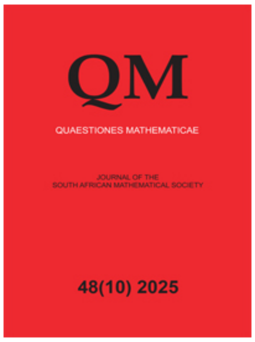

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A CLASS OF CONSTANT COEFFICIENT NONLINEAR EQUATIONS THAT ARE SOLVABLE BY THE HEAT KERNEL

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Abstract. In this paper, we document a new tractable method to solve nonlinear classes of partial differential equations subject to initial-values. The method involves a detailed transformation procedure, underpinned by the Lie pint symmetries of the one dimensional heat transfer equation. We illustrate the approach with the famous Burgers' equation, Potential Burgers' equation and the Hamilton-Jacobi-Bellman equation. This approach identifies new solutions of these nonlinear equations, which enhances our comprehension of its analytical properties and solution behaviour.

Mathematics Subject Classification (2020): 35K05, 35K15, 35Q92.
Key words: Nonlinear equations, symmetries, initial conditions.

1. Introduction. Partial differential equations (PDEs) are foundational models for studying nonlinear phenomena. Nonlinear PDEs have intricate solution processes and are largely difficult to solve. Analytic frameworks are highly sought after for exploring and characterizing their solutions.

Many traditional methods for solving linear PDEs are not applicable to nonlinear PDEs. Establishing the uniqueness of solutions to nonlinear PDEs can be much more complicated than for linear ones. These challenges may be overcome if a connection can be found between linear and nonlinear equations, namely through invertible transformations.

The motivation of this study is to tackle a general nonlinear equation that exhibits complicated behaviour and challenges existing analytical and numerical techniques. This problem offers us an exciting opportunity to develop a novel analytical framework to overcome such challenges in solving nonlinear equations.

In this study, we construct transformations connecting two general classes of nonlinear PDEs with constant coefficients to the heat equation. Consequently, these equations adopt the fundamental solution [15, 9] of the heat equation into their solution process. This enables us to find symmetries that incorporate initial data, and generate meaningful exact solutions. In fact, the solutions of such nonlinear PDEs admit fundamental solutions which are invariant solutions. Physically, the background of such equations are often connected to reaction-diffusion

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processes in the sciences, or commonly a type of Fokker-Planck equation with applications in statistical physics, thermodynamics, optics and finance.

It is well-known that the Hopf-Cole [17, 8] transformation renders the nonlinear Burgers' equation to the heat equation. We extend this idea to a broad class of nonlinear equations, but apply symmetries instead to achieve transformations and solutions. We emphasise that this is an original methodological framework for a class of nonlinear differential equations, along with a new solution for its analysis. Our approach departs from standard techniques found in the literature by incorporating equivalence transformations and introducing a scheme to solve its initial-value problem (IVP), enabling the derivation of more valuable exact solutions. This method not only yields new analytical insights but also enhances the technique available for tackling similar problems in mathematical modelling and applied analysis.

Lie [21] originated an approach to reduce certain equations to simple equations, like the heat equation. Since then, many works have focussed on manipulating linear equations to the heat equation [4, 18, 5, 24, 23]. Numerous explorations of this class of PDEs exist in various contexts. To name a few, in [28] a numerical analysis of a model under this class was performed, in [25] a model linked to brain tumors was investigated, in [3] a bond-pricing equation was considered and in [20] a singular invariant study was performed on a related model. On the other hand, we offer a mathematical treatment of relating these classes of important nonlinear equations to the heat equation. We note that typical studies of such equations ignore initial data, for example [12] or [22].

Lie symmetries stand out as an essential mathematical instrument in both theoretical and applied contexts. We mention several recent studies involving this method: the authors' of [1] leveraged symmetries to improve the generalization of neural networks, [29] analysed the Monge-Ampére equation and [2] employed Lie symmetry analysis on a Ivancevic model.

This paper is structured into various sections. In Section 2, we establish the theory that first linearises the nonlinear classes of PDEs under consideration, followed by a proof to construct its general solution. A lemma is given that indicates how an IVP may be attached to the nonlinear problem. Furthermore, we propose two categories of solutions with the application of symmetries. Section 3 describes some applications of this investigation in the form of relevant PDEs from the literature. We conclude in Section 4.

2. Primary results.

We begin by stating the nonlinear PDEs under study and establishing the theory required in their solution. Integral to this process, is the one dimensional heat equation and its fundamental solution [7, 19, 13]. In this section, we demonstrate how symmetry generators perfectly construct a framework of transformations to create a new systematic approach for the underlying equations.

Theorem 1. *The generalized (1+1) nonlinear equation*

$$\frac{\partial u(x,t)}{\partial t} - A\frac{\partial^2}{\partial x\,\partial x}u(x,t) - B\frac{\partial}{\partial x}u(x,t) - C\left(\frac{\partial}{\partial x}u(x,t)\right)^2 - D = 0, \tag{2.1}$$

where A, B, C and D are arbitrary constants, admits the general solution

$$u(x,t) = \frac{A}{C}\left(\ln\omega(y,\tau) - \gamma(y,\tau)\right), \tag{2.2}$$

where ω is a solution to the classical heat equation

$$\omega_\tau - \omega_{yy} = 0, \tag{2.3}$$

and

$$\gamma(y,\tau) = \begin{cases} \frac{1}{2}\frac{By}{\sqrt{\rho A}} - \frac{CD\tau}{A} + \frac{B^2\tau}{4A\rho}, & A > 0, \\ -\left(\frac{1}{2}\frac{By}{\sqrt{\rho A}} - \frac{CD\tau}{A}\right) + \frac{B^2\tau}{4A\rho}, & A < 0, \end{cases} \tag{2.4}$$

under the invertible transformations

$$y = \frac{x}{\sqrt{\rho A}}, \ \tau = \rho t, \ \rho = \pm 1. \tag{2.5}$$

Proof. The Lie point symmetries of Equation (2.1) are

$$Y_1 = \partial_t, \ Y_2 = \partial_u, \ , Y_3 = \partial_x, \ Y_4 = (Bt + x)\,\partial_u - 2tC\partial_x,$$

$$Y_5 = -4tC\partial_t - 2xC\partial_x + \left(B^2t - 4CDt + Bx\right)\partial_u,$$

$$Y_6 = -4t^2C\partial_t - 4txC\partial_x + \left(B^2t^2 - 4CDt^2 + 2Btx + 2At + x^2\right)\partial_u,$$

$$Y_L = -\frac{A}{C}e^{-\frac{Cu}{A}}f(t,x)\partial_u,$$

where $f(t,x)$ is an arbitrary solution of the linear equation

$$\frac{\partial v(x,t)}{\partial t} - A\frac{\partial^2}{\partial x\,\partial x}v(x,t) - B\frac{\partial}{\partial x}v(x,t) - \frac{CD}{A}v(x,t) = 0. \tag{2.6}$$

Y_L is the infinite-dimensional subalgebra which reflects that (2.1) is linearisable. The transformation from Y_L that linearises (2.1) to (2.6) is

$$v(x,t) = \pm e^{\frac{Cu}{A}}. \tag{2.7}$$

Equation (2.6) is a linear parabolic equation. By application of the transformations (2.5), where ρ is chosen in accordance with the sign of A, followed by a reversible transformation of the dependent variable $v(x,t)$, viz.

$$v(x,t) = \omega(y,\tau)\mathrm{e}^{-\gamma(y,\tau)}, \tag{2.8}$$

reduces Equation (2.6) to the heat equation (2.3). Thus, by (2.7) and (2.8), we obtain the solution (2.2). □

REMARK 2. The transformation (2.7) is equivalent to a Hopf-Cole transformation [17, 8].

Interestingly, (2.1) is connected to a different class of nonlinear PDEs via a simple transformation.

Theorem 3. *The generalized (1+1) nonlinear equation*

$$\frac{\partial\phi(x,t)}{\partial t} - A\frac{\partial^2}{\partial x\,\partial x}\phi(x,t) - B\frac{\partial}{\partial x}\phi(x,t) - 2C\phi(x,t)\frac{\partial}{\partial x}\phi(x,t) = 0, \tag{2.9}$$

where A, B and C are arbitrary constants, admits the general solution

$$\phi(x,t) = \frac{\partial}{\partial x}u(x,t), \tag{2.10}$$

where $u(x,t)$ is a solution of (2.1).

Proof. Taking the antiderivative (2.10) transforms (2.9) into (2.1) with $D = 0$, after an integration with respect to the space variable x. □

Lemma 4. *Suppose that Equation* (2.1) *admits the initial condition*

$$u(x,0) = g(x). \tag{2.11}$$

Hence, Equation (2.9) *will admit the initial condition*

$$\phi(x,0) = g'(x), \tag{2.12}$$

and the transformed initial condition for Equation (2.3) *is*

$$\omega(y,0) \equiv F(y) = \exp\left(\frac{C}{A}g(y) + \gamma(y,0)\right). \tag{2.13}$$

Proof. The results are achieved by an elementary application of (2.10) and (2.2), respectively. □

In what follows, we present the two categories of solutions of the heat equation, $\omega(y,\tau)$, which originate from symmetries. Given the above, $\omega(y,\tau)$ is needed to elucidate solutions of the nonlinear equations Equation (2.1) and Equation (2.9).

Beyond the invariant solutions mentioned earlier, it is particularly valuable to investigate exact solutions in the context of an IVP.

Theorem 5. *Let the heat kernel of* (2.3) *be denoted as [11]*

$$A(y,\tau) := \begin{cases} \frac{1}{\sqrt{4\pi\tau}}e^{-\frac{y^2}{4\tau}}, & (y\in\mathbb{R}, \tau > 0), \\ 0, & (y\in\mathbb{R}, \tau < 0). \end{cases} \tag{2.14}$$

A symmetry of the form

$$\begin{aligned} X &= \left(4c_1\tau^2 + 2c_2\tau + c_6\right)\frac{\partial}{\partial\tau} + \left(4c_1 y\tau + c_2 y + 2c_3\tau + c_4\right)\frac{\partial}{\partial y} \\ &\quad \left(\left(c_1(-2\tau - y^2) - c_3 y + c_5\right)\omega + \int_{\mathbb{R}} A(y-\eta,\tau)\alpha(\eta,0)d\eta\right)\frac{\partial}{\partial\omega}, \end{aligned} \tag{2.15}$$

generates invariant solutions of the heat equation (2.3) *that is subject to the initial condition* (2.13), *and where*

$$\alpha(\eta,0)=\left(c_1\eta^2+c_3\eta-c_5\right)F(\eta)+\left(c_2\eta+c_4\right)F'(\eta)+c_6F''(\eta). \tag{2.16}$$

Proof. Suppose the symmetry generators of the heat equation (2.3), are given in the most general form as

$$X=\xi(y,\tau,\omega)\frac{\partial}{\partial y}+\sigma(y,\tau,\omega)\frac{\partial}{\partial \tau}+\phi(y,\tau\omega)\frac{\partial}{\partial \omega}, \tag{2.17}$$

and by standard calculations of Lie point symmetries, we find

$$\begin{aligned} X &= \left(4c_1\tau^2+2c_2\tau+c_6\right)\frac{\partial}{\partial\tau}+\left(4c_1y\tau+c_2y+2c_3\tau+c_4\right)\frac{\partial}{\partial y} \\ & \left(\left(c_1(-2\tau-y^2)-c_3y+c_5\right)\omega+\alpha(y,\tau)\right)\frac{\partial}{\partial\omega}. \end{aligned} \tag{2.18}$$

The function $\alpha(y,\tau)$ is an arbitrary solution that satisfies (2.3), i.e. $\alpha_\tau=\alpha_{yy}$, and hence by convolution

$$\alpha(y,\tau)=\int_{\mathbb{R}}A(y-\eta,\tau)\alpha(\eta,0)d\eta \quad (y\in\mathbb{R},\tau>0). \tag{2.19}$$

Next, we invoke the invariant surface condition (ISC) [27]

$$\xi\omega_y+\sigma\omega_\tau-\phi=0, \tag{2.20}$$

where it is imposed that $\tau=0$. Hence, for (2.13), this leads to the condition [14]

$$\frac{\phi(y,0,F(y))-\xi(y,0,F(y))F'(y)}{\sigma(y,0,F(y))}=\omega_{yy}, \tag{2.21}$$

which provides the function (2.16). □

REMARK 6. For a discussion on the uniqueness of solutions from convolution, we refer the reader to [30].

For simplicity, we state the following result.

COROLLARY 7. *A solution to* (2.3) *subject to the initial condition* (2.13), *is*

$$\omega(y,\tau)=\int\int_{\mathbb{R}}A(y-\eta,\tau)F'(\eta)d\eta dy. \tag{2.22}$$

Proof. Let $c_4=1$ and $c_j=0,(j=1,\ldots,6,\ j\neq 4)$ in Theorem 5 to obtain the function

$$\alpha(y,0)=F'(y),$$

so that $\alpha(y,\tau)$ can be found by convolution. Moreover from (2.18), we have the symmetry

$$X = \frac{\partial}{\partial y} + \alpha(y,\tau)\frac{\partial}{\partial \omega}, \tag{2.23}$$

and via the ISC (2.20), we obtain that the solution to (2.3) is

$$\omega(y,\tau) = \int \alpha(y,\tau)dy + H(\tau). \tag{2.24}$$

Setting $H(\tau) = 0$ gives the desired result. □

REMARK 8. Equation (2.22) is a selective solution, more exist and may be found by the same process under different choices of c_j. However, complexities in the integration may inhibit obtaining solutions.

In the literature it is common practice to apply symmetry-based methods to find exact solutions independent of any initial-values. This involves the individual Lie point symmetries of (2.3), split from (2.18):

$$\begin{gathered} X_1 = \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial \tau}, \quad X_3 = \omega\frac{\partial}{\partial \omega}, \\ X_4 = y\frac{\partial}{\partial y} + 2\tau\frac{\partial}{\partial \tau}, \quad X_5 = 2\tau\frac{\partial}{\partial y} - y\omega\frac{\partial}{\partial \omega}, \\ X_6 = 4\tau y\frac{\partial}{\partial y} + 4\tau^2\frac{\partial}{\partial \tau} - (y^2+2\tau)\omega\frac{\partial}{\partial \omega}, \\ X_\alpha = \alpha(y,\tau)\frac{\partial}{\partial \omega}. \end{gathered} \tag{2.25}$$

THEOREM 9. *The invariant and non equivalent classes of solutions to the heat equation* (2.3)*, subject to no initial-data, are given in Case I-V in Table 1. The five solutions correspond to the one dimensional optimal system of subalgebras.*

In Table 1, $c \in \mathbb{R}$, C_1, C_2 are arbitrary constants, $U(\mu,\nu,z)$ and $M(\mu,\nu,z)$ are Kummer functions, $W(\mu,\nu,z)$ and $M(\mu,\nu,z)$ are Whittaker functions, $Ai(x)$ and $Bi(x)$ are the Airy Ai and Airy Bi functions, respectively.

Proof. The optimal one dimensional subalgebra is listed in [27] (p. 206). Note that, compared to [27], we have omitted the subalgebra X_3 as it is not useful and of course, the subalgebra $X_2 + X_5$ is easily obtained from a discrete symmetry mapping of $X_2 - X_5$.

Table 1: Non equivalent classes of solutions

Case	Subalgebra	Solution
I	X_1	$\omega(y,\tau) = C_1$
II	$X_2 + cX_3$	$\omega(y,\tau) = \left(C_1\,\mathrm{e}^{\sqrt{c}y} + C_2\,\mathrm{e}^{-\sqrt{c}y}\right)\mathrm{e}^{c\tau}$
III	$X_2 - X_5$	$\omega(y,\tau) = \left(C_1\,\mathrm{Ai}\left(\tau^2+y\right) + C_2\,\mathrm{Bi}\left(\tau^2+y\right)\right)\mathrm{e}^{y\tau+\frac{2\tau^3}{3}}$
IV	$X_2 + X_6 + cX_3$	$\omega(y,\tau) = \dfrac{\left(C_1\,M_{\frac{i}{4}c,\,\frac{1}{4}}\left(\frac{iy^2}{4\tau^2+1}\right)+C_2\,W_{\frac{i}{4}c,\,\frac{1}{4}}\left(\frac{iy^2}{4\tau^2+1}\right)\right)}{\sqrt{y}}\times$ $\mathrm{e}^{\frac{1}{8\tau^2+2}\left(\left(-4\tau^2-1\right)c\arctan\left(\frac{1}{2\tau}\right)-2\tau y^2\right)}$
V	$X_4 + 2cX_3$	$\omega(y,\tau) = \mathrm{e}^{-\frac{y^2}{4\tau}}\left(\frac{\tau}{y^2}\right)^{c-\frac{1}{2}}\times$ $\left(U\left(c+1,\,\frac{3}{2},\,\frac{y^2}{4\tau}\right)C_2 + M\left(c+1,\,\frac{3}{2},\,\frac{y^2}{4\tau}\right)C_1\right)y^{2c}$

The corresponding invariant functions that generate the last column of Table 1 are

I: $m = \tau,\ \omega = z\,(m)$,
II: $m = y,\ \omega = z\,(m)\,\mathrm{e}^{c\tau}$,
III: $m = y + \tau^2$, $\omega = z\,(m)\,\mathrm{e}^{y\tau+\frac{2\tau^3}{3}}$,
IV: $m = \frac{4\tau^2+1}{4y^2}$, $\omega = z\,(m)\,\mathrm{e}^{\frac{y^2}{8\tau^2+2}\left(-\frac{c\left(4\tau^2+1\right)}{y^2}\arctan\left(\frac{1}{2\tau}\right)-2\tau\right)}\frac{1}{\sqrt{y}}$,
V: $m = y\frac{1}{\sqrt{\tau}},\ \omega = z\,(m)\,\tau^c$.

Invariants are substituted into (2.3) to find a reduced equation that is solved for $z(m)$ and hence $\omega(y,t)$. $\square$

3. Applications.

There are possibly a myriad of nonlinear PDEs encompassed by (2.1) and (2.9). We select three examples that exist in various scientific spheres. Most of these nonlinear equations have not been solved subject to initial-values and symmetry methods. Maple was used in the verification of solutions and for obtaining graphical results.

3.1. The nonlinear Hamilton-Jacobi-Bellman equation. This equation was posed by Heath et al. [16] to model mean-variance hedging, an essential component in finance that plays an important role in risk management. The analysis of this equation via symmetry methods has been reported in [10, 26].

The standard form of this equation is defined from (2.1) with the substitutions $A = -\frac{b^2}{2},\ B = -a,\ C = \frac{1}{2},\ D = -k$, where b, a, k are arbitrary constants. For our purposes, Theorem 1 applies with $\rho = -1$, and this model admits a terminal condition with T as the time to expiry. Therefore, it is more appropriate to let $\tau = T - t$, and allow $t = T$ for the IVP.

Consider the IVP with $g(x) = 0$ in Lemma (4). By application of Theorem 5 and Corollary 7, we calculate that

$$\alpha(y,\tau) = \frac{a e^{\frac{a\left(a\tau+\sqrt{2}by\right)}{2b^2}}}{\sqrt{2}b}. \tag{3.1}$$

Hence, the solution to the heat equation is

$$\omega(y,\tau) = e^{\frac{a\left(a\tau+\sqrt{2}by\right)}{2b^2}}, \tag{3.2}$$

and by Theorem 1, Equation (2.1) has solution, subject to (2.11), as

$$u(x,t) = k\,(T-t)\,. \tag{3.3}$$

This solution of the Hamilton-Jacobi- Bellman equation was presented in [26]. A simple check that this solution is subject to (2.11) can be seen if we let $t = T$ in (3.3). We omit the plot of this solution as it is a simple linear function.

Similarly, if we apply Remark 8 with $c_6 = 1$, we find a new solution, viz.

$$\alpha(y,\tau) = \frac{a^2 e^{\frac{a\left(a\tau+\sqrt{2}by\right)}{2b^2}}\left(\operatorname{erf}\left(\frac{\sqrt{2}a\tau+by}{2b\sqrt{\tau}}\right)+1\right)}{4b^2}, \tag{3.4}$$

and the heat equation's solution is then given by

$$\begin{aligned}\omega(y,\tau) = \frac{1}{8b}\Bigg(&4be^{\frac{a\left(a\tau+\sqrt{2}by\right)}{2b^2}}\operatorname{erf}\left(\frac{\sqrt{2}a\tau+by}{2b\sqrt{\tau}}\right)+4be^{\frac{a\left(a\tau+\sqrt{2}by\right)}{2b^2}}\\ &-2\sqrt{2}ay\left(\operatorname{erf}\left(\frac{\sqrt{\frac{y^2}{\tau}}}{2}\right)-1\right)\\ &-4\sqrt{\frac{2}{\pi}}a\sqrt{\tau}e^{-\frac{y^2}{4\tau}}-\frac{4by\left(\operatorname{erf}\left(\frac{\sqrt{\frac{y^2}{\tau}}}{2}\right)-1\right)}{y}\Bigg)\end{aligned} \tag{3.5}$$

and by Theorem 1, the equation (2.1) has solution, subject to (2.11), as

$$\begin{aligned}&u(x,t)\\ = \;&\frac{b^2}{2}\Bigg(2\,\ln(2)+\ln(\pi)-2\,\ln\Bigg(-\frac{1}{b^2x}\Bigg(-\sqrt{\pi}\mathrm{e}^{\frac{1}{2}\,\frac{a(a(T-t)+2\,x)}{b^2}}\\ &\operatorname{erf}\left(\frac{1}{2}\,\frac{(a\,(T-t)+x)\sqrt{2}}{b\sqrt{T-t}}\right)xb^2+x\sqrt{\pi}\left(ax+b^2\right)\operatorname{erf}\left(\frac{1}{2}\,\sqrt{2}\sqrt{\frac{x^2}{b^2\,(T-t)}}\right)\\ &-\sqrt{\pi}\mathrm{e}^{\frac{1}{2}\,\frac{a(a(T-t)+2\,x)}{b^2}}xb^2+\mathrm{e}^{-\frac{1}{2}\,\frac{x^2}{b^2(T-t)}}x\sqrt{2}\sqrt{T-t}ab-x\sqrt{\pi}\left(ax+b^2\right)\Bigg)\\ &\mathrm{e}^{\frac{1}{2}\,\frac{(-T+t)a^2-2\,ax-2\,k(T-t)}{b^2}}\Bigg)\Bigg).\end{aligned} \tag{3.6}$$

Once again, to check that this solution is subject to (2.11), let $t = T$. We plot this solution in Figure 1 and note that $u(x,t)$ in this context, represents a value function linked to minimising risk associated with hedging. The constants a and b are linked to the drift and diffusion that model the dynamics of the underlying asset as inherited from a stochastic model. We observe that under the IVP at fixed time, the value function is decreasing.

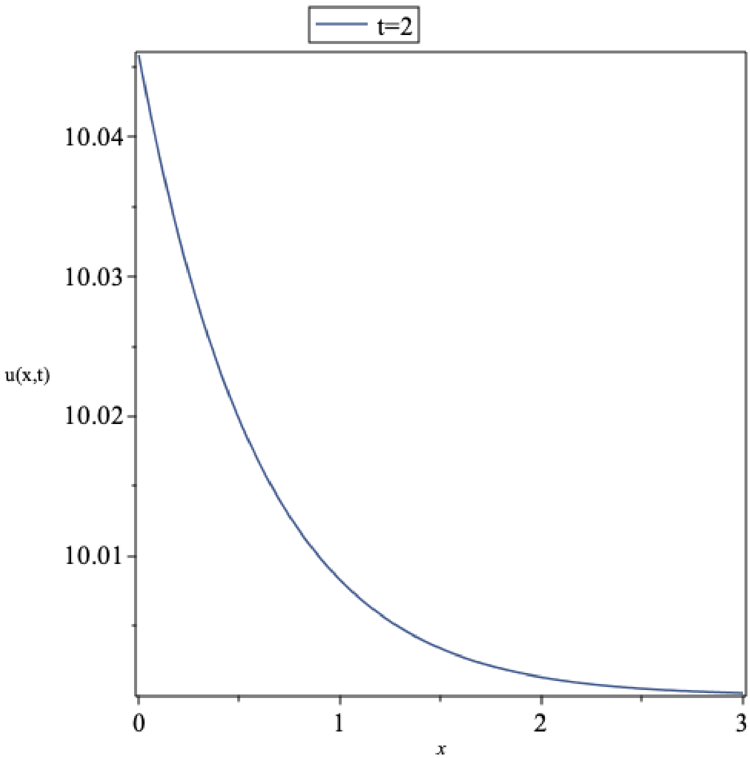


Figure 1: Graphical solution of (3.6) with $b = 1, T = 7, a = 1, k = 2$.

Under non IVP exact solutions, we may use Theorem 9 to generate solutions. We present this example in detail.

Case I:

$$u(x,t) = -b^2 \left(\ln (C_1) + \frac{\left(a^2 + 2\,k\right) t - 2\,ax}{2\,b^2} \right). \tag{3.7}$$

Case II:

$$\begin{aligned} &u(x,t) \\ &= -b^2 \left(\ln \left(C_1\, \mathrm{e}^{2\, \frac{\sqrt{c}\sqrt{2}x}{b}} + C_2 \right) + \frac{1}{2\,b^2} \left(-2\,\sqrt{c}\sqrt{2}xb + \left(-2\,cb^2 + a^2 + 2\,k\right) t - 2\,ax \right) \right). \end{aligned} \tag{3.8}$$

Case III: An application of the invariants of this subalgebra produces the exact solution as

$$\begin{aligned} u(x,t) &= -b^2 \Bigg(\ln \left(C_1 \operatorname{Ai}\left(\frac{t^2 b + \sqrt{2} x}{b} \right) + C_2 \operatorname{Bi}\left(\frac{t^2 b + \sqrt{2} x}{b} \right) \right) \\ &\quad + \frac{-6\sqrt{2} btx - 4 b^2 t^3 + \left(3 a^2 + 6 k\right) t - 6 ax}{6 b^2} \Bigg). \end{aligned} \tag{3.9}$$

Case IV: This subalgebra gives us the exact solution

$$\begin{aligned} &u(x,t) \\ = \; & -\frac{b^2}{4} \Bigg(-\ln(2) + 4 \ln \Bigg(\Bigg(C_1\, M_{\frac{i}{4}c,\, \frac{1}{4}} \left(\frac{-2\, ix^2}{(-2\,t+i)\,(2\,t+i)\, b^2} \right) \\ & + C_2\, W_{\frac{i}{4}c,\, \frac{1}{4}} \left(\frac{-2\, ix^2}{(-2\,t+i)\,(2\,t+i)\, b^2} \right) \Bigg) \times \\ & \mathrm{e}^{\frac{1}{b^2 (8\,t^2+2)} \left(4\, cb^2 \left(\frac{1}{4} + t^2\right) \arctan\left(\frac{1}{2}\, t^{-1}\right) + \left(4\, a^2 + 8\, k\right) t^3 - 8\, at^2 x + \left(a^2 + 4\, x^2 + 2\, k\right) t - 2\, ax\right)} \frac{\sqrt{b}}{\sqrt{x}} \Bigg) \Bigg). \end{aligned} \tag{3.10}$$

Case V:

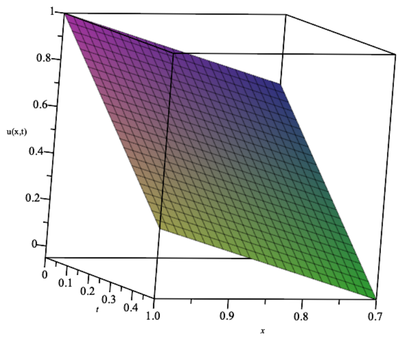
$$\begin{aligned} u(x,t) &= -\frac{b^2}{2} \Bigg(\ln(2) + 2 \ln \Bigg(\frac{x\,(-t)^{-\frac{1}{2}+c}}{b} \mathrm{e}^{\frac{1}{2}\, \frac{\left(a^2+2\,k\right) t^2 - 2\, atx + x^2}{b^2 t}} \\ &\quad \left(U\left(1+c, \frac{3}{2}, -\frac{1}{2}\, \frac{x^2}{b^2 t}\right) C_2 + M\left(1+c, \frac{3}{2}, -\frac{1}{2}\, \frac{x^2}{b^2 t}\right) C_1 \right) \Bigg) \Bigg). \end{aligned} \tag{3.11}$$

Selected solutions appear in Figure 2, which we recall are non IVP solutions of the behaviour of the value function $u(x,t)$. A decrease in the value function means that the risk associated with hedging is reducing as the process evolves, or that the hedging strategy is becoming more effective over time.

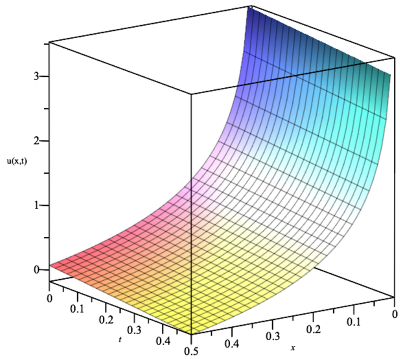
3.2. The nonlinear Potential Burgers' equation. This equation is a well-known variant of the infamous Burgers' equation [6]. It finds applications in the dynamics of fluids, traffic flow, geophysics and nonlinear waves. Under the parameter setting $A = C = 1$, $B = D = 0$ in (2.1), we recover the Potential Burgers' equation. Therefore, by direct calculation, $\gamma(y,\tau) = 0$ in Theorem 1 with $\rho = 1$. Hence all exact solutions are

$$u(x,t) = \ln(\omega(y,\tau)), \tag{3.12}$$

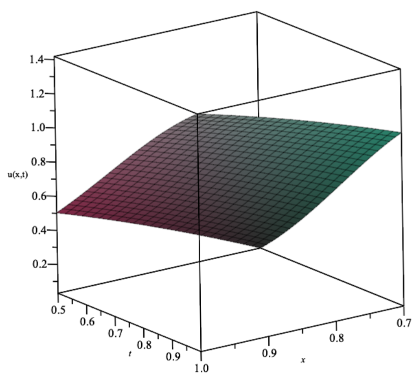
with $y = x, \tau = t$.



(a)



(b)



(c)

Figure 2: Graphical illustration of selected non IVP solutions: (a) Case I with parameter selection $C_1 = b = a = 1$ and $k = 1$; (b) Case II with parameter selection $C_1 = b = a = c = 1, C_2 = -1$ and $k = 1$; (c) Case III with parameter selection $C_1 = b = 2, a = c = C_2 = 1$ and $k = 1$.

A simple example of an IVP with $u(x,0) = x^2$ in Lemma 4, yields the Potential Burgers' IVP solution

$$u(x,t) = \ln\left(\frac{e^{\frac{x^2}{1-4t}}}{\sqrt{1-4t}}\right), \tag{3.13}$$

by application of Theorem 5 and Corollary 7. We plot this solution in Figure 3 with $u(x,t)$ as a gradient equal to a velocity field over space and time. The IVP indicates an increasing function correlating to the spatial variable over fixed time.

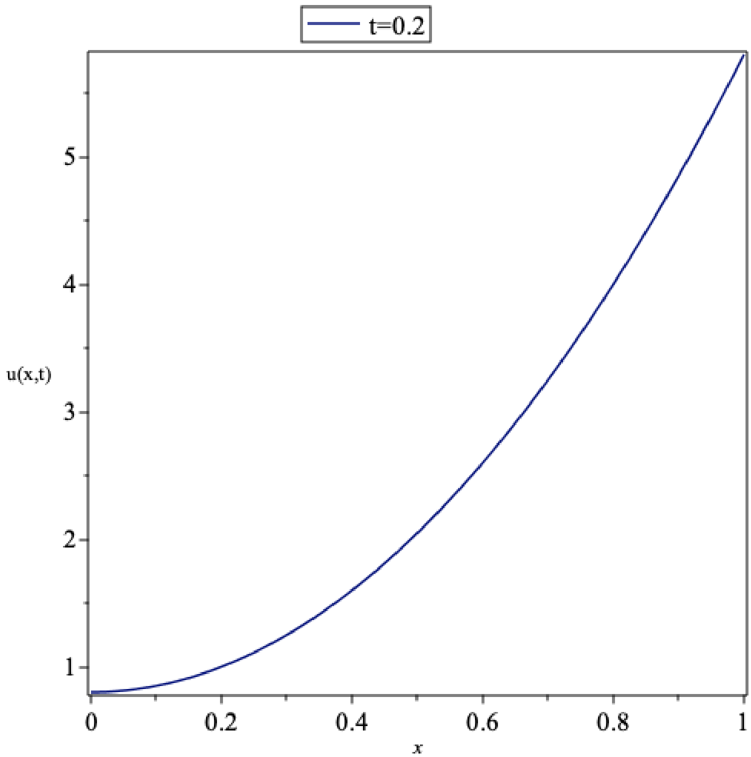


Figure 3: Graphical solution of (3.13).

Table 1 in Theorem 9 provides the exact solutions for non IVPs as given by the formula (3.12).

3.3. The strongly nonlinear Burgers' equation. The standard form of this equation is defined from (2.9) with the substitutions $A = 1$, $B = 0$, $C = -\frac{1}{2}$. For our purposes, Theorem 1 applies with $\rho = 1$ and by direct calculation, $\gamma(y, \tau) = 0$. Hence, by Theorem 3, we have that all solutions of the Burgers' equation are

$$\phi(x,t) = -2\frac{\partial}{\partial x}\ln(\omega(y,\tau)), \tag{3.14}$$

with $y = x, \tau = t$.

An IVP with $\phi(x, 0) = x$ in Lemma 4, yields the Burgers' IVP solution

$$\phi(x,t) = \frac{4x}{4+4t}, \tag{3.15}$$

by application of Theorem 5 and Corollary 7. We plot this solution in Figure 4 where typically in the context of say, fluids, $\phi(x,t)$ is the velocity field amid convection and diffusion. Solution (3.15) represents the velocity field subject to a kinematic viscosity of one and where initially, the velocity field equals the spatial variable.

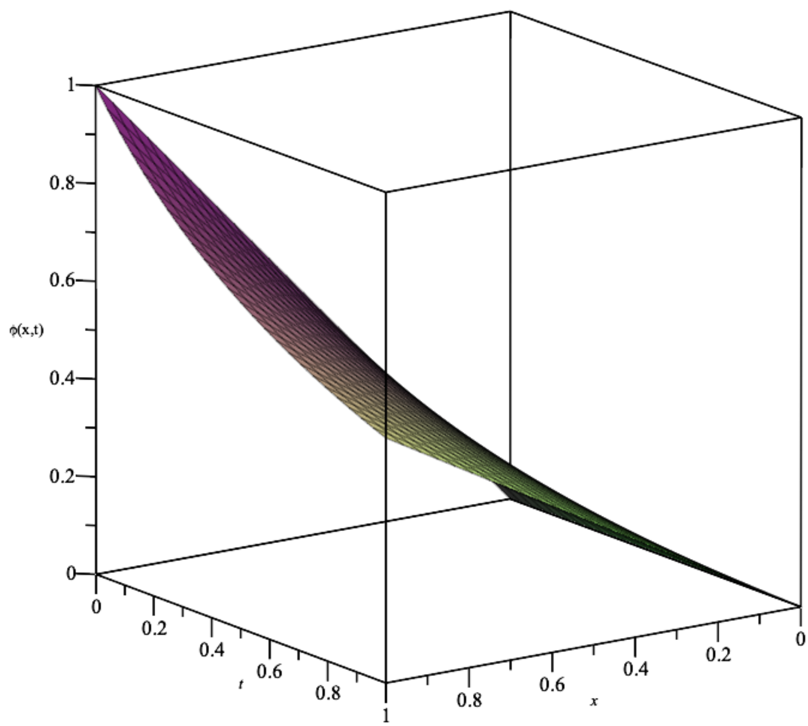


Figure 4: Graphical solution of (3.15).

Table 1 in Theorem 9 provides the exact solutions for non IVPs as per the formula (3.14).

4. Conclusion. In the present paper, we have discussed how a class of inherently nonlinear PDEs may be linearised to a parabolic equation and subsequently, solved with the heat equation. The study identified new independent and dependent variables that lead to the transformations, and removed the difficulty of solving this broad class of nonlinear PDEs subject to a Cauchy problem.

Lie symmetries were at the heart of the approach, whereby new symmetries were found that offer limitless possibilities in deriving solutions sensitive to initial conditions. As a result, the findings of the study demonstrate a potent interplay between linearising transformations and equivalence transformations.

The development of this new solution and accompanying method opens up several promising directions for future research. One natural extension is to explore how the approach can be adapted to broader classes of nonlinear equations, particularly those involving variable coefficients or coupled systems.

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