Investigating the teaching of fractions across the Intermediate Phase (Grade 4 to Grade 6): What range of sub-constructs is made available, and how are these connected?

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Declaration

I declare that this thesis is my own unaided work. It is being submitted for the award of the Degree of Doctor of Philosophy (PhD) in the University of the Witwatersrand. It has not been submitted before for award of any degree or examination in any other university.

Signature of candidate

6th day of April 2021

Abstract

This study was focused on the textbook presentation and teaching of fractions across the Intermediate Phase (Grade 4 to Grade 6) in one South African school. I investigated the teaching of fraction with a focus on the fraction sub-constructs identified in the literature (part-whole, quotient, measure, operator and rate and ratio) that are made available to learners, as well as how these sub-constructs are connected within teaching through the inclusion of what the fraction literature describes as fraction 'unifying elements' (partitioning, unitizing and notation of quantity). The motivation for the study was linked to the importance of fractions in the mathematics curriculum coupled with evidence of an emphasis in fractions teaching on disconnected procedures.

This research contributes to the existing research and literature by considering what fraction knowledge is made available for learners in terms of sub-constructs, unifying elements and cognitive demands. In the study, higher cognitive demand tasks were interpreted on the basis of tasks in which multiple sub-constructs were involved or where sub-constructs were connected with unifying elements. An in-depth analysis of textbook and enacted tasks across the three Intermediate Phase Grades 4-6 in one school focusing on fraction sub-constructs, the unifying elements and cognitive demands provided an understanding of what was made available to learn during fraction instruction.

Both the textbook and enacted task analyses revealed an overreliance on the part-whole sub-construct with pre-partitioned area models, tasks focused on single sub-constructs with little or no reference to the unifying elements, and limited numbers of tasks involving combinations of different sub-constructs. The written tasks made available to learners seldom or never included work with the unifying elements. This resulted in a large proportion of lower cognitive demand tasks in the textbooks and enacted tasks across the Grades. These findings suggest that if fraction instruction is to support learners to develop a connected, robust and complete understanding of fraction concepts, greater emphasis needs to be placed on the different sub-constructs and unifying elements in both textbooks and

enactments. The fraction sub-constructs and the unifying elements play a vital role in developing this connected, robust and complete understanding of fractions.

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List of abbreviations used in the tables to represent the different fraction sub-constructs

PW: Part-whole
M: Measure
R & R: Rate & Ratio
O: Operator
Q: Quotient

CHAPTER ONE

1. INTRODUCTION

This study involves an understanding of the teaching of fractions across the Intermediate Phase (Grade 4 to Grade 6). The study investigates the teaching of fraction with a focus on the range of fraction sub-constructs (identified using the literature) that are made available to learners, as well as how these sub-constructs are connected. This is done by examining the different tasks and representations used by teachers and the tasks and representations presented in the prescribed textbook that are made available to learners in terms of the sub-constructs, unifying elements, and cognitive demands of the tasks. The context of the study is linked to the importance of fractions in the mathematics curriculum. Although fractions have been an important historical component of the curriculum and much research has been done in relation to the topic, research underscores that teachers continue to have difficulty teaching fractions and learners still experience difficulty understanding concepts related to fractions (Anthony & Ding, 2011; Charalambous & Pitta-Pantazi, 2005; Lamon, 2005; McNamara, 2015; Philippou & Christou, 1994; Son, Lo & Watanabe, 2015; Wheeldon, 2008;).

1.1 Rationale for choosing the topic of fractions

1.1.1 Complex nature of fractions

It is well documented that many primary and high school children find fractions challenging. Fractions are problematic for both teachers to teach and learners to learn. Problems have been related to noting that fractions are complex mathematical objects that can be viewed as outcomes of a number of different types of actions. Charalambous and Pitta-Pantazi (2005, p.233) state: “To date there is a consensus among researchers that one of the predominant factors contributing to the complexities of teaching and learning fractions lies in the fact that fractions comprise a multifaceted construct”. Behr et al. (1992, p. 296) delineate this idea further: “…when fractions and rational numbers as applied to real-world problems are looked at from a pedagogical point of view, they take on numerous ‘personalities”.

It is this complex nature of fractions that has led to numerous researchers trying to understand and make sense of the different interpretations of fractions. Behr, Lesh, Post and Silver (1983), Freudenthal (1983), Kieren (1976, 1988), Ohlsson (1988) and Vergnaud (1983), are key among the large number of researchers who have attempted to explain the nature of fractions. Kieren (1976) identified five different sub-constructs or interpretations (part-whole, measure, quotient, operator, ratio), of rational numbers. Although these five sub-constructs can be considered separately, Carpenter, Fennema, and Romberg (1993) identified ‘unifying elements’ or ‘supporting elements’ which are important when looking across all the sub-constructs. The three unifying elements are identification of the unit, partitioning, and the notion of quantity. With these unifying elements the five sub-constructs can be interrelated. Several authors mention key underlying ideas that connect the sub-constructs (Carpenter et al., 1993; Behr et al., 1983 & Kieren, 1988). Some authors refer to these key ideas as unifying elements, others refer to them as underlying concepts. These features will be dealt with in greater detail in Chapter 2. The sub-constructs and unifying elements are briefly described below, drawing particularly on Susan Lamon’s (2012) exemplifications. These concepts are further explained in the literature chapter.

1.1.2 Key fraction sub-constructs

Part-Whole

The part-whole sub-construct is based on partitioning a continuous amount or a set of discrete items into equal-sized groups. For example, three- eighths as parts of a whole is interpreted as three of eight equal-size pieces/groups.

Measure

The measure sub-construct involves successively partitioning the unit (Lamon, 2012). This is done by identifying the unit fraction then dividing the unit using the fixed quantity (unit fraction) repeatedly. For example, when considering $\frac{4}{12}$, one must identify the unit fraction $\frac{1}{12}$ and then use that repeatedly to reach $\frac{4}{12}$. The number line is a common model that can be used for illustrating the measure sub-construct.

Ratio

Hart (1988) defines ratio as "a statement of the numeric relationship between two entities". It is a comparison of any two quantities (Lamon, 2012). There are two types of ratio: part-whole comparisons and part-part comparisons. Part-whole compares the measure of part of a set to the measure of the whole set. Part-part compares the measure of part of a set to another part of the set. For example, in a set of balls containing 6 yellow and 4 green balls, all the following ratios apply: 6 to 4 (yellow to green part-part comparison), 4 to 6 (green to yellow part-part comparison), 6 to 10 (yellow part-whole comparison), and 4 to 10 (green part-whole comparison).

Quotient

The quotient sub-construct may be viewed as the outcome of division actions (Lamon, 2012). For example, $\frac{6}{12}$ is interpreted as the outcome of 12 people sharing 6 pizzas equally. The quotient sub-construct can be further divided into partitive and quotative actions. Dividing 6 pizzas equally amongst 12 people involves partitive dividing or partitioning. An example of quotative division can be represented as follows: A 4 litre can of water is used to fill $\frac{1}{4}$ litre bottles. How many of the smaller bottles can be filled using the water in the can?

Operator

Mack (2000, p. 309) explains that the operator interpretation of rational numbers "represents a multiplicative size transformation where a quantity is reduced to a fraction of its original size by both partitioning the quantity and duplicating various portions of the quantity". For example, $\frac{2}{3}$ of 12 means $\frac{2}{3}$ operates on 12 as a single operation or as a multiplication performed on a division of 12, or as a division performed on a multiplication of 12. It is an operation that can enlarge or expand as well as reduce or contract an object.

1.1.3 Unifying elements

Identifying the unit, notion of quantity and partitioning are unifying elements that appear across the sub-constructs and unify the five sub-constructs. While these unifying elements are implicated in all the sub-constructs their role in the part-whole sub-construct is of great importance (Pitkethly & Hunting, 1996). This is further discussed in chapter 2.

The unifying elements connect the sub-constructs into a unified scheme, and thus, they provide a key route through which fractions learning and teaching of the different sub-constructs can be experienced as coherent and connected. In a South African context where coherence and connections have been highlighted as key problems in instruction – an issue taken up later – exploring the ways in which the unifying elements were drawn into fractions teaching allowed the analysis to consider the nature and extent of coherence and connections in fraction teaching (Carpenter et al., 1993). The unit in fractions is important across all the constructs because to solve fraction problems, the unit that a fraction relates to has to be recognized (Lamon, 2012). We must have an understanding of what the unit is to determine, for example, the answer to the question "How much?' Every fraction depends on some unit. The unit can be defined not only as a single object but as a collection of objects too. A fraction is a relative amount, that is, it tells you how much you have relative to the unit. The composition and recomposition of the unit, also known as, unitizing (Lamon, 1999) helps to transform the unit into 'chunks' that make it possible to solve problems related to the different sub-constructs. For example, with a case of soda (24 cans), it can be 24 individual cans of soda or two twelve –packs or 4 six-packs. Different ways of unitizing or transforming of the unit into different compositions distinguish the sub-constructs from each other with demarcations based on how the unit must be 'acted' upon according to the sub-construct (Carpenter, Fennema, and Romberg, 1993).

Another unifying element known as partitioning can be described as dividing the unit into equal parts/shares. Each sub-construct requires the unit to be partitioned in different ways. For example, the measure sub-construct requires partitioning the unit of measure, while the quotient sub-construct partitions the quantity represented by the numerator of the fraction into the number of parts specified by the denominator (Carpenter et al., 1993). Foregrounding the overlaps and contrasts in these unifying elements when working across different sub-constructs has been noted as helpful and important within instruction.

The notion of quantity also provides unity to the sub-constructs. The notion of quantity works hand in hand with partitioning. Partitioning allows learners to develop counting and partitioning schemes. When the unit is partitioned it results in parts that can be counted. However, these parts cannot be seen as discrete entities but rather a quantity that is represented by a new kind of number (Carpenter et al.,1993). Pitkethly & Hunting (1996, p.4) sum up the role of the unifying elements as follows:

> “the concept of the unit, the process of partitioning, and the concept of quantity not only serve to unify the different rational number subconstructs; they also are closely related to one another. It is units of some kind that are partitioned. The partitioning results in quantities that are assigned a number based on some unit, either the unit that was partitioned or some other unit depending on the construct involved.”

The different ‘personalities’ or sub-constructs of fractions discussed above are central to the investigation in this study. So, I begin with an initial exemplification of elements of this range, drawing from the work of Susan Lamon (2012, p. 257), whose work on rational number has also been important within this study. Each of these five sub-constructs is explained and discussed in the literature chapter, drawing particularly from Lamon’s (2005) extended elaborations.

1.1.4 Fraction sub-constructs and key tasks and representations for use in their teaching

In classrooms, the sub-constructs in focus are interpreted in the context of the tasks used for teaching fractions. The tasks involve the sub-constructs, and different tasks use different representations. Key representations that are useful in fractions teaching include: area/region and sets of objects, number lines and symbolic representations (discussed in the next chapter). Examples of tasks including the different sub-constructs can be seen in the following instances of how $\frac{3}{4}$ can be interpreted:

Part-whole

Represent the following relationship in a drawing.

Three quarters of the learners in my class have black hair.

Quotient

A family of four shared 3 pizzas. How much did each person get?

Operator

Use pictures to show two different ways to take $\frac{3}{4}$ of a set of 12 dots.

Ratio

For every 3 boys in Mr. Zoo's class, there are 4 girls. How many girls are there if there are 15 boys?

Measure

Locate $\frac{3}{4}$ on this number line

Figure 1.1: Number line used in measure interpretation

The methodology used in this study rests on an analysis of the fraction-related tasks presented in the textbook scheme used in the middle-years' (Intermediate Phase, Grades 4-6, learners aged 9-12) teaching in one Johannesburg primary school, and an analysis of teachers' work with their fraction task selections. As noted already, the motivation for my focus on fractions and fractions-related teaching rests on an international literature base noting that the complex nature of fractions brings into question the methodology used for teaching fractions. These issues are exacerbated in a South African terrain where particular problems have been identified with fractions learning, and with highly disconnected mathematics teaching in primary mathematics. This context provides the motivation for this study.

1.2 Motivation for study: Research on fractions teaching internationally and in South Africa

International research highlights that there is a problem with the way fractions are taught (Behr et al., 1992; Charalambous & Pitta, 2005; Lamon, 1999). These authors have noted several aspects of problems with fraction instruction: not enough emphasis placed on learners developing a complete understanding of fractions, and instruction of fraction concepts frequently limited to procedural work with very little or no emphasis on conceptual understanding. Instead of fraction instruction focusing on uniting procedural knowledge and conceptual understanding so that learners develop a rich and complete understanding of fractions, a chasm exists between the two. An over-emphasis on procedural work, without any understanding of why the procedures may work, has been argued to result in a poor understanding of the mathematics (Ma, 1999). Lamon (2012) and Ma (1999) explain that when there is no connection between procedural knowledge and conceptual understanding when teaching fractions, learners develop a superficial understanding of the concepts related to fractions. Despite research highlighting these concerns there remains a gap between the procedural and conceptual development of understanding fractions in primary schools. The aforementioned authors advocate that teaching for conceptual understanding and knowledge must take preference over procedures and algorithms to perform operations on fractions to achieve correct answers. This, certainly does not suggest that teachers must ignore procedural learning, but rather, as indicated by Ma (1999), teachers are required to understand the importance of both procedural learning in conjunction with conceptual understanding in the learning of mathematics.

Teachers in South African classrooms are likely to experience or have experienced similar problems mentioned above when teaching fractions. Note that problems in South Africa are likely to be compounded by evidence of gaps in middle Grades' teachers' mathematical knowledge, including particular weaknesses related to rational number (Venkat & Spaull, 2015), and an apartheid era mathematics curriculum that tended to promote procedural orientations (DoE, 2009). Over the past two decades, the meaning of successful mathematics learning and teaching has experienced a revolution in South Africa. The move to an outcomes based curriculum within Curriculum 2005 (C2005) (DoE, 1997) to overcome the apartheid legacy wanted to prepare citizens for global economic challenges. Curriculum 2005 focused on integrating different knowledge branches and school knowledge with everyday knowledge, particularly in relation to Mathematics and language subjects. The playing out of integration led to concerns about a neglect of conceptual coherence and progression i.e., failure to systematically build conceptual understanding (Chisholm report, 2000). In a context where many teachers were inadequately trained, evidence suggested that the curriculum was interpreted, understood and implemented in a range of ways by different teachers (Chisholm et.al, 2005). This led to the Department of Education introducing a revised curriculum in 2003 for Grade R-9- The Revised National Curriculum Statement, (RNCS) (DoE, 2002). This curriculum, whilst retaining C2005 emphasis on lifelong learners, independent thinkers, and learners who were able to problem solve, and display critical and insightful reasoning and interpretative and communicative skills, inserted a stronger emphasis on content specifications and progression across grades in comparison to C2005.

With the ongoing prevalence of poor performance in mathematics in primary schools, in 2008 the Department of Education introduced the Foundations for Learning Campaign (FFL) (DoE, 2008) into primary schools. The campaign hoped to aid primary school teachers unpack the Learning Outcomes (LO) and Assessment Standards (AS) for mathematics by providing further increased specifications on content, order and sequence of coverage, with the introduction of termly ‘milestones’ (DoE, 2008). It is important to note that when this research project started the Department of Education was in the process of launching their latest curriculum and assessment document called CAPS for introduction in 2012 (DoE, 2011). When the data was collected, the RNCS was the mandatory curriculum with school in many provinces encouraged to use the FFL.

Given its mandatory status, I used the RNCS to identify curriculum and assessment criteria related to fractions. The RNCS clearly define the aims and purpose of teaching and learning Mathematics in relation to developing learners' understandings. It states as one of its aims, that learners need to develop ‘deep conceptual understandings in order to make sense of Mathematics,’ (DoE, 2002, p.5). This suggests that teachers are responsible to help learners develop these ‘deep conceptual understandings’ although how this is to be achieved is not specified. The RNCS provides teachers with a list of learning outcomes and assessment standards. Given the importance of fractions, it is not surprising that RNCS lists numerous fraction-related assessment standards (AS) for each grade (See appendix 5).

The results from the South African Annual National Assessments (ANA) (which ran between 2010 and 2014 across Grades 1-6 and 2012-2014 in Grade 9) indicated a lack of understanding of fractions by Grade 3, 6 and 9 learners. The purpose of the ANA was to assess the mathematical and language skills and knowledge that learners acquired as a result of the teaching and learning they experienced at school. The results were obtained from random samples of Grade 3, 6 and 9 learner scripts. The ANA results for Grades 3, 6 and 9 for the past 3 years (2012a,b, 2013, 2014) indicate fractions and ratios as key areas of concern with regards to Mathematics as learners found it difficult to comprehend these concepts (DBE, 2012b). In 2012 the areas of weakness with regards to fractions and fraction-related topics in primary schools were summarized as follows:

- Relative sizes of fractions (i.e., arranging fractions in increasing/decreasing order)- Grade 3
- Addition of mixed numbers- Grade 6
- Application of knowledge of fractions and percentages in given contexts (word problems) – Grade 6

In 2013 (pg.8; 47-50) the areas of weakness with regards to fractions were summarized as follows:

- Inability to recognize fractions in diagrams and fraction names, and comparing unitary fractions from smallest to biggest- Grade 3
- Learners use wrong strategies when dealing with fractions. They often apply incorrect mathematical rules to manipulate the denominators and numerators- Grade 6
- Learners are unable to relate metres to centimetres, minutes to hours, and litres to millilitres- Grade 6

In 2014 (pp 8–13) the areas of weakness with regards to fractions were summarized as follows:

- Comparing common fractions- Grade 4
- Conversions of units in measurement – Grade 4
- Solve problems on capacity – Grade 4
- Calculations of time in hours and minutes- Grade 5
- Arranging units of measurements from the least to the most – Grade 5
- Conversions of units in measurement – Grade 6

This situation is not unique to South Africa. Charalambous and Pitta-Pantazi (2007), in their research on fractions found that primary school children in Cyprus struggled with certain concepts related to fractions. The children they worked with performed well with tasks related to the part-whole sub-construct but were least competent in the measure sub-construct. A similar study carried out by Leung (2009) on Hong Kong students revealed similar patterns. The Hong Kong students performed well in the part-whole sub-construct (84% of the test items) and more poorly in measure sub-construct (32% of the test items). Studies done in America shows that middle and junior high school children find fractions difficult and challenging. National Assessment of Educational Progress (NAEP) reports show similar trends with both junior and high school learners achieving low scores on fractional computation and displaying little understanding when doing these computations (Braswell, Lutkus, Grigg, Santapau, Tay-Lim, & Johnson,2001; Carpenter et al.,1980; Dossey, Mullis, Lindquist, & Chambers,1988; Perie, Moran, & Lutkus, & Tirre, 2004; Post, 1981; Sowder & Wearne, 2006; Wearne & Kouba, 2000). Thomson and Fleming (2004) reported that achievement by Australian primary school learners in fractions is also below international averages. While the ANAs provide some insights into South African learner performance on fractions and fractions-related tasks, there is limited detail on what fraction instruction across the Intermediate Phase looks like in South Africa. The purpose and aim of this study is to provide insights into this gap.

The international research suggests that there needs to be shift in the way fractions are taught. For this shift to take place we must know what teaching is currently taking place in classrooms. My Masters study focused on a case study of a Grade 7 teacher and how he went about teaching fractions (Govender, 2008). The findings revealed, in line with the literature, that frequently the teaching of fractions was dominated more by procedural working at the cost of conceptual understanding. The task chosen by the teacher had potential for conceptual working. In his enactment of the task, however, he taught procedurally. The mathematical work of demonstrating and relying on mathematical procedures to lead learners to the correct answers formed a large proportion of his procedural teaching. Learners were afforded no opportunity to come up with their own procedures, or to discuss the presented procedures, to solve the task.

In the light of the international research, this is not a unique situation. Teaching fractions is a complex and difficult task and teachers have been found to rely on procedures to solve fractions related problems. The research suggests though, that this choice is to the detriment of learners developing a deep understanding of fractions, with connections between aspects being noted as critical to developing a rich understanding of fractions. Lamon (2012) explains that it is in teaching across the different interpretations (sub-constructs) of fractions that connections can be made that afford learners the opportunity to develop a strong understanding of fractions concepts. These connections are useful because they aid in learners seeing mathematics as a unified body of knowledge instead of a set of disjoint and complex concepts and procedures. Given South African evidence that has identified highly disconnected teaching of primary mathematics, research on how to make connections in fraction instruction has particularly important implications for the teaching of fractions.

These findings lead to questions about how fractions are taught in the middle Grades in South Africa and how concept connection occurs during these years.

1.3 Connections between sub-constructs and procedural and conceptual knowledge of fractions

In this study, I take the view that the development of both conceptual and procedural knowledge in inter-connected ways is important to developing a sound understanding of fractions. The literature base introduced above suggests that these interconnections come about through making connections between the sub-constructs, with the unifying elements providing one route for doing this. While I agree with the literature that suggests procedural and conceptual knowledge should be closely interconnected, evidence from teaching suggests that many teachers appear to hold highly procedural orientations in their work with fractions. Therefore, looking at fractions from a conceptual perspective (which can include working across sub-constructs in tasks that involve connections in various ways that can include attention to unifying elements and procedures) and a purely procedural aspect is useful.

Whilst teacher knowledge of fractions is not the central focus of my study, it is important to draw attention to what literature suggests regarding teacher knowledge and the teaching of fractions, as this body of evidence suggests close interlinks between teacher knowledge and how teachers make knowledge related to fractions available to learners (Ma, 1999). It also highlights many of the limitations that continue to exist in primary mathematics classrooms around the world in relation to what is made available to learn about fractions.

My Masters study led to a broader interest in how fractions are taught in the Intermediate Phase. I became interested in understanding whether connecting procedural and conceptual approaches was a broader issue in South African schools, and how connections, where they existed, were enacted.

Developing fraction knowledge is a long-term process and a complex endeavour because of the web of ideas related to the sub-constructs. Development tends not to occur in a linear, systematic way, but rather in a forward and back, zigzagging manner because of the several factors involved in acquiring a complete understanding of fractions. While making meaningful connections between the sub-constructs when teaching fractions has been argued to help learners acquire a stronger, more flexible, and more efficient ways of working with fractions, the reliance on procedural approaches presents ways of side-lining the complexity of the interpretations and interconnections of fractions while also taking away from learners engaging with and developing thought processes that encourage a deep understanding of fractions.

A general consensus in the research base is that the teaching of fractions must include developing all the sub-constructs of fractions in a wide variety of contexts. This experience is strongly argued to help develop a complete understanding of fractions.

This need for connected understanding underpins my focus on procedural/ conceptual teaching approaches in this study. Literature emphasizes that the multifaceted and inter-connected nature of fractions makes teaching that combines procedural and conceptual knowledge particularly important, to support learners to develop a deeper understanding of the topic.

1.4 Task demands in relation to teaching fractions in connected ways

Lamon (2012), Verschaffel (2006), Pitkethly & Hunting (1996), Bezuk & Cramer (1989), over an extended period, are among the researchers noting that teachers have an important role in helping learners understand the challenging topic of fractions. The careful selection, set-up and arranging of learning activities with appropriate instruction so that learners understand and make sense of the differences and similarities among the various sub-constructs related to fractions is emphasized as particularly important for developing conceptual understanding. This leads to a focus on fractions tasks and their enactment in instruction in this research.

Teaching of fractions is laden with complexities, therefore the tasks used by teachers to represent the content provide the basis for an investigation of what is made available to learners with regards to fraction instruction in Intermediate Phase classrooms. The tasks used to teach fractions, in this study, helped identify the different sub-constructs made available.

In the empirical analysis, unifying elements were seen *within* work with particular sub-constructs but were rarely used to connect between sub-constructs. A further route to thinking about connections was therefore via the different cognitive demands placed on learners when engaging with the tasks. In Stein et al.'s (2000) formulation of cognitive demand, a key feature of higher cognitive demand tasks is a basis in connections between ideas. The level and kind of thinking that learners engage with determines what they learn. Higher cognitive demand tasks allow learners to build connections to underlying concepts and meanings. In Stein et al.'s (2000) work, higher cognitive demand tasks require learners to make connections among various representations and to attach meaning and justification to results obtained when working on fractional tasks in novel ways. As argued above, connections, in turn, form the basis for more conceptual approaches to fractions to be made available to learners.

Literature emphasises that to work proficiently with fractions, tasks provided for learners must contain both procedural and conceptual knowledge. According to Hiebert and Wearne (1986), procedural knowledge involves the formal language of mathematics and the rules, algorithms and procedures required to solve and complete mathematical tasks. Conceptual knowledge comprises of relationships either between existing knowledge and new information, or between distinct facts/procedures in the mathematics terrain. It involves the comprehension of mathematical concepts, operations, and relations. As mentioned above, teachers have two options when teaching fractions, they teach algorithms to achieve correct answers as opposed to teaching fractions using more complex tasks, so that their learners develop a robust understanding of fractions. Opting for the latter, may lead to learners becoming apprehensive because of the uncertainty associated with the tasks (Stein et al, 2000). Choosing the former, they risk the chance of learners developing a superficial understanding of fraction concepts. Stein et al (2000) argue that having a range of task demands helps with promoting conceptual understanding and procedural fluency. They refer to two broad levels of cognitive demands when examining mathematical instructional tasks: lower-level and higher-level cognitive demands. Lower-level cognitive demands include, what they refer to as, “memorization” and “procedures without connections” tasks whereas “procedures with connections” and “doing mathematics” tasks are considered as higher-level cognitive demands. These ideas and concepts are further explained in relation to the teaching of fractions in the literature chapter. Stein et al (2000) defines cognitive demands as a certain kind and level of thinking required to successfully engage, complete and solve a task. They further claim that the kind and level of thinking that learners are exposed to will determine what they learn. In light of fraction instruction, teachers are responsible for what knowledge is made available to learn. They can provide minimal thinking and reasoning tasks, or they can provide tasks that requires extensive engagement in cognitive terms. Tasks selected by teachers must match goals for student learning. If the goal is fluency in certain procedures then memorization tasks may be appropriate. If developing a deeper understanding of fraction concepts is the goal then “doing mathematics” tasks will be more suitable (Stein et al., 2000). Stein et al. (2000) note that even if intended tasks have higher cognitive demand, and therefore more openings for conceptual approaches, teaching often lowers the cognitive demand of such tasks. It is for this reason my research focused on examining the cognitive demands of intended tasks and of enacted tasks in instruction. An analysis of the tasks used to teach fractions, tasks from the prescribed textbook (the intended tasks) and enacted tasks that figured in instruction, provides information of not only what sub-constructs are made available but also the cognitive demands placed on learners thinking when engaging with the tasks. And, if higher cognitive demand tasks are important in bringing connections into play, then studying the cognitive demand of tasks is important for deciding whether fractions tasks are being presented in connected ways that move beyond procedures. This provided a lens to investigate the nature of fractions tasks and teaching in mathematics classrooms across the Intermediate Phase in one primary school.

1.5 Problem Statement and Research Questions

In this study, the key goal was to understand which sub-constructs were made available and how they appeared in tasks. How they appeared was related to how the sub-constructs were connected with an analysis of cognitive demand and the presence of unifying elements providing markers of more conceptual approaches. This leads to the overall question and sub-questions that guide this study:

How are fractions presented in general across the Intermediate Phase classrooms in one school over a time period of one year?

1. What range of sub-constructs and unifying elements did teachers focus on through their task selections to develop an understanding of fractions for their learners across Grades 4-6 in one school?
2. What levels of cognitive demand and work with unifying elements did the teachers make available in their enactment of fractions tasks?

3. What can be said about the fractions knowledge that is made available to learn based on the analysis of sub-constructs, cognitive demand, and unifying elements across the three Grades?

The answers to these questions provided useful illumination of the knowledge made available by teachers across the Intermediate Phase when teaching fractions.

1.6 The importance of fractions in the mathematics curriculum

The context of this study is linked to the importance of fractions in the mathematics curriculum. The consensus amongst researchers is that fractions form an important part of the Intermediate Phase mathematics curriculum since the topic (or concept) supports and acts as a foundation for the development of proportional reasoning, and other important topics addressed later in the senior phase, including algebra, probability, geometry, and trigonometry (Booth & Newton, 2012; Kieran, 1993; Lamon, 1999; Lesh, Post, & Behr, 1988; Litwiller & Bright, 2002; Siegler et al., 2012).

1.6.1 Proportional reasoning

Lamon (2012, p. 4) describes proportional reasoning as "detecting, expressing, analysing, explaining, and providing evidence in support of assertions about proportional relationships". In other words, it involves the use of multiplicative relationships to compare quantities and to predict the value of one quantity based on the values of another. She further explains that *reasoning* is not applying rules and operations to word problems, but rather, it is "mental, free-flowing processes that require conscious analysis of the relationships among quantities" (p. 4). It is more about the use of number sense than following a set of rules to solve proportions. Vergnaud (1979), similarly to Lamon, proposes that learners should actively engage with concepts because often the difficulties they experience with rational numbers arise from not understanding and making sense of concepts rather than focusing on the calculations. He also advises that concepts build on concepts over a period of time. This means that the systematic understanding of concepts takes place in stages over a long period of time and always in relation to other concepts. Interrelated topics like ratio, proportion and percent require a broad understanding of multiplicative structures. This allows for the necessary groundwork for the conceptualisation of the concept of rational number and in turn higher order concepts like algebra.

1.6.2 Fractions in algebra

Both Empson and Levi (2011) and Wu (2001) claim that learners struggle with algebra in the middle and senior years because of their misunderstanding of fractions. If there is a weak fraction foundation, then the learning of algebra will suffer since much of the content in basic algebra relies on the understanding of fractional concepts. For example, when learning to add and subtract fractions working with the concept of like terms is introduced i.e., you can only add or subtract fractions if the denominators are alike. Connecting this with algebra, like terms are terms that have the same variables. The coefficient of like terms may be different. One can only simplify the expression by adding or subtracting if the terms are alike.

Brown and Quinn (2007) explain that when we use a constant in an equation to clear the denominators (multiply the equation by a constant), we are using an understanding of fraction concepts. And, when we are solving proportional equations, we employ constructs that have their bases in equivalent fractions.

The solving of equations forms a large chunk of high school algebra. For learners to successfully engage with equations they require an understanding of fractions and being able to compute with fractions (Wu, 2008). Wu (2008) warns that unless learners are able to work competently with fractions, they have a very little or no chance of learning algebra.

Understanding the network of ideas related to proportional reasoning is a long-term learning process. It starts in early Grades, for example, when learners think of twenty-four as four sixes or six fours instead of thinking of it as one more than twenty- three. Proportional reasoning is later used when they deal with the topic of speed and distance. They reason about how the speed of 100km/h is the same as 50km/30 min. This reasoning continues into higher Grades with linear graphs.

Lamon (2012) describes how fraction teaching often disadvantages learners and robs them of the process of proportional reasoning.

> "A long-term learning process is required for understanding the web of ideas related to proportional reasoning. Current instruction that gives a brief introduction to part-whole fractions and proceeds to introduce computation procedures does not give children the time they need to construct and become comfortable with important ideas and way of thinking" (p. 255).

Proportional reasoning requires time and well thought out tasks and activities that promote learners thinking of numbers in relative terms, rather than absolute terms. For example, learners are using proportional reasoning when they are able to decide and conclude that a group of 5 people growing to a group of 15 people represents a more significant change than a group of 100 people growing to 150, since in the first situation the number trebled but in the second situation it only grew by fifty percent, not even doubling.

Literature notes that ratio and proportionality in particular, depend on a solid fraction understanding and getting learners to reason proportionally without a solid fraction foundation can be problematic. Literature claims that if we want our learners to develop proportional reasoning in the senior phase, we need to focus on correcting how fractions are being taught at primary school. If we fail to address fraction instruction in primary schools, we will continue to produce learners who struggle to reason. Just learning procedures when working with fractions is harmful and detrimental to mathematics learning and understanding.

For this research project the literature and my own teaching and research experiences have prompted me to examine the kinds of tasks, in terms of sub-constructs, cognitive demands and unifying elements, that are used for teaching fractions and how they are used to develop a deeper understanding of fractions.

1.7 Study Location

The purpose of this study was to investigate and provide a description of the range of sub-constructs the teachers selected to develop an understanding of fractions for their learners across Grades 4-6. The study also aimed to investigate what the teachers made available in their enactment of fraction tasks with regards to the levels of cognitive demand and work with the unifying elements and ultimately what fraction knowledge was made available to learn based on the analysis of sub-constructs, cognitive demand, and unifying elements.

A qualitative research approach was used in this study. This approach is primarily concerned with the quality of human understanding, interpretations and intersubjectivity (Denzin & Lincoln, 2008). The purpose of the study was to investigate and provide a description of what took place in three different mathematics classrooms, across three different Grades, focusing on tasks selected and set-up by three different teachers, to gain an understanding of how fractions were taught, with a specific focus on sub-constructs, unifying elements and task demands. This included an analysis of the fraction chapters and fraction related chapters in the prescribed textbook for each Grade.

Middle Grades' teaching of fractions constituted the phenomena of interest in this study, with in-depth analysis of a single case of this phenomenon based on studying the textbooks and teaching used across a one-year period. Examining tasks, used by the teacher and those that appeared in the prescribed textbooks, sub-constructs, unifying elements and different cognitive demands, all formed part of the case study. The key data collection method used was lesson observations, preceded by a curriculum and textbook analysis.

The study was carried out in a well-resourced private primary school. Selecting a relatively wealthy independent school provided a relatively privileged base with regards to well-qualified teachers in terms of teacher knowledge and the availability of resources (Shalem & Hoadley, 2009). The exploratory findings from this context therefore potentially provided insights into fractions instruction in what can be regarded as ‘best case scenarios’ in the South African terrain, with issues and concerns found here likely to be magnified in the broader landscape. The intention was simply to explore for nature of focus and attention to development of conceptual knowledge of fractions, rather than to seek generalizable findings.

1.8 Analytical Framework

The framework used in this study was developed from a comprehensive review of the literature on sub-constructs, unifying elements, representations, and cognitive task demands, and is presented and discussed in Chapter 2.

To answer my first question, what range of sub-constructs did teachers focus on through their task selections to develop an understanding of fractions for their learners across Grades 4-6 in one school, I needed an understanding of what happens in practice and one way of achieving this was by analysing tasks teachers selected and set-up when teaching fractions. This included an analysis of the fraction exercises prescribed in a set of mathematics textbooks (Classroom Mathematics) for the Intermediate Phase (Grade 4 to Grade 6). Teachers refer to textbooks and different resources to find appropriate tasks/activities that would best represent the content and knowledge that must be learnt. These textbooks are artifacts that aid teachers in making knowledge available. Therefore, an analysis of fractions from the mathematics textbook, *Classroom Mathematics* (Scheiber et al., 2004a,b,c), used in the Intermediate Phase (Grade 4 to Grade 6) in the school where this study was located provided an understanding of what sub-constructs and unifying elements were addressed and what cognitive demands were required of learners when engaging with fraction tasks. It provided insight into what fraction knowledge, through sub-constructs, unifying elements, and cognitive demands, were made available for learners.

The second question, what levels of cognitive demand and work with unifying elements did the teachers make available in their enactment of fractions tasks, required analysing the cognitive task demands in conjunction with the unifying elements, within each sub-construct, to gain an understanding of how connections were being made to develop a complete understanding of fractions. Analysing the actual teaching that took place with a focus on the cognitive demands of the tasks and the unifying elements provided a lens for gaining insight into what fraction knowledge was made available to learners and whether/the extent to which it was made available in connected ways.

To answer both questions an analytical framework for tasks was required. For the goal of this study, using Stein et al.'s (2000) understanding of mathematical tasks and how they are enacted within the classroom context was most appropriate and beneficial. Examining Stein et al.'s (2000) work on tasks was useful since it aided in identifying the kinds of knowledge made available to learners. The selection, set-up, and implementation of tasks are the three phases distinguished by Stein et al. (2000). The selection phase focuses on the teacher as she/he selects tasks as they appear in the curriculum and instructional materials. When analysing the selection phase, it is important to examine the criteria, sources, and purpose of the selected tasks. The selection phase is important, particularly for the teachers in my study, since they had a prescribed textbook and other resources (internet, own worksheets) to select tasks from. They had the freedom to select or disregard tasks as they pleased. Focusing on the selection of tasks enabled me to understand the criteria that guided the teachers' selection of certain tasks and the exclusion of others, the source of the tasks and how teachers planned to use the tasks.

The set-up phase can be described as follows:

> "the teacher's communication to students regarding what they expected to do, how they expected to do it, and with what resources. The teacher's setup of a task can be as brief as directing student's attention to a task that appears on the blackboard and telling them to start working on it. Or it can be as long and involved as discussing how students should work on the problem in small groups, working through a sample problem, and discussing the forms of solutions that will be acceptable." (p. 25).

It is during the set-up phase that teachers usually alter the cognitive demands of a task. They either intentionally or unintentionally change the task from how it appeared in the curriculum or instructional printed materials from which they originally took the idea. This phase was important to my study because it allowed me to examine how the teachers set up the tasks in terms of what knowledge they made available to their learners. In other words, through the set-up phase I was able to identify the sub-constructs, unifying elements and cognitive demand of the tasks selected by the teachers to teach fractions. The tasks' cognitive complexities could be assessed through the connections made with the unifying elements. The enacted tasks formed part of the set-up while the intended tasks formed part of the selection phase. The implementation phase, focused on how learners engage with the tasks in class, fell outside the instructional focus of my study.

For this research project, the focus was only on the selection and set-up of the different tasks. While Stein et al.'s (2000) do not refer to tasks in relation to specific topics like fractions, they provide a certain degree of understanding of things to consider with regards to teachers' experiences when selecting, setting up and implementing tasks in any topic (Govender, 2008). Since fraction instruction is laden with complexities because of the nature of fractions, tasks used to represent the content provided initial tools to study the selections and set-ups of fraction knowledge made available to the learners. That is to say, these tasks were important artifacts of practice that illuminated the nature and range of fraction ideas made available for learners. This is discussed later in the literature chapter.

1.9 Overview of thesis

In this chapter I have presented my rationale for embarking on the study. The next chapter focuses on literature related to the nature of fractions (sub-constructs and unifying elements) and the teaching of fractions, procedural and conceptual work related to the teaching of fractions and the theoretical framework that underpins this study, as well as a description of Stein et al.'s (2000) Mathematical Instructional Task framework that will be used together for the data analysis. Chapter three focuses on the research methodology used for this study. In chapter four I present an analysis of the tasks in the fractions and fractions related chapters of the 3 prescribed textbooks used in the Intermediate Phase. The focus of chapter five is on the analysis of the enacted tasks used by the 3 different teachers when teaching fractions. The final chapter, chapter 6, concludes the thesis with a summary of the findings and implications for teaching and further research studies.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

The literature review focuses on research literature relating to the nature of fractions across the various sub-constructs and the unifying elements and considers the teaching of fractions and how tasks and representations provide a key route into understanding which sub-constructs and unifying elements are in focus in textbooks and in instruction. The literature review also focuses on procedural and conceptual ways of working with fractions. The writing on the distinction between procedural and conceptual ways of working with fractions being marked by connections as a feature of conceptual working leads me into a description of Stein et al.'s (2000) Mathematical Instructional Task Framework. This framework views connections as central to higher cognitive demand-oriented tasks, and given this, the cognitive demand focus within the Instructional Task framework came to function as a useful proxy for exploring more procedurally oriented and more conceptually oriented tasks and ways of working with them.

I provide a description of Stein et al.'s (2000) Mathematical Instructional Task Framework and how it feeds into the analytical framework used in this study. The analytical framework used in this study cross references Stein et al.'s (2000) higher and lower levels of cognitive demand with the five fraction sub-constructs and the unifying elements. The framework was used in chapter 4 to present an analysis of the presentation of fractions in the mathematics textbook, *Classroom Mathematics* (Scheiber et al., 2004a,b,c), used in the Intermediate Phase (Grade 4 to Grade 6) in the school where this study was located. Evaluating the textbooks using this analytical framework helped me assess the cognitive demand range of problems presented there. The framework was also used in chapter 5 to present an analysis of the tasks selected and set-up by the three middle-Grades teachers at the school when teaching fractions. As noted in chapter 1, literature advocates that when it comes to instructional approaches for teaching fractions, there is an overemphasis on procedures (Ball, 1993; Bezuk & Cramer, 1989; Lamon, 2012;

Mack, 2001; Philippou & Christou, 1994). Further, the part-whole sub-construct takes up a large chunk of the fraction curricula across different countries and has been the traditional approach to introducing fraction concepts in primary Grades (Baturo, 2004; Clarke et al., 2011; Common Core State Standards for Mathematics, 2010; National Mathematics Advisory Panel 2008; Lamon, 2001; Van de Walle, Karp, & Bay-Williams, 2013). The multifaceted and inter-connected nature of fractions makes teaching that combines procedural and conceptual knowledge particularly important, to support learners to develop a deeper understanding of fractions. Essentially, this literature critiques the fact that instead of teachers first developing an understanding of fractions using the different interpretations and unifying elements, they move instantly to getting learners to solve operations with fraction problems.

In the first half of this chapter, I detail literature on the sub-constructs and unifying elements that I introduced in Chapter 1. Given that the early, but seminal, literature on both sub-constructs and unifying elements was conducted in the eighties and nineties, I begin with detail on these literature bases and provide a chronology of the ways in which the literature on sub-constructs and unifying elements has been developed and taken up over time. This section concludes with my decisions on how this literature was taken up in this study and reasons for these choices. I then go onto overview key tasks and representations that have been described as particularly useful to use in the teaching and learning of fractions in ways that incorporate attention to the sub-constructs and unifying elements in connected ways. I highlight the different representations associated with particular sub-constructs and unifying elements because this assisted with the categorizing of textbook tasks into sub-constructs and unifying elements, and subsequently, for the analysis of the tasks as enacted.

In this chapter, I draw on literature on fractions as mathematical objects, and the teaching of fractions to build an argument that a rich understanding of fractions consists of a fluid and flexible understanding that works across all the sub-constructs of fractions identified in the literature, with the ability to connect between sub-constructs as well. Connections between sub-constructs occurs via tasks focused on the sub-constructs that emphasis these links. The unifying elements provide an important route through which the different sub-constructs can be connected to each other. These connections help make better sense of the different personalities of fractions because, as Lamon (2012, pg. 257) suggests, "…, all of the interpretations do not provide equal access to a deep understanding and no single interpretation is a panacea……interpretations are tightly intertwined….".

This kind of fluid and flexible understanding is supported by the incorporation of tasks that work across the cognitive demand levels as well as working across sub-constructs to provide openings for connections.

2.2 Chronological development of ideas on fraction sub-constructs and unifying elements

In this section I provide a chronological overview of the development of the sub-constructs. The trajectory of literature related to the unifying elements is dealt with later in this section. The development clarifies how the particular interpretations used in this study was determined. While researchers are unable to reach a consensus on an agreed set of sub-constructs of fractions, they do agree on some central concepts. These are measure, quotient, ratio, operator, and a version of the part-whole interpretation. For this study, all the above-mentioned sub-constructs were considered since they all appear in primary school mathematics.

Many researchers, over an extended period, have tried to explain the different interpretations of fractions and their interconnectedness. Kieren (1976, 1980, 1988,1992,1993, 1995), Behr, Lesh, Post and Silver (1983), Vergnaud (1983) and Freudenthal (1983) are key among these. Kieren (1976) argued that to have a complete understanding of fractions an individual must have experience with multiple interpretations of fractions. Kieren (1976) was the first to propose that the concept of fractions consists of several sub-constructs. Along with Vergnaud (1983) and Freudenthal (1983), Kieren (1980) established the notion of the four interrelated sub-constructs of ratio, operator, quotient, and measure. Kieren originally did not see the part-whole as a separate sub-construct because he identified the part-whole relationship as the foundation of rational number knowledge. Kieren (1980) argued that the part-whole relationship encourages the development of the four sub-constructs of measure, quotient, operator, and ratio, and is embedded in all of them. Behr et al.(1983) agreed with Kieren (1976) on the part-whole sub-construct as an essential part in understanding the other four sub-constructs. They, however contested the idea that the part-whole could not form a separate sub-construct. Thus, they distinguished between part-whole, measure, quotient, ratio and operator as sub-constructs of fractions. Behr et al. (1983) suggested that the part-whole sub-construct coupled with partitioning were the most fundamental constructs for rational number development and that both were basic to learning other sub-constructs.

Taken together, Kieren (1980), Vergnaud (1983), Freudenthal (1983) and Behr et al. (1983) agree on some central concepts: quotient, operator, ratio, measure, and a version of the part-whole interpretation of fractions and these have been widely used and cited in research.

A small number of more recent studies have focused on the part-whole sub-construct and partitioning actions since research indicates that challenges experienced when teaching fractions may stem from the limitations of the part-whole sub-construct. It is important to understand these limitations if we want to improve fraction instruction. I provide a brief explanation of the different studies involving the part-whole sub-construct and partitioning and whether it develops an understanding of the other four sub-constructs. Charalambous & Pitta-Pantazi (2005) sought empirical validity for Behr et al.'s(1983) fraction model concentrating on the claim made that the part-whole sub-construct with partitioning were the most fundamental constructs for rational number development and that both were basic to learning other sub-constructs.

Wilkin and Norton (2018) outline a hierarchy of fraction schemes, supported by mental actions that map a progression from part-whole concepts to measure concepts of fractions (pg.2). Their study involved 646 fifth and sixth Graders' performance on fractions, based on the fundamental role of the part-whole sub-construct in developing understanding of the remaining sub-constructs of fractions. Their findings provided empirical support to the important role of the part-whole sub-construct in developing understanding of the remaining four sub-constructs of fractions. This justified the claims made that the traditional instructional approach used to teach fractions in many different countries is the part-whole subconstruct. They emphasized though that the part-whole sub-construct explains different percentages of the variations in student performance on tasks related to each of the four secondary sub-constructs. Their study revealed that almost all the variance in student performance of the ratio and the operator sub-construct and only a very small proportion of the variance of the measure and quotient sub-construct. They provided three reasons for this finding:

- the core idea of comparing quantities is embedded in part-whole, ratio and operator sub-constructs, but not required for developing the measure and quotient sub-construct.
- the measure and the quotient sub-construct could be explained by other concepts that are not included in Behr et al.'s (1983) model, such as the notion of the unit, which research suggests is very important for building meaning in the quotient and measure constructs (Behr et al., 1983; Lamon, 1999; Marshall, 1993;) as a unifying element.
- understanding the part-whole sub-construct does not necessarily mean that a student will not encounter significant difficulties with the concepts of measure and quotient.

Other studies reveal that the part-whole model can inhibit the development of the other sub-constructs, particularly the quotient sub-construct (Charalambous & Pitta-Pantazi ,2007; Pitkethly & Hunting, 1996). For example, if the fraction three-quarters is only understood as three parts of a pizza divided in quarters and not as a fair share of three wholes divided equally amongst four. Charalambous &

Pitta-Pantazi (2005, p. 239) conclude thus on the five sub-constructs of fractions: "the part-whole interpretation of fractions should be considered as a necessary but not sufficient condition for developing and understanding of the remaining notions of fractions."

The Fractions Project (Steffe and Olive, 2010) elaborated on the results from the Rational Number Project (Behr et al. 1983; Kieren 1980) by focusing on mental actions like partitioning that support various sub-constructs. For example, the part-whole sub-construct relies on the mental actions of partitioning and disembedding. Disembedding is taking parts out of a whole as separate units while maintaining their relationship with the whole (Wilkins and Norton, 2018, p. 2). Steffe and Olive (2010) and Wilkins and Norton (2018) argue that the part-whole sub-construct misses the important mental action of iterating: repeating a unit of length or area to produce a connected whole (p.2). Given this finding, Steffe and Olive (2010) and Wilkins and Norton (2018) also concluded that the part-whole sub-construct was necessary but not sufficient for the development of the remaining sub-constructs.

Wilkins and Norton (2018) exemplify attention to the limitation of the part-whole construct in teacher and learners' working with improper fractions, like $\frac{7}{5}$ which often involves reverting to working with mixed numbers ($1\frac{2}{5}$in this case) on the basis that it is impossible to get 7 parts out of five parts. They advocate for awareness of more sophisticated concepts of fractions, e.g., understanding that $\frac{3}{5}$ as a fraction is three times as big as $\frac{1}{5}$. Such examples move learners' conceptualization of fractions from part-whole concepts to measurement concepts. The part-whole sub-construct involves understanding a proper fraction where a certain number of equal parts are taken out of a whole that has been divided into equal parts while the measure sub-construct involves understanding that one of the equal parts of the fraction is a unit fraction and a certain number of unit fractions make up the whole.

Wilkins & Norton's and The Fraction Project (Steffe & Olive, 2010) research findings provide a trajectory of mental actions based on the unifying elements that can lead from part-whole sub-constructs connected to measure and other sub-constructs. This is important for my study since I am interested in analysing through the different sub-constructs how the unifying elements provide a deeper understanding of the fractions.

The findings from Charalambous & Pitta-Pantazi (2005), The Fractions Project (Steffe and Olive, 2010) and Wilkins and Norton (2018) encourage us to examine and explore how the different sub-constructs work together, in practice, to provide a complete understanding of fractions and that focusing only on the part-whole sub-construct leads to a fragmented and partial understanding of fractions. Their work reveals constraints and limitations in the Behr et al. (1983) model of the sub-constructs of fractions, that helped me to understand how fraction teaching can be problematic. More complete understandings of fractions are achieved when teachers allow their learners to work with tasks that allow them to engage with the part-whole, measure, operator, ratio and quotient sub-constructs and the unifying elements. This is very important for my research since I am interested in what teachers select for their learners to develop a strong understanding of fractions. Analysing the tasks selected and enacted using the five sub-constructs will provide insight into what fraction knowledge is made available for learners.

The next section provides a discussion of the different sub-constructs, and illustrates key tasks and activities described as useful for developing awareness of each sub-construct in the literature. It also deals with a range of aspects that underpin a comprehensive understanding of the sub-constructs.

2.3 Sub-constructs

2.3.1 Part-whole sub-construct

As discussed in chapter 1, fractions have various mathematical meanings and symbols such as $\frac{2}{4}$ might be interpreted in a range of ways. Useful descriptions of important aspects of fraction in part-whole and quotient situations provided by Mamede, Nunes & Bryant (2005, p.282) offer an understanding of these two sub-constructs. The denominator in the part-whole situation, describes the number of parts a whole has been divided into, while the numerator suggests the number of parts taken. This is shown in Figure 2.1 where the whole has been divided into four equal parts and two parts have been shaded, which can be represented by the fraction $\frac{2}{4}$.

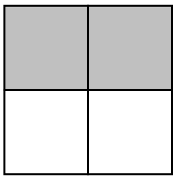


Figure 2.1: A part –whole representation of the fraction $\frac{2}{4}$

From the example above, the fraction represents the shaded part of a whole (square) partitioned or divided into equal-sized pieces. The numerator refers to the number of shaded parts of the partitioned unit, while the denominator refers to the total number of equal-size pieces (parts into which the unit is partitioned). Within any whole unit, the smaller the denominator, the bigger the partitions, resulting in fewer pieces. The bigger the denominator, the smaller the partitions, resulting in more pieces. When solving problems related to the part-whole sub-construct it is important to establish how 'the whole' is considered and what is to be considered as 'the part' of the whole. The result of this relationship between the dividend and the divisor is called the quotient; from the example above, $\frac{2}{4}$ shows that an object has been partitioned or divided into four equal parts and two of those parts are taken. In cases like this where parts are taken from a whole, the numerator must be less than or equal to the denominator (Wong & Evans, 2008).

The part-whole sub-construct applies to both continuous quantities and sets of discrete objects (Sowder, Armstrong et al.,1998; Sowder, Philipp, et al., 1998), so whether the context involves part of a whole or part of a set, it is still considered to be a part-whole sub-construct. When it comes to fraction instruction it is important to use the correct model i.e., either continuous quantities or discrete units to match the context. If the context is related to part of a set, discrete objects would be used and if a context is related to part of a whole, such as pizza, then an area model would be appropriate (Post, Behr, and Lesh 1982).

As noted already, Behr et al (1983) linked the part-whole sub-construct with the process of partitioning – one of the key unifying elements. Lamon (2012), in her extensive writing on rational numbers, explains that the part- whole sub-construct involves the notion of partitioning or dividing either a continuous quantity (including area, length, and volume models) or a set of discrete objects into a number of equal parts (for example, three-sevenths as parts of a whole is interpreted as three of seven equal-size pieces) and equal parts can be composed and recomposed to the initial whole (Lamon, 2005). Partitioning is a unifying element that connects the sub-constructs and is further discussed later in this chapter. The unit refers to the whole in the activities described below.

Activities used to develop the part-whole sub-construct of fractions include:

- partitioning of the unit, with this partitioning sometimes incorporated explicitly, or given implicitly,
- identifying how many equal-sized parts make up a unit,
- unitizing the unit, which is related to equivalent fractions, i.e., mentally chunking the area into different-size pieces, and using this to name equivalent fractions (Lamon, 2012, p. 135).
- providing units of various types i.e., discrete, and continuous units (Lamon,2012) to work with.

Several ideas have developed from research regarding what mastery of the part-whole sub-construct and the partitioning scheme involves (Charalambous & Pitta-Pantazi, 2005). Lamon (2012) suggests the following:

Learners must:

- understand that the parts into which the whole has been partitioned must be of equal size.
- be able to partition a discrete set or a continuous whole into equal parts.
- be able to identify whether the whole has been partitioned into equal parts.
- understand the parts taken together must equal the whole regardless of the size, shape, or arrangement of the equivalent parts.
- understand the more parts the whole is partitioned into, the smaller the produced parts become.
- develop unitizing and reunitizing abilities, which allows learners to reconstruct the whole based on its parts.

2.3.2 Quotient sub-construct

Different from the part-whole sub-construct, the denominator in the quotient situation describes the number of shares and the numerator describes the number of items shared. Mamede et al. (2005) point out that it is important that a connection is made between the meaning of the denominator (divisor) and the numerator (dividend), rather than treating these two quantities separately. In which case $\frac{2}{4}$, means that 2 items (pizza for example) were divided among four children. The answer to the problem of the quantity that each person gets is the quotient. In other words, the division or quotient sub-construct may be understood through partitioning or equal sharing. Partitioning plays an important role in the quotient sub-construct just as in the part-whole and measure sub-constructs (Sowder, Philipp, et al., 1998). The quotient sub-construct therefore opens up routes for fractions to be interpreted either as the operation of dividing or as the outcome of this division, e.g., as $5 \div 7 = \frac{5}{7}$.

Furthermore, this sub-construct can be separated into two kinds of partitioning actions, partitive and quotative dividing. In partitive division, the divisor represents the number of shares. In quotitive division, the divisor represents the size of each share. Graeber and Tanenhaus (1993) explain that partitive division can be thought of as “sharing” or “dealing”. For example, $3 \div 4$ is interpreted as 3 chocolates shared among 4 people with each person getting $\frac{3}{4}$ of a chocolate. The size of the groups is the unknown to be evaluated. Quotitive division can be thought of as a repeated subtraction. For example, Jane has material that measures $9\frac{3}{8}$ m long. Each girl in her sewing class needs a piece $\frac{5}{8}$ m long. How many of the required lengths can she cut? The number of groups is unknown.

Lamon (2012) suggests that for learners to master the quotient sub-construct they:

- must understand and recognize that, unlike in the part-whole sub-construct, there are two different measure spaces for partitive division (e.g., 3 CHOCOLATES shared among four FRIENDS).
- must understand there is no restriction to the size of the fraction. It means that the numerator can be larger, equal to or smaller than the denominator, resulting in the answer being equal to, less or more than the unit (or whole).
- must identify fractions with division and understand what role the dividend and divisor play.
- must develop a thorough understanding of quotitive and partitive division.

Particularly important from the perspective of teaching are the features that differ in quotient sub-construct work from work with the part-whole sub-construct (two different measure spaces for partitive division and no constraints on the size of the numerator in relation to the denominator), that teachers need to be alert to and emphasize in their instruction to support learners to move between the sub-constructs.

2.3.3 Operator sub-construct

The operator sub-construct describes the fraction $\frac{3}{4}$ as a function applied to a number, a set of objects or an object (Behr et al., 1992). The numerator quantity operates on the object, followed by the denominator quantity applied to this result, or vice versa. The numerator produces an extension of the quantity, while the denominator results in a contraction. In figure 2.2 the smaller box was produced by operating on the dimensions of the larger box by a factor of $\frac{3}{4}$.

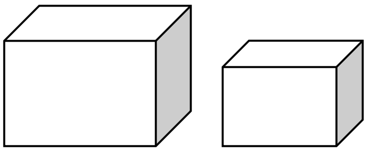


Figure 2.2: Using the operator to shrink the dimensions of a picture. Adapted from Kerslake (1986, p.314)

In other words, the operator sub-construct uses a fraction to operate on a quantity or on the result of a previous operation, (Behr et al., 1983; Kerslake, 1986; Lamon, 2005; Post et al., 1982). It involves both shrinking and enlarging. For example, $\frac{3}{5}$ of 25 would mean multiply 25 by 3 and divide your answer by 5 or divide 25 by 5 and multiply your answer by 3.

Lamon (2012) suggests that for learners to master the operator sub-construct they must:

- interpret a fractional multiplier in a variety of ways (e.g., $\frac{3}{4}$ should be interpreted either as 3 x [$\frac{1}{4}$ of a unit] or $\frac{1}{4}$ x [3 times a unit] (Lamon, 2012).
- name a single fraction to describe a composite operation (when multiplication and division are performed one on the result of the other) (Lamon, 2012)
- identify the effects of an operator and relate inputs and outputs using a rule (e.g., an input of 6 and an output of 13 results from a 13-for-6 operator, an operator that enlarges, the output is $\frac{13}{6}$ of the input.

2.3.4 Measure sub-construct

The concept of unit is central to the measure sub-construct. When working with number lines (finding points a number line), the measure sub-construct is used (Behr et al, 1992). Iterating is an important aspect of the measure sub-construct (Van de Walle et al., 2013; Wilkins &Norton, 2018). For example, two-quarters is constructed by iterating two one-quarters. An illustrative measure task involves giving each child a piece of string that is one meter long and asking them to cut another length of string $\frac{1}{4}$ metre long. When children are given the opportunity to divide string, for example, into identical lengths of various equally sized parts, they gain experience identifying equivalent fractions. For example, they can identify that $\frac{2}{4}$ unit of string is equivalent to $\frac{4}{8}$ unit of string. The measure sub-construct can also facilitate children's understanding and development of addition and subtraction of fractions by helping then identify that adding fractions with like denominators is adding iterations of the unit fraction that consist of the original fractions (Son, Lo, Watanabe, 2015). For example, if one recognises $\frac{3}{9}+\frac{5}{9}$ as 3 pieces of $\frac{1}{9}$and 5 pieces of $\frac{1}{9}$, then combined the answer is 8 pieces of $\frac{1}{9}$ or $\frac{8}{9}$. Understanding this can eradicate the error of adding across numerators and denominators, i.e., $\frac{3}{9}+\frac{5}{9}=\frac{8}{19}$ (Mack,1995) and can be extended to the addition and subtraction of unlike denominators (McNamara, 2015).

The measure sub-construct differs from other sub-constructs since the number of equal parts in a unit varies according to how many times it is partitioned (Lamon, 2005; Van de Walle et al., 2013). Unlike the part-whole sub-construct that highlights how many parts (denominator), the measure sub-construct highlights how much (numerator) (Van de Walle et al., 2013). Successively partitioning the unit in the measure sub-construct differs from the other interpretations in important ways. In the part-whole interpretation, comparing the number of equal parts taken from the whole requires a fixed number of equal parts in a unit. In contrast, the measure sub-construct allows the number of equal parts in the unit to vary and naming the fractional amount depends on how many times you are willing to keep up the partitioning process (Lamon, 2012; Van de Walle et al., 2013). Naik and Subramaniam (2008) explain that when we take a part of a whole, we divide the whole into equal parts to establish a subunit. This subunit is then used to measure out the part that is taken. Lamon (1996) notes that emphasising the counting aspect while missing out the measurement aspect is what has dominated traditional teaching of fractions, with the fact that partitioning involves formation of unit structures underplayed in classrooms (Lamon, 1996).

The focus with the measure construct is on successively partitioning the unit since this allows us to measure with increasing precision (Lamon, 1999; Sowder, Philipp, et al. 1998). The measurements are referred to as 'points', which can be modelled on a number line. The measure sub-construct also allows us to measure any amount by changing the unit of measure, for example, if litres will not work, we can partition into millilitres, and so on. The measure sub-construct develops a strong notion of the unit and subintervals (Lamon 2005; Wilkins & Norton, 2018). Class activities that include measure as a sub-construct would be successive partitioning of a number line, reading meters and gauges. Through successive partitioning, the measure sub-construct is useful for learning about adding fractions (Behr et al., 1992) as equivalences emerge. Understanding equivalence aids learners in determining common denominators when adding and subtracting fractions. This can be achieved before introducing algorithms which results in a deeper understanding of fractions (Lamon, 2012).

The partitioning process help learners to build a sense of the density of rational numbers, relative sizes, relative locations, and fraction equivalence. According to Lamon (2012, p. 213), "the density of rational numbers says that between any two fractions there is an infinite number of fractions and that you can always get as close as you like to any point with a fraction". In her 2005 writing (pp. 173-174) she provides an example of a task directed at this idea drawing from the measure sub-construct interpretation using partitioning. To determine two fractions between $\frac{1}{4}$ and $\frac{1}{3}$ requires the renaming of $\frac{1}{4}$ as $\frac{3}{12}$, and $\frac{1}{3}$ as $\frac{4}{12}$. This is achieved by partitioning the unit interval into twelfths. If the interval between $\frac{3}{12}$ and $\frac{4}{12}$ is partitioned again(into two equal parts), twenty-fourths will be the new subunit and $\frac{3}{12}$ and $\frac{4}{12}$ will be renamed $\frac{6}{24}$ and $\frac{8}{24}$, respectively. So $\frac{7}{24}$ must lie between them. If the interval between $\frac{3}{12}$ and $\frac{4}{12}$ is partitioned again (into three equal parts), the sub-unit will each be $\frac{1}{36}$ and $\frac{3}{12}$ and $\frac{4}{12}$ will be renamed $\frac{9}{36}$ and $\frac{12}{36}$ respectively. Therefore $\frac{10}{36}$ and $\frac{11}{36}$ lie between them.

Lamon suggests that for learners to master the measure sub-construct they must:

- understand the concept of the density of rational number
- be able to measure any distance from the origin using a given unit interval

2.3.5 Rate and ratio sub-construct

Ratio is the fifth and final sub-construct of fractions. Fractions can be used in ratio format. A comparison of any two quantities in a given order (Behr et al., 1983; Charalambous & Pitta-Pantazi, 2006; Lamon 2012, Streefland, 1991) is described as a ratio. The ratio sub-construct differs from the measure, part-whole and operator sub-constructs in that the numerator and the denominator do not refer to the same quantity. Unlike the quotient and part-whole sub-constructs, the ratio sub-construct does not involve the idea of partitioning (Reys et al., 2012). Rate represents an extended ratio, in that a rate is "a ratio that applies not just to the situation at hand, but to a wide range of situations in which two quantities are related in the same way."(Lamon, 2012, p. 235). For example, R3 per meter (m) is a rate that describes the relationship between cost in rands and number of meters in all the following instances R6 for 2m, R24 for 8m, R54 for 18m, and so on.

While the other four sub-constructs add, subtract, multiply and divide according to the same rules, ratios do not follow the same rules that can be used for fractions. For example, $\frac{2}{3}+\frac{3}{4}=\frac{2+3}{3+4}$ when $\frac{2}{3}$ and $\frac{3}{4}$ are ratios $=\frac{5}{7}$ When $\frac{2}{3}$ and $\frac{3}{4}$ is not a ratio, addition is done as follows: $\frac{2}{3}+\frac{3}{4}=\frac{17}{12}$ (Behr et al, 1992). Lamon (2005, pg. 229) exemplifies this using a real world context:

"Yesterday Mary had 3 hits in 5 turns at bat. Today she had two hits in six times at bat. How many hits did she have for a two-day total?

Mary had 3:5 + 2:6 = 5:11 or 5 hits in 11 times at bat. If we were adding fractions, we could not write $\frac{3}{5}+\frac{2}{6}=\frac{5}{11}$" .

Ratio and rate sub-constructs help develop a strong notion of equivalence and proportionality (Behr et al.,1983; Kieren, 1993). Lamon (2005) has argued that children who studied ratio and rate as their primary sub-construct of rational numbers were able to switch between ratio and part-whole comparisons and experienced no difficulty with addition and subtraction of fractions (Lamon, 2005). Rate/ratio fraction understanding also helped these children to develop their own ways of reasoning about multiplication and division of fractions.

Lamon (2005) and Marshall (1993) suggest that to master the ratio sub-construct learners must:

- construct the idea of relative amounts, i.e., comparing measures of two parts of the same set (a part-part comparison) or the measures of any two different quantities (e.g., cups of juice concentrate to cups of water).
- understand the relationship between two quantities and that they change together so that the relationship between them remains invariant.
- realize when two quantities are multiplied by the same nonzero number, the value of the ratio remain the same.

This discussion of the different sub-constructs of fractions was useful for this study in offering insight into the range of sub-constructs necessary to develop an understanding of fractions for learners across Grades 4-6 and to help develop an analytical tool to analyse the different tasks from the textbooks and those used by the teachers when teaching concepts related to fractions. The next section explains the unifying elements that connect the different sub-constructs of fractions.

2.4 Unifying elements

This section discusses the unifying elements described in chapter 1. Prior to detailing the unifying elements in this section, I provide a short historical overview of discussions relating to them, drawing on the work of Behr et al. (1993), Carpenter et al. (1993) and Kieren (1988, 1992, 1993, 1995). This work feeds into Lamon's (2005) writing on the unifying elements, that is used in the illustrations of the unifying elements. Behr et al.'s (1993) model included attention to operations linked to particular sub-constructs, but unifying elements connecting the different sub-constructs were introduced primarily through the work of Kieren (1988, 1992, 1993, 1995) and Carpenter et al. (1993). Carpenter et al. (1993) identified 'unifying elements' or 'supporting elements' which are important when looking across all the sub-constructs. The three unifying elements are identification of the unit, partitioning, and the notion of quantity. The five sub-constructs are interrelated by the unifying elements and understanding of all the sub-constructs is considered important for solving problems related to fractions. In his extended body of work on rational number, Kieren (1988, 1992, 1993, 1995) argued for a redefinition of his own earlier model around four sub-constructs (measure, quotient, operator, and ratio), three underlying concepts (partitioning, equivalence and unit forming), and a description of learning with four levels (ethnomathematics, intuitive, technical symbolic, and axiomatic deductive). This new model was not in opposition to his earlier model, nor to the model used by Behr and associates in the Rational Number Project (Behr et al.,1983; Behr, Harel, Post, & Lesh, 1992); instead Kieren recognized the fraction- related processes of partitioning, equivalence and unit-forming as providing a better categorization of the underlying fraction concepts that were described in the work of the Rational Number Project.

Kieren's model and understanding of 5 sub-constructs and unifying elements are significant for my study because as mentioned in chapter 1, the unifying elements connect the sub-constructs into a unified scheme, and thus, they provide a key route through which fractions learning and teaching of the different sub-constructs can be experienced as coherent and connected. Exploring the ways in which the unifying elements were drawn into fractions teaching allowed the analysis to consider the nature and extent of coherence and connections in fraction teaching (Carpenter et al., 1993). Kieren's model compared to Behr et al's model provided a better understanding of how the 5 sub-constructs are connected and developed through the unifying elements and not just through the part-whole sub-construct. Kieren's model afforded me the opportunity to analyse my data using the unifying elements to build connections between the sub-constructs and cognitive demand levels teachers made available in their selection and enactment of fractions tasks. The part-whole sub-construct, while seen as the seedbed for the other sub-constructs, requires the unifying elements for a more complete understanding of fractions. Thus, exploring the ways in which the unifying elements occurred in fractions instruction provided me with one route into understanding what fraction knowledge was made available for learners.

I will now explain each of the unifying elements, drawing on Lamon's writing, which, in turn, rests on the earlier work overviewed above. Tasks that have been used to illustrate attention to the unifying elements are incorporated in this discussion.

2.4.1 Identification of the Unit

Identifying the unit involves several aspects of the unit. It includes three ideas under the heading of identification of the unit: unitizing, the unit as implicit or explicit and continuous and discrete units.

When working with fractions an important aspect is to identify the unit and to ensure that each fraction is interpreted in terms of that unit (Lamon, 2005). Fractions cannot be compared based on different units. Working with fraction problems using concrete objects and not establishing the unit, leads to confusion for learners. Lamon (2012) explains that every fraction depends on some unit and the answer to the question 'How much?' requires a unit of measurement to determine the proportion in relation to this unit that is required. She uses the example of cookies and explains that if you are told that you can have half of my cookies, but I never tell you how many cookies I have, you will not have any idea of how many you are going to get (p. 97). She further explains that every fraction is a relative amount. In other words, what you have in the form of a fraction is relative to the unit. When teaching fractions, it is important for learners to understand that the unit may be something different in every new context and that an important question to ask is "What is the unit?" (Lamon, 2012, p.98). The whole need not only be a single pizza, cookie, or cake but rather, can be a collection of discrete objects (whole collection of objects, a group regarded as a single entity etc.) before partitioning occurs. When teaching, the meaning of a fraction is derived from the context in which it is used. The context either implicitly or explicitly defines the unit. A problem that tells you exactly what the unit is, is defined as explicit. With implicit units, the unit is not explicitly stated, but there is sufficient information to make sense of what it is (Lamon, 2012).

Lamon (2012) argues that children who are not exposed to the unit and an understanding of how to interpret a problem in terms of the unit, usually never develop a solid fraction sense or reasoning capacity (Lamon, 2012).
Lamon (2012, p. 98) uses the following example to analyse students understanding of the unit.

> "Mr. Mc Donald took six of his basketball players out for pizza. They ordered 2 large pizzas, a cheese, and a pepperoni, and the seven of them each ate 1 slice of each pizza. If each was machine cut into 12 slices, how much of the pizza was eaten?"

The following are responses from 3 students:

- John M. they ate $\frac{14}{24}$ of the pizza.
- Sally J. $\frac{14}{12}$ of the pizza.
- Andy S. 14 pieces

Lamon (2012), explains that from the example we are told that the unit consists of two pizzas or 24 slices. This means that if each of them ate a slice of each pizza, 14 of the 24 slices were eaten. John is correct assuming he did not think that the unit is 1 pizza or 12 slices and incorrectly added $\frac{7}{12} + \frac{7}{12}$ and got $\frac{14}{24}$. Sally did not identify the unit correctly. She thought that 1 pizza or 12 pieces was the unit. According to Sally $\frac{14}{12}$ of the pizza was eaten. This answer is incorrect since it would mean they ate $2\frac{1}{3}$ pizzas, which is more pizza than they had. Andy did not answer the question because instead of answering how much of the pizza was eaten in proportional terms, he answered how many slices were eaten. He counted slices which does not answer the question, which states how much was eaten from the total pizza ordered. In the example above the context of the problem allows students to determine the unit (Lamon,1999). Two pizzas with 12 slices each could have different units. The unit would be a pizza if the questioned asked how much of a pizza was eaten. The unit would be 2 pizza if the question asked how much of the pizza was eaten, which was the case in this example. If learners are never given the opportunity to determine the unit when working with fractions, Lamon would argue that understandings of fractions would be limited. In the empirical analysis, I studied the tasks teachers selected to promote that understanding of the unit and how the unit was presented across the different sub-constructs in the different tasks. I also looked at whether the unit was presented implicitly or explicitly and whether both continuous and discrete units were used. This helped provide insight into the nature of the fraction knowledge that was made available to learners.

2.4.2 Unitizing

Unitizing is a natural mental process involving conceptualising the unit in terms of different size chunks (Lamon, 2005). Lamon explains that unitizing helps children develop a strong notion of the unit and of equivalent fractions. This is because unitizing entails the mental coordination of the number of parts the unit is divided into and the size of the parts. This is an important measurement principle and is an important concept for developing fraction sense. It forces learners to reason up and down which in turn develops proportional reasoning. Unitizing also helps learners move away from the idea that the unit can only be divided into what Lamon (2012) refers to as 'comfortable' or 'nice' numbers, i.e., halves, thirds, and quarters.

Lamon (2005, pp. 125-126) uses the following example of a task focused on the part- whole with unitizing sub-construct to explain how it aids in developing children's understanding of the unit and equivalent fractions:

$\frac{3}{5}$ of the rectangle is shaded, since its area is divided into 5 equal parts and 3 of those parts are shaded compared to the area of all 5 parts.

Looking at it differently, the rectangle could be made up of 20 small squares, then the shaded parts could be named $\frac{12}{20}$.

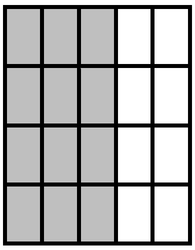


Figure 2.3: Diagram showing $\frac{12}{20}$ after unitizing

The shaded part could also be $\frac{6}{10}$ of the rectangle if the rectangle is composed of small rectangles formed of two of those small rectangles.

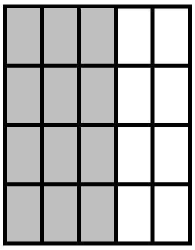


Figure 2.4: Diagram showing $\frac{6}{10}$ after unitizing

In this example, we see that unitizing produces equivalent names for the same amount. Lamon includes examples of unitizing that work across seven different denominations of the unit, going beyond the simple denominations outlined above.

Unitizing is an important unifying element in that it plays a vital role in the processes needed to understand fractions, particularly in sharing and equivalence. It is developed over time and with experience in varied contexts (Lamon, 1996). My research aimed to identify unitizing in the different sub-constructs found in the different tasks used by the teachers to teach fractions.

This categorization of different ways in which 'units' can be configured, coupled with the advice to incorporate the range within fractions instruction, provided a useful lens for studying the ways in which units were configured in the tasks presented in the textbook series and in fraction lesson enactments across Grades 4-6 in this study.

2.4.3 Partitioning

From the literature, the notion of partitioning a whole is central to the part-whole fraction sub-construct. Kieren (1992) describes partitioning as the folding and drawing actions required of children when making equal parts (1995). Such operations generate quantities (Lamon, 1996). In view of the fact that partitioning experiences are so important to the initial development of rational numbers (Ball, 1993; Kieren et al.,1992; Mack, 1991, 1993; Streefland, 1993) learners should be afforded numerous opportunities to partition diverse quantities in different ways to develop an understanding of the representations of fractions (Govender.

Lamon (2012) explains that partitioning activities are best introduced visually: for example, share 3 pizzas among 6 people would entail that children visualize each person's share as a half of a pizza before using their pen or pencil to try and partition the pizzas. Lamon emphasizes the need to incorporate different verbal formulations: Each person gets $\frac{3}{6}$ of a pizza, when 3 pizzas are divided among 6 people. If 6 people share the pizza, each share is $\frac{1}{6}$ of the unit whole.

Unitizing and partitioning (unifying elements) occur within other sub-constructs as well. The example above can be seen as a quotient task rather than a part-whole task, and can therefore, function as a route towards interconnection.

An understanding of partitioning in the different sub-constructs plays a fundamental role in the development of fraction understanding. The measure, part-whole and quotient sub-constructs require the unit to be partitioned in different ways. Partitioning coupled with unitizing and determining the unit worked together to provide an understanding of how the different sub-constructs related to each other to gain insight into what fraction knowledge was made available in fraction instruction.

2.4.4 Notion of quantity

A strong quantitative notion of rational numbers is the basis for a sound understanding of rational number concepts (Behr, Wachsmuth & Post, 1984). Unlike whole numbers, rational numbers are described as being more difficult for learners to conceptualize the size of. The difficulty has been described as arising, at least in part, because rational numbers do not form a fixed succession of numbers as counting numbers do (Lamon, 2012). With rational numbers there is an infinity of other numbers between any two of them. Learners often apply whole number rules when working with fractions and this interferes with their understanding of fractions (Lukhele et al., 1999). When working with fractions, for example ¾, on a number line, learners must see the fraction as a single quantity, rather than as some kind of collection of three and four as separate quantities. This entails a shift from whole number thinking, in which the numbers three and four can be seen as independent values on a number line (Carpenter et al, 1976; Hart, 1989; Howard, 1991; Pitkethly and Hunting, 1996; Streefland, 1991), to a measurement value that can be the outcome of partitioning and re-unitizing, within working with part-whole, measure or quotient tasks. This entails the need to connect partitioning and re-unitizing processes in tasks focused on part-whole, quotient and measure sub-constructs with their outcomes as single quantities in a measurement sense, and further, to perceive the relative size of rational numbers (Lamon, 1993). As mentioned earlier, central to the measure sub-construct is the unit and partitioning process.

As discussed previously, equivalence forms part of a notion of quantity. Being able to work with equivalence and compare different fractions builds a quantitative understanding of fractions. When comparing $\frac{3}{4}$ and $\frac{7}{8}$, learners who know that $\frac{3}{4}$ is equivalent to $\frac{6}{8}$ can reason that $\frac{3}{4}$ is less than $\frac{7}{8}$ since $\frac{6}{8}$ is less than $\frac{7}{8}$, and $\frac{3}{4}$ is equivalent to $\frac{6}{8}$ (Post et al, 1986). This quantitative notion of rational numbers is important for making fractions meaningful for learners (Post et al, 1986). Equivalence appears in all the sub-constructs and research indicates that the concept of equivalence is constructed one sub-construct at a time making its development a recursive process (Ni, 2001).

To conclude, the notion of quantity is developed through partitioning processes and aids the understanding of the density of fractions, their relative sizes, locations, and equivalence. Since all the sub-constructs draw upon equivalence, this element provided a lens to examine tasks to determine whether learners were provided with opportunities to partition and work with equivalence to gain an understanding of how fraction concepts work.

As discussed earlier, for the purpose of this research I take on the view expressed by Kieren (1992) that the sub-constructs are intertwined and interconnected through the unifying elements. This understanding of the interconnectedness of the sub-constructs aided in creating part of the analytical tool.

Somewhat underplayed in the literature, and yet important for my empirical analysis, was the fact that different representations tended to be associated with different fraction sub-constructs and unifying elements. In the following section, I highlight the representations described as key markers of particular sub-constructs/unifying elements because this assisted with the categorizing of textbook tasks into sub-constructs and unifying elements, and subsequently, for the analysis of the tasks as enacted.

2.5 Representations and tasks related to fraction sub-constructs and unifying elements

There are different ways that fractions can be represented when teaching. The literature makes clear that different representations are linked to different sub-constructs. Thus, a key part of supporting learners to develop a complete understanding of fractions involves ensuring that the representations they are given access to include all the sub-constructs and that different representations are included when teaching fractions. Examining the representations therefore provides a key route to determining the sub-constructs made available to learners.

From literature we understand that there are a number of ways that fractions can be represented and several studies (Kong, 2008; Pitta-Pantazi, Gray, & Christou, 2004; Yang & Reys, 2001) show how the links between representations of fractions and the five interpretations of the fraction concepts support learners' fraction understanding. A focus on only one representation and sub-construct limits learners' conceptual understanding of fractions. Kerslake (1986) noted in her study of the different models of fractions children are familiar with that the difficulty with fractions arises because of their limited view of a fraction.

From the literature, the most commonly mentioned representations used to teach fractions include symbolic representations, area/region and set of objects and number line (Duval, 2006). Here, I review the different representations that come into play within certain sub-constructs and how they influence the teaching of fractions.

In fractions, as with other mathematical ideas, working with multiple representations has been noted as very important. Given the various sub-constructs, it is also important to ensure that particular representations recur in the context of different sub-constructs, as meanings and language related to aspects of the representation shift in this move. Drawing explicit attention to these shifts in language and meaning while working with the same representation provides a route into bringing sub-constructs into conversation with each other. The latter point means that in addition to thinking about moves between representations (symbolic/discrete and continuous area/number line/ real-world situations), the teacher also needs to be alert to the ways in which meanings and language associated with the quantities and relationships between the quantities overlap and vary across the tasks/representations used in dealing with different sub-constructs and unifying elements. These differences in meanings and language are dealt with below.

2.5.1 Symbolic Representations

The symbolic representation of fractions is common in most classrooms. The symbolic representation of a fraction uses the notation $\frac{numerator}{denominator}$. Because of the multifaceted nature of fractions, the numerator and denominator take on different meanings according to the type of representation, situation, or sub-construct. For example, in a part-whole representation the denominator represents the number of parts the whole has been partitioned into and the numerator is the number of parts taken (Lamon, 1999; Mamede, Nunes, & Bryant, 2005). In a quotient representation the numerator is the number of items/objects to be shared and the denominator is how many people the items/objects must be shared by (Lamon, 2012). In a ratio representation the numerator is compared to the denominator. In an operator representation the denominator describes what the unit must be partitioned into, once the unit is established and partitioned the numerator describes the amount (multiplier) of the unit. While learners may use the symbolic representation for each sub-construct to provide solutions, this does not guarantee an understanding of the meanings associated with the symbolic representation that work across the sub-constructs. A substantial body of literature has noted that many learners interpret fractions as part-whole and never move beyond this sub-construct. They associate all fraction work with their partial knowledge of the part-whole sub-construct (Clarke, 2011; Lamon, 2001; Van de Walle et al., 2013).

Making sense of symbolic representation requires connections to be made with other models in the teaching of fractions (Cramer & Whitney, 2010; Petit, Laird, & Marsden, 2010), for example, illustrating what $\frac{3}{4}$ looks like in terms of pizzas (area- part-whole), on a number line (length- measure), reducing/enlarging a picture (operator) or filling bags with marbles (quotient). Providing an understanding of symbolic representations with regards to different fraction models and the sub-constructs is not evident in many classrooms and has thus prompted enquiry into how fractions are being taught in mathematics classrooms (Govender, 2008).

2.5.2 Area/ region and set of objects representations

Given that the unit plays an important part in developing a complete understanding of fractions (Lamon, 2012) and it is a unifying element, the literature base notes that representations used to teach fractions should include continuous wholes and discrete units. In discrete sets of objects representations, fractional parts are often identified by different colours or subsets of the whole.

Seeing a rectangular region as a whole compared to a discrete set of circles is easier for learners (Gay, 1997). Partitioning an area model is initially limited to units with a “measure of one,” denoting that that the shape being partitioned represents one unit (Mack, 2001). This limited view encourages the idea that a fraction is always a part less than a whole. To understand improper fractions, learners must reorganize their conception of a fraction as part of a whole. This can be done by separating the unit fraction, iterating it until a whole is formed, then to continue iterating it until a fraction that exceeds the reference whole is formed (Tzur, 1999). The fractions literature base argues that learners must be afforded the opportunity to work in reverse ways to establish the unit i.e., the unit can be represented either implicitly or explicitly. Most importantly, as suggested by Wilkins and Norton (2018), Lamon (2012), Steffe and Olive (2010) and Behr et al (1992), provision must be made for learners to work and engage with the unit and develop an understanding of how to interpret problems in terms of the unit if they are to develop a solid sense of fractions, this however, is often not the case (Alajmi, 2012; Kerslake, 1986; Pantziara & Philippou, 2012; Sowder, 1992). Charalambous & Pitta-Pantazi (2007) in their comparative analysis of the addition and subtraction of fractions in textbooks from three countries, Cyprus, Taiwan, and Ireland, noted limitations in most examples across the three countries:

> "In sum, with only a few exceptions, most textbooks showcased similar procedures in their worked examples; they emphasized the part-whole construct, and favoured area models as a representation" (pg. 27).

Kerslake (1986) concurs with this view of limited access. The area model often uses items such as chocolates, cakes, and pizzas (context-based problem) or circles, squares, rectangles with a fractional part shaded. These 'pre-partitioned' representations have been noted as prevailing in common textbook tasks (Kerslake, 1986; Lamon, 2012; Steffe and Olive,2010; Wilkins and Norton, 2018). The prevalence of pre-partitioned representations is a side-lining of attention to partitioning and unitizing processes and the possibilities of working with different sized parts, with a reduction of fractions as pre-rendered part-whole forms to be identified by counting.

The area/region and set of objects representations can bring into play the part-whole, quotient and ratio sub-constructs and encourage the ideas of fair share and equivalence, but this requires pedagogic attention and tasks to highlight overlaps and differences. The area model can also be used for explaining and making sense of division and multiplication of fractions which is related to the operator sub-construct. The composition of operators naturally leads to fraction multiplication, and the division of fractions can be interpreted as the composition of two operators (Lamon, 2012). Lamon provides the following examples of an area model for multiplication and division and explains how it can be used to teach multiplication and division (pp. 198 & 201). Multiplication of fractions: $\frac{2}{3}$ x $\frac{3}{4}$ means "take $\frac{2}{3}$ of $\frac{3}{4}$ of a unit" and it is the same as taking "$\frac{6}{12}$ or $\frac{1}{2}$ of the unit" (p.198). Using an area model illustrates these compositions in a useful and convenient way. Division of fractions: $\frac{3}{4} \div \frac{2}{3}$, this means how many $\frac{2}{3}$s are there in $\frac{3}{4}$? To solve this, we first need to determine how much is $\frac{2}{3}$ of 1? Our division question now becomes: how many times can we measure an area of $\frac{2}{3}$ out of the shaded area representing $\frac{3}{4}$? We can measure $\frac{2}{3}$ (or 8 small squares) once out of the shaded area and we have 1 small square, so the answer is9/8.

It was interesting in my research to examine the Intermediate Phase textbooks and teaching and see how the area model was used to teach fractions and what connections were made to the different sub-constructs and how the unifying elements came into play, if they did at all.

2.5.3 The number line representation

The number line representation requires children to place fractions on a number line or to identify the fraction that is shown on the number line. Working with number lines requires learners to work with the measure sub-construct. Understanding the unit is central to working with measure and it is a unifying element. This sub-construct develops a strong notion of the unit, and continual partitioning of the unit encourages the understanding of the 'density' of fractions. It also aids in understanding equivalence by showing on the number line that a number can be named in a variety of ways. Number lines help learners see fractions as parts of a whole, part of distance or time. They help learners with comparing fractions and provide an effective alternative to traditional area/region models that are used to teach fractions. Researchers have noted that for learners to work with and understand the number line model they should have prior knowledge of the part-whole sub-construct and partitioning (Bright, Behr, Post & Wachsmuth, 1998; Charalambous & Pitta-Pantazi, 2005). Learners can be taught to interpret $\frac{1}{3}$ as one of three pieces of a cake (part-whole), but as shown in the figure below, also think of $\frac{1}{3}$ as one-third of the distance from zero on a number line (Siegler, Thompson, & Schneider, 2011).

Figure 2.5: Number line showing thirds

Having this knowledge enables them to determine what the whole is in terms of the sub-construct and how to partition the whole. A study of 5th and 7th Grade Finnish students by Hannula (2003) documented results of poor understanding and lack of connections between the part-whole sub-construct, partitioning and measure sub-construct resulting in students inability to use the number line to make sense of the appropriate whole. When learners achieve an understanding of the part-whole fraction sub-construct they can determine the whole when given part of it (Steffe & Olive, 1991). This in turn helps them when working with number lines to determine what the whole is. For example, if a learner understands that $\frac{1}{5}$ means that the whole is made up of 5 equal parts, they will be able to determine on a number line that 5 equal parts, starting from 0 to 1, make up the whole.

By creating and providing connections between the part-whole sub-construct and the measure sub-construct using the unifying elements of partitioning and identification of the unit, learners can transition from using the area model to using the number line resulting in a better understanding of fractions. Similar to the area model, it was interesting to note how the textbooks and teaching in the Intermediate Phase used the number line to teach fraction concepts and how the part-whole and measure sub-construct together with the partitioning and identification of the unit unifying elements came into play to create a complete understanding of fraction concepts.

Representations provide a key route into understanding which sub-constructs and unifying elements are in focus in textbooks and instruction. Following from the discussion of representations, instruction that does the ‘connecting work’ between sub-constructs is critical. Looking for the unifying elements and for attention to the shifts in meaning highlighted above provides one route to looking for connections. Another route is provided by looking at cognitive demand literature which also highlights ‘connections ‘as a key route into more conceptual ways of working. This leads me to the next section in this literature review: Cognitive demand of tasks used to teach fractions. Cognitive demand has been used to demarcate more procedural and more conceptual tasks through looking for connections. Given that in the context of fractions, connections between sub-constructs can occur via representational connections, language, unifying elements – in this section, I introduce Stein et al.’s (2000) cognitive demand model and features, and then present an analytical framework that tailors their features to fractions by bringing in fraction indicators based on sub-construct connections (through representations and language connections) and through unifying elements related to cognitive demand levels.

2.6 Cognitive demand of tasks used to teach fractions

I have already noted that I chose to use Stein et al.’s (2000) cognitive demand framework in this study because attention to connections features centrally in this framework as a marker of conceptually demanding work. In this study I use Stein et al.’s (2000) classification of a mathematical task as a classroom activity that aims to focus learners’ attention on a specific mathematical idea.

I will now discuss Stein *et al.* (2000) understanding of mathematical tasks and part of the analytical framework derived from the literature on tasks.

2.6.1 Stein et al.'s Mathematical Task Framework

Stein et al.'s Mathematical Task Framework (Stein et al., 2000, p. 4) distinguishes between three phases of a task during teaching: selection, set-up, and implementation. The first phase is how the task appears in the curricular or instructional material e.g., textbooks or as created by the teacher. The second phase involves the set-up of the task and this entails the instructions given by the teacher as to how the task must be completed, what is expected of the learners, handing out materials and tools or telling learners "to begin work on a set of problems displayed on the chalkboard" (Stein et al., 1996, p.460). Task implementation, the third phase, involves the work done by learners on the task. All these phases 'are viewed as important influences on what students actually learn' (Stein et al., 2000, p.4). While all three phases are important in studying task practices, for the purpose of my research, my focus will be on the first two phases, respectively.

As discussed in chapter one, focusing on how the tasks appears in the textbooks or what the teacher creates as a task allowed me to examine the range of sub-constructs, unifying elements and representations presented in textbooks and the ways that teachers selected to develop an understanding of fractions for their learners in Grades 4 to 6 and how these sub-constructs are connected. It must be noted that unlike the teachers in Stein et al.'s (2000) study, the teachers involved in my study were given no training or support prior to my data collection. The selection of the tasks by the teacher allowed me to examine the range of sub-constructs and unifying elements the teachers focused on. With regards to the setup of the tasks, my focus was on what happened with the tasks: how the teacher set up the tasks to develop fraction understanding. As mentioned earlier, Stein et al.'s (2000) do not refer to tasks in relation to specific topics like fractions but they do provide a certain degree of understanding of things to consider with regards to teachers' experiences when selecting, setting up and implementing tasks in any topic (Govender, 2008). Since fraction instruction is laden with complexities because of the nature of fractions, the tasks that teachers use to represent the content provide useful insights into what different teachers do as they teach fractions, what representations and manipulatives they use, how they use them and how they explain and connect the concepts related to fractions. This analysis provides avenues for answering my first and second research questions related to the nature and range of fraction tasks in the textbook series (analysis presented in Chapter 4) and in teachers' selections and set-ups of tasks in their instruction (analysis presented in Chapter 5). What fraction knowledge is made available to learn is based on the analysis of sub-constructs, cognitive demand, and unifying elements across the three Grades.

Tasks features and cognitive demand are two interrelated dimensions when examining and analysing tasks during the three phases. Task features include aspects of tasks that mathematics educators deem important for learners' thinking, reasoning, and sense making (Stein, Grover & Henningsen,1996). Task features include aspects such as whether the task is open to multiple solution strategies, whether it lends itself to multiple representations, and whether, in enactment, learners are given the opportunity through the task to offer explanations and justifications. Stein et al. (2000) refers to the level and kind of thinking required of students to successfully engage, complete, and solve a task as cognitive demand (Stein et al., 2000). The authors identify two broadly different levels of cognitive demands of mathematical tasks. Lower-level cognitive demand and higher-level cognitive demands. Memorization and procedures without connections tasks are procedural in nature and considered to offer lower-level cognitive demands while procedures with connections and doing mathematics tasks are conceptual in nature and considered to offer higher-level cognitive demands. For the purposes of my study and given the literature base highlighting memorized procedural orientations in fraction teaching, it made sense to work with the two broad levels of lower and higher cognitive demand.

It is important and necessary to note that the categories established by Stein et al. (2000) are relative to learners' Grades and to individual learners. This means that the level of the task must be appropriate for the Grade. A higher-level cognitive demand task set at a Grade 5 level can, in relation to the curriculum, be classified as a lower-level cognitive demand task in Grade 6 or 7 and vice versa. Tasks can also be classified based on how different learners engage with the task cognitively and thus can be classified differently in relation to different learners. Given this study's focus on fraction tasks and teaching, rather than learning, the curriculum specifications mediated via the textbook scheme, were interpreted as the benchmark for Grade-appropriate tasks. This interpretation was useful for deciding whether a task involved the: 'previously learned facts, rules, formulae, or definitions' that are part of Stein et al.'s (2000) definition of memorization tasks.

While Stein et al.'s (2000) Mathematical Task Framework provided a lens to examine the nature of tasks selected and enacted by teachers in terms of cognitive demands, it was limiting in that the features for the levels of cognitive demands were not specific to fractions. With regards to my research, these generic features of cognitive demand can be linked to the literature reviewed earlier in this chapter to generate indicators for higher and lower cognitive demand fraction tasks. The tailored cognitive demand framework allowed for an analysis what specific fraction knowledge was being made available for learners when engaging with the different tasks. This in turn reveals the conceptual and procedural knowledge made available for learners.

2.7 Analytical tool

An analytical tool was developed to examine the data. Examining the literature on the nature and the teaching of fractions helped develop an analytical tool that allowed me to analyse the data and conclude what sub-constructs and unifying elements the teachers used when teaching fractions. Further, according to literature, a feature of higher cognitive demand tasks is that they include connections and integration with different sub-constructs. This connection between sub-constructs has been identified as being central to developing conceptual understanding, and therefore looking for connections – which the literature suggests can occur through the tasks/representations and the unifying elements, in the context of fractions, provided a route into analysing the cognitive demand of tasks with conceptual tasks associated with higher, or more connected, task demands. Stein et al.'s (2000) framework is particularly relevant to the literature but they include other categories, which, based on their observations, also support connections. These include for example, providing explanations and justifications and descriptors like "have no connections to concepts or meaning that underlie the procedures being used" or "procedures are closely connected to underlying concepts" etc. These pedagogic extensions are useful additions that helped me to look for a range of connection types.

Table 2.1: Cognitive demands in Stein et al.'s Mathematical Tasks categorisation

Level of Cognitive Demand	Explanation of Categorisation
Memorization	• recall of previously learnt information (facts, rules, formulae, definitions) • no understanding required • No procedures to solve problem because of time frame.
Procedures without connections	• limited cognitive demand • algorithmic • no connections to concepts or meanings • focus on producing correct answers • requires no explanations or justifications
Procedures with connections	• use of procedures to develop deeper levels of understanding of mathematical concepts • suggested pathways to follow • procedures are closely connected to underlying concepts • requires some degree of cognitive effort
Doing mathematics tasks	• complex and non-algorithmic thinking • no predictable pathways to solve problems • requires exploration and understanding of mathematical concepts, processes, or relationships • requires considerable cognitive effort • demand self-monitoring of learners' cognitive processes • requires learners to make relevant connections between knowledge and experiences in order to work through tasks • requires critical thinking in order to make decisions regarding possible solutions and strategies.

(Source: Stein et al, 2006, p.16)

As mentioned earlier, these descriptors where generic and limiting as they were not specific to fractions. I therefore developed fraction descriptors, used to analyse the cognitive demands of the tasks, drawing on the literature of fractions.

I worked with the two broad levels of lower and higher cognitive demand. From the fraction literature, we understand that tasks which link the sub-constructs and unifying elements provide connections for understating fractions. Procedures linked to concepts, and other ways of connecting sub-constructs by means of the unifying elements, tasks and/or representational translations, or other kind of connections provide routes into higher cognitive demand. The lower cognitive demand descriptors are marked by statements of facts or recall of procedures. For example, the operations of fractions require that procedures be followed to successfully complete them. This can be done by mindlessly following a procedure of how to get the correct answer or a deeper understanding can be created through making connections between the sub-construct and unifying elements. Tasks that are algorithmic, require limited cognitive effort to complete and do not make connections to the sub-constructs and unifying elements to create meaning that underlies the procedures being used or to mathematical concepts in focus qualifies as a lower-level cognitive demand task.

In relation to the fraction literature, this means that the following kinds of tasks would be included in the lower cognitive demand category:

- Restatement/recall of facts/procedures
- Tasks where unifying elements are side-lined or pre-structured (e.g., pre-partitioned), with no need for any engagement with the structuring
- Following a known or stated procedure
- Recall or procedure located within a single sub-construct
- No requirement for explanation/justification linked to sub-constructs, unifying elements, or other connections
- Single representation with no connection to procedures linking sub-constructs and unifying elements.

In contrast, higher cognitive demand tasks need to involve the following:

- Procedures linked to the unifying elements
- Representations or procedures that connect sub-constructs
- Tasks requiring engagement with structuring using the unifying elements (e.g., partitioning, and unitizing models).

- Providing explanation/justification linked to sub-constructs, unifying elements, or other connections.
- Requires significant cognitive effort when working within and between sub-constructs and connecting the unifying elements

Table 2.2 on page 62 is a revised version of Stein et al.'s (2000) task analysis framework (2000) that includes fraction indicators used to determine cognitive demand of fraction tasks. As noted, tasks offer insight into what sub-constructs and unifying elements are made available to learners. The sub-constructs and unifying elements in turn provide an understanding of the cognitive demand of tasks and how the different sub-constructs are connected to develop conceptual understating of fractions. These connections form the basis for more conceptual approaches to fractions made available to learners. By examining the connections between the sub constructs, unifying elements, and cognitive demands of the tasks I was able to develop an analytical tool that allowed me to look at a task in terms of the sub- constructs, unifying elements, and cognitive demands to determine what fraction knowledge was made available to learners. The coding methodology used in this study does not necessarily mean 'hard' tasks or the kinds of fraction tasks that have been described as 'conceptual' in other research (Charalambous & Pitta-Pantazi, 2010; Lamon, 2012; Ma, 1999), but it does provide for a fine grained analysis of fractions tasks and teaching in a national context where 'traditional' teaching predominates and where performance is lower than in many other parts of the world.

Table 2.2: Revised version of cognitive demands features for Mathematical Tasks categorisation including fraction indicators

<table>
<tr><th>Level of Cognitive Demand</th><th>Explanation of Categorisation</th><th>Fraction Indicators for Lower and Higher Cognitive Demand</th></tr>
<tr><td>Memorization</td><td>recall of previously learnt information (facts, rules, formulae, definitions)
no understanding required
No procedures to solve problem because of time frame.</td><td>Lower Cognitive Demand
Restatement/recall of facts/procedures
Tasks where unifying elements are side-lined or pre-structured (e.g., pre-partitioned), with no need for any engagement with the structuring
Following a known or stated procedure
Recall or procedure located within a single sub-construct
No requirement for explanation/justification linked to sub-constructs, unifying elements, or other connections
Single representation with no connection to procedures linking sub-constructs and unifying elements.</td></tr>
<tr><td>Procedures without connections</td><td>limited cognitive demand
algorithmic
no connections to concepts or meanings
focus on producing correct answers
requires no explanations or justifications</td><td></td></tr>
<tr><td>Procedures with connections</td><td>use of procedures to develop deeper levels of understanding of mathematical concepts
suggested pathways to follow
procedures are closely connected to underlying concepts
requires some degree of cognitive effort.</td><td>Higher Cognitive Demand
Procedures linked to the unifying elements
Representations or procedures that connect sub-constructs
Tasks requiring engagement with structuring using the unifying elements (e.g., partitioning, and unitizing models).
Providing explanation/justification linked to sub-constructs, unifying elements, or other connections.</td></tr>
<tr><td>Doing mathematics tasks</td><td>complex and non-algorithmic thinking
no predictable pathways to solve problems
requires exploration and understanding of mathematical concepts, processes, or relationships
requires considerable cognitive effort
demand self-monitoring of learners’ cognitive processes
requires learners to make relevant connections between knowledge and experiences in order to work through tasks
requires critical thinking in order to make decisions regarding possible solutions and strategies.</td><td></td></tr>
</table>

Below, I exemplify the ways in which the analytical framework was applied to tasks in this study. Two examples of tasks that were given lower and higher cognitive demand coding according to the descriptors from the analytical tool are included. One example of how the tool was applied to enacted tasks in teaching are also presented. It is also worth noting that purely procedure-oriented instruction, such as that indicated by Charalambous et al. (2010) study, could not be coded with reference to the sub-constructs, because fraction meanings were left aside in these pure calculation-oriented explanations. Figures 2.6 and 2.7 are examples from the study conducted by Charalambous and colleagues (2010) on textbooks from three different countries. These tasks were coded as lower demand tasks (procedures without connections) because of the absence of sub-constructs and representations. A similar coding methodology was adopted in this study for tasks that did not present any sub-constructs and representations and were coded as lower demand pure calculation tasks.

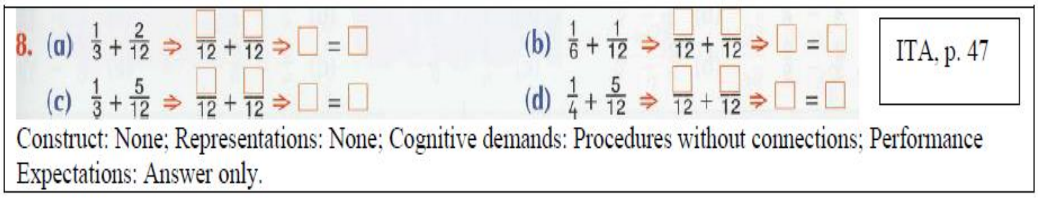


Figure 2 6: Textbook task showing coding for lower demand (Charalambous et al., 2010, p. 70)

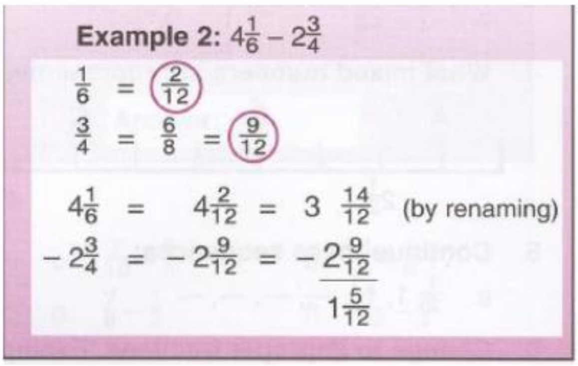


Figure 2.7: Textbook task showing lower demand (Charalambous, et al., 2010, p. 58)

2.7.1 Lower cognitive demand tasks

The task below (figure 2.8) from the Grade 4 textbook, with pre-partitioned part-whole images, with instructions to count the total number of parts and number of shaded parts, and then combine these counts into a fraction, involved working with the definition of unit fractions that the table at the end of the task summarizes. Given that the curriculum specifications in place at the time of this study introduced the basic unit fractions in earlier Grades, this task – with representations located solely within the part-whole sub-construct, and with no need to engage with unifying elements, or with explanations or justifications related, for example, with unequal part representations, was coded as lower cognitive demand.

Working with more fractions

In this section you work with halves, thirds, quarters (or fourths), fifths, sixths, sevenths, eighths, ninths and tenths.

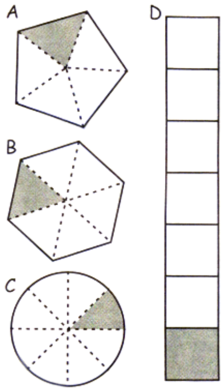
These shapes are all divided into a different number of equal parts.

Exercise 12.7

LO1 AS4.1.3; AS4.1.12 T4

- Each diagram is divided into **equal** parts.
- Look at the diagrams and then answer the questions.

1. a) How many parts is A divided into?
   b) How many parts of A are shaded?
   c) What fraction of A is shaded?
2. a) How many parts is B divided into?
   b) How many parts of B are shaded?
   c) What fraction of B is shaded?
3. a) How many parts is C divided into?
   b) How many parts of C are shaded?
   c) What fraction of C is shaded?
4. a) How many parts is D divided into?
   b) How many parts of D are shaded?
   c) What fraction of D is shaded?

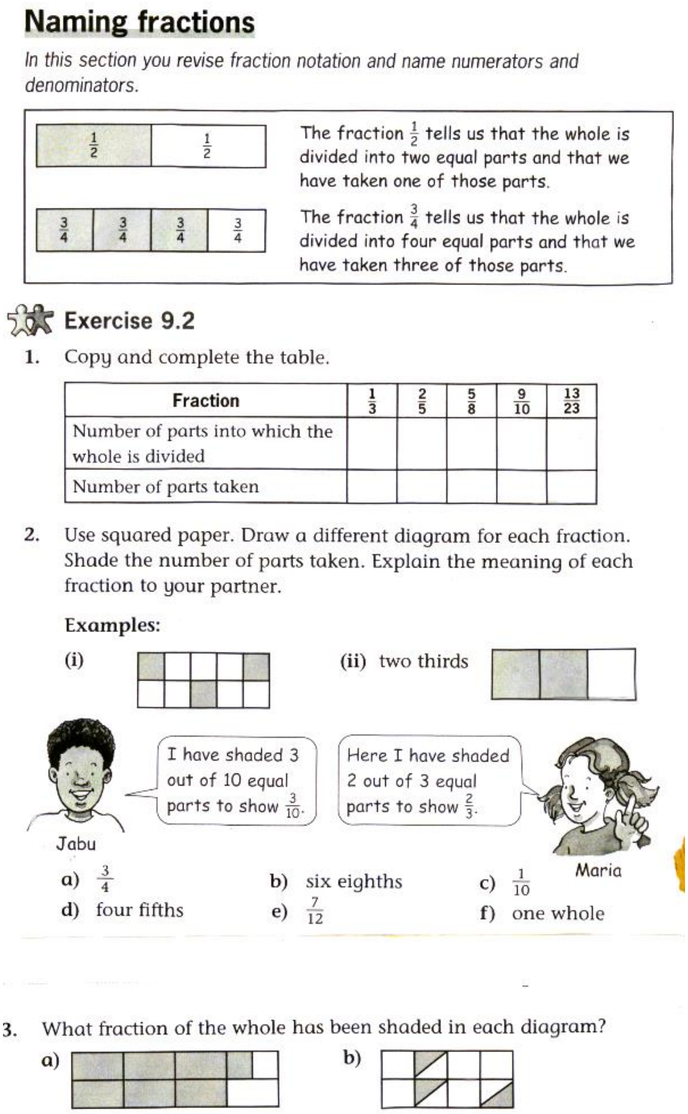


If we divide the whole into	Each part is called
2 equal parts	one half or $\frac{1}{2}$
3 equal parts	one third or $\frac{1}{3}$
4 equal parts	one fourth or one quarter or $\frac{1}{4}$
5 equal parts	one fifth or $\frac{1}{5}$
6 equal parts	one sixth or $\frac{1}{6}$
7 equal parts	one seventh or $\frac{1}{7}$
8 equal parts	one eighth or $\frac{1}{8}$
9 equal parts	one ninth or $\frac{1}{9}$
10 equal parts	one tenth or $\frac{1}{10}$

Figure 2.8: Grade 4 – Exercise 12.7 (*Classroom Mathematics*, p.192)

2.7.2 Higher Cognitive Demand Tasks

The following task (figure 2.9) was taken from *Classroom Mathematics* Grade 6 (Scheiber et al, 2004c, p. 166)


Naming fractions

In this section you revise fraction notation and name numerators and denominators.

The fraction $\frac{1}{2}$ tells us that the whole is divided into two equal parts and that we have taken one of those parts.

The fraction $\frac{3}{4}$ tells us that the whole is divided into four equal parts and that we have taken three of those parts.

Exercise 9.2

1. Copy and complete the table.

Fraction	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{5}{8}$	$\frac{9}{10}$	$\frac{13}{23}$
Number of parts into which the whole is divided					
Number of parts taken					

2. Use squared paper. Draw a different diagram for each fraction. Shade the number of parts taken. Explain the meaning of each fraction to your partner.

Examples:

(i)

(ii) two thirds

Jabu: I have shaded 3 out of 10 equal parts to show $\frac{3}{10}$.

Maria: Here I have shaded 2 out of 3 equal parts to show $\frac{2}{3}$.

a) $\frac{3}{4}$ b) six eighths c) $\frac{1}{10}$

d) four fifths e) $\frac{7}{12}$ f) one whole

3. What fraction of the whole has been shaded in each diagram?

a) b)

Figure 2.9: Grade 6 – Exercise 9.2 (*Classroom Mathematics*, p.166)

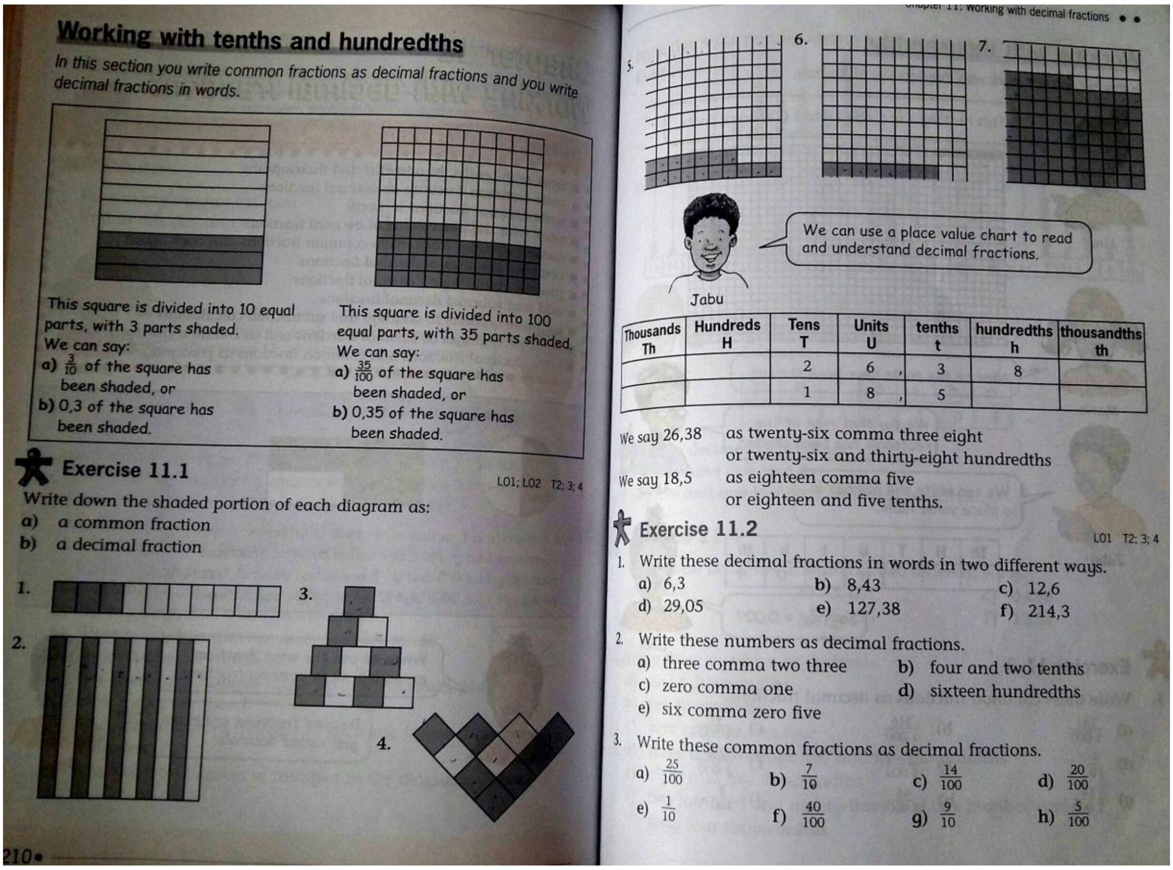
This task was coded as a higher-level cognitive demands task because there is a connection made between the part-whole sub-construct and partitioning. This is done by connecting the area model with the symbolic representation through partitioning. The task requires the whole to be partitioned into different parts in order to name each part. While examples of pre-partitioned wholes are provided to explain what is required of the task, the task encourages and requires different number of partitioning of the whole which creates a notion of quantity and an understanding of the new quantity in relation to the whole. There are also unequal parts in question 3, which requires some visualized or drawn-in partitioning. In contrast with lower-level cognitive demand tasks where the whole is pre-partitioned, this task requires the whole to be partitioned to develop an understanding of the size of the fraction. The task requires a mathematical explanation of the naming and meaning of each fraction and allows for different representations of each fraction. This demands a deeper level of understanding created by connecting the part-whole sub-construct with the unifying elements of partitioning and notion of quantity to name a fraction rather than just being able state how many parts of an area model is shaded or unshaded. The task requires engagement with structuring using the unifying element of partitioning and it also requires and explanation linked to the part-whole sub-construct and partitioning and therefore would be categorized as higher cognitive demand.

2.7.3 Coding of Enacted Tasks

Each lesson was chunked according to episodes, marked by the announcement of the beginning of a new mathematical task, and forming a unit of analysis. This means when a teacher presented a set of similar examples and talked about them in similar ways, it was counted as a new task. I included detail on the time intervals of episodes as this fed into the investigation of balance of emphasis across the sub-constructs.

As with the textbook analysis, the framework helped me to identify the sub-construct/s and unifying element/s and the connections between them to develop understanding of fractions. These connections aided in categorising the enacted task as lower or higher cognitive demand.

Below, I present an example of four enacted tasks from one Grade 6 lesson on decimal fractions. The lesson was chunked into four episodes. Each episode counted as a task and was analysed according to the sub-constructs and unifying elements. This lesson was an introduction to decimal fractions and was based on the following two exercises from the prescribed textbook. The teacher dealt with the two exercises in one lesson.



Chapter 11: Working with decimal fractions

Working with tenths and hundredths

In this section you write common fractions as decimal fractions and you write decimal fractions in words.

This square is divided into 10 equal parts, with 3 parts shaded.
We can say:
a) $\frac{3}{10}$ of the square has been shaded, or
b) 0,3 of the square has been shaded.

This square is divided into 100 equal parts, with 35 parts shaded.
We can say:
a) $\frac{35}{100}$ of the square has been shaded, or
b) 0,35 of the square has been shaded.

Exercise 11.1

LO1; LO2 T2; 3; 4

Write down the shaded portion of each diagram as:
a) a common fraction
b) a decimal fraction

1. 2. 3. 4. 5. 6. 7.

Thousands Th	Hundreds H	Tens T	Units U	tenths t	hundredths h	thousandths th
		2	6 ,	3	8	
		1	8 ,	5		

We say 26,38 as twenty-six comma three eight or twenty-six and thirty-eight hundredths
We say 18,5 as eighteen comma five or eighteen and five tenths.

Exercise 11.2

LO1 T2; 3; 4

1. Write these decimal fractions in words in two different ways.
   a) 6,3 b) 8,43 c) 12,6
   d) 29,05 e) 127,38 f) 214,3
2. Write these numbers as decimal fractions.
   a) three comma two three b) four and two tenths
   c) zero comma one d) sixteen hundredths
   e) six comma zero five
3. Write these common fractions as decimal fractions.
   a) $\frac{25}{100}$ b) $\frac{7}{10}$ c) $\frac{14}{100}$ d) $\frac{20}{100}$
   e) $\frac{1}{10}$ f) $\frac{40}{100}$ g) $\frac{9}{10}$ h) $\frac{5}{100}$

210

Figure 2.10: Grade 6–Exercise 11.1 & 11.2 (*Classroom Mathematics*, p.210-211)

Exercise 11.1 presents area models that are pre-partitioned into tenths and hundredths and the questions require writing the shaded portion as common and decimal fractions. Exercise 11.2 makes use of a place value chart to provide an explanation of the different ways that decimal fractions can be said and written. The textbook task was categorized as lower cognitive demand and is part of the textbook analysis in the next chapter.

The first enacted task began with a discussion of what a decimal fraction is, producing a description that: Decimal fractions are used in everyday life, for example when working with length, mass, money, statistics, and capacity. During this discussion, no reference was made to any sub-construct/s or unifying elements. This task was categorized as lower-cognitive demand because it did not require an explanation/justification linked to sub-constructs, unifying elements, or other connections.

The second enacted task from the same lesson involved an explanation and discussion of how to write and say decimal fractions using words and writing decimals in expanded notation. For example, the teacher used a place value chart to explain how decimal fractions can be written in words and expanded notation, 75.029 can be written as: 'seventy-five comma zero two nine' or 'seventy-five and twenty-nine thousandths' and 7T + 5U + 0t + 2h +9th or 70 + 5 + 0,02 + 0,009 or 70 + 5 + $\frac{2}{100}$ + $\frac{9}{1000}$. The teacher provided several examples of this nature during her explanation. Her explanation followed a procedure of placing the digits under the correct place value by using the comma to separate the whole numbers from the fractions. She explained that for tenths there is only one place after the comma, for hundredths there are two places after the comma and so on. The following excerpt and figure 2.11 exemplifies this.

T: .….so it is 17 hundredths (writes $\frac{17}{100}$ on the board). Which means the seven must be under the what? (points to the 7 in 17).

L: Hundredths

T: So, it is nought comma two spaces (writes 0, _ _). Remember I said to you if it is two nought it is two spaces (points to the two zeros in 100). This is tenths (points to the first open space) and this is hundredths (points to the second open space). Okay, so 7 goes there (fills in 7 in the second space: 0,_7) and the one goes there (fills in 1 in the first space: 0,17).

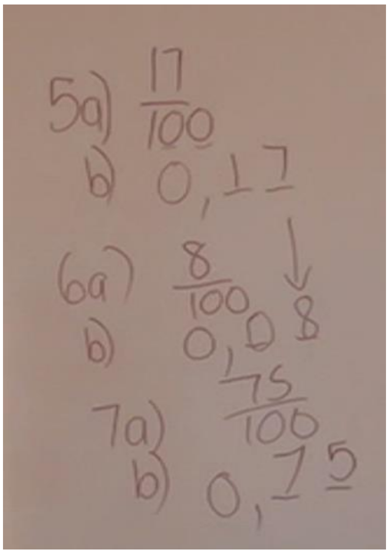


Figure 2. 11: Teacher representing common fractions as decimal fractions

No reference or link was made to any sub-construct/s and unifying element/s. This enacted task was categorized as lower-cognitive demand.

The third enacted task was when a learner asked the question, “Aren’t units less than thousandths?”. This occurred during the discussion of how to write a decimal in expanded notation. The excerpt below contains the teacher’s response to this question. It starts in the twenty-second minute of the lesson.

T: Okay, his question is, aren’t units (points to units on place value chart) less than thousandths (points to thousandths on place value chart)? Okay, so give me an example of a unit?

L: Two

T: Okay, so can you see here (writes at the bottom of the board), two is my unit (writes 2 on the board) and then we going to put two out of a thousandth (points to the place value chart). How do I write two over a thousandth (writes $\frac{2}{1000}$)? Now tell me, if I had to say to you, I am going to give you two sweets (points to the 2 written on the board) or I am going to give you two over a thousandth sweets (points to $\frac{2}{1000}$ written on the board), so in other words, I am going to give you (points to the 2 in $\frac{2}{1000}$)?

L: Two wholes

T: This is two whole (points to the 2 on the board) sweets. This one (points to the 2 in $\frac{2}{1000}$) doesn’t even take up one whole yet. Two of a thousandth is very, very little of a whole (draws a circle in the air to show a whole). Ok why? I will have to give you two thousand over a thousandth to give you two whole sweets (points to $\frac{2}{1000}$ written on board). Okay, because what will one whole sweet be?

L: one thousand over….

T: one thousand over one thousandth (writes $\frac{1000}{1000}$=1 on the board) gives you one whole. I only got 2 (points to $\frac{1000}{1000}$) of 1 (points to the 1 in $\frac{1000}{1000}$ = 1) so it's very different. Okay, so two units (circles 2 on the board) are definitely much bigger than anything on this side (points to the left of the pace value chart). Everything, if you go to the left all units are bigger, if you go to the right the smaller it becomes (points to the right of the place value chart). Okay, so tenths is greater than hundredths. Okay, if I had to take a piece of paper (starts erasing the work on the board), let me quickly show you, can I take this off guys (as she erases the board)?

L: Yes ma'am

From this excerpt we note that the teacher explains why 2 units is greater than 2 thousandths. The teacher used the part-whole sub-construct with partitioning in her explanation. She used the whole, units in this case, to show that a unit is larger than a thousandth by using the place value chart to show the value of each number. The teacher successfully linked the procedure she used earlier, when writing decimals in expanded notation, to partitioning the whole and explaining the difference between a unit and a thousandth. This was categorized as higher cognitive demand. While the use of precise mathematical language for teaching fractions does not form part of my coding, it is important to note the imprecise language used by the teacher in this excerpt, she constantly referred to the '2' separate from the 1000. We know from research that this type of teacher talk often leads to ambiguity and learners perceive that the fraction is made up of two separate whole numbers, instead of recognizing it as one quantity (Brown, 1993; Mack, 1995; Siebert & Gaskin, 2006). Such ambiguity has been noted as a problem in prior South African research (Venkat & Naidoo, 2012).

Continuing with the same discussion, the teacher used the diagram below (figure 2.12) to explain that a tenth is greater than a hundredth. She used an area model and partitioned it into tenths then unitized the partitioned modelled to show that $\frac{2}{100}$ is less than $\frac{2}{10}$. The teacher was able to provide an explanation linked to the part-whole sub-construct using the unifying elements of partitioning and unitizing. She was able create a notion of quantity related tenths and hundredths. This was also categorized as higher cognitive demand.

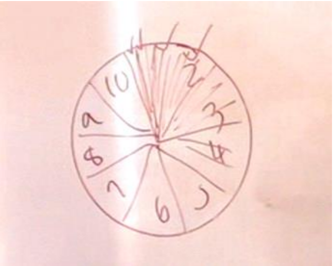


Figure 2.12: Whole partitioned and unitized to demonstrate tenths and hundredths

The fourth and final episode involved learners completing the two exercises from the textbooks. They first worked through 11.1 and then 11.2. As the learner worked through the tasks they interacted with the teacher. During this time, no reference was made to any of the sub-constructs and unifying elements. The tasks were completed by using the examples provided in the textbook and the explanation provided by the teacher earlier in the lesson. Learners used the place value chart and following the procedure of where each digit fits into the chart or counted the number of shaded parts and wrote it as a common and decimal fraction. The exercises from the textbook did not require any work with the unifying elements. This episode was categorized as lower-cognitive demand.

2.8 Summary

Literature explains the link between representations and tasks, the different sub-constructs and unifying elements and an understanding of how the interconnectedness of the sub-constructs support learners' understanding of fractions. Learners often receive a limited number of interpretations and representations from the curriculum with the part-whole and area model the most common sub-construct and representation, respectively. Research has noted struggles for teachers in finding useful, appropriate, and varied representations and models for teaching fractions that reflect their multifaceted nature (Lamon, 1999; Post et al., 1993). They often resort to using representations or models that comprise of regularly shaped objects (circles or squares etc.) that are divided into equal parts or they use the number line (Verschaffel, 2006). Both regularly shaped objects and number lines deal only with continuous wholes rather than discrete units. This leaves learners with the belief that fractions are only part of a whole and only circles and squares etc. can be divided into equal parts. It may also lead to learners experiencing difficulties when dealing with problems that involve sharing, for example, a single chocolate bar among 5 friends or calculating a fraction of discrete units, a $\frac{3}{4}$ packet of biscuits containing 12 biscuits for example. This over reliance on the continuous part-whole model prevents learners from seeing fractions as numbers and restricts the development of other fraction interpretations (Pitkethly and Hunting, 1996). Teachers should use a variety of tasks and representations to teach the different interpretations of fractions. They should select different tasks and representations and include all the sub-constructs and unifying elements when teaching fractions. This link or connection will ensure that learners are exposed to a broad understanding of the fraction concepts.

The next chapter focuses on the methodology that provided the data sources that allowed the analytical framework to be put to use. It is interesting to see what the teachers did as they taught fractions. What representations and manipulatives they used and how did they use them? What sub-constructs and unifying elements came into play. Were all the sub-constructs involved when teaching fractions? This provided an opportunity to map out what the teachers did, to what is prescribed by the literature to come to an understanding of what and how fractions are taught. Stein et al.’s (2000) framework on tasks allows us to examine the sub-constructs made available and whether they are taught in connected ways to provide both procedural and conceptual understanding, by way of the cognitive demands of the tasks in the textbooks and the tasks set- up by teachers.

The focus of this study is not to find solutions or answers to how teachers should teach fractions or how students construct fraction understanding (Herman et al.,2004; Mamede et al., 2005, Pirie, et al., 1994,) but rather, as noted in the previous chapter, it focuses on three teachers and how they teach fractions, and specifically, what range of sub-constructs they focus on through their task selections to develop an understanding of fractions for their learners (Grade 4-6), what levels of cognitive demand and work with unifying elements do the teachers make available in their enactment of fractions tasks and ultimately what fraction knowledge is made available to learn based on the analysis of sub-constructs, cognitive demand and unifying elements across the three Grades . As noted by Govender (2008), despite a substantial amount of research done to establish and understand why the teaching and learning of fractions is so difficult, the difficulty persists and remains a constant challenge for teachers. An in-depth study of what happens in practice could offer some explanation and illuminate why the problems related to the teaching and learning of fractions persist, particularly within the South African context (Govender,2008). Hopefully through this study, more light will be shed on the problem and what needs to be done to improve fraction instruction.

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CHAPTER THREE

DESIGN AND METHODOLOGY

3.1 Introduction

Chapter 3 focuses on the research methods adopted for this study as well as the data collection techniques employed. The sample used for this study and the ethical issues that I have considered are also addressed.

3.2 Methodological Approach

> "A research paradigm is a network of coherent ideas about the nature of the world and of the function of researchers which, adhered to by a group of researchers, conditions the patterns of their thinking and underpins their research actions."
>
> (Bassey, 2003, p. 42).

This statement by Bassey (2003) explains that researchers have different beliefs about the nature of reality when making sense of the world. Two broad research paradigms are proposed by Bassey (2003): the positivist research paradigm and the interpretive research paradigm.

The interpretive researcher sees reality as the construct of the human mind while positivists believe there is a reality 'out there' in the world that exists, whether it is observed or not and irrespective of who observes. Interpretive researchers believe that people perceive and make sense of the world in ways which are often similar, but not fundamentally the same. Consequently, what is real can be understood in different ways. Observers are 'out there' rather than reality being 'out there' (Bassey, 2003). Cohen et al (2002, p.22) agree that: " The interpretive paradigm, in contrast to its normative counterpart, is characterized by a concern for the individual". With this brief explanation of the interpretive research paradigm, I will now discuss it in relation to my study.

My research was classroom-based, and its main purpose was to gain an understanding of how fractions were presented in general across the Intermediate Phase classrooms in one school over a time period of one year. The following questions guide this study:

1. What range of sub-constructs and unifying elements did teachers focus on through their task selections to develop an understanding of fractions for their learners across Grades 4-6 in one school?
2. What levels of cognitive demand and working with unifying elements did the teachers make available in their enactment of fraction tasks?
3. What can be said about the fractions knowledge that is made available to learn based on the analysis of sub-constructs, cognitive demand, and unifying elements across the three Grades?

What takes place in the classroom is dependent on teachers and the range of sub-constructs and unifying elements they focus on through their task selections to develop an understanding of fractions for their learners, and learners' actions and reactions toward the tasks that are offered. These actions, interactions and responses are all interrelated, interdependent, and open to interpretations based on the kinds of tasks and interactions advocated in the literature and the theoretical position taken in the study. The research involves discussing teachers' and textbook interpretations of fractions in relation to what is advocated in mathematics research studies. The latter is also an interpretation of a particular community of mathematics education researchers. In this study, the interpretations of this latter group provide a vantage point from which to think about the teaching of fractions and what tasks are used. The research is therefore based on the interpretations of a community of researchers rather than an objective position. It is for this reason that a positivist paradigm would be inappropriate. Instead, an in-depth, qualitative, and interpretive research paradigm is more suitable.

This study is a single case study in that it focuses on the teaching of fractions in the middle Grades through an in-dept analysis of the teaching of fractions in middle school, based on studying the textbooks and teaching across a one-year period. A case study can be described as "an opportunity for one aspect of a problem to be studied in some depth within a limited timescale" (Bell, 1987) or as portraying "…what it is like to be in a particular situation, to catch the close-up reality and 'thick-description' of participants' lived experiences of, thoughts about and feelings for a situation." (Cohen et al., 2002, p.182).

This study of the teaching of fractions involves an understanding of what takes place across a one-year period, in three different mathematics classrooms across three different Grades, in one primary school when teaching fractions and other topics related to fractions. Examining tasks, used by the teachers and those that appear in the prescribed textbook, and analysing the sub-constructs, unifying elements, representations, and different cognitive demands at play within tasks and teaching all form part of the case study.

An understanding of case study research and my research questions, led to the use of a case study which proved to be most beneficial. This is not a comparative case study, as we would not expect three teachers across three different Grades to be working entirely similarly. The case study provided an opportunity to understand the work in the three different classrooms, examining similarities and differences across the three Grades with regards to the sub-constructs, unifying elements, representations, and cognitive demand used when teaching fractions. It also allowed comparisons and contrasts between the different teachers' choices of tasks. Using case study as a research method, I gained insights into the kinds of sub-constructs and unifying elements teachers use, the cognitive demands of tasks they select and how these are connected when teaching fractions.

3.3 Data Collection Strategies

As already noted, data were drawn from case studies of the three teachers involved in teaching Grade 4, 5 and 6 Mathematics in one Johannesburg independent school as they went about teaching the topic of fractions. To obtain this data, qualitative observation was used to ascertain the range of sub-constructs and unifying elements teachers selected to develop an understanding of fractions for their learners across Grades 4-6. For this focus, all lessons taught by these teachers relating to the topics linked centrally to fractions in the literature (mediated via the South African curriculum and textbooks) across the year – were video recorded. In my analysis, I then analysed the tasks selected and enacted according to the sub-constructs, unifying elements and task demands.

The key data collection methods used for this research were curriculum and textbook analyses followed by video-recorded lesson observations. I will now discuss each of the data sources and how they relate to the research questions in this study.

3.3.1 Analysis of curriculum documents and textbooks

Curriculum and textbooks analysis were important sources of information in my study**.** These analyses were done prior to the observation analysis and drew from the literature on fractions. Looking across the fraction content in the mathematics curriculum for the Intermediate Phase Grades provided information on what was stipulated as needing to be taught at the different Grade levels, as well as what and when concepts needed to be covered in each Grade.

For the analysis of the focal school's prescribed mathematics textbook for Grade 4 to 6, I focused on all the chapters that included reference to the fraction chapters and the fractions related chapters. Common fractions and decimal fractions were separate chapters in the Grades 5 and 6 textbooks. Drawing from the literature, the set included common fractions, decimal fractions, time & measurement, measurement (capacity) and whole number rate & ratio chapters. I summarised both the occurrence of the range of sub-constructs and unifying elements identified in the literature across the Grade levels to be studied, and also the locations of this occurrence. When identifying the sub-constructs and unifying elements I used Lamon's approach involving tasks as the key unit of analysis. I examined each task and categorised it according to the sub-constructs and unifying elements present through different representations, and cognitive demand. The focus was on the tasks in different Grades in the chapters from the textbook, and I used this to begin to understand which sub-constructs and unifying elements were highlighted in each Grade. I studied each chapter, focusing on the tasks, and documented where the different sub-constructs and unifying elements appeared and the tasks' cognitive demands. This analysis is presented in chapter 4. Arranging the information in a table allowed me to analyse how these aspects were distributed across curriculum topics.

The teachers in this study used the textbooks and other resources to select tasks for teaching fractions. An analysis of what tasks were selected from the textbooks also provided an understanding of what sub-constructs and unifying elements were addressed by the teacher and the kinds of cognitive demands that were required of learners when engaging with fraction tasks. This provided insight into what fraction knowledge, through sub-constructs, unifying elements, and cognitive demands, were made available for learners by the teachers. The teachers selected certain textbook tasks and omitted certain other textbook tasks; analysing what tasks were selected and omitted also provided insight to what fraction knowledge was made available to learners.

3.3.2 Observations of fraction-related lessons

A researcher can take on two roles: participatory or non- participatory (Opie, 2004). A non- participant role involves no communication with the subjects when collecting the data and is often associated with structured observation. A participatory role can take up one of three broad forms: observer as participant, participant as observer and complete participant. In this research, I took on a non-participatory role in relation to the lesson observations, with an associate collecting some of this data through video-recording the teaching. I was not present in all the classes because of my own teaching obligations.

To see and understand the work teachers do, we must observe what is actually happening in the classroom during teaching (Ball & Cohen, 1999; Ball & Bass, 2000). Stake (1995, p. 62) states: "During observation, the qualitative case researcher keeps a good record of events to provide a relatively incontestable description for further analysis and ultimate reporting. He or she lets the occasion tell its story, the situation, the problem, resolution or irresolution of the problem". Descriptions of observations were then interpreted using the lenses provided by sub-constructs, unifying elements, and cognitive demand to understand teachers' work during the teaching process in the three different classrooms. This assisted me in understanding and establishing how the teacher's set-up the tasks and what range of sub-constructs and unifying element they made available for their learners. Even though using observational research was time consuming, it provided direct insight into what was happening during the teaching process.

As mentioned above, information was collected in the form of observational data. Video recordings were looked at through the analytical framework based on the literature presented in chapter 2. An analytical tool was developed based on this analytical framework and is discussed in the next chapter.

The observational data that I used was based on lessons that were videotaped and transcribed. The video recordings was an important part of the data collection strategy because it captured everything that took place in the classroom. The recordings were watched repeatedly to make sense of what was taking place in the classrooms. Important information such as body language and gesture, which helped understand what was being communicated, by both the teacher and the learners were captured and could be viewed at any time. Lessons were transcribed verbatim. All teacher talk, learner talk and teacher- learner talk was captured, and I included descriptions of all tasks, representations, and teacher gestures. This process formed the basis for a systematic and comprehensive analysis of what sub-constructs, unifying elements and task demands were made available for learners. Each lesson was chunked according to episodes. Each episode constituted a unit of analysis. The beginning of a new idea marked the beginning of an episode. When a teacher introduced a new mathematical concept, it was considered as a new idea. Each unit began when the teacher announced the new concept (that which was to be learnt) and ended when he/she completed teaching the concept. I included detail on the time intervals of episodes as this fed into the investigation of balance of emphasis across the sub-constructs.

Both data sources, lesson observations and textbook/curriculum analysis, fed into all the research questions. My first research question focused on the range of sub-constructs and unifying elements that appeared in the textbooks and enacted tasks. By analysing the textbook tasks and observing all the lessons I was able to document all the sub-constructs and unifying elements and this aided in answering the question. In order to determine the cognitive demand of the textbook and enacted tasks I focus on connections through analysing for the presence of different sub-constructs and/or sub-constructs with the unifying elements presented in the textbooks and enacted tasks. This aided in answering my second research question. My third and final research question involved a meta-analysis of the findings from the previous two questions across the analysis of the textbook and enacted tasks, and played an important role in helping understand the fraction knowledge made available to learn across the three Grades.

3.4 The sample

The research was carried out at a multilingual, multicultural, well-resourced co – educational private primary school situated in the south of Johannesburg. English was the language of instruction throughout the school. The school was a high performing school and was considered privileged in the broader South African context. The teachers were well qualified with a number of years of teaching experience. This study was therefore set in a site that is associated with high performance and relatively high academic results. In the South African context, this suggested that a study of the nature and range of fractions teaching in this school would likely provide a 'good' case scenario of fraction coverage in relation to the South African context. The school was selected with knowledge of its place in the broader socio-economic and educational context, but primarily as a convenience sampling based on it being the school at which I taught at the time of data collection. Participants in the study were therefore primarily a convenience sample - an accessible population of three Mathematics teachers and their learners all from the above -mentioned school. The fractions teaching of the Grade 4, a

Grade 5 and a Grade 6 mathematics teachers formed the centre of the empirical data collection. There were 20 learners in the Grade 4 class, 25 learners in the Grade 5 class and 24 learners in the Grade 6 class. The teachers were an opportunistic sample, known to me due to my teaching at the school and this gave me easy access to the teachers. These teachers were selected because they were the mathematics teachers for each Grade in the Intermediate Phase.

Mrs. B[1], the Grade 4 teacher, had a four-year higher education teaching diploma. Mrs B's major at college was Mathematics. She had been teaching for 20 years. The majority of her teaching career had been in Early Childhood Development and the Foundation Phase Grades 1-3. However, for the two years prior to my classroom observations during (2010 & 2011) she had been teaching in the Intermediate Phase. In her first year in this phase, she taught all the Intermediate Phase subjects. In the second year the school moved to subject teaching and she taught only Mathematics to the Grade 4 learners. She had taught at two different private schools in Johannesburg over a period of 8 years and had then moved to a government school in Johannesburg and taught there for 9 years. She had been teaching at the focal school for three years at the point of observations.

Mrs B was the only Grade 4 mathematics teacher in the school. She taught four Grade four classes every day on a five-day cycle. Since she had only recently joined the Intermediate Phase, she had no training with regards to the Revised National Curriculum Statement. However, she was well informed and knowledgeable about the Mathematics Curriculum. She collaborated on a regular basis with the Grade 5 and 6 Mathematics teachers so as to keep up with what needed to be done in the school with regards to the mathematics curriculum. Her classroom was well resourced in terms of the South African context with mathematical equipment (unit fractions, geoboards, etc), teaching aids, textbooks as well as a whiteboard, blackboard, and overhead projector.

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[1] The full transcript of each lesson was bound and kept separately.

[1] This abbreviated pseudonym is used for the Grade 4 teacher to protect her identity, and this approach is used for all teachers' and learners' names across this study.

Mr Z and Mrs W were the Grade 5 and 6 Mathematics teachers, respectively. Both Mr Z and Mrs W had a four-year higher education teaching diploma. They both majored in Mathematics and Mrs W was high school trained. Mr Z had 23 years teaching experience, while Mrs W had been teaching for 17 years. Mr Z had taught Mathematics for 13 years. In those 13 years he had taught Grade 5, 6 and 7 Mathematics. He had taught Grade 5 Mathematics for six years. He taught for two years in Zimbabwe and the rest of his teaching took place at both private and government schools in South Africa. The data collection year was his third year at this particular school. Mrs W had taught Mathematics for 17 years. She had taught at a high school, a primary school, and a combined school. Mrs W had taught Grades 8 to 10 Mathematics for 7 years at two different high schools and Grades 6 to 9 Mathematics at a combined primary and high school. She had taught primary school Mathematics (Grade 4 to 7) for ten years. The data collection year was her 10th year teaching Grade 6 Mathematics. She had taught at both private and government schools and was in her second year at the study school.

Like Mrs B, Mr Z and Mrs W were the only Grade 5 and 6 mathematics teachers at the school. They also taught four classes every day on a five-day cycle. Both teachers collaborated on a regular basis with their team of mathematics teachers, across the Intermediate Phase, so as to keep up with what needed to be done in the school with regards to the mathematics curriculum. They were well informed and knowledgeable about the mathematics curriculum and had attended the required training with regards to the Revised National Curriculum Statement, as well as professional development workshops for mathematics. Just like the Grade 4 teacher, their classrooms were well resourced with mathematical equipment (unit fractions, geoboards, etc), teaching aids, textbooks as well as a whiteboards and overhead projectors.

3.5 Ethical considerations

Respect for democracy, respect for truth and respect for persons are all part of research ethics (Bassey, 2003, p. 74). Each one will be briefly described and explained in connection with this study. I will further explain the ethical process embarked on with regards to my research project.

3.5.1 Respect for Democracy

Respect for democracy entails the freedom researchers have i.e., “the freedom to investigate and to ask questions, the freedom to give and receive information, the freedom to express ideas and to criticize the ideas of others; and the freedom to publish findings” (Bassey, 2003, p. 74). With this freedom comes certain responsibilities. It is expected that researchers have respect for the truth and for people. Adhering to this means that researchers have the freedom to do things that will not jeopardize themselves or their careers. In this study there were several factors that pointed to respect for democracy. Firstly, obtaining permissions from the relevant participants was a priority. Permission was obtained from the principal of the school, the teachers who participated in the study as well as the learners and their parents. All participants involved in the study (i.e., the principal, the teacher, the learners, and their parents) completed a consent form. Letters were sent out informing them of the purpose of the research, ensuring them that the learners, school, principal, and teacher would remain anonymous and that they could decide whether they wanted to be a part of this research project. The letter also explained the data collection strategy and how it involved the learners and teacher. It explained how the research findings would be used and confirmed the teacher’s, learners, and parent’s approval of the use of the transcripts and video-recordings by myself, in publication and by other researchers (see appendix 1,2 & 3). Furthermore, I also obtained ethics clearance from the University of the Witwatersrand (See appendix 4). The ethical issues addressed above may also form part of respect for truth and respect for persons.

3.5.2 Respect for Truth

Researchers are expected to be truthful in data collection, analysis and the reporting of findings. In other words, researchers owe it to both their subjects and themselves to be honest. Having integrity and not intentionally or unintentionally deceiving others is important. Trustworthiness plays an important role and will be addressed later in this chapter.

Before conducting the research, I considered what would happen or what I would do if the information gathered from the research was potentially harmful in that it was neither flattering to the school nor the teacher. In order to resolve this dilemma, I made it clear to the principal and teacher from the onset that the aim of my research was not to find fault with the teachers or their teaching strategies, but rather to learn from them.

3.5.3 Respect for Persons

Researchers must take cognisance of the fact that when collecting data, the data initially belongs to their subject/s and therefore they owe it to them to treat them as fellow human beings, with dignity and privacy (Bassey, 2003).

Several ethical issues arose during the research project. The teachers involved in the study were colleagues of mine. I was unsure of how our professional relationship would be influenced by this study, so before I conducted the research, I explained clearly to my colleagues what was expected of them as the teachers and me as the researcher, making sure that they were not in any way threatened, intimidated by me, or insecure about the research. I did this by conversing with them individually, as well as writing an official letter to each one of them.

Time played a vital role during the data collection. The teachers involved in the research had very busy co-curricular timetables and time was a restraining factor. As part of my ethical considerations, we negotiated times that would suit both parties in order to ensure that the research did not become a burden for them. I also ensured that our appointments were diarised and that any cancellations were made in advance by both parties and rescheduled for another time.

With regards to classroom observation, I was very aware of the time the teachers had in order to achieve the different outcomes set out by the Revised National

Curriculum Statement. I therefore ensured that being in the class caused minimal distraction and disruption for the teachers and learners.

‘Rape research’ is described as a phenomena where researchers enter the research field, obtain the necessary information required and forgets to return to the field to thank the participants for their contribution towards the study (Patti Lather cited in Opie, 2004, p. 29). Being mindful of this, I regularly provided feedback to the teachers and planned on making my findings available to all the teachers once the research report was completed.

3.6 Rigour in my research

Sikes (in Opie 2004 p. 17) states, “It is on the match between methodology and procedures and research focus/topic/questions that the credibility of any findings, conclusions and claims depend, so the importance of getting it right cannot be overemphasized.” For my research findings to be credible, I had to ensure that the methodology and procedures I chose were best suited to my research topic and questions. The general aim of the research was to focus on the teaching of fractions. In other words, I hoped to bring thick description of what took place in the different classrooms as the teachers taught fractions together with the application of literature-based lenses in my analytical methodology to understand what sub-constructs, unifying elements and task demands the teachers brought into play to teach fraction and what knowledge was made available for learners.

Bassey (2003, p.75) explains, “reliability is the extent to which a research fact or finding can be repeated, given the same circumstances, and validity is the extent to which a research fact or finding is what it is claimed to be”. While both validity and reliability are vital concepts, some researchers believe they are not so with regards to qualitative research. In fact, they argue that it is problematic to use validity and reliability in qualitative research where subjectivity is often viewed as primary. For example, researchers (Guba & Lincoln, 1984) use the concept of trustworthiness (credibility, transferability, dependability, and conformability) instead. Silverman (2001) though, disagrees with this and shows how qualitative

research can be seen as credible through reliability and validity. I use Silverman's notion of credibility to show how reliability and validity in my research were configured in terms of research questions and the different data source used.
Kirk and Miller (1986, p. 226) maintain that:

> 'Qualitative researchers can no longer afford to beg the issue of reliability. While the forte of field research will always lie in its capability to sort out the validity of propositions, its results will (reasonably) go ignored minus attention to reliability. For reliability to be calculated, it is incumbent on scientific investigators to document his or her procedures.'

The main instrument in qualitative research is the researcher. This means that several complications arise because human beings often have different theoretical orientations, making it difficult for the instrument to be repeatable. All my findings was documented when conducting this research. Every video recording were transcribed verbatim in order to strengthen the reliability of the interpretation of the transcripts. Checking of the interpretation of the transcripts and verifying that the codes were applied uniformly was done by my supervisor. The anonymised transcripts are bound, kept separately and remain available for scrutiny at any time.

I presented the instrument that I used for data collection to my supervisor for her guidance and comments. Once we agreed on the instrument then only was the data collected. I video recorded all lessons. I then transcribed them ensuring that all data was accurately captured. My supervisor helped categorise tasks from the textbooks and lesson observations with me at the analysis stage of my research. She helped look for contradictory evidence through the analysis process and through this, indicators relating to the categories were clarified. We worked constantly between the theoretical and empirical fields. This was done to produce factually accurate information. When analysing both the textbook and teaching, we aimed to be consistent by using the criteria and indicators in a consistent and principled manner, thereby moving beyond idiosyncratic interpretations.

3.7 Methodological approaches to data preparation

As discussed in detail in chapter 5, the lessons were chunked according to episodes based on fraction tasks. Each new mathematical task marked the beginning of a new episode. A new task referred to the introduction of a new fraction concept as explained earlier. Since the systematic, consistent, and concise recording and analysis of the concrete evidence increases the reliability of the research (Silverman, 2001), the data analysis was done coherently, consistently, and systematically. The textbook and the observed lessons data was analysed using the analytical tool that was developed from the framework. The categories derived from the use of this tool were used in a standardized way. I provided a description for each category (sub-constructs, unifying elements and task demands) so as to ensure that another researcher/person would be able to categorise the data in the same way. Allowing my supervisor to analyse the data according to the agreed set of categories and ironing out all differences provided accountability and ensured reliability. This "inter-rater reliability" (Silverman, 2001) proved to be very useful in ensuring the reliability of this research.

Ultimately all methods of data collection are analysed 'quantitatively', in so far as the act of analysis is an interpretation, and therefore of necessity a selective rendering. Whether the data collected are quantifiable or qualitative, the issue of the warrant for their interferences must be confronted (Silverman, 2001, p. 233). Silverman (2001) points out that it is important that researchers both qualitative and quantitative have a 'warrant for their inferences' and that their work is valid. Instead of using triangulation and members' validation, he suggests that in order to validate quantitative research the following methods must be considered: analytic induction, the constant comparative method, deviant-case analysis, comprehensive data treatment and using appropriate tabulations. I will discuss these briefly in relation to my research.

Analytical induction depends on a model of how social life works as well as a set of concepts specific to that model (Silverman, 2001). My primary concern with regards to this research was to describe the present state of what sub-constructs and unifying elements were used to teach fractions and what knowledge was made available and accessible for learners when engaging with the topic of fractions. The interpretations of the data collected were shaped by both an 'involved' theoretical framework, which guaranteed contact with established theory, as well as my experience and knowledge of the classroom which ensured contact with the established practices (Brodie, 1994). This relationship that existed between the practical and the theoretical knowledge aided in the establishment of the validity of the research.

The constant comparative method involves 'simply inspecting and comparing all the data fragments that arise in a single case (Glaser & Strauss in Silverman 2001, p. 239)'. This comparison was made possible by ensuring that the data I collected was assembled in an analysable form. The whole data set from the video recordings was transcribed and chunked into episodes. Each episode is a unit of analysis. I started by analysing relatively small chunks of the data. From this I generated and modified a set of categories in conjunction with the theory. The literature on tasks and representations provided details of indicators to look for related to the concepts of sub-constructs, unifying elements, and cognitive demand. Cognitive demand required additional indicators since the indicators where generic and my data required the application of descriptors to fractions.
The first category - unifying elements was sub-divided into the literature-derived:

(1) unitizing

- (1.1) The unit as implicit or explicit

- (1.2) continuous wholes and discrete objects

(2) partitioning

(3) notation of quantity/equivalence.

The five sub-constructs formed another category (part-whole, quotient, measure, operator, and ratio).

The task demands (lower cognitive demand and higher cognitive demand) formed the third category. Because the features of the task's demands provided by Stein et al. (2000) were generic and not specific to fraction knowledge, I developed fraction features that I used to determine the cognitive demands of the tasks. I relied on the literature to derive meaning for each task demand. This was done in order to create a coding system that was consistent with the literature. I still coded each task demand as either lower cognitive demand or higher cognitive demand. I tested out emerging hypothesis by steadily expanding the amount of data. The data was placed into the appropriate categories and all data was accounted for. This ensured that there was a constant to and fro movement between the different parts of the data. When a teacher revisited tasks through 'going over' or 'marking' a task, it was coded as a separate task as this allowed space for the teacher to adapt/extend his/her explanation in relation to learner responses in ways that could take more and/or different sub-constructs and unifying elements into account. No data was left 'uninspected' or 'unanalysed' and Silverman (2001) refers to this as comprehensive data treatment.

The data could be placed into the different categories because of the concepts and indicators derived from the literature. As mentioned before, these categories were derived partly from the theoretically defined concepts. While the data collected in the study was qualitative, I employed and incorporated quantitative methodology in order to analyse the data and gain insight into the bigger picture. I used a simple counting technique to record how many times the teachers used mathematical tasks that related to the literature, and categories described earlier and how many times they referred to the different sub-constructs and unifying elements. This allowed me to survey all the data ensuring that nothing was overlooked. It also enables the readers to survey the data and get an idea of the data as a whole. As mentioned earlier, the textbook analysis was done in a similar way. Each task was categorised according to the sub-constructs, unifying elements and cognitive demands and a simple counting technique was used, as explained above.

3.8 Summary

This chapter dealt with the research methods employed in this study, ethical issues that were considered when embarking on the project and issues of credibility with regards to reliability, validity, and generalisability.

The next chapter draws attention to the analysis of the prescribed textbook in terms of sub-constructs, unifying elements and task demands. This leads to providing answers to the first critical question of this study i.e., what range of sub-constructs and unifying elements do teachers select to develop an understanding of fractions for their learners across Grades 4-6 in one school? And following that an engagement with questions two and three i.e., and what levels of cognitive demand and work with unifying elements did the teachers make available in their enactment of fraction tasks and what can be said about fraction knowledge that is made available to learners based on the analysis of the sub-constructs, unifying elements, and cognitive demand across three Grades?

CHAPTER FOUR

TEXTBOOK ANALYSIS

4.1 Introduction

In this chapter I provide an analysis of the prescribed textbook, Classroom Mathematics, used in the school where my study was carried out. The textbook was used across Grades four to seven. For this study, the focus was on the Intermediate Phase Grades four, five and six textbooks. Since tasks are central to my study, it was important to look at the intended tasks presented in the textbooks. An analysis of these tasks provided insight into what is expected of learners in terms of conceptual understanding and procedural fluency (Charalambous et al, 2010; Stein et al, 2000). Tasks in their printed form were analyzed using the five sub-constructs and three unifying elements identified in the literature and were incorporated into Stein et al.'s (2000) cognitive demands of tasks presented in the Literature Chapter.

I begin the analysis by providing a tabulated overview of fraction and fraction related chapters covered in each textbook. I then provide a tabulated outline of the topics and concepts covered in each chapter across the three Grades. This is followed by a tabled presentation and discussion of the analysis of all the fraction and fraction-related tasks across chapters and Grades in the textbook series.

After studying the 5 different sub-constructs and the three unifying elements presented in the literature and examining tasks that included the sub-constructs, I used the Intermediate Phase mathematics curriculum (RNCS) to distinguish all concepts related to the topic of fractions. The curriculum presented a fractions strand, but the literature review also identified other sections in the curriculum that included fraction concepts that were not part of the fraction section. For example, the topic of time was not included in the fraction section of the curriculum or the textbook, but it included important concepts related to fractions. Lamon (2012), in her studies, provided time tasks that involved the part-whole, measure and operator sub-constructs. Measurement, rate and ratio, geometric enlargement and money tasks are also referred to in the literature (Wilkins & Norton, 2018; Nagar et al, 2015; Lamon, 1999) as providing important routes into more connected ways of dealing with the fraction sub-constructs and unifying elements.

In the analysis I refer to literature-based tasks related to the topics in focus and draw in, where useful, contrasts with the way tasks are presented in the textbooks, and to note overlaps and deviations from recommended approaches for providing connected access to fractions. All the topics mentioned above included fraction concepts and they appear as individual instructional topics. Using the curriculum to identify these topics was an important process, as I did not want to overlook any topics where fraction concepts were implicitly incorporated. I focused on all chapters that included reference to fraction related activities and not just the fraction chapters. I called these 'fraction related chapters'. Table 4.1 shows all the fraction chapters and the fraction related chapters. Of interest in Table 4.1 is that across all three Grades, all but one of the fraction-related chapters come in ahead of the fraction chapters. This suggests that the other topics, if presented and used in well-connected ways, can lay foundations for fraction-related concepts. The caveat here is that the teachers in this study did not all work through textbook chapters in their presented order.

Table 4.1: Gr 4 – 6 Fraction and Fraction Related Chapters

Grade	Fraction Chapters	Fraction Related Chapters
4	➢ Common fractions (Chapter 12)	➢ Time (Chapter 3) ➢ Mass (Chapter 7) ➢ Capacity (Chapter 10)
5	➢ Common fractions (Chapter 8) ➢ Decimal fractions (Chapter 10)	➢ Time & temperature (Chapter 3) ➢ Mass & capacity (Chapter 5) ➢ Multiplication & division (ratio) (Chapter 7)
6	➢ Common fractions (Chapter 9)	➢ Time & temperature (Chapter 3)

Grade	Fraction Chapters	Fraction Related Chapters
	➢ Decimal fractions (Chapter 11)	➢ Length, mass & capacity (Chapter 5) ➢ Multiplication & division (ratio & rate) (Chapter 7) ➢ 2-D shapes (Chapter 8) ➢ Money (Chapter 13)

The analytic framework developed from the fractions and task demands literature was used to analyse the tasks in the three textbooks. When analysing each textbook, I looked at all the exercises in the fraction and fraction related chapters. Exercises were usually focused on a single idea, but the questions within them varied, and therefore each question within an exercise constituted the 'tasks' that formed the unit of analysis, to which the analytical framework was applied. Tasks were considered in terms of their position within a sequence of tasks when looking for connections. I coded for the sub-constructs and the connections made between sub-constructs and between the sub-construct/s and the unifying elements to determine the cognitive demand of a task.

This approach allowed for an analysis of the fraction sub-constructs and cognitive demand of tasks within each Grade, and across the three Grades in the textbook scheme.

Coding the tasks using the coding framework helped determine the extent to which the textbook tasks presented the full range of fraction sub-constructs and unifying elements, and the extent to which the textbook tasks offered access to higher cognitive demand. This analysis provided routes into understanding the extent to which conceptual working with fractions was required across the tasks presented in the textbook used at the focal school.

4.2 Structure of the three textbooks

A similar pattern of presentation permeated chapters across all three Grades' textbooks. Usually, an idea was explained using different representations (number lines, diagrams, pictures etc.). Thereafter, exercises related to the concept were presented. This pattern appeared across fraction and fraction related chapters. Each chapter included a range of concepts related to the chapter topic. For example, in the Grade 4 textbook, Chapter 3 addressed the topic of time and included exercises with tasks related to the concepts of hours, half hours and quarter hours.

In table 4.2, I summarise the structure of each of the three textbooks, including detail on the chapter focus, and the concepts in focus across the fraction and fraction related exercises within each chapter. Decimal fractions was a separate chapter in the Grade 5 and 6 textbooks.

Table 4.2: ***Classroom Mathematics*** **Grade 4-6 textbook structure**

Grade	Chapter	Topic	Concepts in focus across fraction and fraction related exercises.
4	3	Time	Hours, half-hours, quarter hours (p. 36)
	7	Mass	Measuring in kg (pp. 111 – 112 & 114 –115)
	10	Capacity	Working with ml and litres. (pp. 162–165)
	12	Common fractions	Halves, quarters, thirds & tenths, fifths, sixths, eights, ninths, tenths + adding like fractions, fraction notation, comparing fractions, (pp. 187 –205)
5	3	Time & temperature	Showing time (p. 49) Measuring temperature (p. 63)
	5	Length, mass & capacity	Measuring in g & kg (pp. 107– 108) Measuring in ml & l (pp. 112 & 115–116)
	7	Ratio	Part-part ratio (pp. 164–165)
	8	Common fractions	Naming fractions (p. 172) Part-whole (pp. 174 –177) Equivalent fractions (pp. 178 –181) Number lines (pp. 182–183) Improper fractions (pp. 184 –185) Mixed numbers (pp. 186–187) Adding fractions (same denominator) (pp. 188– 189) Subtracting fractions (same denominator) (pp. 190 –191) Revision (p. 192) Problem Solving (p. 193)
	10	Decimal fractions	Tenths (pp. 212–213) Converting fractions (pp. 214 – 215) Decimal fractions bigger than 1 (pp. 216 – 217) Counting in tenths (pp. 218 – 219) Measuring in mm & cm (pp. 220 – 221) Measuring in metres (pp. 222 – 223) Hundredths (pp. 224 – 227) Comparing & ordering decimals (pp. 228 – 229)

Grade	Chapter	Topic	Concepts in focus across fraction and fraction related exercises.
			Place value (pp. 230 – 231) Check your skills (p. 232) Stretch your skills (p. 233)
6	3	Time & temperature	Multiplication and Division units of time (pp. 46–47)
	5	Length, mass & capacity	Measuring length (p. 83) Conversions- mm, cm, m, km (pg. 85) Conversions- g, kg, t (p. 91) Conversions- ml & l (p. 95) Reading scales (pp. 96–97)
	7	Rate & ratio	Rate (p. 140); Ratio (p. 141)
	8	2-D shapes	Enlarging shapes (pp. 160–161)
	9	Common Fractions	Fractions (p. 165) Naming fractions (pp. 166 – 167) Equivalent fractions (pp. 168 – 169) Simplifying fractions (pp. 170 – 171) Fraction of a quantity (pp. 172 –173) Improper & mixed numbers (pp. 174 – 175) Adding fractions (pp. 176 – 177) Subtracting fractions (pp. 178 – 179) & - like fractions (pp. 180 – 181) & - unlike denominators (pp. 182 – 183) Fractions & percentages (pp. 184 – 185) Check your skills (p. 186) Stretch your skills (p. 187)
	11	Decimal fractions	Tenths & hundredths (pp. 210 – 211) Thousandths (pp. 212 – 213) Place value (pp. 214 – 215) Converting fractions (pp. 216 – 217) Ordering decimals (pp. 218 – 219) Rounding off decimals (pp. 220 – 221) & - decimals (pp. 222–223) Estimating (pp. 226 – 227) Percentages & decimals (pp. 228 – 229) Check your skills (p. 230) Stretch your skills (pp. 230 – 231)
	13	Money	Percentages & sales (pp. 250–251) Check your skills (p. 252)

As noted already, the teachers in the focal school did not work in order through the textbooks.

4.3 Textbook Analysis

The analytical framework was applied to every single task across all exercises in the fraction and fraction related chapters of each of the three textbooks. When coding the tasks, I looked for the 'minimal' fraction work required to complete a task. This approach was taken given the widespread evidence of procedural working with fractions in instruction noted in the literature. Thus, it was important to look for the extent to which tasks *demanded* more than a procedural working to solve the set problem. The general mathematics education literature suggests that connections mark more conceptual task orientations, and the fractions literature combined with Stein et al.'s (2000) framework points to connections between sub-constructs, via unifying elements, producing connections. Making the assumption that children will look for the simplest route to solving problems, I therefore looked for the most 'minimal' route to answering the question in focus. 'Minimal' here was interpreted in terms of the simplest procedures/ combinations of fraction sub-construct and procedures that could be used to answer the question.

In this section, I present a summary table for the outcomes of this coding for the fraction chapter, and then the set of fraction related chapters, for each Grade. Each table is followed by a commentary that highlights the key patterns seen in the results and where useful, examples that illuminate the patterns or points made in the preceding analysis are included. This Grade-by-Grade commentary for the fraction and then the fraction related chapters is followed by a concluding analysis that considers the overall patterns of presentation of fraction concepts in the textbook scheme across the three Grades.

4.3.1 Grade 4: Fraction Chapter

Table 4.3: Grade 4: Fraction Chapter Summary

		PW	O	M	Q	R&R	PW & O	PW & M	PW & Q	PW & R&R	O&M	O&Q	0 & R&R	M & Q	M & R&R	Q & R&R	Lower Cognitive Demand	Higher Cognitive Demand
Fair share	Ex12.1 4 tasks	4/4															4/4	
Halves	Ex 12.2 3 tasks	2/3							1/3								1/3	2/3
Quarters	Ex.12.3 3 tasks	1/3							2/3								1/3	2/3
Thirds	Ex12.4 4 tasks	1/4							3/4								1/4	3/4
Tenths	Ex.12.5 2 tasks	1/2							1/2								1/2	1/2
Partitioning	Ex 12.6 1 task	1/1																1/1
	Ex 12.7 1 task	1/1															1/1	
	Ex 12.8 10 tasks	10/10															8/10	2/10
Writing fractions	Ex 12.9 1 task	1/1															1/1	
	Ex 12.10 1 task	1/1															1/1	
Comparing fractions	Ex 12.11 4 tasks	4/4															1/4	3/4
	Ex 12.12 2 tasks	2/2															2/2	
Using fractions	Ex 12.13 3 tasks	1/3							2/3								1/3	2/3
	Ex 12.14 6 tasks	1/6							5/6								1/6	5/6
Adding like fractions	Ex 12.15 5 tasks	5/5															5/5	
Dividing shapes & numbers	Ex 12.16 3 tasks	3/3																3/3
	Ex 12.17 1 task								1/1									1/1
Checks your skills	Ex. 12.18 5 tasks								5/5									5/5
Stretch your skills	Ex 12.19 1 task	1/1																1/1

Some important overview patterns are seen through this analysis of the Grade 4 ‘Common Fractions’ chapter. Firstly, of the 60 tasks that were coded, 29 of these tasks were categorised as lower cognitive demand while 31 were categorised as higher cognitive demand. 40 of the 60 tasks focused solely on the part-whole sub-construct. The quotient sub-construct was the only other sub-construct that featured in the fractions chapter, and 20 tasks linked the part-whole and quotient sub-constructs across eight exercises in the chapter. This set of part-whole/quotient tasks encompassed the use of part-whole pre-partitioned area models, then moved to connecting the part-whole sub-construct with the quotient sub-construct. The connections were made via the unifying element of partitioning. The following 3 tasks from exercise 12.3 further explains these connections. The exercise was broken up into 3 different tasks. An analysis of each task is provided below.

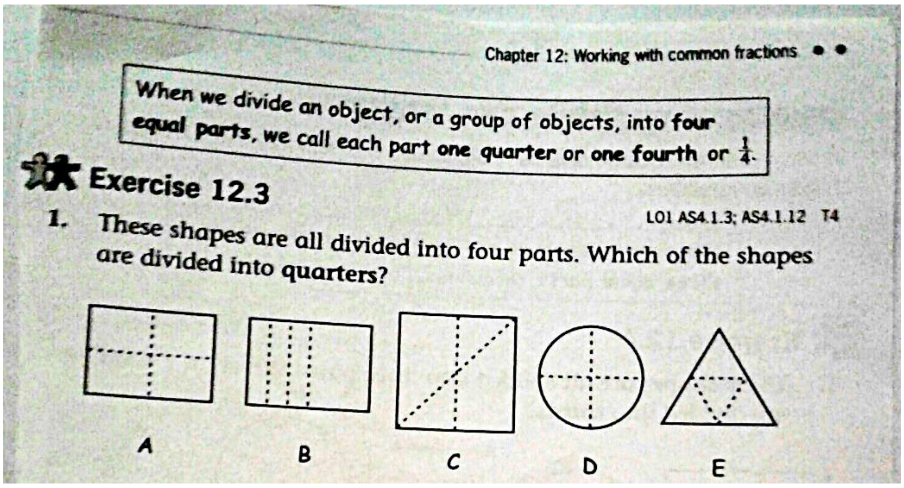
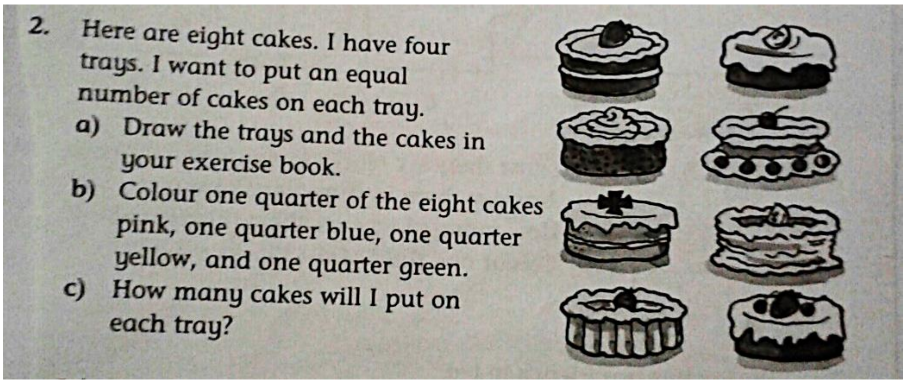


Figure 4.1: Task 1 Exercise 12.3- Grade 4 textbook (*Classroom Mathematics*, p.189)

The box that precedes task 1 provides a description of the partitioning action that produces the pieces that we call ‘quarters’. Given that the literature describes partitioning as a key unifying element, this inclusion of attention to the action of creating equal pieces is considered important. Definitionally though, it is of interest that what constitutes ‘equal parts’ is left somewhat ambiguous, as it is unclear whether the parts produce have to be congruent or of equal size. Ball (1993) notes that there is often an assumption, left unaddressed in textbook examples and teaching, that the pieces produced must be congruent to each other. The examples in Task 1 (figure 4.1) play into this misconception, as there is no example of quarters that presents non- congruent equal parts. Instead, the distinction pointed out is between unequal parts in terms of pieces of different sizes and equal parts considered only in congruent terms. Further, all examples in question 1 focus on the part whole sub-construct with a continuous whole unit with pre-partitioned examples. The partitioning procedure description has to be applied to each example, and the distinction between non-equal sized and equal-sized parts has to be understood but given the partial nature of the description offered coupled with the limited nature of the example space, this question tends to veer towards a simple application of a partial procedure- and hence was classified as a lower cognitive demand question.



2. Here are eight cakes. I have four trays. I want to put an equal number of cakes on each tray.
a) Draw the trays and the cakes in your exercise book.
b) Colour one quarter of the eight cakes pink, one quarter blue, one quarter yellow, and one quarter green.
c) How many cakes will I put on each tray?

Figure 4.2: Task 2 Exercise 12.3- Grade 4 textbook (*Classroom Mathematics*, p.189)

Task 2 (figure 4.2) is of interest because the basic situation presented involves 8 cakes that need to be shared equally across four trays. Fundamentally, this can be seen as a partitive division problem rather than a fraction task. However, the rectangular array arrangement of the cakes in a 4-row x 2 column format, coupled with the partitioning action description that leads the exercise and the 2b instruction to 'colour one quarter of the cakes' suggests that an aim of this task is to use partitioning actions to produce four equal parts and then link the outcomes of this (a quarter) with two cakes. Thus here, partitioning actions based on part-whole ideas can be seen as connected to a partitive division situation- which provides a link to the quotient sub-construct in Lamon's terms. In this example- involving discrete whole quantities here, 2 cakes come to be named as a quarter.

The notion of equivalence is possible to focus on here (2 of the 8 cakes for a quarter), but this is not demanded by the task. There are, however, connections between part-whole and quotient sub-construct via partitioning as a unifying element that push this task into higher cognitive demand terrain in terms of possibilities for linking sub-constructs.

Jabu has 12 chocolates. He wants to share them equally among himself and three friend. a) How many people must get chocolates? b) Draw the chocolates in your exercise book. Then show how to divide them equally. c) What fraction of the chocolates does each one get?

Figure 4.3: Task 3 Exercise 12.3- Grade 4 textbook (*Classroom Mathematics*, p.189)

Question 3 (figure 4.3) reverses the order of question 2, by beginning by asking for the result of a partitive division situation involving 4 equal shares, and subsequently connecting the outcome of equal sharing to a fractional representation (3 chocolates = a quarter). The variation across the parts of this task allows for a range of representations of a quarter across continuous and discrete wholes. There is potential here- if attention is drawn across the examples- for a focus on equivalence as a quarter can be seen in part-whole terms and as 2 out of 8 and as 3 out of 12. Overall, therefore, given these possibilities for connections between sub-constructs and across unifying elements through the parts of this task, I coded this task as providing possibilities for access to higher cognitive demand overall.

While the coding methodology meant that all linked sub-construct tasks were automatically placed in the higher cognitive demand category, there were some tasks focused on a single sub-construct that were also coded as higher cognitive demand. These tasks usually included area models or discrete units that required partitioning and/or unitizing. While these tasks only included one-sub-construct, they also required working with the unifying elements to be completed successfully. For example, partitioning an 8 by 8 grid into quarters in as many ways as possible. This led to connections between the sub-construct and unifying element/s and was therefore categorised as higher cognitive demand task. ways as possible.

4.3.2 Grade 4: Fraction Related Chapters

Table 4.4: Grade 4: Fraction Related Chapters Summary

		PW	O	M	Q	R&R	PW & O	PW & M	PW & Q	PW & R&R	O & M	O & Q	0 & R&R	M & Q	M & R&R	Q & R&R	Lower Cognitive Demand	Higher Cognitive Demand
Time	Ex3.4 1 task	1/1															1/1	
	Ex 3.5 1 task	1/1															1/1	
Mass	Ex7.8 5 tasks							5/5										5/5
Capacity	Ex 10.4 3 tasks							3/3										3/3

An analysis of the fraction related chapters in the Grade 4 textbook reveals that of the 10 tasks coded, 2 of the tasks focused solely on the part-whole sub-construct and 8 on the part-whole/measure sub-constructs. 2 of the 10 tasks were categorised as lower cognitive demand while 8 were categorised as higher cognitive demand. The time fraction related tasks presented clocks that were pre-partitioned into minutes and hours. These tasks required either reading the time off the clocks or drawing different times onto pre-partitioned clocks. With the number of minutes given for each clock number, the tasks did not require any fraction knowledge to be completed and therefore presented features of lower demand tasks. This is further discussed in the Grade 5 analysis as a similar task appear in the Grade 5 textbook.

All the Mass and Capacity tasks included both the part-whole and measure sub-constructs. These tasks required working with sand (mass) and water (capacity) and measuring instruments such as scales and measuring jugs/cups, respectively. Partitioning the water or sand into different fractions (halves and quarters) to either find the mass or capacity of the substance resulted in working with kilograms, grams, litres, and millilitres. The partitioning of the whole (kilogram/litre) and then using the appropriate measuring instrument to establish, for example, that a quarter of a kilogram is equivalent 250g or that half a litre is the same as 500 ml, provided a clear connection between the part-whole and measure sub-construct. The integration of these two sub-constructs through partitioning provided a connection to understanding fraction as a measure.

4.3.3 Grade 5: Fraction Chapter

Table 4.5: Grade 5: Fraction Chapter Summary

		PW	**O**	**M**	**Q**	**R&R**	**PW & O**	**PW & M**	**PW & Q**	**PW & R&R**	**O&M**	**O & Q**	**0 & R&R**	**M & Q**	**M & R&R**	**Q & R&R**	**Lower Cognitive demand**	**Higher Cognitive demand**
Naming fractions	Ex8.1 5 tasks	1/5							4/5									5/5
	Ex8.2 3 tasks	3/3															1/3	2/3
Fraction of a whole	Ex.8.3 6 tasks											6/6						6/6
	Ex8.4 3 tasks						3/3											3/3
	Ex.8.5 5 tasks	5/5															1/5	4/5
Equivalent fractions	Ex 8.6 2 tasks	2/2															2/2	
	Ex 8.7 2 tasks	2/2															2/2	
Beads & number lines	Ex 8.8 6 tasks	2/6		3/6			1/6										4/6	2/6
Improper fractions	Ex 8.9 4 tasks	4/4															3/4	1/4
	Ex 8.10 1 task							1/1										1/1
Mixed numbers	Ex 8.11 5 tasks	5/5															2/5	3/5
	Ex 8.12 1 task							1/1										1/1
Adding fractions (same denominator)	Ex 8.13 3 tasks	3/3															3/3	
	Ex 8.14 2 tasks	2/2															2/2	
Subtracting fractions (same denominator)	Ex 8.15 5 tasks	5/5															3/5	2/5
Check your skills	Ex 8.16 4 tasks	3/4		1/4													2/4	2/4
Stretch your skills	Ex 8.17 3 tasks	1/3	1/3				1/3										1/3	2/3

The overview patterns seen through the analysis of the Grade 5 'Common Fractions' chapter differs from Grade 4. Firstly, the 60 tasks in the Grade 5 chapter covered a wider range of sub-constructs and a wider range of sub-construct connections than seen in Grade 4. 38 tasks focused solely on the part-whole sub-construct, 1 focused solely on the operator sub-construct, 4 focused solely on the measure sub-construct, 2 linked part-whole and measure sub-constructs, 6 linked operator and quotient sub-constructs, 5 linked part-whole and operator sub-constructs and 4 linked part-whole and quotient sub-constructs. The measure tasks included pre-partitioned number lines, with given unit fractions that required completing the number lines by filling in the missing tick marks, while the part-whole/measure sub-constructs tasks included pre-partitioned number lines that required work with the partitioning to determine the unit fraction to complete the number lines. The part-whole/quotient tasks were very similar to those presented in the Grade 4 textbook and discussed in detail in the Grade 4 analysis. The task that included the operator sub-construct required calculating, for example, a quarter or a half or a third of a year. The operator/quotient sub-construct tasks emphasised fair share and made a connection between division and how the whole is operated on. For example, eight learners share a slab of chocolate with 24 blocks equally among them. (a) What fraction of the chocolate does each learner get? (b) How many blocks of chocolate does each learner get? (c) $\frac{1}{8}$ of 24 blocks of chocolate =_____ blocks of chocolates.

The part-whole/operator tasks appeared across 4 exercises. The following task from Exercise 8.3 explains the coding and analysis of these tasks.

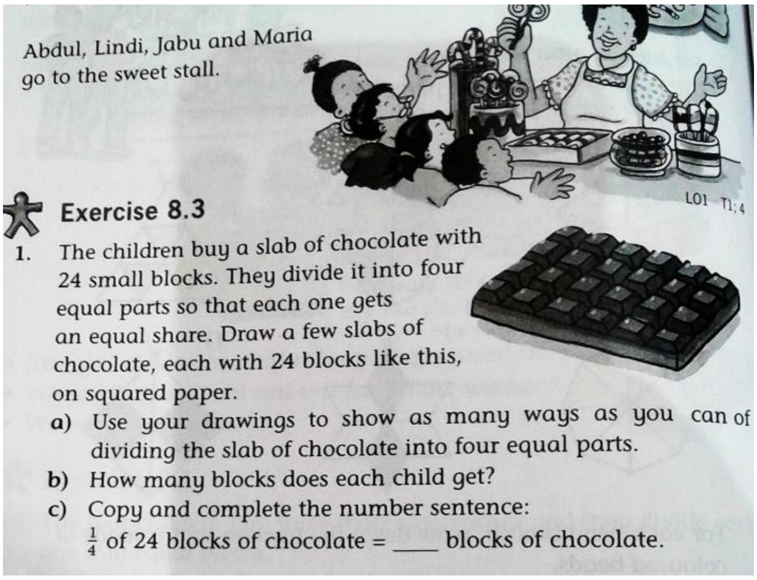


Figure 4.4: Task 1 Exercise 8.3 Grade 5 (*Classroom Mathematics*, p.174)

The first question of this task works with partitioning the 24 blocks of chocolates into four equal parts in as many ways as possible. This is the quotient sub-construct with partitioning and because it required the physical action of partitioning linking the sub-construct and unifying element, it was categorised as higher cognitive demand. This question feeds into the next question and aids in determining the answer, the partitioning action of the whole in different ways produces the answers. The final question makes a connection between how the whole is operated on through partitioning in order to find how many blocks of chocolate are equivalent to one $\frac{1}{4}$ of 24. The partitioning is performed on the same whole in different ways, resulting in the same answer. This task was categorised as a higher cognitive demand task.

4.3.4 Grade 5: Decimal Fractions Chapter

Table 4.6: Grade 5: Decimal Fraction Chapter Summary

		PW	O	M	Q	R&R	PW & O	PW & M	PW & Q	PW & R&R	O&M	O&Q	0 & R&R	M & Q	M & R&R	Q & R&R	Lower Cognitive Demand	Higher Cognitive Demand
Tenths as fractions	Ex 10.1 5 tasks	5/5															5/5	
Converting fractions	Ex 10.2 3 tasks	3/3															3/3	
	Ex 10.3 2 tasks	2/2															1/2	1/2
Decimals bigger than 1	Ex 10.4 5 tasks	4/5															5/5	
Counting in tenths	Ex 10.5 3 tasks			3/3													3/3	
	Ex 10.6 5 tasks			5/5													5/5	
Milli & centimetres	Ex 10.7 5 tasks							5/5										5/5
Metres	Ex 10.8 6 tasks			2/6				4/6									2/6	4/6
Hundredths as fractions	Ex 10.9 3 tasks	3/3															3/3	
	Ex 10.10 4 tasks	4/4															2/4	2/4
Comparing and ordering	Ex 10.11 3 tasks	3/3															3/3	
Place value	Ex 10.12 3 Questions																3/3	
	Ex 10.13 3 tasks																3/3	
Check your skills	Ex 10.14 6 tasks																6/6	
Stretch your skills	Ex 10.15 2 tasks	2/2																2/2

Of the 58 textbooks tasks, 44 were categorised as lower cognitive demand and 14 were categorised as higher cognitive demand. The 58 decimal fraction tasks were located primarily across the part-whole (26 tasks), measure (10 tasks) and part-whole/measure (9 tasks) sub-constructs. 12 tasks made no reference to any sub-construct – I return to these later in this section. The part-whole sub-construct tasks all used pre-partitioned area models, either divided into tenths or hundredths with questions ranging from writing common fractions as decimals or vice versa to counting the number of shaded parts. Five of these tasks were categorised as higher cognitive demand because they included the use of partitioning and unitizing. For example, a task from Exercise 10.10: Show the following fractions on squared paper a) two tenths = 20 hundredths (or 0,2 = 0,20) or a task from Exercise 10.3 that required drawing rectangles on squared paper and then colouring them in to show decimal values such as 1,2 or 3,4.

The measure tasks comprised of pre-partitioned number lines with given unit fractions and required filling in missing fractions. Some tasks included comparing decimal fractions using a number line. The number lines were pre-partitioned into tenths with the decimal fractions displayed on the number line and the questions were as follows: Fill in < or > to make each number sentence true: a) 0.2 ___ 0,5. These tasks, involving reading off results from pre-partitioned number lines, were categorised as lower cognitive demand.

The 9 part-whole/measure sub-constructs tasks included the use of rulers and a section of measuring tapes. These measuring instruments were pre-partitioned and unitized and were used to explain wholes and tenths or hundredths in relation to mm, cm, and m. The representations and explanations used in examples with the use of partitioning and re-unitizing provided connections between the two sub-constructs creating an understanding of decimals in the context of measurement, as illustrated in Figure 4.5. Examples of questions that followed on from these explanations were: Write the distances in tenths of a metre (a) 1,6m and (b) 3,2m etc.

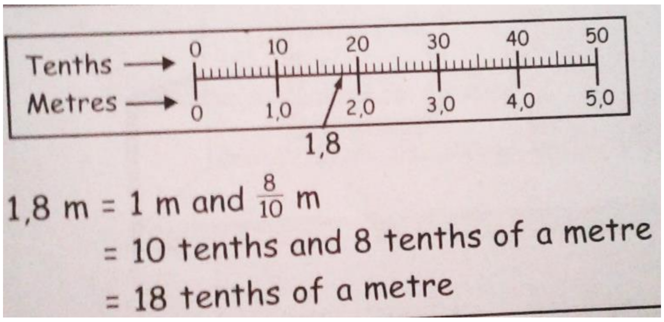


Figure 4.5: Exercise 10.8 Grade 5 (*Classroom Mathematics*, p.222)

The last 4 exercises made no reference to any of the sub-constructs. Some tasks involved working with the place value chart and determining the value of a digit in a number. Other tasks required writing common fractions as decimal fractions and vice versa, without any model or explanation. Set as revision tasks and including questions like: Write seven tenths as a decimal or write 1,1 as a common fraction or mixed number, these tasks did not demand the use of any sub-constructs or unifying elements. The presentation of the place value chart instead, provided a decimal – fraction conversion route through placing numbers in the chart and reading them out as fractions without the need to engage with any fraction concepts. These tasks were categorised as lower-cognitive demand.

4.3.5 Grade 5: Fraction Related Chapters

Table 4.7: Grade 5: Fraction Related Chapters Summary

		PW	O	M	Q	R&R	PW & O	PW & M	PW & Q	PW & R&R	O&M	O&Q	O & R&R	M & Q	M & R&R	Q & R&R	Lower Cognitive Demand	Higher Cognitive Demand
Time	Ex3.6 1 task	1/1															1/1	
Temp	Ex 3.18 1 task			1/1													1/1	
Mass	Ex5.11 1 task		1/1														1/1	
	Ex 5.12 2 tasks			1/2				1/2									1/2	1/2
Capacity	Ex5.14 1 task		1/1														1/1	
Ratio	Ex7.32 3 tasks					3/3											3/3	

Unlike the Grade 4 analysis, the coding of the Grade 5 fraction related chapters showed several different sub-constructs but only one task that was categorised as higher cognitive demand. Of the 9 tasks, 1 included the part-whole sub-construct, 2 tasks included the operator sub-construct and 2 tasks included the measure sub-construct. There were 3 rate and ratio tasks and 1 part-whole/measure sub-constructs task.

As noted earlier, the time tasks could be completed with little or no fraction knowledge. The Grade 5 task (very similar to the Grade 4-time task mentioned earlier) is detailed here to explain the coding and analysis. The following explanation was provided prior to the task, “The big hand takes five minutes to move from the 12 to the 1. It also takes five minutes to move from the 1 to the 2, from the 2 to the 3, …., and so on.”

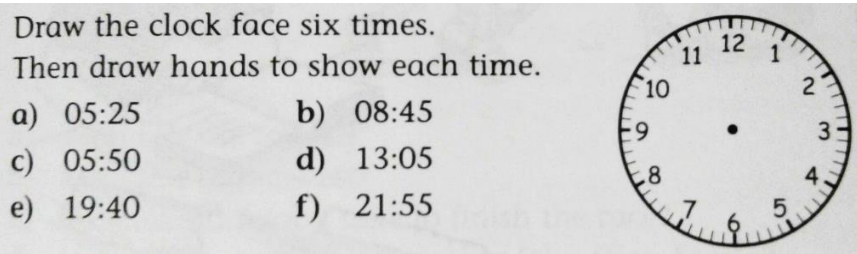
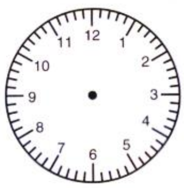


Figure 4.6: Grade 5 Time Task (*Classroom Mathematics,* p.49)

Given the information and marking of 5-minute intervals between clock numbers, placing the big hand requires only counting skills, rather than fraction understanding. While some fraction-related thinking is required to place the small hand in a roughly appropriate position, the pre-amble suggests that the task focus is only on the big hand placement. While the task alludes to the part-whole sub-construct it does not require any fraction knowledge for it to be completed. The task clearly points out that each mark represents five minutes, so they need to count in fives to find the correct position of the long hand. No reference is made to the short hand in the task. This task was therefore coded as lower cognitive demand as it involved ‘knowing’ about 5-minute intervals. The task could have been considered as higher cognitive demand if the task did not provide the value of each mark, since it could then be completed by successively partitioning 60 minutes to find the value of each marking i.e. 1 hour = 60 minutes ($\frac{12}{12}$), so half an hour = 30 mins ($\frac{6}{12}$), so $\frac{1}{4}$ hour = 15 mins ($\frac{3}{12}$), so each mark = 5 mins ($\frac{1}{12}$) or it could be worked out as an operator problem in a measure context ($\frac{1}{12}$ of 60 mins = 5 mins). This would be considered as higher cognitive demand because this result is connected to a prior result and involves a unifying element.

Lamon (2012, p. 220) provides a similar task, however the contrast is that her example explicitly uses fraction sub-constructs to teach the same concept.


9. Answer these questions about the short hand on a clock (the hour hand).

a. What does one full rotation mean?
b. What do the spaces between the longer hash marks represent?
c. What do the spaces between the shorter hash marks represent?

10. Answer these questions about the longer hand on a clock (the minute hand).

a. What does one full rotation mean?
b. What do the spaces between the longer hash marks represent?
c. What do the spaces between the smaller hash marks represent?

Figure 4.7: Time Task by Lamon (2012, p. 220)

Lamon's (2012) tasks contrast with the textbook tasks in their demand of fraction related attention. The follow-up tasks requiring children to mark 'as precisely as possible' times like 3:18 on similar clock faces further entrench this demand.

If we compare the tasks from the textbooks with Lamon's task we see that very little or no fraction knowledge is required to complete the tasks, yet Lamon's task requires work with different sub-constructs and a unifying element. The sub-constructs and how the unit is partitioned creates connections for a deeper understanding of how to work with minutes and hours when working with analogue clock times. While some tasks alluded to a sub-construct it was not required to complete the task.

This is also seen in the 2 measure sub-constructs where the measuring instruments are pre-partitioned with a given fraction unit. These tasks have the potential to be higher cognitive demand but because of the pre-partitioned whole and given unit fraction, little room is left for any fraction work. The part-whole/measure sub-constructs task was different since the measuring scale was pre-partitioned, but the unit fraction had to be calculated by working with the partitioning to find its value. This provided a connection between the whole and the pre-partitioned units to create an understanding of the value of each partitioning and making a connection between the part-whole and measure sub-constructs via partitioning.

The operator tasks were procedural (and therefore lower cognitive demand) in nature and did not require any of the unifying elements to complete them successfully. Like the Grade 4 tasks, the questions required answers to: how many millilitres are there in a $\frac{1}{4}$, $\frac{1}{2}$ or $\frac{3}{4}$ of a litre of milk?

The rate and ratio tasks were introduced by providing an explanation of each of these terms, followed by examples of rate and ratio notation. The questions required following the examples of the symbolic representations of rate and ratio. These ratio tasks only included part-part ratios; they did not make a connection to the part-whole and ratio sub-construct to show that a ratio can also be written as a fraction. Examples from the literature encourage the interpretation of ratios as a part-whole comparison (written in fraction form) and as a ratio (Lamon, 2012; Lo et al, 1997). The connection between the part-whole interpretation and ratio sub-construct provides a comprehensive understanding of ratio. For this reason, the tasks were categorised as lower demand task.

4.3.6 Grade 6: Fraction Chapter

Table 4.8: Grade 6: Fraction Chapter Summary

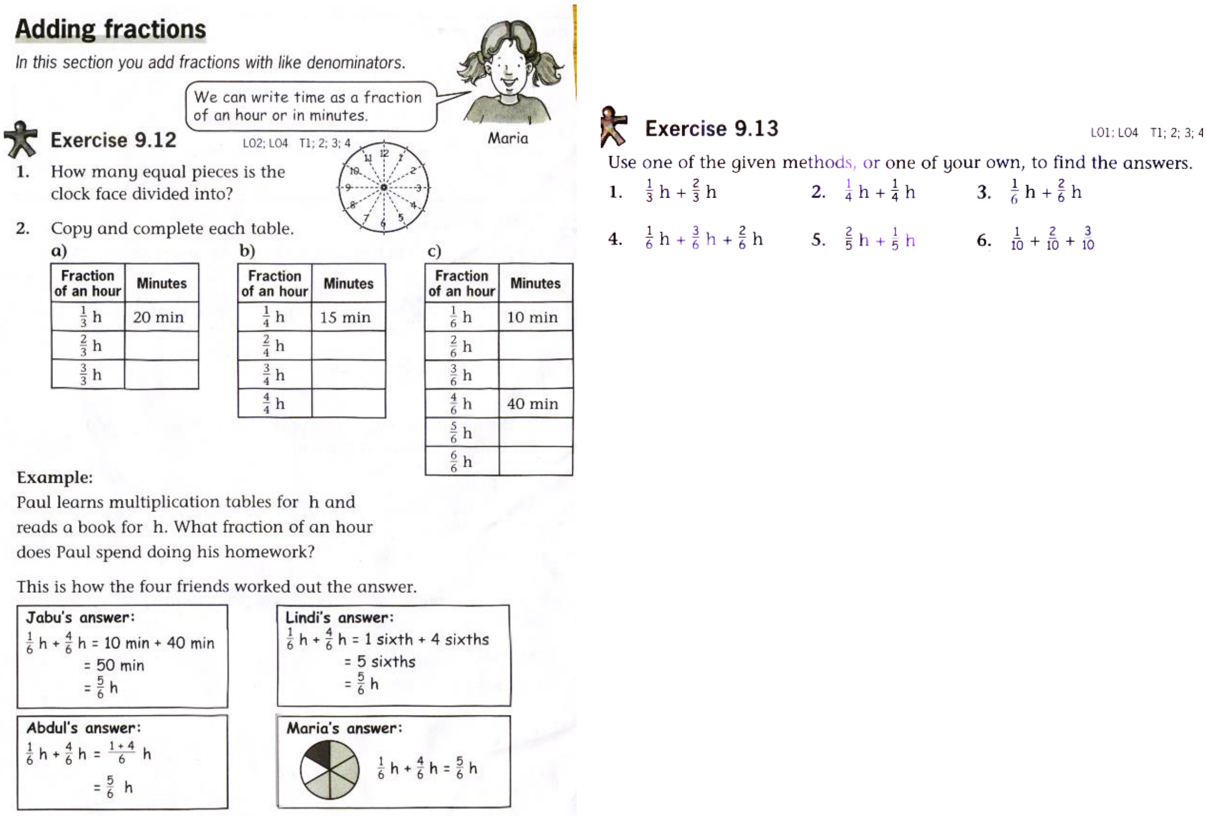
			PW	O	M	Q	R&R	PW & O	PW & M	PW & Q	PW & R&R	O&M	O&Q	O & R&R	M & Q	M & R&R	Q & R&R	Lower Cognitive Demand	Higher Cognitive Demand
	Naming Fractions	Ex9.2 3 tasks	3/3															1/3	2/3
		Ex.9.3 3 tasks	3/3															3/3	
	Equivalent fraction	Ex9.4 2 tasks	2/2															1/2	1/2
		Ex.9.5 2 tasks	2/2															2/2	
		Ex 9.6 2 tasks	2/2															1/2	1/2
	Simplifying fractions	Ex 9.7 7 tasks	6/7							1/7								4/7	3/7
	Fraction of a quantity	Ex 9.8 3 tasks						3/3											3/3
		Ex 9.9 3 tasks						3/3											3/3
	Improper fractions & mixed numbers	Ex 9.10 2 tasks			1/2				1/2									1/2	1/2
		Ex 9.11 3 tasks	3/3															1/3	2/6
	Adding fractions	Ex 9.12 2 tasks	2/2															2/2	
		Ex 9.13 1 task	1/1															1/1	
		Ex 9.14 1 task	1/1															1/1	
	Subtracting fractions	Ex 9.15 1 task	1/1															1/1	
		Ex 9.16 1 task	1/1															1/1	
	Adding & subtracting like fractions	Ex 9.17 11 tasks	11/11															9/11	2/11
	Adding &	Ex 9.18	1/1															1/1	

			PW	O	M	Q	R&R	PW & O	PW & M	PW & Q	PW & R&R	O&M	O&Q	O & R&R	M & Q	M & R&R	Q & R&R	Lower Cognitive Demand	Higher Cognitive Demand
	subtracting unlike fractions	1 task																	
		Ex 9.19 5 tasks	4/5	1/5														5/5	
	Converting fractions & percentages	Ex 9.20 3 tasks	1/3															3/3	
		Ex 9.21 1 task																1/1	
		Ex 9.22 1 task																1/1	
	Check your skills	Ex 9.23 5 tasks	2/5	1/5				1/5	1/5										5/5
	Stretch your skills	Ex 9.24 2 tasks	2/2																2/2

Similar to the Grade 4 and 5 analysis, the Grade 6 coding revealed majority of the tasks focused on the part-whole sub constructs, while the other tasks included a range of specific sub-constructs. The Grade 6 textbook had the most fraction tasks since more content was covered in the curriculum. Of the 116 tasks coded, 79 where solely focused on the part-whole sub-construct, 2 focused solely on the operator sub-construct, 1 focused solely on the measure sub-construct, 7 linked the part-whole/operator sub-constructs, 2 linked the part-whole/measure sub-constructs and 6 linked the part-whole/quotient sub-constructs. 19 of the tasks did not refer to any sub-construct or unifying elements. These tasks very procedural in nature. An example of these type of tasks: Write each percentage as a common fraction and in its simplest form: (1) 12%, (2) 15%...etc. They were categorised as lower cognitive demand.

The operator, measure, part-whole/operator, part-whole/quotient and part-whole/measure sub-constructs tasks were similar to those in the Grade 4 and 5 textbook analysis.

The analysis of the part-whole sub-construct tasks revealed that topics like equivalent, comparing, simplifying, converting, and adding and subtracting fractions used the part-whole sub-construct with pre-partitioned area or linear models. Majority of these tasks were algorithmic and procedural with no connection to sub-construct/s and unifying elements and is demonstrated in the following example from the Grade 6 textbook (pg. 176-177).



Adding fractions

In this section you add fractions with like denominators.

We can write time as a fraction of an hour or in minutes.

Maria

Exercise 9.12 LO2; LO4 T1; 2; 3; 4

1. How many equal pieces is the clock face divided into?
2. Copy and complete each table.

a)

Fraction of an hour	Minutes
$\frac{1}{3}$ h	20 min
$\frac{2}{3}$ h	
$\frac{3}{3}$ h	

b)

Fraction of an hour	Minutes
$\frac{1}{4}$ h	15 min
$\frac{2}{4}$ h	
$\frac{3}{4}$ h	
$\frac{4}{4}$ h	

c)

Fraction of an hour	Minutes
$\frac{1}{6}$ h	10 min
$\frac{2}{6}$ h	
$\frac{3}{6}$ h	
$\frac{4}{6}$ h	40 min
$\frac{5}{6}$ h	
$\frac{6}{6}$ h	

Example:
Paul learns multiplication tables for h and reads a book for h. What fraction of an hour does Paul spend doing his homework?

This is how the four friends worked out the answer.

Jabu's answer:
$\frac{1}{6}$ h + $\frac{4}{6}$ h = 10 min + 40 min
= 50 min
= $\frac{5}{6}$ h

Lindi's answer:
$\frac{1}{6}$ h + $\frac{4}{6}$ h = 1 sixth + 4 sixths
= 5 sixths
= $\frac{5}{6}$ h

Abdul's answer:
$\frac{1}{6}$ h + $\frac{4}{6}$ h = $\frac{1+4}{6}$ h
= $\frac{5}{6}$ h

Maria's answer:
$\frac{1}{6}$ h + $\frac{4}{6}$ h = $\frac{5}{6}$ h

Exercise 9.13 LO1; LO4 T1; 2; 3; 4

Use one of the given methods, or one of your own, to find the answers.

1. $\frac{1}{3}$ h + $\frac{2}{3}$ h
2. $\frac{1}{4}$ h + $\frac{1}{4}$ h
3. $\frac{1}{6}$ h + $\frac{2}{6}$ h
4. $\frac{1}{6}$ h + $\frac{3}{6}$ h + $\frac{2}{6}$ h
5. $\frac{2}{5}$ h + $\frac{1}{5}$ h
6. $\frac{1}{10}$ + $\frac{2}{10}$ + $\frac{3}{10}$

Figure 4.8: Grade 6 Adding Fractions Tasks (*Classroom Mathematics*, p.176-177)

The task is set in a measure context using the concept of time to add fractions with like denominators. From research we know that students must have a deep understanding of both measure and part-whole to support their development of an understanding of addition and subtraction of fractions (Lamon, 2001; Keijzer & Terwel, 2001; Watanabe, 2006). The pre-partitioned clock face with given answers to a $\frac{1}{3}$ of an h, $\frac{1}{4}$ of an h and $\frac{1}{6}$ of an hour (unit fraction) moves the task demand to lower cognitive demand. There is no connection made between the unit fractions and the partitioning of the clock face to show that the 12 partitioned sections can be unitized differently to establish the unit fractions or that the operator sub-construct can be used to determine the value of each unit fraction. The answer to each unit fraction is provided and the tables can be completed by following the procedure of adding the value of each unit fraction. The follow-on task (Figure 4.8: Exercise 9.13) can be completed by following the provided procedures with very little cognitive effort.

4.3.7 Grade 6: Decimal Fractions Chapter

Table 4.9: Grade 6: Decimal Fractions Chapter Summary

		PW	O	M	Q	R & R	PW & O	PW & M	PW & Q	PW & R&R	O & M	O & Q	0 & R&R	M & Q	M & R&R	Q & R&R	Lower Cognitive demand	Higher Cognitive demand
Tenths & hundredths	Ex 11.1 1 task	1/1															1/1	
	Ex 11.2 3 tasks	1/3															3/3	
Thousandths	Ex 11.3 2 tasks	1/2															2/2	
	Ex 11.4 2 tasks																2/2	
Place value	Ex 11.5 1 task																1/1	
	Ex 11.6 3 tasks																3/3	
Converting fractions	Ex 11.7 3 Tasks	3/3															3/3	
	Ex 11.8 3 tasks	3/3															3/3	
Ordering decimals	Ex 11.9 6 tasks	4/6		2/3													4/6	2/6
Rounding off	Ex 11.10 6 tasks																6/6	
Adding & subtracting	Ex 11.11 2 tasks			2/2													2/2	
	Ex 11.12 6 tasks																6/6	
	Ex 11.13 8 tasks																8/8	
Estimating sum & differences	Ex 11.14 2 tasks																2/2	
	Ex 11.15 6 tasks																6/6	
% as decimals and fractions	Ex 11.16 3 tasks	3/3															3/3	
	Ex 11.17 3 tasks	3/3															3/3	
Check your skills	Ex 11.18 4 tasks																4/4	
Stretch your skills	Ex 11.19 2 tasks	2/2																2/2

This decimal chapter coding showed a few similarities with the Grade 5 chapter. Of the 66 tasks, 21 of these tasks focused solely on the part-whole sub-construct and 4 of these tasks focused solely on the measure sub-construct. But, compared to the Grade 5 analysis, 39 tasks did not make use of any sub-construct or unifying elements and there were no tasks with linking sub-constructs. The tasks were pure calculation tasks and therefore categorised as lower cognitive demand, while those that were categorised as higher cognitive demand were problem-solving tasks that included work with the part-whole sub-construct and partitioning to be completed successfully

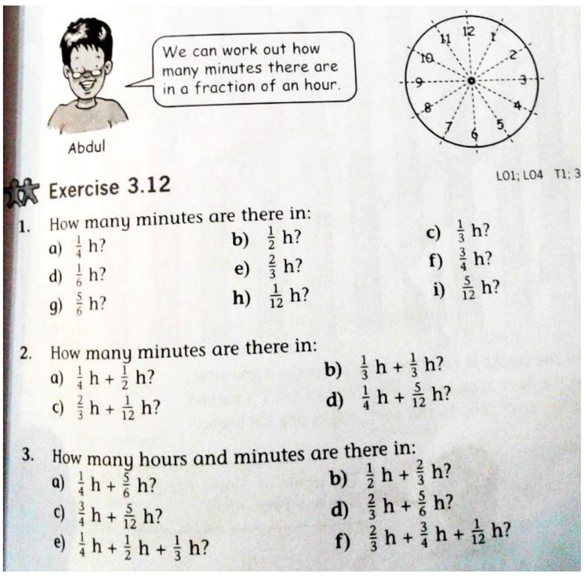
4.3.8 Grade 6: Fraction Related Chapters

Table 4.10: Grade 6: Fraction Related Chapters Summary

		PW	**O**	**M**	**Q**	**R&R**	**PW & O**	**PW & M**	**PW & Q**	**PW & R&R**	**O&M**	**O&Q**	**0 & R&R**	**M & Q**	**M & R&R**	**Q & R&R**	**Lower Cognitive Demand**	**Higher Cognitive Demand**
Time	Ex3.12 3 tasks						3/3											3/3
Length	Ex5.2 1 task			1/1													1/1	
	Ex 5.4 5 tasks			5/5													5/5	
Mass	Ex5.12 5 tasks			5/5													5/5	
Capacity	Ex5.16 6 tasks			6/6													6/6	
Reading scales	Ex.5.17 6 tasks			1/6				5/6									1/6	5/6
Rate	Ex 7.22 3 tasks					3/3											3/3	
Ratio	Ex7.23 3 tasks					3/3											3/3	
Enlarging shapes	Ex8.15 2 tasks		2/2														2/2	
	Ex 8.16 1 task		1/1														1/1	
Money	Percentage- Ex13.4 1 task		1/1														1/1	
	Ex 13.5 2 tasks		2/2														2/2	

Part-whole tasks included pre-partitioned area or linear models used to explain tenths, hundredths, and thousandths. These tasks were similar to those in the Grade 5 textbook except that they included working with thousandths. Tasks that included the measure sub-construct comprised of pre-partitioned number lines with the decimal fractions displayed on them, for example, adding two decimal fractions using the pre-partitioned, labeled number line to calculate: 0,2 + 0,4 =___. These tasks were very similar to those found in the Grade 5 textbook except that the number range included thousandths. These tasks were categorised as lower cognitive demand because the number lines were pre-partitioned and there was no connection made between the partitioning and the measure sub-construct to create meaning of the different sized decimal fractions. The task could be successfully completed by reading off the number line with no understanding of the quantity that is created when two decimal fractions are added together. It is interesting to note that none of the tasks in the Grade 6 fraction related chapters included solely the part-whole sub-construct. Of the 46 tasks, 14 solely focused on the operator sub-construct, 18 focused solely on the measure sub-construct, 6 focused solely on the rate and ratio sub-construct, 3 linked the part-whole/operator sub-constructs and 5 linked the part-whole and measure sub-constructs.

Coding the Grade 6-time task was not as straight forward because while it appeared to be a higher cognitive demand task, possibilities exist that could move it to a lower cognitive demand.


We can work out how many minutes there are in a fraction of an hour.

Abdul

Exercise 3.12

LO1; LO4 T1; 3

1. How many minutes are there in:
   a) $\frac{1}{4}$ h? b) $\frac{1}{2}$ h? c) $\frac{1}{3}$ h?
   d) $\frac{1}{6}$ h? e) $\frac{2}{3}$ h? f) $\frac{3}{4}$ h?
   g) $\frac{5}{6}$ h? h) $\frac{1}{12}$ h? i) $\frac{5}{12}$ h?

2. How many minutes are there in:
   a) $\frac{1}{4}$ h + $\frac{1}{2}$ h? b) $\frac{1}{3}$ h + $\frac{1}{3}$ h?
   c) $\frac{2}{3}$ h + $\frac{1}{12}$ h? d) $\frac{1}{4}$ h + $\frac{5}{12}$ h?

3. How many hours and minutes are there in:
   a) $\frac{1}{4}$ h + $\frac{5}{6}$ h? b) $\frac{1}{2}$ h + $\frac{2}{3}$ h?
   c) $\frac{3}{4}$ h + $\frac{5}{12}$ h? d) $\frac{2}{3}$ h + $\frac{5}{6}$ h?
   e) $\frac{1}{4}$ h + $\frac{1}{2}$ h + $\frac{1}{3}$ h? f) $\frac{2}{3}$ h + $\frac{3}{4}$ h + $\frac{1}{12}$ h?

Figure 4.9: Grade 6 Time Task *(*Classroom Mathematics*, p.47)*

The analogue clock face (in figure 4.9) is partitioned into twelfths and required a calculation to determine the number of minutes in a given fraction of an hour. Question 1 required unitizing the whole (clock face partitioned into twelfths) into halves, thirds, quarters, sixths, and twelfths. If students knew that each part of the clock measured 5 minutes, this was not made explicit, and unitized the 12 parts accordingly, it could be determined for example, how many minutes in a third of an hour. This question dealt with the part-whole sub-construct (partitioning and unitizing), the measure sub-construct by determining the measure of each twelfth, and the operator sub-construct (operating on the 12 parts to determine how many minutes in each fraction). Question 2 and 3 builds on from question 1. Two or three fractions, with unlike denominators, of the whole added together to get an answer in minutes. Question 3 takes it a step further by determining how many minutes makes a whole (hour). The connections created between fraction and time concepts through the sub-constructs and unifying elements moved me into categorising this as a higher cognitive demand task, yet while this task is rich with fractions concepts to help promote a better understanding of time concepts, it can also be solved by following a simple procedure that includes the operator sub-construct and makes no connection to understanding time through fractions. Assuming, Grade 6 students knew that an hour consists of 60 minutes, question could be completed by dividing 60 minutes by 4 and getting the answer 15 minutes. This is a straightforward procedure with no connections to the sub-constructs or unifying elements and therefore could be categorised as a lower demand task. I categorised this task as higher demand because I felt that it promotes a rich connection between time and fractions and begins to establish an understanding of adding fractions with unlike denominators using equivalence in a measure context.

Part-whole/measure sub-constructs tasks entailed pre-partitioned measurement instruments like scales, thermometers, measuring jugs/cylinders. Even though the instruments were pre-partitioned, the unit fraction had to be determined in order to complete the task. This required use of the part-whole and measure sub-construct with the unifying element of partitioning. The integration of these two sub-constructs through partitioning provided a connection to understanding fraction as a measure. As noted in the literature, when the whole is divided into equal parts to determine the unit fraction which is used to measure out, it creates an opportunity for understanding measure (Naik and Subramaniam, 2008; Lamon, 1996).

Operator and measure sub-constructs tasks made no reference to the unifying elements. Questions involved following a procedure, for example: Use the conversion table to complete: ______litre = $\frac{1}{1000}$ of 1 kilolitre or how many grams are there in $\frac{1}{4}$ of a kilogram or measure the given line and write the measurements in: (a) cm, (b) mm and (c) cm, using decimals or tasks required calculating the percentage of an amount using a procedure relating to the operator sub-construct, for example: Calculate 10% of R100.00. The given procedure provided no connection to understanding how the percentage was operating on the amount and therefore was categorises as lower demand.

The tasks were straightforward with no connections to any of the sub-constructs or unifying elements to develop an understanding of the concepts of fractions as an operator or measure.

Rate and ratio tasks were introduced by providing an explanation of each term. This was followed by examples of how rate and ratio are written. The questions were straightforward and required following the examples of the symbolic representations of rate and ratio. The nature of the ratio tasks was like the Grade 5 tasks, except that these tasks also included part-part ratios. While these ratio tasks included both part-part and part-whole ratios, they did not make a connection to the part-whole and ratio sub-construct to show that a ratio can also be written as a fraction. These tasks were categorised as lower cognitive demand.

Two tasks were extremely difficult to code. They dealt with enlarging and reducing 2-D shapes. Different shapes were presented on squared paper and an example of one of the questions were as follows, ‘Reduce the size of each shape by making its sides half as long as the given shape’. While this can be solved by counting the blocks on each side of the shape and halving them accordingly, it also introduces the idea of an operator. The operator sub-construct is very different from the part-whole and quotient sub-construct in that it is the comparison between the quantity resulting from an operation (in this case, the reduced number of blocks on each side of the shape) and the quantity that is acted upon (the initial number of blocks). The operator defines the relationship, and, in this case, the operator is a half (Lamon, 2012). Because this task required the sides to be operated on by a half, I code it as an operator sub-construct. I coded it as a lower cognitive demand because although it does provide for a connection between the operator and what needs to be operated on, it can also be done by just counting the blocks and halving it without much thought of what is happening mathematically

4.4 Overall Patterns

The overall pattern for the fraction chapters across the Grades reveal that there is a preoccupation with the understanding of fractions as part of a whole. The majority of the tasks across the Grades comprised of the part-whole sub-construct with pre-partitioned area or linear models. Grade 4 tasks focused only on the part-whole sub-construct and part-whole/quotient sub-constructs, while the Grades 5 and 6 tasks focused on a wider range of sub-constructs. They included the part-whole, operator, measure, part-whole/operator, part-whole/measure, part-whole/quotient. The quotient/operator sub-constructs were the only other linking sub-constructs that appeared in the Grade 5 tasks. The solely operator tasks were procedural in nature and categorised as lower cognitive demand while the solely measure sub-construct tasks were used to introduce number line representations of fractions, though largely in lower cognitive demand tasks. Some tasks that focused only on the part-whole sub-construct included working with the unifying elements of partitioning and unitizing and were considered as higher cognitive demand. Tasks with linking sub-constructs through the unifying elements were categorised as higher cognitive demand. The tasks where no sub-constructs could be assigned were categorised as lower cognitive demand because the tasks were not accompanied by any representations and took the form of pure calculations. These tasks were found across the fraction and fraction related chapters and were categorised accordingly.

The decimal fraction chapters for Grades 4 and 5 show a very similar pattern. The tasks focused only on the part-whole and only on the measure sub-constructs. A few tasks in the Grade 5 textbook focused on the part-whole/measure sub-constructs. These tasks included the work of partitioning and/or unitizing.

The fraction related chapters reveal a completely different pattern. The solely part-whole sub-construct hardly feature in these tasks. Grade 5 and 6 tasks focus a lot more on the measure and operator sub-constructs. The Grade 4 tasks include the part-whole sub-construct and the part-whole/measure sub-constructs. The part-whole/measure sub-constructs are also seen across the Grade 5 and 6 tasks. The rate and ratio sub-construct appeared in both the Grade 5 and 6 tasks but not in the Grade 4 tasks. The part-whole/operator tasks were only found in the Grade 6 textbook. Majority of these tasks were categorised as lower cognitive demand, except for those with linking sub-constructs.

4.5 Summary

In this chapter I explained how three textbooks across the Intermediate Phase were analysed according to task demands. I did this by providing tabulated results of the tasks according to the sub-constructs and cognitive demands. I also provided examples of how the tasks were coded and analysed. The next chapter deals with the analysis of the lessons carried out by three different teachers across the Intermediate Phase. The lessons comprise of fraction and fraction related topics and are detailed in the following chapter.

CHAPTER FIVE

ANALYSIS OF ENACTED TASKS

5.1 Introduction

In this chapter I begin with a brief description of the lessons taught in Grades 4, 5 and 6. The lessons include work done from fractions chapters and fraction related chapters. As explained previously, fraction chapters were chapters in the textbooks that focused directly on fraction concepts, while fraction related chapters were chapters that foregrounded other mathematical concepts with fraction concepts appearing in the background. Common fractions and decimal fractions were separate chapters in the Grades 5 and 6 textbooks but not in the Grade 4 textbook. I focus on the occurrence of the range of sub-constructs, unifying elements and the task demands identified in the lessons for Grades 4 to 6. This allowed for an analysis of what the teacher made available to the learners through teaching. In other words, the focus is on what is made available to learners in terms of the sub-constructs, unifying elements and the task demands.

Table 5.1 below shows the fraction and fraction related teaching arranged in chronological order for each Grade. The measurement section was moved to later in the year by both the Grade 5 and 6 teachers. After the first measurement lesson, the teachers expressed that they felt that the learners were not yet ready for the measurement section and decided to rearrange their planning and teach decimal fractions before measurement. Details of this shift are also given in Table 5.1.

Table 5.1: Overview of Lessons

GRADE	TOPIC	NUMBER OF LESSONS	DATES
4	Time	1	3 May
	Measurement	1	23 July
	Fractions	6	10, 11, 12, 15, 16, 18 October
5	Time	1	13 March
	Measurement		No recording – interview (28 November) but corrections on the 6 August
	Fractions	5	23 April – 3 May (23, 24, 25 April, 2, 3 May)
	Decimal fractions	5	14 May – 22 May (14, 16, 18, 21, 22 May)
	Ratio	1	6 August
	Rate	1	7 August
6	Time	1	23 April
	Measurement		Field notes, no recording- moved to 4th term
	Rate	1	7 June
	Ratio	1	19 July
	Fractions	11	6, 7, 8, 16, 17, 21, 22, 23, 24, 29, 30 August
	Decimal Fractions	9	22, 23, 24, 25, 26, 29, 30, 31 October, 1, November
	Money – percentages	4	6, 8, 9 November
	Measurement	1	12 November

Grade 4

The Grade four teacher (Mrs. B) taught a total of eight lessons. Six of the lessons were from the fraction chapter and the remaining two were from three fraction related chapters. I observed all eight recorded lessons taught to one of the four Grade 4 classes. The duration of the lessons were approximately 60 minutes long. Recording occurred during the months of May, July, August, and October. The focus of this study is on all the lessons. During the 8 lessons, Mrs. B worked on fair share, fraction notation, equivalent fractions, adding fractions with the same denominator, partitioning a whole, finding a fraction of a number, decimal fractions, time, and measurement.

Grade 5

The Grade five teacher (Mr. Z) taught a total of thirteen lessons. Five lessons where from the Common Fractions chapter and five lessons from the Decimal Fractions chapter. The remaining three lessons were from three different fraction related chapters: Time, Measurement and Whole-Numbers. I observed all thirteen recorded lessons taught to one of the four Grade 5 classes. The lessons were approximately 60 minutes long. Recording occurred during the months of March, April, May, and August. Mr. Z worked on fair share, fraction notation, equivalent fractions, adding fractions with the same denominator, partitioning a whole, finding a fraction of a number, decimal fractions, time, measurement and comparing quantities.

Grade 6

The Grade six teacher (Mrs. W) taught a total of twenty-eight lessons. Twenty of the lessons were from two fraction chapters and the remaining eight were from three fraction related chapters. I observed all twenty- eight recorded lessons taught to one of the four Grade 6 classes. The lessons were approximately 60 minutes long. Recording occurred during the months of April, June, July, August, October, and November. The lessons in Grade 6 focused on types of fractions, comparing fractions, equivalent fractions, simplifying fractions, fractions of quantity, converting fractions, adding, and subtracting fractions with different denominators, decimal fractions, percentages, rate, ratio, money, length, mass, and capacity.

5.2 Coding of Enacted Tasks

As mentioned in chapter 2 and 3, each lesson was chunked according to episodes, marked by the announcement of the beginning of a new mathematical task, and forming a unit of analysis. This meant that when a teacher presented a set of similar examples and talked about them in similar ways, it was counted as a single task. I included detail on the time intervals of episodes in my lesson summaries as this fed into the investigation of balance of emphasis across the sub-constructs.

As with the textbook analysis, the framework allowed for an analysis of the sub-construct/s and unifying element/s and the connections between them to develop understanding of fractions. These connections aided in categorising the enacted tasks as lower or higher cognitive demand.

In this section, I present a summary table for the outcomes of this coding for the fraction teaching, and then the set of fraction related teaching, for each Grade. Each table is followed by a commentary that highlights the key patterns seen in the results and where useful, examples that illuminate the patterns or points made in the preceding analysis are included. This Grade-by-Grade commentary for the common fraction, decimal fraction and then the fraction-related teaching is followed by a concluding analysis that considers the overall patterns of presentation of fraction concepts across the three Grades.

5.2.1 Grade 4: Fraction Enacted Tasks

Table 5.2: Grade 4: Fraction Enacted Tasks Summary

			PW	**O**	**M**	**Q**	**R&R**	**PW & O**	**PW & M**	**PW & Q**	**PW & R&R**	**O &M**	**O & Q**	**0 & R&R**	**M & Q**	**M & R&R**	**Q & R&R**	**Lower Cognitive Demand**	**Higher Cognitive Demand**
Lesson	Episode	Task																	
1	1	fair share- 2 tasks				2/2													2/2
	2	fair share - ex 12.1 – 1 task	1/1															1/1	
	3	halves, quarters, and eighths- paper folding- notation- 2 tasks	2/2																2/2
	4	fraction notation- worksheet (textbook)- 1 task	1/1															1/1	
	5	worksheet on wholes (from textbook)- 1 task	1/1															1/1	
	6	notation - numerator & denominator- 2 tasks	2/2															2/2	
	7	worksheets on pre-partitioned wholes (from textbook)- 2 tasks	2/2															2/2	
Lesson 2	1	recapping fraction notation and working with wholes- 1 task	1/1															1/1	
	2	Classroom Maths (ex 12.9)- 1 task	1/1															1/1	
	3	learners complete previous day’s worksheets on part-whole- 1 task	1/1															1/1	
	4	Classroom Maths (ex12. 10)- 1 task	1/1															1/1	

			PW	O	M	Q	R&R	PW & O	PW & M	PW & Q	PW & R&R	O &M	O & Q	0 & R&R	M & Q	M & R&R	Q & R&R	Lower Cognitive Demand	Higher Cognitive Demand
Lesson 3	1	using a fraction wall- teacher explains equivalent fractions, Ex 12.11- 3 tasks	3/3															1/3	2/3
	2	using a fraction wall- learners complete 2 worksheets-1 task	1/1															1/1	
	3	using a fraction wall learners make up their own equivalent fractions- 1task	1/1															1/1	
Lesson 4	1	adding fractions using a fraction wall - like denominators & types of fractions- 3 tasks	3/3															1/3	2/3
	2	adding like fractions- Classroom Maths ex 12.15- 2 tasks	2/2															2/2	
	3	word problems related to fractions- Classroom Maths ex 12.13 - 1 task								1/1									1/1
	4	word problems related to fractions- Classroom Maths ex 12.14- 1 task								1/1									1/1
Lesson 5	1	Dividing shapes & numbers-Classroom Maths- ex 12.16- 2 tasks	2/2																2/2
	2	Classroom Maths- ex 12.17 and board work (adding & subtracting fractions with like denominators) -2	1/2							1/2									2/2

			PW	O	M	Q	R&R	PW & O	PW & M	PW & Q	PW & R&R	O &M	O & Q	0 & R&R	M & Q	M & R&R	Q & R&R	Lower Cognitive Demand	Higher Cognitive Demand
		tasks																	
	3	Marking exercise 12.17 & board work (adding & subtracting fractions)- 1task	1/1																1/1
Lesson 6	1	Finding a fraction of a number-1 task						1/1											1/1
	2	Learners do problems from board – 1 task		1/1														1/1	
	3	Mark previous examples-.1 task		1/1														1/1	
	4	Decimal fractions -1 task			1/1													1/1	

Some important overview patterns are seen through the analysis of the teaching in Grade 4. Firstly, of the 36 tasks from the six lessons, 21 were categorised as lower cognitive demand, while the remaining 15 where categorised as higher cognitive demand. 28 of the 36 tasks focused solely on the part-whole sub-construct, 1 focused solely on the quotient sub-construct, 2 focused solely on the operator sub-construct, 3 tasks included the part-whole/quotient sub-constructs, 1 included the part-whole/operator sub-constructs, and 1 involved only the measure sub-construct.

While only 4 of the 36 tasks connected between sub-constructs, the remaining 4 higher cognitive demand coded tasks incorporated attention to one or more unifying elements and involved explanations and questioning by the teacher. During these explanations of concepts, the teacher often referred to the part-whole sub-construct with partitioning to aid learners' understanding of the concepts. The following example, from Lesson 1, illustrates the work done by the teacher using the part-whole sub-construct and partitioning unifying element. The lesson involved working with halves, quarters, and eights. In the first episode the teacher introduced the lesson by demonstrating the meaning of fair shares. She divided a chocolate into different parts, first unequally and then equally while explaining and discussing that fair share meant "everyone gets an equal piece" of the chocolate. This is a pre-cursor to the quotient sub-construct and the episode was categorised as higher-cognitive demand since she used partitioning to demonstrate a fair share. During the second episode the children completed Exercise 12 .1 from the textbook and this textbook task was categorised as lower cognitive demand because all the diagrams were pre-partitioned and did not include any work with unifying elements. The next episode in the same lesson required learners to partition a whole into halves, quarters, and eights. This task was done collectively as a group. The teacher demonstrated each step and then allowed the learners to mimic her actions. While demonstrating the partitioning she explained how halves, quarters and eighths are obtained. Figure 5.1 below shows the partitioning work done by the teacher.

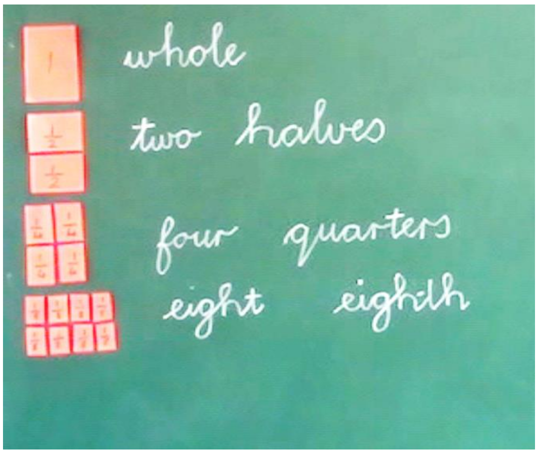


Figure 5.1: Partitioning work done by teacher to explain halves, quarters, and eights

An interesting observation during the lessons was that while the teacher used the part-whole sub-construct with partitioning in her explanations and discussions with the learners, the tasks that she provided for the learners to complete independently did not include work with partitioning. This pattern is seen across all the Grade 4 lessons. The episodes that were categorised as higher cognitive demand were all focused on work done by the teacher, that is, her explanations, demonstrations and questioning included a sub-construct and unifying element, but the follow-up tasks provided for the learners were mostly categorised as lower cognitive demand because they often included pre-partitioned geometric shapes or following a procedure to get to an answer. During lesson 6, the teacher provided an in dept explanation of how to find a fraction of a quantity. She drew an 8x8 grid on the board and got the learners to do the same in their books. The learners on their own, then partitioned the grid into four equal parts to establish what is a quarter of 64. This was followed by a discussion of how the grid could be partitioned in different ways. The partitioning and unitizing of a continuous whole by the learners led me to categorise this task as higher cognitive demand. The next set of examples provided by the teacher involved work with discrete objects. The teacher drew 16 circles on the board and explained that she had 16 smarties and wanted to share them among 4 children. She asked how many smarties each child would get. She wrote $\frac{1}{4}$ of 16 = on the board. In this teaching, she used a quotient related question with an operator sub-construct and partitioning to arrive at the answer. She explained that 16 must be divided into 4 equal groups, and she grouped 4 smarties and explained that a $\frac{1}{4}$ of 16 is 4. With another example on the board: $\frac{1}{3}$ of 12, the teacher drew 12 circles with learners copying this into their books. The teacher explained that "3 times 4 equal 12 so we going to make groups of 4 (she grouped 4 circles) in order to give us 3 groups there (points to the 3 in the denominator)". She then provided 2 more examples: $\frac{3}{4}$ of 8 and $\frac{3}{5}$ of 15, thereby moving to non-unitary fractions. These examples were also worked through using circles made into the number of groups denoted by the denominator, with the children copying these images into their books. Teacher talk alongside these images stressed the procedure of taking the whole numbers and dividing it by the denominator and multiplying it by the numerator. The following examples were provided for the learners to complete:

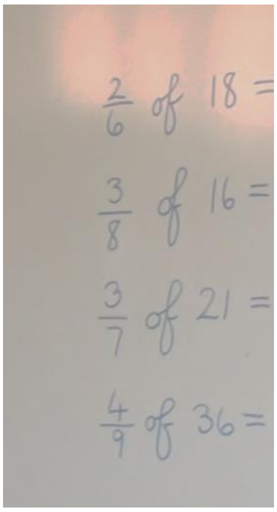


Figure 5.2: Task provided by the teacher.

Of interest was the observation that as children worked through the examples, they did not make use of diagrams. Instead, they followed the procedure provided by the teacher through the previous examples. In her working with individual children, the teacher also only used the procedure without any reference to diagrammatic representations.

Therefore, while work with operator sub-construct problems was introduced with images showing the partitioning, the teacher moved swiftly to learner working involving only the numerical procedure, without any accompanying moves between representations. In her marking of task, shown in Figure 5.3 below, this emphasis on the numerical procedure was again emphasized. In this instructional sequence then, the cognitive demand for learners is reduced through the lack of requirement for learners to connect between representations.

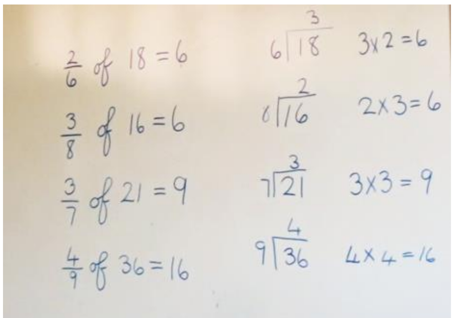


Figure 5.3: Task showing the work done by the teacher

While it could be the case that children had already appropriated the diagrammatic forms in their earlier work, South African evidence of children's difficulties with rational number (ANA, 2012; ANA, 2013; ANA, 2014), suggests that it may have been useful to integrate some openings for moves between representations in at least some of the working that children were asked to do. In the coding framework developed for this study, what was opened up for learners in this task was therefore coded as lower cognitive demand, in spite of the teacher introduction that did include some connecting between representations.

5.2.2 Grade 4: Fraction Related Enacted Tasks

Table 5.3: Grade 4: Fraction Related Enacted Tasks Summary

			PW	**O**	**M**	**Q**	**R&R**	**PW & O**	**PW & M**	**PW & Q**	**PW & R&R**	**O&M**	**O & Q**	**0 & R&R**	**M & Q**	**M & R&R**	**Q & R&R**	**Lower Cognitive Demand**	**Higher Cognitive Demand**
	Episode	Task																	
Lesson 1 – Time	1	analogue and digital time, s, min, hr.-introduction- 1 task			1/1													1/1	
	2	analogue time- making an analogue clock- 1 task	1/1																1/1
	3	working with minutes on the analogue clock face (past and to) -1 task			1/1													1/1	
Lesson 2- Measure ment	1	Measurement instruments- 1 task			1/1													1/1	
	2	Units of measurement - cm & m – 1 task							1/1										1/1
	3	mm, cm, m & km – 1 task							1/1										1/1
	4	Using a ruler to measure accurately- 1 task			1/1													1/1	
	5	Measuring using cm – 1 task							1/1										1/1
	6	Using a ruler to measure-cm – 1 task			1/1													1/1	
	7	Estimating lengths – 1 task			1/1													1/1	

The fraction related lessons comprised of the topics Time and Measurement with 10 tasks altogether. Six of the 10 tasks were categorised as lower cognitive demand and 4 were categorised as higher cognitive demand. Of the 10 tasks, 6 focused solely on the measure sub-construct, 1 solely on the part-whole sub-construct, and 3 on the part-whole/measure sub-constructs.

The part-whole sub-construct task, in the Time lesson, included work with partitioning. It took place during the second episode of the lesson. The teacher referred to a clock on the wall and led a discussion about the different hands on the clock and how long it took each of the hands to move around the face of the clock. She then provided each learner with a pre-partitioned paper clock as shown in figure 5.4.

Figure 5.4: Pre-partitioned clock provided by teacher

The following excerpt from the lesson details the partitioning work done by the teacher and learners.

T: Oh, well done. Right. What I want you to do now is I want you to take your clock like this and I want you to fold it between the 12 and the 6.

L1: So, kind of like a?

T: *[demonstrates by folding clock it in half] Like* that.

L2: Like a pizza

L3: Folding it in half, Ma'am.

T: Exactly. Well done. What have we done? We've folded our clock in.

Class: Half

T: So that is why when the long hand gets down to the 6, we say it's half past, because it's gone halfway round the clock. Right. So, what I want you to do now is I want you to colour in half of your clock.

Here the teacher demonstrated how to partition the face of the clock and then required the learners to do the same. The face of the clock was partitioned into half and later in the lesson into quarters.

T: From the 9, where you've folded your line now, from the 9 to the 3. Right. We've divided our clock in… Fold it across and now we've divided our clock in?

Class: Quarters

T: Quarters. We've got how many quarters? Cara (L57)?

L4: 4

T: 4 quarters. And we've coloured in how much of our clock?

L5: Half

T: Half. Right. Where the long hand is on the 6 what do we say it is?

Class: Half past

T: Half past. When the long hand is on the 3, what do we say it is?

Class: Quarter past

T: And when the long hand is on the 9 what do we have?

Class: Quarter to

While the teacher's focus here is on partitioning the face of the clock rather than partitioning the hour, the inclusion of work on partitioning in the context of the part-whole sub-construct led to this task being categorised as higher cognitive demand. There are missed opportunities for connections here to the measure and operator sub-constructs through asking questions such as: how many minutes does a quarter represent or what fraction of an hour equals 15 minutes? This 'separate' dealing with the part-whole sub-construct reflects both South African literature on limited connections (Venkat & Adler, 2012) and the international literature base on fractions teaching (Wilkins and Norton, 2018; Steffe and Olive, 2010; Charalambous & Pitta-Pantazi. 2007).

In contrast though, the part-whole/measure sub-constructs tasks in the Measurement topic involved work with partitioning, the part-whole and measure sub-constructs. The following is an example of part-whole/measure sub-constructs task that was categorised as higher cognitive demand. The lesson began with a discussion of how objects were measured before units of measurements were invented. Learners were given the opportunity to measure different objects with their hands, feet, fingers, cubits etc. Each measurement was discussed, and this discussion led to the need for different units of measurement (mm, cm, m, and km) when measuring. The following excerpt from the lesson shows a part-whole/measure sub-constructs task. The excerpt is from the second episode of the lesson.

T: *[Cuts a piece of string that is 1 meter long].* And who was this (points to the board)? I think it was Joshua and Nathan, they said about half a cubit. But they didn't know that it was exactly half a cubit because they couldn't measure their arm to see that it was exactly half a cubit. But nowadays we can take a meter string (holds up the string for the class to see) and break it in half and if I cut it (she cuts it in half) what do you think the length of this…. (learners shout out the answer before she can complete sentence)

Class: Half a meter!

T: Half a meter. So, who thinks they know what half a meter is?

L1: 500 cm.

T: 500 cm?

L1: I mean 50cm.

T: Pardon?

L1: 50 cm

T: 50 cm, how many mm?

Class: 500

T: How do you know that (points to the learner who has been answering the questions)?

L1: Because a cm is 10mm, um is, is it 10 or 5mm? I get confused. It is 10mm

T: It is 10 and how long do you know a meter is?

L: 1 000 mm

T: A 1000 mm so what is a half of a thousand mm?

L: 500 mm.

T: So, if you were measuring a half there (points to the board), you would know that it is exactly 500 mm but when you were measuring in cubits there (points to the cupboard) you didn't know that it was exactly half a cubit.

From the excerpt we can identify 2 different sub-constructs and a unifying element. The teacher took the unit, in this case a meter, and physically partitioned it into halves. The unit was now partitioned into two equal parts. Assuming some learners knew that a meter is equivalent to 100 cm she asked what half of the unit would be equivalent to. Once this was established, she continued to 'successively partition' the unit by asking how many millimetres in a meter and then how many millimetres in half a meter. As the lesson progressed, she did the same for a quarter of a meter and when working with other units of length. Here we see that the part-whole and measure sub-construct came into play through the unifying element of partitioning. This created a connection between the unit and how much it measured, and that the unit could be successively partitioned and used to measure any amount. For example, when meters do not work, we can partition into centimetres, when centimetres do not work, we can partition into millimetres, and so on. Because of the connection between the part-whole and the measure sub-constructs through the unifying element of partitioning, this task was categorised as higher cognitive demand.

The measure sub-construct tasks, in both the Time and Measurement sections, included tasks with pre-partitioned clocks and rulers. The teacher provided the learners with pre-partitioned measuring instruments, a clock, and a ruler, to either tell or show the time on a clock or measure the lengths of different objects. There were no physical acts of partitioning when working with these measurement tasks. They required reading of a pre-partitioned measuring instrument and therefore were categorised as lower cognitive demand. As in the fraction chapter, the tasks provided for the learners as consolidation did not include work with any of the unifying elements.

When comparing the textbook and enacted tasks with regards to the fractions related chapters a similar pattern was observed as that for the fraction chapter. The part-whole sub-construct formed a huge chunk of the higher demand tasks. Of the 10 tasks in the textbook 8 were higher demand tasks, focusing on the part-whole/measure sub-constructs, and 2 were lower demand tasks, focusing solely on the part-whole sub-construct, compared to the 4 higher demand and 6 lower demand enacted tasks. The enacted lower demand tasks focused mainly on a single sub-construct (measure-6 tasks), while the higher demand tasks focused on the part-whole/measure (3 tasks) sub-construct and the part-whole (1 tasks) sub-construct with partitioning.

A comparison between the textbook and enacted tasks revealed 31 higher demand tasks in the textbook compared to the 15 enacted tasks. The significantly lower number of higher demand enacted tasks was due to the focus on single sub-constructs (quotient-2 tasks, part-whole-19 tasks, operator-1 task, and measure-1 task) with no connections to the unifying elements. When connections appeared, they included the part-whole sub-construct with another sub-construct (quotient-3 tasks or operator-1 task) or the part-whole (12 tasks) sub-construct with partitioning. The connections that appeared in the textbook tasks always included the part-whole/quotient (20 tasks) sub-constructs or only the part-whole (10 tasks) sub-construct with partitioning and/or unitizing. Higher demand tasks in both the textbook and enacted tasks always included the part-whole sub-construct.

5.2.3 Grade 5: Fraction Enacted Tasks

Table 5.4: Grade 5: Fraction Enacted Tasks Summary

			PW	O	M	Q	R&R	PW & O	PW & M	PW & Q	PW & R&R	O&M	O & Q	0 & R&R	M & Q	M & R&R	Q & R&R	Lower Cognitive Demand	Higher Cognitive Demand
	Episode	Task																	
Lesson 1	1	What is a fraction? – 1 task	1/1																1/1
	2	Worksheet-part-whole- 1 task	1/1															1/1	
	3	Comparing fractions-1 task	1/1																1/1
	4	Comparing fractions-worksheet-1 task	1/1															1/1	
Lesson 2	1	Marking previous day's tasks-part-whole-1 task	1/1															1/1	
	2	Marking comparing fractions -1 task																1/1	
	3	Types of fractions-explanation- 1 task																1/1	
	4	Mixed numbers & improper fractions-explanation- 1 task																1/1	
	5	Converting mixed numbers and improper fractions-explanation- 1 task																1/1	
	6	Worksheet- mixed numbers & improper fractions- 1 task																1/1	
Lesson 3	1	Marking previous day's tasks – 1task																1/1	
	2	Equivalent fractions -explanation- 1 task																1/1	
	3	Worksheet - equivalent fractions-																1/1	

			PW	O	M	Q	R&R	PW & O	PW & M	PW & Q	PW & R&R	O&M	O & Q	0 & R&R	M & Q	M & R&R	Q & R&R	Lower Cognitive Demand	Higher Cognitive Demand
		1 task																	
Lesson 4	1	Mark previous day's task- 1 task.																1/1	
	2	Adding & subtracting fractions- 1 task																1/1	
Lesson 5	1	Marking previous day's task- 1 task																1/1	
	2	Revision of fraction work completed- 1 task																1/1	
	3	Finding a fraction of a whole- 1 task		1/1														1/1	

Some important overview patterns were seen through the analysis of the teaching in Grade 5. Firstly, 16 of the 18 enacted tasks from the five lessons were categorised as lower-cognitive demand, while the remaining 2 were categorised as higher cognitive demand. 5 tasks focused solely on the part-whole sub-construct, 1 focused solely on the operator sub-construct, and the remaining 12 did not included any sub-constructs. Of the 5 part-whole tasks, 2 involved work with a unifying element.

The following example from lesson 1 exemplifies work done by the teacher with the part-whole sub-construct and partitioning unifying element. The lesson began with the teacher drawing a circle on the board and together with the learners, he partitioned the circle into 12 equal parts. During this process of partitioning, they named each part as they partitioned the circle as follows:

T: You are going to draw a line going down that will cross through the centre (places the ruler in position to draw a line to divide the circle in half). From top to bottom, right.

Class: Yes sir.

T: And it must go through the centre like that (draws a line from one side of the circle to the other side passing through the centre). I have divided that circle into… (learners answer before he can finish.

Class: half.

T: Into 2 equal parts. What do we call each part?

L1: Half.

T: (writes $\frac{1}{2}$ on the top and below it he writes $\frac{1}{2} + \frac{1}{2} = 1$)

He continued partitioning the circle into quarters and naming each part as shown in figure 5.5.

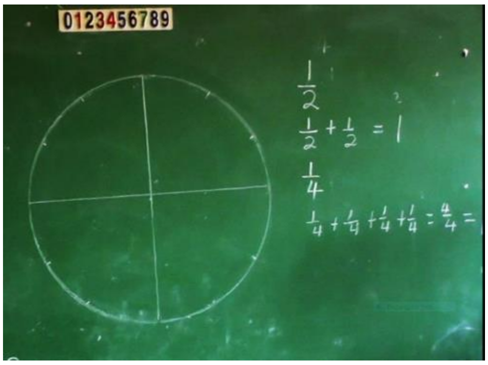


Figure 5.5: Partitioning a circle

The partitioning continued until there were 12 equal parts. The teacher emphasised that each part is called a fraction and then moved into a discussion and definition of a fraction. The discussion revolved around fraction notation i.e., the denominator and numerator with reference to the part-whole diagram on the board and the teacher concluded that, “a fraction is piece (part, segment) of a whole”. This task was categorised as higher cognitive demand because the teacher used the part-whole sub-construct with partitioning to explain the meaning of fraction in part-whole terms. He also made a connection between his diagrammatic representation of a fraction and the written notation.

The next episode in the same lesson involved the completion of a task by the learners. This task was used to consolidate what was discussed in the first episode. The task was categorised as lower cognitive demand since it included only pre-partitioned part-whole area models. None of the questions required partitioning of the area models as modelled by the teacher earlier. Figure 5.6 shows the questions from this task.

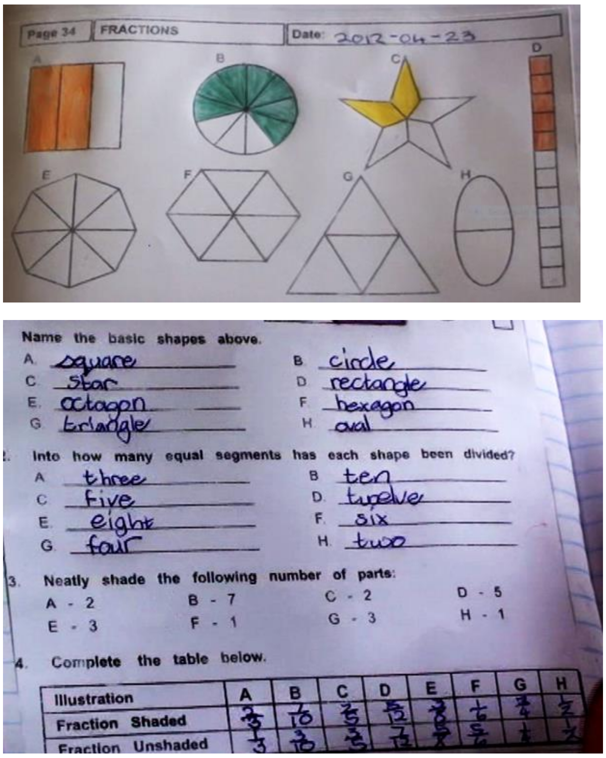


Figure 5.6: Part-whole fraction task

Similar questions followed through 4 of the part-whole sub-construct tasks in the Grade 5 lessons.

During the third episode of the same lesson the teacher explained the comparison of fractions using equivalence. He used the part-whole area model already drawn on the board to explain that $\frac{1}{2} = \frac{6}{12}$ and $\frac{1}{4} = \frac{3}{12}$. When working with the part-whole area model the teacher used unitizing to show these equivalent fractions. By using this visual representation and naming each fraction (see figure 5.7 below), the learners were able to agree that the fractions represented by the diagram and the symbolic statements were equivalent. This task was the only other task categorised as higher cognitive demand because of the presence of the part-whole sub-construct with partitioning and unitizing.

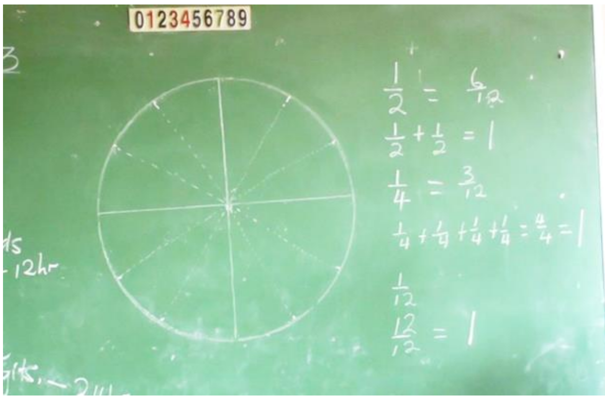


Figure 5.7: Part-whole area model with unitizing to show equivalent fractions.

Following the above explanation of comparing fractions the teacher offered the learners an alternative way to determine equivalent fractions. The following excerpt explains the procedure suggested by the teacher.

T: There is another method that I want to teach you. I want you to listen okay. Another method that you can use (teacher reprimands a learner then continues). Alright, besides using that diagram (points to the diagram) to compare the fractions what you can do is, you can take one over four and six over twelve (write $\frac{1}{4}$ $\frac{6}{12}$ on the board) and all you need to do is cross multiply. Okay, you cross multiply, meaning this twelve times that 1 (draws an arrow from the second fraction's denominator-12 to the first fraction's numerator 1). What is twelve times one?

Class: Twelve

T: Let me write it here (writes 12 above the 1)

The teacher then followed the same procedure with the other numerator and denominator. Noting that 12 was smaller than 24 he stated that this meant that $\frac{1}{4}$ was smaller than $\frac{6}{12}$. He completed one more example using this procedure but did not explain why this always works. A similar task appeared in the textbook for comparing fractions, but an alternative method was offered. The textbook provided a pre-partitioned fraction wall and the task required working with the fraction wall to determine the size of each fraction and then drawing a comparison between them. There was no procedure involved when comparing the different fractions. The final episode of this lesson focused on learners completing a task to consolidate their discussion on comparing fractions. This task included a pre-partitioned fraction wall (partitioned from halves to twelfths), very similar to the textbook task and method mentioned above, used to compare two different fractions. For example: $\frac{2}{8}$ and $\frac{3}{10}$ etc. While the fraction wall was made available to learners, they were encouraged to use the procedure explained by the teacher. This task exemplified the 'pure' calculation that I noted in Chapter 2; there was no reference at all to the notion of fraction, with the four 'digits' in the problem worked with separately as entities for calculation. Such tasks were categorised as lower cognitive demand because they did not include any work with unifying elements and a pre-partitioned fraction wall was provided to solve the problems. As in the Grade 4 analyses, the tasks used by the teacher to explain fraction concepts included the unifying elements, but the tasks provided for the learners to complete required no engagement with the unifying elements, and therefore limited openings for independent engagement with the meaning and understanding of fraction concepts.

The tasks coded as not including any fraction sub-constructs were completely procedural in nature. For example, when converting mixed number to improper fractions, learners were taught the procedure of multiplying the whole number by the denominator and then adding the numerator. No connections were made to create understanding and meaning of mixed numbers and improper fractions through the sub-constructs and unifying elements. The same was done when adding and subtracting fractions, simplifying fractions etc. These tasks were categorised as lower cognitive demand because of their procedural nature and absence of reference to any of the sub-constructs. The tasks used to teach fraction conversions between improper fractions and mixed numbers, fraction addition/subtraction, and simplifying were all focused on pure calculation in these ways. Figure 5.8 is an example of the nature of the tasks provided by the teacher.

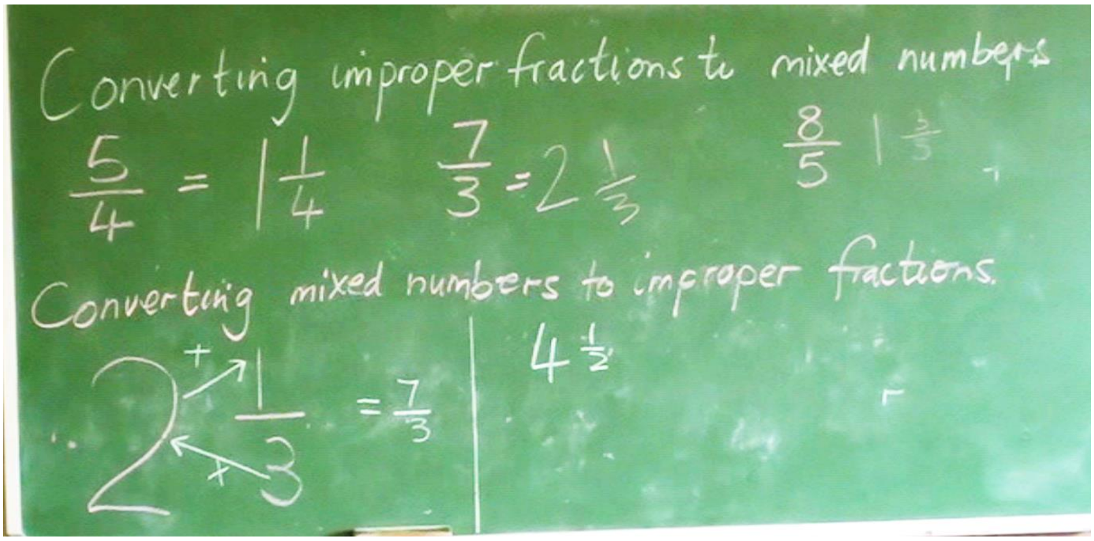


Figure 5.8: Example to tasks provided by the teacher

The operator task in this set was dealt with in similar ways: the teacher's explanation included neither reference to sub-constructs nor to any unifying elements and was therefore categorised as lower cognitive demand. By way of example, for the task: What is a $\frac{1}{4}$ of 24? The teacher's explanation involved telling children to divide 24 by 4 (denominator) and then multiplying the answer by 1 (numerator) as was done in the Grade 4 teacher's explanation.

A comparison between the textbook and enacted tasks revealed 34 higher cognitive demand tasks in the textbook compared to the 2 enacted tasks. The significantly lower number of higher demand enacted tasks was because 12 of the 18 enacted tasks dealt with what has been described as pure calculations. These tasks were disconnected from any sub-constructs and contributed to the high degree of lower cognitive demand calculation tasks.

5.2.4 Grade 5: Decimal Fractions Enacted Tasks

Table 5.5: Grade 5: Decimal Fractions Enacted Tasks Summary

			PW	**O**	**M**	**Q**	**R&R**	**PW & O**	**PW & M**	**PW & Q**	**PW & R&R**	**O&M**	**O & Q**	**0 & R&R**	**M & Q**	**M & R&R**	**Q & R&R**	**Lower Cognitive Demand**	**Higher Cognitive Demand**
	Episode	Task																	
Lesson 1	1	Defining a decimal fraction-Place value and decimal notation-1 task	1/1															1/1	
	2	Writing tenths as decimals-Ex 10.1-task	1/1															1/1	
Lesson 2	1	Recapping place value & notation of a decimal fraction-1 task																1/1	
	2	Converting common fractions to decimal fractions and vice versa-explanation 1 task																1/1	
	3	Exercise 10.3-explanation -1 task	1/1																1/1
	4	Ex 10.2 &10.3 – 3 tasks	3/3															2/3	1/3
Lesson 3	1	Revising previous																1/1	

			PW	O	M	Q	R&R	PW & O	PW & M	PW & Q	PW & R&R	O&M	O & Q	0 & R&R	M & Q	M & R&R	Q & R&R	Lower Cognitive Demand	Higher Cognitive Demand
		lesson – converting- 1 task																	
	2	Decimals bigger than 1-converting to mixed numbers & vice versa- 1 task																1/1	
	3	Ex 10.4 – 3 tasks																3/3	
Lesson 4	1	Revising previous lesson – converting – 1 task																1/1	
	2	Writing measurement as decimals- explanation – 1 task			1/1													1/1	
	3	Ex 10.6 &10.7-7 tasks – 7 tasks			7/7													3/7	4/7
Lesson 5	1	Revising decimal notation & place value. – 1 task																1/1	
	2	Writing hundredths as fractions- explanation – 1 task	1/1															1/1	
	3	Ex 10.9- 1 task	1/1															1/1	

Of the 25 enacted decimal fraction tasks from the 5 lessons, 19 were categorised as lower cognitive demand and 6 were categorised as higher cognitive demand. 8 focused solely on the part-whole sub-construct, 8 focused solely on the measure sub-construct and the remaining 9 focused on pure calculations with no reference to any sub-construct.

As before, enacted tasks that included work with the part-whole sub-construct and partitioning (2 tasks) were categorised as higher cognitive demand and part-whole tasks that included pre-partitioned models (6 tasks) with no connections to the unifying elements were categorised as lower cognitive demand tasks. 2 higher cognitive demand tasks demonstrated work by the teacher that included drawing, partitioning, and unitizing a 10 by 10 square and using it to represent decimal fractions. These enacted tasks also included work with part-whole area models to demonstrate how to write decimal fractions as common fractions and vice versa. Figure 5.12 and 5.13 below show these different representations using the part-whole sub-construct with partitioning when working with decimal fractions.

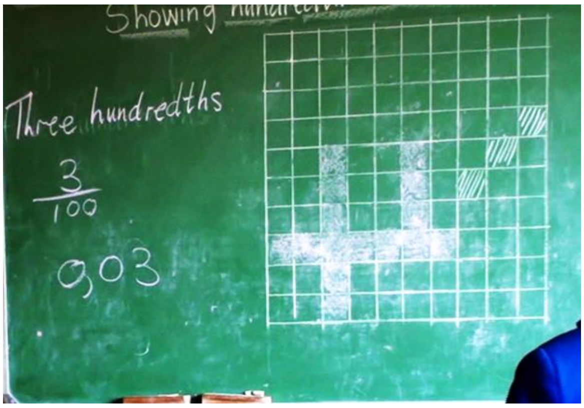


Figure 5.9: Part-whole sub-construct with partitioning to teach decimal fractions

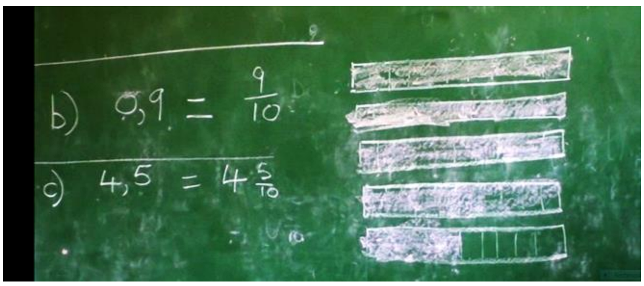


Figure 5.10: Part-whole sub-construct with partitioning used to convert decimals to common fractions

The 8 measure sub-construct tasks either included work with pre-partitioned number lines and/ or measuring instruments like rulers and meter sticks where the unit fraction was provided, or they included pre-partitioned number lines and/or measuring instruments where the unit fraction was implicit and needed to be determined to complete the tasks. 4 enacted tasks that included work with pre-partitioned given unit fractions were categorised as lower cognitive demand while 4 tasks that required finding the unit fraction were categorised as higher cognitive demand. Working with the partitioning to establish the unit fraction provided a connection to the relative size of each partition thus providing connections to the measure sub-construct.

The 10 enacted tasks that made no reference to any of the sub-constructs involved re-writing common fractions as decimal fractions, and vice versa, and simply 'reading off' values with the place value chart to state the value of a digit in a number, to write common fractions as decimal fractions, compare decimals etc. There was no reference to sub-constructs and unifying elements as shown in figure 5.11.

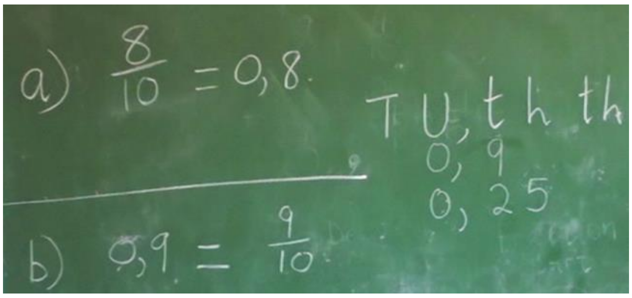


Figure 5.11: Work with place value in place of sub-constructs

A comparison between the textbook and enacted tasks revealed 14 higher demand tasks in the textbook compared to the 6 enacted tasks. The lower number of higher demand enacted tasks was due to the focus on single sub-constructs (part-whole-8 tasks, and measure-8 tasks) with no connections to the unifying elements. 9 of the enacted tasks focused on pure calculations with no reference to any sub-constructs. The connections that appeared in the textbook tasks always included the part-whole/measure (9 tasks) sub-constructs or only the part-whole (5 tasks) sub-construct with partitioning and/or unitizing.

5.2.5 Grade 5: Fraction Related Enacted Tasks

Table 5.6: Grade 5: Fraction Related Enacted Tasks Summary

			PW	**O**	**M**	**Q**	**R&R**	**PW & O**	**PW & M**	**PW & Q**	**PW & R&R**	**O&M**	**O & Q**	**0 & R&R**	**M & Q**	**M & R&R**	**Q & R&R**	**Lower Cognitive demand**	**Higher Cognitive demand**
	Episode	Task																	
Lesson 1 Time	1	Time- units of measurement- 1 task			1/1													1/1	
	2	weeks, months, years- 4 tasks		1/1	1/1			1/1	1/1									2/2	2/2
	3	Analogue time- 3 tasks	1/1																3/3
Lesson 2- Measure ment	1	Units & instruments of measurement- 1 task			1/1													1/1	
	2	Fraction of a quantity (cm, m, l & kl)- 1 task		1/1														1/1	
Lesson 3- Ratio	1	Understanding ratio- 2 tasks						1/1			1/1								2/2
	2	Worksheet on ratio- 1 task									1/1								1/1
Lesson 4- Rate	1	Mark previous task – 1 task									1/1								1/1
	2	Rate – 1 task					1/1											1/1	

Of the 13 enacted fraction related tasks from the 5 lessons 7 tasks were categorised as higher cognitive demand and 6 tasks as lower cognitive demand. 1 task focused solely on the part-whole sub-construct, 2 tasks focused solely on the operator sub-construct, 3 tasks focused solely on the measure sub-construct, 1 task focused solely on the rate and ratio sub-construct, 2 on the part-whole/operator sub-constructs, 1 on the part-whole/measure sub-constructs and 3 tasks on the part-whole/rate and ratio sub-constructs. In comparison to the Grade 5 fraction and decimal fraction chapters the fraction related chapter had more tasks that included two sub-constructs thus pushing up the prevalence of connections and higher cognitive demand.

Of interest in the Grade 5 fraction related chapter analysis is the combination of part -whole/rate and ratio sub-constructs and the part-whole/operator sub-constructs tasks. The rest of the tasks followed the same analysis of similar tasks in the previous sections where the tasks involved single sub-constructs (part-whole, operator, measure and rate and ratio) with no connections to unifying elements or a single sub-construct with partitioning.
The part-whole/operator sub-constructs were of interest here given that operator tasks in the fraction and decimal fraction chapter were dealt with from a pure calculation perspective

In the fractions related task, the teacher worked with a whole (year) and explained how a quarter operated on it to determine the number of months in one quarter of a year.

To do this, he first had to first establish what the whole looked like and then what a quarter of the whole looked like. He referred to the measure, part-whole and operator sub-constructs to help make sense of how many months there are in a quarter of a year. The excerpt is from the second episode of the Time lesson. The first episode dealt with the history of time and units used to measure time.

T: Okay, we've already done one week = seven days. One quarter, this is one quarter of a year. First of all, now, can someone, how do we write one quarter? Yes (points to a learners)

L: one over four.

T: So, one over four (while writing $\frac{1}{4}$ on the board another learner asks a question)

L: Sir, isn't it one over three? (another learner corrects him and tell him that is one third)

T: Thank you for asking that question because, it's (writes)$\frac{1}{4}$. What does that mean? One over four? Gen? (refers to a learner)

L: It means, it is a quarter.

T: A quarter of?

L: Uh, of a year.

T: A quarter of a year (writes 'a quarter of a year' on the board). Right, we know that in a year there are how many months?

L: Twelve.

From the above excerpt we observe that the teacher tried to establish the meaning of a $\frac{1}{4}$ and what the whole was that needed to be operated on. This was done by explicitly getting the learners to state that a year is equal to twelve months. The next excerpt shows how he attempted to explain a quarter of a year in months and in weeks.

T: Twelve months. It is one quarter of twelve months (he writes ' $\frac{1}{4}$ of 12 months' on the board) Phil, you not paying attention. Right, one quarter of twelve months. Okay, now how do we get one quarter of twelve months? How are we going to calculate one quarter of twelve months? Yes? (refers to a learners)

L: You divide twelve by three…. Four.

T: You divide, okay, twelve divided by four (writes '12 ÷ 4 = 3 months'), which is going to give us three months, okay.

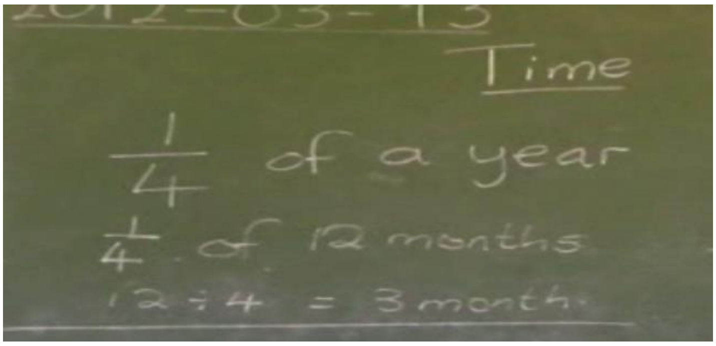


Figure 5.12: Part-whole/operator sub-construct to establish a $\frac{1}{4}$ of a year

The teacher continued working with the operator as a $\frac{1}{4}$ but changed the whole to a different unit. The move to change the whole to a different unit brings the measure sub-construct into play and is exemplified in the excerpt below.

T: Right, the other part that we don't have yet is, err (refers to the work on the board), okay, we know that if we talking in terms of one quarter (underlines the number sentence he wrote previously on the board) of a year, we can also talk in terms of one quarter of how many weeks. How many weeks are there in a year? Yes? (refers to a learner)

L: Fifty- two.

T: There are 52 weeks in a year so one quarter of a year can also be, is, I mean the same as one quarter of fifty-two weeks (writes 'one quarter of 52 weeks' on the board), okay. In essence it means that we are dividing fifty-two weeks by four (writes' 52 weeks ÷4' on the board). Right, can someone come and do that division for me on the board? Amelia.

A learner comes to the board and completes the sum. Once she is done and the answer is determined. The teacher poses the following question:

T:…. Okay, one year is equal to four quarters. Can someone explain that to us, one year is equal to four quarters. How is one year equal to four quarters? Darren? (refers to a learner)

L: One year is equal to four what?

T: Four quarters.

L: One year is equal to four quarters of a year.

T: Ja, but how is that possible? (Calls out a learner's name but inaudible)

L: Because there are four seasons that divide the year into four quarters.

T: Because there are four seasons that divide the year into four quarters. That's one example. One answer that you can give. Yes? (refers to another learner)

L: Sir, there is also a leap year.

T: No, but when, we not talking about leap years here. We talking about how is it possible for us to have four quarters in a year? (cleans the board) Yes, Dave? (refers to a learner)

L: Sir, because you said four quarters, sir.

T: Mmm

L: So, it's three months...

T: There's three months in?

L: (learner shouts out) In one season.

T: (writes '$\frac{1}{4}$= 3 months' and explains) one quarter is equal to three months, right? And how many months are there in a year?

L: Twelve.

T: Twelve months. So, if (writes '$\frac{12}{3}$months') we divide twelve months by three what do you get?

L: (learner chant together) four

T: (completes number sentence by filling in '4') You going to get four. Which means you can divide a year into four equal…. (learner interrupts)

L: Groups.

T: Groups of three months. Okay, that's why we say there are four quarters because we can divide a year (refers to number sentence) into four equal groups of three months. Which each is equivalent to one quarter. Okay.

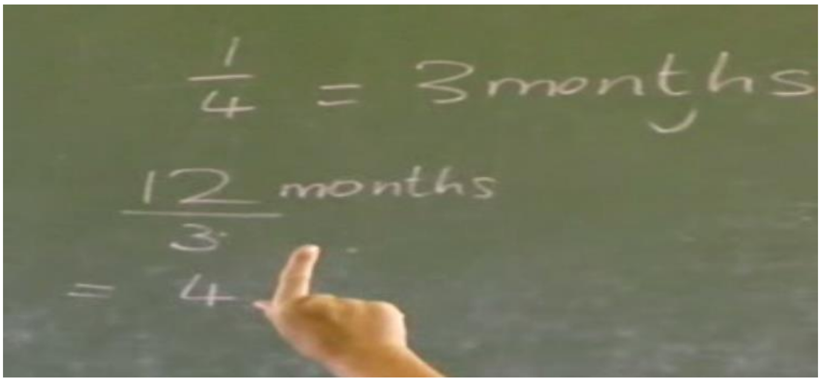


Figure 5.13: Shows $\frac{1}{4}$ of a year is 3 months and there are 4 quarters in a year.

The measure sub-construct together with the part-whole/operator sub-constructs was used to help develop an understanding of a quarter and how a quarter operates on the whole. The combinations of the different sub-constructs and connections between them lead to me to categorising this task as a higher demand task.

The combination of the part-whole/ratio sub-constructs was of interest because it had not been seen in the fractions or decimal fractions chapters. The following exemplifies the connection between the two sub-constructs. Figure 5.14 shows the ratio of toy cars to balls written on the board by the teacher. The teacher wrote 5 toy cars and 20 balls as shown in the figure and used it to explain how to write the comparison of the different toys in ratio notation. He went on to simplify the ratio by using the highest common factor. The teacher then asked what fraction of the toys were cars and what fraction of the toys were balls. Using 2 learners he explained that for every 5 toy cars Mark had, Tayla had 20 balls. So, if the ratio is simplified, for every 1 toy car Mark had, Tayla had 4 balls. He explained that of the 5 toys, Mark had $\frac{1}{5}$ of the toys and Tayla had $\frac{4}{5}$ of the toys. He went back to the original ratio and explained that Mark had 5 of the 25 toys and Tayla had 20 of the 25 toys. This task was categorised as higher demand because of the work done by the teacher with the part-whole/rate and ratio sub-constructs.

Figure 5.14: Rate and ratio sub-construct

When comparing the textbook and enacted tasks with regards to the fraction related chapters there is a greater proportion of higher demand enacted tasks (7 out of 13 tasks) compared to the textbook tasks (1 out of 9 tasks). 6 of the 13 enacted tasks included combinations of the different sub-constructs (part-whole/operator, part-whole/measure and part-whole/rate & ratio) compared to 1 out of 9 textbook tasks (part-whole/measure). The combinations of the sub-constructs in the enacted tasks allowed for greater connections resulting in higher demand tasks. When comparing fraction related enacted tasks across the Grades, the Grade 5 tasks presented more combinations of the different sub-constructs and connections resulting in a greater proportion of higher demand tasks compared to the other two Grades.

5.2.6 Grade 6: Fraction Enacted Tasks

Table 5.7: Grade 6: Fraction Enacted Tasks Summary

			PW	**O**	**M**	**Q**	**R&R**	**PW & O**	**PW & M**	**PW & Q**	**PW & R&R**	**O&M**	**O & Q**	**O & R&R**	**M & Q**	**M & R&R**	**Q & R&R**	**Lower Cognitive Demand**	**Higher Cognitive Demand**
	Episode	Task																	
Lesson 1	1	fair share - exercise on worksheet (textbook)- 1 task				1/1													1/1
	2	Naming fractions- 1 task	1/1															1/1	
	3	Task on naming fractions – 3 tasks	3/3															2/3	1/3
Lesson 2	1	Recapping types of fractions -1 task	1/1															1/1	
	2	Task on naming fractions- 3 tasks	3/3															3/3	
	3	Marking previous day's task- 3	3/3															1/3	2/3
	4	Equivalent fractions- Ex 9.4- 2 tasks	2/2															1/2	1/2
Lesson 3	1	Marking Ex9 4 – 2 tasks	2/2															1/2	1/2
Lesson 4	1	Recapping equivalent fractions - 1 task	1/1																1/1
	2	Equivalent fraction task- 1 task	1/1															1/1	
	3	Simplifying fractions explanation – 1 task																1/1	
	4	Simplifying task- ex 9.7 –6 tasks	6/6															6/6	
	5	Marking Ex9.7- 2 tasks																2/2	
Lesson 5	1	Continue marking Ex 9.7- 4 tasks	1/4															4/4	

			PW	O	M	Q	R&R	PW & O	PW & M	PW & Q	PW & R&R	O&M	O & Q	O & R&R	M & Q	M & R&R	Q & R&R	Lower Cognitive Demand	Higher Cognitive Demand
	2	HCF task- 2 tasks																2/2	
Lesson 6	1	Mark previous tasks- 2 tasks																2/2	
	2	Simplifying fractions worksheet- 1 task																1/1	
	3	Comparing fractions- 1 task	1/1																1/1
Lesson 7	1	Fraction of a quantity- 1 task						1/1											1/1
	2	Ex 9.8- 3 tasks		1/3				2/3										1/3	2/3
	3	Converting mixed number to improper fractions- 1 task	1/1																1/1
	4	Converting improper fractions to mixed number- 1 task	1/1																1/1
	5	Ex 9.11- 3 tasks	3/3															2/3	1/3
Lesson 8	1	Marking Ex 9.8- 3 tasks		1/3				2/3										1/3	2/3
	2	Marking Ex 9.11- 3 tasks	3/3															2/3	1/3
	3	2 Worksheets on converting fractions 2 tasks																2/2	
Lesson 9	1	Adding fractions-Ex 9.14- 1 task	1/1															1/1	
	2	Subtracting fractions- Ex 9.15- 1 task	1/1															1/1	
	3	Simplifying fractions – 1 task																1/1	
Lesson 10	1	Adding unlike fractions- explanation – 2 tasks	2/2															1/1	1/1
	2	Subtracting unlike																1/1	

			PW	O	M	Q	R&R	PW & O	PW & M	PW & Q	PW & R&R	O&M	O & Q	O & R&R	M & Q	M & R&R	Q & R&R	Lower Cognitive Demand	Higher Cognitive Demand
		fractions- explanation – 1 task																	
	3	Ex 9.18 & 9.19 – selected examples- 2 task																2/2	
	4	Marking exercises - 1 task																1/1	
Lesson 11	1	Continue marking previous exercise- 1 task																1/1	
	2	Converting fractions to % -explanation – 1 task	1/1															1/1	
	3	Ex 9.20, 9.21 &9.22- 3 tasks	1/3															3/3	

The following overview patterns were noted through the analysis of the Grade 6 teaching. Firstly, of the 68 tasks from the eleven lessons, 50 were categorised as lower cognitive demand and 18 as higher cognitive demand. 39 tasks focused solely on the part-whole sub-construct, 2 focused solely on the operator sub-construct, 5 focused on the part-whole/operator sub-constructs, 1 focused solely on the quotient sub-construct and the remaining 21 focused on pure calculations with no reference to the sub-constructs.

The part-whole sub-construct with partitioning and/or unitizing was used to teach some of the fraction concepts like adding and subtracting fractions, converting fractions, equivalent fractions, comparing fractions etc

The excerpt below from lesson 2, episode 4 provides an example of how the part-whole sub-construct with partitioning and unitizing was used to teach equivalent fractions to compare fractions. The first episode focused on recapping the different types of fractions, like mixed numbers, improper fractions etc., this was done verbally with the teacher leading the discussion. The second episode focused on learners completing a task by listing the different types of fractions and counting the number of shaded and unshaded parts of different shapes. During the third episode the teacher together with the learners marked the previous day's tasks. This was followed by the fourth episode, during this episode the learners where given Exercise 9.4 from the textbook to complete independently with guidance from the teacher when necessary. The task comprised of two questions relating to equivalent fractions. Many of the learners experienced difficult answering the second question and there were lots of queries. The teacher decided to explain the question to the entire class.

T: Let's look at 2a. It says study the diagrams and then answer the questions. What do you notice about the shaded area in A, B and C? If you had to start at the top of the shaded area and go down and stop (draws three quarter of a circle in the air) for each one. What can you tell me?

L1: They all take up the same amount of space.

T: They all take up the same amount of space. Okay, if you had a clock it would take up that same amount of space. The second one says, write down the fraction of the whole that is shaded in A, B and C. So, in other words write down how much is shaded, what fraction is shaded out of the whole. Okay, so in the first one count how many blocks are in the shaded area. 2, 4, 6 are shaded and how many is that whole divided into?

L2: 8

T: 8, so what is my shaded area called? 6 over 8. And the next one?

Class: 3 over 4

T: 3 out of 4 and the next one?

Class: 9 out of 12

T: 9 out of 12. Okay, so now, those fractions we said, what where those fractions? Give me the fractions (walks to the board to write).

Class: 6 over 8 and 3 over 4 and 9 over 12 (Writes: $\frac{6}{8}$ $\frac{3}{4}$ $\frac{9}{12}$). If the amount of area they take up is the same are they not the same size fraction, then? (Writes: $\frac{6}{8} = \frac{3}{4} = \frac{9}{12}$). They are the same size because it is the same size area that is shaded. So, what they trying to tell you is that 6 over 8 is exactly the same size as 3 over 4 is exactly the same size 9 over 12, accept they taking a cake and they dividing it into either bigger or smaller pieces. So here is my cake and I divide it into quarters (Draws a circle and divides it into quarters). When you have a birthday do you divide your cake into 4 big pieces?

Class: No

T: No, because you have lots of people at your party…..We can't just say these three we going to eat and there is only 1 left (points to 3 parts of the circle). So, we decide to divide it into bigger pieces (divided each quarter in half). Now I have got 8. Whether I eat 3 big pieces (marks of 2 eights 3 times) or whether I eat 1, 2, 3,4, 5, 6 out of the 8 pieces I have divided it into (marks each eight 1to 6). Whether I eat 3 of the 4 big ones or whether

I eat 6 out of the 8 pieces (circles $\frac{3}{4}$ then $\frac{6}{8}$) I am eating the same amount of cake. So, they the same. I can decide to eat a slab of chocolate in 4 different pieces or I can decide to break it into 8 pieces. If I eat 8 of the 8 or 4 of the 4 how much chocolate am, I eating?

Class: The whole chocolate.

T: the whole chocolate. Doesn't matter if I divide it into smaller pieces but you still eating the same amount. This is the big thing about fractions, whether I am eating 3 out 4 pieces or 6 out of 8 pieces I am eating the same amount.

The figure 5.12 shows the diagram that the teacher drew on the board. From the excerpt we see that she started with the part-whole sub-construct and partitioning and then moved to unitizing to explain that $\frac{3}{4}$ of the circle is the same as $\frac{6}{8}$ of the circle. This task was categorised as higher cognitive demand because of the connections made between the sub-construct and unifying element to create meaning and understanding of equivalent fractions.

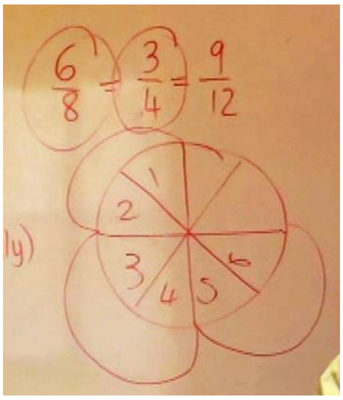


Figure 5.15: Diagram presenting part-whole, partitioning, and unitizing to teach equivalence

When working with the operator sub-construct the teacher worked through examples and used part-whole and partitioning to make meaning of the fraction concept. Figure 5.16 and 5.17 displays this work done by the teacher.

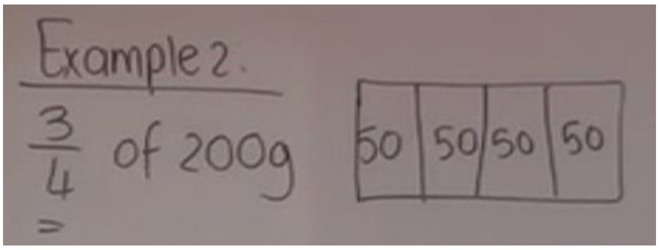


Figure 5.16: Operator/part-whole sub-constructs with partitioning used by the teacher

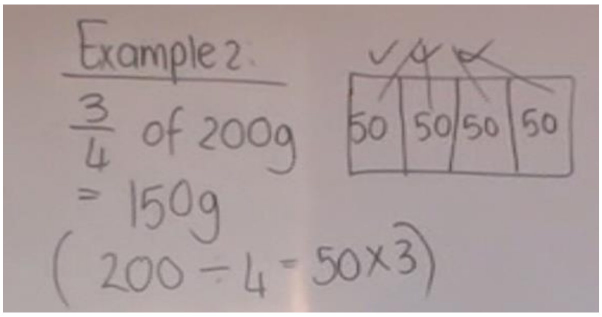


Figure 5.17: Operator/part-whole sub-constructs with partitioning used by the teacher

In these figures we see that the whole is represented as an area model and partitioned accordingly. There is a connection made between the whole and the fraction used to operate on it. It is important to note that the tasks provided for the learners to complete on their own were solely operator tasks and involved questions that were straightforward and could be answered by using a procedure with no connections made to any of the unifying elements, or the operator tasks that required work with partitioning was side stepped and replaced with a procedure. This is also seen in the Grade 4 analysis. While the teacher used partitioning to explain and make meaning of the operator sub-construct, she did not afford the learners the opportunity to engage with partitioning required by the tasks, instead she provided a procedure and encouraged them to follow it when completing the tasks. There was a quick shift to procedural work even though the tasks required engagement with the sub-constructs and unifying elements. The following statement was made by the teacher to encourage the learners to follow the procedure, *"Okay, 3a says the following, use counters or draw diagrams to work out the answers. But we not going to. We not going to use diagrams or use* *counters. We are just going to work it out like the example- looks at the board with the example"*. The example was a procedure where the whole was divided by the denominator and multiplied by the numerator. Given that the process of partitioning plays a vital role in providing connections to create understanding these tasks were categorised as lower cognitive demand.

The quotient task appeared in a worksheet (adapted from the textbook task to make it more personal for the learners by using their names) provided by the teacher. The learners were given time to complete the worksheet independently with guidance and assistance from the teacher when necessary. The main question was, "Five friends go on a picnic. Each friend brings something to share equally with the others. Share the eats equally between them. Draw a diagram to show how you share it", with 5 sub-questions (1) Thecla brings a slab of chocolate with 15 squares, (2) Kiara brings 4 apples etc. The enactment of this task by the teacher when addressing individual learners showed the teacher encouraging the learners to draw diagrams, partition them and unitize them when necessary. For example, learners were encouraged to draw a slab of chocolate, partition it into 15 different parts, then unitize the 15 parts into fifths. This allowed the learners to see and establish that each child gets $\frac{3}{15}$ or $\frac{1}{5}$ of the slab. This process was followed for all the questions and the task was categorised as higher cognitive demand.

Tasks that included work with only the part-whole sub-construct and tasks that did not include any sub-constructs were categorised as lower cognitive demand. As mentioned in the Grade 5 analysis, tasks that did not refer to any sub-constructs were classified as pure calculations. For example, add $\frac{3}{4}$ and $\frac{2}{12}$. The answer was obtained by following a procedure to change $\frac{3}{4}$ to an equivalent fraction so that the denominators were the same and the fractions could be added. There was no meaning or understanding of the procedure used. The teacher emphasised the procedure and made no reference to any of the sub-constructs during the enacted tasks. This was the same for written tasks made available to the learners.

A comparison between the textbook and enacted tasks revealed 30 higher demand tasks in the textbook compared to the 18 enacted tasks. Once again, the lower number of higher demand enacted tasks was because 21 of the 68 enacted tasks dealt with what has been described as pure calculations. These tasks were disconnected from any sub-constructs and contributed to the high degree of lower cognitive demand calculation tasks. The textbook tasks included more combinations of the different sub-constructs (part-whole/operator- 7 tasks, part-whole/measure- 2, part-whole/quotient – 6 tasks) allowing for greater connections thus resulting in more higher demand tasks.

5.2.7 Grade 6: Decimal Fractions Enacted Tasks

Table 5.8: Grade 6: Decimal Fractions Enacted Tasks Summary

			PW	O	M	Q	R&R	PW & O	PW & M	PW & Q	PW & R&R	O&M	O & Q	0 & R&R	M & Q	M & R&R	Q & R&R	Lower Cognitive demand	Higher Cognitive demand
	Episode	Task																	
Lesson 1	1	Defining a decimal- 1 task																1/1	
	2	Decimal notation – 1 task																1/1	
	3	Tenths & hundredths- explanation- 1 task	1/1																1/1
	4	Tenths & hundredths Ex 11.1 – 1 task	1/1															1/1	
	5	Tenths & hundredths Ex 11.2- 2 tasks																2/2	
Lesson 2	1	Working with thousandths- Marking Ex 11.3 & 11.4-homework- 4 tasks																4/4	
	2	Place Value – ex 11.5 & 11.6- 7 tasks																7/7	
Lesson 3	1	Recap place value and decimal notation- 1 task																1/1	
	2	Marking 11.5 & 11.6- 7 tasks																7/7	
	3	Converting DF to CF & vice versa- 1 task																1/1	
	4	Ex 11.7 & 11.8- 4 tasks																4/4	
Lesson 4	1	Marking 11.7 & 11.8 – 4 tasks																4/4	
	2	Converting CF to DF- explanation-1 task																1/1	

			PW	O	M	Q	R&R	PW & O	PW & M	PW & Q	PW & R&R	O&M	O & Q	0 & R&R	M & Q	M & R&R	Q & R&R	Lower Cognitive demand	Higher Cognitive demand
	3	Ordering & comparing decimals – explanation – 2 tasks	1/1															1/1	1/1
	4	Ordering decimals - Ex 11.9- 5 tasks	5/5															5/5	
Lesson 5	1	Marking Ex 11.9- 5 tasks	5/5															5/5	
	2	Rounding off decimals- explanation – 1 task																1/1	
	3	Ex 11.10-5 tasks																5/5	
Lesson 6	1	Recap rounding off- 1 task																1/1	
	2	Mark Ex 11.10- 1 task																1/1	
	3	Complete Ex 11.10- 1 task																1/1	
	4	Marking rest of Ex 11.10-4 tasks																4/4	
Lesson 7	1	Marking homework – Ex 11.11 & 11.2- 2 tasks																2/2	
	2	Percentages as decimals- explanation – 2 tasks																2/2	
	3	Ex 11.16 & 11.17- 3 tasks																3/3	
	4	Marking homework- Ex 11.13- 2 tasks																2/2	
Lesson 8	1	Marking rest of 11.13- 4 tasks																4/4	
	2	Marking Ex 11.16 & 11.17- 2 task	1/2															2/2	
	3	Marking Ex 11.10- 1 task																1/1	
	4	Ex 11.18- 4 tasks																4/4	
Lesson 9	1	Marking 11.18- 3 tasks	1/3															3/3	

Of the 83 tasks from the 9 lessons, 81 were categorised as lower demand tasks and 2 were categorised as higher demand tasks. 15 of the tasks focused solely on the part-whole sub-construct while the remaining tasks focused on pure calculations and did not include any sub-constructs.

Very similar to the Grade 5 analysis, the part-whole subconstructs that included partitioned area models were categorised as lower cognitive demand and part-whole sub-construct tasks where the teacher worked with partitioning and unitizing were categorised as higher cognitive demand. It is important to note that during the enactments of the tasks the teacher relied on the pre-partitioned area models that were represented in the textbook. It is also important to note the tasks that focused on pure calculations and made no reference to any of the sub-constructs. For example: Convert $\frac{1}{4}$ to a decimal fraction. The teacher explained the procedure of how to convert $\frac{1}{4}$ to $\frac{25}{100}$ and then because "there are 2 zeros in a hundred there must be 2 places after the decimal comma". This is further illustrated in figure 5.15 below. This type of procedural work is seen through all the tasks that were categorised as lower cognitive demand.

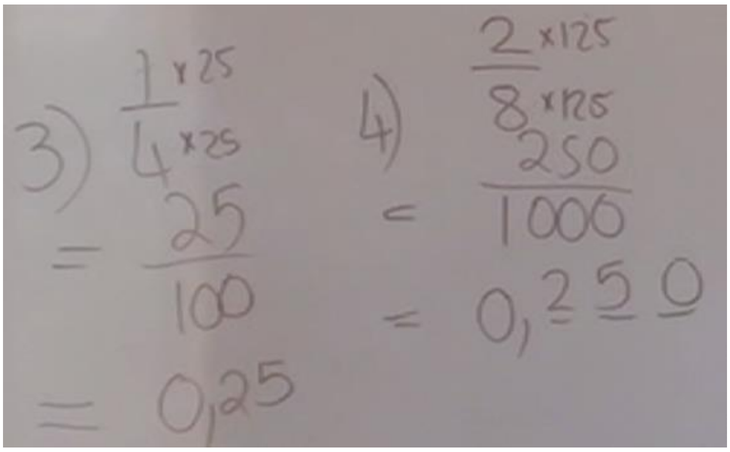


Figure 5.18: Converting to decimal fractions using a procedure

A comparison between the textbook and enacted tasks revealed a very low number of higher demand tasks in both. The high number of lower demand textbook (41 out of 66 tasks) and enacted (68 out of 83) tasks was due to the focus on pure calculations with no connections to any of the sub-constructs.

5.2.8 Grade 6: Fraction Related Enacted Tasks

Table 5.9: Grade 6: Fraction Related Enacted Tasks Summary

			PW	**O**	**M**	**Q**	**R&R**	**PW & O**	**PW & M**	**PW & Q**	**PW & R&R**	**O & M**	**O & Q**	**0 & R&R**	**M & Q**	**M & R&R**	**Q & R&R**	**Lower Cognitive Demand**	**Higher Cognitive Demand**
Lesson	Episode	Task																	
Time	1	Worksheet - time & fractions- 5 tasks						5/5											5/5
	2	Ex 3.11- 3 tasks						3/3											3/3
Rate	1	Defining rate- 1 task					1/1											1/1	
	2	Examples of rate- 2 tasks					2/2											2/2	
	3	Ex 7.2 – 3 tasks					3/3											3/3	
Ratio	1	Recapping rate- 1 task					1/1											1/1	
	2	Marking 7.2- 3 tasks					3/3											3/3	
	3	Defining ratio- 1 task					1/1											1/1	
	4	Ex 7.23 -3 tasks					3/3											3/3	
Money 1	1	Percentages- explanation- 2 task		1/2				1/2										1/2	1/2
	2	Ex 13.4 – 1 task		1/1														1/1	
Money 2	1	Marking Ex 13.4 – 1 task		1/1														1/1	
	2	Discount – explanation -1 task		1/1														1/1	
	3	Ex 13.5 – 2 tasks		2/2														2/2	

			PW	O	M	Q	R&R	PW & O	PW & M	PW & Q	PW & R&R	O & M	O & Q	0 & R&R	M & Q	M & R&R	Q & R&R	Lower Cognitive Demand	Higher Cognitive Demand
Money 3	1	Recapping discount – 3 tasks		3/3														3/3	
Measurement	1	Length-conversions-1 task			1/1													1/1	
	2	Mass – 1 task			1/1													1/1	
	3	Capacity – 1 task			1/1													1/1	

Of the 35 tasks from the 7 lessons, 26 tasks were categorised as lower demand tasks and 9 were categorised as higher demand tasks. 9 of the tasks focused solely on the operator sub-construct, 3 focused solely on the measure sub-construct, 14 focused solely on the rate and ratio sub-construct and 9 focused on the part-whole/operator sub-constructs.

The analysis of the tasks in the fraction related chapter for Grade 6 follows the same pattern of analysis carried out on similar tasks mentioned throughout the analysis. Like the Grade 5 analysis, the rate and ratio sub-construct appeared here for the first time in any of the Grade 6 analysis. The enactment of these tasks was categorised as lower cognitive demand. Work with the rate and ratio sub-construct involved writing a ratio in its simplest form or studying different pictures and writing the ratio of the different objects found in the pictures. In the enactment of these tasks no refence was made to any other sub-construct or unifying element and therefore they were categorised as lower cognitive demand.

A comparison between the textbook and enacted tasks revealed 8 out of 38 tasks were categorised as higher demand tasks in the textbook and 9 out of 35 enacted tasks. The low number of higher demand tasks in both the textbook and enacted tasks was due to the focus on single sub-constructs (operator, rate and ratio, measure). The higher demand tasks in both the textbook and enacted task involved the combination of the part-whole/operator sub-constructs with partitioning.

5.3 Overall patterns

There are several overall patterns that emerge from the analysis of the enacted tasks across the Grades. The Grade 4 and 6 fraction enacted tasks analysis revealed that while the teachers in their enactment of the tasks referred to certain sub-constructs with partitioning and unitizing, the written tasks made available to learners seldom or never included work with the unifying elements. The teachers across the grades offered certain methods during the set-up of the tasks but when it came to the implementation of the tasks, they often recommended strategies or procedures that differed from their methods and approaches. This slippage between what was offered to the learners and what was recommended to them when engaging individually with the tasks resulted in lower demand tasks since tasks took on a pure calculation or procedural focus with no connection made to the sub-constructs and unifying elements.

An interesting pattern noted across the Grades and chapters was that most higher demand tasks were based on the teachers working with the unifying element of partitioning with limited attention to the other unifying elements. Another pattern observed was that from the textbook tasks the part-whole sub-construct with partitioning formed a large proportion of the higher demand tasks. When comparing the tasks demands from the textbooks and the enacted tasks it was noted that the textbook tasks had a greater proportion of higher demand tasks compared to the enacted tasks across the Grades. It was only in the Grade 5 enacted tasks that the proportion of high cognitive demand tasks was greater than that for the textbook tasks.

The Grade 4 and 6 fraction enacted tasks analysis also revealed that most of the work done by the teachers involved the part-whole sub-construct with an area model. 28 of the 36 tasks in Grade 4 focused solely on the part-whole sub-construct while 39 of the 68 tasks in Grade 6 focused solely on the part-whole sub-construct. The Grade 5 analysis revealed something completely different, 5 of the 18 tasks included work with the part-whole sub-construct while 12 of the 18 tasks focused solely on pure calculations with no connections to any sub-constructs or unifying elements

The decimal fractions enacted tasks for both Grade 5 and 6 illustrated that the majority of the tasks involved pure calculations with no reference to any sub-constructs. When a sub-construct was used, it was most often the part-whole with pre-partitioned area models and very seldom included work with partitioning and unitizing.

The Grade 5 fraction related enacted tasks displayed more combinations of the different sub-constructs compared to the Grade 4 and 6 enacted tasks. A comparison between the Grade 5 fraction related enacted tasks and the Grade 5 fraction enacted tasks revealed a very different pattern. The fraction enacted tasks did not include any combinations of the different sub-constructs and focused mostly on pure calculations.

5.4 Summary

In this chapter I provide an analysis of the enacted tasks that took place across the three Grades. This was done by presenting a summary table for the outcomes of the coding for the fraction teaching, and then the set of fraction-related teaching, for each Grade. Each table was followed by a commentary that highlights the key patterns seen in the results and where useful, examples that illuminate the patterns or points made in the preceding analysis were included. This Grade-by-Grade commentary for the fraction and then the fraction-related teaching was followed by a concluding analysis that considers the overall patterns of presentation of fraction concepts across the three Grades.

CHAPTER SIX

FINDINGS, IMPLICATIONS AND LIMITATIONS

6.1 Introduction

In this chapter, I reflect on the extent to which I have achieved the goals of this study. I begin with a summary of the analytical framework I developed for this study. I discuss how the framework enabled me to categorise the cognitive demands of tasks in the different textbooks and the enacted tasks. As I do this, I reveal my findings, discuss the implications of the study for teacher education and development and future research, and acknowledge the limitations of the study.

6.2 Summary of analytical framework

I undertook this study to gain an understanding of how fractions are presented in general across the Intermediate Phase classrooms in one school over a time period of one year. The study was guided by the following 3 questions:

1. What range of sub-constructs and unifying elements did teachers focus on through their task selections to develop an understanding of fractions for their learners across Grades 4-6 in one school?
2. What levels of cognitive demand and working with unifying elements did the teachers make available in their enactment of fraction tasks?
3. What can be said about the fraction knowledge that is made available to learn based on the analysis of sub-constructs, unifying elements, cognitive demand across the three Grades?

To answer these questions, I had to develop a framework that allowed me to identify the sub-constructs, unifying elements and cognitive demands of the textbooks and enacted tasks. The framework was established from an in-depth study of the literature on fractions and tasks demands. Stein et al.'s (2000) framework was particularly relevant for my study and I used it together with literature to develop fraction related indicators for the tasks that were more specific to fractions so that when analysing the different tasks, the focus was on what specific fraction knowledge was being made available for learners when engaging with the different tasks. This in turn revealed the conceptual and procedural knowledge made available for learners. A feature of higher cognitive demand tasks is that they include connections and integration with different sub-constructs. This connection between sub-constructs has been identified as being central to developing conceptual understanding, and therefore looking for connections – which the literature suggests can occur through the tasks/representations, in the context of fractions, provided a route into analysing the cognitive demand of tasks with conceptual tasks associated with higher, or more connected, task demands.

The analytical framework is one contribution of this study. It is worth noting that using this framework as a way of looking at the connectedness of the presentation of fractions differs in important ways from earlier writing on conceptual and procedural ways of working with fractions. Charalambous and colleagues (2010) explained that using Stein et al.'s (2000) Task Analysis Guide (2000) in their study to distinguish between procedures without connections and procedures with connections with respect to cognitive demands did not yield satisfactory results because it involved a high level of inference. The framework used in this study allowed me to look at connections in a somewhat different way - through looking for combinations of sub-constructs and/or work with sub-constructs that included attention to the unifying elements. This way of looking at cognitive demand produced a lower 'bar' for what counted as connections than seen in Charalambous et al's (ibid) paper. Working with lower Grades required a framework that provided a lens for understanding tasks at a lower level of connection for use in the Intermediate Phase to establish an understanding of fraction knowledge made available for the learners. The various task frameworks in the literature (e.g., Charalambous et al, 2010; Son & Senk, 2010; Stein et al., 2000) provide different routes for understanding connections in fraction tasks and instruction, in studies of teacher fraction knowledge, learner understanding of the content, task demands at higher levels of connection, connections between and within sub-constructs and classroom instructions and textbook tasks.

Given that the intentions of this study were to study fraction tasks at lower levels, a framework that catered for examining such tasks on the basis of connections based on the different sub-constructs and unifying elements was necessary. Looking at textbook and enacted tasks in this way provides openings to see what connections within fraction work in the middle Grades can look like, extending work that has tended to look at more complex fractions tasks than typically seen in these Grades.

6.3 Findings from the textbook analysis

The findings from the textbook analysis revealed recurring patterns across the fraction chapters in the three Grades' textbooks and overall, more similarities than differences. The similarities related to the examples used to explain each task, the sub-constructs and representations used.

The unifying elements used across the Grade 4 to 6 tasks included mostly partitioning and very seldom unitizing. Partitioning appeared in two ways across the tasks. Firstly, it appeared frequently with the part-whole sub-construct with pre-partitioned area models. The second way required the physical partitioning and sometimes unitizing of area models within the part-whole sub-construct. There were very few of these tasks across the three Grades that required physical partitioning of an area or other models.

As mentioned above, the part-whole sub-construct with pre-partitioned area models was present in most of the tasks. Although the part-whole sub-construct was the most common construct that appeared in the Grade 4 textbook, in a few tasks the part-whole sub-construct was accompanied by the quotient sub-construct. The majority of the part-whole sub-construct tasks that appeared in the Grade 4 chapter did not include any work with partitioning or unitizing. The representations mostly included pre-partitioned area models. The Grade 5 textbook presented more sub-constructs with greater connections between the sub-constructs. The part-whole sub-construct, however, remained dominant in this textbook too. Here again though, the part-whole sub-construct was accompanied by pre-partitioned area models. The part-whole sub-construct was accompanied by the operator or measure or quotient sub-construct when connected. The Grade 6 textbook followed a similar pattern where the part-whole sub-construct either appeared on its own with pre-partitioned area models or it was accompanied by the operator or measure or quotient sub-constructs to create meaning and make connections.

The decimal fractions chapters in the Grade 5 textbook emphasised the part-whole sub-construct with pre-partitioned area models and when the measure sub-construct was present it was either on its own with pre-partitioned models or accompanied by the part-whole sub-construct. The Grade 6 textbook included a majority focus on the part-whole sub-construct with pre-partitioned area models and only two measure tasks, one that required working with the unifying element of partitioning and the other not. There were no combinations of different sub-constructs. From studies based on fraction coverage in textbooks we know that when tasks include multiple sub-constructs with several different representations and solutions, more opportunities are created for students to develop a multifaceted, connected, and deeper level of understanding of fractions (Charalambous et al., 2010; Stein et al., 2000, Lamon, 2012).

The fraction related chapters included a range of different single sub-constructs with minimal reference to the unifying elements. The combined sub-constructs always included the part-whole sub-construct with either the measure or operator sub-constructs.

From the textbook analysis of the fraction and decimal fraction chapters across the Grades it can be concluded that there was an emphasis on the part-whole sub-construct with pre-partitioned area models and single sub-constructs with no reference to unifying elements and whenever a combination of sub-constructs was present it always included the part-whole sub-construct. Tasks where no sub-constructs could be assigned because of the absence of representations and the focus was on pure calculations were found across all three textbooks.

The overall textbook analysis of the fraction related chapters across the three Grades revealed a different pattern from the fraction and decimal fraction chapters. The part-whole sub-construct hardly featured in these tasks. The focus was more on the measure and operator single sub-constructs with the part-whole sub-construct appearing sporadically. The focus on single sub-constructs (measure, operator and sporadically part-whole) with no unifying elements resulted in a majority of lower demand tasks.

6.4 Findings from the enacted tasks analysis

The enacted tasks analysis revealed that the teachers did not cover all the content from the textbook chapters analysed. While they sometimes used the tasks from the textbooks in their original form and context, they sometimes taught differently from what the textbook recommended, and they sometimes supplemented or omitted certain textbook tasks. The focus therefore was on all the enacted tasks that involved fractions and fraction related concepts. The analysis of the fraction teaching across the grades highlighted that nearly all the conceptual work done by the three teachers involved the part-whole sub-construct with partitioning. Very little work included all the single sub-constructs and where two sub-constructs were present, one was always the part-whole sub-construct. The Grade 5 and 6 teaching, and – in particular, the Grade 5 teaching, revealed a significant number of enacted tasks that did not include any sub-constructs since no representations were incorporated in the teaching and therefore were categorised as lower cognitive demand. The decimal fractions teaching in both Grade 5 and 6 followed a very similar pattern. Most of the enacted tasks did not involve any sub-constructs and those that did include sub-constructs always focused on the part-whole, most of the time with pre-partitioned area models or measuring instruments and very seldom involved work with the unifying elements. The enacted tasks for the fraction related chapter for Grade 4 and 5 included work with the single sub-constructs of measure and operator with no reference to unifying elements. The tasks also included work with the part-whole sub-construct and partitioning, as well as work with the part-whole sub-construct accompanied by either the measure or operator sub-construct. When comparing the enacted tasks with the textbook tasks there was a great proportion of higher demand enacted tasks compared to textbook tasks. The Grade 6 enacted tasks included work with single sub-constructs, operator, measure and rate and ratio, with no reference to unifying elements. Again, when there was a combination of two sub-constructs the part-whole sub-construct was one of them. In this case it was the part-whole and operator sub-constructs.

In summing up the enacted tasks analysis it can be concluded that nearly all the conceptual work done by the teachers involved the part-whole sub-construct with partitioning and very similar to the textbook analysis, single sub-constructs included no work with the unifying elements and that when a combination of sub-constructs were present one was always the part-whole sub-construct.

6.5 Answering the research questions

From the analysis of the textbook and enacted tasks I was able to answer the three questions that guided this study.

1. What range of sub-constructs and unifying elements did teachers focus on through their task selections to develop an understanding of fractions for their learners across Grades 4-6 in one school?

When selecting tasks to teach fractions (fractions enacted tasks) I noted that the Grade 4 and 6 teachers' selections mostly included the part-whole sub-construct (Grade 4- 86%, Grade 6- 65%) while other sub-constructs were side-lined. The Grade 5 analysis revealed that 28% of the tasks selected by the teacher included work with the part-whole sub-construct and 67% of the tasks were pure calculations with no sub-constructs present. Even when tasks in the textbook called on other sub-constructs all 3 teachers defaulted to the part-whole sub-construct. A number of enacted tasks (Grade 4- 56%, Grade 5- 22%, Grade 6- 46%) selections included pre-partitioned area models, number lines or measuring instruments. The Grade 5 and 6 tasks were lower in proportion compared to Grade 4 because the Grade 4 tasks were the only tasks where sub-constructs could be assigned to all tasks while 67% of tasks in Grade 5 and 31% of Grade 6 tasks could not be assigned to a sub-construct. When single sub-constructs were used to create meaning of fraction concepts they were often void of unifying elements and procedural in nature. When tasks did include work with unifying elements, partitioning was the unifying element that was most commonly used. 44% of Grade 4, 11% of Grade 5 and 26% of Grade 6 fraction enacted tasks included work with partitioning. .

The analysis of the decimal fractions enacted tasks for Grade 5 showed that 32% of the tasks selected included the part-whole sub-construct, 32% of the tasks included the measure sub-construct and 36 % of the tasks were pure calculation tasks that could not be assigned to a sub-construct. 24 % of the tasks included working with partitioning. A higher proportion of work with partitioning in the enacted decimal fractions tasks compared to the fraction enacted tasks was noted, In the Grade 6 analysis 18% of the tasks selected involved the part-whole sub-construct while 80% of the tasks were pure calculation tasks with no sub-constructs assigned and 2% of the tasks include the part-whole sub-construct with partitioning None of the Grade 6 tasks involved working with the unifying elements. The analysis revealed that in both Grades a large percentage of tasks selected did not include any work with unifying elements to create connections and develop understanding of decimal fractions and were categorised as lower demand tasks. This was due primarily to the great number of pure calculations tasks. Only 24% of the tasks from Grade 5 allowed for work with partitioning to help create meaning and understanding.

Results from the fraction related enacted tasks showed that teachers selected tasks that included the part-whole and measure sub-constructs. 10% of the Grade 4 tasks focused solely on the part-whole sub-construct with partitioning, 60% of the tasks focused solely on the measure sub-construct with no unifying elements present and 30% of the tasks focused on the part-whole/measure sub-construct with partitioning. A large proportion of tasks did not include the unifying elements that allow for connections and understanding. The Grade 5 fraction related enacted tasks showed a different pattern. A greater percentage of tasks compared to Grades 4 and 6 worked with the unifying element of partitioning to help create meaning. 46% of the tasks were categorised as higher cognitive demand because of connections between different sub-constructs (part-whole/operator, part-whole/measure and part-whole/rate& ratio) and partitioning while only 26% of the Grade 6 tasks included combined sub-constructs (part-whole/operator). 54% of the Grade 5 tasks (part-whole, operator, rate & ratio) and 75% of the Grade 6 tasks (operator, measure and rate and ratio) focused on single sub-constructs with no connections to the unifying elements or other sub-constructs. It was noted that the part-whole sub-construct was once again at the centre of connecting other sub-constructs and that a larger percentage of tasks selected focused on single sub-constructs.

It was interesting to note that none of the fraction related enacted tasks included only pure calculation tasks. Each task was assigned a sub-construct/s unlike in the fraction and decimal fractions tasks. The tasks, however, that were selected for learners to consolidate what was explained and discussed were very different. Learner tasks rarely ever involved work with partitioning and/or unitizing and there appeared to be a disconnect between what was taught and presented compared to what was provided for the learners to engage with in their own working. This points to the likelihood of a further drop in cognitive demand in the move from teacher presentation to learner independent working with fractions.

2. What levels of cognitive demand and working with unifying elements did the teachers make available in their enactment of fraction tasks?

As mentioned above, when the teachers worked more conceptually, the part-whole sub-construct with partitioning and unitizing were most commonly present. They made connections between the sub-construct using representations with unifying elements and this led to my coding of these tasks as higher cognitive demand tasks. When the teachers worked one on one with different learners their approach differed from their whole class discussions. When working with these individual learners who experienced difficulty with some tasks, the teachers almost always resorted to procedures or/and pure calculations with hardly ever referring to the unifying elements they used during the whole class discussions. The teachers moved quickly to procedures and did not provide sufficient time for reasoning and problem-solving that could be achieved through working with different sub-construct and the unifying elements. When enacted tasks presented single sub-constructs, they required little or no work with the unifying elements and once again these tasks were of a lower cognitive demand since no connections were provided between the sub-construct and unifying element. 58% of the Grade 4 fraction enacted tasks, 89% of the Grade 5 fraction enacted tasks and 74% of the Grade 6 fraction enacted tasks were categorised as lower demand tasks. Majority of these fraction enacted tasks from Grades 5 and 6 involved either procedural work with no connections to the unifying elements or they comprised of pure calculations with no sub-constructs present.

76% of the Grade 5 decimal fractions enacted tasks and 98% of the Grade 6 decimal fractions enacted tasks were categorised as lower cognitive demand. Once again, these tasks made no connections to the unifying elements and/or they included only pure calculations with no reference to any sub-constructs.

60% of the Grade 4 fraction related enacted tasks, 46% of the Grade 5 fraction-related enacted tasks and 74% of the Grade 6 fraction related enacted tasks were categorised as lower demand task. The Grade 5 tasks included more work with the unifying elements and a combination of different sub-constructs and therefore recorded a lower percentage of lower demand tasks.

An overall observation of the cognitive demands and working with the unifying elements in the enacted tasks revealed that a larger proportion of teaching involved procedural work with single sub-constructs or pure calculations with very little or no connections to unifying elements resulting in a large percentage of lower demand tasks.

3. What can be said about fraction knowledge that is made available to learn based on the analysis of sub-constructs, unifying elements, cognitive demand across the three Grades?

Based on the analysis of the sub-constructs, unifying elements and cognitive demand across the three Grades it can be concluded that the fraction knowledge provided was predominantly focused on the part-whole sub-construct with a limited emphasis on partitioning resulting in a high percentage of tasks being categorised as lower demand tasks. The fraction knowledge was largely comprised of a brief introduction to the part-whole sub-construct and then proceeded to introduce computation procedures. There was limited exposure to working with all the sub-constructs and unifying elements, leading to what can be described as an impoverished understanding of fraction concepts. These findings tend to offer overlap with what has been found in the international studies of fraction task presentation and instruction (Geller, et al., 2017; Gabriel, 2016; Lamon, 2012, Charalambous et al, 2010). In contrast to the literature base's advocacy for working with all sub-constructs, unifying elements and various representations to create more opportunities for learners to develop a more robust and connected understanding of fractions, the findings suggested that there is still an over-emphasis on the part-whole sub-construct when teaching fractions. When single sub-constructs are used in fraction instruction without the unifying elements the fraction knowledge made available for learners is superficial and does not provide connections that allow for a deeper understanding of fractions. When a single sub-construct or two or more sub-constructs with the unifying elements appear in a task it creates for a richer and more robust understanding of fractions because of the connections it provides with the unifying elements. The data reveals that tasks limited to single sub-constructs with pre-partitioned wholes and no unifying elements create lower demand tasks and ultimately an impoverished understanding of fractions.

6.6 Implications of this study

The purpose of this study was to understand how fractions are presented in the Intermediate Phase in order to improve teaching practices. In this section, I consider the implications of this study.

This study suggests that there needs to be an overhaul or redesign of fraction instruction. Textbook writers and curriculum developers need to include a range of different sub-constructs and unifying elements in textbook tasks and enacted tasks need to be tweaked to include more attention to connections between sub-constructs and unifying elements. If the part-whole sub-construct is going to be used it must include more working with the unifying elements and less emphasis on pre-partitioned area models. Textbooks and enactments need to include a combination of the different sub-constructs to allow for a complete understanding of fractions. For example, tasks used to teach the addition of fractions do not have to rely solely on pre-partitioned area models as seen in current textbooks and instruction but instead can be introduced using the operator and quotient sub-constructs with equivalence, which is created through partitioning and unitizing. Instead of moving so rapidly to teaching procedures when working with fractions, greater emphasis must be placed on working with partitioning and unitizing of different representations/models since it forms such an important foundation for understanding fractions. Wilkin and Norton (2018, p10) remind us that through careful selection of tasks and activities that focus on actions like partitioning, iterating and the coordination of multiple levels of units, that support the development of fraction concepts, children can build more robust concepts of fractions. Without this knowledge and understanding obtained through working with the unifying elements, fraction work takes on a procedural focus that aims at obtaining correct answers with little or no meaning to understanding fractions.

Fraction knowledge cannot be boxed and contained within certain chapters. From this study it has been noted that there is an overlap and connection between the different chapters in the textbooks. Fractions cover a large amount of the mathematics content when studying its different strands. It is not confined to the part-whole sub-construct and procedures found in fraction chapters of textbooks but when examining it on a deeper level we notice that for example, measure forms a huge chunk of the curriculum and is an important sub-construct of fractions yet it is found in a separate chapter. This is the same for rate and ratio. If fractions is to be understood in its entirety than the lines that keep the mathematical content separate needs to be removed and connections need to be made to help create a solid and firm foundation for fraction leaning.

The findings from what actually took place in the classrooms may have implications for improving Intermediate Phase teachers' preparation for teaching mathematics. Understanding the different sub-constructs and unifying elements and knowing how they help create meaning and understanding of fractions will provide pre-service teachers with the knowledge and skills required to teach fractions for conceptual understanding. The evidence from this study may enlighten professionals involved in teacher training and development and help them plan better and improve instruction for prospective teachers. This in turn will produce Intermediate Phase teachers with a deeper and greater understanding of fractions, thus allowing them to become more effective and resourceful mathematics teachers.

6.7 Limitations of the study

One of the major limitations of this study is the time it has taken for me complete it. As noted in chapter one, during the process of completing this study there was a change in curriculum in South Africa from the RNCS to CAPS. The data was collected under the RNCS. However, in my subsequent teaching with the CAPS curriculum, I have noticed that there is limited change in the content or approaches to fractions teaching. This suggests, that in my school at least, the findings of this study remain valid.

This study was limited to the teaching of fractions only. To get a fair reflection of what takes place in practice it is important to study both the teaching and learning that occurs. It would be interesting to investigate how learners work with and make sense of the different sub-constructs and unifying elements made available to them during fraction instruction. This will require extensive work with learners and could prove to be beneficial in the progress of establishing a fraction curriculum that allows for a connected, robust and complete understanding of fractions.

The framework developed in this research is in its early stages. The sub-constructs can be interpreted separately but when combined create a meaningful understanding of fractions. Future research needs to focus on more connections between these sub-constructs. This research only just scratched the surface, but it did provide a foundation for describing the interwoven connections among the sub-constructs. Future research should build on these connections and aim to find further connections.

6.8 Summary

This research contributes to the existing research and literature by relating what fraction knowledge is made available for learners in terms of sub-constructs, unifying elements and cognitive demands. This study enables us to determine what fraction instruction looks like in the Intermediate Phase in South Africa in order to develop best practice. The teaching of fractions is complex. This research reveals that teachers continue to struggle to teach fractions in a meaningful way. It is important that teachers are provided with opportunities and encouraged to improve fraction instruction so that learners develop a connected, robust and complete understanding of fractions. Fraction sub-constructs and the unifying elements play a vital in role in developing this connected, robust and complete understanding of fractions.

As a mathematics teacher in the Intermediate Phase this study has afforded me the opportunity to assess my own teaching of fractions. I have always been dissatisfied with what I have offered my learners in terms of fraction knowledge because I knew that there was more to fractions than the procedural tasks that I was offering them. While completing this study I have gathered a wealth of skills and fraction knowledge that I have already started implementing in my practise. I am aware that if I want to see change and improvement of fraction instruction at my school in the Intermediate Phase I will need buy-in and commitment from all the mathematics teachers in the phase. Our work will begin by investigating and reviewing tasks and activities that encourage work across sub-constructs and with unifying elements and incorporate these into our current fraction instruction. This is an exciting opportunity and one that I hope to embark on in the near future.

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ANNEXURES

Appendix 1: Principal consent letter and form

Dear Mrs. [redacted]

As you are aware, I am currently completing my PhD at the University of the Witwatersrand and am required to conduct a research project.

For the purpose of this study, I require a Grade 4, 5 and 6 Mathematics educator, as well as a group of Grade four, five and six learners who will participate in my study. The study aims to investigate the teaching of fractions across the Intermediate Phase (Grade 4 to Grade 6): what range of sub-constructs are made available, and how are these connected.

With your consent, and permission from the management, governing body and parents, I would like Mrs. [redacted], Mrs. [redacted] and Mr. [redacted] to be participants in the study. These educators have informally indicated that they would be willing to participate in the project.

I plan to observe lessons that are dedicated to the teaching of fractions. I plan to videotape these lessons as well as to have access to copies of some of the materials produced by the learners during these lessons. Lessons will continue as normal and as scheduled, with my presence in the back of the classroom. I also plan on conducting approximately 2 interviews with each educator. I will interview them before they introduce the concept of fractions and after they have taught it. I will be collecting data for approximately 3 weeks (i.e., 5 days per week, each lesson is an hour long). This works out to be approximately 15 hours in each classroom. The interviews will be about 45 minutes each. The total time required of the educator is approximately 17 hours.

All data collected will only be used for research purposes. There is a possibility that the research could be reported at appropriate conferences or in relevant journals. I assure you that anonymity and confidentiality will be protected in all written and verbal reports by making use of a pseudonym to refer to the school, teacher and learners. Video extracts, where anonymity cannot be provided, will only be used with consent from the learners and parents/guardians. Upon completion of the project, all data collected will be archived and securely stored at the University of the Witwatersrand for a maximum period of five years.

Please note that if consent is not granted, I will respect your decision. In addition, if at any point you wish to withdraw your consent, you may do so without any penalty or prejudice.

Thank you for your support.

Yours sincerely

Sharon Govender

CONSENT FORM (PRINCIPAL):

I, ____ (please print full name,) the principal of [redacted] give consent to the following:

1. Allowing Ms Govender to conduct her research at [redacted]

YES [ ] NO [ ] please tick

2. Videotaping of lessons on mathematics fractions in which an educator of the school might appear as part of the videotext.

YES [ ] NO [ ] please tick

3. Copies made of classwork, homework or assessment that learners might produce as part of these lessons.

YES [ ] NO [ ] please tick

4. Tape recording of interviews with researcher.

YES [ ] NO [ ] please tick

Signed: ____________________________________

Date: ______________________________________

Appendix 2: Teacher consent letter and form

Dear Mr. [redacted]

As you are aware, I am currently completing my PhD at the University of the Witwatersrand and am required to conduct a research project.

For the purpose of this study, I require a Grade 4/5/6 Mathematics educator as well as a group of Grade 4/5/6 learners who will participate in my study. The study aims to investigate the teaching of fractions across the Intermediate Phase (Grade 4 to Grade 6): what range of sub-constructs are made available, and how are these connected.

With your consent, and permission from Mrs. [redacted], management, the governing body and parents, I would like you and Grade 4/5/6 to be participants in the study.

In this phase of the project the focus will be on classroom teaching of fractions in Grade 4/5/6, as well as interviews with you at two points. The interviews will be conducted before you introduce the concept of fractions and after you have taught it. I will be collecting data for approximately 3 weeks (i.e., 5 days per week, each lesson is an hour long). This works out to be approximately 15 hours in the classroom. The interviews will be about 45 minutes each. The total time required of you is approximately 17 hours.

I plan to observe lessons that are dedicated to the teaching of fractions. I plan to videotape these lessons as well as to have access to copies of some of the materials produced by your learners during these lessons. Since you are the teacher of these learners in these classes, I ask for your consent to allow me where necessary to have access to copies of materials that your learners might produce. Lessons will continue as normal and as scheduled, with my presence in the back of the classroom.

All data collected will only be used for research purposes. There is a possibility that the research could be reported at appropriate conferences or in relevant journals. I assure you that anonymity and confidentiality will be protected in all written and verbal reports by making use of a pseudonym to refer to the school, teacher and learners. Video extracts, where anonymity cannot be provided, will only be used with your and your learners' consent. Upon completion of the project, all data collected will be archived and securely stored at the University of the Witwatersrand for a maximum period of five years. The findings of my study will be communicated with you, if you so desire, upon completion of my study. Please note that the results of the study will not be shared with the school's principal.

Please note that if consent is not granted, I will respect your decision. In addition, if at any point you wish to withdraw your consent, you may do so without any penalty or prejudice.

Please complete the form attached and return it to Ms. Govender at your earliest convenience. I will be happy to answer any questions or queries that you might have.

Looking forward to hearing from you. Thank you for your support.

Yours sincerely
Ms. Govender

CONSENT FORM (TEACHER):

I, ____ (please print full name), a Grade 5 teacher, give consent to the following:

1. Videotaping of lessons on mathematics fractions in which I might appear as part of the videotext.

YES [ ]   NO [ ]   please tick

2. Copies made of classwork, homework or assessment that my learners might produce as part of these lessons.

YES [ ]   NO [ ]   please tick

3. Tape recording of interviews with researcher.

YES [ ]   NO [ ]   please tick

Signed: _______________________________________

Date: ____

Appendix 3: Learner consent letter and form

Dear Parents/Guardians

I am currently completing my PhD at the University of the Witwatersrand in Johannesburg. As part of my study, I am investigating the teaching of fractions across the Intermediate Phase (Grade 4 to Grade 6): what range of sub-constructs are made available, and how are these connected.

This letter is to request your consent for your child/ward to participate in the above-mentioned research project.

In this phase of the project the focus will be on classroom teaching of fractions in Grade 4,5 and 6. I plan to observe lessons that are dedicated to the teaching of fractions. I plan to videotape these lessons as well as to have access to copies of some of the materials produced by your child/ward during these lessons. Since you are the parent/guardian of the learners in these classes, I ask for your consent to allow your child/ward to appear as part of the videotext and where necessary to have access to copies of materials that your child/ward might produce. Lessons will continue as normal and as scheduled, with my presence in the back of the classroom.

All data collected will only be used for research purposes. There is a possibility that the research could be reported at appropriate conferences or in relevant journals. I assure you that anonymity and confidentiality will be protected in all written and verbal reports by making use of a pseudonym to refer to the school, teacher and learners. Video extracts, where anonymity cannot be provided, will only be used with you and your child/wards' consent. Upon completion of the project, all data collected will be archived and securely stored at the University of the Witwatersrand for a maximum period of five years. I will be collecting data for approximately 3 weeks (i.e., 5 days per week, each lesson is an hour long). This works out to be approximately 15 hours in the classroom. The total time required of the learners is approximately 15 hours.

Please note that if consent is not granted, I will respect your decision. Therefore, your child/ward together with any other children not participating in the project will be seated on one side of the classroom and will not be videotaped. Furthermore, any text that your child/ward might produce will not be used in the project. In addition, if at any point you wish to withdraw your consent, you may do so without any penalty or prejudice.

Please complete the form attached and return it to Ms. Govender at your earliest convenience. I will be happy to answer any questions or queries that you might have.

Looking forward to hearing from you. Thank you for your support.

Yours sincerely
Ms. Govender

CONSENT FORM (PARENTS/GUARDIANS):

I, ____ (please print full name,) parent/guardian of ____________________________________,
give consent to the following:

1. Videotaping of lessons on mathematics fractions in which my child/ward might appear as part of the videotext.

YES [ ]   NO [ ]   please tick

2. Copies made of classwork, homework or assessment that my child/ward might produce as part of these lessons.

YES [ ]   NO [ ]   please tick

Signed: ______________________________________

Date: ____

Appendix 4: Ethics Clearance issued by the University of the Witwatersrand

Wits School of Education

27 St Andrews Road, Parktown, Johannesburg, 2193 • Private Bag 3, Wits 2050, South Africa

Tel: +27 11 717-3064 • Fax: +27 11 717-3100 • E-mail: enquiries@educ.wits.ac.za • Website: www.wits.ac.za

Student number: 9309109G

2011ECE149**C**

14 December 2011

Ms. Sharon Govender

Sharongovender0@gmail.com

Dear Ms. Govender

Re: Application for Ethics: Doctor of Philosophy

Thank you very much for your ethics application. The Ethics Committee in Education of the Faculty of Humanities, acting on behalf of the Senate has considered your application for ethics clearance for your proposal entitled:

Investigating the teaching of fractions across the Intermediate Phase (Grade 4 to Grade 6): What ranges of sub-constructs are made available, and how are these connected?

The committee recently met and I am pleased to inform you that clearance was granted. The committee was delighted about the ways in which you have taken care of and given consideration to the ethical dimensions of your research project. Congratulations to you and your supervisor!

Please use the above protocol number in all correspondence to the relevant research parties (schools, parents, learners etc.) and include it in your research report or project on the title page.

The Protocol Number above should be submitted to the Graduate Studies in Education

Committee upon submission of your final research report.

All the best with your research project.

Yours sincerely

M Mabeta

Matsie Mabeta
Wits School of Education

(011) 717 3416

Cc Supervisor: Prof. H Venkatakrishnan (via email)

Appendix 5; Table showing the fraction – related assessment standards that appear in the RNCS (DoE, 2002) across grades (Grades 4-6).

	Grade 4	Grade 5	Grade 6
AS	• Recognise, represents, describes, compares numbers: - common fractions (different denominators), - Common fractions in diagrammatic form - Decimal fractions in the context of measurement. -the equivalence of division and factions • Solve problems involving: - comparing two or more quantities of the same kind (ratio) - comparing two or more quantities of different same kinds (rate) - using a number line - doubling and halving • Estimate, calculate a problem solve: -Addition of common fractions in context.	• Counting forward & backwards in fractions. • Recognise, represents, describes, compares numbers: - common fractions (different denominators), - Common fractions in diagrammatic form - Decimal fractions in the context of measurement. -the equivalence of division and factions • Solve problems involving: - comparing two or more quantities of the same kind (ratio) - comparing two or more quantities of different same kinds (rate) - using a number line - doubling and halving • Estimate, calculate and solve problems: - Addition of common fractions in context. - Addition & subtraction of	• Counting forward & backwards in decimal fractions. • Recognise, represents, describes, compares numbers: - common fractions (different denominators), - Common fractions in diagrammatic form - Decimal fractions in the context of measurement. -the equivalence of division and factions • Solve problems involving: - comparing two or more quantities of the same kind (ratio) - comparing two or more quantities of different same kinds (rate) - using a number line - doubling and halving • Estimate, calculate and solve problems: - Addition of common

		common fractions with the same denominators & whole numbers with common fractions (mixed numbers) - finding fractions of hole numbers which results in whole numbers - equivalent fractions	fractions in context. - Addition & subtraction of common fractions with denominators which are multiples of each other & whole numbers with common fractions (mixed numbers) - Addition & subtraction of common fractions with the same denominators & whole numbers with common fractions (mixed numbers) - finding fractions of whole numbers which results in whole numbers - equivalent fractions - addition & subtraction of positive decimals • Place value - decimal fractions • Transformation - enlargements & reductions of shapes