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RESEARCH

Proximal Point Algorithms with Inertial Extrapolation for Quasi-convex Pseudo-monotone Equilibrium Problems

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Abstract

In this paper, we study the proximal point algorithm with inertial extrapolation to approximate a solution to the quasi-convex pseudo-monotone equilibrium problem. In the proposed algorithm, the inertial parameter is allowed to take both negative and positive values during implementations. The possibility of the choice of negative values for the inertial parameter sheds more light on the range of values of the inertial parameter for the proximal point algorithm. Under standard assumptions, we prove that the sequence of iterates generated by the proposed algorithm converges to a solution of the equilibrium problem when the bifunction is strongly quasi-convex in its second argument. Sublinear and linear rates of convergence are also given under standard conditions. Numerical results are reported for both cases of negative and positive inertial factor of the proposed algorithm and comparison with related algorithm is discussed.

Keywords Equilibrium problems · Proximal point algorithms · Inertial technique · Quasi-convexity · Strong quasi-convexity

Mathematics Subject Classification 90C25 · 90C30 · 90C60 · 68Q25 · 49M25 · 90C22

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1 Introduction

Suppose $K \in \mathbb{R}^n$ is a closed and convex subset and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ a bifunction. The equilibrium problem is the problem of finding a point $\bar{x} \in K$ such that

$$f(\bar{x}, y) \geq 0, \quad \forall\, y \in K. \tag{1}$$

Let S denote the solution set of problem (1). It is well known that any point $\bar{x}$ that solves (1) is called an equilibrium point of f.

Equilibrium problems involve several other mathematical problems such as minimization problems, complementary problems, Kakutani fixed point problems, saddle point problems, Nash equilibria, and variational inequality problems, among others (see Bigi et al. 2019; Blum and Oettli 1994; Facchinei and Pang 2003; Iiduka 2010; Iiduka and Yamada 2009; Konnov 2001; Maingé 2010; Muu and Oettli 1992 and other references therein). The equilibrium problem (1) can also be recognized as a Ky Fan inequality. This is due to the fact that Fan (1972) obtained the first result for the existence of the solution of the equilibrium problem (1). It is well known that Muu and Oettli (1992) established the notion of "equilibrium problems" which was later advanced further by Blum (1994). Several authors have generalized many results with regard to establishing the existence of the solution to an equilibrium problem (see Ansari et al. 1999; Brézis et al. 2008; Cotrina and García 2018; Hieu 2017a, b; Oettli 1997; Peng et al. 2009; Qin et al. 2009; Rehman et al. 2019a, b; Yuan 1999 and other references therein).

Over the years, the equilibrium problem (1) has been studied in the convex case with emphasis on the existence of solutions and optimality conditions. Also, some recent existence solutions, necessary and sufficient optimality conditions have been established in the nonconvex case (see Cotrina and García 2018; Oettli 1997 and other references therein). However, Iusem and Lara (2019) obtained an existence result for the pseudo-monotone equilibrium problem in the general quasi-convex case.

When solving the equilibrium problem (1), two effective techniques are well used due to their numerical efficiency. These techniques are the proximal point algorithm (Moudafi 1999) and the principle of auxiliary problems (Mastroeni 2000). Our main focus in this work is the proximal point algorithm. The proximal point algorithm was introduced by Martinet (1970) to solve a monotone variational inequality problem. Rockafellar (1976) improved this work for monotone operators. Moudafi (1999) generalized the proximal point algorithm introduced by Martinet (1970) from solving monotone variational inequality problems to solving monotone equilibrium problems. Several authors have extended the proximal point algorithm to solve equilibrium problems in the convex case (see Iusem and Sosa 2003; Konnov 2001; Quoc et al. 2008 and other references therein). We note that some of these algorithms assume a generalized monotonicity concept on the bifunction. The limitation of these algorithms in the literature lies in the fact that none of these algorithms has weakened the assumption of convexity on the bifunction in its second argument. To overcome this limitation, Iusem and Lara (2022) proposed and studied a proximal point algorithm for solving quasi-convex pseudo-monotone equilibrium problems.

For more details on the importance of extending the analysis of the proximal point algorithm to the case of quasi-convex bifunctions, rather than the convex bifunctions see Iusem and Lara (2022) and other references therein.

Algorithms with inertial extrapolations have attracted the interest of several researchers due to the increase in the rate of convergence of these iterative algorithms (please, see Nesterov 1983; Polyak 1964). These inertial extrapolation algorithms are based on a discrete analogue of a second-order dissipative dynamical system and are known for their efficiency in improving the rate of convergence of the base iterative methods.

2 Motivation

Equilibrium problem (1) has been applied to modeling, error estimation, operations research, economics, and visualization of industrial problems. For instance, it was noted that the equilibrium conditions of the traffic management problem had the structure of (1) (see Siddiqi 2018, Chapter 9). Siddiqi (2018, Chapter 9) identified network equilibrium conditions with an equilibrium problem (1). Since the equilibrium theorem is very well established in economics, game theorem, and other areas of social sciences, problems from these fields can be modeled as equilibrium problems (1). Stella and Anna Nagurney (see, Siddiqi 2018, Chapter 9) developed a general multinetwork equilibrium model with elastic demands and asymmetric interactions formulating the equilibrium conditions as a variational inequality problem. They studied a projection method for computing the equilibrium pattern. Their work includes several traffic network equilibrium models, spatial price equilibrium models, and general economic equilibrium problems with production. There are a series of papers on the successful application of equilibrium problems in the setting of finite-dimensional spaces to qualitative analysis and computation of perfectly and imperfectly competitive problems, for example, Dupis and Nagurney (1993), Nagurney (1993), Nagurney and Zhang (1995). Also, it was shown in Wilmott et al. (1993) that the American option pricing of financial mathematics and engineering is modeled by an evolution equilibrium problem.

A significant amount of the activities in urban areas concern the movement of people and goods between different locations in the transportation infrastructure, and a smooth and efficient transportation system is essential for economic health and the quality of life within the urban region (see, Wang et al. 2022). When analyzing the present infrastructure for future investments and operating policies, a careful study of the transportation system is among the planning process's most important components. Traffic flow models are usually developed to control such movements (see, for example, Wang et al. 2022). For instance, consider a traffic flow network with N number of nodes that are connected by oriented edges (Konnov 2007, Chapter 6). We denote the set of edges of the network and the set of oriented pairs of the nodes by D and W, respectively. Let $w = (a, b) \in W$, where a and b represent the original node and the destination node, respectively, then P_w is the set of all paths from node a to node b and $Q = \cup_{w \in W} P_w$ is the set of all paths in the network. For each path $p \in Q$, let z_p be the path flow. We associate each $w \in W$ with a positive number d_w, which denotes the flow demand from a to b. The feasible set of flows K is defined by

$$K = \prod_{w \in W} K_w = \{z \in \mathbb{R}^{|Q|} \ : \ \sum_{p \in P_w} z_p = d_w, \ \forall w \in W; z_p \geq 0, \ \forall p \in P_w\},$$

where

$$K_w = \{z \in \mathbb{R}^{|P_w|} \ : \ \sum_{p \in P_w} z_p = d_w, \ z_p \geq 0, \ \forall p \in P_w\}.$$

We can define the value of edge flow h_d for each edge $d \in D$ (if the flow vector z is known) by

$$h_d = \sum_{p \in Q} \xi_{pd} z_p = \sum_{p \in P_w} \sum_{w \in W} \xi_{pd} z_p,$$

where

$$\xi_{pd} = \begin{cases} 1 & \text{if } d \text{ belongs to path p,} \\ 0 & \text{otherwise.} \end{cases}$$

Also, we can define the value of costs for each edge $d \in D$ by $t_d = K_d(h_d)$. Then, the value of costs for each path p can be found (see Hieu et al. 2022) by

$$F_p(z) = \sum_{d \in D} \xi_{pd} t_d.$$

A feasible flow vector $\bar{x} \in K$ is called the equilibrium vector if it satisfies

$$\forall q \in P_w, \ z_q^* > 0 \implies F_{z_q^*}(\bar{x}) = \min_{p \in P_w} F_p(\bar{x}), \ \forall w \in W. \tag{2}$$

This means that when the traffic network is at equilibrium, among all paths of P_w, the path with traffic has the lowest cost. It was shown that (see, for example, Konnov 2007, Theorem 6.1) problem (2) is equivalent to equilibrium problem (1).

Typically (as a numerical example similar to that given in Hieu et al. 2022), we can consider a traffic network with five nodes and eight edges di ($i = 1, 2, \dots, 8$), with cost function K_d, given by

$$K_d(h) = \begin{cases} \alpha_d h + \beta_d & \text{if } \ 0 \leq h \leq \kappa_d, \\ \gamma_d h + \alpha_d \kappa_d + \beta_d - \gamma_d \kappa_d & \text{if } \ h > \kappa_d, \end{cases} \tag{3}$$

where $\gamma_d, \alpha_d, \kappa_d, \beta_d$ are some given real values. The formula (3) can be explained as follows:

1. When the traffic h increases but is still within the allowable limit of edge (see, $h \leq \kappa_d$), the costs increase with a low rate α_d. We can see that β_d is the minimal cost on edge d.

2. Otherwise, when the traffic h exceeds the allowable limit κ_d, the costs increases with huge rate γ_d (traffic jam).

Now, let $W = \{(1,5)\}$ and $d_{(1,5)} = 100$. The set $P_{(1,5)}$ includes five elements: $p_1 = d1 \to d6$, $p_2 = d3 \to d8$, $p_3 = d2 \to d7$, $p_4 = d2 \to d5 \to d8$ and $p_5 = d2 \to d4 \to d6$ (where di ($i = 1, 2, \ldots, 8$) are the edges). Since the cost function K_d increases, F is monotone, where $F(z)$ is a vector in $\mathbb{R}^{|Q|}$ with components $F_p(z)$. By setting $f(\bar{x}, z) = \langle F(\bar{x}), z - \bar{x}\rangle$, one can employ a proximal point algorithm to compute numerically.

Our Contribution In this paper, we propose and study a proximal point type algorithm with an inertial extrapolation step for solving a pseudo-monotone equilibrium problem when the bifunction is quasi-convex. Our proposed algorithm allows us to relax the convexity in the second argument of the bifunction. Also, the inertial factor in the proposed algorithm is allowed to take values in $(-1, 1)$, unlike what is known in the literature of inertial-type algorithms where the inertial factors are taken in [0, 1). We obtain the global convergence of our proposed algorithm to a solution of the equilibrium problem (1). We also obtain sublinear and linear rates of convergence of the proposed algorithm using standard assumptions. Finally, numerical experiments to show the efficiency of our algorithm with related algorithm is given.

Outline We organize the rest of the paper as follows; Section 3 contains basic definitions, lemmas, and results needed in subsequent sections. In Section 4, we present the necessary assumptions, our new algorithm, and its convergence results. In Section 5, we present some numerical results and comparisons to show the effectiveness of our proposed algorithm. We conclude the paper with a recap of what we achieved in Section 6.

3 Preliminaries

In this section, we recall some basic definitions, lemmas, and results that will be needed in our convergence analysis. Let the inner product and the Euclidean norm of $\mathbb{R}^n$ be denoted by $\langle \cdot, \cdot \rangle$ and $|| \cdot ||$, respectively.

Definition 2.1 Let C be a nonempty set in $\mathbb{R}^n$. A bifunction $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is said to be

(i) *strongly monotone on* C, if there exists a constant $c > 0$ such that

$$f(x, y) + f(y, x) \leq -c\|x - y\|^2, \quad \forall\, x, y \in C,$$

(ii) *monotone on* C, if

$$f(x, y) + f(y, x) \leq 0, \ \forall\, x, y \in C,$$

(iii) *pseudo-monotone on* C, if

$$f(x,y) \geq 0 \implies f(y,x) \leq 0, \ \forall\, x,y \in C,$$

(iv) *quasi-monotone on* C if

$$f(x,y) > 0 \implies f(y,x) \leq 0, \ \ \forall\, x,y \in C,$$

(v) satisfying a Lipschitz-like condition on C if there exist constants $a_1 > 0$ and $a_2 > 0$ such that

$$f(x,y) + f(y,z) \geq f(x,z) - a_1\|x-y\|^2 - a_2\|y-z\|^2, \quad \forall x,y,z \in C.$$

Note that $(i) \implies (ii) \implies (iii) \implies (iv)$ but the converses are not always true (see Hadjisavvas et al. 2005 and other references therein).

Definition 2.2 The domain of a function $h : \mathbb{R}^n \to \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is defined by $\operatorname{dom}\ h := \{x \in \mathbb{R}^n : h(x) < +\infty\}$. The function $h : \operatorname{dom}\ h \subseteq \mathbb{R}^n \to \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is said to be

(i) *proper* if $\operatorname{dom}\ h \neq \emptyset$ and $h(x) > -\infty, \ \forall\, x \in \mathbb{R}^n$,
(ii) *lower semicontinuous (lsc)* at $\bar{x} \in \operatorname{dom}\ h$ if for any sequence $\{x_k\} \in$ dom h with $x_k \to \bar{x}$

$$h(\bar{x}) \leq \liminf_{k\to+\infty} h(x_k).$$

We denote the epigraph of the function h by $\operatorname{epi}\ h := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : h(x) \leq t\}$. We denote the sublevel set of the function h at the height $\lambda \in \mathbb{R}$ by $S_\lambda(h) := \{x \in \mathbb{R}^n : h(x) \leq \lambda\}$ and the set of all minimal points of the function h by $\arg\min_{\mathbb{R}^n} h$.

Lemma 2.3 *Let $x,y,z \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Then*

(i) $\langle x-z,\ y-x\rangle = \frac{1}{2}\|z-y\|^2 - \frac{1}{2}\|x-z\|^2 - \frac{1}{2}\|y-x\|^2$,
(ii) $\|(1-\alpha)x - \alpha y\|^2 = (1-\alpha)\|x\|^2 - \alpha\|y\|^2 + \alpha(1-\alpha)\|x-y\|^2$.

Definition 2.4 A function $h : \operatorname{dom}\ h \subseteq \mathbb{R}^n \to \bar{\mathbb{R}}$ is called

(i) convex if for all $t \in [0,1]$ and $x,y \in \operatorname{dom}\ h$,

$$h(tx + (1-t)y) \leq t\, h(x) + (1-t)h(y),$$

if the inequality is strict, then h is called strictly convex,

(ii) strongly convex on dom h if there exists $\delta \in (0,+\infty)$ such that for all $x,y \in \operatorname{dom}\ h$ and all $t \in [0,1]$,

$$h(ty + (1-t)x) \leq th(y) + (1-t)h(x) - t(1-t)\frac{\delta}{2}\|x-y\|^2,$$

(iii) semistrictly quasi-convex with $h(x) \neq h(y)$, if for all $t \in [0,1]$ and $x,y \in \operatorname{dom}\ h$,

$$h(tx + (1-t)y) < \max\{h(x), h(y)\},$$

(iv) quasi-convex, if for all $t \in [0,1]$ and $x, y \in \operatorname{dom} h$,

$$h(tx + (1-t)y) \le \max\{h(x), h(y)\},$$

if the inequality is strict, then h is called strictly quasi-convex,

(v) strongly quasi-convex on dom h if there exists $\delta \in (0, +\infty)$ such that for all $x, y \in \operatorname{dom} h$ and all $t \in [0,1]$,

$$h(tx + (1-t)y) \le \max\{h(x), h(y)\} - t(1-t)\frac{\delta}{2}\|x-y\|^2.$$

Remark 2.5 Every strictly convex function is strictly quasi-convex and semi-strictly quasi-convex, and every lsc and semi-strictly quasi-convex function is quasi-convex (see Cambini and Martein 2009, Theorem 2.3.3). In (ii) and (v), we have that h is strongly convex and strongly quasi-convex respectively with modulus $\delta > 0$. Also, we note that every strongly convex function is strongly quasi-convex with the same modulus and every strongly quasi-convex function is strictly quasi-convex.

Let K be a closed and convex set in $\mathbb{R}^n$ and $h : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a proper function with $K \subseteq \operatorname{dom} h$. Then the proximity operator on K of parameter $\beta > 0$ of h at $x \in \mathbb{R}^n$ is defined as $\operatorname{prox}_{\beta h} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, where

$$\operatorname{prox}_{\beta h}(K, x) = \arg\min_{y \in K} \left\{ h(y) + \frac{1}{2\beta}\|y-x\|^2 \right\}.$$

If $K = \mathbb{R}^n$, then $\operatorname{prox}_{\beta h}(\mathbb{R}^n, x) = \operatorname{prox}_{\beta h}(x)$.

When studying the proximal point algorithm, the properties of proximity are very important.

Lemma 2.6 *Iusem and Lara (2022) Let K be a closed and convex set in $\mathbb{R}^n$, $h : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a proper, lsc, strongly quasi-convex function with modulus $\delta \ge 0$ and such that $K \subseteq \operatorname{dom} h$, $\beta > 0$ and $x \in K$. If $\bar{x} \in \operatorname{Prox}_{\beta h}(K, x)$, then $\forall y \in \mathbb{R}, \forall \lambda \in [0,1]$*

$$h(\bar{x}) - \max\left\{h(y), h(\bar{x})\right\} \le \frac{\lambda}{\beta}\langle \bar{x} - x,\ y - \bar{x}\rangle + \frac{\lambda}{2}\left(\frac{\lambda}{\beta} - \delta + \lambda\delta\right)\|y-\bar{x}\|^2. \quad (4)$$

4 Main Results

In this section, we present our proposed algorithm and give its convergence analysis. We begin by giving the following assumptions under which our convergence result is obtained.

Assumption 3.1

(A1) $\forall\, x \in \mathbb{R}^n$, the function $f(x, \cdot)$ is lower semicontinuous and $\forall\, y \in \mathbb{R}^n$, the function $f(\cdot, y)$ is upper semicontinuous.
(A2) f is pseudo-monotone on $\mathbb{R}^n$.
(A3) f is lower semicontinuous jointly with both arguments.
(A4) $\forall\, x \in \mathbb{R}^n$, the function $f(x, \cdot)$ is strongly quasi-convex on $\mathbb{R}^n$ with modulus $\gamma > 0$.
(A5) f satisfies the Lipschitz condition: there exists $\eta > 0$ such that

$$f(x,z) - f(x,y) - f(y,z) \le \eta\big(\|x-y\|^2 + \|y-z\|^2\big), \quad \forall\, x, y, z \in \mathbb{R}^n.$$

(A6) The Lipschitz constant η and the modulus of quasi-convexity γ are such that $12\eta < \gamma$.

Remark 3.2 An example satisfying Assumption (A6) is given as a numerical example in Section 5. Kindly see Iusem and Lara (2022, Examples 4.1 and 4.2) for more examples of bifunctions satisfying Assumption (A6).

Now, we state an additional assumption that is standard in the theory of equilibrium problems. This assumption is excluded from Assumption 3.1 because it is a consequence of $(A2)$ and $(A5)$. This assumption will be used in results that do not require $(A5)$.

(A0) $f(x,x) = 0$ for all $x \in K$.

Remark 3.3 Iusem and Lara (2022) We note that $(A2)$ and $(A5)$ implies $(A0)$.

We now present our algorithm as follows:

Algorithm 1 Proximal Point Algorithm with Inertial

1: Choose $\theta \in (-1,1)$, $x_0, y_0 \in \mathbb{R}^n$, $\beta > 0$ and set $k = 0$.
2: Given x_k, y_k, take

$$x_{k+1} \in \arg\min_{x \in K} \left\{ f(y_k, x) + \frac{1}{2\beta}\|y_k - x\|^2 \right\}. \tag{5}$$

If $x_{k+1} = y_k$, then **Stop:** $\{x_{k+1}\} = S$. Otherwise, compute

$$y_{k+1} = x_{k+1} + \theta(x_{k+1} - x_k). \tag{6}$$

3: Set $k \leftarrow k+1$, and **go to Step 2**.

Remark 3.4 If $\theta = 0$, our Algorithm 1 reduces to Iusem and Lara (2022, Algorithm 1).

For the rest of this paper, we assume that Assumption 3.1 holds in our convergence analysis.

Proposition 3.5 *Suppose K is a closed and convex subset of $\mathbb{R}^n$ and f enjoys* $(A0)$, $(A1)$, $(A2)$ *and* $(A4)$. *Then S is a singleton.*

Proof The proof of this proposition is the same as the proof of Proposition 3.1 of Iusem and Lara (2022). □

Proposition 3.6 *Assume that $\{x_k\}$ is generated by Algorithm* 1 *and f enjoys* $(A0)$, $(A1)$ *and* $(A4)$. *Suppose also that $x_{k+1} = y_k$. Then, $\{x_{k+1}\} = S$.*

Proof From (5) of Algorithm 1, we obtain $\forall x \in K$,

$$f(y_k, x_{k+1}) + \frac{1}{2\beta}\|y_k - x_{k+1}\|^2 \leq f(y_k, x) + \frac{1}{2\beta}\|y_k - x\|^2. \tag{7}$$

Since $y_k = x_{k+1}$, we then obtain further that

$$0 \leq f(x_{k+1}, x) + \frac{1}{2\beta}\|x_{k+1} - x\|^2, \quad \forall x \in K. \tag{8}$$

Take $\bar{x} \in S$ and let $y \in K\backslash\{\bar{x}\}$, $\lambda \in [0, 1]$, and take $x = \lambda y + (1 - \lambda)x_{k+1}$. We then obtain that (noting that f is strongly quasi-convex on $\mathbb{R}^n$ with modulus $\gamma > 0$ in its second argument, and $x_{k+1} = y_k$)

$$\begin{aligned} 0 &\leq f(x_{k+1}, \lambda y + (1 - \lambda)x_{k+1}) + \frac{1}{2\beta}\|\lambda(x_{k+1} - y)\|^2 \\ &\leq \max\left\{f(x_{k+1}, y), 0\right\} + \frac{\lambda}{2}\left(\frac{\lambda}{\beta} - \gamma + \lambda\gamma\right)\|x_{k+1} - y\|^2, \quad \forall y \in \mathbb{R}^n, \ \lambda \in (0, 1]. \end{aligned}$$

Let us take λ such that $0 < \lambda < \frac{\gamma\beta}{1+\gamma\beta}$.

Then, we obtain $\forall k$,

$$\frac{\lambda}{\beta} - \gamma + \lambda\gamma < 0.$$

Therefore, $\forall\, y \in K\backslash\{x_{k+1}\}$

$$0 < \max\left\{f(x_{k+1}, y), 0\right\} = f(x_{k+1}, y).$$

Thus, $x_{k+1} \in S$. □

4.1 Convergence Analysis for Nonnegative Inertial Parameter

We first give the convergence result of Algorithm 1 for the case when the inertial factor $\theta \geq 0$.

Assumption 3.7 For the convergence analysis of this subsection, we assume that

(i) $0 \leq \theta < \frac{1}{3}$;
(ii) $\frac{1}{\gamma - 8\eta} < \beta < \frac{1-3\theta}{4\eta(1-\theta)}$.

We next give the following two lemmas.

Lemma 3.8 *Assume that f enjoys (A_i), $i = 1, 2, 4, 5$ in Assumption 3.1 and take $\bar{x} \in S$. Let $\{x_k\}$ be generated by Algorithm 1. Then, either (15) or (18) holds.*

Proof Observe that f satisfies $(A0)$ because f enjoys (A_i), $i = 1, 2, 4, 5$. Therefore, S is a singleton by Proposition 3.5. Using Lemma 2.6 with $\gamma > 0$ in (5), we obtain $\forall x \in K,\ \lambda \in [0, 1]$ that

$$f(y_k, x_{k+1}) - \max\left\{f(y_k, x_{k+1}), f(y_k, x)\right\} \leq \frac{\lambda}{\beta}\langle x_{k+1} - y_k,\ x - x_{k+1}\rangle + \frac{\lambda}{2}\left(\frac{\lambda}{\beta} - \gamma + \gamma\lambda\right)\|x - x_{k+1}\|^2. \tag{9}$$

Substituting $x = \bar{x} \in S$ in (9), we obtain $\forall\ \lambda \in [0, 1]$,

$$f(y_k, x_{k+1}) - \max\left\{f(y_k, x_{k+1}), f(y_k, \bar{x})\right\} \leq \frac{\lambda}{\beta}\langle x_{k+1} - y_k,\ \bar{x} - x_{k+1}\rangle + \frac{\lambda}{2}\left(\frac{\lambda}{\beta} - \gamma + \gamma\lambda\right)\|x_{k+1} - \bar{x}\|^2. \tag{10}$$

We now consider these two cases:

Case I: If $f(y_k, x_{k+1}) \geq f(y_k, \bar{x})$, then we obtain from (10) that

$$0 \leq \frac{1}{\beta}\langle x_{k+1} - y_k,\ \ \bar{x} - x_{k+1}\rangle + \frac{1}{2}\left(\frac{\lambda}{\beta} - \gamma + \gamma\lambda\right)\|x_{k+1} - \bar{x}\|^2,\ \ \forall\ \lambda \in (0, 1]. \tag{11}$$

Let $\lambda = \frac{1}{2}$ and by Lemma 2.3 (i), we have from (11) that

$$\frac{(1 + \gamma\beta)}{2}\|x_{k+1} - \bar{x}\|^2 \leq \|y_k - \bar{x}\|^2 - \|x_{k+1} - y_k\|^2. \tag{12}$$

By Lemma 2.3 (ii), we obtain

$$\begin{aligned}\|y_k - \bar{x}\|^2 &= \|(1+\theta)(x_k - \bar{x}) - \theta(x_{k-1} - \bar{x})\|^2 \\ &= (1+\theta)\|x_k - \bar{x}\|^2 - \theta\|x_{k-1} - \bar{x}\|^2 \\ &\quad + (1+\theta)\theta\|x_k - x_{k-1}\|^2.\end{aligned} \tag{13}$$

Also, we obtain

$$\begin{aligned}\|x_{k+1} - y_k\|^2 &= \|x_{k+1} - (x_k + \theta(x_k - x_{k-1}))\|^2 \\ &= \|x_{k+1} - x_k\|^2 - 2\theta\langle x_{k+1} - x_k,\ x_k - x_{k-1}\rangle + \theta^2\|x_k - x_{k-1}\|^2 \\ &\geq \|x_{k+1} - x_k\|^2 - 2\theta\|x_{k+1} - x_k\|\|x_k - x_{k-1}\| + \theta_2\|x_k - x_{k-1}\|^2 \\ &\geq \|x_{k+1} - x_k\|^2 - \theta\Big[\|x_{k+1} - x_k\|^2 + \|x_k - x_{k-1}\|^2\Big] \\ &\quad + \theta^2\|x_k - x_{k-1}\|^2 \\ &= (1-\theta)\|x_{k+1} - x_k\|^2 + (\theta^2 - \theta)\|x_k - x_{k-1}\|^2.\end{aligned} \tag{14}$$

Substituting (13) and (14) into (12), we obtain

$$\begin{aligned}&\frac{(1+\gamma\beta)}{2}\|x_{k+1} - \bar{x}\|^2 - \theta\|x_k - \bar{x}\|^2 + (1-\theta)\|x_{k+1} - x_k\|^2 \\ \leq\ &\|x_k - \bar{x}\|^2 - \theta\|x_{k-1} - \bar{x}\|^2 + (1-\theta)\|x_k - x_{k-1}\|^2 \\ &+ (3\theta - 1)\|x_k - x_{k-1}\|^2.\end{aligned} \tag{15}$$

Case II: Suppose $f(y_k, x_{k+1}) < f(y_k, \bar{x})$. Then we obtain from (10) that (also noting $(A5)$)

$$\begin{aligned}0 &\leq \frac{\lambda}{\beta}\langle x_{k+1} - y_k,\ \bar{x} - x_{k+1}\rangle + \frac{\lambda}{2}\left(\frac{\lambda}{\beta} - \gamma + \gamma\lambda\right)\|x_{k+1} - \bar{x}\|^2 + f(y_k, \bar{x}) \\ &\quad - f(y_k, x_{k+1}) \\ &\leq \frac{\lambda}{\beta}\langle x_{k+1} - y_k,\ \bar{x} - x_{k+1}\rangle + \frac{\lambda}{2}\left(\frac{\lambda}{\beta} - \gamma + \gamma\lambda\right)\|x_{k+1} - \bar{x}\|^2 \\ &\quad + f(x_{k+1}, \bar{x}) + \eta\big(\|x_{k+1} - y_k\|^2 + \|x_{k+1} - \bar{x}\|^2\big).\end{aligned} \tag{16}$$

Since $\bar{x} \in S$ and f is pseudo-monotone, we obtain that $f(x_{k+1}, \bar{x}) \leq 0$. Hence, (16) implies that

$$\begin{aligned}0 &\leq \frac{\lambda}{\beta}\langle x_{k+1} - y_k,\ \bar{x} - x_{k+1}\rangle + \frac{\lambda}{2}\left(\frac{\lambda}{\beta} - \gamma + \gamma\lambda\right)\|x_{k+1} - \bar{x}\|^2 \\ &\quad + \eta\big(\|x_{k+1} - y_k\|^2 + \|x_{k+1} - \bar{x}\|^2\big), \quad \forall\ \lambda \in [0, 1].\end{aligned}$$

By Lemma 2.3 (i), we obtain

$$0 \le \frac{\lambda}{2\beta}\|y_k - \bar{x}\|^2 + \frac{\lambda}{2}\left(\frac{\lambda}{\beta} - \gamma + \gamma\lambda - \frac{1}{\beta}\right)\|x_{k+1} - \bar{x}\|^2$$
$$- \frac{\lambda}{2\beta}\|x_{k+1} - y_k\|^2 + \eta\big(\|x_{k+1} - y_k\|^2 + \|x_{k+1} - \bar{x}\|^2\big).$$

Taking $\lambda = \frac{1}{2}$, we have

$$\left(\frac{1+\gamma\beta}{8\beta} - \eta\right)\|x_{k+1} - \bar{x}\|^2 \le \frac{1}{4\beta}\|y_k - \bar{x}\|^2 - \left(\frac{1}{4\beta} - \eta\right)\|x_{k+1} - y_k\|^2. \quad (17)$$

Using (13) and (14) in (17), we get

$$\begin{aligned}\left(\frac{1+\gamma\beta}{8\beta} - \eta\right)\|x_{k+1} - \bar{x}\|^2 &\le \frac{1}{4\beta}\Big[(1+\theta)\|x_k - \bar{x}\|^2 - \theta\|x_{k-1} - \bar{x}\|^2 \\ &\quad + (1+\theta)\theta\|x_k - x_{k-1}\|^2\Big] \\ &\quad - \left(\frac{1}{4\beta} - \eta\right)\Big[(1-\theta)\|x_{k+1} - x_k\|^2 + (\theta^2 - \theta)\|x_k - x_{k-1}\|^2\Big] \\ &= \frac{1}{4\beta}\Big[(1+\theta)\|x_k - \bar{x}\|^2 - \theta\|x_{k-1} - \bar{x}\|^2\Big] \\ &\quad + \left[\frac{1}{4\beta}(1+\theta)\theta - \left(\frac{1}{4\beta} - \eta\right)(\theta^2 - \theta)\right]\|x_k - x_{k-1}\|^2 \\ &\quad - \left(\frac{1}{4\beta} - \eta\right)(1-\theta)\|x_{k+1} - x_k\|^2.\end{aligned} \quad (18)$$

Therefore, at least one of (15) and (18) holds. □

Lemma 3.9 *Suppose* $(A_i), i = 1, 2, 4, 5, 6$ *in Assumptions* 3.1 *and* 3.7 *are fulfilled. Then* $\{x_k\}$ *generated by Algorithm* 1 *is bounded.*

Proof By Assumption 3.7, we obtain $\frac{1}{\gamma - 8\eta} < \beta < \frac{1-3\theta}{4\eta(1-\theta)} < \frac{1}{4\eta}$. Also by Proposition 3.5, we have that S is a singleton. Let $\{\bar{x}\} = S$. Then $\forall k \ge 1$, we have by Lemma 3.8, one of (15) and (18) holds. Let us, therefore, consider these two cases:

Case A: Suppose (15) holds. Since $\frac{1}{\gamma} < \frac{1}{\gamma - 8\eta} < \beta$, we obtain

$$\begin{aligned}&\|x_{k+1} - \bar{x}\|^2 - \theta\|x_k - \bar{x}\|^2 + (1-\theta)\|x_{k+1} - x_k\|^2 \\ &\quad < \frac{1+\gamma\beta}{2}\|x_{k+1} - \bar{x}\|^2 - \theta\|x_k - \bar{x}\|^2 + (1-\theta)\|x_{k+1} - x_k\|^2 \\ &\quad \le \|x_k - \bar{x}\|^2 - \theta\|x_{k-1} - \bar{x}\|^2 + (1-\theta)\|x_k - x_{k-1}\|^2 \\ &\qquad + (3\theta - 1)\|x_k - x_{k-1}\|^2.\end{aligned}$$

Define, $\forall k \ge 1$,

$$a_k = \|x_k - \bar{x}\|^2 - \theta\|x_{k-1} - \bar{x}\|^2 + (1-\theta)\|x_k - x_{k-1}\|^2.$$

Then

$$a_{k+1} - a_k \le (3\theta - 1)\|x_k - x_{k-1}\|^2. \tag{19}$$

Now, observe that

$$\begin{aligned} a_k = & \ \|x_k - \bar{x}\|^2 - \theta\|x_{k-1} - \bar{x}\|^2 + (1-\theta)\|x_k - x_{k-1}\|^2 \\ \ge & \ \|x_k - \bar{x}\|^2 - 2\theta\|x_k - x_{k-1}\|^2 - 2\theta\|x_k - \bar{x}\|^2 + (1-\theta)\|x_k - x_{k-1}\|^2 \\ = & \ (1-2\theta)\|x_k - \bar{x}\|^2 + (1-3\theta)\|x_k - x_{k-1}\|^2 \\ \ge & \ 0, \end{aligned} \tag{20}$$

since $\theta < \frac{1}{3}$. Thus, $\lim\limits_{k\to\infty} a_k$ exists and

$$\lim_{k\to\infty} \|x_k - x_{k-1}\| = 0. \tag{21}$$

Also,

$$\|x_{k+1} - y_k\| \le \|x_{k+1} - x_k\| + \theta\|x_k - x_{k-1}\| \to 0, \ k \to \infty \tag{22}$$

and

$$\|y_k - x_k\| = \theta\|x_k - x_{k-1}\| \to 0, \ k \to \infty. \tag{23}$$

Since $\lim\limits_{k\to\infty} a_k$ exists, we have from (20) that $\{x_k\}$ is bounded. This concludes **Case A**.

Case B: Suppose (18) holds. Then, we obtain from (18) that

$$\begin{aligned} \left(\frac{1+\gamma\beta}{8\beta} - \eta\right)\|x_{k+1} - \bar{x}\|^2 \le & \ \frac{1}{4\beta}\left[(1+\theta)\|x_k - \bar{x}\|^2 - \theta\|x_{k-1} - \bar{x}\|^2\right] \\ & + \left[\frac{1}{4\beta}(1+\theta)\theta - \left(\frac{1}{4\beta} - \eta\right)(\theta^2 - \theta)\right]\|x_k - x_{k-1}\|^2 \\ & - \left(\frac{1}{4\beta} - \eta\right)(1-\theta)\|x_{k+1} - x_k\|^2. \end{aligned} \tag{24}$$

Therefore,

$$\begin{aligned} & \left(\frac{1+\gamma\beta}{8\beta} - \eta\right)\|x_{k+1} - \bar{x}\|^2 - \frac{\theta}{4\beta}\|x_k - \bar{x}\|^2 + \left(\frac{1}{4\beta} - \eta\right)(1-\theta)\|x_{k+1} - x_k\|^2 \\ & \le \frac{1}{4\beta}\|x_k - \bar{x}\|^2 + \left[\frac{1}{4\beta}(1+\theta)\theta - \left(\frac{1}{4\beta} - \eta\right)(\theta^2 - \theta)\right]\|x_k - x_{k-1}\|^2 \\ & - \frac{\theta}{4\beta}\|x_{k-1} - \bar{x}\|^2. \end{aligned} \tag{25}$$

Since $\frac{1}{\gamma} < \frac{1}{\gamma - 8\eta} < \beta$, we have

$$\begin{aligned} \frac{1}{4\beta} = & \ \left(\frac{1+\gamma\beta}{8\beta} - \frac{\gamma\beta - 1}{8\beta}\right) \\ < & \ \frac{1+\gamma\beta}{8\beta} - \eta. \end{aligned} \tag{26}$$

Using (26) in (25), we obtain

$$\begin{aligned}\frac{1}{4\beta}\|x_{k+1}-\bar{x}\|^2-\frac{\theta}{4\beta}\|x_k-\bar{x}\|^2+\left(\frac{1}{4\beta}-\eta\right)(1-\theta)\|x_{k+1}-x_k\|^2\\ \leq \frac{1}{4\beta}\|x_k-\bar{x}\|^2-\frac{\theta}{4\beta}\|x_{k-1}-\bar{x}\|^2\\ +\left(\frac{1}{4\beta}-\eta\right)(1-\theta)\|x_k-x_{k-1}\|^2\\ +\left[\frac{1}{4\beta}(1+\theta)\theta-\left(\frac{1}{4\beta}-\eta\right)(1-\theta)^2\right]\|x_k-x_{k-1}\|^2.\end{aligned}$$

Define

$$\begin{aligned}d_k :=& \frac{1}{4\beta}\left[\|x_k-\bar{x}\|^2-\theta\|x_{k-1}-\bar{x}\|^2\right]\\ &+\left(\frac{1}{4\beta}-\eta\right)(1-\theta)\|x_k-x_{k-1}\|^2.\end{aligned}$$

Then

$$\begin{aligned}d_{k+1} &\leq d_k+\left[\frac{1}{4\beta}(1+\theta)\theta-\left(\frac{1}{4\beta}-\eta\right)(1-\theta)^2\right]\|x_k-x_{k-1}\|^2\\ &= d_k-t_1,\end{aligned} \tag{27}$$

where

$$t_1 := \left(\frac{1}{4\beta}-\eta\right)(1-\theta)^2-\frac{1}{4\beta}(1+\theta)\theta>0,$$

since $\beta<\frac{1-3\theta}{4\eta(1-\theta)}\leq\frac{1-3\theta}{4\eta(1-\theta)^2}$. Now,

$$\begin{aligned}d_k &= \frac{1}{4\beta}\|x_k-\bar{x}\|^2-\frac{\theta}{4\beta}\|x_{k-1}-\bar{x}\|^2+\left(\frac{1}{4\beta}-\eta\right)(1-\theta)\|x_k-x_{k-1}\|^2\\ &\geq \frac{1}{4\beta}\left[\|x_k-\bar{x}\|^2-2\theta\|x_{k-1}-x_k\|^2-2\theta\|x_k-\bar{x}\|^2\right]\\ &\quad+\left(\frac{1}{4\beta}-\eta\right)(1-\theta)\|x_k-x_{k-1}\|^2\\ &= \frac{(1-2\theta)}{4\beta}\|x_k-\bar{x}\|^2+\left[\left(\frac{1}{4\beta}-\eta\right)(1-\theta)-\frac{2\theta}{4\beta}\right]\|x_k-x_{k-1}\|^2\\ &\geq 0,\end{aligned} \tag{28}$$

since $\theta<\frac{1}{2}$ and $\beta<\frac{1-3\theta}{4\eta(1-\theta)}$. Therefore, $\{d_k\}$ converges. Also, we have just like (21)–(23), that

$$\lim_{k\to\infty}\|x_k-x_{k-1}\|=\lim_{k\to\infty}\|x_{k+1}-y_k\|=\lim_{k\to\infty}\|y_k-x_k\|=0. \tag{29}$$

Consequently, we have from (27) that $\{x_k\}$ is bounded. This concludes **Case B**. □

We now prove the following theorem.

Theorem 3.10 *Suppose $K \subset \mathbb{R}^n$ is a closed and convex subset. Assume that both Assumptions* 3.1 *and* 3.7 *are satisfied. Then $\{x_k\}$ generated by Algorithm* 1 *converges to a point $\bar{x} \in S$.*

Proof We have by Lemma 3.9 that $\{x_k\}$ is bounded. Suppose $\hat{x} \in K$ is a cluster point of $\{x_k\}$. Then there exists $\{x_{k_i}\} \subset \{x_k\}$ such that $x_{k_i} \to \hat{x},\ i \to \infty$. We obtain from (5) that $\forall x \in K$,

$$f(y_k, x_{k+1}) + \frac{1}{2\beta}\|y_k - x_{k+1}\|^2 \leq f(y_k, x) + \frac{1}{2\beta}\|y_k - x\|^2$$

and this implies that

$$f(y_k, x_{k+1}) \leq f(y_k, x) + \frac{1}{2\beta}\|y_k - x\|^2, \quad \forall x \in K.$$

Suppose $x = \lambda y + (1-\lambda)\hat{x}$, when $y \in K,\ \lambda \in [0,1]$. Then,

$$\begin{aligned} f(y_k, x_{k+1}) &\leq f(y_k, \lambda y + (1-\lambda)\hat{x}) + \frac{1}{2\beta}\|\lambda(y_k - y) + (1-\lambda)(y_k - \hat{x})\|^2 \\ &\leq \max\left\{f(y_k, y), f(y_k, \hat{x})\right\} - \frac{\lambda(1-\lambda)\gamma}{2}\|y - \hat{x}\|^2 \\ &\quad + \frac{\lambda}{2\beta}\|y_k - y\|^2 + \frac{(1-\lambda)}{2\beta}\|y_k - \hat{x}\|^2 - \frac{\lambda(1-\lambda)}{2\beta}\|y - \hat{x}\|^2 \\ &= \max\left\{f(y_k, y), f(y_k, \hat{x})\right\} + \frac{\lambda}{\beta}\langle \hat{x} - y_k,\ \ y - \hat{x}\rangle \\ &\quad + \frac{1}{2\beta}\|y_k - \hat{x}\|^2 + \frac{\lambda}{2}\left(\frac{\lambda}{\beta} - \gamma + \gamma\lambda\right)\|y - \hat{x}\|^2. \end{aligned} \tag{30}$$

Taking $\limsup_{i\to\infty}$ in (29) and using $(A3)$, we obtain (noting (22), (23) and (28)) $\forall y \in K, \lambda \in [0,1]$,

$$\begin{aligned} 0 = f(\hat{x}, \hat{x}) &\leq \liminf_{i\to\infty} f(y_{k_i}, x_{k_{i+1}}) \leq \limsup_{i\to\infty} f(y_{k_i}, x_{k_{i+1}}) \\ &\leq \max\left\{f(\hat{x}, y), 0\right\} + \frac{\lambda}{2}\left(\frac{\lambda}{\beta} - \gamma + \gamma\lambda\right)\|y - \hat{x}\|^2. \end{aligned}$$

Now, since $\gamma > 0$, we take $\lambda < \frac{\beta\gamma}{1+\beta\gamma}$. Thus, $\frac{\lambda}{\beta} - \gamma + \gamma\lambda < 0$. Therefore,

$$0 < \max\left\{f(\hat{x}, y), 0\right\} = f(\hat{x}, y), \ \forall y \in K\backslash\{\hat{x}\}.$$

Hence, $\hat{x} \in S$. Now, since f is strongly quasi-convex in its second argument, S is a singleton by Proposition 3.5. Therefore, $\{x_k\}$ converges to $\bar{x} \in S$. ◻

In the next result, we give a non-asymptotic $\mathcal{O}(1/k)$ convergence rate of our proposed Algorithm 1 when the inertial factor $\theta \geq 0$.

Theorem 3.11 *Suppose $K \subset \mathbb{R}^n$ is a closed and convex subset. Assume that both Assumptions 3.1 and 3.7 are satisfied and $x_{-1} = x_0 = x_1 \in \mathbb{R}^n$. Let $\{x_k\}$ be generated by Algorithm 1. Then, for any $\bar{x} \in S$ and $k > 1$, it holds that*

$$\min_{0\leq j\leq k-1} \|x_{j+1} - y_j\|^2 \leq \frac{(2+2\theta^2)(1-\theta)}{kM^{**}} \|x_1 - \bar{x}\|^2,$$

where $M^{**} = \min\{3\theta - 1, 4\beta\left(\frac{1}{4\beta} - \eta\right)(1-\theta)^2 - (1+\theta)\theta\}$.

Proof **Case A:** Suppose (15) holds. Then, from the definition of $\{a_k\}$, we get

$$\begin{aligned} a_1 &= \|x_1 - \bar{x}\|^2 - \theta\|x_0 - \bar{x}\|^2 + (1-\theta)\|x_1 - x_0\|^2 \\ &= (1-\theta)\|x_1 - \bar{x}\|^2. \end{aligned}$$

From (19), we obtain

$$(3\theta - 1)\sum_{j=1}^{k} \|x_j - x_{j-1}\|^2 \leq a_1 - a_{k+1}.$$

This implies that

$$\begin{aligned} \sum_{j=1}^{k} \|x_j - x_{j-1}\|^2 &\leq \frac{1}{3\theta - 1} a_1 \\ &= \frac{1}{3\theta - 1}(1-\theta)\|x_1 - \bar{x}\|^2. \end{aligned} \tag{31}$$

Thus,

$$\min_{1\leq j\leq k} \|x_j - x_{j-1}\|^2 \leq \frac{1-\theta}{(3\theta - 1)k} \|x_1 - \bar{x}\|^2.$$

From (22), we obtain

$$\begin{aligned} \|x_{k+1} - y_k\|^2 &\leq \Big(\|x_{k+1} - x_k\| + \theta\|x_k - x_{k-1}\|\Big)^2 \\ &\leq 2\Big(\|x_{k+1} - x_k\|^2 + \theta^2\|x_k - x_{k-1}\|^2\Big). \end{aligned}$$

Hence,

$$\begin{aligned}\min_{0\le j\le k-1}\|x_{j+1}-y_j\|^2 &\le 2\min_{0\le j\le k-1}\|x_{j+1}-x_j\|^2+2\theta^2\min_{0\le j\le k-1}\|x_j-x_{j-1}\|^2\\ &\le \frac{(2+2\theta^2)(1-\theta)}{(3\theta-1)k}\|x^1-\bar{x}\|^2.\end{aligned}\tag{32}$$

This concludes **Case A**.

Case B: Suppose (18) holds. Then, from the definition of $\{d_k\}$, we obtain

$$d_1=\frac{1}{4\beta}(1-\theta)\|x_1-\bar{x}\|^2.$$

From (26), we obtain

$$\left(\frac{1}{4\beta}-\eta\right)(1-\theta)^2-\frac{1}{4\beta}(1+\theta)\theta\sum_{j=1}^{k}\|x_j-x_{j-1}\|^2\le d_1-d_{k+1}.$$

This implies that

$$\begin{aligned}\sum_{j=1}^{k}\|x_j-x_{j-1}\|^2 &\le \frac{1}{\left(\frac{1}{4\beta}-\eta\right)(1-\theta)^2-\frac{1}{4\beta}(1+\theta)\theta}d_1\\ &= \frac{(1-\theta)}{4\beta\left(\frac{1}{4\beta}-\eta\right)(1-\theta)^2-(1+\theta)\theta}\|x_1-\bar{x}\|^2.\end{aligned}\tag{33}$$

Thus,

$$\min_{1\le j\le k}\|x_j-x_{j-1}\|^2\le\frac{1-\theta}{\left[4\beta\left(\frac{1}{4\beta}-\eta\right)(1-\theta)^2-(1+\theta)\theta\right]k}\|x_1-\bar{x}\|^2.$$

Similar to the arguments in **Case A**, we arrive at

$$\min_{0\le j\le k-1}\|x_{j+1}-y_j\|^2\le\frac{(2+2\theta^2)(1-\theta)}{\left[4\beta\left(\frac{1}{4\beta}-\eta\right)(1-\theta)^2-(1+\theta)\theta\right]k}\|x^1-\bar{x}\|^2.\tag{34}$$

This concludes **Case B**. Therefore, our proof is complete. □

4.2 Convergence Analysis for Negative Inertial Parameter

In this subsection, we obtain convergence analysis of Algorithm 1 when the inertial parameter $\theta<0$.

Assumption 3.12 Here, we assume that

(i) $-1 < \theta < 0$;
(ii) $\frac{1}{\gamma-8\eta} < \beta < \frac{1}{4\eta}$.

Lemma 3.13 *Suppose* $(A_i), i = 1,2,4,5,6$ *are satisfied in Assumptions* 3.1 *and* 3.12 *holds. Then* $\{x_k\}$ *generated by Algorithm* 1 *is bounded.*

Proof Just like in Lemmas 3.8 and 3.9, we consider these two cases:

Case I: Assume $f(y_k, x_{k+1}) \geq f(y_k, \bar{x})$. Then, from (12) and (13), we obtain

$$\begin{aligned}\frac{(1+\gamma\beta)}{2}\|x_{k+1}-\bar{x}\|^2 &\leq \|y_k-\bar{x}\|^2 - \|x_{k+1}-y_k\|^2\\ &= (1+\theta)\|x_k-\bar{x}\|^2 - \theta\|x_{k-1}-\bar{x}\|^2\\ &\quad + (1+\theta)\theta\|x_k-x_{k-1}\|^2 - \|x_{k+1}-y_k\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{(1+\gamma\beta)}{2}\|x_{k+1}-\bar{x}\|^2 - \theta\|x_k-\bar{x}\|^2 - (1+\theta)\theta\|x_k-x_{k-1}\|^2\\ &\leq \|x_k-\bar{x}\|^2 - \theta\|x_{k-1}-\bar{x}\|^2 - \|x_{k+1}-y_k\|^2\\ &= \|x_k-\bar{x}\|^2 - \theta\|x_{k-1}-\bar{x}\|^2 - (1+\theta)\theta\|x_{k-1}-x_{k-2}\|^2\\ &\quad - \|x_{k+1}-y_k\|^2 + (1+\theta)\theta\|x_{k-1}-x_{k-2}\|^2.\end{aligned} \tag{35}$$

Since $\frac{1}{\gamma} < \frac{1}{\gamma-8\eta} < \beta$, we have

$$\begin{aligned}&\|x_{k+1}-\bar{x}\|^2 - \theta\|x_k-\bar{x}\|^2 - (1+\theta)\theta\|x_k-x_{k-1}\|^2\\ &\leq \|x_k-\bar{x}\|^2 - \theta\|x_{k-1}-\bar{x}\|^2 - (1+\theta)\theta\|x_{k-1}-x_{k-2}\|^2\\ &\quad - \|x_{k+1}-y_k\|^2 + (1+\theta)\theta\|x_{k-1}-x_{k-2}\|^2.\end{aligned} \tag{36}$$

Let

$$\begin{aligned}c_k &:= \|x_k-\bar{x}\|^2 - \theta\|x_{k-1}-\bar{x}\|^2 - (1+\theta)\theta\|x_{k-1}-x_{k-2}\|^2\\ &\geq 0.\end{aligned}$$

Then we have

$$c_{k+1} \leq c_k - \|x_{k+1}-y_k\|^2 + (1+\theta)\theta\|x_{k-1}-x_{k-2}\|^2.$$

Therefore, $\lim\limits_{k\to\infty} c_k$ exists and $\{c_k\}$ is bounded. Also,

$$\lim_{k\to\infty}\|x_{k+1}-y_k\| = 0, \quad \text{and} \quad \lim_{k\to\infty}\|x_{k-1}-x_{k-2}\| = 0. \tag{37}$$

Thus, we get that

$$\|x_k-y_k\| \leq \|x_{k+1}-y_k\| + \|x_k-x_{k+1}\| \to 0, k \to \infty.$$

Since $\{c_k\}$ is bounded, we have that $\{x_k\}$ is bounded.

Case II: Let $f(y_k, x_{k+1}) < f(y_k, \bar{x})$. Then we have from (17) that

$$\begin{aligned}\Big(\frac{1+\gamma\beta}{8\beta}-\eta\Big)\|x_{k+1}-\bar{x}\|^2 &\leq \frac{1}{4\beta}\|y_k-\bar{x}\|^2-\Big(\frac{1}{4\beta}-\eta\Big)\|x_{k+1}-y_k\|^2\\ &\leq \frac{1}{4\beta}\Big[(1+\theta)\|x_k-\bar{x}\|^2-\theta\|x_{k-1}-\bar{x}\|^2\\ &\quad+(1+\theta)\theta\|x_k-x_{k-1}\|^2\Big]-\Big(\frac{1}{4\beta}-\eta\Big)\|x_{k+1}-y_k\|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}&\Big(\frac{1+\gamma\beta}{8\beta}-\eta\Big)\|x_{k+1}-\bar{x}\|^2-\frac{\theta}{4\beta}\|x_k-\bar{x}\|^2\\ &\leq \frac{1}{4\beta}\|x_k-\bar{x}\|^2-\frac{\theta}{4\beta}\|x_{k-1}-\bar{x}\|^2\\ &\quad+\frac{(1+\theta)\theta}{4\beta}\|x_k-x_{k-1}\|^2-\Big(\frac{1}{4\beta}-\eta\Big)\|x_{k+1}-y_k\|^2.\end{aligned}\tag{38}$$

Observe that since $\frac{(1+\theta)\theta}{4\beta}=\eta(1+\theta)\theta+\Big(\frac{1}{4\beta}-\eta\Big)\theta(1+\theta)$ and $-\Big(\frac{1}{4\beta}-\eta\Big)\theta(1+\theta) > 0$, we obtain from (37) that

$$\begin{aligned}&\Big(\frac{1+\gamma\beta}{8\beta}-\eta\Big)\|x_{k+1}-\bar{x}\|^2-\frac{\theta}{4\beta}\|x_k-\bar{x}\|^2\\ &-\Big(\frac{1}{4\beta}-\eta\Big)\theta(1+\theta)\|x_k-x_{k-1}\|^2\\ &\leq \frac{1}{4\beta}\|x_k-\bar{x}\|^2-\frac{\theta}{4\beta}\|x_{k-1}-\bar{x}\|^2\\ &-\Big(\frac{1}{4\beta}-\eta\Big)\theta(1+\theta)\|x_{k-1}-x_{k-2}\|^2-\Big(\frac{1}{4\beta}-\eta\Big)\|x_{k+1}-y_k\|^2\\ &+\eta(1+\theta)\theta\|x_k-x_{k-1}\|^2.\end{aligned}\tag{39}$$

Using (25) in (38), we have

$$\begin{aligned}&\frac{1}{4\beta}\|x_{k+1}-\bar{x}\|^2-\frac{\theta}{4\beta}\|x_k-\bar{x}\|^2-\Big(\frac{1}{4\beta}-\eta\Big)\theta(1+\theta)\|x_k-x_{k-1}\|^2\\ &\leq \frac{1}{4\beta}\|x_k-\bar{x}\|^2-\frac{\theta}{4\beta}\|x_{k-1}-\bar{x}\|^2-\Big(\frac{1}{4\beta}-\eta\Big)\theta(1+\theta)\|x_{k-1}-x_{k-2}\|^2\\ &-\Big(\frac{1}{4\beta}-\eta\Big)\|x_{k+1}-y_k\|^2+\eta(1+\theta)\theta\|x_k-x_{k-1}\|^2.\end{aligned}\tag{40}$$

Letting

$$\begin{aligned}e_k := \frac{1}{4\beta}\|x_k-\bar{x}\|^2-\frac{\theta}{4\beta}\|x_{k-1}-\bar{x}\|^2\\ -\Big(\frac{1}{4\beta}-\eta\Big)\theta(1+\theta)\|x_{k-1}-x_{k-2}\|^2,\end{aligned}$$

we have from (39) that

$$e_{k+1} \leq e_k - \left(\frac{1}{4\beta} - \eta\right)\|x_{k+1} - y_k\|^2 + \eta(1+\theta)\theta\|x_k - x_{k-1}\|^2.$$

Consequently, $\lim_{k\to\infty} e_k$ exists and $\{e_k\}$ is bounded. Also,

$$\lim_{k\to\infty}\|x_{k+1} - y_k\| = 0, \lim_{k\to\infty}\|x_k - x_{k-1}\| = 0,$$

and

$$\lim_{k\to\infty}\|x_k - y_k\| = 0.$$

We finally have the boundedness of $\{x_k\}$ since $\{e_k\}$ is bounded. □

Now, following similar arguments as in the proof of Theorem 3.10, we can prove the following theorem.

Theorem 3.14 *Suppose $K \subset \mathbb{R}^n$ is a closed and convex subset with Assumptions* 3.1 *and* 3.12 *satisfied. Then $\{x_k\}$ generated by Algorithm* 1 *converges to a point $\bar{x} \in S$.*

In the next result, we give a non-asymptotic $\mathcal{O}(1/k)$ convergence rate of our proposed Algorithm 1 when the inertial factor $\theta < 0$.

Theorem 3.15 *Suppose $K \subset \mathbb{R}^n$ is a closed and convex subset. Assume that both Assumptions* 3.1 *and* 3.12 *are satisfied and $x_{-1} = x_0 = x_1 \in \mathbb{R}^n$. Let $\{x_k\}$ be generated by Algorithm* 1*. Then, for any $\bar{x} \in S$ and $k > 1$, it holds that*

$$\min_{0\leq j\leq k-1}\|x_{j+1} - y_j\|^2 \leq \frac{(2+2\theta^2)(1-\theta)}{4\beta\eta|\theta|(1+\theta)k}\|x_1 - \bar{x}\|^2.$$

Proof It follows from the arguments in the proof of Theorem 3.11. □

We next obtain linear convergence result when $\theta = 0$ in Algorithm 1 under the use of error bounds. We refer to Pang (1997), Solodov (2003), Tseng (1995) for more details and applications on error bounds. Let $\{x_k\}$ be a sequence in K which converges to $\bar{x} \in K$. This convergence is said to be Q–linear, if there is $q \in (0, 1)$ such that $\|x_{k+1} - \bar{x}\| < q\|x_k - \bar{x}\|$ for all k large enough. Furthermore, the convergence is R–linear, if for all k large enough, $\|x_k - \bar{x}\| \leq \zeta_k$ and $\{\zeta_k\}$ converges Q–linearly to zero.

Suppose $\beta > 0$, we define the natural residual

$$r(z, \beta) := x - \arg\min_{x\in K}\left\{f(y, x) + \frac{1}{2\beta}\|y - x\|^2\right\}.$$

Clearly, $z \in S$ if and only if $r(z, \beta) = 0$. We say that the equilibrium problem (1) satisfies an error bound condition if there exist positive constants c_1 and c_2 such that

$$\mathrm{dist}(x, S) \le c_1 \|r(z, \beta)\|, \forall x \in \mathbb{R}^n \text{ with } \|r(z, \beta)\| \le c_2. \tag{41}$$

We now obtain the following linear convergence result for equilibrium problem (1).

Theorem 3.16 *Suppose $K \subset \mathbb{R}^n$ is a closed and convex subset. Suppose Assumptions* 3.1, 3.12 *and error bound* (40) *are satisfied. Given $x_{-1} = x_0 = x_1 \in \mathbb{R}^n$, let $\{x_k\}$ be generated by Algorithm* 1. *Then, $\{x_k\}$ converges to a solution of equilibrium problem* (1) *at least R−linearly with $\theta = 0$.*

Proof With $\theta = 0$, we have from (19) and (26) respectively that

$$\|x_{k+1} - \bar{x}\|^2 \le \|x_k - \bar{x}\|^2 - \|x_{k+1} - x_k\|^2 \tag{42}$$

and

$$\frac{1}{4\beta}\|x_{k+1} - \bar{x}\|^2 \le \frac{1}{4\beta}\|x_k - \bar{x}\|^2 - \left(\frac{1}{4\beta} - \eta\right)\|x_{k+1} - x_k\|^2. \tag{43}$$

By Theorems 3.10 and 3.14, we have that $\{x_k\}$ generated by Algorithm 1 converges to a point $\bar{x} \in S$. Then we have $\|r(x_k, \beta)\| \to 0, k \to \infty$. Setting $\bar{x} = P_S(x_k)$, we have from (41) that

$$\begin{aligned}
\mathrm{dist}(x_{k+1}, S)^2 \le\ & \mathrm{dist}(x_k, S)^2 - \|x_{k+1} - x_k\|^2 \\
=\ & \mathrm{dist}(x_k, S)^2 - \|r(x_k, \beta)\| \\
\le\ & \mathrm{dist}(x_k, S)^2 - \frac{1}{c_1}\mathrm{dist}(x_k, S)^2 \\
=\ & \left(1 - \frac{1}{c_1}\right)\mathrm{dist}(x_k, S)^2.
\end{aligned} \tag{44}$$

Since $\mathrm{dist}(x_k, S)^2 \ge 0$, we have that $0 \le 1 - \frac{1}{c_1}$. Similarly, we have from (26) that

$$\begin{aligned}
\frac{1}{4\beta}\mathrm{dist}(x_{k+1}, S)^2 &\le \frac{1}{4\beta}\mathrm{dist}(x_k, S)^2 - \left(\frac{1}{4\beta} - \eta\right)\|x_{k+1} - x_k\|^2 \\
&= \frac{1}{4\beta}\mathrm{dist}(x_k, S)^2 - \left(\frac{1}{4\beta} - \eta\right)\|r(x_k, \beta)\| \\
&\le \frac{1}{4\beta}\mathrm{dist}(x_k, S)^2 - \frac{1}{c_1}\left(\frac{1}{4\beta} - \eta\right)\mathrm{dist}(x_k, S)^2 \\
&= \left[1 - \frac{4\beta}{c_1}\left(\frac{1}{4\beta} - \eta\right)\right]\frac{1}{4\beta}\mathrm{dist}(x_k, S)^2.
\end{aligned} \tag{45}$$

Similarly, we have that $0 \le 1 - \frac{4\beta}{c_1}\left(\frac{1}{4\beta} - \eta\right)$ since $\mathrm{dist}(x_k, S)^2 \ge 0$. We then the desired conclusion from (43) and (44). $\square$

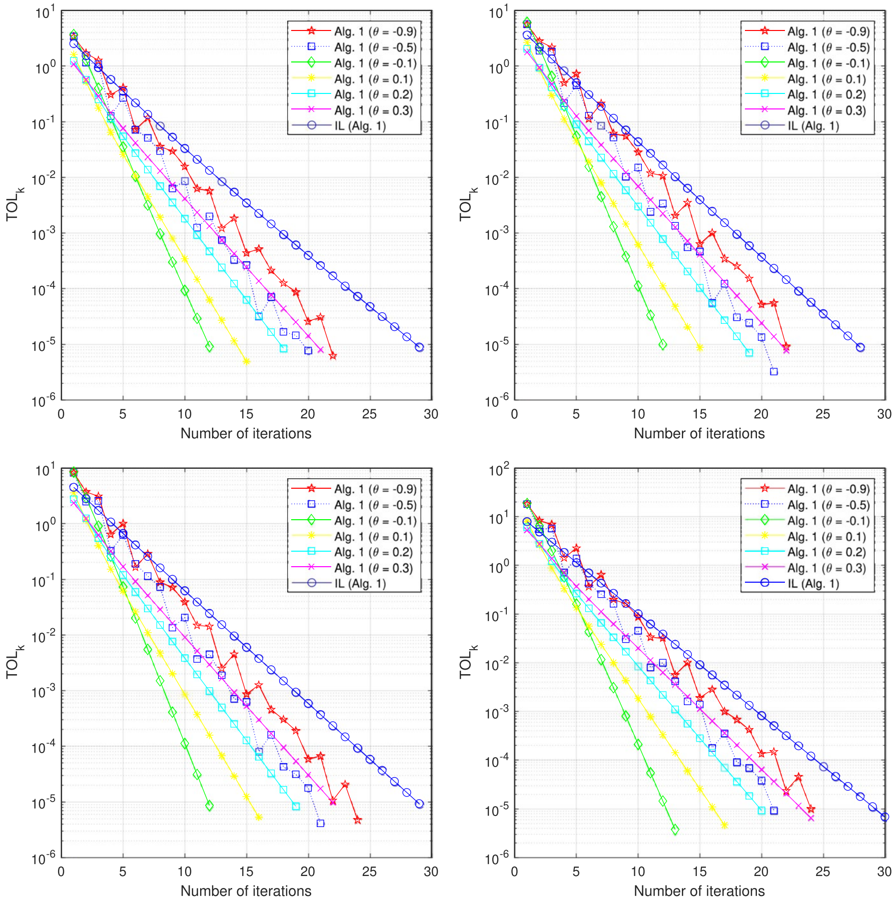

Fig. 1 The behavior of TOL_k with $\epsilon = 10^{-5}$: Top Left: $n = 20$; Top Right: $n = 50$; Bottom left: $n = 100$; Bottom Right: $n = 500$

Table 1 Comparison of algorithms with $\epsilon = 10^{-5}$

Algorithms	$n = 20$		$n = 50$		$n = 100$		$n = 500$	
	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter
Alg. 1 ($\theta = -0.9$)	0.0094	22	0.0283	22	0.0480	24	0.4099	24
Alg. 1 ($\theta = -0.5$)	0.0094	20	0.0278	21	0.0429	21	0.3594	21
Alg. 1 ($\theta = -0.1$)	0.0074	12	0.0245	12	0.0347	12	0.2985	13
Alg. 1 ($\theta = 0.1$)	0.0082	15	0.0270	15	0.0409	16	0.3153	17
Alg. 1 ($\theta = 0.2$)	0.0090	18	0.0277	19	0.0415	19	0.3844	20
Alg. 1 ($\theta = 0.3$)	0.0093	21	0.0284	22	0.0435	22	0.4261	24
IL (Alg. 1) in Iusem and Lara (2022)	1.0149	29	1.0301	28	1.0467	29	2.5456	30

Remark 3.17

(a) Our results are obtained in the setting of finite dimensional spaces. It would be interesting to generalize them to infinite dimensional Hilbert spaces, as this will further enhance the applicability and impact of our proposed algorithm. For the convergence properties of Algorithm 1 to hold in infinite dimensional Hilbert spaces, we need to ensure the unique existence of the solution to (1) in an infinite dimensional Hilbert space (i.e., establish Proposition 3.5 in infinite dimensional Hilbert spaces). One way to ensure this is to establish that $f(x, .)$ is 2-supercoercive in an infinite dimensional Hilbert space (see, for example, Lara 2022, Theorem 1). Furthermore, we also need to establish Lemma 2.6 in an infinite dimensional Hilbert space, but this is straightforward as seen in the proof of Lara (2022, Proposition 7). Therefore, to generalize our results in Section 4 to infinite dimensional Hilbert spaces, the main difficulty would be to establish Proposition 3.5 in such settings. This is beyond the scope of this paper and would be considered in our future research.

(b) In the proof arguments of Lemmas 3.8, 3.9 and Theorem 3.10, we see that the condition $0 \leq \theta < \frac{1}{3}$ is needed for our convergence analysis. We do not know if this condition "$0 \leq \theta < \frac{1}{3}$" could be weakened to "$0 \leq \theta < 1$" in the proof of Lemmas 3.8, 3.9 and Theorem 3.10. Part of clue to this could be by borrowing proof ideas from Attouch and Cabot (2020). However, we investigate this scenario in our future projects.

5 Numerical Result

We give the numerical implementation of our proposed Algorithm 1 and compare it with Iusem and Lara (2022, Algorithm 1) using an example of an equilibrium problem that is not a variational inequality problem. As stated in Iusem and Lara (2022, Section 4), Assumption 3.1 is satisfied for this numerical test.

Let $A, B \in \mathbb{R}^{n \times n}$ be symmetric and positive definite matrices. Then, we define

$$f(x, y) = y^t A y - x^t A x + x^t B(y - x). \tag{46}$$

One can easily verify that f satisfies $(A_i), i = 1, 2, 3$ in Assumption 3.1. We denote the largest eigenvalue of B by $\rho(B)$ and the smallest eigenvalues of A by $\mu(A)$. By the positive definiteness assumption, we have that both $\rho(B)$ and $\mu(A)$ are positive. Since $f(x, \cdot)$ is a quadratic function with matrix A, we have that $f(x, \cdot)$ is strongly convex with modulus $\mu(A)$, and hence strongly quasi-convex with the same modulus. Thus, assumption $(A4)$ holds. Also, it is easy to verify that f is Lipschitz continuous with constant $\frac{\rho(B)}{2}$, so that assumption $(A5)$ holds. blueAssumption $(A6)$ holds if we assume that $6\rho(B) < \mu(A)$. Since f is not affine on y, we have that the problem is indeed not a variational inequality problem but a mixed variational inequality problem.

The computations for the numerical implementation are performed using Matlab 2016 (b) which is running on a personal computer with an Intel(R) Core(TM) i5-2600 CPU at 2.30GHz and 8.00 Gb-RAM. During the computations, we randomly generate the starting points x_0 and y_0 for $n = 20, 50, 100$, and 500. We choose different choices of $\theta \in \{-0.9, -0.5, -0.1, 0.1, 0.2, 0.3\}$ and we set $\eta = \frac{\rho(B)}{2}$. Furthermore, we choose $\beta = \frac{0.99(1-3\theta)}{4\eta(1-\theta)}$ when $\theta \geq 0$ and $\beta = \frac{0.99}{4\eta}$ when $\theta < 0$. We then use the stopping criterion; $\text{TOL}_k := \|x_{k+1} - y_k\| < \varepsilon$, where ε is the predetermined error.

In the table, "CPU" and "Iter" mean the CPU time in seconds and the number of iterations, respectively. Also, in the table and figure, Alg. 1 and IL (Alg. 1) represent Algorithm 1 and Algorithm 1 in Iusem and Lara (2022), respectively (Fig. 1 and Table 1).

Remark 4.1 The experiments show the benefit gained (in terms of number of iterations and CPU time) over Algorithm 1 in Iusem and Lara (2022) by introducing the inertial parameter θ in our algorithm. We observe that as θ increases towards 0 for $\theta \in (-1, 0)$ and as θ decreases towards 0 for $\theta \in (0, \frac{1}{3})$, the CPU time and the number of iterations decrease. Therefore, our proposed algorithm performs better in terms of CPU time and number of iterations if θ is very close to 0, with $\theta \neq 0$.

6 Conclusion

This paper introduced and studied a proximal point algorithm with inertial extrapolation to solve a quasi-convex pseudo-monotone equilibrium problem. We proved that the sequence of iterates generated by the proposed algorithm converges to a solution to the problem. Furthermore, we obtain both sublinear and linear rates of convergence of the proposed algorithm using standard assumptions. Numerical comparisons are investigated to show the superiority and efficiency of our proposed algorithm and the effect of both negative and positive inertial factor. In the future, we shall study the convergence of the stochastic version of the proposed algorithm of this paper.

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Code Availability The Matlab codes employed to run the numerical experiments are available on request.

Declarations

Ethical Approval and Consent to Participate All the authors gave the ethical approval and consent to participate in this article.

Consent for Publication All the authors gave consent for the publication of identifiable details to be published in the journal and article.

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