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Conservation Laws of some 'high-order' nonlinear PDEs using variational and 'partial-variational' principles

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Abstract

The construction of conserved vectors using Noether's theorem via a knowledge of a Lagrangian (or via the recently developed concept of partial Lagrangians) is well known. The formulae to determine these for higher-order flows is somewhat cumbersome and becomes more so as the order increases. We carry out these for a class of fourth, fifth and sixth order PDEs. In the latter case, we involve the fifth-order KdV equation using the concept of 'weak' Lagrangians analogous to the third-order KdV case. Then we considered the case of a mixed 'high-order' equations working on the Shallow Water Wave and Regularized Long Wave equations. These mixed type equations have not been dealt with thus far using this technique. We finally, analyse the conserved flows of some multi-variable equations that arises in relativity.

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Introduction

The work of Sophus Lie on transformation groups have had far reading and wide ranging consequences in the analysis of differential equations. The applications are now in broad use such as in mathematical physics, fluid mechanics, relativity and financial mathematics. The notion of symmetries entered the area of conservation law in variational equations through the work of Emmy Noether. The variational techniques that employed here are applications of recent works that provide slightly modified approaches to tackling the problem of conservation laws.

We discuss the use of these variational techniques on higher order partial differential equations (PDEs). The importance of investigating these sorts of equations, are due to their appearance in different branches of science and engineering, like plasma physics, fluid dynamics, quantum theory, nonlinear optics and solid state physics.

In the first chapter, we introduce the preliminary mathematics that is needed to tackle our investigation.

In chapter two, we discuss the role of this technique in attaining conservation laws for the fifth-order KdV, and fourth-order Boussinesq equations. For these and any high order partial differential equations (PDEs), finding conservation laws by first principles can be extremely tedious. The important point of consideration is the bulky formulae that is required due to the order of the Lagrangians and related functions.

In the third chapter we consider higher order mixed derivatives and their conservation laws using this technique. The Shallow Water Wave and Regularized Long Wave Equations are examined due to their highest derivative term being mixed. These sorts of equations have not been studied before using this technique.

In the fourth chapter we consider the classical wave equation in some Lorentzian space-time backgrounds with a point in mind that the wave equation there may naturally inherit nonlinearity from geometry. In this study we look at the case of multi-independent variables that are present in the Relativistic Wave Equations due to space-time variables.

Chapter 1

Preliminaries

1.1 Introduction

In this chapter, we introduce the preliminaries and results that are needed to tackle our investigation for higher-order partial differential equations.

1.2 Main Operators

We first introduce to the reader, the universal space A of differential functions. A locally analytic function $f(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(k)})$ of a finite number of variables is called a *differential function* of order k. The space A is the vector space of all differential functions of all finite orders and forms an algebra.

A total derivative converts any differential function of order k to a differential function of order $k + 1$. Hence, the space A is closed under total derivations. There are other operators on A and some of the important ones which we will utilize are explained below.

The summation convention is adopted throughout. Let $x = (x^1, \cdots, x^n)$ be the independent variable with co-ordinates x^i, and $u = (u^\alpha, \cdots, u^m)$ the dependent variable with co-ordinates u^α. The derivatives of the u with respect to x are

$$u_i{}^\alpha = D_i(u^\alpha), \quad u^\alpha_{ij} = D_{ij}(u^\alpha), \quad \cdots, \tag{1.1}$$

where

$$D = \frac{\partial}{\partial x^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij} \frac{\partial}{\partial u^\alpha_j} + \cdots, \quad i = 1, \cdots, n \tag{1.2}$$

is the *total differential operator*. The collection of all first derivatives $u_i{}^\alpha$ is denoted by $u_{(1)}$. Similarly, the collections of all higher-order derivatives are denoted by $u_{(2)}, u_{(3)}, \cdots$. Following Lie, in group analysis it is expedient to consider all variables $x, u, u_{(1)}, u_{(2)}, u_{(3)}, \cdots$ as functionally independent connected only by the differential relations (1.1). Consequently, the u^α are referred to as differential variables.

Intrinsic to the modern group analysis of differential equations is the universal space A defined above and as follows.

We denote by z the sequence

$$z = (x, u, u_{(1)}, u_{(2)}, \cdots) \tag{1.3}$$

with elements z^ν,$\nu \geq 1$, for example,

$z^i = x^i, \quad 1 \leq i \leq n, \quad z^{n+\alpha} = u^\alpha, \quad 1 \leq \alpha \leq m,$

with the remaining elements representing the derivatives of u. However, in application one invariably utilizes only infinite subsequences of z which are donated by $[z]$. A locally analytic function $f(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(k)})$ of a finite number of variables is called a differential function of order k and for brevity is written as $f([z])$. The space A is the vector space of all differential function of all finite orders. A total

derivative (1.2) converts any differential function of order k to a differential function of order $k+1$.

Hence, the space A is closed under total derivations D_i. The main operators introduced below are correctly defined in the space A. Precisely, this means that the operators defined as formal sums truncate when they act on differential functions.

Definition 1: The *Euler-Lagrange operator* is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}, \quad \alpha = 1,\cdots,m. \tag{1.4}$$

The operator (1.4) is sometimes referred to as the Euler operator, named after Euler (1744) who first introduced it in a geometrical manner of the one-dimensional case. Also, it is called the *Lagrange operator*, bearing the name of Lagrange (1762) who considered the multidimensional case and established its use in a variational sense (see for example, [1] for a history of the calculus of variations). Following Lagrange, equation (1.4) is frequently referred to as a variational derivative. In the modern literature, the terminology Euler-Lagrange and variational derivative are used interchangeably as (1.4) usually arises in considering a variational problem.

Definition 2: The *Lie-Bäcklund operator* is given by

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} \quad \xi^i, u^\alpha \in A. \tag{1.5}$$

This operator is in fact an abbreviated form of the following infinite formal sum,

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \zeta^\alpha_i\frac{\partial}{\partial {u_i}^\alpha} + \zeta^\alpha_{ij}\frac{\partial}{\partial u^\alpha_{ij}} + \cdots, \tag{1.6}$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned}\zeta_i^\alpha &= D_i(W^\alpha) + \xi^j u_{ij}^\alpha \\ \zeta_{i_1 i_2}^\alpha &= D_{i_1} D_{i_2}(W^\alpha) + \xi^j u_{i_1 i_2 j}^\alpha \\ &\cdots\end{aligned} \tag{1.7}$$

In (1.7), W^α is the *Lie characteristic function* given by

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha. \tag{1.8}$$

One can write the Lie-Bäcklund operator (1.6) in form

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + D_i(W^\alpha)\frac{\partial}{\partial {u_i}^\alpha} + D_{i_1} D_{i_2}(W^\alpha)\frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \cdots. \tag{1.9}$$

Definition 3: The *Noether operator* associated with a Lie-Bäcklund operator X is defined by

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta {u_i}^\alpha} + \sum_{s\geq 1} D_{i_1} \cdots D_{i_s}(W^\alpha)\frac{\partial}{\partial u_{i i_1 \cdots i_s}^\alpha}, \quad i = 1, \cdots, n \tag{1.10}$$

where the Euler-Lagrange operator with respects to derivatives of u^α are obtained from (1.4) by replacing u^α by the corresponding derivatives, for example,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s\geq 1}(-1)^s D_{j_1} \cdots D_{j_s}(W^\alpha)\frac{\partial}{\partial u_{i j_1 \cdots j_s}^\alpha}, \quad i = 1, \cdots, n, \quad \alpha = 1, \cdots, m. \tag{1.11}$$

The operator (1.10) is named the Noether operator and was given in recognition of the Noether contribution. As consequence of the operator (1.10), the proof of Noether's theorem becomes purely algebraic and independent of variational calculus. The algebraic proof is based on the identity presented in the next section.

1.3 Noether Identity

Theorem 1: The Euler-Lagrange, Lie-Bäcklund and Noether operators are connected by the operator identity

$$X + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i N^i. \tag{1.12}$$

Here, $D_i(\xi^i)$ is a differential function which is a sum of functions obtained by total derivations D_i of differential functions ξ_i. That is, $D_i(\xi^i)$ is a divergence of the vector $\xi = (\xi^1, \cdots, \xi^n)$, in other words, $div\xi$ whereas, $D_i N^i$ is an operator obtained as a sum of products of operators D_i on N^i, that is, it is scalar product of vector operators $D = (D_1, \cdots, D_n)$ and $N = (N^1, \cdots, N^n)$. The identity (1.12) is called the Noether identity because of its close relation to the Noether theorem.

1.4 Noether Generators

Consider a kth order differential equation

$$E^\alpha(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(k)}) = 0, \quad \alpha = 1, \cdots, m. \tag{1.13}$$

Definition 4: A conserved vector of (1.13) is tuple $T = (T^1, \cdots, T^n)$,
$T^j = T^j(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(k)}) \in A, \quad j = 1, ...n$, such that

$$D_i(T^i) = 0 \tag{1.14}$$

is satisfied for all solutions of (1.13).

REMARK. When Definition 4 is satisfied, (1.14) is called a conservation law for (1.13).

We now discuss conservation law of Euler-Lagrange equations. That is, differential equations of the form

$$\frac{\delta L}{\delta u^{\alpha}} = 0, \quad \alpha = 1, \cdots, m, \tag{1.15}$$

where $L = L(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(l)}) \in A$, $l \leq k$, k being the order of (1.15), is a Lagrangian and $\frac{\delta}{\delta u^{\alpha}}$ is the Euler-Lagrange operator defined by (1.4).

Definition 5: A Lie-Bäcklund operator X of the form (1.6) is called a Noether symmetry corresponding to a Lagrangian $L \in A$ if there exists a vector $B = (B^1, \cdots, B^n)$, $B^i \in A$, such that

$$X(L) + LD_i(\xi^i) = D_i(B^i). \tag{1.16}$$

If equation (1.16), $B^i = 0, \quad i = 1, \cdots, n$, then X is referred to as strict Noether symmetry corresponding to a Lagrangian $L \in A$.

Theorem 2: For a any Noether symmetry X corresponding to a given Lagrangian $L \in A$, there corresponds a vector $T = (T^1, \cdots, T^n), \quad T^i \in A$, defined by

$$T^i = N^i(L) - B^i, \quad i = 1, \cdots, n, \tag{1.17}$$

which is a conserved vector of equation (1.15), that is, $D_i(T^i) = 0$ on the solutions of (1.15).

1.5 Noether-Type Generators

Consider a kth order differential system

$$E^{\alpha}(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(k)}) = 0, \quad \alpha = 1, \cdots, m \tag{1.18}$$

which is of maximal rank and locally solvable. The following definition is well-known.

REMARK. When Definition 4 is satisfied, (1.14) is called the local conservation law for (1.18). Also $D_i T^i = Q^\alpha E_\alpha$ is referred to as the characteristic form of conservation law (1.18) and the function $Q = (Q^1, \cdots, Q^n)$ the associated characteristic form of the conservation law.

Suppose that equations (1.18) are written as

$$E_\alpha \equiv E^0_\alpha + E^1_\alpha = 0, \quad \alpha = 1, \cdots, m. \tag{1.19}$$

We now introduce the definition of Partial Lagrangian.

Definition 6: If there exists a function $L = L(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(l)}) \in A, l \leq k$ and non-zero functions $f^\beta_\alpha \in A$ such that (1.19) can be written as $\delta L/\delta u^\alpha = f^\beta_\alpha E^1_\beta$ then, provided $E^1_\beta \neq 0$, L is called a partial Lagrangian of equation (1.19) otherwise it is the standard Lagrangian. It is known that differential equations of the form $\frac{\delta L}{\delta u^\alpha} = 0, \quad \alpha = 1, \cdots, m$, are Euler-Lagrange equations. We term differential equations of the form

$$\frac{\delta L}{\delta u^\alpha} = f^\beta_\alpha E^1_\beta, \tag{1.20}$$

as Euler-Lagrange-type equations.

Definition 7: A Lie-Bäcklund or generalized operator X of the form (1.6) is called a Noether-type symmetry operator corresponding to a partial Lagrangian $L \in A$ if there exists a vector $B = (B^1, \cdots, B^n), \quad B^i \in A, \quad B^i \neq N^i L + C^i, C^i$ constants, such that

$$X(L) + LD_i(\xi^i) = W^\alpha \frac{\delta L}{\delta u^\alpha} + D_i(B^i), \tag{1.21}$$

where $W = (W^1, \cdots, W^m), W^\alpha \in A$ is characteristics of X.

If the B^i's are identically zero , then the Lie-Bäcklund operator X is called a *strict Noether-type symmetry operator*.

Note that for Euler-Lagrange equations $\delta L/\delta u^\alpha = 0$, if (1.21) is satisfied, X is a *Noether symmetry generator* corresponding to a standard Lagrangian L.

Recall also that for a Noether symmetry generator X corresponding to a standard L, X is said to leave the functional invariant up to gauge B. It is easy to see from (1.21) that if X and Y are Noether-type operators, then so is a linear combination of these operators. Indeed the Noether-type symmetry operators span a vector space.

Theorem 3: A Lie-Bäcklund symmetry operator X of the form (1.9) is a *Noether-type symmetry operator* of a partial Lagrangian L corresponding to an *Euler-Lagrange-type system* of the form (1.20) if and only if the characteristic $W = (W^1, \cdots, W^m), W^\alpha \in A$, of X is also the characteristic of the conservation law $D_i T^i = 0$, where

$$T^i = N^i(L) - B^i, \quad i = 1, \cdots, n, \tag{1.22}$$

of the Euler-Lagrange-type Equations (1.20).

Proof: We use identity (1.12) and act with it on L to obtain

$$XL + D_i(\xi^i)L = W^\alpha \frac{\delta L}{\delta u^\alpha} + D_i N^i L \tag{1.23}$$

Since X is a Noether-type symmetry operator of an L corresponding to an Euler-Lagrange system, we can by use of (1.20) replace the left hand side of the last equation(1.23) with $W^\alpha \delta L/\delta u^\alpha + D_i B^i$ which in turn can be replaced by $W^\alpha f^\beta_\alpha E^1_\beta + D_i B^i$ by utilizing (1.20). We immediately get

$$W^\alpha f^\beta_\alpha E^1_\beta + D_i B^i = W^\alpha \frac{\delta L}{\delta u^\alpha} + D_i N^i L. \tag{1.24}$$

From this we have

$$D_i(B^i - N^i L) = W^\alpha (\frac{\delta L}{\delta u^\alpha} - f^\beta_\alpha E^1_\beta) \tag{1.25}$$

and thus

$$D_i T^i = W^\alpha (\frac{\delta L}{\delta u^\alpha} - f^\beta_\alpha E^1_\beta) \tag{1.26}$$

as a consequence of (1.21) is a conservation law with conserved components $T^i = B^i - N^i(L)$ of the system (1.20) with characteristic W. The steps are reversible. This proves the result. □

A further detailed analysis of the operators is completely given below for the scalar case in two dimensions, viz., (t, x). This discussion is peculiar to our work in the sequel as the Lagrangian and conserved flows are of a high order (third-order). The proofs and finer details of the results are obtainable in [2]. Suppose $X = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \phi(t, x, u)\partial_u$ is a Noether point symmetry generator with gauge (f, g). Then the conserved flow (T^t, T^x) is given by

$$\begin{aligned}
T^t &= L\tau + W\tfrac{\delta L}{\delta u_t} + D_t(W)\tfrac{\delta L}{\delta u_{tt}} + D_x(W)\tfrac{\delta L}{\delta u_{tx}} \\
&\quad + D_tD_t(W)\tfrac{\delta L}{\delta u_{ttt}} + D_tD_x(W)\tfrac{\delta L}{\delta u_{tx}} + D_xD_x(W)\tfrac{\delta L}{\delta u_{txx}} - f \\
&= L\tau + W(\tfrac{\partial L}{\partial u_t} - D_t\tfrac{\partial L}{\partial u_{tt}} - D_x\tfrac{\partial L}{\partial u_{tx}} + D_t^2\tfrac{\partial L}{\partial u_{ttt}} + D_x^2\tfrac{\partial L}{\partial u_{txx}} + D_tD_x\tfrac{\partial L}{\partial u_{ttx}}) \\
&\quad + D_t(W)\tfrac{\delta L}{\delta u_{tt}} + D_x(W)\tfrac{\delta L}{\delta u_{tx}} \\
&\quad + D_tD_t(W)\tfrac{\delta L}{\delta u_{ttt}} + D_tD_x(W)\tfrac{\delta L}{\delta u_{tx}} + D_xD_x(W)\tfrac{\delta L}{\delta u_{txx}} - f, \\
T^x &= L\xi + W\tfrac{\delta L}{\delta u_x} + D_t(W)\tfrac{\delta L}{\delta u_{xt}} + D_x(W)\tfrac{\delta L}{\delta u_{xx}} \\
&\quad + D_tD_t(W)\tfrac{\delta L}{\delta u_{xtt}} + D_tD_x(W)\tfrac{\delta L}{\delta u_{xxt}} + D_xD_x(W)\tfrac{\delta L}{\delta u_{xxx}} - g \\
&= L\xi + W(\tfrac{\partial L}{\partial u_x} - D_t\tfrac{\partial L}{\partial u_{xt}} - D_x\tfrac{\partial L}{\partial u_{xx}} + D_t^2\tfrac{\partial L}{\partial u_{xtt}} + D_x^2\tfrac{\partial L}{\partial u_{xxx}} + D_tD_x\tfrac{\partial L}{\partial u_{txx}}) \\
&\quad + D_t(W)\tfrac{\delta L}{\delta u_{xt}} + D_x(W)\tfrac{\delta L}{\delta u_{xx}} \\
&\quad + D_tD_t(W)\tfrac{\delta L}{\delta u_{xtt}} + D_tD_x(W)\tfrac{\delta L}{\delta u_{xxt}} - g,
\end{aligned} \tag{1.27}$$

where

$$\frac{\delta}{\delta v} = \frac{\partial}{\partial v} - D_t\frac{\partial}{\partial v_t} - D_x\frac{\partial}{\partial v_x} + D_t^2\frac{\partial}{\partial v_{tt}} + D_x^2\frac{\partial}{\partial v_{xx}} + D_tD_x\frac{\partial}{\partial v_{tx}} - \dots. \tag{1.28}$$

A range of literature pertaining to conservation laws is now available mainly presenting the various methods involved, see [**?**, **?**, **?**, **?**, 9].

1.6 Illustrative example

For simplicity we have looked at point type symmetry operators and we have restricted the gauge terms to be independent of derivatives. One can equally well try to obtain true Lie-Bäcklund type symmetry operators and our method still applies. However, the calculations in this case are quite tedious and best left for a computer algebra package. The illustrative example [2], is on the classical heat equation. Although simple, it is considered a paradigm for evolution equations and is frequently utilized as benchmark for ones approach.

Consider the (1+1) linear heat equation

$$u_t = u_{xx}. \tag{1.29}$$

If we invoke the partial Lagrangian $L = u_x^2/2, \delta L/\delta u = -u_{xx}$ so that (1.29) can be written as $u_t = -\delta L/\delta u$ and, therefore, $\delta L/\delta u$ can be replaced by $-u_t$ in (1.29) to determine the Noether-type operators, by Definition 7, $X = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u}$ corresponding to L. That is,

$$\zeta_x u_x + (D_t\tau + D_x\xi)(\frac{1}{2}u_x^2) = (\eta - \tau u_t - \xi u_x)(-u_t) + D_t B^1 + D_x B^2. \tag{1.30}$$

Expansion of the total derivative operators as well as ζ_x and then separation of the

derivatives of u yield the over-determined linear system

$$\begin{array}{lcl} u_x^3 & : & \xi_u = 0, \\ u_x^2 & : & \eta_u = 0, \\ u_x^2 u_t & : & \tau_u = 0, \\ u_x u_t & : & \xi = -\tau_x, \\ u_t^2 & : & \tau = 0, \\ u_x & : & \eta_x = B_u^2, \\ u_t & : & \eta = B_u^1, \\ 1 & : & B_t^1 + B_x^2 = 0. \end{array} \tag{1.31}$$

The calculations reveal that $X = \eta(t,x)\frac{\partial}{\partial u}$ where η satisfies the equation

$$\eta_t + \eta_{xx} = 0 \tag{1.32}$$

and $B^1 = \eta u + f(t,x), \quad B^2 = \eta_x u + g(t,x)$,where $f_t + g_x = 0$. We set $f = g = 0$. The corresponding conserved vector components, by Theorem 3, are $T^1 = \eta u$ and $T^2 = -\eta u_x + \eta_x u$. The corresponding conservation law $D_t T^1 + D_x T^2 = 0$ is $\eta(u_t - u_{xx}) = 0$ with characteristic η which is the characteristic of the Noether-type symmetry operator X.

Thus, if for example

(i)$\eta = 1, \quad T^1 = u, \quad T^2 = -u_x,$
(ii)$\eta = t - \frac{1}{2}x^2, \quad T^1 = (t - \frac{1}{2}x^2)u, \quad T^2 = -(t - \frac{1}{2}x)u_x - ux.$

Chapter 2

Higher order PDEs

2.1 Introduction

The fifth-order KdV, and fourth-order Boussinesq equations are well known examples from mathematical physics purported to be of 'high' order. For these and any high order partial differential equations (PDEs), finding conservation laws by first principles can be extremely tedious. Thus, one needs to resort to alternate methods appealing to the underlying symmetry generators of the equations. If this means the variational route, then there may be problems such as the existence and determination of a Lagrangian. For the two cases cited here, we construct 'weak' or 'partial' Lagrangian and successfully construct conservation laws. The point of emphasis is the cumbersome formulae that is required in the determination of the conserved flows due to the order of the Lagrangians and related functions.

2.2 The fifth-order KdV equation

The particular case that we investigate is the well known generalized fifth-order KdV, (also known as the KdV-5 equation)

$$v_{xxxxx} + \alpha v_x v_{xx} + \frac{\beta}{2} v v_{xxx} + \gamma v^2 v_x + v_t = 0, \tag{2.1}$$

where α, β, γ are arbitrary non-zero constants.

For a variety of combinations of the parameters, (2.1) has been studied using a number of methods, analytical and numerical. Inc [3] and Abbasandy & Zakaria [4] made a detailed numerical study using the Adomian decomposition and homotopy analysis methods, respectively. Several works on the soliton solutions and various analytical methods have been done. For e.g., Lax [5] ($\beta/2 = 10, \alpha = 20, \gamma = 30$), Sawada-Kotera [6] ($\beta/2 = 5, \alpha = 5, \gamma = 5$), Ito [7] ($\beta/2 = 3, \alpha = 6, \gamma = 2$). The well known Kaup-Kuperschmidt equation is based on the case $\beta/2 = -15, \alpha = -15, \gamma = 45$.

The standard third-order KdV equation (KdV-3) is an evolution equation but its differential consequence admits a Lagrangian [8] and, thus, the KdV equation itself is construed as a variational equation. We show that one can do this for (2.1) by which some interesting results regarding conservation laws via Noether's theorem are obtained.

2.2.1 The sixth-order expansion of the KdV-5 equation

This analogous study of the KdV-5 equation has not, to the knowledge of the author, been done before. This may be due to the cumbersome forms of the extended Euler-Lagrange operators that need to be used.

If equation (2.1) is differentiated by x or if $v = u_x$ in (2.1), we get the sixth-order equation

$$u_{xxxxxx} + \alpha u_{xx}u_{xxx} + \frac{\beta}{2}u_x u_{xxxx} + \gamma u_x^2 u_{xx} + u_{xt} = 0 \tag{2.2}$$

which has a partial Lagrangian

$$L = -[\frac{1}{2}u_{xxx}^2 + \frac{1}{2}u_x u_t + \frac{\gamma}{12}u_x^4 + \frac{\beta}{8}u_x^2 u_{xxx}] \tag{2.3}$$

so that

$$\frac{\delta L}{\delta u} = u_{xt} + \frac{\beta}{2}u_x u_{xxxx} + \gamma u_x^2 u_{xx} + u_{xxxxxx} = (\beta - \alpha)u_{xx}u_{xxx}. \tag{2.4}$$

Now applying the partial Lagrangian to the Noether-type Identity (1.21), the following expression

$$X^{[3]}(L) + L(D_t\tau + D_x\xi) = (\phi - u_t\tau - u_x\xi)\frac{\delta L}{\delta u} + (D_t f + D_x g), \tag{2.5}$$

where f and g are gauge functions. The governing equations are obtained from (2.5), by separating the equations by coefficients. These coefficients are derivatives of the dependent variable u.

The separation of monomials are listed as

$$
\begin{aligned}
u_t u_x^2 u_{xxx} &: \tau_u,\\
u_t u_x^2 u_{xx} u_{xxx} &: \tau_{uu},\\
u_x^2 u_{xx} u_{xxx} &: \xi_{uu},\\
u_x u_{xxx}^2 &: \xi_u,\\
u_x^3 u_{xxx} &: \phi_{uuu},\\
u_{xxx} u_{xxt} &: \tau_x,\\
u_{xxx}^2 &: \tfrac{5}{2}\xi_x - \tfrac{1}{2}\tau_t - \phi_u,\\
u_x^2 u_{xxx} &: \tfrac{1}{2}[\tfrac{5\beta}{4}\xi_x - \tfrac{1}{4}\beta(\xi_x + \tau_t) - \tfrac{3\beta}{4}\phi_u - 6\phi_{xuu}],\\
u_x u_{xx} u_{xxx} &: (\beta - \alpha)\xi,\\
u_t u_{xx} u_{xxx} &: (\beta - \alpha)\tau,\\
u_{xx} u_{xxx} &: (\beta - \alpha)\phi,\\
u_x u_{xxx} &: -\tfrac{3}{4}\beta\phi_x\phi + \xi_{xxx} - 3\phi_{xxu},\\
u_{xxx} &: -\phi_{xxx},\\
u_x^3 u_{xx} &: -\tfrac{3}{8}\beta\phi_{uu},\\
u_x^2 u_{xx} &: \tfrac{1}{2}(\tfrac{3}{4}\beta\xi_{xx} - \tfrac{3}{4}\beta\phi_{xu}),\\
u_x^4 &: \tfrac{9}{4}\xi_x - \tfrac{9}{12}\tau_t - \tfrac{9}{3}\phi_u,\\
u_x^3 &: -\tfrac{9}{3}\phi_u + \tfrac{1}{8}\beta\xi_{xxx},\\
u_x^2 &: \xi_t,\\
u_t u_x &: \tfrac{1}{2}\xi_x + \tfrac{1}{2}(-\xi_x - \tau_t) + \tfrac{1}{2}\tau_t - \phi_u,\\
u_t &: -f_u - \tfrac{1}{2}\phi_x,\\
u_x &: -g_u - \tfrac{1}{2}\phi_t,\\
1 &: -f_t - g_x.
\end{aligned}
\tag{2.6}
$$

From the governing equations (2.6), it can be observed that there are two cases appearing (i) $\alpha \neq \beta$ and (ii) $\alpha = \beta$.

In the case(i), we obtain no symmetry generators due to ξ, τ and ϕ are equal to zero, therefore trivial solutions.

The case(ii), leads to a nontrivial solution. That is, the partial Lagrangian is, in fact, a Lagrangian of (2.2) due to $\alpha = \beta$, where $\frac{\delta L}{\delta u} = (\beta - \alpha)u_{xx}u_{xxx} = 0$, which

therefore changes the Noether-type Identity to Noether Identity,

$$X^{[3]}(L) + L(D_t\tau + D_x\xi) = (D_t f + D_x g). \tag{2.7}$$

The generators are the corresponding Noether symmetries, viz.,

$$X = \partial_t \quad (W = -u_t), \qquad X = \xi\partial_x \quad (W = -\xi u_x). \tag{2.8}$$

We now list the corresponding conserved vectors which are obtained from the given formula (1.27).

Note that the formula (1.27) is for a second order Lagrangian, the equations that we are dealing with are of third order, so there has to be a further expansion to the formula to obtain the conserved quantities.

(1) $X = \partial_t \quad (W = -u_t)$

$$\begin{aligned}
T^t &= -(\frac{1}{2}u_{xxx}^2 + \frac{1}{2}u_x u_t + \frac{\gamma}{12}u_x^4 + \frac{\beta}{8}u_x^2 u_{xxx})(1) + (-u_t)(-\frac{1}{2}u_x), \\
&= -\frac{1}{2}u_{xxx}^2 - \frac{1}{2}u_x u_t - \frac{\gamma}{12}u_x^4 - \frac{\beta}{8}u_x^2 u_{xxx} + \frac{1}{2}u_x u_t, \\
&= -\frac{1}{2}u_{xxx}^2 - \frac{\gamma}{12}u_x^4 - \frac{\beta}{8}u_x^2 u_{xxx},
\end{aligned}$$

$$\begin{aligned}
T^x &= (-u_t)(-\frac{1}{2}u_t - \frac{\gamma}{3}u_x^3 - \frac{\beta}{4}u_x u_{xxx} + \mathrm{D}_x^2(-u_{xxx} - \frac{\beta}{8}u_x^2)) \\
&+ \mathrm{D}_x(-u_t)(-\mathrm{D}_x(-u_{xxx} - \frac{\beta}{8}u_x^2)) + \mathrm{D}_x^2(-u_t)(-u_{xxx} - \frac{\beta}{6}u_x^2), \\
&= (u_t)(\frac{1}{2}u_t + u_{xxxxx} + \frac{\beta}{4}u_{xx}^2 + \frac{\beta}{4}u_x u_{xxx}) \\
&+ (-u_{tx})(u_{xxxx} + \frac{\beta}{4}u_x u_{xx}) + (u_{txx})(u_{xxx} + \frac{\beta}{8}u_x^2), \\
&= \frac{1}{2}u_t^2 + u_t u_{xxxxx} + \frac{\beta}{4}u_t u_{xx}^2 + \frac{\beta}{4}u_t u_x u_{xxx} - u_{tx}u_{xxxx} - \frac{\beta}{4}u_{tx}u_x u_{xx} \\
&+ u_{txx}u_{xxx} + \frac{\beta}{8}u_{txx}u_x^2.
\end{aligned}$$

Thus,

$$\begin{aligned} D_tT^t + D_xT^x &= D_t(-\frac{1}{2}u_{xxx}^2 - \frac{\gamma}{12}u_x^4 - \frac{\beta}{8}u_x^2u_{xxx}) \\ &+ D_x(\frac{1}{2}u_t^2 + u_tu_{xxxxx} + \frac{\beta}{4}u_tu_{xx}^2 + \frac{\beta}{4}u_tu_xu_{xxx}) \\ &+ D_x(-u_{tx}u_{xxxx} - \frac{\beta}{4}u_{tx}u_xu_{xx} + u_{txx}u_{xxx} + \frac{\beta}{8}u_{txx}u_x^2), \\ &= u_t(u_{xt} + \beta/2u_xu_{xxxx} + \beta u_{xx}u_{xxx} + \gamma u_x^2u_{xx} + u_{xxxxxx}), \\ &= 0. \end{aligned}$$

(2) $X = \xi\partial_x \quad (W = -\xi u_x)$

$$\begin{aligned} T^t &= -\xi u_x(-\frac{1}{2}u_x), \\ &= \frac{1}{2}\xi u_x^2, \\ T^x &= -\xi(\frac{1}{2}u_{xxx}^2 + \frac{\gamma}{12}u_x^4 + \frac{\beta}{8}u_x^2u_{xxx}) + \xi u_x(u_{xxxxx} + \frac{\beta}{4}u_{xx}^2 + \frac{\beta}{4}u_xu_{xxx}) \\ &- \xi u_{xx}(u_{xxxx} + \frac{\beta}{4}u_xu_{xx}) + \xi u_{xxx}(u_{xxx} + \frac{\beta}{8}u_x^2) + \xi u_x(\frac{\gamma}{3}u_{xxx}^3 + \frac{\gamma}{4}u_xu_{xxx}) \end{aligned}$$

Thus,

$$\begin{aligned} D_tT^t + D_xT^x &= D_t(\frac{1}{2}\xi u_x^2) + D_x(-\xi(\frac{1}{2}u_{xxx}^2 + \frac{\gamma}{12}u_x^4 + \frac{\beta}{8}u_x^2u_{xxx})) \\ &+ D_x(\xi u_x(u_{xxxxx} + \frac{\beta}{4}u_{xx}^2 + \frac{\beta}{4}u_xu_{xxx}) - \xi u_{xx}(u_{xxxx} + \frac{\beta}{4}u_xu_{xx})) \\ &+ D_x(\xi u_{xxx}(u_{xxx} + \frac{\beta}{8}u_x^2) + \xi u_x(\frac{\gamma}{3}u_{xxx}^3 + \frac{\gamma}{4}u_xu_{xxx})), \\ &= \xi u_x(u_{xt} + \beta/2u_xu_{xxxx} + \beta u_{xx}u_{xxx} + \gamma u_x^2u_{xx} + u_{xxxxxx}), \\ &= 0. \end{aligned}$$

REMARK. The conserved vector in (1) is of 'nonlocal' type for the fifth-order KdV equation (2.1) when we substitute back to v since, if $v = u_x$, $u_t = \int v_t \mathrm{d}x$.

2.2.2 Traveling Wave reduction of the KdV-5 Equation

We now reduce the KdV-5 equation to a fourth-order equation by taking the sum of the symmetries from the previous result (2.8). By using the characteristic method of solving PDEs, we obtain $y = x - ct$, that is a traveling wave equation. By taking $v = w$, we find that $v_t = -cw'$ and $v_x = w'$. Then substituting the derivatives into the main equation we end up with

$$w'''' + \frac{\beta}{2}ww''' + \alpha w'w'' + \gamma w^2 w' - cw' = 0. \tag{2.9}$$

We then integrate Equation(2.9), which becomes

$$w'''' + \frac{\beta}{2}ww'' - \frac{\beta}{4}w'^2 + \frac{\alpha}{2}w'^2 + \frac{\gamma}{3}w^3w' - cw = k, \tag{2.10}$$

which also can be written as

$$w'''' + \frac{\beta}{2}ww'' + (\frac{\alpha}{2} - \frac{\beta}{4})w'^2 + \frac{\gamma}{3}w^3w' - cw = k, \tag{2.11}$$

for the study of different cases, which has partial Lagrangian

$$L = \frac{1}{2}w''^2 + (\frac{\alpha}{2} - \frac{\beta}{4})ww'^2 + \frac{\gamma}{12}w^4 - \frac{1}{2}cw^2 - kw \tag{2.12}$$

which has

$$\frac{\delta L}{\delta w} = (\alpha - \beta)ww''. \tag{2.13}$$

(i) For $\alpha \neq \beta$, equation (2.11) has partial Lagrangian (2.12), applying the Noether-type Identity which is

$$X^{[2]}(L) + L(D_y\sigma) = (\eta - w'\sigma)\frac{\delta L}{\delta w} + (D_t f) \tag{2.14}$$

The separation of monomials are:

$$\begin{array}{lcl} w_y^3 w_{yy} & : & \sigma_{ww}, \\ w_y^2 w_{yy} & : & \eta_{ww} - 2\sigma_{yw}, \\ w_y w_{yy} & : & 2\eta_{yw} - \sigma_{yy} + w(\alpha - \beta), \\ w_y w_{yy}^2 & : & \sigma_w, \\ w_{yy}^2 & : & \eta_w - \frac{3}{2}\sigma_y, \\ w_{yy} & : & \eta_{yy} - w\eta(\alpha - \beta), \\ w_y^2 & : & \eta(\frac{\alpha}{2} - \frac{\beta}{4}) + w\eta_w(\alpha - \frac{\beta}{2}) - w\sigma_y(\frac{\alpha}{2} - \frac{\beta}{4}), \\ w_y & : & f_w + (\alpha - \frac{\beta}{2})w\eta_y, \\ 1 & : & -cw\eta + \frac{1}{3}w^3\gamma + \frac{\gamma}{12}w^4\sigma_y - \frac{c}{2}w^2\sigma_y - f_y. \end{array} \tag{2.15}$$

We obtain no symmetry generators since σ and η are equal to zero - this leads to a trivial solution.

(ii) For $\alpha = \beta$, equation (2.11) transforms to

$$w'''' + \frac{\beta}{2}ww'' + \frac{\beta}{4}w'^2 + \frac{\gamma}{3}w^3w' - cw = k, \tag{2.16}$$

which has partial Lagrangian

$$L = \frac{1}{2}w''^2 - \frac{\beta}{4}ww'^2 + \frac{\gamma}{12}w^4 - \frac{1}{2}cw^2 - kw. \tag{2.17}$$

The separation of monomials are:

$$\begin{array}{lcl} w_y^3 w_{yy} & : & \sigma_{ww}, \\ w_y^2 w_{yy} & : & \eta_{ww} - 2\sigma_{yw}, \\ w_y w_{yy} & : & 2\eta_{yw} - \sigma_{yy}, \\ w_y w_{yy}^2 & : & \sigma_w, \\ w_{yy}^2 & : & \eta_w - \frac{3}{2}\sigma_y, \\ w_{yy} & : & \eta_{yy}, \\ w_y^2 & : & -\frac{\beta}{4}\eta - \frac{1}{2}w\beta\eta_w + \frac{1}{4}w\beta\sigma_y, \\ w_y & : & -f_w - \frac{1}{2}w\beta\eta_y, \\ 1 & : & -cw\eta + \frac{1}{3}w^3\gamma + \frac{\gamma}{12}w^4\sigma_y - \frac{c}{2}w^2\sigma_y - f_y. \end{array} \tag{2.18}$$

This leads to nontrivial solution, that is, the partial Lagrangian is, in fact, a Lagrangian of (2.10) and the generators are the corresponding Noether symmetries, viz.,

$$\partial_y \quad (W = w') \tag{2.19}$$

with the corresponding conserved vector.

$$\begin{aligned} T &= \frac{1}{2}w''^2 - \frac{\beta}{4}ww'^2 + \frac{\gamma}{12}w^4 - \frac{1}{2}cw^2 + \frac{\beta}{2}ww'^2 + w'w''' - w''^2, \\ &= -\frac{1}{2}w''^2 + \frac{\beta}{4}ww'^2 + \frac{\gamma}{12}w^4 - \frac{1}{2}cw^2 + w'w''', \end{aligned}$$

$$\begin{aligned} D(T) &= -w''w''' + \frac{\beta}{4}(w'^3 + 2ww'w'') + \frac{\gamma}{3}w^3w' - cww' + w''w''' + w'w'''', \\ &= \frac{\beta}{4}(w'^3 + 2ww'w'') + \frac{\gamma}{3}w^3w' - cww' + w'w'''', \\ &= w'(w'''' + \frac{\beta}{4}w'^2 + \frac{\beta}{2}ww'' + \frac{\gamma}{3}w^3 - cw), \\ &= 0. \end{aligned}$$

2.3 The fourth-order Boussinesq equation

The Boussinesq equation which models the behaviour of long waves is sometimes written as the fourth-order

$$u_{xxxx} + uu_{xx} + u_x^2 + u_{tt} = 0. \tag{2.20}$$

The highest derivative in the equation (2.20) is a singular independent variable derivative term. Its Noether type symmetries, $X = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \phi(t,x,u)\partial_u$, via the partial Lagrangian

$$L = \frac{1}{2}u_{xx}^2 - \frac{1}{2}u_t^2 - \frac{1}{2}uu_x^2, \tag{2.21}$$

which has

$$\frac{\delta L}{\delta u} = -\frac{1}{2}u_x^2 \tag{2.22}$$

is determined by (1.20). In this case, XL is a second prolongation of X, viz.,

$$\begin{aligned}
XL = &-\tfrac{1}{2}\phi u_x^2 + u_t u_x \xi_t + u_t^2 u_x \xi_u + u u_x^3 \xi_u + u u_x^2 \xi_x + \\
&u_t^2 \tau_t + u_t^3 \tau_u + u u_t u_x^2 \tau_u + u u_t u_x \tau_x - u_t \phi_t - \\
&u_t^2 \phi_u - u u_x^2 \phi_u - u u_x \phi_x - 2 u_x \tau_u u_{x,t} u_{x,x} - \\
&2\tau_x u_{x,t} u_{x,x} - 3 u_x \xi_u u_{x,x}^2 - 2\xi_x u_{x,x}^2 - \\
&u_t \tau_u u_{x,x}^2 + \phi_u u_{x,x}^2 - u_x^3 u_{x,x} \xi_{u,u} - 2 u_x^2 u_{x,x} \xi_{x,u} - \\
&u_x u_{x,x} \xi_{x,x} - u_t u_x^2 u_{x,x} \tau_{u,u} - 2 u_t u_x u_{x,x} \tau_{x,u} - \\
&u_t u_{x,x} \tau_{x,x} + u_x^2 u_{x,x} \phi_{u,u} + 2 u_x u_{x,x} \phi_{x,u} + \\
&u_{x,x} \phi_{x,x}.
\end{aligned} \tag{2.23}$$

The separation of monomials gives rise to

$$\begin{array}{lcl}
u_t u_{xx} u_x^2 & : & -\tau_{uu}, \\
u_t u_{xx} u_x & : & -2\tau_{xu}, \\
u_{xx} u_x^3 & : & -\xi_{uu}, \\
u_{xx} u_x^2 & : & -2\xi_{xu} + \phi_{uu}, \\
u_{xx} u_x & : & -\xi_{xx} + 2\phi_{xu}, \\
u_{xx} u_{xt} & : & \tau_x, \\
u_{xx}^2 u_x & : & \xi_u, \\
u_{xx}^2 & : & -\frac{3}{2}\xi_x + \frac{1}{2}\tau_t + \phi_u, \\
u_{xx} & : & \phi_{xx}, \\
u_t u_x^2 & : & -\frac{1}{2}\tau + \frac{1}{2}u\tau_u, \\
u_t u_x & : & \xi_t, \\
u_t^3 & : & \frac{1}{2}\tau_u, \\
u_x^3 & : & -\frac{1}{2}\xi + \frac{1}{2}u\xi_u, \\
u_x^2 & : & \frac{1}{2}u\xi_x - \frac{1}{2}u\tau_t - u\phi_u, \\
u_t & : & -f_u - \phi_t, \\
u_x & : & -g_u - u\phi_x, \\
1 & : & -f_t - g_x.
\end{array} \tag{2.24}$$

The over-determined system has solution

$$\begin{array}{l}
\tau = 0, \quad \xi = 0, \quad \phi = A + Bt + Cx + Dxt, \\
f = -(B + Dx)u + a(x,t), \quad g = -\frac{1}{2}(C + Dt)u^2 + b(x,t)
\end{array} \tag{2.25}$$

where $a_t + b_x = 0$ and A, B, C and D are arbitrary constants. If we choose, for example, $A = D = 0$ (Noether type symmetry $X = (Bt + Cx)\partial_u$, $W = (Bt + Cx)$, $f = -Bu$ and $g = -\frac{1}{2}Cu^2$), we obtain, via a truncated version of (1.27), i.e.,

$$\begin{array}{l}
T^t = L\tau + W\frac{\partial L}{\partial u_t} + [D_j W - W D_j]\frac{\partial L}{\partial u_{tj}} - f, \\
T^x = L\xi + W\frac{\partial L}{\partial u_x} + [D_j W - W D_j]\frac{\partial L}{\partial u_{xj}} - g,
\end{array} \tag{2.26}$$

the conserved density and flux

$$\begin{array}{l}
T^t = -(Bt + Cx)u_t + Bu \\
T^x = -(Bt + Cx)uu_x + Cu_{xx} - (Bt + Cx)u_{xxx} + \frac{1}{2}Cu^2.
\end{array} \tag{2.27}$$

so that $D_t T^t + D_x T^x = -(Bt + Cx)(u_{xxxx} + uu_{xx} + u_x^2 + u_{tt})$.

2.4 A fourth-order non-linear equation

The Lagrangian, $L = \frac{1}{2}u_{xx}^2 - uu_x^2$, of

$$u_{xxxx} + 2uu_{xx} + u_x^2 = 0, \tag{2.28}$$

has been discussed in [9] constructed by the homotopy formula since the Frechet derivative of $u_{xxxx} + 2uu_{xx} + u_x^2$, viz., $D_x^4 + 2uD_x^2 + 2u_xD_x + 2u_{xx}$ is self adjoint. As before, (1.27) yields the Noether symmetries which are the translations ∂_t and ∂_x. The symmetry $x\partial_x - 2u\partial_u$ is not variational with regard to this Lagrangian. The conservation laws via translations, via (1.27), are, respectively to ∂_t and ∂_x,

$$\begin{aligned} T^t &= \tfrac{1}{2}u_{xx}^2 - uu_x^2, \\ T^x &= 2uu_tu_x - u_{xt}u_{xx} + u_tu_{xxx} \end{aligned} \tag{2.29}$$

and

$$\begin{aligned} T^t &= 0, \\ T^x &= \tfrac{1}{2}u_{xx}^2 + uu_x^2 - u_{xx}u_{xxx} + u_xu_{xxx}. \end{aligned} \tag{2.30}$$

Note. If one uses L as a partial Lagrangian for the evolution equation $u_t = u_{xxxx} + 2uu_{xx} + u_x^2$, so that $\frac{\delta L}{\delta u} = u_t$, we obtain no Noether type symmetries. In fact, direct calculations do not yield any either. The point symmetry generators of the equation are, in addition to translations, $4t\partial_t + x\partial_x - 2u\partial_u$.

2.5 Discussion and conclusion

We see that the conserved flows for high-order equations (with Lagrangians and, equivalently, partial Lagrangians of order greater than one in derivatives) support a formula similar to the well known Noether's theorem with the proviso that the higher-order cases have more terms giving rise to the appropriate order of the conserved flow. Also, in the KdV-5 evolution equation, we resorted to variational techniques usually adopted for the KdV-3 equation.

Chapter 3

Higher order PDEs with mixed derivative term

3.1 Introduction

In the previous chapter we had observed the variational and 'partial-variational' technique being applied on higher order equations where no derivatives were of mixed type. When considering the partial Lagrangian formula, a special point of consideration is the term of the highest derivative; the highest derivative term of the equation must be derived from the partial Lagrangian. This brings us to some interesting investigation when we consider the equations in which the highest derivative term is mixed. That is, the mixed derivative term involves differentiation by more than one of the independent variables.

In this chapter we consider the Shallow Water Wave and Regularized Long Wave equations which has been discussed in [10]. In addition to the above, their importance also lies in many areas of physics, and real world applications, e.g., tsunamis are characterized with long periods and wave lengths as a result they behave as

shallow-water waves.

In this chapter, we write (T^t, T^x as T^1, T^2).

3.2 The Shallow Water Wave equation

The shallow water wave equation (SWW), models simplest water waves that reasonably approximates the behavior of real ocean waves,

$$u_{xxxt} + \alpha u_x u_{tx} + \beta u_t u_{xx} - u_{tx} - u_{xx} = 0, \tag{3.1}$$

where α and β are arbitrary constants.

From the equation (3.1), there are two cases that emerge, viz., (i) $\alpha \neq \beta$ and (ii) $\alpha = \beta$.

In case (i) $\alpha \neq \beta$, we refer to this equation as shallow water wave-1 (SSW-1), and for case(ii) $\alpha = \beta$, equation (3.1), α is replaced by β, we refer to this case as shallow water wave-2 (SSW-2), which becomes,

$$u_{xxxt} + \beta u_x u_{tx} + \beta u_t u_{xx} - u_{tx} - u_{xx} = 0. \tag{3.2}$$

3.2.1 Shallow Water Wave-1

Here, we use the partial Lagrangian

$$L = \frac{1}{2} u_{tx} u_{xx} + \frac{1}{2} u_x^2 + \frac{1}{2} u_x u_t - \frac{1}{2} \beta u_t u_x^2, \tag{3.3}$$

for which

$$\frac{\delta L}{\delta u} = (2\beta - \alpha) u_{tx} u_x. \tag{3.4}$$

The separation of monomials gives rise to

$$\begin{array}{lcl}
u_x u_{tx}^2 & : & \tau_u, \\
u_{tx}^2 & : & \tau_x, \\
u_x u_{xx}^2 & : & \xi_u, \\
u_{xx}^2 & : & \xi_t, \\
u_{tx} u_{xx} & : & \eta_u - \xi_x, \\
u_t u_x u_{tx} & : & (2\beta - \alpha)\tau, \\
u_x^2 u_{tx} & : & (2\beta - \alpha)\xi, \\
u_x u_{tx} & : & (2\beta - \alpha)\eta, \\
u_t u_x^2 & : & \xi_x - 3\eta_u, \\
u_t u_x & : & \eta_u - \beta\eta_x, \\
u_x^2 & : & \eta_u - \frac{1}{2}\beta\eta_t - \frac{1}{2}\xi_x + \frac{1}{2}\tau_t, \\
u_x & : & -g_u + \frac{1}{2}\eta_t + \eta_x, \\
u_t & : & -f_u + \frac{1}{2}\eta_x, \\
1 & : & f_t + g_x.
\end{array} \tag{3.5}$$

From equation (3.5), we observe there are two cases that emerge, (a) $\alpha = 2\beta$ and (b) $\alpha \neq 2\beta$.

Subcase(a): $\alpha = 2\beta$

(1) $X = \partial_t, \quad W = -u_t$

$T^1 = \frac{1}{2}u_x^2 + \frac{1}{2}u_t u_{xxx}$

$T^2 = -u_t u_x - \frac{1}{2}u_t^2 + u_t^2 u_x \beta + u_t u_{xxt} - \frac{1}{2}u_{xt}^2 - \frac{1}{2}u_{xx}u_{tt}$

The total divergence is given below.

$$\begin{aligned} D_t(T^1) + D_x(T^2) &= D_t(\tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_t u_{xxx}) \\ &+ D_x(-u_t u_x - \tfrac{1}{2}u_t^2 + u_t^2 u_x \beta + u_t u_{xxt} - \tfrac{1}{2}u_{xt}^2 - \tfrac{1}{2}u_{xx}u_{tt}), \\ &= u_x u_{tx} + \tfrac{1}{2}u_{xxx}u_{tt} + \tfrac{1}{2}u_t u_{xxxt} - u_x u_{tx} - u_t u_{xx} - \tfrac{1}{2}u_{xx}u_{ttx} \\ &- u_t u_{tx} + 2\beta u_t u_x u_{tx} + \beta u_t^2 u_{xx} + u_t u_{xxxt} - \tfrac{1}{2}u_{xxx}u_{tt}, \\ &= u_t(u_{xxxt} + \alpha u_x u_{tx} + \beta u_t u_{xx} - u_{tx} - u_{xx}) + \tfrac{1}{2}u_t u_{xxxt} - \tfrac{1}{2}u_{xx}u_{xtt}, \\ &= \tfrac{1}{2}u_t u_{xxxt} - \tfrac{1}{2}u_{xx}u_{xtt}. \end{aligned} \tag{3.6}$$

We observe that extra terms emerge. By making an adjustment, these terms can be absorbed into the conservation law. The adjustment of these extra terms can be done by finding differentiable functions that form the extra terms, when they are differentiated,

$$\begin{aligned} D_t(T^1) + D_x(T^2) &= \tfrac{1}{2}u_t u_{xxxt} - \tfrac{1}{2}u_{xx}u_{xtt}, \\ &= \tfrac{1}{2}D_t(u_t u_{xxx}) - \tfrac{1}{2}D_x(u_{xx}u_{tt}). \end{aligned} \tag{3.7}$$

Then by taking these differentials across and adding them to the conserved flows, this satisfies the conservation law

$$D_t(T^1 - \tfrac{1}{2}u_t u_{xxx}) + D_x(T^2 + \tfrac{1}{2}u_{xx}u_{tt}) = 0. \tag{3.8}$$

The modified conserved quantities are now labeled $\tilde{T^i}$, where $D_t(\tilde{T^1}) + D_x(\tilde{T^2}) = 0$.

$$\begin{aligned} \tilde{T^1} &= T^1 - \tfrac{1}{2}u_t u_{xxx} \\ &= \tfrac{1}{2}u_x^2 \\ \tilde{T^2} &= T^2 + \tfrac{1}{2}u_{xx}u_{tt} \\ &= -u_t u_x - \tfrac{1}{2}u_t^2 + u_t^2 u_x \beta + u_t u_{xxt} - \tfrac{1}{2}u_{xt}^2 \end{aligned} \tag{3.9}$$

The same consequences apply for the results below.

(2) $X = \partial_x, \quad W = -u_x$

$T^1 = -\frac{1}{2}u_x^2 + \frac{1}{2}u_x^3\beta - \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_x u_{xxx}$

$T^2 = -\frac{1}{2}u_x^2 + \frac{1}{2}u_t u_x^2\beta + u_x u_{xxt} - \frac{1}{2}u_{xx}u_{xt}$

$$\begin{aligned} \tilde{T^1} &= T^1 - \tfrac{1}{2}u_{xx}^2 \\ &= -\tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_x^3\beta - \tfrac{1}{2}u_{xx}^2 \\ \tilde{T^2} &= T^2 - \tfrac{1}{2}u_x u_{xxt} \\ &= -\tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_t u_x^2\beta + u_x u_{xxt} \end{aligned} \tag{3.10}$$

Subcase(b): $\alpha \neq 2\beta$

(1) $X = \partial_u, \quad B^1 = \frac{1}{2}u_x^2(2\beta - \alpha), \quad B^2 = 0, \quad W = 1$

$T^1 = \frac{1}{2}u_x - \frac{1}{2}\beta u_x^2 - \frac{1}{2}u_{xxx} + \frac{1}{2}u_x^2(2\beta - \alpha)$

$T^2 = u_x + \frac{1}{2}u_t - u_t u_x\beta - u_{xxt}$

The total divergence is given by,

$$\begin{aligned} D_t(T^1) + D_x(T^2) &= D_t(\tfrac{1}{2}u_x - \tfrac{1}{2}\beta u_x^2 - \tfrac{1}{2}u_{xxx} + \tfrac{1}{2}u_x^2(2\beta - \alpha)) \\ &+ D_x(u_x + \tfrac{1}{2}u_t - u_t u_x\beta - u_{xxt}), \\ &= \tfrac{1}{2}u_{tx} - u_x u_{tx}\beta - \tfrac{1}{2}u_{xxxt} + u_x u_{tx}(2\beta - \alpha) \\ &+ u_{xx} + \tfrac{1}{2}u_{tx} - u_x u_{tx}\beta - u_t u_{xx}\beta - u_{xxxt}, \\ &= (u_{xxxt} + \alpha u_x u_{tx} + \beta u_t u_{xx} - u_{tx} - u_{xx}) - \tfrac{1}{2}u_{xxxt}, \\ &= -\tfrac{1}{2}u_{xxxt}. \end{aligned} \tag{3.11}$$

From the equation (3.11), u_{xxxt} has two derivative consequences,

$$\begin{aligned} u_{xxxt} &= D_t(u_{xxx}) \\ &= D_x(u_{xxt}) \end{aligned} \tag{3.12}$$

this leads to two pairs of conserved quantities.

(i)

$$\begin{aligned} \tilde{T_1^1} &= T^1 + \tfrac{1}{2}u_{xxx} \\ &= \tfrac{1}{2}u_x - \tfrac{1}{2}\beta u_x^2 + \tfrac{1}{2}u_x^2(2\beta - \alpha) \\ \tilde{T_1^2} &= T^2 \\ &= u_x + \tfrac{1}{2}u_t - u_t u_x \beta - u_{xxt} \end{aligned} \tag{3.13}$$

(ii)

$$\begin{aligned} \tilde{T_2^1} &= T^1 \\ &= \tfrac{1}{2}u_x - \tfrac{1}{2}\beta u_x^2 - \tfrac{1}{2}u_{xxx} + \tfrac{1}{2}u_x^2(2\beta - \alpha) \\ \tilde{T_2^2} &= T^2 + \tfrac{1}{2}u_{xxt} \\ &= u_x + \tfrac{1}{2}u_t - u_t u_x \beta - \tfrac{1}{2}u_{xxt} \end{aligned} \tag{3.14}$$

(2) $X = \partial_x, \quad B^1 = \frac{1}{3}u_x^3(2\beta - \alpha), \quad B^2 = 0, \quad W = -u_x$

$T^1 = \frac{1}{2}u_x^2 + \frac{1}{2}u_x^3\beta - \frac{1}{2}u_{xx}^2 - \frac{1}{3}u_x^3(2\beta - \alpha) - \frac{1}{2}u_x u_{xxx}$

$T^2 = \frac{1}{2}u_x^2 + \frac{1}{2}u_t u_x^2 \beta + u_x u_{xxt} - \frac{1}{2}u_{xx}u_{xt}$

$$\begin{aligned} \tilde{T^1} &= T^1 + \tfrac{1}{2}u_{xx}^2, \\ &= \tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_x^3\beta - \tfrac{1}{2}u_{xx}^2 - \tfrac{1}{3}u_x^3(2\beta - \alpha) \\ \tilde{T^2} &= T^2 - \tfrac{1}{2}u_x u_{xxt}, \\ &= \tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_t u_x^2 \beta + u_x u_{xxt} \end{aligned} \tag{3.15}$$

3.2.2 Shallow Water Wave-2

For equation (3.2), we use the partial Lagrangian

$$L = \frac{1}{2}u_{tx}u_{xx} + \frac{1}{2}u_x^2 + \frac{1}{2}u_x u_t - \frac{1}{2}\beta u_t u_x^2, \tag{3.16}$$

which has

$$\frac{\delta L}{\delta u} = \beta u_{tx} u_x. \tag{3.17}$$

The separation of monomials gives rise to

$$\begin{array}{lcl}
u_x u_{tx}^2 & : & \tau_u, \\
u_{tx}^2 & : & \tau_x, \\
u_x u_{xx}^2 & : & \xi_u, \\
u_{xx}^2 & : & \xi_t, \\
u_{tx} u_{xx} & : & \eta_u - \xi_x, \\
u_t u_x u_{tx} & : & \beta\tau, \\
u_x^2 u_{tx} & : & \beta\xi, \\
u_x u_{tx} & : & \beta\eta, \\
u_t u_x^2 & : & \xi_x - 3\eta_u, \\
u_t u_x & : & \eta_u - \beta\eta_x, \\
u_x^2 & : & \eta_u - \frac{1}{2}\beta\eta_t - \frac{1}{2}\xi_x + \frac{1}{2}\tau_t, \\
u_x & : & -g_u + \frac{1}{2}\eta_t + \eta_x, \\
u_t & : & -f_u + \frac{1}{2}\eta_x, \\
1 & : & f_t + g_x.
\end{array} \tag{3.18}$$

From equation (3.18), it is clear that $\beta \neq 0$ or $\beta = 0$.

If $\beta \neq 0$ then it is a trivial solution, and if $\beta = 0$, then equation (3.2) changes to

$$u_{xxxt} - u_{tx} - u_{xx} = 0, \tag{3.19}$$

and the partial lagrangian (3.16) becomes a standard Lagrangian

$$L = \frac{1}{2} u_{tx} u_{xx} + \frac{1}{2} u_x^2 + \frac{1}{2} u_x u_t, \tag{3.20}$$

and the conserved quantities are as follows:

(i) $X = \partial_t, \quad W = -u_t$

$T^1 = \frac{1}{2}u_x^2 + \frac{1}{2}u_t u_{xxx}$

$T^2 = -u_t u_x - \frac{1}{2}u_t^2 + u_t u_{xxt} - \frac{1}{2}u_{xt}^2 - \frac{1}{2}u_{xx}u_{tt}$

$$\begin{aligned} \tilde{T^1} &= T^1 - \tfrac{1}{2}u_t u_{xxx} \\ &= \tfrac{1}{2}u_x^2 \\ \tilde{T^2} &= T^2 + \tfrac{1}{2}u_{xx}u_{tt} \\ &= -u_t u_x - \tfrac{1}{2}u_t^2 + u_t u_{xxt} - \tfrac{1}{2}u_{xt}^2 \end{aligned} \tag{3.21}$$

(ii) $X = \partial_x, \quad W = -u_x$

$T^1 = -\frac{1}{2}u_x^2 - \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_x u_{xxx}$

$T^2 = -\frac{1}{2}u_x^2 + u_x u_{xxt} - \frac{1}{2}u_{xx}u_{xt}$

$$\begin{aligned} \tilde{T^1} &= T^1 - \tfrac{1}{2}u_{xx}^2 \\ &= -\tfrac{1}{2}u_x^2 - \tfrac{1}{2}u_{xx}^2 \\ \tilde{T^2} &= T^2 - \tfrac{1}{2}u_x u_{xxt} \\ &= -\tfrac{1}{2}u_x^2 + u_x u_{xxt} \end{aligned} \tag{3.22}$$

REMARK. From the conserved quantities attained above, we have some interesting outcomes, were extra terms are found when applying the conservation law. These extra terms are adjusted by merely taking the extra terms into derivative functions that can be absorbed into the conservation law, therefore producing new conserved quantities.

3.3 The Regularized Long Wave Equation

The Regularized Long Wave Equation (RLW), models soliton waves and is sometimes referred to as the Benjamin-Bona-Mahoney Equation. The regularized long wave (RLW) equation is an important nonlinear wave equation. Solitary waves are wave packets or pulses, which propagate in nonlinear dispersive media. Due to dynamical balance between the nonlinear and dispersive effects these waves retain a stable waveform. A soliton is a very special type of solitary wave, which also keeps its waveform after collision with other solitons. RLW is an alternative description of nonlinear dispersive waves to the more Korteweg de Vries (KdV) equation,

$$v_{txx} + \alpha v^2 v_x + v_t + v_x = 0. \tag{3.23}$$

The RLW equation is a third order equation and for our purposes of investigation, we modify this equation to compared it with the equations we have dealt with thus far, we do this by differentiating the equation by a spatial variable, x and a time variable t.

We refer to the modified RLW equation that is differentiated by t or let $v = u_t$ as RLW-1,

$$u_{xxtt} + \alpha u_t^2 u_{tx} + u_{tt} + u_{tx} = 0. \tag{3.24}$$

We refer to the modified RLW equation that is differentiated by x or let $v = u_x$ as RLW-2,

$$u_{xxxt} + \alpha u_x^2 u_{xx} + u_{tx} + u_{xx} = 0. \tag{3.25}$$

3.3.1 Regularized Long Wave-1

Here, we use the partial Lagrangian

$$L = \frac{1}{2}u_{tx}^2 - \frac{1}{2}u_t u_x - \frac{1}{2}u_t^2, \tag{3.26}$$

for which

$$\frac{\delta L}{\delta u} = -\alpha u_t^2 u_{tx}. \tag{3.27}$$

Using Noether's Identity we substitute into the expression below, which we use to find our determining equations and then separate by monomials,

$$X^{[2]}L + L(D_t\xi^1 + D_x\xi^2) = (\eta - u_t\xi^1 - u_x\xi^2)\frac{\delta L}{\delta u} + D_tB^1 + D_xB^2. \tag{3.28}$$

The separation of monomials are:

$$\begin{array}{lcl}
u_{xx}u_{tx} & : & \xi_t + u_t\xi_u, \\
u_{tt}u_{tx} & : & \tau_x + u_x\tau_u, \\
u_{tx}^2 & : & \eta_u - \frac{1}{2}\xi_x - \frac{1}{2}\tau_t, \\
u_x u_t^2 u_{tx} & : & \alpha\xi, \\
u_t u_t^2 u_{tx} & : & \alpha\tau, \\
u_t^2 u_{tx} & : & \alpha\eta, \\
u_t u_x u_{tx} & : & \eta_{uu}, \\
u_t u_{tx} & : & \eta_{tu}, \\
u_x u_{tx} & : & \eta_{xu}, \\
u_{tx} & : & \eta_{tx}, \\
u_x u_t & : & \eta_u, \\
u_t^2 & : & ,\eta_u + \frac{1}{2}\xi_x + \frac{1}{2}\tau_t \\
u_x & : & -g_u - \frac{1}{2}\eta_t, \\
u_t & : & -f_u - \eta_t - \frac{1}{2}\eta_x, \\
1 & : & f_t + g_x.
\end{array} \tag{3.29}$$

From equation (3.29), it is clear that $\alpha \neq 0$ or $\alpha = 0$.

If $\alpha \neq 0$ then it is a trivial solution, and if $\alpha = 0$, then equation (3.24) changes to

$$u_{xxtt} + u_{tt} + u_{tx} = 0 \tag{3.30}$$

and the partial Lagrangian becomes a standard Lagrangian. By solving the over-determined system, we get the symmetries. From the symmetries we calculate the conserved vectors:

(i) $X = \partial_t, \quad W = -u_t$

$T^1 = -\frac{1}{2}u_{tx}^2 + \frac{1}{2}u_t^2 + u_t u_{txx}$

$T^2 = \frac{1}{2}u_t^2 + u_t u_{ttx} - u_{tt}u_{tx}$

Checking if the conservation law holds, i.e.,

$$\begin{aligned} D_t(T^1) + D_x(T^2) &= D_t(-\tfrac{1}{2}u_{tx}^2 + \tfrac{1}{2}u_t^2 + u_t u_{txx}) + D_x(\tfrac{1}{2}u_t^2 + u_t u_{ttx} - u_{tt}u_{tx}), \\ &= -u_{tx}u_{ttx} + u_t u_{tt} + u_{tt}u_{txx} + u_t u_{ttxx} + u_t u_{tx} \\ &+ u_{tx}u_{ttx} + u_t u_{ttxx} - u_{tx}u_{ttx} + u_{tt}u_{txx}, \\ &= u_t u_{ttxx} - u_{tx}u_{ttx}. \end{aligned} \tag{3.31}$$

We observe that extra terms emerge. By making an adjustment, these terms can be absorbed into the conservation law. The adjustment of these extra terms can be done by finding differentiable functions that form the extra terms, when they are differentiated.

Thus,

$$\begin{aligned} D_t(T^1) + D_x(T^2) &= u_t u_{ttxx} - u_{tx}u_{ttx}, \\ &= D_t(u_t u_{txx}) - D_x(u_{tt}u_{tx}). \end{aligned} \tag{3.32}$$

Then by taking these differentials across and adding them to the conserved flows,

this satisfies the conservation law,

$$D_t(T^1 - u_t u_{txx}) + D_x(T^2 + u_{tt}u_{tx}) \;=\; 0. \tag{3.33}$$

The modified conserved quantity are now labeled $\tilde{T^i}$, where $D_t(\tilde{T^1}) + D_x(\tilde{T^2}) = 0$.

$$\begin{aligned} \tilde{T^1} &= T^1 - u_t u_{txx} \\ &= -\tfrac{1}{2}u_{tx}^2 + \tfrac{1}{2}u_t^2 \\ \tilde{T^2} &= T^2 + u_{tt}u_{tx} \\ &= \tfrac{1}{2}u_t^2 + u_t u_{ttx} \end{aligned} \tag{3.34}$$

The same consequences apply for the results below.

(ii) $X = \partial_x, \quad W = -u_x$

$T^1 = \frac{1}{2}u_x^2 + u_t u_x + u_x u_{txx} - u_{tx}^2$

$T^2 = -\frac{1}{2}u_{tx}^2 - \frac{1}{2}u_t^2 + u_x u_{ttx}$

$$\begin{aligned} \tilde{T^1} &= T^1 + u_{tt}u_{tx} \\ &= \tfrac{1}{2}u_x^2 + u_t u_x + u_x u_{txx} - u_{tx}^2 + u_{tt}u_{tx} \\ \tilde{T^2} &= T^2 - u_x u_{ttx} \\ &= -\tfrac{1}{2}u_{tx}^2 - \tfrac{1}{2}u_t^2 \end{aligned} \tag{3.35}$$

(iii) $X = \partial_u, \quad W = 1$

$T^1 = -\frac{1}{2}u_x - u_t - u_{txx}$

$T^2 = -\frac{1}{2}u_t - u_{ttx}$

The divergence is,

$$\begin{aligned} D_t(T^1) + D_x(T^2) &= D_t(-\tfrac{1}{2}u_x - u_t - u_{txx}) + D_x(-\tfrac{1}{2}u_t - u_{ttx}), \\ &= -\tfrac{1}{2}u_{tx} - u_{tt} - u_{ttxx} - \tfrac{1}{2}u_{tx} - u_{ttxx}, \\ &= -u_{ttxx}. \end{aligned} \tag{3.36}$$

From the equation (3.36), u_{ttxx} has two derivative consequences,

$$\begin{aligned} u_{ttxx} &= D_t(u_{txx}) \\ &= D_x(u_{ttx}) \end{aligned} \tag{3.37}$$

which leads to two pairs of conserved quantities

$$\begin{aligned} \tilde{T_1^1} &= T^1 + u_{txx} \\ &= -\tfrac{1}{2}u_x - u_t \\ \tilde{T_1^2} &= T^2 \\ &= -\tfrac{1}{2}u_t - u_{ttx} \end{aligned} \tag{3.38}$$

$$\begin{aligned} \tilde{T_2^1} &= T^1 \\ &= -\tfrac{1}{2}u_x - u_t - u_{txx} \\ \tilde{T_2^2} &= T^2 + u_{ttx} \\ &= -\tfrac{1}{2}u_t \end{aligned} \tag{3.39}$$

(iv) $X = t\partial_u, \quad W = t, \quad f = -u, \quad g = -\frac{1}{2}u$

$T^1 = -\frac{1}{2}tu_x - tu_t - tu_{txx} - u$

$T^2 = -\frac{1}{2}tu_t - tu_{ttx} + u_{tx} - \frac{1}{2}u$

Checking if the conservation law holds,

$$\begin{aligned} D_t(T^1) + D_x(T^2) &= D_t(-\tfrac{1}{2}tu_x - tu_t - tu_{txx} - u) + D_x(-\tfrac{1}{2}tu_t - tu_{ttx} + u_{tx} - \tfrac{1}{2}u) \\ &= -tu_{ttxx}. \end{aligned} \tag{3.40}$$

From the equation (3.40), u_{ttxx} has two derivative consequences,

$$\begin{aligned} tu_{ttxx} &= D_x(tu_{ttx}) \\ &= D_t(tu_txx) - D_x(u_{tx}) \end{aligned} \tag{3.41}$$

this leads to two pairs of conserved quantities

$$\begin{aligned} \tilde{T_1^1} &= T^1 \\ &= -\tfrac{1}{2}tu_x - tu_t - tu_{txx} - u \\ \tilde{T_1^2} &= T^2 + tu_{ttx} \\ &= -\tfrac{1}{2}tu_t + u_{tx} - \tfrac{1}{2}u \end{aligned} \tag{3.42}$$

$$\begin{aligned} \tilde{T_2^1} &= T^1 + tu_{txx} \\ &= -\tfrac{1}{2}tu_x - tu_t - u \\ \tilde{T_2^2} &= T^2 - u_{tx} \\ &= -\tfrac{1}{2}tu_t - tu_{ttx} - \tfrac{1}{2}u \end{aligned} \tag{3.43}$$

3.3.2 Regularized Long Wave-2

Here, we use the partial Lagrangian

$$L = \frac{1}{2}u_{xx}u_{tx} - \frac{1}{2}u_tu_x - \frac{1}{2}u_x^2 \tag{3.44}$$

for which

$$\frac{\delta L}{\delta u} = -\alpha u_x^2 u_{tx}. \tag{3.45}$$

Using Noether's Identity we substitute into the expression below, which we use to find our determining equations and then separate by monomials,

$$X^{[2]}L + L(D_t\xi^1 + D_x\xi^2) = (\eta - u_t\xi^1 - u_x\xi^2)\frac{\delta L}{\delta u} + D_tB^1 + D_xB^2. \tag{3.46}$$

The separation of monomials are:

$$\begin{array}{lcl}
u_{xx}u_{tx} & : & \xi_t - u_t\xi_u, \\
u_{tt}u_{tx} & : & \tau_x - u_x\tau_u, \\
u_{tx}u_{xx} & : & \eta_u - \xi_x, \\
u_x^3u_{tx} & : & \alpha\xi, \\
u_tu_x^2u_{tx} & : & \alpha\tau, \\
u_x^2u_{tx} & : & \alpha\eta, \\
u_tu_xu_{xx} & : & \eta_{uu}, \\
u_xu_{xx} & : & \eta_{tu}, \\
u_tu_{xx} & : & \eta_{xu}, \\
u_{xx} & : & \eta_{tx}, \\
u_xu_{tx} & : & \eta_{xu} - \frac{1}{2}\xi_{xx}, \\
u_{tx} & : & \eta_{xx}, \\
u_xu_t & : & \eta_u, \\
u_x^2 & : & -\eta_u + \frac{1}{2}\xi_x - \frac{1}{2}\tau_t, \\
u_x & : & -g_u - \frac{1}{2}\eta_t - \eta_x, \\
u_t & : & -f_u - \eta_x, \\
1 & : & f_t + g_x.
\end{array} \tag{3.47}$$

From equation (3.29), it is clear that $\alpha \neq 0$ or $\alpha = 0$.

If $\alpha \neq 0$ then it is a trivial solution, and if $\alpha = 0$, then equation (3.24) changes to

$$u_{xxtt} + u_{tt} + u_{tx} = 0 \tag{3.48}$$

and the partial Lagrangian becomes a standard Lagrangian. By solving the over-determined system, we get the Noether symmetries. From the symmetries we calculate the conserved vectors:

(i) $X = \partial_t, \quad W = -u_t$

$T^1 = -\frac{1}{2}u_x^2 + \frac{1}{2}u_t u_{xxx}$

$T^2 = \frac{1}{2}u_t^2 + u_t u_x + u_t u_{txx} - \frac{1}{2}u_{tx}^2 - \frac{1}{2}u_{tt}u_{xx}$

$$\begin{aligned} \tilde{T^1} &= T^1 - \tfrac{1}{2}u_t u_{xxx} \\ &= -\tfrac{1}{2}u_x^2 \\ \tilde{T^2} &= T^2 + \tfrac{1}{2}u_{tt}u_{xx} \\ &= \tfrac{1}{2}u_t^2 + u_t u_x + u_t u_{txx} - \tfrac{1}{2}u_{tx}^2 \end{aligned} \tag{3.49}$$

(ii) $X = \partial_x, \quad W = -u_x$

$T^1 = \frac{1}{2}u_x^2 + \frac{1}{2}u_x u_{xxx} - \frac{1}{2}u_{xx}^2$

$T^2 = \frac{1}{2}u_x^2 + u_x u_{txx} - \frac{1}{2}u_{tx}u_{xx}$

$$\begin{aligned} \tilde{T^1} &= T^1 + \tfrac{1}{2}u_{xx}^2 \\ &= \tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_x u_{xxx} \\ \tilde{T^2} &= T^2 - \tfrac{1}{2}u_x u_{txx} \\ &= \tfrac{1}{2}u_x^2 - \tfrac{1}{2}u_{tx}u_{xx} + \tfrac{1}{2}u_x u_{txx} \end{aligned} \tag{3.50}$$

(iii) $X = x\partial_u, \quad W = x, \quad f = -\frac{1}{2}u, \quad g = -u$

$T^1 = -\frac{1}{2}xu_x - \frac{1}{2}xu_{xxx} + \frac{1}{2}u_{xx} + \frac{1}{2}u$

$T^2 = -\frac{1}{2}xu_t - xu_x - xu_{txx} + \frac{1}{2}u_{tx} + u$

$$\begin{aligned}
\tilde{T^1} &= T^1 \\
&= -\tfrac{1}{2}xu_x - \tfrac{1}{2}xu_{xxx} + \tfrac{1}{2}u_{xx} + \tfrac{1}{2}u \\
\tilde{T^2} &= T^2 - \tfrac{1}{2}u_{tx} + \tfrac{1}{2}xu_{txx} \\
&= -\tfrac{1}{2}xu_t - xu_x - \tfrac{1}{2}xu_{txx} + u
\end{aligned} \tag{3.51}$$

3.4 Discussion and conclusion

We used the new modified approach of the Noether identity to find symmetries and then conservation laws for the high order equations. We know that when considering the use of partial lagrangian, we have to take into account the highest derivative of the equation. Where the highest derivative of the equation must be derived from the partial Lagrangian. This brings us to some interesting investigation where we considered the equations with higher order derivative in the equation that are singular independent variable and also those that are mixed (multi-independent variable).

Using the variational technique on the Shallow Water Wave equation, we get conserved flows that produce extra terms when the conservation law is applied. These extra terms are adjusted and then merged with conservation law to form new conserved quantities. These extra terms also occur in the second equation, the Regularized Long Wave equation. An interesting observation, in all results is the mixed derivative equations always produces extra terms. These extra terms always consisted of the product of the characteristic function and the highest derivative term of the equation in question.

In this chapter, we write (T^t, T^x, T^y, T^z) as (T^1, T^2, T^3, T^4).

Chapter 4

Relativistic Wave Equations

4.1 Introduction

The wave equation has extensively been studied in literature from the point of view of its Lie point symmetries. A comprehensive symmetry analysis of the equation is discussed by Cantwell [11], Ibragimov [12] and Bluman and Kumei [13]. It is well known that in three space dimensions the linear wave equation admits 16-dimensional Lie algebra of point symmetries excluding the 'infinite symmetry' [14]. The dimension of the algebra of the Lie point symmetry of the wave equation reduces with introduction of nonlinearities there. Realistically speaking, one would expect that a genuinely interesting wave equation will possess nonlinearities. Whereas nonlinearities make the wave equation represent physically plausible situations, the difficult part is to justify introduction of nonlinearities? Generally nonlinearities in the wave equation are introduced by keeping in mind physical considerations such as properties of material.

In this study we try a geometric approach to introduce nonlinearity in the wave equation. With this point in mind and the fact that geometric considerations may

be of interest to other areas of applied sciences, we use two back ground geometries to write the wave equation in; one a plane symmetric static geometry and other spherically symmetric non-static one. Both geometries are of Lorentzian nature with signature -2 in 4-space-time dimensions. We will write the wave equation in these two geometries one by one and obtain a 'nonlinear' form of the wave equations. We will then discuss Lie point symmetries of these wave equations and compare them with some other conventional symmetries possessed by the background Lorentzian metrics [15, 16].

The plan of the chapter is as follows. In the next section we derive the wave equation in a plane symmetric static space-time metric, find its Lie point symmetries and perform some reductions. In section 3 we derive the wave equation in spherically symmetric non-static metric and present our analysis there. We conclude the work in the last section by giving a brief comparison of the symmetries of the wave equations with that of some well known conventional symmetries of the back ground geometries.

4.2 Wave equations in General Background

4.2.1 Wave equation in a plane symmetric static space-time background

In order to write the wave equation in a Lorentzian geometry, we use the well know d'Alembertian operator $\Box$ to write the wave equation. In four Lorentzian metric g_{ab} in which a time translational invariance exists, this operator acting on a the wave (or mode) function $u(t, \mathbf{x})$ is given by,

$$\Box u(\mathbf{x}, t) = g^{00}\partial_0^2 + \frac{1}{2}g^{ij}[g^{00}(\partial_i g_{00})\partial_j + \partial_i\partial_j - \Gamma^k_{ij}\partial_k]u(\mathbf{x}, t) = 0, \tag{4.1}$$

where $\Gamma^k_{ij} = \frac{1}{2}g^{km}(\partial_j g_{im} + \partial_i g_{jm} - \partial_m g_{ij})$ represents Christoffel symbol, g^{ij} inverse of the metric g_{ij} and $\mathbf{x}$ three space variables x, y, z. Since we are interested in the

present study to motivate to studying wave equation in some given back ground Lorentzian geometries, we chose a particular metric,

$$g_{ij} = ((x/a)^2, -1, -1, -1), \tag{4.2}$$

with i and j take values from $0, ..., 3$ respectively. Further, 0 is used to represent t coordinate whereas $1, ..., 3$ respectively represent x, y and z coordinates. In the metric given by (4.2), the wave equation (4.1) takes the form,

$$u_{tt} = \frac{x^2}{a^2}(u_{xx} + u_{yy} + u_{zz}) + \frac{x}{a^2}u_x \tag{4.3}$$

We now derive infinitesimal symmetry generators. The one parameter Lie point transformations which leave (4.1) invariant are given by [17, 11]

$$\widetilde{\mathbf{x}} = \mathbf{x} + \epsilon\xi_i(\mathbf{x}, u) + O(\epsilon^2) \ for \ i = 0, \ldots, 3 \tag{4.4}$$

where $\mathbf{x}$ and ξ_i respectively represent variables on which the wave equation depends and corresponding components of the tangent vector X. Using (4.4), the expressions for the derivatives (of the transformed 'dependent' variables with respect to the transformed 'independent' variables) become:

$$\widetilde{u_j} = u_j + \epsilon\phi^j(\mathbf{x}, u) + O(\epsilon^2) \ for \ j = 1, 2, \ldots \tag{4.5}$$

where $u_j = \frac{\partial u}{\partial x^1}$ and $u_j = \frac{\partial^2 u}{\partial x^1 \partial x^2}$ respectively, for $j = 1$ and $j = 1, 2$. To solve (4.3), we start by writing symmetry generator corresponding to the variables x, y, t and u

$$X = m\frac{\partial}{\partial x} + n\frac{\partial}{\partial y} + p\frac{\partial}{\partial z} + q\frac{\partial}{\partial t} + s\frac{\partial}{\partial u} \tag{4.6}$$

where $m,\ n,\ p$ and q and s are the components of the tangent vector X computed for $i = 0, \ldots, 3$ at $\epsilon = 0$. Now prolonging the above generator to second order [13, 11, 17] and using symmetry criterion $X^2[u_{tt} - \frac{x^2}{a^2}(u_{xx} + u_{yy} + u_{zz}) - \frac{x}{a^2}u_x] \mid_{(4.3)} = 0$ for partial differential equations to the wave equation. Using this condition, replacing u_{tt} into the resulting expression and then comparing coefficients of all possible derivatives and products of derivative of u, gives rise to an over-determined system of partial differential equations,

$$
\begin{aligned}
&m_u = 0 = n_u = p_u = q_u = q_{u,u} = s_{u,u} \\
&a^2 m_t - x^2 q_x = 0, \quad a^2 n_t - x^2 q_y = 0, \quad a^2 p_t - x^2 q_z = 0 \\
&m - x m_x + x q_t = 0, \quad m - x n_y + x q_t = 0, \quad m - x p_z + x q_t = 0 \\
&m_y + n_x = 0, \quad m_z + p_x = 0, \quad n_z + p_y = 0 \\
&m + 2x q_t - x m_x + a^2 m_{t,t} - x^2 (m_{x,x} + m_{y,y} + m_{z,z} - 2 s_{x,u}) = 0 \\
&x n_x - a^2 n_{t,t} + x^2 (n_{x,x} + n_{y,y} + n_{z,z} - 2 s_{y,u}) = 0 \\
&x p_x - a^2 p_{t,t} + x^2 (p_{x,x} + p_{y,y} + p_{z,z} - 2 s_{z,u}) = 0 \\
&x q_x - a^2 q_{t,t} + x^2 (q_{x,x} + q_{y,y} + q_{z,z} - 2 s_{t,u}) = 0 \\
&x s_x - a^2 s_{t,t} + x^2 (s_{x,x} + s_{y,y} + s_{z,z}) = 0
\end{aligned}
\tag{4.7}
$$

One way to solve the above system is to follow the ab-initio method [18]. However, keeping in mind that this is a routine calculation, we use an algebraic software [19] to solve this system giving rise to a Lie algebra of point symmetry generators spanned by 15 linearly independent Lie point symmetries with an additional infinite-dimensional one, $\phi(t,x,y,z)\partial_u$ where ϕ satisfies the wave equation (4.3), viz.,

$$
\begin{aligned}
&X_0 = \tfrac{\partial}{\partial t}, \quad X_1 = \tfrac{\partial}{\partial y}, \quad X_2 = \tfrac{\partial}{\partial z}, \quad X_4 = u\tfrac{\partial}{\partial u} \\
&X_3 = x\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial y} + z\tfrac{\partial}{\partial z} - u\tfrac{\partial}{\partial u} \\
&X_5 = z\tfrac{\partial}{\partial y} - y\tfrac{\partial}{\partial z}, \qquad X_6 = e^{\frac{t}{a}}(\tfrac{\partial}{\partial x} - \tfrac{a}{x}\tfrac{\partial}{\partial t}), \quad X_7 = e^{-\frac{t}{a}}(\tfrac{\partial}{\partial x} + \tfrac{a}{x}\tfrac{\partial}{\partial t}) \\
&X_8 = e^{\frac{t}{a}}(z\tfrac{\partial}{\partial x} - x\tfrac{\partial}{\partial z} - a\tfrac{z}{x}\tfrac{\partial}{\partial t}), \quad X_9 = e^{-\frac{t}{a}}(z\tfrac{\partial}{\partial x} - x\tfrac{\partial}{\partial z} + a\tfrac{z}{x}\tfrac{\partial}{\partial t}) \\
&X_{10} = e^{\frac{t}{a}}(y\tfrac{\partial}{\partial x} - x\tfrac{\partial}{\partial y} - a\tfrac{y}{x}\tfrac{\partial}{\partial t}), \quad X_{11} = e^{-\frac{t}{a}}(y\tfrac{\partial}{\partial x} - x\tfrac{\partial}{\partial y} + a\tfrac{y}{x}\tfrac{\partial}{\partial t}) \\
&X_{12} = x\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial y} + z\tfrac{\partial}{\partial z} + u\tfrac{\partial}{\partial u} \\
&X_{13} = 2xy\tfrac{\partial}{\partial x} + (y^2 - x^2 - z^2)\tfrac{\partial}{\partial y} + 2yz\tfrac{\partial}{\partial z} - 2uy\tfrac{\partial}{\partial u} \\
&X_{14} = 2xz\tfrac{\partial}{\partial x} + 2yz\tfrac{\partial}{\partial y} + (z^2 - x^2 - y^2)\tfrac{\partial}{\partial z} - 2uz\tfrac{\partial}{\partial u}
\end{aligned}
\tag{4.8}
$$

Reducing (4.3) to an ordinary differential equation (ODE) using similarity methods would require three-dimensional subalgebras of (4.8). From this equation one can notice that the three generators $\{X_5, X_6, X_{12}\}$ generate a subalgebra whose commutators are,

$$[X_5, X_6] = 0 \quad [X_6, X_{12}] = X_6, \quad [X_5, X_{12}] = 0.$$

The invariants of X_5 can be shown to be $\alpha = y^2 + z^2, t, x, u$ by which (4.3) becomes

$$u_{tt} = \frac{x^2}{a^2}(u_{xx} + 4\alpha u_{\alpha\alpha} + 4u_\alpha) + \frac{x}{a^2}u_x \tag{4.9}$$

and X_6 in these variables remain with invariants $\beta = t + a\ln x, \alpha, u$. Equation (4.9) then reduces to

$$\alpha u_{\alpha\alpha} + u_\alpha = 0. \tag{4.10}$$

Lastly, X_{12} in the final set of variables is $a\partial_\beta + 2\alpha\partial_\alpha + u\partial_u$ with invariants $\gamma = \ln\alpha - \frac{2}{a}\beta, u$ so that (4.10) becomes the ode

$$u_{\gamma\gamma} = 0. \tag{4.11}$$

Thus $u = A\gamma + B$ for some constant A and B so that

$$u = A[\ln(y^2 + z^2) - \frac{2}{a}(t + a\ln x)] + B$$

which is invariant under the subalgebra formed by rotation X_5, X_6 and dilation X_{12}.

4.2.2 Wave equation in flat Friedmann space-time background

The metric of the flat Friedmann metric in Cartesian coordinates is given by [15, 16]

$$ds^2 = dt^2 - t^{4/3}(dx^2 + dy^2 + dz^2). \tag{4.12}$$

The wave equation, in Cartesian coordinates, on this manifold lead to

$$t^{4/3}u_{tt} + 2t^{1/3}u_t - (u_{xx} + u_{yy} + u_{zz}) = 0. \tag{4.13}$$

The Lie point symmetry generator

$$X = p\frac{\partial}{\partial x} + q\frac{\partial}{\partial y} + r\frac{\partial}{\partial z} + n\frac{\partial}{\partial t} + f\frac{\partial}{\partial u} \tag{4.14}$$

is obtained by $X[(t^{4/3}u_{tt} + 2t^{1/3}u_t - (u_{xx} + u_{yy} + u_{zz})|_{(4.13)} = 0$. Following a procedure similar to the first case, the terms in the resulting expression can be separated.

First separating by quadratic and cubic terms in the second derivatives of u, it is immediately found that p, q, r are n are independent of u whilst f is linear in u. Proceeding with further separations one obtains the over estimated system

$$\begin{aligned}
&nt^{1/3} - t^{4/3}n_t - t^{7/3}f_{tu} + \tfrac{1}{2}t^{7/3}n_{tt} - \tfrac{1}{2}tn_{xx} - \tfrac{1}{2}tn_y - \tfrac{1}{2}tn_{zz} = 0\\
&\tfrac{1}{3}\tfrac{n}{t} - n_t + 2p_x + tf_{tu} - \tfrac{1}{2}tn_{tt} + \tfrac{1}{2t^{1/3}}(n_{xx} + n_{yy} + n_{zz}) = 0\\
&\tfrac{1}{3}\tfrac{n}{t} - n_t + 2q_y + tf_{tu} - \tfrac{1}{2}tn_{tt} + \tfrac{1}{2t^{1/3}}(n_{xx} + n_{yy} + n_{zz}) = 0\\
&\tfrac{1}{3}\tfrac{n}{t} - n_t + 2r_z + tf_{tu} - \tfrac{1}{2}tn_{tt} + \tfrac{1}{2t^{1/3}}(n_{xx} + n_{yy} + n_{zz}) = 0\\
&2n_y - 2t^{4/3}q_t = 0\\
&2n_x - 2t^{4/3}p_t = 0\\
&2n_z - 2t^{4/3}r_t = 0\\
&2p_z + 2r_x = 0\\
&2p_y + 2q_x = 0\\
&2q_z + 2r_y = 0\\
&-2t^{1/3}p_t - 2f_{xu} - t^{4/3}p_{tt} + p_{xx} + p_{yy} + p_{zz} = 0\\
&-2t^{1/3}q_t - 2f_{yu} - t^{4/3}q_{tt} + q_{xx} + q_{yy} + q_{zz} = 0\\
&-2t^{1/3}r_t - 2f_{zu} - t^{4/3}r_{tt} + r_{xx} + r_{yy} + r_{zz} = 0\\
&t^{4/3}f_{tt} + 2t^{1/3}f_t - (f_{xx} + f_{yy} + f_{zz}) = 0
\end{aligned} \tag{4.15}$$

The solution of this over determined system leads to a 10-dimensional algebra of symmetry generators

$$\begin{aligned}
&X_0 = \tfrac{\partial}{\partial x}, \quad X_1 = \tfrac{\partial}{\partial y}, \quad X_2 = \tfrac{\partial}{\partial z},\\
&X_3 = x\tfrac{\partial}{\partial y} - y\tfrac{\partial}{\partial x}, \quad X_4 = x\tfrac{\partial}{\partial z} - z\tfrac{\partial}{\partial x}, \quad X_5 = y\tfrac{\partial}{\partial z} - z\tfrac{\partial}{\partial y},\\
&X_6 = -2xy\tfrac{\partial}{\partial x} + (-9t^{2/3} + x^2 - y^2 - z^2)\tfrac{\partial}{\partial y} - 2zy\tfrac{\partial}{\partial z} - 6ty\tfrac{\partial}{\partial t} + 6yu\tfrac{\partial}{\partial u},\\
&X_7 = (-9t^{2/3} - x^2 + y^2 + z^2)\tfrac{\partial}{\partial x} - 2xy\tfrac{\partial}{\partial y} - 2zx\tfrac{\partial}{\partial z} - 6tx\tfrac{\partial}{\partial t} + 6xu\tfrac{\partial}{\partial u},\\
&X_8 = -2xz\tfrac{\partial}{\partial x} - 2yz\tfrac{\partial}{\partial y} + (-9t^{2/3} + x^2 + y^2 - z^2)\tfrac{\partial}{\partial z} - 6tz\tfrac{\partial}{\partial t} + 6zu\tfrac{\partial}{\partial u},\\
&X_9 = x\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial y} + z\tfrac{\partial}{\partial z} + 2t\tfrac{\partial}{\partial t}
\end{aligned} \tag{4.16}$$

(excluding the infinite one, $\phi(t, x, y, z)\partial_u$ where ϕ satisfies the wave equation (4.13)).

We consider a reduction to an ordinary differential equation via the three-dimensional sub-algebra representing rotation in xy, dilation and translation in z, viz., $X_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$, $X_9 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + 2t\frac{\partial}{\partial t}$ and $X_2 = \frac{\partial}{\partial z}$ whose commutators are zero.

The generator X has invariants $\alpha = x^2 + y^2$, t, z and u so that, with simultaneously applying the translation in z, (4.13) becomes

$$t^{4/3}u_{tt} + 2t^{1/3}u_t - (4u_{\alpha\alpha} + 4u_\alpha) = 0. \tag{4.17}$$

Equation (4.17) inherits the generator Y which in the new coordinates is $2\alpha\frac{\partial}{\partial\alpha} + 3t\frac{\partial}{\partial t}$ with invariants $\gamma = \frac{\alpha}{t^{2/3}}$ and u which reduces (4.17) to the linear ode

$$-(\frac{2}{9}\gamma + 4)u_\gamma + 4(\frac{1}{9}\gamma^2 - \gamma)u_{\gamma\gamma} = 0 \tag{4.18}$$

which, for further analysis, may be written

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}[4(\frac{1}{9}\gamma^2 - \gamma)u_\gamma] = \frac{10}{9}\gamma u_\gamma. \tag{4.19}$$

4.3 Conservation laws for Wave Equations

We, firstly, rewrite the definitions of Noether conservation laws in a way that is peculiar to this chapter. A current $T = (T^1, \ldots, T^n)$ is conserved if it satisfies

$$D_i T^i = 0 \tag{4.20}$$

along the equation in question. The Euler-Lagrange equations, if they exist, associated with the equation are the system $\delta L/\delta u^\alpha = 0$, $\alpha = 1, \ldots, m$, where $\delta/\delta u^\alpha$ is the Euler-Lagrange operator given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\cdots D_{i_s}\,\frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}\,, \quad \alpha = 1, \ldots, m. \tag{4.21}$$

L is referred to as a Lagrangian and a Noether symmetry operator X of L arises from a study of the invariance properties of the associated functional

$$\mathcal{L} = \int_\Omega L(x, u, u_{(1)}, \ldots, u_{(r)})\mathrm{d}x \tag{4.22}$$

defined over Ω.

In the examples below, T^1 is the conserved density and T^2, T^3 and T^4 are the fluxes in the x, y and z directions, respectively.

4.3.1 Wave equation in a plane symmetric static space-time background

Equation (4.3) has a Lagrangian given by

$$L = \frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] \tag{4.23}$$

which, when substituted into Noether's Identity , yields a subalgebra of Noether symmetries of (4.8). Noethers Identity equation, for this L, after splitting by monomials, become

$$\begin{array}{lcl}
u_t^3 & : & \tau_u, \\
u_x^3 & : & \xi_u, \\
u_y^3 & : & \eta_u, \\
u_z^3 & : & \zeta_u, \\
u_t^2 & : & \frac{1}{2}[-\frac{\alpha^2}{x^2}\xi + \frac{\alpha^2}{x}\zeta_z + \frac{\alpha^2}{x}\eta_y + \frac{\alpha^2}{x}\xi_x - \frac{\alpha^2}{x}\tau_t] + \frac{\alpha^2}{x}\phi_u, \\
u_x^2 & : & -\frac{1}{2}\xi - \frac{1}{2}x\zeta_z - \frac{1}{2}x\eta_y + \frac{1}{2}x\xi_x - \frac{1}{2}x\tau_t - x\phi_u, \\
u_y^2 & : & -\frac{1}{2}\xi - \frac{1}{2}x\zeta_z + \frac{1}{2}x\eta_y - \frac{1}{2}x\xi_x - \frac{1}{2}x\tau_t - x\phi_u, \\
u_z^2 & : & -\frac{1}{2}\xi + \frac{1}{2}x\zeta_z - \frac{1}{2}x\eta_y - \frac{1}{2}x\xi_x - \frac{1}{2}x\tau_t - x\phi_u, \\
u_x u_t & : & -\frac{\alpha^2}{x}\xi_t + x\tau_x, \\
u_y u_t & : & -\frac{\alpha^2}{x}\eta_t + x\tau_y, \\
u_z u_t & : & -\frac{\alpha^2}{x}\zeta_t + x\tau_z, \\
u_y u_x & : & x\eta_x + x\xi_y, \\
u_z u_x & : & x\zeta_x + x\xi_z, \\
u_z u_y & : & x\zeta_y + x\eta_z, \\
u_t & : & -a_u + \frac{\alpha^2}{x}\phi_t, \\
u_x & : & -b_u + \phi_x, \\
u_y & : & -c_u + \phi_y, \\
u_z & : & -d_u + \phi_z, \\
1 & : & a_t + b_x + c_y + d_z,
\end{array} \tag{4.24}$$

where a, b, c, d are gauge functions that are dependent on (t, x, y, z, u).

The generators X_4, X_{13} and X_{14} are not variational via this Lagrangian and, hence, will not contribute a conservation law. We note that on the 'flat' Minkowski manifold, the only non-variational symmetry is $u\partial_u$. Nevertheless, we can construct twelve conservation laws via Noether's theorem, viz.,

$$X_0 = \partial_t \tag{4.25}$$

$$\begin{aligned}
T^1 &= \frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [-u_t][\frac{a^2}{x}u_t], \\
T^2 &= [-u_t][-xu_x], \\
T^3 &= [-u_t][-xu_y], \\
T^4 &= [-u_t][-xu_z]
\end{aligned}$$

$$X_1 = \partial_x \tag{4.26}$$

$$\begin{aligned}
T^1 &= [-u_x][\frac{a^2}{x}u_t], \\
T^2 &= \frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [-u_x][-xu_x], \\
T^3 &= [-u_x][-xu_y], \\
T^4 &= [-u_x][-xu_z]
\end{aligned}$$

$$X_2 = \partial_z \tag{4.27}$$

$$\begin{aligned}
T^1 &= [-u_z][\frac{a^2}{x}u_t], \\
T^2 &= [-u_z][-xu_x], \\
T^3 &= [-u_z][-xu_y],
\end{aligned}$$

$$T^4 = \frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [-u_z][-xu_z]$$

$$X_3 = x\partial_x + y\partial_y + z\partial_z - u\partial_u \tag{4.28}$$

$$\begin{aligned}
T^1 &= [-u - xu_x - yu_y - zu_z][\frac{a^2}{x}u_t], \\
T^2 &= [x]\frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [-u - xu_x - yu_y - zu_z][-xu_x], \\
T^3 &= [y]\frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [-u - xu_x - yu_y - zu_z][-xu_y], \\
T^4 &= [z]\frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [-u - xu_x - yu_y - zu_z][-xu_z]
\end{aligned}$$

$$X_5 = z\partial_y - y\partial_z \tag{4.29}$$

$$\begin{aligned}
T^1 &= [yu_z - zu_y][\frac{a^2}{x}u_t], \\
T^2 &= [yu_z - zu_y][-xu_x], \\
T^3 &= [z]\frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [yu_z - zu_y][-xu_y], \\
T^4 &= [-y]\frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [yu_z - zu_y][-xu_z]
\end{aligned}$$

$$X_6 = \frac{1}{x}e^{t/a}\partial_t - \frac{1}{a}e^{t/a}\partial_x \tag{4.30}$$

$$T^1 = [\frac{1}{x}e^{t/a}]\frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [-\frac{1}{x}e^{t/a}u_t + \frac{1}{a}e^{t/a}u_x][\frac{a^2}{x}u_t],$$

$$
\begin{aligned}
T^2 &= [-\frac{1}{a}e^{t/a}]\frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [-\frac{1}{x}e^{t/a}u_t + \frac{1}{a}e^{t/a}u_x][-xu_x],\\
T^3 &= [-\frac{1}{x}e^{t/a}u_t + \frac{1}{a}e^{t/a}u_x][-xu_y],\\
T^4 &= [-\frac{1}{x}e^{t/a}u_t + \frac{1}{a}e^{t/a}u_x][-xu_z]
\end{aligned}
$$

$$X_7 = \frac{1}{x}e^{-t/a}\partial_t + \frac{1}{a}e^{-t/a}\partial_x \tag{4.31}$$

$$
\begin{aligned}
T^1 &= [\frac{1}{x}e^{-t/a}]\frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [-\frac{1}{x}e^{-t/a}u_t - \frac{1}{a}e^{-t/a}u_x][\frac{a^2}{x}u_t],\\
T^2 &= [\frac{1}{a}e^{-t/a}]\frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [-\frac{1}{x}e^{-t/a}u_t - \frac{1}{a}e^{-t/a}u_x][-xu_x],\\
T^3 &= [-\frac{1}{x}e^{-t/a}u_t - \frac{1}{a}e^{-t/a}u_x][-xu_y],\\
T^4 &= [-\frac{1}{x}e^{-t/a}u_t - \frac{1}{a}e^{-t/a}u_x][-xu_z]
\end{aligned}
$$

$$X_8 = \frac{a}{x}ze^{t/a}\partial_t - ze^{t/a}\partial_x + xe^{t/a}\partial_z \tag{4.32}$$

$$
\begin{aligned}
T^1 &= [\frac{a}{x}ze^{t/a}]\frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [-\frac{a}{x}ze^{t/a}u_t + ze^{t/a}u_x - xe^{t/a}u_z][\frac{a^2}{x}u_t],\\
T^2 &= [-ze^{t/a}]\frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [-\frac{a}{x}ze^{t/a}u_t + ze^{t/a}u_x - xe^{t/a}u_z][-xu_x],\\
T^3 &= [-\frac{a}{x}ze^{t/a}u_t + ze^{t/a}u_x - xe^{t/a}u_z][-xu_y],\\
T^4 &= [xe^{t/a}]\frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [-\frac{a}{x}ze^{t/a}u_t + ze^{t/a}u_x - xe^{t/a}u_z][-xu_z]
\end{aligned}
$$

$$X_9 = -\frac{a}{x}ze^{-t/a}\partial_t - ze^{-t/a}\partial_x + xe^{-t/a}\partial_z \tag{4.33}$$

$$
\begin{aligned}
T^1 &= [-\frac{a}{x}ze^{-t/a}]\frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [\frac{a}{x}ze^{-t/a}u_t + ze^{-t/a}u_x - xe^{-t/a}u_z][\frac{a^2}{x}u_t],\\
T^2 &= [-ze^{-t/a}]\frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [\frac{a}{x}ze^{-t/a}u_t + ze^{-t/a}u_x - xe^{-t/a}u_z][-xu_x],\\
T^3 &= [\frac{a}{x}ze^{-t/a}u_t + ze^{-t/a}u_x - xe^{-t/a}u_z][-xu_y],\\
T^4 &= [xe^{-t/a}]\frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [\frac{a}{x}ze^{-t/a}u_t + ze^{-t/a}u_x - xe^{-t/a}u_z][-xu_z]
\end{aligned}
$$

$$X_{10} = \frac{a}{x}ye^{t/a}\partial_t - ye^{t/a}\partial_x + xe^{t/a}\partial_y \tag{4.34}$$

$$
\begin{aligned}
T^1 &= [\frac{a}{x}ye^{t/a}]\frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [-\frac{a}{x}ye^{t/a}u_t + ye^{t/a}u_x - xe^{t/a}u_y][\frac{a^2}{x}u_t],\\
T^2 &= [-ye^{t/a}]\frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [-\frac{a}{x}ye^{t/a}u_t + ye^{t/a}u_x - xe^{t/a}u_y][-xu_x],\\
T^3 &= [xe^{t/a}]\frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [-\frac{a}{x}ye^{t/a}u_t + ye^{t/a}u_x - xe^{t/a}u_y][-xu_y],\\
T^4 &= [-\frac{a}{x}ye^{t/a}u_t + ye^{t/a}u_x - xe^{t/a}u_y][-xu_z]
\end{aligned}
$$

$$X_{11} = \frac{a}{x}ye^{-t/a}\partial_t - ye^{-t/a}\partial_x + xe^{-t/a}\partial_y \tag{4.35}$$

$$
\begin{aligned}
T^1 &= [\frac{a}{x}ye^{-t/a}]\frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [-\frac{a}{x}ye^{-t/a}u_t + ye^{-t/a}u_x - xe^{-t/a}u_y][\frac{a^2}{x}u_t],\\
T^2 &= [-ye^{-t/a}]\frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [-\frac{a}{x}ye^{-t/a}u_t + ye^{-t/a}u_x - xe^{-t/a}u_y][-xu_x],\\
T^3 &= [xe^{-t/a}]\frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [-\frac{a}{x}ye^{-t/a}u_t + ye^{-t/a}u_x - xe^{-t/a}u_y][-xu_y],\\
T^4 &= [-\frac{a}{x}ye^{-t/a}u_t + ye^{-t/a}u_x - xe^{-t/a}u_y][-xu_z]
\end{aligned}
$$

$$X_{12} = x\partial_x + y\partial_y + z\partial_z + u\partial_u \tag{4.36}$$

$$\begin{aligned}
T^1 &= [u - xu_x - yu_y - zu_z][\frac{a^2}{x}u_t], \\
T^2 &= [x]\frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [u - xu_x - yu_y - zu_z][-xu_x], \\
T^3 &= [y]\frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [u - xu_x - yu_y - zu_z][-xu_y], \\
T^4 &= [z]\frac{1}{2}[\frac{a^2}{x}u_t^2 - xu_x^2 - xu_y^2 - xu_z^2] + [u - xu_x - yu_y - zu_z][-xu_z]
\end{aligned}$$

4.3.2 Wave equation in flat Friedmann space-time background

A Lagrangian for (4.13) is

$$L = \frac{1}{2}[t^2u_t^2 - t^{2/3}(u_x^2 + u_y^2 + u_z^2)]. \tag{4.37}$$

It turns out that the wave is even more 'non-flat' than the previous case as the Lie algebra of Lie symmetries with basis (4.16) is reduced to an algebra of Noether symmetries containing only the translations and rotations in space (conservation of linear and angular momenta). The remaining symmetries are non variational with regard to (4.37). Nevertheless, the conserved flows are

$$X_0 = \partial_x \tag{4.38}$$

$$\begin{aligned}
T^1 &= [-u_x][t^2u_t], \\
T^2 &= \frac{1}{2}[t^2u_t^2 - t^{2/3}(u_x^2 + u_y^2 + u_z^2)] + [-u_x][-t^{2/3}u_x], \\
T^3 &= [-u_x][-t^{2/3}u_y], \\
T^4 &= [-u_x][-t^{2/3}u_z]
\end{aligned}$$

$$X_1 = \partial_y \tag{4.39}$$

$$\begin{aligned}
T^1 &= [-u_y][t^2 u_t], \\
T^2 &= [-u_y][-t^{2/3} u_x], \\
T^3 &= \frac{1}{2}[t^2 u_t^2 - t^{2/3}(u_x^2 + u_y^2 + u_z^2)] + [-u_y][-t^{2/3} u_y], \\
T^4 &= [-u_y][-t^{2/3} u_z]
\end{aligned}$$

$$X_2 = \partial_z \tag{4.40}$$

$$\begin{aligned}
T^1 &= [-u_z][t^2 u_t], \\
T^2 &= [-u_z][-t^{2/3} u_x], \\
T^3 &= [-u_z][-t^{2/3} u_y], \\
T^4 &= \frac{1}{2}[t^2 u_t^2 - t^{2/3}(u_x^2 + u_y^2 + u_z^2)] + [-u_z][-t^{2/3} u_z]
\end{aligned}$$

$$X_3 = -y\partial_x + x\partial_y \tag{4.41}$$

$$\begin{aligned}
T^1 &= [y u_x - x u_y][t^2 u_t], \\
T^2 &= [-y]\frac{1}{2}[t^2 u_t^2 - t^{2/3}(u_x^2 + u_y^2 + u_z^2)] + [y u_x - x u_y][-t^{2/3} u_x], \\
T^3 &= [x]\frac{1}{2}[t^2 u_t^2 - t^{2/3}(u_x^2 + u_y^2 + u_z^2)] + [y u_x - x u_y][-t^{2/3} u_y], \\
T^4 &= [y u_x - x u_y][-t^{2/3} u_z]
\end{aligned}$$

$$X_4 = z\partial_x - x\partial_z \tag{4.42}$$

$$
\begin{aligned}
T^1 &= [xu_z - zu_x][t^2u_t],\\
T^2 &= [z]\frac{1}{2}[t^2u_t^2 - t^{2/3}(u_x^2+u_y^2+u_z^2)] + [xu_z - zu_x][-t^{2/3}u_x],\\
T^3 &= [xu_z - zu_x][-t^{2/3}u_y],\\
T^4 &= [-x]\frac{1}{2}[t^2u_t^2 - t^{2/3}(u_x^2+u_y^2+u_z^2)] + [xu_z - zu_x][-t^{2/3}u_z]
\end{aligned}
$$

$$
X_5 = -z\partial_y + y\partial_z \tag{4.43}
$$

$$
\begin{aligned}
T^1 &= [zu_y - yu_z][t^2u_t],\\
T^2 &= [zu_y - yu_z][-t^{2/3}u_x],\\
T^3 &= [-z]\frac{1}{2}[t^2u_t^2 - t^{2/3}(u_x^2+u_y^2+u_z^2)] + [zu_y - yu_z][-t^{2/3}u_y],\\
T^4 &= [y]\frac{1}{2}[t^2u_t^2 - t^{2/3}(u_x^2+u_y^2+u_z^2)] + [zu_y - yu_z][-t^{2/3}u_z]
\end{aligned}
$$

4.4 Discussion and conclusion

We have considered the classical wave equation in some Lorentzian space-time backgrounds with a point in mind that the wave equation there may naturally inherit nonlinearity from geometry. In this connection we have considered two space-time metrics which respectively represent a plane symmetric static metric [20] and flat Friedmann metric of signature -2. For both cases we have given solutions each to show how wave equation there can be either solved or reduced to ordinary differential equations by using the method of invariants. Yielding additional conservation laws that were not given previously.

In his book [14] Ibragimov suggests that in three flat space dimensions the linear wave equation admits 16-dimensional Lie algebra of point symmetries excluding the

'infinite symmetry'. In this study we show that the wave equations admits fewer symmetries when it is solved on general Lorentzian manifolds. In particular we have shown that the wave equations in plane symmetric static space-time admits 15 Lie point symmetries which are one less than 16 Lie pint symmetries of the wave equation in 3 cartesian space dimensions suggested in [14]. It is presumably a shift away effect from 'flatness' from Minkowski manifold to other manifolds leading to reduction in symmetry as well as solutions of the wave equation. This shift away from flatness of Minkowski manifolds is more clear in flat Friedmann metric case where only 10 Lie point symmetries of the wave equation are recovered.

The shift away from flatness is by 7 symmetry generators [14] and 7 solutions at least. In fact, the alternatives to the Minkowski case do not lend themselves to the variational case as conveniently as does the Minkowski case. Also, it should be noted that these alternatives to the wave equation on the Minkowski manifold are not achievable via a simple point transformation of variables on the Minkowski version. It is hoped that solving wave equation in curved space-time background may provide some other useful relationship of solutions with those of gravitational waves solutions in general relativity.

Conclusion

In the second chapter we see that the conserved flows for high-order equations (with Lagrangians and, equivalently, partial Lagrangians of order greater than one in derivativates) support a formula similar to the well known Noether's theorem with the provisor that the higher-order cases have more terms in the Euler operator giving rise to the appropriate order of the conserved flow. Also, in the fifth-order KdV evolution equation, we resorted to variational techniques usually adopted for the third-order KdV equation.

In general, it would be cumbersome to determine the conserved vector for such a high-order equation using first principles. We used the modified approach of the Noether identity to find symmetries and then conservation laws for the high order equations. We know that when considering the use of partial Lagrangian, we have to take into account the highest derivative of the equation, where the highest derivative of the equation must be derived from the partial Lagrangian.

In the third chapter, we considered the equations with the highest order derivative being mixed. Using the variational technique on the Shallow Water Wave equation, we get conserved flows that produce extra terms when the conservation law is applied. These extra terms are adjusted and then merged with conservation law to form new conserved quantities. These extra terms also occur in the the Regularized Long Wave equation. A similar procedure was applied to the method in multi-dimensional case in space using examples for relativity. In chapter 4, we study the conservation laws of the wave equation on non flat manifolds.

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