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ORIGINAL ARTICLE

Deconstructing South African Grade 1 learners' awareness of number in terms of cardinality, ordinality and relational understandings

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Abstract
The cardinal and ordinal aspects of number have been widely written about as key constructs that need to be brought together in children's understanding in order for them to appreciate the idea of numerosity. In this paper, we discuss similarities and differences in the ways in which understandings not only of ordinality, cardinality but also additive and multiplicative relations have been theorized. We examine how the connections between these can be considered through a focus on number line representations and children positioning and comparing numbers. The responses of a cohort of South African Grade 1 learners' (6- and 7-year-olds) to a numerical magnitude estimation task and to a numerical comparison task are analysed and the findings compared to those in the international literature, some of which argue that children's early, informal, understandings of cardinality and ordinality are underpinned by an intuitive logarithmic model relating number order and size. A main finding presented here is that the responses from learners in this study exhibited a better fit with an exponential model of the relationship between cardinality and ordinality. These findings raise questions about whether some of the findings in previous research are as universal as sometimes claimed.

Keywords Cardinality · Ordinality · Relational understanding · Log-to-linear shift

1 Introduction

Verschaffel (2017), in a plenary presentation surveying research into young children's learning about number, and, specifically, their perceptions of number magnitude, distinguished differences in the foci of cognitive research and mathematics education research. He noted that the purely psychological concerns arising from cognitive studies typically focus on number magnitude in ways that foreground cardinality in continuous quantity contexts such as length. Such studies have, however, paid limited attention to ordinal understandings of number, and even less attention to the range of additional concerns that feature more prominently within mathematics education studies of young children learning about number. Key among these additional concerns, according to Verschaffel, include attention to pattern and, of particular relevance to this paper, quantitative relations arising from mathematical structure. In the context of young children learning about number, quantitative relations can be considered as arising in two ways: as parts and wholes connected by additive relationships (for example, 10 is 8 more than 2), or as parts and wholes connected in ratio-based multiplicative relationships (for example, 10 is twice as large as 5, see for example, Askew 2018).

Our motivation for the work reported on here lies firmly in this breadth of concerns within the mathematics education sphere. The research is located in a South African schooling context where there is wide evidence of delays in mathematical and language development for large numbers of children, causes of which include poverty, poor early childhood nutrition and stimulation, and lack of access to quality schooling (Fleisch 2008). Well into the upper primary years there is evidence of children's persistent use of concrete counting (fingers, objects or tally marks) and, in multiplicative problem contexts, use of repeated addition-based solution methods, applying such methods to number ranges that render the approaches not only highly inefficient but also error-prone (Schollar 2008). Schollar suggests that heavy emphasis in foundation phase classes in South

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Africa (Grades 1–3) on recitation of oral counts and returning to unit counting to check answers may, in part, account for these delays in learners developing more sophisticated methods. What this local evidence base does not provide, however, is any disaggregation of the ordinal, cardinal, and relational (additive and multiplicative) understandings of number amongst young learners that might need to be built on and developed in moving learners away from concrete and additive approaches.

A key question then, arising from the above observations and explored below in the background literature, is what is the nature of the connection between developing understandings of cardinality and ordinality, and the contribution that informal knowledge of additive and multiplicative relations might make to developing these understandings?

In this paper, we begin to explore this question by presenting and discussing findings from the responses of a sample of South African Grade 1 learners (6- and 7-year-olds) to two tasks adapted from the intervention studies of Siegler (2009)—tasks that bring into focus the nature of young children's ability to co-ordinate ordinal and cardinal aspects of number, and what this reveals about their relational understandings of quantities in terms of whether this coordinating appears to be based in additive or in multiplicative relations. The findings enable us to flag which coordinating aspects appear to be foregrounded, which that are less well developed, and, therefore, what it might be important to focus on within instruction.

2 Background literature

The writing on developing number understanding in young children, on number sense, commonly points out not only the importance of children having a range of understandings about number but also the importance of deep interconnections between these understandings (McIntosh et al. 1992). Ordinal, cardinal and relational understandings are widely considered to be central to this range of understandings and so we begin with review of the literature on these constructs, and how various writers theorize the connections and distinctions between them.

Linked awareness of ordinal and cardinal aspects of number are widely understood to be critical to children becoming both fluent in early oral counting, and also in developing an appreciation of numerosity. These constructs figure prominently within Gelman and Gallistel's (1978) list of principles required for counting discrete quantities, with the 'stable order' sequence of number names needing to be operationalized in coordination with 'order irrelevance' principle (that the order in which objects are counted does not make a difference to their overall numerosity). These ordinal principles need to be coordinated with the cardinality principle, that is, the awareness of a shift from the last number in a count not merely being a 'label' for the last object counted, but to also then represent the total number of objects in the set being counted. Several longitudinal studies have confirmed that both reciting the ordinal number sequence and using this sequence to enumerate collections are important predictors of later mathematical performance (e.g. Aunola et al. 2004).

The importance of awareness of arithmetical relations has also been identified as predictive of later mathematical performance, with some evidence pointing to understandings of quantities and relations not only correlating with later performance but also being distinct from proficiency in whole number sequence counting skills (Aunio et al. 2006). The nature, however, of the relations in focus is quite varied across different studies. In the psychological literature, a large number of studies have looked at additive comparisons in pre-verbal/symbolic number settings, accumulating evidence that very young infants (less than a year old) can discriminate between two auditory or visual quantities in small number ranges (Kobayashi et al. 2005) or when the multiplicative ratio between two quantities is large (Xu et al. 2005).

Of particular interest in relation to these findings is Dehaene's (1997) claim that findings from such studies of pre-symbolic understandings, and other studies focusing on the symbolic number terrain, suggest that human beings have an innate sense of a mental number line, that we are drawn to an 'automatic association between numbers and space […] It is as if numbers were mentally aligned on a segment, with each location corresponding to a particular quantity' (p. 70). Tasks involving children having to position numerals number lines with the upper and lower bounds of the line marked but no intermediate markings, explore children's intuitive, spatial, understandings of quantity, in ways that reflect Dehaene's theoretical position. While 'pure' ordinal recitation of numerals need not entail any spatial dimension at all (learners, for example, could rely on rote memorization to order number names, as in the case of learning the alphabet sequence), in the number line context learners demonstrating an ordinal understanding of number that is based in relational understanding between successive numbers would be revealed through a strictly increasing positioning of number as distances from zero. Thus, in this task context, the notion of ordinality builds in a cardinal orientation to number, in which larger numbers are associated with 'more', or greater distances from zero.

In line with Dehaene's theory, Booth and Siegler (2008), in using bounded empty number line tasks across a variety of number ranges (0–10; 0–100; 0–1000), focus on the evidence of 'linearity' in children's positioning of numerals on a number line: the equi-spacing of numerals through iterating some implicit unit. Evidence of linearity in children's responses, they argue, corresponds to an awareness of the

structure of an interval, and involves positioning intermediate numbers on the line on the basis of this structure—recognizing, for example, that 5 is halfway along the 0–10 line, and spacing other numbers equally across the line segment. Their findings show that what they describe as 'extent of linearity' in children's responses to positioning numerals along a bounded but unmarked number line (in the 1–100 range in their study) not only correlates with broader mathematical understandings but is also predictive of children's later scores on unfamiliar addition tasks.

Dehaene (1997) argues that this linear and iterative awareness may be preceded by an intuitive 'relative' rather than 'absolute' awareness of magnitude: that the, metaphorical, distance between pairs of larger numbers is commonly perceived as smaller than the equivalent gap between pairs of smaller numbers, based on findings showing that the greater the size of the numbers being compared, the longer it takes to decide which is the larger (after allowing for the longer time to read the numbers). A range of studies supporting this relative rather than absolute hypothesis shows that young learners' perceptions of the magnitude of symbolic numbers fit, initially, with a logarithmic model that moves, as learners develop, to a more linear model. This result has proved consistent across both smaller (0–10 and 0–20) and larger (e.g. 0–1000) number ranges (e.g. Siegler et al. 2009). The logarithmic model thus suggests a multiplicative, ratio-based, rather than additive, iterative, foundation to this 'mental compression' of larger numbers that Dehaene (1997) describes (p. 65), in that as a ratio, 105, say, is 'closer' to 106 than 5 is to 6, whilst the linear difference between each pair is the same. The 'logarithmic to linear' shift of awareness is of interest to us, particularly because a number of studies are predicated on testing for the progression from a logarithmic to a linear patterning as the two basic options for ways of understanding number (e.g. Booth and Siegler 2008). This progression thus involves the (implicit) awareness of multiplicative relations being superseded by additive awareness.

This seeming 'innateness' of awareness of multiplicative relations seen in responses to bounded number line tasks arising from the cognitive psychology studies is interesting to examine from a mathematics education perspective. The evidence of multiplicative reasoning between numbers preceding additive reasoning between numbers stands in stark contrast with results from mathematics education research that point to the slow, and often limited, development of multiplicative reasoning over time (Clark and Kamii 1996), with substantial differences identified between additive reasoning and multiplicative reasoning (Nunes and Bryant 1996). Further, within school-based tasks, there is also evidence of children's preferences for incorrectly treating number relationships additively, rather than multiplicatively, in instances where non-integer ratio relationships between quantities are operated upon (van Dooren et al. 2010).

In the mathematics education literature there are two broadly opposing positions on the relationship between awareness of ordinality/cardinality and awareness of relations in young children's learning about number. A widely adopted position, with accompanying frameworks, takes ordinal/cardinal understandings as coming together in one-to-one counting, and developed and seen in concrete enumeration tasks. Thus, studies of cardinality in the literature on early years' education, rather than examining the ordering of distances, have more usually explored children's enumeration of discrete quantities (e.g. Fuson and Hall 1983). Such enumeration involves, for example, counting the number of fruits in a bowl containing apples, oranges and bananas. While each fruit counted increases the count by one, there is no need here for the unit being counted having to be of the 'same' magnitude as the other counted units. Essentially, in this kind of cardinal counting, there is no need for the existence of an 'anchor' unit that is iterated, as is the case in the number line tasks.

From this position of importance of concrete counting, later moves towards the use of relations, either through drawing on properties such as commutativity or associativity, or through working with 'benchmark' numbers (five and ten), abridge the processes of unit counting (Wright et al. 2006). Both property-based relations and benchmark-based relations involve awareness of the relationships between numbers being used to make strategic choices about how to arrive at arithmetic results—an underpinning feature of what Anghileri (2000) describes as number sense. From this position, counting, bringing together ordinal and cardinal understandings of collections of discrete objects, forms the springboard for work that later makes use of the properties and relationships that underpin arithmetic structures. Further, in many curricula, including South Africa's, understanding of additive relations is presumed to develop in advance of understanding multiplicative relations.

In contrast to this trajectory, literature following the Davydov approach argues for attention, from the start of schooling, to relations underpinned by multiplicative structuring (Schmittau 2003). Units created or selected through multiplicative relations (how many times larger one quantity is than another) can then be iterated for measurement activities (the sum of the units), thereby bringing additive relations into play through arising out of multiplicative relations. From this position, a very different instructional task sequence, based initially in non-numerical comparisons of length or weight or other continuous quantities, is presented, with learning about number emerging in the form of ratio-based measurement comparisons. Ordinal and cardinal counting, in this approach, are seen as 'spontaneous concepts' that children pick up at least some elements of in the context of their everyday activities. Since these understandings are spontaneously deployed in the context

of measurement activities, they are not viewed as aspects that need to be explicitly taught. And given the evidence, discussed above, from cognitive studies of a possible preference for a sense of multiplicative relations preceding additive relations, this Davydov approach would seem to be more in line with what children are inclined towards.

We are interested in these debates about the multiplicative/additive relations inter-connections for a number of reasons. Firstly, Dehaene suggests that young children's ability to compare pre-symbolic quantities appears slightly later than awareness of quantities, but still in the early infant years (1997, p. 51). This kind of comparison involves recognizing same/more/less relations, which can be accomplished through a base in ordinal or cardinal number understandings.

Second, while the evidence emanating from the bounded number line tasks discussed above tends to make little explicit reference to additive relational understandings, some studies have worked with unbounded number line tasks in which a unit of length one is indicated on a line, with children having to position a range of other larger numbers on the line (e.g. Cohen and Sarnecka 2014). Such tasks can be completed with a base in additive relational understandings of number. Interestingly, Cohen and Sarnecka suggest that the logarithmic pattern of number positioning found in the literature is a consequence of the bounded nature of the number line, rather than a feature of initial ways of seeing number relations.

The logarithmic pattern that emerges in both numerical magnitude estimation tasks and numerical (and pre-symbolic numerical) comparison tasks suggests a presence of some ratio-oriented structuring of number relationships with a multiplicative, or scalar, structure. Dehaene describes the initial logarithmic pattern as related to two effects: a 'magnitude effect' ('for an equal distance, we have a harder time discriminating large numerosities, such as 90 and 100, than two small ones, such as 10 and 20'); and a 'distance effect' ('we more easily distinguish two distant numerosities, such as 80 and 100, than two closer numbers such as 80 and 81'—p. 61). Over time, responses to numerical magnitude estimation tasks, in particular, move towards more linear magnitude representations, pointing to an 'over-riding' of the earlier 'mental compression' of larger numbers, a result that appears to be stable across smaller and larger number ranges, further contributing to the logarithmic to linear claim.

3 Research focus

We now turn to how this study contributes to the exploration of these different possible networks of connections. The above overview suggests a possible 'nested' set of understandings in relation to magnitude tasks, in which cardinal understandings are rolled in with ordinal appreciation, contained within additive and multiplicative relational awareness. Our choice of how to examine the nature of these connections arises from the evidence discussed above that the extent to which learners can place numerals along the bounded 0–10 number line, both in relation to successive numbers and in relation to the overall length of the line, can provide insights not only into their understandings of cardinality and ordinality, but also of additive and multiplicative relations We examine this through the findings from children's responses on two tasks in focus. The first task was a version of the bounded number line task (Siegler and Ramani 2009), as we were interested in whether children displayed any sense of the idea of a unit that is iterated (without being given such a unit, as in the unbounded version of the task), that is, whether they displayed awareness of the additive relation between numbers up to 10. Alongside this, another focus of our analysis is whether there was evidence of (intuitive) awareness of the multiplicative relation, demonstrated by the children working with an implicit unit that was in proportion to the length of the overall line, rather than selecting an arbitrary anchor unit without reference to the overall length.

Children's responses to a second numerical comparison task provided a secondary source of evidence that allowed us to explore distance and magnitude effects in the 0–10 number range. The second task involved children being given pairs of numbers and asked to say which was the larger. Patterns in these young South African learners' responses to this numerical comparison enable us to further explore whether choices appear to be based more in either linear or more logarithmic reasoning, in other words, whether the children were treating the relationship between two numbers as addtive or multiplicative.

The research questions that, thus, interest us are:

1.What do children's responses in the context of relative magnitude-focused tasks indicate about their awareness of the ordinality/cardinality of number, and how this relates to their implicit awareness of additive and multiplicative relations?

2.What do the findings suggest in terms of foci of interventions to develop these understandings?

The data drawn up below was initially intended to figure as a baseline dataset that would allow us to assess the extent to which an intervention based on Siegler and Ramani's (2009) linear number track game would be useful. Contextual constraints in the focal school made the intervention difficult to run as described in the original research. We did, however, analyse the pre-intervention assessments and this pointed to significant divergences in the children's responses from findings previously presented in the literature. Thus, our focus in this paper is on the outcomes of the initial testing, conducted in individual interview format, and the questions these outcomes raise for thinking about these children's

understandings of number—cardinality, ordinality and relations—and in particular, whether some of the findings in the research outlined above are as universal as sometimes claimed, and implications for teaching.

In the following section, we detail the methodology and data sources that we drew on to explore a sample of South African Grade 1 learners' responses to the numerical magnitude estimation task and numerical comparison task, before going on to present and discuss the findings for what they indicate about ordinal, cardinal and relational understandings of number.

4 Methodology

The data sources that we draw on were collected as part of a small-scale exploratory investigation of South African Grade 1 children's understanding of number concepts in the range 1–10. Age appropriate Grade 1 children in South Africa turn seven during the course of the year, but Grade 1 classes also frequently contain some older learners 'retained' or held back in the grade due to very poor prior attainment in curricular concepts. The 1–10 number range was chosen as it is the curriculum expectation for the first two terms of schooling in Grade 1 and the research was carried out early in the second term.

The school that this exploratory study was located in was a suburban government school in Johannesburg that was part of a larger research and development project that we are involved in—the Wits Maths Connect—Primary project. The school serves a historically disadvantaged learner population in the post-apartheid South African terrain, but is also relatively privileged in terms of its school buildings and facilities, and class sizes (42, 43 and 44 learners were assessed across the three classes in the grade, with these numbers smaller than the class sizes that are common in township and rural schools). This sample, in the broader South African context, tends to perform more strongly than learner samples drawn from township and rural schools—a feature that we were alert to in thinking about implications for teaching. The Grade 1 cohort that were assessed had not been involved in any project initiatives, and therefore, provided a convenience sample for exploring understandings of number—ordinal, cardinal and relational. In this paper, we draw on the pre-test responses from learners for whom we had pre- and post-test data on the two tasks—resulting in data for 109 learners.

Members of our research team administered the assessment in school to all the learners individually, who were withdrawn from their lesson for approximately 10 min for each assessment. Here we report on two tasks from the assessment, adapted from Siegler and Ramani's (2009) paper detailing the orally-administered assessment they used to assess the efficacy of a linear number track intervention programme, a version of a task that has been widely administered in psychological studies of number sense. The first task is based on the bounded 0–10 numerical magnitude estimation task outlined earlier. With 0 at the left-hand end of a blank number line and 10 at the right-hand end of this line, a numeral was printed above the line. Each child was presented with all the integers between 0 and 10 on different slips of paper, but with the length of the 0–10 number line segment being the same in each case. The numerals were presented to each child in the same, non-sequential order: 3, 6, 9, 2, 5, 8, 1, 4, 7. The researcher asked the child to read the numeral printed above the line. All the children in our sample could correctly read all the numerals from 1 to 9 in the order in which they were presented. Given success in reading the numeral, each child was then asked to point to where on the number line they thought the stated number should be marked. The researcher checked the position by also pointing, asking 'here?' and upon the child's agreement, marking the place on the line.

In our administration of the task, we chose for the child to position each subsequent (but not in numerical order) numeral on a clean number line, with previous markings turned over, and therefore not visible to the child. Our interest in setting the task in this way arose from the desire to examine children's cardinal, ordinal, additive and relational awareness of number, with evidence of all of these understandings possible to discern through the task in this form. Through the numerals only being presented to children one at a time, in a non-sequential order and with previous positioning not being visible, then any correct ordinal positioning could not be arrived at simply through rote memorization of the order of number names, and by not starting with the numeral 1, children were not going to be 'tuned in' to creating a unit that they might iterate.

To examine the correctness of the ordinal aspects of the child's positioning of the numerals, we transcribed each individual's separate numeral positions onto a single number line to capture the order in which the numerals were placed. To examine the cardinal aspects of the child's positioning we then recorded learners' empirical positioning for the numerals in comparison to where they should actually have been positioned on the bounded number line. This enabled us to examine two things. Firstly, to calculate the value of the position that the child had actually located on the line. Second, the children's spacing between the numerals could then be examined for evidence of any mathematical model apparently underlying that positioning (linear or logarithmic). By plotting the median actual plotted value of all the learners' responses against the given numeral we were able to examine a number of lines of best fit. This then provided insight into the implicit relational understanding of the connections between

the numbers, and whether these appeared to be predominantly multiplicative or additive.

The second task that we discuss comprised a series of numerical comparison series of problems. Children were shown pairs of numbers in turn, with the interviewer saying, in association with pointing to, for example a 1 and 6 card: 'Hamsa had one egg. Corin had six eggs. Which is more, one or six? Eighteen such pairs were offered: nine of the items within the 1–9 range having a difference of three or less (5, 2; 8, 9; 4, 5; 3, 5; 7, 6; 5, 6; 9, 6; 6, 4; 2, 1), five items within the 1–18 range with a difference greater than 3 but less than 8 (8, 1; 2, 9; 1, 7; 12, 17; 14, 9), two items within the 19–35 range with a difference of three or less (25, 26; 34, 31) and two items within this range with a larger difference (19, 23; 32, 28). The order in which the numbers was presented on the cards—larger/smaller or smaller/larger—was randomly split 50/50 across the 18 pairs. These pairs were selected so that we could compare the ordering facilities with that on the number line and also begin to examine the children's responses both for magnitude and distance effects (measured in terms of correct answers, rather than time taken to answer). We recorded the results for this task by circling the number indicated in each learner's response on a record sheet, with our subsequent data capturing noting the correctness, or not, of this response. We subsequently collated facilities for each item and considered these facilities in relation to the magnitude and distance effects noted in the literature.

Our interviews were not strictly 'clinical' as they were not scripted for the wording to be exactly the same for each administration, although on these two items, children were asked every question—to position all the numerals from 1 to 9 and to compare all 18 pairs of numbers. Our analyses in this paper make use of exploratory data analysis approaches and techniques, with this choice linked to our probing for possible patterns in a dataset that was marked by key demarcations from the findings in the cognition studies literature outlined above.

5 Findings

5.1 Numerical magnitude estimation task

As noted above, individual learner's responses were measured from zero along the line and then the value of the actual number that had been marked calculated. For example, Table 1 shows one learner's positioning of the numbers 1–9 compared to the numbers (to two decimal places) they actually positioned on the line were as follows:

The first thing to note, for this child, is that the location of successive numbers is strictly increasing. Given that this positioning arose from a task in which the numerals were not only presented in a non-sequential order but were each positioned independently of any other, with prior positioning not being visible, this points to this child 'holding' onto an ordinal image of these numbers across the individual markings made. If the child had simply had a rote memorization of the order, then, under the circumstances of the assessment it would have been highly unlikely that she could produce this strict ordering of the distances from zero.

In examining the learners' responses overall, and reported on below, we are not only interested in whether they display such understanding of a strictly increasing ordering, but also whether they display a sense of an anchor unit that is iterated, indicative of awareness of the additive relation. For instance, for the example given in Table 1, the 0–4 positioning can be modelled as the result of iterating a segment of approximately 9 mm, but this pattern breaks down beyond this range, with a particularly large gap between the positioning of 6 and 7 showing the breakdown of any linearity in this child's positioning of the numerals.

This particular child's response also shows considerable underestimating of the position of numbers up to 6, with, as noted, a large gap between 6 and 7, and then 7–9 being overestimated, but closer to their actual positions than any of the numbers up to 6. Were these positions to fit with the oft-reported logarithmic model, then we would expect the reverse of this with smaller numbers more spread-out and the 'bunching' to be seen at the upper end.

The pattern demonstrated by this one learner was seen to play out more generally across the sample. Figure 1 presents box and whisker plots showing the number value of where all learners ($n = 109$) actually plotted each of the numerals 1, 2, 3, etc.

We look first at what the results show about the children's facility in ordering the numerals. Whilst the child whose results are shown in Table 1 succeeded in positioning all the numbers from 1 to 9 in a correct, strictly increasing distance from zero, this was not the case for all learners. Figure 1 shows that median values plotted did increase in value in line with the numeral order. So, the results from this assessment item show that, overall, the sample of learners did display the ability to position numerals independently of each other along the bounded, unmarked number line and for this positioning, in general, to be in the correct order. However, as the number of outliers show for the positioning of the smaller

Table 1 Numeral plotted against actual value on 1–10 number line

Numeral	1	2	3	4	5	6	7	8	9
Number actually positioned	0.17	0.51	0.91	1.27	1.48	2.87	7.80	8.65	9.45

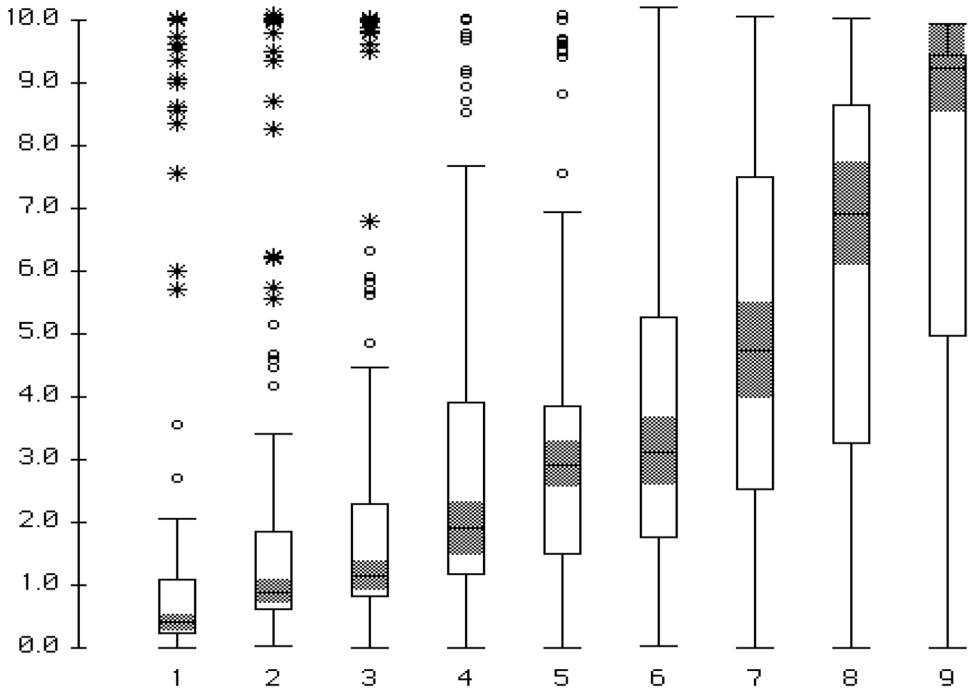


Fig. 1 Box & whisker plot of learners' numeral positioning for 1–9

numerals, at the level of the individual learner, errors in ordering the numerals did occur.

Looking at where the most common mis-orderings occurred, Table 2 shows the percentages of the learners who mis-ordering (that is positioned one in front of the other) pairs of consecutive numbers. This shows that the fewest errors occurred for consecutive numbers at the upper and lower ends of the range, with around a fifth of learners each positioning 2 before 1 and 9 before 8. The most frequent error was in ordering 5 and 6 with almost two-fifths of the learners marking the position of 6 on the number line in front of where they marked 5. Given emphasis in the literature on the importance of 5 as a benchmark number this finding suggests that many of these learners had yet to develop that sense of the importance of 5.

Turning to consider the actual number of errors that learners made, then the median number of ordering errors made was 2 with some learners mis-ordering up to 6 of the numbers. We show, in Fig. 2, on the x-axis, the number of ordinal (rank ordering) errors each learner made, from zero (all numerals positioned in the correct order) to the maximum of 6 numerals being incorrectly ordered. These are plotted against the total absolute magnitude of the error of marked numbers in relation to actual number positions, that is the sum of the measured differences between the learner's placement of each numeral and the actual placement of that number. This graph shows that good ordinal understanding (i.e. fewer ordinal errors) was associated with lower error magnitudes. This suggests that awareness of order, and being able to hold this awareness in mind even when previous markings are not visible, may be linked to having a sense of the need to iterate a unit. Without such an awareness of an implicit unit, it would be possible to position the numerals in the correct order and still make large error magnitudes. But the data shows instead that an increased in the number of ordinal errors is associated an increase in the size of the total error magnitude. Thus, the data indicates that better awareness of ordinality appears to develop alongside better awareness of cardinality, alongside having better awareness of multiplicative relations in the 0–10 range.

We turn now to consider what the data reveals about the learners understanding of cardinality, as revealed by the distances from zero at which they positioned the numerals. Figure 1 also shows that the mean intervals between consecutive numerals were small for the numerals 1–3, gradually increasing from 4 to 6 and then greatly increasing from 7 to 9. As noted, if this data were to fit with the literature discussed, that is, were the learners' modelling to fit with a logarithmic,

Table 2 Errors in ordering consecutive pairs of numerals

Order incorrect	1 & 2	2 & 3	3 & 4	4 & 5	5 & 6	6 & 7	7 & 8	8 & 9
Percentage of children ordering incorrectly (n = 106)	22	34	28	26	37	32	23	20

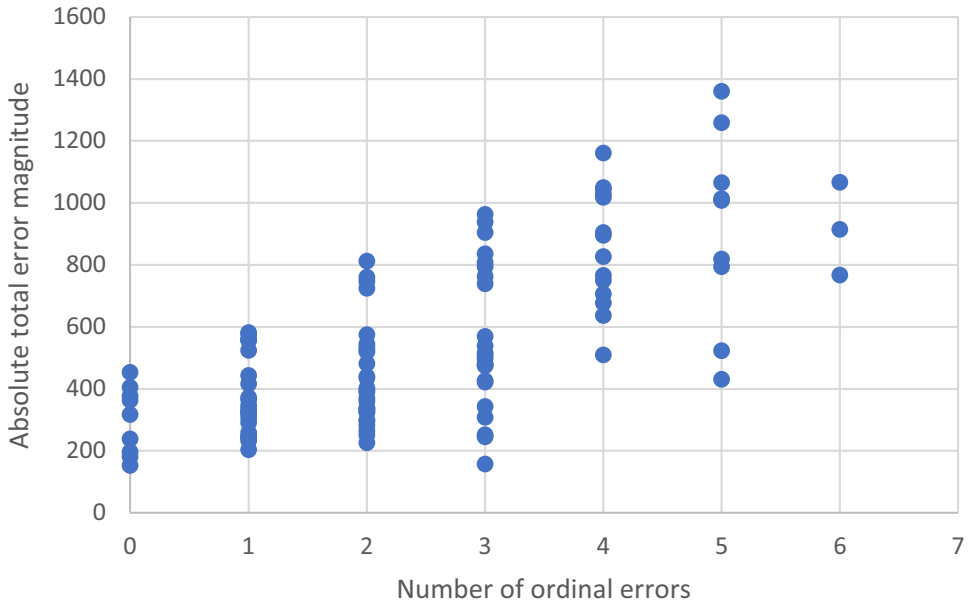


Fig. 2 Numeral positioned against median value plotted, with exponential line of best fit

multiplicative scaling, then the medians should actually get closer together as the numerals get larger, the exact reverse of the pattern in Fig. 1, where the intervals become more stretched out rather than compressed as the numerals get larger. The pattern of medians looks more exponential than logarithmic here. Figure 1 also indicates increasing variance in positioning as the numeral to be positioned gets larger, as shown by the boxes. Figure 3 presents the median values plotted: the line of best fit for this graph is indeed exponential with $R^2 = 0.98$. In comparison, fitting a logarithmic line to the graph returns a value only of $R^2 = 0.686$, showing that this data does not align with the oft-reported findings of a logarithmic underlying model. Fitting a linear line of best fit returns a value of $R^2 = 0.896$ respectively, but this relatively high figure is a consequence of some (condensed) linearity in the positioning of the small numerals, the fit of the line not being good across the full range.

This finding of an exponential best fit not only challenges the claim that a logarithmic reasoning is universal in young learners but also brings into question the argument put forward in Cohen and Sarnecka's (2014) cognition study mentioned earlier which claimed that the compression of larger

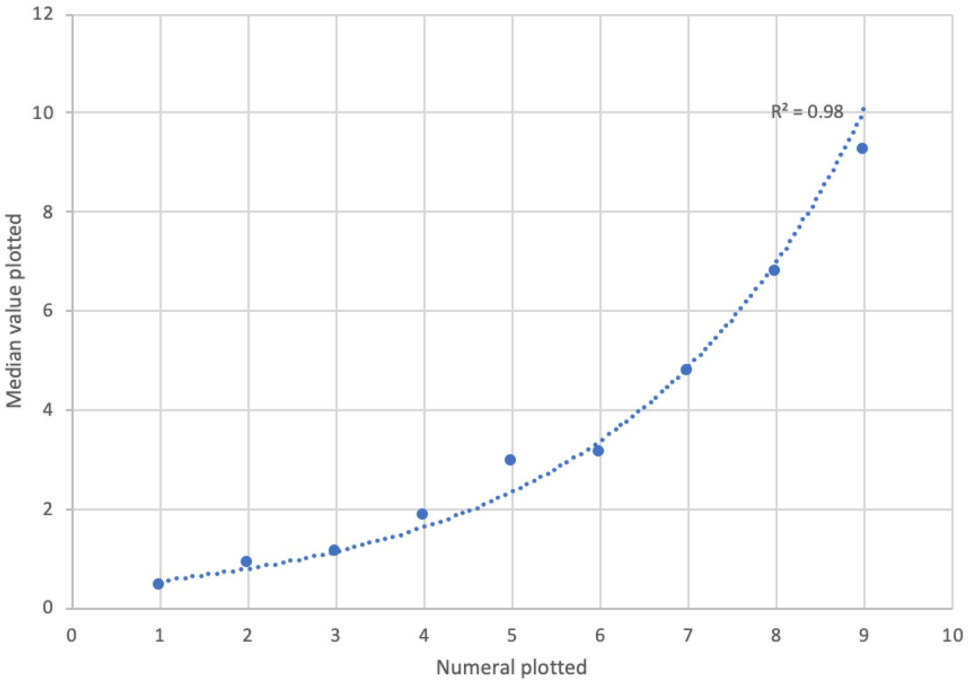


Fig. 3 Number of learners making ordinal errors against total magnitude of errors

numbers was an effect of the upper bounding of the number line with 10.

5.2 Numerical comparison task

This item presents openings for using awareness of ordinality to consider a cardinally framed question in the context of presentation of two numbers and asking: Which is more? Figure 4 shows the percentages of learners correctly identifying the larger of various pairs of numbers, in relation to the size of the difference between the two numbers in each pair.

The x-axis in Fig. 4 organizes the pairs according to the size of the difference between the two numbers, from a difference of 1 to a maximum difference of 7, and the y-axis presents the facility on each item. As noted above there is evidence in the literature of a magnitude effect in making comparison; the larger the pair of numbers being compared, the harder it is to make the comparison. As might be expected, in our data the pair 2, 1 had the highest facility (94%) and the pair 32, 28 the lowest at 28%. The pattern of response in pairs of numbers with difference 1 indicate a magnitude effect—lower number ranges have higher facility rates (although 8, 9 is anomalous). Similar patterns are shown for pairs with a difference of 2 or a difference of 3. So, in line with other findings, our data suggest a slight magnitude effect: facilities decrease as the numbers get larger, with six of the seven items where the overall facility was below 76% involving numbers greater than 10.

Less in evidence was a clear distance effect—facilities on correctly identifying the larger of a pair of numbers as the differences between the numbers increased were not necessarily higher, although, as Fig. 4 shows, there was a slow increase in facility with increasing difference between values.

The higher than might be expected facility for the 34, 31 pair in comparison to 32, 28, suggests an additional complexity coming in when learners had to consider numbers on different sides of a multiple of ten—either side of 30 in the case of 32 and 28—as opposed to a pair of numbers within a multiple of ten range—from 30 to 39 in the case of 34 and 31.

6 Discussion

Our findings diverge from the model of development of the logarithmic to linear shift in the underlying model for positioning numbers on the line, as discussed in the (cognitive psychological) literature, a development which is not absolute, but is age dependent as well as dependent on the number range. Here, both the number range (which was deliberately small) and the age of the learners (slightly older than in other studies) would suggest that we would have found more evidence for a linear model for positioning the numerals. From our observations of children's behaviours during the assessment we speculate as to why our learners' responses do not fit with the logarithmic-to-linear model proposed in

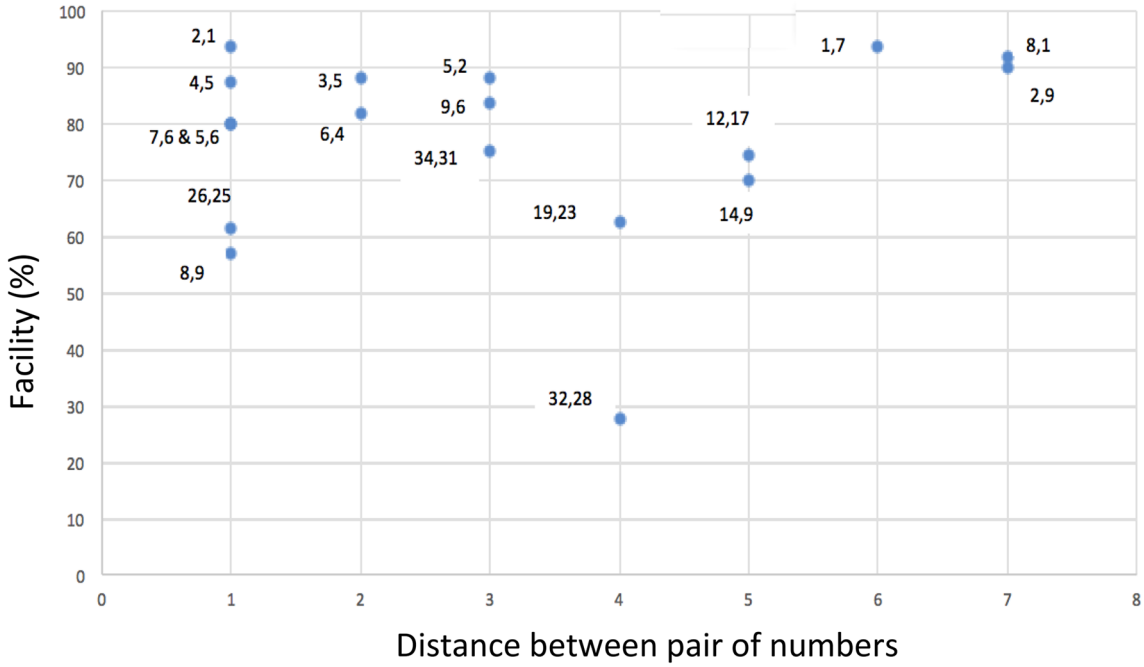


Fig. 4 Facility of larger number identification in relation to difference between numbers (n = 109)

the literature, and although their responses were closer to a linear model, have an exponential model as the best fit. During the interviews we frequently observed, across the sample, children counting segments in 1 s from 0 in a more or less iterative fashion, using a small unit up until the positioning of 8 (hence the evidence of some sense of linearity). In contrast, when asked to mark 9, many children shifted their eyes immediately to attend to the 10 end of the line, and then positioning 9 relative to 10 rather than 0. Thus, the high median marking for 9 resulting in a large distance between 8 and 9, and consequently 9 being the only number that learners tended to overestimate rather than underestimate.

Our main finding, thus, is that, for these learners at least, the evidence does not point towards the argument that young learners 'naturally' treat ordinality in a logarithmic fashion which later becomes replaced with a scalar, linear positioning of numbers. The mainly non-overlapping medians in Fig. 1 suggest these children had a reasonable ordinal sense of numbers but without a consistent sense of how this relates to cardinality, with numbers up to 7 or 8 all being associated with being very close to 1 and then a flip to 9 being associated with ten. Contrary to findings that the positioning of numerals gets compressed as learners position numbers approaching ten, due either to the logarithmic understanding, or the placing of ten on the line having a 'capping' effect on the space available for placing larger numbers, these learners spread out the numerals more as they got larger. We note, further, that in the interview, 9 was the third numeral the children were asked to position. So, despite that positioning bringing their attention to the upper bound of the line, they still reverted to 'small steps from zero' for subsequent numerals to 6. This 'small steps' sense and the number of learners placing 9 closer to 10, in our sample, seemed not be a result of the capping of the end of the line requiring larger numbers to be 'squeezed in' but, as indicated above, more a result of using a very small unit to 'step back' from 10. This result suggests these children did attend to the 'bounds' of the number line but in a sense different to that suggested by the findings of Cohen and Sarnecka (2014) in their argument that the capping of the line leads to the logarithmic result.

This data points to a foregrounding of linked ordinal/cardinal awareness in children's early work with number—a finding backed up by the extensive evidence of concrete (and cardinal) unit counting in South Africa when faced with additive problems to solve. However, the numerical magnitude estimation results suggest that this awareness stretches neither to additive structure in terms of unit iteration, nor to a multiplicative structure that produces linearity.

With regard to the ordering of pairs of numbers, the learners were more in line with the literature-based findings of evidence of a magnitude effect: ordering larger pairs of numbers is more difficult than ordering a pair of smaller numbers with the same difference. But there was less evidence of a distance effect—pairs of numbers with a larger difference were not necessarily easier to discriminate than pairs with a smaller difference.

As noted in the findings, the facility of identifying the larger of 2 and 1 in this assessment showed that the vast majority of the children could answer this correctly (94%) whereas on the positioning on the number line task, a fifth of the children mis-ordered this pair of numerals. The difference in these two results suggest that some of the children could order these small numbers when presented in the context that invited them to consider the numbers in a discrete cardinality sense ('Hamsa had two eggs. Corin had one egg. Which is more, two or one?) but had less awareness of the cardinality in the continuous number line sense.

It is also possible that the high facility on comparing 2 and 1 was simply a consequence of familiarity with the order counting order, in turn possibly a result of the heavy emphasis in foundation phase classes in South Africa on frequent recitation of oral counts. As the majority of counting in this year group is counting in ones or twos, the higher facilities of correctly identifying the larger number when the difference in the pair was 1 or 2, than was the case for differences of 3, may be a consequence of recognition of the pattern as part of the recitation, 'four, six', for example, being a more common pairing of numbers spoken and heard than, say, 'four, seven'. Whatever might be an underlying cause of the difference in the findings on ordering 2 and 1 across the two task is worthy of further investigation.

The anomaly in the comparing numbers findings is the result for 8, 9 which is considerably lower than that for the other pairs of numbers that had a difference of 1. But as noted above, 8, 9 was also something of a 'break-point' in the number line task, with 9 being positioned much closer to 10 than expected, and 8 closer to zero. Rather than treating 8 and 9 as both being part of an ordinal sequence where the cardinal gap between each number in the sequence is the same, it is as if for these learners there are two cardinal sequences in play—the one that operates up to 7 or 8 and a different one that comes into play at 9—perhaps involving a backward count from 10, another frequently rehearsed counting pattern, and consistent with the 'step back from ten' approach noted above. In that case, if 8 and 9 operate under different 'rules', deciding which is the larger is not straightforward.

This was a small-scale study and although it raises some interesting findings, there are limitations to the study that could be explored in further research. First, although the school and its catchment area were typical of many previously disadvantaged schools in South Africa the results are likely to be somewhat better would be obtained from learners attending township or rural schools. Future research could usefully gather data from such learners to gather more representative data that confirmatory statistics could be run

on. We also, deliberately, restricted our attention to the 0–10 number range as that was in line with curriculum expectations. It would be interesting to see if the exponential model we found also was the case in similar tasks with a greater number range. It would also be helpful to examine the impact of working with non-bounded number lines. Recent studies have explored the effect of different markings on the number line, for example, coming responses on number only bounded at each end, at each end with an additional midpoint benchmark and lines with benchmarks at every quartile (Peeters et al. 2016). Given our findings with respect to the learners' apparent lack of attention to five as a benchmark, such variations would be interesting to explore.

With regard to the number comparison task, it would be valuable to extend this to comparison of non-symbolic quantities (Lyons and Beilock 2013).

7 Conclusion

Taken together these findings indicate that for these learners generally had a sound understanding of the ordinality of numbers up to 8, underpinned by some sense of cardinality. However, the iteration of a small unit breaks down after reaching 3 or 4, suggesting an early falling away of understanding of additive structure in a spatial sense. Further, the selected unit size was independent of the length of the number line that the numbers were to be positioned on. Coupled with the widespread underestimation of the position of 5 on the 0–10 line, the data pointed to very limited evidence of awareness of the multiplicative structure of the cardinality of number. However, the finding of an association between a strong ordinal sense of number and extent of linearity in the numerical magnitude estimation tasks suggests that providing access to linear magnitude representations of number, as Siegler (2009) advocates, may support progress towards relational (additive and multiplicative) understandings. Such resources may allow learners to connect the relative strengths in ordinal awareness seen in the second task with spatial representations of number with some sense of proportion built into them.

In our continuing work with schools, we have been working with teachers on introducing learners to partially marked number lines (0, 10, 20, 30 … 100) with the intermediate value positions marked, but not the numerals. Working with the learners on using the labelled positions as benchmark values from which to locate intermediate values, without counting up in ones, is showing promise in helping learners begin to work with iterating units. In moving to lines where only the benchmark numbers are labelled with no intermediate marks between these, we have seen learners beginning to talk about the relations in multiplicative terms, such as five being 'half-way' between 0 and 10. Future research will examine the outcomes of these approaches.

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