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Row–column duality and combinatorial topological strings

Adrian Padellaro[1], **Rajath Radhakrishnan**[1,2,*] **and Sanjaye Ramgoolam**[1,3]

[1] Centre for Theoretical Physics, School of Physical and Chemical Sciences, Queen Mary University of London, London E1 4NS, United Kingdom
[2] International Centre for Theoretical Physics, Strada Costiera 11, 34151 Trieste, Italy
[3] National Institute for Theoretical Physics, School of Physics and Centre for Theoretical Physics, University of the Witwatersrand, Wits 2050, South Africa

E-mail: rradhakr@ictp.it

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Abstract

Integrality properties of partial sums over irreducible representations, along columns of character tables of finite groups, were recently derived using combinatorial topological string theories (CTST). These CTST were based on Dijkgraaf-Witten theories of flat G-bundles for finite groups G in two dimensions, denoted G-TQFTs. We define analogous combinatorial topological strings related to two dimensional topological field theories (TQFTs) based on fusion coefficients of finite groups. These TQFTs are denoted as $R(G)$-TQFTs and allow analogous integrality results to be derived for partial row sums of characters over conjugacy classes along fixed rows. This relation between the G-TQFTs and $R(G)$-TQFTs defines a row-column duality for character tables, which provides a physical framework for exploring the mathematical analogies between rows and columns of character tables. These constructive proofs of integrality are complemented with the proof of similar and complementary results using the more traditional Galois theoretic framework for integrality properties of character tables. The partial row and column sums are used to define generalised partitions of the integer row and column sums, which are of interest in combinatorial representation theory.

Supplementary material for this article is available online

Keywords: topological quantum field theory, representation theory, finite groups, galois theory, character theory

* Author to whom any correspondence should be addressed.

Contents

1. Introduction

Quantum field theories have led to the discovery of new mathematical structures, and have given us deep insights into several areas of mathematics. Some examples include the study of low-dimensional topology using Chern–Simons theory [1–3], the discovery of mirror symmetry from string theory [4], the discovery of vertex operator algebras from conformal field theory [5–7] and many more [8].

In this paper we continue this tradition by studying the implications of string-theoretic constructions for finite group representation theory. G-TQFTs are two-dimensional topological field theories (TQFTs) based on a finite group G [9–12]. The partition functions compute topological invariants of flat G-bundles over two-manifolds. Using representation theoretic Fourier transforms over the centres of the group algebras $\mathcal{Z}(\mathbb{C}(\mathcal{G}))$ these partition functions are also expressible in terms of representation theory data. In combinatorial representation theory, properties of group representations are studied by introducing combinatorial objects (see [13] for an overview of combinatorial representation theory). Recently [14], combinatorial topological string theories (G-CTSTs) based on G-TQFTs were used to obtain constructions of representation theoretic data. Finite algorithms based on combinatorial amplitudes in the G-CTST were given for computing the integers

$$\frac{|G|}{d_R}, \tag{1.1}$$

where d_R is the dimension of irreducible representation R of G. Algorithms for constructing characters from G-CTST amplitudes were given, and related to the Burnside algorithm for computing characters.

These results were developed further in [15] where combinatorial observables in G-CTST, constructed from the combinatorics of flat G-bundles on surfaces of arbitrary genus and one boundary labelled by a fixed conjugacy class C, were used to derive integrality properties of partial sums of characters of C over irreducible representations of G. These are partial sums along columns of the character table of G. For example, proposition 3.4.1-II [15] states that sums over irreducible characters of a given conjugacy class, for representations of fixed dimension, are integers. It follows immediately that the sum n_C, over irreducible characters along a column C of the character table, is an integer (see proposition 2.1 and a positive integrality property for normalized characters in proposition 2.2 in this paper). It is a known fact in representation theory that the sum n_R, of irreducible characters of a group G along a row R, is a positive integer and it is an open problem to find a combinatorial interpretation of this positive integer [16, problem 12].

This raises the question of whether there is a CTST/TQFT approach to study the sums along rows of character tables of finite groups G. In this paper, we give a positive answer to this question by introducing a related family of CTSTs, based on fusion algebras $R(G)$ where results complementary to those coming from G-CTSTs can be derived, with the role of columns switched with that of rows. We therefore refer to the relation between G-CTSTs and $R(G)$-CTSTs as a row–column duality. The integrality property of the partial column sums for a column corresponding to conjugacy class C, restricted by fixing the dimension of the irreps being summed, was generalised to obtain integrality for restrictions involving additional conjugacy classes which have the property that they have integer characters for all rows (e.g. propositions 3.4.1-III and 3.4.1-IV in [15]). These integrality properties were obtained from the counting of flat G-bundles on surfaces with one boundary labelled by C and multiple additional boundaries labelled by these additional conjugacy classes. Following the theme of row–column duality we will obtain analogous results for partial row sums.

The algebra structure of $\mathcal{Z}(\mathbb{C}(\mathcal{G}))$ is defined using the product of conjugacy classes of G and its decomposition into conjugacy classes. Similarly, the fusion algebra $R(G)$ is defined using the tensor product of representations of G and its decomposition into irreducible representations. Both the product of conjugacy classes as well as the tensor product of representations play a crucial role in the fusion rules of line operators in three-dimensional G-TQFTs [9]. In [17], the Arad–Herzog conjecture in finite group theory [18] was studied in the context of three-dimensional topological quantum field theories (TQFTs). It was shown that this conjecture implies special fusion rules for line operators in these TQFTs. These special fusion rules were then independently proved giving evidence for the validity of the conjecture. In closely related work [19], special fusion rules of line operators in three-dimensional TQFTs based on finite and Lie groups were studied, relating them to the representation theory of these groups.

The more traditional approach to integrality properties of finite group character tables uses Galois theory and analogies between rows and columns have been investigated in this framework [20]. In this paper, we will complement the constructive TQFT/CTST approaches to properties of character tables with the Galois theoretic approaches. We reproduce the integrality properties of the partial row/column sums using Galois theory. We also derive some new results motivated by these TQFT/CTST investigations.

Every character table has at least one integer row and one integer column : namely, the trivial representation, and the the identity element. We study, using the Galois theory methods, the conditions under which the number of integer rows equals the number of integer columns. We refine this analysis to study the number of rows and columns of the character table belonging to a field extension of $\mathbb{Q}$. We capture the number theoretic properties of the rows and columns in two graphs, one defined using rows of the character table and the other using columns. We find sufficient conditions under which the graphs are isomorphic.

The plan of the paper is as follows. We start with a review of the construction of a G-CTST in section 2, which is followed by a review of the relevant integrality results in [15]. Following this, we introduce the $R(G)$-CTST and illustrate how it is used to prove dual integrality results. A G-TQFT is defined through a semi-simple commutative Frobenius algebra with a basis labelled by conjugacy classes with a product on the classes inherited from group multiplication, and an appropriate Frobenius pairing. A $R(G)$-TQFT is defined through a basis of elements labelled by irreducible representations (irreps) with a product given by a tensor product multiplicities of irreps and an appropriate Frobenius pairing (A closely related TQFT with the same product but different pairing is defined in [21]). The idempotent (projector) basis for G-TQFTs is naturally labelled by irreducible representations, while the idempotent basis for a $R(G)$-TQFT is naturally labelled by conjugacy classes. The properties of the idempotent basis for $R(G)$-TQFT imply that fusion data can be used to construct the characters of G. This is analogous to Burnside's algorithm for constructing characters from structure constants of the centre of the group algebra. Section 3 builds on section 2, where we prove refined integrality results for character sums using CTST. The refined integrality results are used in the next subsection, where we give generalized integer partitions of the integers n_R and n_C. These consist of the expression of n_R or n_C as a sum of zeroes with a multiplicity along with a positive integer equipped with a partition and the negative of another positive integer equipped with a partition. These expressions are obtained by inspecting the column C adding up to n_C in terms of level sets of a function defined by considering all the columns other than C which are all integers. For the row R the generalised partition is obrained from the row-column dual of the construction used for C. As mentioned, Galois theory is a powerful toolbox for studying number and field theoretic properties of characters. Therefore, we have dedicated section 4 to the Galois theory actions on conjugacy classes and representations of finite groups, and their consequences of interest in relation to CTST. We start with a review of the background and then prove a set of generalizations of the previously mentioned integrality results obtained from G-CTST and $R(G)$-CTST. In section 5 we use Galois theory to study the number of integer rows and integer columns of character tables. In section 6 we describe an algorithm which starts from the G-TQFT and arrives at the $R(G)$-TQFT. The same algorithm applied to the $R(G)$-TQFT produces the G-TQFT. The problem of deducing the integrality results in section 2 from this construction of the duality involves some challenges which we discuss.

In appendix A we prove that the fusion algebra defines a TQFT by explicitly exhibiting the Frobenius algebra structure. Appendix B contains a proof of the Vandermonde matrix lemma used to prove the refined integrality results in section 3. We have put some additional constructive formulas and algorithms based on the $R(G)$-CTST in appendix C. We explore an interesting connection between Harada's conjecture and row-column duality in appendix D. The supplementary material contains a table of groups of order up to 100, the number of integer rows and columns in the character table of these groups, and the isomorphism class of their Galois groups. We also compute this data for sporadic groups.

2. Row–column duality

In this section we will review the combinatorial construction of partition functions in a Dijkgraaf-Witten theory based on finite group G, following the presentation in [14]. Closely related discussions have appeared in [22, 23] motivated by the investigation of low-dimensional models of wormhole physics [24]. We follow this with a review the derivation from [15] of integrality properties of partial sums along a column of the character table of G. We then give a combinatorial description of a dual TQFT, based on the fusion algebra $R(G)$, which gives integrality results for partial sums along rows rather than columns.

2.1. G-CTST

The Dijkgraaf-Witten theory for a finite group G can be described as follows. Let $\mathbb{C}(G)$ be the group algebra of a finite group G and $\mathcal{Z}(\mathbb{C}(G))$ the subalgebra of central elements which commute with all $\mathbb{C}(G)$, also known as the center of $\mathbb{C}(G)$. The center of a finite group algebra has two useful bases. The first one is based on the set

$$\mathrm{Cl}(G) = \{C_1, \ldots, C_K\}, \tag{2.1}$$

of distinct conjugacy classes of G, which can be used to label a set of basis elements of the center since

$$|\mathrm{Cl}(G)| = \dim \mathcal{Z}(\mathbb{C}(G)). \tag{2.2}$$

The sums over elements of each conjugacy class are central elements

$$T_C = \sum_{g \in C} g. \tag{2.3}$$

The set

$$\{T_{C_1}, \ldots, T_{C_K}\}, \tag{2.4}$$

form a basis of the center.

The second basis is labelled by irreducible representations of G. It is a standard result in representation theory that

$$|\mathrm{Cl}(G)| = |\mathrm{Irr}(G)|, \tag{2.5}$$

where

$$\mathrm{Irr}\,(G) = \{R_1, \ldots, R_K\}\,, \tag{2.6}$$

is a complete set of non-isomorphic irreducible representations of G. This implies that

$$|\mathrm{Irr}\,(G)| = \dim \mathcal{Z}\,(\mathbb{C}\,(G))\,, \tag{2.7}$$

and in particular there exists a basis for $\mathcal{Z}(\mathbb{C}(G))$ labelled by irreducible representations of G. The representation basis is constructed using irreducible characters χ^R of G

$$P_R = \frac{d_R}{|G|}\sum_{g\in G}\chi^R\left(g^{-1}\right)g = \frac{d_R}{|G|}\sum_{g\in G}\overline{(\chi^R\,(g))}g, \tag{2.8}$$

where $\overline{\chi^R(g))}$ is the complex conjugate of $\chi^R(g)$ and $d_R = \chi^R(1)$ the dimension of R. The two bases are related by

$$P_R = \frac{d_R}{|G|}\sum_{C\in\mathrm{Cl}(G)}\overline{(\chi^R_C)}T_C \tag{2.9}$$

and

$$T_C = \sum_{R\in\mathrm{Irr}(G)}\frac{\chi^R\,(T_C)}{d_R}P_R, \tag{2.10}$$

where

$$\chi^R_C \equiv \chi^R\,(g)\,\text{for any}g\in C. \tag{2.11}$$

Note that the structure constants $f^{C_3}_{C_1C_2}$ in the conjugacy class basis, defined by

$$T_{C_1}T_{C_2} = \sum_{C_3\in\mathrm{Cl}(G)}f^{C_3}_{C_1C_2}T_{C_3}, \tag{2.12}$$

are integers. They have an expression in terms of characters

$$f^{C_3}_{C_1C_2} = \frac{|C_1|\,|C_2|}{|G|}\sum_{R\in\mathrm{Irr}(G)}\frac{1}{d_R}\chi^R_{C_1}\chi^R_{C_2}\overline{\chi^R_{C_3}}. \tag{2.13}$$

The structure constants in the representation basis are diagonal,

$$P_{R_1}P_{R_2} = \delta_{R_1R_2}P_{R_1}. \tag{2.14}$$

so that the projectors form an orthogonal set and further satisfy the completeness property

$$\sum_{R\in\mathrm{Irr}(G)}P_R = 1. \tag{2.15}$$

To define the partition functions in a Dijkgraaf-Witten theory we introduce the delta function on G

$$\delta\,(g) = \begin{cases} 1 & \text{if g is the identity,} \\ 0 & \text{otherwise.}\end{cases} \tag{2.16}$$

In the representation basis we have

$$\frac{1}{|G|}\delta\left(P_R\right) = \frac{d_R}{|G|^2}\sum_{g\in G}\overline{\left(\chi^R\left(g\right)\right)}\delta\left(g\right) = \frac{d_R^2}{|G|^2}. \tag{2.17}$$

We use the following short-cut to arrive at the genus h partition functions. Define the handle creation operator

$$\Pi = \sum_{R\in\mathrm{Irr}(G)}\frac{|G|^2}{d_R^2}P_R. \tag{2.18}$$

Because

$$T_C P_R = |C|\,\frac{\chi_C^R}{d_R}P_R, \tag{2.19}$$

the eigenvalues of the integer matrix f_C of structure constants are given by normalized characters. In particular, using (2.9) we have

$$\Pi = \sum_{R\in\mathrm{Irr}(G)}\frac{|G|}{d_R}\sum_{C\in\mathrm{Cl}(G)}\overline{(\chi_C^R)}T_C = \sum_{C\in\mathrm{Cl}(G)}\frac{|G|}{|C|}\overline{\mathrm{tr}\left(f_C\right)}T_C = \sum_{C\in\mathrm{Cl}(G)}\mathrm{Sym}C\,\mathrm{tr}\left(f_C\right)T_C, \tag{2.20}$$

where

$$\mathrm{Sym}C = \frac{|G|}{|C|}, \tag{2.21}$$

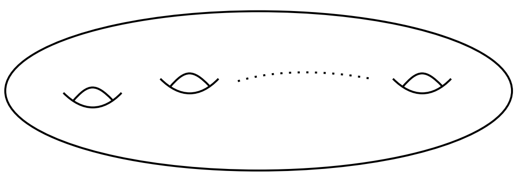
is the size of the centralizer of $g\in C$. Therefore, it is possible to express Π as an integer linear combination of conjugacy class basis elements.

Inserting powers of the handle creation operator into the delta function gives

$$\frac{1}{|G|}\delta\left(\Pi^h\right) = \sum_{R\in\mathrm{Irr}(G)}\left(\frac{|G|}{d_R}\right)^{2h-2}. \tag{2.22}$$

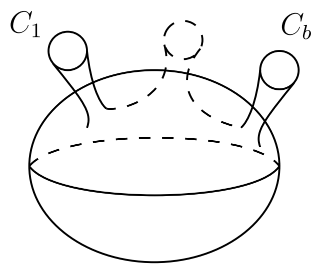
This is the partition function of a manifold of genus h with no boundaries,

$$Z_h^G = \frac{1}{|G|}\delta(\Pi^h) = \tag{2.23}$$



In general, we are interested in partition functions of manifolds with boundaries labelled by conjugacy classes. For example, the genus zero partition function with b boundaries labeled by conjugacy classes $C_1,\ldots,C_b$ is equal to

$$Z^G_{h=0,C_1;\ldots;C_b} = \frac{1}{|G|}\delta(T_{C_1}\ldots T_{C_b}) = \tag{2.24}$$



This can be re-expressed in terms of characters of G using (2.10), (2.14) and (2.17)

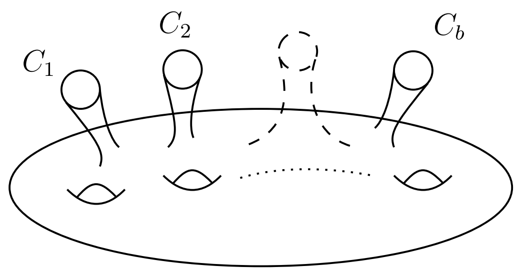
$$Z^G_{h=0,C_1;\ldots;C_b} = \sum_{R\in\mathrm{Irr}(G)} \frac{d_R^2}{|G|^2}\frac{\chi^R(T_{C_1})}{d_R}\ldots\frac{\chi^R(T_{C_b})}{d_R} \tag{2.25}$$

$$= \sum_{R\in\mathrm{Irr}(G)} \frac{d_R^2}{|G|^2}|C_1|\frac{\chi^R_{C_1}}{d_R}\ldots|C_b|\frac{\chi^R_{C_b}}{d_R}. \tag{2.26}$$

To compute genus h partition functions with b boundaries we insert handle creation operators

$$Z^G_{h;C_1;\ldots;C_b} = \frac{1}{|G|}\delta\left(\Pi^h T_{C_1}\ldots T_{C_b}\right) = \sum_{R\in\mathrm{Irr}(G)}\left(\frac{|G|}{d_R}\right)^{2h-2}\frac{\chi^R(T_{C_1})}{d_R}\ldots\frac{\chi^R(T_{C_b})}{d_R} \tag{2.27}$$

$$= \tag{2.28}$$



2.2. Integral partial column sums from G-CTST

We now review relevant integrality results from section 3.4 of [15], based on the above construction of G-CTST.

Let $D\in\mathrm{Cl}(G)$ be a conjugacy class such that χ^R_D is an integer for all $R\in\mathrm{Irr}(G)$. That is, D defines an integer column in the character table of G. Note that this implies that

$$\frac{\chi^R(T_D)}{d_R} = |D|\frac{\chi^R_D}{d_R} \tag{2.29}$$

is rational for all $R \in \mathrm{Irr}(G)$, which will be important in what follows. Fix an irreducible representation $S \in \mathrm{Irr}(G)$ and define the set

$$\begin{aligned} [(S,D)] &= \left\{ R \in \mathrm{Irr}\,(G)\,\text{s.t.}\ \frac{\chi^R\,(T_D)}{d_R} = \frac{\chi^S\,(T_D)}{d_S} \right\} \\ &= \left\{ R \in \mathrm{Irr}\,(G)\,\text{s.t.}\ |D|\,\frac{\chi^R_D}{d_R} = |D|\,\frac{\chi^S_D}{d_S} \right\} \\ &= \left\{ R \in \mathrm{Irr}\,(G)\,\text{s.t.}\ \frac{\chi^R_D}{d_R} = \frac{\chi^S_D}{d_S} \right\}. \end{aligned} \tag{2.30}$$

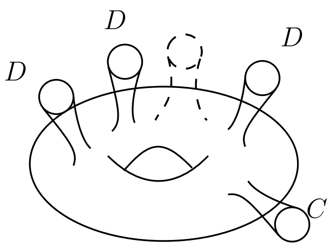
It was proved in [15] (proposition 3.4.1-II) that

$$\sum_{R\in[(S,D)]} |C|\,\frac{\chi^R_C}{d_R} \in \mathbb{Z}, \tag{2.31}$$

for every $C \in \mathrm{Cl}(G)$.

The proof uses the family of partition functions

$$Z^G_{h=1;D^b,C} = \sum_{R\in\mathrm{Irr}(G)} \left(\frac{\chi^R(T_D)}{d_R}\right)^b \frac{\chi^R(T_C)}{d_R} = \tag{2.32}$$



for b a non-negative integer. The sum over irreducible representations is organized in terms of level sets of the normalized characters $\frac{\chi^R_D}{d_R}$

$$Z^G_{h=1;D^b,C} = \sum_{R'} \left(\frac{\chi^{R'}\,(T_D)}{d_{R'}}\right)^b \sum_{R\in[(R',D)]} \frac{\chi^R\,(T_C)}{d_R}, \tag{2.33}$$

where the first sum is over a set $\{R'_1,\ldots,R'_k\}$ of irreducible representations such that

$$\frac{\chi_D^{R'_1}}{d_{R'_1}},\ldots,\frac{\chi_D^{R'_k}}{d_{R'_k}}, \tag{2.34}$$

are all distinct. Now we restrict to the partition functions with $b = 1,\ldots,k$. We defined (2.30) as the set of irreps having a fixed normalized character $|D|\frac{\chi^R_D}{d_R}$ which implies that the conjugacy class D determines a set partition of the set of irreps $\mathrm{Irr}(G)$:

$$[(R'_1,D)] \cup [(R'_2,D)] \cup \ldots \cup [(R'_k,D)] = \mathrm{Irr}\,(G)\,. \tag{2.35}$$

Equation (2.33) can be understood as a matrix equation

$$y = Vx \quad ; \quad \text{with indices explicit } y_b = \sum_{R'} V_{bR'} x_{R'} \tag{2.36}$$

where

$$y_b = Z^G_{h=1;D^b,C}, \tag{2.37}$$

$$x_{R'} = \sum_{R \in [(R',D)]} \frac{\chi^R(T_C)}{d_R} = \sum_{R \in [(R',D)]} |C| \frac{\chi^R_C}{d_R} \tag{2.38}$$

$$V_{bR'} = \left(\frac{\chi^{R'}(T_D)}{d_{R'}}\right)^b = \left(|D| \frac{\chi^{R'}_D}{d_{R'}}\right)^b. \tag{2.39}$$

Since we assumed that χ^R_D is an integer for all R, the matrix V is a $k \times k$ Vandermonde matrix with rational entries. Consequently, the inverse matrix V^{-1} is a rational matrix as well and we have

$$x = V^{-1} y. \tag{2.40}$$

The r.h.s. of this equation is rational vector, while the l.h.s. is a vector of normalized characters which are known to be algebraic integers. An algebraic integer is rational if and only if it is an integer and therefore

$$x_S = \sum_{R \in [(S,D)]} \frac{\chi^R(T_C)}{d_R} = \sum_{R \in [(S,D)]} |C| \frac{\chi^R_C}{d_R} \in \mathbb{Z}. \tag{2.41}$$

which is the result (2.31).

Integrality can also be shown for sums over irreducible representations with fixed dimension. Define the sets

$$[d_{R'}] = \left\{ R \in \mathrm{Irr}(G) \, \text{s.t.} \, \frac{|G|^2}{d_R^2} = \frac{|G|^2}{d_{R'}^2} \right\} \tag{2.42}$$

$$= \{ R \in \mathrm{Irr}(G) \, \text{s.t.} \, d_R = d_{R'} \}. \tag{2.43}$$

A slight variation of the above construction, using insertions of handle creation operators, was used in [15] to prove that (see [15, proposition 3.4.1-II])

$$\sum_{R \in [d_{R'}]} \chi^R_C \in \mathbb{Z}. \tag{2.44}$$

Since the sum of characters along any column is equivalent to a sum of partial sums over irreducible representations of fixed dimension the following proposition follows.

Proposition 2.1. *Column sums of character tables are integers*

$$\sum_{R\in \mathrm{Irr}(G)} \chi^R_C \in \mathbb{Z}, \quad \forall C \in \mathrm{Cl}(G). \tag{2.45}$$

Interestingly, the above sum is not necessarily positive, but a simple G-CTST proof gives the following proposition.

Proposition 2.2. *Let $\mathbb{Z}^+$ be the set of positive integers, then*

$$\sum_{R\in \mathrm{Irr}(G)} \frac{\chi^R(T_C)}{d_R} \in \mathbb{Z}^+, \quad \forall C \in \mathrm{Cl}(G). \tag{2.46}$$

Proof of proposition 2.2. Consider the amplitude

$$Z^G_{h=1;C} = \frac{1}{|G|}\delta(\Pi\ T_C) = \sum_{R\in \mathrm{Irr}(G)} \frac{\chi^R(T_C)}{d_R}. \tag{2.47}$$

the l.h.s. is a positive rational number for all $C \in \mathrm{Cl}(G)$, while the r.h.s. is a sum of algebraic integers. Therefore, it is a positive integer, which proves the proposition.

The equation (2.31) involves sums over sets of irreducible representations (irreps) which are level sets of a function of irreps defined by a conjugacy class S. A natural question to ask is if similar integrality results exist for sums over level sets of conjugacy classes. In other words, are there similar integrality results to the above obtained by exchanging the role of rows and columns of character tables. Analogies between rows and columns have motivated a number of investigations on number-theoretic properties of characters [20]. In the next section we will introduce a new TQFT based on fusion algebras. A similar algebra was defined in [21], but with different Frobenius structure. We will find that the one used in the next section is the natural setting for deriving integrality results in the context of CTST which are dual under an exchange of rows and columns to the above integrality results. This motivates the notion of a physical row-column duality in the context of CTST which provides a physical framework for thinking about mathematical analogies based on the exchange of rows and columns of character tables.

2.3. R(G)-CTST

In this section we give a formulae for the partition functions in the $R(G)$-TQFT in terms of fusion coefficients (tensor product multiplicities) of irreps of G. This is analogous to formulae in the the previous subsection for G-TQFT with fusion coefficients replacing class algebra structure constants in the central algebra $\mathcal{Z}(\mathbb{C}(\mathcal{G}))$. To define a TQFT, it is useful to appeal to the language of Frobenius algebras. In appendix A we show that the set of distinct (isomorphism classes) of irreducible representations of G can be given the structure of a commutative Frobenius algebra, using fusion coefficients, and therefore defines $R(G)$-TQFT. The appendix contains expressions for genus h partition functions with n boundaries, derived using the Frobenius structure. In this section we show that equivalent expressions can be computed using a handle creation operator Θ, counit ε and fusion of irreducible representations.

The algebra underlying the $R(G)$-TQFT is a vector space of dimension

$$\dim R(G) = |\mathrm{Irr}(G)|, \tag{2.48}$$

where $\mathrm{Irr}(G)$ is a complete set of non-isomorphic irreducible representations of G. It has a basis

$$\{a_{R_1}, \ldots, a_{R_K}\}, \tag{2.49}$$

where the structure constants

$$N_{R_1 R_2}^{R_3} = \frac{1}{|G|} \sum_{g \in G} \chi^{R_1}(g) \chi^{R_2}(g) \overline{\chi^{R_3}(g)}. \tag{2.50}$$

are fusion coefficients and

$$a_{R_1} a_{R_2} = \sum_{R_3 \in \mathrm{Irr}(G)} N_{R_1 R_2}^{R_3} a_{R_3}. \tag{2.51}$$

The trivial representation, denoted R_0, corresponds to the identity in this algebra: $a_{R_0} a_R = a_R$.

The fusion algebra has a basis of orthogonal idempotents (projectors)

$$A_C = \frac{1}{\mathrm{Sym}C} \sum_R \chi_C^{\overline{R}} a_R, \tag{2.52}$$

where $\mathrm{Sym}\, C = |G| / |C|$ as in (2.21). The orthogonal idempotent property is expressed in

$$A_{C_1} A_{C_2} = \delta_{C_1 C_2} A_{C_1} \tag{2.53}$$

This proved as follows

$$\begin{aligned}
A_{C_1} A_{C_2} &= \frac{1}{\mathrm{Sym}C_1} \cdot \frac{1}{\mathrm{Sym}C_2} \sum_R \chi_{C_1}^{\bar{R}} a_R \sum_S \chi_{C_2}^{\bar{S}} a_S \\
&= \frac{1}{\mathrm{Sym}C_1} \cdot \frac{1}{\mathrm{Sym}C_2} \sum_{R,S} \chi_{C_1}^{\bar{R}} \chi_{C_2}^{\bar{S}} a_R . a_S \\
&= \frac{1}{\mathrm{Sym}C_1} \frac{1}{\mathrm{Sym}C_2} \sum_{R,S,T} \chi_{C_1}^{\bar{R}} \chi_{C_2}^{\bar{S}} N_{RS}^T a_T \\
&= \frac{1}{\mathrm{Sym}C_1\ \mathrm{Sym}C_2} \sum_{R,S,T} \chi_{C_1}^{\bar{R}} \chi_{C_2}^{\bar{S}} \sum_C \chi_C^R \chi_C^S \chi_C^{\bar{T}} \cdot \frac{|C|}{|G|} . a_T \\
&= \frac{1}{\mathrm{Sym}C_1\ \mathrm{Sym}C_2} \sum_{T,C} \delta_{C_1,C} \delta_{C_2,C} \mathrm{Sym}C_1\ \mathrm{Sym}C_2 . \frac{|C|}{|G|} . \chi_C^{\bar{T}} a_T \\
&= \frac{1}{\mathrm{Sym}C_1} \sum_T \chi_{C_1}^{\bar{T}} a_T . \delta_{C_1,C_2} \\
&= A_{C_1} \delta_{C_1,C_2}.
\end{aligned} \tag{2.54}$$

where we have used an expression for the fusion coefficient as a sum over characters, along with character orthogonality relations. The equation (2.53) also shows that the structure constants of the multiplication are diagonal in the idempotent basis.

The orthogonal projectors obey the completeness relation

$$\sum_C A_C = a_{R_0}, \tag{2.55}$$

where R_0 is the trivial representation since

$$\begin{aligned}
\sum_C A_C &= \sum_C \frac{1}{\mathrm{Sym}C} \sum_R \chi_C^{\bar{R}} a_R \\
&= \sum_R \sum_C \frac{|C| \chi_C^{\bar{R}}}{|G|} a_R \\
&= \sum_R \sum_g \frac{\chi^R \left(g^{-1}\right)}{|G|} \; a_R \\
&= \sum_R \sum_g \frac{\chi^R \left(g^{-1}\right)}{|G|} \chi^{R_0} \left(g\right) \; a_R \\
&= \sum_R \delta^{R,R_0} a_{R_0} = a_{R_0}.
\end{aligned} \tag{2.56}$$

The idempotents are also eigenvectors of the operation of multiplication by a_R :

$$a_R A_C = \chi_C^R A_C, \,. \tag{2.57}$$

This is derived as follows

$$\begin{aligned}
a_R A_C &= \frac{1}{\mathrm{Sym}C} \sum_S \chi_C^{\bar{S}} a_R a_S \\
&= \frac{1}{\mathrm{Sym}C} \sum_{S,T} \chi_C^{\bar{S}} N_{RS}^T a_T \\
&= \frac{1}{\mathrm{Sym}C} \sum_{S,T} \chi_C^{\bar{S}} \sum_{C'} \chi_{C'}^R \chi_{C'}^S \chi_{C'}^{\bar{T}} \frac{|C'|}{|G|} a_T \\
&= \frac{1}{\mathrm{Sym}C} \sum_{C',T} \left(\sum_S \chi_C^{\bar{S}} \chi_{C'}^S \right) \chi_{C'}^R \chi_{C'}^{\bar{T}} . \frac{|C'|}{|G|} .a_T \\
&= \frac{1}{\mathrm{Sym}C} \sum_{C',T} (\delta_{C,C'} \mathrm{Sym}C') \chi_{C'}^R \chi_{C'}^{\bar{T}} . \frac{|C'|}{|G|} .a_T \\
&= \sum_T \frac{1}{\mathrm{Sym}C} \chi_C^R \chi_C^{\bar{T}} a_T \\
&= \chi_C^R \frac{1}{\mathrm{Sym}C} \sum_T \chi_C^{\bar{T}} a_T \\
&= \chi_C^R A_C.
\end{aligned} \tag{2.58}$$

Comparing (2.57) and (2.51), we see that the matrix N_R corresponding to left multiplication of a_R has eigenvalues corresponding to characters χ_C^R for $C \in \mathrm{Cl}(G)$. Therefore, the fusion algebra gives an algorithm analogous to the Burnside's algorithm [25–27], which constructs characters of G from the structure constant matrices f_C of the centre of the group algebra. As discussed in section 3.1 of [14], G-CTST amplitudes on genus one surfaces with boundaries provide the combinatorial data from multiplication of conjugacy classes in $\mathcal{Z}(\mathbb{C}(\mathcal{G}))$ which enters the Burnside construction of the normalized characters $\frac{\chi^R(T_C)}{d_R}$. Similarly the corresponding amplitudes in $R(G)$-CTST give the positive integer fusion data which can be used to construct the characters $\chi^R(T_C)$.

To define the partition functions of the fusion TQFT we introduce a counit on $R(G)$ (see (A.11))

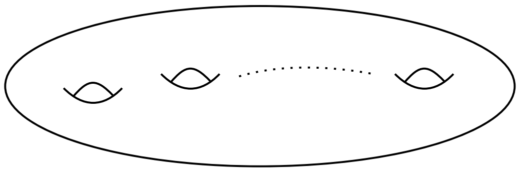
$$\varepsilon(a_R) = \begin{cases} 1 & \text{if } R = R_0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.59}$$

In the idempotent basis we have

$$\varepsilon(A_C) = \frac{1}{\mathrm{Sym}C} \sum_R \chi_C^{\bar{R}} \varepsilon(a_R) = \frac{1}{\mathrm{Sym}C} \chi_C^{R_0} = \frac{1}{\mathrm{Sym}C} \tag{2.60}$$

Just as in the G-TQFT, it is possible to construct a handle creation operator Θ in the fusion algebra such that

$$Z_h^{R(G)} = \varepsilon(\Theta^h) = \tag{2.61}$$



We can define the handle creation operator

$$\Theta = \sum_{S \in \mathrm{Irr}(G)} N_{S\bar{S}}^R a_R \tag{2.62}$$

which reflects the fact that a torus with a hole (labelled with state R) is obtained by gluing two boundaries from a sphere with three boundaries. The gluing operation is associated with the sum over S. This can be used to obtain an expression for Θ as a sum over the projectors A_C. It is useful to observe

$$\begin{aligned} \sum_S N_{S\bar{S}}^R &= \frac{1}{|G|} \sum_S \sum_{g \in G} \chi^S(g) \chi^{\bar{S}}(g) \chi^{\bar{R}}(g) \\ &= \sum_{C \in \mathrm{Cl}(G)} \chi_C^{\bar{R}}(g) \end{aligned} \tag{2.63}$$

where the second line is obtained by using the orthogonality of characters to simplify the sum over S. It then follows, using the formula (2.52) for the projector A_C that

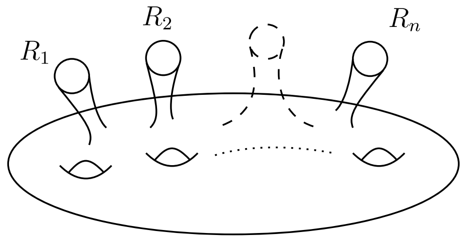
$$\Theta = \sum_C (\mathrm{Sym}C) A_C. \tag{2.64}$$

From (2.60) we have

$$Z_h^{R(G)} = \epsilon(\Theta^h) = \sum_C \epsilon\left((\mathrm{Sym}C)^h A_K\right) = \sum_C (\mathrm{Sym}C)^{h-1}, \tag{2.65}$$

which agrees with (A.28) derived from the Frobenius structure. Genus h partition functions with n boundaries are given by

$$Z^{R(G)}_{h,R_1,\dots,R_n} = \varepsilon(\Theta^h a_{R_1}\dots a_{R_n}) = \tag{2.66}$$



From (2.57) we have

$$\begin{aligned}
\varepsilon\left(\Theta^h a_{R_1}\dots a_{R_n}\right) &= \sum_{C\in\mathrm{Cl}(G)} (\mathrm{Sym}C)^h\,\varepsilon\left(A_C a_{R_1}\dots a_{R_n}\right)\\
&= \sum_{C\in\mathrm{Cl}(G)} (\mathrm{Sym}C)^h\,\chi^{R_1}_C\dots\chi^{R_n}_C\,\varepsilon\left(A_C\right)\\
&= \sum_{C\in\mathrm{Cl}(G)} (\mathrm{Sym}C)^{h-1}\,\chi^{R_1}_C\dots\chi^{R_n}_C,
\end{aligned} \tag{2.67}$$

which agree with (A.41) derived from the Frobenius structure.

2.4. Integral partial row sums from R(G)-CTST

The $R(G)$-CTST will allow us to derive integrality results for sums of characters along rows. The proof mirrors that of section 2.2. The input of the integrality properties of multiplication in $\mathcal{Z}(\mathbb{C}(\mathcal{G}))$ in the basis of class central elements T_C (2.12) exploited in section 2.2 will now be replaced by the integrality properties of fusion coefficients which play a role as building blocks for amplitudes in $R(G)$-TQFT.

Theorem 1. *Let $S\in\mathrm{Irr}(G)$ be an irreducible representation such that χ^S_C is an integer for all $C\in\mathrm{Cl}(G)$. This defines an integer row in the character table of G. Now, fix a conjugacy class D and define the set*

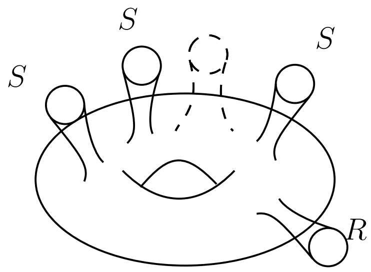
$$\langle (S,D)\rangle = \left\{C\in\mathrm{Cl}\,(G)\, s.t.\ \chi^S_C = \chi^S_D\right\}. \tag{2.68}$$

Now consider any $R\in\mathrm{Irr}(G)$ with $R\neq S$. We will show that

$$\sum_{C\in\langle (S,D)\rangle} \chi^R_C \in \mathbb{Z}. \tag{2.69}$$

Proof of theorem 1. Consider the vector of partition functions

$$Z^{R(G)}_{h=1;S^b,R} = \sum_{C\in \mathrm{Cl}(G)} \left(\chi^S_C\right)^b \chi^R_C = \tag{2.70}$$



for b a non-negative integer. We organize the sum into level sets

$$Z^{R(G)}_{h=1;S^b,R} = \sum_{C'} \left(\chi^S_{C'}\right)^b \sum_{C\in\langle(S,C')\rangle} \chi^R_C, \tag{2.71}$$

where the first sum is over a set $\{C'_1,\ldots,C'_k\}$ of conjugacy classes such that

$$\chi^S_{C'_1},\ldots,\chi^S_{C'_k} \tag{2.72}$$

are all distinct and

$$\langle(S,C'_1)\rangle \cup \langle(S,C'_2)\rangle \cup \cdots \cup \langle(S,C'_k)\rangle = \mathrm{Cl}(G)\,. \tag{2.73}$$

Now we restrict to the partition functions with $b = 1,\ldots,k$. Equation (2.71) is a matrix equation

$$y = Vx, \tag{2.74}$$

where

$$y_b = Z^{R(G)}_{h=1;S^b,R} \tag{2.75}$$

$$x_{C'} = \sum_{C\in\langle(S,C')\rangle} \chi^R_C \tag{2.76}$$

$$V_{bC'} = \left(\chi^S_{C'}\right)^b. \tag{2.77}$$

The matrix V is an integer $k\times k$ Vandermonde matrix and its inverse V^{-1} is a rational matrix. We have

$$x = V^{-1}y. \tag{2.78}$$

The l.h.s. is a vector of algebraic integers and the r.h.s. is a vector of rational numbers. Therefore

$$x_D = \sum_{C\in\langle(S,D)\rangle} \chi^R_C \in \mathbb{Z}, \tag{2.79}$$

for all $R\in\mathrm{Irr}(G)$, which concludes the proof of theorem 1.

Proposition 2.3. *A useful special case to consider is $S = R_0$, the trivial representation, and any $D \in \mathrm{Cl}(G)$. Then*

$$\langle (R_0, D) \rangle = \mathrm{Cl}(G), \tag{2.80}$$

and it follows that

$$\sum_{C \in \mathrm{Cl}(G)} \chi^R_C \in \mathbb{Z}, \tag{2.81}$$

for any $R \in \mathrm{Irr}(G)$.

In fact, the $R(G)$-CTST readily gives the following positivity property.

Proposition 2.4. *The sum*

$$\sum_{C \in \mathrm{Cl}(G)} \chi^R_C \in \mathbb{Z}^+, \tag{2.82}$$

for any $R \in \mathrm{Irr}(G)$.

Proof of proposition 2.4. Consider the partition function the expressions for $Z^{R(G)}_{h=1,R}$ obtained by specialising (2.67) to

$$Z^{R(G)}_{h=1,R} = \epsilon(\Theta a_{\bar{R}}) = \sum_{C \in \mathrm{Cl}(G)} \chi^R_C. \tag{2.83}$$

We also have

$$Z^{R(G)}_{h=1,R} = \sum_{S \in \mathrm{Irr}(G)} N^R_{S\bar{S}} \tag{2.84}$$

using the identity (2.63). Since the fusion coefficients $N^R_{S\bar{S}}$ are positive, the proposition follows.

3. Row–column duality: refined integrality properties of rows/columns and generalised partitions

In this section we will extend the techniques used in the previous section to prove refinements of the character integrality results in [15, section 3.4]. In particular, we relax the assumption that entire rows/columns are integer. As we review in section 4, these results can also be proven using Galois theory. Here, we give a combinatorial topological string theory derivation and interpretation of these results. We end this section by highlighting a new partition structure in the integer row and column sums that emerges from our considerations.

3.1. Refined integral column sums from G-CTST

In this subsection we will use the following lemma, which we prove in appendix B.

Lemma 1. *Let Z be a diagonalizable $K \times K$ matrix of non-negative integers with eigenvalues $\lambda_1 \ldots, \lambda_K$. Let the set of distinct eigenvalues be $\{z_1, \ldots, z_k\}$. Note that $k \leqslant K$ generally. Define the $k \times k$ Vandermonde matrix*

$$V_{b,i} = z_i^b. \tag{3.1}$$

In general, the eigenvalues $z_1,\ldots,z_k$ *are algebraic integers. But suppose* z_j *is rational, then the inverse Vandermonde* V^{-1} *has rational entries in the jth row:*

$$\left(V^{-1}\right)_{j,b} \in \mathbb{Q} \quad \forall b \in \{1,\ldots,k\}. \tag{3.2}$$

We are interested in the special case of z_j *being an integer.*

The first character integrality result, which we will now prove, is the following.

Theorem 2. *Let D be a conjugacy class and S an irreducible representation of G such that*

$$\frac{\chi^S(T_D)}{d_S} = |D|\frac{\chi^S_D}{d_S} \in \mathbb{Z}. \tag{3.3}$$

Define the subset of $\mathrm{Irr}(G)$

$$[(S,D)] = \left\{R \in \mathrm{Irr}(G)\, s.t.\ \frac{\chi^R(T_D)}{d_R} = \frac{\chi^S(T_D)}{d_S}\right\}. \tag{3.4}$$

$$= \left\{R \in \mathrm{Irr}(G)\, s.t.\ |D|\frac{\chi^R_D}{d_R} = |D|\frac{\chi^S_D}{d_S}\right\}. \tag{3.5}$$

Then the sum

$$\sum_{R\in[(S,D)]}\frac{\chi^R(T_C)}{d_R} = \sum_{R\in[(S,D)]}|C|\frac{\chi^R_C}{d_R} \in \mathbb{Z}, \quad \forall C \in \mathrm{Cl}(G). \tag{3.6}$$

is an integer for all conjugacy classes. Note that, compared to (2.31), we have dropped the assumption that χ^R_D *is an integer for all* $R \in \mathrm{Irr}(G)$.

Proof of theorem 2 (Constructive). To prove the above result, consider the l-dimensional integer vector $y = (y_1,\ldots,y_l)$ of amplitudes

$$y_b(D,C) = Z^G_{h=1,D^b,C} = \sum_R\left(\frac{\chi^R(T_D)}{d_R}\right)^b\frac{\chi^R(T_C)}{d_R}, \quad b = 1,\ldots,l. \tag{3.7}$$

where l is a non-negative integer. We re-write the sum as a sum over level sets

$$y_b(D,C) = \sum_{R'}\left(\frac{\chi^{R'}(T_D)}{d_{R'}}\right)^b\sum_{R\in[(R',D)]}\frac{\chi^R(T_C)}{d_R}, \tag{3.8}$$

where the first sum is over a set of irreducible representations $R'_1,\ldots,R'_k$ such that

$$\frac{\chi^{R'_1}(T_D)}{d_{R'_1}},\ldots,\frac{\chi^{R'_k}(T_D)}{d_{R'_k}}, \tag{3.9}$$

are all distinct and

$$[(R'_1,D)] \cup [(R'_2,D)] \cup \cdots \cup [(R'_k,D)] = \mathrm{Irr}(G). \tag{3.10}$$

Now we restrict to the partition functions with $b = 1, \ldots, k$. Define the vector

$$x_{R'}(D,C) = \sum_{R \in [(R',D)]} \frac{\chi_C^R(T_C)}{d_R}, \tag{3.11}$$

and the $k \times k$ Vandermonde matrix

$$V_{b,R'}(D) = \left(|D| \frac{\chi_D^{R'}}{d_{R'}}\right)^b = \left(\frac{\chi^{R'}(T_D)}{d_{R'}}\right)^b. \tag{3.12}$$

Note that

$$T_C P_R = \frac{\chi^R(T_C)}{d_R} P_R = |C| \frac{\chi_C^R}{d_R} P_R, \tag{3.13}$$

and therefore normalized characters are the eigenvalues of the integer structure constant matrix f_C, as required to use the inverse Vandermonde lemma. Now equation (3.8) reads

$$y(D,C) = V(D)\, x(D,C). \tag{3.14}$$

Since $V(D)$ is a Vandermonde matrix of distinct normalized characters it is invertible and

$$x(D,C) = V^{-1}(D)\, y(D,C). \tag{3.15}$$

In terms of the inverse Vandermonde, equation (3.6) reads,

$$x_S(D,C) = \sum_{b=1}^{K} \left(V^{-1}\right)_{Sb}(D)\, y_b(D,C). \tag{3.16}$$

By lemma (3.2), $(V^{-1})_{Sb}(D)$ is rational for all $b = 1, \ldots, k$. Since y_b is an integer vector, the r.h.s is rational. The l.h.s is a sum of algebraic integers. A sum of algebraic integers that is also rational is necessarily an integer. This concludes the constructive proof of theorem 2. In section 4 we give a Galois theoretic proof of the same proposition.

The sum in (3.6) was specified by a single pair (S,D). We now prove a refinement of the above proposition.

Theorem 3. *Consider two pairs* $(S_1,D_1),(S_2,D_2)$ *with the property*

$$\frac{\chi^{S_1}(T_{D_1})}{d_{S_1}}, \quad \frac{\chi^{S_2}(T_{D_2})}{d_{S_2}} \in \mathbb{Z}. \tag{3.17}$$

As we will prove,

$$\sum_{R \in [(S_1,D_1)] \cap [(S_2,D_2)]} \frac{\chi^R(T_C)}{d_R} \in \mathbb{Z}, \quad \forall C \in \mathrm{Cl}(G), \tag{3.18}$$

where the sum is over the set

$$[(S_1,D_1)] \cap [(S_2,D_2)] = \left\{ R \in \mathrm{Irr}(G)\, s.t.\, \frac{\chi^R(T_{D_1})}{d_R} = \frac{\chi^{S_1}(T_{D_1})}{d_{S_1}} \text{and } \frac{\chi^R(T_{D_2})}{d_R} = \frac{\chi^{S_2}(T_{D_2})}{d_{S_2}} \right\}. \tag{3.19}$$

Proof of theorem 3 (Constructive). To prove this, consider the matrix $y(D_1, D_2, C)$ of partition functions

$$y_{b,b'}(D_1, D_2, C) = Z^G_{h=1, D_1^b, D_2^{b'}, C} = \sum_{R \in \mathrm{Irr}(G)} \left(\frac{\chi^R(T_{D_1})}{d_R} \right)^b \left(\frac{\chi^R(T_{D_2})}{d_R} \right)^{b'} \frac{\chi^R(T_C)}{d_R}, \quad b, b' = 1, \ldots, l. \tag{3.20}$$

for some non-negative integer l. We partition the sum into a sum over level sets

$$y_{b,b'}(D_1, D_2, C) = \sum_{R_1'} \left(\frac{\chi^{R_1'}(T_{D_1})}{d_{R_1'}} \right)^b \sum_{R \in \left[\left(R_1', D_1\right)\right]} \left(\frac{\chi^R(T_{D_2})}{d_R} \right)^{b'} \frac{\chi^R(T_C)}{d_R}. \tag{3.21}$$

Now we restrict to the partition functions with $b = 1, \ldots, k$ where k is the number of irreducible representations R with distinct $\frac{\chi^R(T_{D_1})}{d_R}$. The $k \times k$ matrix

$$\left(\frac{\chi^{R_1'}(T_{D_1})}{d_{R_1'}} \right)^b, \tag{3.22}$$

is a Vandermonde matrix with integer row for $R_1' = S_1$, by assumption in the theorem. The inverse Vandermonde lemma argument in the proof of theorem 2 implies that

$$\sum_{R \in [(S_1, D_1)]} \left(\frac{\chi^R(T_{D_2})}{d_R} \right)^{b'} \frac{\chi^R(T_C)}{d_R} \in \mathbb{Z}. \tag{3.23}$$

We further partition this sum into level sets as

$$\sum_{R \in [(S_1, D_1)]} \left(\frac{\chi^R(T_{D_2})}{d_R} \right)^{b'} \frac{\chi^R(T_C)}{d_R} = \sum_{R_2'} \left(\frac{\chi^{R_2'}(T_{D_2})}{d_{R_2'}} \right)^{b'} \sum_{R \in [(S_1, D_1)] \cap \left[\left(R_2', D_2\right)\right]} \frac{\chi^R(T_C)}{d_R}, \tag{3.24}$$

where the sum over R_2' is a set of irreducible representations such that

$$\frac{\chi^{R_2'}(T_{D_2})}{d_{R_2'}} \tag{3.25}$$

are all distinct, and the union of $[(R_2', D_2)]$ is all of $[(S_1, D_1)]$. Now we restrict to the partition functions with $b' = 1, \ldots, k'$ where k' is the number of irreducible representations R with distinct $\frac{\chi^R(T_{D_2})}{d_R}$. Again, it follows by the Vandermonde argument that

$$\sum_{R \in [(S_1, D_1)] \cap [(S_2, D_2)]} \frac{\chi^R(T_C)}{d_R} \in \mathbb{Z}, \quad \forall C \in \mathrm{Cl}(G). \tag{3.26}$$

It is straight-forward to generalize the argument to arbitrary numbers of pairs (S, D) with integer normalized characters. This is proven using Galois theory in (4.38).

Another generalization involves the insertion of handle creation operators to fix d_R in the sums. This can be used to engineer integral sums over set

$$\left\{R \in \operatorname{Irr}(G) \, \text{s.t.} \frac{\chi^R(T_D)}{d_R} = \frac{\chi^S(T_D)}{d_S} \text{and } d_R = d_S\right\} = \tag{3.27}$$

$$\left\{R \in \operatorname{Irr}(G) \, \text{s.t.} \, |D| \frac{\chi^R_D}{d_R} = |D| \frac{\chi^S_D}{d_S} \text{and } d_R = d_S\right\} = \tag{3.28}$$

$$\left\{R \in \operatorname{Irr}(G) \, \text{s.t.} \chi^R_D = \chi^S_D, \text{and } d_R = d_S\right\}. \tag{3.29}$$

giving the following proposition.

Proposition 3.1. *The following sum over the subset of irreducible representations defined above*

$$\sum_R \chi^R_C \tag{3.30}$$

is an integer for all $C \in \mathrm{Cl}(G)$.

As we will see in section 4, this is a special case of the Galois result (4.37) where the ordered list of group elements $H = (h_1, e)$ with $h_1 \in D$ and the ordered list of integers $N = (\chi^S_D, d_S)$.

3.2. Refined integral row sums from R(G)-CTST

In this section we will use $R(G)$-CTST to derive results dual to those above. We will prove the following integrality of row sums.

Theorem 4. *Let* $(S, D) \in \operatorname{Irr}(G) \times \mathrm{Cl}(G)$ *such that*

$$\chi^S_D \in \mathbb{Z}. \tag{3.31}$$

Define the following subset of $\mathrm{Cl}(G)$

$$\langle (S,D) \rangle = \left\{C \in \mathrm{Cl}(G) \, s.t. \; \chi^S_C = \chi^S_D\right\}. \tag{3.32}$$

Then the sums

$$\sum_{C \in \langle (S,D) \rangle} \chi^R_C \in \mathbb{Z}, \quad \forall R \in \operatorname{Irr}(G). \tag{3.33}$$

are integers.

Proof of theorem 4 (Constructive). To prove this, we follow steps similar to those in section 3.1. Consider the K-dimensional integer vector

$$y_b(S,R) = Z^{R(G)}_{h=1,S^b,R} = \sum_C \left(\chi^S_C\right)^b \chi^R_C, \quad b = 1, \dots, l. \tag{3.34}$$

where l is a non-negative integer. We partition the sum into level sets

$$y_b(S,R) = \sum_{C'} \left(\chi^S_{C'}\right)^b \sum_{C\in\langle(S,C')\rangle} \chi^R_C, \tag{3.35}$$

where the first sum is over a set of conjugacy classes $C'_1,\ldots,C'_k$ such that

$$\chi^S_{C'_1},\ldots,\chi^S_{C'_k}, \tag{3.36}$$

are all distinct and

$$\langle(S,C'_1)\rangle \cup \langle(S,C'_2)\rangle \cup \cdots \cup \langle(S,C'_k)\rangle = \mathrm{Cl}(G). \tag{3.37}$$

Now we restrict to the partition functions with $b = 1,\ldots,k$. Define the vector

$$x_{C'}(S,R) = \sum_{C\in\langle(S,C')\rangle} \chi^R_C, \tag{3.38}$$

and Vandermonde matrix

$$V_{b,C'}(S) = \left(\chi^S_{C'}\right)^b. \tag{3.39}$$

Then

$$y(S,R) = V(S)\,x(S,R), \tag{3.40}$$

and since $V(S)$ is a Vandermonde matrix, it is invertible and

$$x(S,R) = V^{-1}(S)\,y(S,R). \tag{3.41}$$

By the Vandermonde lemma $(V^{-1}(S))_{Db}$ is rational for all $b = 1,\ldots,K$. The l.h.s of equation (3.33) reads

$$x_D(S,R) = \sum_{b=1}^{K} \left(V^{-1}(S)\right)_{Db} y_b(S,R). \tag{3.42}$$

The l.h.s. is a sum of algebraic integers while the r.h.s. is a sum of rationals, and therefore they are integers.

The sum can be further refined by fixing several pairs $(S_1,D_1),(S_2,D_2),\ldots$. We omit the constructive proof of this because it mirrors the analogous proof given in the previous subsection. The Galois theoretic proof of this generalization, and the above special case is found in section 4 (see **Proof of theorem 4*** around equation (4.53)).

3.3. Generalized partitions of the integer row sums

Consider the character table χ_C^R of a finite group G and fix a row S. The sum

$$\sum_C \chi_C^S = n_S, \tag{3.43}$$

is an integer, as proved in equation (2.81). We will now see that this integer has further combinatorial structure. We call a triplet $[p,q,r]$ where p,q are integer partitions, and r is a list of zeroes satisfying

$$\sum_i p_i - \sum_j q_j + \sum_k r_k = n_S. \tag{3.44}$$

a generalized partition of n_S and write

$$[p,q,r] \vDash n_S. \tag{3.45}$$

The previous results on integrality of character sums can be used to give a generalized integer partition of n_S.

To see this, construct the table $\chi_C^{R\neq S}$, equal to the original character table but with row S removed. From this table, construct $\chi_C^{R\neq S,\mathbb{Z}}$ by keeping only the rows that are entirely integer. The table $\chi_C^{R\neq S,\mathbb{Z}}$ defines an equivalence relation on conjugacy classes by

$$C \sim_S C' \text{if} \chi_C^{R\neq S,\mathbb{Z}} = \chi_{C'}^{R\neq S,\mathbb{Z}} \text{for all rows.} \tag{3.46}$$

The equivalence relation partitions the set of conjugacy classes into equivalence classes $\kappa_1,\ldots,\kappa_l$. The equivalence classes κ_i are level sets of the vector-valued function

$$f^R(C) = \chi_C^{R\neq S,\mathbb{Z}}, \tag{3.47}$$

on conjugacy classes, defined by the integer rows of the character table with row S removed. They define a set partition of the set of conjugacy classes such that

$$n_S = \sum_{C\in \mathrm{Cl}(G)} \chi_C^S = \sum_{i=1}^{l} \sum_{C\in\kappa_i} \chi_C^S. \tag{3.48}$$

Let $C_1 \in \kappa_1,\ldots,C_l \in \kappa_l$ be a set of representatives. The equivalence classes κ_i are equal to the sets

$$\kappa_i = \bigcap_{\substack{R\neq S \\ \text{R integer row}}} \langle (R, C_i) \rangle. \tag{3.49}$$

Consequently, the sums

$$\sum_{C\in\kappa_i} \chi_C^S, \tag{3.50}$$

are integers for all $S \in \mathrm{Irr}(G)$.

We illustrate the procedure on the character table of $\mathbb{Z}_6$,

$$\begin{array}{c|cccccc} & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \\ \hline R_1 & 1 & 1 & 1 & 1 & 1 & 1 \\ R_2 & 1 & \zeta_3 & -\zeta_3-1 & \zeta_3+1 & -1 & -\zeta_3 \\ R_3 & 1 & -\zeta_3-1 & \zeta_3 & \zeta_3 & 1 & -\zeta_3-1 \\ R_4 & 1 & 1 & 1 & -1 & -1 & -1 \\ R_5 & 1 & \zeta_3 & -\zeta_3-1 & -\zeta_3-1 & 1 & \zeta_3 \\ R_6 & 1 & -\zeta_3-1 & \zeta_3 & -\zeta_3 & -1 & \zeta_3+1. \end{array} \tag{3.51}$$

With $S = R_1$ we have

$$\chi_C^{R\neq R_1} = \begin{array}{c|cccccc} & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \\ \hline R_2 & 1 & \zeta_3 & -\zeta_3-1 & \zeta_3+1 & -1 & -\zeta_3 \\ R_3 & 1 & -\zeta_3-1 & \zeta_3 & \zeta_3 & 1 & -\zeta_3-1 \\ R_4 & 1 & 1 & 1 & -1 & -1 & -1 \\ R_5 & 1 & \zeta_3 & -\zeta_3-1 & -\zeta_3-1 & 1 & \zeta_3 \\ R_6 & 1 & -\zeta_3-1 & \zeta_3 & -\zeta_3 & -1 & \zeta_3+1 \end{array} \tag{3.52}$$

and

$$\chi_C^{R\neq R_1,\mathbb{Z}} = \begin{array}{c|cccccc} & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \\ \hline R_4 & 1 & 1 & 1 & -1 & -1 & -1. \end{array} \tag{3.53}$$

This leads to the following partition of the conjugacy classes

$$\{C_1, C_2, C_3, C_4, C_5, C_6\} = \{C_1, C_2, C_3\} \cup \{C_4, C_5, C_6\} = \kappa_1 \cup \kappa_2. \tag{3.54}$$

In particular, we have

$$\sum_C \chi_C^{R_1} = 6 = \sum_{C=C_1,C_2,C_3} \chi_C^{R_1} + \sum_{C=C_4,C_5,C_6} \chi_C^{R_1} = 3+3. \tag{3.55}$$

For $S = R_2$ we have

$$\chi_C^{R\neq R_2} = \begin{array}{c|cccccc} & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \\ \hline R_1 & 1 & 1 & 1 & 1 & 1 & 1 \\ R_3 & 1 & -\zeta_3-1 & \zeta_3 & \zeta_3 & 1 & -\zeta_3-1 \\ R_4 & 1 & 1 & 1 & -1 & -1 & -1 \\ R_5 & 1 & \zeta_3 & -\zeta_3-1 & -\zeta_3-1 & 1 & \zeta_3 \\ R_6 & 1 & -\zeta_3-1 & \zeta_3 & -\zeta_3 & -1 & \zeta_3+1 \end{array} \tag{3.56}$$

and

$$\chi_C^{R\neq R_2,\mathbb{Z}} = \begin{array}{c|cccccc} & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \\ \hline R_1 & 1 & 1 & 1 & 1 & 1 & 1 \\ R_4 & 1 & 1 & 1 & -1 & -1 & -1\ . \end{array} \tag{3.57}$$

This gives the same partition of the conjugacy classes

$$\{C_1, C_2, C_3, C_4, C_5, C_6\} = \{C_1, C_2, C_3\} \cup \{C_4, C_5, C_6\} = \kappa_1 \cup \kappa_2 \tag{3.58}$$

and therefore

$$\sum_C \chi_C^{R_2} = 0 = \sum_{C=C_1,C_2,C_3} \chi_C^{R_2} + \sum_{C=C_4,C_5,C_6} \chi_C^{R_2} = 0 + 0. \tag{3.59}$$

Notably, the partition depends on the choice of S. For example, let $S = R_4$, then

$$\chi_C^{R \neq R_4} = \begin{array}{c|cccccc} & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \\ \hline R_1 & 1 & 1 & 1 & 1 & 1 & 1 \\ R_2 & 1 & \zeta_3 & -\zeta_3 - 1 & \zeta_3 + 1 & -1 & -\zeta_3 \\ R_3 & 1 & -\zeta_3 - 1 & \zeta_3 & \zeta_3 & 1 & -\zeta_3 - 1 \\ R_5 & 1 & \zeta_3 & -\zeta_3 - 1 & -\zeta_3 - 1 & 1 & \zeta_3 \\ R_6 & 1 & -\zeta_3 - 1 & \zeta_3 & -\zeta_3 & -1 & \zeta_3 + 1 \end{array} \tag{3.60}$$

and

$$\chi_C^{R \neq R_4, \mathbb{Z}} = \begin{array}{c|cccccc} & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \\ \hline R_1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \tag{3.61}$$

such that the induced partition is trivial

$$\{C_1, C_2, C_3, C_4, C_5, C_6\} = \{C_1, C_2, C_3, C_4, C_5, C_6\} = \kappa_1. \tag{3.62}$$

Therefore

$$\sum_C \chi_C^{R_4} = 0 = \sum_{C=C_1,C_2,C_3,C_4,C_5,C_6} \chi_C^{R_4} = 0. \tag{3.63}$$

In the language of generalized partitions, we have found that

$$[[3,3], [], []] \vDash n_{R_1}, \quad [[], [], [0,0]] \vDash n_{R_2}, \quad [[], [], [0]] \vDash n_{R_4}. \tag{3.64}$$

We now give some examples of character table partitions for a selection of groups. For each row we give: the value of the characters on each subset κ_i that partitions the columns, the corresponding generalized partition $[p, q, r]$ and lastly the integer row sum n_S. In the familiar example of $\mathbb{Z}_6$ we find

(3.65)

S	κ_1	κ_2	p	q	r	n_S
R_1	$[1,1,1]$	$[1,1,1]$	$[3,3]$	$[]$	$[]$	6
R_2	$[1,\zeta_3,-\zeta_3-1]$	$[\zeta_3+1,-1,-\zeta_3]$	$[]$	$[]$	$[0,0]$	0
R_3	$[1,-\zeta_3-1,\zeta_3]$	$[\zeta_3,1,-\zeta_3-1]$	$[]$	$[]$	$[0,0]$	0
R_4	$[1,-1,1,-1,1,-1]$	$[]$	$[]$	$[]$	$[0]$	0
R_5	$[1,\zeta_3,-\zeta_3-1]$	$[-\zeta_3-1,1,\zeta_3]$	$[]$	$[]$	$[0,0]$	0
R_6	$[1,-\zeta_3-1,\zeta_3]$	$[-\zeta_3,-1,\zeta_3+1]$	$[]$	$[]$	$[0,0]$	0

For the Mathieu group M11 we find

S	κ_1	κ_2	κ_3	κ_4	κ_5	κ_6	κ_7	κ_8	p	q	r	n_S
R_1	$[1]$	$[1]$	$[1]$	$[1]$	$[1]$	$[1]$	$[1,1]$	$[1,1]$	$[1,1,1,1,1,1,2,2]$	$[]$	$[]$	10
R_2	$[10]$	$[2]$	$[1]$	$[2]$	$[0]$	$[-1]$	$[0,0]$	$[-1,-1]$	$[10,2,1,2]$	$[-1,-2]$	$[0,0]$	12
R_3	$[10]$	$[-2]$	$[1]$	$[0]$	$[0]$	$[1]$	$[\zeta_8^3+\zeta_8, -\zeta_8^3-\zeta_8]$	$[-1,-1]$	$[10,1,1]$	$[-2,-2]$	$[0,0,0]$	8
R_4	$[10]$	$[-2]$	$[1]$	$[0]$	$[0]$	$[1]$	$[-\zeta_8^3-\zeta_8, \zeta_8^3+\zeta_8]$	$[-1,-1]$	$[10,1,1]$	$[-2,-2]$	$[0,0,0]$	8
R_5	$[11]$	$[3]$	$[2]$	$[-1]$	$[1]$	$[0]$	$[-1,-1]$	$[0,0]$	$[11,3,2,1]$	$[-1,-2]$	$[0,0]$	14
R_6	$[16]$	$[0]$	$[-2]$	$[0]$	$[1]$	$[0]$	$[0,0]$	$[\zeta_{11}^9+\zeta_{11}^5+\zeta_{11}^4+\zeta_{11}^3+\zeta_{11}, -\zeta_{11}^9-\zeta_{11}^5-\zeta_{11}^4-\zeta_{11}^3-\zeta_{11}-1]$	$[16,1]$	$[-2,-1]$	$[0,0,0,0]$	14
R_7	$[16]$	$[0]$	$[-2]$	$[0]$	$[1]$	$[0]$	$[0,0]$	$[-\zeta_{11}^9-\zeta_{11}^5-\zeta_{11}^4-\zeta_{11}^3-\zeta_{11}-1, \zeta_{11}^9+\zeta_{11}^5+\zeta_{11}^4+\zeta_{11}^3+\zeta_{11}]$	$[16,1]$	$[-2,-1]$	$[0,0,0,0]$	14
R_8	$[44]$	$[4]$	$[-1]$	$[0]$	$[-1]$	$[1]$	$[0,0]$	$[0,0]$	$[44,4,1]$	$[-1,-1]$	$[0,0,0]$	47
R_9	$[45]$	$[-3]$	$[0]$	$[1]$	$[0]$	$[0]$	$[-1,-1]$	$[1,1]$	$[45,1,2]$	$[-3,-2]$	$[0,0,0]$	43
R_{10}	$[55]$	$[-1]$	$[1]$	$[-1]$	$[0]$	$[-1]$	$[1,1]$	$[0,0]$	$[55,1,2]$	$[-1,-1,-1]$	$[0,0]$	55

(3.66)

and for $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$

S	κ_1	κ_2	κ_3	κ_4	κ_5	κ_6	κ_7	κ_8	p	q	r	n_S
R_1	$[1,1]$	$[1]$	$[1,1]$	$[1]$	$[1,1]$	$[1]$	$[1,1,1,1]$	$[1,1]$	$[2,1,2,1,2,1,4,2]$	$[]$	$[]$	15
R_2	$[1,1]$	$[-1]$	$[-1,-1]$	$[1]$	$[1,1]$	$[1]$	$[-1,-1,-1,-1]$	$[1,1]$	$[2,1,2,1,2]$	$[-1,-2,-4]$	$[]$	1
R_3	$[1,1]$	$[-1]$	$[1,1]$	$[1]$	$[1,1]$	$[-1]$	$[1,1,1,1]$	$[1,1]$	$[2,2,1,2,4,2]$	$[-1,-1]$	$[]$	11
R_4	$[1,1]$	$[1]$	$[-1,-1]$	$[1]$	$[1,1]$	$[-1]$	$[-1,-1,-1,-1]$	$[1,1]$	$[2,1,1,2,2]$	$[-2,-1,-4]$	$[]$	1
R_5	$[2,-2,2]$	$[0]$	$[0,0]$	$[2,-2,2,-2]$	$[0]$	$[0,0,0,0]$	$[]$	$[]$	$[2]$	$[]$	$[0,0,0,0,0]$	2
R_6	$[2,2]$	$[0]$	$[-2,-2]$	$[2]$	$[-1,-1]$	$[0]$	$[1,1,1,1]$	$[-1,-1]$	$[4,2,4]$	$[-4,-2,-2]$	$[0,0]$	2
R_7	$[2,2]$	$[0]$	$[2,2]$	$[2]$	$[-1,-1]$	$[0]$	$[-1,-1,-1,-1]$	$[-1,-1]$	$[4,4,2]$	$[-2,-4,-2]$	$[0,0]$	2
R_8	$[2,-2]$	$[0]$	$[-\zeta_8^3-\zeta_8, \zeta_8^3+\zeta_8]$	$[0]$	$[2,-2]$	$[0]$	$[-\zeta_8^3-\zeta_8, -\zeta_8^3-\zeta_8, \zeta_8^3+\zeta_8, \zeta_8^3+\zeta_8]$	$[0,0]$	$[]$	$[]$	$[0,0,0,0,0,0,0,0,0]$	0
R_9	$[2,-2]$	$[0]$	$[\zeta_8^3+\zeta_8, -\zeta_8^3-\zeta_8]$	$[0]$	$[2,-2]$	$[0]$	$[\zeta_8^3+\zeta_8, \zeta_8^3+\zeta_8, -\zeta_8^3-\zeta_8, -\zeta_8^3-\zeta_8]$	$[0,0]$	$[]$	$[]$	$[0,0,0,0,0,0,0,0,0]$	0
R_{10}	$[2,2]$	$[0]$	$[0,0]$	$[-2]$	$[-1,-1]$	$[0]$	$[-\zeta_{12}^3+2\zeta_{12}, \zeta_{12}^3-2\zeta_{12}, -\zeta_{12}^3+2\zeta_{12}, \zeta_{12}^3-2\zeta_{12}]$	$[1,1]$	$[4,2]$	$[-2,-2]$	$[0,0,0,0]$	2
R_{11}	$[2,2]$	$[0]$	$[0,0]$	$[-2]$	$[-1,-1]$	$[0]$	$[\zeta_{12}^3-2\zeta_{12}, -\zeta_{12}^3+2\zeta_{12}, \zeta_{12}^3-2\zeta_{12}, -\zeta_{12}^3+2\zeta_{12}]$	$[1,1]$	$[4,2]$	$[-2,-2]$	$[0,0,0,0]$	2
R_{12}	$[2,-2]$	$[0]$	$[-\zeta_8^3-\zeta_8, \zeta_8^3+\zeta_8]$	$[0]$	$[-1,1]$	$[0]$	$[-\zeta_{24}^7+\zeta_{24}^3-\zeta_{24}, \zeta_{24}^7+\zeta_{24}^5, \zeta_{24}^7-\zeta_{24}^3+\zeta_{24}, -\zeta_{24}^7-\zeta_{24}^5]$	$[\zeta_{12}^3-2\zeta_{12}, -\zeta_{12}^3+2\zeta_{12}]$	$[]$	$[]$	$[0,0,0,0,0,0,0,0,0]$	0
R_{13}	$[2,-2]$	$[0]$	$[-\zeta_8^3-\zeta_8, \zeta_8^3+\zeta_8]$	$[0]$	$[-1,1]$	$[0]$	$[\zeta_{24}^7+\zeta_{24}^5, -\zeta_{24}^7+\zeta_{24}^3-\zeta_{24}, -\zeta_{24}^7-\zeta_{24}^5, \zeta_{24}^7-\zeta_{24}^3+\zeta_{24}]$	$[-\zeta_{12}^3+2\zeta_{12}, \zeta_{12}^3-2\zeta_{12}]$	$[]$	$[]$	$[0,0,0,0,0,0,0,0,0]$	0
R_{14}	$[2,-2]$	$[0]$	$[\zeta_8^3+\zeta_8, -\zeta_8^3-\zeta_8]$	$[0]$	$[-1,1]$	$[0]$	$[-\zeta_{24}^7-\zeta_{24}^5, \zeta_{24}^7-\zeta_{24}^3+\zeta_{24}, \zeta_{24}^7+\zeta_{24}^5, -\zeta_{24}^7+\zeta_{24}^3-\zeta_{24}]$	$[-\zeta_{12}^3+2\zeta_{12}, \zeta_{12}^3-2\zeta_{12}]$	$[]$	$[]$	$[0,0,0,0,0,0,0,0,0]$	0
R_{15}	$[2,-2]$	$[0]$	$[\zeta_8^3+\zeta_8, -\zeta_8^3-\zeta_8]$	$[0]$	$[-1,1]$	$[0]$	$[\zeta_{24}^7-\zeta_{24}^3+\zeta_{24}, -\zeta_{24}^7-\zeta_{24}^5, -\zeta_{24}^7+\zeta_{24}^3-\zeta_{24}, \zeta_{24}^7+\zeta_{24}^5]$	$[\zeta_{12}^3-2\zeta_{12}, -\zeta_{12}^3+2\zeta_{12}]$	$[]$	$[]$	$[0,0,0,0,0,0,0,0,0]$	0

(3.67)

3.4. Generalized partitions of the integer column sums

Similar considerations are relevant for column sums. For a fixed conjugacy class D, the sum

$$\sum_{R\in\mathrm{Irr}(G)} \chi_D^R = n_D \tag{3.68}$$

is an integer, as we proved in proposition 2.1. We use a completely analogous set of rules to partition the columns. First construct the table $\chi^R_{C\neq D}$, corresponding to the character table without the column D. Further, restrict to only integer columns, yielding the table $\chi^R_{C\neq D,\mathbb{Z}}$. The table defines an equivalence relation on rows

$$R\sim_D R' \mathrm{if} \chi^R_{C\neq D,\mathbb{Z}} = \chi^{R'}_{C\neq D,\mathbb{Z}} \text{ for all columns,} \tag{3.69}$$

that partitions column D into subsets $\rho_1,\dots,\rho_l$ such that

$$n_D = \sum_{R\in\mathrm{Irr}(G)} \chi_D^R = \sum_{i=1}^{l}\sum_{R\in\rho_i}\chi_D^R. \tag{3.70}$$

For D equal to the conjugacy class of the identity, the sums

$$\sum_{R\in\rho_i}\chi_D^R = \sum_{R\in\rho_i} d_R, \tag{3.71}$$

are manifestly integers. For D different from the identity element, we can prove that they are integers as follows. Let $R_1\in\rho_1,\dots,R_l\in\rho_l$ be a set of representatives. The equivalence classes are sets

$$\rho_i = \left\{R\in\mathrm{Irr}(G)\,\text{s.t.}\chi_C^R = \chi_C^{R_i} \text{ for all integer integer columns } C\neq D\right\}. \tag{3.72}$$

In particular, the column of the identity conjugacy class is always integer and therefore included in the above set. This forces $d_R = d_{R_i}$. Consequently, the equivalence classes can be understood as intersections of subsets

$$\rho_i = \left[\bigcap_{\substack{C\neq D\\ C\text{integer column}}} [(R_i,C)]\,0\right]\cap[d_{R_i}]. \tag{3.73}$$

The intersection with $[d_{R_i}]$ is necessary to force $d_R = d_{R_i}$. It follows that

$$\sum_{R\in\rho_i}\chi_D^R\in\mathbb{Z}. \tag{3.74}$$

We give examples of generalized partitions of columns for the same groups as above. Note that the rows now label columns in the character tables. For $\mathbb{Z}_6$ we have

D	ρ_1	ρ_2	p	q	r	n_D
C_1	$[1,1,1]$	$[1,1,1]$	$[3,3]$	$[]$	$[]$	6
C_2	$[1,\zeta_3,-\zeta_3-1]$	$[\zeta_3+1,-1,-\zeta_3]$	$[]$	$[]$	$[0,0]$	0
C_3	$[1,-\zeta_3-1,\zeta_3]$	$[\zeta_3,1,-\zeta_3-1]$	$[]$	$[]$	$[0,0]$	0
C_4	$[1,-1,1,-1,1,-1]$	$[]$	$[]$	$[]$	$[0]$	0
C_5	$[1,\zeta_3,-\zeta_3-1]$	$[-\zeta_3-1,1,\zeta_3]$	$[]$	$[]$	$[0,0]$	0
C_6	$[1,-\zeta_3-1,\zeta_3]$	$[-\zeta_3,-1,\zeta_3+1]$	$[]$	$[]$	$[0,0]$	0

(3.75)

for the Mathieu group M11 we have

D	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8	p	q	r	n_D
C_1	[1]	[10]	[10, 10]	[11]	[16, 16]	[44]	[45]	[55]	[1, 10, 20, 11, 32, 44, 45, 55]	[]	[]	218
C_2	[1]	[2]	$[-2,-2]$	[3]	[0, 0]	[4]	$[-3]$	$[-1]$	[1, 2, 3, 4]	$[-4,-3,-1]$	[0]	2
C_3	[1]	[1]	[1, 1]	[2]	$[-2,-2]$	$[-1]$	[0]	[1]	[1, 1, 2, 2, 1]	$[-4,-1]$	[0]	2
C_4	[1]	[2]	[0, 0]	$[-1]$	[0, 0]	[0]	[1]	$[-1]$	[1, 2, 1]	$[-1,-1]$	[0, 0, 0]	2
C_5	[1]	[0]	[0, 0]	[1]	[1, 1]	$[-1]$	[0]	[0]	[1, 1, 2]	$[-1]$	[0, 0, 0, 0]	3
C_6	[1]	$[-1]$	[1, 1]	[0]	[0, 0]	[1]	[0]	$[-1]$	[1, 2, 1]	$[-1,-1]$	[0, 0, 0]	2
C_7	[1]	[0]	$[\zeta_8^3+\zeta_8,-\zeta_8^3-\zeta_8]$	$[-1]$	[0, 0]	[0]	$[-1]$	[1]	[1, 1]	$[-1,-1]$	[0, 0, 0, 0]	0
C_8	[1]	[0]	$[-\zeta_8^3-\zeta_8,\zeta_8^3+\zeta_8]$	$[-1]$	[0, 0]	[0]	$[-1]$	[1]	[1, 1]	$[-1,-1]$	[0, 0, 0, 0]	0
C_9	[1]	$[-1]$	$[-1,-1]$	[0]	$[\zeta_{11}^9+\zeta_{11}^5+\zeta_{11}^4+\zeta_{11}^3+\zeta_{11},-\zeta_{11}^9-\zeta_{11}^5-\zeta_{11}^4-\zeta_{11}^3-\zeta_{11}-1]$	[0]	[1]	[0]	[1, 1]	$[-1,-2,-1]$	[0, 0, 0]	-2
C_{10}	[1]	$[-1]$	$[-1,-1]$	[0]	$[-\zeta_{11}^9-\zeta_{11}^5-\zeta_{11}^4-\zeta_{11}^3-\zeta_{11}-1,\zeta_{11}^9+\zeta_{11}^5+\zeta_{11}^4+\zeta_{11}^3+\zeta_{11}]$	[0]	[1]	[0]	[1, 1]	$[-1,-2,-1]$	[0, 0, 0]	-2

(3.76)

and for $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$

D	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8	ρ_9	p	q	r	n_D
C_1	[1]	[1]	[1]	[1]	[2]	[2, 2]	[2, 2]	[2, 2]	[2, 2, 2, 2]	[1, 1, 1, 1, 2, 4, 4, 4, 8]	[]	[]	26
C_2	$[1,-1]$	$[-1,1]$	[0]	[0, 0]	[0, 0]	[0, 0]	[]	[]	[0, 0, 0, 0]	[]	[]	[0, 0, 0, 0, 0, 0, 0]	0
C_3	[1]	$[-1]$	[1]	$[-1]$	[0]	$[-2,2]$	$[-\zeta_8^3-\zeta_8,\zeta_8^3+\zeta_8]$	[0, 0]	$[-\zeta_8^3-\zeta_8,-\zeta_8^3-\zeta_8,\zeta_8^3+\zeta_8,\zeta_8^3+\zeta_8]$	[1, 1]	$[-1,-1]$	[0, 0, 0, 0, 0]	0
C_4	[1]	[1]	[1]	[1]	$[-2]$	$[2,2,-2,-2]$	[0, 0]	[0, 0, 0, 0]	[]	[1, 1, 1, 1]	$[-2]$	[0, 0, 0]	2
C_5	[1]	[1]	[1]	[1]	[2]	[2, 2]	$[-2,-2]$	[2, 2]	$[-2,-2,-2,-2]$	[1, 1, 1, 1, 2, 4, 4]	$[-4,-8]$	[]	2
C_6	[1]	[1]	[1]	[1]	[2]	$[-1,-1]$	[2, 2]	$[-1,-1]$	$[-1,-1,-1,-1]$	[1, 1, 1, 1, 2, 4]	$[-2,-2,-4]$	[]	2
C_7	$[1,-1]$	$[1,-1]$	[0]	[0, 0]	[0, 0]	[0, 0]	[0, 0, 0, 0]	[]	[]	[]	[0, 0, 0, 0, 0, 0, 0]	0	
C_8	[1]	$[-1]$	[1]	$[-1]$	[0]	$[-2,2]$	$[\zeta_8^3+\zeta_8,-\zeta_8^3-\zeta_8]$	[0, 0]	$[\zeta_8^3+\zeta_8,\zeta_8^3+\zeta_8,-\zeta_8^3-\zeta_8,-\zeta_8^3-\zeta_8]$	[1, 1]	$[-1,-1]$	[0, 0, 0, 0, 0]	0
C_9	[1]	$[-1]$	[1]	$[-1]$	[0]	$[1,-1]$	$[-\zeta_8^3-\zeta_8,\zeta_8^3+\zeta_8]$	$[-\zeta_{12}^3+2\zeta_{12},\zeta_{12}^3-2\zeta_{12}]$	$[-\zeta_{24}^7+\zeta_{24}^3-\zeta_{24},\zeta_{24}^7+\zeta_{24}^5,-\zeta_{24}^7-\zeta_{24}^5,\zeta_{24}^7-\zeta_{24}^3+\zeta_{24}]$	[1, 1]	$[-1,-1]$	[0, 0, 0, 0, 0]	0
C_{10}	[1]	[1]	[1]	[1]	$[-2]$	$[-1,-1]$	[0, 0]	[1, 1]	$[\zeta_{12}^3-2\zeta_{12},-\zeta_{12}^3+2\zeta_{12},-\zeta_{12}^3+2\zeta_{12},\zeta_{12}^3-2\zeta_{12}]$	[1, 1, 1, 1, 2]	$[-2,-2]$	[0, 0]	2
C_{11}	[1]	[1]	[1]	[1]	[2]	$[-1,-1]$	$[-2,-2]$	$[-1,-1]$	[1, 1, 1, 1]	[1, 1, 1, 1, 2, 4]	$[-2,-4,-2]$	[]	2
C_{12}	[1]	$[-1]$	[1]	$[-1]$	[0]	$[1,-1]$	$[-\zeta_8^3-\zeta_8,\zeta_8^3+\zeta_8]$	$[\zeta_{12}^3-2\zeta_{12},-\zeta_{12}^3+2\zeta_{12}]$	$[\zeta_{24}^7+\zeta_{24}^5,-\zeta_{24}^7+\zeta_{24}^3-\zeta_{24},\zeta_{24}^7-\zeta_{24}^3+\zeta_{24},-\zeta_{24}^7-\zeta_{24}^5]$	[1, 1]	$[-1,-1]$	[0, 0, 0, 0, 0]	0
C_{13}	[1]	$[-1]$	[1]	$[-1]$	[0]	$[1,-1]$	$[\zeta_8^3+\zeta_8,-\zeta_8^3-\zeta_8]$	$[-\zeta_{12}^3+2\zeta_{12},\zeta_{12}^3-2\zeta_{12}]$	$[\zeta_{24}^7-\zeta_{24}^3+\zeta_{24},-\zeta_{24}^7-\zeta_{24}^5,\zeta_{24}^7+\zeta_{24}^5,-\zeta_{24}^7+\zeta_{24}^3-\zeta_{24}]$	[1, 1]	$[-1,-1]$	[0, 0, 0, 0, 0]	0
C_{14}	[1]	[1]	[1]	[1]	$[-2]$	$[-1,-1]$	[0, 0]	[1, 1]	$[-\zeta_{12}^3+2\zeta_{12},\zeta_{12}^3-2\zeta_{12},\zeta_{12}^3-2\zeta_{12},-\zeta_{12}^3+2\zeta_{12}]$	[1, 1, 1, 1, 2]	$[-2,-2]$	[0, 0]	2
C_{15}	[1]	$[-1]$	[1]	$[-1]$	[0]	$[1,-1]$	$[\zeta_8^3+\zeta_8,-\zeta_8^3-\zeta_8]$	$[\zeta_{12}^3-2\zeta_{12},-\zeta_{12}^3+2\zeta_{12}]$	$[-\zeta_{24}^7-\zeta_{24}^5,\zeta_{24}^7-\zeta_{24}^3+\zeta_{24},-\zeta_{24}^7+\zeta_{24}^3-\zeta_{24},\zeta_{24}^7+\zeta_{24}^5]$	[1, 1]	$[-1,-1]$	[0, 0, 0, 0, 0]	0

(3.77)

4. Galois theory for character tables

In this section we will review Galois theory of field extensions and its particular application to character theory. By focusing on cyclotomic fields, which are the field extensions of $\mathbb{Q}$ relevant to character theory, we are able to give a concrete exposition of their Galois theory. In particular, this includes explicit descriptions of their Galois groups, associated actions as automorphisms of the fields, and the fundamental theorem of Galois Theory for cyclotomic fields. The Galois action on cyclotomic fields gives rise to Galois actions on characters of G and we use this to study integrality properties of characters.

Consider a finite group G and a representation R of G. From lemma 2.15 in [28], we know that the matrix $R(g)$ for some element $g \in G$ is similar to a diagonal matrix with eigenvalues ε_i. If N_g is the order of g such that $g^{N_g} = 1$ it follows that

$$R\left(g^{N_g}\right) = R\left(g\right)^{N_g} = 1, \tag{4.1}$$

and the eigenvalues satisfy $\epsilon_i^{N_g} = 1$. Therefore, ε_i are N_g^{th} roots of unity. Because the character of $R(g)$ is given by

$$\chi^R\left(g\right) = \sum_i \epsilon_i, \tag{4.2}$$

it lies in the cyclotomic field $\mathbb{Q}(e^{\frac{2\pi i}{N_g}})$. This is a number field containing elements of the form

$$a + b\, e^{\frac{2\pi i}{N_g}} \tag{4.3}$$

where $a, b \in \mathbb{Q}$, and their combinations using the field operations $+, \times$. Since roots of unity are algebraic integers, the character of a representation evaluated on $g \in G$ is an algebraic integer in $\mathbb{Q}(e^{\frac{2\pi i}{N_g}})$.

In general, let E be the exponent of the group G, that is, the smallest integer E such that $g^E = 1$ for all $g \in G$. Then the above argument shows that the characters of all representations of G lie in the cyclotomic field $\mathbb{Q}(e^{\frac{2\pi i}{E}})$. In fact, from Brauer's theorem (theorem 10.3 in [28]) it is known that any representation R of a group G can be realized in the field $\mathbb{Q}(e^{\frac{2\pi i}{E}})$. In other words, it is always possible to choose a basis such that the matrix elements of $R(g)$ for any $g \in G$ is a linear combination of E^{th} roots of unity with rational coefficients.

In the next subsection, we will go through properties of cyclotomic fields and associated Galois actions. Then we will show that elements invariant under the Galois group are necessarily rational. Following this discussion, we will study Galois action on group representations and characters. This will lead to interesting integrality properties of characters.

4.1. Cyclotomic fields and Galois actions

We found that in the study of group representations and their characters, the cyclotomic field $\mathbb{Q}(e^{\frac{2\pi i}{E}})$ plays a crucial role[4]. As we will now explain, $\mathbb{Q}(e^{\frac{2\pi i}{E}})$ can be understood as a vector space. An element $\xi \in \mathbb{Q}(e^{\frac{2\pi i}{E}})$ is called a primitive E^{th} root of unity if

$$\xi^E = 1 \text{ and } \xi^k \neq 1 \text{ for } k = 1, \ldots, E-1. \tag{4.4}$$

[4] For a discussion on general field extensions and Galois actions, see chapters 13 and 14 of [29].

$\mathbb{Q}(e^{\frac{2\pi i}{E}})$ can be understood as a vector space over $\mathbb{Q}$ of dimension $\phi(E)$, where $\phi(E)$ is the Euler's totient function (see section 13.6 corollary 42 in [29]). In general, the set of elements $\{1, e^{\frac{2\pi i}{E}}, e^{\frac{2\pi i2}{E}}, \ldots, e^{\frac{2\pi i(\phi(E)-1)}{E}}\}$ form a basis of $\mathbb{Q}(e^{\frac{2\pi i}{E}})$ (for example, see [30]). Therefore, a general element of this field is of the form

$$a = \sum_{k=0}^{\phi(E)-1} a_k \xi_k \tag{4.5}$$

where a_k are rational and $\xi_k = e^{\frac{2\pi ik}{E}}$.

We now define the Galois group and its action on the field $\mathbb{Q}(e^{\frac{2\pi i}{E}})$. The group $\mathbb{Z}_E^\times$ is the subset of integers less than E that are also co-prime to E. The product is given by multiplication of integers modulo E. The Galois group of the number field $\mathbb{Q}(e^{\frac{2\pi i}{E}})$ is the group of automorphisms of this field which fixes the rationals. In other words, it is a bijective map $\sigma : \mathbb{Q}(e^{\frac{2\pi i}{E}}) \to \mathbb{Q}(e^{\frac{2\pi i}{E}})$ such that for $a, b \in \mathbb{Q}(e^{\frac{2\pi i}{E}})$

$$\sigma(a+b) = \sigma(a) + \sigma(b)\ ,\ \sigma(a \times b) = \sigma(a) \times \sigma(b) \text{ and } \sigma(a) = a\ \forall\, a \in \mathbb{Q}\,. \tag{4.6}$$

The Galois group of $\mathbb{Q}(e^{\frac{2\pi i}{E}})$ is isomorphic to $\mathbb{Z}_E^\times$ (see section 14.5 theorem 26 in [29]). Let $\sigma \in \mathbb{Z}_E^\times$ and $a \in \mathbb{Q}(e^{\frac{2\pi i}{E}})$, the action of σ on a is given by

$$\sigma(a) = \sigma\left(\sum_{k=0}^{\phi(E)-1} a_k \xi_k\right) = \sum_{k=0}^{\phi(E)-1} a_k \xi_k^\sigma\,. \tag{4.7}$$

4.2. Elements invariant under Galois group

In the cyclotomic field $\mathbb{Q}(e^{\frac{2\pi i}{E}})$ the elements fixed by the full Galois group $\mathbb{Z}_E^\times$ are precisely the rationals. To prove this, we will use the Fundamental Theorem of Galois Theory. Stating this theorem requires the definition of field extensions of Galois type (Galois extensions), which we now give.

The field F is called an extension of $\mathbb{Q}$ if $\mathbb{Q} \subset F$ and the operations in F (multiplication, addition) correspond to operations in $\mathbb{Q}$ when restricted to the subset $\mathbb{Q}$. In other words, if $\mathbb{Q}$ is a subfield of F. For example, the real numbers are an extension of $\mathbb{Q}$.

A field extension F of $\mathbb{Q}$ is called algebraic if for every $\alpha \in F$,

$$\text{there exists a } P_\alpha(x) \in \mathbb{Q}[x] \text{ such that } P_\alpha(\alpha) = 0.$$

In fact, for an algebraic field extension F and $\alpha \in F$ there exists a unique polynomial of lowest degree for which α is a root [29]. This polynomial is called the minimal polynomial associated with α.

An extension F is called separable if for every $\alpha \in F$ and minimal polynomial $P_\alpha(x)$ of degree d

$$|\{\beta\,|\,P_\alpha(\beta) = 0\}| = d.$$

That is, all roots are distinct. F is called a normal extension if for every $\alpha \in F$ and minimal polynomial $P_\alpha(x)$

$$P_\alpha(\beta) = 0 \text{ implies that } \beta \in F.$$

That is, all roots lie in F. A separable and normal extension is called a Galois extension. This assumption on a field extension allows one to prove the fundamental theorem of Galois theory (theorem 14 in [29]).

To apply the fundamental theory of Galois theory to the cyclotomic fields, we first show that it is a Galois extension. The field $\mathbb{Q}(e^{\frac{2\pi i}{E}})$ is the minimal field containing all the roots of the polynomial

$$p(x) = x^E - 1. \tag{4.8}$$

It follows that $\mathbb{Q}(e^{\frac{2\pi i}{E}})$ is a Galois extension of $\mathbb{Q}$ (see theorem 13 in [29]). Therefore, we are allowed to use the fundamental theorem of Galois theory, which says

Fundamental theorem of Galois Theory for cyclotomic fields: There is a one-to-one correspondence between the subgroups of $\mathbb{Z}_E^\times$ and the subfields of $\mathbb{Q}(e^{\frac{2\pi i}{E}})$ given by

$$K \subset \mathbb{Q}\left(e^{\frac{2\pi i}{E}}\right) \to H \subset \mathbb{Z}_E^\times \text{ which fixes all elements in K.} \tag{4.9}$$

$$H \subset \mathbb{Z}_E^\times \to \text{Subfield } K \subset \mathbb{Q}\left(e^{\frac{2\pi i}{E}}\right) \text{ of fixed points under H-action.} \tag{4.10}$$

In particular, this theorem also implies that the trivial subgroup of $\mathbb{Z}_E^\times$ fixes the full field $\mathbb{Q}(e^{\frac{2\pi i}{E}})$ and the subfield fixed by the full Galois group is the field of rationals.

4.3. Galois action on representations and characters

In this section, we will study how the Galois group acts on irreducible representations and conjugacy classes [31, 32]. Given a group G and a representation R, recall that $R(g)$ can be realized as a matrix with components in the field $\mathbb{Q}(e^{\frac{2\pi i}{E}})$. An element σ in the Galois group of this field acts on the matrix $R(g)$ component-wise resulting in a new matrix $\sigma(R(g))$. We know that $R(g)$ satisfies

$$R(g)R(h) = R(gh) \ \forall g,h \in G. \tag{4.11}$$

This is an algebraic relation

$$\sum_j R_{ij}(g)R_{jk}(h) - R_{ik}(gh) = 0 \tag{4.12}$$

between matrix elements. Since $\sigma(0) = 0$ for all σ and using (4.6) we have

$$\sigma\left(\sum_j R_{ij}(g)R_{jk}(h) - R_{ik}(gh)\right) = \sum_j \sigma(R_{ij}(g))\sigma(R_{jk}(h)) - \sigma(R_{ik}(gh)) = 0, \tag{4.13}$$

or equivalently

$$\sigma(R(g))\sigma(R(h)) = \sigma(R(gh)) \ \forall g,h \in G. \tag{4.14}$$

Therefore, $\sigma(R)$ is also a representation of the group G.

If R is an irreducible representation, so is $\sigma(R)$. This is understood as follows. The character of the representation R is an algebraic combination of elements of the matrix R. Therefore, under a Galois action on the representation R, using (4.6) we get

$$\chi^{\sigma(R)}(g) = \sum_i \sigma(R_{ii}(g)) = \sigma\Big(\sum_i R_{ii}(g)\Big) = \sigma\left(\chi^R(g)\right). \tag{4.15}$$

By character orthogonality, a representation R is irreducible if and only if

$$\frac{1}{|G|}\sum_{g\in G}\chi^R(g)\,\chi^R\left(g^{-1}\right) = 1. \tag{4.16}$$

Now assume that R is irreducible. The character of $\sigma(R(g))$ is simply $\sigma(\chi^R(g))$ and

$$\frac{1}{|G|}\sum_{g\in G}\sigma\left(\chi^R(g)\right)\sigma\left(\chi^R\left(g^{-1}\right)\right) = \sigma\left(\frac{1}{|G|}\sum_{g\in G}\chi^R(g)\,\chi^R\left(g^{-1}\right)\right) = \sigma(1) = 1. \tag{4.17}$$

where in the first equality we used (4.6). Very similar manipulations can be used to show that the Galois action preserves orthogonality of irreducible characters. This implies that two non-isomorphic irreducible representations remain non-isomorphic after the action of the Galois group. This discussion implies that the Galois action on the set of irreducible representations is a permutation (bijection) of the irreducible representations.

Recall that the character of a representation evaluated on an element $g\in G$ can be written as

$$\chi^R(g) = \sum_i \epsilon_i \tag{4.18}$$

where ε_i are E^{th} roots of unity. Then, using (4.6) the action of $\sigma\in\mathbb{Z}_E^\times$ is

$$\sigma\left(\chi^R(g)\right) = \sum_i \epsilon_i^{m_\sigma} = \chi^R\left(g^{m_\sigma}\right). \tag{4.19}$$

where m_σ is an integer coprime to the order of g corresponding to the Galois action σ. For ease of notation, from now on we will write g^{m_σ} as g^σ. The action of the Galois group $g\to g^\sigma$ takes conjugacy classes to conjugacy classes since

$$\left(hgh^{-1}\right)^\sigma = hg^\sigma h^{-1}. \tag{4.20}$$

Therefore, Galois group also acts on conjugacy classes as a permutation. If g is in the conjugacy class $C\in\mathrm{Cl}(G)$, and $g^\sigma\in C'$ we write $\sigma(C) = C'$ for the action on conjugacy classes.

To summarize, equations (4.15) and (4.19) are two equivalent ways in which we can describe the action of the Galois group on characters. The former action is on the irreducible representations by a permutation while the latter action permutes the conjugacy classes. This permutation of the conjugacy classes may not be an automorphism of the group.

4.4. Integrality of characters

We now study the following question: Given the fundamental theorem of Galois theory for cyclotomic fields, and the action of the Galois group on representations, can we deduce when a character is integer valued? As we will now prove, $\chi^R(g)$ is an integer if g^σ belongs to the same conjugacy class as g for all σ co-prime to E.

The proof goes as follows. From our discussion in the previous section (see equation (4.19)), we know that the Galois action on $\chi^R(g)$ gives $\chi^R(g^\sigma)$. Suppose g^σ belongs to the same conjugacy class as g for all σ, then

$$\chi^R(g^\sigma) = \chi^R(g). \tag{4.21}$$

In other words, $\chi^R(g)$ is invariant under the full Galois group $\mathbb{Z}_E^\times$. Then from our discussion in section 4.2, we know that $\chi^R(g)$ is a rational number. Since $\chi^R(g)$ is an algebraic integer, $\chi^R(g)$ being rational implies that $\chi^R(g)$ is an integer. In fact, it is sufficient to consider a subgroup $\mathbb{Z}_{N_g}^\times \subset \mathbb{Z}_E^\times$ of integers co-prime to N_g, the order of g. We will not prove this here.

A famous example of this is the symmetric group S_N. In S_N, both g and g^σ for any σ co-prime to the order of g are in the same conjugacy class. Therefore, all characters are valued in the integers (see page 103, corollary 2 and following examples in [33]).

4.5. Integrality of sums of characters along columns

As we just learned, the integrality of particular characters requires checking non-trivial properties of powers of group elements. This depends strongly on the group G. In this section we will find that there exists combinations of characters that are integral for general G.

Proposition 4.1. *For any $n \in \mathbb{N}$, define the following sum of irreducible characters*

$$L_n(g) = \sum_R \chi^R(g), \tag{4.22}$$

where the sum is over the set

$$\{R \in \mathrm{Irr}(G) \,|\, d_R = n\} \tag{4.23}$$

of irreducible representations of dimension equal to n.

We will now use Galois theory to show that the sum is an integer for any n, g.

Proof of proposition 4.1. To show this, consider the Galois action on $L_n(g)$ by an element $\sigma \in \mathbb{Z}_E^\times$,

$$\sigma(L_n(g)) = \sum_R \sigma\left(\chi^R(g)\right). \tag{4.24}$$

Using (4.15), we get

$$\sigma(L_n(g)) = \sum_R \chi^{\sigma(R)}(g), \tag{4.25}$$

where $\sigma(R)$ is the permutation action of irreducible representations. Since d_R is an integer, we have

$$d_{\sigma(R)} = \sigma\left(\chi^R(1)\right) = \chi^R(1) = d_R, \tag{4.26}$$

for every $\sigma \in \mathbb{Z}_E^\times$. Therefore, $\sigma(R)$ and R have the same dimension. In particular, the set in equation (4.23) is invariant under the action of $\mathbb{Z}_E^\times$. Equivalently, it is a permutation of terms in the sum (4.22). Therefore, $L_n(g)$ is invariant under the action of the Galois group $\mathbb{Z}_E^\times$. From our discussion in section 4.2, we know that $L_n(g)$ has to be rational. Since $L_n(g)$ is a sum of algebraic integers, it is in fact an integer. We have proved that $L_n(g)$ is an integer for any n and $g \in G$. For $n = d_{R'}$ for some $R' \in \mathrm{Irr}(G)$, this is the Galois theory proof of equation (2.44).

More generally, we can consider the following sum of characters.

Proposition 4.2. *Fix an element $h \in G$, and an integer n. Define the sum of characters*

$$L_{n,h}(g) = \sum_R \chi^R(g), \tag{4.27}$$

where the sum is over the set

$$\left\{R \in \mathrm{Irr}(G) \mid \chi^R(h) = n\right\}. \tag{4.28}$$

If there are no representations satisfying the constraint in the sum above, we define the sum to be zero. We will show that $L_{n,h}(g)$ is an integer for any n, g, h.

In fact, the proof is the same as above.

Proof of proposition 4.2. Consider the Galois action on $L_{n,h}(g)$ by an element $\sigma \in \mathbb{Z}_E^\times$. We have

$$\sigma(L_{n,h}(g)) = \sum_R \sigma\left(\chi^R(g)\right), \tag{4.29}$$

and using (4.15) we get

$$\sigma(L_{n,h}(g)) = \sum_R \chi^{\sigma(R)}(g). \tag{4.30}$$

Since n is an integer we have

$$\sigma\left(\chi^R(h)\right) = \chi^R(h) = n, \tag{4.31}$$

or equivalently $\sigma(R)$ satisfies $\chi^{\sigma(R)}(h) = n$. The Galois action leaves the set in (4.28) invariant. That is, it only permutes the terms in equation (4.27) for $L_{n,h}(g)$. Therefore, $L_{n,h}(g)$ is invariant under the action of the Galois group $\mathbb{Z}_E^\times$. From our discussion in section 4.2, we know that $L_{n,h}(g)$ has to be rational. Since $L_{n,h}(g)$ is a sum of algebraic integers, it is in fact an integer. We have proved that $L_{n,h}(g)$ is an integer for any n and $g, h \in G$.

Proof of theorem 2 (Galois). A small variation of this is to consider a sum over the set

$$\left\{R \in \mathrm{Irr}(G) \mid \frac{\chi^R(h)}{d_R} = q\right\}, \tag{4.32}$$

where R is an irreducible representation, d_R its dimension and $q \in \mathbb{Q}$. Since

$$\sigma\left(\frac{\chi^R(h)}{d_R}\right) = \sigma\left(\frac{\chi^R(h)}{\chi^R(1)}\right) = \frac{\sigma\left(\chi^R(h)\right)}{\sigma\left(\chi^R(1)\right)} = \frac{\chi^{\sigma(R)}(h)}{\chi^{\sigma(R)}(1)} = \sigma(q) = q, \tag{4.33}$$

the sum

$$\sum_R \frac{\chi^R(g)}{d_R}, \tag{4.34}$$

is invariant under Galois action and therefore an integer. Picking $h \in D, S \in \mathrm{Irr}(G)$ such that

$$q = \frac{\chi_D^S}{d_S} \in \mathbb{Z} \tag{4.35}$$

gives theorem 2.

Proposition 4.3. *Even more generally, for a given set of group elements* $H = \{h_1, h_2, \ldots, h_r\}$ *and integers* $N = \{n_1, n_2, \ldots, n_r\}$ *we can define the sum of characters*

$$L_{N,H}(g) = \sum_R \chi^R(g) \tag{4.36}$$

where the sum is over the set

$$\left\{R \in \mathrm{Irr}(G) \,|\, \chi^R(h_1) = n_1, \ldots, \chi^R(h_r) = n_r\right\}. \tag{4.37}$$

If there are no representations satisfying the constraint in the sum above, we define the sum to be zero.

It is straightforward to generalize the arguments used above to show that $L_{N,H}(g)$ is an integer for any choice of H, N and $g \in G$.

Proof of theorem 3* (Galois). As before, we can make contact with section 3 by consider the set

$$\left\{R \in \mathrm{Irr}(G) \,|\, \frac{\chi^R(h_1)}{d_R} = q_1, \ldots, \frac{\chi^R(h_r)}{d_R} = q_r\right\}. \tag{4.38}$$

The sum

$$\sum_R \frac{\chi^R(g)}{d_R}, \tag{4.39}$$

is an integer for $q_1, \ldots, q_r \in \mathbb{Q}$. Picking $h_i \in D_i$ and $S_i \in \mathrm{Irr}(G)$ such that

$$q_i = \frac{\chi_{D_i}^{S_i}}{d_{S_i}} \in \mathbb{Z}, \tag{4.40}$$

gives the generalization of theorem 3. The special case of $r = 2$ is theorem 3.

4.6. Integrality of sums of characters along rows

We can also prove integrality of sums of characters along rows.

Proposition 4.4. *Consider the following sum of characters. Let R_1 be a fixed representation of the group and let n be fixed integer. We define the sum of characters*

$$K_{n,R_1}(R) := \sum_{g,\chi^{R_1}(g)=n} \chi^R(g). \tag{4.41}$$

If there are no conjugacy classes satisfying the constraint in the sum above, we define the sum to be zero. We will show that $K_{n,R_1}(R)$ is an integer for any n, R_1, R.

Proof of proposition 4.4. Consider the Galois action on $K_{n,R_1}(R)$ by an element $\sigma \in \mathbb{Z}_E^{\times}$. We have

$$\sigma(K_{n,R_1}(R)) := \sum_g \sigma\left(\chi^R(g)\right). \tag{4.42}$$

where the sum is over the set

$$\left\{g \in G \,|\, \chi^{R_1}(g) = n\right\}. \tag{4.43}$$

Using (4.19), we get

$$\sigma(K_{n,R_1}(R)) := \sum_g \chi^R(g^\sigma). \tag{4.44}$$

Since n is an integer, we have

$$\sigma\left(\chi^{R_1}(g)\right) = \chi^R(g^\sigma) = n. \tag{4.45}$$

Therefore, g^σ satisfies $\chi^{R_1}(g^\sigma) = n$. The Galois action on $K_{n,R_1}(R)$ is a permutation of terms in the sum (4.41). Therefore, $K_{n,R_1}(R)$ is invariant under the action of the Galois group $\mathbb{Z}_E^{\times}$. From our discussion in section 4.2, we know that $K_{n,R_1}(R)$ has to be rational. Since $K_{n,R_1}(R)$ is a sum of algebraic integers, it is in fact an integer.

We have proved that $K_{n,R_1}(R)$ is an integer for any n and representations R, R_1 of G.

Proposition 4.5. *More generally, for a given set of group representations $\Pi = \{R_1, R_2, \ldots, R_r\}$ and integers $N = \{n_1, n_2, \ldots n_r\}$ we can define the sum of characters*

$$K_{N,\Pi}(R) := \sum_g \chi^R(g) \tag{4.46}$$

where the sum is over the set

$$\left\{g \in G \,|\, \chi^{R_i}(g) = n_i\right\}. \tag{4.47}$$

If there are no group elements satisfying the constraint in the sum above, we define the sum to be zero. This sum is an integer for any choice of Π, N and representation R of G.

It is straightforward to generalize the arguments used above to show that $K_{N,\Pi}(R)$ is an integer.

Proposition 4.6. *For a fixed integer n, another combination of characters that we have considered is*

$$M_n(R) := \sum_C \chi_C^R \tag{4.48}$$

where the sum is over the set

$$\{C \in \mathrm{Cl}(G) \,|\, |C| = n\} \tag{4.49}$$

If no such conjugacy classes exist, we define the sum to be zero. We will show that $M_n(R)$ is an integer for any n and representation R.

Proof of proposition 4.6. To see this, we again use Galois theory. Consider the Galois action on $M_n(R))$ by an element $\sigma \in \mathbb{Z}_E^\times$. We have

$$\sigma(M_n(R)) := \sum_C \sigma\left(\chi_C^R\right) \tag{4.50}$$

Using (4.19), we get

$$\sigma(M_n(R)) := \sum_C \chi_{\sigma(C)}^R \tag{4.51}$$

We will show that $|C| = |\sigma(C)|$. Indeed,

$$hgh^{-1} = l \implies hg^\sigma h = l^\sigma \text{and } hg^\sigma h = l^\sigma \implies hg^{\sigma\tau}h = l^{\sigma\tau} \implies hgh^{-1} = l \tag{4.52}$$

where τ is the inverse of σ in $\mathbb{Z}_E^\times$, and therefore $g^{\sigma'\tau} = g$. This shows that the conjugacy classes containing $|C|$ and $\sigma(C)$ contains the same number of elements. Therefore, the Galois action by σ on (4.48) is just a permutation of the terms in the sum (4.48). Therefore, $M_n(R)$ is invariant under the action of the Galois group $\mathbb{Z}_E^\times$. From our discussion in section 4.2, we know that $M_n(R)$ has to be rational. Since $M_n(R)$ is a sum of algebraic integers, it is in fact an integer.

We now prove theorem 4 and its generalizations.

Proof of theorem 4* (Galois). Using the above, we can prove the generalization of theorem 4. Consider the set

$$\left\{C \in \mathrm{Cl}(G) \,|\, \chi_C^{S_1} = \chi_{D_1}^{S_1}, \ldots, \chi_C^{S_r} = \chi_{D_r}^{S_r}\right\}, \tag{4.53}$$

where $S_i, \in \mathrm{Irr}(G), D_i \in \mathrm{Cl}(G)$ is a collection of irreducible representation such that $\chi_{D_i}^{S_i} \in \mathbb{Z}$. It follows that

$$\sum_{C \in \mathrm{Cl}(G)} \chi_C^R \tag{4.54}$$

is an integer for all $R \in \mathrm{Irr}(G)$, where sum is over the set of conjugacy classes in (4.53). Theorem 4 is the special case of $r = 1$.

5. Counting rows and columns in a number field

In this section, we will find sufficient conditions for the number of rows of the character table belonging to a number field F to be equal to the number of columns of the character table belonging to F. We will begin with a review of general group actions on character tables. We will state and prove Brauer's Lemma on Character tables. We will then use this lemma to show that if the Galois group of a number field containing the character table is cyclic, then the number of integer rows and columns of the character table are equal [31]. We will introduce the minimal number field F_G containing the character table of G and derive necessary conditions for its Galois group to be cyclic. We will refine these results by showing that the the Galois group of F_G being cyclic implies that the number of rows in a subfield of F_G is equal to the number of columns in the same subfield. We will introduce two decorated graphs which capture the number-theoretic properties of rows and columns of the character table, respectively. Finally, we will end this section with various explicitly worked out examples which illustrates our results.

5.1. Counting integer rows and columns

For a group with a cyclic Galois group, the character table has equal number of integer rows and columns. The main results that goes into proving this statement are Brauer's lemma on character tables and the Fundamental Theorem of Galois theory. Let us recall Brauer's lemma on character tables and its proof [31, 34] (see also chapter 18 of [1998]). Before stating the lemma and proving it, let us fix some notation. For a group G, let $\mathrm{Cl}(G)$ be the set of conjugacy classes of G. Let $g_i \in \mathrm{Cl}(G)$ be some representatives of the conjugacy classes of G where $i \in I$ belongs to some index set. Similarly, let $\mathrm{Irr}(G)$ be the set of equivalence classes of irreducible representations of G and let $R_i \in \mathrm{Irr}(G)$ be some representatives where i again belongs to the same index set I. Given this choice of representatives, let X be the character matrix with components

$$X_{ij} = \chi^{R_i}(g_j)\,. \tag{5.1}$$

Brauer's lemma on character tables: Let A and G be groups and suppose A acts on the set $\mathrm{Irr}(G)$ and $\mathrm{Cl}(G)$. Suppose this action satisfies

$$\chi^{R_i}(a(g_j)) = \chi^{a(R_i)}(g_j) \;\; \forall i,j \in I \tag{5.2}$$

where $a \in A$. Then a fixes the same number of irreducible representations as conjugacy classes.

Note that the statement of this lemma is slightly different from that in [31]. Indeed, Brauer's lemma on character table is usually proven assuming the condition

$$\chi^{a(R_i)}(a(g_j)) = \chi^{R_i}(g_j) \;\; \forall i,j \in I \tag{5.3}$$

which is different from what we have in (5.2). Action of the automorphism group on the character table satisfies (5.3) instead of (5.2). For an automorphism $\alpha \in \mathrm{Aut}(G)$ acting on the group, the action on the irreducible representations is defined as

$$\chi^{\alpha(R)}(g) := \chi^{R}(\alpha^{-1}(g))) \tag{5.4}$$

The inverse action of α on g is crucial to get the equality

$$\chi^{(\alpha\beta)(R)}(g) = \chi^{R}\left((\alpha\beta)^{-1}(g)\right) = \chi^{R}\left(\beta^{-1}\left(\alpha^{-1}(g)\right)\right) = \chi^{\beta(R)}\left(\alpha^{-1}(g)\right) = \chi^{\alpha(\beta(R))}(g) \tag{5.5}$$

for $\alpha, \beta \in \mathrm{Aut}(G)$. On the other hand, using (4.15) and (4.19), the action of the Galois group satisfies

$$\chi^{\sigma(R)}(g) = \chi^{R}(g^{\sigma}). \tag{5.6}$$

Therefore, when A is the Galois group action on the characters it satisfies (5.2). To prove Brauer's Lemma on Character tables, let us introduce the matrices $P(a)$ and $Q(a)$ where $a \in A$. The matrix $P(a)$ is defined as follows

$$(P(a))_{ij} = 1 \text{if } \chi^{a(R_i)} = \chi^{R_j} \tag{5.7}$$

$$(P(a))_{ij} = 0 \text{ otherwise} \tag{5.8}$$

The matrix $Q(a)$ is defined as follows

$$(Q(a))_{ij} = 1 \text{if } K_{g_i} = K_{a(g_j)} \tag{5.9}$$

$$(Q(a))_{ij} = 0 \text{ otherwise }. \tag{5.10}$$

We have

$$(P(a)X)_{ij} = \sum_k (P(a))_{ik} X_{kj} = \chi^{a(R_i)}(g_j). \tag{5.11}$$

Similarly, we find that

$$(XQ(a))_{ij} = \sum_k X_{ik} (Q(a))_{kj} = \chi^{R_i}(a(g_j)). \tag{5.12}$$

Therefore, using (5.2) we find that $P(a)X = XQ(a)$. Therefore, $P(a) = XQ(a)X^{-1}$. Hence $\mathrm{Tr}(P(a))$ $=\mathrm{Tr}(Q(a))$. Trace of $P(a)/Q(a)$ precisely gives the number of irreps/conjugacy classes left invariant by the $a \in A$ action. If A is cyclic, then we can apply this lemma to a generator of this cyclic group showing that A acts trivially on equal number of irreps and conjugacy classes.

If A is the Galois group of the Cyclotomic field $\mathbb{Q}(e^{\frac{2\pi i}{E}})$, then the action of this Galois group on the irreducible representations and conjugacy classes satisfies (5.2) (see section 4.3). We are interested in the case where the Galois group $\mathbb{Z}_E^{\times}$ of $\mathbb{Q}(e^{\frac{2\pi i}{E}})$ is cyclic. This is the case if and only if the exponent of the group E is one of the following form: $1, 2, 4, p^k, 2p^k$ where p is an odd prime and k is a positive integer [36]. In particular, groups of these orders have a cyclic Galois group.

While the character table is guaranteed to be contained in the cyclotomic field $\mathbb{Q}(e^{\frac{2\pi i}{E}})$, this field is often much bigger than the minimal field containing the character table. For example, in the case of the group S_n, the character table contains only integers while the dimension of the field $\mathbb{Q}(e^{\frac{2\pi i}{E}})$ as a vector space increases with n. Therefore, it is convenient to define the field F_G for a group G which is the minimal field containing the character table of G. F_G is a subfield of $\mathbb{Q}(e^{\frac{2\pi i}{E}})$. Moreover, F_G is a Galois extension of the rationals. This follows from the Fundamental Theorem of Galois Theory.

Consider the tower of field extensions $F/I/\mathbb{Q}$, where F is a Galois extension and let N be the subgroup of $\mathrm{Gal}(F)$ which fixes the field I.

$$N = \{\sigma \in \mathrm{Gal}(F) \,|\, \sigma(\alpha) = \alpha \;\forall \alpha \in I\}. \tag{5.13}$$

A consequence of the Fundamental Theorem of Galois Theory (see chapter 14.2, theorem 14 (4) in [29]) is that the intermediate field I is a Galois extension if and only if the subgroup N is normal in Gal(F). Then the Galois group of I is

$$\mathrm{Gal}(I) = \mathrm{Gal}(F)/N. \tag{5.14}$$

In our case, $F = \mathbb{Q}(e^{\frac{2\pi i}{E}})$ is a cyclotomic field for which all subgroups of the Galois group are normal. Therefore, all intermediate fields are Galois extensions. In particular, F_G is a Galois extension with Galois group

$$\mathrm{Gal}(F_G) = \mathbb{Z}_E^\times/N \tag{5.15}$$

for some normal subgroup N. If $\mathbb{Z}_E^\times$ is cyclic, then its quotient by a normal subgroup is also cyclic. Therefore, if the exponent of the group G is of the form $1, 2, 4, p^k, 2p^k$, where p is an odd prime and k is a positive integer, then $\mathrm{Gal}(F_G)$ is cyclic. For the rest of this section, we will assume that $\mathrm{Gal}(F_G)$ is cyclic.

From Brauer's lemma on character tables applied to the case of $\mathrm{Gal}(F_G)$ acting on irreducible representations and conjugacy classes, it is clear that the number of conjugacy classes left invariant by the full Galois group is same as the number of irreducible representations left invariant by it. From the Fundamental Theorem of Galois theory (see section (4.2) for a discussion), if an element of the field extension F_G is left invariant under the full Galois group, then it should be rational. Since characters are algebraic integers, if $\chi_{R_i}(g_j)$ is invariant under the Galois group then it should be an integer. Therefore, we get the following theorem.

Equal number of integer rows and columns: For a finite group G, let F_G be the minimal field containing the character table of G. If the Galois group of F_G is cyclic, then the character table of G contains equal number of integer rows and integer columns.

A version of this result is stated and proved in [31, theorem 3.3].

5.2. *Counting rows and columns in a number field*

Given the results in the previous section, it is natural to wonder whether the number of rows and columns of the character table belonging to a given subfield of F_G is the same. This is indeed the case. To understand this, note that Brauer's lemma on character tables is not just about invariance under the full cyclic group A, but invariance under the action of any element $a \in A$. In fact, if $\chi_{R_i}(g_j)$ is invariant under the action of $a \in A$, then it is invariant under the full cyclic subgroup generated by a. Therefore, it is natural to consider invariants under subgroups of A. Then Brauer's lemma tells us that the number of rows and columns of the character table invariant under a given subgroup, say B, of the cyclic group A is the same.

When $A = \mathrm{Gal}(F_G)$ is the Galois group of F_G, then the Fundamental Theorem of Galois theory states that subgroups of $\mathrm{Gal}(F_G)$ and subfields of F_G are in one-to-one correspondence. For every subgroup of $H \subset \mathrm{Gal}(F_G)$, the corresponding subfield is the set of elements in F_G fixed by the action of H. In the opposite direction, given a subfield, the corresponding subgroup of the Galois group is that which fixes the chosen subfield. Therefore, by identifying columns and rows of the character table invariant under a subgroup of the Galois group, we are equivalently looking at rows and columns of the character table belonging to a subfield of F_G. Using Brauer's lemma in this case we get the following theorem.

Equal number of rows and columns in a subfield: For a finite group G, let F_G be the minimal field containing the character table of G. If the Galois group of F_G is cyclic, then the number

Table 1. Character table of $\mathbb{Z}_6$.

	1	g	g^2	g^3	g^4	g^5
R_1	1	1	1	1	1	1
R_2	1	$e^{\frac{2\pi i}{6}}$	$e^{\frac{2\pi i2}{6}}$	-1	$e^{\frac{2\pi i4}{6}}$	$e^{\frac{2\pi i5}{6}}$
R_3	1	$e^{\frac{2\pi i}{3}}$	$e^{\frac{2\pi i2}{3}}$	1	$e^{\frac{2\pi i}{3}}$	$e^{\frac{2\pi i2}{3}}$
R_4	1	-1	1	-1	1	-1
R_5	1	$e^{\frac{2\pi i2}{3}}$	$e^{\frac{2\pi i}{3}}$	1	$e^{\frac{2\pi i2}{3}}$	$e^{\frac{2\pi i}{3}}$
R_6	1	$e^{\frac{2\pi i5}{6}}$	$e^{\frac{2\pi i4}{6}}$	-1	$e^{\frac{2\pi i2}{6}}$	$e^{\frac{2\pi i}{6}}$

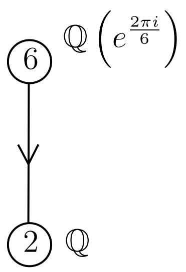


Figure 1. Graph $G1$ for $\mathbb{Z}_6$.

of rows in the character table with entries in a subfield $I \subset F_G$ is the same as the number of columns with entries in I.

This equality can be depicted using a decorated graph. Given the character table of a group, we define a graph with vertices labelled by the field F_G and its subfields. Two vertices labelled by fields F, K have a directed edge pointing from K to F if F is a subfield of K. Once we have this graph, we can further decorate the vertices in the following two ways

1. $G1$: Label the vertex corresponding to the field F as the number of rows of the character table contained in that field.
2. $G2$: Label the vertex corresponding to the field F as the number of columns of the character table contained in that field.

Then our results above show that the decorated graphs G1 and G2 are the same. Let us study a few examples to see this explicitly.

5.2.1. $\mathbb{Z}_6$. Consider the cyclic group $\mathbb{Z}_6$ with character table given in table 1. We find that all entries in the character table are contained in the field $\mathbb{Q}(e^{\frac{2\pi i}{6}})$. Since the primitive 6^{th} roots of unity are negatives of primitive 3^{rd} roots of unity, $\mathbb{Q}(e^{\frac{2\pi i}{6}}) \cong \mathbb{Q}(e^{\frac{2\pi i}{3}})$. Therefore, the only non-trivial subfield is $\mathbb{Q}$. The Galois group of $\mathbb{Q}(e^{\frac{2\pi i}{6}})$ is $\mathbb{Z}_2$. The character table has some rows and columns belonging to the subfield $\mathbb{Q}$. The graph $G1$ defined above constructed for this particular case is given in figure 1. The number in the vertices of the graph is the number of rows of the character table contained in the field attached to the vertex. It is easily verified that graph $G2$ constructed as defined above is same as this graph.

5.2.2. $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$. The group $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ can be represented by the following generators and relations

$$\mathbb{Z}_7 \rtimes \mathbb{Z}_3 = \langle a, b | a^7 = b^3 = 1, bab^{-1} = a^4. \rangle \tag{5.16}$$

It has the character table given in table 2. We find that all entries in the character table are contained in the field $\mathbb{Q}(e^{\frac{2\pi i}{3}}, i\sqrt{7})$. $i\sqrt{7}$ is contained in the cyclotomic field $\mathbb{Q}(e^{\frac{2\pi i}{7}})$. This follows from the quadratic Gauss sum [1998]

$$\sum_{n=0}^{p-1} e^{\frac{2\pi i n^2}{p}} = \begin{cases} \sqrt{p} & \text{if } p = 1 \bmod 4 \\ i\sqrt{p} & \text{if } p = 3 \bmod 4 \end{cases} \tag{5.17}$$

where p is prime. Therefore, the smallest cyclotomic field containing the entries of the character table is $\mathbb{Q}(e^{\frac{2\pi i}{21}})$. The Galois group of this number field is $\mathbb{Z}_{21}^\times \simeq \mathbb{Z}_7^\times \times \mathbb{Z}_3^\times \simeq \mathbb{Z}_6 \times \mathbb{Z}_2$. Let σ_1 and σ_2 be generators of $\mathbb{Z}_6$ and $\mathbb{Z}_2$ factors of the Galois group, respectively with action on the number field given by

$$\sigma_1\left(e^{\frac{2\pi i}{21}}\right) = e^{\frac{2\pi i 10}{21}},\ \sigma_2\left(e^{\frac{2\pi i}{21}}\right) = e^{\frac{2\pi i 8}{21}} \tag{5.18}$$

σ_2 acts trivially on $i\sqrt{7}$ while σ_1 acts as $\sigma_1(i\sqrt{7}) = -i\sqrt{7}$ and σ_1 acts trivially on $e^{\frac{2\pi i}{3}}$ while σ_2 acts as $\sigma_2(e^{\frac{2\pi i}{3}}) = e^{\frac{2\pi i 2}{3}}$. Therefore, both σ_1 and σ_2 have order two actions on the character table. Let us find the rows fixed by σ_1 and σ_2.

$$\sigma_1 \text{ fixes the rows } R_1, R_2, R_3 \tag{5.19}$$

$$\sigma_2 \text{ fixes the rows } R_1, R_4, R_5 \tag{5.20}$$

Similarly, let us find the columns fixed by σ_1 and σ_2.

$$\sigma_1 \text{ fixes the columns } 1, a, b^2 \tag{5.21}$$

$$\sigma_2 \text{ fixes the columns } 1, b, a^3. \tag{5.22}$$

Note that σ_1 fixes equal number of rows and columns. The same is true for σ_2. This follows from Brauer's lemma on character tables as σ_1 and σ_2 generate cyclic subgroups of the Galois group. The rows and columns fixed by the full Galois group are

$$\{\text{rows fixed by } \sigma_1\} \cap \{\text{rows fixed by } \sigma_2\} = R_1 \tag{5.23}$$

$$\{\text{columns fixed by } \sigma_1\} \cap \{\text{columns fixed by } \sigma_2\} = 1. \tag{5.24}$$

The graph $G1$ defined above constructed for this particular case is given in figure 2. It is easily verified that graph $G2$ constructed as defined above is same as this graph.

This example shows that even if the Galois group is not cyclic, the number of integer rows and columns in the character table can be equal.

5.2.3. $\mathbb{Z}_{24} \rtimes \mathbb{Z}_2$. Let us consider the group $\mathbb{Z}_{24} \rtimes \mathbb{Z}_2$ of order 48 which is represented as a Polycyclic group with 5 generators $f1, f2, f3, f4, 5$ in GAP (SmallGroup(48,6)) [38, 39]. The character table is given by table 3 where $A = -2i, B = \sqrt{3}, C = -e^{\frac{2\pi i}{24}} - e^{\frac{2\pi i 11}{24}}, D = -e^{\frac{2\pi i 17}{24}} - e^{\frac{2\pi i 19}{24}}$.

A belongs to the field $\mathbb{Q}(e^{\frac{2\pi i}{4}})$, B belongs to the field $\mathbb{Q}(e^{\frac{2\pi i}{12}})$ and C, D belongs to the field $\mathbb{Q}(e^{\frac{2\pi i}{24}})$. Therefore, all entries in the character table are contained in the field $\mathbb{Q}(e^{\frac{2\pi i}{24}})$. The character table has some rows and columns belonging to the subfields $\mathbb{Q}(e^{\frac{2\pi i}{12}}), \mathbb{Q}(e^{\frac{2\pi i}{4}})$ and $\mathbb{Q}$.

Table 2. Character table of $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$.

	1	a	b	b^2	a^3
R_1	1	1	1	1	1
R_2	1	$e^{\frac{2\pi i 2}{3}}$	1	$e^{\frac{2\pi i}{3}}$	1
R_3	1	$e^{\frac{2\pi i}{3}}$	1	$e^{\frac{2\pi i 2}{3}}$	1
R_4	3	0	$\frac{-1+i\sqrt{7}}{2}$	0	$\frac{-1-i\sqrt{7}}{2}$
R_5	3	0	$\frac{-1-i\sqrt{7}}{2}$	0	$\frac{-1+i\sqrt{7}}{2}$

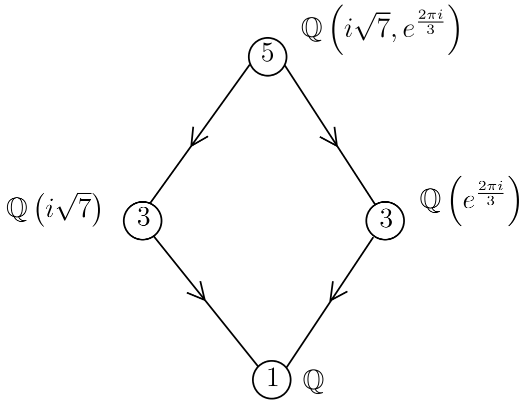


Figure 2. Graph $G1$ for $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$.

Table 3. Character table of $\mathbb{Z}_{24} \rtimes \mathbb{Z}_2$.

	1	$f1$	$f2$	$f3$	$f4$	$f5$	$f1f2$	$f2f4$	$f2f5$	$f3f5$	$f4f5$	$f2f3f5$,	$f2f4f5$	$f3f4f5$	$f2f3f4f5$
R_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
R_2	1	−1	−1	1	1	1	1	−1	−1	1	1	−1	−1	1	−1
R_3	1	−1	1	1	1	1	−1	1	1	1	1	1	1	1	1
R_4	1	1	−1	1	1	1	−1	−1	−1	1	1	−1	−1	1	−1
R_5	2	0	0	−2	2	2	0	0	0	−2	2	0	0	−2	0
R_6	2	0	−2	2	2	−1	0	−2	1	−1	−1	1	1	−1	1
R_7	2	0	2	2	2	−1	0	2	−1	−1	−1	−1	−1	−1	−1
R_8	2	0	A	0	−2	2	0	−A	A	0	−2	A	−A	0	−A
R_9	2	0	−A	0	−2	2	0	A	−A	0	−2	−A	A	0	A
R_{10}	2	0	0	−2	2	−1	0	0	B	1	−1	−B	B	1	−B
R_{11}	2	0	0	−2	2	−1	0	0	−B	1	−1	B	−B	1	B
R_{12}	2	0	A	0	−2	−1	0	−A	C	−B	1	D	−C	B	−D
R_{13}	2	0	A	0	−2	−1	0	−A	D	B	1	C	−D	−B	−C
R_{14}	2	0	−A	0	−2	−1	0	A	−D	B	1	−C	D	−B	C
R_{15}	2	0	−A	0	−2	−1	0	A	−C	−B	1	−D	C	B	D

The graph $G1$ defined above constructed for this particular case is given in figure 3. It is easily verified that graph $G2$ constructed as defined above is same as this graph. Note that in this case the Galois group of the cyclotomic field $\mathbb{Q}(e^{\frac{2\pi i}{24}})$ is $\mathbb{Z}_{24}^{\times} \simeq \mathbb{Z}_8^{\times} \times \mathbb{Z}_3^{\times} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ which is not cyclic. Let $\sigma_1, \sigma_2, \sigma_3$ be the three generators of the cyclic factors which act as

$$\sigma_1\left(e^{\frac{2\pi i}{24}}\right) = e^{\frac{2\pi i 13}{24}},\ \sigma_2\left(e^{\frac{2\pi i}{24}}\right) = e^{\frac{2\pi i 11}{24}},\ \sigma_3\left(e^{\frac{2\pi i 5}{24}}\right) = e^{\frac{2\pi i 13}{24}}. \tag{5.25}$$

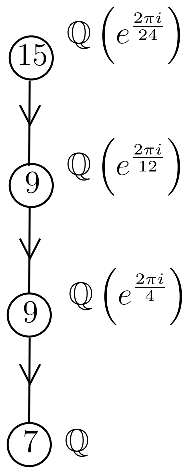


Figure 3. Graph $G1$ for $\mathbb{Z}_{24} \rtimes \mathbb{Z}_2$.

Table 4. Character table of $Q_8 \rtimes \mathbb{Z}_4$.

	1	$f1$	$f2$	$f3$	$f4$	$f5$	$f1f2$	$f1f4$	$f2f4$	$f3f4$	$f4f5$	$f1f2f4$	$f1f2f5$	$f1f2f4f5$
R_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
R_2	1	−1	1	1	1	1	−1	−1	1	1	1	−1	−1	−1
R_3	1	1	−1	1	1	1	−1	1	−1	1	1	−1	−1	−1
R_4	1	−1	−1	1	1	1	1	−1	−1	1	1	1	1	1
R_5	1	A	1	1	−1	1	A	−A	−1	−1	−1	−A	A	−A
R_6	1	−A	1	1	−1	1	−A	A	−1	−1	−1	A	−A	A
R_7	1	A	−1	1	−1	1	−A	−A	1	−1	−1	A	−A	A
R_8	1	−A	−1	1	−1	1	A	A	1	−1	−1	−A	A	−A
R_9	2	0	0	−2	2	2	0	0	0	−2	2	0	0	0
R_{10}	2	0	0	−2	−2	2	0	0	0	2	−2	0	0	0
R_{11}	2	0	0	0	2	−2	B	0	0	0	−2	B	−B	−B
R_{12}	2	0	0	0	2	−2	−B	0	0	0	−2	−B	B	B
R_{13}	2	0	0	0	−2	−2	C	0	0	0	2	−C	−C	C
R_{14}	2	0	0	0	−2	−2	−C	0	0	0	2	C	C	−C

Similar to the previous example, we can verify that equal number of rows and columns are left invariant by the full Galois group even though the Galois group is not cyclic. However, as the next example shows, there are groups for which the number of integer rows and the number of integer columns of the character table are different.

5.2.4. $Q_8 \rtimes \mathbb{Z}_4$. The character table for a semi-direct product of Q_8 and $\mathbb{Z}_4$ denoted in the GAP library as SmallGroup(32,10) is given by table 4, where $A = i, B = e^{\frac{2\pi i}{8}} + e^{\frac{2\pi i 3}{8}} = i\sqrt{2}$ and $C = \sqrt{2}$ [38, 39]. We see that the minimal field containing the character table is $F_G = \mathbb{Q}(e^{\frac{2\pi i}{8}}) = \mathbb{Q}(i, \sqrt{2})$. The Galois group for this field is $\mathbb{Z}_2 \times \mathbb{Z}_2$. We will denote the elements of this group as $\sigma_1, \sigma_3, \sigma_5, \sigma_7$. These elements act as follows

$$\sigma_a\left(e^{\frac{2\pi i}{8}}\right) = e^{\frac{2\pi i a}{8}},\ a = 1, 3, 5, 7 \tag{5.26}$$

σ_3, σ_5 and σ_7 generate three $\mathbb{Z}_2$ subgroups of the full Galois group. These subgroups fixes the following subfields of $F_G = \mathbb{Q}(i, \sqrt{2})$. σ_3 fixes the field $\mathbb{Q}(i\sqrt{2})$, σ_5 fixes the field $\mathbb{Q}(i)$ and σ_7 fixes the field $\mathbb{Q}(\sqrt{2})$. From this information, we can look at the rows and columns of the character table fixed under the action of the generators of $\mathbb{Z}_2 \times \mathbb{Z}_2$, say σ_5 and σ_7

$$\sigma_5 \text{ fixes the rows } R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8, R_9, R_{10} \tag{5.27}$$

$$\sigma_7 \text{ fixes the rows } R_1, R_2, R_3, R_4, R_9, R_{10}, R_{13}, R_{14}. \tag{5.28}$$

Similarly, let us find the columns fixed by σ_5 and σ_7

$$\sigma_5 \text{ fixes the columns 1, f 1, f2, f3, f4, f5, f 1f4, f2f4, f3f4, f4f5} \tag{5.29}$$

$$\sigma_7 \text{ fixes the columns 1, f2, f3, f4, f5, f2f4, f3f4, f4f5.} \tag{5.30}$$

We find that σ_5 fixes the same number of rows and columns implying that the number of rows and columns contained in the field $\mathbb{Q}(i)$ are equal. Similarly, we find that σ_7 fixes the same number of rows and columns implying that the number of rows and columns contained in the field $\mathbb{Q}(\sqrt{2})$ are equal. This is consistent with Brauer's lemma on character tables. Indeed, the subgroup of the full Galois group generated by σ_5 is isomorphic to the cyclic group $\mathbb{Z}_2$. Therefore, Brauer's lemma applied to this group tells us that the number of rows and number of columns columns fixed by σ_5 are equal. This implies that the number of rows and number of columns columns contained in $\mathbb{Q}(i)$ are equal. Similar argument holds for σ_7 and σ_3.

Since σ_5 and σ_7 generate the full Galois group, the rows fixed under the action of the full Galois group are given by

$$\{\text{rows fixed by}\sigma_5\} \cap \{\text{rows fixed by}\sigma_7\} = R_1, R_2, R_3, R_4, R_9, R_{10} \tag{5.31}$$

which are precisely the 6 integer rows of the character table. Similarly, the columns fixed under under the action of the full Galois group are given by

$$\{\text{columns fixed by}\sigma_5\} \cap \{\text{columns fixed by}\sigma_7\} = 1, f2, f3, f4, f5, f2f4, f3f4, f4f5 \tag{5.32}$$

which are precisely the 8 integer columns of the character table. Therefore, this example explicitly shows that for non-cyclic Galois groups, the number of rows and number of columns fixed by the Galois group action may not be equal. The graphs $G1$ in figure 4 and $G2$ in figure 5 below illustrates the result above.

6. Constructing row-column duality

In this section we will give an algorithm for going from the G-CTST to the $R(G)$-CTST and vice versa.

In the G-CTST we have a basis of elements T_C, where $C \in \mathrm{Cl}(G)$ is a conjugacy class of G. The size of the conjugacy class is $|C|$

$$T_C = \sum_{g \in C} g. \tag{6.1}$$

The Frobenius form

$$g\left(T_{C_1} T_{C_2}\right) = \frac{|C_1|}{|G|} \delta_{C_1, S(C_2)}, \tag{6.2}$$

where $S(C_2)$ is the conjugacy class which contains the inverses of the elements in C_1. For real groups such as S_n, $S(C) = C$.

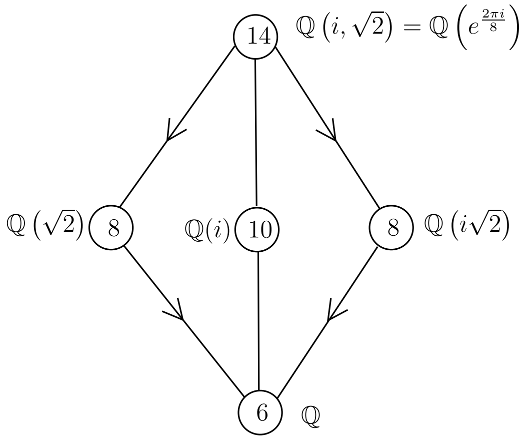


Figure 4. Graph $G1$ for $Q_8 \rtimes \mathbb{Z}_4$.

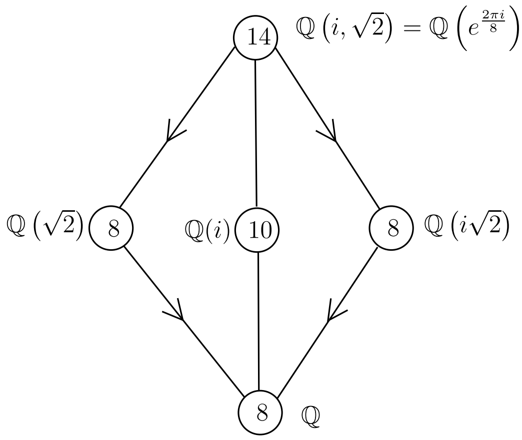


Figure 5. Graph $G2$ for $Q_8 \rtimes \mathbb{Z}_4$.

The amplitude for three circles going to the vacuum is

$$Z^G_{h=0;C_1,C_2C_3} = \frac{1}{|G|}\delta\left(T_{C_1}T_{C_2}T_{C_3}\right) = \frac{1}{|G|^2}\sum_R \frac{|C_1||C_2||C_3|}{d_R}\chi^R_{C_1}\chi^R_{C_2}\chi^R_{C_3}, \tag{6.3}$$

where χ^R_C is the character of a fixed element $g \in C$.

It is convenient to define $\widehat{T_C} = T_C\sqrt{\frac{|G|}{|C|}}$. On this, the bilinear form is

$$g\left(\widehat{T_{C_1}}, \widehat{T_{C_2}}\right) = \delta_{C_1, S(C_2)}, \tag{6.4}$$

and the three-to-null amplitude is

$$\widehat{Z}^G_{h=0;C_1,C_2C_3} = \frac{1}{|G|}\delta\left(\widehat{T_{C_1}}\widehat{T_{C_2}}\widehat{T_{C_3}}\right) = \frac{1}{\sqrt{|G|}}\sum_R \frac{\sqrt{|C_1||C_2||C_3|}}{d_R}\chi^R_{C_1}\chi^R_{C_2}\chi^R_{C_3}. \tag{6.5}$$

We also know that the eigenvectors of T_C—viewed as an operator by multiplication in the centre of the group algebra, are labelled by a label R and their eigenvalues are given by

$$E_R(T_C) = \frac{|C|}{d_R}\chi^R_C. \tag{6.6}$$

Note that we can take the point of view that we have a bunch of commuting matrices, with non-degenerate pairing. R is just a label for the common eigenvectors. It follows, for the rescaled versions,

$$E_R\left(\widehat{T_C}\right) = \frac{\sqrt{|G||C|}}{d_R}\chi^R_C. \tag{6.7}$$

As we will now prove,

$$\mathrm{Max}_R\left|E_R\left(\widehat{T_C}\right)\right| = \sqrt{|G||C|}. \tag{6.8}$$

In particular, we will prove that

$$\left|\chi^R_C\right| \leqslant d_R, \tag{6.9}$$

from which the result follows. Consider an irreducible representation $D^R(g)$ of an element $g \in G$ with order k ($g^k = 1$). The representation is similar to a diagonal matrix $\mathrm{Diag}(\lambda_1, \ldots, \lambda_{d_R})$. Since $g^k = 1$ we have $(D^\lambda(g))^k = 1$ and therefore λ_i are kth roots of unity. It follows that $|\lambda_i| = 1$. Now consider

$$\left|\chi^R(g)\right| = |\lambda_1 + \ldots \lambda_{d_R}| \leqslant |\lambda_1| + \cdots + |\lambda_{d_R}| = d_R, \tag{6.10}$$

where we used the triangle inequality in the second to last step. Equation (6.8) follows.

The handle creation operator in the idempotent basis is

$$\Pi = \sum_R \frac{|G|^2}{d_R^2} P_R. \tag{6.11}$$

Let us call

$$\Pi_R = \frac{|G|^2}{d_R^2}. \tag{6.12}$$

Let us now define a dualisation of the eigenvalues by using the components of the handle operator and the maximum of the eigenvalues

$$\begin{aligned}\widetilde{E}_R\left(\widehat{T}_C\right) &= \frac{1}{\sqrt{\Pi_R}} E_R\left(\widehat{T}_C\right) \frac{1}{\mathrm{Max}_R\left|E_R\left(\widehat{T}_C\right)\right|} \\ &= \frac{d_R}{|G|}\frac{\sqrt{|G||C|}\chi^R_C}{d_R}\frac{1}{\sqrt{|G||C|}} \\ &= \frac{\chi^R_C}{|G|}.\end{aligned} \tag{6.13}$$

Define the constructive dual amplitudes, by using the dualised eigenvalues, fixing a triple of eigenvectors R_1, R_2, R_3 and summing over the label $C \in \mathrm{Cl}(G)$:

$$\widetilde{Z}^G_{h=0;R_1,R_2,R_3} = \sum_C |G|.\widetilde{E}_{R_1}(\widehat{T}_C)\widetilde{E}_{R_2}(\widehat{T}_C)\widetilde{E}_{R_3}(\widehat{T}_C).(\mathrm{Max}_R(\left|E_R(\widehat{T}_C)\right|)^2. \tag{6.14}$$

Now plug in all the ingredients of the duality formula

$$\begin{aligned}\widetilde{Z}^{G}_{h=0;R_1,R_2,R_3} &= \sum_R |G|.\frac{\chi^{R_1}_C}{|G|}\frac{\chi^{R_2}_C}{|G|}\frac{\chi^{R_3}_C}{|G|}.|C||G| \\ &= \sum_R \frac{|C|}{|G|}\chi^{R_1}_C\chi^{R_2}_C\chi^{R_3}_C. \end{aligned} \tag{6.15}$$

Note that the dualized amplitudes are exactly the amplitudes of the $R(G)$-CTST

$$\widetilde{Z}^{G}_{h=0;R_1,R_2,R_3} = Z^{R(G)}_{h=0;R_1,R_2,R_3}. \tag{6.16}$$

The structure constants of the dual ($R(G)$) theory have been constructed using a formula involving an appropriate form of the eigenvalues of the class-algebra theory. We hope that the same construction (in terms of eigenvalues) starting from the dual theory should produce back the class algebra theory.

So let us start now with the $R(G)$ theory. Here we have

$$\delta(a_{R_1}a_{R_2}) = \delta_{R_1,S(R_2)} \tag{6.17}$$

where $S(R_2)$ is the complex conjugate of R_2. For real groups such as S_n, $S(R_2)=R_2$. The three-to-null amplitude is

$$Z^{R(G)}_{h=0;R_1,R_2,R_3} = \sum_C \frac{|C|}{|G|}\chi^{R_1}_C\chi^{R_2}_C\chi^{R_3}_C. \tag{6.18}$$

Look at the eigenvalues

$$E_C(a_R) = \chi^R_C. \tag{6.19}$$

The maximum of the absolute value is

$$\mathrm{Max}_C|E_C(a_R)| = d_R, \tag{6.20}$$

using the same argument as in the previous case. The handle creation eigenvalues are

$$\Pi_C = \frac{|G|}{|C|}. \tag{6.21}$$

Use same formula as above for rescaling the eigenvalue

$$\begin{aligned}\widetilde{E}_C(a_R) &= \frac{1}{\sqrt{\Pi_C}}.E_C(a_R).\frac{1}{\mathrm{Max}_C(|E_K(a_R)|)} \\ &= \frac{\sqrt{|C|}}{\sqrt{|G|}}\frac{\chi^R_C}{d_R}. \end{aligned} \tag{6.22}$$

The dualized amplitude is defined as

$$\widetilde{Z}^{R(G)}_{h=0;C_1,C_2,C_3} = \sum_R |G|.\widetilde{E}_{C_1}(a_R)\widetilde{E}_{C_2}(a_R)\widetilde{E}_{C_3}(a_R).(\mathrm{Max}_C|E_C(a_R)|)^2. \tag{6.23}$$

Note that this is the same dualization formula as (6.14) by taking the $R(G)$ as the starting point (rather than the conjugacy class algebra as the starting point). Simplify

$$\begin{aligned}\widetilde{Z}^{R(G)}_{h=0;C_1,C_2,C_3} &= \sum_R |G|.\frac{\sqrt{|C|}}{\sqrt{|G|}}\frac{\chi^R_{C_1}}{d_R}\frac{\sqrt{|C|}}{\sqrt{|G|}}\frac{\chi^R_{C_2}}{d_R}\frac{\sqrt{|C_3|}}{\sqrt{|G|}}\frac{\chi^R_{C_3}}{d_R}.d_R^2 \\ &= \sum_R \frac{1}{\sqrt{|G|}}\sqrt{|C_1|}\sqrt{|C_2|}\sqrt{|C_3|}\frac{\chi^R_{C_1}\chi^R_{C_2}\chi^R_{C_3}}{d_R}. \end{aligned} \tag{6.24}$$

Note that the amplitude defined by dualising the $R(G)$ theory is exactly the one of the G-CTST

$$\widetilde{Z}^{R(G)}_{h=0;C_1,C_2,C_3} = \widehat{Z}^{G}_{h=0;C_1,C_2,C_3}. \tag{6.25}$$

So we have applied a duality transformation—for any system of k commuting $k\times k$ matrices with non-degenerate bilinear form—defined by (6.23).

We remark that the above dualisation process lands at the amplitude in the normalized basis for the conjugacy class algebra. To get the amplitude in the original, unnormalized, basis we use the combination

$$\sum_R |G|.\frac{\widetilde{E}_{C_1}(a_R)\widetilde{E}_{C_2}(a_R)\widetilde{E}_{C_3}(a_R)}{\sqrt{\Pi_{C_1}\Pi_{C_2}\Pi_{C_3}}}.\left(\mathrm{Max}_C\,|E_C(a_R)|\right)^2 = \sum_R \frac{1}{|G|}|C_1||C_2||C_3|\frac{\chi^R_{C_1}\chi^R_{C_2}\chi^R_{C_3}}{d_R}. \tag{6.26}$$

This is equal to the unnormalized amplitude $Z_{h=0;C_1,C_2,C_3}$.

These dualisation procedures have some intriguing properties. Let us call the dualisation procedure (6.14)

$$S_1\left(Z^G\right) = Z^{R(G)}, \tag{6.27}$$

and the dualisation (6.23)

$$S_2\left(Z^{R(G)}\right) = \widehat{Z}^G. \tag{6.28}$$

Taking the point of view that we have a set of commuting matrices with non-degenerate bilinear pairing, we observed that S_1, S_2 are actually the same procedure. However, since

$$S_2\left(S_1\left(Z^G\right)\right) = \widehat{Z}^G \neq Z^G, \tag{6.29}$$

they are not exact inverses. On the other hand, let the dualisation (6.26)

$$S_3\left(Z^{R(G)}\right) = Z^G. \tag{6.30}$$

It has the property that

$$S_3\left(S_1\left(Z^G\right)\right) = Z^G, \tag{6.31}$$

but S_3 is not the same procedure as S_1. It would be interesting to investigate if there is a dualisation procedure that satisfies both properties in the future.

7. Conclusions

In this paper we have introduced a new class of combinatorial topological string theories ($R(G)$-CTSTs) based on fusion algebras $R(G)$ of a finite group G. They are dual to G-CTSTs based on Dijkgraaf–Witten theories (G-TQFTs). G-CTSTs have been used to give constructive proofs of integrality properties of characters, otherwise only accessible by Galois theoretic techniques. Integrality properties derived from G-CTSTs concern column sums or sums over irreducible normalized characters for a fixed conjugacy class. Integrality properties derived from $R(G)$-CTSTs concern row sums, or fixed irreducible characters summed over conjugacy classes. In this sense, G-CTSTs and $R(G)$-CTSTs are dual. We have made a start towards an explicit construction of the duality map, but there remain issues to resolve in using this construction in getting the map of integrality results from rows to columns.

We reviewed the application of Galois theory to character tables and showed how it can be used to derive refined integrality properties of character sums. Naturally, we asked if such refinements can be constructed using CTST techniques. We answered this question in the affirmative for large classes of refined sums. The derivation of these refinements in the context of CTSTs are novel for G-CTSTs as well as $R(G)$-CTSTs. The result relies on a lemma on rationality of inverse Vandermonde matrices, which we proved in appendix B.

Following the observation of the refined integer sums, we defined generalized partitions of the integer row and column sums of character tables. The row sums of characters are positive integers while the column sums are integers, not necessarily positive. The row sums are the subject of a combinatorial construction question highlighted in [16]. We showed that the row and column sums naturally come equipped with the structure of generalised partitions, determined by integrality properties of the character table. A refinement of the combinatorial question is to find combinatorial constructions for these generalised partitions. It is also worth observing that the column sums of the normalized characters are positive integers. These would also admit generalised partitions along similar lines.

Two very natural extensions of our work involve extending the type of TQFTs considered. G-CTSTs have been extended to twisted G-CTSTs defined by twisted Dijkgraaf-Witten theories (twisted G-TQFTs) in [15]. These are defined using a group G and a 3-cocycle in $H^3(G, U(1))$. It would be interesting to extend our notion of Row-Column duality of CTSTs to twisted G-TQFTs. Such an extension will provide a physical setting to analyse number theoretic properties of characters of projective representations of a group.

The TQFTs considered in this paper were of the closed type. They compute topological invariants of manifolds with closed boundaries. A TQFT that computes topological invariants of manifolds which can have corners/open boundaries is called an open-closed TQFT. For closed TQFTs determined by semi-simple Frobenius algebras, which includes G-TQFTs, there exists a classification of the possible consistent extensions to an open-closed TQFT [40]. An extension of the closed G-TQFT is described by the full non-commutative group algebra $\mathbb{C}(G)$. It would be interesting to consider the open-closed extensions of $R(G)$-TQFTs and their implications for row–column duality. A closely related direction is to consider extended operators in 2D TQFTs. It would be interesting to study the number theoretic properties of the group G contained in the correlation functions of line operators in the G-TQFT and $R(G)$-TQFT. Properties of the gauge group that can be constructed from topological line and surface operators of higher-dimensional quantum field theories were studied in [41].

The Galois group naturally acts on the rows and columns of the character table. Invariants under this action correspond to integer rows and columns, respectively. The action of the Galois groups on irreducible representations and conjugacy classes can be used to define symmetries (in the sense used in [2021, 43]) of the G-TQFT and $R(G)$-TQFT respectively [44]. It will be interesting to study states in the Hilbert space of the G-TQFT and $R(G)$-TQFT which are invariant under this symmetry and relate them to the various sums of characters studied in this paper.

We explained how Galois theory gives a description of sufficient conditions of the number of integer rows to agree with the number of integer columns of the character table of a group G. More generally, we considered the number of rows and columns belonging to particular extensions of the rationals. Formulating a CTST approach to these kinds of integrality questions is an interesting problem.

The G-CTST construction in subsection 3.1 proves that sums of normalized characters over sets of irreducible representations

$$\left\{ R \in \mathrm{Irr}(G) \mid \frac{\chi^R_{D_i}}{d_R} = \frac{\chi^{S_i}_{D_i}}{d_{S_i}} \right\} \tag{7.1}$$

are integral. Meanwhile, Galois theory allowed us to prove integrality over sums of irreducible representations

$$\left\{ R \in \mathrm{Irr}(G) \mid \chi^R_{D_i} = n_i \right\}, \tag{7.2}$$

for integers n_i. A good future problem is to reconstruct such level sets within the framework of CTST.

Another problem in finite group representation theory which has a direct link with CTST constructions is Harada's conjecture, which we now describe. The ratio of determinants of the matrix of structure constants in G-TQFT and $R(G)$-TQFT is given by the product of conjugacy class sizes divided by the product of dimensions of irreducible representations

$$\frac{\prod_C |C|}{\prod_R d_R}. \tag{7.3}$$

This is explained in appendix D. The integrality of this ratio is the subject of Harada's conjecture. A better understanding of the construction of row-column duality (developing section 6) may provide a physical perspective on the conjecture. A broader question is to characterize the invariants of the row-column duality between G-CTST and $R(G)$-CTST.

A similar philosophy to the present paper of using stringy constructions to approach problems in combinatorial representation theory was taken in [45]. Bipartite graphs connected to string theory were used to give a realisation of Kronecker coefficients, which are tensor product multiplicities for symmetric groups, in terms of the counting of null vectors of combinatorially constructed integer matrices. The construction of such null vectors is amenable to known combinatorial algorithms for integer matrices. The null vector equations are related to eigenvalue problems which can be solved using efficient quantum algorithms based on quantum phase estimation [46] (see also closely related work [47]). This is along the lines of the use of quantum algorithms for efficient solution of linear algebra problems [48]. The CTST algorithms of the present paper, which are based on matrix equations, would similarly be amenable to quantum algorithms. Finite groups arise as hidden symmetries in correlator computations for matrix and tensor models (see for example recent work [49, 50]) with continuous group gauge symmetries. This gives an avenue for wider applications in holography and gauge-string duality for algorithms related to finite group representation theory.

Data availability statement

No new data were created or analysed in this study.

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Appendix A. Frobenius structure of *R*(*G*)-TQFT

We define a two-dimensional TQFT using the algebra of representations of a finite group G, denoted $R(G)$. As a vector space, $R(G)$ has a basis a_R labelled by the set $\mathrm{Irr}(G)$ of isomorphism classes of irreducible representations of G. The product $\mu : R(G)\otimes R(G)\to R(G)$ is given by

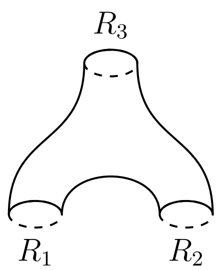
$$\mu\left(a_{R_1},a_{R_2}\right)=\sum_{R_3} N^{R_3}_{R_1R_2}a_{R_3}, \tag{A.1}$$

where

$$N^{R_3}_{R_1R_2}=\frac{1}{|G|}\sum_{g\in G}\chi^{R_1}\left(g\right)\chi^{R_2}\left(g\right)\overline{\chi^{R_3}\left(g\right)}. \tag{A.2}$$

We introduce the graphical notation

$$N^{R_3}_{R_1R_2} = \tag{A.3}$$



The algebra is commutative since $R_1\otimes R_2\cong R_2\otimes R_1$. It is also unital, the trivial representation, denoted R_0, plays the role of the identity element. Equivalently, we have a unit $\eta:\mathbb{C}\to R(G)$ determined by

$$\eta(1) = a_{R_0} = \tag{A.4}$$

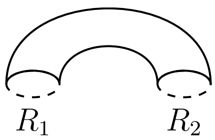
and $\eta(c) = c\eta(1)$ for $c \in \mathbb{C}$. This definition of an algebra is the one used in [21, definition 2.1.18].

The algebra $R(G)$ is turned into a Frobenius algebra by adding a non-degenerate bilinear form $g : R(G) \otimes R(G) \to \mathbb{C}$ to its structure (see [21, definition 2.2.5]). We use the definition

$$g_{R_1 R_2} = g\left(a_{R_1}, a_{R_2}\right) = \frac{1}{|G|} \sum_{g \in G} \chi^{R_1}(g)\, \chi^{R_2}(g) = \delta_{R_1 \overline{R_2}}, \tag{A.5}$$

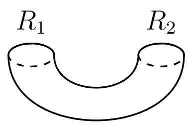
where $\overline{R}$ is the complex conjugate representation of R. Graphically, this corresponds to

$$g_{R_1 R_2} = \tag{A.6}$$

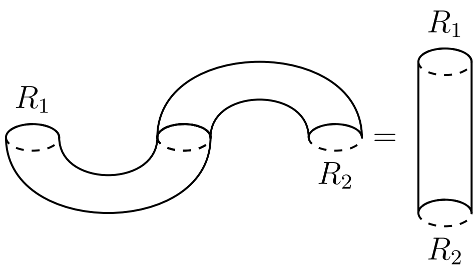


It has an inverse

$$g^{R_1 R_2} = g^{-1}(a_{R_1}, a_{R_2}) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi^{R_1}(g)\chi^{R_2}(g)} = \delta^{\overline{R_1} R_2} = \tag{A.7}$$



Graphically,

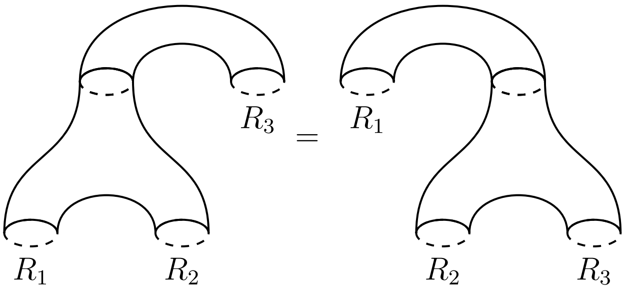


$$= \delta^{R_1}_{R_2}. \tag{A.8}$$

Note that the definition (A.5) gives a bilinear form satisfying

$$g\left(a_{R_1} a_{R_2}, a_{R_3}\right) = g\left(a_{R_1}, a_{R_2} a_{R_3}\right), \tag{A.9}$$

or

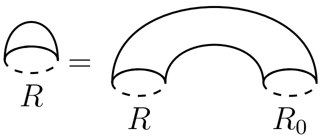


(A.10)

The bilinear form defines a counit $\varepsilon : R(G) \to \mathbb{C}$ by

$$\varepsilon(a_R) = g(a_R, a_{R_0}) = \delta_{R\overline{R_0}} = \delta_{RR_0} = \tag{A.11}$$

where $R_0 = \overline{R_0}$ is the trivial representation (see [21, 2.2.3]). Graphically we have,



(A.12)

A Frobenius algebra has a canonical [21, definition 2.3.16] coproduct $\Delta : R(G) \to R(G) \otimes R(G)$ defined as follows,

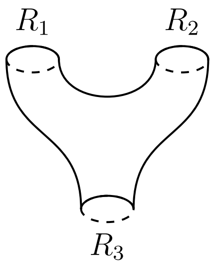
$$\Delta(a_{R_3}) = \sum_{R_1, R_2} N_{R_3}^{R_1 R_2} a_{R_1} \otimes a_{R_2}, \tag{A.13}$$

where

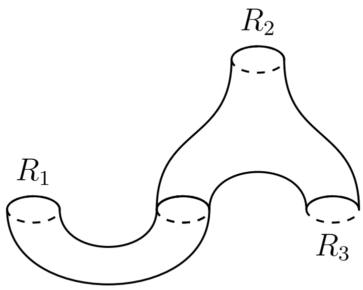
$$N_{R_3}^{R_1 R_2} = g^{R_1 R_4} N_{R_4 R_3}^{R_2} = N_{\overline{R_1} R_3}^{R_2}. \tag{A.14}$$

In terms of cobordisms we have

$$N^{R_1 R_2}_{R_3} =$$



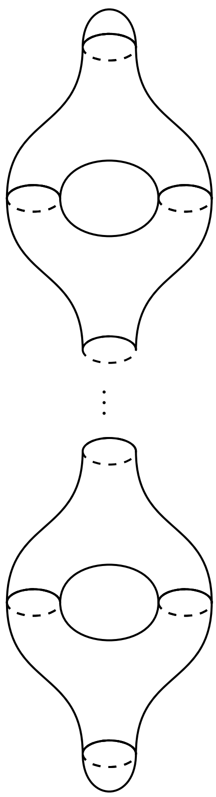
$$=$$



(A.15)

General genus h partition functions come from iterated use of the coproduct Δ followed by the product μ. To get a closed manifold we cap off both ends with a unit η and counit ε,

$$Z^{R(G)}_h = \varepsilon \circ (\mu \circ \Delta)^h \circ \eta(1) = \tag{A.16}$$



To evaluate this, note that

$$\mu \circ \Delta\left(a_{R_1}\right) = \sum_{R_2,R_3} N_{R_1}^{R_2R_3} \mu\left(a_{R_2} \otimes a_{R_3}\right) \tag{A.17}$$

$$= \sum_{R_2,R_3,R_4} N_{R_1}^{R_2R_3} N_{R_2R_3}^{R_4} a_{R_4} \tag{A.18}$$

$$= \sum_{R_2,R_3,R_4} N_{\overline{R_2}R_1}^{R_3} N_{R_2R_3}^{R_4} a_{R_4}. \tag{A.19}$$

We define

$$N_{R_1}^{R_4} = \sum_{R_2,R_3} N_{R_1}^{R_2R_3} N_{R_2R_3}^{R_4}, \tag{A.20}$$

which is equal to

$$N_{R_1}^{R_4} = \frac{1}{|G|^2} \sum_{R_2,R_3,g,h} \chi^{R_1}\left(g\right) \overline{\chi^{R_2}\left(g\right) \chi^{R_3}\left(g\right)} \chi^{R_2}\left(h\right) \chi^{R_3}\left(h\right) \overline{\chi^{R_4}\left(h\right)} \tag{A.21}$$

$$= \sum_{R_2,R_3,C_1,C_2} \frac{|C_1|\,|C_2|}{|G|^2} \chi_{C_1}^{R_1} \overline{\chi_{C_1}^{R_2} \chi_{C_1}^{R_3}} \chi_{C_2}^{R_2} \chi_{C_2}^{R_3} \overline{\chi_{C_2}^{R_4}} \tag{A.22}$$

$$= \sum_{C_1,C_2} \chi_{C_1}^{R_1} \left(\delta_{C_1C_2}\right)^2 \overline{\chi_{C_2}^{R_4}} \tag{A.23}$$

$$= \sum_{C} \chi_{C}^{R_1} \overline{\chi_{C}^{R_4}}. \tag{A.24}$$

Using the above, we can write the genus h partition function as

$$Z_h^{R(G)} = \sum_{R_1,\ldots,R_{h-1}} N_{R_0}^{R_1} N_{R_1}^{R_2} \ldots N_{R_{h-1}}^{R_0}, \tag{A.25}$$

which is equal to

$$\sum_{\substack{C_1,\ldots,C_h \\ R_1,\ldots,R_{h-1}}} \chi_{C_1}^{R_0} \overline{\chi_{C_1}^{R_1}} \chi_{C_2}^{R_1} \overline{\chi_{C_2}^{R_2}} \ldots \chi_{C_{h-2}}^{R_{h-2}} \overline{\chi_{C_{h-1}}^{R_{h-1}}} \chi_{C_h}^{R_{h-1}} \overline{\chi_{C_h}^{R_0}}. \tag{A.26}$$

Summing over the representations and using

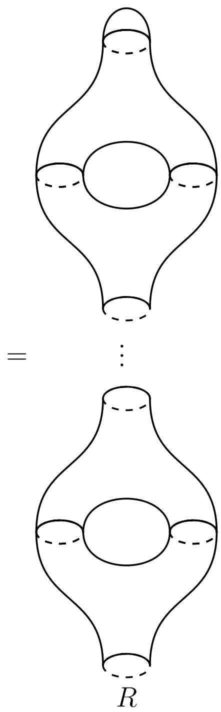
$$\sum_{R} \chi_{C_1}^{R} \chi_{C_2}^{R} = \frac{|G|}{|C_1|} \delta_{C_1C_2} \tag{A.27}$$

together with $\chi_C^{R_0} = 1$ gives

$$Z_h^{R(G)} = \sum_{C_1,\ldots,C_h} \frac{\delta_{C_1C_2}\,|G|}{|C_1|} \ldots \frac{\delta_{C_{h-1}C_h}\,|G|}{|C_h|} = \sum_{C} \left(\frac{|G|}{|C|}\right)^{h-1} = \sum_{C} \mathrm{Sym}C^{h-1}. \tag{A.28}$$

The Frobenius structure can also be used to compute partition functions of closed manifolds with boundary. As a simple example, the genus h partition function with a single in-boundary labelled by R is given by

$$Z^{R(G)}_{h;R} = \varepsilon \circ (\mu \circ \Delta)^h (a_R) = \sum_{R_1,\dots,R_{h-1}} N^{R_1}_{R} N^{R_2}_{R_1} \dots N^{R_0}_{R_{h-1}} = \sum_{C} \left(\frac{|G|}{|C|}\right)^{h-1} \chi^R_C \tag{A.29}$$



(A.30)

The partition function with n in-boundaries is given by

$$Z^{R(G)}_{h;R_1,\dots,R_n} = \varepsilon \circ (\mu \circ \Delta)^h \mu^{n-1} (a_{R_1} \otimes \cdots \otimes a_{R_n}) . \tag{A.31}$$

To simplify this we first need to evaluate

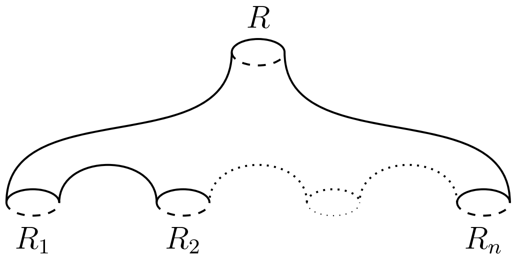
$$\mu^{n-1} (a_{R_1} \otimes \cdots \otimes a_{R_n}) = N^{R_1'}_{R_1 R_2} N^{R_2'}_{R_1' R_3} \dots N^{R_{n-1}'}_{R_{n-2}' R_n} a_{R_{n-1}'} . \tag{A.32}$$

We define

$$N^{R}_{R_1 R_2 \dots R_n} = N^{R_1'}_{R_1 R_2} N^{R_2'}_{R_1' R_3} \dots N^{R}_{R_{n-2}' R_n} . \tag{A.33}$$

and graphically represent it by the cobordism

$$N^{R}_{R_1 R_2 \dots R_n} = \tag{A.34}$$



Note that

$$N^{R_2'}_{R_1 R_2 R_3} = N^{R_1'}_{R_1 R_2} N^{R_2'}_{R_1' R_3} \tag{A.35}$$

$$= \frac{1}{|G|^2} \sum_{g_1, g_2} \chi^{R_1}(g_1) \chi^{R_2}(g_1) \overline{\chi^{R_1'}(g_1)} \chi^{R_1'}(g_2) \chi^{R_3}(g_2) \overline{\chi^{R_2'}(g_2)} \tag{A.36}$$

$$= \sum_{C_1, C_2} \frac{|C_1||C_2|}{|G|^2} \chi^{R_1}_{C_1} \chi^{R_2}_{C_1} \overline{\chi^{R_1'}_{C_1}} \chi^{R_1'}_{C_2} \chi^{R_3}_{C_2} \overline{\chi^{R_2'}_{C_2}} \tag{A.37}$$

$$= \frac{1}{|G|} \sum_{g} \chi^{R_1}(g) \chi^{R_2}(g) \chi^{R_3}(g) \overline{\chi^{R_2'}(g)}. \tag{A.38}$$

Therefore,

$$N^{R}_{R_1 R_2 \dots R_n} = \frac{1}{|G|} \sum_{g} \chi^{R_1}(g) \chi^{R_2}(g) \dots \chi^{R_n}(g) \overline{\chi^{R}(g)}. \tag{A.39}$$

Going back to (A.31) we have

$$Z^{R(G)}_{h;R_1 \dots R_n} = \sum_{R} N^{R}_{R_1 R_2 \dots R_n} \varepsilon \circ (\mu \circ \Delta)^h (a_R) = \sum_{R} N^{R}_{R_1 R_2 \dots R_n} Z_{h;R} \tag{A.40}$$

$$= \sum_{C} \left(\frac{|G|}{|C|} \right)^{h-1} \chi^{R_1}_{C} \chi^{R_2}_{C} \dots \chi^{R_n}_{C}. \tag{A.41}$$

Appendix B. Inverse Vandermonde lemma

In equations (3.2), (3.16) and (3.42) we used the following lemma to prove integrality of character sums.

Lemma. *Let Z be a diagonalizable $K \times K$ matrix of non-negative integers with eigenvalues $\lambda_1 \dots, \lambda_K$. Let the set of distinct eigenvalues be $\{z_1, \dots, z_k\}$. Note that $k \leqslant K$ generally. Define the $k \times k$ Vandermonde matrix*

$$V_{b,i} = z_i^b. \tag{B.1}$$

In general, the eigenvalues $z_1,\ldots,z_k$ are algebraic integers. But suppose z_j is rational, then the inverse Vandermonde V^{-1} has rational entries in the jth row:

$$\left(V^{-1}\right)_{j,b} \in \mathbb{Q} \quad \forall b \in \{1,\ldots,k\}. \tag{B.2}$$

In this paper we use this lemma for the special case of z_j being an integer.

Before we go into the technical details involved in proving this lemma, let us outline the proof. We will use the fact that the matrix elements $(V^{-1})_{i,b}$ of a Vandermonde matrix can be written in terms of elementary symmetric polynomials (see section 1.2.3 exercise 40 of [51])

$$\left(V^{-1}\right)_{j,b} = \frac{(-1)^{k-b} e_{k-b}\left(z_1,\ldots,z_{j-1},z_{j+1},\ldots z_k\right)}{z_j \prod_{i\neq j}\left(z_j - z_i\right)}, \tag{B.3}$$

where e_m is the mth elementary symmetric polynomial. Given this, it remains to prove that the these elementary symmetric polynomials are rational. The strategy will be to investigate the polynomial

$$q_{Z,j}(x) = \prod_{i\neq j}(x - z_i) = \sum_{i=0}^{k-1}(-1)^i e_i\left(z_1,\ldots,z_{j-1},z_{j+1},\ldots,z_k\right) x^{k-1-i}. \tag{B.4}$$

By proving that $q_{Z,j}(x)$ has rational coefficients, it follows that the corresponding elementary symmetric polynomials are rational numbers. The polynomial $q_{Z,j}(x)$ is closely related to the determinant

$$\det(x - Z) = \prod_{i=1}^{K}(x - \lambda_i) = \prod_{i=1}^{k}(x - z_i)^{m_i}, \tag{B.5}$$

where m_i is the multiplicity of the eigenvalue λ_i. The polynomial corresponding to setting $m_i = 1$ for all i is called the minimal polynomial of Z

$$p_Z(x) = \prod_{j=1}^{k}(x - z_j). \tag{B.6}$$

The minimal polynomial differs from $q_{Z,j}(x)$ by a linear factor

$$p_Z(x) = (x - z_j)\, q_{Z,j}(x), \tag{B.7}$$

with $z_j \in \mathbb{Q}$. As we will prove in the remainder of this appendix, $p_Z(x)$ has rational coefficients, and comparing the coefficients on both sides gives that $q_{Z_j}(x)$ has rational coefficients as well. It follows that

$$e_i\left(z_1,\ldots,z_{j-1},z_{j+1},\ldots,z_k\right) \in \mathbb{Q}. \tag{B.8}$$

Lastly,

$$z_j\left(z_j - z_1\right)\ldots\left(z_j - z_{j-1}\right)\left(z_j - z_{j+1}\right)\ldots\left(z_j - z_k\right) = z_j q_{Z,j}\left(z_j\right) \in \mathbb{Q}, \tag{B.9}$$

which concludes the proof of the lemma in equation (B.2).

We will now prove that the minimal polynomial $p_Z(x)$ has rational coefficients. The non-negative integers are contained in the field $\mathbb{Q}$ of rationals and therefore Z is an element of the set $M_k(\mathbb{Q})$ of $k \times k$ matrices with rational coefficients. Let F be a finite field extension of the rationals. The minimal polynomial of a matrix X over a field F is uniquely defined as the monic polynomial $p_X^{(F)}(x) \in F[x]$ of lowest degree such that

$$p_X^{(F)}(X) = 0. \tag{B.10}$$

The minimal polynomial divides the characteristic polynomial (see [52, theorem 3.3.1])

$$P_X(x) = \det(x - X). \tag{B.11}$$

That is, there exists a monic polynomial $h(x) \in F[x]$ such that

$$P_X(x) = h(x)\, p_X^{(F)}(x). \tag{B.12}$$

In particular, the characteristic polynomial $P_X(x)$ satisfies

$$P_X(X) = 0 \tag{B.13}$$

and therefore the minimal polynomial of a matrix exists. The minimal polynomial of a matrix is unique. To show this, suppose $p_{X,1}^{(F)}(x)$ and $p_{X,2}^{(F)}(x)$ are minimal polynomials of the matrix X. We have

$$p_{X,2}^{(F)}(x) = q(x)\, p_{X,1}^{(F)}(x) + r(x) \tag{B.14}$$

for some polynomials $q(x), r(x) \in F[x]$ and the polynomial $r(x)$ has degree less than that of $p_{X,1}^{(F)}(x)$. Evaluating on the matrix X, we get

$$r(X) = 0. \tag{B.15}$$

Therefore, if $r(X)$ is a non-trivial polynomial, then we have a contradiction with the assumption that $p_{X,1}^{(F)}(x)$ is a polynomial of least degree evaluating to zero on X. Therefore, $r(x) = 0$. This shows that $p_{X,1}^{(F)}(x)$ divides the polynomial $p_{X,2}^{(F)}(x)$. Since $p_{X,1}^{(F)}(x)$ and $p_{X,2}^{(F)}(x)$ are polynomials of the same degree, $q(x)$ must be a constant polynomial. Since $p_{X,1}^{(F)}(x)$ and $p_{X,2}^{(F)}(x)$ are also monic, we must have $q(x) = 1$ and therefore,

$$p_{X,2}^{(F)}(x) = p_{X,1}^{(F)}(x). \tag{B.16}$$

In our particular case, we have $X = Z$ is a matrix with integer entries. Let us show that the minimal polynomial $p_Z^{(F)}(x)$ has rational coefficients. Let

$$p_Z^{(F)}(x) = x^k + b_{k-1}x^{k-1} + \cdots + b_0, \tag{B.17}$$

be the expansion of $p_Z^{(F)}(x)$ with coefficients in F. We will now show that

$$b_i \in \mathbb{Q}. \tag{B.18}$$

Let $r_j^{(i)}$ be the jth row of Z^i, then

$$p_Z^{(F)}(Z) = Z^k + b_{k-1}Z^{k-1} + \cdots + b_0 = 0 \tag{B.19}$$

implies that

$$r_j^{(k)} + \sum_{i=0}^{k-1} b_i r_j^{(i)} = 0 \tag{B.20}$$

for all $j = 1, \ldots, K$ where K is the size of the matrix Z. That is, the vectors $r_j^{(0)}, \ldots, r_j^{(k)}$ are linearly dependent over the field F. Let f_l be a basis for the field F such that

$$b_i = \sum_l \beta_{i,l} f_l \tag{B.21}$$

where $f_0 = 1$ and $\beta_{i,l} \in \mathbb{Q}$. Substituting this into (B.20) gives

$$r_j^{(k)} + \sum_{i=0}^{k-1} \sum_l \beta_{i,l} f_l r_j^{(i)} = 0. \tag{B.22}$$

Note that $r_j^{(i)}$ are rational numbers. Therefore, the above equation contains a sum over the basis elements f_l with rational coefficients. Since f_l are linearly independent over the rationals, this implies vanishing of all coefficients in front of f_l for every l. In particular, using the fact that $f_0 = 1$ we have

$$r_j^{(k)} + \sum_{i=0}^{k-1} \beta_{i,0} f_0 r_j^{(i)} = \left(r_j^{(k)} + \sum_{i=0}^{k-1} \beta_{i,0} r_j^{(i)} \right) f_0 = 0, \tag{B.23}$$

and vanishing of the coefficients in front of f_0 implies

$$r_j^{(k)} + \sum_{i=0}^{k-1} \beta_{i,0} r_j^{(i)} = 0. \tag{B.24}$$

Since (B.24) holds for all rows $j = 1, \ldots, K$, we have

$$Z^k + \beta_{k-1,0} Z^{k-1} + \cdots + \beta_{0,0} = 0. \tag{B.25}$$

That is, the monic polynomial $p(x) \in \mathbb{Q}$ defined by

$$p(x) = x^k + \beta_{k-1,0} x^{k-1} + \cdots + \beta_{0,0}, \tag{B.26}$$

satisfies $p(Z) = 0$. However, since the minimal polynomial is unique, we must have $p_X^{(F)}(x) = p(x)$. Therefore, if the matrix is defined over the rationals, the minimal polynomial has rational coefficients. This justifies the assumption in the above outline of the proof, and therefore we have proven the lemma.

Appendix C. Class sizes and characters from $R(G)$-CTST

In this appendix we construct polynomials $F(x)$, from $R(G)$-CTST amplitudes, with roots at $x = \mathrm{Sym}C_1, \ldots, \mathrm{Sym}C_k$, where $\mathrm{Sym}C = \frac{|G|}{|C|}$. Thus, solving the polynomial is a constructive method for finding conjugacy class sizes from CTST. The polynomial $F(x)$ contains contributions from manifolds with more than one genus. The roots of $F(x)$ are directly related to the eigenvalues of the handle creation operator. This is analogous to the G-CTST construction [14, 15] of dimension of irreducible representations, which uses the fact that the eigenvalues of the G-TQFT handle creation operator are $\frac{|G|^2}{d_R^2}$.

The handle creation operators can be used to construct level sets of characters with fixed eigenvalues of the handle creation operator, or equivalently, fixed conjugacy class sizes. Using the previously constructed conjugacy class data, we give explicit formulas for these character sums in terms of elementary symmetric polynomials in $z_i = \mathrm{Sym}C_i$ and $R(G)$-CTST amplitudes with one boundary but varying genus. This is dual to the G-CTST considerations in [15], where the handle creation operator can be used to construct level sets of characters with the same dimension.

By considering surfaces with a single boundary labelled by R_2 and b boundaries labelled by R_1, we construct level set sums of $\chi_C^{R_2}$ with fixed values of $\chi_C^{R_1}$. This is the row-column dual fo the construction in [15], where surfaces with a single boundary labelled by C_2 and b boundaries labelled by C_1 gave rise to level sets of normalized characters of C_2.

C.1. Class sizes from R(G)-CTST

To construct classes sizes from $R(G)$-CTST amplitudes, define

$$z_C = \frac{|G|}{|C|} = \mathrm{Sym}C \tag{C.1}$$

and the polynomial

$$F(x) = \prod_{C \in \mathrm{Cl}(G)} (x - z_C)\,. \tag{C.2}$$

The polynomial can be expanded into elementary symmetric functions e_k,

$$F(x) = \sum_k (-1)^k x^{K-k} e_k(z_{C_1}, \ldots, z_{C_K})\,, \tag{C.3}$$

where $K = |\mathrm{Cl}(G)|$. Elementary symmetric functions evaluated on z_K can be expressed in terms of the partition functions $Z_h^{R(G)}$, see [14, equation (2.17)]

$$e_k(z_{C_1}, \ldots, z_{C_K}) = \sum_{p \vdash k} \frac{(-1)^{k-\sum_i p_i}}{\mathrm{Aut}(p)} \prod_i \left(Z_{h=i+1}^{R(G)}\right)^{p_i}. \tag{C.4}$$

The sum is over integer partitions $p = (p_1, p_2, \ldots, p_k)$ of k

$$\sum_i i p_i = k, \text{and } \mathrm{Aut}(p) = \prod i^{p_i} p_i!. \tag{C.5}$$

Substituting (C.4) into (C.3) gives

$$F(x) = \sum_k (-1)^k x^{K-k} \sum_{p \vdash k} \frac{(-1)^{k-\sum_i p_i}}{\mathrm{Aut}(p)} \prod_i \left(Z^{R(G)}_{h=i+1}\right)^{p_i}, \tag{C.6}$$

and solving $F(x) = 0$ gives the conjugacy class sizes from $R(G)$-CTST data.

C.2. Reducible characters over level sets of class sizes from genus h fusion data

In this section we will probe the irreducible representation S using surfaces labelled by a boundary S with varying genus. The genus one partition function is

$$\begin{aligned} Z_{h=1;S} &= \sum_R N_{R,\bar{R},S} = \frac{1}{|G|} \sum_R N_{R\bar{R}S} \\ &= \frac{1}{|G|} \sum_g \chi^R(g)\,\chi^R\left(g^{-1}\right)\chi^S(g) \\ &= \sum_C \frac{|C|\mathrm{Sym}C}{|G|} \chi^S_C = \sum_C \chi^S_C \end{aligned} \tag{C.7}$$

and for higher genus we have

$$\begin{aligned} Z^{R(G)}_{h;S} &= \sum_{R_1,\cdots,R_{h+1}} N_{R_1\cdots R_{h+1},\bar{R}_1,\cdots,\bar{R}_{h+1};S} \\ &= \frac{1}{|G|} \sum_g \sum_{R_1\cdots R_{h+1}} \chi^{R_1}(g)\,\chi^{R_2}(g)\cdots\chi^{R_{h+1}}(g)\,\chi^{\bar{R}_1}(g)\,\chi^{\bar{R}_2}(g)\cdots\chi^{\bar{R}_{h+1}}(g)\,\chi^S(g) \\ &= \frac{1}{|G|} \sum_C |C|\left(\mathrm{Sym}C\right)^h \chi^S_C \\ &= \sum_C \left(\mathrm{Sym}C\right)^{h-1} \chi^S_C. \end{aligned} \tag{C.8}$$

Define the following equivalence relation on $\mathrm{Cl}(G)$

$$C_1 \sim C_2 \text{ if } \mathrm{Sym}C_1 = \mathrm{Sym}C_2. \tag{C.9}$$

It partitions $\mathrm{Cl}(G)$ into subsets $\kappa_1, \ldots, \kappa_l$ such that

$$\mathrm{Cl}(G) = \kappa_1 \cup \cdots \cup \kappa_l. \tag{C.10}$$

We introduce the notation

$$z_i = \mathrm{Sym}C, \text{for any } C \in \kappa_i. \tag{C.11}$$

Now we can rewrite equation (C.8) as

$$Z^{R(G)}_{h;S} = \sum_{i=1}^{l} z_i^{h-1} \sum_{C \in \kappa_i} \chi^S_C. \tag{C.12}$$

Define the l-vector

$$y_h(S) = (Z_{2;S}, \ldots, Z_{l+1;S}), \tag{C.13}$$

the $l \times l$ Vandermonde matrix

$$V_{h,i} = z_i^h \tag{C.14}$$

and l-vector

$$x_i(S) = \sum_{C \in \kappa_i} \chi_C^S. \tag{C.15}$$

Equation (C.8) reads

$$y(S) = Vx(S) \tag{C.16}$$

The inverse of the Vandermonde matrix is

$$\left(V^{-1}\right)_{i,h} = \frac{(-1)^{l-h} e_{l-h}(z_1 \ldots, z_{i-1}, z_{i+1}, \ldots, z_l)}{z_i \prod_{j \neq i}(z_i - z_j)}. \tag{C.17}$$

This immediately gives a formula for level set sums of characters:

$$\sum_{C \in \kappa_i} \chi_C^S = \frac{\sum_{h=1}^{l} (-1)^{l-h} e_{l-h}(z_1 \ldots, z_{i-1}, z_{i+1}, \ldots, z_l) Z_{h+1;S}}{z_i \prod_{j \neq i}(z_i - z_j)}. \tag{C.18}$$

C.3. Constructing sums of characters of R_2 over level sets of characters of R_1

Similar arguments allow us to obtain sums of characters of level sets of another character. Fix an irreducible representation R_1 and consider the partition functions

$$\begin{aligned} Z^{R(G)}_{h=1;R_1^b,R_2} &= \sum_R N_{R,\bar{R},R_1^b,R_2} = \frac{1}{|G|} \sum_g \sum_R \chi^R(g) \chi^{\bar{R}}(g) \left(\chi^{R_1}(g)\right)^b \chi^{R_2}(g) \\ &= \frac{1}{|G|} \sum_K |K|.\mathrm{Sym}K. \left(\chi_K^{R_1}\right)^b \chi_K^{R_2} \\ &= \sum_K \left(\chi_K^{R_1}\right)^b \chi_K^{R_2}. \end{aligned} \tag{C.19}$$

Define the equivalence relation

$$C_1 \sim_{R_1} C_2 \text{if} \chi_{C_1}^{R_1} = \chi_{C_2}^{R_1}, \tag{C.20}$$

on $\mathrm{Cl}(G)$, which partitions conjugacy classes into subsets $\kappa_1, \ldots, \kappa_l$. Introduce

$$z_i^{R_1} = \chi_C^{R_1} \text{for any} C \in \kappa_i. \tag{C.21}$$

We can re-write the partition functions as

$$Z^{R(G)}_{h=1;R_1^b;R_2} = \sum_{i=1}^{l} \left(z_i^{R_1}\right)^b \sum_{C\in\kappa_i} \chi_C^{R_2}. \tag{C.22}$$

As before, define the l-vector

$$y = \left(Z_{h=1,R_1^1,R_2}, \dots, Z_{h=1,R_1^l,R_2}\right), \tag{C.23}$$

the $l\times l$ Vandermonde matrix

$$V_{b,i}(R_1) = \left(z_i^{R_1}\right)^b \tag{C.24}$$

and l-vector

$$x_i(R_2) = \sum_{C\in\kappa_i} \chi_C^{R_2}. \tag{C.25}$$

Then equation (C.22) reads

$$y(R_1,R_2) = V(R_1)\,x(R_2). \tag{C.26}$$

The inverse of $V(R_1)$ is

$$\left(V^{-1}\right)_{i,b} = \frac{(-1)^{l-b} e_{l-b}\left(z_1^{R_1},\dots,z_{i-1}^{R_1},z_{i+1}^{R_1},\dots,z_l^{R_1}\right)}{z_i^{R_1}\prod_{j\neq i}\left(z_i^{R_1}-z_j^{R_1}\right)}, \tag{C.27}$$

which gives

$$\sum_{C\in\kappa_i}\chi_C^{R_2} = \frac{\sum_{b=1}^{l}(-1)^{l-b} e_{l-b}\left(z_1^{R_1},\dots,z_{i-1}^{R_1},z_{i+1}^{R_1},\dots,z_l^{R_1}\right) Z_{h=1;R_1^b;R_2}}{z_i^{R_1}\prod_{j\neq i}\left(z_i^{R_1}-z_j^{R_1}\right)}. \tag{C.28}$$

Appendix D. Harada's conjecture and row-column duality of TQFTs

From class algebra

$$T_{C_1}T_{C_2} = \sum_{C_3} f_{C_1C_2}^{C_3} T_{C_3}. \tag{D.1}$$

Define the matrix $\hat{f}_C$ by

$$(f_C)_{C_1}^{C_2} = f_{CC_1}^{C_3}. \tag{D.2}$$

We know

$$T_C P_R = \frac{\chi^R(T_C)}{d_R} P_R. \tag{D.3}$$

Thus the eigenvalues of $\hat{f}_C$ are $\frac{\chi^R(T_C)}{d_R}$ The determinant is

$$\det\left(\hat{f}_C\right) = \prod_R \frac{\chi^R(T_C)}{d_R} = \prod_R \frac{|C|}{d_R}\chi_C^R. \tag{D.4}$$

This gives an integer for every column of the character table. The product

$$X = \prod_C \det\left(\hat{f}_C\right) = \prod_C \prod_R \frac{|C|}{d_R} \chi^R_C, \tag{D.5}$$

of these integers is an integer associated with the whole character table.

In the fusion algebra theory we have

$$a_R a_S = N^T_{RS} a_T. \tag{D.6}$$

Define the matrix $\hat{N}_R$ for every row by the equation

$$\left(\hat{N}_R\right)^T_S = N^T_{RS}. \tag{D.7}$$

We have

$$a_R A_C = \chi^R_C A_C. \tag{D.8}$$

This the eigenvalues of $\hat{N}_R$ are χ^R_C. Take the determinant

$$\det\left(\hat{N}_R\right) = \prod_C \chi^R_C. \tag{D.9}$$

This is an integer for every row. Take the product over all R to get

$$\tilde{X} = \prod_R \det\left(\hat{N}_R\right) = \prod_R \prod_C \chi^R_C. \tag{D.10}$$

Comparing the two we get

$$\frac{X}{\tilde{X}} = \prod_C \prod_R \frac{|C|}{d_R}. \tag{D.11}$$

This ratio is conjectured to be an integer by Harada (see notes at the end of chapter 4 of [20] and the paper [53]).

ORCID iDs

Rajath Radhakrishnan https://orcid.org/0000-0003-1158-3320
Sanjaye Ramgoolam https://orcid.org/0000-0002-1211-4780

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