a ring and give some examples.

Definition 2.1.1. A semigroup is an associative binary system, i.e. we say $(S, *)$ is a semigroup if for all $a, b, c \in S$, we have $a * b \in S$ and $a * (b * c) = (a * b) * c$.

Definition 2.1.2. If for all $a \in S$, there exists a unique element, namely e, such that $e * a = a * e = a$, called an identity element.

Definition 2.1.3. A monoid is a semigroup $(S, *)$ with an identity element $e \in S$.

We now give the definition of a group as given by Allenby [4].

Definition 2.1.4. A group is a monoid $(G, *)$ in which every element $a \in G$ has an *inverse* $a^{-1} \in G$.

In other words, a group is a binary system $(G, *)$ satisfying the following axioms:

G1. Associativity: for all $a, b, c \in G$ we have $(a * b) * c = a * (b * c)$;

G2. Existence of neutral or identity element: $\exists\ e \in G$ such that $e * a = a = a * e$ for all $a \in G$;

G3. Existence of inverses: for each $a \in G\ \exists\ a^{-1} \in G$ such that $a * a^{-1} = e = a^{-1} * a$.

A group $(G, *)$ is *commutative or abelian* if $a * b = b * a$ for all $a, b \in G$.

Remark 2.1.1. The unit element e and for every $a \in G$, then the inverse element a^{-1} is unique.

Example 2.1.1. The set of integers $\mathbb{Z}$ is an abelian group; i.e. if $(\mathbb{Z}, +)$ then for all $a, b, c \in \mathbb{Z}$ we have:

- $a + (b + c) = (a + b) + c$;
- $a + (-a) = 0 = -a + a$;
- $a + b = b + a$;
- $a + 0 = a = 0 + a$.

Before we define a ring lets define what is meant by a cyclic group see [13].

Definition 2.1.5. Given a group G and element $a \in G$, if $G = \{a^n | n \in \mathbb{Z}\}$ then a is a generator of G. And a group G is cyclic if there is some element a in G that generates G.

Definition 2.1.6. A subgroup H of a group G is a non-empty subset of G that forms a group under the binary operation of G. This is denoted by $H \leq G$.

Theorem 2.1.1. *Let G be a group and let $a \in G$. Then*

$$H = \{a^n \mid n \in \mathbb{Z}\}$$

is a subgroup of G and is the smallest subgroup of G that contains a. Then H is called a cyclic *subgroup, often denoted by $H \equiv \langle a \rangle$.*

For example, let $\mathbb{Z}_3$ be the group. We know that $\mathbb{Z}_3 = \{0, 1, 2\}$, then $\langle 1 \rangle = \langle 2 \rangle = \mathbb{Z}_3$. Also the group $\mathbb{Z}$ under addition is a cyclic group and both $\langle 1 \rangle$ and $\langle -1 \rangle$ are generators for the group.

Theorem 2.1.2. *Let $\langle G, *\rangle$ be a group and H be a nonempty subset of G. Then H is a subgroup of G, denoted by $H \leq G$, if and only for all $a, b \in H$ we have both*

(i) $a * b \in H$ and

(ii) $a^{-1} \in H$.

Definition 2.1.7. If G is a finite group, then the order of G denoted by $|G|$ is the number of elements of G.

Theorem 2.1.3. *Any group of prime order is cyclic.*

Definition 2.1.8. Let $H \leq G$. If, for all $g \in G$, we have $H = g^{-1}Hg$, where $g^{-1}Hg$ is defined to be the subset$\{g^{-1}hg : h \in H\}$ of G, then H is called a normal subgroup of G, denoted by $H \lhd G$.

Definition 2.1.9. A map ϕ of a group G into a group G' is a homomorphism if $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G$.

Before defining homomorphism in rings, we discuss homomorphism in groups.

Definition 2.1.10. Let $(G, \circ)$ and $(S, *)$ be groups. The map $\phi : G \to S$ is called a homomorphism from $(G, \circ)$ to $(S, *)$, if for all $a, b \in G$, $\phi(a \circ b) = \phi(a) * \phi(b)$, see Allenby [4].

Theorem 2.1.4. *Let ϕ be a homomorphism of a group G into a group S.*

1. *If e is the identity in G, then $\phi(e)$ is the identity e' is in S.*
2. *If $a \in G$, then $\phi(a^{-1}) = \phi(a)^{-1}$.*
3. *If H is a subgroup of G, then $\phi(H)$ is a subgroup of S.*
4. *If K' is a subgroup of S, then $\phi^{-1}(K')$ is a subgroup of G, Thus we can clearly say that ϕ preserves the identity, inverses and subgroups.*

The theorem has been proved by Fraliegh in [13]

Definition 2.1.11. Let $\phi : G \to S$ be a homomorphism of groups. The subgroup $\phi^{-1}(\{e'\})$ consisting of all elements of G mapped by ϕ into the identity e' of S is the kernel of ϕ, denoted by $ker(\phi)$. A group homomorphism $\phi : G \to S$ is a one-to-one map if and only if the $ker(\phi) = \{e\}$.

Definition 2.1.12. A homomorphism $\phi : G \to S$ is called monomorphism if the map ϕ is injective, epimorphism if the map ϕ is surjective and isomorphism if the map ϕ is bijective.

Definition 2.1.13. Two groups G and S are isomorphic if there is an isomorphism from group G to group S, denoted by $G \cong S$.

Definition 2.1.14. A homomorphism of a group G into itself is an endomorphism and an isomorphism $\phi : G \to G$ is an automorphism.

Example 2.1.2.

1. The map $\phi : \langle \mathbb{Z}, + \rangle \to \langle 2\mathbb{Z}, \cdot \rangle$ given by $\phi(z) = 2z$ is an isomorphism between the groups.

2. $\langle \mathbb{Z}, + \rangle \cong \langle 2\mathbb{Z}, \cdot \rangle$, this example has been proved by Allenby in [4] and Fraleigh in [13].

A ring may be defined as an abelian group under addition with associative multiplication which satisfies the left-and-right distributive laws:

Definition 2.1.15. A ring $(R, +, \cdot)$ is a set R together with two binary operations $+$ and $\cdot$, which we call addition and multiplication, defined on R such that the following axioms are satisfied:

R_1. $(R, +)$ is an abelian group.

R_2. $(R, \cdot)$ is a semigroup.

R_3. For all $a, b, c \in R$, the left distributive law, $a(b+c) = (ab) + (ac)$ and the right distributive law, $(a+b)c = (ac) + (bc)$, hold.

A ring $(R, +, \cdot)$ is **commutative** if it is commutative under multiplication, i.e., if the semigroup $(R, \cdot)$ is commutative.

The sets of integers $\mathbb{Z}$, real numbers $\mathbb{R}$, rational numbers $\mathbb{Q}$ and complex numbers $\mathbb{C}$ are all examples of commutative rings.

Example 2.1.3 (Residue Class Ring Mod n)**.** Let n be a positive integer and let $\mathbb{Z}_n = \{\overline{0}, \overline{1}, \ldots, \overline{n-1}\}$ with addition and multiplication modulo n. Then $\mathbb{Z}_n$ is a commutative ring.

Example 2.1.4. (Direct product of rings).

If $R_1, \ldots, R_n$ are rings, then their direct product is the set of all $(r_1, r_2, \ldots, r_n) \in R_1 \times R_2 \times \cdots \times R_n$ with component-wise addition and multiplication. Thus for any $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in R_1 \times \cdots \times R_n$ we have

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \cdots, a_n + b_n),$$

$$(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = (a_1 b_1, \cdots, a_n b_n).$$

Definition 2.1.16. A non-zero element x in a commutative ring, R, is called a *zero divisor* if there exists a non-zero $y \in R$ such that $xy = 0$. Thus if $a, b \in R$, where $a \neq 0$ and $b \neq 0$ but $ab = 0$, we say that a and b are zero divisors. If $ab = ba = 1$, we say that a is a *unit* or that a is *invertible.* More generally if $a \neq 0$, $b \neq 0$ and $ab \neq 0$, we say that a and b are *non-zero divisors.*

Note that 0 is neither a zero divisor nor a non-zero divisor.

Definition 2.1.17. A commutative ring with identity which has no zero divisors is called an *integral domain.*

Definition 2.1.18. A division ring is a ring R, with identity $1 \in R$, such that for each non-zero $x \in R$, $\exists\, a \in R$ satisfying $a \cdot x = 1 = x \cdot a$.

Definition 2.1.19. If $S \subseteq R$ is a subset of a ring R, such that $(S, +)$ is a subgroup of $(R, +)$ and for all $s_1, s_2 \in S$ the product $s_1 \cdot s_2$ lies in S then S is called a subring of R denoted by $S \leq R$.

Definition 2.1.20 (A subring test)**.** A subset $S \subseteq R$ of a ring R is a subring of R if $0 \in S$, $1 \in S$ and if $s, t \in S$, then $s + t, st$ and $-s$ are all in S.

The set of integers $\mathbb{Z}$ is a subring of $\mathbb{R}$, but the set of even integers $2\mathbb{Z}$ is not a subring of $\mathbb{Z}$.

Definition 2.1.21. The characteristic of a ring R with 1, denoted by $char R$, is the smallest positive integer n such that $n \cdot 1 = 1 + 1 + 1 + 1 + \cdots + 1 = 0$. If there is no such n then R is said to have a characteristic 0.

Example 2.1.5. The characteristic of $\mathbb{Z}_2$ is 2 and that of $\mathbb{Z}_3$ is 3.

Definition 2.1.22. A nonzero ring with unity is called a division ring or a skew field if every nonzero element has an inverse. A commutative division ring is called a field, see McCoy [20].

Example 2.1.6.

1. The set $\mathbb{Z}$ is an integral domain but not a field, we know that a field is commutative division ring, i.e. a field is a commutative ring in which every nonzero element is a unit.
2. The set $\mathbb{Z}$ is a commutative ring.
3. The set of $\mathbb{Q}$ rationals in which $\frac{a}{b} = \frac{a'}{b'}$ if and only if $ab' = ba'$, forms a ring. Also the sets $\mathbb{R}$ and $\mathbb{C}$ are rings.

4. The set of natural numbers is not a ring under the usual addition and multiplication, since the existence of additive inverses fails, i.e. there is no $n \in \mathbb{N}$ for which $1 + n = 0$
5. The set of even integers $2\mathbb{Z}$ is a commutative ring without an identity.
6. Polynomials with real coefficients, form a commutative ring with identity under the usual addition and multiplication.

Definition 2.1.23. The center of a ring R is defined to be

$$Z(R) := \{r \in R | rs = sr \text{ for all } s \in R\}.$$

Proposition 2.1.5. *A ring R is commutative if and only if $Z(R) = R$.*

2.2 Homomorphism and Isomorphism of Rings

By a homomorphism of a ring R into a ring S we mean a function $\phi : R \to S$ which preserves the ring operations, i.e.

$$\begin{gathered} \phi(a+b) = \phi(a) + \phi(b) \\ \phi(ab) = \phi(a) \cdot \phi(b) \quad \forall\ a, b \in R \\ \text{with } \phi(1_R) = 1_S. \end{gathered}$$

If there exists a homomorphism ϕ of R onto S so that $\phi(R) = S$, we may say that R is homomorphic to S or that S is a homomorphic image of R.

Proposition 2.2.1. *If $\phi : R \to S$ is a homomorphism, then the following are true.*

1. *If 0 is the zero of R, then $\phi(0)$ is the zero of S.*
2. *If $a \in R$, then $\phi(-a) = -(\phi(a))$.*
3. *$\phi(R)$ is a subring of S.*
4. *If R has a unity 1, then the ring $\phi(R)$ has $\phi(1)$ as a unity.*
5. *If R is a commutative ring, then the ring $\phi(R)$ is also commutative.*

Definition 2.2.1. A homomorphism that is one-to-one and onto is called an isomorphism. An injective homomorphism is called a *monomorphism* and a surjective homomorphism is called an *epimorphism*. Thus an *isomorphism* is a homomorphism which is both a monomorphism and an epimorphism. A homomorphism from R to itself is called an *endomorphism*, and a bijective endomorphism is called an *automorphism*. We write $R \cong S$ to indicate that R is isomorphic to S.

Definition 2.2.2. Let I be a subset of a ring R. Then an additive subgroup of R having the property that $ar \in I$ for $a \in I, r \in R$ is called a *left ideal* of R. If instead we have $ra \in I$ for $a \in I, r \in R$, we say that we have a *right ideal*. An ideal that is both a right and a left ideal is called a *two-sided ideal*, or simply an ideal, of R. For any ring R, both R and $\{0\}$ are ideals of R.

Definition 2.2.3. An ideal I of R is called *proper* if $I \neq R$. We say that it is non-trivial if $I \neq R$ and $I \neq 0$.

Definition 2.2.4. An ideal I of a commutative ring R is called a *principal ideal* if there exists an $a \in I$ such that $I = \{ab | b \in R\}$. We say that I is generated by a and write $I = (a)$.

Let $a_1, a_2, \cdots, a_n \in R$. The smallest ideal I of R containing these elements is denoted by $(a_1, a_2, \cdots, a_n)$, we say that I is generated by $(a_1, a_2, \cdots, a_n)$.

Definition 2.2.5. If $\phi : R \to S$ is a ring homomorphism we define the kernel of ϕ as $ker\, \phi = \{r \in R, \phi(r) = 0\}$.

Proposition 2.2.2. *If $\phi : R \to S$ is a ring homomorphism and $ker\, \phi = \{r \in R, \phi(r) = 0\}$, then it can be proved that: see [13]*

- *kernel ϕ is an additive subgroup of R.*
- *For $a \in ker\, \phi$ and $r \in R$ we have*
  $\phi(ra) = \phi(r)\phi(a) = 0,$
  $\phi(ar) = \phi(a)\phi(r) = 0,$
  showing that ar and ra are in $ker\phi$.
- *Then $ker\, \phi$ has to be proper, i.e., $ker\, \phi \neq R$, since $\phi(1) = 1 \neq 0$.*

The kernel of a ring homomorphism $\phi : R \to S$ is an ideal of R.

Definition 2.2.6. Let H be a subgroup of G whose left and right cosets coincide. And the coset of H form a group G/H under the binary operation $(aH)(bH) = (ab)H$. The group G/H is the factor group or quotient group of G modulo H.

Theorem 2.2.3. *Let R, S be commutative rings and $\phi : R \to S$ an epimorphism. Let $\psi : R \to R/ker\phi$ be the natural homomorphism $\psi(r) = r + ker\phi$. There exists an isomorphism $\gamma : R/ker\, \phi \to S$ such that $\phi = \gamma \circ \psi$.*

Theorem 2.2.4. *If $\phi : R \to S$ is a ring homomorphism with kernel K, then the image of ϕ is isomorphic to R/K:*

$$im(\phi) \cong R/ker(\phi)$$

Note that a subring $I \leq R$ is an ideal of R if, for all $i \in I$ and $x \in R$, the products $i \cdot x$ and $i \cdot y$ lie in I. If R is commutative then one needs only verify that $x \cdot i \in I$ for all $x \in R, i \in I$. Let $I \triangleleft R$, then the multiplication of *cosets* in R/I is well-defined and makes the group, R/I, into a ring called the quotient of R modulo I. The sets of cosets of two-sided ideal I given by

$$R/I = \{r + I, r \in R\}$$

is a ring with identity $1_R + I$ and a zero element $0_R + I$ called a **quotient ring**.

Definition 2.2.7. An ideal, $I \triangleleft R$, is called prime if $xy \in I$ implies that either $x \in I$ or $y \in I$. An ideal, $I \triangleleft R$, is called maximal if $J \triangleleft R$ and $I \leq J$ implies that either $I = J$ or $J = R$. A prime ideal P of R is an ideal such that whenever $ab \in P$, with $a, b \in R$, then $a \in P$ or $b \in P$ or both. A maximal ideal M of R is an ideal which is not contained in any proper ideal.

Suppose R is a commutative ring with identity. Then P is a prime ideal if and only if R/P is an integral domain.

Let R be a commutative ring with identity. Then R is a field if and only if its only ideals are 0 and R.

Proposition 2.2.5. *If F is a field and R is a ring which is isomorphic to F then R is also a field. In order to show that R is a field we have to show that R is commutative, has a non-zero element and that all non-zero elements have inverses.*

Suppose R is a commutative ring with identity. Then M is a maximal ideal if and only if R/M is a field.

Theorem 2.2.6. *A finite integral domain is a field.*

For example if p is prime, then $\mathbb{Z}_p$ is a field.

Theorem 2.2.7. *Let R be a ring with identity, $1 \neq 0$ and let S be the subset of R given by $S = \{n1 \mid n \in \mathbb{Z}\}$. Then S is a subring of R, and*

(i) if R has a characteristic 0, then $S \cong \mathbb{Z}$,

(ii) if R has a characteristic $m \neq 0$, then $S \cong \mathbb{Z}_m$.

2.3 Chinese Remainder Theorem

The Chinese Remainder Theorem traditionally was formulated for number theory but it has been extended to rings and integers. In the ring of integers, we use the Euclidean algorithm to compute the greatest common divisors of integers. It gives a simple method of finding the solutions of a system of simultaneous congruences $x_i \cong a_i(\mod m_i),\ i = 1, 2, \cdots, k$, provided the moduls m_i are relatively prime, see Oystein in [23].

Theorem 2.3.1 (The Chinese Remainder Theorem)**.** *The solution of the simultaneous congruences given by* $x \cong a_i(\mod m_i)$ $i = 1, 2, \cdots, k$ *can be presented in the form*

$$x \cong a_1 c_1 \frac{M}{m_1} + \cdots + a_k c_k \frac{M}{m_k} (\mod M),$$

where the c_i *form a set of integers satisfying the condition*

$$c_1 \frac{M}{m_1} + \cdots + c_k \frac{M}{m_k} = 1.$$

The theorem has been proved by many authors, see Oystein in [23].

Theorem 2.3.2. *Let* $R_1, \ldots, R_n$ *be rings and let* R *be their direct product, with a natural projections* $\varphi_i : R \to R_i$ *and natural embedding* $\psi_i : R_i \to R$. *Then* $A_i = Im\psi_i$ *is an ideal in* R *and we have the direct sum*

$$R = A_1 \oplus \cdots \oplus A_n \tag{2.1}$$

where

$$A_i A_j = 0 \ \text{if} \ i \neq j,\ A_i A_j \subseteq A_i \tag{2.2}$$

Conversely, if R *can be written as a direct sum of ideals as in Equation (2.1), then the ideals satisfy Equation (2.2) and* R *is a direct product of rings isomorphic to the* A_i*, regarded as rings.*

The theorem was proved by many authors, V.P Snaith proved it in [27].

Example 2.3.1. If $R = \mathbb{Z}$, the Chinese Remainder Theorem simplifies to say that if $n = \prod_i n_i$, where the n_i are pairwise coprime, then

$$\mathbb{Z}/n\mathbb{Z} \simeq \prod_i \mathbb{Z}/n_i\mathbb{Z}.$$

Thus we can say, for instance, that

$$\mathbb{Z}/6\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$$

The Chinese Remainder Theorem does not hold for non-commutative rings. We are going to use the Chinese Remainder Theorem to prove a property of cyclic groups of Derived Partitions of a positive integer.

Chapter 3

The Group of Derived Partitions

In this chapter, we define a derived partition of a partition $\lambda \vdash n$. Note that $\lambda \vdash n$ means "λ is a partition of n". We also show that the algebraic structures outlined in Chapter 2 apply for derived partitions.

We will generally express partitions in weakly increasing order of magnitude in this chapter. Thus

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k),\ 1 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k. \tag{3.1}$$

This may also be written in the form

$$\lambda = (\lambda_1^{v_1}, \lambda_2^{v_2}, \ldots, \lambda_t^{v_t}),\ 0 < \lambda_1 < \lambda_2 < \ldots < \lambda_t, \tag{3.2}$$

where $v_i > 0$ denotes the multiplicity of λ_i, and $t \leq k$ with $v_1 + v_2 + \cdots + v_t = k$.

A third notation for λ is to express it as an infinite integer sequence with finitely many nonzero terms. First we write

$$\lambda = (1^{f_1}, 2^{f_2}, 3^{f_3}, \ldots),\ \text{where } f_i \geq 0\ \forall\ i. \tag{3.3}$$

Clearly $f_i = 0$ for $i > f_c$, when c is the largest part of λ; generally, $f_i = 0$ for all parts $i \notin \lambda$.

Since the part sizes are fixed by the natural numbers 1,2,3,. . . , then

$$\lambda = [f_1, f_2, f_3, \ldots] = \{f_i\}_{i=1}^{\infty},\ \text{where } f_i = 0\ \forall\ i \notin \lambda. \tag{3.4}$$

For instance, consider $\lambda = (1, 1, 1, 2, 2, 3, 6) = (1^3, 2^2, 3, 6)$. Then

$$\lambda = (1^3, 2^2, 3^1, 4^0, 5^0, 6^1, 7^0, 8^0, 9^0, \ldots) \equiv [3, 2, 1, 0, 0, 1, 0, 0, 0, \ldots]$$

3.1 Additive Group of Derived Partitions

Subpartitions of a partition shall be called derived partitions according to the following definition.

Definition 3.1.1. Given a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3 \cdots, \lambda_k) \vdash n$, a *derived partition* (or subpartition) of λ is any partition π of a nonnegative integer $m \leq n$ whose parts form a sub-multiset of the multiset of the parts of λ.

The set of derived partitions of λ will be denoted by $D(\lambda)$.

The empty partition of 0 is denoted by (0) (or $\emptyset$), and may be defined in terms of the notations (3.2) and (3.3), by

$$(0) = (\lambda_1^0, \ldots \lambda_k^0) = (1^0, 2^0, \ldots). \tag{3.5}$$

Lemma 3.1.1. *Let* $\lambda \vdash n > 0$. *Then* $(0) \in D(\lambda)$.

Proof. The empty partition (0) has no nonzero parts that form a partition; so there is no part in (0) that is not in a derived partition. Thus $(0) \in D(\lambda)$. Alternatively, if $\lambda = (\lambda_1^{v_1}, \lambda_2^{v_2}, \ldots, \lambda_t^{v_t})$, then any $\pi \in D(\lambda)$ has the form $\pi = (\lambda_1^{w_1}, \lambda_2^{w_2}, \ldots, \lambda_t^{w_t})$, where $0 \leq w_i \leq v_i \; \forall \, i$. In particular the choice of multiplicities, $w_1 = \cdots = w_t = 0$, proves the lemma. □

From Lemma 3.1.1, we deduce that (0) is a derived partition of every partition. Therefore,

$$(0) \in D(\lambda) = \emptyset.$$

Example 3.1.1. Consider the partition $\lambda = (1, 2^2, 3) \vdash 8$. Then

$$D((1, 2^2, 3)) = \\ \{(0), (1), (2), (2^2), (3), (1, 2), (1, 2^2), (1, 3), (2, 3), (2^2, 3), (1, 2, 3), \lambda\}.$$

From Example 3.1.1, the cardinality of $D((1, 2^2, 3))$, by actual counting, is $|D((1, 2^2, 3))| = 12$.

Lemma 3.1.2. *The set* $D(\lambda)$ *is finite for any partition* $\lambda \vdash n$.

Proof. The finiteness of $D(\lambda)$ follows from the fact that λ is a partition of a finite integer n, and so it has a finite number of subpartitions, or derived partitions. □

We wish to study the set of derived partitions of a given partition and understand its algebraic properties.

Now let $\lambda = (1^{N_1}, 2^{N_1}, \ldots) = \{N_i\}_{i=1}^{\infty}$. Then for any $\pi \in D(\lambda)$ we can write

$$\pi(1^{f_1}, 2^{f_1}, \ldots) = \{f_i\}_{i=1}^{\infty}, \quad 0 \leq f_i \leq N_i, \, i \geq 1.$$

Thus $0 \leq f_i \leq N_i < N_i + 1$. Therefore we can define algebraic operations based solely on the multiplicities f_i of individual parts i. For brevity we will sometimes write $\{f_i\}$ instead of $\{f_i\}_{i=1}^{\infty}$, when the meaning is clear from the context.

Definition 3.1.2. Let $\{f_i\}, \{g_i\} \in D(\lambda)$ with $\lambda = (1^{N_1}, 2^{N_1}, \ldots) = \{N_i\}_{i=1}^{\infty}$. We define

$$\{f_i\} + \{g_i\} = \{h_i\} \iff f_i + g_i \equiv h_i \pmod{N_i + 1}.$$

In other words

$$(1^{f_1}, 2^{f_2}, \ldots) + (1^{g_1}, 2^{g_2}, \ldots) = (1^{f_1+g_1}, 2^{f_2+g_2}, \ldots) = (1^{h_1}, 2^{h_2}, \ldots)$$

where $(f_1+g_1, f_2+g_2, \ldots) \equiv (h_1, h_2, \ldots) \pmod{(N_1+1, N_2+1, \ldots)}$, termwise, where $0 \leq h_i < N_i + 1$.

Observe that the notations in Definition 3.1.2 may also be written in compact form as

$$\{f_i\} + \{g_i\} \equiv \{f_i + g_i \pmod{N_i + 1}\}$$

or

$$\{f_i\} + \{g_i\} \equiv \{f_i + g_i \pmod{N_i + 1}\} \equiv \{h_i \pmod{N_i + 1}\},$$

where $f_i + g_i \equiv h_i \pmod{N_i + 1}$.

We now establish some important properties of the set of derived partitions $D(\lambda)$ with respect to the operation of addition.

A derived partition π of λ is any subpartition of λ. This statement may be rephrased by saying that λ is a *superpartition* of π. Thus π is a derived partition of λ if and only if λ is a superpartition of π. Every partition is a superpartition of (0).

Since $\lambda \in D(\lambda)$ is the unique superpartition of all members of $D(\lambda)$, we will sometimes refer to λ as the *(special) maximal partition* in $D(\lambda)$.

Lemma 3.1.3 (Identity Element)**.** *$D(\lambda)$ has an additive identity, denoted by 0_λ. The identity element 0_λ is unique.*

Proof. From (3.5) and Definition 3.1.2, the identity element is given by $0_\lambda = (0)$, that is,

$$0_\lambda = (0) = (1^0, 2^0, \ldots) = (1^{N_1+1}, 2^{N_2+1} \cdot \cdots) = \{N_i + 1\},$$

since $0 \equiv N_i + 1 \pmod{N_i + 1}$ for all i.

Then for any $\pi \in D(\lambda)$ with $\pi = (1^{f_1}, 2^{f_2}, \ldots) = \{f_i\}$, we have

$$\{f_i\} + \{N_i + 1\} \equiv \{f_i + N_i + 1 \pmod{N_i + 1}\} \equiv \{f_i \pmod{N_i + 1}\}$$

and

$$\{N_i + 1\} + \{f_i\} \equiv \{N_i + 1 + f_i \pmod{N_i + 1}\} \equiv \{f_i \pmod{N_i + 1}\}.$$

Hence $\pi + 0_\lambda = 0_\lambda + \pi = \pi$.

For uniqueness, assume that $D(\lambda)$ has another identity element, say $0'_\lambda = \{d_i\}$. Then

$$\begin{aligned} 0_\lambda + 0'_\lambda &= 0'_\lambda \quad \text{using } 0_\lambda \text{ as } \textit{identity}; \\ &= 0_\lambda \quad \text{using } 0'_\lambda \text{ as } \textit{identity}; \\ &\implies 0_\lambda = 0'_\lambda. \end{aligned}$$

Hence the identity element is unique. □

Remark 3.1.1. [Remark on the Uniqueness of the Zero Element] If λ has one distinct part, $\lambda = (\lambda_1^{N_1})$, the uniqueness of $0_\lambda = (\lambda_1^{N_1+1})$ is trivially clear.

On the other hand, if λ has the form $\lambda = (\lambda^{N_1}, \ldots, \lambda^{N_t})$, $t > 1$, then we see that $0_\lambda = (\lambda^{N_1+1}, \ldots, \lambda^{N_t+1})$, uniquely. This is because the residue class of each $N_i + 1$ is congruent to 0 modulo $N_i + 1$ for each i.

On containment, $0_\lambda = (0) \in D(\lambda)$ by definition. However, technically $(\lambda^{N_1+1}, \ldots, \lambda^{N_t+1}) \notin D(\lambda)$. In any case, if we assume that also $(\lambda^{N_1+1}, \ldots, \lambda^{N_t+1}) \in D(\lambda)$, then $(\lambda^{N_1+1}, \ldots, \lambda^{N_t+1}) \equiv (\lambda^0, \ldots, \lambda^0)$ since $N_i + 1 \equiv 0 \pmod{N_i + 1}$ for all i.

This reasoning is consistent with the isomorphism of $D(\lambda)$ with the direct product of additive groups of residue classes modulo $N_i + 1$, $i = 1, 2, \ldots, t$; see Theorem 3.1.6.

Lemma 3.1.4. *Every element of $D(\lambda)$ has a unique additive inverse.*

Proof. We show that for each $\pi \in D(\lambda)$ there exists $\overline{\pi} \in D(\lambda)$ such that $\pi + \overline{\pi} = \overline{\pi} + \pi = (0)$. Let $\pi = (1^{f_1}, 2^{f_1}, \ldots)$. We set $\overline{\pi} = -\pi$, where

$$-\pi = -(1^{f_1}, 2^{f_2}, \ldots) = (1^{N_1+1-f_1}, 2^{N_2+1-f_2}, \ldots) = \{N_i + 1 - f_i\}.$$

Therefore,

$$\begin{aligned} \pi + \overline{\pi} &= \{f_i\} + \{N_i + 1 - f_i\} \\ &\equiv \{f_i + N_i + 1 - f_i \pmod{N_i + 1}\} \\ &\equiv \{N_i + 1 \pmod{N_i + 1}\} \\ &= 0_\lambda. \end{aligned}$$

Similarly, $\overline{\pi} + \pi = 0_\lambda$.

For uniqueness, assume that π has another inverse, so that $\pi + \beta = 0_\lambda$. Then $\pi + \beta = \pi + \overline{\pi}$. Adding $-\pi$ to both sides of this equation gives

$$-\pi + \pi + \beta = -\pi + \pi + \overline{\pi} \implies 0_\lambda + \beta = 0_\lambda + \overline{\pi}.$$

That is, $\beta = \overline{\pi}$. Hence additive inverses are unique. $\square$

Now we prove that the set of derived partitions $D(\lambda)$ forms a group under addition.

Theorem 3.1.5. *Let $\lambda = (1^{N_1}, 2^{N_2}, \ldots) = \{N_i\}$. Then $(D(\lambda), +)$ is a finite abelian group. In order words, $D(\lambda)$ satisfies the following properties, given any π, $\alpha \in D(\lambda)$:*

1. ***Closure**. $\pi + \alpha \in D(\lambda)$.*
2. ***Associativity**. $(\pi + \alpha) + \beta = \pi + (\alpha + \beta)$.*
3. ***Identity Element**. $\exists!\, 0_\lambda \in D(\lambda)$, where $0_\lambda = (0)$, such that*
   $\pi + (0) = (0) + \pi = \pi$.
4. ***Inverses**. $\exists!\, \overline{\pi} \in D(\lambda)$ such that $\pi + \overline{\pi} = \overline{\pi} + \pi = 0_\lambda$.*
5. ***Commutativity**. $\pi + \alpha = \alpha + \pi$.*

Proof. We establish the group axioms of closure, associativity, identity element and inverses, as stated in 2.1.4. Assume that π, α, $\beta \in D(\lambda)$, where
$\pi = (1^{f_1}, 2^{f_2}, \ldots) = \{f_i\}$,
$\alpha = (1^{a_1}, 2^{a_2}, \ldots) = \{a_i\}$,
$\beta = (1^{b_1}, 2^{b_2}, \ldots) = \{b_i\}$.
Then

1. **Closure**.

   $$\pi + \alpha = \{f_i\} + \{a_i\} \equiv \{f_i + a_i \pmod{N_i + 1}\} \equiv \{h_i \pmod{N_i + 1}\},$$

   where $0 \le h_i \le N_i$. Since $\{N_i\} \in D(\lambda)$ and $h_i \le N_i$, it follows that $\pi + \alpha \in D(\lambda)$.

2. **Associativity**.

+	(0)	(1)	(2)	(2^2)	(1,2)	$(1,2^2)$
(0)	(0)	(1)	(2)	(2^2)	(1,2)	$(1,2^2)$
(1)	(1)	(0)	(1,2)	$(1,2^2)$	(2)	(2^2)
(2)	(2)	(1,2)	(2^2)	(0)	$(1,2^2)$	(1)
(2^2)	(2^2)	$(1,2^2)$	(0)	(2)	(1)	(1,2)
(1,2)	(1,2)	(2)	$(1,2^2)$	(1)	(2^2)	(0)
$(1,2^2)$	$(1,2^2)$	(2^2)	(1)	(1,2)	(0)	(2)

Table 3.1: The Cayley Table for $(D((1,2^2)),+)$

Let $\{f_i\} + \{a_i\} = \{h_i\} \in D(\lambda)$ which is guaranteed by Closure. Then

$$\begin{aligned}(\{f_i\} + \{a_i\}) + \{b_i\} &\equiv \{h_i\} + \{b_i\} \equiv \{h_i + b_i \pmod{N_i + 1}\}\\ &\equiv \{(f_i + a_i) + b_i \pmod{N_i + 1}\}\\ &\equiv \{f_i + (a_i + b_i) \pmod{N_i + 1}\}\\ &\equiv \{f_i\} + \{a_i + b_i\} \pmod{N_i + 1}\\ &\equiv \{f_i\} + (\{a_i\} + \{b_i\}) \pmod{N_i + 1}.\end{aligned}$$

Hence $(\pi + \alpha) + \beta = \pi + (\alpha + \beta)$.

3. **Identity Element**. See Lemma 3.1.3.

4. **Inverses**. See Lemma 3.1.4.

   Hence $D(\lambda)$ is a group under addition.

5. **Commutativity**. The set $D(\lambda)$ is clearly commutative under +.

Lastly, $D(\lambda)$ is a finite set, by Lemma 3.1.2.
Hence, the theorem is proved. □

Example 3.1.2. Given $\lambda = (1, 2, 2) = (1, 2^2)$. Then

$$D(\lambda) = \{(0), (1), (2), (2^2), (1, 2), (1, 2^2)\}.$$

The Cayley table of $D((1, 2^2))$ is shown in Table 3.1. From this table, $(D((1, 2^2)), +)$ is clearly closed under addition, and is commutative since the entries in the table are symmetrical with respect to the main diagonal.

1. In each column of the Table there is an identity partition $(0) = 0_\lambda$, showing that every partition in $D((1, 2^2))$ has an inverse.

Note that the zero element of $0_\lambda \in D((1,2^2))$ corresponds uniquely to the superpartition $(1^2, 2^3)$ via the residue classes of 0 modulo 2, and 0 modulo 3.
2. Take for instance, $(1,2),(2),(1,2^2) \in D(1,2^2)$, then we have

$$\begin{aligned}(1,2)+((2)+(1,2^2)) &= (1,2)+(1,2^3)\\ &= (1,2)+(1) = (1^2,2) = (2),\end{aligned}$$

$$\begin{aligned}((1,2)+(2))+(1,2^2) &= (1,2^2)+(1,2^2)\\ &= (1^2,2^4) = (2).\end{aligned}$$

This is true for all the partitions you pick in $D((1,2^2))$. Therefore $D((1,2^2))$ is associative. Hence $(D((1,2^2)),+)$ is an abelian group.

3.1.1 A Special Isomorphism

Consider the set of residue classes modulo a positive integer m:

$$\mathbb{Z}/m\mathbb{Z} = \{0+m\mathbb{Z}, 1+m\mathbb{Z}, 2+m\mathbb{Z}, \ldots, (m-1)+m\mathbb{Z}\}. \tag{3.6}$$

We may use the alternate notation $\mathbb{Z}_m \cong \mathbb{Z}/m\mathbb{Z}$ to abbreviate (3.6) to

$$\mathbb{Z}_m = \{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{(m-1)}\}, \tag{3.7}$$

where it is understood that $\overline{r}$ corresponds to $r+m\mathbb{Z}$ for all $r = 0,1,\ldots,m-1$. It is well known that $(\mathbb{Z}_m,+)$ is an abelian group, see for example, Fraleigh [13] with the following properties: The identity element of $(\mathbb{Z}_m,+)$ is $\overline{0} \equiv m \pmod{m}$ and the inverse of $\overline{a} \in \mathbb{Z}_m$ is $\overline{a} \equiv m-a \in \mathbb{Z}_m$.

Reducing the direct product definition in Example 2.1.4 to only abelian groups we deduce the following definition of direct product of any r groups $G_1, G_2, \ldots, G_r$:

$$G_1 \oplus G_2 \oplus \ldots \oplus G_r = \{(g_1, g_2, \ldots, g_r) \mid g_1 \in G_1, \ldots, g_r \in G_r\}. \tag{3.8}$$

We now prove our main theorem.

Theorem 3.1.6. *Let $\lambda = (\lambda_1^{n_1}, \cdots, \lambda_r^{n_r}) \vdash n$, $0 < \lambda_1 < \cdots < \lambda_r$ and $n_i > 0$ for all i. Then there is an isomorphism*

$$D(\lambda) \cong \mathbb{Z}_{n_1+1} \oplus \mathbb{Z}_{n_2+1} \oplus \cdots \oplus \mathbb{Z}_{n_r+1}. \tag{3.9}$$

Note that we use $+$ to denote addition in $D(\lambda)$ and use $\vee$ to denote addition in $\mathbb{Z}_{n_1+1} \oplus \cdots \oplus \mathbb{Z}_{n_r+1}$.

Proof. Note that each $\pi \in D(\lambda)$ has the form $\pi = (\lambda_1^{a_1}, \lambda_2^{a_2}, \cdots, \lambda_r^{a_r})$, where $0 \leq a_i \leq n_i$, $1 \leq i \leq r$, such that $a_i = 0$ whenever $\lambda_i \notin \pi$.

Define $\Omega : D(\lambda) \longmapsto \mathbb{Z}_{n_1+1} \oplus \cdots \oplus \mathbb{Z}_{n_r+1}$ by

$$\Omega(\pi) = (\overline{a_1}, \overline{a_2}, \ldots, \overline{a_r}).$$

Now let also $\beta \in D(\lambda)$ with $\beta = (\lambda_1^{b_1}, \cdots, \lambda_r^{b_r})$. Then we have

$$\begin{aligned}\Omega(\pi + \beta) &= \Omega\left((\lambda_1^{a_1}, \cdots, \lambda_r^{a_r}) + (\lambda_1^{b_1}, \cdots, \lambda_r^{b_r})\right)\\ &= \Omega((\lambda_1^{a_1+b_1}, \cdots, \lambda_r^{a_r+b_r}))\\ &= \Omega(\lambda_1^{h_1}, \cdots, \lambda_r^{h_r})\\ &= (\overline{h_1}, \cdots, \overline{h_r}),\end{aligned}$$

where $h_i \equiv a_i + b_i \pmod{n_i + 1}$ for $i = 1, \ldots, r$ in $D(\lambda)$.
Secondly,

$$\begin{aligned}\Omega(\pi) \vee \Omega(\beta) &= \Omega((\lambda_1^{a_1}, \cdots, \lambda_r^{a_r})) \vee \Omega((\lambda_1^{b_1}, \cdots, \lambda_r^{b_r}))\\ &= (\overline{a_1}, \ldots, \overline{a_r}) \vee (\overline{b_1}, \ldots, \overline{b_r})\\ &= (\overline{a_1 + b_1}, \ldots, \overline{a_r + b_r})\\ &= (\overline{h_1}, \ldots, \overline{h_r}),\end{aligned}$$

where $h_i \equiv a_i + b_i \pmod{n_i + 1}$ for $i = 1, \ldots, r$ in $\mathbb{Z}_{n_1+1} \oplus \cdots \oplus \mathbb{Z}_{n_r+1}$.
So

$$\Omega(\pi + \beta) = \Omega(\pi) \vee \Omega(\beta),$$

that is, Ω is a homomorphism.

Next, we have

$$\begin{aligned}\Omega(\pi) = \Omega(\beta) &\Longrightarrow (\overline{a_1}, \ldots, \overline{a_r}) = (\overline{b_1}, \ldots, \overline{b_r})\\ &\Longrightarrow \overline{a_1} = \overline{b_1}, \ldots, \overline{a_r} = \overline{b_r}\\ &\Longrightarrow \lambda_1^{a_1} = \lambda_2^{a_2}, \ldots, \lambda_r^{a_r} = \lambda_r^{b_r}\\ &\Longrightarrow (\lambda_1^{a_1}, \lambda_2^{a_2}, \cdots, \lambda_r^{a_r}) = (\lambda_1^{b_1}, \lambda_2^{b_2}, \cdots, \lambda_r^{b_r})\\ &\Longrightarrow \pi = \beta.\end{aligned}$$

So Ω is a monomorphism.

Next, we have to show that Ω is an epimorphism, i.e. Ω maps $D(\lambda)$ onto $\mathbb{Z}_{n_1+1} \oplus \cdots \oplus \mathbb{Z}_{n_r+1}$. Assume that $(u_1, \ldots, u_r) \in \mathbb{Z}_{n_1+1} \oplus \cdots \oplus \mathbb{Z}_{n_r+1}$. Then since each u_i is reduced modulo $n_i + 1$ it follows that $u_1 \leq n_1, \ldots, u_r \leq n_r$ which implies that $(\lambda_1^{u_1}, \ldots, \lambda_r^{u_r}) \in D(\lambda)$. So Ω is a epimorphism.

Lastly, note that since $(0) = (\lambda_1^{n_1+1}, \ldots, \lambda_r^{n_r+1})$, we have

$$\Omega((0)) = (\overline{n_1 + 1}, \ldots, \overline{n_r + 1}) \equiv \overline{0} \in \mathbb{Z}_{n_1+1} \oplus \cdots \oplus \mathbb{Z}_{n_r+1}.$$

Hence we conclude that Ω is an isomorphism. $\square$

3.2 Groups Generated by Parts of Partitions

Let k, m be positive integers. We say that the set of derived partitions of (k^m) is *generated* by k. We denote this set by $\langle k\rangle_m$, that is,

$$\langle k\rangle_m = \{(0), (k), (k^2), \ldots, (k^m)\}. \tag{3.10}$$

It is clear that

$$\langle k\rangle_m = D((k^m)). \tag{3.11}$$

Equation (3.10) can be extended to a group generated by positive integers k_1 and k_2 to give

$$\langle k_1, k_2\rangle_{(m_1,m_2)} = \langle k_1\rangle_{m_1} \cup \langle k_2\rangle_{m_2} \cup \langle k_1\rangle_{m_1}\langle k_2\rangle_{m_2},$$

where each partition in $\langle k_1\rangle\langle k_2\rangle$ is of the form $(k_1^{t_1}, k_2^{t_2})$ with $t_1, t_2 \geq 1$.

Again we see that

$$\langle k_1, k_2\rangle_{(m_1,m_2)} = D((k_1^{m_1}, k_2^{m_2})). \tag{3.12}$$

Remark 3.2.1. The rigorous distinction between the two sets is that any member $\lambda \in D((k_1^{m_1}, k_2^{m_2}))$ is a 2-tuple of the form $\lambda = (k_1^a, k_2^b)$ such that a or b or both may be 0, whereas a member of $\langle k_1, k_2\rangle_{(m_1,m_2)}$ is simply (0) or a t-tuple with non-zero multiplicities for $t = 1, 2$.

For example, consider the partition $\lambda = (1^2, 2)$. Then we have

$$D((1^2, 2)) = \{(0, 0), (1, 0), (1^2, 0), (0, 2), (1, 2), (1^2, 2)\}$$

and

$$\langle 1, 2\rangle_{(2,1)} = \{(0), (1), (1^2), (2), (1, 2), (1^2, 2)\}.$$

By Remark 3.2.1 the following general statement should be intuitively clear; but we give a formal proof.

Proposition 3.2.1. *The group of derived partitions of λ is the same as the group of partitions generated by the distinct parts of λ.*

Proof. Let $\lambda = (k_1^{m_1}, k_2^{m_2}, \cdots, k_r^{m_r})$. We prove the proposition by induction on r.

When $r = 1$ the result is given by Equation (3.11), and when $r = 2$ the result follows from Equation (3.12) and Remark 3.2.1.

When $r = 3$ we have

$$\begin{aligned}\langle k_1, k_2, k_3\rangle_{(m_1,m_2,m_3)} &= \langle k_1, k_2\rangle_{(m_1,m_2)} \cup \langle k_3\rangle_{m_3} \cup \langle k_1, k_2\rangle_{(m_1,m_2)}\langle k_3\rangle_{m_3}\\ &= D((k_1^{m_1}, k_2^{m_2})) \cup D((k_3^{m_3})) \cup \langle k_1, k_2\rangle_{(m_1,m_2)}\langle k_3\rangle_{m_3}.\end{aligned} \tag{3.13}$$

The union of two sets $D((k_1^{m_1}, k_2^{m_2})) \cup D((k_3^{m_3}))$ is a proper subset of $D((k_1^{m_1}, k_2^{m_2}, k_3^{m_3}))$ because it does not contain partitions of the forms $(k_i^{u_i}, k_3^{u_3}), i = 1, 2$ and $(k_1^{u_1}, k_2^{u_2}, k_3^{u_3})$ for $u_i > 0$. These missing partitions are provided by $\langle k_1, k_2\rangle_{(m_1,m_2)}\langle k_3\rangle_{m_3}$ which represents a Cartesian product pairing of non-empty partitions of the forms $(k_i^{u_i}), i = 1, 2$ or $(k_1^{u_1}, k_2^{u_2})$, with $(k_3^{u_3}), u_3 > 0$. Hence it follows from Equation (3.13) that

$$\langle k_1, k_2, k_3\rangle_{(m_1,m_2,m_3)} = D((k_1^{m_1}, k_2^{m_2}, k_3^{m_3})).$$

Now assume that the statement is true for $r = n$, that is,

$$\langle k_1, k_2, \cdots, k_r\rangle_{(m_1,m_2,\cdots,m_n)} = D((k_1^{m_1}, k_2^{m_2}, \cdots, k_n^{m_n})). \tag{3.14}$$

We show that it is true for $r = n + 1$.

$$\begin{aligned}\langle k_1, \cdots, k_{n+1}\rangle_{(m_1,\cdots,m_{n+1})} &= \langle k_1, \cdots, k_n\rangle_{(m_1,\cdots,m_n)} \cup \langle k_{n+1}\rangle_{m_{n+1}}\\ &\qquad \cup \langle k_1, \cdots, k_{n+1}\rangle_{(m_1,\cdots,m_n)}\langle k_{n+1}\rangle_{m_{n+1}}\\ &= D((k_1^{m_1}, \ldots, k_n^{m_n})) \cup D((k_{n+1}^{m_{n+1}}))\\ &\qquad \cup \langle k_1, \cdots, k_{n+1}\rangle_{(m_1,\cdots,m_n)}\langle k_{n+1}\rangle_{m_{n+1}},\end{aligned}$$

where the first equality follows from the case $r = 2$ and the second equality follows from the inductive hypothesis.

The union of two sets $D((k_1^{m_1}, \ldots, k_n^{m_n})) \cup D((k_{n+1}^{m_{n+1}}))$ is a proper subset of $D((k_1^{m_1}, \ldots, k_{n+1}^{m_{n+1}}))$ because it does not contain partitions of the forms

$$(k_{j_1}^{u_{j_1}}, k_{j_2}^{u_{j_2}}, \ldots, k_{j_n}^{u_{j_n}}, k_{n+1}^{u_{n+1}}),\ u_{j_i} > 0, \tag{3.15}$$

for all choices of t-sequences $(j_1, \ldots, j_t), 1 \leq j_1 < j_2 < \cdots < j_t$, from $\{1, \ldots, n\}$, where $t = 1, 2, \ldots, n$.

The missing partitions described in (3.15) are restored by

$$\langle k_1, \cdots, k_{n+1}\rangle_{(m_1,\cdots,m_n)}\langle k_{n+1}\rangle_{m_{n+1}}$$

which represents a Cartesian product pairing of non-empty partitions of the forms

$$(k_{j_1}^{u_{j_1}}, k_{j_2}^{u_{j_2}}, \ldots, k_{j_n}^{u_{j_n}}) \text{ with } (k_{n+1}^{u_{n+1}}), u_{n+1} > 0.$$

Hence, it follows from Equation (3.13) that

$$\langle k_1, k_2, \cdots, k_r\rangle_{(m_1,m_2,\cdots,m_n)} = D((k_1^{m_1}, k_2^{m_2}, \cdots, k_n^{m_n})).$$

Therefore, the statement is true for $r = n + 1$ whenever it holds for $r = n$, and since it is true for $r = 1$, it is true for all r. □

From the proof of Proposition 3.2.1 we deduce that the order of $D(\lambda)$, when $\lambda = (k_1^{m_1}, \cdots, k_r^{m_r})$, is the same as the number of distinct sublists that can be formed from the list of parts of λ, which equals the number of positive-integer divisors of the number $k_1^{m_1} \times \cdots \times k_r{}^{m_r}$, if the $k_i^{m_i}$ are treated as prime powers. We record this observation in the next result.

Corollary 3.2.2. *Let $\lambda = (k_1^{m_1}, k_2^{m_2}, \cdots, k_r^{m_r})$, $m_i > 0\ \forall\ i$. The order of the group $D(\lambda)$ is given by*

$$|D(\lambda)| = (m_1 + 1)(m_2 + 1) \cdots (m_r + 1). \tag{3.16}$$

Example 3.2.1. Consider $\lambda = (1^2, 2^3, 3) \vdash 11$. Then $|D(\lambda)| = (2+1)(3+1)(1+1) = 24$; where the enumerated partitions are:

$D(\lambda) = \{(1), (1^2), (2), (2^2), (2^3), (3), (1,2), (1,2^2), (1,2^3), (1,3), (2,3),$
$(1^2,2), (1^2,3), (2^3,3), (2^3,3), (1,2,3), (1^2,2,3), (1,2^2,3), (1,2^3,3),$
$(1^2,2^2,3), (1^2,2^3,3), (1^2,2^2), (1^2,2^3)\} \cup \{0\}$.

If λ is a partition into r distinct parts, then $|D(\lambda)| = 2^r$.

3.2.1 Cyclic Groups of Derived Partitions

Let $\lambda = (\lambda_1^{N_1}, \ldots, \lambda_k^{N_k}) \vdash n$. The partition obtained by adding λ to itself s times will be denoted by λ^s (instead of $s \cdot \lambda$), that is,

$$\underbrace{\lambda + \lambda + \cdots + \lambda}_{s \text{ times}} = \lambda^s = (\lambda_1^{N_1 \cdot s}, \ldots, \lambda_k^{N_k \cdot s}). \tag{3.17}$$

We will also use the following notation to denote the set of all partitions λ^s for all integers $s \geq 0$:

$$\langle\lambda\rangle := \{(\lambda_1^{N_1 \cdot s}, \ldots, \lambda_k^{N_k \cdot s}) \mid s \geq 0\}. \tag{3.18}$$

Lemma 3.2.3. *$(\langle\lambda\rangle, +)$ is a commutative additive group, where $+$ is defined as in the set $D(\lambda)$.*

Proof. Let $\pi, \beta \in \langle\lambda\rangle$, where $\pi = (\lambda_1^{N_1 \cdot s}, \ldots, \lambda_k^{N_k \cdot s})$ and $\beta = (\lambda_1^{N_1 \cdot t}, \ldots, \lambda_k^{N_k \cdot t})$, then

$$\pi + \beta = (\lambda_1^{N_1 \cdot (s+t)}, \ldots, \lambda_k^{N_k \cdot (s+t)}) = (\lambda_1^{N_1}, \ldots, \lambda_k^{N_k})^{(s+t)} \in \langle\lambda\rangle.$$

So $\langle\lambda\rangle$ is closed under $+$, using Definition 3.1.2 and also Theorem 3.1.5. When $s = 0$ we have $(\lambda_1^{N_1 \cdot 0}, \ldots, \lambda_k^{N_k \cdot 0}) = (0) \in \langle\lambda\rangle$.

Since $N_i \cdot s = N_i \cdot s + s - s = (N_i + 1) \cdot s - s \equiv N_i + 1 - s \pmod{N_i + 1}$, it follows that

$$\overline{\pi} = (\lambda_1^{N_1+1-s}, \dots, \lambda_k^{N_k+1-s}).$$

Thus the inverse is also in $\langle\lambda\rangle$. Lastly we have to show that $(\langle\lambda\rangle, +)$ is commutative, i.e.

$$\pi + \beta = (\lambda_1^{N_1\cdot(s+t)}, \dots, \lambda_k^{N_k\cdot(s+t)}) = (\lambda_1^{N_1}, \dots, \lambda_k^{N_k})^{(s+t)} \in \langle\lambda\rangle.$$

and

$$\beta + \pi = (\lambda_1^{N_1\cdot(t+s)}, \dots, \lambda_k^{N_k\cdot(t+s)}) = (\lambda_1^{N_1}, \dots, \lambda_k^{N_k})^{(t+s)} \in \langle\lambda\rangle.$$

$\Longrightarrow$

$$(\lambda_1^{N_1}, \dots, \lambda_k^{N_k})^{(s+t)} = (\lambda_1^{N_1}, \dots, \lambda_k^{N_k})^{(t+s)}.$$

Thus $(\langle\lambda\rangle, +)$ is commutative. $\square$

Definition 3.2.1. Let $\lambda = (\lambda_1^{N_1}, \dots, \lambda_k^{N_k}) \vdash n$.
Then $\langle\lambda\rangle$ will be referred to as the *cyclic group generated by* λ.
The group $D(\lambda)$ will be called *cyclic* if $D(\lambda) = \langle\lambda\rangle$.

When $k = 1$ we have a special case of Theorem 3.1.6 and obtain the following Corollary

Corollary 3.2.4. *There exists an isomorphism $\langle(\lambda^m)\rangle \cong \mathbb{Z}_{m+1} \cong C_{m+1}$ where C_{m+1} is the standard cyclic group of order $m+1$.*

Example 3.2.2. Let $\lambda = (2^7)$, then

$$\langle(2^7)\rangle = \{(0), (2), (2^2), (2^3), (2^4), (2^5), (2^6), (2^7)\} = D(2^7)$$

Each partition in the set $D(2^7)$ which has an odd multiplicity is a generator of $D(2^7)$. Thus $D(2^7)$ is a cyclic group and the order is $m + 1 = 8$.

Example 3.2.3. Let $\lambda = (1^2, 2)$. Then

$$\langle(1^2, 2)\rangle = \{(1^2, 2), (1), (2), (1^2), (1, 2), (0)\} = D((1^2, 2)).$$

Thus $D((1^2, 2)$ is a cyclic group

The next theorem characterizes all partitions λ that satisfy $D(\lambda) = \langle\lambda\rangle$.

Theorem 3.2.5. *Let $\lambda = (\lambda_1^{N_1}, \dots, \lambda_k^{N_k}) \vdash n$. Then $D(\lambda) = \langle\lambda\rangle$ if and only if λ has one distinct part or the $N_i + 1$ are pairwise coprime.*

Proof. The case $k = 1$ follows immediately from Corollary 3.2.4. So we assume that $k \geq 2$. Note that both $\langle\lambda\rangle$ and $D(\lambda)$ share the same zero element, namely, $(\lambda_1^{N_1+1}, \ldots, \lambda_k^{N_k+1}) \equiv (0)$. Moreover, $\lambda^s \in D(\lambda)$, $s > 0$, by Equation (3.17), since $N_i \cdot s \equiv N_i + 1 - s \pmod{N_i + 1}$ and $N_i + 1 - s \leq N_i$ for all i. It follows that $\langle\lambda\rangle \subseteq D(\lambda)$.

For the reverse inclusion let $(0) \neq \pi = (\lambda_1^{a_1}, \ldots, \lambda_k^{a_k}) \in D(\lambda)$ with $0 \leq a_i \leq N_i\ \forall\ i$. Then $\pi \in \langle\lambda\rangle$ if $\exists\ s > 0$ such that

$$(\lambda_1^{a_1}, \ldots, \lambda_k^{a_k}) = (\lambda_1^{N_1 \cdot s}, \ldots, \lambda_k^{N_k \cdot s}) \tag{3.19}$$

$$\begin{aligned} &\iff N_i \cdot s \equiv a_i \pmod{N_i + 1},\ 1 \leq i \leq k \\ &\iff s \equiv -a_i \pmod{N_i + 1},\ 1 \leq i \leq k. \end{aligned} \tag{3.20}$$

Since $\gcd(N_i + 1, N_j + 1) = 1$ for $i \neq j$, the system of k congruences (3.20) has a common solution modulo $M = (N_1 + 1)(N_2 + 1) \cdots (N_k + 1)$ by the Chinese Remainder Theorem, see Theorem 2.3.1, and also Example 2.3.1.

This implies that the s appearing on the right-hand-side of the equation

$$(\lambda_1^{a_1}, \ldots, \lambda_k^{a_k}) = (\lambda_1^{N_1 \cdot s}, \ldots, \lambda_k^{N_k \cdot s})$$

exists; so $\pi \in \langle\lambda\rangle$. Therefore $D(\lambda) \subseteq \langle\lambda\rangle$. Hence $D(\lambda) = \langle\lambda\rangle$.

Conversely, we note that if $\gcd(N_i + 1, N_j + 1) > 1$ for some indices i, j, the system (3.20) either may not have a solution, or there may be a solution modulo $\operatorname{lcm}(N_1 + 1, \ldots, N_k + 1)$ which is less than M. This is because $\operatorname{lcm}(N_1 + 1, \ldots, N_k + 1)$ becomes smaller when the $N_i + 1$ are not pairwise coprime. Since $|D(\lambda)| = M$, it follows that λ cannot generate the entire group, that is, we must have $|\langle\lambda\rangle| < |D(\lambda)|$. So $D(\lambda)$ is not cyclic. $\square$

Example 3.2.4. Using Example 3.2.1 we can now verify Theorem 3.2.5. In addition we find that $\langle(1^2, 2^2, 3)\rangle \neq D((1^2, 2^2, 3))$ since, according to the theorem $N_1 = 2, N_2 = 2, N_3 = 1$ and $\gcd(2 + 1, 2 + 1) = \gcd(3, 3) = 3 > 1$. In fact we have,

$$\langle(1^2, 2^2, 3)\rangle = \{(1^2, 2^2, 3), (1, 2), (3), (1^2, 2^2), (1, 2, 3), (0)\}.$$

So $|\langle(1^2, 2^2, 3)\rangle| = 6$, but $|D((1^2, 2^2, 3))| = (2 + 1)(2 + 1)(1 + 1) = 18$.

Lastly, one can also verify that the multiplicities of $\lambda = (1^6, 2^4, 4^2, 5^3)$ satisfy the requirement $\gcd(N_i + 1, N_j + 1) = 1$, $i \neq j$. So by Theorem 3.2.5 we have $D((1^6, 2^4, 4^2, 5^3)) = \langle(1^6, 2^4, 4^2, 5^3)\rangle$.

3.3 Subgroups

Definition 3.3.1. Let $A \subseteq D(\lambda)$. Then $(A, +)$ is a subgroup of $(D(\lambda), +)$ if A forms a group under the operation $+$ with the same identity element 0_λ.

If $(A, +)$ is a subgroup of $(D(\lambda), +)$, we also write $(A, +) \leq (D(\lambda), +)$, or simply $A \leq D(\lambda)$ when the group operation is clear.

Example 3.3.1. Let $\lambda = (1^2, 2^2, 3) \vdash 9$, and consider the set

$A = \{(0), (3), (1, 2), (1^2, 2^2), (1, 2, 3), (1^2, 2^2, 3)\}$.

We show that $A \leq D((1^2, 2^2, 3))$. Note that each $\alpha \in A$ has the form $\alpha = (1^a, 2^b, 3^c)$, where $0 \leq a \leq 2$, $0 \leq b \leq 2$ and $0 \leq c \leq 1$. Thus $\alpha \in A \implies \alpha \in D((1^2, 2^2, 3))$. Hence $A \subseteq D((1^2, 2^2, 3))$.

1. Closure.
A is closed under $+$. For example, $(1, 2) + (3) = (1, 2, 3) \in A$, $(1^2, 2^2) + (1^2, 2^2, 3) = (1^4, 2^4, 3) \equiv (1, 2, 3) \in A$, etc. All the sums are shown in Table 3.2.

2. Associativity.

$$((1, 2) + (1^2, 2^2)) + (1, 2, 3) = (0) + (1, 2, 3) = (1, 2, 3)$$

and

$$\begin{aligned} (1, 2) + ((1^2, 2^2) + (1, 2, 3)) &= (1, 2) + (1^3, 2^3, 3) \\ &= (1, 2) + (3) = (1, 2, 3) \end{aligned}$$

Thus $((1, 2) + (1^2, 2^2)) + (1, 2, 3) = (1, 2) + ((1^2, 2^2) + (1, 2, 3))$, etc.

3. Identity. This is clear looking at the second row and second column of Table 3.2.

4. Additive Inverses.

$$\begin{aligned} &(1, 2) + (1^2, 2^2) = (1, 2, 3^0) + (1^2, 2^2, 3^0) = (1^4, 2^4, 3^0) \equiv (0) \\ &(1^2, 2^2, 3) + (1, 2, 3) = (1^3, 2^3, 3^2) \equiv (0), \end{aligned}$$

where the converse additions follow similarly, etc.

Hence, $A \leq D((1^2, 2^2, 3))$ as desired.

Definition 3.3.2. Let $\lambda \vdash n$. A subgroup A of $D(\lambda)$ is called *cyclic* if there is a partition $\pi \in D(\lambda)$ such that $A = \langle \pi \rangle$.

Thus, cyclic subgroups of $D(\lambda)$ have the form $\langle \pi \rangle \subseteq D(\lambda)$.

Definition 3.3.3. If $D(\lambda)$ is a group, then the cyclic subgroup consisting of $D(\lambda)$ itself is the improper cyclic subgroup of $D(\lambda)$

$+$	(0)	$(1{,}2)$	(3)	$(1^2 2^2)$	(123)	$(1^2 2^2 3)$
(0)	(0)	(12)	(3)	$(1^2 2^2)$	(123)	$(1^2 2^2 3)$
(12)	(12)	$(1^2 2^2)$	(123)	(0)	$(1^2 2^2 3)$	(3)
(3)	(3)	(123)	(0)	$(1^2 2^2 3)$	(12)	$(1^2 2^2)$
$(1^2 2^2)$	$(1^2 2^2)$	(0)	$(1^2 2^2 3)$	(12)	(3)	(123)
(123)	(123)	$(1^2 2^2 3)$	(12)	(3)	$(1^2 2^2)$	(0)
$(1^2 2^2 3)$	$(1^2 2^2 3)$	(3)	$(1^2 2^2)$	(123)	(0)	(12)

Table 3.2: Cayley table for the subgroup $A \leq D((1^2, 2^2, 3))$

We have already discussed the improper cyclic subgroup $\langle\lambda\rangle = D(\lambda)$ as defined in Definition 3.3.3, that is, $\langle\lambda\rangle \leq D(\lambda)$, and determined when equality holds in general, see Theorem 3.2.5.

For Example, we found that $\langle(1^2, 2)\rangle = D((1^2, 2))$ (that is, an improper cyclic subgroup), and $\langle(1^3, 2)\rangle < D((1^3, 2))$ (that is, a proper cyclic subgroup).

When $\pi \in D(\lambda)$, any of the following three situations may arise:

(1) $D(\pi) = \langle\pi\rangle \leq D(\lambda)$;

(2) $D(\pi) \neq \langle\pi\rangle \leq D(\lambda)$;

(3) $\langle\pi\rangle \neq D(\pi) \leq D(\lambda)$.

Proposition 3.3.1 (The Subgroup Rule)**.** *Let* $\lambda = (\lambda_1^{N_1}, \ldots, \lambda_k^{N_k}) \vdash n$, *and let* $\pi = (\lambda_1^{a_1}, \ldots, \lambda_k^{a_k}) \in D(\lambda)$. *Then*

$$D(\pi) \leq D(\lambda) \iff a_i \neq 0 \implies a_i = N_i. \text{ for } 1 \leq i \leq k. \tag{3.21}$$

In other words, $D(\pi)$ *is a subgroup if and only if the parts of* π *have maximal multiplicities relative to the corresponding multiplicities of* λ.

Proof. Assume that $D(\pi) \leq D(\lambda)$. Then $0_\pi = 0_\lambda$, that is,

$$(\lambda_1^{a_1+1}, \ldots, \lambda_k^{a_k+1}) = (\lambda_1^{N_1+1}, \ldots, \lambda_k^{N_k+1})$$

$$\implies a_1 = N_1, a_2 = N_2, \ldots, a_k = N_k.$$

Conversely, assume that $a_x < N_x$ for some index $x \in \{1, \ldots, k\}$, where $a_x \neq 0$. Let $\lambda_x^{a_x} \in \pi$ such that $a_x < y \leq N_x$. Therefore, $\lambda_x^y \notin D(\pi)$. However, if $\alpha, \beta \in D(\pi)$ with $\lambda_x^{a_x} \in \alpha$ and $\lambda_x^{y-a_x} \in \beta$, then $\lambda_x^y \in \alpha + \beta$, which implies that $\alpha + \beta \notin D(\pi)$. So the subset $D(\pi)$ is not closed under addition, and so $D(\pi) \not\leq D(\lambda)$. □

The respective situations (1), (2), (3), may now be described precisely as follows:

(1a) If the condition (3.21) is fulfilled together with Theorem 3.2.5, then we have $D(\pi) = \langle \pi \rangle \leq D(\lambda)$.

(2a) If the condition (3.21) is violated, then $D(\pi) \not\leq D(\lambda)$ and we have $D(\pi) \neq \langle \pi \rangle \leq D(\lambda)$.

(3a) If the condition (3.21) is fulfilled and Theorem 3.2.5 is violated, then we have $\langle \pi \rangle \neq D(\pi) \leq D(\lambda)$, where also $\langle \pi \rangle \leq D(\lambda)$.

Example 3.3.2. We illustrate the Proposition 3.3.1, and the three situations with the following examples. Let $\lambda = (1^3, 2^3, 3^2) \vdash 15$.

(1b) Note that the examples in Section 3.2.1 in which $\langle \pi \rangle = D(\pi)$ are (trivial) examples of this class of partitions.

In addition let $\pi = (1^3, 3^2)$. Then

$$\langle (1^3, 3^2) \rangle = D((1^3, 3^2)) \leq D((1^3, 2^3, 3^2)),$$

where

$$\begin{aligned}\langle (1^3, 3^2) \rangle = \{&(1^3, 3^2), (1^2, 3), (1), (3^2), (1^3, 3), (1^2), (1, 3^2), (3), (1^3),\\ &(1^2, 3^2), (1, 3), (0)\}.\end{aligned}$$

(2b) Let $\pi = (2^2, 3)$. Then

$$D((2^2, 3)) \not\leq D((1^3, 2^3, 3^2)) \text{ and } \langle (2^2, 3) \rangle \leq D((1^3, 2^3, 3^2)),$$

where

$$D((2^2, 3)) = \{(0), (2), (3), (2^2), (2, 3), (2^2, 3)\},$$
$$\langle (2^2, 3) \rangle = \{(2^2, 3), (3^2), (2^2), (3), (2^2, 3^2), (0)\}.$$

Notice that

$(2) + (2^2) = (2^3) \notin D((2^2, 3))$ and $(3) + (2, 3) = (2, 3^2) \notin D((2^2, 3))$.

In a similar manner one can verify that

$$D((2^2, 3^2)) \not\leq D((1^3, 2^3, 3^2)) \text{ and } \langle (2^2, 3^2) \rangle \leq D((1^3, 2^3, 3^2)),$$

where $|D((2^2, 3^2))| = 9$ and $|\langle (2^2, 3^2) \rangle| = 6$.
It is found that $(2) + (2^2) = (2^3) \notin D((2^2, 3^2))$.

(3b) Let $\pi = (1^3, 2^3)$. Then

$$D((1^3,2^3)) \leq D((1^3,2^3,3^2)),\ \langle(1^3,2^3)\rangle \leq D((1^3,2^3,3^2))$$
$$\text{and}\quad D((1^3,2^3)) \neq \langle(1^3,2^3)\rangle,$$

where

$$D((1^3,2^3)) = \{(0),(1),(2),(1^2),(1,2),(2^2),(1^3),(1^2,2),(1,2^2),$$
$$(2^3),(1^3,2),(1^2,2^2),(1,2^3),(1^3,2^2),(1^2,2^3),(1^3,2^3)\};$$

$$\langle(1^3,2^3)\rangle = \{(1^3,2^3),(1^2,2^2),(1,2),(0)\}.$$

We see that $|D((1^3,2^3))| = 16$ and $|\langle(1^3,2^3)\rangle| = 4$.

Chapter 4

The Group of Pseudo-Perfect Partitions

In this chapter, we will discuss two special groups of derived partitions which are connected to perfect partitions.

4.1 Perfect Partitions

The definition of a perfect partition first appeared in the works of Percy Alexander MacMahon [18, 19]. Subsequently other mathematicians studied and found several properties of perfect partitions, see for example, Agarwal-Subbarao [1, 17, 24].

Definition 4.1.1. A *perfect partition* is a partition in which the parts contain exactly one partition of every positive integer smaller than n.

That is, if $\lambda \vdash n$ is a perfect partition, then $D(\lambda)$ contains exactly one partition of each number from 0 to n. This implies that $|D(\lambda)| = n + 1$.

For example, $(1^3, 4) \vdash 7$ is a perfect partition since

$$D((1^3, 4)) = \{(0), (1), (1^2), (1^3), (4), (1, 4), (1^2, 4), (1^3, 4)\}$$

which contains $|D((1^3, 4))| = 8$ partitions consisting of one partition of $0, 1, \ldots, 7$, respectively. All of the perfect partitions of 7 are listed below

$$(1^7), (1, 2^3), (1^3, 4), (1, 2, 4).$$

The enumeration of perfect partitions is facilitated by the enumeration of ordered factorizations of integers into products of factors greater than 1. In an *ordered factorization* the sequence order matters. For example, invoking the Fundamental Theorem of Arithmetic, there is only one factorization of 12 into three factors, namely $2 \cdot 2 \cdot 3$. But there are three ordered factorizations, namely, $2 \cdot 2 \cdot 3$, $2 \cdot 3 \cdot 2$ and $3 \cdot 2 \cdot 2$.

Let $H(n,k)$ denote the number of ordered factorizations of n into k factors > 1, and let the prime factorization of n be $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. MacMahon [19] proved the following computing formula for $H(n,k)$ and clearer derivation was given by Andrews [7]:

$$H(n,k) = \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} \prod_{j=1}^{k} \binom{\alpha_j + k - i - 1}{\alpha_j}.$$

Theorem 4.1.1 (MacMahon)**.** *The number of perfect partitions of n is the same as the number of ordered factorizations of $n+1$ without 1's.*

Proof. This theorem can be proved with an easy bijection of MacMahon. Let $n+1 = g_1 g_2 \cdots g_r$, $g_i > 1$, and r a positive number, be an ordered factorization of $n+1$. Then the corresponding perfect partition of n is given by

$$\lambda = \left(1^{g_1-1}, g_1^{g_2-1}, (g_1g_2)^{g_3-1}, \ldots, (g_1g_2\cdots g_{r-1})^{g_r-1}\right). \tag{4.1}$$

The weight $w(\lambda)$ of λ may be found by directly adding the parts in (4.1):

$$\begin{aligned}
w(\lambda) &= 1(g_1-1) + g_1(g_2-1) + (g_1g_2)(g_3-1) + \cdots + (g_1g_2\cdots g_{r-1})(g_r-1) \\
&= g_1g_2 - 1 + (g_1g_2)(g_3-1) + \cdots + (g_1g_2\cdots g_{r-1})(g_r-1) \\
&= g_1g_2g_3 - 1 + (g_1g_2g_3)(g_4-1) + \cdots + (g_1g_2\cdots g_{r-1})(g_r-1) \\
&\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
&= g_1g_2g_3\cdots g_{r-1}g_r - 1 \\
&= (n+1) - 1 = n.
\end{aligned}$$

On the other hand, we can calculate the order of $D(\lambda)$ based on (4.1) as follows.

$$\begin{aligned}
|D(\pi)| &= (g_1 - 1 + 1)(g_2 - 1 + 1)(g_3 - 1 + 1)\ldots(g_r - 1 + 1) \\
&= g_1g_2g_3\ldots g_r \\
&= n+1,
\end{aligned}$$

since $g_1g_2g_3\ldots g_r$ is a factorization of $n+1$. $\square$

Example 4.1.1. We give the ordered factorizations of 8 which correspond to the perfect partitions of 7 in Table 4.1.

Perfect Partition of 7	(1^7)	$(1,2^3)$	$(1^3,4)$	$(1,2,4)$
Ord. Factorization of 8	8	$2\cdot 4$	$4\cdot 2$	$2\cdot 2\cdot 2$

Table 4.1: Perfect Partitions of 7 and ordered factorizations of 8

We conclude this section with the question: "When is a perfect partition self-conjugate?" The following theorem may be deduced from the proof of MacMahon's theorem, Theorem 4.1.1.

Theorem 4.1.2. *Let n be a positive integer and let $n+1 = g_1 g_2 \cdots g_r$ be an ordered factorization of $n+1$ with $g_i > 1$, $i = 1, \ldots, r$.*
(i) *The perfect partition*

$$\lambda = \left(1^{g_1-1}, g_1^{g_2-1}, (g_1 g_2)^{g_3-1}, \ldots, (g_1 g_2 \cdots g_{r-1})^{g_r-1}\right).$$

is self-conjugate if and only if $g_r = r = 2$.
(ii) *Every odd number n has a unique self-conjugate perfect partition of the form $(1^{g-1}, g)$, where $n = 2g-1$. An even number has no self-conjugate perfect partition.*

Proof. (i) Note that since λ is a perfect partition it must contain 1 as the smallest part which means that the multiplicity of the largest part of λ is 1. From the bijective proof of Theorem 4.1.1, it follows that the corresponding ordered factorization of $n+1$ must have 2 as the last factor, that is, $g_r = 2$. Since $n+1 = g_1 g_2 \cdots g_{r-1} \cdot 2$, λ must be a partition of an odd number n.

If we adapt the self-conjugation pattern in (1.2) to the increasing partition λ in (4.1) we obtain

$$g_1 g_2 \cdots g_j = g_r + g_{r-1} + \cdots + g_{r-j} - j + 1, \; 1 \le j \le r-1,$$

where if we set $g_0 = 1$, we get $g_0 = g_r - 1 = 2 - 1 = 1$.

But when $j = 1$, we get

$$g_1 = g_r + g_{r-1} - 2 = 2 + g_{r-1} - 2 = g_{r-1} \implies r = 2.$$

Hence $g_r = r = 2$.
(ii) Since $g_r = r = 2$ our ordered factorization has the form $n+1 = g \cdot 2$. From (4.1) the corresponding perfect partition of n is $(1^{g-1}, g)$, and we see that $n = 2g-1$. □

4.2 Group of Pseudo-Perfect Partitions

We generalize the concept of perfect partitions with the following definition.

Definition 4.2.1. A partition π of n is called *pseudo-perfect* if whenever the parts contain the partition of m, they contain exactly one partition of m.

Thus every perfect partition is pseudo-perfect but not conversely in general. For instance $(1, 2, 4)$ and $(1, 3^2)$ are both pseudo-perfect partitions of 7 of which $(1, 2, 4)$ is perfect. Note that
$D((1, 3^2)) = \{(0), (1), (3), (1, 3), (3^2), (1, 3^2)\}$ has order 6 while perfect partitions of 7 has order 8.

$$\text{If } \pi \vdash n \text{ is pseudo-perfect, then } |D(\pi)| \leq n + 1.$$

Whenever a partition π is a pseudo-perfect partition then $D(\pi)$ is a *group of pseudo-perfect partitions*. Thus $D((1, 2, 4))$ and $D((1, 3^2))$ are both groups of pseudo-perfect partitions.

Given a pseudo-perfect partition $\pi \vdash n$, let us call the group $D(\pi)$ 'large' if π is a perfect partition since $D(\pi)$ now has the greatest order of $n + 1$. Thus $D((1, 2, 4))$ is large and $D((1, 3^2))$ is not.

Lastly, we note that the cyclic group $\langle k^m \rangle$ is a group of pseudo-perfect partitions which is large if and only if $k = 1$.

Now we want to look at infinite groups of pseudo-perfect partitions, $D(\pi_\infty)$.

4.3 Infinite Group of Pseudo-Perfect Partitions

The perfect partition of infinity π_∞ is defined as

$$\lambda_\infty = (1^{g_1-1}, g_1^{g_2-1}, (g_1 g_2)^{g_3-1}, \cdots, (g_1 g_2 \cdots g_{r-1})^{g_r-1}, \cdots) \tag{4.2}$$

for any arbitrary sequence $\{g_j\}_{j=1}^{\infty}$, $2 \leq g_j < \infty$ for all j. This is obtained when r tends to infinity in (4.1).

Clearly the set $D(\lambda_\infty)$ of derived partitions generated by the parts of λ_∞ contains exactly one pseudo-perfect partition of every positive integer n.

We show that $D(\lambda_\infty)$ forms an infinite group of pseudo-perfect partitions.

Since each term of the sequence $\{g_j\}_{j=1}^{\infty}$ is a finite positive integer it follows that each part of λ_∞ is not only a finite integer but also appears a finite number of times.

Thus since the group operation is defined partwise, we have that $D(\lambda_\infty)$ is a group in which the modulus is M_r for any r, $1 \le r < \infty$, where $\{M_r - 1\}_{r=1}^{\infty} = \lambda$ as in

$$\lambda = \left(1^{g_1-1}, g_1^{g_2-1}, (g_1g_2)^{g_3-1}, \ldots, (g_1g_2\cdots g_{r-1})^{g_r-1}\right).$$

We observe that there exists a unique increasing chain of pseudo-perfect partitions associated with $D(\lambda_\infty)$ which is given by

$$D((0)) \subset D(\Lambda_1) \subset D(\Lambda_2) \cdots \subset D(\Lambda_k) \cdots$$

where Λ_k is given by

$$\Lambda_k = \left(1^{g_1-1}, g_1^{g_2-1}, (g_1g_2)^{g_3-1}, \ldots, (g_1g_2\cdots g_{k-1})^{g_k-1}\right).$$

Example 4.3.1. Consider a random infinite sequence, say, 2,2,2,3,4,4,..., that is,

$$\{g_i\}_{i=1}^{\infty} = \{2, 2, 2, 3, 4, 4, \cdots\}.$$

Applying equation (4.2) gives

$$\begin{aligned}\pi_\infty &= (1, 2, (2\cdot 2)^{2-1}(2\cdot 2\cdot 2)^{3-1}(2\cdot 2\cdot 2\cdot 3)^{4-1}(2\cdot 2\cdot 2\cdot 3\cdot 4)^{4-1}, \ldots)\\ &= (1, 2, 4, 8^2, (24)^3, (96)^3, \ldots).\end{aligned}$$

We note that each partial segment is a perfect partition, that is,

$$(1),\ (1, 2),\ (1, 2, 4),\ (1, 2, 4, 8^2),\ (1, 2, 4, 8^2, (24)^3), \ldots$$

with respective weights $1, 3, 7, 23, 95, \ldots$.

The increasing chain of groups of pseudo-perfect partitions in this case starts with

$$D((0)) \subset D(\pi_1) \subset D(\pi_2) \subset D(\pi_3) \subset \ldots \subset D(\pi_k) \ldots,$$

with respective orders $1, 2, 4, 8, 24, 96, \ldots$.

The $D(\pi_i)$ are specified below.

$$\begin{aligned}
&\{(0)\} \subset \{(0), (1)\} \subset \{(0), (1), (2), (1, 2)\}\\
&\subset \{(0), (1), (2), (1, 2), (4), (1, 4), (2, 4), (1, 2, 4)\}\\
&\subset \{(0), (1), (2), (12), (4), (1, 4), (2, 4), (1, 2, 4), (8), (1, 8), (2, 8), (1, 2, 8),\\
&\qquad (4, 8), (1, 4, 8), (2, 4, 8), (1, 2, 4, 8), (8^2), (1, 8^2), (2, 8^2), (1, 2, 8^2),\\
&\qquad (4, 8^2), (1, 4, 8^2), (2, 4, 8^2), (1, 2, 4, 8^2)\}\\
&\subset \{(0), (1), (2), (1, 2), (4), (1, 4), (2, 4), (1, 2, 4), (8), (1, 8), (2, 8), (1, 2, 8),\\
&\qquad (4, 8), (1, 4, 8), (2, 4, 8), (1, 2, 4, 8), (8^2), (1, 8^2), (2, 8^2), (1, 2, 8^2),\\
&\qquad (4, 8^2), (1, 4, 8^2), (2, 4, 8^2), (1, 2, 4, 8^2), (24), (1, 24), (2, 24),\\
&\qquad (1, 2, 24), (4, 24), \ldots, (1, 2, 4, 8^2, 24^3)\}\\
&\subset \ldots.
\end{aligned}$$

From the above example we obtain the following ascending chain of proper subgroups.

$$D((0)) < D(\pi_1) < D(\pi_2) < D(\pi_3) < D(\pi_4) < \cdots$$

which implies that the order of $D(\pi_\infty)$ is infinite.

Finally, we note that $\bigcap_{k=1}^{\infty} D(\pi_k) = \{(0)\}$, the trivial subgroup of $D(\pi_\infty)$.

Summary.

(1) For every infinite sequence of integers, λ_∞ contains a unique partition of every positive integer n.

(2) If we truncate the sequence in (1) and apply, say $h_1, h_2, \ldots, h_k$, $h_k > 1$, then we obtain a (perfect) partition which contains a unique partition of each integer nonnegative $m \le n = h_1 h_2 \cdots h_k - 1$.

Chapter 5

The Ring of Derived Partitions

In Chapter 2, we outlined the essential theory of rings. In this chapter, we will extend the group of derived partitions by defining a natural ring structure. We will also study few basic properties that arise.

5.1 The Multiplicative Monoid

Definition 5.1.1. Let $\{f_i\}, \{g_i\} \in D(\lambda)$. We denote the multiplication operation in $D(\lambda)$ by $*$ and define

$$\{f_i\} * \{g_i\} = \{h_i\} \iff f_i g_i \equiv h_i \pmod{N_i + 1},$$

where $0 \leq h_i \leq N_i, i \geq 1$. Note that $h_i = 0$ if and only if $f_i \equiv 0 \pmod{(N_i+1)}$ or $g_i \equiv 0 \pmod{(N_i+1)}$.

In other words

$$(1^{f_1}, 2^{f_2}, \ldots) * (1^{g_1}, 2^{g_2}, \ldots) = (1^{f_1 g_1}, 2^{f_2 g_2}, \ldots) = (1^{h_1}, 2^{h_2}, \ldots)$$

where $(f_1 g_1, f_2 g_2, \ldots) \equiv (h_1, h_2, \ldots) \pmod{(N_i+1)}$ termwise.

Now we consider the ordered pair $(D(\lambda), *)$, where

$$\lambda = (\lambda_1^{N_1}, \lambda_2^{N_2}, \ldots, \lambda_t^{N_t}),\ 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_t,\ N_i > 0\ \forall\ i.$$

Assume that π, α, $\beta \in D(\lambda)$, where, in order to avoid zero exponents, we have

$$\begin{aligned} \pi &= (\lambda_1^{f_1}, \lambda_2^{f_2}, \ldots, \lambda_t^{f_t}),\ 0 < f_i \leq N_i\ \forall\ i, \\ \alpha &= (\lambda_1^{a_1}, \lambda_2^{a_2}, \ldots, \lambda_t^{a_t}),\ 0 < a_i \leq N_i\ \forall\ i, \\ \beta &= (\lambda_1^{b_1}, \lambda_2^{b_2}, \ldots, \lambda_t^{b_t}),\ 0 < b_i \leq N_i\ \forall\ i. \end{aligned} \tag{5.1}$$

Proposition 5.1.1. *$(D(\lambda), *)$ is a (finite) commutative monoid, that is, a commutative multiplicative semigroup with identity element.*

Proof. From Definition 5.1.1, we know that $\pi * \alpha \in D(\lambda)$ for all $\pi, \alpha \in D(\lambda)$. So $(D(\lambda), *)$ has the **closure** property.

Since $(D(\lambda), *)$ is closed, we know that $\pi * \beta = \gamma$ for some $\gamma \in D(\lambda)$. Let $\gamma = (\lambda_1^{h_1}, \ldots, \lambda_t^{h_t})$, $0 < h_i \leq N_i$. Then

$$\begin{aligned}
(\pi * \alpha) * \beta &= \Big((\lambda_1^{f_1}, \ldots, \lambda_t^{f_t}) * (\lambda_1^{a_1}, \cdots, \lambda_t^{a_t}) \Big) * (\lambda_1^{b_1}, \ldots, \lambda_t^{b_t}) \\
&\equiv (\lambda_1^{h_1}, \ldots, \lambda_t^{h_t}) * (\lambda_1^{b_1}, \ldots, \lambda_t^{b_t}) \\
&\equiv (\lambda_1^{h_1 b_1 \pmod{N_1+1}}, \ldots, \lambda_t^{h_t b_t \pmod{N_1+1}}) \\
&\equiv (\lambda_1^{f_1(a_1 b_1) \pmod{N_1+1}}, \ldots, \lambda_t^{f_t(a_t b_t) \pmod{N_1+1}}) \\
&\equiv (\lambda_1^{f_1}, \cdots, \lambda_t^{f_t}) * (\lambda_1^{(a_1 b_1) \pmod{N_1+1}}, \ldots, \lambda_t^{(a_t b_t) \pmod{N_1+1}}).
\end{aligned}$$

Hence $(\pi * \alpha) * \beta = \pi * (\alpha * \beta)$ and **associativity** holds.

The **multiplicative identity** is $1_\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$. To see this, we have

$$\begin{aligned}
\pi * 1_\lambda &= (\lambda_1^{f_1}, \ldots, \lambda_t^{f_t}) * (\lambda_1, \lambda_2, \ldots, \lambda_t) \\
&= (\lambda_1^{f_1 \cdot 1}, \ldots, \lambda_t^{f_t \cdot 1}) \\
&= (\lambda_1^{f_1}, \ldots, \lambda_t^{f_t}) \\
&= \pi.
\end{aligned}$$

Similarly, we can show that $1_\lambda * \pi = \pi$.

The **commutativity** property holds immediately because

$$\begin{aligned}
\pi * \alpha &= (\lambda_1^{f_1 a_1}, \ldots, \lambda_t^{f_t a_t}) \\
&= (\lambda_1^{a_1 f_1}, \ldots, \lambda_t^{a_t f_t}) \\
&= \alpha * \pi.
\end{aligned}$$

Hence $(D(\lambda), *)$ is a (finite) commutative monoid. □

The following result shows that multiplicative inverses do not exist for all partitions in $(D(\lambda), *)$.

Proposition 5.1.2. *Let $\lambda_i = (\lambda_1^{N_1}, \ldots, \lambda_k^{N_k})$. A partition $\pi = (\lambda_1^{f_1}, \cdots, \lambda_t^{f_t}) \in D(\lambda)$ has a multiplicative inverse if and only if $\lambda_i > 0$ and gcd $(f_i, N_i{+}1) = 1$, for all i.*

Proof. A partition $\pi \in (D(\lambda), *)$ is invertible if $\exists\ \alpha \in (D(\lambda), *)$ such that $\pi * \alpha = 1_\lambda$. The inverse of $\pi = (\lambda_1^{f_1}, \cdots, \lambda_t^{f_t})$ is given by $\alpha = (\lambda_1^{a_1}, \ldots, \lambda_t^{a_t})$ provided that the a_i satisfy

$$f_i a_i \equiv 1 \pmod{N_i + 1}. \tag{5.2}$$

But the congruence (5.2) has a unique solution only if $\gcd(f_i, N_i + 1) = 1$. Thus if f_i and $N_i + 1$ are not relatively prime for some index i, then π has no multiplicative inverse. Hence the proof. □

The foregoing discussion is summarized in the following theorem.

Theorem 5.1.3. *Let $\lambda = (\lambda_1^{N_1}, \cdots, \lambda_t^{N_t})$, $0 < \lambda_1 < \ldots < \lambda_t$. Then $(D(\lambda), *)$ is a finite commutative monoid.*

Example 5.1.1. Let $\lambda = (2^2, 5)$ and show that $D((\lambda), *)$ is a commutative monoid.

$$D(\lambda) = \{(0), (2), (2^2), (5), (2, 5), (2^2, 5)\},$$

with $0_\lambda = (2^3, 5^2) \equiv (0)$ and $1_\lambda = (2, 5) \equiv (0)$. The Cayley Table is shown in Table 5.1.

$*$	(0)	(2)	(2^2)	(5)	(2,5)	$(2^2, 5)$
(0)	(0)	(0)	(0)	(0)	(0)	(0)
(2)	(0)	(2)	(2^2)	(0)	(2)	(2^2)
(2^2)	(0)	(2^2)	(2)	(0)	(2^2)	(2)
(5)	(0)	(0)	(0)	(5)	(5)	(5)
(2,5)	(0)	(2)	(2^2)	(5)	(2,5)	$(2^2, 5)$
$(2^2, 5)$	(0)	(2^2)	(2)	(5)	$(2^2, 5)$	$(2, 5)$

Table 5.1: The Cayley Table for $(D(2^2, 5), *)$

Note, for instance, that

$$\begin{aligned}
&(2^2) * (2) = (2^{2 \cdot 1}) = (2^2), \\
&(2^2) * (2^2) = (2^4) \equiv (2)\ (\because\ 4 \equiv 1 \pmod 3), \\
&(2^2) * (5) = (2^2, 5^0) * (2^0, 5) = (2^{2 \cdot 0}, 5^{1 \cdot 0}) = (0), \\
&\text{etc.}
\end{aligned}$$

From the table we observe the following:

1. $D(2^2, 5)$ is closed under $*$.

2. Associativity holds in $(D(2^2,5),*)$. For example,

$$((2)*(2^2))*(2^2,5) = (2^2)*(2^2,5) = (2,5^0) = (2) \in D(2^2,5)$$

and

$$(2)*((2^2)*(2^2,5)) = (2)*(2,5^0) = (2)*(2) = (2) \in D(2^2,5).$$

Thus $((2)*(2^2))*(2^2,5) = (2)*((2^2)*(2^2,5))$. Similarly we can show that all triples of partitions in the set $D(2^2,5)$ behave the same way. Hence associative holds.

3. Commutativity holds in $(D(2^2,5),*)$ since the entries in the Cayley Table are symmetrical with respect to the main diagonal.

4. The multiplicative identity of $(D(2^2,5),*)$ is $1_\lambda = (2,5)$. This is because

$$(2,5)*\beta = \beta*(2,5) = \beta \ \forall \beta \in D((2^2,5)).$$

For example,

$$(2,5)*(2^2) = (2,5)*(2^2,5^0) = (2^{1\cdot 2},5^{1\cdot 0}) = (2^2,5^0) = (2^2),$$

$$\text{and similarly} \quad (2^2)*(2,5) = (2^2), \ \text{etc}$$

.

Hence $D((2^2,5))$ is commutative monoid.

We note that from Table 5.1, that $D((2^2,5))$ has only two invertible elements, namely $(2,5)$ and $(2^2,5)$, each of which is its own inverse.

5.2 A Ring Structure

We can now prove the following theorem.

Theorem 5.2.1. *Let* $\lambda = (\lambda_1^{N_1},\cdots,\lambda_t^{N_t})$, $0 < \lambda_1 < \cdots < \lambda_t$. *Then* $(D(\lambda),+,*)$ *is a finite commutative ring with identity.*

Proof. From Theorem 3.1.5 $(D(\lambda),+)$ is an abelian group. From Theorem 5.1.3 $(D(\lambda),*)$ is a commutative monoid.

It remains to prove the distributive laws. Let $\pi, \alpha, \beta \in D(\lambda)$, where these elements are defined in (5.1). Then

$$\begin{aligned}
\pi * (\alpha + \beta) &= (\lambda_1^{f_1}, \ldots, \lambda_t^{f_t}) * \big((\lambda_1^{a_1}, \ldots, \lambda_t^{a_t}) + (\lambda_1^{b_1}, \ldots, \lambda_t^{b_t})\big) \\
&= (\lambda_1^{f_1}, \ldots, \lambda_t^{f_t}) * (\lambda_1^{a_1+b_1 \pmod{N_1+1}}, \ldots, \lambda_t^{a_t+b_t \pmod{N_t+1}}) \\
&= (\lambda_1^{f_1\cdot(a_1+b_1) \pmod{N_1+1}}, \ldots, \lambda_t^{f_t\cdot(a_t+b_t) \pmod{N_t+1}}) \\
&= (\lambda_1^{f_1\cdot a_1+f_1\cdot b_1 \pmod{N_1+1}}, \ldots, \lambda_t^{f_t\cdot a_t+f_t\cdot b_t \pmod{N_t+1}}) \\
&\equiv (\lambda_1^{f_1\cdot a_1}, \ldots, \lambda_t^{f_t\cdot a_t}) + (\lambda_1^{f_1\cdot b_1}, \ldots, \lambda_t^{f_t\cdot b_t}) \\
&= \pi * \alpha + \pi * \beta.
\end{aligned}$$

Hence, the left distributive law holds. The right distributive law follows from the commutativity of $(D(\lambda), *)$.

This completes the proof. $\square$

From the Definition 2.1.16 of a zero divisor of a ring we have the following definition:

Definition 5.2.1. The partition $\alpha \in D(\lambda) \setminus (0)$ is called a zero divisor of the ring $(D(\lambda), +, *)$ if there exists a partition $\beta \in D(\lambda) \setminus (0)$ such that $\alpha * \beta = (0)$.

Proposition 5.2.2. *The ring $(D(\lambda), +, *)$ contains a zero divisor if λ has two or more distinct parts.*

Proof. If $0_\lambda \neq (\lambda_i) \in D(\lambda)$ and $0_\lambda \neq (\lambda_j) \in D(\lambda)$ with $i < j$, then

$$\begin{aligned}
(\lambda_j) * (\lambda_j) &= (\lambda_1^0, \ldots, \lambda_j, \ldots, \lambda_t^0) * (\lambda_1^0, \ldots, \lambda_i, \ldots, \lambda_t^0) \\
&= (\lambda_1^{0\cdot 0}, \ldots, \lambda_i^{1\cdot 0}, \ldots, \lambda_j^{1\cdot 0}, \lambda_t^{0\cdot 0}) \\
&= (\lambda_1^0, \lambda_2^0 \ldots, \ldots, \lambda_t^0) \\
&= 0_\lambda.
\end{aligned}$$

Thus λ_i and λ_j are both zero divisors of the given ring. $\square$

As a consequence of Proposition 5.2.2, if λ contains more than one distinct part, then $(D(\lambda), +, *)$ cannot form an integral domain.

However if $\lambda = (k^{p-1})$, $k > 0$, where p is a prime number, then $(D(\lambda), +, *)$ has no zero divisors. In fact from Corollary 3.2.4, we obtain an isomorphism with the finite field $\mathbb{Z}_p$, i.e.,

$$D(k^{p-1}) \cong \mathbb{Z}_p. \tag{5.3}$$

Example 5.2.1. Let $\lambda = (1^3, 2^2, 3, 4) \vdash 14$ and let $\pi = (1^3, 2^2) \in D(\lambda)$. Then $D(\pi)$ is given by

$$\{(0),(1),(1^2),(1^3),(2),(2^2),(1,2),(1,2^2),(1^2,2),(1^2,2^2),(1^3,2),(1^3,2^2)\}.$$

We know that $D(1^3, 2^2)$ is an abelian group under $+$, and by Theorem 5.1.3 is a monoid. Now we have to show that the distributive laws hold.

Let $(1^3), (1^2, 2), (1, 2^2) \in D(1^3, 2^2)$. Then we have

$$\begin{aligned}(1^3) * ((1^2,2) + (1,2^2)) &= (1^3,2^0) * ((1^3,2^3)) \\ &= (1^{9 \pmod 4}, 2^{0 \pmod 3}) \\ &= (1),\end{aligned}$$

$$\begin{aligned}(1^3) * (1^2,2) + (1^3) * (1,2^2) &= (1^3,2^0) * (1^2,2) + (1^3,2^0) * (1,2^2) \\ &= (1^{6 \pmod 4}, 2^{0 \pmod 3}) + (1^{3 \pmod 4}, 2^{0 \pmod 4}) \\ &= (1^2,2^0) + (1^3,2^0) \\ &= (1^{5 \pmod 4}, 2^{0 \pmod 3}) \\ &= (1,2^0) = (1).\end{aligned}$$

Since the binary operation $*$ is commutative, the two distributive laws are fulfilled.

In a similar manner we can show that any three elements of $D(1^3, 2^2)$ satisfy the distributive laws.

The multiplicative identity of $D(1^3, 2^2)$ is clearly $1_\pi = (1, 2)$. From this we can conclude that $(D(1^3, 2^2), +, *)$ is a ring which is commutative with an identity element.

However, $D(1^3, 2^2)$ is not an integral domain since it has zero divisors such as (1^3) and (2^2). Note that

$$(1^3) * (2^2) = (1^3, 2^0) * (1^0, 2^2) = (1^{3 \cdot 0}, 2^{0 \cdot 2}) = (1^0, 2^0) = 0_\pi.$$

Example 5.2.2. Consider the two rings, $D(1^7)$ and $D(1^{10})$. The ring $D(1^7)$ is not an integral domain. Indeed $(1^2), (1^4) \in D(1^7)$ and $(1^2) * (1^4) = (1^8) \equiv (1^0)$, which shows that (1^2) and (1^4) are zero divisors. However, it can be proved that $D(1^{10})$ is both an integral domain and a field. These facts follow at once from the isomorphism (5.3).

5.3 A Direct Product of Rings

In Subsection 3.1.1 we prove the following isomorphism see Theorem 3.1.6:

$$(D(\lambda), +) \cong \bigoplus_{i=1}^{r} (\mathbb{Z}_{n_i+1}, \vee). \tag{5.4}$$

In this section, we will extend this to an isomorphism between the ring $(D(\lambda), +, *)$ and the direct product of the residue class ring.

We discuss the direct product of rings $\mathbb{Z}_m$ instead of direct sum. Then the isomorphism extends to

$$(D(\lambda), +, *) \cong \bigotimes_{i=1}^{r} (\mathbb{Z}_{n_i+1}, \vee, \wedge), \tag{5.5}$$

where we use $\wedge$ to denote multiplication on $\bigotimes_{i=1}^{r} (\mathbb{Z}_{n_i+1}, \vee, \wedge)$.

It is known that $\bigoplus_{i=1}^{r} (\mathbb{Z}_{n_i+1}, \vee)$ also forms a multiplicative monoid under $\wedge$ since each factor $\mathbb{Z}_{n_i+1}$ does.

Theorem 5.3.1. *Let $\lambda = (\lambda_1^{n_1}, \cdots, \lambda_r^{n_r}) \vdash n$, $0 < \lambda_1 < \cdots < \lambda_r$ and $n_i > 0$ for all i. Then there is an isomorphism of rings given by*

$$D(\lambda) \cong \mathbb{Z}_{n_1+1} \otimes \mathbb{Z}_{n_2+1} \otimes \cdots \otimes \mathbb{Z}_{n_r+1}, \tag{5.6}$$

where we use $+$ and $\times$ to denote addition and multiplication in $D(\lambda)$ and use $\vee$ and $\wedge$ to denote addition and multiplication in $\mathbb{Z}_{n_1+1} \otimes \cdots \otimes \mathbb{Z}_{n_r+1}$.

Proof. We will use the assumptions in the proof of Theorem 3.1.6. We extend the same mapping to identify elements on both sides via multiplication. Define $\Omega : D(\lambda) \longmapsto \mathbb{Z}_{n_1+1} \oplus \cdots \oplus \mathbb{Z}_{n_r+1}$ by

$$\Omega(\pi) = (\overline{a_1}, \overline{a_2}, \ldots, \overline{a_r}).$$

From Theorem 3.1.6 it has been shown that Ω preserves the addition operation in both groups, and is one-to-one and onto. From the definition of a homomorphism we know that if $D(\lambda)$ and $\bigotimes_{i=1}^{r} (\mathbb{Z}_{n_i+1}, \vee, \wedge)$ are rings then a map $\Omega : D(\lambda) \to \bigotimes_{i=1}^{r} (\mathbb{Z}_{n_i+1}, \vee, \wedge)$ is a homomorphism if $\Omega(\pi + \beta) = \Omega(\pi) + \Omega(\beta)$ and $\Omega(\pi\beta) = \Omega(\pi)\Omega(\beta)$ holds. By the same reasoning used in Theorem 3.1.6, Ω preserves the addition operation in rings.

It now remains to show that Ω preserves multiplication, and that the multiplicative identities are compatible. The latter would be true if Ω takes the multiplicative identity of the domain to the multiplicative identity of the range.

Let $\beta \in D(\lambda)$ with $\beta = (\lambda_1^{b_1}, \cdots, \lambda_r^{b_r})$. Then we have

$$\begin{aligned}\Omega(\pi) \wedge \Omega(\beta) &= (\overline{a_1}, \cdots, \overline{a_r}) \wedge (\overline{b_1}, \cdots, \overline{b_r}) \\ &= (\overline{a_1} \cdot \overline{b_1}, \cdots, \overline{a_r} \cdot \overline{b_r}) \\ &\equiv (\overline{a_1 \cdot b_1}, \cdots, \overline{a_r \cdot b_r}) \\ &= \Omega(\pi * \beta).\end{aligned}$$

So Ω is a homomorphism. Assume that $(u_1, \cdots, u_r) \in \mathbb{Z}_{n_1+1} \otimes \cdots \otimes \mathbb{Z}_{n_r+1}$. Since each u_i is reduced modulo $m_i + 1$, it follows that $u_1 \leq m_1, \cdots, u_r \leq m_r$ which implies that $\lambda_1^{m_1+1}, \cdots, \lambda_r^{m_r+1} \in D(\lambda)$. So Ω is an epimorphism.

Lastly, note that since $1_\lambda = (\lambda_1, \ldots, \lambda_r)$, we have

$$\Omega(1_\lambda) = (1, \ldots, 1) \in \mathbb{Z}_{n_1+1} \otimes \cdots \otimes \mathbb{Z}_{n_r+1}.$$

Hence, we conclude that Ω is a ring isomorphism. $\square$

Chapter 6

k-Complete Partitions and Miscellaneous Structures

Hoggatt and King in [16] defined an arbitrary sequence $\{f_i\}_{i=1}^{\infty}$ of positive integers to be complete, if and only if every positive integer n can be represented in the form $n = \sum_{i=1}^{\infty} \alpha_i f_i$, where α_i is either 0 or 1. When we translate it to partition, SeungKyung Park [25] defines a complete partition of an integer as follows:

Definition 6.0.1 (S. Park)**.** A complete partition of an integer n is a weakly increasing partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$, with $\lambda_1 = 1$, such that each integer i, $1 \leq i \leq n$, can be expressed as a sum of elements of λ, that is, each i can be expressed as $\sum_{j=1}^{k} \alpha_j \lambda_j$, where α_j is either 0 or 1.

For example, of the 11 partitions of $n = 6$, 5 are complete partitions, namely, $(1^6), (1^3 3), (1^4 2), (123), (1^2 2^2)$. SeungKyung Park [26] subsequently extended the idea to r-complete partitions whereby λ is said to be r-complete if each i with $1 \leq i \leq rn$ can be expressed as $\sum_{j=1}^{k} \alpha_j \lambda_j$, where $\alpha_j \in S = \{0, 1, 2, \ldots, r\}$. It follows that the complete partitions correspond to 1-complete partitions under the latter definition. For example, (1^4) and $(1^2 2)$ are 1-complete partitions of $n = 4$, and $(1^4), (1^2 2)$ and $(1, 3)$ are 2-complete partitions.

In this chapter, we consider a more inclusive type of complete partitions. The "k-complete" partitions discussed here are complete by virtue of the fact that they contain $p(i)$ representations of every positive integer $i < k$, where $p(N)$ is the number of all partitions of N. In addition, we give a brief discussions of two types of completeness by containment of perfect partitions called 'perfect-partition-complete' and 'full-perfect-partition-complete' partitions.

6.1 k-Complete Partitions

Definition 6.1.1. A partition π of a positive integer is called k-complete if the parts of π contain the parts of all partitions of every integer from 1 to k.

Alternatively, we say that a partition π is k-complete if the group $D(\pi)$ of derived partitions of π contains all partitions of every positive integer from 1 to k.

For instance (1) is 1-complete, (1,1,1,2) and (1,1,2,3) are both 2-complete while (1,1,1,2,3) is 3-complete (and therefore 1- and 2-complete). A perfect partition is at most 1-complete. Thus

$$D((1,1,1,2)) = \{\emptyset, (1), (1,1), (2), (1,1,1), (1,2), (1,1,2), (1,1,1,2)\}$$

which is seen to contain every partition λ of n for $n = 1, 2$. But $(1,1,1,2)$ is not 3-complete because $(3) \notin D((1,1,1,2))$.

Let $C_k(n)$ denote the set of k-complete partitions of n, and let $c_k(n) = |C_k(n)|$. Any partition $\pi \in C_k(n)$ is j-complete, where $j = 1, 2, \ldots, k$.

$C_k(n)$ is largest when $k = 1$ since $C_1(n)$ is the set of all partitions of n that contain 1 as a part. The union of all k-complete partitions of n is $C_1(n)$.

There is a strictly decreasing chain of sets of k-complete partitions of n, namely

$$C_1(n) \supset C_2(n) \supset C_3(n) \supset \cdots .$$

For any integer $n \geq k$ there is a finite number of k-complete partitions. But for any $k > 0$ there is an infinite number of k-complete partitions. For instance, any partition of n of the form $(2^a, 1^b)$, $a > 0, b > 1$, is 2-complete since it contains $(2, 1^2)$.

Table 6.1 shows all the partitions in the sets $C_i(1), C_i(2), \ldots, C_i(10)$ for $i = 2, 3$.

Lemma 6.1.1. *Let π be a k-complete partition, and let m be an integer with $1 \leq m \leq k$. Then π contains all positive integer divisors of m.*

Proof. If $d \mid m$ and $d \notin \pi$, then the partition $(d^{m/d}) \notin D(\pi)$. This contradicts the fact that π is k-complete: the weight of $(d^{m/d})$ is $(m/d) \cdot d = m \leq k$, showing that π does not contain at least one partition of m. □

6.1.1 Initial k-Complete Partitions

From Table 6.1 we notice the presence of some k-complete partitions which occur uniquely for certain minimal weights n, and are as economical with respect to multiplicity of parts as possible. This motivates the next definition.

n	k	$C_k(n)$
1	2	$\emptyset$
2	2	$\emptyset$
3	2	$\emptyset$
4	2	$(2, 1^2)$
5	2	$(2, 1^3)$
6	2	$(2, 1^4), (2^2, 1^2)$
7	2	$(2, 1^5), (2^2, 1^3), (3, 2, 1^2)$
7	3	$\emptyset$
8	2	$(2, 1^6), (2^2, 1^4), (3, 2, 1^3), (2^3, 1^2), (4, 2, 1^2)$
8	3	$(3, 2, 1^3)$
9	2	$(2, 1^7), (2^2, 1^5), (3, 2, 1^4), (2^3, 1^3), (4, 2, 1^3), (3, 2^2, 1^2), (5, 2, 1^2)$
9	3	$(3, 2, 1^4)$
10	2	$(2, 1^8), (2^2, 1^6), (3, 2, 1^5), (2^3, 1^4), (4, 2, 1^4), (3, 2^2, 1^3), (5, 2, 1^3),$ $(2^4, 1^2), (3^2, 2, 1^2), (4, 2^2, 1^2), (6, 2, 1^2)$
10	3	$(3, 2, 1^5), (3, 2^2, 1^3)$
10	4	$\emptyset$

Table 6.1: The sets of $k-$complete partitions, $C_k(n)$ for $1 \leq n \leq 10$ and $k = 2, 3$

Definition 6.1.2. The k-complete partition with least weight is called the *initial k-complete partition.*

Let the initial k-complete partition be denoted by π_k, and let the initial k-complete partition of n be denoted by $\pi_k(n)$.

For example, the initial k-complete partitions for $k = 1, 2, 3$ are respectively

$$\pi_1 = (1),\ \pi_2 = (2, 1^2),\ \pi_3 = (3, 2, 1^3).$$

From Lemma 6.1.1 we see that the initial k-complete π_k partition depends on the divisors of all $m \in \{1, \ldots, k\}$. By definition π_k does not contain all the divisors of $k + 1$, in particular it does not contain $k + 1$. The following lemma gives further characterizations of π, from which it may be computed. The proof may be deduced from Lemma 6.1.1.

Lemma 6.1.2. *The initial k-complete partition π_k satisfies the following properties:*

(i) The partition π_k is equivalent to the multiset union of the sets of divisors of the integers $1, 2, \ldots, k$. Thus if $V(n)$ denotes the set of divisors

of n, then

$$\pi_k \equiv \bigcup_{m=1}^{k} V(m). \tag{6.1}$$

(ii) *Let $pV(m)$ denote the set of partitions of m into the divisors of m. Then π is obtained by selecting the (possibly repeated) parts with maximal multiplicity from the set $pV(1) \cup pV(2) \cup \cdots \cup pV(k)$.*

Proof.

(i) If $1 \leq m \leq k$, then $V(m) \subseteq \pi_k$, by Lemma 6.1.1. Thus $\bigcup_{m=1}^{k} V(m) \subseteq \pi_k$. Also π_k is completely determined by partitions of $1, 2, \ldots, m$, and the minimality of the weight of π_k implies $k+1 \notin \pi_k$, that is, $V(k+1) \not\subseteq \pi_k$. Therefore, $\pi_k \subseteq \bigcup_{m=1}^{k} V(m)$. Hence the result.

(ii) Note that any $\lambda \in pV(m)$ has the form $\lambda = (d^{m/d})$.

So we can write $pV(m) = \{(1^{e_m}), (2^{e_m/2}), \ldots, (k^{e_m/k})\}$, where $e_m \equiv m$, $e_m/d = 0$ if $d \nmid m$ and $\tau(m) \leq k$ for $m = 1, 2, \ldots, k$. Then

$$\pi_k \equiv \left(t^{\max(e_t/t,\, e_{t+1}/t,\, \ldots,\, e_k/t)} \mid t = 1, 2, \ldots k\right). \tag{6.2}$$

In fact this expression may be deduced from part (i) since a finite union of multisets consists of distinct elements with maximal multiplicities. This reasoning may be clarified by arranging the divisors of each m, $1 \leq m \leq k$, in a row labelled with $1, 2, \ldots, k$. In the mth row, write d in column d if $d|m$ and write $\emptyset$ if $d \nmid m$. When all the k rows are filled, collect the divisors by columns. The following table illustrates this display for $k = 4$.

m	1	2	3	4
1	1	$\emptyset$	$\emptyset$	$\emptyset$
2	1	2	$\emptyset$	$\emptyset$
3	1	$\emptyset$	3	$\emptyset$
4	1	2	$\emptyset$	4
$pV(m)$	1^4	2^2	3	4

□

Further illustrations of Lemma 6.1.2 may be deduced from Table 6.2.

Example 6.1.1. Since $V(m)$, $m = 1, 2, 3, 4$ are given by $\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 4\}$ respectively, part (i) of Lemma 6.1.2 gives

$$\pi_4 \equiv (1) \cup (1, 2) \cup (1, 3) \cup (1, 2, 4) = (1, 1, 1, 1, 2, 2, 3, 4).$$

On the other hand, since $Vp(m)$, $m = 1, 2, 3, 4$ are given by $\{1\}, \{1^2, 2\}$, $\{1^3, 3\}, \{1^4, 2^2, 4\}$ respectively, part (ii) of Lemma 6.1.2 gives

$$\pi_4 \equiv (1^{\max(1,2,3,4)}, 2^{\max(1,1,2)}, 3^{\max(1,0)}, 4^{\max(1)}) = (1^4, 2^2, 3, 4).$$

k	divisors	$pV(k)$	π_k	w_k
1	1	(1)	(1)	1
2	1,2	$(1^2, 2)$	$(1^2, 2)$	4
3	1,3	$(1^3, 3)$	$(1^3, 2, 3)$	8
4	1,2,4	$(1^4, 2^2, 4)$	$(1^4, 2^2, 3, 4)$	15
5	1,5	$(1^5, 5)$	$(1^5, 2^2, 3, 4, 5)$	21
6	1,2,3,6	$(1^6, 2^3, 3^2, 6)$	$(1^6, 2^3, 3^2, 4, 5, 6)$	33

Table 6.2: Some initial k-complete partitions for $1 \leq k \leq 6$

Note that $\pi_k \in C_k(n)$ if and only if $\pi_k = \pi_k(n)$, and this holds if and only $n \in \{w_1, w_2, \ldots\}$. From Table 6.2 we know so far the values of $w_1, w_2, \ldots, w_6$.

The general formula is given in the next Proposition.

Proposition 6.1.3. *The weight w_k of the initial k-complete partition π_k is given by*

$$w_k = \sum_{j=1}^{k} \sigma(j) = \sum_{r=1}^{k} r \left\lfloor \frac{k}{r} \right\rfloor, \tag{6.3}$$

where $\sigma(N)$ is the sum of divisors of N.

Proof. By Lemma 6.1.1, π_k contains all divisors of each $m \in \{1, \ldots, k\}$. Since π_k has the least weight among k-complete partitions, it does not contain all divisors of $k+1$. In fact the largest part of π_k is k. Hence $w_k = \sigma(1)+\sigma(2)+ \cdots + \sigma(k)$. This proof confirms that

$$\pi_k \equiv \bigcup_{m=1}^{k} V(m).$$

.

The second equality is a well-known identity, see for example [8]. However, we can use the same hypothesis to derive the second sum. Note that for each integer m, the quantity $\lfloor \frac{k}{m} \rfloor$ counts the multiples of m not exceeding k, or equivalently, the copies of m in the multiset of divisors of $1, 2, \ldots, k$. So the full contribution of m to w_k is $m\lfloor \frac{k}{m} \rfloor$, and to π_k is $\lfloor \frac{k}{m} \rfloor$ copies of m

or $m^{\lfloor \frac{k}{m} \rfloor}$. Hence π_k has the following explicit form, which is compatible with Equation (6.2).

$$\pi_k = (1^k, 2^{\lfloor k/2 \rfloor}, 3^{\lfloor k/3 \rfloor}, \ldots, \lfloor k/2 \rfloor^2, \lfloor k/2 \rfloor + 1, \lfloor k/2 \rfloor + 2, \ldots, k). \tag{6.4}$$

By summing the parts of π_k in Equation (6.4) we obtain an independent derivation of the second equality in Equation (6.3). $\square$

From Equation (6.3) we have $w_k = \sum_{j=1}^{k-1} \sigma(j) + \sigma(k)$, which implies the recurrence

$$w_k = w_{k-1} + \sigma(k),\ k > 0,\ \text{with } w_0 = 0. \tag{6.5}$$

The sequence $\{w_k\}_{k \geq 1}$ begins as follows:

$$1, 4, 8, 15, 21, 33, 41, 56, 69, 87, 99, 127, 141, 165, 189, 220, 238, 277, \ldots$$

This agrees with sequence *A*024916 in Sloane [28], wherein a comment describes the k-th term as

“the total number of parts in all partitions of the positive integers less than or equal to k into equal parts”. (6.6)

The following result may be deduced from the proof of Proposition 6.1.3. Specifically, we sum up the multiplicities of the general initial k-complete partition

$$\pi_k = (1^k, 2^{\lfloor k/2 \rfloor}, 3^{\lfloor k/3 \rfloor}, \ldots, \lfloor k/2 \rfloor^2, \lfloor k/2 \rfloor + 1, \lfloor k/2 \rfloor + 2, \ldots, k).$$

.

Corollary 6.1.4. *The length ℓ_k of the initial k-complete partition π_k is given by*

$$\ell_k = \sum_{j=1}^{k} \tau(j) = \sum_{r=1}^{k} \left\lfloor \frac{k}{r} \right\rfloor, \tag{6.7}$$

where $\tau(N)$ is the number of divisors of N.

Apart from

$$\pi_k \equiv \bigcup_{m=1}^{k} V(m),$$

$$\pi_k \equiv \left(t^{\max(e_t/t,\, e_{t+1}/t,\, \ldots,\, e_k/t)} \mid t = 1, 2, \ldots k \right)$$

and

$$\pi_k = (1^k, 2^{\lfloor k/2 \rfloor}, 3^{\lfloor k/3 \rfloor}, \ldots, \lfloor k/2 \rfloor^2, \lfloor k/2 \rfloor + 1, \lfloor k/2 \rfloor + 2, \ldots, k),$$

π_{k+1} may be obtained (recursively) from π_k using the following algorithm:

"Insert the divisors of $k+1$ into π_k" .

For example, since the divisors of 4 are $1, 2, 4$, we have

$$\pi_4 = \pi_3 \cup (1, 2, 4) = (3, 2, 1^3) \cup (1, 2, 4) = (4, 3, 2^2, 1^4).$$

Similarly,

$$\pi_5 = \pi_4 \cup (1, 5) = (4, 3, 2^2, 1^4) \cup (1, 5) = (5, 4, 3, 2^2, 1^5).$$

Since $C_1(n)$ is the set of partitions of n containing 1, we have $c_1(n) = p(n-1)$. This fact is extended in the following statement.

Theorem 6.1.5. *Let w_k be the weight of the initial k-complete partition π_k. The number of k-complete partitions of n is given by $p(n - w_k)$, where $p(N)$ denotes the number of all partitions of N:*

$$c_k(n) = p(n - w_k). \tag{6.8}$$

Proof. The number of partitions of n containing a positive integer r is given by $p(n-r)$. This is because we can remove one copy of r from a partition of n containing r to obtain an arbitrary partition of $n - r$.

Now by definition, the multiset of parts of any k-complete partition λ of n contains the multiset of the parts of π_k, that is, contains a partition of w_k. So we may replace the parts of π_k in λ with w_k to obtain an arbitrary partition of n containing w_k. Therefore $C_k(n)$ is equinumerous with the set of partitions of n containing w_k. Hence the result. $\square$

Theorem 6.1.5 implies the generating function see for example, Andrews [7]

$$\sum_{n=1}^{\infty} c_k(n) q^n = q^{w_k} \prod_{n=1}^{\infty} \frac{1}{1-q^n}. \tag{6.9}$$

Remark 6.1.1. Denote the set of partitions of N by $P(N)$, i.e., $p(N) = |P(N)|$. The proof of Theorem 6.1.5 guarantees that the set $C_k(n)$ may be obtained by inserting the parts of π_k into partitions $\pi \in P(n-w_k)$ where π may be empty. For example, $C_2(8) = (2,1^2) \cup P(4)$ gives:

$$\begin{aligned} C_2(8) &= (2,1^2) \cup \{(1^4),(2,1^2),(3,1),(2^2),(4)\} \\ &= \{(2,1^6),(2^2,1^4),(3,2,1^3),(2^3,1^2),(4,2,1^2)\}, \end{aligned}$$

compare with Table 6.1.

On the other hand $c_k(n) = p(n-w_k)$ implies

$$c_k(n) - c_k(n-1) = p(n-w_k) - p(n-1-w_k) = p(n-w_k)_{\geq 2},$$

where $p(N)_{\geq 2}$ is the number of partitions of N without 1's. This gives the following recurrence on which the construction of the rows of Table 6.1 is based

$$c_k(n) = c_k(n-1) + p(n-w_k)_{\geq 2}. \tag{6.10}$$

Thus to obtain $C_k(n)$ we either insert 1 into each $\pi \in C_k(n-1)$ or merge the parts of π_k with the parts of any partition of $n-w_k$ that does not contain 1.

We can recover the generating function Equation (6.9) from the recurrence in Equation (6.10) as follows.

$$\sum_{n=1}^{\infty} c_k(n-1)q^n = \sum_{n=2}^{\infty} c_k(n-1)q^n = q\sum_{n=1}^{\infty} c_k(n)q^n = q^{w_k+1}\prod_{n=1}^{\infty}\frac{1}{1-q^n}$$

and

$$\sum_{n=1}^{\infty} p(n-w_k)_{\geq 2}q^n = q^{w_k}\prod_{n=2}^{\infty}\frac{1}{1-q^n}.$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty}\left(c_k(n-1)+p(n-w_k)_{\geq 2}\right)q^n &= q^{w_k+1}\prod_{n=1}^{\infty}\frac{1}{1-q^n} + q^{w_k}\prod_{n=2}^{\infty}\frac{1}{1-q^n} \\ &= q\cdot q^{w_k}\prod_{n=1}^{\infty}\frac{1}{1-q^n} + q^{w_k}\prod_{n=1}^{\infty}\frac{1}{1-q^n}(1-q) \\ &= \left(q+(1-q)\right)q^{w_k}\prod_{n=1}^{\infty}\frac{1}{1-q^n} \\ &= q^{w_k}\prod_{n=1}^{\infty}\frac{1}{1-q^n}. \end{aligned}$$

6.1.2 Strictly k-Complete Partitions

A partition π is said to be strictly k-complete if π is k-complete but not $(k+1)$-complete. Thus $(3, 2, 1^2)$ is strictly 2-complete and $(4, 3, 2, 1^3)$ is strictly 3-complete. The partition $(7, 6, 5^2, 4, 3, 2^2, 1)$ is strictly 1-complete. Any perfect partition is strictly 1-complete.

By definition every initial k-complete partition is strictly k-complete.

The number $s_k(n)$ of strictly k-complete partitions of n is given by

$$s_k(n) = |C_k(n) \setminus C_{k+1}(n)| = |C_k(n)| - |C_{k+1}(n)|.$$

Thus Equation (6.8) implies

$$s_k(n) = p(n - w_k) - p(n - w_{k+1}).$$

Hence using Equations (6.9) and (6.5) we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} s_k(n) q^n &= (q^{w_k} - q^{w_{k+1}}) \prod_{n=1}^{\infty} \frac{1}{1-q^n} \\ &= q^{w_k}(1 - q^{\sigma(k+1)}) \prod_{n=1}^{\infty} \frac{1}{1-q^n}. \end{aligned} \tag{6.11}$$

Thus if $f_t(n)$ denotes the number of partitions of n that do not contain t as a part, then

$$\sum_{n=0}^{\infty} s_k(n) q^n = q^{w_k} \sum_{n=0}^{\infty} f_{\sigma(k+1)}(n) q^n. \tag{6.12}$$

Hence we have proved the following partition theorem. But we also give a combinatorial proof as follows.

Theorem 6.1.6. *The number of strictly k-complete partitions of n is equal to the number of partitions of $n - w_k$ which do not contain $\sigma(k+1)$ as a part.*

Combinatorial Proof of Theorem 6.1.6. Let $S_k(n)$ denote the set of strictly k-complete partitions of n. In view of $c_k(n) = p(n - w_k)$, the set of k-complete partitions of n is equinumerous with the set $P(n - w_k)$ of partitions of $n - w_k$, with a generally disjoint union:

$$C_k(n) = P(n - w_k) = S_k(n) \cup C_{k+1}(n).$$

Recall that the parts of each $\pi \in C_{k+1}(n)$ consists of all the divisors of $1, 2, \ldots, k, k+1$. Thus if $n < w_{k+1}$, then $C_{k+1}(n) = \emptyset$ and we simply have that

$S_k(n) = P(n - w_k)$ and clearly no contained partition in the right-hand-side can contain $\sigma(k+1)$.

However, if $n \geq w_{k+1}$, we may replace the members of the set of divisors of $k+1$ in $\pi \in C_{k+1}(n)$ with their sum, $\sigma(k+1)$. This transformation induces a bijection between $C_{k+1}(n)$ and the set of partitions of $n - w_k$ containing $\sigma(k+1)$. This then produces $P(n - w_k) = S_k(n)$ provided we exclude partitions of $n - w_k$ containing the set of divisors of $k+1$ which is in turn identified with partitions of $n - w_k$ containing $\sigma(k+1)$. Hence the result follows. $\square$

For example, from Table 6.1 we see that $C_2(7) = S_2(7) \cup C_3(7) = S_2(7)$ since $C_3(7) = \emptyset$; but

$$C_2(8) = S_2(8) \cup C_3(8) \equiv \{(2,1^6),(2^2,1^4),(2^3,1^2),(4,2,1^2)\} \cup \{(3,2,1^3)\}.$$

Remark 6.1.2. Referring to Theorem 6.1.6, note that a partition $\pi \in S_k(n)$ may contain $\sigma(k+1)$ as a part. For example, when $n = 10$ and $k = 2$, then $\sigma(3) = 1 + 3 = 4$. We find in Table 6.1 that

$$(4,2,1^4),(4,2^2,1^2) \in S_2(10) = C_2(10) \setminus C_3(10),$$

where $C_3(10) = \{(3,2,1^5),(3,2^2,1^3)\}$.
However, by replacing 1 and 3 with 4 we find that

$$C_3(10) = \{(3,2,1^5),(3,2^2,1^3)\} \longmapsto \{(4,2,1^4),(4,2^2,1^2)\} \in S_2(10).$$

6.1.3 Further Properties of the Group $D(\pi_k)$

Consider the group $D(\pi_k)$ of derived partitions of the initial k-complete partition. If λ is any k-complete partition, then by definition the multiset of parts of λ contains the multiset of the parts of π_k. Thus $\pi_k \in D(\lambda)$ which implies that $D(\pi_k) \subseteq D(\lambda)$.

Hence the intersection of all groups of derived partitions of k-complete partitions is $D(\pi_k)$.

Therefore, if λ is any k-complete partition, then $|D(\pi_k)| \leq |D(\lambda)|$.

We compare the sum $p(0) + p(1) + \cdots + p(k)$ with the order of the group $D(\pi_k)$. Note that $p(0) = 1$ counting the empty partition $\emptyset$ of 0, and $p(0) + p(1) = 1 + 1 = 2$ including the partition (1). On the other hand, $D(\emptyset) = \{\emptyset\}$ and $D((1)) = \{\emptyset, (1)\}$. Hence $p(0) + p(1) + \cdots + p(k) = |D(\pi_k)|$ when $k = 0$ and $k = 1$. But when $k \geq 3$, we obtain the following strong inequality

$$p(0) + p(1) + \cdots + p(k) < |D(\pi_k)|, \;\; k > 1. \tag{6.13}$$

Note that $D(\pi_k)$ contains all partitions of $0, 1, \ldots, k$ including a partition of w_k. Since the left side of (6.13) counts only partitions of $0, 1, \ldots, k$, the result follows. For example, $p(0)+p(1)+p(2) = 1+1+2 = 4$, and $|D(\pi_2)| = |D((1^2, 2))| = 3 \cdot 2 = 6$ since

$$D((1^2, 2)) = \{\emptyset, (1), (1, 1), (2), (1, 2), (1, 1, 2)\}.$$

We may compute the value of $|D(\pi_k)|$ using the explicit form in

$$\pi_k = (1^k, 2^{\lfloor k/2 \rfloor}, 3^{\lfloor k/3 \rfloor}, \ldots, \lfloor k/2 \rfloor^2, \lfloor k/2 \rfloor + 1, \lfloor k/2 \rfloor + 2, \ldots, k).$$

to obtain

$$|D(\pi_k)| = \left\lfloor \frac{k+1}{1} \right\rfloor \left\lfloor \frac{k+2}{2} \right\rfloor \cdots \left\lfloor \frac{k+k}{k} \right\rfloor = \prod_{j=1}^{k} \left\lfloor \frac{k+j}{j} \right\rfloor, \; k \geq 0. \tag{6.14}$$

Therefore the inequality (6.13) may be restated as follows:

$$p(0) + p(1) + \cdots + p(k) < \prod_{j=1}^{k} \left\lfloor \frac{k+j}{j} \right\rfloor, \; k > 1. \tag{6.15}$$

In fact the difference between the two quantities increases so rapidly that the following limit converges very fast to 0, see Table 6.3.

$$\lim_{k \to \infty} \frac{p(0) + p(1) + \cdots + p(k)}{\prod_{j=1}^{k} \lfloor \frac{k+j}{j} \rfloor} = 0. \tag{6.16}$$

k	$p(0) + \cdots + p(k)$	$\lvert D(\pi_k)\rvert$	$(p(0) + \cdots + p(k))/\lvert D(\pi_k)\rvert$
3	7	16	0.3750000000
4	12	60	0.1833333333
5	19	144	0.1250000000
6	30	672	0.04315476190
7	45	1536	0.02864583333
8	67	6480	0.01018518519
9	97	19200	0.005000000000
10	139	76032	0.001815025253
11	195	165888	0.001169463735
12	272	1048320	0.0002585088523

Table 6.3: The sum $p(0) + \cdots + p(k)$ versus $|D(\pi_k)|$ for some values of k

6.2 Perfect-Partition-Complete Partitions

A partition of a positive integer is called *perfect-partition-complete for n* (*pp-complete* for short) if the parts of λ contain all perfect partitions of n.

In other words, a partition λ is pp-complete for n if $\pi \in D(\lambda)$ for every perfect partition $\pi \vdash n$.

As with k-complete partitions, there is an infinite number of pp-complete partitions for any positive integer n. For example, since the perfect partitions of 3 are (1^3) and $(1, 2)$ we find that the pp-complete partition for $n = 3$ with least weight is $(1^3, 2)$. It is clear that any partition of the form

$$(1^a, 2^b, 3^{c_1}, 4^{c_2}, \ldots),\ a \geq 3, b \geq 1, c_1 \geq 0, c_2 \geq 0, \ldots,$$

is also a pp-complete partition for $n = 3$.

The pp-complete partition with least weight that contains all the perfect partitions of k will be called the kth *initial pp-complete* partition.

We denote the kth initial pp-complete partition by Λ_k, $k = 1, 2, 3, \ldots$. Then the partitions $\Lambda_1, \ldots, \Lambda_{12}$ are given in Table 6.4 (see the third column). The perfect partitions π of k contained in Λ_k are also given in the table (see the second column).

For example, the following set inclusion holds for $k = 7$:

$$\{(1^7), (1, 2^3), (1^3, 4), (1, 2, 4)\} \subset D((1^7, 2^3, 4)).$$

Many perfect partitions have the form $(1^{x-1}, x^y)$ for a fixed $x > 0$. Consequently $x \in \Lambda_{xy+x-1}$. For example, when $x = 3$, we find that $3 \in \Lambda_{3y+2}$, $y \geq 1$. This is the origin of the following statement.

Proposition 6.2.1. *Let k and t be positive integers with $k > t$. Then Λ_k contains t as a part if and only if $k \equiv t - 1$ (mod t).*

Proof. We have

$$\begin{aligned} t \in \Lambda_k &\iff \exists \text{ a perfect partition } \pi \vdash k \text{ with } t \in \pi \\ &\iff \exists \text{ a factorization of the form } k + 1 = t \cdot v,\ v > 0 \\ &\iff \exists \text{ a perfect partition of the form } \pi = (1^{t-1}, t^{v-1}) \vdash k. \end{aligned}$$

The weight of π is $t - 1 + t(v - 1) = tv - 1 = k$. Hence $k \equiv t - 1$ (mod t). $\square$

The following theorems concerning the initial pp-complete partitions may be proved in a similar manner to the proofs of analogous theorems in Section 6.1.

k	Perfect Partitions of k	Λ_k	$w(\Lambda_k)$
1	(1)	(1)	1
2	(1^2)	(1^2)	2
3	$(1^3), (1,2)$	$(1^3, 2)$	5
4	(1^4)	(1^4)	4
5	$(1^5), (1,2^2), (1^2,3)$	$(1^5, 2^2, 3)$	12
6	(1^6)	(1^6)	6
7	$(1^7), (1,2^3), (1^3,4), (1,2,4)$	$(1^7, 2^3, 4)$	17
8	$(1^8), (1^2,3^2)$	$(1^8, 3^2)$	14
9	$(1^9), (1,2^4), (1^4,5)$	$(1^9, 2^4, 5)$	22
10	(1^{10})	(1^{10})	10
11	$(1^{11}), (1,2^5), (1,2^2,6), (1,2,4^2),$ $(1^3,4^2), (1^5,6), (1^2,3^3), (1^2,3,6)$	$(1^{11}, 2^5, 3^3, 4^2, 6)$	44
12	(1^{12})	(1^{12})	12

Table 6.4: The kth initial pp-complete partitions for $1 \le k \le 12$

Theorem 6.2.2. *Let $\ell(\Lambda_k)$ denote the length of Λ_k. Then*

$$\ell(\Lambda_k) = \sigma(k+1) - \tau(k+1). \tag{6.17}$$

Proof. We give a constructive proof based on Theorem 4.1.1 which established a bijection between ordered factorizations $n+1 = g_1 g_2 \cdots g_r$, $g_i > 1$ and increasing perfect partitions $\lambda \vdash n$, where

$$\lambda = \left(1^{g_1-1}, g_1^{g_2-1}, (g_1 g_2)^{g_3-1}, \ldots, (g_1 g_2 \cdots g_{r-1})^{g_r-1}\right).$$

Let $\Lambda_k = (\lambda_1^{e_1}, \lambda_2^{e_2}, \ldots, \lambda_r^{e_r})$, $\lambda_1 < \cdots < \lambda_k$, $e_i > 0\ \forall\ i$. Then we observe that, in general, the e_i are maximal among the multiplicities of distinct parts of perfect partitions of k. Such maximal multiplicities arise from perfect partitions of k whose corresponding ordered factorizations have at most two factors.

To see this, note that the trivial factorization $k+1 = 1\cdot(k+1)$ gives the maximal multiplicity in $(1^k) \in \Lambda_k$. Thus necessarily, $e_1 = k$.

The 2-factorization $k+1 = a_1 \cdot a_2$, $a_i > 1$ corresponds to $(1^{a_1-1}, a_1^{a_2-1}) \vdash k$ which gives the maximal string $a_1^{a_2-1}$ for all first factors a_1. Correspondingly, the $a_2 - 1$ give the multiplicities in Λ_k from left to right. We find that each string $\lambda_i^{e_i}$ originated from an expression of the form $\lambda_i \cdot (e_i + 1) = k+1$.

This may be clarified with a quick example. Let $k = 7$. Then $\{a_1 a_2 \mid a_1 a_2 = 8\} = \{2\cdot 4,\ 4 \cdot 2\}$. Then the partitions corresponding to these two factorizations, together with the 'trivial' case, $8 = 1 \cdot 8$, suffice to determine Λ_7:

$$\begin{aligned}\{1 \cdot 8\} \cup \{2 \cdot 4,\ 4 \cdot 2\} &\mapsto \{(1^{8-1}), (1, 2^{4-1}), (1^3, 4^{2-1})\} \\ &= \{(1^7), (1, 2^3), (1^3, 4)\} \\ &\subset D((1^7, 2^3, 4)),\ \text{with } \Lambda_7 = (1^7, 2^3, 4).\end{aligned}$$

More directly we can write:

$$\begin{aligned}(1 \cdot 8, 2 \cdot 4,\ 4 \cdot 2,\ 8 \cdot 1) &\mapsto (1^{8-1}, 2^{4-1}, 4^{2-1}, 8^{1-1}) \\ &= (1^7, 2^3, 4, \emptyset) \\ &= (1^7, 2^3, 4).\end{aligned}$$

To summarize the construction in general, let the increasing sequence of divisors of $k+1$ be

$$d_1, d_2, d_3, \ldots, d_{\tau-1}, d_\tau, \tag{6.18}$$

where $d_1 = 1$, $d_\tau = k+1$ and $\tau = \tau(k+1)$, with $d_i \cdot d_{\tau-i+1} = k+1$ for all i.

Then Λ_k has the explicit form:

$$\Lambda_k = \left(d_1^{d_\tau - 1}, d_2^{d_{\tau-1}-1}, d_3^{d_{\tau-2}-1}, \ldots, d_{\tau-2}^{d_3-1}, d_{\tau-1}^{d_2-1}, d_\tau^{d_1-1}\right) \tag{6.19}$$

Therefore, the number of parts is given by

$$\begin{aligned}\ell(\Lambda_k) &= (d_\tau - 1) + (d_{\tau-1} - 1) + (d_{\tau-2} - 1) + \cdots + (d_2 - 1) + (d_1 - 1) \\ &= (d_\tau + d_{\tau-1} + d_{\tau-2} + \cdots + d_2 + d_1) + (-1)\tau(k+1) \\ &= \sigma(k+1) - \tau(k+1).\end{aligned}$$

□

For example, let $k = 7$. Then

$$\sigma(8) - \tau(8) = (1 + 2 + 4 + 8) - |\{1, 2, 4, 8\}| = 15 - 4 = 11,$$

and from Table 6.4 we can verify that Λ_7 has 11 parts.

Theorem 6.2.3. *Let $w(\Lambda_k)$ denote the weight of Λ_k. Then*

$$w(\Lambda_k) = (k+1)\tau(k+1) - \sigma(k+1). \tag{6.20}$$

Proof. In order to obtain the weight we sum the parts of Λ_k directly from the explicit general form (6.19).

$$\begin{aligned}
w(\Lambda_k) &= (d_\tau - 1)1 + (d_{\tau-1} - 1)d_2 + \cdots + (d_2 - 1)d_{\tau-1} + (d_1 - 1)d_\tau \\
&= d_\tau 1 + d_{\tau-1}d_2 + \cdots + d_2 d_{\tau-1} + d_1 d_\tau \\
&\quad - 1 - d_2 - \cdots - d_{\tau-1} - d_\tau \\
&= \sum_{i=1}^{\tau(k+1)} d_i d_{\tau+1-i} - \sigma(k+1) \\
&= \sum_{i=1}^{\tau(k+1)} (k+1) - \sigma(k+1) \qquad \because\ d_i d_{\tau+1-i} = k+1 \\
&= (k+1)\tau(k+1) - \sigma(k+1).
\end{aligned}$$

□

The sequence $\{w(\Lambda_k)\}_{k\geq 1}$ begins as follows:

$$1, 2, 5, 4, 12, 6, 17, 14, 22, 10, 44, 12, 32, 36, 49, \ldots$$

This agrees with sequence *A*094471 in Sloane [28]. A comment also describes the kth term as (cf. (6.6))

"The sum of all parts minus the total number of parts of all partitions of $k+1$ into equal parts". (6.21)

We recall from Elementary Number Theory that the product of all divisors of n is equal to n raised to the power of half its number of divisors, see Apostol [8].

Theorem 6.2.4. *The order of the group of derived partitions of Λ_k is given by*

$$|D(\Lambda_k)| = \prod_{d|(k+1)} d = (k+1)^{\tau(k+1)/2}. \tag{6.22}$$

Proof. Give a partition $\lambda = (\lambda_1^{e_1}, \ldots, \lambda_k^{e_k})$, where $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k$ and $e_i > 0$ for all i, we know that $|D(\lambda)| = (e_1+1)(e_2+1)\cdots(e_k+1)$.

The first equality in the theorem follows immediately on applying this formula to

$$\Lambda_k = \left(d_1^{d_\tau - 1}, d_2^{d_{\tau-1}-1}, d_3^{d_{\tau-2}-1}, \ldots, d_{\tau-2}^{d_3-1}, d_{\tau-1}^{d_2-1}, d_\tau^{d_1-1}\right).$$

□

For example, let $k = 7$. Then $\tau(8)/2 = 4/2 = 2$, so according to Theorem (6.2.4) $|D(\Lambda_7)| = 8^2 = 64$. From Table 6.4 we can verify that

$$|D(\Lambda_7)| = |D((1^7, 2^3, 4))| = (7+1)(3+1)(1+1) = (8)(4)(2) = 64.$$

6.3 Full-Perfect-Partition-Complete Partitions

A partition λ of a positive integer is said to be *full-perfect-partition-complete* for n (*fpp-complete* for short) if the parts of λ contain all perfect partitions of every integer from 1 to n.

In other words, λ is fpp-complete for n if $\pi \in D(\lambda)$ for every perfect partition π of $m \in \{1, 2, \ldots, n\}$.

Again we concentrate on the kth *initial fpp-complete* partition, to be denoted by Π_k for $k = 1, 2, 3, \ldots$.

Since each Λ_k contains all perfect partitions of k and Π_k contains all perfect partitions of all integers from 1 to k, it follows that Π_k contains all the partitions Λ_k, $k = 1, \ldots, k$. So we can write:

$$\Pi_k = \Lambda_1 \hat{U} \Lambda_2 \hat{U} \cdots \hat{U} \Lambda_k, \tag{6.23}$$

where $A\hat{U}B$ stands for taking every distinct part with maximum multiplicity between A and B. Note that the binary operation $\hat{U}$ is associative.

The partitions $\Pi_1, \ldots, \Pi_{12}$ are given in Table 6.5, see the second column. The corresponding lengths $\ell(\Pi_k)$ and weights $w(\Pi_k)$ appear in the third and fourth columns respectively.

k	Π_k	$\ell(\Pi_k)$	$w(\Pi_k)$
1	(1)	1	1
2	(1^2)	2	2
3	$(1^3, 2)$	4	5
4	$(1^4, 2)$	5	6
5	$(1^5, 2^2, 3)$	8	12
6	$(1^6, 2^2, 3)$	9	13
7	$(1^7, 2^3, 3, 4)$	12	20
8	$(1^8, 2^3, 3^2, 4)$	14	24
9	$(1^9, 2^4, 3^2, 4, 5)$	17	32
10	$(1^{10}, 2^4, 3^2, 4, 5)$	18	33
11	$(1^{11}, 2^5, 3^3, 4^2, 5, 6)$	23	49
12	$(1^{12}, 2^5, 3^3, 4^2, 5, 6)$	24	50

Table 6.5: The kth initial fpp-complete partitions for $1 \le k \le 12$

By comparing Table 6.5 and Table 6.4 when $k = 7$ we find, for instance, that Equation (6.23) gives

$$\begin{aligned}\Pi_7 &= \Lambda_1\hat{U}\Lambda_2\hat{U}\cdots\hat{U}\Lambda_7\\ &= (1)\hat{U}(1^2)\hat{U}(1^3,2)\hat{U}(1^4)\hat{U}(1^5,2^2,3)\hat{U}(1^3)\hat{U}(1^7,2^3,4)\\ &= (1^7,2^3,3,4).\end{aligned}$$

We observe the following immediate properties of Π_k, from Table 6.5:

(1) Π_k contains the integers $1, 2, \ldots, \lfloor \frac{k+1}{2} \rfloor$, for all $k \geq 5$.

(2) The sequences $\{\ell(\Pi_k)\}_{k\geq 1}$ and $\{w(\Pi_k)\}_{k\geq 1}$ are monotone increasing.

(3) The integer $k+1$ is prime if and only if $\ell(\Pi_k) = \ell(\Pi_{k-1}) + 1$ and $w(\Pi_k) = w(\Pi_{k-1}) + 1$.

The following theorems about the initial fpp-complete partitions may be proved in a similar manner to the proofs of analogous theorems in Section 6.1.

Theorem 6.3.1. *Let $\ell(\Pi_k)$ denote the length of Π_k. Then*

$$\ell(\Pi_k) = -k + \sum_{j=1}^{k} \tau(j+1). \tag{6.24}$$

Proof. Note that Π_k has the form

$$\Pi_k = (1^{e_1}, 2^{e_2}, 3^{e_3}, \ldots, h^{e_h}, \ldots, \lfloor (k+1)/2 \rfloor^1), \tag{6.25}$$

where $1 \leq h \leq \lfloor (k+1)/2 \rfloor$ and $e_h > 0$ for all h.

In order to determine the e_i we consider the system of sets of proper divisors of 2,3,...,k+1, and let $dV(r)$ denote the set of divisors of r. Let

$$DV(k) = \{dV(r) \mid 2 \leq r \leq \lfloor (k+1)/2 \rfloor\}.$$

Clearly $1 \in dV(r)$ for every r, and $\lfloor (k+1)/2 \rfloor \in dV(r)$ for some r. The parts of Π_k are obtained with the following rule:

For each h select all occurrences of h as a proper divisor in $DV(k)$, the number of occurrences is the multiplicity e_h.

Note that $1 \in dV(r)$ is a proper divisor for all r. So the multiplicity of 1 in Π_k is k.

But $2 \le h \in dV(r)$ is a proper divisor if and only if $r = jh$, where $2 \le j \le \lfloor (k+1)/h \rfloor$. This implies that

$$h \in dV(r) \iff r = 2h,\ 3h,\ \ldots,\ h\lfloor (k+1)/h \rfloor$$

which gives exactly $\lfloor (k+1-h)/h \rfloor$ instances. Therefore the multiplicity of h in Π_k is $e_h = \lfloor (k+1-h)/h \rfloor$.

It follows that the general form of Π_k is

$$\Pi_k = (1^k, 2^{\lfloor (k-1)/2 \rfloor}, 3^{\lfloor (k-2)/3 \rfloor}, \ldots, \lfloor (k+1)/3 \rfloor^2, \lfloor (k+4)/3 \rfloor, \ldots, \lfloor (k+1)/2 \rfloor). \tag{6.26}$$

For the length we add the multiplicities in Equation (6.26) and obtain

$$\begin{aligned}
\ell(\Pi_k) &= \sum_{h=1}^{s_k} \left\lfloor \frac{k+1-h}{h} \right\rfloor, \quad s_k = \left\lfloor \frac{k+1}{2} \right\rfloor \\
&= -s_k + \sum_{h=1}^{s_k} \left\lfloor \frac{k+1}{h} \right\rfloor \\
&= -s_k + \sum_{h=1}^{k+1} \left\lfloor \frac{k+1}{h} \right\rfloor - \sum_{h=s_k+1}^{k+1} \left\lfloor \frac{k+1}{h} \right\rfloor \\
&= -s_k + \sum_{h=1}^{k+1} \left\lfloor \frac{k+1}{h} \right\rfloor - \sum_{h=s_k+1}^{k+1} 1 \\
&= -s_k + \sum_{h=1}^{k+1} \left\lfloor \frac{k+1}{h} \right\rfloor - \left\lfloor \frac{k+2}{2} \right\rfloor \\
&= -\left(\left\lfloor \frac{k+1}{2} \right\rfloor + \left\lfloor \frac{k+2}{2} \right\rfloor \right) + \sum_{h=1}^{k+1} \left\lfloor \frac{k+1}{h} \right\rfloor \\
&= -(k+1) + \sum_{h=1}^{k+1} \left\lfloor \frac{k+1}{h} \right\rfloor \\
&= -k + \sum_{h=2}^{k+1} \left\lfloor \frac{k+1}{h} \right\rfloor
\end{aligned}$$

Thus on applying the identity $\sum_{j=1}^{k} \tau(j) = \sum_{r=1}^{k} \lfloor \frac{k}{r} \rfloor$, see Equation (6.7), we obtain the desired formula.

$$\ell(\Pi_k) = -k + \sum_{j=2}^{k+1} \tau(j). \tag{6.27}$$

□

For example, let $k = 7$. Then

$$\begin{aligned}\ell(\Pi_7) &= -7 + \tau(2) + \tau(3) + \tau(4) + \tau(5) + \tau(6) + \tau(7) + \tau(8) \\ &= -7 + 2 + 2 + 3 + 2 + 4 + 2 + 4 \\ &= 12.\end{aligned}$$

From Table 6.5 we confirm that $\ell(\Pi_7) = \ell((1^7, 2^3, 3, 4)) = 12$.

Theorem 6.3.2. *The weight of* Π_k *is given by*

$$w(\Pi_k) = \sum_{j=1}^{k} (\sigma(j+1) - j - 1)\,. \tag{6.28}$$

Proof. In order to get the weight we sum the parts of Π_k in Equation (6.26):

$$\begin{aligned}w(\Pi_k) &= \sum_{h=1}^{s_k} h \left\lfloor \frac{k+1-h}{h} \right\rfloor, \quad s_k = \left\lfloor \frac{k+1}{2} \right\rfloor \\ &= \sum_{h=1}^{s_k} h \left\lfloor \frac{k+1}{h} \right\rfloor - \sum_{h=1}^{s_k} h \\ &= \sum_{h=1}^{k+1} h \left\lfloor \frac{k+1}{h} \right\rfloor - \sum_{s_k+1}^{k+1} h \left\lfloor \frac{k+1}{h} \right\rfloor - \sum_{h=1}^{s_k} h \\ &= \sum_{h=1}^{k+1} h \left\lfloor \frac{k+1}{h} \right\rfloor - \sum_{h=s_k+1}^{k+1} h \cdot 1 - \sum_{h=1}^{s_k} h \\ &= \sum_{h=1}^{k+1} h \left\lfloor \frac{k+1}{h} \right\rfloor - \sum_{h=1}^{k+1} h.\end{aligned}$$

Thus on applying the identity $\sum_{j=1}^{k} \sigma(j) = \sum_{r=1}^{k} r \left\lfloor \frac{k}{r} \right\rfloor$, see Equation (6.3), and noting that $\sigma(1) - 1 = 0$, we obtain the desired formula:

$$w(\Pi_k) = \sum_{j=2}^{k+1} (\sigma(j) - j). \tag{6.29}$$

□

The sequence $\{w(\Pi_k)\}_{k \geq 1}$ begins as follows:

$$1, 2, 5, 6, 12, 13, 20, 24, 32, 33, 49, 50, 60, 69, 84, \ldots$$

This agrees with sequence $A153485$ in Sloane [28]. A comment also describes the kth term as

> "The sum of all aliquot divisors of all positive integers $\leq k+1$".

This interpretation coincides with Equation (6.28) since the phrase "aliquot divisors" usually stands for "proper divisors".

Theorem 6.3.3. *The order of the group of derived partitions of* Π_k *is given by*

$$|D(\Pi_k)| = \prod_{h=1}^{s_k} \left\lfloor \frac{k+1}{h} \right\rfloor, \quad s_k = \left\lfloor \frac{k+1}{2} \right\rfloor. \tag{6.30}$$

Proof. Given a partition $\lambda = (\lambda_1^{e_1}, \ldots, \lambda_k^{e_k})$, where $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k$ and $e_i > 0$ for all i, we know that $|D(\lambda)| = (e_1+1)(e_2+1)\cdots(e_k+1)$.

The result follows on applying this formula to

$\Pi_k = (1^k, 2^{\lfloor (k-1)/2 \rfloor}, \ldots, \lfloor (k+1)/3 \rfloor^2, \lfloor (k+4)/3 \rfloor, \ldots, \lfloor (k+1)/2 \rfloor)$.

That is,

$$\begin{aligned}
|D(\Pi_k)| &= (k+1) \cdot (\lfloor \frac{k-1}{2} \rfloor + 1) \cdots (2+1) \cdot (1+1) \cdot \cdots \cdot (1+1) \\
&= \frac{k+1}{1} \cdot \left\lfloor \frac{k+1}{2} \right\rfloor \cdot \left\lfloor \frac{k+1}{3} \right\rfloor \cdots \left\lfloor \frac{k+1}{k-1} \right\rfloor \cdot \left\lfloor \frac{k+1}{k} \right\rfloor, \ k > 1, \\
&= \prod_{h=1}^{s_k} \left\lfloor \frac{k+1}{h} \right\rfloor.
\end{aligned}$$

□

Chapter 7

Concluding Remarks

The main objective of the thesis was to study certain algebraic structures in integer partitions. We defined a derived partition of a partition, and showed that the set of derived partitions forms a group under addition, and a monoid under multiplication. A ring isomorphism was established with the direct product of the rings of residue classes modulo a positive integer.

In Chapter 3, we demonstrated that the set of derived partitions forms a group under addition. We also classified the types of additive cyclic groups of derived partitions. In Chapter 4, we discussed a special type of the group of derived partitions, namely, the group of pseudo-perfect partitions. We showed that the set $D(\lambda_\infty)$ of derived partitions of infinity forms an infinite group of pseudo-perfect partitions.

In Chapter 5, we discussed the ring of a derived partitions of n. A ring isomorphism was established between the $(D(\lambda), +, *)$ and the direct product of the residue class ring modulo $n + 1$. In Chapter 6, we discussed k-complete partitions and analysed the k-complete partitions with least weights which are also known as the initial $k-$complete partitions. We also discussed perfect-partition-complete partitions and full-perfect-partition complete partitions, and proved some theorems related to these partitions.

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