Investigation of Higher Spin Theory Through Holography and Mathematical Methods in Free Conformal Field Theory

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Declaration

I declare that this Dissertation is my own, unaided work. It is being submitted for the Degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

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(Signature of candidate)

22$^{\mathbf{nd}}$ day of **May** 20**17** in **Johannesburg**

Abstract

In this dissertation, we will develop novel methods for determining the spectrum of primary operators in a free conformal field theory (vector model) and to construct some of these primaries. To count the spectrum of primaries, we use group theoretic techniques to obtain character formulas for any number of fields as representations of $SO(4,2)$. More precisely, we will construct generating functions that can be expanded to any order in the conformal scaling dimension Δ to yield the complete spectrum of primaries constructed out of n scalar fields. We also develop efficient methods to construct these primaries by using a polynomial description. Finally, these primary operators, which are higher spin currents in the free conformal vector model, correspond to higher spin gauge fields in Vasiliev higher spin theory through the holographic duality.

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Finally, I would like to thank both my parents/sister and Irma's family. They have helped keep me motivated and were always willing to help in any way possible.

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Abbreviations and Conventions

We will often use the following abbreviations:

• Conformal field theory or **CFT**,
• Representation or irreducible representation as **rep** or **irrep**, respectively.

Furthermore, we will use capital letters to denote a group. For example, $SO(4)$. We will use lower case letters in Fraktur style with boldface for algebras. For example, $\mathfrak{su}(2)$.

1 Motivation

In this dissertation, we want to study quantum gravity. A theory of quantum gravity is crucial to our understanding of the origin of the universe through the big bang model and for the evaporation of black holes through Hawking radiation. Naive attempts to merge general relativity with quantum mechanics has proven extremely troublesome. However, string theory has emerged as a strong candidate for a theory of quantum gravity. More recently, we have seen the conjectured holographic duality [1] prove to be invaluable to the study of quantum gravity. As we will argue in the following paragraphs, a promising approach to learning about quantum gravity is to study possible new symmetries. These symmetries are considered to be broken/Higgsed at lower energies, but are believed to be made manifest at higher energies (trans-Planckian energies since this is the domain of quantum gravity). It is thus reasonable to expect that new symmetries will emerge in studies of quantum gravity-symmetries that are broken at energies below the Planck scale (we will provide strong motivation for this in the following paragraphs)[2].

With respect to the fields that transform under higher spin symmetry, we mean any gauge field with spin $s > 2$. Here, we can start to see a possible connection to string theory; indeed, string theory contains higher spin states. This indicates that string theory should admit an interpretation as a higher spin theory with spontaneously broken higher spin symmetries [2]. In the last few years we have seen the holographic duality successfully tested in the framework of higher spin gauge theories (see, for example, the three-point functions calculated by Giombi, Yin [3]).

Consider Figure (1), which represents parameter space that has been divided up in terms of regimes. Here, area $A1$ represents the domain of perturbative conformal field theory while area $A2$ represents the domain of classical supergravity (see Remark (2) in Chapter (3) for the duality between type IIB string theory and $\mathcal{N} = 4$ Super Yang Mills theory). The y-axis is labeled with the string coupling constant g_s, which is equivalent to $\hbar$ of quantum gravity. The x-axis is labeled by a dimensionless ratio λ between the radius of curvature R_{AdS} of AdS spacetime and the string length l_s (note that string length can be related to the tension T of the string through $T \propto 1/l_s^2$, and thus we see that the string tension defines the length of the string). Furthermore, λ is also the well known t' Hooft coupling constant $\lambda = g_{YM}^2 N$ in Yang-Mills theory which is equivalent to $\hbar$ of conformal field theory (CFT). Currently, we are limited to analyzing the two small regions A_1 and A_2 of parameter space. If we are to completely understand quantum gravity, we must be able to understand all of the parameter space. Thus, our problem is to study the parameter space outside these two regions.

Now, there have been several methods that have met some success in understanding quantum gravity outside of the two small regions. We briefly mention each of these methods:

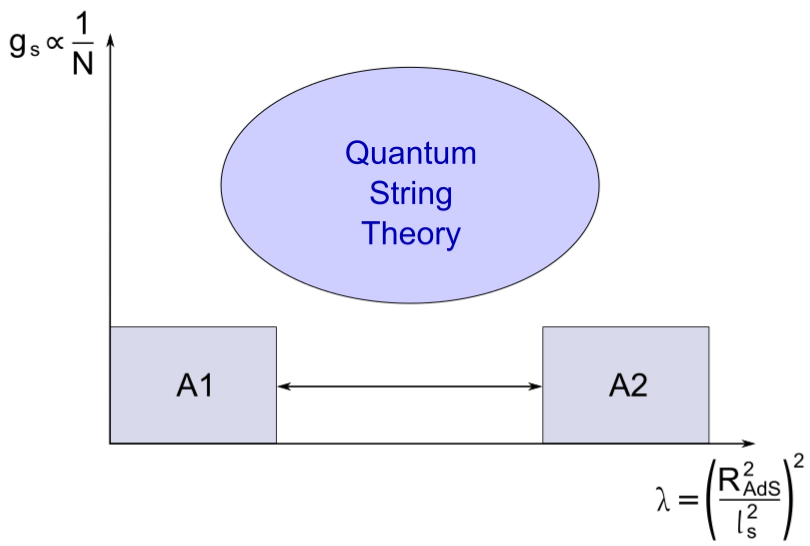


Figure 1: Comparison of the free parameters in the dual theories

- **Non-Renormalized Quantities** Some observables have the same value at strong and weak coupling. For example, the dimensions of a certain operator (representing some observable) would not change under varying the CFT coupling constant (thus, we would have no loop corrections) and therefore one would be able to compare the shaded areas. Crucially, this method depends on the use of sypersymmetry; in particular, it allows one to trust some of the weak coupling calculations even at strong coupling. These quantities are guaranteed by non-renormalization theorems. An example of non-renormalized quantities are the dimensions of BPS operators. For more information, see for example [4].

- **Localization** In some special circumstances, it is possible to evaluate the partition function and even certain correlation functions exactly, even for interacting theories. Critically, this depends on the use of supersymmetry. For some observables, one can reduce the path integral to a finite dimensional integral. As a very simple example (we will be very brief here, the precise derivation can be found at [5]) consider the following partition function and action in a 0-dimensional spacetime

$$\mathcal{Z} = \int \frac{d\phi d\psi_1 d\psi_2}{\sqrt{2\pi}} e^{-S(\phi,\psi_i)}, \tag{1}$$

where

$$S(\phi, \psi_1, \psi_2) = \frac{1}{2}\partial h(\phi)^2 - \psi_1\psi_2\partial^2 h(\phi), \tag{2}$$

and $h(\phi) \in \mathbb{R}$ is some polynomial in ϕ. Here, the derivatives are with respect to the fields ϕ; in other words, $\partial h(\phi) = \partial h(\phi)/\partial\phi$. Also, ϕ is a bosonic field and ψ_1, ψ_2 are fermionic fields that obey the rules of Grassmann variables. Under the supersymmetric transformations

$$\delta\phi = \epsilon_1\psi_1 + \epsilon_2\psi_2, \qquad \delta\psi_1 = \epsilon_2\partial h, \qquad \delta\psi_2 = -\epsilon_1\partial h, \tag{3}$$

(where ϵ_i are fermionic parameters), the action is invariant. If we now consider the supersymmetric variation of the operator $\mathcal{O}_g = \partial g\psi_1$ (for some $g = g(\phi)$) and set the parameters to $\epsilon_1 = -\epsilon_2 = \epsilon$, then under the invariance of the action we find the following correlation function

$$0 = \langle\delta\mathcal{O}_g\rangle = \epsilon\langle\partial g\partial h - \partial^2 g\psi_1\psi_2\rangle. \tag{4}$$

The important point to notice here is that the term $\partial g\partial h - \partial^2 g\psi_1\psi_2$ is the first-order change in the action under the deformation $h \to h + g$, as long as g does not alter the behaviour of h as $|\phi| \to \infty$. If one chooses $g \propto h$, then the deformation simply rescales $h \to (1+\lambda)h$ and therefore $\mathcal{Z}[h]$ is independent of the overall scale of h. Through iterating this procedure, we can rescale h by a larger factor; in particular, we can imagine rescaling h as $h \to (1+\lambda)h \to (1+\lambda)^2 h \to \cdots \to \Lambda h$, where Λ is related to some power of $(1+\lambda)$ such that $|\Lambda| >> 1$ $(\lambda > 0)$. Consequently, the bosonic part of the action becomes $(\partial h)^2/2 \to \Lambda^2(\partial h)^2/2$; and as $\Lambda \to \infty$, the factor e^{-S} exponentially suppresses any contribution to $\mathcal{Z}$ except from an infinitesimal neighbourhood of the critical points of h where $\partial h = 0$. This is known as localization. These critical points are in fact saddle-points. We thus emphasize here that we have reduced the integral to a finite dimensional integral; that is, we have reduced the integral Z to a saddle-point evaluation of Z, which is exact. Near such critical points, say ϕ_*, the partition function simplifies to a particularly simple form

$$\mathcal{Z}[h] = \sum_{\phi_*:\partial h|_{\phi_*}=0} \mathrm{sgn}(\partial^2 h|_{\phi_*}), \tag{5}$$

which clearly depends only on the degree of the polynomial $\partial^2 h(\phi)$ (the precise form of h is irrelevant). If h is a polynomial of odd degree, then ∂h reduces the degree by one and therefore $\partial h = 0$ will have an even number of roots. Furthermore, $\partial^2 h$ will alternate sign between each pair of roots and since we have an even number of roots, $\mathcal{Z}[h] = 0$ identically. Similarly, if h is a polynomial of even degree, then $\partial h = 0$ will have an odd number of roots. Consequently, $\mathcal{Z}[h] = \pm 1$, with the sign depending on whether $h \to \pm\infty$ as $|\phi| \to \infty$.

Under the AdS/CFT correspondence, one can study the duality by looking at operators that can be computed exactly by localization. One such operator with a well established holographic dual is the Wilson loop, which has local invariance under supersymmetric transformations in $\mathcal{N} = 4$ SYM. (see [6][7][8]).

• **Integrability** This relates the planar limit of a CFT (spin chain in $d = 2$-dimensions) to a string world sheet theory ($d = 2$). Consider $\mathcal{N} = 4$ SYM, which has a rich set of symmetries due to supersymmetry and conformal symmetry. Due to the latter, there are no massive particles whose mass spectrum we might hope to compute. Instead, we can replace the characteristic quantity of mass by scaling dimension. In the planar limit of $\mathcal{N}=4$ SYM (by planar limit, we mean the t' Hooft limit $N \to \infty$ while keeping the gauge coupling $\lambda = g_{YM}^2 N$ finite which suppresses Feynman graphs that have crossing lines; this reduces the complexity of graphs from factorial to exponential growth such that the radius of convergence of the perturbative series grows to a finite size), we can essentially solve for the spectrum exactly. More precisely, one can express the scaling dimension of some local operator as a function of the coupling constant λ. This function is given as the solution of a set of integral equations which follow from the so-called Thermodynamic Bethe Ansatz or related techniques (Y-system). In a certain limit, these equations simplify to a set of algebraic equations called the asymptotic Bethe equations. Through the AdS/CFT correspondence, this translates to integrability of the string world sheet model (the dual theory for planar $\mathcal{N}=4$ SYM at arbitrary coupling λ is free type IIB superstrings on $AdS_5 \times S^5$ at arbitrary tension T). The spectrum of planar scaling dimenions for local operators as a function of λ as predicted by integrability is dual to the energy spectrum of free string states. Integrability makes coincident predictions from both models and agreement can be found over the complete range of couplings. Furthermore, integrability can be viewed as a hidden symmetry which is not well understood. More concretely, the type of quantum integrable systems that is studied is usually formulated in terms of deformed universal enveloping algebras of affine Lie algebras called quasi-triangular Hopf algebras. However, the gauge/string theory integrable model appears to be based on some unconventional superalgebra which remains largely to be understood. For more information, see for example [9].

Could there be a more general symmetry which governs the theory as one moves from strong to weak coupling? The points mentioned above certainly suggests that symmetries are being exploited. Concretely, we ask (1) Is this a good question? (2) Could the answer be yes?

Consider then the following arguments:

• **Uniqueness** In a quantum field theory, we can describe interactions through perturbative methods which can be graphically represented by Feynman graphs. Here, particles propagate along worldlines and interaction occurs where worldlines meet, thus creating a vertex. Information about interactions are inserted at these vetrices. Each vertex increases in number with increasing order of perturbation; consequently, the number of Feynman graphs grow factorially at each order of perturbation theory and examining higher orders in the perturbation becomes a complicated task. In addition, if one thinks locally and performs a cut of the Feynman graph along an axis, then we obtain a new scattering process. Now, in string theory, particles are described by $1-$dimensional extended objects that trace out a worldsheet in spacetime. For example, consider a free closed string propagating in spacetime (see Figure (2)). The string traces out a cylindrical worldsheet in spacetime. If we turn on interactions, notice that no vertices emerge. More precisely, every part of the diagram looks like a free propagating string *locally* (this can be seen by comparing the red areas in Figure (2); indeed, one cannot locally distinguish which diagram the red area originated from). Thus, once the free string theory is determined, then the interacting theory is fixed. It is only *globally* that we identify a scattering process. Consequently, the number of distinct topologies is very small compared to the number of distinct Feynman graphs at various orders of perturbation theory. In essence, a single string graph contains all the equivalent quantum field theory graphs at each respective order in perturbation theory [10][11]. Furthermore, in quantum field theory, we must specify coupling strengths, the type of fields in the theory, and which of the fields interact. This information is contained within the Lagrangian of the theory. Thus, there are a plethora of Lagrangians we can write down for different choices of coupling strengths, field types, and which fields interact. In contrast, for the string, we know that once the free theory is specified, then the interacting theory is determined. This interacting theory is unique and we refer to this property as *uniqueness*.

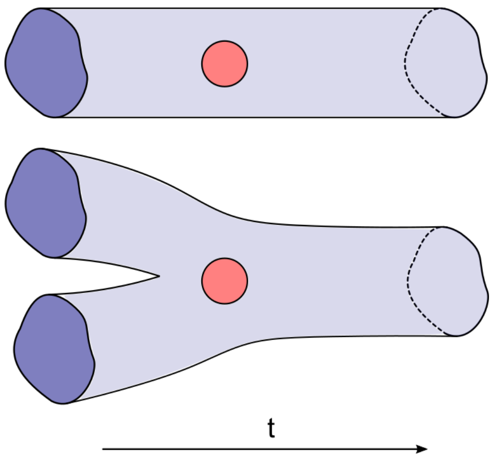


Figure 2: (Top) A Free Closed String Propagating Through Spacetime, Thus Tracing Out A Worldsheet. (Bottom) A Scattering Process Between Two Closed Strings

Uniqueness is usually a consequence of symmetry. This motivates us to conclude that there may be a more general gauge symmetry which is most likely broken/Higgsed at lower energies. A logical next step, therefore, would be to look for a phase where the hidden symmetry is made manifest. To contrast this with the previously mentioned methods used to study the correspondence, note that uniqueness is general and thus it holds for all observables in the theory.

• **Large gauge invariance of open string field theory** A further motivation for this hidden gauge symmetry comes from the stringy version of Chern-Simons theory. In 1992, Witten [12] was able to show that one can formulate Chern-Simons gauge theory in terms of open string theory. This gauge theory exhibits an enormous gauge symmetry. Chern-Simons theory is topological since it is defined on oriented manifolds without any choice of metric. Thus, it admits diffeomorphism groups as symmetries as well as the usual isometries [13]. Note that these open string symmetries are directly inherited by closed strings through the *open-closed string duality*.

• **Enlarged symmetry at high energy scattering** Furthermore, there are hints of enlarged symmetry in high energy scattering of closed strings through simple relations between distinct amplitudes. Starting with the work of Gross in 1988 [14], it was demonstrated that one can derive linear relations between string scattering amplitudes through the analysis of the high-energy limit of string scattering. This was performed in the so-called fixed angle regime or Gross Regime. This hinted at broken symmetries that were made manifest at high energies. Later work exploited a second regime, called the Regge regime, that showed an infinite number of recurrence relations among Regge string amplitudes. An interesting link has also been discovered between string amplitudes of the two regimes, which is considered crucial to probing high-energy spacetime symmetries of string theory [15][16][17] .

• **Gauge-gravity dualities in the tensionless limit** The tensionless limit of string theory leads to a huge number of symmetries. The limit is taken as $(El_s^2)^{-1} = TR_{AdS}^2 = R_{AdS}^2/l_s^2 \longrightarrow 0$, where E is the scattering energy. Here, all the massive tower of string states become compressed to a common zero point energy. In the corresponding dual theory, this corresponds to free massless gauge theories. These theories have a large number of conserved currents $J^{(s)}$ of spin $s > 2$ (which are symmetric traceless tensors of rank s) which implies a large number of symmetries. These conserved currents of the boundary gauge theory couple to bulk sources (which are fields in the bulk) that must therefore have a gauge invariance $A^{(s)} \sim A^{(s)} + \partial\Lambda^{(s-1)}$. Recently, apart from the gauge symmetries, an infinite number of global symmetries was also realized in the tensionless limit of bosonic strings [18][19][20][21].

We certainly consider these points to be sufficient motivation for the question; in other words, there is likely an interesting symmetry that governs the theory as we move from weak coupling to strong coupling.

Under consideration of these ideas, we will study free conformal field theory (for more information on conformal field theories and conserved currents, we refer the reader to [22]). It is expected that free theories have the maximum number of conserved quantities and thus have the maximum number of symmetries. To argue this, we will consider the conserved currents in *free CFT vector models* in $d = 3$-dimensions. Let $\phi^i(x)$, where $i = 1, .., N$, be complex fields described by the action

$$S = \int d^d x \partial_\mu \phi_i^* \partial^\mu \phi_i, \tag{6}$$

where summation over repeated indices is implied. The field ϕ lives in the $\square$ representation of $U(N)$, a global symmetry, with dimension $D_{\Box} = N$ (see Appendix A.3 for more discussion on Young diagrams). From this action, one can extract many conserved traceless currents. For spin $s = 1$, we have

$$J_\mu = \phi_i^* \partial_\mu \phi_i - \phi_i \partial_\mu \phi_i^*, \tag{7}$$

and for $s = 2$, we simply have the stress-energy tensor of the theory

$$J_{\mu\nu} = T_{\mu\nu} = \partial_{(\mu}\phi_i^* \partial_{\nu)}\phi_i - \frac{1}{4}\frac{1}{d-2}\big((d-2)\partial_\mu\partial_\nu + \eta_{\mu\nu}\partial^2\big)\phi_i^*\phi_i. \tag{8}$$

For a general spin s current $J_{\mu_1 \dots \mu_s}$, one can make use of Gegenbauer polynomials

$$J_{\mu_1\dots\mu_s} = \sum_{k=0}^{s} \frac{(-1)^k}{k!(k+\frac{d-s}{2})!(s-k)!(s-k+\frac{d-s}{2})!}\partial_{\mu_1}\dots\partial_{\mu_k}\phi_i^*\partial_{\mu_{k+1}}\dots\partial_{\mu_s}\phi^i - \text{traces}. \tag{9}$$

where summation over i is implied. Here, $J_{(s)}$ is a $U(N)$ singlet and thus does not transform under $U(N)$ transformations. The case for $s = 0$ gives the scalar $J_{(0)} = \phi_i^* \phi^i$ which is not a higher spin current. These higher spin currents have conformal dimension $\Delta = d - 2 + s$ and thus, for $d = 3$, $\Delta = 1 + s$. The s term tells us how many derivatives are acting on the fields. This conformal dimension saturates the unitarity bounds in d-dimensional CFT (see [23] for more on restrictions due to unitarity bounds); in other words, in order to ensure a unitary theory, our conformal dimension Δ is bounded below through $\Delta \geq d - 2 + s$. As a consequence of this, the representation includes null states (for a more thorough discussion of this, see the discussion after equation (200)). The conservation law corresponds to these null states. From the action, the scalar field ϕ has dimension $[\phi] = (d-2)/2$. Rewriting the conformal dimension as $\Delta = s + 2 \times (d-2)/2$, we see that for $d = 2$, we can have many more conserved currents. For $d > 2$, the conserved currents are all bilinear in the fields (also, the maximal symmetry the dual string theory can have is Vasiliev gauge theory). Clearly, we obtain the maximum number of conservation laws when we saturate the most unitarity bounds. Furthermore, to saturate a unitarity bound, ϕ must have a free field dimension which implies we must have a free CFT.

Importantly for this dissertation, note that we therefore have a current for each spin $s = 1, 2, 3...$ and a single $s = 0$ scalar. These are referred to as single trace operators (note that this is language borrowed from matrix-models - we are not, of course, performing an actual trace here). Later, when considering AdS_4 bosonic Vasiliev theory, we will see that this fact precisely reproduces the spectrum of gauge fields in Vasiliev theory via the holographic duality.

From [24], the full list of single trace primaries is exhausted by the singlet higher spin currents and the scalar operator $J_{(0)}$. In other words, the complete set of single trace primaries is

$$J_{(0)} + \sum_{s=1}^{\infty} J_{(s)},$$
$$(\Delta, s) = (d-2, 0) + \sum_{s=1}^{\infty} (d-2+s, s). \tag{10}$$

In addition, Vasiliev constructed a theory of gravity that is unique in that it is completely determined by its gauge symmetries. More precisely, in the tensionless limit, the bulk string theory must have a closed subsector of massless fields of $s \geq 2$ and Vasiliev was able to construct the classical dynamics of such a subsector of interacting gauge fields of spin $s \geq 2$ in AdS_{d+1}. We can thus study free CFT's and match the conserved currents to Vasiliev's gauge fields. This seems like the most promising direction to study these hidden symmetries and explore their implications.

Our approach to the problem. In this dissertation, we determine the spectrum of primary operators in free conformal field theory and explain how to construct these primary operators. Through the holographic duality, this will give the spectrum of the dual gravity theory. As further avenues of research, it would be of enormous value if we can understand the implications of the symmetry and to understand how the symmetry is broken at lower energies. We will show that we have been able to count the primaries in the free CFT for any number of fields and we have determined how to construct these primaries.

As a brief display of the results derived in later sections in this dissertation, we have shown that the number $N_{[\Delta, j_1, j_2]}$ of primary operators, of dimension Δ and spin (j_1, j_2) built out of n scalar fields ϕ can be obtained by expanding the generating function

$$G_n(s, x, y) = \sum_{n, j_1, j_2} N_{[n, j_1, j_2]} s^n x^{j_1} y^{j_2}, \tag{11}$$

where the generating function $G_n(s, x, y)$ is given by (here it is important that $n \geq 3$ since there are short representations that appear in the $n = 2$ case)

$$G_n(s, x, y) = \left[(1 - \frac{1}{x})(1 - \frac{1}{y}) Z_n(s, x, y)(1 - s\sqrt{xy})(1 - s\sqrt{\frac{x}{y}})(1 - s\sqrt{\frac{y}{x}})(1 - \frac{s}{\sqrt{xy}}) \right]_{\geq}, \tag{12}$$

where $Z_n(s, x, y)$ is defined by

$$\prod_{q=0}^{\infty} \prod_{a=-\frac{q}{2}}^{\frac{q}{2}} \prod_{b=-\frac{q}{2}}^{\frac{q}{2}} \frac{1}{1 - ts^{q+1} x^a y^b} = \sum_{n=0}^{\infty} t^n Z_n(s, x, y). \tag{13}$$

For example, for $n = 3$, the generating function has the form

$$G_3(s, x, y) \quad = s^3 + s^5 xy + s^6 x^{\frac{3}{2}} y^{\frac{3}{2}} + s^7 x^2 y^2 + s^7 x^2 + s^7 y^2 + s^8 x^{\frac{5}{2}} y^{\frac{5}{2}} + s^8 x^{\frac{3}{2}} y^{\frac{5}{2}} + s^8 x^{\frac{5}{2}} y^{\frac{3}{2}}$$

$$+2s^9x^3y^3 + s^9xy^3 + s^9x^3y + s^{10}x^{\frac{7}{2}}y^{\frac{7}{2}} + s^{10}x^{\frac{7}{2}}y^{\frac{5}{2}} + s^{10}x^{\frac{5}{2}}y^{\frac{7}{2}} + s^{10}x^{\frac{7}{2}}y^{\frac{3}{2}}$$
$$+s^{10}x^{\frac{3}{2}}y^{\frac{7}{2}} + ... \tag{14}$$

A similar generating function was also found for the free fermion CFT. From here, we can construct these primaries by phrasing the problem in terms of polynomials (the corresponding tensor is easily extracted from the polynomial), which are generally easier to manipulate than tensors. Importantly for the construction, we have used a polynomial representation of the conformal group which is given by

$$K_\mu = \frac{\partial}{\partial x^\mu}, \tag{15}$$

$$P_\mu = x^2\partial_\mu - 2x_\mu x \cdot \partial - 2x_\mu, \tag{16}$$

(the other generators can be determined using the conformal algebra commutators). Using spherical harmonic polynomials, we can construct polynomials corresponding to primary operators using a harmonic condition as well as the four equations

$$\sum_I \frac{\partial}{\partial x^I_\mu}\Psi = 0, \qquad I = 1, ..., n. \tag{17}$$

For any polynomial in $z{\cdot}x$, we can translate back into a primary using $(z{\cdot}P)^k \leftrightarrow i^k 2^k k!(z{\cdot}x)^k$, where z is any null polarization. As an example, for the $n = 3$ case with dimension $\Delta = 7$ and spin $(2, 0)$, we have the polynomial

$$\Psi = \Big(w^{(3)}(\bar{z}^{(2)} - \bar{z}^{(1)}) + w^{(2)}(\bar{z}^{(1)} - \bar{z}^{(3)}) + w^{(1)}(\bar{z}^{(3)} - \bar{z}^{(2)})\Big)^2, \tag{18}$$

which is translation invariant and thus a primary.

In Section (2), we will discuss higher spin theory and write down AdS_4 bosonic Vasiliev theory. We will then state the correspondence between free CFT and higher spin theory through the holographic duality in Section (3). In Section (4) we will develop our methods for determining the spectrum of primaries. Finally, in Section (5), we will develop methods to construct these primaries.

2 Higher Spin Theory

In this section, we will develop higher spin theory following [24]. At the end of this section, we will state the equations of motion for AdS_4 bosonic Vasiliev theory.

Higher spin gauge theories might be useful tools in studying quantum gravity. The spectrum of higher spin theory always includes a spin $s = 2$ field, which corresponds to the graviton. If one introduces a field of spin $s > 2$, then an infinite tower of higher spin fields necessarily follows, thus generalizing Einstein's theory of gravitation. In addition to quantum gravity and string theory, it is possible that higher spin gauge theories will also shed more light on the holographic duality [2].

It is interesting to note that Vasiliev's higher spin theory was constructed several years before the holographic duality. At the time, flat spacetime no-go theorems (such as the Mandula-Coleman theorem) severly restricted the construction of an S-matrix for higher-spin theories. This theorem could be surpassed by assuming a non-zero cosmological constant; consequently, this leads to the study of higher spin theory in AdS (or dS) spacetimes. Consistent cubic vertices of higher spin fields in AdS/dS backgrounds were constructed explicitly by Fradkin and Vasiliev [25] and, later, a fully non-linear classical theory of higher spins in AdS/dS backgrounds was realized by Vasiliev (see [26][27][28][29]). From the view point of the AdS/CFT correspondence, it is natural that consistent theories of massless higher-spin theories do exist and they have precisely the right structure to be dual to simple vector model CFT's at the the boundary of AdS [24].

2.1 Higher Spin Theory: Flat Spacetime

In this section, we develop some of the theory for free higher spin fields on a $d+1$-dimensional flat spacetime and we discuss why we cannot construct a higher spin theory with interactions in flat spacetime. We start with reviewing some familiar examples.

We first consider the spin $s = 1$ gauge field from Yang-Mills theory. Recall the mode expansion for a gauge field A_μ in theory which is given by

$$A_\mu(x) = \int \frac{d^{d-1}k}{(2\pi)^{d-1}2|\bar{k}|}(a_\mu(k)e^{-ik\cdot x} + a^\dagger_\mu(k)e^{ik\cdot x}). \tag{19}$$

To quantize the theory, one would impose the following Lorentz invariant commutator

$$[a_\mu(k), a^\dagger_\nu(p)] = \eta_{\mu\nu}(2\pi)^{d-1}2\omega_k\delta(\bar{k} - \bar{p}). \tag{20}$$

However, this would lead to unitarity violations in quantum mechanics; in particular, one finds necessarily that negative normed states would emerge in the theory. These negative normed states correspond to unphysical polarizations in the theory. By choosing a suitable gauge fixing (for example, the Coulomb gauge), one can essentially remove these unphysical polarizations, at the cost of manifest Lorentz invariance. Under the gauge symmetry group $SU(N)$, the gauge field varies as

$$\delta A_\mu(x) = \partial_\mu \epsilon, \tag{21}$$

where ϵ must be hermitian and traceless. Note, that this is only the linearized gauge transformation. This gauge symmetry leaves the linearized equation of motion

$$\partial_\mu F^{\mu\nu} = 0 = \Box A_\mu - \partial_\mu \partial^\nu A_\nu. \tag{22}$$

invariant. Here, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the usual field strength tensor and the equation of motion can be determined from the free theory action

$$S = \int d^d x \frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \tag{23}$$

using the Euler-Lagrange equations.

Similarly, we can consider the spin $s = 2$ case which corresponds to a gravitational field. If we perturb the metric $g_{\mu\nu}(x)$ as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{24}$$

(we can think of this as a ripple on spacetime) then we can linearize the Einstein field equations with zero cosmological constant

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0, \tag{25}$$

to obtain

$$\Box h_{\mu\nu} - \partial_\mu \partial^\rho h_{\rho\nu} - \partial_\nu \partial^\rho h_{\rho\mu} + \partial_\mu \partial_\nu h^\rho_\rho = 0. \tag{26}$$

This equation is invariant under the linearized gauge transformation

$$\delta h_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu. \tag{27}$$

For $s > 2$, Fronsdal was able to develop consistent equations of motion for the free theory of higher spin fields in flat spacetime [30]. Here, we consider gauge fields $\varphi_{\mu_1 \mu_2 \ldots \mu_s}(x)$ where

$s > 2$ and the fields are symmetric and traceless. The linearized gauge transformation of the higher spin field takes the form

$$\delta\varphi_{\mu_1...\mu_s} = \partial_{(\mu_1}\epsilon_{\mu_2...\mu_s)}. \tag{28}$$

The brackets that surround the indices mean that the indices are to be symmetrized; that is, we take the sum of all possible permutations of the indices and divide by the number of terms. We require the equation of motion $G_{\mu_1...\mu_s}$ to be gauge invariant as usual. We will use the free theory action

$$S = \int d^{d+1}x(\varphi^{\mu_1...\mu_s}G_{\mu_1...\mu_s} - \frac{1}{4}s(s-1)\varphi_\mu{}^{\mu\mu_3...\mu_s}G^\nu{}_{\nu\mu_3...\mu_s}). \tag{29}$$

which implies the equation of motion

$$G_{\mu_1...\mu_s} = \Box\varphi_{\mu_1...\mu_s} - s\partial_{(\mu_1}\partial^\mu\varphi_{\mu_2...\mu_{s-1})\mu} + \frac{s(s-1)}{2}\partial_{(\mu_1}\partial_{\mu_2}\varphi^{\ \ \ \ \ \ \mu}_{\mu_3...\mu_s)\mu} = 0. \tag{30}$$

Now, the gauge invariance of S places an interesting restriction on $\varphi_{\mu_1...\mu_s}(x)$

$$\delta S = 0 \text{ under } \delta\varphi_{\mu_1...\mu_s} = \partial_{(\mu_1}\epsilon_{\mu_2...\mu_s)} \qquad \Rightarrow \qquad \varphi_\mu{}^\mu{}_\nu{}^{\nu\mu_5...\mu_s} = 0, \qquad (s \geq 4). \tag{31}$$

In other words, the higher spin gauge field must be double traceless. In addition, for the equation of motion to be gauge invariant, we find that under gauge transformation

$$\delta G_{\mu_1...\mu_s} = \frac{s(s-1)(s-2)}{2}\partial_{(\mu_1}\partial_{\mu_2}\partial_{\mu_3}\epsilon_{\mu_4...\mu_s)\mu}{}^\mu, \tag{32}$$

which vanishes if $\epsilon_{\mu_1...\mu_{s-1}}$ is traceless. Now, similar to the spin $s = 1$ case, the higher- spin gauge field suffers from the same symptoms when we want to quantize. The mode expansion of the gauge field is given by

$$\varphi_{\mu_1...\mu_s}(x) = \int \frac{d^dk}{(2\pi)^d 2|\bar{k}|}(a_{\mu_1...\mu_s}(k)e^{-ik\cdot x} + a^\dagger_{\mu_1...\mu_s}(k)e^{ik\cdot x}), \tag{33}$$

and quantization can be imposed by the commutator

$$[a_{\mu_1...\mu_s}(k), a^\dagger_{\nu_1...\nu_s}(p)] \propto \eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}...\eta_{\mu_s\nu_s} + ..., \tag{34}$$

where the 'dots' mean all other possible permutations (pairings) of the μ's and ν's. Again, we find negative norm states. For higher spin gauge theory, we can remove the unphysical polarizations using the de-Donder gauge fixing

$$\partial^\mu\varphi_{\mu\mu_2...\mu_s} - \frac{s-1}{2}\partial_{(\mu_2}\varphi_{\mu_3...\mu_s)\mu}{}^\mu = 0, \tag{35}$$

which ensures that $\varphi_{\mu_1...\mu_s}$ is transverse and traceless. This gauge fixing does not remove all gauge freedoms but leaves a residual gauge transformation by a gauge parameter satisfying

$$\Box\epsilon_{\mu_1...\mu_{s-1}} = 0. \tag{36}$$

Using this residual gauge freedom, we can fix ${\varphi^\mu}_{\mu\mu_3\ldots\mu_s} = 0$ on shell. Under this gauge fixing, $G_{\mu_1\ldots\mu_s}$ reduces to the Pauli-Fierz equations [24]

$$\begin{aligned}
&\Box\varphi_{\mu_1\ldots\mu_s} = 0,\\
&\partial^\mu\varphi_{\mu\mu_2\ldots\mu_s} = 0,\\
&{\varphi^\mu}_{\mu\mu_3\ldots\mu_s} = 0.
\end{aligned} \tag{37}$$

The next step in developing higher spin theory in flat spacetime would be to turn on interactions; however, any higher spin theory with interactions runs into severe difficulties. This is a consequence of a theorem of Coleman and Mandula [31][32] which states that there exists no non-trivial S-matrix for $s > 2$. More precisely, the theorem states that any conserved higher spin charges $Q_{\mu_1\ldots\mu_{s-1}}$ must commute with the scattering matrix S, since the S-matrix must be trivial and therefore no particle scattering can occur. Thus, we cannot hope to formulate consistent equations of motions for interacting higher spin theory. There is, however, a caveat to the theorem: the theorem only holds for flat spacetimes.

2.2 Higher Spin Theory: AdS Spacetime

In this section, we develop some ideas about higher spin theory in AdS spacetime; in particular, we will generalize the results of Fronsdal from the previous section. We will discuss both the metric-like description and the frame-like description. Due to the "yes-go" results of Fradkin and Vasiliev [26][25], one can write down consistent fully non-linear equations of motion for higher spin fields in the interacting theory. In this spacetime, we will always have theories of gravity (that is, theories of $s = 2$ fields) and, interestingly, if one includes $s = 3$ fields then an infinite tower of higher spin fields necessarily follows.

In order to introduce gravity interactions, we look at curved spacetimes. Consequently, we will replace partial derivatives with covariant derivatives

$$\partial_\mu \to \nabla_\mu = \partial_\mu + \Gamma_\mu, \tag{38}$$

where Γ_μ is the usual Christoffel symbol. This derivative will act on a vector V^α as follows

$$\nabla_\mu V^\alpha = \partial_\mu V^\alpha + \Gamma^\alpha_{\mu\beta}V^\beta. \tag{39}$$

We also replace the flat space metric $\eta_{\mu\nu}$ with the metric $g_{\mu\nu}$ for curved spaces. However, it turns out that this is not sufficient to generalize the Fronsdal equations to general backgrounds due to complications with the Riemann tensor: due to the non-commutative nature of covariant derivatives, the gauge variation of the Fronsdal equations for $s > 3$ involves the full Riemann tensor which is unconstrained for a general background. Consequently, we must remove the extra terms that appear in the Fronsdal equations under a gauge variation so that the equations remain gauge invariant. To simplify the problem, we focus on maximally symmetric spacetimes; in this dissertation, we will focus on AdS backgrounds (these spaces exhibit a constant negative curvature) which has a particularly simple Riemann tensor

$$R_{\mu\nu\rho\sigma} = -\frac{1}{R_{AdS}^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), \tag{40}$$

where R_{AdS} is the radius of curvature in AdS space. We first proceed with the metric-like description (this name arises due to considering the higher spin gauge field as a generalization of the metric). In AdS spacetimes in $d+1$-dimensions, the Fronsdal equation generalizes to

$$G^{AdS_{d+1}}_{\mu_1\ldots\mu_s} = G_{\mu_1\ldots\mu_s}(\partial_\mu \to \nabla_\mu) - \frac{1}{R_{AdS}^2}\big((3-(d+1)-s)(2-s)-s\big)\varphi_{\mu_1\ldots\mu_s} = 0. \tag{41}$$

The notation in the first term is an instruction to replace each derivative in equation (30) with a covariant derivative. The second term exactly cancels the extra terms introduced when trying to vary the equations under a gauge transformation. Due to this extra term, $G^{AdS_{d+1}}_{\mu_1\ldots\mu_s}$ is invariant under the gauge transformation

$$\delta\varphi_{\mu_1\ldots\mu_s} = \nabla_{(\mu_1}\epsilon_{\mu_2\ldots\mu_s)}. \tag{42}$$

This describes the consistent propagation of all massless fields for all integer spins in AdS. We can impose again de-Donder gauge fixing to ensure a transverse and traceless gauge field. The generalized Fronsdal equation reduces to the generalized Pauli- Fierz equations

$$\begin{aligned} \left(\nabla^2 - \frac{(s-2)(s+(d+1)-3)-s}{R_{AdS}^2}\right)\varphi_{\mu_1\ldots\mu_s} &= 0, \\ \nabla^\mu\varphi_{\mu\mu_2\ldots\mu_s} &= 0, \\ \varphi^\mu{}_{\mu\mu_3\ldots\mu_s} &= 0. \end{aligned} \tag{43}$$

Now, the metric for AdS_{d+1} can be written

$$ds^2_{AdS_{d+1}} = \frac{d^2z + d\bar{x}\cdot d\bar{x}}{z^2}, \qquad 0 \le z < \infty. \tag{44}$$

For small z, we thus approach the boundary of AdS. Here, the symmetric traceless transverse tensor obeying the wave equation

$$(\nabla^2 - \frac{(s-2)(s+(d+1-3)-s)}{R_{AdS}^2})\varphi_{\mu_1\ldots\mu_s} = 0, \tag{45}$$

has the following behaviour on the boundary

$$\varphi_{i_1\ldots i_s} \sim z^{\Delta-s}\alpha_{i_1\ldots i_s}(\bar{x}), \tag{46}$$

with

$$\Delta(\Delta - d) - s = (2 - d - s)(2 - s). \tag{47}$$

Solving for the two roots for Δ gives: $\Delta = 2 - s$ and $\Delta = s + d - 2$. The second Δ value is precisely the dimension of a conserved current in the dual CFT_d theory (the first Δ corresponds to gauging the higher spin symmetry at the boundary). In the case of a massive scalar field, the boundary behaviour is given by

$$\varphi(z, \vec{x}) \sim z^{\Delta} A(\vec{x}). \tag{48}$$

Under the AdS/CFT correspondence (see Chapter (3)), the relationship between the mass m and the conformal weight Δ is given by

$$m^2 = \frac{\Delta(\Delta - d)}{R_{AdS}^2}, \tag{49}$$

where $\Delta = d - 2$ [3][24].

The metric-like formulation of higher spin fields is of limited utility since a fully non-linear quantum theory of interacting higher spin fields is not yet known at present (see [24] for recent progress on this problem). For Vasiliev non-linear higher spin theory, we will use the frame-like formulation as in [3][24], which generalizes the vielbein approach to gravity. In this formalism, we will use differential forms.

First, consider $s = 2$. The vielbein is given by

$$e_\mu^a, \tag{50}$$

and is related to the metric through

$$g_{\mu\nu}(x) = e_\mu^a(x)\eta_{ab}e_\nu^b(x). \tag{51}$$

Notice, that we are using Greek symbols for curved indices and Latin symbols for flat indices. Now, let $\Lambda_c^a(x) \in SO(1, d)$ (which is the Lorentz group). By definition, this transformation leaves the flat metric invariant

$$\Lambda_c^a \eta_{ab} \Lambda_d^b = \eta_{cd}. \tag{52}$$

This implies that the vielbein is not a unique choice; more precisely,

$$g_{\mu\nu}(x) = e_\mu^c(x)\eta_{cd}e_\nu^d(x) = (e_\mu^c(x)\Lambda_c^a)\eta_{ab}(\Lambda_d^b e_\nu^d(x)). \tag{53}$$

We thus have a local $SO(1, d)$ gauge symmetry. Indeed, the vielbein has $(d + 1)^2$ components while the metric only has $(d + 1)(d + 2)/2$ components. The remaining $(d + 1)d/2$

components can be accounted for by the freedom of local Lorentz rotations, which act as gauge symmetries. The gauge field that transforms under these symmetries is ω_μ^{ab}, which is known as the *spin connection*, and can be written in terms of the vielbein as [33]

$$\omega_\mu^{ab} = \frac{1}{2}e^{\nu a}\left(\partial_\mu e_\nu^b - \partial_\nu e_\mu^b\right) - \frac{1}{2}e^{\nu b}\left(\partial_\mu e_\nu^a - \partial_\nu e_\mu^a\right) - \frac{1}{2}e^{\rho a}e^{\sigma b}\left(\partial_\rho e_{\sigma c} - \partial_\sigma e_{\rho c}\right)e_\mu^c. \tag{54}$$

This replaces the metric tensor $g_{\mu\nu}$ (which determines the gravitational field) with the fields $e_\mu^a, (\omega_\mu)_b^a$ (note, that the flat indices are raised/lowered using the flat space metric η_{ab}).

We would like to combine e^a and ω^{ab} into a single one-form $W = -idx^\mu W_\mu$. We can view W as the gauge field for the Lie algebra of $SO(2,d)$ which can be argued as follows: the vielbein and spin connection together gives $(d+1)^2 - (d+1)d/2 = (d+1)(d+2)/2$ one-forms. This is the dimension of our Lie algebra. Now, since ω^{ab} is a gauge field under $SO(1,d)$ transformations, the algebra must containt the Lorentz generator M_{ab} satisfying the usual commutator relations for a Lorentz algebra.

$$[M_{ab}, M_{cd}] = i\eta_{bc}M_{ad} - i\eta_{bc}M_{ad} + i\eta_{bd}M_{ca} - i\eta_{ad}M_{cb}. \tag{55}$$

Also, e^a transforms as a vector under $SO(1,d)$ and thus the corresponding generator P_a must transform as a vector. This fixes the commutator between P_a, M_{bc} since we know from the previous commutator how M_{bc} acts on each index. Thus,

$$[P_a, M_{bc}] = i\eta_{bc}P_a - i\eta_{ac}P_b. \tag{56}$$

Finally, to close the algebra, observe that $[P_a, P_b]$ is antisymmetric (since, $[P_a, P_b] = -[P_b, P_a]$) and therefore we can only have M_{ab} on the right hand side of the equation. Futhermore, to satisfy dimensional analysis, we need to add a coefficient with dimension L^{-2} since $[P_aP_b] = L^{-2}$ and $[M_{ab}] = L^0$. Thus, since we are working in AdS, our commutator is

$$[P_a, P_b] = \frac{i}{R_{AdS}^2}M_{ab}. \tag{57}$$

Remark 1. *Observe that if $R_{AdS} \to \infty$, then $[P_a, P_b] = 0$; in other words, we obtain the usual commutator in flat space (Poincare algebra). Furthermore, if we rewrite the factor i/R_{AdS}^2 as $-i/R_{AdS}^2$ (that is, we change the sign of the cosmological constant), then we have the corresponding commutator for dS space.*

Recall from Yang-Mills theory that the gauge field A_μ lives in the algebra since we can write the field in terms of components of the Lie algebra generators; in other words,

$$(A_\mu)_b^a = A_\mu^\alpha (T^\alpha)_b^a \qquad \text{where} \qquad [T^\alpha, T^\beta] = f^{\alpha\beta\gamma}T^\gamma. \tag{58}$$

Similarly, we can write the one-form W as components of the $SO(d,2)$ generators. By expressing the vielbein and the spin connection as one forms

$$e^a = e_\mu^a dx^\mu, \qquad \omega^{ab} = \omega_\mu^{ab} dx^\mu, \tag{59}$$

we can write the the gauge field as

$$\begin{aligned} W &= -idx^\mu W_\mu \\ &= -i\big(dx^\mu e^a_\mu P_a + \frac{1}{2}dx^\mu \omega^{ab}_\mu M_{ab}\big) \\ &= -i\big(e^a P_a + \frac{1}{2}\omega^{ab} M_{ab}\big). \end{aligned} \tag{60}$$

Under the Cartan structure equation,

$$\begin{aligned} dW + W\wedge W &= -i\big((de^a + \omega^a{}_b \wedge e^b)P_a + \frac{1}{2}(d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} + \frac{1}{R^2_{AdS}} e^a \wedge e^b)M_{ab}\big) \\ &= -i\big((de^a + \omega^a{}_b \wedge e^b)P_a + \frac{1}{2}(R^{ab} + \frac{1}{R^2_{AdS}} e^a \wedge e^b)M_{ab}\big). \end{aligned} \tag{61}$$

Here, we have a *flat connection* $dW + W\wedge W = 0$. To argue this, consider the group element g of some Lie group G. Then, using the generators of the Lie algebra $T \in \mathfrak{g}$, we can always write the group element as an exponent

$$g = e^{i\alpha(x)T}. \tag{62}$$

Thus,

$$dg = e^{i\alpha(x)T} i(\partial_\mu\alpha)T dx^\mu \qquad \Rightarrow \qquad g^{-1}dg = i(\partial_\mu\alpha)T dx^\mu. \tag{63}$$

Observe that this is precisely the form of the gauge field W in equation (60). In other words, W can be written as $W = g^{-1}dg$, with the generators concretely given by P_a, M_{ab}. Consequently,

$$\begin{aligned} dW &= dg^{-1}\wedge dg + g^{-1}d^2 g \\ &= dg^{-1}\wedge dg, \end{aligned} \tag{64}$$

where the second term vanishes since $d^2 = 0$. Furthermore,

$$\begin{aligned} W\wedge W &= g^{-1}dg \wedge g^{-1}dg \\ &= -dg^{-1}g\wedge g^{-1}dg \\ &= -dg^{-1}\wedge dg. \end{aligned} \tag{65}$$

where we used $d(g^{-1}g) = 0 = dg^{-1}g + g^{-1}dg$. Thus, we have $dW + W\wedge W = 0$. Therefore, the terms in equation (61) in brackets must go to zero and, in particular, we find that

$$R^{ab} = -\frac{1}{R^2_{AdS}} e^a \wedge e^b. \tag{66}$$

The first term set to zero $T^a = de^a + \omega^a{}_b \wedge e^b = 0$ is a zero torsion constraint. Now, we can rewrite the above equation as follows

$$\begin{aligned} R^{ab} \quad &= -\frac{1}{R_{AdS}^2} e_\mu^a e_\nu^b dx^\mu \wedge dx^\nu \\ &= -\frac{1}{2R_{AdS}^2} \left(e_\mu^a e_\nu^b dx^\mu \wedge dx^\nu - e_\mu^a e_\nu^b dx^\nu \wedge dx^\mu\right) \\ &= -\frac{1}{2R_{AdS}^2} \left(e_\mu^a e_\nu^b - e_\mu^b e_\nu^a\right) dx^\mu \wedge dx^\nu, \end{aligned} \tag{67}$$

where we used equation (59) in the first line, the antisymmetry of the wedge product in the second line, and relabeled dummy indices to achieve the third line. The Riemann tensor two-form can be written as

$$R^{ab} = \frac{1}{2} R_{\mu\nu}^{ab} dx^\mu \wedge dx^\nu. \tag{68}$$

Equating the above expressions thus yields

$$R_{\mu\nu}^{ab} = -\frac{1}{R_{AdS}^2} \left(e_\mu^a e_\nu^b - e_\mu^b e_\nu^a\right). \tag{69}$$

Finally, the Riemann tensor written in terms of the Riemann tensor two-form is given by

$$R_{\mu\nu\rho\sigma} = R_{\mu\nu}^{ab} e_{\rho a} e_{\sigma b}. \tag{70}$$

Using equation (51), we therefore find that

$$\begin{aligned} R_{\mu\nu\rho\sigma} \quad &= -\frac{1}{R_{AdS}^2} \left(e_\mu^a e_\nu^b e_{\rho a} e_{\sigma b} - e_\mu^b e_\nu^a e_{\rho a} e_{\sigma b}\right) \\ &= -\frac{1}{R_{AdS}^2} \left(\eta_{ac}\eta_{db} e_\mu^a e_\nu^b e_\rho^c e_\sigma^d - \eta_{ac}\eta_{db} e_\mu^b e_\nu^a e_\rho^c e_\sigma^d\right) \\ &= -\frac{1}{R_{AdS}^2} \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}\right). \end{aligned} \tag{71}$$

This is simply the Riemann tensor in equation (40), which follows naturally from the flat connection condition. Finally, the gauge transformations for W are

$$\begin{aligned} \delta W &= d\epsilon + [W, \epsilon] \\ \epsilon &= -i(e^a P_a + \frac{1}{2}\epsilon^{ab} M_{ab}), \end{aligned} \tag{72}$$

where $\epsilon^a, \epsilon^{ab}$ are the gauge parameters for local translations and local Lorentz transformations, respectively [24].

From here we can generalize the frame-like approach to arbitrary s. First, we introduce the linearized relation between vielbein and metric fluctuations

$$\begin{aligned} g_{\mu\nu} &= g_{\mu\nu}^{(0)} + \varphi_{\mu\nu} \\ e_\mu^a &= e_{(0)\mu}^a + \hat{e}_\mu^a. \end{aligned} \tag{73}$$

Then equation (51) yields to first order in the fluctuation

$$\varphi_{\mu\nu} = \hat{e}_{\mu,\nu} + \hat{e}_{\nu,\mu}, \tag{74}$$

where $\hat{e}_{\mu,\nu} \equiv \hat{e}^a_\mu e^b_{(0)\nu}\eta_{ab}$. Observe that the totally symmetric Fronsdal field $\varphi_{\mu\nu}$ is precisely equal to the symmetric part of the vielbein fluctuation. From here, we can generalize to the higher spin case. The vielbein is now given by the one-form

$$e^{a_1\ldots a_{s-1}} = e_\mu^{a_1\ldots a_{s-1}} dx^\mu. \tag{75}$$

Here, the vielbein is totally symmetric and traceless in the fiber indices

$$\eta_{ab}e^{aba_3\ldots a_{s-1}} = 0. \tag{76}$$

The Fronsdal field $\varphi_{\mu\nu}$ is then related to the symmetric part of the higher spin vielbein through

$$\varphi_{\mu_1\ldots\mu_s} = e_{(\mu_1,\mu_2\ldots\mu_s)}, \tag{77}$$

where the fiber indices are lowered with the background vielbein $e^a_{(0)\mu}$. Observe here that $\varphi_{(s)}$ is automatically traceless as is required by Fronsdal's approach.

Now, the frame-like higher spin field stated above has more components than the Fronsdal field. We can decompose the tensor $e_\mu^{a_1\ldots a_{s-1}}$ according to representations of $SO(d)$. If $[i, j, k, \ldots]$ represents a Young diagram with i boxes in the first row, j boxes in the second row, k boxes in the third row, and so forth, then the tensor decomposes according to the following tensor product

$$[1, 0, \ldots] \otimes [s-1, 0, \ldots] = [s, 0, \ldots] + [s-2, 0, \ldots] + [s-1, 1, 0, \ldots]. \tag{78}$$

Clearly, the first two representations make up a totally symmetric, double traceless field. Indeed, if we have a symmetric double traceless spin s tensor (which we will denote with a hat symbol)

$$\hat{T}^{c_1c_2\ldots c_s}, \qquad \hat{T}^{aabbc_5\ldots c_s} = 0, \tag{79}$$

and a symmetric traceless spin s tensor (which we will denote without a hat symbol)

$$T^{a_1\ldots a_s}, \qquad T^{aaa_3\ldots a_s} = 0, \tag{80}$$

then we can decompose $\hat{T}^{c_1\ldots c_s}$ into

$$\hat{T}^{c_1\ldots c_s} = T^{c_1c_2\ldots c_s} + \delta^{(c_1c_2}T^{c_3\ldots c_s)}. \tag{81}$$

The first term is precisely the symmetric spin s representation and the second term is the symmetric spin $s-2$ representation. The last representation above, which is hook-type, corresponds to gauge redundancies (similar to the local Lorentz group for $s = 2$) and thus we need a corresponding gauge field $\omega_\mu^{a_1...a_{s-1},b}$, which generalizes the spin connection. Interestingly, the spin connection here has its own gauge redundancy (due to the fact that the analog version of the torsion constraint does not determine the spin connection, as opposed to the $n = 2$ case) which requires us to add another higher spin connection $\omega_\mu^{a_1...a_{s-1},bc}$. We thus find a tower of spin connection-like fields in the frame-like formulation of higher spin fields

$$\begin{aligned} &e_\mu^{a_1...a_s} \\ &\omega_\mu^{a_1...a_s,b_1...b_t}, \qquad t = 1, 2, ..., s-1, \end{aligned} \tag{82}$$

with a corresponding tower of torsion-like constraints

$$\begin{aligned} &R^{a_1...a_{s-1}} = 0 \\ &R^{a_1...a_{s-1},b} = 0 \\ &\vdots \\ &R^{a_1...a_{s-1},b_1...b_{s-1}} = e_{(0)c} \wedge e_{(0)d} C^{a_1...a_{s-1},b_1...b_{s-1}d} \end{aligned} \tag{83}$$

that allow one to solve for the spin connections in terms of the higher spin vielbein. Here, $R^{a_1...a_{s-1},b_1...b_t}, t = 0, ..., s-1$ are curvature two-forms and $C^{a_1...a_{s-1},b_1...b_{s-1}d}$ is the higher spin generalization of the Weyl tensor which transforms in the $[s, s, 0, ...]$ representation and is constructed out of $s-$derivatives of the Fronsdal field. These equations are equivalent to the Fronsdal equations (41) in terms of the symmetric field given by equation (77).

In the $s = 2$ case, we were able to combine e^a, ω^{ab} into a single gauge field in the adjoint of $SO(d, 2)$. Here, we can do the same by writing the tower of fields into a single irreducible representation of $SO(d, 2)$, with $A, B = 0, ..., d + 1$,

$$\{e^{a_1...a_{s-1}}, \omega^{a_1...a_{s-1},b_1...b_t}|_{t=1,...,s-1}\} \to \omega^{A_1...A_{s-1},B_1...B_{s-1}}, \tag{84}$$

which is in the $[s-1, s-1, 0, ...]$ representation of the AdS algebra (or the conformal algebra). Then, each gauge field is associated to a generator $T_{A_1...A_{s-1},B_1...B_{s-1}}$ in the same representation. These generators live in the *universal enveloping algebra* $\mathfrak{u}$(d,2). This algebra consists of all polynomials of all the generators of $SO(d, 2)$ quotiented by the Lie algebra. For example, if we consider $SO(4, 2)$ which is generated by $P_\mu, M_{\mu\nu}, K_\mu, D$, then the universal enveloping algebra of $SO(4, 2)$ consists of elements of the form

$$P_{\mu_1}...P_{\mu_k}, \qquad P_\mu M_{\rho\sigma} K_\gamma..., \qquad etc. \tag{85}$$

Clearly, this is an infinite algebra. Furthermore, we must divide out by the Lie algebra by using the commutators of $SO(d, 2)$. In the example for the univeral enveloping algebra for $SO(4, 2)$, we can replace terms such as

$$P_\mu K_\nu - K_\nu P_\mu, \tag{86}$$

with the relevant commutator. These generators of the higher spin algebra becomes a gauge algebra from the AdS point of view. We can now combine all the higher spin fields into a single one-form

$$W = -i\sum_s \omega^{A_1\ldots A_{s-1},B_1\ldots B_{s-1}} T_{A_1\ldots A_{s-1},B_1\ldots B_{s-1}}. \tag{87}$$

We can expand this to linear order in the fluctuations ω_1 around the AdS background

$$W = -i\big(\omega_0^{AB}T_{AB} + \sum_s \omega_1^{A_1\ldots A_{s-1},B_1\ldots B_{s-1}} T_{A_1\ldots A_{s-1},B_1\ldots B_{s-1}}\big). \tag{88}$$

The linearized curvatures in the torsion-like constraints above may then be obtained according to the gauge theory rules

$$R_1 = (dW + W\wedge W)_{linearized} = d\omega_1 + [\omega_0,\omega_1], \tag{89}$$

where the commutators $[T_{(2)}, T_{(s)}]$ needed to perform this calculation are fixed by the $SO(d,2)$ algebra. In a similar manner, the linearized gauge transformations are given by

$$\delta\omega_1 = d\epsilon + [\omega_0,\epsilon], \tag{90}$$

where $\omega_0 = \omega_0^{AB}T_{AB}, \omega_1 = \sum_s \omega_1^{A_1\ldots A_{s-1},B_1\ldots B_{s-1}} T_{A_1\ldots A_{s-1},B_1\ldots B_{s-1}}$, and the gauge parameter is $\epsilon = \sum_s \epsilon^{A_1\ldots A_{s-1},B_1\ldots B_{s-1}} T_{A_1\ldots A_{s-1},B_1\ldots B_{s-1}}$.

Thus, we have generalized the higher spin fields in the frame-like description of the one-form W [24].

2.3 Twistor Variables and the $\star$-Product Realization of the Higher Spin Algebra

We briefly develop some new mathematics which will be used to formulate AdS_4 bosonic Vasiliev theory. Specifically, we will focus on using the group $SO(4)$. This simplifies computations considerably since we can decompose $SO(4)$ into $SU(2)\times SU(2)$. Now, a totally symmetric tensor that transforms under $SU(2)$, transforms in an irreducible representation (irrep). Furthermore, a tensor that transforms under $SO(4)$ decomposes into a sum of $SU(2)$ irreps. For example, if we have a two index tensor $T_{\mu\nu}$, then each index transforms under $SU(2)\times SU(2)$. This representation decomposes as follows

$$(\frac{1}{2},\frac{1}{2})\otimes(\frac{1}{2},\frac{1}{2}) = (0,0)\oplus(1,0)\oplus(0,1)\oplus(1,1). \tag{91}$$

Here, a symmetric traceless tensor would transform in the $(1,1)$ representation and an anti-symmetric tensor would transform in the $(1,0)\oplus(0,1)$ representation. The $(0,0)$ representation is a trace (scalar). Thus, to construct these $SU(2)$ irreps, we must construct a *traceless* tensor. Traceless tensors simplify calculations and this fact makes $SU(2)$ simpler to work with than $SO(4)$.

Under the isomorphism $SO(4) = SU(2) \times SU(2)$, we can replace a vector index by a bispinor index. The explicit map is given by

$$(\sigma^a)_{\alpha\dot{\alpha}} = (\mathbf{1}, \vec{\sigma})_{\alpha\dot{\alpha}}, \qquad a = 0, ..., 3, \qquad \alpha, \dot{\alpha} = 1, 2. \tag{92}$$

Here $\vec{\sigma}$ are the usual Pauli matrices. We can therefore associate a vector v^a to a bispinor $v_{\alpha\dot{\alpha}}$ as

$$v^a \to v_{\alpha\dot{\alpha}} = v^a(\sigma_a)_{\alpha\dot{\alpha}}. \tag{93}$$

This is the correspondance between the fundamental representation $[1, 0]$ of $SO(4)$ and the $(1/2, 1/2)$ representation of $SU(2) \times SU(2)$. Furthermore, we can raise and lower spinor indices as follows

$$v^\alpha = \epsilon^{\alpha\beta} v_\beta, \qquad v_\alpha = v^\beta \epsilon_{\beta\alpha}, \tag{94}$$

where

$$\epsilon^{\alpha\beta} = \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{95}$$

The case for dotted indices is similar. Using the ϵ tensor, we can also define the following matrix

$$(\bar{\sigma}^a)^{\dot{\alpha}\alpha} = \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} (\sigma_a)_{\beta\dot{\beta}}, \tag{96}$$

where we may sometimes drop the 'bar' notation since it is understood how the matrices above are related. See [24] for useful relations that follow from this. Moreover, we can also write the antisymmetric tensor F_{ab} in the $[1, 1]$ representation of $SO(4)$ in the corresponding representation $(1, 0) \oplus (0, 1)$ of $SU(2) \times SU(2)$ as

$$F_{ab} \to F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}. \tag{97}$$

Observe that $F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}$ are related to F_{ab} through its selfdual and anti-selfdual parts. Thus, as an example of the $s = 2$ case, the vielbein and spin connection in spinoral notation is

$$e^a, \omega^{ab} \to e^{\alpha\dot{\alpha}}, \omega^{\alpha\beta}, \bar{\omega}^{\dot{\alpha}\dot{\beta}}. \tag{98}$$

From here, we can write the higher spin equations (82) in terms of spinorial language.

Consider a representation $[k,l]$ of $SO(4)$; in other words, a representation corresponding to a Young diagram with k boxes in the first row and l boxes in the second row. Let J_{ij} be a generator of $SO(4)$. Then, the expressions

$$\begin{aligned} A_i &= \frac{1}{2}\big(\epsilon_{ijk}J_{jk} + J_{i4}\big), \\ B_i &= \frac{1}{2}\big(\epsilon_{ijk}J_{jk} - J_{i4}\big), \end{aligned} \tag{99}$$

relates $SO(4)$ to $SU(2)\times SU(2)$; indeed, this representation is precisely

$$\left(\frac{k+l}{2}, \frac{k-l}{2}\right). \tag{100}$$

However, observe that under the parity transformation $x^4 \to -x^4$, we have $A_i \leftrightarrow B_j$. Thus, to ensure invariance under this transformation, the $SU(2)\times SU(2)$ reps are related to the $[k,l]$ reps of $SO(4)$ under

$$\left(\frac{k+l}{2}, \frac{k-l}{2}\right) \oplus \left(\frac{k-l}{2}, \frac{k+l}{2}\right). \tag{101}$$

Therefore, under the general map between $SO(4)$ and $SU(2)\times SU(2)$ representations

$$[k,l] \to \left(\frac{k+l}{2}, \frac{k-l}{2}\right) \oplus \left(\frac{k-l}{2}, \frac{k+l}{2}\right), \tag{102}$$

tensors become

$$v^{a_1\ldots a_k, b_1\ldots b_l} \to v^{\alpha_1\ldots\alpha_{k+l},\dot\alpha_1\ldots\dot\alpha_{k-l}}, \bar v^{\alpha_1\ldots\alpha_{k-l},\dot\alpha_1\ldots\dot\alpha_{k+l}}. \tag{103}$$

Thus, the tower of fields in equation (82) in the frame-like description corresponds to the collection of multispinors

$$\omega^{\alpha_1\ldots\alpha_{s-1+n},\dot\alpha_1\ldots\dot\alpha_{s-1-n}}, \qquad n = -(s-1), ..., (s-1), \tag{104}$$

where the spin s vielbein $e^{a_1\ldots a_{s-1}}$ corresponds to the $n=0$ case above; in other words, $e^{\alpha_1\ldots\alpha_{s-1},\dot\alpha_1\ldots\dot\alpha_{s-1}}$. All other fields are spin connection-like.

Next, we introduce the twistor-like variables $y_\alpha, \bar y_{\dot\alpha}$ which are in the spinor representations $(1/2,0)$ and $(0,1/2)$. Since $y_\alpha y^\alpha = \epsilon^{\alpha\beta}y_\alpha y_\beta = 0$, these variables are bosonic. We then define the $\star$-product as follows

$$f(y,\bar y)\star g(y,\bar y) = f(y,\bar y)e^{\epsilon^{\alpha\beta}\frac{\overleftarrow{\partial}}{\partial y^\alpha}\frac{\overrightarrow{\partial}}{\partial y^\beta} + \epsilon^{\dot\alpha\dot\beta}\frac{\overleftarrow{\partial}}{\partial y^{\dot\alpha}}\frac{\overrightarrow{\partial}}{\partial y^{\dot\beta}}} g(y,\bar y), \tag{105}$$

where the infix notation means $f\overleftarrow{\partial}_\alpha g = (\partial_\alpha f)g$ and $f\overrightarrow{\partial}_\alpha g = f\partial_\alpha g$ (in other words, the arrows indicate on which object the derivative acts). Then for the monomials $y, \bar y$ we have

$$\begin{aligned} y_\alpha \star y_\beta &= y_\alpha y_\beta + \epsilon_{\alpha\beta} \\ \bar y_{\dot\alpha} \star \bar y_{\dot\beta} &= \bar y_{\dot\alpha}\bar y_{\dot\beta} + \epsilon_{\dot\alpha\dot\beta} \\ y_\alpha \star \bar y_{\dot\alpha} &= y_\alpha \bar y_{\dot\alpha}. \end{aligned} \tag{106}$$

We can also represent the $\star$-product as an integral

$$f \star g = \int d^2u d^2v d^2\bar{u} d^2\bar{v} e^{uv+\bar{u}\bar{v}} f(y+u, \bar{y}+\bar{u}) g(y+v, \bar{y}+\bar{v}), \tag{107}$$

where $uv \equiv u^\alpha v_\alpha$, $\bar{u}\bar{v} \equiv \bar{u}^{\dot{\alpha}} \bar{v}_{\dot{\alpha}}$ and a choice of integration contour and measure normalization is assumed so that $1 \star f = f \star 1 = f$. For example, we assume that

$$\int d^2u e^{uv} = \delta^2(v). \tag{108}$$

Now, consider the following bilinears in $y, \bar{y}$

$$P_{\alpha\dot{\alpha}} = y_\alpha \bar{y}_{\dot{\alpha}}, \qquad M_{\alpha\beta} = y_\alpha y_\beta, \qquad \bar{M}_{\dot{\alpha}\dot{\beta}} = \bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}}. \tag{109}$$

These bilinears satisfy an algebra under $\star$-commutators (where $[A, B]_\star = A \star B - B \star A$, see [24] for useful identities) that, up to normalization, is precisely the $SO(3,2)$ algebra; in other words, $[M, M]_\star \sim M, [M, P]_\star \sim P, [P, P]_\star \sim M$. Here, $P_{\alpha\dot{\alpha}}$ corresponds to the local translations P_a; and $M_{\alpha\beta}, \bar{M}_{\dot{\alpha}\dot{\beta}}$ the selfdual and anti-selfdual parts of the Lorentz generators M_{ab}. Note that we have set the AdS scale $R^2_{AdS} = 1$ in our definition of the $\star$-product. This is the utility in using the $\star$-product; our commutation relations reduces to the familiar forms of the commutators given in equations (55)-(57). Later, when we apply the $\star$-product commutator to the generators of the higher spin algebra, we will find that their commutators will also have a surprisingly simple form, despite the fact that the higher spin generators have complicated structures (recall, that the generators live in the universal enveloping algebra $\mathfrak{u}(d, 2)$).

From here, we can realize the $s = 2$ gauge field in equation (60) as a function of homogeneous degree 2 in $y, \bar{y}$

$$W_{s=2} = e_{\alpha\dot{\alpha}}(x) y^\alpha \bar{y}^{\dot{\alpha}} + \omega_{\alpha\beta}(x) y^\alpha y^\beta + \bar{\omega}_{\dot{\alpha}\dot{\beta}}(x) \bar{y}^{\dot{\alpha}} \bar{y}^{\dot{\beta}}. \tag{110}$$

To generalize to higher spins, we take a one form $W(x|y, \bar{y}) = dx^\mu W_\mu(x|y, \bar{y})$ which is a general function of the twistor variables

$$W(x|y, \bar{y}) = \sum_{n,m=0}^{\infty} \frac{1}{n!m!} W^{(n,m)}_{\alpha_1 \ldots \alpha_n, \dot{\alpha}_1 \ldots \dot{\alpha}_m} y^{\alpha_1} \cdots y^{\alpha_n} \bar{y}^{\dot{\alpha}_1} \cdots \bar{y}^{\dot{\alpha}_m}. \tag{111}$$

We refer to this one-form as a *master field*. The Taylor expansion of the master field encodes the physical higher spin gauge fields of all spins. Indeed, the spin s gauge fields described by equation (104) can now be written as

$$W^{(s-1+n, s-1-n)}_{\alpha_1 \ldots \alpha_{s-1+n}, \dot{\alpha}_1 \ldots \dot{\alpha}_{s-1-n}}, \qquad -(s-1) \leq n \leq s-1, \tag{112}$$

which are the terms of total degree $2(s-1)$ in $y, \bar{y}$ in the master field. These monomials are interpreted as the generators of the higher spin algebra

$$T_s(y, \bar{y}) = \{y^{\alpha_1} \cdots y^{\alpha_n} \bar{y}^{\dot{\alpha}_1} \ldots \bar{y}^{\dot{\alpha}_m}\}|_{n+m=2(s-1)}. \tag{113}$$

These generators can be rewritten in the $SO(3,2)$ representation $T_{A_1\ldots A_{s-1},B_1\ldots B_{s-1}}$, which was previously mentioned. The higher spin algebra is therefore the algebra of the generators under $\star$-commutators, with the $\star$-product defined as above. Schematically, the commutator has the following form

$$[T_{s_1}, T_{s_2}]_\star = \sum_{s=|s_1-s_2|+2}^{s_1+s_2-2} T_s, \tag{114}$$

which is very simple and follows from the fact that the higher spin generators consists of polynomials. As some examples, the commutator above yields $[T_2, T_2]_\star \sim T_2$, $[T_3, T_3]_\star \sim T_4 + T_2$, and $[T_4, T_4] \sim T_6 + T_4 + T_2$. Observe that for each commutator for $s > 2$, we must introduce another generator; in other words, we will have an infinite tower of generators.

Finally, we can the relate the Fronsdal field to the higher spin vielbein through the component $W^{(s-1,s-1)}$; that is,

$$W_\mu^{\alpha_1\ldots\alpha_{s-1},\dot{\alpha}_1\ldots\dot{\alpha}_{s-1}} \to e_\mu^{a_1\ldots a_{s-1}} \to \varphi_{\mu_1\mu_2\ldots\mu_s} = e_{(\mu_1,\mu_2\ldots\mu_s)}. \tag{115}$$

Now, $W(x|y,\bar{y})$ encodes all gauge fields with $s \geq 1$. From the holographic duality (see Section (3)), we should include a scalar field (in other words, an $s = 1$ field). The correct master zero-form is

$$B(x|y,\bar{y}) = \sum_{n,m=0}^{\infty} \frac{1}{n!m!} B^{(n,m)}_{\alpha_1\ldots\alpha_n,\dot{\alpha}_1\ldots\dot{\alpha}_m} y^{\alpha_1}\cdots y^{\alpha_n}\bar{y}^{\dot{\alpha}_1}\cdots\bar{y}^{\dot{\alpha}_m}. \tag{116}$$

We emphasize here for clarity that the main difference between equation (111) and equation (116) is the fact that $W^{(n,m)}_{\alpha_1\ldots\alpha_n,\dot{\alpha}_1\ldots\dot{\alpha}_m}$ is a one-form and $B^{(n,m)}_{\alpha_1\ldots\alpha_n,\dot{\alpha}_1\ldots\dot{\alpha}_m}$ is a zero-form; more precisely, there is a 'hidden index' which can be seen by seen by writing the one-form explicitly $W^{(n,m)}_{\alpha_1\ldots\alpha_n,\dot{\alpha}_1\ldots\dot{\alpha}_m} = dx^\mu W^{(n,m)}_{\mu\alpha_1\ldots\alpha_n,\dot{\alpha}_1\ldots\dot{\alpha}_m}$ (there is no such index for $B^{(n,m)}_{\alpha_1\ldots\alpha_n,\dot{\alpha}_1\ldots\dot{\alpha}_m}$ since it is a is a zero-form).

Thus, we have found two master fields $W(x|y,\bar{y}), B(x|,y,\bar{y})$ that encode all physical fields and are sufficient to write down the equations of motion for the free theory.

Thus, in conclusion, we have achieved tremendous simplification of the higher spin algebra by using spinor notation and the $\star$-product. In addition, we have been able to construct two master fields, namely, the one-form W and the zero-form B. We can now proceed to formulate Vasiliev theory.

2.4 AdS_4 Bosonic Vasiliev Theory

In this section, we develop AdS_4 bosonic Vasiliev theory. The work in this section was originally pioneered by Vasiliev in [26][27] [28][29].

Note that we study AdS_4 as it provides the most simplistic bosonic version of the model. In addition, [34] conjectured that Vasiliev's minimal bosonic theory in AdS_4 is dual to either the $d = 3$-dimensional field theory of N massless scalar fields or the critical $O(N)$ vector model (depending on the choice of the boundary for the bulk scalar field)[3]. Note also that if we chose to rather study AdS_3, then the dual theory is given by CFT_2. This is a special type of CFT with an infinite dimensional algebra. CFT_d's with $d \geq 3$ are considered to be more generic CFT's and thus $d = 3$ is the simplest CFT (dual theory) we can study here.

First, we introduce a technical trick that will be useful to us. To construct the fully non-linear theory, we will double the auxiliary twistor space by introducing additional variables $z, \bar{z}$

$$(y_\alpha, \bar{y}_{\dot{\alpha}}) \to (y_\alpha, \bar{y}_{\dot{\alpha}}, z_\alpha, \bar{z}_{\dot{\alpha}}). \tag{117}$$

The benefit of this trick, as will be seen later, is that it allows us to write the equations of motion of Vasiliev theory in an extremely simple form. Note, that the new variables are purely auxiliary and the physical fields will be encoded in the $(z, \bar{z})$-independent part of the master fields we derive below. Consequently, we must extend the definition of our $\star$-product to

$$\begin{aligned} f(Y,Z) \star g(Y,Z) = f(Y,Z) exp\Big[&\epsilon^{\alpha\beta}\Big(\overleftarrow{\partial}_{y^\alpha} + \overleftarrow{\partial}_{z^\alpha}\Big)\Big(\overrightarrow{\partial}_{y^\beta} - \overrightarrow{\partial}_{z^\beta}\Big) \\ &+ \epsilon^{\dot{\alpha}\dot{\beta}}\Big(\overleftarrow{\partial}_{\bar{y}^{\dot{\alpha}}} + \overleftarrow{\partial}_{\bar{z}^{\dot{\alpha}}}\Big)\Big(\overrightarrow{\partial}_{\bar{y}^{\dot{\beta}}} - \overrightarrow{\partial}_{\bar{z}^{\dot{\beta}}}\Big)\Big] g(Y,Z), \end{aligned} \tag{118}$$

where $(Y, Z) \equiv (y_\alpha, \bar{y}_{\dot{\alpha}}, z_\alpha, \bar{z}_{\dot{\alpha}})$. As before, we can represent the extended $\star$-product as an integral

$$\begin{aligned} &f(Y,Z) \star g(Y,Z) \\ &= \int d^4U d^4V e^{uv+\bar{u}\bar{v}} f(y+u, \bar{y}+\bar{u}, z+u, \bar{z}+\bar{u}) g(y+v, \bar{y}+\bar{v}, z-v, \bar{z}-\bar{v}), \end{aligned} \tag{119}$$

where $d^4U = d^2u d^2\bar{u}$ (the other cases are similar). Just as in the previous case, we assume an integration contour and measure normalization such that $f \star 1 = 1 \star f = 1$. This star product admits the operators $\kappa = e^{z^\alpha y_\alpha}, \bar{\kappa} = e^{\bar{z}^{\dot{\alpha}} \bar{y}_{\dot{\alpha}}}$, known as the *Kleinian operators*, which satisfies $\kappa \star \kappa = 1$ and $\bar{\kappa} \star \bar{\kappa} = 1$. As before, more properties are shown in [24].

From here, we can introduce the full set of master fields necessary to formulate the Vasiliev equations. As in [24], we list them as follows

- **A Master one-form** The master one-form is $W(x|y, \bar{y}, z, \bar{z}) = dx^\mu W_\mu(x|y, \bar{y}, z, \bar{z})$, which is to be viewed as a gauge field of the higher spin algebra (observe that this is the same one-form we stated earlier except is is now dependent on the enlarged twistor space variables). The gauge transformation is

$$\delta W = d_x \epsilon + [W, \epsilon]_\star. \tag{120}$$

Here, the gauge parameter $\epsilon = \epsilon(y, \bar{y}, z, \bar{z})$ is a function of all twistor space variables.

• **A Master zero-form** The zero-form $B(x|y, \bar{y}, z, \bar{z})$ is postulated to transform in the twisted-adjoint representation. It transforms under gauge symmetry as

$$\begin{aligned} \delta B &= -\epsilon \star B + B \star \pi(\epsilon) \\ \pi(\epsilon) &= \epsilon(x| - y, \bar{y}, -z, \bar{z}) = \kappa \star \epsilon \star \kappa. \end{aligned} \tag{121}$$

Using the Kleinian operators, one can map the twisted adjoint to the ordinary adjoint by the $\star$-product. It follows from the above equations then, that $\Phi = B \star \kappa$ transforms in the adjoint as follows

$$\delta \Phi = -\epsilon \star \Phi + \Phi \star \epsilon. \tag{122}$$

• **An additional gauge field** This additional gauge field is a one-form in the $(z, \bar{z})$-direction and is given by

$$S(x|y, \bar{y}, z, \bar{z}) = dz^\alpha S_\alpha(x|y, \bar{y}, z, \bar{z}) + d\bar{z}^{\dot{\alpha}} \bar{S}_{\dot{\alpha}}(x|y, \bar{y}, z, \bar{z}), \tag{123}$$

which transforms under gauge symmetry as

$$\delta S = d_Z \epsilon + [S, \epsilon]_\star, \tag{124}$$

where $d_Z = dz^\alpha \frac{\partial}{\partial z^\alpha} + d\bar{z}^{\dot{\alpha}} \frac{\partial}{\partial \bar{z}^{\dot{\alpha}}}$.

As some remarks, note that the physical degrees of freedom are contained in W and B projected to the $z = \bar{z} = 0$ subspace. The additional gauge field S is purely auxiliary.

We will impose necessary constraints on the master fields which will be important for the consistency of the non-linear equations. These constraints are

$$\begin{aligned} W(Y, Z) &= W(-Y, -Z), \qquad & B(Y, Z) &= B(-Y, -Z), \\ S_\alpha(Y, Z) &= -S_\alpha(-Y, -Z), \qquad & \bar{S}_{\dot{\alpha}} &= -\bar{S}_{\dot{\alpha}}(-Y, -Z). \end{aligned} \tag{125}$$

The gauge parameter has a similar constraint $\epsilon(Y, Z) = \epsilon(-Y, -Z)$. These constraints ensure that the physical spectrum includes only integer spins. To see why, we project to the $z = \bar{z} = 0$ subspace and consider the one-form $W(Y, Z)$ as a simple example. The projected one-form is then precisely equation (111); in other words,

$$W(Y, Z)\big|_{z=\bar{z}=0} = W(x|y, \bar{y}). \tag{126}$$

Now, the statement $W(Y,Z) = W(-Y,-Z)$ is translated to the statement $W(x|y,\bar{y}) = W(x|-y,-\bar{y})$. From the form of equation (111), this implies that $(-1)^{n+m} = 1 \Rightarrow n+m = 2k, k \in \{0,1,2,...\}$. The labels n,m are related to the spin s through $n+m = 2(s-1)$ (see comments after equation (111)). Thus, $2(s-1) = 2k \Rightarrow s = k+1 \Rightarrow s = \{1,2,3,...\}$. Thus, the constraint ensures that the physical spectrum only includes integer spins.

Before we introduce the equations of motion, we will define a one-form that will allow us to write the equations in a compact form. The one-form in (x,Z) space is given by

$$\mathcal{A}(x|Y,Z) = dx^\mu W_\mu(x|Y,Z) + dz^\alpha S_\alpha(x|Y,Z) + d\bar{z}^{\dot{\alpha}} \bar{S}_{\dot{\alpha}}(x|Y,Z), \tag{127}$$

which transforms under the gauge symmetry as

$$\delta\mathcal{A} = d\epsilon + [\mathcal{A},\epsilon]_\star, \qquad d \equiv d_x + d_Z. \tag{128}$$

Finally, the Vasiliev non-linear equations of motion are given by

$$\begin{aligned} &d\mathcal{A} + \mathcal{A}\star\mathcal{A} = f_\star(B\star\kappa)dz^2 + \bar{f}_\star(B\star\bar{\kappa})d\bar{z}^2, \\ &dB + \mathcal{A}\star B - B\star\pi(\mathcal{A}) = 0. \end{aligned} \tag{129}$$

More explicitly in terms of the master fields, the equations are given by

$$\begin{aligned} &d_x W + W\star W = 0, \\ &d_Z W + d_x S + W\star S + S\star W = 0, \\ &d_Z S + S\star S = f_\star(B\star\kappa)dz^2 + \bar{f}_\star(B\star\bar{\kappa})d\bar{z}^2, \\ &d_x B + W\star B - B\star\pi(W) = 0, \\ &d_Z B + S\star B - B\star\pi(S) = 0. \end{aligned} \tag{130}$$

Here, $f(X)$ is an analytic function of X, and $\bar{f}$ is its complex conjugate. Moreover, $f_\star(X)$ is the corresponding $\star$-function obtained by replacing all products of X in the Taylor series of $f(X)$ by $\star$-products. The symbol π is defined in general as the operation that switches the signs of (y,z,dz) while preserving the signs of $(\bar{y},\bar{z},d\bar{z})$.

Note, that we can recover Fronsdal's equation from the master fields by expanding the fields perturbatively around the AdS_4 vacuum. This allows us to solve Vasiliev's equations order by order. The precise argument can be found in [26]. Also, observe that equation (129) is gauge manifestly invariant under the transformations in equation (121) and equation (128).

In conclusion, we have constructed the exact non-linear equations of motion of Vasiliev theory. These equations admit a vacuum solution which corresponds to AdS. As a consequence of the construction, the theory necessarily includes an infinite tower of higher spin massless fields with integer-valued spins (note, that one can add further constraints to the theory which results in a spectrum of even spins, thus simplifying the theory further- this is known as minimal bosonic higher spin theory).

3 Holography

In this section, we discuss holography which will relate the free conformal field theory to the higher spin theory. More precisely, we will use important results from holography which are important for the rest of the dissertation. We will first state the correspondence between the CFT living on the boundary of AdS and the higher spin gauge fields in AdS. From here, we use bi-local fields to state how the product of scalar fields in $2+1$-dimensions decomposes into an infinite sum of gravity fields in $3+1$-dimensions. Finally, we will prove an important result known as factorization, which is shown to exist in the large-N limit. For more information about holography itself, we refer the reader to [1][35][36][37].

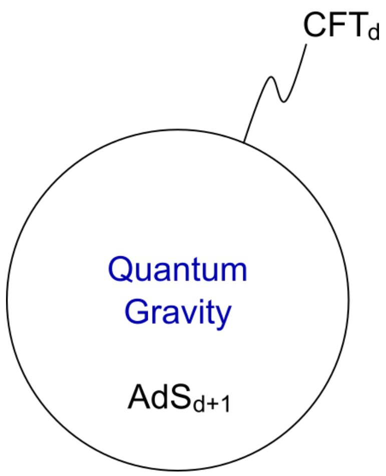


Figure 3: Holography

According to the duality, a singlet sector CFT should be dual to some kind of gravitational theory in AdS which becomes weakly coupled in the large N limit. More precisely, we expect that single trace operators in the CFT should be dual to single particle states in AdS. For each conserved spin s current $J_{(s)}$ there is a dual massless higher spin gauge field $\varphi_{(s)}$ in AdS (the massless property of the gauge theory corresponds to the conserved property of the CFT operator) and the scalar operator $J_{(0)}$ is dual to a bulk scalar field. Schematically,

$$J_{(s)}, \qquad \Delta = d-2+s \qquad \Leftrightarrow \qquad \text{massless gauge field } \varphi_{(s)}, \qquad \delta\varphi_{\mu_1\ldots\mu_s} = \nabla_{(\mu_1}\epsilon_{\mu_2\ldots\mu_s)}, \tag{131}$$

and

$$J_{(0)}, \qquad \Delta = d-2 \qquad \Leftrightarrow \qquad \varphi, \qquad m^2 = \frac{\Delta(\Delta-d)}{R^2_{AdS}}. \tag{132}$$

Here, m^2 is the mass of the bulk field and the above expression relates the dimension of the scalar operator to the field mass. In our case of a free CFT_d, the dual scalar operator has

$\Delta = d-2$ so that $m^2 = -2(d-2)/R_{AdS}^2$; consequently, for CFT_3, $m^2 = -2/R_{AdS}^2$. In flat spacetimes, negative masses signal instabilities (tachyon condensation). However, in curved space this is not a problem since we are above the so-called *Breitenlohner-Freedman bound* which is given by $m^2 > -9/(4R_{AdS}^2)$ [38]. Futhermore, from the above duality, we note that the stress-energy tensor $T_{\mu\nu}$ is dual to a graviton in the bulk theory (in other words, $T_{\mu\nu} \Leftrightarrow g_{\mu\nu}$).

Remark 2. *The first example of the correspondence was given by [1] for AdS_5/CFT_4. In its strongest form, the correspondence can be stated as follows: $\mathcal{N}=4$ Super Yang-Mills (SYM) theory with gauge group $SU(N)$ and dimensionless Yang-Mills coupling constant g_{YM} is dynamically equivalent to type IIB superstring theory with string length l_s (a dimensionful constant which which sets the size of fluctuations of the string worldsheet) and coupling constant g_s (a dimensionless constant which measures the string interaction strength relevant for string splitting and joining) on $AdS_5 \times S^5$ with radius of curvature R_{AdS} and N units of $F_{(5)}$ flux on S^5 (by dynamically equivalent, we mean that the physics in one description is mapped precisely to the physics in the dual description). The parameters on the gauge field theory side are g_{YM} and N; the parameters on the string theory side are g_s and R_{AdS}^2/l_s^2. The parameters are mapped to each other as follows*

$$g_{YM}^2 = 2\pi g_s, \qquad 2g_{YM}^2 N = R_{AdS}^4/l_s^4. \tag{133}$$

Notice that the second term contains precisely the t' Hooft coupling constant $\lambda = g_{YM}^2 N$. Observe that λ is a ratio between the string length and the radius of curvature; in other words, λ compares the size of a string to the size of the space the string lives in. A large space and small string will result in a large value for λ. [37][35].

Using these facts, we can compare correlation functions. To do this, we can make use of the so called GKPW (Gubser, Klebanov, Polyakov, Witten) dictionary (see [39] and [36]). In the CFT theory, the path integral can be written

$$Z_{CFT}[J] = \int [D\phi] e^{-S_{CFT} + \int J\mathcal{O}}, \tag{134}$$

where J denotes source terms and the measure, for the $d=3$ vector model, is given by $[D\phi] = D\phi^i D\phi_i^*$. On the AdS side, the path integral can be written

$$Z_{gravity}[\Phi_0] = \int_{AdS_4} [d\Phi dg_{\mu\nu} ...] e^{-S_{gravity}}. \tag{135}$$

In order to define the gravity path integral, we need to give the boundary condition for Φ on the boundary of AdS_4. This condition is given by

$$\Phi|_{\partial AdS_4} = \Phi_0. \tag{136}$$

Now, the $\hbar$ analogy to quantum gravity is the string coupling given by $g_s \sim 1/N$. Moving to the large N limit causes this coupling to vanish and thus we reduce to classical gravity. In this case, the gravity path integral reduces to a simpler form

$$Z_{gravity}[\Phi_0] = e^{-S_{gravity}}|_{\Phi|_{\partial AdS_4}=\Phi_0}. \tag{137}$$

One can thus calculate correlators in both dual theories. Indeed, under the duality, we should expect the three-point functions

$$\langle J_{(s_1)} J_{(s_2)} J_{(s_3)} \rangle \qquad \Leftrightarrow \qquad \frac{\delta}{\delta \Phi_{J_{(s_1)}}} \frac{\delta}{\delta \Phi_{J_{(s_2)}}} \frac{\delta}{\delta \Phi_{J_{(s_3)}}} e^{-S_{gravity}}, \tag{138}$$

to be in perfect agreement and [3] finds exactly this.

Now, we can rewrite the vector model in terms of bi-local fields. We can perform a loop expansion in powers of $\hbar$ for the path integral in equation (134) (which can be seen by restoring $\hbar$ in the exponent to give $\frac{i}{\hbar}S_{CFT}$); alternatively, by making use of bi-local fields, we can perform a loop expansion in $1/N$. In other words, we have a loop expansion in the $\hbar$ of the quantum gravity. Thus, this change of variables seems to be performing the reorganization from CFT degrees of freedom into gravity degrees of freedom.

In vector model Lagrangians (such as in [40]) there is an implicit dependence in N; more precisely, there is a factor of N in the summation over repeated indices

$$\phi_i^* \phi_i \equiv \sum_{i=1}^{N} \phi_i^* \phi_i. \tag{139}$$

We would like to write this dependence on N in an *explicit form*.

One method to achieve this is to introduce an *auxiliary field* σ as is demonstrated in [41] (see Remark (3)). The use of an auxiliary field is well demonstrated in the *Gross-Neveu model* as well as the $\mathbb{C}P^{N-1}$ *model*. The use of an auxiliary field does, however, have limitations. As stated in [40], this method is only viable for Lagrangians consisting of quadratic plus quartic terms and the local auxiliary field is a poor substitute for the composite (bi-local) meson field.

Remark 3. *Using an auxiliary field, we can transform the Lagrangian density as*

$$\mathscr{L} \longrightarrow \mathscr{L} + \frac{1}{2}\frac{N}{g_0}(\sigma - \frac{1}{2}\frac{g_0}{N}\phi^a\phi^a)^2. \tag{140}$$

This transformation does not affect the physics being studied (as it should be) since, by the Euler-Lagrange equations, we find that σ is not determined by a true equation of motion; rather, it is given by an equation of constraint

$$\sigma = \frac{1}{2}\frac{g_0}{N}\phi^a\phi^a. \tag{141}$$

This transformation leads to the new Lagrangian density

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{1}{2}\mu_0^2\phi^a\phi^a + \frac{1}{2}\frac{N}{g_0}\sigma^2 - \frac{1}{2}\sigma\phi^a\phi^a, \tag{142}$$

where μ_0^2 is a mass parameter. From this, [41] derives an effective action S_{eff} defined through the equation

$$\mathrm{e}^{iS_{eff}(\sigma)} = \int\prod_a[d\phi^a]\mathrm{e}^{iS(\phi^a,\sigma)}, \tag{143}$$

where they have integrated over all configurations of the ϕ's only. This effective action allows one to write the N dependence explicitly

$$S_{eff}(\sigma, N) = NS_{eff}(\sigma, 1). \tag{144}$$

The use of bi-local fields solve the shortcomings of the auxiliary field method, although here, we will only show that we can achieve an effective action with explicit N dependence. This explanation follows directly from [40].

The time ordered expectation value of invariant operators is given by

$$\langle F[\sigma]\rangle \equiv \langle\prod_{i=1}^m\sigma(x_i, y_i)\rangle = \frac{\int D\phi^* D\phi\mathrm{e}^{iS}\prod_{i=1}^m\sigma(x_i, y_i)}{\int D\phi^* D\phi\mathrm{e}^{iS}}, \tag{145}$$

where $\sigma(x, y) = \phi^{*a}(x)\phi^a(y)$ is the bi-local field. Starting from the well-known identity

$$\int Dx\frac{\delta}{\delta x}f[x] = 0, \tag{146}$$

(where $f[x]$ is some functional that vanishes at the boundary and for which the functional derivative exists) we obtain

$$\begin{aligned}
0 =& \int D\phi^* D\phi\frac{\delta}{\delta\phi^a(x)}[\phi^a(y)F[\sigma]\mathrm{e}^{iS}]\\
=& \int D\phi^* D\phi[F[\sigma]\frac{\delta\phi^a(y)}{\delta\phi^a(x)} + \phi^a(y)\frac{\delta F[\sigma]}{\delta\phi^a(x)} + i\phi^a(y)F[\sigma]\frac{\delta S}{\delta\phi^a(x)}]\mathrm{e}^{iS}\\
=& \int D\phi^* D\phi N\delta^d(x-y)F[\sigma]\mathrm{e}^{iS} + \int D\phi^* D\phi\phi^a(y)\frac{\delta F[\sigma]}{\delta\phi^a(x)}\mathrm{e}^{iS}\\
&+ \int D\phi^* D\phi i\phi^a(y)F[\sigma]\frac{\delta S}{\delta\phi^a(x)}\mathrm{e}^{iS}\\
=& \langle N\delta^d(x-y)F[\sigma]\rangle + \langle\phi^a(y)\frac{\delta F[\sigma]}{\delta\phi^a(x)}\rangle + i\langle\phi^a(y)F[\sigma]\frac{\delta S}{\delta\phi^a(x)}\rangle.
\end{aligned} \tag{147}$$

We now perform a change of variables from the original fields ϕ^a to the bi-local fields $\sigma(x, y)$ and add a Jacobian J associated with this change of variables; more precisely,

$$\int D\phi^* D\phi F[\sigma]\mathrm{e}^{iS} = \int D\sigma JF[\sigma]e^{iS}. \tag{148}$$

We can now write an effective action S_{eff} as

$$\mathrm{e}^{iS_{eff}} = \mathrm{e}^{i(S - i\ln J)}. \tag{149}$$

In order to write S_{eff} in an explicit form of the parameter N, we need to determine an expression for J. Using the identity above again, we have

$$0 = \int D\sigma \int d^d z \frac{\delta}{\delta\sigma(z,x)}[\sigma(z,y)F[\sigma]e^{iS_{eff}}]. \tag{150}$$

We again apply the product rule as in the previous application of the identity and follow the same procedure. The first term makes use of

$$\begin{aligned}\frac{\delta\sigma(z,y)}{\delta\sigma(z,x)} &= \frac{\delta(\phi^{a*}(z)\phi^a(y))}{\delta(\phi^{b*}(z)\phi^b(x))}\\ &= \frac{\delta(\phi^{1*}(z)\phi^1(y) + ... + \phi^{N*}(z)\phi^N(y))}{\delta(\phi^{1*}(z)\phi^1(x) + ... + \phi^{N*}(z)\phi^N(x))}\\ &= \delta^d(z-z)\delta^d(x-y)\\ &= \delta^d(x-y)\delta^d(0).\end{aligned} \tag{151}$$

Thus the first term will be given by

$$\begin{aligned}\int D\sigma \int d^d z F[\sigma]\frac{\delta\sigma(z,y)}{\delta\sigma(z,x)}\mathrm{e}^{iS_{eff}} &= \int D\sigma F[\sigma]\delta^d(x-y)\delta^d(0)\mathrm{e}^{iS_{eff}} \int d^d z\\ &= \int D\sigma F[\sigma]\delta^d(x-y)\delta^d(0)L^d \mathrm{e}^{iS_{eff}}\\ &= \langle \delta^d(x-y)\delta^d(0)L^d F[\sigma]\rangle,\end{aligned} \tag{152}$$

where L^d (note d is not a power here) is defined by the volume of the system we are studying

$$L^d = \int d^d z. \tag{153}$$

As stated, the rest of the terms are obtained exactly as before with the application of the identity. This yields

$$\begin{aligned}&\langle \delta^d(x-y)\delta^d(0)L^d F[\sigma]\rangle + \langle \int d^d z \sigma(z,y)\frac{\delta F[\sigma]}{\delta\sigma(z,x)}\rangle +\\ &\langle \int d^d z\sigma(z,y)F[\sigma]\frac{\delta lnJ}{\delta\sigma(z,x)}\rangle + i\langle \int d^d z\sigma(z,y)F[\sigma]\frac{\delta S}{\delta\sigma(z,x)}\rangle = 0.\end{aligned} \tag{154}$$

Now according to equation (148), equation (147) and equation (154) must be in agreement. To see this, we apply the chain rule

$$\begin{aligned}\frac{\delta}{\delta\phi^a(x)} &= \int d^d z \int d^d y \frac{\delta\sigma(z,y)}{\delta\phi^a(x)}\frac{\delta}{\delta\sigma(z,y)}\\ &= \int d^d z \int d^d y \phi^{b*}(z)\frac{\delta\phi^b(y)}{\delta\phi^a(x)}\frac{\delta}{\delta\sigma(z,y)}\\ &= \int d^d z \int d^d y \phi^{a*}(z)\delta^d(x-y)\frac{\delta}{\delta\sigma(z,y)}\\ &= \int d^d z \phi^{a*}(z)\frac{\delta}{\delta\sigma(z,x)},\end{aligned} \tag{155}$$

to equation (147) to yield the equivalent expression

$$\langle N\delta^d(x-y)F[\sigma]\rangle + \langle\int d^dz\sigma(z,y)\frac{\delta F[\sigma]}{\delta\sigma(z,x)}\rangle + i\langle\int d^dz\sigma(z,y)F[\sigma]\frac{\delta S}{\delta\sigma(z,x)}\rangle = 0. \quad (156)$$

Comparing equation (154) and equation (156), we obtain a differential equation for the Jacobian J

$$\int d^dz\sigma(z,y)\frac{\delta \ln J}{\delta\sigma(z,x)} = (N - L^d\delta^d(0))\delta^d(x-y), \quad (157)$$

with solution

$$\ln J = (N - L^d\delta^d(0))\mathrm{Tr}(\ln\sigma), \quad (158)$$

where the trace acts on functional space. Observe here that the Jacobian is independent of the action. Substituting this result into the expression for S_{eff} given by equation (149) yields

$$S_{eff} = S - i\ln J = -iN\mathrm{Tr}(\ln\sigma) + S + iL^d\delta^d(0)\mathrm{Tr}(\ln\sigma). \quad (159)$$

Finally, we rescale $\sigma \to N\sigma$ so that $S \to NS$, to obtain an explicit effective action

$$S_{eff} = -iN\mathrm{Tr}(\ln\sigma) + NS + iL^d\delta^d(0)\mathrm{Tr}(\ln\sigma) = NS_0 + S_1. \quad (160)$$

Note that this rescaling is possible since [40] assumes that coupling constants of the theory have been rescaled appropriately so as to yield a systematic $1/N$ expansion. An equivalent statement to this is that a rescaling of the fields exists under which $S \to NS$. Thus, equation (134) becomes

$$Z_{CFT}[J] = \int [D\sigma(x,y)]e^{-S_{\text{eff}} + \int J\mathcal{O}}. \quad (161)$$

Furthermore, it was recently concretely shown that this duality does indeed exist in this case. For scalar fields $\phi^a(t,\mathbf{x})$ on $2+1$-dimensional spacetime with $O(N)$ symmetry, where $\mathbf{x} = (x^1, x^2)$, we have the bi-local collective fields

$$\begin{aligned}\sigma(t,\mathbf{x},\mathbf{y}) &= \sum_{a=1}^{N}\phi^a(t,\mathbf{x})\phi^a(t,\mathbf{y}) \\ &= \sum_s \mathrm{e}^{is\theta}\sigma_s(t,\mathbf{z}),\end{aligned} \quad (162)$$

where $\mathbf{z} = (z^1, z^2, z^3)$ and s corresponds to spin number with even parity. Thus, the product of the scalar fields in $2+1$-dimensions decomposes into an infinite sum of fields σ_s in $3+1$-dimensions. Each field σ_s (corresponding to an even spin s) is a gravity field [42][43].

As a final topic for this section, we prove a result that has a non-trivial consequence. In particular, we will show that for *distinct* scalar fields $\phi^a \phi^b \dots \phi^z$ in the large-N limit,

$$\begin{aligned} &\langle \phi^a(x_1)\phi^a(x_2)\phi^b(x_3)\phi^b(x_4)\dots\phi^z(x_n)\phi^z(x_{n+1})\rangle = \\ &\langle \phi^a(x_1)\phi^a(x_2)\rangle\langle \phi^b(x_3)\phi^b(x_4)\rangle\dots\langle \phi^z(x_n)\phi^z(x_{n+1})\rangle. \end{aligned} \tag{163}$$

That is, the expectation value in equation (163) can be written as the product of expectation values for each pair of fields in the large-N limit. This important result is known as *factorization.*

Here, we will work in 0-dimensions and use the action $S = \alpha\phi^a\phi^a + g(\phi^a\phi^a)^2$. The result immediately extends to higher dimensional spacetimes. As a simpler case, we will show that

$$\langle \phi^a(x_1)\phi^a(x_2)\phi^b(x_3)\phi^b(x_4)\rangle = \langle \phi^a(x_1)\phi^a(x_2)\rangle\langle \phi^b(x_3)\phi^b(x_4)\rangle, \tag{164}$$

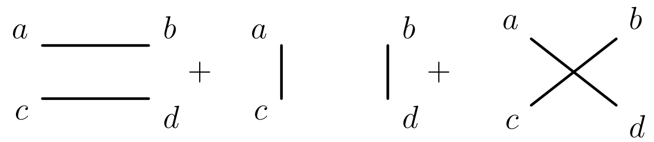
in the large-N limit. The two-point correlation function was calculated in the 0-dimensional model (see Appendix (A.1)). We therefore need to determine the four-point function in the interacting theory.

Similar to equation (513) of Appendix (A.1), we can write the four-point function as a perturbative expansion of correlation functions in the free theory

$$\begin{aligned} &\langle \phi^a(x_1)\phi^b(x_2)\phi^c(x_3)\phi^d(x_4)\rangle_{int} = \\ &\langle \phi^a\phi^b\phi^c\phi^d\rangle_{free} - g\langle \phi^a\phi^b\phi^c\phi^d\phi^e\phi^e\phi^f\phi^f\rangle_{free} + \dots. \end{aligned} \tag{165}$$

We calculated the four-point function in the free theory which is given by equation (518). In terms of Feynman diagrams, the four-point function in the free theory can be represented as

$$\langle \phi^a\phi^b\phi^c\phi^d\rangle_{free} =$$

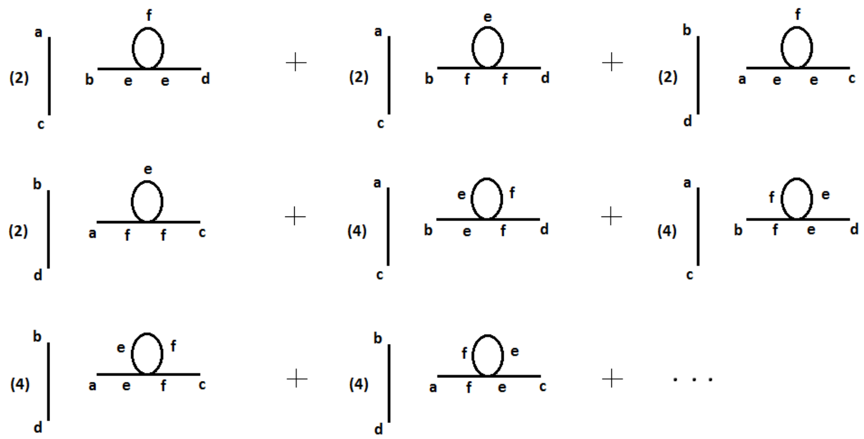


(166)

Applying the Feynman rules previously determined will give the correct expression given by equation (518). Note that the cross term above *does not intersect.*

The first order correction can be represented through Feynman diagrams as four external points (labeled a, b, c, and d) and a vertex (labeled d, d, e, and e). The number of Feynman diagrams for the first order correction is very large ($7!! = 105$ Feynman diagrams); therefore, we will only consider a few of the important diagrams. After renormalization (therefore, ignoring vacuum diagrams), we have diagrams of the form

$$\langle \phi^a\phi^b\phi^c\phi^d\phi^e\phi^e\phi^f\phi^f\rangle_{free} = \tag{167}$$



$$\begin{aligned}
&= \frac{2}{(2\alpha)^4}(\delta^{ac}\delta^{be}\delta^{ff}\delta^{ed}) + \frac{2}{(2\alpha)^4}(\delta^{ac}\delta^{bf}\delta^{ee}\delta^{fd}) \\
&+ \frac{2}{(2\alpha)^4}(\delta^{bd}\delta^{ae}\delta^{ff}\delta^{ec}) + \frac{2}{(2\alpha)^4}(\delta^{bd}\delta^{af}\delta^{ee}\delta^{fc}) \\
&+ \frac{4}{(2\alpha)^4}(\delta^{ac}\delta^{be}\delta^{ef}\delta^{fd}) + \frac{4}{(2\alpha)^4}(\delta^{ac}\delta^{bf}\delta^{fe}\delta^{ed}) \\
&+ \frac{4}{(2\alpha)^4}(\delta^{bd}\delta^{ae}\delta^{ef}\delta^{fc}) + \frac{4}{(2\alpha)^4}(\delta^{bd}\delta^{af}\delta^{fe}\delta^{ec}) + \ldots .
\end{aligned} \tag{168}$$

The other terms are similar. Thus, up to first order,

$$\begin{aligned}
\langle \phi^a\phi^b\phi^c\phi^d\rangle_{int} = & \frac{1}{(2\alpha)^2}(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}) \\
& - \frac{g}{(2\alpha)^4}(16N + 32)\delta^{ac}\delta^{bd} \\
& - \frac{g}{(2\alpha)^4}(16N + 32)\delta^{ab}\delta^{cd} \\
& - \frac{g}{(2\alpha)^4}(16N + 32)\delta^{ad}\delta^{bc} + \ldots \\
& = [\frac{1}{(2\alpha)^2} - \frac{g}{(2\alpha)^4}(16N + 32) + \ldots] \\
& \times(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}).
\end{aligned} \tag{169}$$

As before, we rescale the coupling constant to the t'Hooft coupling constant and add the term K_i to represent uncalculated factors in front of higher order terms. In the large-N limit, we therefore obtain in leading order of terms in $1/N$

$$\begin{aligned}\langle\phi^a\phi^b\phi^c\phi^d\rangle_{int} = [&\frac{1}{(2\alpha)^2} - \frac{16}{(2\alpha)^4}\lambda - \frac{32}{(2\alpha)^4}\frac{\lambda}{N} \\ &+ \frac{K_1}{(2\alpha)^6}\lambda^2 + \frac{K_2}{(2\alpha)^6}\frac{\lambda^2}{N} + \dots] \\ &\times (\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}).\end{aligned} \tag{170}$$

For meson type a and meson type b with $a \neq b$, we obtain

$$\begin{aligned}\langle\phi^a\phi^a\phi^b\phi^b\rangle_{int} = [&\frac{1}{(2\alpha)^2} - \frac{16}{(2\alpha)^4}\lambda - \frac{32}{(2\alpha)^4}\frac{\lambda}{N} \\ &+ \frac{K_1}{(2\alpha)^6}\lambda^2 + \frac{K_2}{(2\alpha)^6}\frac{\lambda^2}{N} + \dots]\delta^{aa}\delta^{bb},\end{aligned} \tag{171}$$

and

$$\begin{aligned}\langle\phi^a\phi^a\rangle_{int} = [&\frac{1}{2\alpha} - \frac{8}{(2\alpha)^3}\lambda - \frac{16}{(2\alpha)^3}\frac{\lambda}{N} + \frac{K_1}{(2\alpha)^5}\lambda^2 \\ &+ \frac{K_2}{(2\alpha)^5}\frac{\lambda^2}{N} + \dots]\delta^{aa},\end{aligned} \tag{172}$$

$$\begin{aligned}\langle\phi^b\phi^b\rangle_{int} = [&\frac{1}{2\alpha} - \frac{8}{(2\alpha)^3}\lambda - \frac{16}{(2\alpha)^3}\frac{\lambda}{N} + \frac{K_1}{(2\alpha)^5}\lambda^2 \\ &+ \frac{K_2}{(2\alpha)^5}\frac{\lambda^2}{N} + \dots]\delta^{bb}.\end{aligned} \tag{173}$$

Analysis of the above expression shows that we can recover the terms of the four-point function from the product of the two two-point functions. For example, the second term in the series for the four-point function is given by the two two-point functions as

$$-\frac{16}{(2\alpha)^4}\delta^{aa}\delta^{bb}\lambda = -\frac{1}{2\alpha}\frac{8}{(2\alpha)^3}\delta^{aa}\delta^{bb}\lambda - \frac{1}{2\alpha}\frac{8}{(2\alpha)^3}\delta^{bb}\delta^{aa}\lambda. \tag{174}$$

We therefore conclude that in the large-N limit, equation (164) holds. In addition, the proof demonstrated here for the simpler case extends directly to arbitrary distinct fields, as in equation (163); and, since only the terms change when considering the same graphs in higher dimensions, but the explicit contribution of N from Kronecker delta functions remain the same, we conclude that factorization also holds for $3+1$-dimensions.

The result of factorization leads to a very important and non-trivial statement about the large-N limit of a vector model. To understand this result, we draw an analogy with statistical mechanics. Consider the expectation value of some physical parameter O (such as energy E) in some macrostate with many different microstates

$$\langle O \rangle = \sum_i \mu_i O(i). \tag{175}$$

where i indexes all possible configurations (microstates) of the system, μ_i is the probability to be in state i, and $O(i)$ is the value of O in state i. Assuming we can write the expectation value for n parameters in terms of factorization yields

$$\begin{gathered} \langle O_1 O_2 \dots O_n \rangle = \langle O_1 \rangle \langle O_2 \rangle \dots \langle O_n \rangle \\ \Rightarrow \sum_i \mu_i O_1(i) O_2(i) \dots O_n(i) = \\ \sum_{i_1} \mu_{i_1} O_1(i_1) \sum_{i_2} \mu_{i_2} O_2(i_2) \dots \sum_{i_n} \mu_{i_n} O_n(i_n). \end{gathered} \tag{176}$$

The above statement holds if and only if

$$\mu_i = \left\{ \begin{array}{ll} 1 & : i = i* \\ 0 & : i \neq i* \end{array} \right. . \tag{177}$$

That is, factorization implies that *only one configuration contributes*. Returning to quantum field theory, equation (163) implies that only one configuration of the vector model contributes in the large-N limit. This single configuration must correspond to the classical limit of some quantum theory. In fact, the AdS/CFT correspondance states that this classical theory is precisely the dual gravitational theory.

4 Spectrum of Primaries

4.1 Objective

In this chapter, we will demonstrate how we can use group characters to determine the spectrum of primaries. More precisely, we will develop a character formula that will yield the number of primaries built using any number of scalar fields. Conformal field theories are invariant under $SO(d, 2)$. We will look specifically at $SO(4, 2)$.

The problem of determining the spectrum of primaries for a product of fields is effectively a group theory problem. Indeed, distinct primary states in conformal field theory belong to distinct irreducible representations of the conformal group $SO(4, 2)$; all other states in an irreducible representation are descendants of the primary states. These primaries are constructed out of a product of fields. We therefore want to understand how the product of representations decomposes into a sum of representations. If we can then determine the irreducible representations in this sum, we have determined the spectrum of primaries in the theory.

4.2 The Conformal Group $SO(4, 2)$

In this subsection, we will briefly discuss the conformal group generators and the conformal algebra that is closed under the commutators of these generators. These commutators will be used extensively for later results in the dissertation.

The full set of generators for the conformal group in $d-$dimensional Minkowski space is given by the following generators: $\{P_\mu, M_{\mu\nu}, D, K_\mu\}$. Here, P_μ (the momentum operator) generates translations, $M_{\mu\nu}$ (the Lorentz operator) generates rotations/boosts, D (the dilatation operator) generates scaling, and K_μ (the special conformal operator) generates special conformal transformations. Furthermore, these generators closes the Lie algebra through the following commutators

$$\begin{aligned}
&[P_\alpha, M_{\mu\nu}] = i\eta_{\alpha\mu}P_\nu - i\eta_{\alpha\nu}P_\mu \qquad [K_\alpha, M_{\mu\nu}] = i\eta_{\alpha\mu}K_\nu - i\eta_{\alpha\nu}K_\mu \\
&[D, P_\mu] = iP_\mu \qquad [D, K_\mu] = -iK_\mu \qquad [D, M_{\mu\nu}] = 0 \\
&[P_\mu, K_\nu] = 2i(\eta_{\mu\nu}D + M_{\mu\nu}) \\
&[M_{\alpha\beta}, M_{\mu\nu}] = i\eta_{\mu\beta}M_{\alpha\nu} - i\eta_{\nu\beta}M_{\alpha\mu} + i\eta_{\mu\alpha}M_{\nu\beta} - i\eta_{\nu\alpha}M_{\mu\beta}.
\end{aligned} \tag{178}$$

All other possible commutators are zero. Here, we work with a metric with mostly negative signature $(+, -, -, -)$.

Remark 4. *The algebra of the conformal group can be rewritten in terms of an antisymmetric operator T_{AB}, $A, B = -1, 0, 1, 2, ..., d$, which is defined through*

$$\begin{aligned} T_{\mu\nu} &= M_{\mu\nu} \qquad T_{\mu -1} = \frac{1}{2}(P_\mu + K_\mu) \\ T_{-1d} &= D \qquad L_{\mu d} = \frac{1}{2}(P_\mu - K_\mu). \end{aligned} \tag{179}$$

The resulting algebra is

$$[T_{AB}, T_{MN}] = i\eta_{MB}T_{AN} - i\eta_{NB}T_{AM} - i\eta_{MA}T_{BN} + i\eta_{NA}T_{BM}. \tag{180}$$

From this, it is clear why the conformal group is given by $SO(4,2)$, since clearly the above commutator is the Lorentz algebra $\mathfrak{so}(d,2)$ in $(d+2)$-dimensions.

As stated in Remark 4, $SO(4,2)$ is the Lorentz group in $(4+2)$-dimensions. Now, $SO(4,2) = SO(2) \times SO(4)$; that is, $SO(2) \times SO(4)$ is the maximal compact subgroup of $SO(4,2)$. Furthermore, $SO(4) = SU(2)_L \times SU(2)_R$ (here, the equality is given under isomorphism between the two groups; we say that $SU(2) \times SU(2)$ forms a double cover of $SO(4)$). The generator for $SO(2)$ is the dilatation operator D and the generator for $SU(2)$ is J_i, $i = 1, 2, 3$. Now, the Cartan subalgebra of $\mathfrak{so}(4,2)$ is the largest set of commuting generators. Here, we will use $\{D, J_{3L}, J_{3R}\}$ as the Cartan subalgebra. Since these generators are diagonalizable, we can use them to label states; more precisely, we can label a state as $|\Delta, m_L, m_R\rangle$, where $-j_{3L} \leq m_L \leq j_{3L}$ and $-j_{3R} \leq m_R \leq j_{3R}$.

These generators also label the irreducible representations of $SO(4,2)$. Indeed, since there is a unique primary in the irreducible representation, we can use the labels of the primary to label the irreducible representation. The scalar field itself is a primary operators with $\Delta = 1$ and $j_L = j_R = 0$. Consequently we say that the scalar field ϕ is in representation $[1, 0, 0]$.

4.3 Group Characters

In this subsection, we will prove a few basic results using group characters. In later subsections, we will make use of these results to determine the spectrum of primaries.

We start by defining a group character. Let G be any group that admits a matrix representation and let $g \in G$. Then, if $\Gamma_R(g)$ is any matrix representation R of G, the corresponding character $\chi_R(g)$ is given by

$$\chi_R(g) = \mathrm{Tr}\Gamma_R(g). \tag{181}$$

Here, Tr denotes the trace operation acting on a matrix; that is, we sum the diagonal elements of the matrix.

Using this definition and the properties of traces, we can prove some useful relations that will be used later. The relations are as follows
1) If R is the tensor product of two representations $R = r_1 \otimes r_2$, then

$$\chi_R(g) = \chi_{r_1}(g)\chi_{r_2}(g). \tag{182}$$

2) If R is the direct sum of two representations $R = r_1 \otimes r_2$, then

$$\chi_R(g) = \chi_{r_1}(g) + \chi_{r_2}(g). \tag{183}$$

These relations can be argued as follows. For relation 1), if $R = r_1 \otimes r_2$ then $g \in G$ can be represented as

$$\Gamma_R(g) = \Gamma_{r_1}(g) \otimes \Gamma_{r_2}(g). \tag{184}$$

Then,

$$\chi_R = \mathrm{Tr}\Gamma_R(g) = \mathrm{Tr}(\Gamma_{r_1}(g) \otimes \Gamma_{r_2}(g)). \tag{185}$$

Since the Γ's are matrices, we can make use of the Kronecker product, defined as:

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & & \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}B & A_{n2}B & \cdots & A_{nn}B \end{pmatrix}. \tag{186}$$

From this, observe that

$$\begin{aligned} \mathrm{Tr}(A \otimes B) &= A_{11}TrB + A_{22}TrB + ... + A_{nn}TrB \\ &= (A_{11} + A_{22} + ... + A_{nn}\mathrm{Tr}B \\ &= (\mathrm{Tr}A)(\mathrm{Tr}B). \end{aligned} \tag{187}$$

For relation 2), if $R = r_1 \otimes r_2$, then $\forall g \in G$ (using the definition of the direct sum for matrices),

$$\begin{aligned} \Gamma_R(g) &= \Gamma_{r_1}(g) \oplus \Gamma_{r_2}(g) \\ &= \begin{pmatrix} \Gamma_{r_1} & 0 \\ 0 & \Gamma_{r_2} \end{pmatrix} \\ &\Rightarrow \mathrm{Tr}\Gamma_R(g) = \mathrm{Tr}\Gamma_{r_1}(g) + \mathrm{Tr}\Gamma_{r_2}(g) \\ &\Rightarrow \chi_R(g) = \chi_{r_1}(g) + \chi_{r_2}(g). \end{aligned} \tag{188}$$

Ultimately, we would like to take products of fields as primaries. Thus, we want to understand how the product of group representations decomposes. Using group characters we can construct equations of the form

$$\chi_A(g)\chi_B(g) = \sum_a \chi_a(g). \tag{189}$$

From relation 1) above, this tells us that

$$A \otimes B = \oplus_a a, \tag{190}$$

in other words, how the product of representations decompose.

4.4 Example Using $SU(2)$ Characters

As a simple example, we consider the group $SU(2)$, which is generated by the operators $\{J_1, J_2, J_3\} \in \mathfrak{su}(2)$. Again, we use the Cartan subalgebra, which in this case corresponds to a single generator J_3, say. Thus, the typical group element we will consider is $g = e^{i\theta J_3} \in SU(2)$.

Now, recall from undergraduate non-relativistic quantum mechanics that the wave function of a particle with spin j can have $2j+1$ different spin states. Futhermore, if we take two particles with spin j_1 and spin j_2, say, the total number of spin states has range $|j_1 - j_2| \leq j \leq j_1 + j_2$. We can easily recover this result using the idea of group characters.

For simplicity, define $x = e^{i\theta}$. Then, if a particle is in the spin j irreducible representation of $SU(2)$, the corresponding character has the form

$$\begin{aligned}\chi_j(g) &= x^j + x^{j-1} + \cdots + x^{-j+1} + x^{-j} \\ &= \frac{x^{j+\frac{1}{2}} - x^{-j-\frac{1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}.\end{aligned} \tag{191}$$

In the first line, the trace was taken over all possible spin states $|j, m = -j\rangle, ..., |j, m = j\rangle$ through $\chi_j(g) = \sum_{m=-j}^{j} \langle j, m | x^J | j, m\rangle$. In the second line, we argued as follows: Let $S = \sum_{n=-j}^{j} x^n$. Then, we can write

$$S = x^{\frac{1}{2}} S_{-\frac{1}{2}}, \tag{192}$$

and

$$S = x^{-\frac{1}{2}} S_{\frac{1}{2}}, \tag{193}$$

where $S_{-\frac{1}{2}} = \sum_{n-=-j-\frac{1}{2}}^{j-\frac{1}{2}} x^n$ and $S_{\frac{1}{2}} = \sum_{n-=-j+\frac{1}{2}}^{j+\frac{1}{2}} x^n$. Now, observe that

$$S_{\frac{1}{2}} - S_{-\frac{1}{2}} = x^{j+\frac{1}{2}} - x^{j-\frac{1}{2}}. \tag{194}$$

Also,

$$S_{\frac{1}{2}} - S_{-\frac{1}{2}} = \frac{S}{x^{-\frac{1}{2}}} - \frac{S}{x^{\frac{1}{2}}} = S(x^{\frac{1}{2}} - x^{-\frac{1}{2}}). \tag{195}$$

Equating these expressions gives

$$S = \frac{x^{j+\frac{1}{2}} - x^{j-\frac{1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}. \tag{196}$$

Now, consider two particles; let one particle be in the spin j_1 irredicible representation of $SU(2)$ and let the other particle be in the spin j_2 irredicible representation of $SU(2)$. Taking the tensor product of the two representations and using the above result, we find that (assume for the following argument that $j_2 > j_1$)

$$\begin{aligned}
\chi_{j_1}(x)\chi_{j_2}(x) &= (x^{j_1} + x^{j_1-1} + \cdots + x^{-j_1+1} + x^{-j_1})\frac{x^{j_2+\frac{1}{2}} - x^{-j_2-\frac{1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}} \\
&= \sum_{k=-j_1}^{j_1} \frac{x^{j_2+k+\frac{1}{2}} - x^{-j_2+k-\frac{1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}} \\
&= \sum_{k=-j_1}^{j_1} \frac{x^{j_2+k+\frac{1}{2}} - x^{-j_2-k-\frac{1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}} \\
&= \sum_{k=j_2-j_1}^{j_2+j_1} \frac{x^{k+\frac{1}{2}} - x^{-k-\frac{1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}} \\
&= \sum_{k=j_2-j_1}^{j_2+j_1} \chi_k(x).
\end{aligned} \tag{197}$$

From equation (182), this implies that

$$j_1 \otimes j_2 = \oplus_{|j_1-j_2|}^{j_1+j_2}\ j. \tag{198}$$

In other words, the product of representations for two particles decomposes into a direct sum of representations. This is precisely the result derived in an undergraduate quantum mechanics course.

4.5 Group Characters of $SO(4,2)$

The previous section is a concrete example of the strategy we are going to use in this section. That is, if we have the characters of $SO(4,2)$ then we can easily compute the products of irreducible representations in the conformal field theory. More precisely, if we suppose that the free scalar field belongs to the irreducible representation V of $SO(4,2)$, then we want to decompose the character $\chi_{\mathrm{Sym}(V^{\otimes n})}$ into a sum over characters of irreducible representations. Note that symbol "Sym" in the subscript denotes 'symmetric' since we must respect the bosonic statistics of the field. This can be achieved using Young projectors, which will be seen later in this section when we consider products of fields.

As stated previously, the Cartan subalgebra consists of the generators $\{D, J_L, J_R\}$. As in the previous example, we can make the same simplification by defining $s = e^t, x = e^{i\theta_L}$, and $y = e^{i\theta_R}$. Then the group element we will look at is given by $g = s^D x^{J_L} y^{J_R} \in SO(4,2)$. In other words, our characters will be functions of three variables. The scalar field is thus labeled by three numbers $[1,0,0]$. Note that $SO(4,2)$ is not compact and can thus have null states.

For a scalar field in the representation V, the group character for $SO(4,2)$ can be determined by using the work of [44]. For a representation with labels $[1,0,0]$ (in other words, a scalar field), the character of a unitary irreducible representation of $SO(4,2)$ is given by

$$\begin{aligned}\mathcal{A}^4_{[1,0,0]} &= sP^4(s,x,y)\\ &= \frac{s}{(1-s\tilde{x})(1-s\tilde{y})(1-s\tilde{x}^{-1})(1-s\tilde{y}^{-1})},\end{aligned} \tag{199}$$

where $\tilde{x} = \sqrt{xy}$ and $\tilde{y} = \sqrt{xy^{-1}}$. We can show that this formula is equivalent to

$$\chi_V(s,x,y) = s(1-s^2)\sum_{p,q=0}^{\infty} s^{2p+q}\chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}}(y). \tag{200}$$

Here, the factor $(1-s^2)$ *removes* null states and its descendants. This is due to the fact that $SO(4,2)$ is not compact, and thus can have null states; in fact, the free scalar field saturates the unitarity bound. This can be readily seen from dimensional analysis. From the action in equation (6) for $d = 4$, the dimension of ϕ is $[\phi] = L^{-1}$; from [23], the unitarity bound is given by $\epsilon \geq 1$. The corresponding null state is precisely the equation of motion $\partial_\mu\partial^\mu\phi = 0$. To derive this bound we have examined the norm of descendents of the primary. If we try to derive a bound for the primary itself we find $\epsilon \geq 0$ which is consistent with the fact that the identity, which is the primary of the trivial representation, has dimension zero and no descendents.

To show that (199) and (200) are equivalent, we show that the terms are equal for arbitrary powers of s^n. If the terms are equal for arbitrary $n \in \mathbb{N}$, then the relation above must hold. We start with equation (200). For terms of order s^n, we have

$$\sum_{p,q} s^{2p+q+1}\chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}}(y) - \sum_{p,q} s^{2p+q+3}\chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}}(y), \tag{201}$$

where we must have $2p+q+1=n$ and $2p+q+3=n$. Using this, we can restrict the values of the q in each character χ; in particular, we have

$$q = n - 1 - 2p, \tag{202}$$

$$q = n - 3 - 2p. \tag{203}$$

Apparently, for a given n and p, q can only take on two values. For example,

$$\begin{aligned} &p = 0 : q = n-1, q = n-3 \\ &p = 1 : q = n-3, q = n-5 \\ &p = 2 : q = n-5, q = n-7 \\ &\vdots \end{aligned} \tag{204}$$

In addition, observe that due to the minus in equation (201), only $q = n-1$ survives for arbitrary $n \in \mathbb{N}$. This is due to the fact that $1-s^2$ removes the null state and its descendants. We are therefore left with

$$\begin{aligned} &s^n[\chi_{\frac{n-1}{2}}(x)\chi_{\frac{n-1}{2}}(y)] \\ &= s^n[(x^{\frac{n-1}{2}} + ... + x^{\frac{1-n}{2}})(y^{\frac{n-1}{2}} + ... + y^{\frac{1-n}{2}})] \\ &= s^n \sum_{i,k,l,m} \{[x^{i/2}y^{k/2}x^{-l/2}y^{-m/2}]^{(n-1)}\}, \end{aligned} \tag{205}$$

where the bracket $[x^{i/2}y^{k/2}x^{-l/2}y^{-m/2}]^{(n-1)}$ means: $\forall i,k,l,m \in \mathbb{N}$, $\frac{1}{2}(i+k+l+m) = n-1$ (in other words, each term should have powers that add up to $n-1$).

Next, we consider equation (199). Here, we have

$$\begin{aligned} &\frac{s}{(1-s\tilde{x})(1-s\tilde{y})(1-s\tilde{x}^{-1})(1-s\tilde{y}^{-1})} \\ &= s(1+s\tilde{x}+s^2\tilde{x}^2+...)(1+s\tilde{y}+s^2\tilde{y}^2+...) \\ &\times(1+s\tilde{x}^{-1}+s^2\tilde{x}^{-2}+...)(1+s\tilde{y}^{-1}+s^2\tilde{y}^{-2}+...) \\ &= s\sum_{i=0}^{\infty}(s\tilde{x})^i \sum_{k=0}^{\infty}(s\tilde{y})^k \sum_{l=0}^{\infty}(s\tilde{x}^{-1})^l \sum_{m=0}^{\infty}(s\tilde{y}^{-1})^m. \end{aligned} \tag{206}$$

Now, consider terms of order s^n. We must therefore have $s^{i+j+k+l+1} = s^n \Rightarrow i+j+k+l = n-1$. Using the expressions for $\tilde{x}$ and $\tilde{y}$, we find that terms of order s^n must have powers distributed as

$$s^n \sum_{i,k,l,m} \{[x^{i/2}y^{k/2}x^{-l/2}y^{-m/2}]^{(n-1)}\}. \tag{207}$$

This is precisely the same result as equation (205). We conclude that the relation is true.

Now, we can check that equation (200) does indeed produce the correct group character for $SO(4,2)$. If we expand this equation, we obtain (up to order s^4)

$$\begin{aligned}\chi_V(s,x,y) &= s(1-s^2)\sum_{p,q}^{\infty} s^{2p+q}\chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}}(y)\\ &= s\sum_{q}^{\infty} s^q\chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}}(y)\\ &= s(1+s\chi_{\frac{1}{2}}(x)\chi_{\frac{1}{2}}(y)+s^2\chi_1(x)\chi_1(y)+s^3\chi_{\frac{3}{2}}(x)\chi_{\frac{3}{2}}(y)+\cdots)\\ &= s\Big(1+s(x^{\frac{1}{2}}+x^{-\frac{1}{2}})(y^{\frac{1}{2}}+y^{-\frac{1}{2}})+s^2(x^1+1+x^{-1})(y^1+1+y^{-1})\\ &+s^3(x^{\frac{3}{2}}+x^{\frac{1}{2}}+x^{-\frac{1}{2}}+x^{-\frac{3}{2}})(y^{\frac{3}{2}}+y^{\frac{1}{2}}+y^{-\frac{1}{2}}+y^{-\frac{3}{2}})+\cdots\Big).\end{aligned} \tag{208}$$

Lets consider the character from its definition. In this case, since $g = s^D x^{J_L} y^{J_R}$, the character is given by the trace

$$\mathrm{Tr}(g) = \mathrm{Tr}(s^D x^{J_L} y^{J_R}) = \sum_{\Delta,m_L,m_R}\langle\Delta,m_L,m_R|e^{tD+i\theta_L J_{3,L}+i\theta_R J_{3,R}}|\Delta,m_L,m_R\rangle. \tag{209}$$

(Note, that here we mean the trace of the matrix, say Γ, representing g. For simplicity however, we will simply write $\mathrm{Tr}(g)$). For the scalar field ϕ, the group element acts as follows

$$\begin{aligned}g|\phi\rangle &= s^D x^{J_L} y^{J_R}|1,0,0\rangle\\ &= e^t|1,0,0\rangle\\ &= s|\phi\rangle.\end{aligned} \tag{210}$$

Thus the contribution to the trace from this state is given by

$$\langle\phi|g|\phi\rangle = s = s\chi_0(x)\chi_0(y). \tag{211}$$

For the descendant $\partial_\mu\phi$ of the scalar field, there is a range of states

$$|\partial_\mu\phi\rangle \leftrightarrow \left\{\left|2,\frac{1}{2},\frac{1}{2}\right\rangle,\left|2,\frac{1}{2},-\frac{1}{2}\right\rangle,\left|2,-\frac{1}{2},\frac{1}{2}\right\rangle,\left|2,-\frac{1}{2},-\frac{1}{2}\right\rangle\right\}. \tag{212}$$

Acting on these states with g gives

$$\begin{aligned}g|\partial_\mu\phi\rangle \leftrightarrow &\left\{s^2x^{1/2}y^{1/2}\left|2,\frac{1}{2},\frac{1}{2}\right\rangle, s^2x^{1/2}y^{-1/2}\left|2,\frac{1}{2},-\frac{1}{2}\right\rangle, s^2x^{-1/2}y^{1/2}\left|2,-\frac{1}{2},\frac{1}{2}\right\rangle,\right.\\ &\left. s^2x^{-1/2}y^{-1/2}\left|2,-\frac{1}{2},-\frac{1}{2}\right\rangle\right\}.\end{aligned} \tag{213}$$

Thus the contribution to the trace from these states is given by

$$\langle\partial_\mu\phi|g|\partial_\mu\phi\rangle = s^2(x^{\frac{1}{2}}+x^{-\frac{1}{2}})(y^{\frac{1}{2}}+y^{-\frac{1}{2}}) = s^2\chi_{\frac{1}{2}}(x)\chi_{\frac{1}{2}}(y). \tag{214}$$

Similarly, for the descendant $\partial_\mu\partial_\nu\phi$, we have

$$\begin{aligned}g|\partial_\mu\partial_\nu\phi\rangle \leftrightarrow \{&s^3xy|3,1,1\rangle, s^3x|3,1,0\rangle, s^3xy^{-1}|3,1,-1\rangle, s^3y|3,0,1\rangle, s^3y^{-1}|3,0,-1\rangle,\\ &s^3|3,0,0\rangle, s^3x^{-1}y|3,-1,1\rangle, s^3x^{-1}|3,-1,0\rangle, s^3x^{-1}y^{-1}|3,-1,-1\rangle\},\end{aligned} \tag{215}$$

giving the contribution

$$\begin{aligned}\langle\partial_\mu\partial_\nu\phi|g|\partial_\mu\partial_\nu\phi\rangle &= s^3(xy+x+xy^{-1}+y+y^{-1}+1+x^{-1}y+x^{-1}+x^{-1}y^{-1})\\ &= s^3\chi_1(x)\chi_1(y).\end{aligned} \tag{216}$$

Comparing these results with the terms in the expansion of equation (200), we see that the two expressions contain identical terms. This result holds for higher orders as well. Thus,

$$\mathrm{Tr}(e^{tD+i\theta_L J_{3,L}+i\theta_R J_{3,R}}) = \chi_V(s,x,y). \tag{217}$$

4.6 Product of Two Operators

Using the results of the previous sections, we can now consider products of fields. That is, we want to take a product of two copies of the representation V that the scalar field belongs to.

Here, we need to project to the symmetric subspace to satisfy bosonic statistics. Thus, the character we want to compute in this case is

$$\chi_{Sym(V^{\otimes 2})} = \sum_{i,j}\langle i|\otimes\langle j|g\otimes g\,P|i\rangle\otimes|j\rangle, \tag{218}$$

where P projects us onto the symmetric subspace as follows

$$P|i\rangle\otimes|j\rangle = \frac{1}{2}\left[|i\rangle\otimes|j\rangle + |j\rangle\otimes|i\rangle\right]. \tag{219}$$

We have shortened the notation here to sums over i, j for states $|i\rangle$ and $|j\rangle$.

To construct this character, consider first the following argument: the matrix representation M (for some group element $g \in G$ in representation V) acting on the basis $|i\rangle$ can be constructed through the inner product

$$\langle i|M|j\rangle = M_{ij}, \tag{220}$$

and thus, the trace of the matrix is given by

$$\sum_i\langle i|M|i\rangle = \sum_i M_{ii}. \tag{221}$$

Now, the tensor product $M \otimes M$ can be computed through

$$\langle i_1 i_2 | M \otimes M | j_1 j_2 \rangle = M_{i_1 j_1} M_{i_2 j_2}, \tag{222}$$

and thus the trace yields

$$\begin{aligned} \mathrm{Tr}(M^{\otimes 2}) &= \sum_{i_1, i_2} \langle i_1, i_2 | M \otimes M | i_1, i_2 \rangle \\ &= \sum_{i_1, i_2} M_{i_1 i_1} M_{i_2 i_2} \\ &= \mathrm{Tr}(M)^2. \end{aligned} \tag{223}$$

This is, of course, the character of g in representation $V^{\otimes 2}$; in other words, $\chi_{V^{\otimes 2}}(g)$. When we move to the symmetric subspace of $V \otimes V$, the character becomes more complicated. We can project to the symmetric subspace using the projection operator

$$P = \frac{1}{2!} \sum_{\sigma \in S_2} \sigma, \tag{224}$$

where S_2 is the symmetric group. The operator σ acts by exchanging indices that labels a state; in other words, if $\sigma = (12)$ then $\sigma | i_1, i_2 \rangle = (12) | i_1, i_2 \rangle = | i_2, i_1 \rangle$ (similar for other permutations in the symmetric group S_2). In this example, the trace of $M^{\otimes 2}$ over the symmetric subspace can be calculated as

$$\begin{aligned} \chi_{Sym(V^{\otimes 2})} &= \mathrm{Tr}(M^{\otimes 2})_{\mathrm{Sym}} \\ &= \sum_{i_1, i_2} \left\langle i_1, i_2 \middle| \left(M \otimes M \frac{1}{2!} \sum_{\sigma \epsilon S_2} \sigma \right) \middle| i_1, i_2 \right\rangle. \end{aligned} \tag{225}$$

Expanding this and permuting indices according to S_2 yields

$$\begin{aligned} \chi_{Sym(V^{\otimes 2})} &= \frac{1}{2} \sum_{i_1, i_2} (M_{i_1 i_1} M_{i_2 i_2} + M_{i_1 i_2} M_{i_1 i_2}) \\ &= \frac{1}{2} (\mathrm{Tr}(M)^2 + \mathrm{Tr}(M^2)). \end{aligned} \tag{226}$$

Similarly, for the product $M^{\otimes 3}$, the character is given by

$$\begin{aligned} \chi_{Sym(V^{\otimes 3})} &= \mathrm{Tr}(M^{\otimes 3})_{\mathrm{Sym}} \\ &= \frac{1}{3!} \sum_{i_1, i_2, i_3} \left\langle i_1, i_2, i_3 \middle| M^{\otimes 3} \Big(1 + (12) + (23) + (13) + (123) + (132) \Big) \middle| i_1, i_2, i_3 \right\rangle \\ &= \frac{1}{3!} \sum_{i_1, i_2, i_3} (M_{i_1 i_1} M_{i_2 i_2} M_{i_3 i_3} + M_{i_1 i_2} M_{i_2 i_1} M_{i_3 i_3} + M_{i_1 i_1} M_{i_2 i_3} M_{i_3 i_2} \\ &+ M_{i_1 i_3} M_{i_2 i_2} M_{i_3 i_1} + M_{i_1 i_2} M_{i_2 i_3} M_{i_3 i_1} + M_{i_1 i_3} M_{i_2 i_1} M_{i_3 i_2}) \\ &= \frac{1}{3!} (\mathrm{Tr}(M)^3 + 3\mathrm{Tr}(M^2)\mathrm{Tr}(M) + 2\mathrm{Tr}(M^3)). \end{aligned} \tag{227}$$

These are the so-called *Schur polynomials.*

For our case then, the relevant character to calculate is

$$\chi_{Sym(V^{\otimes 2})} = \frac{1}{2}\left(\mathrm{Tr}(g)^2 + \mathrm{Tr}(g^2)\right). \tag{228}$$

Now, $\mathrm{Tr}(g)^2 = \chi_V(s,x,y)^2$. We need to therefore determine the character $\mathrm{Tr}(g^2)$. Using our previous arguments and the fact that $g^2 = s^{2D}x^{2J_L}y^{J_R}$, we find the following

$$g^2|\phi\rangle = s^{2D}x^{2J_L}y^{2J_R}|1,0,0\rangle = s^2|\phi\rangle, \tag{229}$$

$$\begin{aligned} g^2|\partial_\mu\phi\rangle \leftrightarrow \Big\{ & s^4x^1y^1\Big|2,\frac{1}{2},\frac{1}{2}\Big\rangle, s^4x^1y^{-1}\Big|2,\frac{1}{2},-\frac{1}{2}\Big\rangle, s^4x^{-1}y^1\Big|2,-\frac{1}{2},\frac{1}{2}\Big\rangle, \\ & s^4x^{-1}y^{-1}\Big|2,-\frac{1}{2},-\frac{1}{2}\Big\rangle\Big\}. \end{aligned} \tag{230}$$

The contribution to the trace from these states is given by

$$\langle\phi|g^2|\phi\rangle = s^2, \tag{231}$$

$$\begin{aligned} \langle\partial_\mu\phi|g|\partial_\mu\phi\rangle &= s^4(xy + xy^{-1} + x^{-1}y + x^{-1}y^{-1}) \\ &= s^4(x + x^{-1})(y + y^{-1}) \\ &= s^4\chi_{\frac{1}{2}}(x^2)\chi_{\frac{1}{2}}(y^2). \end{aligned} \tag{232}$$

Comparing this with previous results, it clear that

$$\mathrm{Tr}(g^2) = \chi_V(s^2,x^2,y^2). \tag{233}$$

The character to calculate is

$$\chi_{Sym(V^{\otimes 2})} = \frac{1}{2}(\chi_V(s,x,y)^2 + \chi_V(s^2,x^2,y^2)). \tag{234}$$

From here, we can compute the character for product of two operators explicitly. Using the expression from [44]

$$P(s,x,y) = \frac{1}{(1-sx^{1/2}y^{1/2})(1-sx^{1/2}y^{-1/2})(1-sx^{-1/2}y^{1/2})(1-sx^{-1/2}y^{-1/2})}, \tag{235}$$

and expanding, we obtain

$$P(s,x,y) = \sum_{p,q=0}^{\infty} s^{2p+q}\chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}}(y). \tag{236}$$

Using the trivial identity

$$\frac{1}{1-t^2} = \frac{1}{1-t}\frac{1}{1+t}, \tag{237}$$

for each of the four factors in $P(s,x,y)$, we find

$$\begin{aligned} P(s^2,x^2,y^2) &= P(s,x,y)P(-s,x,y) \\ &= \sum_{p,q=0}^{\infty} (-s)^{2p+q} \chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}}(y) P(s,x,y). \end{aligned} \tag{238}$$

Therefore,

$$\begin{aligned} \chi_V(s^2,x^2,y^2) &= s^2(1-s^4)P(s^2,x^2,y^2) \\ &= s^2(1-s^4)P(s,x,y)P(-s,x,y) \\ &= s^2(1-s^4)P(s,x,y)\sum_{p,q=0}^{\infty}(-s)^{2p+q}\chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}}(y) \\ &= s^2(1+s^2)P(s,x,y)\sum_{q=0}^{\infty}(-s)^{q}\chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}}(y) \\ &= s^2P(s,x,y) - \sum_{q=0}^{\infty}(-1)^q\left[s^{3+q}\chi_{\frac{q+1}{2}}(x)\chi_{\frac{q+1}{2}}(y) - s^{4+q}\chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}}(y)\right]P(s,x,y) \\ &= s^2P(s,x,y) - \sum_{d=3}^{\infty}(-1)^{d+1}\ \left[s^{d}\chi_{\frac{d-2}{2}}(x)\chi_{\frac{d-2}{2}}(y) - s^{d+1}\chi_{\frac{d-3}{2}}(x)\chi_{\frac{d-3}{2}}(y)\right]P(s,x,y). \end{aligned} \tag{239}$$

Moreover,

$$\begin{aligned} (\chi_V(s,x,y))^2 &= \chi_{[1,0,0]} \times \chi_{[1,0,0]} \\ &= \sum_{q=0}^{\infty} s^{q+1}\chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}}(y) \times s(1-s^2)P(s,x,y) \\ &= s^2P(s,x,y) + \sum_{q=0}^{\infty}\left[s^{3+q}\chi_{\frac{q+1}{2}}(x)\chi_{\frac{q+1}{2}}(y) - s^{4+q}\chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}}(y)\right]P(s,x,y) \\ &= s^2P(s,x,y) + \sum_{d=3}^{\infty}\left[s^{d}\chi_{\frac{d-2}{2}}(x)\chi_{\frac{d-2}{2}}(y) - s^{d+1}\chi_{\frac{d-3}{2}}(x)\chi_{\frac{d-3}{2}}(y)\right]P(s,x,y). \end{aligned} \tag{240}$$

Thus, our character is given explicitly by

$$\begin{aligned} \chi_{Sym(V^{\otimes 2})} &= \frac{1}{2}\left((\chi_V(s,x,y))^2 + \chi_V(s^2,x^2,y^2)\right) \\ &= s^2P(s,x,y) + \sum_{d=1}^{\infty}\left[s^{2d+2}\chi_{\frac{2d}{2}}(x)\chi_{\frac{2d}{2}}(y) - s^{2d+3}\chi_{\frac{2d-1}{2}}(x)\chi_{\frac{2d-1}{2}}(y)\right]P(s,x,y). \end{aligned} \tag{241}$$

Using [44], this can be written as

$$Sym(D_{[100]} \otimes D_{[100]}) = \mathcal{A}_{[200]} + \sum_{k_1=1} D_{[2k_1+2,\frac{2k_1}{2},\frac{2k_1}{2}]}. \tag{242}$$

Here, $\mathcal{A}_{[200]}$ is the representation for a spinless scalar field of dimension 2; the corresponding primary is ϕ^2. This representation has no null states. The term $D_{[2k_1+2,\frac{2k_1}{2},\frac{2k_1}{2}]}$ is the representation for conserved currents. As we will see later, this will agree with our construction of primaries. These representations do have null states.

4.7 Product of Many Operators

In this section, we will generalize the results from the previous section. In particular, we will generalize (233) to any n. Using this, we will be able to calculate the character $\chi_{\text{Sym}V^{\otimes n}}$ for arbitrary product of representations V.

From Section (4.4), we found that the character for $SU(2)$ in the spin j representation can be written

$$\begin{aligned}\chi_j(x) &= x^j + x^{j-1} + \cdots + x^{-j+1} + x^{-j} \\ &= \frac{x^{j+\frac{1}{2}} - x^{-j-\frac{1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}.\end{aligned} \tag{243}$$

Consequently,

$$\begin{aligned}\chi_{\frac{k}{2}}(x^n) &= x^{\frac{nk}{2}} + x^{\frac{n(k-2)}{2}} + \cdots + x^{-\frac{n(k-2)}{2}} + x^{-\frac{nk}{2}} \\ &= \left(\frac{x^{\frac{nk}{2}} + x^{\frac{n(k-2)}{2}} + \cdots + x^{-\frac{n(k-2)}{2}} + x^{-\frac{nk}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}\right)(x^{\frac{1}{2}} - x^{-\frac{1}{2}}) \\ &= \sum_{l=0,1,\ldots}^{\lfloor k/2 \rfloor} \chi_{\frac{kn}{2}-nl}(x) - \sum_{l=0,1,\ldots}^{\lfloor (k-1)/2 \rfloor} \chi_{\frac{kn}{2}-nl-1}(x),\end{aligned} \tag{244}$$

where the last line can be found by multiplying the numerator out and collecting terms. This implies that

$$\begin{aligned}P(s^n, x^n, y^n) &= \sum_{p,q=0}^{\infty} s^{2np+nq} \chi_{\frac{q}{2}}(x^n)\chi_{\frac{q}{2}}(y^n) \\ &= \frac{1}{1-s^{2n}} \sum_{q=0}^{\infty} s^{nq} \chi_{\frac{q}{2}}(x^n)\chi_{\frac{q}{2}}(y^n)\end{aligned}$$

$$= \frac{1}{1-s^{2n}} \sum_{q=0}^{\infty} s^{nq} \left[\sum_{l=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn}{2}-nl}(x) - \sum_{l=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn}{2}-nl-1}(x) \right]$$
$$\times \left[\sum_{l=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn}{2}-nl}(y) - \sum_{l=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn}{2}-nl-1}(y) \right], \tag{245}$$

and therefore,

$$\chi_V(s^n, x^n, y^n) = P(s^n, x^n, y^n) s^n (1-s^{2n})$$
$$= s^n \sum_{q=0}^{\infty} s^{nq} \left[\sum_{l=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn}{2}-nl}(x) - \sum_{l=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn}{2}-nl-1}(x) \right]$$
$$\times \left[\sum_{l=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn}{2}-nl}(y) - \sum_{l=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn}{2}-nl-1}(y) \right]. \tag{246}$$

In practice, we would like to write the result as $SU(2)$ characters multiplied by a factor of $P(s,x,y)$ since then we can easily translate the terms into a sum of $SO(4,2)$ characters as in [45]. Towards this result, observe that

$$1 = P(s,x,y)(1-sx^{1/2}y^{1/2})(1-sx^{1/2}y^{-1/2})(1-sx^{-1/2}y^{1/2})(1-sx^{-1/2}y^{-1/2})$$
$$= P(s,x,y)\big[1+s^4 - s(1+s^2)\chi_{\frac{1}{2}}(x)\chi_{\frac{1}{2}}(y) + s^2(\chi_1(x)+\chi_1(y))\big]. \tag{247}$$

Using this, we find the lengthy expression

$$\chi_V(s^n, x^n, y^n) = P(s^n, x^n, y^n) s^n (1-s^{2n})$$
$$= s^n \Bigg[(1+s^4) \sum_{q=0}^{\infty} s^{nq} \left[\sum_{l_1=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn}{2}-nl_1}(x) - \sum_{l_1=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn}{2}-nl_1-1}(x) \right]$$
$$\times \left[\sum_{l_2=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn}{2}-nl_2}(y) - \sum_{l_2=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn}{2}-nl_2-1}(y) \right]$$
$$-s(1+s^2) \sum_{q=0}^{\infty} s^{nq} \left[\sum_{l_1=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn+1}{2}-nl_1}(x) - \sum_{l_1=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn-1}{2}-nl_1-1}(x) + \sum_{l_1=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn-1}{2}-nl_1}(x) \right.$$
$$\left. - \sum_{l_1=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn+1}{2}-nl_1-1}(x) \right] \times \left[\sum_{l_2=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn+1}{2}-nl_2}(y) - \sum_{l_2=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn-1}{2}-nl_2-1}(y) \right.$$
$$\left. + \sum_{l_2=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn-1}{2}-nl_2}(y) - \sum_{l_2=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn+1}{2}-nl_2-1}(y) \right]$$
$$+s^2 \sum_{q=0}^{\infty} s^{nq} \left[\sum_{l_1=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn}{2}-nl_1}(x) - \sum_{l_1=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn}{2}-nl_1-1}(x) \right]$$

$$\times\left[\sum_{l_2=0,1,\ldots}^{\lfloor q/2\rfloor}\chi_{\frac{qn}{2}-nl_2+1}(y)-\sum_{l_2=0,1,\ldots}^{\lfloor (q-1)/2\rfloor}\chi_{\frac{qn}{2}-nl_2}(y)+\sum_{l_2=0,1,\ldots}^{\lfloor q/2\rfloor}\chi_{\frac{qn}{2}-nl_2}(y)\right.$$
$$\left.-\sum_{l_2=0,1,\ldots}^{\lfloor (q-1)/2\rfloor}\chi_{\frac{qn}{2}-nl_2-1}(y)+\sum_{l_2=0,1,\ldots}^{\lfloor q/2\rfloor}\chi_{\frac{qn}{2}-nl_2-1}(y)-\sum_{l_2=0,1,\ldots}^{\lfloor (q-1)/2\rfloor}\chi_{\frac{qn}{2}-nl_2-2}(y)\right]$$
$$+s^2\sum_{q=0}^{\infty}s^{nq}\left[\sum_{l_1=0,1,\ldots}^{\lfloor q/2\rfloor}\chi_{\frac{qn}{2}-nl_1}(y)-\sum_{l_1=0,1,\ldots}^{\lfloor (q-1)/2\rfloor}\chi_{\frac{qn}{2}-nl_1-1}(y)\right]$$
$$\times\left[\sum_{l_2=0,1,\ldots}^{\lfloor q/2\rfloor}\chi_{\frac{qn}{2}-nl_2+1}(x)-\sum_{l_2=0,1,\ldots}^{\lfloor (q-1)/2\rfloor}\chi_{\frac{qn}{2}-nl_2}(x)+\sum_{l_2=0,1,\ldots}^{\lfloor q/2\rfloor}\chi_{\frac{qn}{2}-nl_2}(x)\right.$$
$$\left.\left.-\sum_{l_2=0,1,\ldots}^{\lfloor (q-1)/2\rfloor}\chi_{\frac{qn}{2}-nl_2-1}(x)+\sum_{l_2=0,1,\ldots}^{\lfloor q/2\rfloor}\chi_{\frac{qn}{2}-nl_2-1}(x)-\sum_{l_2=0,1,\ldots}^{\lfloor (q-1)/2\rfloor}\chi_{\frac{qn}{2}-nl_2-2}(x)\right]\right]P(s,x,y). \tag{248}$$

We can now calculate characters for higher order products of operators. The relevant Schur polynomials for the characters for three, four, and five fields are

$$\chi_{Sym(V^{\otimes 3})}=\frac{1}{6}\left((\chi_V(s,x,y))^3+3\chi_V(s,x,y)\chi_V(s^2,x^2,y^2)+2\chi_V(s^3,x^3,y^3)\right), \tag{249}$$

$$\begin{aligned}\chi_{Sym(V^{\otimes 4})}=\frac{1}{24}\Big(&(\chi_V(s,x,y))^4+6(\chi_V(s,x,y))^2\chi_V(s^2,x^2,y^2)+8\chi_V(s^3,x^3,y^3)\chi_V(s,x,y)\\&+3(\chi_V(s^2,x^2,y^2))^2+6\chi_V(s^4,x^4,y^4)\Big),\end{aligned} \tag{250}$$

$$\begin{aligned}\chi_{Sym(V^{\otimes 5})}=\frac{1}{120}\Big(&(\chi_V(s,x,y))^5+10(\chi_V(s,x,y))^3\chi_V(s^2,x^2,y^2)+20\chi_V(s^3,x^3,y^3)(\chi_V(s,x,y))^2\\&+30\chi_V(s^4,x^4,y^4)\chi_V(s,x,y)+15(\chi_V(s^2,x^2,y^2))^2\chi_V(s,x,y)\\&+20\chi_V(s^2,x^2,y^2)\chi_V(s^3,x^3,y^3)+24\chi_V(s^5,x^5,y^5)\Big).\end{aligned} \tag{251}$$

Using the results we derived, one can easily calculate the characters using mathematics software such as Mathematica (see Appendix (A.4)). The results are

$$\begin{aligned}\chi_{Sym(V^{\otimes 3})}=&\,\mathcal{A}_{[3,0,0]}+\mathcal{A}_{[5,1,1]}+\mathcal{A}_{[6,\frac{3}{2},\frac{3}{2}]}+\mathcal{A}_{[7,2,2]}+\mathcal{A}_{[7,0,2]}+\mathcal{A}_{[7,2,0]}\\&+\mathcal{A}_{[8,\frac{5}{2},\frac{5}{2}]}+\mathcal{A}_{[8,\frac{3}{2},\frac{5}{2}]}+\mathcal{A}_{[8,\frac{5}{2},\frac{3}{2}]}+2\mathcal{A}_{[9,3,3]}+\mathcal{A}_{[9,1,3]}+\mathcal{A}_{[9,3,1]}\\&+\mathcal{A}_{[10,\frac{7}{2},\frac{7}{2}]}+\mathcal{A}_{[10,\frac{7}{2},\frac{5}{2}]}+\mathcal{A}_{[10,\frac{5}{2},\frac{7}{2}]}+\mathcal{A}_{[10,\frac{7}{2},\frac{3}{2}]}+\mathcal{A}_{[10,\frac{3}{2},\frac{7}{2}]}+\ldots\end{aligned} \tag{252}$$

$$\begin{aligned}\chi_{Sym(V^{\otimes 4})}=&\,\mathcal{A}_{[4,0,0]}+\mathcal{A}_{[6,1,1]}+\mathcal{A}_{[7,\frac{3}{2},\frac{3}{2}]}+\mathcal{A}_{[8,0,0]}+\mathcal{A}_{[8,0,2]}+\mathcal{A}_{[8,2,0]}+\mathcal{A}_{[8,1,1]}+2\mathcal{A}_{[8,2,2]}\\&+\mathcal{A}_{[9,\frac{3}{2},\frac{1}{2}]}+\mathcal{A}_{[9,\frac{5}{2},\frac{1}{2}]}+\mathcal{A}_{[9,\frac{1}{2},\frac{3}{2}]}+\mathcal{A}_{[9,\frac{5}{2},\frac{3}{2}]}+\mathcal{A}_{[9,\frac{1}{2},\frac{5}{2}]}+\mathcal{A}_{[9,\frac{3}{2},\frac{5}{2}]}+\mathcal{A}_{[9,\frac{5}{2},\frac{5}{2}]}\\&+\mathcal{A}_{[10,0,0]}+2\mathcal{A}_{[10,1,1]}+\mathcal{A}_{[10,2,1]}+2\mathcal{A}_{[10,3,1]}+\mathcal{A}_{[10,1,2]}+2\mathcal{A}_{[10,2,2]}+\mathcal{A}_{[10,3,2]}\\&+2\mathcal{A}_{[10,1,3]}+\mathcal{A}_{[10,2,3]}+3\mathcal{A}_{[10,3,3]}+\ldots\end{aligned} \tag{253}$$

$$
\begin{aligned}
\chi_{Sym(V^{\otimes 5})} = {} & \mathcal{A}_{[5,0,0]} + \mathcal{A}_{[7,1,1]} + \mathcal{A}_{[8,\frac{3}{2},\frac{3}{2}]} + \mathcal{A}_{[9,0,0]} + \mathcal{A}_{[9,1,1]} + \mathcal{A}_{[9,2,0]} + \mathcal{A}_{[9,0,2]} + 2\mathcal{A}_{[9,2,2]} \\
& + \mathcal{A}_{[10,\frac{1}{2},\frac{1}{2}]} + \mathcal{A}_{[10,\frac{3}{2},\frac{1}{2}]} + \mathcal{A}_{[10,\frac{1}{2},\frac{3}{2}]} + \mathcal{A}_{[10,\frac{3}{2},\frac{3}{2}]} + \mathcal{A}_{[10,\frac{1}{2},\frac{5}{2}]} + \mathcal{A}_{[10,\frac{5}{2},\frac{1}{2}]} + \mathcal{A}_{[10,\frac{5}{2},\frac{3}{2}]} \\
& + \mathcal{A}_{[10,\frac{3}{2},\frac{5}{2}]} + 2\mathcal{A}_{[10,\frac{5}{2},\frac{5}{2}]} + \ldots
\end{aligned}
\tag{254}
$$

In conclusion, we have formulated the problem of determining the spectrum of primaries into a group theory problem. By using characters, we have successfully developed a method that can count the number of primaries for any number of scalar fields.

4.8 Deriving a Generating Function for the Spectrum of Primaries

In this section, we reproduce the same results as the previous section by deriving generating functions that can be used to extract the spectrum of primaries in the theory.

The Basic Idea. First, to illustrate the idea, consider the following argument. Let M be a diagonal matrix with entries

$$
M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}. \tag{255}
$$

Then, the characters in the $\square, \square\square, \square\square\square$ representations of $SU(2)$ (that is, the totally symmetric matrices) is given by

$$
\chi_{\square}(M) = a + b, \tag{256}
$$

$$
\chi_{\square\square}(M) = a^2 + ab + b^2, \tag{257}
$$

$$
\chi_{\square\square\square}(M) = a^3 + a^2 b + ab^2 + b^3, \tag{258}
$$

$$
\chi_{\square\square\square\square}(M) = a^4 + a^3 b + a^2 b^2 + ab^3 + b^4. \tag{259}
$$

We can of course continue this to higher orders. However, these reults can also be expressed in terms of a generating function given by

$$
\frac{1}{\det(1 - tM)} = \sum_n t^n \chi_{(n)}(M), \tag{260}
$$

which holds for *any* matrix M. We can check this result for our matrix M above. Indeed,

$$
\begin{aligned}
\frac{1}{\det(1 - tM)} &= \frac{1}{(1 - ta)(1 - tb)} \\
&= (1 + ta + t^2 a^2 + t^3 a^3 + \cdots)(1 + tb + t^2 b^2 + t^3 b^3 + \cdots) \\
&= 1 + (a + b)t + (a^2 + ab + b^2)t^2 + (a^3 + a^2 b + ab^2 + b^3)t^3 + \cdots \\
&= 1 + \chi_1 t + \chi_2 t^2 + \chi_3 t^3 + \cdots \\
&= \sum_n t^n \chi_{(n)}(M).
\end{aligned}
\tag{261}
$$

To prove equation (260), we can make use of the following Gaussian integral

$$(I_n)^{j_1\cdots j_n}_{i_1\cdots i_n} = \frac{1}{\pi^N}\int\prod_{i=1}^{N} dx_i dy_i e^{-\sum_k z_k\bar{z}^k}\frac{1}{n!}z_{i_1}\cdots z_{i_n}\bar{z}^{j_1}\cdots\bar{z}^{j_n}. \tag{262}$$

We can evaluate this integral as follows: consider first the generating function

$$I = \frac{1}{\pi^N}\int\prod_{i=1}^{N} dx_i dy_i e^{-\sum_k z_k\bar{z}^k+\sum_k(\bar{j}^k z_k+j_k\bar{z}^k)}. \tag{263}$$

By shifting $z\to z+j$ and $\bar{z}\to\bar{z}+\bar{j}$ and completing the square, we find that

$$I = \frac{1}{\pi^N}\int dz_i d\bar{z}_i e^{-\sum z_k\bar{z}^k+\sum\bar{j}^k j_k}. \tag{264}$$

This is the usual Gaussian integral times some source terms. Evaluating the Gaussian integral thus leaves us with

$$I = e^{\sum_k \bar{j}^k j_k}. \tag{265}$$

Using this result we now find

$$\begin{aligned}(I_n)^{j_1\cdots j_n}_{i_1\cdots i_n} &= \frac{1}{n!}\frac{\delta}{\delta j_{i_1}}\cdots\frac{\delta}{\delta j_{i_n}}\frac{\delta}{\delta\bar{j}^{i_1}}\cdots\frac{\delta}{\delta\bar{j}^{i_n}}(\bar{j}^{k_1}\cdots\bar{j}^{k_n}j_{k_1}\cdots j_{k_n})e^{\sum_k\bar{j}^k j_k}\Big|_{j_i=\bar{j}_i=0}\\ &= \frac{1}{n!}\sum_{\sigma\epsilon S_n}\delta^{j_1}_{i_{\sigma(1)}}\cdots\delta^{j_1}_{i_{\sigma(n)}}\end{aligned} \tag{266}$$

Furthermore, observe that

$$\begin{aligned}(I_n)^{j_1\cdots j_n}_{i_1\cdots i_n}M^{i_1}_{j_1}M^{i_2}_{j_2}\cdots M^{i_n}_{j_n} &= \frac{1}{n!}\sum_{\sigma\epsilon S_n}\delta^{j_1}_{i_{\sigma(1)}}\cdots\delta^{j_1}_{i_{\sigma(n)}}M^{i_1}_{j_1}M^{i_2}_{j_2}\cdots M^{i_n}_{j_n}\\ &= \frac{1}{n!}\sum_{\sigma\epsilon S_n}M^{i_1}_{i_{\sigma(1)}}\cdots M^{i_n}_{i_{\sigma(n)}}\\ &= \frac{1}{n!}\sum_{\sigma\epsilon S_n}\mathrm{Tr}(\sigma M^{\otimes n})\\ &= \chi_R(M).\end{aligned} \tag{267}$$

Continuing with the proof, consider next the integral

$$Z = \frac{1}{\pi^N}\int\prod_{i=1}^{N} dx_i dy_i e^{-\sum_{i,j} z_i(\delta^i_j - tM^i_j)\bar{z}^j}. \tag{268}$$

Notice that we have a matrix in the exponent. We can make use of an identity (often used in quantum field theory) where a matrix is inserted in the exponent. The identity is

$$\frac{1}{\pi^N}\int\prod_{i=1}^{N} dx_i dy_i e^{-\sum_{i,j} z_i O^i_j\bar{z}^j} = \frac{1}{\det O}. \tag{269}$$

Thus, the above integral can also be written in terms of a determinant; more precisely,

$$Z = \frac{1}{\det(1 - tM)}. \tag{270}$$

By expansion, we can compute Z as follows

$$\begin{aligned} Z &= \frac{1}{\pi^N} \int \prod_{i=1}^{N} dx_i dy_i e^{-\sum_i z_i \bar{z}^i} \sum_{n=0}^{\infty} t^n M^{i_1}_{j_1} M^{i_2}_{j_2} \cdots M^{i_n}_{j_n} z_{i_1} \cdots z_{i_n} \bar{z}^{j_1} \cdots \bar{z}^{j_n} \\ &= \sum_n t^n \chi_{(n)}(M). \end{aligned} \tag{271}$$

Comparing equation (270) with equation (271) thus completes the proof.

In our case, we have

$$M = s^D x^{J_{3,L}} y^{J_{3,R}}. \tag{272}$$

In view of equation (200), we must have

$$\frac{1}{\det(1 - tM)} = \prod_{q=0}^{\infty} \prod_{a=-\frac{q}{2}}^{\frac{q}{2}} \prod_{b=-\frac{q}{2}}^{\frac{q}{2}} \frac{1}{1 - t s^{q+1} x^a y^b}. \tag{273}$$

We would like to decompose this result into representations of $SO(4,2)$. A useful approach to this can be found in [46].

To illustrate the idea, we first start with a simpler problem. Suppose we are given a partition function that is a sum of $SU(2)$ characters

$$Z(x) = \mathrm{Tr}\,(x^{J_3}) = \sum_j N_j \chi_j(x) = Z_0 + \sum_{k=1}^{\infty} Z_k (x^k + x^{-k}). \tag{274}$$

To obtain the last equality, we made use of the fact that all the $SU(2)$ characters have an inversions symmetry; in other words, the characters are invariant under $x \to 1/x$. We would like to determine the numbers N_j. To do this, we will make use of character orthogonality. Rewriting the $SU(2)$ character as

$$\chi_j(x) = \frac{\sin\left((j + \frac{1}{2})\theta\right)}{\sin\left(\frac{\theta}{2}\right)}, \tag{275}$$

we can prove the following orthogonality relations

$$\begin{aligned} &\int_0^{4\pi} d\theta \frac{\sin^2 \frac{\theta}{2}}{2\pi} \chi_j(x) \chi_k(x) = \delta_{jk}, \\ &\int_0^{4\pi} d\theta \frac{\sin^2 \frac{\theta}{2}}{2\pi} \chi_j(x) (x^k + x^{-k}) = \delta_{j,k} - \delta_{j+1,k}, \\ &\int_0^{4\pi} d\theta \frac{\sin^2 \frac{\theta}{2}}{2\pi} \chi_j(x) = \delta_{j,0}. \end{aligned} \tag{276}$$

Indeed, for positive integers j and k ($j, k \geq 0$),

$$\begin{aligned}
\int_0^{4\pi} d\theta \frac{\sin^2\frac{\theta}{2}}{2\pi}\chi_j(x)\chi_k(x) &= \int_0^{4\pi} \sin\frac{\sin^2\frac{\theta}{2}}{2\pi}\left(\frac{\sin((j+1/2)\theta)}{\sin\frac{\theta}{2}}\right)\left(\frac{\sin((k+1/2)\theta)}{\sin\frac{\theta}{2}}\right)\\
&= \int_0^{4\pi} d\theta \frac{\sin((j+1/2)\theta)}{2\pi}\sin((k+1/2)\theta)\\
&= \int_0^{4\pi} \frac{d\theta}{2\pi}\left(\frac{e^{i(j+\frac{1}{2})\theta}-e^{-i(j+\frac{1}{2})\theta}}{2i}\right)\left(\frac{e^{i(k+\frac{1}{2})\theta}-e^{-i(k+\frac{1}{2})\theta}}{2i}\right)\\
&= -\int_0^{4\pi} \frac{d\theta}{8\pi}\left(e^{i(j+k+1)\theta}-e^{i(j-k)\theta}-e^{i(k-j)\theta}+e^{-(j+k+1)\theta}\right)\\
&= -\frac{1}{8\pi}\left(4\pi\delta_{j+1,-k}-4\pi\delta_{j,k}-4\pi\delta_{j,k}+4\pi\delta_{-j-1,k}\right)\\
&= -\frac{1}{8\pi}\left(-4\pi\delta_{j,k}-4\pi\delta_{j,k}\right)\\
&= \delta_{j,k}, \qquad (277)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{4\pi} d\theta \frac{\sin^2(\frac{\theta}{2})}{2\pi}\chi_j(x)(x^k+x^{-k})\\
&= \int_0^{4\pi} d\theta \frac{\sin^2(\frac{\theta}{2})}{2\pi}\frac{\sin((j+1/2)\theta)}{\sin(\frac{\theta}{2})}(e^{ik\theta}+e^{-ik\theta})\\
&= \int_0^{4\pi}\frac{d\theta}{2\pi}\sin\frac{\theta}{2}\sin((j+\frac{1}{2})\theta)e^{ik\theta}+\int_0^{4\pi}\frac{d\theta}{2\pi}\sin\frac{\theta}{2}\sin((j+\frac{1}{2})\theta)e^{-ik\theta}\\
&= \int_0^{4\pi}\frac{d\theta}{2\pi}\left(\frac{e^{-i\theta/2}-e^{-i\theta/2}}{2i}\right)\left(\frac{e^{i(j+1/2)\theta}-e^{-i(j+1/2)\theta}}{2i}\right)e^{ik\theta}\\
&+\int_0^{4\pi}\frac{d\theta}{2\pi}\left(\frac{e^{-i\theta/2}-e^{-i\theta/2}}{2i}\right)\left(\frac{e^{i(j+1/2)\theta}-e^{-i(j+1/2)\theta}}{2i}\right)e^{-ik\theta}. \qquad (278)
\end{aligned}$$

Multiplying out the terms in the brackets, we obtain

$$\begin{aligned}
\int_0^{4\pi} d\theta \frac{\sin^2(\frac{\theta}{2})}{2\pi}\chi_j(x)(x^k+x^{-k}) &= -\int_0^{4\pi}\frac{d\theta}{8\pi}\left(e^{i(j+k+1)\theta}-e^{i(k-j)\theta}-e^{i(k+j)\theta}+e^{-i(j-k+1)\theta}\right)\\
&-\int_0^{4\pi}\frac{d\theta}{8\pi}\left(e^{i(j-k+1)\theta}-e^{-i(k+j)\theta}-e^{i(k-j)\theta}+e^{-i(j+k+1)\theta}\right)\\
&= -\frac{1}{8\pi}\left(4\pi\delta_{j+1,-k}-4\pi\delta_{j,k}-4\pi\delta_{j,-k}+4\pi\delta_{j+1,k}\right)\\
&-\frac{1}{8\pi}\left(4\pi\delta_{j+1,k}-4\pi\delta_{j,-k}-4\pi\delta_{j,k}+4\pi\delta_{j+k,-k}\right)\\
&= \delta_{j,k}-\delta_{j+1,k}. \qquad (279)
\end{aligned}$$

Finally,

$$\begin{aligned}
\int_0^{4\pi} d\theta \frac{\sin^2(\frac{\theta}{2})}{2\pi}\chi_j(x) &= \int_0^{4\pi} \frac{d\theta}{2\pi}\sin\frac{\theta}{2}\sin((j+1/2)\theta) \\
&= \frac{1}{2}\int_0^{4\pi}\frac{d\theta}{2\pi}\Big(\cos(j\theta)-\cos((j+1)\theta)\Big) \\
&= \delta_{j,0}.
\end{aligned} \tag{280}$$

Using these relations, it is simple to see that

$$\begin{aligned}
\int_0^{4\pi} d\theta \frac{\sin^2\frac{\theta}{2}}{2\pi}\chi_j(x)Z(x) &= \int_0^{4\pi} d\theta \frac{\sin^2\frac{\theta}{2}}{2\pi}\chi_j(x)\sum_k N_k\chi_k(x) \\
&= N_j,
\end{aligned} \tag{281}$$

and

$$\begin{aligned}
\int_0^{4\pi} d\theta \frac{\sin^2\frac{\theta}{2}}{2\pi}\chi_j(x)Z(x) &= \int_0^{4\pi} d\theta \frac{\sin^2\frac{\theta}{2}}{2\pi}\chi_j(x)\left(Z_0+\sum_{k=1}^{\infty} Z_k(x^k+x^{-k})\right) \\
&= Z_j - Z_{j+1} \qquad (j>0) \\
&= Z_0 \qquad (j=0).
\end{aligned} \tag{282}$$

Next, we define the generating function

$$G(x) = \sum_j x^j N_j, \tag{283}$$

and the "regular part of a function" as

$$\left[\sum_n a_n x^n\right]_{\geq} = \sum_{n=0}^{\infty} a_n x^n, \tag{284}$$

in other words, we discard all negative powers of x. Then, we can write

$$G(x) = \left[(1-\frac{1}{x})Z(x)\right]_{\geq}, \tag{285}$$

which is the result we wanted for the $SU(2)$ case.

Generating function for the bosonic field case. From the $SU(2)$ example, we can turn to the $SO(4,2)$ case. Here, the role of $Z(s,x,y)$ is played by $\chi_{\mathrm{Sym}(V^{\otimes n})}(s,x,y)$; more precisely,

$$Z(s,x,y) = \chi_{Sym(V^{\otimes n})}(s,x,y) = \sum_{\Delta,j_1,j_2} N_{[\Delta,j_1,j_2]}\chi_{[\Delta,j_1,j_2]}(s,x,y). \tag{286}$$

For now, we will restrict to the case that $n \geq 3$; this is due to the fact that the characters which contribute in the $n=2$ case saturates a unitarity bound and thus contain null states.

In other words, the theory for $n = 2$ has short multiplets. As we will see, naive application of (288) leads to smaller integer coefficients in the expansion of the generating function. In fact, these coefficients will be negative.

For $n \geq 3$ [46],

$$\chi_{[\Delta,j_1,j_2]}(s,x,y) = \frac{s^{\Delta}\chi_{j_1}(x)\chi_{j_2}(y)}{(1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})(1-s\sqrt{\frac{y}{x}})(1-\frac{s}{\sqrt{xy}})}, \tag{287}$$

and thus

$$Z(s,x,y)(1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})(1-s\sqrt{\frac{y}{x}})(1-\frac{s}{\sqrt{xy}}) = \sum_{\Delta,j_1,j_2} N_{[\Delta,j_1,j_2]} s^{\Delta}\chi_{j_1}(x)\chi_{j_2}(y). \tag{288}$$

Observe that the right hand side of this equation is precisely the sum of products of $SU(2)$ characters and, therefore, we can use the $SU(2)$ example above to derive the generating function. To achieve this, we rewrite equation (288) as follows (where we have shortened the left hand side to Z')

$$\begin{aligned} Z'(s,x,y) &= \sum_{\Delta,j_1,j_2} N_{[\Delta,j_1,j_2]} s^{\Delta}\chi_{j_1}(x)\chi_{j_2}(y) \\ &= \sum_{\Delta} s^{\Delta} \sum_{j_1} N_{[\Delta,j_1]}\chi_{j_1}(x) \sum_{j_2} N_{[\Delta,j_2]}\chi_{j_2}(y), \end{aligned} \tag{289}$$

where we relate these two lines through

$$N_{[\Delta,j_1]} \cdot N_{[\Delta,j_2]} = N_{[\Delta,j_1,j_2]}. \tag{290}$$

Continuing, we find that

$$Z'(s,x,y) = \sum_{\Delta} s^{\Delta} Z(x)Z(y), \tag{291}$$

where

$$\begin{aligned} Z(x) &= \sum_{j_1} N_{[\Delta,j_1]}\chi_{j_1}(x), \\ Z(y) &= \sum_{j_2} N_{[\Delta,j_2]}\chi_{j_2}(y). \end{aligned} \tag{292}$$

Now, the generating function for the $SO(4,2)$ version of equation (283) is defined by

$$G(s,x,y) = \sum_{n,j_1,j_2} N_{[n,j_1,j_2]} s^n x^{j_1} y^{j_2}. \tag{293}$$

Rewriting this equation in a similar manner allows us to apply equation (285) to each of the $SU(2)$ characters in our expression

$$\begin{aligned} G(s,x,y) &= \sum_{n,j_1,j_2} s^n x^{j_1} y^{j_2} \\ &= \sum_n s^n G(x) G(y) \\ &= \sum_n s^n \big[(1-\frac{1}{x})Z(x)\big]_{x\geq} \big[(1-\frac{1}{y})Z(y)\big]_{y\geq}, \end{aligned} \tag{294}$$

where the subscript on each bracket means we only consider the regular part of the function in the brackets with respect to x and y. From this, we find that

$$\begin{aligned} G(s,x,y) &= \big[(1-\frac{1}{x})(1-\frac{1}{y})\sum_n s^n Z(x)Z(y)\big]_{x\geq y\geq} \\ &= \big[(1-\frac{1}{x})(1-\frac{1}{y})Z'(s,x,y)\big]_{\geq}, \end{aligned} \tag{295}$$

Substituting the expression for $Z'(s,x,y)$ from the left hand side of equation (288) gives the result

$$G(s,x,y) = \left[(1-\frac{1}{x})(1-\frac{1}{y})Z(s,x,y)(1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})(1-s\sqrt{\frac{y}{x}})(1-\frac{s}{\sqrt{xy}})\right]_{\geq}, \tag{296}$$

which is the generating function we wanted to derive.

Thus, we can state our final result: the number $N_{[\Delta,j_1,j_2]}$ of primary operators, of dimension Δ and spin (j_1,j_2) built out of n scalar fields ϕ can be obtained by expanding the generating function

$$G_n(s,x,y) = \sum_{n,j_1,j_2} N_{[n,j_1,j_2]} s^n x^{j_1} y^{j_2}. \tag{297}$$

The generating function is given by (we emphasize here that, by our construction, $n \geq 3$)

$$G_n(s,x,y) = \left[(1-\frac{1}{x})(1-\frac{1}{y})Z_n(s,x,y)(1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})(1-s\sqrt{\frac{y}{x}})(1-\frac{s}{\sqrt{xy}})\right]_{\geq}, \tag{298}$$

where $Z_n(s,x,y)$ is defined by

$$\prod_{q=0}^{\infty}\prod_{a=-\frac{q}{2}}^{\frac{q}{2}}\prod_{b=-\frac{q}{2}}^{\frac{q}{2}} \frac{1}{1-ts^{q+1}x^a y^b} = \sum_{n=0}^{\infty} t^n Z_n(s,x,y). \tag{299}$$

Again, using MATHEMATICA, the generating function for $n = 3, 4$, and 5 thus yields

$$\begin{aligned}G_3(s,x,y) =& \, s^3 + s^5xy + s^6x^{\frac{3}{2}}y^{\frac{3}{2}} + s^7x^2y^2 + s^7x^2 + s^7y^2 + s^8x^{\frac{5}{2}}y^{\frac{5}{2}} + s^8x^{\frac{3}{2}}y^{\frac{5}{2}} + s^8x^{\frac{5}{2}}y^{\frac{3}{2}} \\ &+ 2s^9x^3y^3 + s^9xy^3 + s^9x^3y + s^{10}x^{\frac{7}{2}}y^{\frac{7}{2}} + s^{10}x^{\frac{7}{2}}y^{\frac{5}{2}} + s^{10}x^{\frac{5}{2}}y^{\frac{7}{2}} + s^{10}x^{\frac{7}{2}}y^{\frac{3}{2}} \\ &+ s^{10}x^{\frac{3}{2}}y^{\frac{7}{2}} + ...\end{aligned} \tag{300}$$

$$\begin{aligned}G_4(s,x,y) =& \, s^4 + s^6xy + s^7x^{\frac{3}{2}}y^{\frac{3}{2}} + 2s^8x^2y^2 + s^8x^2 + s^8xy + s^8y^2 + s^8 \\ &+ s^9x^{\frac{5}{2}}y^{\frac{5}{2}} + s^9x^{\frac{5}{2}}y^{\frac{3}{2}} + s^9x^{\frac{5}{2}}y^{\frac{1}{2}} + s^9x^{\frac{3}{2}}y^{\frac{5}{2}} + s^9x^{\frac{3}{2}}y^{\frac{1}{2}} + s^9x^{\frac{1}{2}}y^{\frac{5}{2}} + \\ &+ s^9x^{\frac{1}{2}}y^{\frac{3}{2}} + ...\end{aligned} \tag{301}$$

$$\begin{aligned}G_5(s,x,y) =& \, s^5 + s^7xy + s^8x^{\frac{3}{2}}y^{\frac{3}{2}} + 2s^9x^2y^2 + s^9x^2 + s^9y^2 + s^9xy + s^9 \\ &+ s^{10}x^{\frac{3}{2}}y^{\frac{3}{2}} + s^{10}x^{\frac{3}{2}}y^{\frac{5}{2}} + s^{10}x^{\frac{3}{2}}y^{\frac{1}{2}} + s^{10}x^{\frac{5}{2}}y^{\frac{3}{2}} + 2s^{10}x^{\frac{5}{2}}y^{\frac{5}{2}} \\ &+ s^{10}x^{\frac{5}{2}}y^{\frac{1}{2}} + s^{10}x^{\frac{1}{2}}y^{\frac{3}{2}} + s^{10}x^{\frac{1}{2}}y^{\frac{5}{2}} + s^{10}x^{\frac{1}{2}}y^{\frac{1}{2}} + ...\end{aligned} \tag{302}$$

Comparing these expression to our answers in the previous section, we do indeed find agreement.

So far, we have excluded the $n = 2$ case. If we naively compute $G_2(s, x, y)$, then we obtain the following terms

$$G_2(s,x,y) = s^2 + s^4xy - s^5\sqrt{x}\sqrt{y} + s^6x^2y^2 - s^7x^{3/2}y^{3/2} + ... \tag{303}$$

Observe here that we have negative coefficients. This is due to the fact that we have null states which have not been subtracted correctly. From [46], the condition for a short multiplet is $\Delta = f(j_1) + f(j_2)$ with $f(j) = 0$ if $j = 0$ or $f(j) = j + 1$ if $j > 0$. Observe that there is a primary with $\Delta = 2$ and $j_1 = j_2 = 0$, which is not short. We also have a primary with $\Delta = 4$ and $j_1 = j_2 = 1$. This is short and the $-s^5\sqrt{x}\sqrt{y}$ is precisely the null state for this primary. Furthermore, it is clear that we also have a primary with $\Delta = 6$ and $j_1 = j_2 = 2$. This is also short and the $-s^7x^{3/2}y^{3/2}$ is precisely the null state for this primary. Removing the $\Delta = 2$ and $j_1 = j_2 = 0$ primary, subtracting the null states (this is done by dividing by $1 - s/\sqrt{xy}$) and then putting the original primary back in we find the correct generating function for $n = 2$

$$G_2(s,x,y) = \left[\left(\frac{(1-\frac{1}{x})(1-\frac{1}{y})Z_2(s,x,y)}{P(s,x,y)} - s^2\right)\frac{1}{1-\frac{s}{\sqrt{xy}}}\right]_{\geq}, \tag{304}$$

$$P(s,x,y) = \frac{1}{(1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})(1-s\sqrt{\frac{y}{x}})(1-\frac{s}{\sqrt{xy}})}. \tag{305}$$

Indeed, this generating function yields

$$G_2(s,x,y) = s^2 + s^4xy + s^6x^2y^2 + s^8x^3y^3 + s^{10}x^4y^4 + s^{12}x^5y^5 + ... \tag{306}$$

which is the correct result.

Generating function for the free fermion field case. We can extend these results to the case of fermion fields. Here, the fields are Grassman fields, and are thus antisymmetric when exchanged (in other words, they change sign under exchange). Consequently, we need to project to the antisymmetric product of representations.

First, we need to adapt our Schur polyomials that expresses our characters. The formula is exactly the same as in equation (225), except that permuting indices in a term an odd number of times will change the sign of the term. For example, consider the matrix

$$M = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}. \tag{307}$$

Then, the character in the $\substack{\Box\\\Box\\\Box}$ representation is given by

$$\begin{aligned}
\chi_{\substack{\Box\\\Box\\\Box}}(M) =& \frac{1}{6}\sum_{\sigma\in S_3} \chi_{\substack{\Box\\\Box\\\Box}}(\sigma) M^{i_1}_{i_{\sigma(1)}} M^{i_2}_{i_{\sigma(2)}} M^{i_3}_{i_{\sigma(3)}} \\
&= \frac{1}{6}\big[\chi_{\substack{\Box\\\Box\\\Box}}(1) M^{i_1}_{i_1} M^{i_2}_{i_2} M^{i_3}_{i_3} - \chi_{\substack{\Box\\\Box\\\Box}}((12)) M^{i_1}_{i_2} M^{i_2}_{i_1} M^{i_3}_{i_3} \\
&- \chi_{\substack{\Box\\\Box\\\Box}}((13)) M^{i_1}_{i_3} M^{i_2}_{i_2} M^{i_3}_{i_1} - \chi_{\substack{\Box\\\Box\\\Box}}((23)) M^{i_1}_{i_1} M^{i_2}_{i_3} M^{i_3}_{i_2} \\
&+ \chi_{\substack{\Box\\\Box\\\Box}}((123)) M^{i_1}_{i_3} M^{i_2}_{i_1} M^{i_3}_{i_2} + \chi_{\substack{\Box\\\Box\\\Box}}((132)) M^{i_1}_{i_2} M^{i_2}_{i_3} M^{i_3}_{i_1}\big] \\
&= \frac{1}{6}\big[(Tr(M))^3 - 3Tr(M^2)Tr(M) + 2Tr(M^3)\big] \\
&= abc.
\end{aligned} \tag{308}$$

Similarly,

$$\chi_{\Box}(M) = a + b + c, \tag{309}$$

$$\chi_{\substack{\Box\\\Box}}(M) = ab + ac + bc. \tag{310}$$

As a short notation, we will use 1^n to denote a Young diagram with a single column of n boxes. Then, we have the following relation

$$\det(1 + tM) = (1 + ta)(1 + tb)(1 + tc) = \sum_{n=0}^{\infty} t^n \chi_{(1^n)}(M). \tag{311}$$

We can argue this formula by making use of the following integral

$$\int \prod_{i=1}^{N} d\psi_i d\bar{\psi}^i e^{-\sum_{i,j}\psi_i(\delta^i_j + tM^i_j)\bar{\psi}^j}. \tag{312}$$

We emphasize here that the fields are Grassman variables; in other words $\psi_i\psi_j = -\psi_j\psi_i$. First, consider the $N = 3$ case. We rewrite the summation in the exponent to matrix multiplication

$$\begin{aligned}\sum_{i,j=1}^{3}\psi_i(\delta^i_j + tM^i_j)\bar{\psi}^j &= \begin{bmatrix}\psi_1 & \psi_2 & \psi_3\end{bmatrix}\left(\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} + t\begin{bmatrix}a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{bmatrix}\right)\begin{bmatrix}\bar{\psi}_1\\ \bar{\psi}_2\\ \bar{\psi}_3\end{bmatrix}\\ &= \psi(\mathbf{1} + tM)\bar{\psi}.\end{aligned} \tag{313}$$

We then expand the exponential as follows,

$$\begin{aligned}\int \prod_{i=1}^{3} d\psi_i d\psi^i e^{-\sum_{i,j=1}^{N}\psi_i(\delta^i_j + tM^i_j)\bar{\psi}^j} &= \int d\psi_1 d\bar{\psi}_1 d\psi_2 d\bar{\psi}_2 d\psi_3 d\bar{\psi}_3\Big(1 - \psi(\mathbf{1} + tM)\bar{\psi}\\ &+\frac{1}{2!}(\psi(\mathbf{1} + tM)\bar{\psi})^2 - \frac{1}{3!}(\psi(\mathbf{1} + tM)\bar{\psi})^3 + \cdots\Big).\end{aligned} \tag{314}$$

Here, only the fourth term yields a non-zero integral. Thus,

$$\begin{aligned}&\int \prod_{i=1}^{N} d\psi_i d\psi^i e^{-\sum_{i,j=1}^{N}\psi_i(\delta^i_j + tM^i_j)\bar{\psi}^j}\\ &= -\int d\psi_1 d\bar{\psi}_1 d\psi_2 d\bar{\psi}_2 d\psi_3 d\bar{\psi}_3 \frac{1}{3!}(\psi(\mathbf{1} + tM)\bar{\psi})^3\\ &= -\int d\psi_1 d\bar{\psi}_1 d\psi_2 d\bar{\psi}_2 d\psi_3 d\bar{\psi}_3 \frac{1}{3!}\Big(\psi_1(1+ta)\bar{\psi}_1 + \psi_2(1+tb)\bar{\psi}_2 + \psi_3(1+tb)\bar{\psi}_3\Big)^3\\ &= -\int d\psi_1 d\bar{\psi}_1 d\psi_2 d\bar{\psi}_2 d\psi_3 d\bar{\psi}_3 \Big((1+ta)(1+tb)(1+tc)\Big)\psi_1\bar{\psi}_1\psi_2\bar{\psi}_2\psi_3\bar{\psi}_3\\ &= (1+ta)(1+tb)(1+tc)\\ &= 1 + (a+b+c)t + (ab+ac+bc)t^2 + (abc)t^3\\ &= \sum_{n=0}^{\infty} t^n \chi_{(1^n)}(M)\\ &= \det(1+tM).\end{aligned} \tag{315}$$

In general, only the term of the form $-\frac{1}{N!}(\psi(\mathbf{1}+tM)\bar{\psi})^N$ would contribute. Thus, this result generalizes. Observe also that the series above only expands to $n = 3$; in other words, for $n > 3$, the character χ_{1^n} vanishes. This results from the fact that the trace terms in the character formula for χ_{1^n} are no longer independent . This occurs precisely when the rank of the matrix (here, the rank is 3) is less than the number of matrices appearing in each trace term in the Schur polynomial (see [47]).

From here, we can derive the generating function for the free fermion. From [44], the character of a left-handed Weyl Fermion is given by

$$\chi_L(s,x,y) = s^{\frac{3}{2}}(\chi_{\frac{1}{2}}(x) - s\chi_{\frac{1}{2}}(y))P(s,x,y), \tag{316}$$

which can be rewritten as

$$s^{3/2}\big(\chi_{\frac{1}{2}}(x) - s\chi_{\frac{1}{2}}(y)\big)P(s,x,y) = s^{3/2}\big(\chi_{\frac{1}{2}}(x) - s\chi_{\frac{1}{2}}(y)\big)\sum_{p,q=0}^{\infty} s^{2p+q}\chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}}(x). \tag{317}$$

Now, using the equation

$$\chi_{j\otimes j'}(x) = \chi_j(x)\chi_{j'}(x) = \sum_{k=|j-j'|}^{j+j'} \chi_k(x), \tag{318}$$

we obtain

$$\begin{aligned}\chi_L(s,x,y) &= s^{3/2}\sum_{p,q=0}^{\infty} s^{2p+q}\Bigg(\sum_{k=|\frac{q}{2}-\frac{1}{2}|}^{\frac{q}{2}+\frac{1}{2}} \chi_k(x)\chi_{\frac{q}{2}}(y) - s\sum_{k=|\frac{q}{2}-\frac{1}{2}|}^{\frac{q}{2}+\frac{1}{2}} \chi_{\frac{q}{2}}(x)\chi_k(y)\Bigg)\\ &= s^{\frac{3}{2}}\sum_{p,q=0}^{\infty} s^{2p+q}\Big(\chi_{\frac{q}{2}-\frac{1}{2}}(x)\chi_{\frac{q}{2}}(y) + \chi_{\frac{q}{2}+\frac{1}{2}}(x)\chi_{\frac{q}{2}}(y) - s\chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}-\frac{1}{2}}(y) - s\chi_{\frac{q}{2}}(x)\chi_{\frac{q}{2}+\frac{1}{2}}(y)\Big).\end{aligned} \tag{319}$$

Observe that by shifting $q \to 1$ in the first term and the third term, the first term cancels with the last term. Consequently, the second and third terms simplify into

$$\chi_L(s,x,y) = s^{3/2}(1-s^2)\sum_{p,q=0}^{\infty} s^{2p+q}\chi_{\frac{q}{2}+\frac{1}{2}}(x)\chi_{\frac{q}{2}}(y). \tag{320}$$

Summing over the s^p terms then yields

$$\chi_L(s,x,y) = s^{\frac{3}{2}}\sum_{q=0}^{\infty} s^q\chi_{\frac{q+1}{2}}(x)\chi_{\frac{q}{2}}(y). \tag{321}$$

For our case, $M = s^D x^{J_{3,L}} y^{J_{3,R}}$ and therefore

$$\det(1+tM) = \prod_{t=0}^{\infty}\prod_{a=-\frac{q+1}{2}}^{\frac{q+1}{2}}\prod_{b=-\frac{q}{2}}^{\frac{q}{2}}(1+ts^{\frac{3}{2}+q}x^a y^b). \tag{322}$$

The generating function for primary operators in the free fermion conformal field theory is given by

$$G_n(s,x,y) = \left[(1-\frac{1}{x})(1-\frac{1}{y})Z_n(s,x,y)(1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})(1-s\sqrt{\frac{y}{x}})(1-\frac{s}{\sqrt{xy}})\right]_{\geq}, \tag{323}$$

where now $Z_n(s,x,y)$ is defined by

$$\det(1+tM)=\prod_{t=0}^{\infty}\prod_{a=-\frac{q+1}{2}}^{\frac{q+1}{2}}\prod_{b=-\frac{q}{2}}^{\frac{q}{2}}(1+ts^{\frac{3}{2}+q}x^a y^b)=\sum_{n=0}^{\infty}t^n Z_n(s,x,y). \tag{324}$$

As an example, the generating function for the $n=3$ case yields (see Appendix (A.4))

$$G_3(s,x,y)=s^{\frac{11}{2}}x\sqrt{y}+s^{\frac{13}{2}}x^{\frac{5}{2}}+s^{\frac{15}{2}}y^{\frac{3}{2}}+s^{\frac{15}{2}}x^3y^{\frac{3}{2}}+... \tag{325}$$

We can compare this with the approach in the previous sections. Here, the relevant Schur polynomial is given by

$$\begin{aligned}\chi_{asym(V^{\otimes 3})}(s,x,y)=&\ \frac{1}{6}\big[(\chi_L(s,x,y))^3-3\chi_L(s^2,x^2,y^2)\chi_L(s,x,y)\\&+2\chi_L(s^3,x^3,y^3)\big].\end{aligned} \tag{326}$$

Next, we adapt the equations from Section (4.7)

$$\begin{aligned}\chi_L(s^n,x^n,y^n)=&\ s^{3n/2}\sum_{q=0}^{\infty}s^{nq}\chi_{\frac{(q+1)}{2}}(x^n)\chi_{\frac{q}{2}}(y^n)\\&=s^{3n/2}\sum_{q=0}^{\infty}s^{nq}\left[\sum_{l=0,1,...}^{\lfloor(q+1)/2\rfloor}\chi_{\frac{(q+1)n}{2}-nl}(x)-\sum_{l=0,1,...}^{\lfloor q/2\rfloor}\chi_{\frac{(q+1)n}{2}-nl-1}(x)\right]\\&\times\left[\sum_{l=0,1,...}^{\lfloor q/2\rfloor}\chi_{\frac{qn}{2}-nl}(y)-\sum_{l=0,1,...}^{\lfloor(q-1)/2\rfloor}\chi_{\frac{qn}{2}-nl-1}(y)\right].\end{aligned} \tag{327}$$

As before, we need to add a factor of $P(s,x,y)$. Using equation (247), we rewrite the above expression as

$$\begin{aligned}\chi_L(s^n,x^n,y^n)=&\ \chi_L(s^n,x^n,y^n)(1+s^4-s(1+s^2)\chi_{\frac{1}{2}}(x)\chi_{\frac{1}{2}}(y)\\&+s^2(\chi_1(x)+\chi_1(y)))P(s,x,y)\\&=\chi_L(s^n,x^n,y^n)(1+s^4)-s(1+s^2)s^{3n/2}\sum_{q=0}^{\infty}s^{nq}\times\\&\Big[\sum_{l=0,1,...}^{\lfloor(q+1)/2\rfloor}(\chi_{\frac{(q+1)n}{2}-nl-1/2}(x)+\chi_{\frac{(q+1)n}{2}-nl+1/2}(x))\\&-\sum_{l=0,1,...}^{\lfloor q/2\rfloor}(\chi_{\frac{(q+1)n}{2}-nl-3/2}(x)+\chi_{\frac{(q+1)n}{2}-nl-1/2}(x))\Big]\\&\times\Big[\sum_{l=0,1,...}^{\lfloor q/2\rfloor}(\chi_{\frac{qn}{2}-nl-1/2}(y)+\chi_{\frac{qn}{2}-nl+1/2}(y))\end{aligned}$$

$$
\begin{aligned}
&- \sum_{l=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} (\chi_{\frac{qn}{2}-nl-3/2}(y) + \chi_{\frac{qn}{2}-nl-1/2}(y))\Big] \\
&+s^2 s^{3n/2} \sum_{q=0}^{\infty} s^{nq} \times \\
&\Big[\sum_{l=0,1,\ldots}^{\lfloor (q+1)/2 \rfloor} (\chi_{\frac{(q+1)n}{2}-nl-1}(x) + \chi_{\frac{(q+1)n}{2}-nl}(x) + \chi_{\frac{(q+1)n}{2}-nl+1}(x)) \\
&- \sum_{l=0,1,\ldots}^{\lfloor q/2 \rfloor} (\chi_{\frac{(q+1)n}{2}-nl-2}(x) + \chi_{\frac{(q+1)n}{2}-nl-1}(x) + \chi_{\frac{(q+1)n}{2}-nl}(x))\Big] \\
&\quad \times \Big[\sum_{l=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{qn}{2}-nl}(y) - \sum_{l=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} \chi_{\frac{qn}{2}-nl-1}(y)\Big] \\
&+s^2 s^{3n/2} \sum_{q=0}^{\infty} s^{nq} \times \\
&\Big[\sum_{l=0,1,\ldots}^{\lfloor (q+1)/2 \rfloor} \chi_{\frac{(q+1)n}{2}-nl}(x) - \sum_{l=0,1,\ldots}^{\lfloor q/2 \rfloor} \chi_{\frac{(q+1)n}{2}-nl-1}(x)\Big] \\
&\quad \times \Big[\sum_{l=0,1,\ldots}^{\lfloor q/2 \rfloor} (\chi_{\frac{qn}{2}-nl-1}(y) + \chi_{\frac{qn}{2}-nl}(y) + \chi_{\frac{qn}{2}-nl+1}(y)) \\
&- \sum_{l=0,1,\ldots}^{\lfloor (q-1)/2 \rfloor} (\chi_{\frac{qn}{2}-nl-2}(y) + \chi_{\frac{qn}{2}-nl-1}(y) + \chi_{\frac{qn}{2}-nl}(y))\Big].
\end{aligned}
\tag{328}
$$

Using similar arguments, one can derive the other terms in the polynomial. Using MATHEMATICA, we find

$$
\begin{aligned}
\chi_{asym(V^{\otimes 3})}(s,x,y) = & \mathcal{A}_{[\frac{11}{2},1,\frac{1}{2}]} + \mathcal{A}_{[\frac{13}{2},\frac{5}{2},0]} + \mathcal{A}_{[\frac{13}{2},\frac{3}{2},1]} \\
& + \mathcal{A}_{[\frac{15}{2},0,\frac{3}{2}]} + \mathcal{A}_{[\frac{15}{2},2,\frac{3}{2}]} + \mathcal{A}_{[\frac{15}{2},3,\frac{3}{2}]} \\
& + \mathcal{A}_{[\frac{17}{2},\frac{7}{2},1]} + \mathcal{A}_{[\frac{17}{2},\frac{3}{2},2]} + \mathcal{A}_{[\frac{17}{2},\frac{5}{2},2]} \\
& + \mathcal{A}_{[\frac{19}{2},4,\frac{3}{2}]} + + \mathcal{A}_{[\frac{19}{2},1,\frac{5}{2}]} + 2\mathcal{A}_{[\frac{19}{2},3,\frac{5}{2}]} + \mathcal{A}_{[\frac{19}{2},4,\frac{5}{2}]} \\
& + \mathcal{A}_{[\frac{21}{2},\frac{9}{2},0]} + \mathcal{A}_{[\frac{21}{2},\frac{9}{2},2]} + \mathcal{A}_{[\frac{21}{2},\frac{3}{2},3]} + \mathcal{A}_{[\frac{21}{2},\frac{5}{2},3]} + \mathcal{A}_{[\frac{21}{2},\frac{7}{2},3]} \\
& + \mathcal{A}_{[\frac{21}{2},\frac{9}{2},3]} + \ldots.
\end{aligned}
\tag{329}
$$

In conclusion, we have derived a generating function from which we can extract the spectrum of primaries. This is clearly a more efficient method than our previous algorithm for counting primaries.

5 Constructing Higher Spin Currents/ Primaries

In this section, we will develop methods to construct primary operators. First, we will construct symmetric traceless primary operators using an ansatz. After this, we will show how we can construct primaries by rephrasing the problem in the language of polynomials. In some cases, these polynomials turn out to be the hyperspherical Gegenbauer polynomials. Finally, we will use a many-body wavefunction problem to solve for primaries, which will be more general in form than the approach using Gegenbauer polynomials.

5.1 Spin-2 Primary Operator

We start by constructing the symmetric traceless primary built out of two fields. More precisely, we are constructing primaries that are in the (l, l) representations of $SO(4)$, which corresponds to symmetric tensors. Furthermore, these representations are traceless. Thus, to construct the primary, we must subtract the trace. To see a concrete example of this, consider the following decomposition

$$(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2}) = (0, 0) \oplus (1, 0) \oplus (0, 1) \oplus (1, 1). \tag{330}$$

The representation $(1, 0) \oplus (0, 1)$ corresponds to an antisymmetric tensor. Thus, we are left with the representation $(0, 0) \oplus (1, 1)$. The representation $(1, 1)$ is a spin-2 tensor. The representation $(0, 0)$ is a trace (in other words, a scalar) and thus does not transform under $SO(4)$. To construct the irrep $(1, 1)$ (which is the type of primary operators we are considering in this section), we must construct a tensor that is symmetric in its two indices and we must subtract the trace. For this example, this tensor is precisely the stress-energy tensor (8).

The ansatz for the four field primary operator with two indices is given by

$$\begin{aligned}\mathcal{O}_{\nu_1\nu_2} &= c_1(\partial_{\nu_1}\phi\partial_{\nu_2}\phi - \frac{1}{4}\eta_{\nu_1\nu_2}\partial^\mu\phi\partial_\mu\phi)\phi^2 \\ &+ c_2(\partial_{\nu_1}\partial_{\nu_2}\phi)\phi^3.\end{aligned} \tag{331}$$

This operator is indeed traceless since

$$\begin{aligned}\eta^{\nu_1\nu_2}\mathcal{O}_{\nu_1\nu_2} &= c_1(\partial^\mu\phi\partial_\mu\phi - \partial^\mu\phi\partial_\mu\phi)\phi^2 \\ &+ c_2(\partial^\mu\partial_\mu\phi)\phi^3 \\ &= 0,\end{aligned} \tag{332}$$

where we used the fact that $\eta^{\mu\nu}\eta_{\mu\nu} = \delta^{\mu}_{\mu} = 4$ and the equation of motion $\partial^{\nu_1}\partial_{\nu_1}\phi = 0$. In addition, the operator is also symmetric since

$$\begin{aligned}\mathcal{O}_{\nu_2\nu_1} &= c_1(\partial_{\nu_2}\phi\partial_{\nu_1}\phi - \frac{1}{4}\eta_{\nu_2\nu_1}\partial^{\mu}\phi\partial_{\mu}\phi)\phi^2 \\ &+c_2(\partial_{\nu_2}\partial_{\nu_1}\phi)\phi^3 \\ &= c_1(\partial_{\nu_1}\phi\partial_{\nu_2}\phi - \frac{1}{4}\eta_{\nu_1\nu_2}\partial^{\mu}\phi\partial_{\mu}\phi)\phi^2 \\ &+c_2(\partial_{\nu_1}\partial_{\nu_2}\phi)\phi^3 \\ &= \mathcal{O}_{\nu_1\nu_2}.\end{aligned} \tag{333}$$

To show that this operator is primary, we use the special conformal operator K_μ and solve the equation $K_\mu\mathcal{O}_{\nu_1\nu_2} = 0$ for the coefficients c_1 and c_2. We make use of the following commutation relations

$$\begin{aligned}&[P_\mu, K_\nu] = -i(2\eta_{\mu\nu}D + 2M_{\mu\nu}), \\ &[M_{\mu\nu}, P_\alpha] = -i(\eta_{\nu\alpha}P_\mu - \eta_{\mu\alpha}P_\nu), \\ &[D, P_\mu] = -iP_\mu.\end{aligned} \tag{334}$$

We also use

$$D\phi = -i\phi, \qquad K_\mu\phi = 0, \qquad M_{\mu\nu}\phi = 0. \tag{335}$$

Using the equivalent expression in terms of momentum operators $\partial_u \longrightarrow P_\mu$, we therefore obtain

$$\begin{aligned}K_\mu\mathcal{O}_{\nu_1\nu_2} &= c_1(K_\mu P_{\nu_1}\phi P_{\nu_2}\phi + P_{\nu_1}\phi K_\mu P_{\nu_2}\phi - \frac{1}{4}\eta_{\mu_1\mu_2}K_\mu P^\sigma\phi P_\sigma\phi \\ &-\frac{1}{4}\eta_{\nu_1\nu_2}P^\sigma\phi K_\mu P_\sigma\phi)\phi^2 + c_2(K_\mu P_{\nu_1}P_{\nu_2}\phi)\phi^3.\end{aligned} \tag{336}$$

Now,

$$\begin{aligned}K_\mu P_{\nu_1} &= P_{\nu_1}K_\mu + 2i\eta_{\nu_1\mu}D + 2iM_{\nu_1\mu}, \\ K_\mu P_{\nu_2} &= P_{\nu_2}K_\mu + 2i\eta_{\nu_2\mu}D + 2iM_{\nu_2\mu}.\end{aligned} \tag{337}$$

Similarly,

$$\begin{aligned}K_\mu P^\sigma &= \eta^{\sigma\rho}K_\mu P_\rho = \eta^{\sigma\rho}(P_\rho K_\mu + 2i\eta_{\rho\mu}D + 2iM_{\rho\mu}) \\ &= P^\sigma K_\mu + 2i\delta^\sigma_\mu D + 2i\eta^{\sigma\rho}M_{\rho\mu}.\end{aligned} \tag{338}$$

Finally,

$$\begin{aligned}K_\mu P_{\nu_1}P_{\nu_2}\phi &= (P_{\nu_1}K_\mu + 2i\eta_{\nu_1\mu}D + 2iM_{\nu_1\mu})P_{\nu_2}\phi \\ &= P_{\nu_1}K_\mu P_{\nu_2}\phi + 2i\eta_{\nu_1\mu}DP_{\nu_2}\phi + 2iM_{\nu_1\mu}P_{\nu_2}\phi \\ &= P_{\nu_1}(P_{\nu_2}K_\mu + 2i\eta_{\nu_2\mu}D + 2iM_{\nu_2\mu})\phi + 2i\eta_{\nu_1\mu}(P_{\nu_2}D - iP_{\nu_2})\phi \\ &+2i(P_{\nu_2}M_{\nu_1\mu} - i\eta_{\mu\nu_2}P_{\nu_1} + i\eta_{\nu_1\nu_2}P_\mu)\phi \\ &= 4\eta_{\nu_2\mu}P_{\nu_1}\phi + 4\eta_{\nu_1\mu}P_{\nu_2}\phi - 2\eta_{\nu_1\nu_2}P_\mu\phi.\end{aligned} \tag{339}$$

Using these results, we find that

$$\begin{aligned}K_\mu \mathcal{O}_{\nu_1\nu_2} =& c_1(2\eta_{\nu_1\mu}\phi P_{\nu_2}\phi + 2\eta_{\nu_2\mu}P_{\nu_1}\phi\phi - \frac{1}{2}\eta_{\nu_1\nu_2}\delta^\sigma_\mu \phi P_\sigma\phi - \frac{1}{2}\eta_{\nu_1\nu_2}P^\sigma\phi\eta_{\sigma\mu}\phi)\phi^2 \\ &+c_2(4\eta_{\nu_2\mu}P_{\nu_1}\phi + 4\eta_{\nu_1\mu}P_{\nu_2}\phi - 2\eta_{\nu_1\nu_2}P_\mu\phi)\phi^3 \\ &= c_1(2\eta_{\nu_1\mu}\phi P_{\nu_2}\phi + 2\eta_{\nu_2\mu}P_{\nu_1}\phi\phi - \frac{1}{2}\eta_{\nu_1\nu_2}\phi P_\mu\phi - \frac{1}{2}\eta_{\nu_1\nu_2}P_\mu\phi\phi)\phi^2 \\ &+c_2(4\eta_{\nu_2\mu}P_{\nu_1}\phi + 4\eta_{\nu_1\mu}P_{\nu_2}\phi - 2\eta_{\nu_1\nu_2}P_\mu\phi)\phi^3 \\ &= c_1(2\eta_{\nu_1\mu}\phi P_{\nu_2}\phi + 2\eta_{\nu_2\mu}P_{\nu_1}\phi\phi - \eta_{\nu_1\nu_2}P_\mu\phi\phi)\phi^2 \\ &+c_2(4\eta_{\nu_2\mu}P_{\nu_1}\phi + 4\eta_{\nu_1\mu}P_{\nu_2}\phi - 2\eta_{\nu_1\nu_2}P_\mu\phi)\phi^3.\end{aligned} \tag{340}$$

Setting this equation equal to zero, we find the solution $c_2 = -1/2c_1$. Choosing $c_1 = 1$, we have therefore determined that the operator

$$\mathcal{O}_{\nu_1\nu_2} = (\partial_{\nu_1}\phi\partial_{\nu_2}\phi - \frac{1}{4}\eta_{\nu_1\nu_2}\partial^\mu\phi\partial_\mu\phi)\phi^2 - \frac{1}{2}(\partial_{\nu_1}\partial_{\nu_2}\phi)\phi^3 \tag{341}$$

is a primary operator.

Remark 5. *In the beginning of this argument, we assumed the equation of motion holds ie* $\partial^\mu\partial_\mu\phi = 0$. *Interestingly, we still obtain the same coefficients even if this assumption does not hold. In this case, our ansatz for the traceless two index primary operator will be given by*

$$\begin{aligned}\mathcal{O}_{\nu_1\nu_2} =& c_1(\partial_{\nu_1}\phi\partial_{\nu_2}\phi - \frac{1}{4}\eta_{\nu_1\nu_2}\partial^\mu\phi\partial_\mu\phi)\phi^2 \\ &+c_2(\partial_{\nu_1}\partial_{\nu_2}\phi - \frac{1}{4}\eta_{\nu_1\nu_2}\partial^\mu\partial_\mu\phi)\phi^3.\end{aligned} \tag{342}$$

Acting with K_μ *on this new term gives*

$$\begin{aligned}K_\mu P^\alpha P_\alpha\phi =& [K_\mu, P_\alpha]P^\alpha\phi + P^\alpha[K_\mu, P_\alpha]\phi \\ &= i(2\eta_{\mu\alpha}D + 2M_{\alpha\mu})P^\alpha\phi + iP^\alpha(2\eta_{\mu\alpha}D + 2M_{\alpha\mu})\phi \\ &= 4P_\mu\phi + 2i[M_{\alpha\mu}, P^\alpha]\phi + 2P_\mu\phi \\ &= 6P_\mu\phi + 2(P_\mu - 4P_\mu)\phi \\ &= 0.\end{aligned} \tag{343}$$

Thus, the coefficients remain unchanged even if we were to assume that the equation of motion does not hold. This is also true for the spin 3 *primary operator in the next section.*

5.2 Spin-3 Primary Operator

The ansatz for the operator with three indices is given by

$$\begin{aligned}\mathcal{O}_{\nu_1\nu_2\nu_3} =& c_1(\partial_{\nu_1}\phi\partial_{\nu_2}\phi\partial_{\nu_3}\phi - \text{traces})\phi + c_2(\partial_{\nu_1}\partial_{\nu_2}\phi\partial_{\nu_3}\phi + \partial_{\nu_1}\partial_{\nu_3}\phi\partial_{\nu_2}\phi \\ &+\partial_{\nu_2}\partial_{\nu_3}\phi\partial_{\nu_1}\phi - \text{traces})\phi^2 + c_3(\partial_{\nu_1}\partial_{\nu_2}\partial_{\nu_3}\phi)\phi^3.\end{aligned} \tag{344}$$

To write this operator out in full detail, we must first determine the traces. More precisely, we need to ensure that

$$\eta^{\nu_1\nu_2}\mathcal{O}_{\nu_1\nu_2\nu_3} = \eta^{\nu_2\nu_3}\mathcal{O}_{\nu_1\nu_2\nu_3} = \eta^{\nu_3\nu_1}\mathcal{O}_{\nu_1\nu_2\nu_3} = 0. \tag{345}$$

Consider the first term in the above expression. Choosing any of the combinations for indices in the metric in equation (345) will result in raising an index. We must thus subtract three terms to cancel this:

$$\partial_{\nu_1}\phi\partial_{\nu_2}\phi\partial_{\nu_3}\phi - \frac{1}{4}\eta_{\nu_1\nu_2}\partial^\mu\phi\partial_\mu\phi\partial_{\nu_3} - \frac{1}{4}\eta_{\nu_2\nu_3}\partial_{\nu_1}\phi\partial^\mu\phi\partial_\mu\phi - \frac{1}{4}\eta_{\nu_1\nu_3}\partial^\mu\phi\partial_{\nu_2}\phi\partial_\mu\phi. \tag{346}$$

However, if we act with any of the combinations of metric given in equation (345) on equation (346), we are left with a nonzero term. For example, using $\eta^{\nu_1\nu_2}$ reduces equation (346) to

$$\begin{aligned}&\eta^{\nu_1\nu_2}(\partial_{\nu_1}\phi\partial_{\nu_2}\phi\partial_{\nu_3}\phi - \frac{1}{4}\eta_{\nu_1\nu_2}\partial^\mu\phi\partial_\mu\phi\partial_{\nu_3}\phi - \frac{1}{4}\eta_{\nu_2\nu_3}\partial_{\nu_1}\phi\partial^\mu\phi\partial_\mu\phi - \frac{1}{4}\eta_{\nu_1\nu_3}\partial^\mu\phi\partial_{\nu_2}\phi\partial_\mu\phi)\\ &= -\frac{1}{2}\partial_{\nu_3}\phi\partial^\mu\phi\partial_\mu\phi.\end{aligned} \tag{347}$$

Thus, to account for this, we need to again add three new terms. This will consequently lead to an infinite number of terms to add in order to compensate for each different combination of metric indices. In other words, we must have

$$\begin{aligned}&\partial_{\nu_1}\phi\partial_{\nu_2}\phi\partial_{\nu_3}\phi - (\sum_{n=2}^{\infty}\frac{(-1)^n}{2^n})\partial^\mu\phi\partial_\mu\phi\partial_{\nu_3}\phi - (\sum_{n=2}^{\infty}\frac{(-1)^n}{2^n})\partial_{\nu_1}\phi\partial^\mu\phi\partial_\mu\phi\\ &-(\sum_{n=2}^{\infty}\frac{(-1)^n}{2^n})\partial^\mu\phi\partial_{\nu_2}\phi\partial_\mu\phi.\end{aligned} \tag{348}$$

The series in the brackets is well known to converge

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n} = \frac{2}{3} \Rightarrow \sum_{n=2}^{\infty}\frac{(-1)^n}{2^n} = \frac{1}{6}. \tag{349}$$

Therefore, the first terms and its traces is given by

$$\partial_{\nu_1}\phi\partial_{\nu_2}\phi\partial_{\nu_3}\phi - \frac{1}{6}\partial^\mu\phi\partial_\mu\phi\partial_{\nu_3}\phi - \frac{1}{6}\partial_{\nu_1}\phi\partial^\mu\phi\partial_\mu\phi - \frac{1}{6}\partial^\mu\phi\partial_{\nu_2}\phi\partial_\mu\phi. \tag{350}$$

This same argument also applies for the second term. The third term requires no extra trace terms since it will be zero by the equation of motion.

Thus, our ansatz for the traceless and symmetric primary operator is given by

$$\begin{aligned}\mathcal{O}_{\nu_1\nu_2\nu_3} =\ & c_1(\partial_{\nu_1}\phi\partial_{\nu_2}\phi\partial_{\nu_3}\phi - \frac{1}{6}\eta_{\nu_1\nu_2}\partial^\mu\phi\partial_\mu\phi\partial_{\nu_3}\phi - \frac{1}{6}\eta_{\nu_2\nu_3}\partial_{\nu_1}\phi\partial^\mu\phi\partial_\mu\phi - \frac{1}{6}\eta_{\nu_1\nu_3}\partial^\mu\phi\partial_{\nu_2}\phi\partial_\mu\phi)\phi\\ &+c_2(\partial_{\nu_1}\partial_{\nu_2}\phi\partial_{\nu_3}\phi + \partial_{\nu_1}\partial_{\nu_3}\phi\partial_{\nu_2}\phi + \partial_{\nu_2}\partial_{\nu_3}\phi\partial_{\nu_1}\phi - \frac{1}{6}\eta_{\nu_1\nu_2}\partial^\mu\partial_{\nu_3}\phi\partial_\mu\phi\\ &-\frac{1}{6}\eta_{\nu_1\nu_2}\partial_\mu\partial_{\nu_3}\phi\partial^\mu\phi - \frac{1}{6}\eta_{\nu_2\nu_3}\partial_{\nu_1}\partial^\mu\phi\partial_\mu\phi - \frac{1}{6}\eta_{\nu_2\nu_3}\partial_{\nu_1}\partial_\mu\phi\partial^\mu\phi\\ &-\frac{1}{6}\eta_{\nu_1\nu_3}\partial^\mu\partial_{\nu_2}\phi\partial_\mu\phi - \frac{1}{6}\eta_{\nu_1\nu_3}\partial_{\nu_2}\partial_\mu\phi\partial^\mu\phi)\phi^2 + c_3(\partial_{\nu_1}\partial_{\nu_2}\partial_{\nu_3}\phi)\phi^3.\end{aligned} \tag{351}$$

Rewriting this equation in terms of P_μ, we have

$$\begin{aligned}
\mathcal{O}_{\nu_1\nu_2\nu_3} = {} & c_1(P_{\nu_1}\phi P_{\nu_2}\phi P_{\nu_3}\phi - \frac{1}{6}\eta_{\nu_1\nu_2}P^\mu\phi P_\mu\phi P_{\nu_3}\phi - \frac{1}{6}\eta_{\nu_2\nu_3}P_{\nu_1}\phi P^\mu\phi P_\mu\phi - \frac{1}{6}\eta_{\nu_1\nu_3}P^\mu\phi P_{\nu_2}\phi P_\mu\phi)\phi \\
& +c_2(P_{\nu_1}P_{\nu_2}\phi P_{\nu_3}\phi + P_{\nu_1}P_{\nu_3}\phi P_{\nu_2}\phi + P_{\nu_2}P_{\nu_3}\phi P_{\nu_1}\phi - \frac{1}{6}\eta_{\nu_1\nu_2}P^\mu P_{\nu_3}\phi P_\mu\phi \\
& -\frac{1}{6}\eta_{\nu_1\nu_2}P_\mu P_{\nu_3}\phi P^\mu\phi - \frac{1}{6}\eta_{\nu_2\nu_3}P_{\nu_1}P^\mu\phi P_\mu\phi - \frac{1}{6}\eta_{\nu_2\nu_3}P_{\nu_1}P_\mu\phi P^\mu\phi \\
& -\frac{1}{6}\eta_{\nu_1\nu_3}P^\mu P_{\nu_2}\phi P_\mu\phi - \frac{1}{6}\eta_{\nu_1\nu_3}P_{\nu_2}P_\mu\phi P^\mu\phi)\phi^2 + c_3(P_{\nu_1}P_{\nu_2}P_{\nu_3}\phi)\phi^3.
\end{aligned} \tag{352}$$

To determine the coefficients, we again solve the equation $K_\rho\mathcal{O}_{\nu_1\nu_2\nu_3} = 0$, and thus show that our ansatz is indeed a primary operator. We start by letting K_ρ act on the first term

$$\begin{aligned}
& c_1(K_\rho P_{\nu_1}\phi P_{\nu_2}\phi P_{\nu_3}\phi + P_{\nu_1}\phi K_\rho P_{\nu_2}\phi P_{\nu_3}\phi + P_{\nu_1}\phi P_{\nu_2}\phi K_\rho P_{\nu_3}\phi - \frac{1}{6}\eta_{\nu_1\nu_2}K_\rho P^\mu\phi P_\mu\phi P_{\nu_3}\phi \\
& -\frac{1}{6}\eta_{\nu_1\nu_2}P^\mu\phi K_\rho P_\mu\phi P_{\nu_3}\phi - \frac{1}{6}\eta_{\nu_1\nu_2}P^\mu\phi P_\mu\phi K_\rho P_{\nu_3}\phi - \frac{1}{6}\eta_{\nu_2\nu_3}K_\rho P_{\nu_1}\phi P^\mu\phi P_\mu\phi \\
& -\frac{1}{6}\eta_{\nu_2\nu_3}P_{\nu_1}\phi K_\rho P^\mu\phi P_\mu\phi - \frac{1}{6}\eta_{\nu_2\nu_3}P_{\nu_1}\phi P^\mu\phi K_\rho P_\mu\phi - \frac{1}{6}\eta_{\nu_1\nu_3}K_\rho P^\mu\phi P_{\nu_2}\phi P_\mu\phi \\
& -\frac{1}{6}\eta_{\nu_1\nu_3}P^\mu\phi K_\rho P_{\nu_2}\phi P_\mu\phi - \frac{1}{6}\eta_{\nu_1\nu_3}P^\mu\phi P_{\nu_2}\phi K_\rho P_\mu\phi)\phi \\
& = c_1(2\eta_{\nu_1\rho}\phi P_{\nu_2}\phi P_{\nu_3}\phi + P_{\nu_1}\phi 2\eta_{\nu_2\rho}\phi P_{\nu_3}\phi + P_{\nu_1}\phi P_{\nu_2}\phi 2\eta_{\nu_3\rho}\phi - \frac{2}{3}\eta_{\nu_1\nu_2}\phi P_\rho\phi P_{\nu_3}\phi \\
& -\frac{1}{3}\eta_{\nu_1\nu_2}P^\mu\phi P_\mu\phi\eta_{\nu_3\rho}\phi - \frac{1}{3}\eta_{\nu_2\nu_3}\eta_{\nu_1\rho}\phi P^\mu\phi P_\mu\phi - \frac{2}{3}\eta_{\nu_2\nu_3}P_{\nu_1}\phi\phi P_\rho\phi - \frac{2}{3}\eta_{\nu_1\nu_3}P_\rho\phi P_{\nu_2}\phi\phi \\
& -\frac{1}{3}\eta_{\nu_1\nu_3}P^\mu\phi\eta_{\nu_2\rho}\phi P_\mu\phi)\phi.
\end{aligned} \tag{353}$$

Similarly, for the second term we get

$$\begin{aligned}
& c_2(K_\rho P_{\nu_1}P_{\nu_2}\phi P_{\nu_3}\phi + P_{v_1}P_{\nu_2}\phi K_\rho P_{\nu_3}\phi + K_\rho P_{\nu_1}P_{\nu_3}\phi P_{\nu_2}\phi + P_{\nu_1}P_{\nu_3}\phi K_\rho P_{\nu_2}\phi + K_\rho P_{\nu_2}P_{\nu_3}\phi P_{\nu_1}\phi \\
& +P_{\nu_2}P_{\nu_3}\phi K_\rho P_{\nu_1}\phi - \frac{1}{6}\eta_{\nu_1\nu_2}K_\rho P^\mu P_{\nu_3}\phi P_\mu\phi - \frac{1}{6}\eta_{\nu_1\nu_2}P^\mu P_{\nu_3}\phi K_\rho P_\mu\phi - \frac{1}{6}\eta_{\nu_1\nu_2}K_\rho P_\mu P_{\nu_3}\phi P^\mu\phi \\
& -\frac{1}{6}\eta_{\nu_1\nu_2}P_\mu P_{\nu_3}\phi K_\rho P^\mu\phi - \frac{1}{6}\eta_{\nu_2\nu_3}K_\rho P_{\nu_1}P^\mu\phi P_\mu\phi - \frac{1}{6}\eta_{\nu_2\nu_3}P_{\nu_1}P^\mu\phi K_\rho P_\mu\phi - \frac{1}{6}\eta_{\nu_2\nu_3}K_\rho P_{\nu_1}P_\mu\phi P^\mu\phi \\
& -\frac{1}{6}\eta_{\nu_2\nu_3}P_{\nu_1}P_\mu\phi K_\rho P^\mu\phi - \frac{1}{6}\eta_{\nu_1\nu_3}K_\rho P^\mu P_{\nu_2}\phi P_\mu\phi - \frac{1}{6}\eta_{\nu_1\nu_3}P^\mu P_{\nu_2}\phi K_\rho P_\mu\phi - \frac{1}{6}\eta_{\nu_1\nu_3}K_\rho P_{\nu_2}P_\mu\phi P^\mu\phi \\
& -\frac{1}{6}\eta_{\nu_1\nu_3}P_{\nu_2}P_\mu\phi K_\rho P^\mu\phi)\phi^2 \\
& = c_2(4\eta_{\nu_2\rho}P_{\nu_1}\phi P_{\nu_3}\phi + 4\eta_{\nu_1\rho}P_{\nu_2}\phi P_{\nu_3}\phi - 2\eta_{\nu_1\nu_2}P_\rho\phi P_{\nu_3}\phi + 2P_{\nu_1}P_{\nu_2}\phi\eta_{\nu_3\rho}\phi + 4\eta_{\nu_3\rho}P_{\nu_1}\phi P_{\nu_2}\phi \\
& +4\eta_{\nu_1\rho}P_{\nu_3}\phi P_{\nu_2}\phi - 2\eta_{\nu_1\nu_3}P_\rho\phi P_{\nu_2}\phi + 2P_{\nu_1}P_{\nu_3}\phi\eta_{\nu_2\rho}\phi + 4\eta_{\nu_3\rho}P_{\nu_2}\phi P_{\nu_1}\phi + 4\eta_{\nu_2\rho}P_{\nu_3}\phi P_{\nu_1}\phi \\
& -2\eta_{\nu_2\nu_3}P_\rho\phi P_{\nu_1}\phi + 2P_{\nu_2}P_{\nu_3}\phi\eta_{\nu_1\rho}\phi - \frac{4}{3}\eta_{\nu_1\nu_2}\eta_{\nu_3\rho}P^\mu\phi P_\mu\phi - \frac{2}{3}\eta_{\nu_1\nu_2}P_{\nu_3}\phi P_\rho\phi - \frac{2}{3}\eta_{\nu_1\nu_2}P_\rho P_{\nu_3}\phi\phi \\
& -\frac{2}{3}\eta_{\nu_1\nu_2}\eta_{\mu\rho}P_{\nu_3}\phi P^\mu\phi + \frac{1}{3}\eta_{\nu_1\nu_2}\eta_{\mu\nu_3}P_\rho\phi P^\mu\phi - \frac{1}{3}\eta_{\nu_2\nu_3}P_{\nu_1}\phi P_\rho\phi - \frac{4}{3}\eta_{\nu_2\nu_3}\eta_{\nu_1\rho}P^\mu\phi P_\mu\phi \\
& -\frac{2}{3}\eta_{\nu_2\nu_3}P_{\nu_1}P_\rho\phi\phi - \frac{2}{3}\eta_{\nu_2\nu_3}\nu_{\mu\rho}P_{\nu_1}\phi P^\mu\phi + \frac{1}{3}\eta_{\nu_2\nu_3}\eta_{\nu_1\mu}P_\rho\phi P^\mu\phi - \frac{4}{3}\eta_{\nu_1\nu_3}\eta_{\nu_2\rho}P^\mu\phi P_\mu\phi \\
& -\frac{1}{3}\eta_{\nu_1\nu_3}P_{\nu_2}\phi P_\rho\phi - \frac{2}{3}\eta_{\nu_1\nu_3}P_\rho P_{\nu_2}\phi\phi - \frac{2}{3}\eta_{\nu_1\nu_3}\eta_{\mu\rho}P_{\nu_2}\phi P^\mu\phi + \frac{1}{3}\eta_{\nu_1\nu_3}\eta_{\nu_2\mu}P_\rho\phi P^\mu\phi)\phi^2.
\end{aligned} \tag{354}$$

For the third term, we obtain the following

$$
\begin{aligned}
K_\mu P_{\nu_1} P_{\nu_2} P_{\nu_3}\phi &= (P_{\nu_1} K_\mu + 2i\eta_{\nu_1\mu} D + 2iM_{\nu_1\mu}) P_{\nu_2} P_{\nu_3}\phi \\
&= P_{\nu_1}(P_{\nu_2} K_\mu + 2i\eta_{\nu_2\mu} D + 2iM_{\nu_2\mu}) P_{\nu_3}\phi + 2i\eta_{\nu_1\mu}(P_{\nu_2} D - iP_{\nu_2}) P_{\nu_3}\phi \\
&+2i(P_{\nu_2} M_{\nu_1\mu} - i\eta_{\mu\nu_2} P_{\nu_1} + i\eta_{\nu_1\nu_2} P_\mu) P_{\nu_3}\phi \\
&= P_{\nu_1} P_{\nu_2} K_\mu P_{\nu_3}\phi + 2i\eta_{\nu_2\mu} P_{\nu_1} D P_{\nu_3}\phi + 2iP_{\nu_1} M_{\nu_2\mu} P_{\nu_3}\phi + 2i\eta_{\nu_1\mu} P_{\nu_2} D P_{\nu_3}\phi \\
&+2\eta_{\nu_1\mu} P_{\nu_2} P_{\nu_3}\phi + 2iP_{\nu_2} M_{\nu_1\mu} P_{\nu_3}\phi + 2\eta_{\mu\nu_2} P_{\nu_1} P_{\nu_3}\phi - 2\eta_{\nu_1\nu_2} P_\mu P_{\nu_3}\phi \\
&= P_{\nu_1} P_{\nu_2}(P_{\nu_3} K_\mu + 2i\eta_{\nu_3\mu} D + 2iM_{\nu_3\mu})\phi + 2i\eta_{\nu_2\mu} P_{\nu_1}(P_{\nu_3} D - iP_{\nu_3})\phi \\
&+2iP_{\nu_1}(P_{\nu_3} M_{\nu_2\mu} - i\eta_{\mu\nu_3} P_{\nu_2} + i\eta_{\nu_2\nu_3} P_\mu)\phi + 2i\eta_{\nu_1\mu} P_{\nu_2}(P_{\nu_3} D - iP_{\nu_3})\phi \\
&+2\eta_{\nu_1\mu} P_{\nu_2} P_{\nu_3}\phi + 2iP_{\nu_2}(P_{\nu_3} M_{\nu_1\mu} - i\eta_{\mu\nu_3} P_{\nu_1} + i\eta_{\nu_1\nu_3} P_\mu)\phi + 2\eta_{\mu\nu_2} P_{\nu_1} P_{\nu_3}\phi \\
&-2\eta_{\nu_1\nu_2} P_\mu P_{\nu_3}\phi \\
&= 2\eta_{\nu_3\mu} P_{\nu_1} P_{\nu_2}\phi + 2\eta_{\nu_2\mu} P_{\nu_1} P_{\nu_3}\phi + 2\eta_{\nu_2\mu} P_{\nu_1} P_{\nu_3}\phi + 2\eta_{\mu\nu_3} P_{\nu_1} P_{\nu_2}\phi - 2\eta_{\nu_2\nu_3} P_{\nu_1} P_\mu\phi \\
&+2\eta_{\nu_1\mu} P_{\nu_2} P_{\nu_3}\phi + 2\eta_{\nu_1\mu} P_{\nu_2} P_{\nu_3}\phi + 2\eta_{\nu_1\mu} P_{\nu_2} P_{\nu_3}\phi + 2\eta_{\mu\nu_3} P_{\nu_2} P_{\nu_1}\phi - 2\eta_{\nu_1\nu_3} P_{\nu_2} P_\mu\phi \\
&+2\eta_{\mu\nu_2} P_{\nu_1} P_{\nu_3}\phi - 2\eta_{\nu_1\nu_2} P_\mu P_{\nu_3}\phi \\
&= 6\eta_{\nu_3\mu} P_{\nu_1} P_{\nu_2}\phi + 6\eta_{\nu_2\mu} P_{\nu_1} P_{\nu_3}\phi + 6\eta_{\nu_1\mu} P_{\nu_2} P_{\nu_3}\phi - 2\eta_{\nu_2\nu_3} P_{\nu_1} P_\mu\phi - 2\eta_{\nu_1\nu_3} P_{\nu_2} P_\mu\phi \\
&-2\eta_{\nu_1\nu_2} P_\mu P_{\nu_3}\phi,
\end{aligned}
\tag{355}
$$

and thus

$$
\begin{aligned}
c_3(K_\rho P_{\nu_1} P_{\nu_2} P_{\nu_3}\phi)\phi^3 = c_3(&6\eta_{\nu_3\rho} P_{\nu_1} P_{\nu_2}\phi + 6\eta_{\nu_2\rho} P_{\nu_1} P_{\nu_3}\phi + 6\eta_{\nu_1\rho} P_{\nu_2} P_{\nu_3}\phi - 2\eta_{\nu_2\nu_3} P_{\nu_1} P_\rho\phi \\
&- 2\eta_{\nu_1\nu_3} P_{\nu_2} P_\rho\phi - 2\eta_{\nu_1\nu_2} P_\rho P_{\nu_3}\phi)\phi^3.
\end{aligned}
\tag{356}
$$

Collecting terms and equating to zero, we find the following expressions for the coefficients:

$$
\begin{aligned}
c_2 &= -\frac{1}{4} c_1 \\
c_3 &= -\frac{1}{3} c_2.
\end{aligned}
\tag{357}
$$

Thus, choosing $c_1 = 1$, the primary operator is given by

$$
\begin{aligned}
\mathcal{O}_{\nu_1\nu_2\nu_3} &= (\partial_{\nu_1}\phi\partial_{\nu_2}\phi\partial_{\nu_3}\phi - \frac{1}{6}\eta_{\nu_1\nu_2}\partial^\mu\phi\partial_\mu\phi\partial_{\nu_3}\phi - \frac{1}{6}\eta_{\nu_2\nu_3}\partial_{\nu_1}\phi\partial^\mu\phi\partial_\mu\phi - \frac{1}{6}\eta_{\nu_1\nu_3}\partial^\mu\phi\partial_{\nu_2}\phi\partial_\mu\phi)\phi \\
&-\frac{1}{4}(\partial_{\nu_1}\partial_{\nu_2}\phi\partial_{\nu_3}\phi + \partial_{\nu_1}\partial_{\nu_3}\phi\partial_{\nu_2}\phi + \partial_{\nu_2}\partial_{\nu_3}\phi\partial_{\nu_1}\phi - \frac{1}{6}\eta_{\nu_1\nu_2}\partial^\mu\partial_{\nu_3}\phi\partial_\mu\phi \\
&-\frac{1}{6}\eta_{\nu_1\nu_2}\partial_\mu\partial_{\nu_3}\phi\partial^\mu\phi - \frac{1}{6}\eta_{\nu_2\nu_3}\partial_{\nu_1}\partial^\mu\phi\partial_\mu\phi - \frac{1}{6}\eta_{\nu_2\nu_3}\partial_{\nu_1}\partial_\mu\phi\partial^\mu\phi \\
&-\frac{1}{6}\eta_{\nu_1\nu_3}\partial^\mu\partial_{\nu_2}\phi\partial_\mu\phi - \frac{1}{6}\eta_{\nu_1\nu_3}\partial_{\nu_2}\partial_\mu\phi\partial^\mu\phi)\phi^2 + \frac{1}{12}(\partial_{\nu_1}\partial_{\nu_2}\partial_{\nu_3}\phi)\phi^3.
\end{aligned}
\tag{358}
$$

5.3 Spin 3 Primary: Using the Subalgebra

We now recalculate the coefficients of the spin-3 primary operator using the subalgebra. Thus, we are constructing a single state. Now, the primary is a spin multiplet worth of states. Therefore, from the single state we construct, the remaining states follow by performing rotations.

Defining $P = P_1 - iP_2$ and $K = K_1 + iK_2$, we define the subalgebra through the commutators

$$\begin{aligned} &[D, P] = -iP, \qquad [D, K] = iK, \qquad [K, P] = 4iD + 4M_{21}, \\ &[M_{21}, K] = -K, \qquad [M_{21}, P] = P. \end{aligned} \tag{359}$$

As before, $D\phi = -i\phi$ and $M_{21}\phi = 0$. Using the subalgebra, we can prove the following useful relation

$$KP^s\phi = 4s^2P^{s-1}\phi. \tag{360}$$

Now,

$$\begin{aligned} [K, P^s] &= \sum_{k=0}^{s-1} P^{s-1-k}[K, P]P^k \\ &= 4i \sum_{k=0}^{s-1} P^{s-1-k}\big(P^kD + [D, P^k]\big) + 4\sum_{k=0}^{s-1} P^{s-1-k}\big(P^kM_{21} + [M_{21}, P^k]\big) \\ &= 4isP^{s-1}D + 4sP^{s-1}M_{21} + 4i\sum_{k=0}^{s-1} P^{s-1-k}[D, P^k] + 4\sum_{k=0}^{s-1} P^{s-1-k}[M_{21}, P^k], \end{aligned} \tag{361}$$

where we used the fact that $\sum_{k=0}^{s-1} 1 = s$. The commutators in the final line can be calculated as follows

$$\begin{aligned} [D, P^k] &= \sum_{t=0}^{k-1} P^{k-1-t}[D, P]P^t \\ &= -ikP^k, \end{aligned} \tag{362}$$

$$\begin{aligned} [M_{21}, P^k] &= \sum_{t=0}^{k-1} P^{k-1-t}[M_{21}, P]P^t \\ &= kP^k. \end{aligned} \tag{363}$$

Therefore,

$$[K, P^s] = 4isP^{s-1}D + 4sP^{s-1}M_{21} + 2(s-1)sP^{s-1} + 2(s-1)sP^{s-1}, \tag{364}$$

where we used the fact that $\sum_{k=1}^{s-1} k = (s-1)s/2$. Finally,

$$\begin{aligned} KP^s\phi &= [K, P^s]\phi \\ &= 4isP^{s-1}D\phi + 4sP^{s-1}M_{21}\phi + 2(s-1)sP^{s-1}\phi + 2(s-1)sP^{s-1}\phi \\ &= 4sP^{s-1}\phi + 2(s-1)sP^{s-1}\phi + 2(s-1)sP^{s-1}\phi \\ &= 4s^2P^{s-1}\phi. \end{aligned} \tag{365}$$

Using the ansatz for the three index operator

$$\mathcal{O}_{(3)} = c_1(P\phi P\phi P\phi)\phi + c_2(P^2\phi P\phi)\phi^2 + c_3(P^3\phi)\phi^3, \tag{366}$$

and applying K, we find that

$$\begin{aligned} K\mathcal{O}_{(3)} &= c_1(4\phi P\phi P\phi + 4P\phi\phi P\phi + 4P\phi P\phi\phi)\phi + c_2(16P\phi P\phi \\ &+4P^2\phi\phi)\phi^2 + 36c_3P^2\phi\phi^3 \\ &= 0 \end{aligned} \tag{367}$$

from which it follows that

$$\begin{aligned} &16c_1 + 12c_2 = 0, \\ &4c_2 + 36c_3 = 0, \\ &c_2 = -\frac{3}{4}c_1 \\ &c_3 = -\frac{1}{9}c_2. \end{aligned} \tag{368}$$

Previously, we found that $c_2 = -1/4c_1$ and $c_3 = -1/3c_2$ using the full conformal algebra. The difference in the coefficients can be attributed to our specialization to the subalgebra, which results in a change of basis, and therefore a different set of coefficients for the three index operator.

However, to ensure that our results are indeed in agreement, we can perform various checks, which is the topic of the following section.

5.4 Performing Checks On Our Construction

Here, we will briefly digress from our construction of primaries to perform some useful checks on our calculations. We begin with the spin-2 primary.

The spin-2 primary operator. We can check our results as follows. For the two index primary operator we found in the previous sections, we can use the sublagebra to write

$$\begin{aligned}\mathcal{O}_{(2)} &= (\partial_1 - i\partial_2)\phi(\partial_1 - i\partial_2)\phi\phi^2 - \frac{1}{2}(\partial_1^2 - 2i\partial_1\partial_2 - \partial_2^2)\phi\phi^3 \\ &= \partial_1\phi\partial_1\phi\phi^2 - \partial_2\phi\partial_2\phi\phi^2 - i\partial_1\phi\partial_2\phi\phi^2 - i\partial_2\phi\partial_1\phi\phi^2 - \frac{1}{2}\partial_1^2\phi\phi^3 \\ &+ i\partial_1\partial_2\phi\phi^3 + \frac{1}{2}\partial_2^2\phi\phi^3. \end{aligned} \tag{369}$$

Using the full conformal algebra, we must be able to write both the real and imaginary part of this expression as a linear combination of the two index operator determined previously; if the form for our operator is incorrect, then we won't be able to write the above expression's real/imaginary parts as a linear combination. Consider then the real part

$$Re[\mathcal{O}_{(2)}] = \partial_1\phi\partial_1\phi\phi^2 - \partial_2\phi\partial_2\phi\phi^2 - \frac{1}{2}(\partial_1^2 - \partial_2^2)\phi\phi^3 \tag{370}$$

Observe that we can rewrite this using the traceless matrix M, given by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{371}$$

as

$$Re[\mathcal{O}_{(2)}] = M_{ab}\partial_a\phi\partial_b\phi\phi^2 - \frac{1}{2}M_{ab}\partial_a\partial_b\phi\phi^3. \tag{372}$$

Previously, we found (using the full conformal algebra)

$$\mathcal{O}_{\nu_1\nu_2} = (\partial_{\nu_1}\phi\partial_{\nu_2}\phi - \frac{1}{4}\eta_{\nu_1\nu_2}\partial^\mu\phi\partial_\mu\phi)\phi^2 - \frac{1}{2}(\partial_{\nu_1}\partial_{\nu_2}\phi)\phi^3. \tag{373}$$

Then,

$$\begin{aligned}\mathcal{O}_{11} &= (\partial_1\phi\partial_1\phi - \frac{1}{4}\partial^\mu\phi\partial_\mu\phi)\phi^2 - \frac{1}{2}(\partial_1\partial_1\phi)\phi^3, \\ \mathcal{O}_{22} &= (\partial_2\phi\partial_2\phi - \frac{1}{4}\partial^\mu\phi\partial_\mu\phi)\phi^2 - \frac{1}{2}(\partial_2\partial_2\phi)\phi^3. \end{aligned} \tag{374}$$

Clearly,

$$Re[\mathcal{O}_{(2)}] = \mathcal{O}_{11} - \mathcal{O}_{22}. \tag{375}$$

Thus we can be confident of the form of our two index operator.

The spin-3 primary operator. We first do a small check to see if our coefficients, calculated using the full conformal algebra, agree with a specific choice of indices. If the coefficients of our three index operator is correct, we should find the same coefficients for this specific case (since our calculation was done in general). The case we consider has the following restrictions on the indices: $\nu_1 \neq \nu_2$, $\nu_2 \neq \nu_3$, and $\nu_1 \neq \nu_3$. This restriction greatly simplifies the calculation since all terms containing the metric will now be zero; in particular, the operator simplifies to

$$\begin{aligned}\mathcal{O}_{\nu_1\nu_2\nu_3} =& c_1(P_{\nu_1}\phi P_{\nu_2}\phi P_{\nu_3}\phi)\phi + c_2(P_{\nu_1}P_{\nu_2}\phi P_{\nu_3}\phi + P_{\nu_1}P_{\nu_3}\phi P_{\nu_2}\phi + P_{\nu_2}P_{\nu_3}\phi P_{\nu_1}\phi)\phi^2 \\ &+ c_3(P_{\nu_1}P_{\nu_2}P_{\nu_3}\phi)\phi^3. \end{aligned} \tag{376}$$

Apply K_μ, we find that

$$\begin{aligned} K_\mu\mathcal{O}_{\nu_1\nu_2\nu_3} =& c_1(K_\mu P_{\nu_1}\phi P_{\nu_2}\phi P_{\nu_3}\phi + P_{\nu_1}\phi K_\mu P_{\nu_2}\phi P_{\nu_3}\phi + P_{\nu_1}\phi P_{\nu_2}\phi K_\mu P_{\nu_3}\phi)\phi \\ &+ c_2(K_\mu P_{\nu_1}P_{\nu_2}\phi P_{\nu_3}\phi + P_{\nu_1}P_{\nu_2}\phi K_\mu P_{\nu_3}\phi + K_\mu P_{\nu_1}P_{\nu_3}\phi P_{\nu_2}\phi + P_{\nu_1}P_{\nu_3}\phi K_\mu P_{\nu_2}\phi \\ &+ K_\mu P_{\nu_2}P_{\nu_3}\phi P_{\nu_1}\phi + P_{\nu_2}P_{\nu_3}\phi K_\mu P_{\nu_1}\phi)\phi^2 + c_3(K_\mu P_{\nu_1}P_{\nu_2}P_{\nu_3}\phi) \\ =& c_1(2\eta_{\nu_1\mu}\phi P_{\nu_2}\phi P_{\nu_3}\phi + 2\eta_{\nu_2\mu}P_{\nu_1}\phi\phi P_{\nu_3}\phi + 2\eta_{\nu_3\mu}P_{\nu_1}\phi P_{\nu_2}\phi\phi)\phi \\ &+ c_2\big((4\eta_{\nu_2\mu}P_{\nu_1} + 4\eta_{\nu_1\mu}P_{\nu_2}\phi)P_{\nu_3}\phi + 2\eta_{\nu_3\mu}P_{\nu_1}P_{\nu_2}\phi\phi + (4\eta_{\nu_3\mu}P_{\nu_1} + 4\eta_{\nu_1\mu}P_{\nu_3})\phi P_{\nu_2}\phi \\ &+ 2\eta_{\nu_2\mu}P_{\nu_1}P_{\nu_3}\phi\phi + (4\eta_{\nu_3\mu}P_{\nu_2} + 4\eta_{\nu_2\mu}P_{\nu_3})\phi P_{\nu_1}\phi + 2\eta_{\nu_1\mu}P_{\nu_1}P_{\nu_3}\phi\phi\big)\phi^2 \\ &+ c_3(6\eta_{\nu_3\mu}P_{\nu_1}P_{\nu_2}\phi + 6\eta_{\nu_2\mu}P_{\nu_1}P_{\nu_3}\phi + 6\eta_{\nu_1\mu}P_{\nu_2}P_{\nu_3}\phi - 2\eta_{\nu_2\nu_3}P_{\nu_1}P_\mu\phi - 2\eta_{\nu_1\nu_3}P_{\nu_2}P_\mu\phi) \\ =& 0. \end{aligned} \tag{377}$$

Collecting terms as before, we find that

$$\begin{aligned} 2c_1 + 8c_2 = 0, \\ 2c_2 + 6c_3 = 0, \end{aligned} \tag{378}$$

from which it follows that

$$\begin{aligned} c_2 = -\frac{1}{4}c_1, \\ c_3 = -\frac{1}{3}c_2. \end{aligned} \tag{379}$$

This is indeed the same result we found previously.

The three index operator in the subalgebra has the form (with choice $c_1 = 1$)

$$\begin{aligned} \mathcal{O}_{(3)} =& (\partial_1 - i\partial_2)\phi(\partial_1 - i\partial_2)\phi(\partial_1 - i\partial_2)\phi\phi - \frac{3}{4}(\partial_1^2 - 2i\partial_1\partial_2 - \partial_2^2)\phi(\partial_1 - i\partial_2)\phi\phi^2 \\ &+ \frac{1}{12}(\partial_1\partial_1\partial_1 - i\partial_1\partial_1\partial_2 - 2i\partial_1\partial_2\partial_1 - 2\partial_1\partial_2\partial_2 - \partial_2\partial_2\partial_1 + i\partial_2\partial_2\partial_2)\phi\phi^3 \\ =& (\partial_1\phi\partial_1\phi\partial_1\phi - i\partial_1\phi\partial_1\phi\partial_2\phi - 2i\partial_1\phi\partial_2\phi\partial_1\phi - 2\partial_1\phi\partial_2\phi\partial_2\phi - \partial_2\phi\partial_2\phi\partial_1\phi + \\ & i\partial_2\phi\partial_2\phi\partial_2\phi)\phi - \frac{3}{4}(\partial_1\partial_1\phi\partial_1\phi - i\partial_1\partial_1\phi\partial_2\phi - 2i\partial_1\partial_2\phi\partial_1\phi - 2\partial_1\partial_2\phi\partial_2\phi \\ & -\partial_2\partial_2\phi\partial_1\phi + i\partial_2\partial_2\phi\partial_2\phi)\phi^2 + \frac{1}{12}(\partial_1\partial_1\partial_1\phi - i\partial_1\partial_1\partial_2\phi - 2i\partial_1\partial_2\partial_1\phi - \\ & 2\partial_1\partial_2\partial_2\phi - \partial_2\partial_2\partial_1\phi + i\partial_2\partial_2\partial_2\phi)\phi^3, \end{aligned} \tag{380}$$

from which we find that

$$Re[\mathcal{O}_{(3)}]= (\partial_1\phi\partial_1\phi\partial_1\phi - 3\partial_1\phi\partial_2\phi\partial_2\phi)\phi - \frac{3}{4}(\partial_1\partial_1\phi\partial_1\phi - 2\partial_1\partial_2\phi\partial_2\phi - \partial_2\partial_2\phi\partial_1\phi)\phi^2 + \frac{1}{12}(\partial_1\partial_1\partial_1\phi - 3\partial_1\partial_2\partial_2\phi)\phi^3. \tag{381}$$

From the conformal algebra, we found that

$$\begin{aligned}\mathcal{O}_{\nu_1\nu_2\nu_3}=& (\partial_{\nu_1}\phi\partial_{\nu_2}\phi\partial_{\nu_3}\phi - \frac{1}{6}\eta_{\nu_1\nu_2}\partial^\mu\phi\partial_\mu\phi\partial_{\nu_3}\phi - \frac{1}{6}\eta_{\nu_2\nu_3}\partial_{\nu_1}\phi\partial^\mu\phi\partial_\mu\phi - \frac{1}{6}\eta_{\nu_1\nu_3}\partial^\mu\phi\partial_{\nu_2}\phi\partial_\mu\phi)\phi\\ &-\frac{1}{4}(\partial_{\nu_1}\partial_{\nu_2}\phi\partial_{\nu_3}\phi + \partial_{\nu_1}\partial_{\nu_3}\phi\partial_{\nu_2}\phi + \partial_{\nu_2}\partial_{\nu_3}\phi\partial_{\nu_1}\phi - \frac{1}{6}\eta_{\nu_1\nu_2}\partial^\mu\partial_{\nu_3}\phi\partial_\mu\phi\\ &-\frac{1}{6}\eta_{\nu_1\nu_2}\partial_\mu\partial_{\nu_3}\phi\partial^\mu\phi - \frac{1}{6}\eta_{\nu_2\nu_3}\partial_{\nu_1}\partial^\mu\phi\partial_\mu\phi - \frac{1}{6}\eta_{\nu_2\nu_3}\partial_{\nu_1}\partial_\mu\phi\partial^\mu\phi\\ &-\frac{1}{6}\eta_{\nu_1\nu_3}\partial^\mu\partial_{\nu_2}\phi\partial_\mu\phi - \frac{1}{6}\eta_{\nu_1\nu_3}\partial_{\nu_2}\partial_\mu\phi\partial^\mu\phi)\phi^2 + \frac{1}{12}(\partial_{\nu_1}\partial_{\nu_2}\partial_{\nu_3}\phi)\phi^3.\end{aligned} \tag{382}$$

Clearly, we need to calculate $\mathcal{O}_{111}$ and $\mathcal{O}_{122}$:

$$\mathcal{O}_{111}= (\partial_1\phi\partial_1\phi\partial_1\phi - \frac{3}{6}\partial_1\phi\partial^\mu\phi\partial_\mu\phi)\phi - \frac{1}{4}(3\partial_1\partial_1\phi\partial_1\phi - \frac{3}{6}\partial^\mu\partial_1\phi\partial_\mu\phi - \frac{3}{6}\partial_\mu\partial_1\phi\partial^\mu\phi)\phi^2 + \frac{1}{12}(\partial_1\partial_1\partial_1\phi)\phi^3, \tag{383}$$

$$\mathcal{O}_{122}= (\partial_1\phi\partial_2\phi\partial_2\phi - \frac{1}{6}\partial_1\phi\partial^\mu\phi\partial_\mu\phi)\phi - \frac{1}{4}(2\partial_1\partial_2\phi\partial_2\phi + \partial_2\partial_2\phi\partial_1\phi - \frac{1}{6}\partial_1\partial^\mu\phi\partial_\mu\phi - \frac{1}{6}\partial_1\partial_\mu\phi\partial^\mu\phi)\phi^2 + \frac{1}{12}(\partial_1\partial_2\partial_2\phi)\phi^3. \tag{384}$$

Observe that to obtain the first and third line of equation (381), we can simply form the combination $\mathcal{O}_{111} - 3\mathcal{O}_{122}$.

Remark 6. *If our coefficients for the three index operator is correct, then we expect this combination to be a primary operator as well. Indeed, we find that*

$$\begin{aligned}K_\mu(\mathcal{O}_{111} - 3\mathcal{O}_{122})=& (6\eta_{1\mu}\phi P_1\phi P_1\phi - 6\eta_{1\mu}\phi P_2\phi P_2\phi - 12\eta_{2\mu}\phi P_1\phi P_2\phi)\phi\\ &-\frac{3}{4}(8\eta_{1\mu}P_1\phi P_1\phi - 2P_\mu\phi P_1\phi + 2\eta_{1\mu}\phi P_1P_1\phi\\ &-8\eta_{2\mu}P_1\phi P_2\phi - 8\eta_{1\mu}P_2\phi P_2\phi - 4\eta_{2\mu}\phi P_1P_2\phi\\ &-8\eta_{2\mu}P_2\phi P_1\phi + 2P_\mu\phi P_1\phi - 2\eta_{1\mu}\phi P_2P_2\phi)\phi^2\\ &+\frac{1}{12}(18\eta_{1\mu}P_1P_1\phi - 6P_1P_\mu\phi - 36\eta_{2\mu}P_1P_2\phi\\ &-18\eta_{1\mu}P_2P_2\phi + 6P_1P_\mu\phi)\phi^3\\ &= 0.\end{aligned} \tag{385}$$

We therefore find that

$$\begin{aligned}Re[\mathcal{O}_{(3)}]=& (\partial_1\phi\partial_1\phi\partial_1\phi - 3\partial_1\phi\partial_2\phi\partial_2\phi)\phi - \frac{3}{4}(\partial_1\partial_1\phi\partial_1\phi - 2\partial_1\partial_2\phi\partial_2\phi - \partial_2\partial_2\phi\partial_1\phi)\phi^2\\ &+\frac{1}{12}(\partial_1\partial_1\partial_1\phi - 3\partial_1\partial_2\partial_2\phi)\phi^3\\ &= \mathcal{O}_{111} - 3\mathcal{O}_{122},\end{aligned} \tag{386}$$

which is the expected result.

5.5 Constructing Primaries with Mixed Symmetry

From our spectrum of primaries for $n = 3$, we found primaries with representation $(7, 0, 2)$ and $(7, 2, 0)$ in the $n = 3$ case. Here, we attempt to reconstruct these primaries. As in Section 2.3, we will use spinor indices, thus simplifying the treatment of the $SO(4)$ irreps.

Consider the primary with representation $(7, 2, 0)$ or, in terms of the group element, s^7x^2. We have $\Delta = 3+4 = 7$ and thus we will have 4 derivatives acting on the fields. Furthermore, we have spin representation $(2, 0)$ and thus our tensor will have mixed symmetry. Furthermore, note that the operator will be complex and the irrep $(7, 0, 2)$ is its complex conjugate.

Now, we introduce the Pauli matrices $\sigma^m_{\alpha\dot{\beta}}$ with $\sigma^m = (-i\vec{\sigma}, \mathbf{1})$ (see Section (2.3) for more on spinorial notation). Note that we are working in Euclidean space here. The commutators in this case is given by

$$\begin{aligned}
&\big[M_{pq}, M_{rs}\big] = \delta_{qr}M_{ps} + \delta_{ps}M_{qr} - \delta_{qs}M_{pr} - \delta_{pr}M_{qs},\\
&\big[M_{pq}, D\big] = 0, \qquad \big[D, P_p\big] = P_p, \qquad \big[D, K_p\big] = -K_p,\\
&\big[M_{pq}, P_r\big] = \delta_{qr}P_p - \delta_{pr}P_q, \qquad \big[M_{pq}, K_r\big] = \delta_{qr}K_p - \delta_{pr}K_q,\\
&\big[K_p, P_q\big] = 2M_{pq} - 2\delta_{pq}D.
\end{aligned} \tag{387}$$

To obtain a spin-0 state for the y spin, we must contract all the dotted indices, in pairs, using

$$\epsilon^{\dot{\beta}\dot{\rho}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{388}$$

and to obtain a spin-2 state for the x spin we must symmetrize all the undotted indices (see Appendix (A.3) for more information). Now note that

$$T_{\alpha\gamma} = \sigma^{m_1}_{\alpha\dot{\beta}}\sigma^{m_2}_{\gamma\dot{\rho}}\epsilon^{\dot{\beta}\dot{\rho}}\partial_{m_1}\partial_{m_2}\phi, \tag{389}$$

is antisymmetric and thus can not be symmetrized. Therefore, of all the possible terms, only two structures will contribute

$$\begin{aligned}
O_{1\alpha\beta\gamma\rho} &= \phi P_{\alpha\dot{\alpha}}P_{\beta\dot{\beta}}\phi P_{\gamma\dot{\gamma}}P_{\rho\dot{\rho}}\phi\epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon^{\dot{\beta}\dot{\rho}},\\
O_{2\alpha\beta\gamma\rho} &= P_{\alpha\dot{\alpha}}\phi P_{\beta\dot{\beta}}\phi P_{\gamma\dot{\gamma}}P_{\rho\dot{\rho}}\phi\epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon^{\dot{\beta}\dot{\rho}},
\end{aligned} \tag{390}$$

where we use the more convenient notation $P_\mu\sigma^\mu_{\alpha\dot{\alpha}} \equiv P_{\alpha\dot{\alpha}}$. Our primary will thus be constructed out of two terms; consequently, we must determine the coefficient in front of each term such that K_r annihilates this primary. Moreover, we must symmetrize the undotted indices $\alpha\beta\gamma\rho$. A tedious, but straightforward, computation shows that

$$
\begin{aligned}
K_r O_{1\alpha\beta\gamma\rho} = & -4\phi(\sigma_r)_{\alpha\dot\alpha}(P)_{\beta\dot\beta}\phi(P)_{\gamma\dot\gamma}(P)_{\rho\dot\rho}\phi\epsilon^{\dot\alpha\dot\gamma}\epsilon^{\dot\beta\dot\rho} \\
& +2\phi(\sigma_r)_{\alpha\dot\alpha}(\sigma_r)_{\beta\dot\beta}P_r\phi(P)_{\gamma\dot\gamma}(P)_{\rho\dot\rho}\phi\epsilon^{\dot\alpha\dot\gamma}\epsilon^{\dot\beta\dot\rho} \\
& -4\phi(P)_{\alpha\dot\alpha}(\sigma_r)_{\beta\dot\beta}\phi(P)_{\gamma\dot\gamma}(P)_{\rho\dot\rho}\phi\epsilon^{\dot\alpha\dot\gamma}\epsilon^{\dot\beta\dot\rho} \\
& -4\phi(P)_{\alpha\dot\alpha}(P)_{\beta\dot\beta}\phi(\sigma_r)_{\gamma\dot\gamma}(P)_{\rho\dot\rho}\phi\epsilon^{\dot\alpha\dot\gamma}\epsilon^{\dot\beta\dot\rho} \\
& +2\phi(P)_{\alpha\dot\alpha}(P)_{\beta\dot\beta}\phi(\sigma_r)_{\gamma\dot\gamma}(\sigma_r)_{\rho\dot\rho}P_r\phi\epsilon^{\dot\alpha\dot\gamma}\epsilon^{\dot\beta\dot\rho} \\
& -4\phi(P)_{\alpha\dot\alpha}(P)_{\beta\dot\beta}\phi(P)_{\gamma\dot\gamma}(\sigma_r)_{\rho\dot\rho}\phi\epsilon^{\dot\alpha\dot\gamma}\epsilon^{\dot\beta\dot\rho},
\end{aligned} \tag{391}
$$

and

$$
\begin{aligned}
K_r O_{2\alpha\beta\gamma\rho} = & -2(\sigma_r)_{\alpha\dot\alpha}\phi(P)_{\beta\dot\beta}\phi(P)_{\gamma\dot\gamma}(P)_{\rho\dot\rho}\phi\epsilon^{\dot\alpha\dot\gamma}\epsilon^{\dot\beta\dot\rho} \\
& -2(P)_{\alpha\dot\alpha}\phi(\sigma_r)_{\beta\dot\beta}\phi(P)_{\gamma\dot\gamma}(P)_{\rho\dot\rho}\phi\epsilon^{\dot\alpha\dot\gamma}\epsilon^{\dot\beta\dot\rho} \\
& -4(P)_{\alpha\dot\alpha}\phi(P)_{\beta\dot\beta}\phi(\sigma_r)_{\gamma\dot\gamma}(P)_{\rho\dot\rho}\phi\epsilon^{\dot\alpha\dot\gamma}\epsilon^{\dot\beta\dot\rho} \\
& -4(P)_{\alpha\dot\alpha}\phi(\sigma_r)_{\beta\dot\beta}\phi(P)_{\gamma\dot\gamma}(\sigma_r)_{\rho\dot\rho}\phi\epsilon^{\dot\alpha\dot\gamma}\epsilon^{\dot\beta\dot\rho} \\
& +2(P)_{\alpha\dot\alpha}\phi(\sigma_r)_{\beta\dot\beta}\phi(\sigma_n)_{\gamma\dot\gamma}(\sigma_n)_{\rho\dot\rho}P_r\phi\epsilon^{\dot\alpha\dot\gamma}\epsilon^{\dot\beta\dot\rho}.
\end{aligned} \tag{392}
$$

Another similar calculation shows that

$$
K_r(O_{11111} - 4O_{21111}) = 0 = K_r(O_{12222} - 4O_{22222}). \tag{393}
$$

Indeed, we find that

$$
O_{[2,0]\alpha\beta\gamma\rho} = O_{1(\alpha\beta\gamma\rho)} - 4O_{2(\alpha\beta\gamma\rho)}, \tag{394}
$$

(indices inside round brackets are totally symmetrized) is the primary corresponding to s^7x^2. As an example for a specific choice of indices, the primary has the following form

$$
\begin{aligned}
O_{[2,0]1111} = & -2(\partial_2^2\phi + 2i\partial_1\partial_2\phi - \partial_1^2\phi)(\partial_4\phi)^2 + 4\partial_2\phi\partial_2\partial_4\phi\partial_4\phi - 4i\partial_2\phi\partial_2\partial_3\phi\partial_4\phi \\
& +4i\partial_3\phi\partial_2^2\phi\partial_4\phi + 4i\partial_2\partial_4\phi\partial_1\phi\partial_4\phi + 4\partial_2\partial_3\phi\partial_1\phi\partial_4\phi + 4i\partial_2\phi\partial_1\partial_4\phi\partial_4\phi \\
& -4\partial_1\phi\partial_1\partial_4\phi\partial_4\phi + 4\partial_2\phi\partial_1\partial_3\phi\partial_4\phi + 4i\partial_1\phi\partial_1\partial_3\phi\partial_4\phi - 8\partial_3\phi\partial_1\partial_2\phi\partial_4\phi \\
& -4i\partial_3\phi\partial_1^2\phi\partial_4\phi + 2\partial_3^2\phi\partial_2\phi^2 - \phi(\partial_2\partial_4\phi)^2 + \phi\partial_2\partial_3\phi^2 - 2\partial_3^2\phi\partial_1\phi^2 + \phi(\partial_1\partial_4\phi)^2 \\
& -\phi\partial_1\partial_3\phi^2 - 4i\partial_3\phi\partial_2\phi\partial_2\partial_4\phi - 4\partial_3\phi\partial_2\phi\partial_2\partial_3\phi + 2i\phi\partial_2\partial_4\phi\partial_2\partial_3\phi + 2\partial_3\phi^2\partial_2^2\phi \\
& -\phi\partial_3^2\phi\partial_2^2\phi + 4i\partial_3^2\phi\partial_2\phi\partial_1\phi + 4\partial_3\phi\partial_2\partial_4\phi\partial_1\phi - 4i\partial_3\phi\partial_2\partial_3\phi\partial_1\phi + 4\partial_3\phi\partial_2\phi\partial_1\partial_4\phi \\
& -2i\phi\partial_2\partial_4\phi\partial_1\partial_4\phi - 2\phi\partial_2\partial_3\phi\partial_1\partial_4\phi + 4i\partial_3\phi\partial_1\phi\partial_1\partial_4\phi - 4i\partial_3\phi\partial_2\phi\partial_1\partial_3\phi \\
& -2\phi\partial_2\partial_4\phi\partial_1\partial_3\phi + 2i\phi\partial_2\partial_3\phi\partial_1\partial_3\phi + 4\partial_3\phi\partial_1\phi\partial_1\partial_3\phi - 2i\phi\partial_1\partial_4\phi\partial_1\partial_3\phi \\
& +4i\partial_3\phi^2\partial_1\partial_2\phi - 2i\phi\partial_3^2\phi\partial_1\partial_2\phi + \partial_4^2\phi\left(-2\partial_2\phi^2 - 4i\partial_1\phi\partial_2\phi + 2\partial_1\phi^2\right. \\
& \left.+\phi\left(\partial_2^2\phi + 2i\partial_1\partial_2\phi - \partial_1^2\phi\right)\right) - 2\partial_3\phi^2\partial_1^2\phi + \phi\partial_3^2\phi\partial_1^2\phi \\
& +2i\partial_3\partial_4\phi\left(2\partial_2\phi^2 + 4i\partial_1\phi\partial_2\phi - 2\partial_1\phi^2 + \phi\left(-\partial_2^2\phi - 2i\partial_1\partial_2\phi + \partial_1^2\phi\right)\right).
\end{aligned} \tag{395}
$$

A similar argument shows that the primary operator counted by the monomial s^7y^2 is given by

$$O_{[0,2]\dot\alpha\dot\beta\dot\gamma\dot\rho} = \bar{O}_{1(\dot\alpha\dot\beta\dot\gamma\dot\rho)} - 4\bar{O}_{2(\dot\alpha\dot\beta\dot\gamma\dot\rho)}, \tag{396}$$

where

$$\begin{aligned}\bar{O}_{1\dot\alpha\dot\beta\dot\gamma\dot\rho} &= \phi P_{\alpha\dot\alpha}P_{\beta\dot\beta}\phi P_{\gamma\dot\gamma}P_{\rho\dot\rho}\phi\epsilon^{\alpha\gamma}\epsilon^{\beta\rho},\\ O_{2\dot\alpha\dot\beta\dot\gamma\dot\rho} &= P_{\alpha\dot\alpha}\phi P_{\beta\dot\beta}\phi P_{\gamma\dot\gamma}P_{\rho\dot\rho}\phi\epsilon^{\alpha\gamma}\epsilon^{\beta\rho}.\end{aligned} \tag{397}$$

As another example, we construct the primary operator corresponding to $s^8x^{\frac{5}{2}}y^{\frac{3}{2}}$. This operator is built using 5 derivatives. Here, two dotted indices must be contracted and the remaining indices are symmetrized. Concretely, the primary operator in this case is given by

$$O_{[\frac{5}{2},\frac{3}{2}]\alpha\beta\gamma\rho\tau,\dot\alpha\dot\beta\dot\gamma} = O_{1(\alpha\beta\gamma\rho\tau),(\dot\alpha\dot\beta\dot\gamma)} - 2O_{2(\alpha\beta\gamma\rho\tau),(\dot\alpha\dot\beta\dot\gamma)} + 3O_{3(\alpha\beta\gamma\rho\tau),(\dot\alpha\dot\beta\dot\gamma)}, \tag{398}$$

where

$$\begin{aligned}O_{1\alpha\beta\gamma\rho\tau,\dot\alpha\dot\beta\dot\gamma} &= \phi P_{\beta\dot\beta}P_{\gamma\dot\gamma}P_{\rho\dot\rho}\phi P_{\alpha\dot\alpha}P_{\tau\dot\tau}\phi\epsilon^{\dot\rho\dot\tau},\\ O_{2\alpha\beta\gamma\rho\tau,\dot\alpha\dot\beta\dot\gamma} &= P_{\alpha\dot\alpha}\phi P_{\beta\dot\beta}P_{\gamma\dot\gamma}P_{\rho\dot\rho}\phi P_{\tau\dot\tau}\phi\epsilon^{\dot\rho\dot\tau},\\ O_{2\alpha\beta\gamma\rho\tau,\dot\alpha\dot\beta\dot\gamma} &= P_{\alpha\dot\alpha}P_{\beta\dot\beta}\phi P_{\gamma\dot\gamma}P_{\rho\dot\rho}\phi P_{\tau\dot\tau}\phi\epsilon^{\dot\rho\dot\tau}.\end{aligned} \tag{399}$$

All these results can be checked in Mathematica.

5.6 A More Efficient Method Through Gegenbauer Polynomials

In this section, we translate the problem of constructing primaries into a problem of polynomials. In general, polynomials are simpler to work with than tensors. The ideas here stem from [48]. See also [49].

The primary fields of spin l that we are trying to construct are built in terms of symmetric traceless tensors. We can encode the symmetric tensor into a symmetric polynomial. If one considers the polynomial

$$f(z) = f_{a_1...a_l}z^{a_1}...z^{a_l}, \tag{400}$$

then we can see that the coefficients of $f_{a_1...a_l}$ clearly define symmetric tensors. However, we need to ensure that the corresponding tensor is traceless. To do this, we demand that we have harmonic polynomials

$$\frac{\partial}{\partial z^a}\frac{\partial}{\partial z^a}f(z) = 0 = l(l-1)f_{aaa_3...a_l}z^{a_3}...z^{a_l}. \tag{401}$$

Equivalently, we can choose the variables z^a to be a null vector $z^2 = 0$. Thus, these conditions drop terms explicitly proportional to z^2 which ensures the corresponding symmetric tensor is traceless.

We can extract the corresponding tensor from the polynomial by defining the following differential operator

$$D_\mu = \big(\frac{d}{2} - 1 + z\cdot\frac{\partial}{\partial z}\big)\frac{\partial}{\partial z^\mu} - \frac{1}{2}z_\mu\frac{\partial}{\partial z}\cdot\frac{\partial}{\partial z}. \tag{402}$$

In fact, this operator ensures that the corresponding symmetric tensor is traceless so that we do not even have to impose one of the above conditions. This can be seen through the following commutators

$$[D_\mu, D_\nu] = 0, \qquad D^\mu D_\mu = 0. \tag{403}$$

The corresponding tensor is given by

$$f_{\mu_1...\mu_l} = \frac{1}{l!(\frac{d}{2}-l)_l}D_{\mu_1}...D_{\mu_l}f(z), \tag{404}$$

where $(a)_l$ is the so-called *Pochammer symbol* that is defined by

$$(a)_l = \frac{\Gamma(a+l)}{\Gamma(a)}. \tag{405}$$

Thus, if $\mathcal{O}_{\mu_1\mu_2...\mu_l}$ is a conserved spin-l current

$$\partial^\mu\mathcal{O}_{\mu\mu_2...\mu_l} = 0, \tag{406}$$

then the polynomial version of this statement can be written

$$\partial^\mu D_\mu f_{\mathcal{O}}(z) = 0, \tag{407}$$

where $f_{\mathcal{O}}$ is the polynomial corresponding to $\mathcal{O}_{\mu_1\mu_2...\mu_l}$.

We therefore wish to construct a formula that will yield symmetric traceless tensors. In free conformal field theory with N scalar fields, we have an infinite tower of conserved higher spin currents which are bilinears in the scalar fields with a total of s derivatives acting on the fields. Let ζ^μ be a null vector $\zeta^2 = 0$. By projecting indices with this null vector, the conserved currents can be written

$$J^{ij} = \sum_{k=0}^{s} c_{sk} \hat{\partial}^{s-k} \phi^i \hat{\partial}^k \phi^j \tag{408}$$

where $\hat{\partial} \equiv \partial_\mu \zeta^\mu$ is the projected derivative and c_{sk} are coefficients that will be determined. Next, we define two operators

$$\begin{aligned} \alpha &= \zeta \cdot \frac{\partial}{\partial x_1} + \zeta \cdot \frac{\partial}{\partial x_2} = \zeta \cdot \partial_1 + \zeta \cdot \partial_2, \\ \beta &= \zeta \cdot \frac{\partial}{\partial x_1} - \zeta \cdot \frac{\partial}{\partial x_2} = \zeta \cdot \partial_1 - \zeta \cdot \partial_2. \end{aligned} \tag{409}$$

Then the collection of conserved currents can be written as

$$J^{ij}(x, \zeta) = F(\alpha, \beta) : \phi^i(x_1) \phi^j(x_2) : \big|_{x_1 = x_2 = x}, \tag{410}$$

where the coefficients have been effectively coded into the function $F(\alpha, \beta)$. The statement that the current is conserved is given by

$$(\partial_1 + \partial_2) \cdot DF(\alpha, \beta) : \phi^i(x_1) \phi^j(x_2) : \big|_{x_1 = x_2 = x} = 0. \tag{411}$$

One can use this equation to solve for $F(\alpha, \beta)$ by expressing D_μ in terms of α, β. Terms that are multiplied by $\partial_1 \cdot \partial_1$ or $\partial_2 \cdot \partial_2$ can be dropped since we are working in the free theory. The solution is given by

$$F(\alpha, \beta) = \frac{1}{(1 - 2\beta + \alpha^2)^{\frac{d}{2} - \frac{3}{2}}}. \tag{412}$$

To show that this is indeed a solution, we can rewrite equation (411) as a partial differential equation in terms of α and β through use of the chain rule

$$\begin{aligned} \frac{\partial F}{\partial \zeta^\mu} &= \frac{\partial \alpha}{\partial \zeta^\mu} \frac{\partial F}{\partial \alpha} + \frac{\partial \beta}{\partial \zeta^\mu} \frac{\partial F}{\partial \beta} \\ &= [(\partial_1)_\mu + (\partial_2)_\mu] \frac{\partial F}{\partial \alpha} + [(\partial_1)_\mu - (\partial_2)_\mu] \frac{\partial F}{\partial \beta}. \end{aligned} \tag{413}$$

Now,

$$(\partial_1 + \partial_2)^\mu D_\mu F = (\frac{d}{2} - 1 + \zeta \cdot \frac{\partial}{\partial \zeta})(\partial_1 + \partial_2)^\mu \frac{\partial F}{\partial \zeta^\mu} - \frac{1}{2} \alpha \frac{\partial}{\partial \zeta} \cdot \frac{\partial}{\partial \zeta} F, \tag{414}$$

where we have use the fact that $(\partial_1 + \partial_2)^\mu \zeta_\mu = \alpha$. We expand the first term

$$(\partial_1 + \partial_2)^\mu \frac{\partial F}{\partial \zeta^\mu} = 2 \partial_1 \cdot \partial_2 \frac{\partial F}{\partial \alpha}, \tag{415}$$

$$
\begin{aligned}
\zeta\cdot\frac{\partial}{\partial\zeta}(\partial_1+\partial_2)^\mu\frac{\partial F}{\partial\zeta^\mu} &= 2(\partial_1\cdot\partial_2)\zeta\cdot\frac{\partial}{\partial\zeta}\frac{\partial F}{\partial\alpha}\\
&= 2(\partial_1\cdot\partial_2)\zeta^\sigma\frac{\partial}{\partial\alpha}\frac{\partial F}{\partial\zeta^\sigma}\\
&= 2(\partial_1\cdot\partial_2)\zeta^\sigma\frac{\partial}{\partial\alpha}[(\partial_1)_\sigma+(\partial_2)_\sigma]\frac{\partial F}{\partial\alpha}+2(\partial_1\cdot\partial_2)\zeta^\sigma\frac{\partial}{\partial\alpha}[(\partial_1)_\sigma-(\partial_2)_\sigma]\frac{\partial F}{\partial\beta}\\
&= 2(\partial_1\cdot\partial_2)\zeta^\sigma[(\partial_1)_\sigma+(\partial_2)_\sigma]\frac{\partial}{\partial\alpha}\frac{\partial F}{\partial\alpha}+2(\partial_1\cdot\partial_2)\zeta^\sigma[(\partial_1)_\sigma-(\partial_2)_\sigma]\frac{\partial}{\partial\alpha}\frac{\partial F}{\partial\beta}\\
&= 2(\partial_1\cdot\partial_2)\alpha\frac{\partial^2 F}{\partial\alpha^2}+2(\partial_1\cdot\partial_2)\beta\frac{\partial^2 F}{\partial\alpha\partial\beta}.
\end{aligned}
\tag{416}
$$

Thus, the first term becomes

$$
\begin{aligned}
(\frac{d}{2}-1+\zeta\cdot\frac{\partial}{\partial\zeta})(\partial_1+\partial_2)^\mu\frac{\partial F}{\partial\zeta^\mu} &= d(\partial_1\cdot\partial_2)\frac{\partial F}{\partial\alpha}-2(\partial_1\cdot\partial_2)\frac{\partial F}{\partial\alpha}\\
&\quad +2(\partial_1\cdot\partial_2)\alpha\frac{\partial^2 F}{\partial\alpha^2}+2(\partial_1\cdot\partial_2)\beta\frac{\partial^2 F}{\partial\alpha\partial\beta}.
\end{aligned}
\tag{417}
$$

We next expand the second term

$$
\begin{aligned}
\frac{\partial}{\partial\zeta_\mu}\frac{\partial}{\partial\zeta^\mu}F &= \frac{\partial}{\partial\zeta_\mu}[(\partial_1)_\mu+(\partial_2)_\mu]\frac{\partial F}{\partial\alpha}+\frac{\partial}{\partial\zeta_\mu}[(\partial_1)_\mu-(\partial_2)_\mu]\frac{\partial F}{\partial\beta}\\
&= [(\partial_1)_\mu+(\partial_2)_\mu]\frac{\partial}{\partial\alpha}\frac{\partial F}{\partial\zeta_\mu}+[(\partial_1)_\mu-(\partial_2)_\mu]\frac{\partial}{\partial\beta}\frac{\partial F}{\partial\zeta_\mu}\\
&= [(\partial_1)_\mu+(\partial_2)_\mu]\frac{\partial}{\partial\alpha}[(\partial_1)^\mu+(\partial_2)^\mu]\frac{\partial F}{\partial\alpha}+[(\partial_1)_\mu+(\partial_2)_\mu]\frac{\partial}{\partial\alpha}[(\partial_1)^\mu-(\partial_2)^\mu]\frac{\partial F}{\partial\beta}\\
&\quad +[(\partial_1)_\mu-(\partial_2)_\mu]\frac{\partial}{\partial\beta}[(\partial_1)^\mu+(\partial_2)^\mu]\frac{\partial F}{\partial\alpha}+[(\partial_1)_\mu-(\partial_2)_\mu]\frac{\partial}{\partial\beta}[(\partial_1)^\mu-(\partial_2)^\mu]\frac{\partial F}{\partial\beta}\\
&= 2(\partial_1\cdot\partial_2)\frac{\partial^2 F}{\partial\alpha^2}-2(\partial_1\cdot\partial_2)\frac{\partial^2 F}{\partial\beta^2}.
\end{aligned}
\tag{418}
$$

Thus, the second term becomes

$$
-\frac{1}{2}\alpha\frac{\partial}{\partial\zeta}\cdot\frac{\partial}{\partial\zeta}F=-(\partial_1\cdot\partial_2)\alpha\frac{\partial^2 F}{\partial\alpha^2}+(\partial_1\cdot\partial_2)\alpha\frac{\partial^2 F}{\partial\beta^2}.
\tag{419}
$$

We have therefore obtained the following differential operator acting on $F(\alpha,\beta)$

$$
(d\frac{\partial}{\partial\alpha}-2\frac{\partial}{\partial\alpha}+2\alpha\frac{\partial^2}{\partial\alpha^2}+2\beta\frac{\partial^2}{\partial\alpha\partial\beta}-\alpha\frac{\partial^2}{\partial\alpha^2}+\alpha\frac{\partial^2}{\partial\beta^2})F(\alpha,\beta)=0,
\tag{420}
$$

which simplifies to

$$
(d\frac{\partial}{\partial\alpha}-2\frac{\partial}{\partial\alpha}+\alpha\frac{\partial^2}{\partial\alpha^2}+2\beta\frac{\partial^2}{\partial\alpha\partial\beta}+\alpha\frac{\partial^2}{\partial\beta^2})F(\alpha,\beta)=0.
\tag{421}
$$

Using Mathematica, one can show that F is indeed a solution to the conservation equation.

Fascinatingly, $F(\alpha, \beta)$ is precisely the generating function for the *Gegenbaur polynomials* $C_a^{\nu}(x)$ which is given by

$$F(\alpha, \beta) = \sum_s \alpha^s C^{\frac{d}{2}-\frac{3}{2}}(\beta). \tag{422}$$

The Genebauer polynomials are ultraspherical harmonics (they are a generalization of the Legendre polynomials). For more on ultraspherical polynomials, see for example [50][51].

5.7 Extracting Primaries from $F(\alpha, \beta)$

Here, we wish to extract a few primaries from $F(\alpha, \beta)$. We can therefore check if the Gegenbauer polynomials extracted from F do indeed constitute primaries, by acting on these polynomials with the differential operator we found in the previous section (since the operator D converts the polynomial to its corresponding tensor and $(\partial_1 + \partial_2)$ checks current conservation). In this section, we will also demonstrate how to recover the Gegenbauer polynomials from the generating function; however, the explicit calculations were performed using Mathematica.

We can approach this problem in two ways. We will demonstrate one of the ways and leave the second as a remark. Consider first the generating function for Gegenbauer polynomials in [51]

$$\begin{aligned} F(x,t) = \frac{1}{(1-2xt+t^2)^{\alpha}} &= \sum_{n=0}^{\infty} C_n^{\alpha}(x) t^n \\ &= C_0^{\alpha}(x) + tC_1^{\alpha}(x) + t^2 C_2^{\alpha}(x) + ... \end{aligned} \tag{423}$$

where $C_n^{\alpha}(x)$ are the Gegenbauer polynomials. We can recover a specific polynomial by taking the appropriate derivative; in particular, we see that

$$C_n^{\alpha}(x) = \frac{1}{n!} \frac{\partial^n}{\partial t^n} \left. F(x,t) \right|_{t=0}. \tag{424}$$

The polynomials can be determined by this relation using the Mathematica code shown in Appendix (A.5). To specialize to our case, we can simply make the substitution $x = \beta/\alpha$, $t = \alpha$, and $\alpha = (d/2 - 3/2)$ into the polynomials we have obtained.

Remark 7. *Alternatively, we can immediately specialize to our case*

$$\begin{aligned} F(\alpha,\beta) = \frac{1}{(1-2\beta+\alpha^2)^{\frac{d}{2}-\frac{3}{2}}} &= \sum_{n=0}^{\infty} C_n^{\frac{d}{2}-\frac{3}{2}}(\beta/\alpha)\alpha^n \\ &= C_0^{\frac{d}{2}-\frac{3}{2}}(\beta/\alpha) + \alpha C_1^{\frac{d}{2}-\frac{3}{2}}(\beta/\alpha) + \alpha^2 C_2^{\frac{d}{2}-\frac{3}{2}}(\beta/\alpha) + ... \end{aligned} \tag{425}$$

Taking the derivative here with respect to α is problematic; in particular, we can not set $\alpha = 0$ after differentiation. To overcome this, we can demand that the term β/α remains constant in the limit as α goes to 0. Let this constant be A. Then, our equation becomes

$$\frac{1}{(1-2\beta+\beta^2/A^2)^{\frac{d}{2}-\frac{3}{2}}} = C_0^{\frac{d}{2}-\frac{3}{2}}(A) + \frac{1}{A}\beta C_1^{\frac{d}{2}-\frac{3}{2}}(A) + \frac{1}{A^2}\beta^2 C_2^{\frac{d}{2}-\frac{3}{2}}(A) + ... \qquad (426)$$

We can therefore recover the polynomials by taking the derivative with respect to β (A is of course a constant)

$$C_n^{\frac{d}{2}-\frac{3}{2}}(A) = A^n \frac{1}{n!}\frac{\partial^n}{\partial\beta^n}\left.\frac{1}{(1-2\beta+\beta^2/A^2)^{\frac{d}{2}-\frac{3}{2}}}\right|_{\beta=0}. \qquad (427)$$

Substitution of $A = \beta/\alpha$ then leads to the same polynomials determined by the first method.

From here, we can extract the primaries from $F(\alpha,\beta)$ by substituting the expressions for α and β. For example, consider

$$\alpha C_1^{d/2-3/2}(\beta/\alpha) = 2(\frac{d}{2}-\frac{3}{2})\beta. \qquad (428)$$

Then the polynomial is given by

$$\begin{aligned}
&\alpha C_1^{d/2-3/2}(\beta/\alpha) : \phi^i(x_1)\phi^j(x_2) : |_{x_1=x_2=x} \\
&= 2(\frac{d}{2}-\frac{3}{2})(\zeta\cdot\partial_1 - \zeta\cdot\partial_2) : \phi^i(x_1)\phi^j(x_2) : |_{x_1=x_2=x} \\
&= 2(\frac{d}{2}-\frac{3}{2})(: \zeta\cdot\partial_1\phi^i(x_1)\phi^j(x_2) : - : \phi^i(x_1)\zeta\cdot\partial_2\phi^j(x_2) :) |_{x_1=x_2=x} \\
&= 2(\frac{d}{2}-\frac{3}{2})(\zeta\cdot\partial\phi^i(x)\phi^j(x) - \phi^i(x)\zeta\cdot\partial\phi^j(x)).
\end{aligned} \qquad (429)$$

As another example, consider

$$\begin{aligned}
&\alpha^3 C_3^{d/2-3/2}(\beta/\alpha) : \phi^i(x_1)\phi^j(x_2) : |_{x_1=x_2=x} \\
&= (1+\omega)\omega[\frac{4}{3}(2+\omega)\beta^3 - 2\beta\alpha^2] : \phi^i(x_1)\phi^j(x_2) : |_{x_1=x_2=x} \\
&= (1+\omega)\omega[\frac{4}{3}(2+\omega)\{(\zeta\cdot\partial_1)^3 - 3(\zeta\cdot\partial_1)^2\zeta\cdot\partial_2 + 3\zeta\cdot\partial_1(\zeta\cdot\partial_2)^2 - (\zeta\cdot\partial_2)^3\} \\
&-2\{(\zeta\cdot\partial_1)^3 + (\zeta\cdot\partial_1)^2(\zeta\cdot\partial_2) - (\zeta\cdot\partial_1)(\zeta\cdot\partial_2)^2 - (\zeta\cdot\partial_2)^3\}] : \phi^i(x_1)\phi^j(x_2) : |_{x_1=x_2=x},
\end{aligned} \qquad (430)$$

where $\omega = d/2 - 3/2$.

To obtain the corresponding tensor, we use

$$D_\mu = (\frac{d}{2} - 1 + \zeta\cdot\frac{\partial}{\partial\zeta})\frac{\partial}{\partial\zeta^\mu} - \frac{1}{2}\zeta_\mu\frac{\partial}{\partial\zeta}\cdot\frac{\partial}{\partial\zeta}. \qquad (431)$$

Consider the first polynomial on the previous page. Since this polynomial contains a single ζ in each term, we conclude that the corresponding tensor must have one index. We therefore act on the polynomial once with D_μ

$$\begin{aligned}&D_\mu\{\alpha C_1^{d/2-3/2}(\beta/\alpha) : \phi^i(x_1)\phi^j(x_2) : |_{x_1=x_2=x}\}\\&=2(\frac{d}{2}-\frac{3}{2})(D_\mu\zeta\cdot\partial\phi^i(x)\phi^j(x)-\phi^i(x)D_\mu\zeta\cdot\partial\phi^j(x)).\end{aligned}\tag{432}$$

Now,

$$\begin{aligned}D_\mu(\zeta\cdot\partial)&=(\frac{d}{2}-1+\zeta\cdot\frac{\partial}{\partial\zeta})\frac{\partial}{\partial\zeta^\mu}(\zeta\cdot\partial)-\frac{1}{2}\zeta_\mu\frac{\partial}{\partial\zeta}\cdot\frac{\partial}{\partial\zeta}(\zeta\cdot\partial)\\&=(\frac{d}{2}-1)\partial_\mu,\end{aligned}\tag{433}$$

and thus the corresponding tensor is given by

$$\begin{aligned}\mathcal{O}_\mu&=\frac{1}{l!(\frac{d}{2}-1)_l}D_\mu\{\alpha C_1^{d/2-3/2}(\beta/\alpha) : \phi^i(x_1)\phi^j(x_2) : |_{x_1=x_2=x}\}\\&=\frac{1}{l!(\frac{d}{2}-1)_l}D_\mu\big[2(\frac{d}{2}-\frac{3}{2})(\frac{d}{2}-1)\partial_\mu\phi^i(x)\phi^j(x)-\phi^i(x)2(\frac{d}{2}-\frac{3}{2})(\frac{d}{2}-1)\partial_\mu\phi^j(x)\big],\end{aligned}\tag{434}$$

where we used equation (404). Using the special conformal operator K_μ and the following commutation relations

$$\begin{aligned}&[P_\mu,K_\nu]=-i(2\eta_{\mu\nu}D+2M_{\mu\nu}),\\&[M_{\mu\nu},P_\alpha]=-i(\eta_{\nu\alpha}P_\mu-\eta_{\mu\alpha}P_\nu),\\&[D,P_\mu]=-iP_\mu,\end{aligned}\tag{435}$$

with

$$D\phi=-i\frac{d-2}{2}\phi,\qquad K_\mu\phi=0,\qquad M_{\mu\nu}\phi=0,\tag{436}$$

(where we have restored the eigenvalue of D to its general form since we are working in d-dimensions - see the discussion after equation 9) we find that

$$\begin{aligned}K_\nu\mathcal{O}_\mu&=\frac{2}{l!(\frac{d}{2}-1)_l}(\frac{d}{2}-\frac{3}{2})(\frac{d}{2}-1)\big(K_\nu P_\mu\phi^i(x)\phi^j(x)-\phi^i(x)K_\nu P_\mu\phi^j(x)\big)\\&=\frac{2}{l!(\frac{d}{2}-1)_l}(\frac{d}{2}-\frac{3}{2})(\frac{d}{2}-1)(2\eta_{\mu\nu}\phi^i(x)\phi^j(x)-2\eta_{\mu\nu}\phi^i(x)\phi^j(x))\\&=0,\end{aligned}\tag{437}$$

where we have rewritten the operator in terms of the corresponding momentum operators. We therefore conclude that $\mathcal{O}_\mu$ is a primary operator.

To find the corresponding tensor for the second polynomial in our example, notice that it is a degree three polynomial in ζ; we must therefore apply D_μ three times to obtain the corresponding three index tensor. Now,

$$\begin{aligned} D_\nu(\zeta\cdot\partial)^3 &= (\frac{d}{2}-1+\zeta\cdot\frac{\partial}{\partial\zeta})\frac{\partial}{\partial\zeta^\nu}(\zeta\cdot\partial)^3-\frac{1}{2}\zeta_\nu\frac{\partial}{\partial\zeta}\cdot\frac{\partial}{\partial\zeta}(\zeta\cdot\partial)^3 \\ &= (\frac{d}{2}-1+\zeta\cdot\frac{\partial}{\partial\zeta})3(\zeta\cdot\partial)^2\partial_\nu-\frac{6}{2}\zeta_\nu(\zeta\cdot\partial)\partial\cdot\partial \\ &= 3(\frac{d}{2}-1)(\zeta\cdot\partial)^2\partial_\nu+6(\zeta\cdot\partial)^2\partial_\nu-3\zeta_\nu(\zeta\cdot\partial)\partial\cdot\partial, \end{aligned} \tag{438}$$

$$\begin{aligned} D_\mu D_\nu(\zeta\cdot\partial)^3 &= (\frac{d}{2}-1+\zeta\cdot\frac{\partial}{\partial\zeta})\frac{\partial}{\partial\zeta^\mu}D_\nu(\zeta\cdot\partial)^3-\frac{1}{2}\zeta_\mu\frac{\partial}{\partial\zeta}\cdot\frac{\partial}{\partial\zeta}D_\nu(\zeta\cdot\partial)^3 \\ &= (\frac{d}{2}-1+\zeta\cdot\frac{\partial}{\partial\zeta})\big[6(\frac{d}{2}-1)(\zeta\cdot\partial)\partial_\mu\partial_\nu+12(\zeta\cdot\partial)\partial_\mu\partial_\nu-3\eta_{\mu\nu}(\zeta\cdot\partial)\partial\cdot\partial \\ &-3\zeta_\nu\partial_\mu\partial\cdot\partial\big]-\frac{1}{2}\zeta_\mu\big[6(\frac{d}{2}-1)\partial\cdot\partial\partial_\nu+12\partial\cdot\partial\partial_\nu-6\partial_\nu\partial\cdot\partial\big] \\ &= 6(\frac{d}{2})(\frac{d}{2}-1)(\zeta\cdot\partial)\partial_\mu\partial_\nu+12(\frac{d}{2})(\zeta\cdot\partial)\partial_\mu\partial_\nu-3(\frac{d}{2})\eta_{\mu\nu}(\zeta\cdot\partial)\partial\cdot\partial \\ &-3(\frac{d}{2})\zeta_\nu\partial_\mu\partial\cdot\partial-3(\frac{d}{2})\zeta_\mu\partial_\nu\partial\cdot\partial, \end{aligned} \tag{439}$$

where we used the fact that $\zeta\cdot\frac{\partial}{\partial\zeta}$ computes the homogeneity degree and therefore cancels the effect of the $(-1)(\partial/\partial\zeta^\mu)D_\nu(\zeta\cdot\partial)^3$ term. Finally,

$$D_\rho D_\mu D_\nu(\zeta\cdot\partial)^3=(\frac{d}{2}-1+\zeta\cdot\frac{\partial}{\partial\zeta})\frac{\partial}{\partial\zeta^\rho}D_\mu D_\nu(\zeta\cdot\partial)^3-\frac{1}{2}\zeta_\rho\frac{\partial}{\partial\zeta}\cdot\frac{\partial}{\partial\zeta}D_\mu D_\nu(\zeta\cdot\partial)^3. \tag{440}$$

Observe that

$$\begin{aligned} \frac{\partial}{\partial\zeta^\rho}D_\mu D_\nu(\zeta\cdot\partial)^3 &= 6(\frac{d}{2})(\frac{d}{2}-1)\partial_\rho\partial_\mu\partial_\nu+12(\frac{d}{2})\partial_\rho\partial_\mu\partial_\nu-3(\frac{d}{2})\eta_{\mu\nu}\partial_\rho\partial\cdot\partial \\ &-3(\frac{d}{2})\eta_{\rho\nu}\partial_\mu\partial\cdot\partial-3(\frac{d}{2})\eta_{\rho\mu}\partial_\nu\partial\cdot\partial. \end{aligned} \tag{441}$$

Thus, we find that

$$\begin{aligned} D_\rho D_\mu D_\nu(\zeta\cdot\partial)^3 &= 6(\frac{d}{2})(\frac{d}{2}-1)^2\partial_\rho\partial_\mu\partial_\nu+12(\frac{d}{2})(\frac{d}{2}-1)\partial_\rho\partial_\mu\partial_\nu-3(\frac{d}{2})(\frac{d}{2}-1)\eta_{\mu\nu}\partial_\rho\partial\cdot\partial \\ &-3(\frac{d}{2})(\frac{d}{2}-1)\eta_{\rho\nu}\partial_\mu\partial\cdot\partial-3(\frac{d}{2})(\frac{d}{2}-1)\eta_{\rho\mu}\partial_\nu\partial\cdot\partial. \end{aligned} \tag{442}$$

Therefore, we obtain

$$\begin{aligned} D_\rho D_\mu D_\nu(\zeta\cdot\partial_1)^3 &= 6(\frac{d}{2})(\frac{d}{2}-1)^2(\partial_1)_\rho(\partial_1)_\mu(\partial_1)_\nu+12(\frac{d}{2})(\frac{d}{2}-1)(\partial_1)_\rho(\partial_1)_\mu(\partial_1)_\nu \\ &-3(\frac{d}{2})(\frac{d}{2}-1)\eta_{\mu\nu}(\partial_1)_\rho\partial_1\cdot\partial_1-3(\frac{d}{2})(\frac{d}{2}-1)\eta_{\rho\nu}(\partial_1)_\mu\partial_1\cdot\partial_1 \\ &-3(\frac{d}{2})(\frac{d}{2}-1)\eta_{\rho\mu}(\partial_1)_\nu\partial_1\cdot\partial_1. \end{aligned} \tag{443}$$

The case for $D_\rho D_\mu D_\nu(\zeta\cdot\partial_2)^3$ is similar. In addition,

$$\begin{aligned}
D_\nu(\zeta\cdot\partial_1)^2(\zeta\cdot\partial_2) =& (\frac{d}{2}-1+\zeta\cdot\frac{\partial}{\partial\zeta})\frac{\partial}{\partial\zeta^\nu}(\zeta\cdot\partial_1)^2(\zeta\cdot\partial_2)-\frac{1}{2}\zeta_\nu\frac{\partial}{\partial\zeta}\cdot\frac{\partial}{\partial\zeta}(\zeta\cdot\partial_1)^2(\zeta\cdot\partial_2)\\
&=(\frac{d}{2}-1+\zeta\cdot\frac{\partial}{\partial\zeta})\{2(\zeta\cdot\partial_1)(\zeta\cdot\partial_2)(\partial_1)_\nu+(\zeta\cdot\partial_1)^2(\partial_2)_\nu\}\\
&-\frac{1}{2}\zeta_\nu\frac{\partial}{\partial\zeta_\sigma}\{2(\zeta\cdot\partial_1)(\zeta\cdot\partial_2)(\partial_1)_\sigma+(\zeta\cdot\partial_1)^2(\partial_2)_\sigma\}\\
&=(\frac{d}{2}-1)\{2(\zeta\cdot\partial_1)(\zeta\cdot\partial_2)(\partial_1)_\nu+(\zeta\cdot\partial_1)^2(\partial_2)_\nu\}\\
&+\{4(\zeta\cdot\partial_1)(\zeta\cdot\partial_2)(\partial_1)_\nu+2(\zeta\cdot\partial_1)^2(\partial_2)_\nu\}\\
&-\frac{1}{2}\zeta_\nu\{2(\zeta\cdot\partial_2)\partial_1\cdot\partial_1+4(\zeta\cdot\partial_1)\partial_1\cdot\partial_2\}\\
&=(\frac{d}{2}+1)\{2(\zeta\cdot\partial_1)(\zeta\cdot\partial_2)(\partial_1)_\nu+(\zeta\cdot\partial_1)^2(\partial_2)_\nu\}\\
&-\zeta_\nu(\zeta\cdot\partial_2)\partial_1\cdot\partial_1-2\zeta_\nu(\zeta\cdot\partial_1)\partial_1\cdot\partial_2.
\end{aligned}\tag{444}$$

$$\begin{aligned}
D_\mu D_\nu(\zeta\cdot\partial_1)^2(\zeta\cdot\partial_2) =& (\frac{d}{2}-1+\zeta\cdot\frac{\partial}{\partial\zeta})\frac{\partial}{\partial\zeta^\mu}D_\nu(\zeta\cdot\partial_1)^2(\zeta\cdot\partial_2)-\frac{1}{2}\zeta_\mu\frac{\partial}{\partial\zeta}\cdot\frac{\partial}{\partial\zeta}D_\nu(\zeta\cdot\partial_1)^2(\zeta\cdot\partial_2)\\
&=(\frac{d}{2}-1+\zeta\cdot\frac{\partial}{\partial\zeta})[(\frac{d}{2}+1)\{2(\partial_1)_\mu(\zeta\cdot\partial_2)(\partial_1)_\nu+2(\zeta\cdot\partial_1)(\partial_2)_\mu(\partial_1)_\nu\\
&+2(\zeta\cdot\partial_1)(\partial_1)_\mu(\partial_2)_\nu\}-\eta_{\mu\nu}(\zeta\cdot\partial_2)\partial_1\cdot\partial_1-\zeta_\nu(\partial_2)_\mu\partial_1\cdot\partial_1\\
&-2\eta_{\mu\nu}(\zeta\cdot\partial_1)\partial_1\cdot\partial_2-2\zeta_\nu(\partial_1)_\mu\partial_1\cdot\partial_2]-\frac{1}{2}\zeta_\mu\frac{\partial}{\partial\zeta_\sigma}[(\frac{d}{2}+1)\\
&\times\{2(\partial_1)_\sigma(\zeta\cdot\partial_2)(\partial_1)_\nu+2(\zeta\cdot\partial_1)(\partial_2)_\sigma(\partial_1)_\nu+2(\zeta\cdot\partial_1)(\partial_1)_\sigma(\partial_2)_\nu\}\\
&-\eta_{\sigma\nu}(\zeta\cdot\partial_2)\partial_1\cdot\partial_1-\zeta_\nu(\partial_2)_\sigma\partial_1\cdot\partial_1-2\eta_{\sigma\nu}(\zeta\cdot\partial_1)\partial_1\cdot\partial_2-2\zeta_\nu(\partial_1)_\sigma\partial_1\cdot\partial_2]\\
&=(\frac{d}{2})(\frac{d}{2}+1)\{2(\partial_1)_\mu(\zeta\cdot\partial_2)(\partial_1)_\nu+2(\zeta\cdot\partial_1)(\partial_2)_\mu(\partial_1)_\nu\\
&+2(\zeta\cdot\partial_1)(\partial_1)_\mu(\partial_2)_\nu\}-(\frac{d}{2})\{\eta_{\mu\nu}(\zeta\cdot\partial_2)\partial_1\cdot\partial_1+\zeta_\nu(\partial_2)_\mu\partial_1\cdot\partial_1+2\eta_{\mu\nu}\\
&\times(\zeta\cdot\partial_1)\partial_1\cdot\partial_2+2\zeta_\nu(\partial_1)_\mu\partial_1\cdot\partial_2\}-\frac{1}{2}(\frac{d}{2}+1)\{2\zeta_\mu(\partial_1)_\nu\partial_1\cdot\partial_2+2\zeta_\mu\\
&\times(\partial_1)_\mu\partial_1\cdot\partial_2+2\zeta_\mu(\partial_2)_\nu\partial_1\cdot\partial_1\}+\frac{1}{2}\zeta_\mu(\partial_2)_\nu\partial_1\cdot\partial_1+\frac{1}{2}\zeta_\mu(\partial_2)_\nu\partial_1\cdot\partial_2\\
&+\zeta_\mu(\partial_1)_\nu\partial_1\cdot\partial_2+\zeta_\mu(\partial_1)_\nu\partial_1\cdot\partial_2\\
&=(\frac{d}{2})(\frac{d}{2}+1)\{2(\partial_1)_\mu(\zeta\cdot\partial_2)(\partial_1)_\nu+2(\zeta\cdot\partial_1)(\partial_2)_\mu(\partial_1)_\nu\\
&+2(\zeta\cdot\partial_1)(\partial_1)_\mu(\partial_2)_\nu\}-(\frac{d}{2})\{\eta_{\mu\nu}(\zeta\cdot\partial_2)\partial_1\cdot\partial_1+\zeta_\nu(\partial_2)_\mu\partial_1\cdot\partial_1\\
&+2\eta_{\mu\nu}(\zeta\cdot\partial_1)\partial_1\cdot\partial_2+2\zeta_\nu(\partial_1)_\mu\partial_1\cdot\partial_2\}-\frac{1}{2}(\frac{d}{2})\{4\zeta_\mu(\partial_1)_\nu\partial_1\cdot\partial_2\\
&+2\zeta_\mu(\partial_2)_\nu\partial_1\cdot\partial_1\}.
\end{aligned}\tag{445}$$

Finally,

$$\begin{aligned}
D_\rho D_\mu D_\nu(\zeta\cdot\partial_1)^2(\zeta\cdot\partial_2) =& (\frac{d}{2}-1+\zeta\cdot\frac{\partial}{\partial\zeta})\frac{\partial}{\partial\zeta^\rho}D_\mu D_\nu(\zeta\cdot\partial_1)^2(\zeta\cdot\partial_2)\\
&-\frac{1}{2}\zeta_\rho\frac{\partial}{\partial\zeta}\cdot\frac{\partial}{\partial\zeta}D_\mu D_\nu(\zeta\cdot\partial_1)^2(\zeta\cdot\partial_2)\\
&=(\frac{d}{2}-1)\frac{\partial}{\partial\zeta^\rho}D_\mu D_\nu(\zeta\cdot\partial_1)^2(\zeta\cdot\partial_2)\\
&=(\frac{d}{2}-1)(\frac{d}{2})(\frac{d}{2}+1)\{2(\partial_2)_\rho(\partial_1)_\mu(\partial_1)_\nu+2(\partial_1)_\rho(\partial_2)_\mu(\partial_1)_\nu\\
&+2(\partial_1)_\rho(\partial_1)_\mu(\partial_2)_\nu\}-(\frac{d}{2}-1)(\frac{d}{2})\{\eta_{\mu\nu}(\partial_2)_\rho\partial_1\cdot\partial_1+\eta_{\rho\nu}(\partial_2)_\mu\partial_1\cdot\partial_1\\
&+\eta_{\mu\rho}(\partial_2)_\nu\partial_1\cdot\partial_1+2\eta_{\mu\nu}(\partial_1)_\rho\partial_1\cdot\partial_2+2\eta_{\rho\nu}(\partial_1)_\mu\partial_1\cdot\partial_2\\
&+2\eta_{\mu\rho}(\partial_1)_\nu\partial_1\cdot\partial_2\}.
\end{aligned}\tag{446}$$

Similarly,

$$\begin{aligned}
D_\rho D_\mu D_\nu(\zeta\cdot\partial_1)(\zeta\cdot\partial_2)^2 =& (\frac{d}{2}-1)(\frac{d}{2})(\frac{d}{2}+1)\{2(\partial_1)_\rho(\partial_2)_\mu(\partial_2)_\nu+2(\partial_2)_\rho(\partial_1)_\mu(\partial_2)_\nu\\
&+2(\partial_2)_\rho(\partial_2)_\mu(\partial_1)_\nu\}-(\frac{d}{2}-1)(\frac{d}{2})\{\eta_{\mu\nu}(\partial_1)_\rho\partial_2\cdot\partial_2+\eta_{\rho\nu}(\partial_1)_\mu\partial_2\cdot\partial_2\\
&+\eta_{\mu\rho}(\partial_1)_\nu\partial_2\cdot\partial_2+2\eta_{\mu\nu}(\partial_2)_\rho\partial_1\cdot\partial_2+2\eta_{\rho\nu}(\partial_2)_\mu\partial_1\cdot\partial_2\\
&+2\eta_{\mu\rho}(\partial_2)_\nu\partial_1\cdot\partial_2\}.
\end{aligned}\tag{447}$$

Using $\gamma = d/2-1$ and $\phi^i\equiv\phi^i(x_1),\phi^j\equiv\phi^j(x_2)$, we find that the corresponding tensor is given by

$$\begin{aligned}
\mathcal{O}_{\rho\mu\nu} =& \frac{\omega(1+\omega)}{l!(\frac{d}{2}-1)_l}\Bigg([\frac{4}{3}(2+\omega)-2][6(\gamma+1)\gamma^2: (\partial_1)_\rho(\partial_1)_\mu(\partial_1)_\nu\phi^i\phi^j:\\
&+12(\gamma+1)\gamma:(\partial_1)_\rho(\partial_1)_\mu(\partial_1)_\nu\phi^i\phi^j:-3(\gamma+1)\gamma\eta_{\mu\nu}:(\partial_1)_\rho\partial_1\cdot\partial_1\phi^i\phi^j:\\
&-3(\gamma+1)\gamma\eta_{\rho\nu}:(\partial_1)_\mu\partial_1\cdot\partial_1\phi^i\phi^j:-3(\gamma+1)\gamma\eta_{\mu\rho}:(\partial_1)_\nu\partial_1\cdot\partial_1\phi^i\phi^j:]\,|_{x_1=x_2=x}\\
&-[\frac{4}{3}(2+\omega)-2][6(\gamma+1)\gamma^2:\phi^i(\partial_2)_\rho(\partial_2)_\mu(\partial_2)_\nu\phi^j:\\
&+12(\gamma+1)\gamma:\phi^i(\partial_2)_\rho(\partial_2)_\mu(\partial_2)_\nu\phi^j:-3(\gamma+1)\gamma\eta_{\mu\nu}:\phi^i(\partial_2)_\rho\partial_2\cdot\partial_2\phi^j:\\
&-3(\gamma+1)\gamma\eta_{\rho\nu}:\phi^i(\partial_2)_\mu\partial_2\cdot\partial_2\phi^j:-3(\gamma+1)\gamma\eta_{\mu\rho}:\phi^i(\partial_2)_\nu\partial_2\cdot\partial_2\phi^j:]\,|_{x_1=x_2=x}\\
&-[4\omega+10][2\gamma(\gamma+1)(\gamma+2):(\partial_1)_\mu(\partial_1)_\nu\phi^i(\partial_2)_\rho\phi^j:+2\gamma(\gamma+1)(\gamma+2)\\
&\times:(\partial_1)_\rho(\partial_1)_\nu\phi^i(\partial_2)_\mu\phi^j:+2\gamma(\gamma+1)(\gamma+2):(\partial_1)_\rho(\partial_1)_\mu\phi^i(\partial_2)_\nu\phi^j:-\gamma(\gamma+1)\eta_{\mu\nu}\\
&\times:\partial_1\cdot\partial_1\phi^i(\partial_2)_\rho\phi^j:-\gamma(\gamma+1)\eta_{\rho\nu}:\partial_1\cdot\partial_1\phi^i(\partial_2)_\mu\phi^j:-\gamma(\gamma+1)\eta_{\mu\rho}:\partial_1\cdot\partial_1\phi^i(\partial_2)_\nu\phi^j:\\
&-2\gamma(\gamma+1)\eta_{\mu\nu}:(\partial_1)_\rho(\partial_1)_\sigma\phi^i(\partial_2)^\sigma\phi^j:-2\gamma(\gamma+1)\eta_{\rho\nu}:(\partial_1)_\mu(\partial_1)_\sigma\phi^i(\partial_2)^\sigma\phi^j:\\
&-2\gamma(\gamma+1)\eta_{\mu\rho}:(\partial_1)_\nu(\partial_1)_\sigma\phi^i(\partial_2)^\sigma\phi^j:]\,|_{x_1=x_2=x}+[4\omega+10][2\gamma(\gamma+1)\\
&\times(\gamma+2):(\partial_1)_\rho\phi^i(\partial_2)_\mu(\partial_2)_\nu\phi^j:+2\gamma(\gamma+1)(\gamma+2):(\partial_1)_\mu\phi^i(\partial_2)_\rho(\partial_2)_\nu\phi^j:\\
&+2\gamma(\gamma+1)(\gamma+2):(\partial_1)_\nu\phi^i(\partial_2)_\rho(\partial_2)_\mu\phi^j:-\gamma(\gamma+1)\eta_{\mu\nu}:(\partial_1)_\rho\phi^i\partial_2\cdot\partial_2\phi^j:\\
&-\gamma(\gamma+1)\eta_{\rho\nu}:(\partial_1)_\mu\phi^i\partial_2\cdot\partial_2\phi^j:-\gamma(\gamma+1)\eta_{\mu\rho}:(\partial_1)_\nu\phi^i\partial_2\cdot\partial_2\phi^j:-2\gamma(\gamma+1)\eta_{\mu\nu}\\
&:(\partial_1)^\sigma\phi^i(\partial_2)_\rho(\partial_2)_\sigma\phi^j:-2\gamma(\gamma+1)\eta_{\rho\nu}:(\partial_1)^\sigma\phi^i(\partial_2)_\mu(\partial_2)_\sigma\phi^j:-2\gamma(\gamma+1)\eta_{\mu\rho}\\
&:(\partial_1)^\sigma\phi^i(\partial_2)_\nu(\partial_2)_\sigma\phi^j:]\,|_{x_1=x_2=x}\Bigg).
\end{aligned}\tag{448}$$

Setting $x_1 = x_2 = x$, we find that

$$\begin{aligned}
\mathcal{O}_{\rho\mu\nu} = & \frac{\omega(1+\omega)}{l!(\frac{d}{2}-1)_l}\Bigg([\frac{4}{3}(2+\omega)-2][6(\gamma+1)\gamma^2(\partial)_\rho(\partial)_\mu(\partial)_\nu\phi^i\phi^j \\
& +12(\gamma+1)\gamma(\partial)_\rho(\partial)_\mu(\partial)_\nu\phi^i\phi^j - 3(\gamma+1)\gamma\eta_{\mu\nu}(\partial)_\rho\partial\cdot\partial\phi^i\phi^j \\
& -3(\gamma+1)\gamma\eta_{\rho\nu}(\partial)_\mu\partial\cdot\partial\phi^i\phi^j - 3(\gamma+1)\gamma\eta_{\mu\rho} : (\partial)_\nu\partial\cdot\partial\phi^i\phi^j] \\
& -[\frac{4}{3}(2+\omega)-2][6(\gamma+1)\gamma^2\phi^i(\partial)_\rho(\partial)_\mu(\partial)_\nu\phi^j \\
& +12(\gamma+1)\gamma\phi^i(\partial)_\rho(\partial)_\mu(\partial)_\nu\phi^j - 3(\gamma+1)\gamma\eta_{\mu\nu}\phi^i(\partial)_\rho\partial\cdot\partial\phi^j \\
& -3(\gamma+1)\gamma\eta_{\rho\nu}\phi^i(\partial)_\mu\partial\cdot\partial\phi^j - 3(\gamma+1)\gamma\eta_{\mu\rho}\phi^i(\partial)_\nu\partial\cdot\partial\phi^j] \\
& -[4\omega+10][2\gamma(\gamma+1)(\gamma+2)(\partial)_\mu(\partial)_\nu\phi^i(\partial)_\rho\phi^j + 2\gamma(\gamma+1)(\gamma+2)(\partial)_\rho(\partial)_\nu\phi^i(\partial)_\mu\phi^j \\
& +2\gamma(\gamma+1)(\gamma+2)(\partial)_\rho(\partial)_\mu\phi^i(\partial)_\nu\phi^j - \gamma(\gamma+1)\eta_{\mu\nu}\partial\cdot\partial\phi^i(\partial)_\rho\phi^j \\
& -\gamma(\gamma+1)\eta_{\rho\nu}\partial\cdot\partial\phi^i(\partial)_\mu\phi^j - \gamma(\gamma+1)\eta_{\mu\rho}\partial\cdot\partial\phi^i(\partial)_\nu\phi^j \\
& -2\gamma(\gamma+1)\eta_{\mu\nu}(\partial)_\rho(\partial)_\sigma\phi^i(\partial)^\sigma\phi^j - 2\gamma(\gamma+1)\eta_{\rho\nu}(\partial)_\mu(\partial)_\sigma\phi^i(\partial)^\sigma\phi^j \\
& -2\gamma(\gamma+1)\eta_{\mu\rho}(\partial)_\nu(\partial)_\sigma\phi^i(\partial)^\sigma\phi^j] \\
& +[4\omega+10][2\gamma(\gamma+1)(\gamma+2)(\partial)_\rho\phi^i(\partial)_\mu(\partial)_\nu\phi^j + 2\gamma(\gamma+1)(\gamma+2)(\partial)_\mu\phi^i(\partial)_\rho(\partial)_\nu\phi^j \\
& +2\gamma(\gamma+1)(\gamma+2)(\partial)_\nu\phi^i(\partial)_\rho(\partial)_\mu\phi^j - \gamma(\gamma+1)\eta_{\mu\nu}(\partial)_\rho\phi^i\partial\cdot\partial\phi^j \\
& -\gamma(\gamma+1)\eta_{\rho\nu}(\partial)_\mu\phi^i\partial\cdot\partial\phi^j - \gamma(\gamma+1)\eta_{\mu\rho}(\partial)_\nu\phi^i\partial\cdot\partial\phi^j \\
& -2\gamma(\gamma+1)\eta_{\mu\nu}(\partial)^\sigma\phi^i(\partial)_\rho(\partial)_\sigma\phi^j - 2\gamma(\gamma+1)\eta_{\rho\nu}(\partial)^\sigma\phi^i(\partial)_\mu(\partial)_\sigma\phi^j \\
& -2\gamma(\gamma+1)\eta_{\mu\rho}(\partial)^\sigma\phi^i(\partial)_\nu(\partial)_\sigma\phi^j]\Bigg).
\end{aligned} \tag{449}$$

To test whether this tensor is a primary operator, we act on it with the special conformal operator K_σ. We start with a case that simplifies the tensor; in particular, consider the tensor for the case $\mu \neq \nu, \nu \neq \rho$, and $\mu \neq \rho$:

$$\begin{aligned}
\mathcal{O}_{\rho\mu\nu} = & \frac{\omega(1+\omega)}{l!(\frac{d}{2}-1)_l}\Bigg([\frac{4}{3}(2+\omega)-2][6(\gamma+1)\gamma^2(\partial)_\rho(\partial)_\mu(\partial)_\nu\phi^i\phi^j \\
& +12(\gamma+1)\gamma(\partial)_\rho(\partial)_\mu(\partial)_\nu\phi^i\phi^j] - [\frac{4}{3}(2+\omega)-2][6(\gamma+1)\gamma^2\phi^i(\partial)_\rho(\partial)_\mu(\partial)_\nu\phi^j \\
& +12(\gamma+1)\gamma\phi^i(\partial)_\rho(\partial)_\mu(\partial)_\nu\phi^j] - [4\omega+10][2\gamma(\gamma+1)(\gamma+2)(\partial)_\mu(\partial)_\nu\phi^i(\partial)_\rho\phi^j \\
& +2\gamma(\gamma+1)(\gamma+2)(\partial)_\rho(\partial)_\nu\phi^i(\partial)_\mu\phi^j + 2\gamma(\gamma+1)(\gamma+2)(\partial)_\rho(\partial)_\mu\phi^i(\partial)_\nu\phi^j] \\
& +[4\omega+10][2\gamma(\gamma+1)(\gamma+2)(\partial)_\rho\phi^i(\partial)_\mu(\partial)_\nu\phi^j + 2\gamma(\gamma+1)(\gamma+2)(\partial)_\mu\phi^i(\partial)_\rho(\partial)_\nu\phi^j \\
& +2\gamma(\gamma+1)(\gamma+2)(\partial)_\nu\phi^i(\partial)_\rho(\partial)_\mu\phi^j]\Bigg).
\end{aligned} \tag{450}$$

Rewriting this equation in terms of the corresponding momentum operators yields

$$\begin{aligned}\mathcal{O}_{\rho\mu\nu}=&\frac{\omega(1+\omega)}{l!(\frac{d}{2}-1)_l}\bigg([\frac{4}{3}(2+\omega)-2][6(\gamma+1)\gamma^2 P_\rho P_\mu P_\nu\phi^i\phi^j\\&+12(\gamma+1)\gamma P_\rho P_\mu P_\nu\phi^i\phi^j]-[\frac{4}{3}(2+\omega)-2][6(\gamma+1)\gamma^2\phi^i P_\rho P_\mu P_\nu\phi^j\\&+12(\gamma+1)\gamma\phi^i P_\rho P_\mu P_\nu\phi^j]-[4\omega+10][2\gamma(\gamma+1)(\gamma+2)P_\mu P_\nu\phi^i P_\rho\phi^j\\&+2\gamma(\gamma+1)(\gamma+2)P_\rho P_\nu\phi^i P_\mu\phi^j+2\gamma(\gamma+1)(\gamma+2)P_\rho P_\mu\phi^i P_\nu\phi^j]\\&+[4\omega+10][2\gamma(\gamma+1)(\gamma+2)P_\rho\phi^i P_\mu P_\nu\phi^j+2\gamma(\gamma+1)(\gamma+2)P_\mu\phi^i P_\rho P_\nu\phi^j\\&+2\gamma(\gamma+1)(\gamma+2)P_\nu\phi^i P_\rho P_\mu\phi^j]\bigg).\end{aligned}\tag{451}$$

We will explicitly verify that $O_{\rho\mu\nu}$ is a primary operator. We can use the following expressions

$$K_\sigma P_\rho\phi=(d-2)\eta_{\rho\sigma}\phi\tag{452}$$

$$K_\sigma P_\mu P_\nu\phi=d\eta_{\nu\sigma}P_\mu\phi+d\eta_{\mu\sigma}P_\nu\phi-2\eta_{\mu\nu}P_\sigma\phi\tag{453}$$

$$\begin{aligned}K_\sigma P_\rho P_\mu P_\nu\phi=&(d+2)\eta_{\nu\sigma}P_\rho P_\mu\phi+(d+2)\eta_{\mu\sigma}P_\rho P_\nu\phi+(d+2)\eta_{\rho\sigma}P_\mu P_\nu\phi\\&-2\eta_{\mu\nu}P_\rho P_\sigma\phi-2\eta_{\rho\nu}P_\mu P_\sigma\phi-2\eta_{\rho\mu}P_\nu P_\sigma\phi,\end{aligned}\tag{454}$$

which for this case, specializes to

$$K_\sigma P_\rho\phi=(d-2)\eta_{\rho\sigma}\phi\tag{455}$$

$$K_\sigma P_\mu P_\nu\phi=d\eta_{\nu\sigma}P_\mu\phi+d\eta_{\mu\sigma}P_\nu\phi\tag{456}$$

$$K_\sigma P_\rho P_\mu P_\nu\phi=(d+2)\eta_{\nu\sigma}P_\rho P_\mu\phi+(d+2)\eta_{\mu\sigma}P_\rho P_\nu\phi+(d+2)\eta_{\rho\sigma}P_\mu P_\nu\phi.\tag{457}$$

We therefore find that

$$\begin{aligned}K_\sigma\mathcal{O}_{\rho\mu\nu}=&\frac{\omega(1+\omega)}{l!(\frac{d}{2}-1)_l}\bigg([\frac{4}{3}(2+\omega)-2][6(\gamma+1)\gamma^2((d+2)\eta_{\nu\sigma}P_\rho P_\mu\phi^i+(d+2)\eta_{\mu\sigma}P_\rho P_\nu\phi^i\\&+(d+2)\eta_{\rho\sigma}P_\mu P_\nu\phi^i)\phi^j+12(\gamma+1)\gamma((d+2)\eta_{\nu\sigma}P_\rho P_\mu\phi^i+(d+2)\eta_{\mu\sigma}P_\rho P_\nu\phi^i\\&+(d+2)\eta_{\rho\sigma}P_\mu P_\nu\phi^i)\phi^j]-[\frac{4}{3}(2+\omega)-2][6(\gamma+1)\gamma^2\phi^i((d+2)\eta_{\nu\sigma}P_\rho P_\mu\phi^j\\&+(d+2)\eta_{\mu\sigma}P_\rho P_\nu\phi^j+(d+2)\eta_{\rho\sigma}P_\mu P_\nu\phi^j)+12(\gamma+1)\gamma\phi^i((d+2)\eta_{\nu\sigma}P_\rho P_\mu\phi^j\\&+(d+2)\eta_{\mu\sigma}P_\rho P_\nu\phi^j+(d+2)\eta_{\rho\sigma}P_\mu P_\nu\phi^j)]-[4\omega+10][2\gamma(\gamma+1)(\gamma+2)(d\eta_{\nu\sigma}P_\mu\phi^i\\&+d\eta_{\mu\sigma}P_\nu\phi^i)P_\rho\phi^j+2\gamma(\gamma+1)(\gamma+2)P_\mu P_\nu\phi^i((d-2)\eta_{\sigma\rho})\phi^j+2\gamma(\gamma+1)\\&(\gamma+2)(d\eta_{\nu\sigma}P_\rho\phi^i+d\eta_{\rho\sigma}P_\nu\phi^i)P_\mu\phi^j+2\gamma(\gamma+1)(\gamma+2)P_\nu P_\rho\phi^i((d-2)\eta_{\sigma\mu})\phi^j\\&+2\gamma(\gamma+1)(\gamma+2)(d\eta_{\mu\sigma}P_\rho\phi^i+d\eta_{\rho\sigma}P_\mu\phi^i)P_\nu\phi^j+2\gamma(\gamma+1)(\gamma+2)P_\mu P_\rho\phi^i((d-2)\\&\times\eta_{\sigma\nu})\phi^j]+[4\omega+10][2\gamma(\gamma+1)(\gamma+2)((d-2)\eta_{\sigma\rho}\phi^i P_\mu P_\nu\phi^j+2\gamma(\gamma+1)(\gamma+2)\\&\times P_\rho\phi^i(d\eta_{\nu\sigma}P_\mu\phi+d\eta_{\mu\sigma}P_\nu\phi)\phi^j+2\gamma(\gamma+1)(\gamma+2)((d-2)\eta_{\sigma\mu})\phi^i P_\rho P_\nu\phi^j+2\gamma(\gamma\\&+1)(\gamma+2)P_\mu\phi^i(d\eta_{\nu\sigma}P_\rho\phi+d\eta_{\rho\sigma}P_\nu\phi)\phi^j+2\gamma(\gamma+1)(\gamma+2)((d-2)\eta_{\sigma\nu})\phi^i P_\rho P_\mu\phi^j\\&+2\gamma(\gamma+1)(\gamma+2)P_\nu\phi^i(d\eta_{\mu\sigma}P_\rho\phi+d\eta_{\rho\sigma}P_\mu\phi)\phi^j]\bigg).\end{aligned}\tag{458}$$

Observe that terms with a single derivative acting on ϕ^i, ϕ^j cancel each other out. For example, we have

$$-2d\gamma(\gamma+1)(\gamma+2)\eta_{\nu\sigma}P_\mu\phi^i P_\rho\phi^j + 2d\gamma(\gamma+1)(\gamma+2)\eta_{\nu\sigma}P_\mu\phi^i P_\rho\phi^j = 0. \tag{459}$$

Similarly, for terms with two derivatives on ϕ^i or ϕ^j, we find cancellation. For example,

$$[\frac{4}{3}(2+\omega)-2]6(d+2)\gamma^2(\gamma+1)\eta_{\nu\sigma}P_\rho P_\mu\phi^i\phi^j + [\frac{4}{3}(2+\omega)-2]12(d+2)\gamma(\gamma+1)\eta_{\nu\sigma}P_\rho P_\mu\phi^i\phi^j$$
$$-[4\omega+10]2d\gamma(\gamma+1)(\gamma+2)\eta_{\nu\sigma}P_\rho P_\mu\phi^i\phi^j$$

$$\begin{aligned}
&\Rightarrow [4\omega+2]2\gamma(\gamma+1)(\gamma+2)(d+2) - [4\omega+10]2\gamma(\gamma+1)(\gamma+2)(d-2)\\
&= [4\omega+10]2\gamma(\gamma+1)(\gamma+2)(d-2) - [4\omega+10]2\gamma(\gamma+1)(\gamma+2)(d-2)\\
&= 0.
\end{aligned} \tag{460}$$

This completes the demonstration that $O_{\rho\mu\nu}$ is indeed a primary operator.

5.8 An Algorithm to Construct More General Primaries

In this section, we will derive an algorithm for constructing primaries. The utility of this algorithm will be in its ability to provide primaries that are even simpler in form than the approach using Gegenbauer polynomials.

First, we introduce a new representation for the conformal group. This representation is given by taking

$$\begin{aligned}
K_\mu &= \frac{\partial}{\partial x^\mu},\\
P_\mu &= x^2\partial_\mu - 2x_\mu x\cdot\partial - 2x_\mu.
\end{aligned} \tag{461}$$

To determine a D and $M_{\mu\nu}$ that closes the conformal algebra, we will make use of the following commutator:

$$[P_\mu, K_\nu] = i(2\eta_{\mu\nu}D + 2M_{\mu\nu}) \tag{462}$$

Using the above expressions for P_μ and K_ν and making use of a test function (to keep track of derivatives), we find that

$$\begin{aligned}
[P_\mu, K_\nu]f(x) =& (x^2\partial_\mu - 2x_\mu x\cdot\partial - 2x_\mu)\partial_\nu f(x)\\
&-\partial_\nu(x^2\partial_\mu - 2x_\mu x\cdot\partial - 2x_\mu)f(x)\\
&= \partial_\nu(x^2\partial_\mu f(x)) - \partial_\nu x^2\partial_\mu f(x) - 2\partial_\nu(x_\mu x\cdot\partial f(x))\\
&+2\partial_\nu x_\mu x\cdot\partial f(x) + 2x_\mu\partial_\nu f(x) - 2\partial_\nu(x_\mu f(x)) + 2\partial_\nu x_\mu f(x)\\
&-\partial_\nu(x^2\partial_\mu f(x)) + 2\partial_\nu(x_\mu x\cdot\partial f(x)) + 2\partial_\nu(x_\mu f(x))\\
&= -2x_\nu\partial_\mu f(x) + 2\eta_{\mu\nu}x\cdot\partial f(x) + 2x_\mu\partial_\nu f(x) + 2\eta_{\mu\nu}f(x).
\end{aligned} \tag{463}$$

Using the commutator above, it follows that

$$D = -i(x \cdot \partial + 1) \qquad M_{\mu\nu} = -i(x_\mu \partial_\nu - x_\nu \partial_\mu). \tag{464}$$

These operators do indeed close the conformal algebra. For example,

$$\begin{aligned}
[M_{\mu\nu}, M_{\alpha\beta}]f(x) =& [x_\alpha \partial_\beta, x_\mu \partial_\nu]f(x) - [x_\alpha \partial_\beta, x_\nu \partial_\mu]f(x) \\
& -[x_\beta \partial_\alpha, x_\mu \partial_\nu]f(x) + [x_\beta \partial_\alpha, x_\nu \partial_\mu]f(x) \\
& = \eta_{\mu\beta} x_\alpha \partial_\nu f(x) - \eta_{\nu\alpha} x_\mu \partial_\beta f(x) - \eta_{\nu\beta} x_\alpha \partial_\mu f(x) \\
& +\eta_{\mu\alpha} x_\nu \partial_\beta f(x) - \eta_{\mu\alpha} x_\beta \partial_\nu f(x) + \eta_{\nu\beta} x_\mu \partial_\alpha f(x) \\
& +\eta_{\nu\alpha} x_\beta \partial_\mu f(x) - \eta_{\mu\beta} x_\nu \partial_\alpha f(x) \\
& = \eta_{\mu\beta}(x_\alpha \partial_\nu - x_\nu \partial_\alpha) f(x) + \eta_{\nu\alpha}(x_\beta \partial_\mu - x_\mu \partial_\beta) f(x) \\
& +\eta_{\mu\alpha}(x_\nu \partial_\beta - x_\beta \partial_\nu) f(x) + \eta_{\nu\beta}(x_\mu \partial_\alpha - x_\alpha \partial_\mu) f(x) \\
& = i(\eta_{\mu\beta} M_{\nu\alpha} + \eta_{\nu\alpha} M_{\mu\beta} - \eta_{\mu\alpha} M_{\nu\beta} - \eta_{\nu\beta} M_{\mu\alpha}) f(x) \\
& \Rightarrow \qquad [M_{\mu\nu}, M_{\alpha\beta}] = i(\eta_{\mu\beta} M_{\nu\alpha} + \eta_{\nu\alpha} M_{\mu\beta} - \eta_{\mu\alpha} M_{\nu\beta} - \eta_{\nu\beta} M_{\mu\alpha}).
\end{aligned} \tag{465}$$

Similarly, we find that

$$\begin{aligned}
&[M_\mu, P_\alpha] = i(\eta_{\mu\alpha} P_\nu - \eta_{\alpha\nu} P_\mu) \\
&[D, K_\mu] = iK_\mu \\
&[D, P_\mu] = -iP_\mu \\
&[M_{\mu\nu}, K_\alpha] = i(\eta_{\alpha\mu} K_\nu - \eta_{\nu\alpha} K_\mu) \\
&[D, M_{\mu\nu}] = 0.
\end{aligned} \tag{466}$$

Now, consider n fields with corresponding coordinates x^I_μ, $I = 1, 2, ..., n$. Thus, we will have n copies of $SO(4,2)$. For our algorithm, we will use an oscillator realization which replaces $x^J_\mu \to a^{J\dagger}_\mu$ and $\frac{\partial}{\partial x^I_\mu} \to a^I_\mu$. Then, states at dimension $n + k$ in $V^{\otimes n}$ are generated by acting with k oscillator creation operators $a^{I\dagger}_\mu$. Thus, constructing the primaries is the same as constructing degree k polynomials $\Psi(x^I_\mu)$ subject to the constraints

$$\begin{aligned}
K_\mu \Psi &= \sum_I \frac{\partial}{\partial x^I_\mu} \Psi = 0, \\
\mathcal{L}_I \Psi &= \sum_\mu \frac{\partial}{\partial x^I_\mu} \frac{\partial}{\partial x^I_\mu} \Psi = 0.
\end{aligned} \tag{467}$$

The construction of primaries has therefore been framed as a *many-body wavefunction problem* subject to the above constraints. Notice that there are $n + 4$ equations to solve here. The second condition is the null state condition and it comes from the fact that images of states like $P^I_\mu P^I_\mu$ in the Fock space is zero. States like $a^{\dagger\,I}_\mu a^{\dagger\,I}_\mu$ are not in the image of the map from Verma module to Fock space. Setting $a\cdot a$ to zero ensures that there are no traces of the form $a^{\dagger\,I}_\mu a^{\dagger\,I}_\mu$.

Remark 8. *Using*

$$P_\mu = x^2\partial_\mu - 2x_\mu x\cdot\partial - 2x_\mu, \tag{468}$$

we easily find

$$\left[P_\mu, \partial\cdot\partial\right] = -4x_\mu(2 - x\cdot\partial)\partial\cdot\partial \tag{469}$$

(this can be phrased in terms of oscillators too). This immediately implies that if f is a harmonic polynomial, so is $P_\mu f$

$$\partial\cdot\partial f = 0 \qquad \Rightarrow \qquad \partial\cdot\partial\,(P_\mu f) = 0 \tag{470}$$

Operators of this type have been described and studied in detail by [52].

Translating between momenta and polynomials and vice versa. Now, in general, it does not seem to be trivial to translate a polynomial in the x_μ's back into the field language. However, we can simplify things a bit by considering $z\cdot x = z^\mu x_\mu$ with z^μ a null vector $z^\mu z_\mu = 0$. Any polynomial in $z\cdot x$ is automatically related to a traceless symmetric tensor, which is like working with a specific state (value of $\vec{m}$) in the spherical harmonic $Y_{\vec{l},\vec{m}}(x)$ description.

The field ϕ is represented by 1. Thus,

$$\partial_{\mu_1}\cdots\partial_{\mu_k}\phi \leftrightarrow P_{\mu_1}\cdots P_{\mu_k}\cdot 1 \tag{471}$$

By considering a few examples, we observe that $(z^\mu P_\mu)^k\cdot 1 \propto (z\cdot x)^k$. Define the constant of proportionality to be a_k

$$(z^\mu P_\mu)^k\cdot 1 = a_k(z\cdot x)^k. \tag{472}$$

Then

$$(z^\mu P_\mu)^{k+1}\cdot 1 = z^\mu P_\mu a_k(z\cdot x)^k = -2(k+1)a_k(z\cdot x)^{k+1} = a_{k+1}(z\cdot x)^{k+1}. \tag{473}$$

Thus

$$a_{k+1} = -2(k+1)a_k \qquad a_1 = -2, \tag{474}$$

which implies that

$$a_k = (-1)^k 2^k k!, \tag{475}$$

and hence that the translation we are looking for is

$$(z\cdot\partial)^k\phi \leftrightarrow (-1)^k 2^k k!(z\cdot x)^k. \tag{476}$$

Testing the translation formula. Using the above translation, we can translate some of the primaries that we know into polynomials; indeed, this will yield insight into the form of the primary polynomials. The simplest case to consider is $n=2$ fields, which includes *all* of the conserved higher spin currents. Using formula (2.13) of [49], we read off the primaries (note, that these originate from the Gegenbauer polynomials; here, we have set $d=4$ and s is any even integer)

$$O_s=\sum_{k=0}^{s}\frac{(-1)^k((z\cdot\partial)^{s-k}\phi)((z\cdot\partial)^k\phi)}{(k!(s-k)!)^2}. \tag{477}$$

Note, that this corresponds to equation (242). Indeed, each $\mathcal{O}_s$ is in representation $D_{[s+2,\frac{s}{2},\frac{s}{2}]}$. If s is odd, $\mathcal{O}_s=0$ so we have a primary for each representation on the right hand side of (242). Using our translation formula, we find that

$$\begin{aligned} O_s&=\sum_{k=0}^{s}\frac{(-1)^k 2^s(z\cdot x_1)^{s-k}(z\cdot x_2)^k}{k!(s-k)!}\\ &=\frac{2^s}{s!}(z\cdot(x_1-x_2))^s. \end{aligned} \tag{478}$$

Observe that $x_1-x_2=X^{(1)}$ is translation invariant. Notice also that the polynomials corresponding to the higher spin currents are remarkably simple in this language; concretely, the spin s primary is simply $(z\cdot X^{(1)})^s$. Furthermore, note that we are allowing s to be any even integer which therefore gives the complete construction of the primaries for $n=2$; indeed, this choice yields an S_2 invariant. In conclusion, this is significantly simpler than the Gegenbauer polynomial description.

Testing our results on symmetric representations of SO(4). If we repeat the algorithm, with

$$\Psi_{l_1,l_2,\cdots l_n;m_1,m_2,\cdots,m_n}=\prod_{i=1}^{n}(z\cdot x^{(i)})^{l_i}, \tag{479}$$

we are choosing a specific value for the m_i in the product of spherical harmonics. With this choice, we know that the z^μ continue to enforce tracelessness so that we continue to get a primary. Note, that the construction using polynomials in $z\cdot x$ is incomplete; indeed, for $n=3$ we have primaries with $(\Delta,j_L,j_R)=(3,0,0),(5,1,1),(6,\frac{3}{2},\frac{3}{2}),(7,2,2)$ which all correspond to symmetric traceless tensors and hence are captured. There are however also primaries $(\Delta, j_L, j_R) = (7,2,0), (7,0,2)$, which require mixed symmetry tensors.

Consider the $(6, \frac{3}{2}, \frac{3}{2})$ primary. It is given by

$$\begin{aligned}\mathcal{O}=& (z\cdot P)^3|\phi\rangle|\phi\rangle|\phi\rangle - 9(z\cdot P)^2|\phi\rangle(z\cdot P)|\phi\rangle|\phi\rangle \\ &+12(z\cdot P)|\phi\rangle(z\cdot P)|\phi\rangle(z\cdot P)|\phi\rangle.\end{aligned} \tag{480}$$

This can be translated into the operator (this is just to motivate the polynomial)

$$\begin{aligned}\mathcal{O}=& \frac{1}{3}\big(((z\cdot\partial)^3\phi)\phi\phi + \phi((z\cdot\partial)^3\phi)\phi + \phi\phi((z\cdot\partial)^3\phi)\big) \\ &-\frac{9}{6}\big(((z\cdot\partial)^2\phi)(z\cdot\partial\phi)\phi + ((z\cdot\partial)^2\phi)\phi(z\cdot\partial\phi) + (z\cdot\partial\phi)((z\cdot\partial)^2\phi)\phi \\ &+(z\cdot\partial\phi)\phi((z\cdot\partial)^2\phi) + \phi((z\cdot\partial)^2\phi)(z\cdot\partial\phi) + \phi(z\cdot\partial\phi)((z\cdot\partial)^2\phi)\big) \\ &+12(z\cdot\partial)\phi(z\cdot\partial)\phi(z\cdot\partial)\phi.\end{aligned} \tag{481}$$

This then translates into the following polynomial

$$\begin{aligned}\mathcal{O}=& \frac{1}{3}\big((z\cdot x_1)^3 + (z\cdot x_2)^3 + (z\cdot x_3)^3\big) - \frac{9}{6}\big((z\cdot x_1)^2(z\cdot x_2) + (z\cdot x_1)^2(z\cdot x_3) \\ &+(z\cdot x_2)^2(z\cdot x_1) + (z\cdot x_2)^2(z\cdot x_3) + (z\cdot x_3)^2(z\cdot x_1) + (z\cdot x_3)^2(z\cdot x_2)\big) \\ &+12(z\cdot x_1)(z\cdot x_2)(z\cdot x_3).\end{aligned} \tag{482}$$

Thanks to the fact that this is a polynomial in $z\cdot x_i$, it is manifestly traceless and symmetric; in other words, harmonic in all three coordinates. It is also simple to verify that

$$\left(\frac{\partial}{\partial x_1^\mu} + \frac{\partial}{\partial x_2^\mu} + \frac{\partial}{\partial x_3^\mu}\right)\mathcal{O} = 0, \tag{483}$$

so that it is indeed primary. Using the transformation

$$X^{(0)\mu} = x_1^\mu + x_2^\mu + x_3^\mu, \qquad X^{(1)\mu} = x_1^\mu - x_2^\mu, \qquad X^{(2)\mu} = x_2^\mu - x_3^\mu, \tag{484}$$

and its inverse

$$\begin{aligned}x_1^\mu &= \frac{1}{3}\left(X^{(0)\mu} + 2X^{(1)\mu} + X^{(2)\mu}\right), \qquad x_2^\mu = \frac{1}{3}\left(X^{(0)\mu} - X^{(1)\mu} + X^{(2)\mu}\right), \\ x_3^\mu &= \frac{1}{3}\left(X^{(0)\mu} - X^{(1)\mu} - 2X^{(2)\mu}\right),\end{aligned} \tag{485}$$

we easily find

$$\mathcal{O} = (z\cdot X^{(1)} - z\cdot X^{(2)})(2z\cdot X^{(1)} + z\cdot X^{(2)})(z\cdot X^{(1)} + 2z\cdot X^{(2)}). \tag{486}$$

There is no dependence on $X^{(0)}$ in line with the fact that the polynomial must be translation invariant.

The above polynomial has a rather natural explanation: First, recall the characters

$$\chi_{\ydiagram{2,1}}(1) = 2 \qquad \chi_{\ydiagram{2,1}}((\cdot\cdot)) = 0 \qquad \chi_{\ydiagram{2,1}}((\cdot\cdot\cdot)) = -1. \tag{487}$$

The character of the tensor product of reps is the product of the characters, so that for $R = \ydiagram{2,1} \otimes \ydiagram{2,1} \otimes \ydiagram{2,1}$ we have

$$\chi_R(1) = 8 \qquad \chi_R((\cdot\cdot)) = 0 \qquad \chi_R((\cdot\cdot\cdot)) = -1. \tag{488}$$

This rep is reducible and, as usual, irrep r appears n_r times with

$$n_r = \frac{1}{|S_3|} \sum_{\sigma \in S_3} \chi_r(\sigma)\chi_R(\sigma) = \frac{1}{6} \sum_{\sigma \in S_3} \chi_r(\sigma)\chi_R(\sigma). \tag{489}$$

We easily find $n_{\ydiagram{3}} = 1$, $n_{\ydiagram{2,1}} = 3$ and $n_{\ydiagram{1,1,1}} = 1$. This is consistent with the Appendices of [53]. We can easily construct the $\ydiagram{2,1}$ representation which acts on $\begin{bmatrix} z \cdot X^{(1)} \\ z \cdot X^{(2)} \end{bmatrix}$. The result is

$$\Gamma_{\ydiagram{2,1}}(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \Gamma_{\ydiagram{2,1}}(\,(12)\,) = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \qquad \Gamma_{\ydiagram{2,1}}(\,(13)\,) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \tag{490}$$

$$\Gamma_{\ydiagram{2,1}}(\,(23)\,) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \qquad \Gamma_{\ydiagram{2,1}}(\,(123)\,) = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \qquad \Gamma_{\ydiagram{2,1}}(\,(132)\,) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}. \tag{491}$$

The representation R is now obtained by taking the Kronecker product

$$\Gamma_R(\sigma) = \Gamma_{\ydiagram{2,1}}(\sigma) \otimes \Gamma_{\ydiagram{2,1}}(\sigma) \otimes \Gamma_{\ydiagram{2,1}}(\sigma). \tag{492}$$

We can now build the projector

$$P = \sum_{\sigma \in S_3} \Gamma_R(\sigma), \tag{493}$$

which projects $V^{\otimes 3}$ to $V_{\ydiagram{2,1}} \otimes V_{\ydiagram{2,1}} \otimes V_{\ydiagram{2,1}}$ (see Appendix A.3 for more discussion on projectors and Young diagrams); in other words, the subspace of $V^{\otimes 3}$ spanned by mixed symmetric states. This can be seen from the action of the $\Gamma_{\ytableaushort{2,1}}$ matrices. For example,

$$\begin{aligned} \Gamma_{\ydiagram{2,1}}(\,(12)\,) \begin{bmatrix} z \cdot X^{(1)} \\ z \cdot X^{(2)} \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} z \cdot X^{(1)} \\ z \cdot X^{(2)} \end{bmatrix} \\ &= \begin{bmatrix} -z \cdot X^{(1)} \\ z \cdot X^{(1)} + z \cdot X^{(2)} \end{bmatrix}, \end{aligned} \tag{494}$$

which is in the subspace $V_{\yng(2,1)}$. Applying P to $\begin{bmatrix} z\cdot X^{(1)} \\ z\cdot X^{(2)} \end{bmatrix} \otimes \begin{bmatrix} z\cdot X^{(1)} \\ z\cdot X^{(2)} \end{bmatrix} \otimes \begin{bmatrix} z\cdot X^{(1)} \\ z\cdot X^{(2)} \end{bmatrix}$ results in

$$\begin{bmatrix} 0 \\ P \\ P \\ -P \\ P \\ -P \\ -P \\ 0 \end{bmatrix} \tag{495}$$

where

$$P = (z\cdot X^{(1)} - z\cdot X^{(2)})(2z\cdot X^{(1)} + z\cdot X^{(2)})(z\cdot X^{(1)} + 2z\cdot X^{(2)}). \tag{496}$$

Thus, the form of the above polynomial is determined by requiring that we have an S_3 invariant.

5.9 Constructing Polynomials Using Complex Coordinates

In the previous section, we used our algorithm with a specific value for the m_i in the product of spherical harmonics to find polynomials that correspond to symmetric traceless tensors. This construction thus produced polynomials that corresponded to primaries such $(3,0,0),(5,1,1),(6,\frac{3}{2},\frac{3}{2})$. However, the construction failed to capture primaries with mixed symmetry, such as $(7,2,0)$ and its conjugate $(7,0,2)$. In this section, we will construct the corresponding polynomial to the primary $(7,2,0)$ considered in Section 5.5 by making use of our translation formula and complex coordinates.

Consider then $n=3$ fields with dimension $\Delta=7$ with spin representation $(2,0)$. Thus, we need to consider polynomials with total degree 4. Recall from Section (5.5) we had constructed the primary

$$O_{[2,0]\alpha\beta\gamma\rho} = O_{1(\alpha\beta\gamma\rho)} - 4O_{2(\alpha\beta\gamma\rho)}, \tag{497}$$

where

$$\begin{aligned} O_{1\alpha\beta\gamma\rho} &= \phi P_{\alpha\dot\alpha} P_{\beta\dot\beta} \phi P_{\gamma\dot\gamma} P_{\rho\dot\rho} \phi \epsilon^{\dot\alpha\dot\gamma}\epsilon^{\dot\beta\dot\rho}, \\ O_{2\alpha\beta\gamma\rho} &= P_{\alpha\dot\alpha}\phi P_{\beta\dot\beta} \phi P_{\gamma\dot\gamma} P_{\rho\dot\rho} \phi \epsilon^{\dot\alpha\dot\gamma}\epsilon^{\dot\beta\dot\rho}. \end{aligned} \tag{498}$$

We can translate the P_μ's into our polynomials, replace $\phi \to 1$ and symmetrize over the coordinates for the three particles. The polynomial is given by

$$\begin{aligned}\Psi=&\frac{1}{3}\big((z\cdot x^{(2)})^2(z\cdot x^{(3)})^2+(z\cdot x^{(1)})^2(z\cdot x^{(3)})^2+(z\cdot x^{(1)})^2(z\cdot x^{(2)})^2\big)\\ &-\frac{4}{3}\big((z\cdot x^{(1)})(z\cdot x^{(2)})(z\cdot x^{(3)})^2+(z\cdot x^{(1)})(z\cdot x^{(2)})^2(z\cdot x^{(3)})+(z\cdot x^{(1)})^2(z\cdot x^{(2)})(z\cdot x^{(3)})\big).\end{aligned} \tag{499}$$

We can change to complex coordinates using

$$z^{(a)}=x_1^{(a)}+ix_2^{(a)} \qquad w^{(a)}=x_3^{(a)}+ix_4^{(a)}, \tag{500}$$

and therefore,

$$\begin{aligned}x_1^{(a)}&=\frac{1}{2}\big(z^{(a)}+\bar{z}^a\big), \qquad & x_2^{(a)}&=-\frac{i}{2}\big(z^{(a)}-\bar{z}^a\big),\\ x_3^{(a)}&=\frac{1}{2}\big(w^{(a)}+\bar{w}^a\big), \qquad & x_4^{(a)}&=-\frac{i}{2}\big(w^{(a)}-\bar{w}^a\big).\end{aligned} \tag{501}$$

The polynomial in complex coordinates is remarkably simple

$$\Psi=\Big(w^{(3)}(\bar{z}^{(2)}-\bar{z}^{(1)})+w^{(2)}(\bar{z}^{(1)}-\bar{z}^{(3)})+w^{(1)}(\bar{z}^{(3)}-\bar{z}^{(2)})\Big)^2. \tag{502}$$

Observe that polynomial is translation invariant in all three particle's coordinates. To test if the polynomial is harmonic, we use

$$\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}+\frac{\partial^2}{\partial x_4^2}, \tag{503}$$

which in complex coordinates is given by

$$4\frac{\partial^2}{\partial z\partial\bar{z}}+4\frac{\partial^2}{\partial w\partial\bar{w}}. \tag{504}$$

Since the polynomial is independent of z and $\bar{w}$, we find that

$$\sum_{a=1}^{3}\big(4\frac{\partial^2}{\partial z^a\partial\bar{z}^a}+4\frac{\partial^2}{\partial w^a\partial\bar{w}^a}\big)\Psi=0. \tag{505}$$

Thus, Ψ is indeed harmonic. From the original primary we had constructed, it was difficult to observe that it was harmonic and translation invariant.

The above polynomial also suggests a natural generalization. Consider the family of polynomials indexed by the integer n

$$\Psi=\Big(w^{(3)}(\bar{z}^{(2)}-\bar{z}^{(1)})+w^{(2)}(\bar{z}^{(1)}-\bar{z}^{(3)})+w^{(1)}(\bar{z}^{(3)}-\bar{z}^{(2)})\Big)^{2n} \tag{506}$$

This polynomials is also translation invariant and, since it is independent of z and $\bar{w}$, it satisfies the harmonic condition above.

In conclusion, the algorithm we have developed has two advantages over the use of Gegenbauer polynomials, namely, it is a more general construction and the polynomials are significantly more simpler in form than the Gegenbauer polynomials. This can be easily observed in the polynomials we have constructed by simply testing for translation invariance and testing whether the polynomials are indeed harmonic. The task of constructing the set of polynomials that are translation invariant and harmonic is a highly non-trivial problem and outside of the scope of this dissertation. However, it does offer a clear problem statement for future research.

6 Conclusion

In this dissertation, we have presented two topics, namely, determining the spectrum of primary operators and constructing some of these operators. To count the spectrum of primaries, we used group theoretic techniques to obtain character formulas for any number of fields as reps of $SO(4,2)$. These character formulas can be expanded to any Δ. In addition, we presented the spectrum of primaries for $n = 3,4,5$ products of fields. We improved on this approach by deriving generating functions. These generating functions prove to be a more efficient method of determining the spectrum of primaries. An aspect not covered in this dissertation was to generalize the generating function approach to the case of supersymmetry.

We then developed methods to construct some of the primaries that were revealed by our generating function. We first constructed symmetric traceless primaries that were in the (l,l) rep of $SO(4)$. This was performed by stating an ansatz and solving for the coefficients under the condition that the ansatz be annihilated by the special conformal generator K_μ. In Appendix (A.2), we presented a general ansatz for the case of four scalar fields using the subalgebra. This was followed by the construction of a mixed symmetric primary in rep $(7,2,0)$. We then presented a different formulation of the problem which was in terms of Gegenbauer polynomials. This method was more efficient than our previous approach to constructing the primaries. Finally, we generalized the polynomial algorithm by formulating the problem as a many-body wavefunction problem. Criticially, this depended on the representation of the conformal group given by equation (461). Here, two constraints were used to derives the final result. In this approach, primary operators corresponded to polynomials that are harmonic and translation invariant. An aspect not covered in this dissertation is to show that the algorithm does indeed produce the primaries in the spectrum we had determined. Furthermore, interesting questions for future study would be to consider correlation functions in the interacting theory at a fixed point such as the *Wilson-Fisher fixed point*, which is conformally invariant.

A Appendices

A.1 The large N limit

Consider the quantum field theory defined by the following Lagrangian density

$$\mathscr{L} = \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2}\mu_0^2 \phi^a \phi^a - \frac{1}{8}\lambda_0 (\phi^a \phi^a)^2, \tag{507}$$

where summation on repeated indices is implied and $a = 1, ..., N$. Consequently, the correlation function can be determined by the following generating function (which has been expanded perturbatively in the coupling constant λ_0)

$$\begin{aligned}&\mathcal{Z}[J^1 \dots J^N] = \\ &\sum_{n=0}^{\infty} \frac{1}{n!}\{-\frac{i}{8}\lambda_0 \int d^d x (\sum_{a=1}^{N} \frac{\delta^2}{\delta(J^a(x))^2})^2\}^n e^{\int d^d x \int d^d y J^a(x) G(x,y) J^a(y)}\end{aligned} \tag{508}$$

where integration is performed over all spacetime and $G(x, y)$ is the Green's function.

Following the example set out in [41], consider the scattering of two mesons of type (or flavor) a into two mesons of type b where $a \neq b$, represented by the four-point correlation function $\langle \phi^a(x_1)\phi^a(x_2)\phi^b(x_3)\phi^b(x_4)\rangle$. This leads to the following first few Feynman diagrams obtained from the perturbative expansion of $\mathcal{Z}$ above.

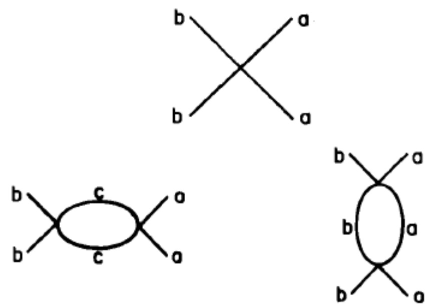


Figure 4: Feynman Diagrams for Quartic Interaction [41]

The first diagram is order $O(\lambda_0)$ in the expansion. The second diagram has an internal index c that must be summed over, leading to a factor of $\sum_{c=1}^{n} \delta^{cc} = N$; consequently, the second digram is order $O(\lambda_0^2 N)$. In contrast, the third diagram has fixed internal indices and is therefore on the order of $O(\lambda_0^2)$.

Observe that a large-N limit would be nonsensical due to the explicit factor of N in the second diagram. To be more precise, note that we can expand the four-point correlation function as an infinite series in λ and N as

$$\langle \phi^a \phi^a \phi^b \phi^b \rangle = \sum_{n,m} \lambda_0^n N^m C_{n,m}, \tag{509}$$

where $C_{n,m}$ is a function of the spacetime coordinates (but independent of λ_0 and N) associated with each term. Now consider a vanishing coupling constant and large-N limit; that is, consider $\lambda_0 \longrightarrow 0$ and $N \longrightarrow \infty$. From simple analysis of the series, we see that each term in the series has a sensible limit if $n \geq 0$ and $m \leq 0$. By 'sensible', we mean none of the terms in the series diverge as $\lambda_0 \to 0$ and $N \to \infty$.

To ensure that we obtain a sensible limit, we simply rescale the coupling constant λ_0. In particular, we define a new coupling constant g_0 (known as the *t' Hooft coupling*) and demand that it remains *fixed* with respect to the limit of large-N and vanishing coupling constant

$$g_0 = \lambda_0 N. \tag{510}$$

This is often called a *double scaling limit* since we scale λ_0 and N to obtain the limit. The four-point correlation function can now be expanded as

$$\begin{aligned} \langle \phi^a \phi^a \phi^b \phi^b \rangle &= \sum_{n,m} \lambda_0^n N^m C_{n,m} \\ &= \sum_{n,m} \lambda_0^n N^n N^{m-n} C_{n,m} \\ &= \sum_{n,m} g_0^n N^{m-n} C_{n,m}. \end{aligned} \tag{511}$$

For $N \longrightarrow \infty$, we see that $N^{m-n} \longrightarrow 0$ since $m - n \leq 0$ and g_0 remains fixed; in other words, through rescaling, we obtain a series that converges to a sensible limit. Using the t' Hooft coupling, the first and second diagram are now of order $O(g_0/N)$ and $O(g_0^2/N)$, respectively; these diagrams are the leading non-trivial order in the $1/N$ expansion. The next order in $1/N$ is the third diagram with $O(g_0^2/N^2)$ which will be negligible in the large-N limit.

The perturbative expansion of a correlation function can be addressed as a problem of combinatorics. To keep track of the parameter N, we need to formulate the correct Feynman rules. An insightful exercise is to consider the theory in a universe consisting of a single spacetime event - a 0-dimensional universe. The rules extracted from this exercise extend naturally to arbitrary dimensions. In addition, the path integral formalism simplifies as fields are represented by real numbers and integrals are reduced to domains over $\mathbb{R}$.

The 0-dimensional model. Consider then the two-point correlation function with the 0-dimensional scalar theory described by the action S given by

$$S = \alpha\phi^a\phi^a + g(\phi^a\phi^a)^2. \tag{512}$$

Note that the coupling constant g here is not the t' Hooft coupling constant.

We can determine the two-point function without the need for source terms. Expanding the expectation value perturbatively in the coupling constant g (here, the fields are represented by real numbers and the integral is performed over $\mathbb{R}$; we therefore drop the i in the exponential), we find that

$$\begin{aligned}
\langle\phi^a(x_1)\phi^b(x_2)\rangle_{int} &= \mathcal{N}\int\prod_{i=1}^{N} d\phi^i \mathrm{e}^{-\alpha\phi^c\phi^c - g\phi^d\phi^d\phi^e\phi^e}\phi^a\phi^b \\
&= \mathcal{N}\int\prod_{i=1}^{N} d\phi^i \mathrm{e}^{-\alpha\phi^c\phi^c}\sum_{m=0}^{\infty}\frac{1}{m!}(-g\phi^d\phi^d\phi^e\phi^e)^m\phi^a\phi^b \\
&= \mathcal{N}\int\prod_{i=1}^{N} d\phi^i \mathrm{e}^{-\alpha\phi^c\phi^c}\phi^a\phi^b \\
&\quad -g\mathcal{N}\int\prod_{i=1}^{N} d\phi^i \mathrm{e}^{-\alpha\phi^c\phi^c}\phi^a\phi^b\phi^d\phi^d\phi^e\phi^e \\
&\quad +\frac{g^2\mathcal{N}}{2}\int\prod_{i=1}^{N} d\phi^i \mathrm{e}^{-\alpha\phi^c\phi^c}\phi^a\phi^b(\phi^d\phi^d\phi^e\phi^e)^2 + \ldots \\
&= \langle\phi^a\phi^b\rangle_{free} - g\langle\phi^a\phi^b\phi^d\phi^d\phi^e\phi^e\rangle_{free} \\
&\quad +\frac{g^2}{2}\langle\phi^a\phi^b(\phi^d\phi^d\phi^e\phi^e)^2\rangle_{free} + \ldots .
\end{aligned} \tag{513}$$

Here, the subscript 'int' in the first line denotes the two-point correlation function in the interacting theory defined by the action given in equation (512); similarly, the subscript '$free$' in the final line denotes the two-point correlation function in the free theory defined by the action $S = \alpha\phi^a\phi^a$. Also, we used the fact that $(\phi^a\phi^a)^2 = (\phi^d\phi^d)(\phi^e\phi^e)$. To summarize, we are able to write the two-point correlation function in the interacting theory as the two-point correlation function in the free theory with additional corrections given as a sum of expectation values in infinite orders in the coupling constant g.

To determine the expansion given by equation (513), we calculate the first few correlation functions of the free theory. We start with the two-point function

$$\begin{aligned}
\langle\phi^a\phi^b\rangle_{free} &= \mathcal{N}\int\prod_{i=1}^{N} d\phi^i \mathrm{e}^{-\alpha\phi^c\phi^c}\phi^a\phi^b \\
&= \mathcal{N}\int d\phi^1 \mathrm{e}^{-\alpha(\phi^1)^2}\int d\phi^2 \mathrm{e}^{-\alpha(\phi^2)^2}\dots \\
&\int d\phi^a \mathrm{e}^{-\alpha(\phi^a)^2}\phi^a\dots\int d\phi^b \mathrm{e}^{-\alpha(\phi^b)^2}\phi^b\dots\int d\phi^N \mathrm{e}^{-\alpha(\phi^N)^2}.
\end{aligned} \tag{514}$$

We emphasize here that there is *no summation* over the repeated indices a and b in the last line. To determine the normalization constant $\mathcal{N}$, observe that for any real field,

$$\begin{aligned}
(\langle 1\rangle_{free})^2 &= \mathcal{N}^2\int_{-\infty}^{\infty} dx \mathrm{e}^{-\alpha x^2}\int_{-\infty}^{\infty} dy \mathrm{e}^{-\alpha y^2} \\
&= \mathcal{N}^2\int_0^{\infty} dr\int_0^{2\pi} d\theta r \mathrm{e}^{-\alpha r^2} \\
&= -\mathcal{N}^2\frac{\pi}{\alpha}\int_0^{\infty} dr\left(\frac{d}{dr}\mathrm{e}^{-\alpha r^2}\right) \\
&= -\mathcal{N}^2\frac{\pi}{\alpha}\mathrm{e}^{-\alpha r^2}\Big|_0^{\infty} \\
&= \mathcal{N}^2\frac{\pi}{\alpha} \Rightarrow \mathcal{N} = \sqrt{\frac{\alpha}{\pi}},
\end{aligned} \tag{515}$$

where we have taken the positive root so that $\langle 1\rangle_{free} = 1$. Since there are N integrals, the normalization constant in equation (514) is $\mathcal{N}^N = (\alpha/\pi)^{N/2}$. Therefore, for any real field x, the two-point function is

$$\begin{aligned}
\langle x^2\rangle &= \mathcal{N}\int_{-\infty}^{\infty} dx \mathrm{e}^{-\alpha x^2}x^2 \\
&= -\sqrt{\frac{\alpha}{\pi}}\frac{d}{d\alpha}\int_{-\infty}^{\infty} dx \mathrm{e}^{-\alpha x^2} \\
&= -\sqrt{\frac{\alpha}{\pi}}\frac{d}{d\alpha}\sqrt{\frac{\pi}{\alpha}} \\
&= \frac{1}{2\alpha}.
\end{aligned} \tag{516}$$

The two-point function given by equation (514) becomes

$$\begin{aligned}
\langle\phi^a\phi^b\rangle_{free} &= \mathcal{N}^2\int d\phi^a \mathrm{e}^{-\alpha(\phi^a)^2}\phi^a\int d\phi^b \mathrm{e}^{-\alpha(\phi^b)^2}\phi^b \\
&= \mathcal{N}^2\int d\phi^a d\phi^b \mathrm{e}^{-\alpha(\phi^a)^2-\alpha(\phi^b)^2}\phi^a\phi^b \\
&= \mathcal{N}^2\int d\phi^a d\phi^b \mathrm{e}^{-\alpha\phi^c\phi^c}\phi^a\phi^b
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{N}^2 \int d\phi^a d\phi^b \mathrm{e}^{-\alpha\phi^c\phi^c} \frac{\delta^{ab}}{2} \phi^c\phi^c \\
&= -\mathcal{N}^2 \frac{\delta^{ab}}{2} \frac{d}{d\alpha} \int d\phi^a d\phi^b \mathrm{e}^{-\alpha\phi^c\phi^c} \\
&= -\mathcal{N}^2 \frac{\delta^{ab}}{2} \frac{d}{d\alpha} \frac{\pi}{\alpha} \\
&= \mathcal{N}^2 \frac{\delta^{ab}}{2} \frac{\pi}{\alpha^2} \\
&= \frac{\delta^{ab}}{2\alpha}
\end{aligned}
\tag{517}
$$

where $\phi^c\phi^c = (\phi^a)^2 + (\phi^b)^2$. Consider now the next term in equation (513). By Wick's theorem, we can break this expectation value up into contractions of its argument. For example, a four-point correlation function in the free theory for fields $\phi^a, \phi^b, \phi^c, \phi^d$ can be written

$$
\begin{aligned}
\langle \phi^a\phi^b\phi^c\phi^d \rangle_{free} &= \phi^a\phi^c\phi^b\phi^d + \phi^a\phi^b\phi^c\phi^d + \phi^b\phi^a\phi^c\phi^d \\
&= \frac{\delta^{ab}}{2\alpha}\frac{\delta^{cd}}{2\alpha} + \frac{\delta^{ac}}{2\alpha}\frac{\delta^{bd}}{2\alpha} + \frac{\delta^{ad}}{2\alpha}\frac{\delta^{bc}}{2\alpha} \\
&= \frac{1}{(2\alpha)^2}(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc})
\end{aligned}
\tag{518}
$$

where we have used the result of equation (517) for each respective contraction. Thus, the four-point function is obtained by pairing the four fields in all possible combinations and simply writing down the expression for the two-point correlation function. The approach to this example extends easily to the six-point function in equation (513), the only difference being more terms that have to be paired. Writing the six-point correlation function in equation (513) immediately in terms of Kronecker deltas, we obtain

$$
\begin{aligned}
\langle \phi^a\phi^b\phi^d\phi^d\phi^e\phi^e \rangle_{free} &= \frac{1}{(2\alpha)^3}(\delta^{ab}\delta^{dd}\delta^{ee} + 2\delta^{ab}\delta^{de}\delta^{de}) \\
&+ \frac{1}{(2\alpha)^3}(2\delta^{ae}\delta^{be}\delta^{dd}) + \frac{1}{(2\alpha)^3}(2\delta^{ad}\delta^{bd}\delta^{ee}) \\
&+ \frac{1}{(2\alpha)^3}(4\delta^{ad}\delta^{be}\delta^{de}) + \frac{1}{(2\alpha)^3}(4\delta^{ae}\delta^{bd}\delta^{de}).
\end{aligned}
\tag{519}
$$

This is the first order correction to the two-point correlation function in the interacting theory. The second order correction is calculated in a similar manner.

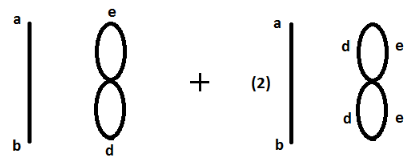
We now have sufficient information to formulate the Feynman rules. Graphically, equation (517) can be represented as

$$a \;\text{———}\; b \quad = \frac{\delta^{ab}}{2\alpha} \tag{520}$$

Figure 5: Propagator

This diagram is called the *propagator*. That is, two points labeled a and b with a solid line connecting the points. The first order correction given by equation (513) can be represented by a *vertex* with four lines attached to it, labeled d, d, e, e, and two external points labeled a, b. The vertex contributes a factor of $-g$. The four arms of the vertex and the two external points are connected by solid lines, with each solid line represented by a Kronecker delta function, and permuted to give all the combinatorics displayed in equation (519). As an example, the first term in equation (519) can be represented graphically as

$$\frac{1}{(2\alpha)^3}(\delta^{ab}\delta^{dd}\delta^{ee} + 2\delta^{ab}\delta^{de}\delta^{de}) = \tag{521}$$



In effect, equation (519) can be represented by 15 Feynman diagrams since there are $(6-1)!! = 5!! = 5 \times 3 \times 1 = 15$ ways of combining the fields into pairs. The next order in the expansion of equation (513) is represented by two vertices and two external points. As with the first order in g, one simply connects the external points and vertices in all possible ways to obtain all of the combinatorics given by the Wick contractions. Once all the diagrams are written down, one simply applies the rules stated in the previous paragraph. This method is applied to all orders of g.

Combining results, equation (513) can therefore be written as

$$\begin{aligned}\langle\phi^a(x_1)\phi^b(x_2)\rangle_{int} = {} & \frac{\delta^{ab}}{2\alpha} - \frac{g}{(2\alpha)^3}[(\delta^{ab}\delta^{dd}\delta^{ee} + 2\delta^{ab}\delta^{de}\delta^{de}) \\ & +(2\delta^{ae}\delta^{be}\delta^{dd}) + (2\delta^{ad}\delta^{bd}\delta^{ee}) \\ & +(4\delta^{ad}\delta^{be}\delta^{de}) + (4\delta^{ae}\delta^{bd}\delta^{de})] + \ldots \end{aligned} \tag{522}$$

Any repeated indices *in a loop diagram* must be summed over and it is these terms that contribute explicit orders of N. For example, $\delta^{ab}\delta^{dd}\delta^{ee} = \delta^{ab}NN = N^2\delta^{ab}$ and $\delta^{ae}\delta^{be}\delta^{dd} =$

$N\delta^{ae}\delta^{be} = N\delta^{ab}$. Applying the same reasoning to the other terms, we obtain

$$\langle \phi^a(x_1)\phi^b(x_2)\rangle_{int} = [\frac{1}{2\alpha} - \frac{g}{(2\alpha)^3}(N^2 + 2N + 4N + 8) + \dots]\delta^{ab}. \tag{523}$$

Using a suitable normalization, terms such as N^2 and $2N$ can be removed from the expansion of the two-point function. In terms of Feynman diagrams, such as those shown in equation (521), these diagrams are called *vacuum diagrams*; they are diagrams that connect to no external points and fold in on themselves. In the free theory, we used the normalization constant $\mathcal{N}$ to force $\langle 1\rangle_{free} = 1$; that is, we rescaled to force the vacuum to have unit norm. Here, we have $\langle 1\rangle_{int} \neq 1$. Normalization removes vacuum diagrams so that in the interacting theory, we again have the vacuum normalized to unit norm. Without summing the vacuum diagrams in the expansion of the two-point correlation function, we have

$$\langle \phi^a(x_1)\phi^b(x_2)\rangle_{int} = [\frac{1}{2\alpha} - \frac{g}{(2\alpha)^3}(4N + 8) + \dots]\delta^{ab}. \tag{524}$$

From our arguments, we can write down the form of the third order term in g in the expansion as well as higher orders. Of course, we would need to determine the factors that are attached to these orders and these factors can be determined using the methodology of the previous pages in this dissertation. Since we are only interested here in N and not these factors, we will use the symbol K_i to mean some appropriate constant for each of the uncalculated terms, so that for the expansion of the two-point correlation function, we have

$$\begin{aligned}\langle \phi^a(x_1)\phi^b(x_2)\rangle_{int} =& [\frac{1}{2\alpha} - \frac{g}{(2\alpha)^3}(4N + 8) + \frac{g^2}{(2\alpha)^5}(K_1N^2 \\ &+K_2N + K_3) + \frac{g^3}{(2\alpha)^7}(K_4N^3 + K_5N^2 + K_6N + K_7) + \dots]\delta^{ab}. \end{aligned}\tag{525}$$

The expansion in equation (525) is not useful for a large-N expansion due to the factors of N appearing in the numerators of terms; in particular, if we let $N \longrightarrow \infty$ and keep g fixed, the limit of the series in equation (525) does not exist. To solve this problem, we rescale the coupling constant g. More precisely, we will rescale g as

$$\lambda = gN^\alpha \tag{526}$$

where we fix λ and $\alpha \in \mathbb{Z} = \{\dots -2, -1, 0, 1, 2, \dots\}$. Using equation (526), we can rewrite equation (525) as

$$\begin{aligned}\langle \phi^a(x_1)\phi^b(x_2)\rangle_{int} =& [\frac{1}{2\alpha} - \frac{8}{(2\alpha)^3}\frac{\lambda}{N^{\alpha-1}} - \frac{16}{(2\alpha)^3}\frac{\lambda}{N^\alpha} + \frac{K_1}{(2\alpha)^5}\frac{\lambda^2}{N^{2\alpha-2}} \\ &+\frac{K_2}{(2\alpha)^5}\frac{\lambda^2}{N^{2\alpha-1}} + \frac{K_3}{(2\alpha)^5}\frac{\lambda^2}{N^{2\alpha}} + \dots]\delta^{ab}. \end{aligned}\tag{527}$$

If we let $\alpha < 1$ then the large-N limit does not exist, as before. Alternatively, if we let $\alpha > 1$, then in the large-N limit

$$\langle \phi^a(x_1)\phi^b(x_2)\rangle_{int} \longrightarrow \frac{1}{2\alpha}, \tag{528}$$

which results in a trivial expansion of the two-point function. Thus we are left with $\alpha = 1$ (in particular, the t'Hooft coupling), which in the large-N limit to leading order in $1/N$, is given by

$$\begin{aligned}\langle \phi^a(x_1)\phi^b(x_2)\rangle_{int} = & [\frac{1}{2\alpha} - \frac{4}{(2\alpha)^3}\lambda - \frac{8}{(2\alpha)^3}\frac{\lambda}{N} + \frac{K_1}{(2\alpha)^5}\lambda^2 \\ & + \frac{K_2}{(2\alpha)^5}\frac{\lambda^2}{N} + \dots]\delta^{ab} \\ & = \mathcal{F}(\lambda, 1/N). \end{aligned} \tag{529}$$

A.2 The General Primary Operator for Four Fields: Construction

The starting ansatz for the general primary operator is given by

$$\begin{aligned}\mathcal{O}^{(l)} = & a_0(P^l\phi)\phi^3 + \sum_{i=1}^{l/2} a_i(P^{l-i}\phi P^i\phi)\phi^2 + \sum_{i=1}^{l/2-1}\sum_{j=i}^{l/2-1} b_{ij}\Theta(i,j)(P^{l-i-j}\phi P^i\phi P^j\phi)\phi \\ & + \sum_{i=1}^{3i<l}\sum_{j=i}^{l/2+\Gamma(i)-i}\sum_{k=j}^{l-(i+j+1)} c_{ijk}\Theta(i,j,k)(P^{l-i-j-k}\phi P^i\phi P^j\phi P^k\phi),\end{aligned} \tag{530}$$

where

$$\Theta(i,j) := \begin{cases} 1 & \text{if } (l-i-j)-j \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Theta(i,j,k) := \begin{cases} 1 & \text{if } (l-i-j-k)-k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Gamma(i) := \begin{cases} 0 & \text{if } i = 1,2 \\ 1 & \text{if } i = 3,4 \\ 2 & \text{if } i = 5,6 \\ \text{etc} & \end{cases}$$

(See equation (550) for the complete ansatz).

We explain how we reached this ansatz. Observe that we must distribute l derivatives over four fields. Thus, we can have four terms with at most l derivatives acting on a single field, two fields, three fields, or four fields. Notice that each P operator is raised to a specific power such that we will always have l derivatives in a given term.

Consider the second term. The highest value our index i can reach is $l-1$ (since $i = l$ will result in all derivatives acting on a single field, which is already accounted for by the first term). We need to write the summation in such a manner that prevents repetition of terms. To see this, observe that for some arbitrary l, we will have terms of the form

$$P^{l-1}\phi P\phi + P^{l-2}\phi P^2\phi + ... + P^{l/2+2}\phi P^{l/2-2}\phi + P^{l/2+1}\phi P^{l/2-1}\phi + P^{l/2}\phi P^{l/2}\phi$$
$$+P^{l/2-1}\phi P^{l/2+1}\phi + P^{l/2-2}\phi P^{l/2+2}\phi + ... + P^2\phi P^{l-2}\phi + P\phi P^{l-1}\phi. \tag{531}$$

Thus, to avoid double counting, we need only sum to $i = l/2$.

Next, consider the third term. We sum here to $l/2 - 1$ again to avoid repetition. If we were to sum to $l/2$, we obtain a term with derivatives acting only on two fields instead of three fields. Also, if we were to sum i to $l/2 - 1$ and j to $l/2$, then $l - i - j = 1$ but we end up double counting terms. For example, for $i = 1$, we obtain the following terms

$$P^{l-1-1}\phi P\phi P\phi + P^{l-1-2}\phi P\phi P^2\phi + ... + P^{l-1-l/2+1}\phi P\phi P^{l/2-1}\phi$$
$$+P^{l-1-l/2}\phi P\phi P^{l/2}\phi. \tag{532}$$

We are therefore forced to sum to $l/2 - 1$ for both i, j. Now notice that when we sum i, j from 1 to $l/2 - 1$, then we still have a repetition of terms: first, observe that for a given i, j we would have the identical terms $P^{l-i-j}\phi P^i\phi P^j\phi$ and $P^{l-j-i}\phi P^j\phi P^i\phi$ (for example, if $l = 6$, we have the two identical terms $P^{6-2-3}\phi P^2\phi P^3\phi$ and $P^{6-3-2}\phi P^3\phi P^2\phi$, where $i = 2, j = 3$ and $i = 3, j = 2$, respectively). To prevent this, we restrict the j index to start from every incremental value of i. One can think of this as analogous to a symmetric square $n \times n$ matrix, where we only consider diagonal elements and elements part of the upper triangle of the matrix, say. If x_{ij} is some coefficient that is symmetric under exchange of indices and we sum each index to n using our method, then we would obtain

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=i}^{n} x_{ij} &= \sum_{j=1}^{n} x_{ij} + \sum_{j=2}^{n} x_{ij} + ... + \sum_{j=n-1}^{n} x_{ij} + x_{nn} \\ &= x_{11} + x_{12} + x_{13} + ... + x_{1n} \\ &\qquad + x_{22} + x_{23} + ... + x_{2n} \\ &\qquad\qquad + x_{33} + ... + x_{3n} \\ &\qquad\qquad\qquad \vdots \\ &\qquad\qquad\qquad ... + x_{nn},\end{aligned} \tag{533}$$

where we have written the series deliberately in this manner to emphasize the matrix analogy. We thus only have x_{ij} appearing once for a given i, j. Unfortunately, this does not prevent the permutation of the $P^{l-i-j}\phi$ part of the term. For example, consider $l = 14$

$$\sum_{i=1}^{6}\sum_{j=i}^{6} P^{14-i-j}\phi P^i\phi P^j\phi$$
$$= P^{12}\phi P\phi P\phi + P^{11}\phi P\phi P^2\phi + P^{10}\phi P\phi P^3\phi + P^9\phi P\phi P^4\phi + P^8\phi P\phi P^5\phi + P^7\phi P\phi P^6\phi$$

$$
\begin{aligned}
&+P^{10}\phi P^2\phi P^2\phi+P^9\phi P^2\phi P^3\phi+P^8\phi P^2\phi P^4\phi+P^7\phi P^2\phi P^5\phi+P^6\phi P^2\phi P^6\phi\\
&+P^8\phi P^3\phi P^3\phi+P^7\phi P^3\phi P^4\phi+P^6\phi P^3\phi P^5\phi+P^5\phi P^3\phi P^6\phi\\
&+P^6\phi P^4\phi P^4\phi+P^5\phi P^4\phi P^5\phi+P^4\phi P^4\phi P^6\phi\\
&+P^4\phi P^5\phi P^5\phi+P^3\phi P^5\phi P^6\phi\\
&+P^2\phi P^6\phi P^6\phi. \qquad (534)
\end{aligned}
$$

We count several terms that appear more than once

$$
\begin{aligned}
&P^6\phi P^2\phi P^6\phi, P^2\phi P^6\phi P^6\phi,\\
&P^6\phi P^3\phi P^5\phi, P^5\phi P^3\phi P^6\phi, P^3\phi P^5\phi P^6\phi,\\
&P^6\phi P^4\phi P^4\phi, P^4\phi P^4\phi P^6\phi.
\end{aligned}
\qquad (535)
$$

If one repeats these sums for various values of l, one will notice that repetition of terms occur when $(l-i-j)-j<0$. We are thus led to the function $\Theta(i,j)$, which was defined previously. This function plays the role of a selection function which kills off terms satisfying the inequality $(l-i-j)-j<0$ and we are left with combinations of values in each term that appears only once in the series above. With this understanding, we can now give a more general argument that shows that any combination of values in the sum appears once and only once: Let a value for i,j be given for some integer l. Then, by our construction of the series, i,j must satisfy

$$
i\le j, \quad l-i-2j\ge 0. \qquad (536)
$$

This can be divided up into four cases

$$
\begin{aligned}
&\text{Case I:} \quad i<j, \quad l-i-2j>0\\
&\text{Case II:} \quad i=j, \quad l-i-2j>0\\
&\text{Case III:} \quad i<j, \quad l-i-2j=0\\
&\text{Case IV:} \quad i=j, \quad l-i-2j=0.
\end{aligned}
\qquad (537)
$$

The six combinations we might find are

$$
\begin{aligned}
&\text{I.} \quad P^{l-i-j}\phi P^i\phi P^j\phi, \quad \text{II.} \quad P^i\phi P^{l-i-j}\phi P^j\phi\\
&\text{III.} \quad P^{l-i-j}\phi P^j\phi P^i\phi, \quad \text{IV.} \quad P^i\phi P^j\phi P^{l-i-j}\phi\\
&\text{V.} \quad P^j\phi P^{l-i-j}\phi P^i\phi, \quad \text{VI.} \quad P^j\phi P^i\phi P^{l-i-j}\phi.
\end{aligned}
\qquad (538)
$$

We start with Case I. By our construction, equation I survives. Equation II will not survive since it would require that $l-i-j\le j\Rightarrow l-i-2j\le 0$. Equation III will not survive since it would require that $j\le i$. For equation IV to survive, it would require that $i-(l-i-j)\ge 0\Rightarrow l-2i-j\le 0\Rightarrow l-i-2j\le 0$ (here, we used the fact that $j>i>0$);

thus, equation IV does not survive. Equation V does not survive since it would require that $l-i-j \le i \Rightarrow l-i-2j \le 0$. Finally, equation VI also would not survive, since it would require that $j-(l-i-j) \le 0 \Rightarrow l-i-2j \le 0$. Thus, only equation I survives our construction of the series.

We next analyze case II. Since $i = j$ in this case, our six possibilities reduces to three possibilities

$$\begin{aligned} &\text{I.} \quad P^{l-j-j}\phi P^j \phi P^j \phi \\ &\text{II.} \quad P^j \phi P^{l-j-j}\phi P^j \phi \\ &\text{III.} \quad P^j \phi P^j \phi P^{l-j-j}\phi. \end{aligned} \tag{539}$$

Equation I survives. Equation II does not survive since $l-3j \le 0$. Equation III also does not survive since $j-(l-2j) \ge 0 \Rightarrow l-3j \le 0$.

We next analyze case III. In this case, $l-i-2j = 0 \Rightarrow l-i-j = j$. As with case II, the six possibilities reduces to three in this case

$$\begin{aligned} &\text{I.} \quad P^j \phi P^i \phi P^j \phi \\ &\text{II.} \quad P^j \phi P^j \phi P^i \phi \\ &\text{III.} \quad P^i \phi P^j \phi P^j \phi. \end{aligned} \tag{540}$$

Equation I survives. Equation II and equation III do not survive since both would imply that $i \ge j$.

Finally, we study case IV. Here, all six possibilities reduces to a single equation

$$P^j \phi P^j \phi P^j \phi. \tag{541}$$

Thus, we have shown that any allowed combination of i, j, and $l-i-j$ appears once and only once according to our construction.

We now explain the final term. In the same manner as the third term, we have restricted the indices j, k; in particular, for every increment of i, the summation over j will start from $j = i$ and for every increment of j, the summation over k will start from $k = j$. We also again added a selection function to prevent $P^{l-i-j-k}\phi$ from permuting with other parts of the fourth term. To explain the upper bounds on the sums, consider the following expressions, where we temporarily ignore our construction of the operator (we only keep the restrictions on j, k's starting values- we therefore expect repetition of terms)

$\mathbf{l = 6}$:

$$\text{i=1:} \quad P^3\phi P\phi P\phi P\phi + P^2\phi P\phi P\phi P^2\phi + P\phi P\phi P\phi P^3\phi + P\phi P\phi P^2\phi P^2\phi, \tag{542}$$

l = 8 :

$$\begin{aligned}
\text{i=1:}\quad & P^5\phi P\phi P\phi P\phi + P^4\phi P\phi P\phi P^2\phi + P^3\phi P\phi P\phi P^3\phi + P^2\phi P\phi P\phi P^4\phi \\
& + P\phi P\phi P\phi P^5\phi + P^3 P\phi P^2\phi P^2\phi + P^2\phi P\phi P^2\phi P^3\phi + P\phi P\phi P^2\phi P^4\phi \\
& + P\phi P\phi P^3\phi P^3\phi \\
\text{i=2:}\quad & + P^2\phi P^2\phi P^2\phi P^2\phi + P\phi P^2\phi P^2\phi P^3\phi,
\end{aligned} \tag{543}$$

l = 10 :

$$\begin{aligned}
\text{i=1:}\quad & P^7\phi P\phi P\phi P\phi + P^6\phi P\phi P\phi P^2\phi + P^5\phi P\phi P\phi P^3\phi + P^4\phi P\phi P\phi P^4\phi \\
& + P^3\phi P\phi P\phi P^5\phi + P^2\phi P\phi P\phi P^6\phi + P\phi P\phi P\phi P^7\phi + P^5\phi P\phi P^2\phi P^2\phi \\
& + P^4\phi P\phi P^2\phi P^3\phi + P^3\phi P\phi P^2\phi P^4\phi + P^2\phi P\phi P^2\phi P^5\phi + P\phi P\phi P^2\phi P^6\phi \\
& + P^3\phi P\phi P^3\phi P^3\phi + P^2\phi P\phi P^3\phi P^4\phi + P\phi P\phi P^3\phi P^5\phi + P\phi P\phi P^4\phi P^4\phi \\
\text{i=2:}\quad & + P^4\phi P^2\phi P^2\phi P^2\phi + P^3\phi P^2\phi P^2\phi P^3\phi + P^2\phi P^2\phi P^2\phi P^4\phi + P\phi P^2\phi P^2\phi P^5\phi \\
& + P^2\phi P^2\phi P^3\phi P^3\phi + P\phi P^2\phi P^3\phi P^4\phi \\
\text{i=3:}\quad & + P^3\phi P^3\phi P^3\phi P\phi.
\end{aligned} \tag{544}$$

Here, we have included each value of the index i to see any pattern clearer. Observe that the upper limit for the index k changes according to the values of i, j. For example, consider the $l = 10$ case. For $i = 1, j = 1$, the index k sums to a maximum value of $10 - 3 = 7$. For $i = 1, j = 2$, the index k sums to a maximum value of $10 - 4 = 6$, and so forth. Furthermore, for $i = 2, j = 2$, the index k now sums to a maximum value of $10 - 5 = 5$. From this, we observe the pattern: for a given i, j, the maximum value the index k can sum to is given by $10 - (i + j + 1)$. This leads us to the general upper limit for k given in equation 530. We note that this expression gives the correct upper summation limits for k for $l = 6, 8$.

For the upper summation limit for j, we only need to consider the value of the index i. We summarize the maximum value j can obtain for a given value of i as follows:

$$\begin{aligned}
l = 8 :\quad & i = 1, j = 8 - 5 = 3, \quad l - (1 + 4) \\
& i = 2, j = 8 - 6 = 2, \quad l - (2 + 4)
\end{aligned} \tag{545}$$

$$\begin{aligned}
l = 10 :\quad & i = 1, j = 10 - 6 = 4, \quad l - (1 + 5) \\
& i = 2, j = 10 - 7 = 3, \quad l - (2 + 5) \\
& i = 3, j = 10 - 7 = 3, \quad l - (3 + 4)
\end{aligned} \tag{546}$$

$$\begin{aligned}
l = 12 :\quad & i = 1, j = 12 - 7 = 5, \quad l - (1 + 6) \\
& i = 2, j = 12 - 8 = 4, \quad l - (2 + 6) \\
& i = 3, j = 12 - 8 = 4, \quad l - (3 + 5)
\end{aligned} \tag{547}$$

$$
\begin{aligned}
l = 16: \quad & i = 1, j = 16 - 9 = 7, \quad l - (1+8) \\
& i = 2, j = 16 - 10 = 6, \quad l - (2+8) \\
& i = 3, j = 16 - 10 = 6, \quad l - (3+7) \\
& i = 4, j = 16 - 11 = 5, \quad l - (4+7) \\
& i = 5, j = 16 - 11 = 5, \quad l - (5+6),
\end{aligned}
\tag{548}
$$

From the column on the right, we observe that the general maximum value the index j can be summed to is given by $l-(i+(l/2-\Gamma(i))) = l/2+\Gamma(i)-i$, where $\Gamma(i)$ was defined previously. From this information, we also infer the maximum value for i; this value is suggested to be the *maximum value* of i such that $3i < l$. For example, if $l = 12$, then the maximum value of i is given by $3i < 12 \Rightarrow i = 3$. Note, that we do not choose $3i \leq 12$, since this would give a value of $i = 4$; by our construction, that would imply values $i = j = k = 4$ for a term with all the derivatives distributed over only three of the four fields.

We note here a caveat in our arguments thus far; we should address the possibility that l might be an odd integer and thus not divisible by 2. The approach to solving this problem is to understand to which integer $l/2$ must change; that is, if $P < l/2 < P+1$ where $P, P+1 \in \mathbb{N}^+/\{0\}$, and l odd, we must determine whether $l = P$ or $l = P+1$ for the appropriate case. To achieve this, we can make use of the floor and ceiling functions defined as

$$
\begin{aligned}
\lfloor x \rfloor &= sup\{m \in \mathbb{N}^+/\{0\} \quad | \quad m \leq x\} \\
\lceil x \rceil &= inf\{m \in \mathbb{N}^+/\{0\} \quad | \quad m \geq x\}.
\end{aligned}
\tag{549}
$$

Thus, we see that if x is an integer, then $\lfloor x \rfloor = \lceil x \rceil = x$. We propose that the complete general operator for positive arbitrary l should be given by

$$
\begin{aligned}
\mathcal{O}^{(l)} =& \; a_0(P^l\phi)\phi^3 + \sum_{i=1}^{\lfloor l/2 \rfloor} a_i(P^{l-i}\phi P^i\phi)\phi^2 + \sum_{i=1}^{\lceil l/2-1 \rceil} \sum_{j=i}^{\lceil l/2-1 \rceil} b_{ij}\Theta(i,j)(P^{l-i-j}\phi P^i\phi P^j\phi)\phi \\
&+ \sum_{i=1}^{3i<l} \sum_{j=i}^{\lfloor l/2+\Gamma(i)-i \rfloor} \sum_{k=j}^{l-(i+j+1)} c_{ijk}\Theta(i,j,k)(P^{l-i-j-k}\phi P^i\phi P^j\phi P^k\phi).
\end{aligned}
\tag{550}
$$

As an example, consider $l = 11$

$$
\begin{aligned}
\mathcal{O}^{(11)} =& \; a_0(P^l\phi)\phi^3 + \sum_{i=1}^{5} a_i(P^{l-i}\phi P^i\phi)\phi^2 + \sum_{i=1}^{5}\sum_{j=i}^{5} b_{ij}\Theta(i,j)(P^{l-i-j}\phi P^i\phi P^j\phi)\phi \\
&+ \sum_{i=1}^{3} \sum_{j=i}^{\lfloor 5.5+\Gamma(i)-i \rfloor} \sum_{k=j}^{11-(i+j+1)} c_{ijk}\Theta(i,j,k)(P^{l-i-j-k}\phi P^i\phi P^j\phi P^k\phi) \\
=& \; a_0(P^{11}\phi)\phi^3 + a_1(P^{10}\phi P\phi)\phi^2 + a_2(P^9\phi P^2\phi)\phi^2 + a_3(P^8\phi P^3\phi)\phi^2 + a_4(P^7\phi P^4\phi)\phi^2
\end{aligned}
$$

$$
\begin{aligned}
&+a_5(P^6\phi P^5\phi)\phi^2+b_{11}P^9\phi P\phi P\phi\phi+b_{12}P^8\phi P\phi P^2\phi\phi+b_{13}P^7\phi P\phi P^3\phi\phi+b_{14}P^6\phi P\phi P^4\phi\phi\\
&+b_{15}P^5\phi P\phi p^5\phi+b_{22}P^7\phi P^2\phi P^2\phi\phi+b_{23}P^6\phi P^2\phi P^3\phi\phi+b_{24}P^5\phi P^2\phi P^4\phi\phi\\
&+b_{33}P^5\phi P^3\phi P^3\phi+b_{34}P^4\phi P^3\phi P^4\phi\phi+c_{111}P^8\phi P\phi P\phi P\phi+c_{112}P^7\phi P\phi P\phi P^2\phi\\
&+c_{113}P^6\phi P\phi P\phi P^3\phi+c_{114}P^5\phi P\phi P\phi P^4\phi+c_{122}P^6\phi P\phi P^2\phi P^2\phi+c_{123}P^5\phi P\phi P^2\phi P^3\phi\\
&+c_{124}P^4\phi P\phi P^2\phi P^4\phi+c_{133}P^4\phi P\phi P^3\phi P^3\phi+c_{222}P^5\phi P^2\phi P^2\phi P^2\phi+c_{223}P^4\phi P^2\phi P^2\phi P^3\phi\\
&+c_{233}P^3\phi P^2\phi P^3\phi P^3\phi, \qquad (551)
\end{aligned}
$$

which is indeed the same result if we were to distribute derivatives as we did earlier in this dissertation. We also note that our general operator does reproduce the form of equation (366)

$$
\begin{aligned}
\mathcal{O}^{(3)} &= a_0(P^3\phi)\phi^3+\sum_{i=1}^{1}a_iP^{3-i}\phi P^i\phi\phi^2+\sum_{i=1}^{1}\sum_{j=1}^{1}b_{ij}P^{3-i-j}\phi P^i\phi P^j\phi\phi\\
&= a_0(P^3\phi)\phi^3+a_1P^2\phi P\phi\phi^2+b_{11}P\phi P\phi P\phi\phi. \qquad (552)
\end{aligned}
$$

The general ansatz we have argued is quite tedious and complicated to manipulate. In addition, it is only valid for primaries constructed out of four scalar fields and for primaries in the (j,j) representation of $SO(4)$.

A.3 Spin, Representations, and Projectors

In this appendix, we will discuss and develop some of the theory behind spin which is applicable to our construction of primary operators. In addition, we will develop some useful representation theory and projection operators.

The Hilbert space of a spin-$\frac{1}{2}$ particle is spanned by two states $|s_\alpha\rangle$, with $\alpha=1,2$. Here, we will choose the following basis

$$
|s_1\rangle=\begin{bmatrix}1\\0\end{bmatrix}, \qquad |s_2\rangle=\begin{bmatrix}0\\1\end{bmatrix}. \qquad (553)
$$

The angular momenta operators on this subspace are given by the Pauli matrices

$$
J_x=\frac{1}{2}\sigma_x \qquad J_y=\frac{1}{2}\sigma_y \qquad J_z=\frac{1}{2}\sigma_z \qquad (554)
$$

We act on each basis of state using J_z to yield

$$
\begin{aligned}
J_z|s_1\rangle &= \frac{1}{2}|s_1\rangle,\\
J_z|s_2\rangle &= -\frac{1}{2}|s_2\rangle, \qquad (555)
\end{aligned}
$$

which shows that the basis states are eigenkets of J_z. We do the same for $\vec{J}\cdot\vec{J} = (J_x)^2 + (J_y)^2 + (J_z)^2 = \frac{3}{4}1_{2\times 2}$ to yield

$$\vec{J}\cdot\vec{J}|s_1\rangle = \frac{3}{4}|s_1\rangle,$$
$$\vec{J}\cdot\vec{J}|s_2\rangle = \frac{3}{4}|s_2\rangle. \tag{556}$$

Thus, the basis states are simultaneous eigenstates of J_z and $\vec{J}\cdot\vec{J}$. Now, $\frac{3}{4} = \frac{1}{2}(\frac{1}{2}+1)$, which means that this representation has spin $j = \frac{1}{2}$.

Using the above eigenkets, we can construct a basis for a two spin-1/2 particle Hilbert space by using a tensor product. Thus, a basis for the two spin-1/2 particle Hilbert space can be given by

$$|s_\alpha s_\beta\rangle = |s_\alpha\rangle \otimes |s_\beta\rangle. \tag{557}$$

Here, the angular momentum operators are given by

$$\vec{J}_1 = \vec{J}\otimes 1_{2\times 2}, \qquad \vec{J}_2 = 1_{2\times 2}\otimes \vec{J}, \tag{558}$$

where $\vec{J}$ are precisely given by the matrices in equation (554). The total angular momentum operators are given by the sum $\vec{J} = \vec{J}_1 + \vec{J}_2$. In particular, $J_z = (J_1)_z \otimes 1_{2\times 2} + 1_{2\times 2}\otimes (J_2)_z$ (we will shorten the notation here by writing $J_z = (J_1)_z + (J_2)_z$). Now, consider the basis given above: $\{|s_1s_1\rangle, |s_1s_2\rangle, |s_2s_1\rangle, |s_2s_2\rangle\}$. These are indeed eigenkets of J_z. For example,

$$J_z|s_1s_1\rangle = (\frac{1}{2}+\frac{1}{2})|s_1s_1\rangle = |s_1s_1\rangle. \tag{559}$$

The other eigenvalues are $0, 0$ and -1. However, this basis has kets that are not eigenkets of $\vec{J}^2$. To see this, observe that

$$J_x|s_1\rangle = \frac{1}{2}|s_2\rangle,$$
$$J_y|s_1\rangle = \frac{i}{2}|s_2\rangle. \tag{560}$$

Thus, for $\vec{J}\cdot\vec{J} = \vec{J_1}^2 + \vec{J_2}^2 + 2\vec{J}_1\vec{J}_2$, we find that $|s_1s_2\rangle$ is not an eigenket. A basis in which the kets are eigenkets of both J_z and $\vec{J}^2$ is given by

$$\{|s_1s_1\rangle, \frac{1}{\sqrt{2}}(|s_1s_2\rangle + |s_2s_1\rangle), \frac{1}{\sqrt{2}}(|s_1s_2\rangle - |s_2s_1\rangle), |s_2s_2\rangle\}, \tag{561}$$

(these are precisely the triplet and singlet states). As an example, we act with the total spin operator on the second basis ket using the expanded expression $\vec{J}^2 = \frac{3}{4}1_{4\times 4} + \frac{3}{4}1_{4\times 4} + 2\big((J_1)_x(J_2)_x + (J_1)_y(J_2)_y + (J_1)_z(J_2)_z\big)$, we have

$$\begin{aligned}\vec{J}^2\big(\frac{1}{\sqrt{2}}(|s_1s_2\rangle + |s_2s_1\rangle)\big) &= \frac{1}{\sqrt{2}}(\frac{6}{4}|s_1s_2\rangle + \frac{6}{4}|s_1s_2\rangle) + \frac{2}{\sqrt{2}}(\frac{1}{4}|s_1s_2\rangle + \frac{1}{4}|s_2s_1\rangle) \\ &= 2\big(\frac{1}{\sqrt{2}}(|s_1s_2\rangle + |s_2s_1\rangle)\big).\end{aligned} \tag{562}$$

For $\vec{J}^2$, the eigenvalues for each basis ket is found to be $2, 2, 0$, and 2; for J_z, the eigenvalues are $1, 0, 0$, and -1. If we relabel the basis kets using the more convenient $|jm\rangle$ notation, we get

$$\{|11\rangle, |10\rangle, |00\rangle, |1-1\rangle\}, \tag{563}$$

from which we can easily write the diagonal matrices for the two operators

$$\sum_{j,m,j'm'} \langle j'm'|J_z|jm\rangle = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{bmatrix} \tag{564}$$

$$\sum_{j,m,j'm'} \langle j'm'|\vec{J}\cdot\vec{J}|jm\rangle = \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 0 & \\ & & & 2 \end{bmatrix}. \tag{565}$$

Notice that we found two spin irreducible representations (irreps), namely, spin 1 and spin 0. The spin 1 irreducible representation is given by the Young diagram $\square\square$, while spin 0 is given by the Young diagram $\begin{smallmatrix}\square\\\square\end{smallmatrix}$. The dimensions of these irreps follows by the usual *factors-over-hooks rule* (see the discussion around equation 584)

$$D_{\square\square} = \frac{\begin{array}{|c|c|}\hline 2 & 3 \\ \hline\end{array}}{\begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array}} = 3, \qquad D_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \frac{\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}}{\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}} = 1. \tag{566}$$

The triplet is a symmetric tensor $S_{\alpha\beta} = \mathrm{Sym}(|s_\alpha\rangle \otimes |s_\beta\rangle)$ with two indices, so it has three components

$$S_{11}, \qquad S_{12} = S_{21}, \qquad S_{22}, \tag{567}$$

which matches the number of states in the spin 1 multiplet. We call this a *spin* 1 *tensor.* The singlet, which is antisymmetric has a single component

$$A_{11} = 0 \qquad A_{12} = -A_{21} \qquad A_{22} = 0, \tag{568}$$

matching the number of states in a spin 0 multiplet. We call this a *spin* 0 *tensor.* The above calculations showed that $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$.

We can perform the same analysis for three spin-1/2 particles, which will reveal an interesting duality. A basis for the three particle Hilbert space can be given by

$$|s_\alpha s_\beta s_\gamma\rangle = |s_\alpha\rangle \otimes |s_\beta\rangle \otimes |s_\gamma\rangle. \tag{569}$$

In this case, the angluar momentum operators are now given

$$\begin{aligned}
\vec{J}_1 &= \vec{J} \otimes 1_{2\times 2} \otimes 1_{2\times 2}, \\
\vec{J}_2 &= 1_{2\times 2} \otimes \vec{J} \otimes 1_{2\times 2}, \\
\vec{J}_3 &= 1_{2\times 2} \otimes 1_{2\times 2} \otimes \vec{J},
\end{aligned} \tag{570}$$

where, as before, $\vec{J}$ is given by the matrices in equation (554). Here, the total angular momentum operator is given by

$$\vec{J} = \vec{J}_1 + \vec{J}_2 + \vec{J}_3. \tag{571}$$

As in the two spin-1/2 particle case, we will present a basis of eigenkets for both J_z and $\vec{J}\cdot\vec{J}$. The kets $|s_1 s_1 s_1\rangle$ and $|s_2 s_2 s_2\rangle$ are the only eigenkets of $\vec{J}\cdot\vec{J}$ in the above basis. With a bit of trial and error, we find that the set of kets

$$\{|b_1\rangle, |b_2\rangle, |b_3\rangle, |b_4\rangle, |b_5\rangle, |b_6\rangle, |b_7\rangle, |b_8\rangle\}, \tag{572}$$

where

$$\begin{aligned}
|b_1\rangle &= |s_1 s_1 s_1\rangle, \\
|b_2\rangle &= \frac{1}{\sqrt{3}}\big(|s_1 s_2 s_2\rangle + |s_2 s_1 s_2\rangle + |s_2 s_2 s_1\rangle\big), \\
|b_3\rangle &= \frac{1}{\sqrt{3}}\big(|s_1 s_1 s_2\rangle + |s_1 s_2 s_1\rangle + |s_2 s_1 s_1\rangle\big), \\
|b_4\rangle &= \frac{1}{\sqrt{2}}\big(|s_1 s_2 s_1\rangle - |s_2 s_1 s_1\rangle\big), \\
|b_5\rangle &= \frac{1}{\sqrt{2}}\big(|s_1 s_2 s_2\rangle - |s_2 s_1 s_2\rangle\big), \\
|b_6\rangle &= \frac{1}{\sqrt{6}}\big(2|s_1 s_1 s_2\rangle - |s_1 s_2 s_1\rangle - |s_2 s_1 s_1\rangle\big), \\
|b_7\rangle &= \frac{1}{\sqrt{6}}\big(|s_1 s_2 s_2\rangle + |s_2 s_1 s_2\rangle - 2|s_2 s_2 s_1\rangle\big), \\
|b_8\rangle &= |s_2 s_2 s_2\rangle,
\end{aligned} \tag{573}$$

is shared as eigenkets by both J_z and $\vec{J}\cdot\vec{J}$; in addition, they form an orthonormal basis. The two operators are diagonalized with their eigenvalues (in this basis) along their diagonals:

$$\sum_{i,j} \langle b_i | J_z | b_j \rangle = \begin{bmatrix}
\frac{3}{2} &&&&&&& \\
& -\frac{1}{2} &&&&&& \\
&& \frac{1}{2} &&&&& \\
&&& \frac{1}{2} &&&& \\
&&&& -\frac{1}{2} &&& \\
&&&&& \frac{1}{2} && \\
&&&&&& -\frac{1}{2} & \\
&&&&&&& -\frac{3}{2}
\end{bmatrix}, \tag{574}$$

$$
\sum_{i,j} \langle b_i | \vec{J} \cdot \vec{J} | b_j \rangle = \begin{bmatrix} \frac{15}{4} & & & & & & & \\ & \frac{15}{4} & & & & & & \\ & & \frac{15}{4} & & & & & \\ & & & \frac{3}{4} & & & & \\ & & & & \frac{3}{4} & & & \\ & & & & & \frac{3}{4} & & \\ & & & & & & \frac{3}{4} & \\ & & & & & & & \frac{15}{4} \end{bmatrix}. \tag{575}
$$

The kets $\{|b_1\rangle, |b_2\rangle, |b_3\rangle, |b_8\rangle\}$ represent spin 3/2 and are symmetric under exchange of the s_α index values. These kets are represented by the Young diagram $\ydiagram{3}$. The dimension of this irrep is given by

$$
D_{\ydiagram{3}} = \frac{\begin{ytableau} 3 & 4 \end{ytableau}}{\begin{ytableau} 3 & 2 & 1 \end{ytableau}} = 4. \tag{576}
$$

The basis kets $\{|b_4\rangle, |b_5\rangle\}$ and $\{|b_6\rangle, |b_7\rangle\}$ are spin 1/2 and mixed symmetric in their exchange of s_α indices. These states are represented by the Young diagram $\ydiagram{2,1}$. The dimension of this irrep is given by

$$
D_{\ydiagram{2,1}} = \frac{\begin{ytableau} 3 & 2 \end{ytableau}}{\begin{ytableau} 3 & 1 \\ 1 \end{ytableau}} = 2. \tag{577}
$$

Above we see a very concrete example of the Schur-Weyl duality; in particular, the space of 3 particles has been organized using representations of $SU(2)$ and S_3. Let the Hilbert space of a single spin particle be denoted V_2. In the analysis above, we are looking at the tensor product of three copies of this space. The Hilbert space $V_2^{\otimes 3}$ of the three particles has thus organized itself as follows

$$
V_2^{\otimes 3} = V_{\ydiagram{3}}^{SU(2)} \otimes V_{\ydiagram{3}}^{S_3} \oplus V_{\ydiagram{2,1}}^{SU(2)} \otimes V_{\ydiagram{2,1}}^{S_3}. \tag{578}
$$

Now, recall the symmetric group dimensions are given by (we will use D for $SU(2)$ dimensions and d for S_n dimensions)

$$
d_{\ydiagram{3}} = \frac{3!}{\begin{ytableau} 3 & 2 & 1 \end{ytableau}} = 1, \qquad d_{\ydiagram{2,1}} = \frac{3!}{\begin{ytableau} 3 & 1 \\ 1 \end{ytableau}} = 2. \tag{579}
$$

Notice the appearance of these numbers in the product

$$
\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} = \mathbf{\frac{1}{2}} \otimes (\mathbf{1} \oplus \mathbf{0}) = \mathbf{\frac{3}{2}} \oplus (2)\mathbf{\frac{1}{2}}. \tag{580}
$$

Here, we have used **bold face** for irreps; and multiplicities are denoted without bold face. Notice that as $SU(2)$ irreps, $\mathbf{\frac{1}{2}}$ is $\ydiagram{2,1}$ and $\mathbf{\frac{3}{2}}$ is $\ydiagram{3}$, thus illustrating equation (578).

All Young diagrams for $SU(2)$ have at most 2 rows. If we have a Young diagram with r_1 boxes in the first row and r_2 boxes in the second row, it corresponds to a spin s of $2s = r_1 - r_2$. As some examples, consider the following Young diagrams

$$\ydiagram{6,2} \qquad \ydiagram{7,1} \qquad \ydiagram{7,7}. \tag{581}$$

The first diagram $\ydiagram{6,2}$ is the spin irrep

$$\begin{aligned} s =& \frac{r_1 - r_2}{2} \\ =& \frac{6-2}{2} \\ =& \ 2. \end{aligned} \tag{582}$$

The second diagram $\ydiagram{7,1}$ is the spin irrep

$$\begin{aligned} s =& \frac{r_1 - r_2}{2} \\ =& \frac{7-1}{2} \\ =& \ 3. \end{aligned} \tag{583}$$

The third diagram $\ydiagram{7,7}$ has spin $s = 0$.

We can verify the results of the spin irreps found above by calculating the dimension of the $SU(2)$ Young diagrams. We use the formula,

$$\begin{aligned} D_{\ydiagram{2,1}} =& \frac{N(N-1)(N+1)}{Hooks} \\ =& \quad 2s+1, \end{aligned} \tag{584}$$

where s is the spin irrep . Furthermore, the factors $N(N-1)(N+1)$ are taken from the Young diagram in the following way

$$\begin{ytableau} N & N{+}1 \\ N{-}1 \end{ytableau} \tag{585}$$

where $N-1$ is the rank of $SU(N)$. Therefore for $SU(2)$, $N = 2$. Using the formula stated above, the dimensions of the following diagrams

$$\begin{gathered} \begin{ytableau} N & N{+}1 & N{+}2 & N{+}3 & N{+}4 & N{+}5 \\ N{-}1 & N \end{ytableau} \\ \begin{ytableau} N & N{+}1 & N{+}2 & N{+}3 & N{+}4 & N{+}5 & N{+}6 \\ N{-}1 \end{ytableau} \end{gathered} \tag{586}$$

$$
\begin{array}{|c|c|c|c|c|c|c|}
\hline
N & N+1 & N+2 & N+3 & N+4 & N+5 & N+6 \\
\hline
N-1 & N & N+1 & N+2 & N+3 & N+4 & N+5 \\
\hline
\end{array}
\tag{587}
$$

respectively given by

$$
\begin{aligned}
D_{\ydiagram{6,2}} =& \frac{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 2}{7\cdot 6\cdot 4\cdot 3\cdot 2\cdot 2} \\
=& 5 \\
=& 2s+1 \\
\Rightarrow s =& 2,
\end{aligned}
\tag{588}
$$

$$
\begin{aligned}
D_{\ydiagram{7,1}} =& \ \frac{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}{8\cdot 6\cdot 4\cdot 3\cdot 2\cdot 2} \\
=& 7 \\
=& \ 2s+1 \\
\Rightarrow s =& \ 3,
\end{aligned}
\tag{589}
$$

and

$$
\begin{aligned}
D_{\ydiagram{7,7}} =& 1 \\
\Rightarrow s =& 0.
\end{aligned}
\tag{590}
$$

We can consider an n-particle Hilbert space and show that the multiplicities are explained by the dimensions of the symmetric group representations as required by Schur-Weyl duality. The tensor product of four spin-$\frac{1}{2}$ particles is

$$
\begin{aligned}
\frac{\mathbf{1}}{\mathbf{2}} \otimes \frac{\mathbf{1}}{\mathbf{2}} \otimes \frac{\mathbf{1}}{\mathbf{2}} \otimes \frac{\mathbf{1}}{\mathbf{2}} =& \frac{\mathbf{1}}{\mathbf{2}} \otimes \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{1} \oplus \mathbf{0}) \\
=& \frac{\mathbf{1}}{\mathbf{2}} \otimes (\frac{\mathbf{3}}{\mathbf{2}} \oplus \frac{\mathbf{1}}{\mathbf{2}} \oplus \frac{\mathbf{1}}{\mathbf{2}}) \\
=& \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{0} \\
=& \mathbf{2} \oplus (3)\mathbf{1} \oplus (2)\mathbf{0}.
\end{aligned}
\tag{591}
$$

The tensor product above can be represented in terms of Young diagrams as follows

$$
\ydiagram{1} \otimes \ydiagram{1} \otimes \ydiagram{1} \otimes \ydiagram{1} = \ydiagram{4} \oplus (3)\ydiagram{3,1} \oplus (2)\ydiagram{2,2}.
\tag{592}
$$

The factors in front of the Young diagrams which are multiplicities, are obtained from computing the dimensions of the symmetric group representation of the corresponding Young diagram. They are computed as follows,

$$
\begin{aligned}
d_{\ydiagram{4}} =& \frac{4!}{4\cdot 3\cdot 2} \\
=& 1,
\end{aligned}
\tag{593}
$$

$$d_{\ydiagram{3,1}} = \frac{4!}{4 \cdot 2 \cdot 1} = 3, \tag{594}$$

and

$$d_{\ydiagram{2,2}} = \frac{4!}{3 \cdot 2 \cdot 2} = 2. \tag{595}$$

As a further example, the tensor product of five spin-$\frac{1}{2}$ particles is

$$\frac{\mathbf{1}}{\mathbf{2}} \otimes \frac{\mathbf{1}}{\mathbf{2}} \otimes \frac{\mathbf{1}}{\mathbf{2}} \otimes \frac{\mathbf{1}}{\mathbf{2}} \otimes \frac{\mathbf{1}}{\mathbf{2}} = \frac{\mathbf{5}}{\mathbf{2}} \oplus (4)\frac{\mathbf{3}}{\mathbf{2}} \oplus (5)\frac{\mathbf{1}}{\mathbf{2}}. \tag{596}$$

Here, the tensor product of five spin-$\frac{1}{2}$ particles represented with $SU(2)$ Young diagrams is given by

$$\square \otimes \square \otimes \square \otimes \square \otimes \square = \ydiagram{5} \oplus (4)\ydiagram{4,1} \oplus (5)\ydiagram{3,2}. \tag{597}$$

As before, the multiplicities in front of the Young diagrams are obtained from computing the dimension of the symmetric group representations

$$d_{\ydiagram{5}} = \frac{5!}{5!} = 1, \tag{598}$$

$$d_{\ydiagram{4,1}} = \frac{5!}{5 \cdot 3 \cdot 2} = 4, \tag{599}$$

and

$$d_{\ydiagram{3,2}} = \frac{5!}{4 \cdot 3 \cdot 2} = 5. \tag{600}$$

Finally, the tensor product of six spin-$\frac{1}{2}$ particles is given by

$$\frac{\mathbf{1}}{\mathbf{2}} \otimes \frac{\mathbf{1}}{\mathbf{2}} \otimes \frac{\mathbf{1}}{\mathbf{2}} \otimes \frac{\mathbf{1}}{\mathbf{2}} \otimes \frac{\mathbf{1}}{\mathbf{2}} \otimes \frac{\mathbf{1}}{\mathbf{2}} = \mathbf{3} \oplus \mathbf{2} \oplus (5)\mathbf{2} \oplus (9)\mathbf{1} \oplus (5)\mathbf{0}. \tag{601}$$

The representation of this tensor product in Young diagrams is given by

$$\square \otimes \square \otimes \square \otimes \square \otimes \square \otimes \square = \ydiagram{6} \oplus (5)\ydiagram{5,1} \oplus (9)\ydiagram{4,2} \oplus (5)\ydiagram{3,3}. \tag{602}$$

The multiplicities are obtained as follows

$$d_{\ydiagram{6}} = \frac{6!}{6!} = 1, \tag{603}$$

$$
\begin{aligned}
d_{\ydiagram{5,1}} =& \frac{6!}{6 \cdot 4 \cdot 3 \cdot 2} \\
=& 5,
\end{aligned}
\tag{604}
$$

$$
\begin{aligned}
d_{\ydiagram{4,2}} =& \frac{6!}{5 \cdot 4 \cdot 2 \cdot 2} \\
=& 9
\end{aligned}
\tag{605}
$$

and

$$
\begin{aligned}
d_{\ydiagram{3,3}} =& \frac{6!}{4 \cdot 3 \cdot 2 \cdot 3 \cdot 2} \\
=& 5.
\end{aligned}
\tag{606}
$$

Clearly, the symmetric group is organizing the multiplicities of the $SU(2)$ irreps that are appearing. Thus, we will also need to discuss more about the symmetric groups and its irreps. Now, we know that Young diagrams with n boxes label the irreps of the symmetric group S_n. Each of these representations is given by a set of matrices $\Gamma_R(\sigma)$ which act on a vector space $V_R^{S_n}$. By decorating the Young diagram of a particular irreducible representation, we can give a convenient labeling for the elements of a complete basis of $V_R^{S_n}$. The decorated Young diagram is called a *Young tableaux* or (in this specific context of labeling a basis for an S_n irreducible representation) a *Young-Yamanouchi symbol.* We will use kets for Young-Yamanouchi symbols, so that if R is the Young diagram, then $|R\rangle$ is one of the Young-Yamanouchi symbols that can be obtained by decorating R.

To obtain a Young-Yamanouchi symbol, we fill the boxes of each Young diagram with the integers $1, 2, 3, ..., n$. Each box is assigned a unique integer so that in the end we fill in n distinct entries. The entries in each row and each column are decreasing (of course, we could also have required that the entries are increasing - these are just conventions) as we move to the right or bottom of the page. As an example, the irreducible representation labeled by $\ydiagram{3,1}$ leads to three distinct Young-Yamanouchi symbols

$$
\begin{ytableau} 4 & 2 & 1 \\ 3 \end{ytableau} \quad \begin{ytableau} 4 & 3 & 1 \\ 2 \end{ytableau} \quad \begin{ytableau} 4 & 3 & 2 \\ 1 \end{ytableau} \; .
\tag{607}
$$

Each such symbol labels a basis vector in $V_{\ydiagram{3,1}}^{S_4}$ so that $\ydiagram{3,1}$ is a 3 dimensional representation.

As more examples, we determine the complete set of Young-Yamanouchi symbols for the following diagrams

$$
\ydiagram{1,1,1,1,1} \qquad \ydiagram{6} \qquad \ydiagram{3,3} \qquad \ydiagram{3,2,1}
\tag{608}
$$

and we check that the number of symbols do indeed match the dimension of the irrep.

1) For the diagram $\ydiagram{1,1,1,1,1}$, we find the following Young - Yamanouchi symbol:

$$\begin{ytableau} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{ytableau} \tag{609}$$

where the dimension of the irrep is given by $d_{\ydiagram{1,1,1,1,1}} = 5!/5! = 1$.

2) For the diagram $\ydiagram{6}$, we find the following Young - Yamanouchi symbol:

$$\begin{ytableau} 6 & 5 & 4 & 3 & 2 & 1 \end{ytableau} \tag{610}$$

where the dimension of the irrep is given by $d_{\ydiagram{6}} = 6!/6! = 1$.

3) For the diagram $\ydiagram{3,3}$, we find the following Young - Yamanouchi symbols:

$$\begin{ytableau} 6 & 5 & 4 \\ 3 & 2 & 1 \end{ytableau} \quad \begin{ytableau} 6 & 4 & 3 \\ 5 & 2 & 1 \end{ytableau} \quad \begin{ytableau} 6 & 5 & 3 \\ 4 & 2 & 1 \end{ytableau}$$
$$\begin{ytableau} 6 & 5 & 2 \\ 4 & 3 & 1 \end{ytableau} \quad \begin{ytableau} 6 & 4 & 2 \\ 5 & 3 & 1 \end{ytableau} \tag{611}$$

where the dimension of the irrep is given by $d_{\ydiagram{3,3}} = 6!/(4 \times 3 \times 3 \times 2 \times 2 \times 1) = 5$.

4) For the last diagram $\ydiagram{3,2,1}$, we find the following Young - Yamanouchi symbols:

$$\begin{ytableau} 6 & 5 & 4 \\ 3 & 2 \\ 1 \end{ytableau} \quad \begin{ytableau} 6 & 5 & 3 \\ 4 & 2 \\ 1 \end{ytableau} \quad \begin{ytableau} 6 & 5 & 2 \\ 4 & 3 \\ 1 \end{ytableau} \quad \begin{ytableau} 6 & 5 & 1 \\ 4 & 3 \\ 2 \end{ytableau} \quad \begin{ytableau} 6 & 4 & 3 \\ 5 & 2 \\ 1 \end{ytableau} \quad \begin{ytableau} 6 & 4 & 2 \\ 5 & 3 \\ 1 \end{ytableau}$$
$$\begin{ytableau} 6 & 4 & 1 \\ 5 & 3 \\ 2 \end{ytableau} \quad \begin{ytableau} 6 & 3 & 2 \\ 5 & 1 \\ 4 \end{ytableau} \quad \begin{ytableau} 6 & 3 & 1 \\ 5 & 2 \\ 1 \end{ytableau} \quad \begin{ytableau} 6 & 5 & 4 \\ 3 & 1 \\ 2 \end{ytableau} \quad \begin{ytableau} 6 & 5 & 3 \\ 4 & 1 \\ 2 \end{ytableau} \quad \begin{ytableau} 6 & 5 & 2 \\ 4 & 1 \\ 3 \end{ytableau}$$
$$\begin{ytableau} 6 & 5 & 1 \\ 4 & 2 \\ 3 \end{ytableau} \quad \begin{ytableau} 6 & 4 & 3 \\ 5 & 1 \\ 2 \end{ytableau} \quad \begin{ytableau} 6 & 4 & 2 \\ 5 & 1 \\ 3 \end{ytableau} \quad \begin{ytableau} 6 & 4 & 1 \\ 5 & 2 \\ 3 \end{ytableau} \tag{612}$$

where the dimension of the irrep is given by $d_{\ydiagram{3,2,1}} = 6!/(5 \times 3 \times 3 \times 1 \times 1 \times 1) = 16$.

Next, we will present a formula for the matrices representing the elements of the symmetric group; in other words, we give a matrix representation of the symmetric group. Concretely, this amounts to giving a set of matrices that obey the equation

$$\Gamma_R(\sigma_1)\Gamma_R(\sigma_2) = \Gamma_R(\sigma_1\sigma_2). \tag{613}$$

The key observation here is the following: on the right hand side of this equation, the product is the usual group composition law for permutations while on the left hand side it is matrix multiplication. Thus, a matrix representation is a map from the elements of the symmetric group to some matrix algebra and this map preserves the structure (in other words, the multiplication table) of the group. A structure preserving map between two algebraic structures is called a *homomorphism.* We can say that a d-dimensional matrix representation of the symmetric group S_n is a group homomorphism from S_n to $GL(V_d)$ (V is a vector space over field F. The general linear group of V, called $GL(V)$ or $\mathrm{Aut}(V)$, is the group of all automorphisms of V. This is the set of all bijective linear transformations $V \to V$, together with function composition as the group operation. For us V_d is real d-dimensional vector space and $GL(V_d)$ is the set of invertible $d \times d$ matrices). From (613) we can immediately prove that the identity element e of the group maps into the identity matrix $\mathbf{1}$. Indeed, since $e \cdot e = e$ we know that

$$\begin{aligned} &\Gamma_R(e)\Gamma_R(e) = \Gamma_R(e) \\ \Rightarrow \qquad &\Gamma_R(e)\Gamma_R(e) - \Gamma_R(e) = \Gamma_R(e)(\Gamma_R(e) - \mathbf{1}) = 0, \end{aligned} \tag{614}$$

which, since $\Gamma_R(e)$ is invertible, implies that $\Gamma_R(e) = \mathbf{1}$. Note also that

$$\Gamma_R(g)\Gamma_R(g^{-1}) = \Gamma_R(g \cdot g^{-1}) = \Gamma_R(e) = \mathbf{1}, \tag{615}$$

so that $\Gamma_R(g^{-1}) = \Gamma_R(g)^{-1}$. Finally, if the map from the group to $GL(V_d)$ is one-to-one, we say the representation is *faithful.*

We will only need to construct the matrices representing *adjacent transpositions*, since any other permutation can be written in terms of adjacent transpositions. Furthermore, we specify these matrices by giving their action on the states labeled by the Young-Yamanouchi symbols. To write the explicit expressions, we need to introduce the *content* of a box in a Young diagram. A box x in row i and column j of R has content $j - i$. Given a Young-Yamanouchi symbol, each box in the Young diagram is labeled by a unique integer from i with $1 \le i \le n$. The content of the box labeled i is c_i. Let $|R_{(k,k+1)}\rangle$ denote the Young-Yamanouchi symbol that is obtained from $|R\rangle$ by swapping the labels of boxes k and $k+1$. We only ever swap boxes whose labels differ by 1. For example,

$$|R\rangle = \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad |R_{(1,2)}\rangle = \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad |R_{(2,3)}\rangle = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \quad . \tag{616}$$

Note that $|R_{(2,3)}\rangle$ is not a valid Young-Yamanouchi symbol. In the end, any illegal Young-Yamanouchi symbols will be multiplied by zero so we need not worry about this. The matrix elements of the adjacent transpositions are now specified by

$$\Gamma_R\left((k,k+1)\right)|R\rangle = \frac{1}{c_k - c_{k+1}}|R\rangle + \sqrt{1 - \frac{1}{(c_k - c_{k+1})^2}}|R_{(k,k+1)}\rangle. \tag{617}$$

As a first example using this formula, we have

$$\Gamma_{\tiny\yng(2,1,1)}((12))\,|\young(43,2,1)\rangle = -|\young(43,2,1)\rangle \tag{618}$$

$$\Gamma_{\tiny\yng(2,1,1)}((12))\,|\young(42,3,1)\rangle = -\frac{1}{3}|\young(42,3,1)\rangle + \frac{\sqrt{8}}{3}|\young(41,3,2)\rangle \tag{619}$$

$$\Gamma_{\tiny\yng(2,1,1)}((12))\,|\young(41,3,2)\rangle = \frac{1}{3}|\young(41,3,2)\rangle + \frac{\sqrt{8}}{3}|\young(42,3,1)\rangle. \tag{620}$$

Choosing

$$|\young(43,2,1)\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad |\young(42,3,1)\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad |\young(41,3,2)\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix}, \tag{621}$$

we find the following matrix representation for the permutation (12)

$$\Gamma_{\tiny\yng(2,1,1)}((12)) = \begin{bmatrix}-1 & 0 & 0\\ 0 & -\frac{1}{3} & \frac{\sqrt{8}}{3}\\ 0 & \frac{\sqrt{8}}{3} & \frac{1}{3}\end{bmatrix}. \tag{622}$$

As another example, we work out the complete matrix representation for the irreducible representation labeled by the Young diagram $\tiny\yng(3,2)$. First, we find the Young-Yamanouchi symbols for this diagram

$$\young(543,21) \qquad \young(532,41) \qquad \young(531,42)$$

$$\young(542,31) \qquad \young(541,32)\,, \tag{623}$$

(as a check, note that $d_{\ydiagram{3,2}} = 5!/4! = 5$). The adjacent transpositions for S_5 are

$$(12), \qquad (23), \qquad (34), \qquad (45). \tag{624}$$

and the content is given by

$$\begin{ytableau} 0 & 1 & 2 \\ -1 & 0 \end{ytableau}. \tag{625}$$

To obtain their matrix representations, we use equation 617. We start with (12):

$$\Gamma_{\ydiagram{3,2}}((12))|\ytableaushort{543,21}\rangle = |\ytableaushort{543,21}\rangle, \tag{626}$$

$$\Gamma_{\ydiagram{3,2}}((12))|\ytableaushort{532,41}\rangle = -\frac{1}{2}|\ytableaushort{532,41}\rangle + \sqrt{\frac{3}{4}}|\ytableaushort{531,42}\rangle, \tag{627}$$

$$\Gamma_{\ydiagram{3,2}}((12))|\ytableaushort{531,42}\rangle = \frac{1}{2}|\ytableaushort{531,42}\rangle + \sqrt{\frac{3}{4}}|\ytableaushort{532,41}\rangle, \tag{628}$$

$$\Gamma_{\ydiagram{3,2}}((12))|\ytableaushort{542,31}\rangle = -\frac{1}{2}|\ytableaushort{542,31}\rangle + \sqrt{\frac{3}{4}}|\ytableaushort{541,32}\rangle, \tag{629}$$

$$\Gamma_{\ydiagram{3,2}}((12))|\ytableaushort{541,32}\rangle = \frac{1}{2}|\ytableaushort{541,32}\rangle + \sqrt{\frac{3}{4}}|\ytableaushort{542,31}\rangle. \tag{630}$$

For (23):

$$\Gamma_{\ydiagram{3,2}}((23))|\ytableaushort{543,21}\rangle = -\frac{1}{3}|\ytableaushort{543,21}\rangle + \sqrt{\frac{8}{9}}|\ytableaushort{542,31}\rangle, \tag{631}$$

$$\Gamma_{\ydiagram{3,2}}((23))|\ytableaushort{532,41}\rangle = |\ytableaushort{532,41}\rangle, \tag{632}$$

$$\Gamma_{\ydiagram{3,2}}((23))|\ytableaushort{542,31}\rangle = \frac{1}{3}|\ytableaushort{542,31}\rangle + \sqrt{\frac{8}{9}}|\ytableaushort{543,21}\rangle, \tag{633}$$

$$\Gamma_{\ydiagram{3,2}}((23))|\ytableaushort{531,42}\rangle = -|\ytableaushort{531,42}\rangle, \tag{634}$$

$$\Gamma_{\ydiagram{3,2}}((23))\left|\,\ytableaushort{541,32}\right\rangle = \left|\,\ytableaushort{541,32}\right\rangle. \tag{635}$$

For (34):

$$\Gamma_{\ydiagram{3,2}}((34))\left|\,\ytableaushort{543,21}\right\rangle = \left|\,\ytableaushort{543,21}\right\rangle, \tag{636}$$

$$\Gamma_{\ydiagram{3,2}}((34))\left|\,\ytableaushort{532,41}\right\rangle = \frac{1}{2}\left|\,\ytableaushort{532,41}\right\rangle + \sqrt{\frac{3}{4}}\left|\,\ytableaushort{542,31}\right\rangle, \tag{637}$$

$$\Gamma_{\ydiagram{3,2}}((34))\left|\,\ytableaushort{531,42}\right\rangle = \frac{1}{2}\left|\,\ytableaushort{531,42}\right\rangle + \sqrt{\frac{3}{4}}\left|\,\ytableaushort{541,32}\right\rangle, \tag{638}$$

$$\Gamma_{\ydiagram{3,2}}((34))\left|\,\ytableaushort{542,31}\right\rangle = -\frac{1}{2}\left|\,\ytableaushort{542,31}\right\rangle + \sqrt{\frac{3}{4}}\left|\,\ytableaushort{532,41}\right\rangle, \tag{639}$$

$$\Gamma_{\ydiagram{3,2}}((34))\left|\,\ytableaushort{541,32}\right\rangle = -\frac{1}{2}\left|\,\ytableaushort{541,32}\right\rangle + \sqrt{\frac{3}{4}}\left|\,\ytableaushort{531,42}\right\rangle. \tag{640}$$

For (45):

$$\Gamma_{\ydiagram{3,2}}((45))\left|\,\ytableaushort{543,21}\right\rangle = \left|\,\ytableaushort{543,21}\right\rangle, \tag{641}$$

$$\Gamma_{\ydiagram{3,2}}((45))\left|\,\ytableaushort{532,41}\right\rangle = -\left|\,\ytableaushort{532,41}\right\rangle, \tag{642}$$

$$\Gamma_{\ydiagram{3,2}}((45))\left|\,\ytableaushort{531,42}\right\rangle = -\left|\,\ytableaushort{531,42}\right\rangle, \tag{643}$$

$$\Gamma_{\ydiagram{3,2}}((45))\left|\,\ytableaushort{542,31}\right\rangle = \left|\,\ytableaushort{542,31}\right\rangle, \tag{644}$$

$$\Gamma_{\ydiagram{3,2}}((45))\left|\,\ytableaushort{541,32}\right\rangle = \left|\,\ytableaushort{541,32}\right\rangle. \tag{645}$$

We choose the basis

$$\left|\,\begin{ytableau} 5 & 4 & 3 \\ 2 & 1 \end{ytableau}\right\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \left|\,\begin{ytableau} 5 & 3 & 2 \\ 4 & 1 \end{ytableau}\right\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \left|\,\begin{ytableau} 5 & 3 & 1 \\ 4 & 2 \end{ytableau}\right\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\left|\,\begin{ytableau} 5 & 4 & 2 \\ 3 & 1 \end{ytableau}\right\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \left|\,\begin{ytableau} 5 & 4 & 1 \\ 3 & 2 \end{ytableau}\right\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{646}$$

Thus, we can write the Γ's as

$$\Gamma_{\ydiagram{3,2}}((12)) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \tag{647}$$

$$\Gamma_{\ydiagram{3,2}}((23)) = \begin{bmatrix} -\frac{1}{3} & 0 & 0 & \frac{2\sqrt{2}}{3} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \frac{2\sqrt{2}}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{648}$$

$$\Gamma_{\ydiagram{3,2}}((34)) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{bmatrix} \tag{649}$$

$$\Gamma_{\ydiagram{3,2}}((45)) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{650}$$

Now, consider the following group product between two elements in S_5: $(12)(21)$. This product corresponds to the identity permutation in S_5: $(12)(21) = (12)(12) = 1$. Thus, by our homomorphism mapping, we must find that

$$\Gamma_{\ydiagram{3,2}}((12))\Gamma_{\ydiagram{3,2}}((12)) = 1_{5\times 5}, \tag{651}$$

which is indeed the case. This result also holds for the other adjacent transpositions. In addition, we can also form more complicated permutations through products of adjacent transpositions. For example, consider the permutation $(13) = (12)(23)(12)$. The matrix representation is given by

$$
\begin{gathered}
\Gamma_{\ydiagram{3,2}}((13)) = \Gamma_{\ydiagram{3,2}}((12))\Gamma_{\ydiagram{3,2}}((23))\Gamma_{\ydiagram{3,2}}((12)) = \\
\begin{bmatrix}
-\frac{1}{3} & 0 & 0 & -\frac{\sqrt{2}}{3} & \sqrt{\frac{2}{3}} \\
0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\
0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\
-\frac{\sqrt{2}}{3} & 0 & 0 & \frac{5}{6} & -\frac{1}{4\sqrt{3}} + \frac{\sqrt{3}}{4} \\
\sqrt{\frac{2}{3}} & 0 & 0 & -\frac{1}{4\sqrt{3}} + \frac{\sqrt{3}}{4} & \frac{1}{2}
\end{bmatrix}
\end{gathered}
\tag{652}
$$

and acting on the first Young- Yamanouchi symbol gives

$$
\begin{aligned}
\Gamma_{\ydiagram{3,2}}((13))|\begin{ytableau} 5 & 4 & 3 \\ 2 & 1 \end{ytableau}\rangle = -\frac{1}{3}|\begin{ytableau} 5 & 4 & 3 \\ 2 & 1 \end{ytableau}\rangle - \frac{\sqrt{2}}{3}|\begin{ytableau} 5 & 4 & 2 \\ 3 & 1 \end{ytableau}\rangle \\
+\sqrt{\frac{2}{3}}|\begin{ytableau} 5 & 4 & 1 \\ 3 & 2 \end{ytableau}\rangle.
\end{aligned}
\tag{653}
$$

Next, we discuss *projection operators* or *projectors*. The equation

$$P^2 = P, \tag{654}$$

defines a projection operator. To see this, we determine the eigenvalues of the projection operator P; in particular, if ψ is an eigenfunction of P with eigenvalue ρ, then

$$P^2\psi = P(\rho\psi) = \rho(P\psi) = \rho^2\psi. \tag{655}$$

But $P^2 = P \Rightarrow \rho^2 = \rho \Rightarrow \rho = 0, 1$. Thus, the definition for P only allows two possible eigenvalues for any eigenfunction ψ. To see why this is a good definition, consider the following: let $\mathcal{H}$ be a Hilbert space and let P be a projection operator defined on this space such that

$$
\begin{aligned}
P :&\mathcal{H} \longrightarrow Y \\
&\psi \mapsto P\psi,
\end{aligned}
\tag{656}
$$

where Y is a subspace of $\mathcal{H}$. Then, for any nonzero $\psi \in \mathcal{H}$ projected into Y, $P(P\psi) = \psi$ since $P : Y \longrightarrow Y$. For any $\psi \notin Y$, we must have $\psi \in Y^\perp$ (where $\langle\psi|\theta\rangle = \langle\theta|\psi\rangle = 0$ for all $\psi \in Y, \theta \in Y^\perp$), so that $P\psi = 0\psi = 0$ since $P : Y^\perp \longrightarrow \{0\}$. These are precisely the eigenvalues we found previously. We therefore see that $P^2 = P$ is a good definition of a projection operator since it captures precisely this idea.

Furthermore, a projection operator can be oblique or orthogonal. A simple illustration of a oblique projection operator in two dimension is the matrix,

$$P = \begin{bmatrix} 0 & 0 \\ \alpha & 1 \end{bmatrix}. \tag{657}$$

This matrix satisfies the property $P^2 = P$

$$\begin{aligned} P^2 = & \begin{bmatrix} 0 & 0 \\ \alpha & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \alpha & 1 \end{bmatrix} \\ = & \begin{bmatrix} 0 & 0 \\ \alpha & 1 \end{bmatrix} \\ =& P. \end{aligned} \tag{658}$$

When $\alpha = 0$, the oblique operator P becomes an orthogonal projection operator

$$P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{659}$$

Geometrically the matrix above projects the point (x, y) to $(0, y)$. Some examples of projection operators in three dimension are

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{660}$$

Here, P_1 projects the point (x, y, z) into $(x, y, 0)$ (the xy plane), P_2 projects the point (x, y, z) into $(0, y, z)$ (the yz plane) and, P_3 projects the point (x, y, z) into $(x, 0, z)$ (the xz plane).

Now, let $|Y\rangle$ be a specific Young-Yammonouchi state, which is one of the states in the carrier space of some irrep R. Consider the operator

$$P_Y = \sum_{\sigma \in S_n} \langle Y|\Gamma_R(\sigma)|Y\rangle\, \sigma, \tag{661}$$

(the precise action of σ will be demonstrated in the paragraphs that follow). We can normalize P_Y to obtain a projection operator as follows

$$\begin{aligned} (P_Y)^2 =& \sum_{\sigma,\rho \in S_n} \langle Y|\Gamma_R(\sigma)|Y\rangle\langle Y|\Gamma_R(\rho)|Y\rangle\, \sigma\rho \\ =& \sum_{\sigma,\psi \in S_n} \langle Y|\Gamma_R(\sigma)|Y\rangle\langle Y|\Gamma_R(\sigma^{-1}\psi)|Y\rangle\, \psi \\ =& \sum_{\sigma,\psi \in S_n} \langle Y|\Gamma_R(\sigma)|Y\rangle\langle Y|\Gamma_R(\sigma^{-1})\Gamma_R(\psi)|Y\rangle\, \psi \\ =& \sum_{\sigma,\psi \in S_n} \langle Y|_a\Gamma_R(\sigma)_{ab}|Y\rangle_b\langle Y|_c\Gamma_R(\sigma^{-1})_{c\alpha}\Gamma_R(\psi)_{\alpha\beta}|Y\rangle_\beta\, \psi \end{aligned}$$

$$
\begin{aligned}
&= \frac{|\mathcal{G}|}{d_R} \sum_{\psi \in S_n} \delta_{cb} \delta_{a\alpha} \langle Y|_a |Y\rangle_b \langle Y|_c \Gamma_R(\psi)_{\alpha\beta} |Y\rangle_\beta \, \psi \\
&= \frac{|\mathcal{G}|}{d_R} \sum_{\psi \in S_n} \langle Y|_\alpha |Y\rangle_c \langle Y|_c \Gamma_R(\psi)_{\alpha\beta} |Y\rangle_\beta \, \psi \\
&= \frac{|\mathcal{G}|}{d_R} \sum_{\psi \in S_n} \langle Y|_c |Y\rangle_c \langle Y|_\alpha \Gamma_R(\psi)_{\alpha\beta} |Y\rangle_\beta \, \psi \\
&= \frac{|\mathcal{G}|}{d_R} \sum_{\psi \in S_n} \langle Y|\Gamma_R(\psi)|Y\rangle \, \psi, \qquad (662)
\end{aligned}
$$

where we used the fundamental orthogonality relation

$$
\sum_{g \in \mathcal{G}} \Gamma_R(g^{-1})_{ab} \Gamma_S(g)_{\alpha\beta} = \frac{|\mathcal{G}|}{d_R} \delta_{RS} \delta_{\alpha\beta} \delta_{b\alpha}. \qquad (663)
$$

Thus, if we have

$$
P_Y = \frac{d_R}{|\mathcal{G}|} \sum_{\sigma \in S_n} \langle Y|\Gamma_R(\sigma)|Y\rangle \, \sigma, \qquad (664)
$$

then $(P_Y)^2 = P_Y$.

Now, we define an action on the space of states of the three particle Hilbert space, where each particle is spin $\frac{1}{2}$. We can define the states in this Hilbert space as

$$
|s_{\alpha_1} s_{\alpha_2} s_{\alpha_3}\rangle = |s_{\alpha_1}\rangle \otimes |s_{\alpha_2}\rangle \otimes |s_{\alpha_3}\rangle, \qquad (665)
$$

and we define the action of the symmetric group S_3 on this space as follows

$$
\sigma |s_{\alpha_1} s_{\alpha_2} s_{\alpha_3}\rangle = |s_{\alpha_{\sigma(1)}} s_{\alpha_{\sigma(2)}} s_{\alpha_{\sigma(3)}}\rangle. \qquad (666)
$$

Using the above projection operators, we compute the states

$$
P_{\young(32,1)} |s_{\alpha_1} s_{\alpha_2} s_{\alpha_3}\rangle \qquad P_{\young(31,2)} |s_{\alpha_1} s_{\alpha_2} s_{\alpha_3}\rangle \qquad P_{\young(321)} |s_{\alpha_1} s_{\alpha_2} s_{\alpha_3}\rangle. \qquad (667)
$$

and we compute the value of the spin operators acting on these states. The Young diagram $\yng(2,1)$ has two Young-Yamanouchi states

$$
\young(32,1) \qquad \young(31,2). \qquad (668)
$$

The complete matrix irreducible representation for the Young diagram $\yng(2,1)$, for a (12) permutation is computated using the formula

$$
\Gamma_R(k, k+1)|R\rangle = \frac{1}{c_k - c_{k+1}} |R\rangle + \sqrt{1 - \frac{1}{(c_k - c_{k+1})^2}} |R_{(k,k+1)}\rangle, \qquad (669)
$$

to yield

$$\Gamma_{\ydiagram{2,1}}(12)\left|\,\begin{ytableau} 3 & 2 \\ 1 \end{ytableau}\right\rangle = -\frac{1}{2}\left|\,\begin{ytableau} 3 & 2 \\ 1 \end{ytableau}\right\rangle + \frac{\sqrt{3}}{2}\left|\,\begin{ytableau} 3 & 1 \\ 2 \end{ytableau}\right\rangle, \tag{670}$$

and

$$\Gamma_{\ydiagram{2,1}}(12)\left|\,\begin{ytableau} 3 & 1 \\ 2 \end{ytableau}\right\rangle = \frac{1}{2}\left|\,\begin{ytableau} 3 & 1 \\ 2 \end{ytableau}\right\rangle + \frac{\sqrt{3}}{2}\left|\,\begin{ytableau} 3 & 2 \\ 1 \end{ytableau}\right\rangle. \tag{671}$$

Let

$$\left|\,\begin{ytableau} 3 & 2 \\ 1 \end{ytableau}\right\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \left|\,\begin{ytableau} 3 & 1 \\ 2 \end{ytableau}\right\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{672}$$

Then

$$\Gamma_R(12) = \Gamma_{\ydiagram{2,1}}(12) = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}. \tag{673}$$

The complete matrix irreducible representation elements for the permutation (23) is

$$\Gamma_{\ydiagram{2,1}}(23)\left|\,\begin{ytableau} 3 & 2 \\ 1 \end{ytableau}\right\rangle = \left|\,\begin{ytableau} 3 & 2 \\ 1 \end{ytableau}\right\rangle, \tag{674}$$

$$\Gamma_{\ydiagram{2,1}}(23)\left|\,\begin{ytableau} 3 & 1 \\ 2 \end{ytableau}\right\rangle = -\left|\,\begin{ytableau} 3 & 1 \\ 2 \end{ytableau}\right\rangle. \tag{675}$$

Since we have already chosen the basis, the matrix representation for $\Gamma_R(23)$ is

$$\Gamma_R(23) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{676}$$

The irreducible matrix representation elements for the permutation (13) is first decomposed into $(13) = (12)(23)(12)$, and the representation is

$$\begin{aligned} \Gamma_R(13) =& \Gamma_R(12)\Gamma_R(23)\Gamma_R(12) \\ =& \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \\ =& \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}. \end{aligned} \tag{677}$$

Now we are left with computations involving the irreducible matrix representation for the permutations (123) and (132). We can decompose these two permutations in the following way

$$(123) = (13)(32), \qquad (132) = (12)(23). \tag{678}$$

Therefore

$$\begin{aligned}\Gamma_R(123) =&\Gamma_R(13)\Gamma_R(32)\\ =&\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1}\\ =&\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \end{aligned} \tag{679}$$

and

$$\begin{aligned}\Gamma_R(132) =&\Gamma_R(12)\Gamma_R(23)\\ =&\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\\ =&\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}. \end{aligned} \tag{680}$$

We can thus use these results to compute the following states

$$P_{\begin{smallmatrix}\boxed{3}\boxed{2}\\\boxed{1}\end{smallmatrix}}|s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle \qquad P_{\begin{smallmatrix}\boxed{3}\boxed{1}\\\boxed{2}\end{smallmatrix}}|s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle \qquad P_{\boxed{3}\boxed{2}\boxed{1}}|s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle. \tag{681}$$

Starting with $P_{\begin{smallmatrix}\boxed{3}\boxed{2}\\\boxed{1}\end{smallmatrix}}$, we compute

$$P_{\begin{smallmatrix}\boxed{3}\boxed{2}\\\boxed{1}\end{smallmatrix}}|s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle = \sum_{\sigma\epsilon S_3}\langle Y|\Gamma_R|Y\rangle\sigma|s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle. \tag{682}$$

We find that

$$\begin{aligned}P_{\begin{smallmatrix}\boxed{3}\boxed{2}\\\boxed{1}\end{smallmatrix}}|s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle =&\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}|s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle + \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}(12)|s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle\\ &+\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}(23)|s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle + \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}(13)|s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle\\ &+\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}(123)|s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle + \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}(132)|s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle\\ &= |s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle - \frac{1}{2}|s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}\rangle + |s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}\rangle - \frac{1}{2}|s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}\rangle - \frac{1}{2}|s_{\alpha_2}s_{\alpha_3}s_{\alpha_1}\rangle\\ &-\frac{1}{2}|s_{\alpha_3}s_{\alpha_1}s_{\alpha_2}\rangle. \end{aligned} \tag{683}$$

Similarly,

$$\begin{aligned}P_{\begin{smallmatrix}\boxed{3}\boxed{1}\\\boxed{2}\end{smallmatrix}}|s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle =&|s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle + \frac{1}{2}|s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}\rangle - |s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}\rangle + \frac{1}{2}|s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}\rangle - \frac{1}{2}|s_{\alpha_2}s_{\alpha_3}s_{\alpha_1}\rangle\\ &-\frac{1}{2}|s_{\alpha_3}s_{\alpha_1}s_{\alpha_2}\rangle. \end{aligned} \tag{684}$$

The next Young diagram we consider is $\square\square\square$. The space of this Young diagram representation is 1 dimensional. We let

$$\left|\,\boxed{3}\boxed{2}\boxed{1}\,\right\rangle = 1. \tag{685}$$

The irreducible matrix elements are

$$\Gamma_{\square\square\square}(12)|\boxed{3}\boxed{2}\boxed{1}\rangle = |\boxed{3}\boxed{2}\boxed{1}\rangle = 1 \tag{686}$$

$$\Gamma_{\square\square\square}(23)|\boxed{3}\boxed{2}\boxed{1}\rangle = |\boxed{3}\boxed{2}\boxed{1}\rangle = 1 \tag{687}$$

$$\Gamma_{\square\square\square}(13)|\boxed{3}\boxed{2}\boxed{1}\rangle = |\boxed{3}\boxed{2}\boxed{1}\rangle = 1 \tag{688}$$

$$\Gamma_{\square\square\square}(123)|\boxed{3}\boxed{2}\boxed{1}\rangle = |\boxed{3}\boxed{2}\boxed{1}\rangle = 1 \tag{689}$$

$$\Gamma_{\square\square\square}(132)|\boxed{3}\boxed{2}\boxed{1}\rangle = |\boxed{3}\boxed{2}\boxed{1}\rangle = 1. \tag{690}$$

Thus

$$\begin{aligned} P_{\boxed{3}\boxed{2}\boxed{1}}|s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle =& |s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle + (12)|s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle + (23)|s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle \\ &+(13)|s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle + (123)|s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle + (132)|s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle \\ =& |s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle + |s_{\alpha_2}s_{\alpha_1}s_{\alpha_3}\rangle + |s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}\rangle + |s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}\rangle \\ &+|s_{\alpha_2}s_{\alpha_3}s_{\alpha_1}\rangle + |s_{\alpha_3}s_{\alpha_1}s_{\alpha_2}\rangle. \end{aligned} \tag{691}$$

The states

$$P_{\substack{\boxed{3}\boxed{2}\\ \boxed{1}\phantom{\boxed{2}}}}|s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle, \qquad P_{\substack{\boxed{3}\boxed{1}\\ \boxed{2}\phantom{\boxed{1}}}}|s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle, \tag{692}$$

are anti-symmetric under permutation of $s_{\alpha_1}, s_{\alpha_2}$ and s_{α_3}. The state

$$P_{\boxed{3}\boxed{2}\boxed{1}}|s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle, \tag{693}$$

is symmetric under permutation of $s_{\alpha_1}, s_{\alpha_2}$ and s_{α_3}.

In order to compute the total angular momentum for the states above, we express the states explicitly

$$P_{\substack{\boxed{3}\boxed{1}\\ \boxed{2}\phantom{\boxed{1}}}}|s_1s_1s_1\rangle = 0 \tag{694}$$

$$P_{\substack{\boxed{3}\boxed{1}\\ \boxed{2}}}|s_1 s_1 s_2\rangle = \frac{3}{2}|s_1 s_1 s_2\rangle - \frac{3}{2}|s_1 s_2 s_1\rangle \tag{695}$$

$$P_{\substack{\boxed{3}\boxed{1}\\ \boxed{2}}}|s_1 s_2 s_1\rangle = \frac{3}{2}|s_1 s_2 s_1\rangle - \frac{3}{2}|s_1 s_1 s_2\rangle \tag{696}$$

$$P_{\substack{\boxed{3}\boxed{1}\\ \boxed{2}}}|s_1 s_2 s_2\rangle = 0 \tag{697}$$

$$P_{\substack{\boxed{3}\boxed{1}\\ \boxed{2}}}|s_2 s_1 s_1\rangle = 0 \tag{698}$$

$$P_{\substack{\boxed{3}\boxed{1}\\ \boxed{2}}}|s_2 s_1 s_2\rangle = \frac{3}{2}|s_2 s_1 s_2\rangle - \frac{3}{2}|s_2 s_2 s_1\rangle \tag{699}$$

$$P_{\substack{\boxed{3}\boxed{1}\\ \boxed{2}}}|s_2 s_2 s_1\rangle = \frac{3}{2}|s_2 s_2 s_1\rangle - \frac{3}{2}|s_2 s_1 s_2\rangle \tag{700}$$

$$P_{\substack{\boxed{3}\boxed{1}\\ \boxed{2}}}|s_2 s_2 s_2\rangle = 0. \tag{701}$$

Similarly,

$$P_{\substack{\boxed{3}\boxed{2}\\ \boxed{1}}}|s_1 s_1 s_1\rangle = 0, \tag{702}$$

$$P_{\substack{\boxed{3}\boxed{2}\\ \boxed{1}}}|s_1 s_1 s_2\rangle = \frac{1}{2}|s_1 s_1 s_2\rangle + \frac{1}{2}|s_1 s_2 s_1\rangle - |s_2 s_1 s_1\rangle \tag{703}$$

$$P_{\substack{\boxed{3}\boxed{2}\\ \boxed{1}}}|s_1 s_2 s_1\rangle = \frac{1}{2}|s_1 s_2 s_1\rangle - |s_2 s_1 s_1\rangle + \frac{1}{2}|s_1 s_1 s_2\rangle \tag{704}$$

$$P_{\substack{\boxed{3}\boxed{2}\\ \boxed{1}}}|s_1 s_2 s_2\rangle = 2|s_1 s_2 s_2\rangle - |s_2 s_2 s_1\rangle - |s_2 s_1 s_2\rangle \tag{705}$$

$$P_{\substack{\boxed{3}\boxed{2}\\ \boxed{1}}}|s_2 s_1 s_1\rangle = 2|s_2 s_1 s_1\rangle - |s_1 s_1 s_2\rangle - |s_1 s_2 s_1\rangle \tag{706}$$

$$P_{\substack{\boxed{3}\boxed{2}\\ \boxed{1}}}|s_2 s_1 s_2\rangle = \frac{1}{2}|s_2 s_1 s_2\rangle - |s_1 s_2 s_2\rangle + \frac{1}{2}|s_2 s_2 s_1\rangle \tag{707}$$

$$P_{\young(32,1)}|s_2s_2s_1\rangle = \frac{1}{2}|s_2s_2s_1\rangle + \frac{1}{2}|s_2s_1s_2\rangle - |s_1s_2s_2\rangle \tag{708}$$

$$P_{\young(32,1)}|s_2s_2s_2\rangle = 0. \tag{709}$$

Finally,

$$P_{\young(321)}|s_1s_1s_1\rangle = 6|s_1s_1s_1\rangle \tag{710}$$

$$P_{\young(321)}|s_1s_1s_2\rangle = 2|s_1s_1s_2\rangle + 2|s_1s_2s_1\rangle + 2|s_2s_1s_1\rangle \tag{711}$$

$$P_{\young(321)}|s_1s_2s_1\rangle = 2|s_1s_2s_1\rangle + 2|s_2s_1s_1\rangle + 2|s_1s_1s_2\rangle \tag{712}$$

$$P_{\young(321)}|s_1s_2s_2\rangle = 2|s_1s_2s_2\rangle + 2|s_2s_1s_2\rangle + 2|s_2s_2s_1\rangle \tag{713}$$

$$P_{\young(321)}|s_2s_1s_1\rangle = 2|s_1s_1s_2\rangle + 2|s_1s_2s_1\rangle + 2|s_2s_1s_1\rangle \tag{714}$$

$$P_{\young(321)}|s_2s_1s_2\rangle = 2|s_1s_2s_2\rangle + 2|s_2s_1s_2\rangle + 2|s_2s_2s_1\rangle \tag{715}$$

$$P_{\young(321)}|s_2s_2s_1\rangle = 2|s_1s_2s_2\rangle + 2|s_2s_1s_2\rangle + 2|s_2s_2s_1\rangle \tag{716}$$

$$P_{\young(321)}|s_2s_2s_2\rangle = 6|s_2s_2s_2\rangle. \tag{717}$$

We are now in a position where we can compute the total angular momentum of the states explicitly stated above. We will ignore the zero states. The total angular momentum on the first state is

$$J^2\left(P_{\young(31,2)}|s_1s_1s_2\rangle\right) = \frac{3}{2}(\frac{3}{4})(|s_1s_1s_2\rangle - |s_1s_2s_1\rangle). \tag{718}$$

The term inside the brackets is the expectation value of the total angular momentum operator. The expectation value is 3/4, which means the spin of the operator is 1/2. Looking at the other operators, we find a similar result

$$J^2\left(P_{\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}|s_1 s_2 s_1\rangle\right) = \frac{3}{2}(\frac{3}{4})(|s_1 s_2 s_1\rangle - |s_1 s_1 s_2\rangle), \tag{719}$$

$$J^2\left(P_{\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}|s_2 s_1 s_2\rangle\right) = \frac{3}{2}(\frac{3}{4})(|s_2 s_1 s_2\rangle - |s_2 s_2 s_1\rangle), \tag{720}$$

$$J^2\left(P_{\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}|s_2 s_2 s_1\rangle\right) = \frac{3}{2}(\frac{3}{4})(|s_2 s_2 s_1\rangle - |s_2 s_1 s_2\rangle). \tag{721}$$

Similarly, the following states also have spin 1/2

$$J^2\left(P_{\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}|s_1 s_1 s_2\rangle\right) = \frac{3}{4}\left(P_{\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}|s_1 s_1 s_2\rangle\right). \tag{722}$$

$$J^2\left(P_{\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}|s_1 s_2 s_1\rangle\right) = \frac{3}{4}\left(P_{\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}|s_1 s_2 s_1\rangle\right) \tag{723}$$

$$J^2\left(P_{\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}|s_2 s_1 s_2\rangle\right) = \frac{3}{4}\left(P_{\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}|s_2 s_1 s_2\rangle\right) \tag{724}$$

$$J^2\left(P_{\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}|s_1 s_1 s_2\rangle\right) = \frac{3}{4}\left(P_{\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}|s_1 s_1 s_2\rangle\right) \tag{725}$$

$$J^2\left(P_{\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}|s_1 s_2 s_2\rangle\right) = \frac{3}{4}\left(P_{\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}|s_1 s_2 s_2\rangle\right) \tag{726}$$

$$J^2\left(P_{\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}|s_2 s_1 s_1\rangle\right) = \frac{3}{4}\left(P_{\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}|s_2 s_1 s_1\rangle\right), \tag{727}$$

The last state to consider is

$$P_{\begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline\end{array}}|s_{\alpha_1} s_{\alpha_2} s_{\alpha_3}\rangle. \tag{728}$$

Acting on these states with J^2 yields

$$J^2\left(P_{\begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline\end{array}}|s_1 s_1 s_1\rangle\right) = 6(\frac{15}{4})|s_1 s_1 s_1\rangle \tag{729}$$

$$J^2\left(P_{\boxed{3}\boxed{2}\boxed{1}}|s_2s_2s_2\rangle\right) = 6(\frac{15}{4})|s_2s_2s_2\rangle \tag{730}$$

$$J^2\left(P_{\boxed{3}\boxed{2}\boxed{1}}|s_1s_1s_2\rangle\right) = 2(\frac{15}{4})(|s_1s_1s_2\rangle + |s_2s_1s_1\rangle + |s_1s_2s_1\rangle), \tag{731}$$

which implies these states have spin-3/2. All these results can be checked using Mathematica (see Appendix A.6).

Recall the fact that the $SU(2)$ representations decomposed as follows $\mathbf{1/2} \otimes \mathbf{1/2} = \mathbf{3/2} \oplus (2)\mathbf{1/2}$. Thus, we see the above projection operators project a state $|s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\rangle$ to the corresponding subspace; more precisely, the projectors $P_{\begin{smallmatrix}\square&\square\\ \square&\end{smallmatrix}}$ projected to the subspace representing spin-1/2 and $P_{\square\square\square}$ projected to the subspace representing spin-3/2.

A.4 Mathematica Sample Code: Character Formulas

To determine the spectrum of primaries, one can make use of the following code

```
(* su(2) product rule for the Xs*)
XRR := Table [
   X[j1 ] X[j2] −> Sum [X [l], {l , Abs[j1 − j2], j1 + j2}], {j1 , 0 ,
      12, 1/2}, { j2 , 0 , 12, 1/2 }] ;
FXRR := Flatten [ XRR ] ;
(* rule for the product of Ys *)
YRR := XRR /. X −> Y
TRR := Flatten [ Append [ XRR , YRR ] ]
(* remove the scalar factors *)
Zer := { X[0] −> 1 , Y[0] −> 1 , X[−1] −> 0, Y[−1] −> 0, X[−1/2] −> 0,
   Y[−1/2] −> 0}
TRRZ := Flatten [ Append [ TRR , Zer ] ]
(* check the su(2) product is working*)
Y[1/2] Y[1] X[3/2] X[3/2] //. TRRZ

(* this is chi(s^n,x^n,y^n) *)
chi [n_] := (s^n Expand[
      Sum[s^(n q) (Sum[X[q n/2 − n l], {l, 0, IntegerPart[q/2]}] −
          Sum[X[q n/2 − n l − 1], {l, 0,
            IntegerPart[(q − 1)/2]}]) (Sum[
           Y[q n/2 − n l], {l, 0, IntegerPart[q/2]}] −
          Sum[Y[q n/2 − n l − 1], {l, 0,
            IntegerPart[(q − 1)/2]}]), {q, 0, 10/n}]]) //. TRRZ;
(* This is the inverse of P *)
IP = 1 + s^4 − s (1 + s^2) X[1/2] Y[1/2] + s^2 X[1] + s^2 Y[1];
```

The spectrum for primaries built out of three fields can be determined from the code

```
(*  Sym (V^3) *)
f111[n_] :=
  Expand[Coefficient[s (1 − s^2) (chi [1]  chi [1]) ,  s^n]]  //. TRRZ;
f21[n_] := Expand[Coefficient[s (1 − s^2) (chi [2]) ,  s^n]]  //. TRRZ;
f3[n_] := Expand[Coefficient[chi[3] IP,  s^n]]  //. TRRZ;
sym3[n_] := (f111[n] + 3 f21[n] + 2 f3[n])/6 //. TRRZ;
Aa3[n_] := (f111[n] − 3 f21[n] + 2 f3[n])/6 //. TRRZ;
M3[n_] := (2 f111[n] − 2 f3[n])/3 //. TRRZ;

(* Here, we calculate to order s^7 *)
n = 7;
Expand[Simplify[f111[n]]]
Expand[Simplify[f21[n]]]
Expand[Simplify[f3[n]]]
Expand[Simplify[sym3[n]]]
```

Finally, we can implement the generating function using the following code

```
(* ns is for Sym(V^n) and nd sets the dimension *)
ns = 3; nd = 8;
(* build 1/det() *)
G = Product[
   Product[Product[1 − t s^(q + 1) x^a y^b, {a, −q/2, q/2}], {b, −q/2,
      q/2}], {q, 0, nd − ns}];
(* build generating function *)
H = Series[
   Coefficient[
    Series[(1 − 1/x) (1 − 1/y) (1 − s Sqrt[x y]) (1 −
        s Sqrt[x/y]) (1 − s Sqrt[y/x]) (1 − s/Sqrt[x y])/G, {t, 0,
      ns}], t^ns], {s, 0, nd}];
(* extract positive powers *)
K = Sum[Coefficient[Series[X (H /. {x −> X^2, y −> Y^2}), {X, 0, 20}],
      X^n] X^(n − 1), {n, 1, 20}];
Expand[Sum[
   Coefficient[Series[Y K, {Y, 0, 20}], Y^n] Y^(n − 1), {n, 1,
    20}] /. {X −> Sqrt[x], Y −> Sqrt[y]}]
```

A.5 Mathematica Sample Code: Gegenbauer Polynomials

To determine the Gegenbauer polynomials, we used the following input code

```
  f := 1/(1 − 2 x*t + t^{2})^(\[Omega]);
k = 4; (* since the summation starts from 0, this determines the \
 first 5 polynomials *)
(* In this loop, we use the formula determined earlier in this \
 dissertation − note, here we added the term t^{n} *)
```

```
For[n = 0, n < k + 1, n++,
 Print[(1/(n!))*(t^{n})*(D[f, {t, n}] /. t -> 0)]]
```

and thus received the following output

```
{1}
{2 t x \[Omega]}
{1/2 t^2 (-2 \[Omega]-4 x^2 (-1-\[Omega]) \[Omega])}
{1/6 t^3 (12 x (-1-\[Omega]) \[Omega]+8 x^3 (-2-\[Omega]) (-1-\[Omega]) \[Omega])}
{1/24 t^4 (-12 (-1-\[Omega]) \[Omega]-48 x^2 (-2-\[Omega]) (-1-\[Omega]) \[Omega]-16 x^4
    (-3-\[Omega]) (-2-\[Omega]) (-1-\[Omega]) \[Omega])}
```

Finally, we tested if these polynomials do indeed correspond to primaries. As an example, we used the following code

```
(* By using the substitution x = \[Beta]/\[Alpha], \[Omega] = d/2 - \
3/2, and t = \[Alpha], we obtain the polynomials for our case. To \
test if these polynomials do lead to primary operators, we use the \
differential operator derived earlier in this dissertation . As an \
example, consider the 4th polynomial above *)
g = 1/6 t^3 (12 x (-1 - \[Omega]) \[Omega] +
     8 x^3 (-2 - \[Omega]) (-1 - \[Omega]) \[Omega]);
x = \[Beta]/\[Alpha];
t = \[Alpha];
\[Omega] = d/2 - 3/2;
d*D[g, \[Alpha]] - 2 D[g, \[Alpha]] + \[Alpha]*D[g, {\[Alpha], 2}] +
 2 \[Beta]*D[g, \[Alpha], \[Beta]] + \[Alpha]*D[g, {\[Beta], 2}]
```

Note, that the differential operator we are referring to is equation 421.

A.6 Mathematica Sample Code: Total Angular Momentum Operator

```
(* Basis vectors for 3 spin=1/2 particles *)
bas = {{1, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0}, {0, 0, 1,
    0, 0, 0, 0, 0}, {0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, 1, 0, 0,
    0}, {0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0,
    0, 0, 0, 0, 1}};

s1s1s1 = bas [[1]];
s2s2s2 = bas [[8]];
s1s1s2 = bas [[2]];
s1s2s1 = bas [[3]];
s1s2s2 = bas [[4]];
s2s1s1 = bas [[5]];
s2s1s2 = bas [[6]];
s2s2s1 = bas [[7]];
```

```
(* Next, we build the total spin operator's matrix *)
ident = {{1, 0}, {0, 1}};
j1 = {1/2 {{0, 1}, {1, 0}}, I/2 {{0, -1}, {1, 0}},
   1/2 {{1, 0}, {0, -1}}};
j2 = {1/2 {{0, 1}, {1, 0}}, I/2 {{0, -1}, {1, 0}},
   1/2 {{1, 0}, {0, -1}}};
j3 = {1/2 {{0, 1}, {1, 0}}, I/2 {{0, -1}, {1, 0}},
   1/2 {{1, 0}, {0, -1}}};

j13 = Table[
   KroneckerProduct[KroneckerProduct[j1[[a]], ident], ident], {a, 1,
    3}];
j23 = Table[
   KroneckerProduct[KroneckerProduct[ident, j2[[a]]], ident], {a, 1,
    3}];
j33 = Table[
   KroneckerProduct[KroneckerProduct[ident, ident], j3[[a]]], {a, 1,
    3}];

tsop = ((9/4)*
     KroneckerProduct[KroneckerProduct[ident, ident], ident] +
    2*(j13[[1]].j23[[1]] + j13[[2]].j23[[2]] + j13[[3]].j23[[3]]) +
    2*(j13[[1]].j33[[1]] + j13[[2]].j33[[2]] + j13[[3]].j33[[3]]) +
    2*(j23[[1]].j33[[1]] + j23[[2]].j33[[2]] + j23[[3]].j33[[3]]));
```

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