Partial Differential Equations in Applied Mathematics 12 (2024) 100962

Contents lists available at ScienceDirect

Partial Differential Equations in Applied Mathematics

journal homepage: www.elsevier.com/locate/padiff

Lie group classification and conservation laws of a $(2+1)$-dimensional nonlinear damped Klein–Gordon Fock equation

Faiza Arif [a], Adil Jhangeer [b,c,*], F.M. Mahomed [d], F.D. Zaman [a]

[a] Abdus Salam School of Mathematical Sciences, Government College University, Lahore 54600, Pakistan
[b] IT4Innovations, VSB-Technical University of Ostrava, Ostrava-Poruba, Czech Republic
[c] Department of Mathematics, Namal University, Talagang Road, Mianwali 42250, Pakistan
[d] DSI-NRF Centre of Excellence in Mathematical and Statistical Sciences, School of Computer Science and Applied Mathematics, University of the Witwatersrand, Johannesburg, Wits 2050, South Africa

ARTICLE INFO

Keywords:
Mathematical model
Lie symmetry analysis
Equivalence transformations
Reductions
Traveling wave solutions
Partial Lagrangian approach

ABSTRACT

In this article, the $(2+1)$-dimensional nonlinear damped Klein–Gordon Fock equation is studied using the classical Lie symmetry method. A complete Lie group classification is conducted to derive the specific forms of the arbitrary smooth functions included in the equation, resulting in two distinct cases. Using the similarity transformation method, the reductions of the considered equation in the form of ordinary differential equations are obtained. Several invariant solutions including the traveling wave solutions and soliton solutions of the $(2+1)$-dimensional nonlinear damped Klein–Gordon Fock equation are uncovered. Also, the results are represented through 2D and 3D graphs with their physical interpretations. Notably, using the partial Lagrangian method, the conservation laws are derived, which also yield two separate cases with several subcases. These results offer better insights into the solution properties of nonlinear damped Klein–Gordon Fock equation and other complex nonlinear wave equations.

1. Introduction

Nonlinear partial differential equations are significant in the areas of engineering, nonlinear science, and mathematics, providing key components for a variety of real-life phenomena. In this context, the nonlinear damped Klein–Gordon Fock equation (KGE) is pivotal for comprehending energy dissipation phenomena and complex wave behavior.

In the literature, the following $(2+1)$-dimensional KGE:

$$u_{tt} + p(u) = u_{xx} + u_{yy} \tag{1}$$

was studied by Magalakwe *et al.* via the Lie symmetry method and they obtained its conservation laws using Ibragimov's theorem.[1]

The present study deals with the $(2+1)$-dimensional nonlinear damped KGE, which reads as follows

$$u_{tt} + \beta(u)\, u_t = \alpha(u) + u_{xx} + u_{yy}, \qquad \beta(u) \neq 0, \tag{2}$$

here, $u(x,t,y)$ represents the amplitude of the wave, while both $\beta(u)$ and $\alpha(u)$ are smooth, arbitrary functions. The addition of the damping term, denoted by $\beta(u)\, u_t$, in Eq. (2) models phenomena like, soliton formations, self interactions of the wave and significantly imparts more complexity and nonlinear dynamics absent in Eq. (1). Moreover, the inclusion of this damping term changes the overall analysis and solutions found in Ref. 1. The nonlinearity of the equation is due to the inclusion of the damping term $\beta(u)\, u_t$ and $\alpha(u)$, allows for the study of the nonlinear dynamics described by the physical system. Eq. (2) has numerous applications in the scientific areas, such as wave propagation, energy dissipation, quantum physics, nonlinear dynamics, and applied mathematics research.[2] Grasping its solutions is essential as it allows modeling energy transfer and improving damping processes in several engineering applications, as well as it reveals the nonlinear effects, energy dissipation, and system stability. The damped KGE has some applications in signal processing, nonlinear optics, and energy dissipation, as discussed in Refs. 3, 4.

Initially, within the domain of group analysis for differential equations (DEs), S. Lie[5] studied the KGE of the form $\frac{d^2u}{dxdy} = G(u)$. Further, the Lie symmetry classification of KGE in general Riemannian space was carried out by Ref. 6 Preeti et al. employed the group theoretical approach to study the Klein–Gordon Zakharov equation. They find its Lie symmetries, reduced forms and obtained its series solutions.[7]

Identifying an exact solution to a nonlinear DE is a crucial and fundamental aspect of nonlinear science. Solving such equations often requires advanced mathematical techniques and computational methods. Researchers apply a variety of approaches, such as analytical

* Corresponding author at: IT4Innovations, VSB-Technical University of Ostrava, Ostrava-Poruba, Czech Republic.
E-mail address: adil.jhangeer@gmail.com (A. Jhangeer).

https://doi.org/10.1016/j.padiff.2024.100962
Received 9 August 2024; Received in revised form 2 October 2024; Accepted 9 October 2024
Available online 18 October 2024

Nomenclature

Abbreviations

DEs	Differential equations
KGE	Klein–Gordon Fock equation
ODEs	Ordinary differential equations
PDEs	Partial differential equations

Other Symbols

u	Amplitude of wave
$\beta(u)$	Unknown arbitrary smooth function
$\alpha(u)$	Unknown arbitrary smooth function
X	Symmetry operator
$\mathcal{D}_i$	Total derivative operator
$X^{[2]}$	Second-order prolongation
$\mathcal{A}$, $\mathcal{F}$	Constants
$\mathcal{A}_1$, $\mathcal{F}_1$	Constants
γ, σ, k	Constants
$Ai(\mu)$	Airy function of the first kind
$Bi(\mu)$	Airy function of the second kind
$F_1(a, b; c; s)$	Hypergeometric function
$\mathcal{L}$	Lagrangian
T^i	Conserved vector components, $i = 1, 2, 3$

methods, numerical methods and software tools, to understand the dynamics of various kinds of DEs. Exact solutions of various nonlinear partial differential equations (PDEs) derived via group-theoretical approaches, are documented in references such as Refs. 8, 9. Nevertheless, Lie symmetry analysis method, renowned for its effectiveness, excels in uncovering reductions, exact invariant solutions and equivalences to more simple forms for both linear and nonlinear differential equations, making it different from other methods. Unlike numerical or analytical approaches that solve equations either directly or by approximations, Lie symmetry method is based on algebraic structure to derive the conserved quantities and symmetries or transformation groups of differential equations, keeping the structure of the DEs. This can lead to lessen the complexity of the equation and find the exact solution of differential equations or new perception on the dynamics of the system that might not be obtained through other methods. For a better understanding of this method, readers are directed to the renowned works authored by Ovsiannikov,[10] Ibragimov,[11] Olver[12] and Bluman and Kumei.[13]

The following $(1+1)$-dimensional KGE

$$u_{tt} = \alpha(u) + u_{xx} \tag{3}$$

was studied for various forms of the function α, for instance, in Refs. 14, 15. Eq. (3), named after Oskar Klein and Walter Gordon, represents a second-order equation in both time and space, comes under the class of nonlinear evolution equations. It models the dynamics of scalar particles in quantum theory. Also, the nonlinear damped KGE is significant in applied mathematics as it acts as a test tool for the development of the numerical, semi-analytical, and analytical methods, to study the stability, uniqueness and existence of solutions, as well as construction of numerical methods to understand the more complex nonlinear PDEs.[16] The partial differential equation represented in Eq. (2) extends the $(1+1)$-dimensional KGE (3) by including the term $\beta(u)\,u_t$ and a third dimension. The presence of these terms models the self-interaction of wave and give rise to solitons and complex wave dynamics.

The Lie symmetry approach aid in studying the nonlinear damped KGE by finding its Lie symmetries, which then simplifies the equation by reducing its number of independent variables, leading to the identification of exact invariant solutions. These invariant solutions offer complete insight into the properties and dynamics described by the equation. More recently, Azad et al.[17] proposed a complete group classification and reductions for Eq. (3) (See Tables 1 and 2). Xiao-Yan et al.[18] studied the $(2+1)$-dimensional nonlinear KGE by using the Lie symmetry method and derived its reduced forms. Moreover, Khalique et al. conducted a Lie symmetry analysis of the nonlinear KGE, leading to the derivation of stationary solutions.[19] The main purpose of this study is to find the comprehensive Lie group classification of Eq. (2), perform reductions and derive conservation laws. Ferenc et al. discussed the special dynamics of the Klein–Gordon equation in three different frameworks.[20] Moreover, using the Nikiforov–Uvarov approach, analytical solutions of Klein–Gordon equation featuring exponential potential were obtained by Ahmadov et al..[21]

Conservation laws in the analysis of PDEs provide essential information, such as identification of conserved components for all the solutions, revelation of non-local symmetries, assessment of integrability and the provision of indicators to check accuracy and stability of numerical methods. Additionally, abundance of conservation laws signifies the integrability of the differential and difference equations, see, for example, Refs. 22, 23.

There exist numerous approaches to derive the conservation laws of DEs, including the multiplier approach, the direct method[24] and Noether's theorem[25] applicable to variational problems. However, despite the efficiency of these methods in finding conservation laws, our chosen approach is the partial Lagrangian method, initially presented by Kara and Mahomed.[26] This method offers an efficient way to discover conservation laws, even when a typical Lagrangian is absent. Unlike Noether's theorem, this approach works for differential equations both with Lagrangian and without a Lagrangian formulation. Applying the Lie symmetry approach, Usman et al. studied the $(2+1)$-dimensional elastic wave equation and identified its conservation laws through the Noether and partial Noether approaches.[27] Zaman et al. conducted a comprehensive Lie group classification of the non-linear damped KGE featuring power-law nonlinearity. They also determined the optimal system, performed reductions and identified associated conservation laws.[28] Imran et al. derived the soliton solutions of third-order KGE.[29]

The structure of this paper is outlined in the following manner: Section 2 focuses on finding the comprehensive Lie symmetries of the (2+1)-dimensional nonlinear damped KGE by systematically deriving the specific forms of the smooth, unknown arbitrary functions $\beta(u)$ and $\alpha(u)$. Subsequently, Section 3 deals with the reductions of Eq. (2), covering all the cases introduced in Section 2. Also, Section 3 features the 2D and 3D graphs of traveling wave solutions. Finally, Section 4 presents the conservation laws, derived using the partial Noether method.

2. Complete Lie group classification

In this section, we consider the Lie group properties for Eq. (2) and summarize the various cases pertaining to the arbitrary functions $\beta(u)$ and $\alpha(u)$. For this purpose, we take one-parameter Lie group of point transformations that keep the invariance of Eq. (2). The transformation formula are widely recognized and can be found, e.g., in Ref. 10

$$\bar{x}^i = x^i + \varepsilon\,\tau^i(u, x, y, t) + \mathcal{O}(\varepsilon^2), \qquad i = 1, \ldots, 3,$$
$$\bar{\varphi} = \bar{u} + \varepsilon\,\varphi(u, x, y, t) + \mathcal{O}(\varepsilon^2),$$

here, $\tau^i = \frac{\partial \bar{x}^i}{\partial \varepsilon}\Big|_{\varepsilon=0}$ characterizes the symmetry operator as follows

$$X = \tau^1 \frac{\partial}{\partial t} + \tau^2 \frac{\partial}{\partial x} + \tau^3 \frac{\partial}{\partial y} + \varphi \frac{\partial}{\partial u}. \tag{4}$$

If the following invariance condition

$$X^{[2]}\big(u_{tt} + \beta(u)\,u_t - u_{xx} - u_{yy} - \alpha(u)\big)\Big|_{\text{Eq. (2)}=0} = 0, \tag{5}$$

Table 1
Generalizations from existing models in the literature.

	Present study	Magalakwe et al.[1]	Azad et al.[17]
Equation	$u_{tt} + \beta(u)u_t = \alpha(u) + u_{xx} + u_{yy}, \quad \beta \neq 0$	$u_{tt} + p(u) = u_{xx} + u_{yy}$	$u_{tt} = \alpha(u) + u_{xx}$
Key extension	Inclusion of a damping term i.e., $\beta(u)u_t$		
Function forms	• Case 1. $\beta(u) = \mathcal{A}(u+\gamma)^k$, $\alpha(u) = \mathcal{F}(u+\gamma)^{1+2k}, \ k \neq 0, 1$. • Case 2. $\beta(u) = \mathcal{A}_1 e^{\sigma u}$, $\alpha(u) = \mathcal{F}_1 e^{2\sigma u}, \quad \sigma \neq 0$.	• $p(u) = \sigma + \delta u$ • $p(u) = \sigma + \delta u^n$ • $p(u) = \sigma + \delta e^{un}$	• $\alpha(u) = \frac{\ln(au+b)}{a} + c$ • $\alpha(u) = (au+b)^n + c$ • $\alpha(u) = a e^{bu} + c$ • $\alpha(u) = au^2 + bu + c$
Reductions	Yes	No	Yes
Solutions	Yes	Yes	Yes
Soliton solutions	Yes	No	No
Graphical representations	All solutions are represented through 2D and 3D graphs with physical interpretations.	No	No
Conservation laws	By partial Lagrangian method	By Ibragimov's theorem	No

Table 2
Comparison of reduced forms and some solutions from existing models in the literature.

	Present study	Magalakwe et al.[1]
Traveling wave solutions	$\left(1-(p^2+q^2)\right)\omega''(s) + \beta\omega'(s) - \alpha = 0.$ Solution: $u_1(x,y,t) = \frac{\alpha}{\beta}(t - px - qy) + \frac{(p^2+q^2-1)}{\beta} c_1 e^{\frac{\beta}{(p^2+q^2-1)}(t-px-qy)} + c_2.$ $\left(1-(p^2+q^2)\right)\omega''(s) + \beta\omega'(s) - \alpha\omega = 0.$ Solution: $u_2(x,y,t) = c_1 e^{\frac{1}{2}(t-px-qy)\nu_1} + c_2 e^{\frac{1}{2}(t-px-qy)\nu_2}.$ $\left(1-(p^2+q^2)\right)\omega''(s) + \beta\omega\omega'(s) - \alpha = 0.$ Solution: $u_3(x,y,t) = -\frac{1}{\beta c_2\left(Ai(\mu)+Bi(\mu)\right)}\Big(\frac{\alpha\beta}{\mu_1}\left(2(p^2+q^2-1)Ai'(\mu)+Ai'(\mu)\right)\Big).$	
For arbitrary case	$\vartheta_{ss} + \alpha(\vartheta) = 0.$ Solution: $\int\left(C_1 - 2\int\alpha(u)\,du\right)^{-\frac{1}{2}}du = C_2 \pm y.$ $s\vartheta_{ss} + \vartheta_s + \frac{1}{2}\alpha = 0.$ Solution: $u_4(x,y,t) = -\frac{\alpha}{4}(y^2+x^2) + c_1\log(\frac{1}{2}(x^2+y^2)) + c_2.$ $\vartheta_{ss} + \beta\vartheta_s = \alpha.$ Solution: $u(x,y,t) = \frac{\alpha}{\beta}t + c_1\frac{1}{\beta}e^{-\beta t} + c_2.$	
For case 1	$\vartheta_{ss} + \mathcal{F}(\vartheta+\gamma)^{1+2k} = 0.$ Solution: $C_2 \pm y = \int\left[C_1 - 2\int\mathcal{F}(\gamma+u)^{1+2k}du\right]^{-\frac{1}{2}}du.$	$2cf_{ss} - \delta f(s)^n = 0.$ Solution: $u(x,y,t) = (\frac{2}{n-1})^{\frac{2}{n-1}}\left[\sqrt{\frac{\delta}{c(n+1)}}\left((c-1)t - cx + y\right) + C_2\right]^{\frac{2}{1-n}}.$
For case 2	$\vartheta_{ss} + \mathcal{F}_1 e^{2\sigma\vartheta} = 0.$ Solution: $u(x,y,t) = \frac{1}{\sigma}\log\left[\pm\frac{\sqrt{\sigma C_1\left(1-\tanh^2\left(\sqrt{\sigma^2 C_1(y+C_2)^2}\right)\right)}}{\sqrt{\mathcal{F}_1}}\right].$ $s\vartheta_{ss} + \vartheta_s + \frac{1}{2}\mathcal{F}_1 e^{2\sigma\vartheta} = 0.$ Solution: $u = \frac{1}{\sigma}\log\left[\pm\frac{\imath\sqrt{8C_1\mathcal{F}_1\sigma^3+1}\sqrt{\tanh^2\left(\frac{1}{2}\sqrt{8C_1\mathcal{F}_1\sigma^3+1}\left(C_2\pm\log(s)\right)-1\right)}}{\sqrt{2\mathcal{F}_1\sigma s}}\right].$	$2cF_{ss} - \delta e^{nF(s)} = 0.$ Solution: $u = \frac{1}{n}\ln\left[\frac{cC_1}{n\delta}\left(\tanh^2\left(\frac{1}{2}\sqrt{C_1}(C_2 + ((c-1)t - cx + y))\right) - 1\right)\right].$

is satisfied, then operator in Eq. (4) refers to a Lie point symmetry generator of Eq. (2). Here, $X^{[2]}$ represents the required second-order prolongation that aligns with the order of Eq. (2), defined as

$$X^{[2]} = X + \psi^i \frac{\partial}{\partial u_i} + \psi^{ij}\frac{\partial}{\partial u_{ij}}, \tag{6}$$

where

$$\psi^i = \mathcal{D}_i(\varphi) - u_j\mathcal{D}_i\tau^j,$$
$$\psi^{ij} = \mathcal{D}_j(\psi^i) - u_{ji}\mathcal{D}_i\tau^j, \quad i,j = 1,2,3,$$

herein, $\mathcal{D}_i$ refers to the total derivative operator:

$$\mathcal{D}_i = \frac{\partial}{\partial x^i} + u_i\frac{\partial}{\partial u} + \dots,$$

where $(x^1, x^2, x^3) = (t, x, y)$. The following subsequent system of determining PDEs are obtained from expansion of Eq. (5), where the coefficients of the independent partial derivatives are compared and then set equal to zero,

$$\varphi_{uu} = 0,\ \tau^1_u = 0,\ \tau^2_u = 0,\ \tau^3_u = 0, \tag{7}$$

$$\tau^2_{yy} + \tau^2_{xx} - \tau^2_{tt} - \beta(u)\tau^2_t - 2\varphi_{xu} = 0, \tag{8}$$

$$\tau^3_{yy} + \tau^3_{xx} - \tau^3_{tt} - \beta(u)\,\tau^3_t - 2\varphi_{yu} = 0, \tag{9}$$

$$\tau^1_{yy} + \tau^1_{xx} - \tau^1_{tt} + \beta(u)\,\tau^1_t + 2\varphi_{tu} + \varphi\,\beta_u = 0, \tag{10}$$

$$\tau^2_x - \tau^1_t = 0, \tag{11}$$

$$\tau^3_y - \tau^1_t = 0, \tag{12}$$

$$\tau^1_x - \tau^2_t = 0, \tag{13}$$

$$\tau^1_y - \tau^3_t = 0, \tag{14}$$

$$\tau^2_y + \tau^3_x = 0, \tag{15}$$

$$-\varphi\,\alpha_u + \beta(u)\,\varphi_t + (\varphi_u - 2\tau^1_t)\,\alpha(u) + \varphi_{tt} - \varphi_{xx} - \varphi_{yy} = 0. \tag{16}$$

Eq. (7) implies that

$$\varphi = f(x,y,t)\,u + g(x,y,t).$$

From Eqs. (11) and (12), we have

$$\tau^3_y - \tau^2_x = 0. \tag{17}$$

Further, Eqs. (11)–(15) yield the following

$$\tau^2_{tt} - \tau^2_{xx} = 0, \quad \tau^2_{yy} + \tau^2_{xx} = 0, \quad \tau^3_{tt} - \tau^3_{yy} = 0, \quad \tau^3_{yy} + \tau^3_{xx} = 0,$$
$$\tau^1_{tt} = \tau^1_{yy} = \tau^1_{xx}.$$

After certain manipulations, we obtain the following updated forms of Eqs. (8)–(10)

$$\tau^2_{tt} + \beta(u)\,\tau^2_t + 2\,f_x = 0, \tag{18}$$

$$\tau^3_{tt} + \beta(u)\,\tau^3_t + 2\,f_y = 0, \tag{19}$$

$$\tau^1_{yy} + \beta(u)\,\tau^1_t + 2\,f_t + \big(f(x,y,t)\,u + g(x,y,t)\big)\,\beta_u = 0. \tag{20}$$

Moreover, Eq. (16) now become

$$-(f\,u + g)\,\alpha_u + \beta(u)\,(f_t\,u + g_t) + (f - 2\tau^1_t)\,\alpha(u) + (f_{tt} - f_{xx} - f_{yy})\,u$$
$$+ (g_{tt} - g_{xx} - g_{yy}) = 0. \tag{21}$$

If β and α are arbitrary in u, then we deduce from Eqs. (18) and (19)

$$\tau^2_t = 0, \quad \tau^3_t = 0, \quad f = f(t).$$

It then follows from Eqs. (13) and (14) that

$$\tau^1 = \tau^1(t),$$

implying that

$$\tau^1 = a_1\,t + a_2. \tag{22}$$

Using the aforementioned results in Eq. (20) provides

$$f = 0, \quad g = 0, \quad a_1 = 0,$$

as well as

$$\tau^1(t) = 0, \quad \tau^2(x) = 0, \quad \tau^3(y) = 0.$$

For arbitrary functions $\beta(u)$ and $\alpha(u)$, we derive the four-dimensional Lie algebra constituting the principal algebra of Eq. (2) and is spanned by

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = \frac{\partial}{\partial y},$$
$$X_4 = y\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial y}, \qquad (\beta \neq 0).$$

Now, in order to extend the principal algebra, we have

$$\tau^1 = \tau^1(t), \quad \tau^2 = \tau^2(x,y), \quad \tau^3 = \tau^3(x,y).$$

From Eq. (20), we find

$$2\,f(t)_t + \beta(u)\,a_1 + \big(f(t)\,u + g(x,y,t)\big)\,\beta_u = 0. \tag{23}$$

Two cases arise for which the principal algebra extends.

Table 3
Commutation relations for arbitrary functions β and α.

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	0	0
X_2	0	0	0	$-X_3$
X_3	0	0	0	X_2
X_4	0	X_3	$-X_2$	0

Table 4
Commutation relations for Case 1.

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	0	0	0	$-k\,X_1$
X_2	0	0	0	$-X_3$	$-k\,X_2$
X_3	0	0	0	X_2	$-kX_3$
X_4	0	X_3	$-X_2$	0	0
X_5	$k\,X_1$	$k\,X_2$	$k\,X_3$	0	0

Case 1: $f \neq 0$

Differentiating Eq. (23) *w.r.t.* x and y, and then comparing the coefficients of u, we obtain

$$g = g(t).$$

Substituting this in Eq. (23) and differentiating the obtained equation *w.r.t.* t, yields

$$2\,f_{tt} + \big(f_t\,u + g(t)_t\big)\,\beta_u = 0. \tag{24}$$

It is clear that, β_u and $u\,\beta_u$ are not related, this implies that

$$f_t = 0, \qquad g_t = 0.$$

Integrating this *w.r.t.* t, we get

$$f = a_6, \qquad g = a_7.$$

Using these values in Eqs. (23) and (21) leads to the subsequent forms of the functions $\beta(u)$ and $\alpha(u)$, which read as follows

$$\beta(u) = \mathcal{A}\,(u+\gamma)^k, \quad \alpha(u) = \mathcal{F}\,(u+\gamma)^{1+2k}, \quad k \neq 0, 1.$$

Using the other equations in the determining system, results in the subsequent infinitesimals

$$\varphi = (u+\gamma)\,a_6, \qquad \tau^1 = -k\,a_6\,t + a_2,$$
$$\tau^2 = -k\,a_6\,x + a_3\,y + a_4, \quad \tau^3 = -k\,a_6\,y - a_3\,x + a_5.$$

The algebra for these forms of functions is five-dimensional and is spanned by the principal algebra, as well as the operator:

$$X_5 = -k\,t\,\frac{\partial}{\partial t} - k\,x\,\frac{\partial}{\partial x} - k\,y\,\frac{\partial}{\partial y} + (\gamma + u)\,\frac{\partial}{\partial u}.$$

Herein, a_j where $j = 1,\dots,6$. Also $\mathcal{A}$, $\mathcal{F}$ and k represent the constants.

Case 2: $f = 0$

For $f = 0$, following the same steps as in Case 1, this case results in other form of the functions, $\beta(u)$ and $\alpha(u)$

$$\beta(u) = \mathcal{A}_1\,e^{\sigma u}, \quad \alpha(u) = \mathcal{F}_1\,e^{2\sigma u}, \quad \sigma \neq 0.$$

The principal algebra in this case expands to five-dimensions along with the operator:

$$X_5 = -\sigma\,x\,\frac{\partial}{\partial x} - \sigma\,t\,\frac{\partial}{\partial t} - \sigma\,y\,\frac{\partial}{\partial y} + \frac{\partial}{\partial u}.$$

Here, σ, $\mathcal{A}_1$, and $\mathcal{F}_1$ represent constants.

The commutator tables for both the cases listed above are presented in Tables 3 and 4, which aid in performing the reductions of Eq. (2).

The results in the table for Case 2 overlap with the table for Case 1, except for the change in symmetry X_5 and the possibility of replacing k by σ.

2.1. Equivalence transformations

Equivalence transformations of an equation basically refer to certain transformations that can be applied to the equation while maintaining its fundamental properties and leaving its family and solutions invariant.[11] The general method for finding the equivalence transformation can be found, for example, in the book Ref. 30. Eq. (2) possesses the following equivalence transformations

$$\bar{t} = e_1 t + e_2, \quad \bar{x} = e_1 x + b_1, \quad \bar{y} = e_1 y + b_2, \quad \bar{u} = c_1 u + c_2, \tag{25}$$

where c_1, c_2, e_1, e_2, b_1, and b_2 represent arbitrary constants. We demonstrate this to be the case. Indeed

$$\mathcal{D}_i = \mathcal{D}_i \zeta^j \bar{\mathcal{D}}_{\bar{j}}, \qquad \bar{x}^i = \zeta^i(x),$$

where $\bar{x}^1 = \bar{t} = \zeta^1$, $\bar{x}^2 = \bar{x} = \zeta^2$, $\bar{x}^3 = \bar{y} = \zeta^3$, and $\mathcal{D}_i$ represents the total derivative transform on ζ^i. For the above transforms in (x, y, t)–space, we have

$$\begin{aligned}
\mathcal{D}_x &= \mathcal{D}_x \zeta^1 \bar{\mathcal{D}}_{\bar{t}} + \mathcal{D}_x \zeta^2 \bar{\mathcal{D}}_{\bar{x}} + \mathcal{D}_x \zeta^3 \bar{\mathcal{D}}_{\bar{y}},\\
\mathcal{D}_y &= \mathcal{D}_y \zeta^1 \bar{\mathcal{D}}_{\bar{t}} + \mathcal{D}_y \zeta^2 \bar{\mathcal{D}}_{\bar{x}} + \mathcal{D}_y \zeta^3 \bar{\mathcal{D}}_{\bar{y}},\\
\mathcal{D}_t &= \mathcal{D}_t \zeta^1 \bar{\mathcal{D}}_{\bar{t}} + \mathcal{D}_t \zeta^2 \bar{\mathcal{D}}_{\bar{x}} + \mathcal{D}_t \zeta^3 \bar{\mathcal{D}}_{\bar{y}}.
\end{aligned}$$

After certain manipulations, we deduce

$$\bar{\mathcal{D}}_{\bar{x}} = \frac{1}{e_1}\mathcal{D}_x, \qquad \bar{\mathcal{D}}_{\bar{y}} = \frac{1}{e_1}\mathcal{D}_y, \qquad \bar{\mathcal{D}}_{\bar{t}} = \frac{1}{e_1}\mathcal{D}_t.$$

The implication of the above three operators on $\bar{u}$ yields

$$\begin{aligned}
&\bar{u}_{\bar{t}} = \frac{c_1}{e_1} u_t, \quad \bar{u}_{\bar{x}} = \frac{c_1}{e_1} u_x, \quad \bar{u}_{\bar{y}} = \frac{c_1}{e_1} u_y,\\
&\bar{u}_{\bar{t}\bar{t}} = \frac{c_1}{e_1^2} u_{tt}, \quad \bar{u}_{\bar{x}\bar{x}} = \frac{c_1}{e_1^2} u_{xx}, \quad \bar{u}_{\bar{y}\bar{y}} = \frac{c_1}{e_1^2} u_{yy}.
\end{aligned}$$

Eq. (2), under the aforementioned equivalence transformations, transforms into the following form

$$\bar{u}_{\bar{t}\bar{t}} + \bar{\beta}(\bar{u})\,\bar{u}_{\bar{t}} = \bar{\alpha}(\bar{u}) + \bar{u}_{\bar{x}\bar{x}} + \bar{u}_{\bar{y}\bar{y}},$$

subject to equivalence conditions

$$\alpha(u) = \frac{e_1^2}{c_1}\bar{\alpha}(\bar{u}) \quad \text{and} \quad \beta(u) = e_1\,\bar{\beta}(\bar{u}).$$

By using equivalence transformations one can simplify $\alpha(u)$ and $\beta(u)$ in both cases.

Case 1

Here, if we set $e_1 = c_1 = 1$ and $c_2 = \gamma$, then $\bar{\beta} = \mathcal{A}\,\bar{u}^k$ and $\bar{\beta} = \mathcal{F}\,\bar{u}^{1+2k}$.

Case 2

In this case, if we take $e_1 = 1$, $c_1 = \sigma$ and $c_2 = 0$, then $\bar{\beta} = \mathcal{A}_1 e^{\bar{u}}$ and $\bar{\alpha} = \tilde{\mathcal{F}}_1 e^{2\bar{u}}$.

Thus, one can take $\gamma = 0$ and $\sigma = 1$ in the extended algebras.

3. Reductions

In the following section, we reduce Eq. (2) into the ordinary differential equations (ODEs), using two-dimensional sub-algebras and identify invariant solutions in some cases. For the reduction process, we use commutator tables 3 and 4. The following steps are involved in this process:

(i) Selection of commuting generators: The commuting symmetries share common properties allowing to simplify the system using one symmetry at a time. Therefore, identify the commuting generators from the commutator table.

(ii) Starting point: From commuting generators, select one generator which simplifies the equation directly and more easily. After reduction of the equation with one symmetry generator apply the other commuting symmetry generator to reduce the resulting equation into more simpler form. For instance, from Table 3, the generators X_1 and X_2 commute, $[X_1, X_2] = 0$, either X_1 or X_2 can be used as a starting point to initiate the reductions of Eq. (2). If we start reduction with X_1, then X_2 can later be applied to reduce the resulting equation further, and vice-versa.

(iii) Using the characteristic method, write the characteristic equation of the chosen symmetry and solve to obtain the similarity variables, which ultimately reduces the PDES into ODES.

We also briefly outline the steps involved in reducing Eq. (2).

3.1. Reductions for arbitrary functions

Traveling wave solutions

For obtaining the traveling wave solutions of Eq. (2), we consider the linear combination of X_1, X_2, and X_3 *i.e.*, given by

$$\mathcal{Z} = \frac{\partial}{\partial t} + p\,\frac{\partial}{\partial x} + q\,\frac{\partial}{\partial y}.$$

Corresponding to this symmetry generator, we obtain the following similarity variables

$$u = \omega(s), \qquad s = t - px - qy. \tag{26}$$

Eq. (26) reduces Eq. (2) into the following ODE

$$\left(1-(p^2+q^2)\right)\omega''(s) + \beta(\omega)\,\omega'(s) - \alpha(\omega) = 0. \tag{27}$$

Herein, p and q are constants. We now propose the following ansatz to obtain the solutions of Eq. (27), which ultimately leads to the solutions of Eq. (2).

Ansatz 1. If $\beta(\omega) = \beta$ and $\alpha(\omega) = \alpha$ are constant.

Eq. (27) admits the following solution for this case

$$\omega(s) = \frac{\alpha}{\beta}\,s + \frac{(p^2+q^2-1)}{\beta}\,c_1\,e^{\frac{\beta}{(p^2+q^2-1)}\,s} + c_2.$$

From here, we find the subsequent exact solution to Eq. (2), given by

$$u_1(x,y,t) = \frac{\alpha}{\beta}(t - px - qy) + \frac{(p^2+q^2-1)}{\beta}\,c_1\,e^{\frac{\beta}{(p^2+q^2-1)}(t-px-qy)} + c_2.$$

Graphically, this solution can be represented as follows:

The plots of solution u_1 in Fig. 1 represents a traveling wave that has a linear as well as exponential behavior due to the presence of combination of the linear and exponential components in the solution. The amplitude of the wave grows exponentially for various time values t. The exponential behavior is due to the dominance of the exponential term $e^{\frac{2}{7}(t-2x-2y)}$, and this causes the amplitude of the wave to rise non-linearly, whereas the first term $(t - 2x - 2y)$ causes the amplitude of wave to make linear shifts. It can be seen from the graphs that at time $t = 0$, the amplitude of wave varies slowly and linearly across xy–plane and with the increase in time, for instance, at time $t = 1$, the wave starts to grow exponentially over the region.

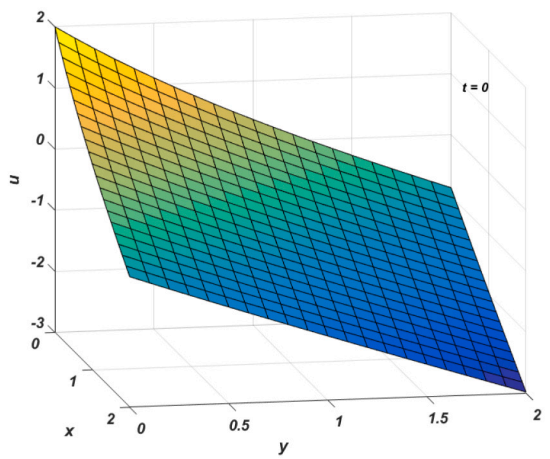
Ansatz 2. If $\beta(\omega) = \beta$ is constant and $\alpha(\omega) = \alpha\,\omega$.

The following solution of Eq. (27) is obtained in this case

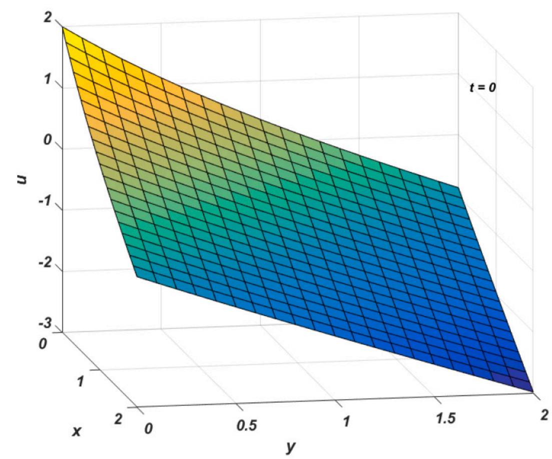
$$\omega(s) = c_1\,e^{\frac{1}{2}s\nu_1} + c_2\,e^{\frac{1}{2}s\nu_2},$$

where

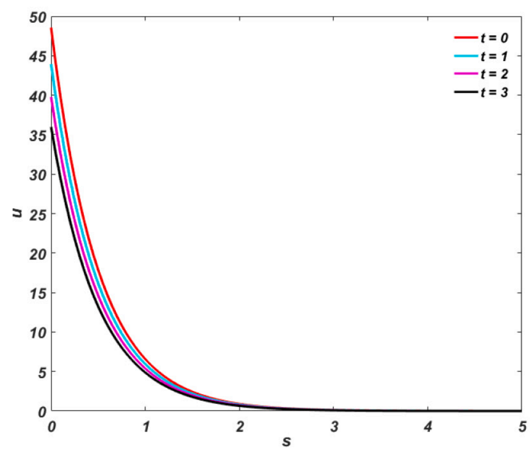
$$\nu_1 = \frac{\beta - \sqrt{\beta^2 - 4\alpha(p^2+q^2-1)}}{(p^2+q^2-1)}, \quad \text{and}$$
$$\nu_2 = \frac{\beta + \sqrt{\beta^2 - 4\alpha(p^2+q^2-1)}}{(p^2+q^2-1)}.$$



(a) For time $t = 0$.



(b) For time $t = 1$.



(c) $2D$ plots of u_1 for different time values t.

Fig. 1. $3D$ and $2D$ plots of u_1 for different time values t, with parameters $p = q = c_1 = c_2 = 2$ and $\alpha = \beta = 2$.

Subsequently, we obtain

$$u_2(x, y, t) = c_1\, e^{\frac{1}{2}(t - px - qy)\, v_1} + c_2\, e^{\frac{1}{2}(t - px - qy)\, v_2}.$$

This solution can be visualized graphically as:

These plots represents the exponential decay in the amplitude of the wave over time. As time progresses, the amplitude of wave expands over the spatial region, whereas, the second exponential term in solution becomes more dominant over increase in time values, showing exponential behavior in graph. So, the amplitude of wave decays exponentially with the increase in time (see Fig. 2).

Ansatz 3. If $\beta(\omega) = \beta\omega$ and $\alpha(\omega) = \alpha$ is constant.

Eq. (2) admits the ensuing solution

$$u_3(x, y, t) = -\frac{1}{\beta\, c_2 \big(Ai(\mu) + Bi(\mu)\big)} \Big(\frac{\alpha\,\beta}{\mu_1} \big(2(p^2 + q^2 - 1)\, Ai'(\mu) + Ai'(\mu)\big) \Big),$$

where

$$\mu_1 = \sqrt[3]{2}\,(p^2 + q^2 - 1)^2\, \Big(\frac{\alpha\,\beta}{(p^2 + q^2 - 1)}\Big)^{\frac{2}{3}},$$

and

$$\mu = \Big(\frac{\alpha\,\beta}{(p^2 + q^2 - 1)}\Big)^{-\frac{2}{3}}\, \frac{2^{\frac{1}{3}}}{(p^2 + q^2 - 1)} \big(\alpha\,\beta\,(t - px - qy) + c_1\,\beta\,(p^2 + q^2 - 1)\big).$$

Reductions under X_1 and X_2

From Table 3, it can be seen that the generators X_1 and X_2 commute, $[X_1, X_2] = 0$. Therefore, either X_1 or X_2 can serve as a starting point to initiate the reductions of Eq. (2). For our intent, we choose to begin the reduction with $X_1 = \frac{\partial}{\partial t}$. The associated characteristic equation for this generator is as follows

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{1} = \frac{du}{0}.$$

According to standard procedure, we integrate this equation in order to derive the similarity invariants, as follows

$$z = x, \quad r = y, \quad u = \omega(z, r).$$

Using these similarity transformations, Eq. (2) can be reformulated in the form

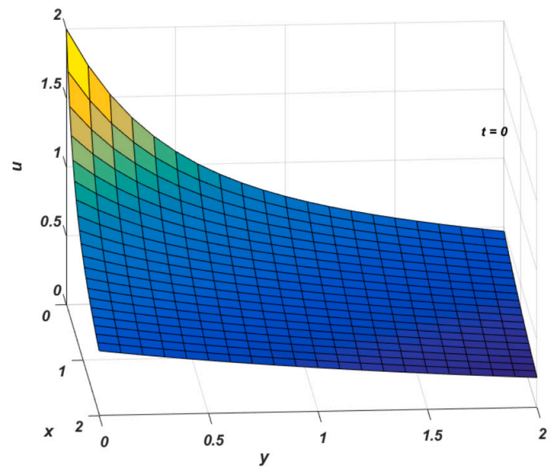
$$\omega_{zz} + \omega_{rr} + \alpha(\omega) = 0. \tag{28}$$

Utilizing X_2, we further reduce the partial differential Eq. (28). To proceed, we express X_2 in new variables, z and r, as defined above. Thus, X_2 now becomes

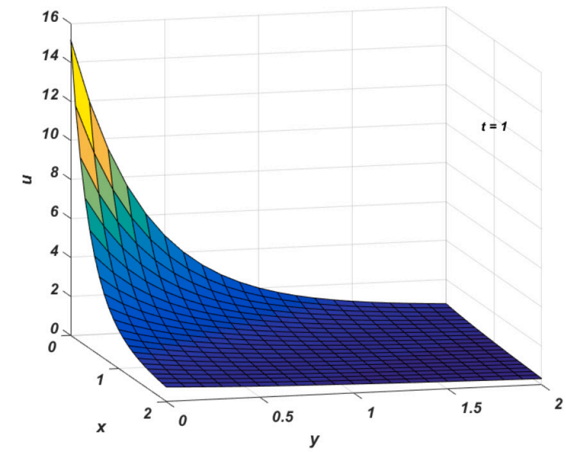
$$X_2 = Y_2 = \frac{\partial}{\partial z}.$$

The characteristic equation for this generator can be described as follows

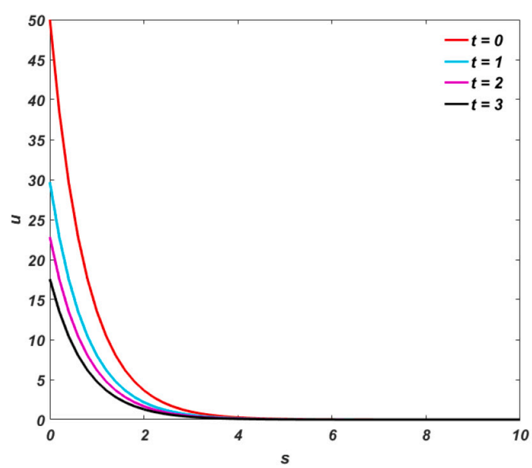
$$\frac{dz}{1} = \frac{dr}{0} = \frac{d\omega}{0},$$



(a) For time $t = 0$.



(b) For time $t = 1$.



(c) $2D$ plots of u_2 for different time values t.

Fig. 2. $3D$ and $2D$ plots of u_2 for different time values t, with parameters $p = q = c_1 = c_2 = 1$, $\alpha = 2$ and $\beta = 4$.

which yields the following similarity variables

$$s = r, \quad \omega = \vartheta(s),$$

resulting in the subsequent ordinary differential equation

$$\vartheta_{ss} + \alpha(\vartheta) = 0, \tag{29}$$

having solution

$$\int \left(C_1 - 2\int \alpha(\vartheta)\, d\vartheta\right)^{-\frac{1}{2}} d\vartheta = C_2 \pm s.$$

Substituting the original variables back, we obtain

$$\int \left(C_1 - 2\int \alpha(u)\, du\right)^{-\frac{1}{2}} du = C_2 \pm y.$$

Next, X_1 and X_3 yield the same reduction as given in Eq. (29), with similarity invariants

$$s = z, \quad \omega = \vartheta(s).$$

Reductions under X_1 and X_4

From Table 3, it is evident that $[X_1,\ X_4] = 0$. Therefore, we can either start with X_1 or X_4. We begin with X_1 and derive Eq. (28). To reduce this equation into an ODE, we now consider X_4, which in terms of new variables, become

$$X_4 = Y_4 = r\frac{\partial}{\partial z} - z\frac{\partial}{\partial r},$$

giving rise to the following invariants

$$s = \frac{1}{2}(r^2 + z^2), \qquad \omega = \vartheta(s).$$

Using these similarity variables, Eq. (28) reduces to ODE

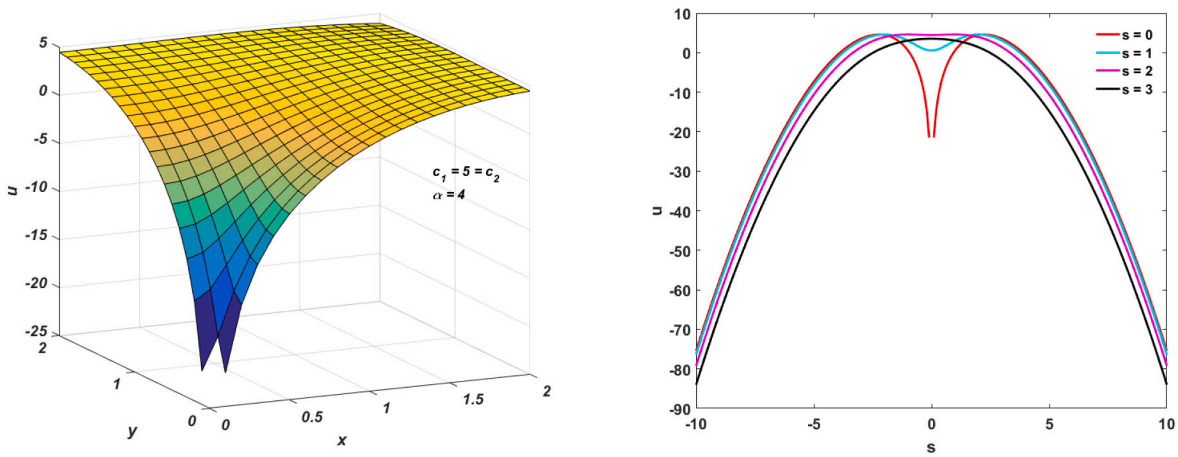
$$s\,\vartheta_{ss} + \vartheta_s + \frac{1}{2}\alpha(\vartheta) = 0.$$

If $\alpha(\vartheta) = \alpha$, in this case, Eq. (2) yields the following solution

$$u_4(x, y, t) = -\frac{\alpha}{4}(y^2 + x^2) + c_1 \log\left(\frac{1}{2}(x^2 + y^2)\right) + c_2,$$

herein, c_1 and c_2 represent constants. Graphically, this solution can be represented as follows:

Fig. 3 shows slow and sharp variations in the amplitude of the wave. It can be observe that the solution accounts for the parabolic behavior of the wave which is due to the term $(x^2 + y^2)$ present in the solution. The wave amplitude reduces with increasing distance from the origin. The slower variations are due to the logarithmic terms.



(a) $3D$ plot of u_4. (b) $2D$ plot of u_4.

Fig. 3. $3D$ and $2D$ plots of u_4 with parameters $\alpha = 4$ and $c_1 = 5 = c_2$.

Reductions under X_2 and X_3

Here from Table 3, we have $[X_2, X_3] = 0$. Hence, we can initiate the reduction process either with X_2 or X_3. We start with X_3 and obtain the following PDE

$$\omega_{rr} + \beta(\omega)\,\omega_r = w_{zz} + \alpha(\omega), \tag{30}$$

which is subjected to the similarity variables associated with X_3, viz.

$$z = x, \quad r = t, \quad u = \omega(z,r).$$

Now, X_2 in new variables, defined by

$$X_2 = Y_2 = \frac{\partial}{\partial z},$$

results in the following similarity variables

$$s = r, \quad \omega = \vartheta(s).$$

Corresponding to these invariants, Eq. (30) now takes on the following form

$$\vartheta_{ss} + \beta(\vartheta)\,\vartheta_s = \alpha(\vartheta).$$

For $\alpha(\vartheta) = \alpha\,(constant)$ and $\beta(\vartheta) = \beta\,(constant)$ the above equation admits the subsequent solution

$$u(x,y,t) = \frac{\alpha}{\beta}\,t + c_1\,\frac{1}{\beta}\,e^{-\beta t} + c_2.$$

3.2. Reductions for Case 1

In this case, Eq. (2) now become

$$u_{tt} + \mathcal{A}\,(u+\gamma)^k\,u_t = u_{xx} + u_{yy} + \mathcal{F}(u+\gamma)^{1+2k}. \tag{31}$$

Reductions under X_1 and X_2

From Table 4, we observe that $[X_1, X_2] = 0$. Consequently we can begin the reduction process either with X_1 or X_2. Let us begin with X_1 and we derive the following PDE

$$\omega_{zz} + \omega_{rr} + \mathcal{F}(\omega + \gamma)^{1+2k} = 0, \tag{32}$$

which is subject to the similarity variables

$$z = x, \quad r = y, \quad u = \omega(z, r).$$

Now, X_2, in new variables, undergoes the transformations

$$s = r, \quad \omega = \vartheta(s),$$

resulting in Eq. (32) being simplified to the following ODE

$$\vartheta_{ss} + \mathcal{F}\,(\vartheta + \gamma)^{1+2k} = 0. \tag{33}$$

The exact invariant solution associated with Eq. (2) is

$$C_2 \pm y = \int \left[C_1 - 2\int \mathcal{F}\,(\gamma + u)^{1+2k} du\right]^{-\frac{1}{2}} du,$$

or alternatively,

$$[(y + C_2)^2,\, u(x,y,t)] = \frac{(\gamma+u)^2\left(1 - \frac{\mathcal{F}(\gamma+u)^{2+2k}}{C_1(k+1)}\right) F_1\left(\frac{1}{2}, \frac{1}{2k+2}; 1 + \frac{1}{2k+2}; \frac{\mathcal{F}(\gamma+u)^{2+2k}}{C_1(k+1)}\right)^2}{C_1 - \frac{\mathcal{F}(\gamma+u)^{2+2k}}{k+1}},$$

where $F_1(a,b;c;s)$ represents the hypergeometric function. By setting $\mathcal{F} = -\frac{\delta}{2c}$ and $1+2k = n$ in Eq. (33), we obtain the reduced ODE outlined in Ref. 1.

Reductions under X_1 and X_3

For $[X_1, X_3] = 0$, we have the following invariants

$$s = z, \quad \omega = \vartheta(s),$$

resulting in Eq. (33).

Reductions under X_1 and X_4

We start the reduction process with X_4. The transformations associated with this generator

$$z = \frac{1}{2}(x^2 + y^2), \quad r = t, \quad u = \omega(z, r),$$

lead to the reduction of Eq. (31) to the PDE

$$\omega_{rr} + \mathcal{A}(\omega + \gamma)^k\,\omega_r = 2\,\omega_z + 2\,z\,\omega_{zz} + \mathcal{F}\,(\omega + \gamma)^{1+2k}. \tag{34}$$

Now, when X_1 is expressed in terms of new variables

$$X_1 = Y_1 = \frac{\partial}{\partial r},$$

it yields the following invariants

$$s = z, \quad \omega = \vartheta(s),$$

yielding the reduction of Eq. (34) to the resultant ODE

$$s\,\vartheta_{ss} + \vartheta_s + \frac{1}{2}\,\mathcal{F}\,(\vartheta + \gamma)^{1+2k} = 0. \tag{35}$$

Reductions under X_1 and X_5

We start with X_1 and obtain Eq. (32). Now, X_5, in new variables takes the form

$$X_5 = Y_5 = -k\,z\,\frac{\partial}{\partial z} - k\,r\,\frac{\partial}{\partial r} + (\omega+\gamma)\,\frac{\partial}{\partial \omega}.$$

Corresponding to this, the similarity transformations

$$s = \frac{z}{r}, \quad \omega = \vartheta\, z^{-\frac{1}{k}} - \gamma,$$

yield the following reduced form

$$s^2(1+s^2)\,\vartheta_{ss} + 2\,s\,(s^2 - \frac{1}{k})\,\vartheta_s + \mathcal{F}\vartheta^{1+2k} + \frac{1+k}{k^2}\,\vartheta = 0. \quad (36)$$

Reductions under X_2 and X_3

We initiate with X_2. Associated with this generator the similarity transformations are

$$z = y, \quad r = t, \quad u = \omega(z, r).$$

These transformations result in Eq. (31) being transformed into the following PDE

$$\omega_{rr} + \mathcal{A}(\omega+\gamma)^k\,\omega_r = \omega_{zz} + \mathcal{F}\,(\omega+\gamma)^{1+2k}. \quad (37)$$

Now, when expressing X_3 in terms of new variables, it undergoes the transformations

$$s = r, \quad \omega = \vartheta(s),$$

giving rise to the reduction

$$\vartheta_{ss} + \mathcal{A}\,(\vartheta+\gamma)^k\,\vartheta_s = \mathcal{F}\,(\vartheta+\gamma)^{1+2k}, \quad (38)$$

having the following solution

$$[(t + C_2)^2, u(x,y,t)] = \int_1^{u(x,y,t)} \left(\frac{1}{\sqrt{C_1 + 2\int_1^{\mathcal{K}_2}(-\mathcal{A}(\gamma+\mathcal{K}_1^k) + \mathcal{F}\,(\gamma+\mathcal{K}_1)^{1+2k})\,d\mathcal{K}_1}} \right) d\mathcal{K}_2.$$

Reductions under X_2 and X_5

From Table 4, it is evident that the provided generators satisfy $[X_2,\ X_5] = -k\,X_2$. This implies that reduction in this case should commence with X_2. For X_2, we derive Eq. (37). Now, X_5 in terms of new variables, has the transformations

$$s = \frac{z}{r}, \quad \omega = \vartheta(s)\,z^{-\frac{1}{k}} - \gamma,$$

which give the following reduction

$$s^2(1-s^2)\,\vartheta_{ss} - 2\,s\,(s^2 - \frac{1}{k})\,\vartheta_s + \mathcal{A}s^2\,\vartheta^k\,\vartheta_s + \mathcal{F}\vartheta^{1+2k} + \frac{1+k}{k^2}\,\vartheta = 0. \quad (39)$$

Reductions under X_3 and X_5

Based on Table 4, we find $[X_3,\ X_5] = -k\,X_3$, indicates that the reduction must begin with X_3. For X_3, we obtain Eq. (37), subject to the subsequent similarity transformations

$$z = x, \quad r = t, \quad u = \omega(z, r).$$

Now, expressing X_5 in terms of new variables yields the same reduction as provided in Eq. (39).

Reductions under X_4 and X_5

We start with X_4, and corresponding to this generator, we obtain the same PDE as given in Eq. (34). Now, X_5, in terms of new variables, possesses the following invariants

$$s = \frac{z}{r^2}, \quad \omega = \vartheta\,z^{-\frac{1}{k}} - \gamma,$$

which consequently results in the subsequent ODE

$$s^2(2s-1)\,\vartheta_{ss} + s\,(3s - 1 + \frac{1}{k})\,\vartheta_s - \mathcal{A}s^{\frac{3}{2}}\,\vartheta^k\,\vartheta_s - \mathcal{F}\vartheta^{1+2k} - \frac{1}{4k^2}\,\vartheta = 0.$$

3.3. Reductions for Case 2

Eq. (2) in this particular case becomes

$$u_{tt} + \mathcal{A}_1\,e^{\sigma u}\,u_t = u_{xx} + u_{yy} + \mathcal{F}_1 e^{2\sigma u}. \quad (40)$$

Reductions under X_1 and X_2

Based on Table 4, we find that $[X_1,\ X_2] = 0$. Consequently we can begin the reduction process either with X_1 or X_2. Let us start with X_1, resulting in the PDE

$$\omega_{zz} + \omega_{rr} + \mathcal{F}_1\,e^{2\sigma\omega} = 0, \quad (41)$$

which is subject to the following transformations

$$z = x, \quad r = y, \quad u = \omega(z, r).$$

Now, X_2, in new variables possesses the invariants

$$s = r, \quad \omega = \vartheta(s),$$

resulting in the following ODE

$$\vartheta_{ss} + \mathcal{F}_1\,e^{2\sigma\vartheta} = 0. \quad (42)$$

The exact invariant solution associated with Eq. (2) for this case, is represented as follows

$$u(x,y,t) = \frac{1}{\sigma}\,\log\,[\pm\frac{\sqrt{\sigma\,C_1\,(1 - \tanh^2\left(\sqrt{\sigma^2\,C_1\,(\,y + C_2\,)^2}\right))}}{\sqrt{\mathcal{F}_1}}].$$

When we take $\mathcal{F}_1 = -\frac{\delta}{2c}$ and $2\sigma = n$ in Eq. (42), it simplifies to the reduced ODE discussed in Ref. 1.

A graphical comparison of this solution u and the solution v presented in the paper Ref. 1 is represented in Fig. 4.

Reductions under X_1 and X_3

For $[X_1,\ X_3] = 0$, we observe the following invariants

$$s = z, \quad \omega = \vartheta(s),$$

leading to the derivation of Eq. (42).

Reductions under X_1 and X_4

We start with X_1. The transformations associated with this generator yield Eq. (41). Now, when X_4 is expressed in terms of new variables

$$X_4 = Y_4 = r\,\frac{\partial}{\partial z} - z\,\frac{\partial}{\partial r},$$

it gives the following invariants

$$s = \frac{1}{2}(r^2 + z^2), \quad \omega = \vartheta(s).$$

This results in the reduction of Eq. (41) to the following ODE

$$s\,\vartheta_{ss} + \vartheta_s + \frac{1}{2}\,\mathcal{F}_1\,e^{2\sigma\vartheta} = 0. \quad (43)$$

The exact invariant solution to Eq. (2) is in Box I

Reductions under X_1 and X_5

We take X_1 and derive Eq. (41). Now, X_5, in terms of new variables assumes the form

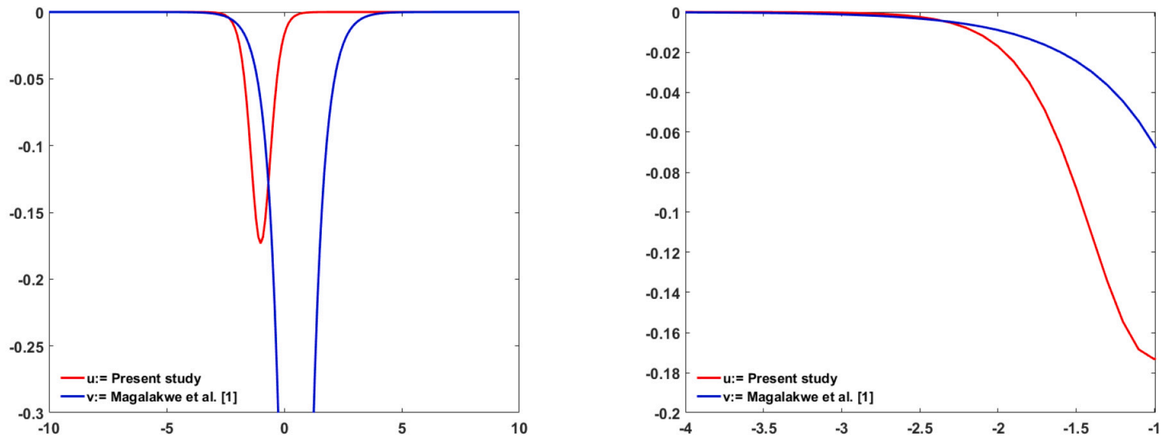
$$X_5 = Y_5 = -\sigma\,z\,\frac{\partial}{\partial z} - \sigma\,r\,\frac{\partial}{\partial r} + \frac{\partial}{\partial \omega}.$$

Corresponding to this, we introduce similarity transformations

$$s = \frac{z}{r}, \quad \omega = -\frac{1}{\sigma}\,\ln z + \ln\vartheta.$$

These transformations lead to the following reduced version of Eq. (41)

$$s^2(1+s^2)\,(\vartheta_{ss}\,\vartheta - \vartheta_s^2) + 2\,s^3\,\vartheta_s\,\vartheta + (\frac{1}{\sigma} + \mathcal{F}_1\,\vartheta^{2\sigma})\,\vartheta^2 = 0. \quad (44)$$



(a) (b)

Fig. 4. $2D$ comparison plots of $u=\frac{1}{2}\log\left(\sqrt{2(1-\tanh^2(\sqrt{4(y+1)^2}))}\right)$ from present study and $v=\frac{1}{3}\ln\left(\tanh^2\left(\frac{1}{2}(1+2t-3x+y)-1\right)\right)$ from Ref. 1.

$$u(x,y,t)=\frac{1}{\sigma}\,\log\Big[\pm\frac{\iota\sqrt{8\,C_1\,\sigma^3\,\mathcal{F}_1\,+\,1}\,\sqrt{\tanh^2\left(\frac{1}{2}\,\sqrt{8\,C_1\,\mathcal{F}_1\,\sigma^3\,+\,1}\,\left(C_2\,\pm\,\log(\frac{1}{2}(x^2+y^2))\right)-1\right)}}{\sqrt{2\,\mathcal{F}_1\,\sigma\,\frac{1}{2}(x^2+y^2)}}\Big].$$

Box I.

Reductions under X_2 and X_3

We commence with X_2. For this generator, we introduce the following similarity variables

$$z\,=\,y,\quad r\,=\,t,\quad u\,=\,\omega(z,\,r).$$

These transformations reduce Eq. (40) into the following PDE

$$\omega_{rr}\,+\,\mathcal{A}_1\,e^{\sigma\,\omega}\,\omega_r\,=\,\omega_{zz}\,+\,\mathcal{F}_1\,e^{2\,\sigma\,\omega}. \tag{45}$$

Now, X_3, in terms of new variables, yields the invariants

$$s\,=\,r,\quad \omega\,=\,\vartheta(s),$$

which lead to the following reduction of Eq. (45)

$$\vartheta_{ss}\,+\,\mathcal{A}_1\,e^{\sigma\,\vartheta}\,\vartheta_s\,=\,\mathcal{F}_1\,e^{2\,\sigma\,\omega}. \tag{46}$$

Corresponding to this equation the solution of Eq. (2) reads

$$u(x,y,t)=\frac{1}{\sigma}\,\ln[-\frac{\sigma}{\mathcal{F}_1}\,(C_1\,+\delta^2\,\ln|\kappa|-\delta\,\kappa)],$$

where $\delta=-\mathcal{F}_1/\mathcal{A}_1$ and

$$t=-\frac{\delta}{\sigma}\int\kappa^{-1}\,(C_1+\delta^2\,\ln|\kappa|\,-\delta\,\kappa)^{-1}d\kappa\,+C_2.$$

Reductions under X_2 and X_5

By Table 4, we have $[X_2,\ X_5]\,=\,-k\,X_2$. Thus, we begin with X_2. For X_2, we obtain Eq. (45). Now, X_5 in terms of new variables, has the similarity variables

$$s\,=\,\frac{z}{r},\quad \omega\,=-\frac{1}{\sigma}\,\ln z\,+\,\ln\vartheta.$$

These invariants lead to the following reduction of Eq. (45)

$$s^2(s^2-1)\,(\vartheta_{ss}\,\vartheta-\vartheta_s^2)+2\,s^3\,\vartheta_s\,\vartheta-s^2\,\mathcal{A}_1\,\vartheta^{\sigma+1}\,\vartheta_s+(\frac{1}{\sigma}-\mathcal{F}_1\,\vartheta^{2\,\sigma})\,\vartheta^2\,=\,0. \tag{47}$$

Reductions under X_3 and X_5

From Table 4, we find that $[X_3,\ X_5]\,=\,-k\,X_3$. This implies reduction to initiate with X_3. For X_3, we obtain Eq. (45), subject to the subsequent transformations

$$z\,=\,x,\quad r\,=\,t,\quad u\,=\,\omega(z,\,r).$$

Now, expressing X_5 in terms of new variables, yields the same reduction as presented in Eq. (47).

Reductions under X_4 and X_5

We start with X_4, and we obtain the following PDE

$$\omega_{rr}\,+\,\mathcal{A}_1\,e^{\sigma\,\omega}\,\omega_r\,=\,2\omega_z\,+\,2\,z\,\omega_{zz}\,+\,\mathcal{F}_1\,e^{2\,\sigma\,\omega},$$

subject to the invariants

$$z\,=\,\frac{1}{2}\,(x^2+y^2),\quad r\,=\,t,\quad u\,=\,\omega(z,\,r).$$

Now, X_5, in terms of new variables, possesses the following similarity transformations

$$s\,=\,\frac{z}{r^2},\quad \omega\,=-\frac{1}{2\,\sigma}\,\ln z\,+\,\ln\vartheta.$$

From these transformations, we obtain the following ODE

$$s^2(2s-1)\,(\vartheta_{ss}\,\vartheta-\vartheta_s^2)+s\,(3s-1)\,\vartheta_s\,\vartheta-\mathcal{A}_1\,s^{\frac{3}{2}}\,\vartheta^{\sigma+1}\,\vartheta_s\,-\,\frac{1}{2}\,\mathcal{F}_1\,\vartheta^{2(1+\sigma)}\,=\,0.$$

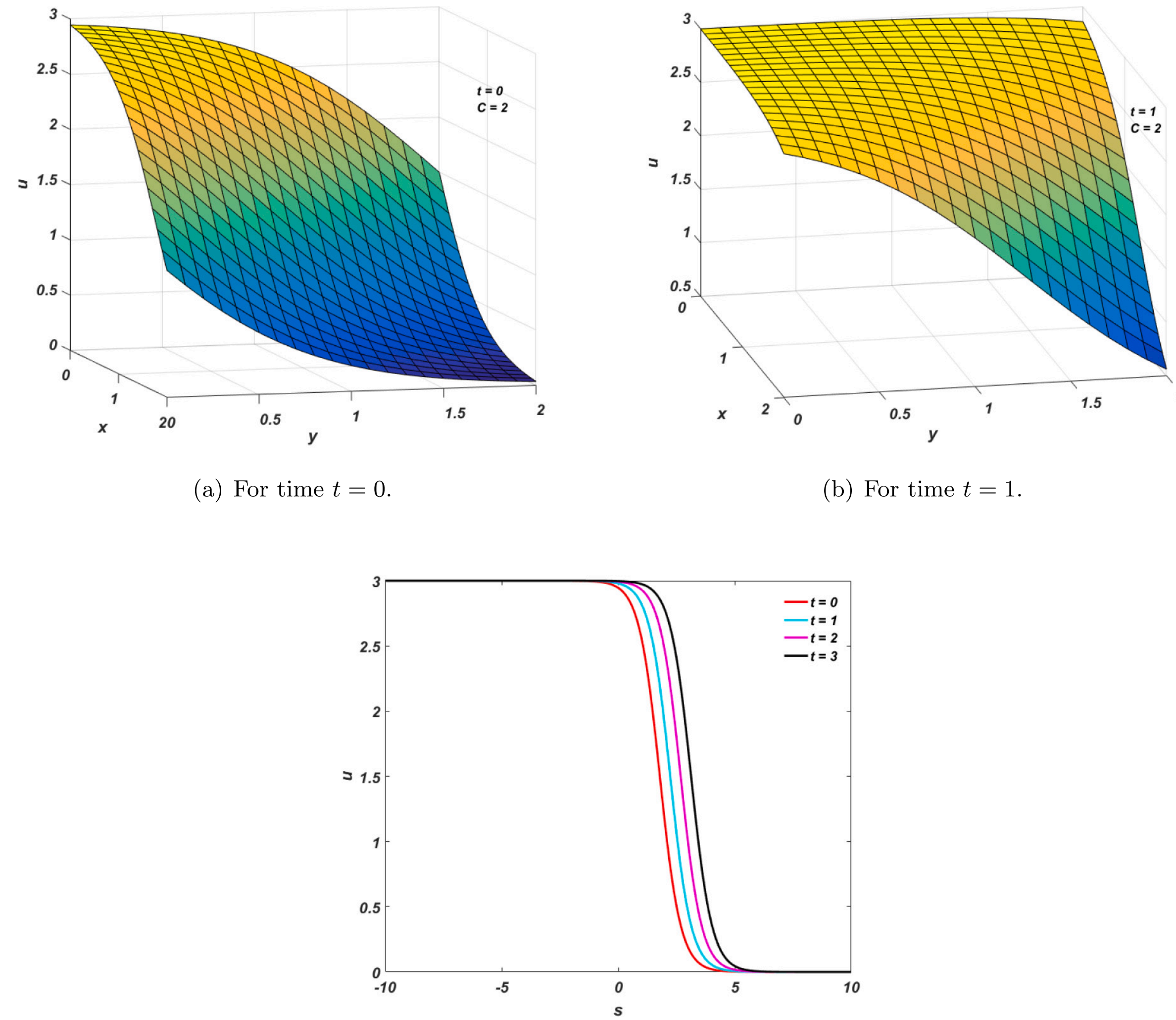
3.4. Examples of soliton solutions

This section deals with several examples of soliton solutions for Eq. (2).[31–33]

Example 1. If $\beta(u)\,=\,c_3$ and $\alpha(u)=c_1\,u-c_2\,u^2$.

Eq. (2) admits the following soliton solution for $\beta(u)\,=\,c_3$ and $\alpha(u)=c_1\,u-c_2\,u^2$.

$$u_5(x,y,t)\,=\,\frac{3}{2}\left(1+\tanh(\frac{3}{2}\,t-\frac{\sqrt{5}}{2}\,x-y+C)\right).$$



(a) For time $t = 0$.

(b) For time $t = 1$.

(c) $2D$ plots of u_5 for different time values t.

Fig. 5. $3D$ and $2D$ plots of u_5 for different time values t, with parameter $C = 2$.

Herein, we consider $c_1 = c_2 = 1 = c_3$. This solution is represented graphically as:

Fig. 5 shows a traveling wave that constantly changes diagonally over time depending on tanh function. Its amplitude is modulated by the factor $\frac{3}{2}$ and experiences constant phase shifts by C. The shifts in amplitude occur near the line $\frac{3}{2}t - \frac{\sqrt{5}}{2}x - y + 2$. The graphs shows that as time changes the wave shifts diagonally, maintaining its shape and amplitude, which is the nature of solitons.

Example 2. If $\beta(u) = c_3$ and $\alpha(u) = c_2 u^3 + c_1 u$.

The soliton solution to Eq. (2) for $\beta(u) = c_3$ and $\alpha(u) = c_2 u^3 + c_1 u$ is given as

$$u_6(x, y, t) = \frac{1}{\sqrt{2}}\left(1 + \tanh\left(\frac{3}{2}t - \frac{\sqrt{6}}{2}x - y - C\right)\right),$$

herein, $c_1 = 2$, $c_2 = -1$, and $c_3 = 1$. This solution can be shown graphically as follows:

Fig. 6 illustrates a traveling wave that continuously evolves over time according to the tanh function. The amplitude of the wave is being scaled and modulated by the factors $\frac{1}{\sqrt{2}}$ and $\frac{3}{2}$, respectively. Moreover, the amplitude of the wave experiences constant phase shifts by C, suggesting that the traveling wave propagates across xy−plane maintaining its shape.

Example 3. If $\beta(u) = u$ and $\alpha(u) = c_1 u - c_2 u^2$.

Eq. (2) has the subsequent soliton solution for $\beta(u) = u$ and $\alpha(u) = c_1 u - c_2 u^2$.

$$u_7(x, y, t) = -\frac{1}{2}\left(1 + \tanh\left(t - \frac{1}{2}x - y - C\right)\right).$$

Herein, we assume $c_1 = -1$ and $c_2 = 1$. The graphs of this solution at different times t can be seen as:

Fig. 7 depicts a traveling wave evolving over different time values. The amplitude of the wave is being scaled by the factor $(-\frac{1}{2})$, and it make smooth transitions across the region due to hyperbolic tangent function. Moreover, it experiences constant phase shifts by C.

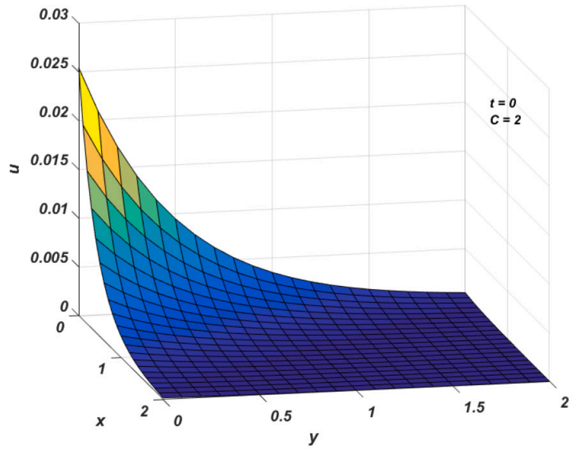
Example 4. If $\beta(u) = u$ and $\alpha(u) = c_1 u + c_2 u^3$.

For $\beta(u) = u$ and $\alpha(u) = c_1 u + c_2 u^3$, Eq. (2) yields the following solution

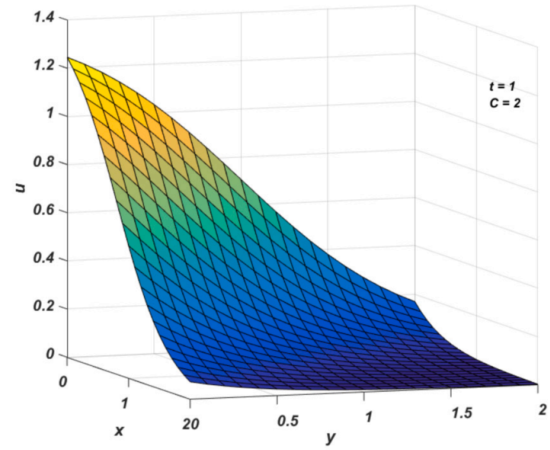
$$u_8(x, y, t) = \frac{1}{\sqrt{4}} \tanh\left(\frac{3}{\sqrt{4}}t - x - y - C\right),$$

herein, $c_1 = \frac{1}{4}$ and $c_2 = -1$. Graphically, this solution is represented as:

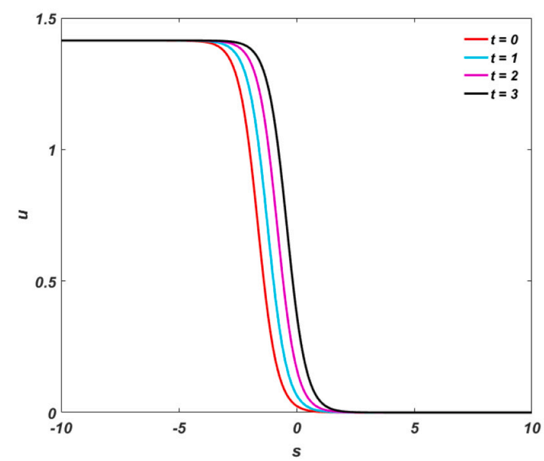
Fig. 8 shows a traveling wave that changes over different time values, characterized by a tanh function. Its amplitude is being scaled by the factor $\frac{1}{\sqrt{4}}$ and it underwent phase shifts by C.



(a) For time $t = 0$.



(b) For time $t = 1$.



(c) $2D$ plots of u_6 for different time values t.

Fig. 6. $3D$ and $2D$ plots of u_6 for different time values t, with parameter $C = 2$.

Example 5. If $\beta(u) = u$ and $\alpha(u) = c_1\,u + c_2\,u^2 - c_3\,u^3$.

For $\beta(u) = u$ and $\alpha(u) = c_1\,u + c_2\,u^2 - c_3\,u^3$, we obtain

$$u_9(x, y, t) = \frac{1}{\sqrt{2}}\left(1 - \tanh(\sqrt{2}\,t - x - y - C)\sqrt{8}\right),$$

herein, $c_1 = \frac{7}{4}$ and $c_2 = 1 = c_3$. This solution can be visualized as follows:

Fig. 9 depicts a stable traveling wave that evolves diagonally over different time values with sharp change in amplitude. The amplitude of the wave is being scaled by the factor $\frac{1}{\sqrt{2}}$ and it underwent diagonal phase shifts by C. This amplitude remains almost constant on either side of the wavefront, revealing the soliton behavior.

4. Conservation laws by partial Lagrangian approach

Conservation laws, play a major role in Lie symmetry analysis as well as other approaches. These are expressed as divergences, signify the conservation of specific physical quantities owing to the inherited symmetries of a system defined by Des. The identification of these conservation laws is, among other things, beneficial for the linearization of Pdes, double reduction, and determination of non-local symmetries in differential equations.

The following section discusses the conservation laws associated with the $(2 + 1)$- dimensional non-linear damped Kge through the partial Lagrangian approach, which is employed even in cases where the conventional Lagrangian for a system is not straightforward or non-existent, unlike Noether's theorem, which requires the existence of Lagrangian of a system. This scenario arises, for instance, in scalar evolution equations and even in the case of the simple classical heat equation. To explore conservation laws of Eq. (2), we opt for the partial Noether's method, as the existence of the damping term $\beta(u)\,u_t$ prevents the standard Lagrangian from being formulated for this equation, that means $\frac{\delta \mathcal{L}}{\delta u} \neq 0$.

The partial Lagrangian of Eq. (2) is as follows

$$\mathcal{L} = \frac{1}{2}\,u_t^2 - \frac{1}{2}\,u_x^2 - \frac{1}{2}\,u_y^2 + \int \alpha(u)\,du, \tag{48}$$

where

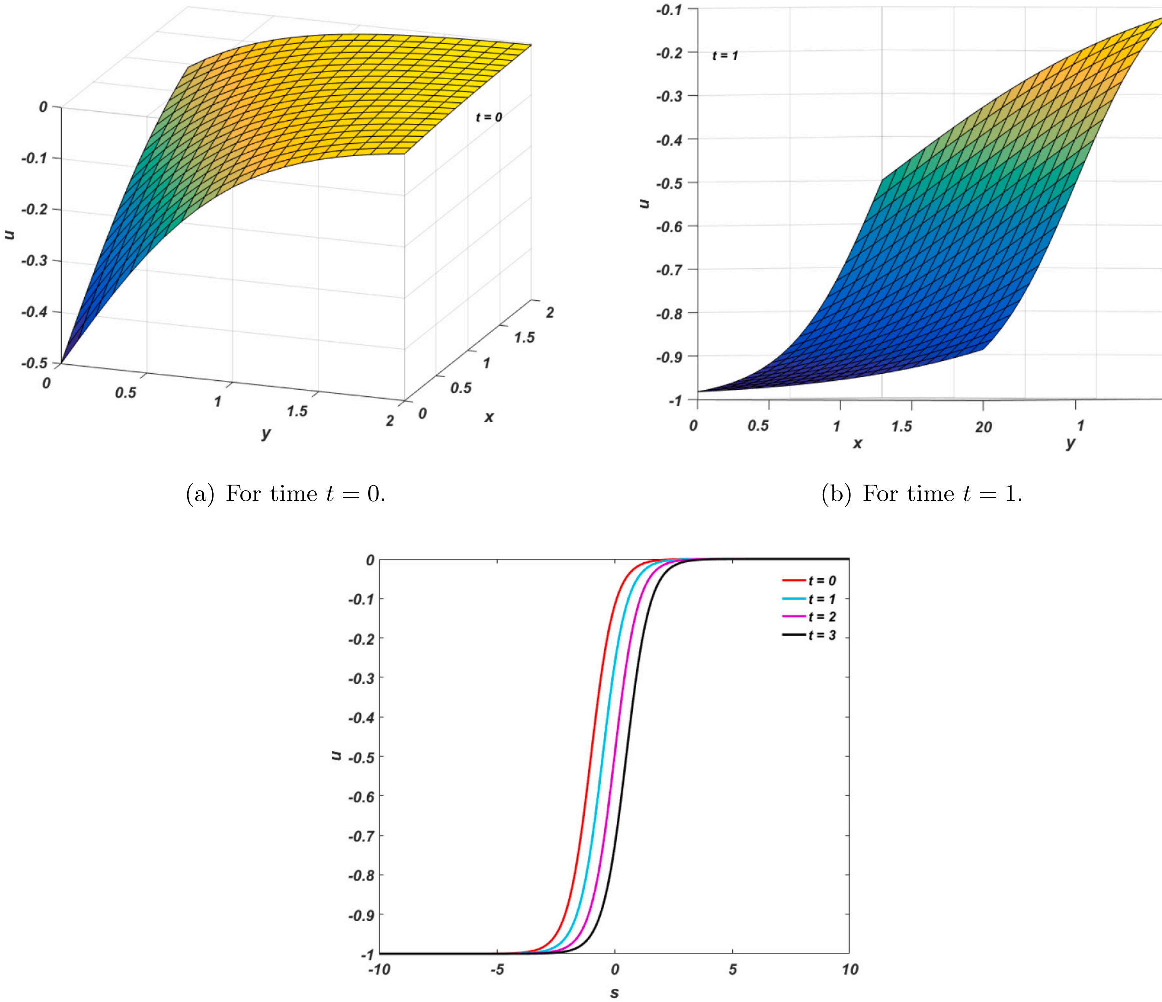
$$\frac{\delta \mathcal{L}}{\delta u} = \alpha(u) + u_{xx} + u_{yy} - u_{tt} = \beta(u)\,u_t.$$

Associated with the partial Lagrangian (48), the operator defined in Eq. (4) is termed as partial Noether operator of Eq. (2) if the following criterion

$$X^{[1]}\,\mathcal{L} + (D_t\,\tau^1 + D_x\,\tau^2 + D_y\,\tau^3)\,\mathcal{L} = W\,\frac{\delta \mathcal{L}}{\delta u} + D_t\,\mathcal{B}^1 + D_x\,\mathcal{B}^2 + D_y\,\mathcal{B}^3, \tag{49}$$

holds, where

$$W = \varphi - \tau^1\,u_t - \tau^2\,u_x - \tau^3\,u_y,$$



(a) For time $t = 0$.

(b) For time $t = 1$.

(c) $2D$ plots of u_7 for different time values t.

Fig. 7. $3D$ and $2D$ plots of u_7 for different time values t, with parameter $C = 2$.

$\mathcal{B}^1$, $\mathcal{B}^2$ & $\mathcal{B}^3$ denote gauge terms that depend on (t, x, y, u). A subsequent set of equations is derived from Eq. (49)

$$\tau^1_u = 0,\ \tau^2_u = 0,\ \tau^3_u = 0, \tag{50}$$

$$\mathcal{B}^2_u + \varphi_x = 0, \tag{51}$$

$$\varphi_t - \varphi\,\beta(u) - \mathcal{B}^1_u = 0, \tag{52}$$

$$\mathcal{B}^3_u + \varphi_y = 0, \tag{53}$$

$$-2\,\varphi_u + \tau^2_x - \tau^1_t - \tau^3_y = 0, \tag{54}$$

$$-2\,\varphi_u - \tau^2_x - \tau^1_t + \tau^3_y = 0, \tag{55}$$

$$2\,\varphi_u + 2\,\tau^1\,\beta(u) + \tau^2_x - \tau^1_t + \tau^3_y = 0, \tag{56}$$

$$\tau^1_x - \tau^2_t + \tau^2\,\beta(u) = 0, \tag{57}$$

$$\tau^1_y - \tau^3_t + \tau^3\,\beta(u) = 0, \tag{58}$$

$$\tau^3_x + \tau^2_y = 0, \tag{59}$$

$$\varphi\,\alpha(u) + (\tau^1_t + \tau^2_x + \tau^3_y)\int \alpha(u)\,du = \mathcal{B}^1_t + \mathcal{B}^2_x + \mathcal{B}^3_y. \tag{60}$$

From Eqs. (54) and (55), we have

$$\tau^3_y - \tau^2_x = 0. \tag{61}$$

Consequently, we obtain

$$\varphi = -\frac{1}{2}\,\tau^1_t\,u + \mathcal{M}(t, x, y). \tag{62}$$

Using Eqs. (61) and (62) in Eq. (56) yields

$$\tau^2_x - \tau^1_t + \beta\,\tau^1 = 0. \tag{63}$$

Next, we examine the subsequent cases for arbitrary smooth functions $\beta(u)$ and $\alpha(u)$:

Case 1: $\beta(u)$ is an arbitrary function of u

If $\beta(u)$ is an arbitrary function of u, i.e. $\beta(u) \neq \text{Constant}$, then from Eqs. (57), (58) and (63), we determine

$$\tau^1 = 0, \quad \tau^2 = 0, \quad \tau^3 = 0.$$

From here, we obtain

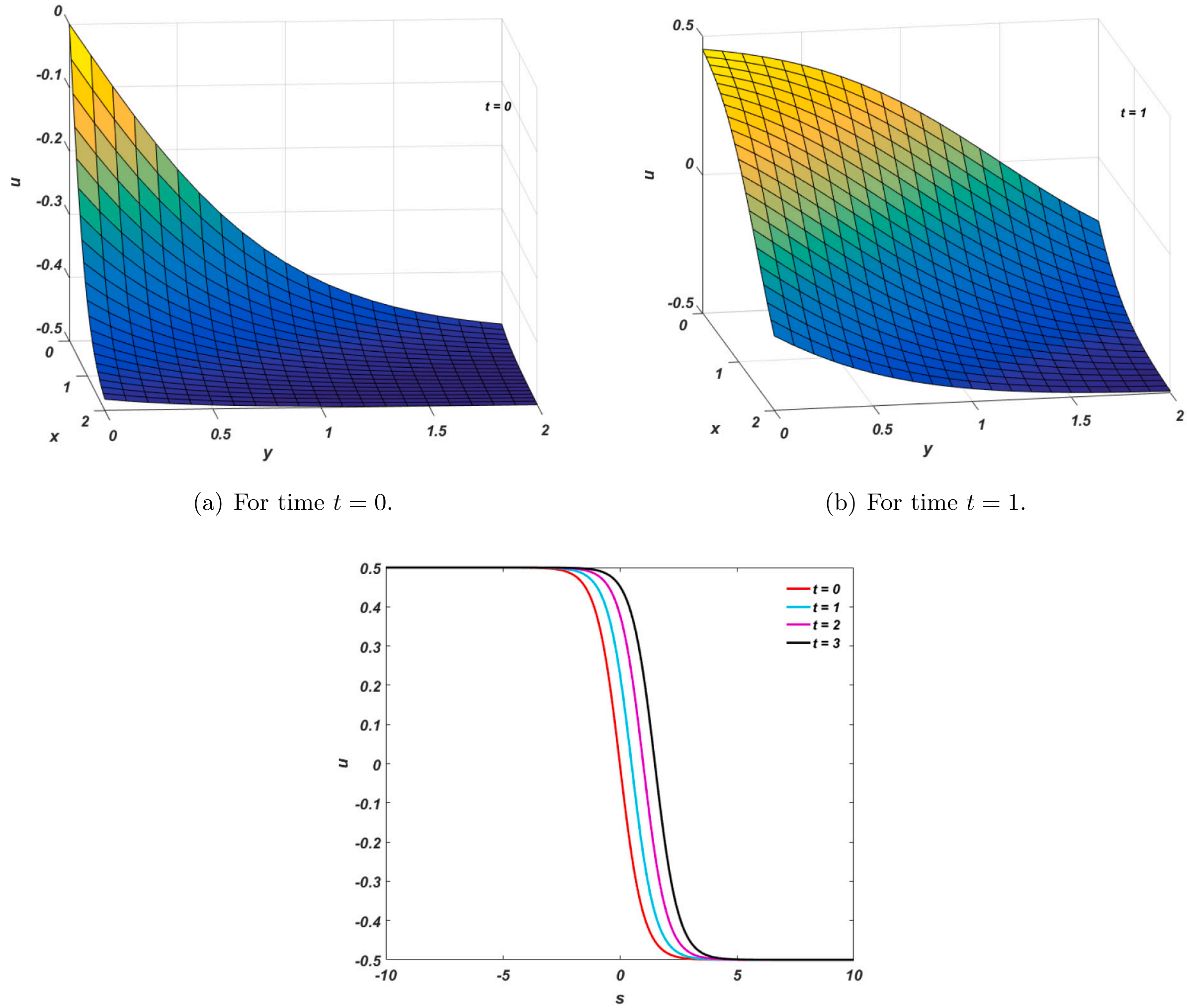
$$\varphi = \mathcal{M}(t, x, y). \tag{64}$$

Now, Eqs. (51)–(53) give rise to the following

$$\mathcal{B}^1 = \mathcal{M}_t\,u - \mathcal{M}\int \beta(u)\,du + \mathcal{H}^1(t, x, y), \tag{65}$$

$$\mathcal{B}^2 = -\mathcal{M}_x\,u + \mathcal{H}^2(t, x, y), \tag{66}$$

$$\mathcal{B}^3 = -\mathcal{M}_y\,u + \mathcal{H}^3(t, x, y). \tag{67}$$



(a) For time $t = 0$.

(b) For time $t = 1$.

(c) $2D$ plots of u_8 for different time values t.

Fig. 8. $3D$ and $2D$ plots of u_8 for different time values t, with parameter $C = 2$.

Using these equations in Eq. (60), results in

$$\mathcal{M}\,\alpha(u) = (\mathcal{M}_{tt} - \mathcal{M}_{xx} - \mathcal{M}_{yy})\,u - \mathcal{M}_t \int \beta(u)\,du + \mathcal{H}^1_t + \mathcal{H}^2_x + \mathcal{H}^3_y. \quad (68)$$

Subcase 1.1

If $\beta(u)$, $\alpha(u)$ and u are not related, then from Eq. (68)

$$\mathcal{M} = 0,$$

and additionally

$$\mathcal{H}^1_t + \mathcal{H}^2_x + \mathcal{H}^3_y = 0.$$

Therefore, no operator is obtained in this case.

Subcase 1.2

If $\beta(u)$ is an arbitrary function of u and $\alpha(u) = 0$, Eq. (68) then yields

$$\mathcal{M}_t = 0, \quad (69)$$

$$\mathcal{M}_{xx} + \mathcal{M}_{yy} = 0, \quad (70)$$

$$\mathcal{H}^1_t + \mathcal{H}^2_x + \mathcal{H}^3_y = 0. \quad (71)$$

After solving Eqs. (69) and (70), we obtain

$$\mathcal{M} = a_1 + a_2\,x + a_3\,y + a_4\,x\,y + a_5\,(x^2 - y^2).$$

To find the conserved vector components, we utilize the following identity

$$T^i = \mathcal{B}^i - \tau^i\,\mathcal{L} - (\varphi - \tau^1\,u_t - \tau^2\,u_x - \tau^3\,u_y)\,\frac{\partial \mathcal{L}}{\partial u_i}, \quad (72)$$

where $i = (1, 2, 3) = (t, x, y)$. The conserved vectors for this case are

$$\begin{aligned} T^t &= -\left(a_1 + a_2\,x + a_3\,y + a_4\,x\,y + a_5\,(x^2 - y^2)\right)(u_t + \int \beta(u)\,du), \\ T^x &= -(a_2 + a_4\,y + 2\,a_5\,x)\,u \\ &\quad + \left(a_1 + a_2\,x + a_3\,y + a_4\,x\,y + a_5\,(x^2 - y^2)\right)u_x, \\ T^y &= -(a_3 + a_4\,x - 2\,a_5\,y)\,u \\ &\quad + \left(a_1 + a_2\,x + a_3\,y + a_4\,x\,y + a_5\,(x^2 - y^2)\right)u_y. \end{aligned}$$

Here, a_p represents constants and $p = (1, \ldots, 5)$. Consequently for constants a_p, there exist five independent conserved quantities.

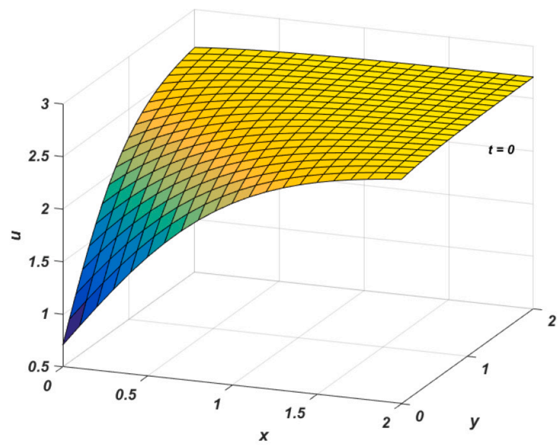
Subcase 1.3

If $\alpha(u) = \alpha_1 + \alpha_2\,u$, and $\beta(u)$ is not linear in u, it consequently follows from Eq. (68) that

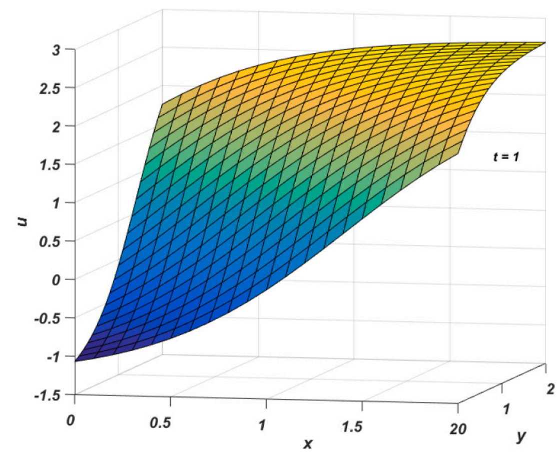
$$\mathcal{M}_t = 0, \quad (73)$$

$$\alpha_2\,\mathcal{M} + \mathcal{M}_{xx} + \mathcal{M}_{yy} = 0, \quad (74)$$

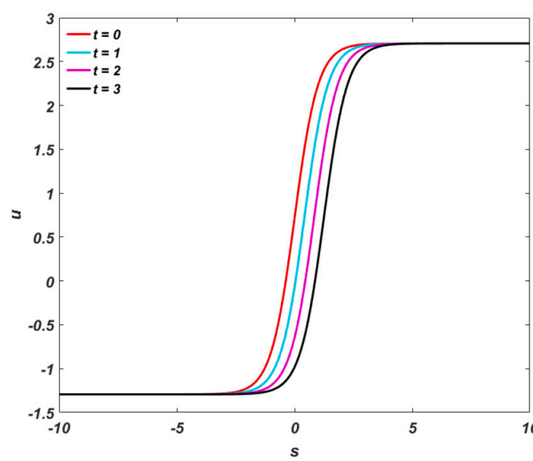
$$\mathcal{H}^1_t + \mathcal{H}^2_x + \mathcal{H}^3_y = \alpha_1\,\mathcal{M}. \quad (75)$$



(a) For time $t = 0$.



(b) For time $t = 1$.



(c) $2D$ plots of u_9 for different time values t.

Fig. 9. $3D$ and $2D$ plots of u_9 for different time values t, with parameter $C = 2$.

The application of the method of separation of variables to solve Eq. (74) gives rise to three subcases:

Subcase 1.3.1: $\lambda^2 + \alpha_2 > 0$

In this subcase

$$\mathcal{M} = a_1 \cos(\rho x) \cos(\rho y) + a_2 \cos(\rho x) \sin(\rho y) + a_3 \sin(\rho x) \cos(\rho y) + a_4 \sin(\rho x) \sin(\rho y),$$

and the conserved vectors are given as

$$T^t = -\big(a_1 \cos(\rho x) \cos(\rho y) + a_2 \cos(\rho x) \sin(\rho y) + a_3 \sin(\rho x) \cos(\rho y) + a_4 \sin(\rho x) \sin(\rho y)\big) (u_t + \int \beta(u)\, du),$$

$$T^x = (\cos(\rho x) + \sin(\rho x))\Big(\big(a_1 \cos(\rho y) + a_2 \sin(\rho y)\big) (\rho u + u_x) + \big(a_3 \cos(\rho y) + a_4 \sin(\rho y)\big) (u_x - \rho u)\Big),$$

$$T^y = (\cos(\rho y) + \sin(\rho y))\Big(\big(a_1 \cos(\rho x) + a_3 \sin(\rho x)\big) (u_y + \rho u) + \big(a_2 \cos(\rho x) + a_4 \sin(\rho x)\big) (u_y - \rho u)\Big),$$

herein, $\rho = \sqrt{\lambda^2 + \alpha_2}$ and λ is a separation constant.

Subcase 1.3.2: $\lambda^2 + \alpha_2 = 0$

Here, the following components of the conserved vectors are derived

$$T^t = -\big(a_1 x y + a_2 x + a_3 y + a_4\big) (u_t + \int \beta(u)\, du),$$
$$T^x = -(a_1 y + a_2) u + \big(a_1 x y + a_2 x + a_3 y + a_4\big) u_x,$$
$$T^y = -(a_1 x + a_3) u + \big(a_1 x y + a_2 x + a_3 y + a_4\big) u_y.$$

Subcase 1.3.3: $\lambda^2 + \alpha_2 < 0$

The following conserved vectors are derived for this particular subcase

$$T^t = -\big(a_1 e^{\rho(x+y)} + a_2 e^{\rho(x-y)} + a_3 e^{\rho(y-x)} + a_4 e^{-\rho(x+y)}\big) (u_t + \int \beta(u)\, du + \mathcal{H}^1,$$

$$T^x = -\big(a_1 e^{\rho(x+y)} + a_2 e^{\rho(x-y)} - a_3 e^{\rho(y-x)} - a_4 e^{-\rho(x+y)}\big) \rho u + \big(a_1 e^{\rho(x+y)} + a_2 e^{\rho(x-y)} + a_3 e^{\rho(y-x)} + a_4 e^{-\rho(x+y)}\big) u_x + \mathcal{H}^2,$$

$$T^y = -\big(a_1 e^{\rho(x+y)} - a_2 e^{\rho(x-y)} + a_3 e^{\rho(y-x)} - a_4 e^{-\rho(x+y)}\big) \rho u + \big(a_1 e^{\rho(x+y)} + a_2 e^{\rho(x-y)} + a_3 e^{\rho(y-x)} + a_4 e^{-\rho(x+y)}\big) u_y + \mathcal{H}^3.$$

Here, $\rho = \sqrt{\lambda^2 + \alpha_2}$, with λ serving as a separation constant.

Subcase 1.4

If $\alpha(u) = k\int \beta(u)\,du$, then from Eq. (68) we find

$$k\,\mathcal{M} + \mathcal{M}_t = 0, \tag{76}$$

$$\mathcal{M}_{tt} - \mathcal{M}_{xx} - \mathcal{M}_{yy} = 0, \tag{77}$$

$$\mathcal{H}_t^1 + \mathcal{H}_x^2 + \mathcal{H}_y^3 = 0. \tag{78}$$

From Eq. (76), we obtain

$$\mathcal{M} = C(x,y)\,e^{-kt}.$$

Using Eq. (76) in Eq. (77) and solving the resulting equation through the method of separation of variables gives rise to three subcases:

Subcase 1.4.1: $k^2 - \lambda^2 > 0$

The following conserved vectors are derived for this particular subcase

$$\begin{aligned} T^t &= -e^{-kt}\left(a_1\,e^{\varrho(x+y)} + a_2\,e^{\varrho(x-y)} + a_3\,e^{\varrho(y-x)} + a_4\,e^{-\varrho(x+y)}\right) \\ &\quad \times\,(k\,u + u_t + \int \beta(u)\,du) + \mathcal{H}^1, \\ T^x &= e^{-kt}\Big(\left(a_1\,e^{\varrho(x+y)} + a_2\,e^{\varrho(x-y)}\right)(u_x - \rho\,u) \\ &\quad + \left(a_3\,e^{\varrho(y-x)} + a_4\,e^{-\varrho(x+y)}\right)(u_x + \rho\,u)\Big) + \mathcal{H}^2, \\ T^y &= e^{-kt}\Big(\left(a_1\,e^{\varrho(x+y)} + a_3\,e^{\varrho(y-x)}\right)(u_y - \rho\,u) \\ &\quad + \left(a_2\,e^{\varrho(x-y)} + a_4\,e^{-\varrho(x+y)}\right)(u_y + \rho\,u)\Big) + \mathcal{H}^3. \end{aligned}$$

Subcase 1.4.2: $k^2 - \lambda^2 = 0$

In this subcase, the components of the conserved vectors are

$$\begin{aligned} T^t &= -e^{-kt}\left(a_1\,x\,y + a_2\,x + a_3\,y + a_4\right)(k\,u + u_t + \int \beta(u)\,du), \\ T^x &= e^{-kt}\Big(-(a_1\,y + a_2)\,u + \left(a_1\,x\,y + a_2\,x + a_3\,y + a_4\right)u_x\Big), \\ T^y &= e^{-kt}\Big(-(a_1\,x + a_3)\,u + \left(a_1\,x\,y + a_2\,x + a_3\,y + a_4\right)u_y\Big). \end{aligned}$$

Subcase 1.4.3: $k^2 - \lambda^2 < 0$

$$\begin{aligned} T^t &= -e^{-kt}\,\big(a_1\cos(\varrho\,x)\cos(\varrho\,y) + a_2\cos(\varrho\,x)\sin(\varrho\,y) + \\ &\quad a_3\sin(\varrho\,x)\cos(\varrho\,y) + a_4\sin(\varrho\,x)\sin(\varrho\,y)\big)\,(k\,u + u_t + \int \beta(u)\,du), \end{aligned}$$

$$\begin{aligned} T^x &= e^{-kt}\,\left(\cos(\varrho\,x) + \sin(\varrho\,x)\right)\Big(\left(a_1\cos(\varrho\,y) + a_2\sin(\varrho\,y)\right)(u_x + \varrho\,u) + \\ &\quad \left(a_3\cos(\varrho\,y) + a_4\sin(\varrho\,y)\right)(u_x - \varrho\,u)\Big), \end{aligned}$$

$$\begin{aligned} T^y &= e^{-kt}\,\left(\cos(\varrho\,y) + \sin(\varrho\,y)\right)\Big(\left(a_1\cos(\varrho\,x) + a_3\sin(\varrho\,x)\right)(u_y + \varrho\,u) + \\ &\quad \left(a_2\cos(\varrho\,x) + a_4\sin(\varrho\,x)\right)(u_y - \varrho\,u)\Big), \end{aligned}$$

wherein, $\varrho = \sqrt{k^2 - \lambda^2}$ and λ is a separation constant.

Case 2: $\beta(u) = Constant = q$

In this case, Eqs. (51)–(53) yield

$$\mathcal{B}^1 = \frac{1}{4}(q\,\tau_t^1 - \tau_{tt}^1)\,u^2 + (\mathcal{M}_t - q\,\mathcal{M})\,u + \mathcal{H}^1(t,x,y), \tag{79}$$

$$\mathcal{B}^2 = \frac{1}{4}\,\tau_{tx}^1\,u^2 - \mathcal{M}_x\,u + \mathcal{H}^2(t,x,y), \tag{80}$$

$$\mathcal{B}^3 = \frac{1}{4}\,\tau_{ty}^1\,u^2 - \mathcal{M}_y\,u + \mathcal{H}^3(t,x,y). \tag{81}$$

By using Eqs. (62) and (79)–(81) in Eq. (60) and taking derivatives of the obtained equation thrice *w.r.t.* u, yields

$$(-\frac{1}{2}\,\tau_t^1\,u + \mathcal{M})\,\alpha_{uuu} + (-\frac{1}{2}\tau_t^1 + \tau_x^2 + \tau_y^3)\,\alpha_{uu} = 0. \tag{82}$$

Here, two subcases arise:

Subcase 2.1: $\alpha_{uu} = 0$

In this subcase, $\alpha(u)$ takes the form

$$\alpha(u) = a\,u + b.$$

Consequently, we derive an infinite number of conserved vector components, which are as follows

$$\begin{aligned} T^t &= (\mathcal{M}_t - q\,\mathcal{M})\,u + \frac{1}{2}\,(a_3\,y + a_5\,x + a_6)\,e^{qt}\,(u_t^2 + u_x^2 + u_y^2 \\ &\quad - a\,u^2 - 2\,b\,u - 2\,c + q\,u\,u_t) - \\ &\quad \mathcal{M}\,u_t + e^{qt}\,u_t\left((-a_1\,y + a_5\,t + a_4)\,u_x + (a_1\,x + a_3\,t + a_2)\,u_y\right) \\ &\quad + \mathcal{H}^1, \end{aligned}$$

$$\begin{aligned} T^x &= \frac{1}{4}\,a_5\,q\,e^{qt}\,u^2 - \mathcal{M}_x\,u + \mathcal{M}\,u_x + \frac{1}{2}\,(-a_1\,y + a_5\,t + a_4)\,e^{qt}\,(-u_t^2 \\ &\quad - u_x^2 + u_y^2 - a\,u^2 - 2\,b\,u - 2\,c) - \\ &\quad e^{qt}\left((\frac{1}{2}\,q\,u\,u_x + u_t\,u_x)(a_3\,y + a_5\,x + a_6)\right. \\ &\quad \left. - (a_1\,x + a_3\,t + a_2)\,u_x\,u_y\right) + \mathcal{H}^2, \end{aligned}$$

$$\begin{aligned} T^y &= \frac{1}{4}\,a_3\,q\,e^{qt}\,u^2 - \mathcal{M}_y\,u + \mathcal{M}\,u_y + \frac{1}{2}\,(a_1\,x + a_3\,t + a_2) \\ &\quad \times e^{qt}\,(-u_t^2 + u_x^2 - u_y^2 - a\,u^2 - 2\,b\,u - 2\,c) - \\ &\quad e^{qt}\left((\frac{1}{2}\,q\,u\,u_y + u_t\,u_y)(a_3\,y + a_5\,x + a_6)\right. \\ &\quad \left. - (-a_1\,y + a_5\,t + a_4)\,u_x\,u_y\right) + \mathcal{H}^3, \end{aligned}$$

subject to

$$a\,\mathcal{M} + \frac{1}{2}\,q\,e^{qt}\,(a_3\,y + a_5\,x + a_6)\,b = \mathcal{M}_{tt} - q\,\mathcal{M}_t - \mathcal{M}_{yy}. \tag{83}$$

The following ansatz can be applied to Eq. (83):

1. If $\mathcal{M} = \mathcal{M}(x)$, then

$$\mathcal{M}(x) = -\frac{1}{2\,a}\,b\,q\,e^{qt}\,(a_3\,y + a_5\,x + a_6).$$

2. If $\mathcal{M} = \mathcal{M}(y)$, then we have

$$\begin{aligned} \mathcal{M}(y) &= a_7\,\sin(\sqrt{a}\,y) + a_8\,\cos(\sqrt{a}\,y) \\ &\quad - \frac{1}{2\,a}\,b\,q\,e^{qt}\,(a_3\,y + a_5\,x + a_6). \end{aligned}$$

3. If $\mathcal{M} = \mathcal{M}(t)$, then we obtain

$$\begin{aligned} \mathcal{M}(t) &= a_7\,e^{\frac{1}{2}\,(q + \sqrt{4\,a + q^2})t} + a_8\,e^{\frac{1}{2}\,(q + \sqrt{4\,a - q^2})t} \\ &\quad - \frac{1}{2\,a}\,b\,q\,e^{qt}\,(a_3\,y + a_5\,x + a_6). \end{aligned}$$

Subcase 2.2: $\alpha_{uu} \neq 0$

From Eq. (82), we arrive at

$$(-\frac{1}{2}\,\tau_t^1\,u + \mathcal{M})\,\frac{\alpha_{uuu}}{\alpha_{uu}} + (-\frac{1}{2}\tau_t^1 + \tau_x^2 + \tau_y^3) = 0. \tag{84}$$

Differentiating Eq. (84) *w.r.t.* u, we obtain the following equation

$$-\frac{1}{2}\,\tau_t^1\,\Big(u\,\frac{\alpha_{uuu}}{\alpha_{uu}}\Big)_u + \mathcal{M}\,\Big(\frac{\alpha_{uuu}}{\alpha_{uu}}\Big)_u = 0. \tag{85}$$

From here, we consider different subcases:

Subcase 2.2.1: $\left(\frac{\alpha_{uuu}}{\alpha_{uu}}\right)_u = 0$

Following this subcase, we consider

$$\frac{\alpha_{uuu}}{\alpha_{uu}} = d.$$

Substituting back into Eq. (85), we determine

$$-\frac{1}{2}\,\tau_t^1\,d = 0.$$

This equation gives rise to new subcases:

Subcase 2.2.1.1: $d \neq 0$

For this subcase

$$\tau^1 = \tau^1(x, y),$$

and $\alpha(u)$ take the following form

$$\alpha(u) = d_2 e^{d u} + d_3 u + d_4.$$

The conserved vectors obtained are as follows

$$\begin{aligned} T^t &= e^{qt}\big((-a_1 y + a_3)u_x + (a_1 x + a_2)u_y\big)u_t + \mathcal{H}^1, \\ T^x &= \frac{1}{2}e^{qt}\Big((-a_1 y + a_3)(-u_t^2 - u_x^2 + u_y^2 - 2\frac{d_2}{d}e^{du} - d_3 u^2 - 2d_4 u) \\ &\quad - 2(a_1 x + a_2)u_y u_x\Big) + \mathcal{H}^2, \\ T^y &= \frac{1}{2}e^{qt}\Big((a_1 x + a_2)(-u_t^2 + u_x^2 - u_y^2 - 2\frac{d_2}{d}e^{du} - d_3 u^2 - 2d_4 u) \\ &\quad - 2(-a_1 y + a_3)u_y u_x\Big) + \mathcal{H}^3. \end{aligned}$$

Subcase 2.2.1.2: $d = 0$

In this subcase, $\alpha(u)$ takes the following form

$$\alpha(u) = \frac{1}{2}\alpha_1 u^2 + \alpha_2 u + \alpha_3.$$

Furthermore, the conserved vectors found are

$$\begin{aligned} T^t &= e^{qt}\big((-a_1 y + a_3)u_x + (a_1 x + a_2)u_y\big)u_t + \mathcal{H}^1, \\ T^x &= \frac{1}{2}e^{qt}\Big((-a_1 y + a_3)(-u_t^2 - u_x^2 + u_y^2 - \frac{1}{3}\alpha_1 u^3 - \alpha_2 u^2 - 2\alpha_3 u) \\ &\quad - 2(a_1 x + a_2)u_y u_x\Big) + \mathcal{H}^2, \\ T^y &= \frac{1}{2}e^{qt}\Big((a_1 x + a_2)(-u_t^2 + u_x^2 - u_y^2 - \frac{1}{3}\alpha_1 u^3 - \alpha_2 u^2 - 2\alpha_3 u) \\ &\quad - 2(-a_1 y + a_3)u_y u_x\Big) + \mathcal{H}^3. \end{aligned}$$

Subcase 2.2.2: $\left(\frac{\alpha_{uuu}}{\alpha_{uu}}\right)_u \neq 0$

In this subcase, we have from Eq. (85)

$$-\frac{1}{2}\tau_t^1 \frac{\left(u\frac{\alpha_{uuu}}{\alpha_{uu}}\right)_u}{\left(\frac{\alpha_{uuu}}{\alpha_{uu}}\right)_u} + \mathcal{M} = 0. \tag{86}$$

Now, we assume

$$\left(u\frac{\alpha_{uuu}}{\alpha_{uu}}\right)_u = \alpha_1\left(\frac{\alpha_{uuu}}{\alpha_{uu}}\right)_u.$$

Eq. (86) yields the following expression

$$-\frac{1}{2}\tau_t^1 \alpha_1 + \mathcal{M} = 0.$$

After some manipulations, we deduce the result

$$\frac{\alpha_{uuu}}{\alpha_{uu}} = \frac{\alpha_2}{u - \alpha_1}. \tag{87}$$

Now, we need to consider various cases from here.

Subcase 2.2.2.1: $\alpha_1 \neq 0$

In this subcase, the following form is obtained for the function $\alpha(u)$

$$\alpha(u) = \frac{\alpha_3}{(\alpha_2 + 1)(\alpha_2 + 2)}(u - \alpha_1)^{\alpha_2 + 2} + \alpha_4 u + \alpha_5.$$

Case A: $\alpha_2 + 1 \neq 0$

For this case, the following conserved vectors are obtained

$$\begin{aligned} T^t &= e^{qt}\big((-a_1 y + a_3)u_x + (a_1 x + a_2)u_y\big)u_t + \mathcal{H}^1, \\ T^x &= \frac{1}{2}e^{qt}\Big((-a_1 y + a_3)(-u_t^2 - u_x^2 + u_y^2 \\ &\quad - 2\int \alpha(u)\,du) - 2(a_1 x + a_2)u_y u_x\Big) + \mathcal{H}^2, \\ T^y &= \frac{1}{2}e^{qt}\Big((a_1 x + a_2)(-u_t^2 + u_x^2 - u_y^2 - 2\int \alpha(u)\,du) \\ &\quad - 2(-a_1 y + a_3)u_y u_x\Big) + \mathcal{H}^3. \end{aligned}$$

Case B: $\alpha_2 + 1 = 0$

We obtain the following conserved components

$$\begin{aligned} T^t &= (\mathcal{M}_t - q\mathcal{M})u + \frac{1}{2}(a_3 y + a_5 x + a_6)e^{qt}(u_t^2 + u_x^2 + u_y^2 \\ &\quad - 2\int \alpha(u)\,du + q u u_t) - \\ &\quad \mathcal{M}u_t + e^{qt}u_t\big((-a_1 y + a_5 t + a_4)u_x + (a_1 x + a_3 t + a_2)u_y\big) \\ &\quad + \mathcal{H}^1, \end{aligned}$$

$$\begin{aligned} T^x &= \frac{1}{4}a_5 q e^{qt}u^2 - \mathcal{M}_x u + \mathcal{M}u_x + \frac{1}{2}(-a_1 y + a_5 t + a_4) \\ &\quad \times e^{qt}(-u_t^2 - u_x^2 + u_y^2 - 2\int \alpha(u)\,du) - \\ &\quad e^{qt}\big((\frac{1}{2}q u u_x + u_t u_x)(a_3 y + a_5 x + a_6) \\ &\quad - (a_1 x + a_3 t + a_2)u_x u_y\big) + \mathcal{H}^2, \end{aligned}$$

$$\begin{aligned} T^y &= \frac{1}{4}a_3 q e^{qt}u^2 - \mathcal{M}_y u + \mathcal{M}u_y + \frac{1}{2}(a_1 x + a_3 t + a_2) \\ &\quad \times e^{qt}(-u_t^2 + u_x^2 - u_y^2 - 2\int \alpha(u)\,du) - \\ &\quad e^{qt}\big((\frac{1}{2}q u u_y + u_t u_y)(a_3 y + a_5 x + a_6) \\ &\quad - (-a_1 y + a_5 t + a_4)u_x u_y\big) + \mathcal{H}^3, \end{aligned}$$

where

$$\mathcal{M} = \frac{1}{2}q e^{qt}\alpha_1(a_3 y + a_5 x + a_6).$$

Subcase 2.2.2.2: $\alpha_1 = 0$

In this subcase, we obtain the form for the function $\alpha(u)$

$$\alpha(u) = \frac{\alpha_3}{(\alpha_2 + 1)(\alpha_2 + 2)}u^{\alpha_2 + 2} + \alpha_4 u + \alpha_5.$$

Case C: $\alpha_2 + 1 \neq 0$

This case leads to the following conserved components

$$T^t = e^{qt}\big((-a_1 y + a_3)u_x + (a_1 x + a_2)u_y\big)u_t + \mathcal{H}^1,$$

$$\begin{aligned} T^x &= \frac{1}{2}e^{qt}\Big((-a_1 y + a_3)\big(-u_t^2 - u_x^2 + u_y^2 \\ &\quad - 2\big(\frac{\alpha_3}{(\alpha_2 + 1)(\alpha_2 + 2)(\alpha_2 + 3)}u^{\alpha_2 + 3} + \\ &\quad \frac{1}{2}\alpha_4 u^2 + \alpha_5 u\big)\big) - 2(a_1 x + a_2)u_y u_x\Big) + \mathcal{H}^2, \end{aligned}$$

$$\begin{aligned} T^y &= \frac{1}{2}e^{qt}\Big((a_1 x + a_2)\big(-u_t^2 + u_x^2 - u_y^2 \\ &\quad - 2\big(\frac{\alpha_3}{(\alpha_2 + 1)(\alpha_2 + 2)(\alpha_2 + 3)}u^{\alpha_2 + 3} + \\ &\quad \frac{1}{2}\alpha_4 u^2 + \alpha_5 u\big)\big) - 2(-a_1 y + a_3)u_y u_x\Big) + \mathcal{H}^3. \end{aligned}$$

Case D: $\alpha_2 + 1 = 0$

We derive the following conserved vectors for this subcase

$$\begin{aligned} T^t &= \frac{1}{2}(a_3 y + a_5 x + a_6)e^{qt}(u_t^2 + u_x^2 + u_y^2 - 2\int \alpha(u)\,du + q u u_t) + \\ &\quad e^{qt}u_t\big((-a_1 y + a_5 t + a_4)u_x + (a_1 x + a_3 t + a_2)u_y\big) + \mathcal{H}^1, \end{aligned}$$

$$\begin{aligned} T^x &= \frac{1}{4}a_5 q e^{qt}u^2 + \frac{1}{2}(-a_1 y + a_5 t + a_4)e^{qt} \\ &\quad \times(-u_t^2 - u_x^2 + u_y^2 - 2\int \alpha(u)\,du) - \\ &\quad e^{qt}\big((\frac{1}{2}q u u_x + u_t u_x)(a_3 y + a_5 x + a_6) \\ &\quad - (a_1 x + a_3 t + a_2)u_x u_y\big) + \mathcal{H}^2, \end{aligned}$$

$$T^y = \frac{1}{4}a_3 q e^{qt}u^2 + \frac{1}{2}(a_1 x + a_3 t + a_2)e^{qt}$$

$$\times\,(-u_t^2 + u_x^2 - u_y^2 - 2\int \alpha(u)\,du) - e^{q\,t}\left((\frac{1}{2}\,q\,u\,u_y + u_t\,u_y)(a_3\,y + a_5\,x + a_6) - (-a_1\,y + a_5\,t + a_4)\,u_x\,u_y\right) + \mathcal{H}^3.$$

5. Conclusion

The Lie symmetry approach was used to examine the (2 + 1)-dimensional nonlinear damped KGE. A comprehensive Lie group classification was conducted for the arbitrary functions, namely, $\beta(u)$ and $\alpha(u)$ appearing in Eq. (2). This classification resulted in the derivation of an extended Lie algebra for each distinct case that arose during our analysis. Following this, we utilized the similarity transformations method to systematically reduce the (2 + 1)-dimensional nonlinear damped KGE to ordinary differential equations, considering each case individually. Several invariant solutions of the equation were identified. Additionally, the conservation laws were derived by employing the partial Lagrangian approach. Throughout our analysis, we have considered all possible cases to determine conserved vectors associated with the equation.

Concisely, the application of Lie symmetry approach on the nonlinear damped KGE resulted in the derivation of its Lie symmetries, specific forms of smooth, arbitrary functions $\beta(u)$ and $\alpha(u)$ involved in the equation, reduction to (1+1)-dimensional KGE and then subsequently to ODEs, and identification of its solutions e.g., soliton and traveling wave solutions.

Regarding future work, further exploration will involve finding the solution of the set of reduced ODEs to further understand the behavior of the invariant solutions. This approach, known for its conciseness and efficacy, can similarly be employed to address other nonlinear PDEs . We studied the conservation laws from the mathematical aspect, they not only help in understanding the structure of the differential equations but also advances in simplifying the process of solving nonlinear partial differential equations, as they lead to the double reduction theory.[34] In future, the discovered conservation laws of the nonlinear KGE can be used to find the solutions and reductions of more complex systems. The discovered conservation laws offer valuable tools for studying the dynamics and stability of physical systems, making this work relevant to a range of scientific endeavours that can be performed.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

This article has been produced with the financial support of the European Union under the REFRESH – Research Excellence For Region Sustainability and High-tech Industries project number CZ .10.03.01/00/22_003/0000048 via the Operational Programme Just Transition.

Appendix

A.1. Reduction under X_1 and X_2

In this appendix, we outline the detailed steps require to reduce Eq. (2) into an ODE, for the arbitrary functions $\beta(u)$ and $\alpha(u)$. It can be seen from Table 3 that the generators X_1 and X_2 commute, $[X_1,\ X_2] = 0$. Therefore, as explained earlier, either X_1 or X_2 can be used as a starting point to begin the reductions of Eq. (2).

We choose to initiate with $X_1 = \frac{\partial}{\partial t}$, for which the corresponding characteristic equation is expressed as follows

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{1} = \frac{du}{0}.$$

According to standard procedure, we integrate this equation to obtain the respective similarity variables, as follows

$$z = x,\quad r = y,\quad u = \omega(z, r).$$

Differentiating these similarity variables twice $w.r.t.$ x, we obtain

$$u_x = \omega_z,\qquad u_{xx} = \omega_{zz}.$$

Similarly, differentiating these similarity variables twice $w.r.t.$ y and t, we get

$$u_y = \omega_r,\qquad u_{yy} = \omega_{rr},$$
$$u_t = 0 = u_{tt}.$$

Using these derivatives, Eq. (2) can be reformulated in the form

$$\omega_{zz} + \omega_{rr} + \alpha(\omega) = 0. \tag{88}$$

Since, we begin with X_1. Now, we apply the other commuting generator, that is X_2, to further reduce the PDE (88). To proceed, we express X_2 in new variables, z and r, as defined above. Thus, X_2 now becomes

$$X_2 = Y_2 = \frac{\partial}{\partial z}.$$

The characteristic equation for this generator can be expressed as follows

$$\frac{dz}{1} = \frac{dr}{0} = \frac{d\omega}{0},$$

which yields the following similarity variables

$$s = r,\quad \omega = \vartheta(s),$$

differentiating these similarity variables twice $w.r.t.$ r and z, we get

$$\omega_r = \vartheta_s,\qquad \omega_{rr} = \vartheta_{ss},$$
$$\omega_z = 0 = \omega_{zz}.$$

Substitution of these derivatives in Eq. (88) results in the subsequent ODE

$$\vartheta_{ss} + \alpha(\vartheta) = 0.$$

A.2. Reduction under X_1 and X_4

Herein, we explain the steps require to reduce Eq. (31) into an ODE. From Table 4, it can be seen that the generators X_1 and X_4 commute, $[X_1,\ X_4] = 0$. Thus, either X_1 or X_4 can serve as a initiating point to start the reductions of Eq. (31).

We begin with

$$X_4 = y\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial y}.$$

The associated similarity transformations are as follows

$$z = \frac{1}{2}(x^2 + y^2),\quad r = t,\quad u = \omega(z,\ r),$$

where the characteristic equation for this symmetry is

$$\frac{dx}{y} = \frac{dy}{-x} = \frac{dt}{0} = \frac{du}{0},$$

differentiating these similarity variables twice $w.r.t.$ t, x, and y, we get

$$u_t = \omega_r,\qquad u_{tt} = \omega_{rr},$$
$$u_x = x\,\omega_z,\qquad u_{xx} = \omega_z + x^2\,\omega_{zz},$$
$$u_y = y\,\omega_z,\qquad u_{yy} = \omega_z + y^2\,\omega_{zz}.$$

By putting these derivatives into Eq. (31), we get the following reduced PDE

$$\omega_{rr} + \mathcal{A}(\omega+\gamma)^k\,\omega_r = 2\,\omega_z + 2\,z\,\omega_{zz} + \mathcal{F}\,(\omega+\gamma)^{1+2k}. \tag{89}$$

Now, when X_1 is expressed in terms of new variables

$$X_1 = Y_1 = \frac{\partial}{\partial r},$$

it yields the following invariants

$$s = z, \quad \omega = \vartheta(s),$$

where the associated characteristic equation is

$$\frac{dz}{0} = \frac{dr}{1} = \frac{d\omega}{0},$$

differentiating these similarity variables twice $w.r.t.$ z and r, we obtain

$$\omega_z = \vartheta_s, \quad \omega_{zz} = \vartheta_{ss},$$
$$\omega_r = 0 = \omega_{rr}.$$

This transforms Eq. (89) to the following ODE

$$s\,\vartheta_{ss} + \vartheta_s + \frac{1}{2}\mathcal{F}\,(\vartheta+\gamma)^{1+2k} = 0. \tag{90}$$

Data availability

No data was used for the research described in the article.

References

1. Magalakwe G, Muatjetjeja B, Khalique CM. Symmetry analysis, nonlinearly self-adjoint and conservation laws of a generalized $(2+1)$-dimensional Klein–Gordon equation. *Malays J Math Sci*. 2019;13(2):123–138.
2. Peskin Michael E. *An Introduction to Quantum Field Theory*. CRC Press; 2018.
3. Mohamad H. Energy asymptotics for the strongly damped Klein–Gordon equation. *Partial Differ Equ Appl*. 2022;3(6):71.
4. Williams LK, Nayfeh AH. Subharmonic resonances of nonlinear Klein–Gordon equations. *Nonlinear Dynam*. 1997;13(1):29–48.
5. Lie S. Diskussion der differentialgleichung $\frac{d^2z}{dxdy} = F(z)$. *Arch fur Math*. 1881;6:112–124. (Reprinted in: Lie S., Gesammelte Abhandlungen, 3, B.G. Teubner, Leipzig and H. Aschehoug & Co, Kristiania, 469–478.).
6. Paliathanasis A, Tsamparlis M, Mustafa MT. Symmetry analysis of the Klein–Gordon equation in Bianchi I spacetimes. *Int J Geom Methods Mod Phys*. 2015;12(03):1550033.
7. Devi P, Guleria A. Classical symmetries of the Klein–Gordon–Zakharov equations with time-dependent variable coefficients. *Arab J Math*. 2024;13(1):103–119.
8. Polyanin AD, Zaitsev VF. *Handbook of Nonlinear Partial Differential Equations: exact Solutions, Methods, and Problems*. Chapman and Hall/CRC; 2003.
9. Polyanin AD. Construction of functional separable solutions in implicit form for non-linear Klein–Gordon type equations with variable coefficients. *Int J Non-Linear Mech*. 2019;114:29–40.
10. Ovsiannikov LV. *Group Analysis of Differential Equations, 1(3)*. New York: Academic Press; 1982:82.
11. Ibragimov NH. *CRC Handbook of Lie Group Analysis of Differential Equations. Vols. 1–3*. Boca Raton: CRC Press; 1994–1996.
12. Olver PJ. *Applications of Lie Groups to Differential Equations. Vol. 107*. Springer Science & Business Media; 1993.
13. Bluman GW, Kumei S. *Symmetries and Differential Equations. Vol. 81*. Springer Science & Business Media; 2013.
14. Lahno V, Zhdanov R. Group classification of nonlinear wave equations. *J Math Phys*. 2005;46(5).
15. Barone A. Theory and applications of the sine-Gordon equation. *Riv Nuovo Cimento*. 1971;1(2):227–267.
16. Lin Y, Cui M. A new method to solve the damped nonlinear Klein–Gordon equation. *Sci China A*. 2008;51(2):304–313.
17. Azad H, Mustafa MT, Ziad M. Group classification, optimal system and optimal reductions of a class of Klein Gordon equations. *Commun Nonlinear Sci Numer Simul*. 2010;15(5):1132–1147.
18. Tang Xiao-Yan, Sen-Yue L. Symmetry analysis of $(2+1)$–dimensional nonlinear Klein–Gordon equations. *Chin Phys Lett*. 2002;19(1):1.
19. Khalique CM, Biswas A. Analysis of non-linear Klein–Gordon equations using Lie symmetry. *Appl Math Lett*. 2010;23(11):1397–1400.
20. Márkus F, Gambár K. Symmetry breaking and dynamic transition in the negative mass term Klein–Gordon equations. *Symmetry*. 2024;16(2):144.
21. Ahmadov AI, Nagiyev SM, Ikot AN, Tarverdiyeva VA. Analytical solutions for the Klein–Gordon equation with combined exponential type and ring-shaped potentials. *Sci Rep*. 2024;14(1):5527.
22. Dorodnitsyn VA. *Applications of Lie Groups to Difference Equations*. Boca Raton: CRC Press; 2011.
23. Cheviakov, Dorodnitsyn, Kaptsov. Invariant conservation law-preserving discretizations of linear and nonlinear wave equations. *J Math Phys*. 2020;61.
24. Anco Stephen C, Bluman G. Direct construction method for conservation laws of partial differential equations Part I: Examples of conservation law classifications. *European J Appl Math*. 2002;13(5):545–566.
25. Noether AE. Invariante variations probleme. *Nachr Akad Wiss Göttingen Math-Phys KI II*. 1918;23:5–237.
26. Kara AH, Mahomed FM. Noether-type symmetries and conservation laws via partial Lagrangians. *Nonlinear Dynam*. 2006;45(3):367–383.
27. Usman M, Zaman FD. Lie symmetry analysis and conservation laws of non-linear $(2+1)$ elastic wave equation. *Arab J Math*. 2023;12(1):265–276.
28. Zaman FD, Mahomed FM, Arif F. Lie symmetry classification, optimal system, and conservation laws of damped Klein–Gordon equation with power law non-linearity. *Math Comput Appl*. 2023;28(5):96.
29. Ahmad I, Faridi WA, Iqbal M, Majeed Z, Tchier F. Exploration of soliton solutions in nonlinear optics for the third order Klein-Fock-Gordon equation and nonlinear Maccari's system. *Internat J Theoret Phys*. 2024;63(6):157.
30. Meleshko SV. *Methods for Constructing Exact Solutions of Partial Differential Equations. Mathematical and Analytical Techniques with Applications to Engineering*. Springer; 2005.
31. Abdeljabbar A, Roshid HO, Aldurayhim A. Bright, dark, and rogue wave soliton solutions of the quadratic nonlinear Klein–Gordon equation. *Symmetry*. 2022;14(6):1223.
32. Rehman HU, Iqbal I, S. Aiadi S, Mlaiki N, Saleem MS. Soliton solutions of Klein–Fock–Gordon equation using sardar subequation method. *Mathematics*. 2022;10(18):3377.
33. Lv Y, Liu Q. Travelling wave solutions of a class of generalized Klein–Gordon equations. 2019.
34. Bokhari AH, Al-Dweik AY, Zaman FD, Kara AH, Mahomed FM. Generalization of the double reduction theory. *Nonlinear Anal Real World Appl*. 2010;11(5):3763–3769.